Ph.D. Thesis in Mathematical Analysis

SISSA - International School for Advanced Studies

Area of Mathematics

Positive solutions to indefinite problems: a topological approach

Candidate:
Guglielmo Feltrin

Supervisor:
Prof. Fabio Zanolin

Academic Year 2015–2016

To my parents Mario and Tiziana
and my brother Ludovico

Declaration

Il presente lavoro costituisce la tesi presentata da Guglielmo Feltrin, sotto la direzione del Prof. Fabio Zanolin, al fine di ottenere l'attestato di ricerca post-universitaria Doctor Philosophiæ presso la Scuola Internazionale Superiore di Studi Avanzati di Trieste (SISSA), Curriculum in Analisi Matematica, Area di Matematica. Ai sensi dell'art. 1, comma 4, dello Statuto della SISSA pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di Dottore di Ricerca in Matematica.

Trieste, Anno Accademico 2015–2016.

Abstract

The present Ph.D. thesis is devoted to the study of positive solutions to indefinite problems. In particular, we deal with the second order nonlinear differential equation

$$u'' + a(t)g(u) = 0,$$

where $g\colon [0,+\infty[\, \to [0,+\infty[$ is a continuous nonlinearity and $a\colon [0,T] \to \mathbb{R}$ is a Lebesgue integrable sign-changing weight. We analyze the Dirichlet, Neumann and periodic boundary value problems on $[0,T]$ associated with the equation and we provide existence, nonexistence and multiplicity results for positive solutions.

In the first part of the manuscript, we investigate nonlinearities $g(u)$ with a superlinear growth at zero and at infinity (including the classical superlinear case $g(u) = u^p$, with $p > 1$). In particular, we prove that there exist $2^m - 1$ positive solutions when $a(t)$ has m positive humps separated by negative ones and the negative part of $a(t)$ is sufficiently large. Then, for the Dirichlet problem, we solve a conjecture by Gómez-Reñasco and López-Gómez (JDE, 2000) and, for the periodic problem, we give a complete answer to a question raised by Butler (JDE, 1976).

In the second part, we study the super-sublinear case (i.e. $g(u)$ is superlinear at zero and sublinear at infinity). If $a(t)$ has m positive humps separated by negative ones, we obtain the existence of $3^m - 1$ positive solutions of the boundary value problems associated with the parameter-dependent equation $u'' + \lambda a(t)g(u) = 0$, when both $\lambda > 0$ and the negative part of $a(t)$ are sufficiently large.

We propose a new approach based on topological degree theory for locally compact operators on open possibly unbounded sets, which applies for Dirichlet, Neumann and periodic boundary conditions. As a byproduct of our method, we obtain infinitely many subharmonic solutions and globally defined positive solutions with complex behavior, and we deal with chaotic dynamics. Moreover, we study positive radially symmetric solutions to the Dirichlet and Neumann problems associated with elliptic PDEs on annular domains. Furthermore, this innovative technique has the potential and the generality needed to deal with indefinite problems with more general differential operators. Indeed, our approach apply also for the non-Hamiltonian equation $u'' + cu' + a(t)g(u) = 0$. Meanwhile, more general operators in the one-dimensional case and problems involving PDEs will be subjects of future investigations.

Publications

- G. Feltrin, F. Zanolin, Multiple positive solutions for a superlinear problem: a topological approach, J. Differential Equations 259 (2015) 925–963. Referred to as [84].

- G. Feltrin, F. Zanolin, Existence of positive solutions in the superlinear case via coincidence degree: the Neumann and the periodic boundary value problems, Adv. Differential Equations 20 (2015) 937–982. Referred to as [83].

- G. Feltrin, Existence of positive solutions of a superlinear boundary value problem with indefinite weight, Discrete Contin. Dyn. Syst. (Dynamical systems, differential equations and applications. 10th AIMS Conference. Suppl.) (2015) 436–445. Referred to as [80].

- A. Boscaggin, G. Feltrin, F. Zanolin, Pairs of positive periodic solutions of nonlinear ODEs with indefinite weight: a topological degree approach for the super-sublinear case, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016) 449–474. Referred to as [31].

- G. Feltrin, A note on a fixed point theorem on topological cylinders, submitted. Referred to as [79].

- G. Feltrin, F. Zanolin, Multiplicity of positive periodic solutions in the superlinear indefinite case via coincidence degree, submitted. Referred to as [82].

- A. Boscaggin, G. Feltrin, F. Zanolin, Positive solutions for super-sublinear indefinite problems: high multiplicity results via coincidence degree, Trans. Amer. Math. Soc., to appear. Referred to as [32].

- A. Boscaggin, G. Feltrin, Positive subharmonic solutions to nonlinear ODEs with indefinite weight, submitted. Referred to as [30].

- G. Feltrin, Multiple positive solutions of a Sturm-Liouville boundary value problem with conflicting nonlinearities, submitted. Referred to as [78].

- G. Feltrin, F. Zanolin, An application of coincidence degree theory to cyclic feedback type systems associated with nonlinear differential operators, submitted. Referred to as [81].

The papers are listed in order of submission.

Contents

Introduction

In the present manuscript we present some recent existence and multiplicity results for positive solutions to boundary value problems associated with second order nonlinear indefinite differential equations, obtained during my Ph.D. studies in [30, 31, 32, 78, 79, 80, 81, 82, 83, 84].

More precisely, we deal with the ordinary differential equation

$$u'' + q(t)g(u) = 0, \tag{I-1}$$

where $g\colon [0,+\infty[\, \to [0,+\infty[$ is a continuous nonlinearity and $q\colon [0,T] \to \mathbb{R}$ is a Lebesgue integrable sign-changing weight.

The terminology "indefinite", meaning that $q(t)$ is of non-constant sign, was probably introduced in [11] dealing with a linear eigenvalue problem and with [107] it has become very popular also in nonlinear differential problems, especially when $g(u)$ is a superlinear function (i.e. $g(u) \sim u^p$ with $p > 1$).

Our main goal is to find positive solutions (in the Carathéodory sense) of the Dirichlet, Neumann and periodic boundary value problems associated with an equation of the form (I-1).

The study of indefinite ODEs was initiated in 1965 by Waltman (see [179]), considering oscillatory solutions for

$$u'' + q(t)u^{2n+1} = 0, \quad n \geq 1,$$

and starting from the Eighties (see [14, 107]) a great deal of attention has been devoted to the investigation of nonlinear boundary value problems with a sign-indefinite weight, especially in connection to partial differential equations of the form

$$-\Delta u = q(x)g(u). \tag{I-2}$$

It is worth noting that equations of this type arise in many models concerning mathematical ecology, differential geometry and mathematical physics, for which only non-negative solutions make sense. However, the investigation of these problems has its own interest from the point of view of the application of the methods of nonlinear analysis to ordinary differential equations, partial differential equations, and dynamical systems.

A strong motivation also comes from the search of stationary solutions of parabolic equations arising in different contexts, such as population dynamics and reaction-diffusion processes (see, for instance, [1] for a recent survey in this direction).

As an example, in [120, 143] (see also [106, ch. 10]) the authors introduced the following model for two types of genes (alleles) A_1, A_2

$$\begin{cases} u_t = d\Delta u + q(x)u^2(1-u) & \text{in } \Omega \times]0,+\infty[\\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times]0,+\infty[, \end{cases} \tag{I-3}$$

where $\Omega \subseteq \mathbb{R}^N$ and $q(x)$ is a sign-changing weight. The function $u(t,x)$ represents the frequency of the allele A_1 at time t and place x in the habitat Ω (therefore one have to assume $0 \leq u \leq 1$). The term Δu is the effect of population dispersal, $d > 0$ is the ratio of the migration rate to the intensity of selection, and the Neumann boundary condition means that there is no flux of genes across the boundary $\partial\Omega$ (see [143] for more details). In order to investigate the steady-state (stationary) solutions of (I-3) we have to deal with positive solutions of the Neumann boundary value problem

$$\begin{cases} d\Delta u + q(x)u^2(1-u) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

which evidently involves an equation of the form (I-2).

Another possible example in this direction is given by differential equations involving Ricker's nonlinearity

$$g(u) := e^{k(1-u)}.$$

This function plays an important role in models for population dynamics of fisheries, and more precisely in the study of the salmon's proliferation (see, for instance, [160]).

Superlinear problems: Dirichlet boundary conditions

We start our investigations from nonlinearities $g(s)$ which have a superlinear growth at zero and at infinity. Boundary value problems of this form are usually named of *superlinear indefinite* type (cf. [19]).

In the past twenty years a great deal of existence and multiplicity results have been reached in this context, mainly with respect to Dirichlet problems of the form

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{I-4}$$

For instance, we mention [4, 7, 19, 20] and also [153, 154] for a more complete list of references concerning different aspects related to the study of superlinear indefinite problems, including the case of non-positive oscillating solutions.

Typically, the right-hand side of (I-4) takes the form

$$f(x,s) = \lambda s + q(x)g(s),$$

with λ a real parameter. In some cases also the weight function $q(x)$ depends on a parameter which plays the role of strengthening or weakening the positive (or negative) part of the coefficient $q(x)$ (see [14, 118]). Accordingly, for the one-dimensional case, sometimes $q(t)$ is expressed as depending on a parameter $\mu > 0$ in this manner:

$$q(t) = a_\mu(t) := a^+(t) - \mu a^-(t), \tag{I-5}$$

where $a\colon [0,T] \to \mathbb{R}$ and, as usual, $a^+(t)$ and $a^-(t)$ are the positive part and the negative part of $a(t)$, respectively.

The starting point for our investigation is [92]. In this paper, Gaudenzi, Habets and Zanolin proved the existence of at least three positive solutions for the two-point boundary value problem associated with

$$u'' + a_\mu(t)u^p = 0, \quad p > 1, \tag{I-6}$$

when $a_\mu(t)$ has two positive humps separated by a negative one, provided that $\mu > 0$ is sufficiently large. Their technique of proof, based on the shooting method, has been generalized in [94] in order to provide the existence of seven positive solutions for (I-6) (for the Dirichlet problem and with μ large) when $a(t)$ has three positive humps separated by two negative ones. Generally speaking, this fact suggests the existence of $2^m - 1$ positive solutions (for μ large) when the weight function exhibits m positive humps separated by $m-1$ negative ones. It is interesting to observe that already in [100], for the one-dimensional case and

$$f(t,s) = \lambda s + a(t)u^p, \quad p > 1,$$

Gómez-Reñasco and López-Gómez conjectured this result (for $\lambda < 0$ sufficiently small) based on some numerical evidence.

More recently, Bonheure, Gomes and Habets in [27] extended the multiplicity theorem in [94] to the PDEs setting and they obtained a result about $2^m - 1$ positive solutions for the problem

$$\begin{cases} -\Delta u = q(x)u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{I-7}$$

using a variational technique. In this context, $\Omega \subseteq \mathbb{R}^N$ is an open bounded domain of class $\mathcal{C}^1$ and $1 < p < (N+2)/(N-2)$ if $N \geq 3$. Also for (I-7) the multiplicity result holds for $q(x) = a_\mu(x)$ (as in (I-5)) with $\mu > 0$ sufficiently large. Further progresses in this direction have been achieved in [98, 99].

On the other hand, if we have a positive weight function $a(x)$, it is known that the existence of at least one positive solution to

$$\begin{cases} -\Delta u = a(x)g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{I-8}$$

is guaranteed for a general class of functions $g(s)$ (including $g(s) = s^p$, with $p > 1$, as a particular case). Indeed, for $f(x,s) = a(x)g(s)$, the superlinear conditions at zero and at infinity can be generalized to suitable hypotheses of crossing the first eigenvalue (see [9, 146], where results in this direction were obtained using a variational and a topological approach, respectively). For instance, if $a(x) \equiv 1$, existence theorems of positive solutions can be obtained, as in [43, 65], provided that

$$\limsup_{s\to 0^+} \frac{g(s)}{s} < \lambda_1 < \liminf_{s\to+\infty} \frac{g(s)}{s}$$

(further technical assumptions on $g(s)$ and on the domain Ω can be required for $N > 1$). In [63, ch. 3], De Figueiredo obtained sharp existence results of positive solutions for (I-4) by assuming

$$f(x,0) \equiv 0, \quad \lim_{s\to 0^+} \frac{f(x,s)}{s} = m_0(x), \quad \lim_{s\to+\infty} \frac{f(x,s)}{s} = m_\infty(x).$$

The assumption of crossing the first eigenvalue is expressed by a hypothesis of the form $\mu_1(m_0) > 1 > \mu_1(m_\infty)$, where $\mu_1(m)$ is the first eigenvalue of $-\Delta u = \lambda m(x)u$. The different conditions at zero and at infinity imply a change of the value of the fixed point index (for an associated operator) between small and large balls in the cone of positive functions of a suitable Banach space X. Hence the existence of a nontrivial fixed point is guaranteed by the non-zero index (or degree) on some open set $D \subseteq X$, with $0 \notin D$.

In the ODEs case various technical growth conditions on the nonlinearity can be avoided. Existence theorems of positive solutions have been obtained in [76] for the superlinear case and in [18, 113, 124, 144] for "crossing the first eigenvalue" type conditions. In this direction, an existence result has been produced in [93] for the one-dimensional version of (I-8). In this case the weight function has non-constant sign and may vanish on some subintervals of $[0,T]$. It is also assumed that the set where $a(t) > 0$ is the union of m pairwise disjoint intervals J_i. Using an approach based on the theory of not well-ordered upper and lower-solutions, the existence of a positive solution is guaranteed provided that

$$\limsup_{s\to 0^+} \frac{g(s)}{s} < \lambda_0 \quad \text{and} \quad \liminf_{s\to +\infty} \frac{g(s)}{s} > \max_{i=1,\dots,m} \lambda_1^i, \tag{I-9}$$

where λ_0 is the first eigenvalue of $\varphi'' + \lambda a^+(t)\varphi = 0$ on $[0, T]$ and λ_1^i is the first eigenvalue of $\varphi'' + \lambda a^+(t)\varphi = 0$ on J_i.

A question, which naturally arises by a comparison between the above recalled existence theorem and the multiplicity results in [27, 94], concerns the possibility of producing a theorem on the existence of $2^m - 1$ positive solutions when $a(t) = a_\mu(t)$ is positive on m intervals separated by $m - 1$ intervals of negativity and $g(s)$ satisfies a condition like (I-9).

A first goal of this thesis is to provide an affirmative answer to this question and thus a solution for the conjecture by Gómez-Reñasco and López-Gómez (see also the final remarks of this introduction). In this manner, we extend [93] and [94] at the same time and, moreover, we are able to prove that the multiplicity results in [94] hold for a broad class of nonlinearities which include $g(s) = s^p$, with $p > 1$, as a special case. More precisely, the following result holds.

Theorem. *Suppose that $a\colon [0, T] \to \mathbb{R}$ is a continuous function and there are $2m$ points*

$$0 = \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \ldots < \sigma_m < \tau_m = T,$$

such that $\tau_1, \sigma_2, \tau_2, \dots, \sigma_m$ are simple zeros of $a(t)$ and, moreover,

- *$a(t) \geq 0$, for all $t \in [\sigma_i, \tau_i]$, $i = 1, \dots, m$;*
- *$a(t) \leq 0$, for all $t \in [\tau_i, \sigma_{i+1}]$, $i = 1, \dots, m - 1$.*

Assume that $g\colon [0, +\infty[\, \to [0, +\infty[$ is a continuous function with $g(0) = 0$, $g(s) > 0$ for $s > 0$ and, moreover, satisfying (I-9). *Then there exists $\mu^* > 0$ such that, for each $\mu > \mu^*$, problem*

$$\begin{cases} u'' + a_\mu(t)g(u) = 0 \\ u(0) = u(T) = 0 \end{cases} \tag{I-10}$$

has at least $2^m - 1$ positive solutions.

The assumptions on the sign of $a(t)$ do not prevent the possibility that $a(t)$ is identically zero on some subintervals of $[\sigma_i, \tau_i]$ or $[\tau_i, \sigma_{i+1}]$. The hypothesis that $\tau_1, \sigma_2, \tau_2, \dots, \sigma_m$ are simple zeros of $a(t)$ is considered here only in order to provide a simpler statement of our theorem and, indeed, this condition can be significantly relaxed.

Figure 1 adds a graphical explanation to our result in a case in which the weight function has two positive humps separated by a negative one.

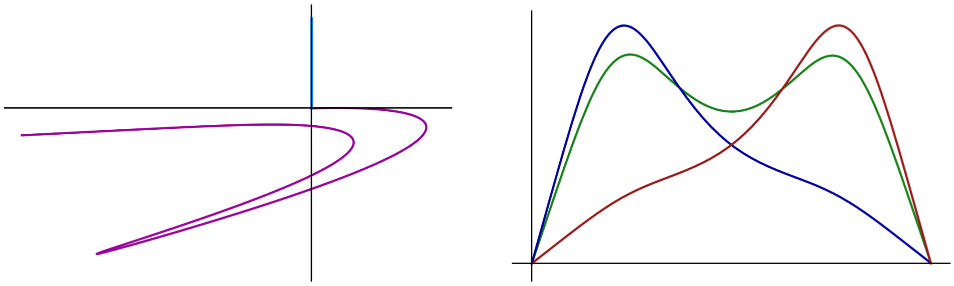

Figure 1: An example of three positive solutions for problem (I-10) on $[0, 1]$. For this numerical simulation we have chosen $a(t) = \sin(3\pi t)$, $\mu = 0.5$ and $g(s) = \max\{0, 100\, s \arctan |s|\}$. On the left we have shown the image of the segment $\{0\} \times [0, 4.4]$ through the Poincaré map in the phase-plane (u, u'). It intersects the negative part of the u'-axis in three points. This means that there are three positive initial slopes from which depart three solutions at $t = 0$ which are positive on $]0, 1[$ and vanish at $t = 1$. On the right we have shown the graphs of these three solutions.

We stress that in our example $g(s)/s \not\to +\infty$ as $s \to +\infty$, so that it shows a case of applicability of our theorem which is not contained in [27, 92, 94].

To prove our result we use a different approach with respect to [94] and [27], where a shooting method and a variational technique were employed, respectively. Indeed, our proof is based on topological degree theory and is in the frame of the classical approach for the search of fixed points founded on the fixed point index for positive operators. First, we write (I-10) as an equivalent fixed point problem in the Banach space $\mathcal{C}([0, T])$ of the continuous functions defined on $[0, T]$. Then, using the additivity/excision property of the Leray-Schauder degree, we localize nontrivial fixed points on suitable open domains of $\mathcal{C}([0, T])$. In general, our open domains are unbounded and therefore we apply an extension of the degree theory for locally compact operators (cf. [147, 148]).

Superlinear problems: Neumann and periodic boundary conditions

A second natural topic is the study of superlinear indefinite problems involving different boundary conditions, as, for instance, the Neumann and the periodic ones.

Among the several authors who have continued the line of research initiated in [179], for the periodic problem, we recall the relevant contributions of Butler [49], Terracini and Verzini [175], and Papini [152], who proved the existence of periodic solutions with a large number of zeros to superlinear indefinite equations of the form (I-1). The presence of chaotic dynamics for superlinear indefinite ODEs was discovered in [175] in the study of

$$u'' + Ku + q(t)u^3 = h(t)$$

(with the constant K and the function $h(t)$ possibly equal to zero). In this framework we also mention [52], where a more general case of (I-1), adding the friction term cu', has been considered (see [154, § 1] for a brief historical survey about this subject). A typical feature of these results lies on the fact that the solutions which have been obtained therein have a large number of zeros in the intervals where $q(t) > 0$. This fact was explicitly observed also by Butler in the proof of [49, Corollary], where the author pointed out that the equation

$$u'' + q(t)|u|^{p-1}u = 0, \quad p > 1,$$

has infinitely many T-periodic solutions, assuming that $q(t)$ is a continuous T-periodic function with only isolated zeros and which is somewhere positive. It was also noted that all these solutions oscillate (have arbitrarily large zeros) if $\int_0^T q(t)\,dt \geq 0$. Since condition

$$\int_0^T q(t)\,dt < 0 \tag{I-11}$$

implies the existence of non-oscillatory solutions (cf. [48]), it was raised the question (see [49, p. 477]) whether there can exist non-oscillatory periodic solutions if (I-11) holds.

A second aim of the thesis is to provided a solution to Butler's open question, by showing the existence of positive (i.e. non-oscillatory) T-periodic solutions under the average condition (I-11).

Accordingly, we study the second order nonlinear boundary value problem on $[0,T]$

$$\begin{cases} u'' + a(t)g(u) = 0 \\ \mathscr{B}(u,u') = \underline{0}. \end{cases} \tag{I-12}$$

As linear boundary operator we take

$$\mathscr{B}(u,u') = (u'(0), u'(T)) \quad \text{ or } \quad \mathscr{B}(u,u') = (u(T)-u(0), u'(T)-u'(0)),$$

namely we consider the Neumann or the periodic boundary conditions.

A feature to take into account concerns the fact that we have to impose additional conditions on the weigth function, with respect to Dirichlet problems. Indeed, assuming that $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $g(0) = 0$ and $g(s) > 0$ for $s > 0$, two conditions turn out to be necessary for the existence of positive solutions. First, integrating the equation, it is easy to notice that $a(t)$ must change its sign. Second, assuming that $g(s)$ is continuously differentiable with $g'(s) > 0$ for $s > 0$, dividing the equation by $g(u)$ and integrating, we obtain that

$$\int_0^T a(t)\,dt < 0$$

(as observed in [15], in the context of the Neumann problem, and also in [36]).

Accordingly, our existence theorem reads as follows.

Theorem. *Let* $a\colon [0,T] \to \mathbb{R}$ *be a continuous function such that* $a(t_0) > 0$ *for some* $t_0 \in [0,T]$. *Assume that* $\{t \in [0,T]\colon a(t) > 0\} = \bigcup_{i=1}^{k} J_i$, *where the sets* J_i *are pairwise disjoint intervals. Let* $g\colon [0,+\infty[\, \to [0,+\infty[$ *be a smooth function with* $g(0) = 0$, $g(s) > 0$ *for* $s > 0$ *and, moreover, such that*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad \liminf_{s\to+\infty} \frac{g(s)}{s} > \max_{i=1,\ldots,k} \lambda_1^i$$

(with λ_1^i *as above). Then there is at least a positive solution of* (I-12) *provided that* $\int_0^T a(t)\,dt < 0$.

The condition on the average of $a(t)$ is new with respect to the Dirichlet case, but, as already observed, in relation to the Neumann and the periodic boundary conditions it becomes necessary in a way. Moreover, still in comparison with our result for the Dirichlet problem, we note also that under our hypotheses the nonlinearity is pushed below the principal eigenvalue $k_0 = 0$ of the Neumann/periodic problem at zero.

We underline that the hypothesis of smoothness for $g(s)$ has been chosen only to simplify the presentation and it can be improved by requiring $g(s)$ continuous on $[0,+\infty[$ and continuously differentiable on a right neighborhood of $s = 0$, or $g(s)$ continuous on $[0,+\infty[$ and regularly oscillating at zero.

Pursuing the line of research initiated by Gómez-Reñasco and López-Gómez in [100], in analogy with the celebrated papers by Dancer [59, 60] providing multiplicity of solutions to elliptic BVPs by playing with the shape of the domain, it is natural to consider the Neumann and periodic problems associated with

$$u'' + a_\mu(t)g(u) = 0 \tag{I-13}$$

and to conjecture (similarly to the Dirichlet problem) the existence of at least $2^m - 1$ positive solutions (for $\mu > 0$ large) when $a_\mu(t)$ has m positive humps separated by negative ones.

A first contribution in this direction, for the Neumann problem, was given by Boscaggin in [28], where he apply a shooting method to prove the existence of at least three positive solutions to (I-6) when $\mu > 0$ is sufficiently large, as done in [92] for the Dirichlet problem.

At the best of our knowledge, the first work addressing the same questions in the periodic setting is due to Barutello, Boscaggin and Verzini, who in [17], using a variational approach, achieved multiplicity of positive periodic solutions for

$$u'' + a_\mu(t)u^3 = 0.$$

The following theorem describe our multiplicity result which can be seen as a further investigation about Butler's problem, in the sense that we show that, if the weight is negative enough, multiple positive periodic solutions appear (depending on the number of positive humps in the weight function which are separated by negative ones). Figure 2 shows a possible example for the Neumann problem.

Theorem. *Suppose that* $a\colon [0,T] \to \mathbb{R}$ *is a continuous function and there are* $2m+1$ *points*

$$0 = \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \ldots < \sigma_m < \tau_m < \sigma_{m+1} = T,$$

such that $\tau_1, \sigma_2, \tau_2, \ldots, \sigma_m, \tau_m$ *are simple zeros of* $a(t)$ *and, moreover,*

- $a(t) \geq 0$*, for all* $t \in [\sigma_i, \tau_i]$*,* $i = 1, \ldots, m$*;*
- $a(t) \leq 0$*, for all* $t \in [\tau_i, \sigma_{i+1}]$*,* $i = 1, \ldots, m$*.*

Assume that $g\colon [0,+\infty[\to [0,+\infty[$ *is a continuous function with* $g(0) = 0$*,* $g(s) > 0$ *for* $s > 0$ *and, moreover, such that*

$$\lim_{s \to 0^+} \frac{g(s)}{s} = 0 \qquad \textit{and} \qquad \liminf_{s \to +\infty} \frac{g(s)}{s} > \max_{i=1,\ldots,m} \lambda_1^i$$

(with λ_1^i *as above). Then there exists* $\mu^* > 0$ *such that for each* $\mu > \mu^*$ *the periodic (or Neumann) boundary value problem associated with* (I-13) *has at least* $2^m - 1$ *positive solutions.*

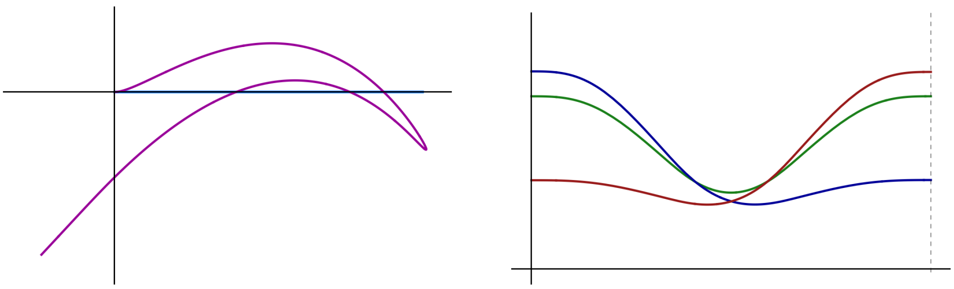

Figure 2: An example of three positive solutions for equation (I-13) with Neumann boundary conditions on $[0,1]$. For this numerical simulation we have chosen $a(t) = \sin(3\pi t)$, $\mu = 7$ and $g(s) = \max\{0, 100\, s \arctan|s|\}$. On the left we have shown the image of the segment $[0, 0.15] \times \{0\}$ through the Poincaré map in the phase-plane (u, u'). It intersects the positive part of the u-axis in three points. On the right-hand side of the figure, we see the graphs of the three positive solutions.

We notice that in the above theorem we do not impose that the mean value of $a(t)$ is negative, since this condition is implicitly assumed taking

$$\mu^* > \frac{\int_0^T a^+(t)\, dt}{\int_0^T a^-(t)\, dt}.$$

Moreover, using again the fact that we can choose μ^* sufficiently large, we avoid any additional regularity conditions on $g(s)$ (such as those needed in our existence theorem or when employing variational methods, as in [17]). It is interesting to observe that increasing the value of μ yields both abundance of solutions and no extra assumptions on $g(s)$.

For Dirichlet boundary conditions, thanks to the fact that the operator $u \mapsto -u''$ is invertible, we can write (I-10) as an equivalent fixed point problem in a suitable Banach space and apply directly some degree theoretical arguments, as previously explained. With respect to periodic and Neumann problems, the linear differential operator $u \mapsto -u''$ has a nontrivial kernel made up of the constant functions. In such a situation the operator is not invertible and we cannot proceed in the same manner as described above. A possibility, already exploited in [19], is that of perturbing the linear differential operator to a new one which can be inverted and then recover the original equation via a limiting process and some careful estimates on the solutions. In our case, we find it very useful to apply the coincidence degree theory developed by J. Mawhin (cf. [130]), which allows to study equations of the form $Lu = Nu$, where L is a linear operator with nontrivial kernel and N is a nonlinear one. The use of the coincidence degree in the search for positive (periodic) solutions is a widely used technique. For instance, in [90, 163] a coincidence theory on positive cones was initiated and developed, with applications to the search of nontrivial non-negative periodic solutions. Our approach, however, is different and uses the classical technique of extending the nonlinearity on the negative real numbers and, subsequently, proving that the nontrivial solutions are positive, via a maximum principle. In fact, our proof combines the approach that we have introduced dealing with the superlinear Dirichlet problem with Mawhin's coincidence degree theory.

Focusing the attention on the T-periodic problem, we observe that every T-periodic coefficient $a\colon \mathbb{R} \to \mathbb{R}$ can be though as a kT-periodic function, for any integer $k \geq 2$. Then, applying our multiplicity theorem to the interval $[0, kT]$, we obtain $2^{mk} - 1$ positive kT-periodic solutions of (I-13) if the negative part of the weight is sufficiently strong (and $a(t)$ has m positive humps in a T-periodicity interval). In this context, a typical problem which occurs is that of proving the minimality of the period, that is, to ensure the presence of subharmonic solutions of order k.

An important feature of our multiplicity result is that all the constants appearing in the statement and in the proof (in particular μ^*) can be explicitly estimated (depending on $g(s)$ and $a(t)$). In particular, it turns out that, whenever the theorem is applied to an interval of the form $[0, kT]$, these constants can be chosen independently on k. Therefore, our topological approach allow us to detect infinitely many subharmonic solutions if $\mu > \mu^*$. Moreover, taking advantage of the combinatorial concept of Lyndon words, we can also give a precise estimate on the number of subharmonics of a given order. Finally, via a diagonal argument, we are able to produce globally bounded solutions defined on the real line, not necessarily periodic and exhibiting a chaotic behavior. Similar results are obtained also in [17].

In this framework, it appears a quite natural question if the sharp mean value assumption $\int_0^T a(t)\,dt < 0$ (without conditions on $\mu > 0$), besides implying the existence of a positive T-periodic solution to (I-13), also guarantees the existence of positive subharmonic solutions. Combining our topological approach with an application of the Poincaré-Birkhoff theorem (cf. [35, 39]) and an estimate of the Morse index of the positive T-periodic solutions of (I-13), we are able to give an affirmative answer to this question.

Super-sublinear problems

A third natural topic is the investigation of indefinite equations involving nonlinearities $g(s)$ satisfying different growth conditions at zero and at infinity.

It is worth noting that, if $g(s)$ is concave with a *sublinear* growth at infinity, there exists a unique classical positive solution for problem (I-8) (cf. [44, 172]). A simple way to avoid the concavity of the function $g(s)$ is to impose a superlinear growth at zero. Hence, as a further investigation, we deal with the *super-sublinear case*, namely with nonlinearities $g(s)$ such that

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to +\infty} \frac{g(s)}{s} = 0. \tag{I-14}$$

Results in this direction have already appeared in the literature. When (I-14) is considered together with Dirichlet boundary conditions $u(0) = u(T) = 0$, for instance, it is well known that two positive solutions to

$$u'' + \lambda a(t) g(u) = 0 \tag{I-15}$$

exist if $\lambda > 0$ is large enough. This is a classical result, on a line of research initiated by Rabinowitz in [158] (dealing with the Dirichlet problem associated with a super-sublinear elliptic PDE on a bounded domain, see also [6] for previous related results) and later developed by many authors. Actually, typical versions of this theorem do not take into account an indefinite weight function (that is, they are stated for $a^- \equiv 0$), but nowadays standard tools (such as critical point theory, fixed point theorems for operators on cones, dynamical systems techniques) permit to successfully handle also this more general situation. We refer to [39] for the precise statement in the indefinite setting as well as to the introduction in [36] for a more complete presentation and bibliography on the subject.

Concerning the periodic problem, analogous results on pairs of positive solutions have been provided in [101] for equations of the form

$$u'' - ku + \lambda a(t) g(u) = 0,$$

with $k > 0$. However, less results seem to be available when $k = 0$. Indeed, dealing with the periodic problem associated with equation (I-15), we notice that the differential operator has a nontrivial kernel and that additional conditions on the weight function $a(t)$ are necessary for the existence of positive solutions (as observed in the superlinear case). These two facts make it unclear how to apply the methods based on the theory of positive operators for cones in Banach spaces.

A first contribution in the framework of (indefinite) periodic problems associated with (I-15) was obtained in [36]. More precisely, taking advantage of the variational (Hamiltonian) structure of the equation, critical point theory for the action functional

$$J_\lambda(u) := \int_0^T \left[\frac{1}{2}(u')^2 - \lambda a(t) G(u) \right] dt$$

was used to prove the existence of at least two positive T-periodic solutions for (I-15), with λ positive and large, by assuming $a^+ \not\equiv 0$ on some interval and $\int_0^T a(t)\,dt < 0$. Roughly speaking, the negative mean value guarantees both that the functional J_λ is coercive and bounded from below and that the origin is a strict local minimum. When $\lambda > 0$ is sufficiently large (so that $\inf J_\lambda < 0$) one gets two nontrivial critical points: a global minimum and a second one from a mountain pass geometry. To perform the technical estimates, in [36] some further conditions on $g(s)$ and $G(s) := \int_0^s g(\xi)\,d\xi$ (implying (I-14)) were imposed. For example, the superlinearity assumption at zero is expressed by

$$\lim_{s \to 0^+} \frac{g(s)}{s^\alpha} = \ell_\alpha > 0,$$

for some $\alpha > 1$. Notice that assumptions of this kind have been used also in previous works dealing with indefinite superlinear problems, like [3, 19].

Our first main contribution in this framework is to provide an existence result for pairs of positive solutions to BVPs associated with equation (I-15) under a minimal set of hypotheses which is less restrictive than the one needed when using variational methods. Our approach is still based on the topological degree.

We now state our existence result for pairs of positive solutions in the context of the T-periodic problem and we stress that the same result is valid also for Dirichlet and Neumann boundary conditions.

Theorem. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a continuous T-periodic function such that $\int_0^T a(t)\,dt < 0$ and $a(t_0) > 0$ for some $t_0 \in \mathbb{R}$. Let $g\colon [0,+\infty[\, \to [0,+\infty[$ be a smooth function with $g(0) = 0$, $g(s) > 0$ for $s > 0$ and, moreover, satisfying* (I-14)*. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ equation* (I-15) *has at least two positive T-periodic solutions.*

The smoothness assumption can be relaxed with hypotheses of regular oscillation of $g(s)$ at zero or at infinity, or with the condition of continuous differentiability of $g(s)$ in a neighborhood of $s = 0$ and near infinity.

The above result seems to be optimal from the point of view of the multiplicity of solutions, in the sense that no more than two positive solutions can be expected for a general weight and in the sense that there are examples of functions $g(s)$ satisfying all the assumptions of our theorem and such that there are no positive T-periodic solutions if $\lambda > 0$ is small or if the average condition is not satisfied (cf. [36, § 2]). In this regard, sharp existence results of exactly two solutions (at least for the Dirichlet problem and with a positive constant weight) are described and surveyed in [115, 150, 151] (more specifically, see [151, Theorem 6.19]).

Our second main contribution for super-sublinear problems is the investigation of high multiplicity of positive solutions to the Dirichlet, Neumann and periodic problems associated with

$$u'' + \big(\lambda a^{+}(t) - \mu a^{-}(t)\big) g(u) = 0, \tag{I-16}$$

in dependence of the nodal properties of the weight function $a(t)$ and of the parameters $\lambda, \mu > 0$.

The following result (stated for the periodic problem) holds.

Theorem. *Suppose that $a\colon [0,T] \to \mathbb{R}$ is a continuous function and there are $2m+1$ points*

$$0 = \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \ldots < \sigma_m < \tau_m < \sigma_{m+1} = T,$$

such that $\tau_1, \sigma_2, \tau_2, \ldots, \sigma_m, \tau_m$ are simple zeros of $a(t)$ and, moreover,

- *$a(t) \geq 0$, for all $t \in [\sigma_i, \tau_i]$, $i = 1, \ldots, m$;*
- *$a(t) \leq 0$, for all $t \in [\tau_i, \sigma_{i+1}]$, $i = 1, \ldots, m$.*

Assume that $g\colon [0,+\infty[\,\to [0,+\infty[$ is a continuous function with $g(0) = 0$, $g(s) > 0$ for $s > 0$ and, moreover, satisfying (I-14). *Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ there exists $\mu^*(\lambda) > 0$ such that for each $\mu > \mu^*(\lambda)$ equation* (I-16) *has at least $3^m - 1$ positive T-periodic solutions.*

Figure 3 below illustrates an example of existence of eight positive solutions for the Dirichlet problem associated with (I-16) when the weight function possesses two positive humps separated by a negative one.

Our proof makes use of a topological degree argument in the same spirit as when dealing with the superlinear case, in the sense that we define some open sets where an operator associated with our boundary value problem has nonzero degree.

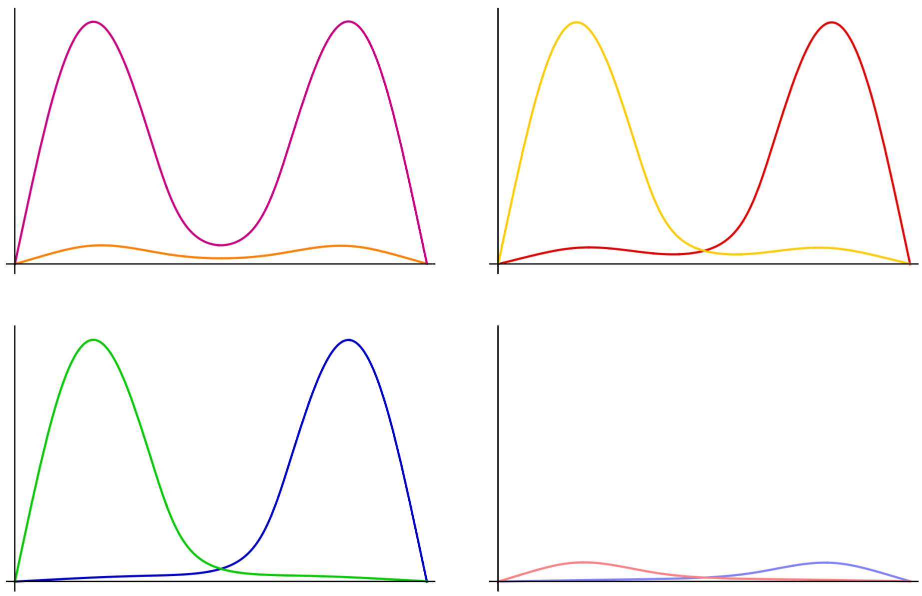

Figure 3: The figure shows an example of $8 = 3^2 - 1$ positive solutions to the Dirichlet problem associated with (I-16). For this simulation we have chosen the interval $[0, T]$ with $T = 3\pi$, the weight function $a_{\lambda,\mu}(t) := \lambda \sin^+(t) - \mu \sin^-(t)$, so that $m = 2$ is the number of positive humps separated by a negative one, and the super-sublinear nonlinearity $g(s) = s^2/(1 + s^2)$. Evidence of multiple positive solutions, agreeing with our result, is obtained for $\lambda = 3$ and $\mu = 10$.

The existence of $3^m - 1$ positive T-periodic solutions of (I-16) comes from the possibility of prescribing the behavior of the solution in each interval of positivity $[\sigma_i, \tau_i]$ of the weight function $a(t)$ among three possible ones: either the solution is "very small" on $[\sigma_i, \tau_i]$, or it is "small", or it is "large". We remark that this fact is strictly connected to the existence of three solutions for the Dirichlet problem associated with $u'' + \lambda a^+(t)g(u) = 0$ on $[\sigma_i, \tau_i]$ when $\lambda > 0$ is sufficiently large: the trivial one and two (positive) solutions given by Rabinowitz's theorem (cf. [158]).

As explained for the superlinear case, our topological approach allows us to deal with subharmonic solutions. Indeed, we notice that all the constants involved in the proof (in particular λ^* and $\mu^*(\lambda)$) depend only on $g(s)$ and on the local properties of $a(t)$. Therefore, when we apply our multiplicity theorem to the interval $[0, kT]$, these constants can be chosen independently on the integer k. Exploiting this crucial fact, we obtain infinitely many subharmonic solutions and globally defined positive bounded solutions with complex behavior. Moreover, in a dynamical system perspective, we also prove the presence of a Bernoulli shift as a factor within the set of positive bounded solutions of (I-16).

Some remarks and future perspectives

First of all, we stress that all the statements of the theorems presented in this introduction are easier versions of the results that will be discussed along the thesis. For instance, in the present manuscript we will deal with coefficients $a(t)$ that are Lebesgue integrable functions and, moreover, the growth conditions on $g(s)$ will be relaxed.

Secondly, a relevant aspect of our results is that a minimal set of assumptions on the nonlinearity $g(s)$ is required. Indeed, only positivity, continuity and the hypotheses on the limits $g(s)/s$ for $s \to 0^+$ and $s \to +\infty$ are needed. In particular, no supplementary power-type growth conditions at zero or at infinity are necessary.

Another advantage in using a topological degree approach lies on the fact that our results are stable with respect to small perturbations. In this manner, for example, we can also extend our theorems to equations like

$$u'' + \varepsilon f(t, u, u') + q(t)g(u) = 0,$$

with $|\varepsilon| < \varepsilon_0$, where ε_0 is a sufficiently small constant. The stability with respect to small perturbations is not generally guaranteed when using different approaches, such as variational or symplectic techniques which require some special structure (e.g. an Hamiltonian). Concerning this aspect, we stress that our results work equally well with respect to the presence or not of the friction term cu' (c can be zero or non-zero, indifferently). More precisely, we can deal with second order differential equations of the form

$$u'' + cu' + q(t)g(u) = 0.$$

Clearly, for $c \neq 0$, we lose the Hamiltonian structure if we pass to the natural equivalent system in the phase-plane

$$u' = y, \qquad y' = -cy - q(t)g(u).$$

With respect to Dirichlet and Neumann condition, via a change of variable, it is easy to reduce our study to the case of an equation without the friction term cu'. However, this is no more guaranteed for the periodic problem, but some technical estimates ensure the applicability of our topological method also for this more complicated case.

Finally, we stress that our topological technique has the potential and the generality needed to deal with indefinite problems with more general differential operators, as in the case of problems involving PDEs.

Recently, applications of topological degree methods in the study of ordinary and partial differential equations involving nonlinear differential operators, like the p-Laplacian, the ϕ-Laplacian, the curvature or the Minkowski ones, have shown a tremendous growth and thus strongly motivated the search of new topological tools. For instance, concerning non-autonomous ODEs, Manásevich and Mawhin developed in [123] new continuation theorems dealing with the second order vector nonlinear equation

$$\big(\phi(u')\big)' + f(t,u,u') = 0,$$

where $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism onto $\mathbb{R}^n$ with $\phi(0)=0$ (notice that, for $p>1$, if $\phi(s) := |s|^{p-2}s$ for $s\neq 0$ and $\phi(0)=0$, the differential operator $(\phi(u'))'$ is the one-dimensional p-Laplacian one).

With this purposes, in [81] we have extended the theory introduced in [123] by considering coincidence equations involving operators defined in product spaces and then investigating new continuations theorems which are designed to have as a natural application the study of cyclic feedback type systems. Consequently, in our paper [81] we have presented the abstract setting and the starting point for the study of indefinite equations involving new differential operators, as in the case of the ϕ-Laplacian one

$$\big(\phi(u')\big)' + q(t)g(u) = 0$$

or as in the search of radial solutions to quasilinear elliptic equations of the form

$$\operatorname{div}\big(A(|\nabla u|)\nabla u\big) + q(x)g(u) = 0,$$

where $A(s)$ is a real positive continuous function defined for all $s>0$ (cf. [88]). Nevertheless, for the sake of brevity, this contribution is not included in the present thesis.

The topological technique described in this thesis does not appear to be restricted to the one-dimensional case and potentially could be also applied to indefinite problems associated with elliptic partial differential equations. Certainly, there should be some technical issues to be addressed and that will be the subjects of a future research.

Another object of investigation is represented by the dynamical aspects related to the indefinite problems presented above. In this manuscript our topological approach, based on degree theory, allows us to give a contribution to this issue, however a more classical way to verify the existence of complex dynamics is the study of the Poincaré map. In this perspective, in [79] we provide a fixed point theorem for maps satisfying a property of "stretching a space along paths", which is a generalization in the context of absolute retracts of a result obtained by Papini and Zanolin in [155] for the detection of chaotic dynamics. We propose a proof based on fixed point index theory. Our abstract theorem will be a useful tool for pursuing investigations in this direction, however, for the sake of concision and since it is not strictly connected to the theory discussed here, we do not include this result in the thesis.

Plan of the thesis

We divide the thesis into two parts. In the first part we deal with the superlinear case, while in the second part we investigate the super-sublinear one.

The first part starts with the study of the Dirichlet boundary value problem. In Chapter 1 we present our existence and multiplicity results, which are contained in [84]. Next, in Chapter 2 we show how the topological approach introduced in the previous chapter can be applied to the investigation of more general indefinite equations: first, only for the existence result, by considering sign-changing nonlinearities $g(s)$ (cf. [80]); secondly, by treating nonlinearities of the form $f(t,s) := \sum_{i=1}^{m} \alpha_i a_i(t) g_i(s) - \sum_{j=0}^{m+1} \beta_j b_j(t) k_j(s)$ (cf. [78]). Chapter 3 and Chapter 4 are devoted to the Neumann and periodic boundary conditions. More precisely, in Chapter 3 we show existence of positive solutions (cf. [83]), while in Chapter 4 we investigate the problem of multiplicity (cf. [82]). We conclude the first part with Chapter 5, which is dedicated to subharmonic solutions and chaotic dynamics (cf. [30, 82]).

The second part contains three chapters: Chapter 6 devoted to the existence problem (cf. [31]), Chapter 7 where we present an high multiplicity theorem (cf. [32]), and Chapter 8 containing the investigations on subharmonics and chaotic dynamics (cf. [32]).

We conclude the thesis with three appendices, where we recall the definitions and the main properties of the Leray-Schauder topological degree and of Mawhin's coincidence degree (for locally compact operators defined on open possibly unbounded sets), and we present some useful tools (maximum principles and a change of variable).

Notation

We conclude this introduction by presenting some symbols and some notations used in this manuscript.

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ the usual numerical sets. Moreover, let $\mathbb{R}^+ := [0,+\infty[$ be the set of non-negative real numbers and let $\mathbb{R}^+_0 :=]0,+\infty[$ be the set of positive real numbers.

The cardinality of a finite set $\mathcal{I} \subseteq \mathbb{N}$ is denoted by the symbol $\#\mathcal{I}$. Given a bounded interval $J \subseteq \mathbb{R}$, we indicate by $|J|$ the length or measure of J.

In the thesis we deal with normed linear spaces. Usually we denote by $(X, \|\cdot\|)$ or simply by X a normed linear space, where $\|\cdot\|$ is its norm. The symbols $B(x_0,r)$ and $B[x_0,r]$, where $x_0 \in X$ and $r > 0$, represent the open and closed balls centered at x_0 with radius r respectively, i.e.

$$B(x_0,r) := \bigl\{x \in X \colon \|x - x_0\| < r\bigr\}, \quad B[x_0,r] := \bigl\{x \in X \colon \|x - x_0\| \leq r\bigr\}.$$

Moreover, given a subset $A \subseteq X$, we denote by $\overline{A}$ or by $\mathrm{cl}(A)$ its closure, with $\mathrm{int}(A)$ its inner part and with ∂A its boundary. If $A, B \subseteq X$, with $A \setminus B$ we mean the relative complement of B in A.

We indicate by Id or Id_X the identity on the space X. Given a function f, $f|_A$ represents the restriction of the function f in A, where A is a subset of the domain of f.

Dealing with functions between $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$, we denote by $\mathcal{C}^k(A, B)$ (with $k \in \mathbb{N}$) the space of functions $f\colon A \to B$ such that f is continuous and its derivatives $f', f'', \ldots, f^{(k)}$ exist and are continuous. Given an interval $J \subseteq \mathbb{R}$, a real number $p \geq 1$ and $k \in \mathbb{N}$, the symbols $L^p(J, \mathbb{R}^n)$ and $W^{k,p}(J, \mathbb{R}^n)$ indicate the classical L^p-space and Sobolev space, respectively. Given a function $a \in L^1(J, \mathbb{R})$, we denote by

$$a^+(t) := \frac{a(t) + |a(t)|}{2} \quad \text{and} \quad a^-(t) := \frac{-a(t) + |a(t)|}{2}, \quad t \in J,$$

the positive part and the negative part of $a(t)$, respectively. Furthermore, $a(t) \succ 0$ on a given interval means that $a(t) \geq 0$ almost everywhere with $a \not\equiv 0$ on that interval; moreover, $a(t) \prec 0$ stands for $-a(t) \succ 0$.

Part I

Superlinear indefinite problems

Chapter 1

Dirichlet boundary conditions

In this chapter we study the problem of existence and multiplicity of positive solutions for the nonlinear two-point boundary value problem

$$\begin{cases} u'' + f(t,u) = 0 \\ u(0) = u(T) = 0, \end{cases} \tag{1.0.1}$$

where $f\colon [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ is a Carathéodory function such that $f(t,0) \equiv 0$. We allow $t \mapsto f(t,s)$ to change its sign in order to cover the case of scalar equations with indefinite weight. Roughly speaking, our main assumptions require that $f(t,s)/s$ is below the first eigenvalue as $s \to 0^+$ and above the first eigenvalue as $s \to +\infty$. In particular, we can deal with the situation in which $f(t,s)$ has a superlinear growth at zero and at infinity. We propose a new approach based on the topological degree which provides multiplicity of solutions. Next, applications are given for

$$f(t,s) = a(t)g(s),$$

with $g(s)/s \to 0$ as $s \to 0^+$ and $g(s)/s \to +\infty$ as $s \to +\infty$, and with a weight function $a(t)$ which is allowed to change its sign on the interval $[0,T]$, so that we deal with a *superlinear indefinite* problem (covering the classical superlinear equation with $g(s) = s^p$ for $p > 1$). We prove the existence of $2^m - 1$ positive solutions to $u'' + a(t)g(u) = 0$, when $a(t)$ has m positive humps separated by negative ones and $a^-(t)$ is sufficiently large. In this way, we solve the conjecture proposed by R. Gómez-Reñasco and J. López-Gómez in [100].

The plan of the chapter is the following. In Section 1.1 we introduce the key ingredients. In more detail we illustrate the problem, we define an equivalent fixed point problem and we list the hypotheses we are going to assume on $f(t,s)$. Moreover, we prove some preliminary technical lemmas that permit to compute the topological degree on suitable small and large balls. In Section 1.2 we present existence theorems. Using the lemmas of the previous section, we prove that there exists at least a positive solution for (1.0.1). In Theorem 1.2.2 we weaken the hypothesis on the growth of $f(t,s)/s$ at infinity, by assuming the linear growth of $f(t,s)$ in the subintervals where the growth condition is not valid. Section 1.3 is devoted to the proof of our main result, namely Theorem 1.3.1, which deals with the multiplicity of solutions. By employing the Leray-Schauder topological degree, we deduce the existence of $2^m - 1$ positive solutions of our boundary value problem. In Section 1.4 we analyze boundary value problems of the form (1.0.1) with $f(t,s) = a(t)g(s)$, as a special case. We discuss the results obtained in the previous sections in this particular context and, in this way, we obtain the existence and multiplicity theorems we look for. We finish that section with some remarks concerning radially symmetric solutions for elliptic partial differential equations in annular domains. Possible generalizations on the weight function $a(t)$ are considered, too.

1.1 Preliminary results

In this section we collect some classical and basic facts which are then applied in the proofs of our main results.

Let $I \subseteq \mathbb{R}$ be a nontrivial compact interval. Let $f\colon I \times \mathbb{R}^+ \to \mathbb{R}$ be an L^1-Carathéodory function, that is

- $x \mapsto f(t,s)$ is measurable for each $s \in \mathbb{R}^+$;
- $s \mapsto f(t,s)$ is continuous for a.e. $t \in I$;
- for each $r > 0$ there is $\gamma_r \in L^1(I, \mathbb{R}^+)$ such that $|f(t,s)| \leq \gamma_r(t)$, for a.e. $t \in I$ and for all $|s| \leq r$.

Without loss of generality, in the sequel we suppose

$$I := [0,T]$$

and we stress that different choices of I can be made. We study the two-point boundary value problem

$$\begin{cases} u'' + f(t,u) = 0 \\ u(0) = u(T) = 0. \end{cases} \tag{1.1.1}$$

A *solution* of (1.1.1) is an absolutely continuous function $u\colon [0,T] \to \mathbb{R}^+$ such that its derivative $u'(t)$ is absolutely continuous and $u(t)$ satisfies (1.1.1) for a.e. $t \in [0,T]$. We look for *positive solutions* of (1.1.1), that is solutions $u(t)$ such that $u(t) > 0$ for every $t \in \,]0,T[$.

We suppose

$$(f^*) \qquad f(t,0) = 0, \quad \text{a.e. } t \in I.$$

Using a standard procedure, we extend $f(t,s)$ to a function $\tilde{f}\colon I \times \mathbb{R} \to \mathbb{R}$ defined as

$$\tilde{f}(t,s) = \begin{cases} f(t,s), & \text{if } s \geq 0; \\ 0, & \text{if } s \leq 0; \end{cases}$$

and we study the modified boundary value problem

$$\begin{cases} u'' + \tilde{f}(t,u) = 0 \\ u(0) = u(T) = 0. \end{cases} \tag{1.1.2}$$

As is well known, by a maximum principle (cf. Lemma C.1.1), all the possible solutions of (1.1.2) are non-negative and hence solutions of (1.1.1).

In view of Lemma C.1.1, from now on, we suppose

(f_0^-) there exists a function $q_- \in L^1(I, \mathbb{R}^+)$ such that

$$\liminf_{s \to 0^+} \frac{f(t,s)}{s} \geq -q_-(t), \quad \text{uniformly a.e. } t \in I.$$

For the sequel we need to introduce a suitable notation concerning the first eigenvalue of a linear problem with non-negative weight. Let $J := [t_1, t_2] \subseteq I$ be a compact subinterval and $q \in L^1(J, \mathbb{R}^+)$ with $q \not\equiv 0$, namely $q > 0$ a.e. on a set of positive measure. We denote by $\mu_1^J(q)$ the first (positive) eigenvalue of

$$\begin{cases} \varphi'' + \mu q(t)\varphi = 0 \\ \varphi|_{\partial J} = 0. \end{cases}$$

Sturm theory (in the Carathéodory setting) guarantees that $\mu_1^J(q)$ is a simple eigenvalue with an associated eigenfunction φ which is positive in the interior of J and such that $\varphi'(t_1) > 0 > \varphi'(t_2)$ (see, for instance, [62, Theorem 5.1, pp. 456–457]). As a notational convention, when $J = I = [0,T]$, we denote the first eigenvalue simply by $\mu_1(q)$.

Our approach to the search of positive solutions of (1.1.1) is based on Leray-Schauder topological degree (cf. Appendix A). Accordingly, we transform problem (1.1.2) into an equivalent fixed point problem for an associated operator which is the classical one defined by means of the Green function $G(t,s)$ for the operator $u \mapsto -u''$ with the two-point boundary conditions, i.e.

$$G(t,s) := \frac{1}{T} \begin{cases} t(T-s), & \text{if } 0 \leq t \leq s \leq T; \\ s(T-t), & \text{if } 0 \leq s \leq t \leq T. \end{cases}$$

Namely, we define $\Phi\colon \mathcal{C}(I) \to \mathcal{C}(I)$ by

$$(\Phi u)(t) := \int_I G(t,\xi)\tilde{f}(\xi, u(\xi))\, d\xi, \quad t \in I. \tag{1.1.3}$$

The operator Φ is completely continuous in $\mathcal{C}(I)$ endowed with the sup-norm $\|\cdot\|_\infty$ (see, for instance, [93, Proposition 2.1]).

Our goal is to find multiple fixed points for Φ using a degree theoretic approach. To this aim, we present now some technical lemmas which are stated in a form that is suitable to be subsequently applied for the computation of the topological degree via homotopy procedures.

Lemma 1.1.1. *Let $f\colon I \times \mathbb{R}^+ \to \mathbb{R}$ be an L^1-Carathéodory function satisfying (f^*) and (f_0^-). Suppose*

(f_0^+) *there exists a measurable function $q_0 \in L^1(I, \mathbb{R}^+)$ with $q_0 \not\equiv 0$, such that*

$$\limsup_{s \to 0^+} \frac{f(t,s)}{s} \leq q_0(t), \quad \textit{uniformly a.e. } t \in I,$$

and

$$\mu_1(q_0) > 1.$$

Then there exists $r_0 > 0$ such that every solution $u(t) \geq 0$ of the two-point boundary value problem

$$\begin{cases} u'' + \vartheta f(t,u) = 0, \quad 0 \leq \vartheta \leq 1, \\ u(0) = u(T) = 0 \end{cases} \tag{1.1.4}$$

satisfying $\max_{t \in I} u(t) \leq r_0$ is such that $u(t) = 0$, for all $t \in I$.

Proof. Let $\varepsilon > 0$ be such that

$$\hat{\mu} := \mu_1(q_0 + \varepsilon) > 1.$$

The existence of ε is ensured by the continuity of the first eigenvalue as a function of the weight and by hypothesis (f_0^+) (see [58], [63, p. 44] or [183]). By condition (f_0^+), there exists $r_0 > 0$ such that

$$\frac{f(t,s)}{s} \leq q_0(t) + \varepsilon, \quad \text{a.e. } t \in I, \ \forall\, 0 < s \leq r_0.$$

Let φ be a positive eigenfunction of

$$\begin{cases} \varphi'' + \hat{\mu}[q_0(t) + \varepsilon]\varphi = 0 \\ \varphi(0) = \varphi(T) = 0. \end{cases}$$

Then $\varphi(t) > 0$, for all $t \in \,]0,T[$, and $\varphi'(0) > 0 > \varphi'(T)$.

In order to prove the statement of our lemma, suppose, by contradiction, that there exist $\vartheta \in [0,1]$ and a solution $u(t) \geq 0$ of (1.1.4) such that $\max_{t\in I} u(t) = r$ for some $0 < r \leq r_0$. Notice that, by the choice of r_0, we have that

$$\vartheta f(t,u(t)) \leq (q_0(t)+\varepsilon)u(t), \quad \text{a.e. } t \in I.$$

Using a Sturm comparison argument, we obtain

$$\begin{aligned}
0 &= \Big[u'(t)\varphi(t) - u(t)\varphi'(t)\Big]_{t=0}^{t=T} \\
&= \int_0^T \frac{d}{dt}\Big[u'(t)\varphi(t) - u(t)\varphi'(t)\Big]\,dt \\
&= \int_0^T \Big[u''(t)\varphi(t) - u(t)\varphi''(t)\Big]\,dt \\
&= \int_0^T \Big[-\vartheta f(t,u(t))\varphi(t) + u(t)\hat{\mu}(q_0(t)+\varepsilon)\varphi(t)\Big]\,dt \\
&= \int_0^T \Big[(q_0(t)+\varepsilon)u(t) - \vartheta f(t,u(t))\Big]\varphi(t)\,dt \\
&\quad + (\hat{\mu}-1)\int_0^T (q_0(t)+\varepsilon)u(t)\varphi(t)\,dt \\
&> 0,
\end{aligned}$$

a contradiction. □

A direct application of Lemma 1.1.1 permits to compute the degree on small neighborhoods of the origin.

Lemma 1.1.2. *Let $f\colon I\times\mathbb{R}^+ \to \mathbb{R}$ be an L^1-Carathéodory function satisfying (f^*), (f_0^-) and (f_0^+). Then there exists $r_0 > 0$ such that*

$$\deg_{LS}(Id-\Phi, B(0,r), 0) = 1, \quad \forall\, 0 < r \leq r_0.$$

Proof. Let r_0 be as in Lemma 1.1.1 and let us fix $r \in \,]0,r_0]$. If $u \in \mathcal{C}(I)$ satisfies $u = \vartheta\Phi(u)$, for some $0 \leq \vartheta \leq 1$, then u is a solution of the equation $u'' + \vartheta \tilde{f}(t,u) = 0$ with $u(0) = u(T) = 0$. Now, either $u = 0$, or (according to Lemma C.1.1) $u(t) > 0$ for each $t \in \,]0,T[$. Therefore, $u(t)$ is a solution of (1.1.4). Hence, Lemma 1.1.1 and the choice of r imply that $\|u\|_\infty \neq r$. This, in turn, implies that

$$u \neq \vartheta\Phi(u), \quad \forall\,\vartheta \in [0,1],\ \forall\, u \in \partial B(0,r).$$

By the homotopic invariance property of the topological degree, we conclude that

$$\deg_{LS}(Id-\Phi, B(0,r), 0) = \deg_{LS}(Id, B(0,r), 0) = 1.$$

This concludes the proof. □

As a next step, we give a result which will be used to compute the degree on large balls. It follows from Lemma 1.1.3 below, where we assume suitable conditions on $f(t,s)/s$ for $s > 0$ and large.

Lemma 1.1.3. *Let $f\colon I \times \mathbb{R}^+ \to \mathbb{R}$ be an L^1-Carathéodory function. Suppose that there exists a closed interval $J \subseteq I$ such that*

$$f(t,s) \geq 0, \quad \text{a.e. } t \in J, \ \forall\, s \geq 0;$$

and there is a measurable function $q_\infty \in L^1(J,\mathbb{R}^+)$ with $q_\infty \not\equiv 0$, such that

$$\liminf_{s\to+\infty} \frac{f(t,s)}{s} \geq q_\infty(t), \quad \text{uniformly a.e. } t \in J. \tag{1.1.5}$$

Suppose

$$\mu_1^J(q_\infty) < 1.$$

Then there exists $R_J > 0$ such that for each L^1-Carathéodory function $h\colon I \times \mathbb{R}^+ \to \mathbb{R}$ with

$$h(t,s) \geq f(t,s), \quad \text{a.e. } t \in J, \ \forall\, s \geq 0,$$

every solution $u(t) \geq 0$ of the two-point boundary value problem

$$\begin{cases} u'' + h(t,u) = 0 \\ u(0) = u(T) = 0 \end{cases} \tag{1.1.6}$$

satisfies $\max_{t\in J} u(t) < R_J$.

We stress that the constant R_J does not depend on the function $h(t,s)$. Notice that our assumptions are "local" (in the spirit of [64] and [149]), in the sense that we do not require their validity on the whole domain.

Proof. Just to fix a notation along the proof, we set $J := [t_1,t_2]$. By contradiction, suppose that there is not a constant R_J with those properties. So, for all $n > 0$ there exists $\tilde{u}_n \geq 0$ solution of (1.1.6) with $\max_{t\in J} \tilde{u}_n(t) =: \hat{R}_n > n$.

Let $q_n(t)$ be a monotone non-decreasing sequence of non-negative measurable functions such that

$$f(t,s) \geq q_n(t)s, \quad \text{a.e. } t \in J, \ \forall\, s \geq n,$$

and $q_n \to q_\infty$ uniformly almost everywhere in J. The existence of such a sequence comes from condition (1.1.5).

Fix

$$\varepsilon < \frac{1-\mu_1^J(q_\infty)}{2}.$$

Hence, there exists an integer $N > 0$ such that $q_n \not\equiv 0$ for each $n \geq N$ and, moreover,

$$\nu_n := \mu_1^J(q_n) \leq 1 - \varepsilon, \quad \forall\, n \geq N.$$

Now we fix N as above and denote by φ the positive eigenfunction of

$$\begin{cases} \varphi'' + \nu_N q_N(t)\varphi = 0 \\ \varphi(t_1) = \varphi(t_2) = 0, \end{cases}$$

with $\|\varphi\|_\infty = 1$. Then $\varphi(t) > 0$, for all $t \in \,]t_1, t_2[$, and $\varphi'(t_1) > 0 > \varphi'(t_2)$.

For each $n \geq N$, let $J'_n \subseteq J$ be the maximal closed interval, such that

$$\tilde{u}_n(t) \geq N, \quad \forall\, t \in J'_n.$$

By the concavity of the solution in the interval J and the definition of J'_n, we also have that

$$\tilde{u}_n(t) \leq N, \quad \forall\, t \in J \setminus J'_n.$$

Another consequence of the concavity of $\tilde{u}_n$ on J ensures that

$$\tilde{u}_n(t) \geq \frac{\hat{R}_n}{t_2 - t_1} \min\bigl\{t - t_1, t_2 - t\bigr\}, \quad \forall t \in J,$$

(see [93, p. 420] for a similar estimate). Hence, if we take $n \geq 2N$, we find that $\tilde{u}_n(t) \geq N$, for all t in the well-defined closed interval

$$A_n := \left[t_1 + \frac{N}{\hat{R}_n}(t_2 - t_1), t_2 - \frac{N}{\hat{R}_n}(t_2 - t_1)\right] \subseteq J'_n.$$

By construction, $|J \setminus J'_n| \leq |J \setminus A_n| \to 0$ as $n \to \infty$.

Using a Sturm comparison argument, for each $n \geq N$, we obtain

$$\begin{aligned}
0 &\geq \tilde{u}_n(t_2)\varphi'(t_2) - \tilde{u}_n(t_1)\varphi'(t_1) = \Bigl[\tilde{u}_n(t)\varphi'(t) - \tilde{u}'_n(t)\varphi(t)\Bigr]_{t=t_1}^{t=t_2} \\
&= \int_{t_1}^{t_2} \frac{d}{dt}\Bigl[\tilde{u}_n(t)\varphi'(t) - \tilde{u}'_n(t)\varphi(t)\Bigr]\,dt \\
&= \int_J \Bigl[\tilde{u}_n(t)\varphi''(t) - \tilde{u}''_n(t)\varphi(t)\Bigr]\,dt \\
&= \int_J \Bigl[-\tilde{u}_n(t)\nu_N q_N(t)\varphi(t) + h(t, \tilde{u}_n(t))\varphi(t)\Bigr]\,dt \\
&= \int_J \Bigl[h(t, \tilde{u}_n(t)) - \nu_N q_N(t)\tilde{u}_n(t)\Bigr]\varphi(t)\,dt \\
&\geq \int_J \Bigl[f(t, \tilde{u}_n(t)) - \nu_N q_N(t)\tilde{u}_n(t)\Bigr]\varphi(t)\,dt \\
&= \int_{J'_n} \Bigl[f(t, \tilde{u}_n(t)) - q_N(t)\tilde{u}_n(t)\Bigr]\varphi(t)\,dt + (1 - \nu_N)\int_{J'_n} q_N(t)\tilde{u}_n(t)\varphi(t)\,dt \\
&\quad + \int_{J \setminus J'_n} \Bigl[f(t, \tilde{u}_n(t)) - \nu_N q_N(t)\tilde{u}_n(t)\Bigr]\varphi(t)\,dt.
\end{aligned}$$

Recalling that

$$f(t,s) \geq q_N(t)s, \quad \text{a.e. } t \in J, \ \forall\, s \geq N,$$

we know that

$$f(t,\tilde{u}_n(t)) - q_N(t)\tilde{u}_n(t) \geq 0, \quad \text{a.e. } t \in J'_n, \ \forall\, n \geq N.$$

Then, using the Carathéodory assumption, which implies that

$$|f(t,s)| \leq \gamma_N(t), \quad \text{a.e. } t \in J, \ \forall\, 0 \leq s \leq N,$$

where γ_N is a suitably non-negative integrable function, we obtain

$$\begin{aligned}
0 &\geq \int_{J'_n} \Big[f(t,\tilde{u}_n(t)) - q_N(t)\tilde{u}_n(t) \Big] \varphi(t)\,dt + (1-\nu_N) \int_{J'_n} q_N(t)\tilde{u}_n(t)\varphi(t)\,dt \\
&\quad + \int_{J\setminus J'_n} \Big[f(t,\tilde{u}_n(t)) - \nu_N q_N(t)\tilde{u}_n(t) \Big] \varphi(t)\,dt \\
&\geq \varepsilon N \int_{J'_n} q_N(t)\varphi(t)\,dt + \int_{J\setminus J'_n} \Big[-\gamma_N(t) - N\nu_N q_N(t) \Big]\,dt \\
&= \varepsilon N \int_J q_N(t)\varphi(t)\,dt - \varepsilon N \int_{J\setminus J'_n} q_N(t)\varphi(t)\,dt \\
&\quad - \int_{J\setminus J'_n} \Big[\gamma_N(t) + N\nu_N q_N(t) \Big]\,dt.
\end{aligned}$$

Passing to the limit as $n \to \infty$ and using the dominated convergence theorem, we obtain

$$0 \geq \varepsilon N \int_J q_N(t)\varphi(t)\,dt > 0,$$

a contradiction. □

Remark 1.1.1. We note that the boundary condition in problem (1.1.6) has no role in the proof of Lemma 1.1.3. In fact, the key point is that we deal with non-negative solutions of the equation $u'' + h(t,u) = 0$. Consequently, the same thesis holds also with any other boundary condition. ◁

Now, we assume a specific sign condition on the function $f(t,s)$. Such condition will play an important role in all our applications.

(H) *There exist $m \geq 1$ intervals $I_1^+, \ldots, I_m^+$, closed and pairwise disjoint, such that*

$$f(t,s) \geq 0, \quad \textit{for a.e. } t \in \bigcup_{i=1}^m I_i^+ \textit{ and for all } s \geq 0;$$

$$f(t,s) \leq 0, \quad \textit{for a.e. } t \in I \setminus \bigcup_{i=1}^m I_i^+ \textit{ and for all } s \geq 0.$$

If $m = 1$, condition (H) simply requires that there exists a compact subinterval $J \subseteq I$ such that for each $s \geq 0$ it holds that $f(t,s) \geq 0$ for a.e. $t \in J$ and $f(t,s) \leq 0$ for a.e. $t \in I \setminus J$ (the possibility that $J = I$ is not excluded).

An immediate consequence of Lemma 1.1.3 is the following result.

Lemma 1.1.4. *Let $f\colon I \times \mathbb{R}^+ \to \mathbb{R}$ be an L^1-Carathéodory function satisfying (H) as well as (f^*) and (f_0^-). Assume also*

(f_∞) *for all $i = 1, \ldots, m$ there exists a measurable function $q_\infty^i \in L^1(I_i^+, \mathbb{R}^+)$ with $q_\infty^i \not\equiv 0$, such that*

$$\liminf_{s \to +\infty} \frac{f(t,s)}{s} \geq q_\infty^i(t), \quad \textit{uniformly a.e. } t \in I_i^+,$$

and

$$\mu_1^{I_i^+}(q_\infty^i) < 1.$$

Then there exists $R^* > 0$ such that

$$\deg_{LS}(Id - \Phi, B(0,R), 0) = 0, \quad \forall\, R \geq R^*.$$

Proof. Define

$$R^* := \max_{i=1,\ldots,m} R_{I_i^+} > 0,$$

with $R_{I_i^+} > 0$ defined as in Lemma 1.1.3. Let us also fix a radius $R \geq R^*$.

We denote by $\mathbb{1}_A$ the indicator function of the set $A := \bigcup_{i=1}^m I_i^+$. We set $v(t) := \int_I G(t,s)\mathbb{1}_A(s)\,ds$ and we consider in $\mathcal{C}(I)$ the operator equation

$$u = \Phi(u) + \alpha v, \quad \text{for } \alpha \geq 0.$$

Clearly, any nontrivial solution $u(t)$ of the above equation is a solution of $u'' + \tilde{f}(t,u) + \alpha \mathbb{1}_A(t) = 0$ with $u(0) = u(T) = 0$. By the first part of Lemma C.1.1 we know that $u(t) \geq 0$ for all $t \in I$. Hence, u is a non-negative solution of (1.1.6) with

$$h(t,s) = f(t,s) + \alpha \mathbb{1}_A(t),$$

for $\alpha \geq 0$. By definition, we have that $h(t,s) \geq f(t,s)$ for a.e. $t \in A$ and for all $s \geq 0$, and also $h(t,s) = f(t,s) \leq 0$ for a.e. $t \in I \setminus A$ and for all $s \geq 0$. By the convexity of the solutions of (1.1.6) in the intervals of $I \setminus A$, we obtain

$$\max_{t \in I} u(t) = \max_{t \in A} u(t)$$

and, as an application of Lemma 1.1.3 on each of the intervals I_i^+, we conclude that

$$\|u\|_\infty < R^* \leq R.$$

As a consequence,

$$u \neq \Phi(u) + \alpha v, \quad \text{for all } u \in \partial B(0,R) \text{ and } \alpha \geq 0,$$

and thus the thesis follows from the second part of Theorem A.2.1. $\square$

1.2 Existence results

In this section we present some existence results. We essentially reconsidered in an explicit topological degree setting the existence results obtained in [93] by means of lower and upper solutions techniques.

An immediate consequence of Lemma 1.1.2 and Lemma 1.1.4 is the following existence theorem which generalizes [93, Theorem 4.1].

Theorem 1.2.1. *Let $f\colon I \times \mathbb{R}^+ \to \mathbb{R}$ be an L^1-Carathéodory function satisfying (f^*), (f_0^-), (f_0^+) and (H) with (f_∞). Then there exists at least a positive solution of the two-point boundary value problem* (1.1.1).

Proof. Let us take r_0 as in Lemma 1.1.2 and R^* as in Lemma 1.1.4. Clearly $0 < r_0 < R^* < +\infty$. Then

$$
\begin{aligned}
&\deg_{LS}(Id - \Phi, B(0, R^*) \setminus B[0, r_0], 0) = \\
&= \deg_{LS}(Id - \Phi, B(0, R^*), 0) - \deg_{LS}(Id - \Phi, B(0, r_0), 0) = \\
&= 0 - 1 = -1 \neq 0.
\end{aligned}
$$

Hence a nontrivial fixed point of the operator Φ exists and the claim follows from Lemma C.1.1. □

The next result is a straightforward consequence of Theorem 1.2.1.

Corollary 1.2.1. *Let $f\colon I \times \mathbb{R}^+ \to \mathbb{R}$ be an L^1-Carathéodory function satisfying*

$$
\lim_{s \to 0^+} \frac{f(t,s)}{s} = 0, \quad \text{uniformly a.e. } t \in I.
$$

Assume (H) and suppose that, for each $i \in \{1, \ldots, m\}$, there exists a compact interval $J_i \subseteq I_i^+$ such that

$$
\lim_{s \to +\infty} \frac{f(t,s)}{s} = +\infty, \quad \text{uniformly a.e. } t \in J_i.
$$

Then there exists at least a positive solution of the two-point boundary value problem (1.1.1).

Proof. The assumption $f(t,s)/s \to 0$ as $s \to 0^+$ clearly implies (f^*), (f_0^-) and (f_0^+) with $q_0(t) \equiv K_0$, where $0 < K_0 < (\pi/L)^2$. Moreover, setting

$$
q_\infty^i(t) = \begin{cases} K_\infty, & \text{for } t \in J_i; \\ 0, & \text{for } t \in I_i^+ \setminus J_i; \end{cases}
$$

with $K_\infty > \max_{i=1,\ldots,m} (\pi/|J_i|)^2$, we have (f_∞) satisfied as well. The conclusion follows from Theorem 1.2.1. □

Hypothesis (f_∞) requires to control from below the growth of $f(t,s)/s$ at infinity, on *each* of the intervals I_i^+. In this context, a natural question which can be raised is whether a condition like (f_∞) can be assumed only on *one* of the intervals. As a partial answer we provide a result where we consider a weaker condition in place of hypothesis (f_∞), namely we assume the condition only on a closed subinterval $J \subseteq I$, as in Lemma 1.1.3. In order to achieve an existence result, we add a supplementary condition of almost linear growth of $f(t,s)$ in $I \setminus J$.

Theorem 1.2.2. *Let $f\colon I \times \mathbb{R}^+ \to \mathbb{R}$ be an L^1-Carathéodory function satisfying (f^*), (f_0^-), (f_0^+). Let $J \subseteq I$ be a closed subinterval such that*

$$f(t,s) \geq 0, \quad \text{a.e. } t \in J, \ \forall\, s \geq 0.$$

Assume the following conditions:

(f_∞^J) *there exists a measurable function $q_\infty \in L^1(J,\mathbb{R}^+)$ with $q_\infty \not\equiv 0$, such that*

$$\liminf_{s \to +\infty} \frac{f(t,s)}{s} \geq q_\infty(t), \quad \text{uniformly a.e. } t \in J,$$

and

$$\mu_1^J(q_\infty) < 1;$$

(G) *there exist $a, b \in L^1(I \setminus J, \mathbb{R}^+)$ and $C > 0$ such that*

$$|f(t,s)| \leq a(t) + b(t)s, \quad \text{a.e. } t \in I \setminus J, \ \forall\, s \geq C.$$

Then there exists at least a positive solution of the two-point boundary value problem (1.1.1).

Proof. As in Lemma 1.1.3, set $J := [t_1, t_2]$. We define the set

$$\Omega_J := \bigl\{ u \in \mathcal{C}(I) \colon |u(t)| < R_J, \text{ for all } t \in J \bigr\},$$

where $R_J > 0$ is as in Lemma 1.1.3. Note that Ω_J is open and not bounded (unless we are in the trivial case $J = I$).

Define $\lambda_J := \mu_1^J(1) = (\pi/|J|)^2$. Along the proof, we denote by φ the positive eigenfunction of

$$\begin{cases} \varphi'' + \lambda_J \varphi = 0 \\ \varphi(t_1) = \varphi(t_2) = 0, \end{cases}$$

with $\|\varphi\|_\infty = 1$. Then $\varphi(t) > 0$, for all $t \in \,]t_1, t_2[$, and $\varphi'(t_2) < 0 < \varphi'(t_1)$.

We denote by $\mathbb{1}_J$ the indicator function of the interval J. We set $v(t) := \int_I G(t,s)\varphi(s)\mathbb{1}_J(s)\,ds$ and we define $F\colon \mathcal{C}(I) \times [0,+\infty[\,\to \mathcal{C}(I)$, as

$$F(u,\alpha) := \Phi(u) + \alpha v.$$

To reach the conclusion as in Theorem 1.2.1, we have to prove that the triplet $(Id - \Phi, \Omega_J, 0)$ is admissible and

$$\deg_{LS}(Id - \Phi, \Omega_J, 0) = 0.$$

To this end we show that conditions (i), (ii), (iii) of Theorem A.2.2 are satisfied. It is obvious that $F(u, 0) = \Phi(u)$, for all $u \in \mathcal{C}(I)$. Hence (i) is valid.

Preliminary to the proof of (ii) and (iii), we observe that any nontrivial solution $u \in \mathcal{C}(I)$ of the operator equation

$$u = \Phi(u) + \alpha v, \quad \text{for } \alpha \geq 0,$$

is a solution of $u'' + \tilde{f}(t, u) + \alpha\varphi(t)\mathbb{1}_J(t) = 0$ with $u(0) = u(T) = 0$. By the first part of Lemma C.1.1 we know that $u(t) \geq 0$ for all $t \in I$. Hence u is a non-negative solution of (1.1.6) with

$$h(t, s) = f(t, s) + \alpha\varphi(t)\mathbb{1}_J(t).$$

By definition, we have that $h(t, s) \geq f(t, s)$ for a.e. $t \in J$ and for all $s \geq 0$, and also $h(t, s) = f(t, s)$ for a.e. $t \in I \setminus J$ and for all $s \geq 0$.

Proof of (ii). Fix $\alpha \geq 0$. Suppose that there exist $u \in \overline{\Omega_J}$ and $\zeta \in [0, \alpha]$ satisfying $u = F(u, \zeta)$. Clearly $u \in \Omega_J$, by the choice of R_J and by Lemma 1.1.3. We first prove that $|u'(t)|$ is bounded on J. Using the fact that

$$|u''(t)| = |f(t, s) + \zeta\varphi(t)\mathbb{1}_J(t)| \leq \gamma_{R_J}(t) + \alpha, \quad \text{a.e. } t \in J,$$

we obtain that for all $y_1, y_2 \in J$

$$|u'(y_1) - u'(y_2)| \leq \left| \int_{y_1}^{y_2} (\gamma_{R_J}(\xi) + \alpha)\, d\xi \right| \leq \|\gamma_{R_J}\|_{L^1(J)} + \alpha(t_2 - t_1).$$

Now we show that there exists $\hat{t} \in J$ such that

$$|u'(\hat{t})| \leq \frac{R_J}{t_2 - t_1}.$$

By contradiction, suppose that $|u'(t)| > R_J/(t_2 - t_1)$, for every $t \in J$. Without loss of generality, suppose $u' > 0$ on J (the opposite case is analogous). Then

$$R_J = \frac{R_J}{t_2 - t_1}(t_2 - t_1) < \int_{t_1}^{t_2} u'(\xi)\, d\xi = u(t_2) - u(t_1) \leq u(t_2) \leq R_J,$$

a contradiction.

Then, for all $t \in J$, we have

$$|u'(t)| \leq |u'(\hat{t})| + |u'(t) - u'(\hat{t})| \leq \frac{R_J}{t_2 - t_1} + \|\gamma_{R_J}\|_{L^1(I)} + \alpha(t_2 - t_1) =: K,$$

where K is a constant depending on J, R_J and α. As a consequence,

$$\|(u(t), u'(t))\| \leq K^* := R_J + K, \quad \forall\, t \in J,$$

where we use $\|(\xi_1, \xi_2)\| = |\xi_1| + |\xi_2|$ as a standard norm in $\mathbb{R}^2$.

Now, recalling the Carathéodory condition on $|f(t,s)|$, we rewrite hypothesis (G) in this form:

(G') *there exist* $a_1, b_1 \in L^1([0,t_1], \mathbb{R}^+)$, $a_2, b_2 \in L^1([t_2,T], \mathbb{R}^+)$, *such that, for all* $s \geq 0$,

$$\begin{aligned} |f(t,s)| &\leq a_1(t) + b_1(t)s, \quad a.e.\ t \in [0,t_1]; \\ |f(t,s)| &\leq a_2(t) + b_2(t)s, \quad a.e.\ t \in [t_2,T]. \end{aligned}$$

Suppose $t_2 < T$. For all $t \in \,]t_2, T]$ we have $\mathbb{1}_J(t) = 0$ and then

$$\begin{aligned} &\|(u(t), u'(t))\| = |u(t)| + |u'(t)| = u(t) + |u'(t)| = \\ &= u(t_2) + \int_{t_2}^{t} u'(\xi)\, d\xi + \left| u'(t_2) + \int_{t_2}^{t} \big[-f(\xi, u(\xi)) - \zeta\varphi(\xi)\mathbb{1}_J(\xi)\big]\, d\xi \right| \\ &\leq \|(u(t_2), u'(t_2))\| + \int_{t_2}^{t} |u'(\xi)|\, d\xi + \int_{t_2}^{t} |f(\xi, u(\xi))|\, d\xi \\ &\leq K^* + \int_{t_2}^{t} |u'(\xi)|\, d\xi + \int_{t_2}^{t} \big[a_2(\xi) + b_2(\xi)u(\xi)\big]\, d\xi \\ &\leq K^* + \|a_2\|_{L^1([t_2,T])} + \int_{t_2}^{t} (b_2(\xi) + 1)\|(u(\xi), u'(\xi))\|\, d\xi. \end{aligned}$$

Define

$$R^2_\alpha := \left(K^* + \|a_2\|_{L^1([t_2,T])}\right) e^{\|1+b_2\|_{L^1([t_2,T])}}$$

(observe that K^* depends on α). By Gronwall's inequality, we have

$$0 \leq u(t) \leq \|(u(t), u'(t))\| \leq R^2_\alpha, \quad \forall\, t \in [t_2, T].$$

If $t_1 > 0$, we achieve a similar upper bound (denoted by R^1_α) for $u(t)$ on $[0, t_1]$. The proof is analogous and therefore is omitted. We conclude that F satisfies condition (ii) of Theorem A.2.2, with $R_\alpha := \max\{R^1_\alpha, R^2_\alpha\}$.

Proof of (iii). Let us fix a constant $\alpha_0 > 0$ with

$$\alpha_0 > \frac{\lambda_J R_J (t_2 - t_1) + \|\gamma_{R_J}\|_{L^1(J)}}{\|\varphi\|^2_{L^2(J)}}.$$

Suppose by contradiction that there exist $\tilde{u} \in \overline{\Omega_J}$ and $\tilde{\alpha} \geq \alpha_0$ such that

$\tilde{u} = F(\tilde{u}, \tilde{\alpha})$. Then, we obtain

$$\begin{aligned}
0 &\geq \tilde{u}(t_2)\varphi'(t_2) - \tilde{u}(t_1)\varphi'(t_1) = \Big[\tilde{u}(t)\varphi'(t) - \tilde{u}'(t)\varphi(t)\Big]_{t=t_1}^{t=t_2} \\
&= \int_{t_1}^{t_2} \frac{d}{dt}\Big[\tilde{u}(t)\varphi'(t) - \tilde{u}'(t)\varphi(t)\Big]\,dt = \int_J \Big[\tilde{u}(t)\varphi''(t) - \tilde{u}''(t)\varphi(t)\Big]\,dt \\
&= \int_J \Big[-\lambda_J \tilde{u}(t)\varphi(t) + f(t, \tilde{u}(t))\varphi(t) + \tilde{\alpha}(\varphi(t))^2\Big]\,dt \\
&\geq -\lambda_J R_J (t_2 - t_1) - \|\gamma_{R_J}\|_{L^1(J)} + \tilde{\alpha}\|\varphi\|^2_{L^2(J)} > 0,
\end{aligned}$$

a contradiction. Hence F satisfies (iii).

We have thus verified all the conditions in Theorem A.2.2. This concludes the proof. $\square$

1.3 Multiplicity results

In this section we propose an approach based on the additivity and excision properties of the Leray-Schauder degree, in order to provide sharp multiplicity results for positive solutions of the Dirichlet problem (1.1.1) and, moreover, to have more precise information about the localization of the solutions.

Throughout the section we suppose that $f(t,s)$ *is an* L^1*-Carathéodory function satisfying* (f^*), (f_0^-), (f_0^+) *and* (H) *with* (f_∞). Recall also that, in view of the discussion in Section 1.1, the positive solutions of (1.1.1) are the nontrivial fixed points of the completely continuous operator $\Phi \colon \mathcal{C}(I) \to \mathcal{C}(I)$ defined in (1.1.3).

We introduce now some notation. Let $\mathcal{I} \subseteq \{1, \ldots, m\}$ be a subset of indices (possibly empty) and let r, R be two fixed constants with

$$0 < r \leq r_0 < R^* \leq R,$$

where r_0 and R^* are the constants obtained from Lemma 1.1.1 and from Lemma 1.1.4, respectively. We define two families of open and possibly unbounded sets

$$\Omega^{\mathcal{I}} := \left\{ u \in \mathcal{C}(I) \colon \max_{t \in I_i^+} |u(t)| < R,\, i \in \mathcal{I}; \right.$$
$$\left. \max_{t \in I_i^+} |u(t)| < r,\, i \in \{1, \ldots, m\} \setminus \mathcal{I} \right\}$$

and

$$\Lambda^{\mathcal{I}} := \left\{ u \in \mathcal{C}(I) \colon r < \max_{t \in I_i^+} |u(t)| < R,\, i \in \mathcal{I}; \right.$$
$$\left. \max_{t \in I_i^+} |u(t)| < r,\, i \in \{1, \ldots, m\} \setminus \mathcal{I} \right\}.$$

See Figure 1.1 for the representations of the sets $\Omega^{\mathcal{I}}$ and $\Lambda^{\mathcal{I}}$ in the easiest case $m = 2$.

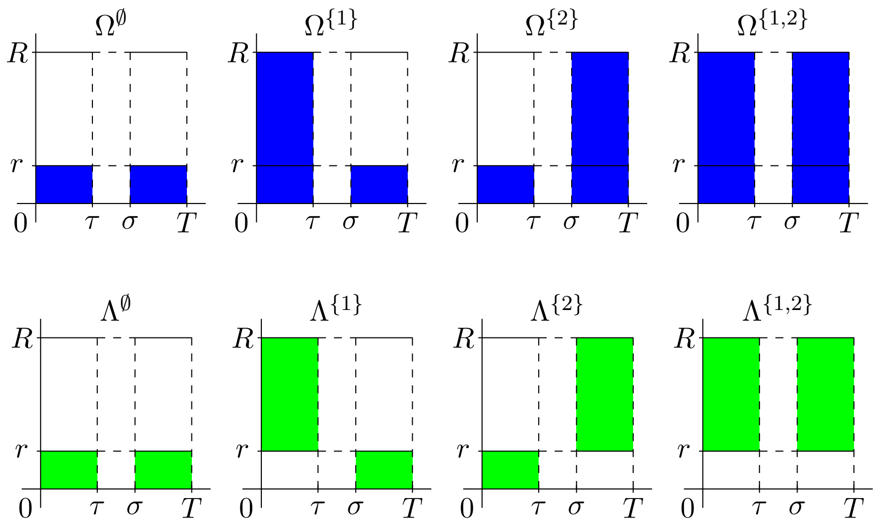

Figure 1.1: The figure represents the families of sets $\Omega^{\mathcal{I}}$ and $\Lambda^{\mathcal{I}}$, when $m = 2$ and the subintervals of positivity $I_1^+ := [0,\tau]$ and $I_2^+ := [\sigma, T]$ are arranged as in the figure. The sets $\Omega^{\mathcal{I}}$ are made up of the continuous functions on $[0,T]$ which are in the blue area on the intervals I_1^+ and I_2^+, while in the remaining interval (of negativity) the functions have no constraints. The sets $\Lambda^{\mathcal{I}}$ are made up of the functions in $\Omega^{\mathcal{I}}$ such that the maximum on I_i^+, $i = 1, 2$, is in the green area. Since a maximum principle ensures that the solutions of (1.1.1) are non-negative, in the figure we consider only the non-negative functions belonging to $\Omega^{\mathcal{I}}$ and $\Lambda^{\mathcal{I}}$, respectively.

We note that, for each $\mathcal{I} \subseteq \{1, \dots, m\}$, we have

$$\Omega^{\mathcal{I}} = \bigcup_{\mathcal{J}\subseteq\mathcal{I}} \Lambda^{\mathcal{J}} \cup \bigcup_{i\in\mathcal{I}} \Big\{ u \in \Omega^{\mathcal{I}} \colon \max_{t\in I_i^+} |u(t)| = r \Big\}$$

and the union $\bigcup_{\mathcal{J}\subseteq\mathcal{I}} \Lambda^{\mathcal{J}}$ is disjoint, since $\Lambda^{\mathcal{J}'} \cap \Lambda^{\mathcal{J}''} = \emptyset$, for $\mathcal{J}' \neq \mathcal{J}''$.

We also observe that for $\mathcal{I} = \emptyset$ we have

$$\Lambda^{\emptyset} = \Omega^{\emptyset} = \Big\{ u \in \mathcal{C}(I) \colon \max_{t\in I_i^+} |u(t)| < r, \, i \in \{1, \dots, m\} \Big\} \supseteq B(0, r).$$

By the maximum principle (cf. Lemma C.1.1) any solution $u \in \mathrm{cl}(\Lambda^{\emptyset})$ of the operator equation $u = \Phi(u)$ is a (non-negative) solution of (1.1.1) such that $0 \leq u(t) \leq r$, for all $t \in \bigcup_{i=1}^m I_i^+$. On the other hand, we know that $u(t)$ is convex in each interval contained in $I \setminus \bigcup_{i=1}^m I_i^+$ and thus we conclude that $0 \leq u(t) \leq r$, for all $t \in I$, so that $u \in B[0, r]$. Lemma 1.1.1, Lemma 1.1.2 and the choice of $r \in \,]0, r_0]$ then imply

$$\deg_{LS}(Id - \Phi, \Lambda^{\emptyset}, 0) = \deg_{LS}(Id - \Phi, B(0, r), 0) = 1. \tag{1.3.1}$$

The above relation shows that even if $\Lambda^{\mathcal{I}}$ is an unbounded open set, then, at least for $\mathcal{I} = \emptyset$, the topological degree is well-defined. The next result is the key lemma to provide the existence of nontrivial fixed points (and hence multiplicity results) whenever the topological degree is defined on the sets $\Lambda^{\mathcal{I}}$ and $\Omega^{\mathcal{I}}$.

Lemma 1.3.1. *Let $\mathcal{I} \subseteq \{1, \ldots, m\}$ be a set of indices. Suppose that for all $\mathcal{J} \subseteq \mathcal{I}$ the triplets $(Id - \Phi, \Lambda^{\mathcal{J}}, 0)$ and $(Id - \Phi, \Omega^{\mathcal{J}}, 0)$ are admissible with*

$$\deg_{LS}(Id - \Phi, \Omega^{\mathcal{J}}, 0) = 0, \quad \forall\, \emptyset \neq \mathcal{J} \subseteq \mathcal{I}. \tag{1.3.2}$$

Then

$$\deg_{LS}(Id - \Phi, \Lambda^{\mathcal{I}}, 0) = (-1)^{\#\mathcal{I}}. \tag{1.3.3}$$

Proof. First of all, we notice that, in view of (1.3.1), the conclusion is trivially satisfied when $\mathcal{I} = \emptyset$. Suppose now that $m := \#\mathcal{I} \geq 1$. We are going to prove our claim by using an inductive argument. More precisely, for every integer k with $0 \leq k \leq m$, we introduce the property $\mathscr{P}(k)$ which reads as follows.

$\mathscr{P}(k)$: *The formula*

$$\deg_{LS}(Id - \Phi, \Lambda^{\mathcal{J}}, 0) = (-1)^{\#\mathcal{J}}$$

holds for each subset $\mathcal{J}$ of $\mathcal{I}$ having at most k elements.

In this manner, if we are able to prove $\mathscr{P}(m)$, then (1.3.3) follows.

Verification of $\mathscr{P}(0)$. See (1.3.1).

Verification of $\mathscr{P}(1)$. If $\mathcal{J} = \emptyset$ the result is already proved in (1.3.1). If $\mathcal{J} = \{j\}$, with $j \in \mathcal{I}$, we have

$$\begin{aligned}\deg_{LS}(Id - \Phi, \Lambda^{\mathcal{J}}, 0) &= \deg_{LS}(Id - \Phi, \Lambda^{\{j\}}, 0)\\ &= \deg_{LS}(Id - \Phi, \Omega^{\{j\}} \setminus \Lambda^{\emptyset}, 0)\\ &= 0 - 1 = -1 = (-1)^{\#\mathcal{J}}.\end{aligned}$$

Verification of $\mathscr{P}(k-1) \Rightarrow \mathscr{P}(k)$, for $1 \leq k \leq m$. Assuming the validity of $\mathscr{P}(k-1)$ we have that the formula is true for every subset of $\mathcal{I}$ having at most $k-1$ elements. Therefore, in order to prove $\mathscr{P}(k)$, we have only to check that the formula is true for an arbitrary subset $\mathcal{J}$ of $\mathcal{I}$ with $\#\mathcal{J} = k$. First, we write $\Omega^{\mathcal{J}}$ as the disjoint union

$$\Omega^{\mathcal{J}} = \Lambda^{\mathcal{J}} \cup \bigcup_{\mathcal{K} \subsetneq \mathcal{J}} \Lambda^{\mathcal{K}} \cup \bigcup_{i \in \mathcal{J}} \Big\{ u \in \Omega^{\mathcal{J}} \colon \max_{t \in I_i^+} |u(t)| = r \Big\}$$

and, since the degree is well-defined on the sets $\Lambda^{\mathcal{K}}$, we observe that there is no fixed point of Φ with $u \in \bigcup_{i \in \mathcal{J}} \big\{ u \in \Omega^{\mathcal{J}} \colon \max_{I_i^+} |u| = r \big\}$. Then, by the inductive hypothesis, we obtain

$$\begin{aligned}
&\deg_{LS}(Id-\Phi,\Lambda^{\mathcal{J}},0)=\\
&=\deg_{LS}(Id-\Phi,\Omega^{\mathcal{J}},0)-\sum_{\mathcal{K}\subsetneq\mathcal{J}}\deg_{LS}(Id-\Phi,\Lambda^{\mathcal{K}},0)\\
&=0-\sum_{\mathcal{K}\subsetneq\mathcal{J}}(-1)^{\#\mathcal{K}}=-\sum_{\mathcal{K}\subseteq\mathcal{J}}(-1)^{\#\mathcal{K}}+(-1)^{\#\mathcal{J}}.
\end{aligned}$$

Observe now that

$$\sum_{\mathcal{K}\subseteq\mathcal{J}}(-1)^{\#\mathcal{K}}=0,$$

due to the fact that in a finite set there are so many subsets with even cardinality as there are with odd cardinality. Thus we conclude that

$$\deg_{LS}(Id-\Phi,\Lambda^{\mathcal{J}},0)=(-1)^{\#\mathcal{J}}.$$

Therefore $\mathscr{P}(k)$ is proved. □

In order to apply Lemma 1.3.1 we have to check assumption (1.3.2). To this aim, we introduce a third family of unbounded sets, defined as follows

$$\Gamma^{\mathcal{I}}:=\Big\{u\in\mathcal{C}(I)\colon \max_{t\in I_i^+}|u(t)|<r,\ i\in\{1,\dots,m\}\setminus\mathcal{I}\Big\},$$

where $\mathcal{I}\subseteq\{1,\dots,m\}$. See Figure 1.2 for the representation of $\Gamma^{\mathcal{I}}$ when $m=2$.

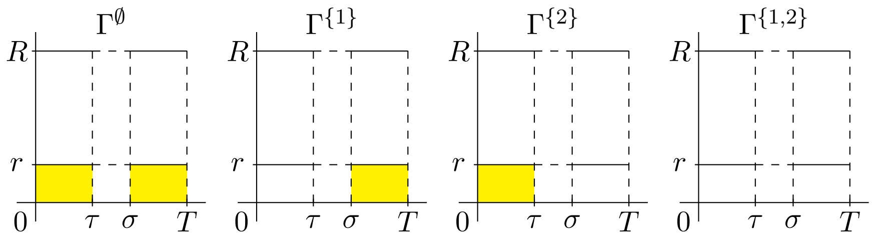

Figure 1.2: The figure represents the family of sets $\Gamma^{\mathcal{I}}$, when $m=2$ and the subintervals of positivity $I_1^+:=[0,\tau]$ and $I_2^+:=[\sigma,T]$ are arranged as in the figure. The sets $\Gamma^{\mathcal{I}}$ are made up of the continuous functions on $[0,T]$ such that the maximum on some of I_i^+, $i=1,2$, is in the yellow area, while in the remaining interval the functions have no constraints. Notice that $\Gamma^{\emptyset}=\Omega^{\emptyset}=\Lambda^{\emptyset}$, while $\Gamma^{\{1,2\}}=\mathcal{C}(I)$, because there are not constraints for the continuous functions. Since a maximum principle ensures that the solutions of (1.1.1) are non-negative, we consider only the non-negative function in $\Gamma^{\mathcal{I}}$.

It is also convenient to consider L^1-Carathéodory functions $h\colon I \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\begin{aligned} &h(t,s) \geq f(t,s), \quad \text{a.e. } t \in \bigcup_{i\in\mathcal{I}} I_i^+,\ \forall\, s \geq 0; \\ &h(t,s) = f(t,s), \quad \text{a.e. } t \in I \setminus \bigcup_{i\in\mathcal{I}} I_i^+,\ \forall\, s \geq 0. \end{aligned} \tag{1.3.4}$$

Using a standard procedure, for any given $h(t,s)$ as above, we define a completely continuous operator $\Psi_h\colon \mathcal{C}(I) \to \mathcal{C}(I)$ as

$$(\Psi_h u)(t) := \int_I G(t,\xi)\tilde{h}(\xi,u(\xi))\,d\xi, \quad t \in I,$$

where

$$\tilde{h}(t,s) = \begin{cases} h(t,s), & \text{if } s \geq 0; \\ h(t,0), & \text{if } s \leq 0. \end{cases}$$

Notice that $h(t,0) = 0$, for a.e. $t \notin \bigcup_{i\in\mathcal{I}} I_i^+$, while $h(t,0) \geq 0$, for a.e. $t \in \bigcup_{i\in\mathcal{I}} I_i^+$. In this manner, the first part of Lemma C.1.1 applies for $\tilde{h}(t,s)$.

The next result provides sufficient conditions for (1.3.2).

Lemma 1.3.2. *Let $\mathcal{I} \subseteq \{1,\ldots,m\}$, with $\mathcal{I} \neq \emptyset$. Suppose that the triplet $(Id - \Psi_h, \Gamma^{\mathcal{I}}, 0)$ is admissible for every Carathéodory function $h\colon I \times \mathbb{R}^+ \to \mathbb{R}$ satisfying* (1.3.4). *Then*

$$\deg_{LS}(Id - \Phi, \Omega^{\mathcal{I}}, 0) = 0.$$

Proof. The proof combines some arguments previously developed along the proofs of Lemma 1.1.4 and Theorem 1.2.2. In order to simplify the notation we set $A := \bigcup_{i\in\mathcal{I}} I_i^+$.

For each index $i \in \mathcal{I}$, we define $\lambda_{I_i^+} := \mu_1^{I_i^+}(1) = (\pi/|I_i^+|)^2$ and we denote by φ_i the positive eigenfunction of

$$\begin{cases} \varphi'' + \lambda_{I_i^+}\varphi = 0 \\ \varphi|_{\partial I_i^+} = 0, \end{cases}$$

with $\|\varphi\|_\infty = 1$. Then $\varphi_i(t) \geq 0$, for all $t \in I_i^+$. We denote by $\mathbb{1}_{I_i^+}$ the indicator function of the interval I_i^+.

Next, we define

$$v(t) := \int_I G(t,s)\biggl(\sum_{i\in\mathcal{I}} \varphi_i(s)\mathbb{1}_{I_i^+}(s)\biggr)\,ds$$

and introduce the operator $F\colon \mathcal{C}(I) \times [0,+\infty[\to \mathcal{C}(I)$, as

$$F(u,\alpha) := \Phi(u) + \alpha v.$$

To prove our claim, we check (ii) and (iii) of Theorem A.2.2 (clearly, $F(u,0) = \Phi(u)$, so that (i) is trivially satisfied).

Proof of (ii). Fix $\alpha \geq 0$. By the definition of $v(t)$ and the first part of Lemma C.1.1, any nontrivial solution $u \in \mathrm{cl}(\Omega^{\mathcal{I}})$ of

$$u = F(u, \zeta), \quad \text{for } \zeta \in [0, \alpha],$$

is a non-negative solution of

$$\begin{cases} u'' + h(t,u) = 0 \\ u(0) = u(T) = 0 \end{cases} \tag{1.3.5}$$

with

$$h(t,s) := f(t,s) + \zeta \sum_{i \in \mathcal{I}} \varphi_i(t) \mathbb{1}_{I_i^+}(t).$$

The hypothesis $u \in \mathrm{cl}(\Omega^{\mathcal{I}})$ implies that $0 \leq u(t) \leq r$ for all $t \in I_i^+$, if $i \notin \mathcal{I}$, and $0 \leq u(t) \leq R$ for all $t \in I_i^+$, if $i \in \mathcal{I}$.

We first note that $0 \leq u(t) < R$, for every $t \in A$, by the choice of $R \geq R^*$. If $\mathcal{I} = \{1, \ldots, m\}$, clearly $u \in \Omega^{\mathcal{I}}$. Otherwise, the admissibility of the triplets $(Id - \Psi_h, \Gamma^{\mathcal{I}}, 0)$ implies that any Ψ_h has no fixed points on $\partial \Gamma^{\mathcal{I}}$. Then each non-negative solution of (1.3.5) satisfies $0 \leq u(t) < r$, for all $t \in \bigcup_{i \notin \mathcal{I}} I_i^+$. We deduce that $u \in \Omega^{\mathcal{I}}$.

Since $h(t,s) = f(t,s) \leq 0$ for a.e. $t \in I \setminus \bigcup_{i=1}^m I_i^+$ and for all $s \geq 0$, by convexity, we conclude that

$$\|u\|_\infty = \max_{t \in I} u(t) \leq R.$$

Then (ii) is proved with $R_\alpha := R$.

Proof of (iii). Let us fix a constant $\alpha_0 > 0$ with

$$\alpha_0 > \max_{i \in \mathcal{I}} \frac{\lambda_{I_i^+} RT + \|\gamma_R\|_{L^1(I)}}{\|\varphi_i\|^2_{L^2(I_i^+)}}.$$

Suppose, by contradiction, that there exist $\tilde{u} \in \mathrm{cl}(\Omega^{\mathcal{I}})$ and $\tilde{\alpha} \geq \alpha_0$ such that $\tilde{u} = F(\tilde{u}, \tilde{\alpha}) = \Phi(\tilde{u}) + \tilde{\alpha} v$. Since $\Phi(0) = 0$, we have $\tilde{u} \not\equiv 0$ and, as in the previous step, $\tilde{u}(t)$ is a non-negative solution of (1.3.5) for

$$h(t,s) := f(t,s) + \tilde{\alpha} \sum_{i \in \mathcal{I}} \varphi_i(t) \mathbb{1}_{I_i^+}(t).$$

By assumption, $\mathcal{I} \neq \emptyset$. So, let us fix an index $k \in \mathcal{I}$ and set $I_k^+ := [t_1, t_2]$.

Arguing as in the proof of Theorem 1.2.2, we obtain

$$
\begin{aligned}
0 &\geq \tilde{u}(t_2)\varphi_k'(t_2) - \tilde{u}(t_1)\varphi_k'(t_1) = \Big[\tilde{u}(t)\varphi_k'(t) - \tilde{u}'(t)\varphi_k(t)\Big]_{t=t_1}^{t=t_2} \\
&= \int_{I_k^+} \frac{d}{dt}\Big[\tilde{u}(t)\varphi_k'(t) - \tilde{u}'(t)\varphi_k(t)\Big]\,dt = \int_{I_k^+}\Big[\tilde{u}(t)\varphi_k''(t) - \tilde{u}''(t)\varphi_k(t)\Big]\,dt \\
&= \int_{I_k^+}\Big[-\lambda_{I_k^+}\tilde{u}(t)\varphi_k(t) + f(t,\tilde{u}(t))\varphi_k(t) + \tilde{\alpha}(\varphi_k(t))^2\Big]\,dt \\
&\geq -\lambda_{I_k^+} R(t_2-t_1) - \|\gamma_R\|_{L^1(I_k^+)} + \tilde{\alpha}\|\varphi_k\|^2_{L^2(I_k^+)} > 0,
\end{aligned}
$$

a contradiction. Hence F satisfies (iii). $\square$

Putting together Lemma 1.3.1 and Lemma 1.3.2 we can obtain results of multiplicity of positive solutions provided that we are able to show that the topological degree on certain open sets is well-defined. With this respect, observe that from Lemma 1.1.3 we know that there are no positive solutions $u(t)$ with $\max_{t\in I_i^+} u(t) \geq R$. Thus, we only have to show that the level r is not achieved by the solutions $u(t)$ of $u = \Psi_h(u)$ for t in some of the intervals I_i^+.

Theorem 1.3.1. *Let $f\colon I\times\mathbb{R}^+ \to \mathbb{R}$ be an L^1-Carathéodory function satisfying (f^*), (f_0^-), (f_0^+) and (H) with (f_∞). Suppose that for every Carathéodory function $g\colon I\times\mathbb{R}^+\to\mathbb{R}$ satisfying* (1.3.4) *and for every $\emptyset \neq \mathcal{I} \subseteq \{1,\ldots,m\}$ the triplet $(Id-\Psi_h,\Gamma^{\mathcal{I}},0)$ is admissible. Then there exist at least 2^m-1 positive solutions of the two-point boundary value problem* (1.1.1)*.*

Proof. First of all, we claim that the triplet $(Id-\Phi,\Lambda^{\mathcal{I}},0)$ is admissible for all $\mathcal{I}\subseteq\{1,\ldots,m\}$. Indeed, if $\mathcal{I}=\emptyset$, this is clear from (1.3.1). If $\mathcal{I}\neq\emptyset$, the claim follows since all possible fixed points of Φ are contained in $B(0,R)$ (as already observed) and they can not achieve the radius r by virtue of the admissibility of $(Id-\Psi_h,\Gamma^{\mathcal{I}},0)$ for all $\emptyset\neq\mathcal{I}\subseteq\{1,\ldots,m\}$.

From Lemma 1.3.1 and Lemma 1.3.2 it follows that

$$
\deg_{LS}(Id-\Phi,\Lambda^{\mathcal{I}},0)\neq 0, \quad \text{for all } \mathcal{I}\subseteq\{1,\ldots,m\}.
$$

Notice that $0\notin\Lambda^{\mathcal{I}}$, for all $\emptyset\neq\mathcal{I}\subseteq\{1,\ldots,m\}$, and the sets $\Lambda^{\mathcal{I}}$ are pairwise disjoint. We obtain the claim using the fact that the number of nonempty subsets of a set with m elements is 2^m-1. $\square$

1.4 A special case: $f(t,s)=a(t)g(s)$

In this section we provide an application of the existence and multiplicity results obtained in Section 1.2 and Section 1.3 to the search of positive solutions for a two-point boundary value problem of the form

$$\begin{cases} u'' + a(t)g(u) = 0 \\ u(0) = u(T) = 0, \end{cases} \tag{1.4.1}$$

where we suppose that $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that

$$(g_*) \qquad g(0) = 0, \qquad g(s) > 0 \quad \text{for} \quad s > 0.$$

With the aim of providing a simplified exposition of our main result, we suppose that the weight function $a\colon I \to \mathbb{R}$ is *continuous*. The more general case of an L^1-weight function can be treated as well with minor modifications in the statements of the theorems (this will be briefly discussed in Section 1.4.4). Since we are looking for *positive solutions* of (1.4.1), in order to avoid trivial situations, we suppose that

$$a^+(t) := \max\{a(t), 0\} \not\equiv 0. \tag{1.4.2}$$

In the context of continuous functions this is just the same as to assume $a(x_0) > 0$ for some $x_0 \in I$. As usual, we also set $a^-(t) := \max\{-a(t), 0\}$, so that $a(t) = a^+(t) - a^-(t)$.

In order to enter the general setting of the previous sections for

$$f(t,s) := a(t)g(s), \tag{1.4.3}$$

we suppose that

$$g_0 := \limsup_{s \to 0^+} \frac{g(s)}{s} < +\infty \quad \text{and} \quad g_\infty := \liminf_{s \to +\infty} \frac{g(s)}{s} > 0.$$

In such a situation, we can extend $g(s)$ to the negative real line, by setting $g(s) = 0$, for every $s \leq 0$. Then, Lemma C.1.1 ensures that any nontrivial solution $u(t)$ of (1.4.1) satisfies $u(t) > 0$ for all $t \in \,]0,T[$ with $u'(0) > 0 > u'(T)$. Notice also that $f(t,s)$ defined as in (1.4.3) satisifies (f_0^-) for $q_-(t) := g_0\, a^-(t)$.

Now, we translate condition (f_0^+) in the new setting. From

$$\limsup_{s \to 0^+} \frac{f(t,s)}{s} = \limsup_{s \to 0^+} \frac{a(t)g(s)}{s} \leq g_0\, a^+(t), \quad \text{uniformly for all } t \in I,$$

we immediately conclude that (f_0^+) holds if and only if

$$g_0 < \lambda_0,$$

where $\lambda_0 > 0$ is the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a^+(t)\varphi = 0, \quad \varphi(0) = \varphi(T) = 0.$$

As a next step, we look for an equivalent formulation of conditions (H) and (f_∞) for $f(t,s)$ as in (1.4.3). Accordingly, we consider the following hypothesis on the weight function.

(a_*) *There exists a finite sequence of* $2m+2$ *points in* $[0,T]$ *(possibly coincident)*

$$0 = \tau_0 \leq \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \ldots < \sigma_m < \tau_m \leq \sigma_{m+1} = T$$

such that

- $a(t) \geq 0$, *for all* $t \in [\sigma_i, \tau_i]$, $i = 1, \ldots, m$;
- $a(t) \leq 0$, *for all* $t \in]\tau_i, \sigma_{i+1}[$, $i = 0, 1, \ldots, m$.

By assuming (a_*) we implicitly suppose that $a(t)$ vanishes at the points $x = \tau_1, \sigma_2, \ldots, \tau_{m-1}, \sigma_m$. With a usual convention, if $\tau_0 = \sigma_1$ (or $\tau_m = \sigma_{m+1}$) the assumption $a(t) \leq 0$ on the first open interval (or on the last one, respectively) is vacuously satisfied.

Remark 1.4.1. The sign condition on the weight function allows the possibility that $a(t)$ may identically vanish in some subintervals of I (even infinitely many). Figure 1.3 shows a possible graph which is in agreement with assumption (a_*). $\lhd$

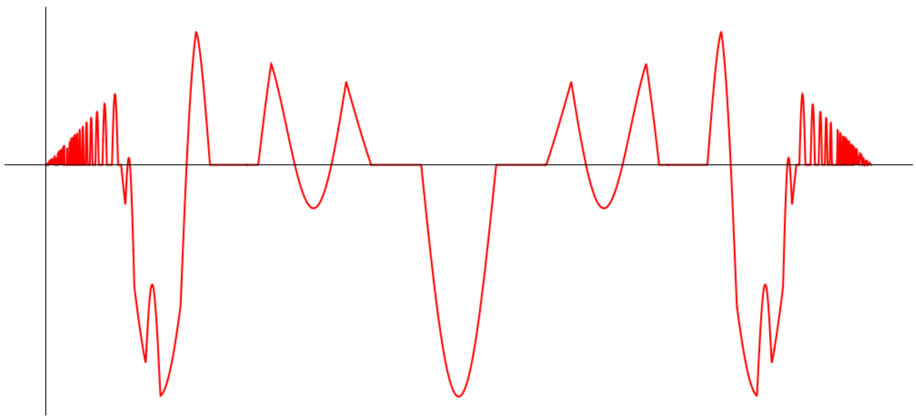

Figure 1.3: The figure shows the graph of the continuous function $a(t) := \max\{0, t(1-t)\sin(3/t(1-t))\} - \max\{0, -\sin(11t\pi)/4\}$ on $]0,1[$ and defined as 0 at the endpoints. This is an example of weight function that satisfies (a_*) for an obvious choice of the points σ_i and τ_i and, moreover, it has infinitely many humps.

Given any $a(t)$ satisfying (a_*), consistently with the notation introduced in Section 1.1, we set

$$I_i^+ := [\sigma_i, \tau_i], \quad i = 1, \ldots, m.$$

For such a choice of the weight function $a(t)$, we have that (H) is satisfied for $f(t,s)$ as in (1.4.3). Moreover, for every $i = 1, \ldots, m$, we obtain

$$\liminf_{s \to +\infty} \frac{f(t,s)}{s} = \liminf_{s \to +\infty} \frac{a(t)g(s)}{s} \geq g_\infty \, a(t), \quad \text{uniformly for all } t \in I_i^+.$$

Thus we conclude that (f_∞) holds provided that

$$a(t) \not\equiv 0 \ \text{ on } I_i^+ \quad \text{and} \quad g_\infty > \lambda_1^i, \quad \forall\, i = 1, \dots, m,$$

where $\lambda_1^i > 0$ is the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a(t)\varphi = 0, \quad \varphi|_{\partial I_i^+} = 0.$$

Notice that, as a consequence of Sturm theory (see, for instance, [55, 183]), we know that

$$\lambda_0 \leq \lambda_1^i, \quad \forall\, i = 1, \dots, m.$$

1.4.1 Existence of positive solutions

Now we are in a position to present some corollaries of the existence results in Section 1.2 for problem (1.4.1). In this context, Theorem 1.2.1 implies the following one.

Theorem 1.4.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and let $a\colon I \to \mathbb{R}$ be a continuous function satisfying (a_*). Moreover, suppose that*

$$g_0 < \lambda_0$$

and $a(t) \not\equiv 0$ on I_i^+, for each $i = 1, \dots, m$, with

$$g_\infty > \max_{i=1,\dots,m} \lambda_1^i.$$

Then problem (1.4.1) has at least a positive solution.

As an obvious corollary of Theorem 1.4.1, we have that if $g_0 = 0$ and $g_\infty = +\infty$, then a positive solution always exists, provided that $a(t) \not\equiv 0$ on I_i^+ (see Corollary 1.2.1 and also compare to [93, Corollary 4.2]).

Remark 1.4.2. First of all, we observe that Theorem 1.4.1 (as well as the more general Theorem 1.2.1) applies in a trivial manner if $a(t) \geq 0$ (and $a(t) \not\equiv 0$) on I. Indeed, as already remarked after the introduction of condition (H), such hypothesis is satisfied also when $f(t,s) \geq 0$ for a.e. $t \in I$ and for each $s \geq 0$.

In the case of a sign-changing weight function $a(t)$, namely when $a^+(t) \not\equiv 0$ and also $a^-(t) \not\equiv 0$, the choice of the intervals I_i^+ is mandatory when the set $a^{-1}(0) = \{t \in I \colon a(t) = 0\}$ is made by a finite number of simple zeros. In such a situation, $a(t) > 0$ on $]\sigma_i, \tau_i[$ and $a(t) < 0$ on $]\tau_i, \sigma_{i+1}[$. The choice of the intervals I_i^+ is also determined if $a^{-1}(0)$ is finite. However, generally speaking, there is some arbitrariness in the choice of the way in which we separate the intervals of non-negativity to the intervals of non-positivity of $a(t)$. This happens, for instance, when $a^{-1}(0)$ contains an interval. In such a situation, the manner in which we define the intervals I_i^+ affects the computation of the eigenvalues λ_i and hence the lower bound for g_∞.

With this respect we exhibit a simple example. Let us consider the following weight function

$$a_\varepsilon(t) = \begin{cases} 1, & \text{if } t \in \left[0, \dfrac{\pi}{2} - \varepsilon\right] \cup \left[\dfrac{\pi}{2} + \varepsilon, \pi\right]; \\ 0, & \text{if } t \in \left]\dfrac{\pi}{2} - \varepsilon, \dfrac{\pi}{2} + \varepsilon\right[; \\ -1, & \text{if } t \in \left]\pi, 2\pi\right]; \end{cases} \tag{1.4.4}$$

where $0 < \varepsilon < \pi/2$ is fixed (cf. Figure 1.4). For convenience, we have chosen for our example a (discontinuous) step function, however, our argument can be adapted in the continuous case via a smoothing procedure on $a_\varepsilon(t)$.

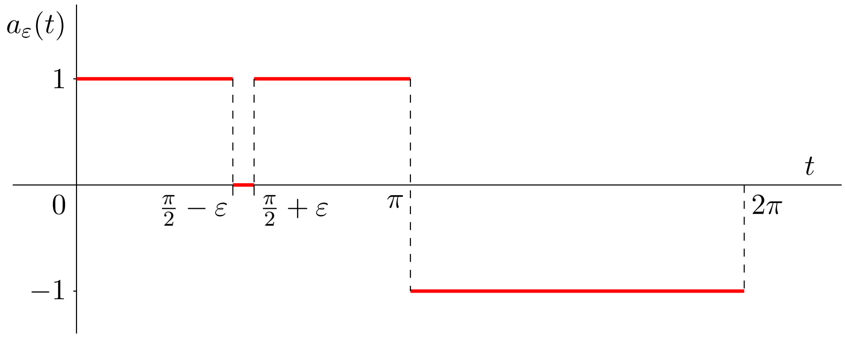

Figure 1.4: The figure shows the graph of the function $a_\varepsilon \colon [0, 2\pi] \to \mathbb{R}$ defined in (1.4.4).

For this weight function we can take $I_1^+ = [\sigma_1, \tau_1] = [0, \frac{\pi}{2} - \varepsilon]$ and $I_2^+ = [\sigma_2, \tau_2] = [\frac{\pi}{2} + \varepsilon, \pi]$. In this situation, we have $a_\varepsilon(t) = 0$ on $]\tau_1, \sigma_2[=]\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon[$ and $a_\varepsilon(t) < 0$ on $]\tau_2, \sigma_3] =]\pi, 2\pi]$. Moreover,

$$\lambda_1^1 = \lambda_1^2 = \left(\frac{2\pi}{\pi - 2\varepsilon}\right)^2 > 4.$$

On the other hand, for the same weight function, we can also take $I_1^+ = [\sigma_1, \tau_1] = [0, \pi]$ as unique interval of non-negativity. To compute λ_1^1, we have to determine the first eigenvalue of $\varphi'' + \lambda a_\varepsilon(t)\varphi = 0$ with $\varphi(0) = \varphi(\pi) = 0$. For $\varepsilon > 0$ very close to zero, we find that λ_1^1 is close to 1 (and for sure less than 4). As a consequence, with this second choice of the interval, we provide a better lower bound for g_∞.

The above example shows that Theorem 1.4.1 is a slightly more general version of [93, Theorem 4.1], in the sense that we can improve the lower bound on g_∞ (at least for some particular weight functions which vanish on their intervals of non-negativity). $\lhd$

Another way to improve the lower bound on g_∞ of Theorem 1.4.1 is feasible by applying Theorem 1.2.2. However, this requires to impose a further growth assumption on $g(s)$.

Theorem 1.4.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and let $a\colon I \to \mathbb{R}$ be a continuous function satisfying (a_*). Moreover, suppose that*

$$g_0 < \lambda_0$$

and $a(t) \not\equiv 0$ on I_i^+, for each $i = 1, \ldots, m$, with

$$g_\infty > \min_{i=1,\ldots,m} \lambda_1^i \quad \textit{and} \quad \limsup_{s\to+\infty} \frac{g(s)}{s} < +\infty.$$

Then problem (1.4.1) has at least a positive solution.

Remark 1.4.3. We have stated Theorem 1.4.2 in a form which is suitable for a comparison with Theorem 1.4.1. Actually, the result holds even we do not assume (a_*), but we just suppose that there exists an interval $J \subseteq I$ where $a(t) \geq 0$ and $a(t) \not\equiv 0$, and $g_\infty > \lambda_1^J$, where λ_1^J is the first eigenvalue of the eigenvalue problem $\varphi'' + \lambda a(t)\varphi = 0$, $\varphi|_{\partial J} = 0$. $\lhd$

We conclude this section with a direct corollary of Theorem 1.4.1.

Corollary 1.4.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and let $a\colon I \to \mathbb{R}$ be a continuous function satisfying (a_*). Moreover, suppose that*

$$g_0 = 0 \quad \textit{and} \quad g_\infty > 0.$$

Then, there exists $\nu^* > 0$ such that, for each $\nu > \nu^*$, the boundary value problem

$$\begin{cases} u'' + \nu a(t) g(u) = 0 \\ u(0) = u(T) = 0 \end{cases}$$

has at least a positive solution.

1.4.2 Multiplicity of positive solutions

Now we show how the main results of Section 1.3 can be applied when $f(t,s) = a(t)g(s)$. To this aim, besides (1.4.2), we also suppose

$$a^-(t) \not\equiv 0. \tag{1.4.5}$$

Consistently with assumption (a_*), we select, without loss of generality, the endpoints of the intervals I_i^+ in such a manner that

> *$a(t) \not\equiv 0$ on each of the subintervals $[\sigma_i, \tau_i]$ and $a(t) \not\equiv 0$ on all left neighborhoods of σ_i and on all right neighborhoods of τ_i.*

In order to explain the rule that we have decided to follow so as to determine the endpoints of the intervals, let us consider the following weight function on the interval $I = [0, 7\pi]$

$$a(t) = \begin{cases} \sin t, & \text{if } t \in [0,\pi] \cup [3\pi, 4\pi] \cup [6\pi, 7\pi]; \\ 0, & \text{if } t \in [\pi, 3\pi] \cup [4\pi, 6\pi]; \end{cases} \tag{1.4.6}$$

(cf. Figure 1.5). Among the various possibilities that one could adopt to choose the endpoints of the intervals according to condition (a_*), the following choice would fit with the above convention: $\sigma_1 = 0$, $\tau_1 = 3\pi$, $\sigma_2 = 4\pi$, $\tau_2 = 7\pi$.

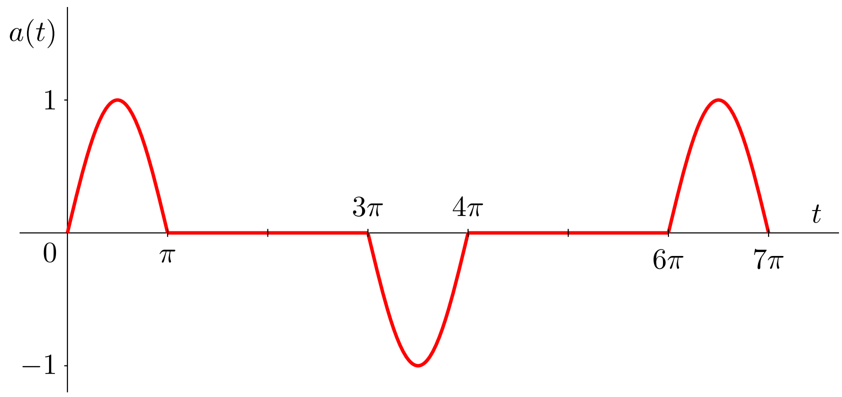

Figure 1.5: The figure shows the graph of the continuous function $a(t)\colon [0, 7\pi] \to \mathbb{R}$ defined in (1.4.6).

To discuss another example, let us consider a function with a graph as that of Figure 1.3. It is clear that it satisfies (1.4.2) and (1.4.5), provided that we adopt a suitable choice of the points σ_i and τ_i. Typically, we shall proceed in the following manner: if there is an interval where $a(t) \equiv 0$ between an interval where $a(t) > 0$ and an interval where $a(t) < 0$, we choose σ_i and τ_i in such a way that $a(t) < 0$ on $]\tau_i, \sigma_{i+1}[$ and we merge the interval where $a(t) \equiv 0$ to an adjacent interval where $a(t) \geq 0$.

We need also to introduce a further notation. For any weight function $a(t)$ satisfying (a_*) (with the endpoints σ_i, τ_i chosen as described above), we set

$$a_\mu(t) := a^+(t) - \mu a^-(t), \quad t \in [0, T],$$

where $\mu > 0$ is a parameter. Notice that $a(t) = a_\mu(t)$ for $\mu = 1$ and, moreover, for every $\mu > 0$, it holds that $a_\mu(t)$ satisfies (a_*) with the same σ_i and τ_i chosen for $a(t)$.

The introduction of the parameter μ is made only with the purpose to clarify the role of the negative humps of $a(t)$ in order to produce multiplicity results. In other words, when we require that $\mu > 0$ is sufficiently large, we have a more precise manner to express the intuitive fact that the negative humps of $a(t)$ are great enough.

Now we are in a position to present our main multiplicity result for the boundary value problem

$$\begin{cases} u'' + a_\mu(t)g(u) = 0 \\ u(0) = u(T) = 0. \end{cases} \tag{1.4.7}$$

Recall that we are assuming that $a\colon I \to \mathbb{R}$ is a continuous weight function satisfying (1.4.2), (1.4.5) and (a_*) with the convention described above.

Theorem 1.4.3. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and let $a\colon I \to \mathbb{R}$ be a continuous function satisfying (a_*). Moreover, suppose that*

$$g_0 < \lambda_0 \quad \textit{and} \quad a(t) \not\equiv 0 \textit{ on each } I_i^+ \textit{ with } g_\infty > \max_{i=1,\ldots,m} \lambda_1^i.$$

Then there exists $\mu^* > 0$ such that, for each $\mu > \mu^*$, problem (1.4.7) has at least $2^m - 1$ positive solutions.

Proof. From $g_0 < \lambda_0$, we can choose $\delta > 0$ such that $g_0 < \lambda_0 - \delta$. Let $r_0 > 0$ be as in Lemma 1.1.2 and fix $0 < r \le r_0$ such that

$$\frac{g(s)}{s} < \lambda_0 - \delta, \quad \forall\, 0 < s \le r. \tag{1.4.8}$$

Let $R^* > 0$ as in Lemma 1.1.4 and fix $R \ge R^*$.

Let $\mathcal{I} \subseteq \{1, \ldots, m\}$. Using the notation introduced in Section 1.3, consider the open and unbounded set $\Gamma^{\mathcal{I}}$. Moreover, consider an arbitrary L^1-Carathéodory function $h\colon I \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\begin{aligned} h(t,s) &\ge a(t)g(s), \quad \text{a.e. } t \in \bigcup_{i\in\mathcal{I}} I_i^+,\ \forall\, s \ge 0; \\ h(t,s) &= a(t)g(s), \quad \text{a.e. } t \in I \setminus \bigcup_{i\in\mathcal{I}} I_i^+,\ \forall\, s \ge 0; \end{aligned} \tag{1.4.9}$$

and, as usual, define the completely continuous operator $\Psi_h\colon \mathcal{C}(I) \to \mathcal{C}(I)$,

$$(\Psi_h u)(t) := \int_I G(t,\xi)\tilde{h}(\xi, u(\xi))\, d\xi, \quad t \in I,$$

where

$$\tilde{h}(t,s) := \begin{cases} h(t,s), & \text{if } s \ge 0; \\ h(t,0), & \text{if } s \le 0. \end{cases}$$

We know that every fixed point of Ψ_h is a non-negative solution of

$$\begin{cases} u'' + h(t,u) = 0 \\ u(0) = u(T) = 0. \end{cases} \tag{1.4.10}$$

To prove the claim, we use Theorem 1.3.1. In particular we have to show that the triplet $(I - \Psi_h, \Gamma^{\mathcal{I}}, 0)$ is admissible for each Carathéodory function h satisfying (1.4.9) and for each $\emptyset \neq \mathcal{I} \subseteq \{1, \ldots, m\}$.

By the choice of $R \geq R^*$ and by the convexity of the solution of (1.4.10) on each interval contained in $I \setminus \bigcup_{i=1}^m I_i^+$, we know that every fixed point u of Ψ_h is contained in the open ball $B(0, R)$, with $R > 0$ independent of the particular choice of $h(t, s)$ (see Lemma 1.1.3). Consequently it is sufficient to prove that Ψ_h has no fixed points in $\partial(\Gamma^{\mathcal{I}} \cap B(0, R))$, for μ sufficiently large.

First, we note that if $\mathcal{I} = \{1, \ldots, m\}$ there is nothing to prove, since all fixed points in $\Gamma^{\mathcal{I}} = \mathcal{C}(I)$ are contained in $B(0, R)$. Then fix $\emptyset \neq \mathcal{I} \subsetneq \{1, \ldots, m\}$. By contradiction, suppose that there is a fixed point u of Ψ_h in $\partial(\Gamma^{\mathcal{I}} \cap B(0, R))$. Due to what we have just remarked, this is equivalent to assuming the existence of a solution u of (1.4.10) with

$$\max_{t \in I_k^+} u(t) = r, \qquad \text{for some index } k \in \{1, \ldots, m\} \setminus \mathcal{I},$$

and such that $\max_{t \in I} u(t) < R$. Clearly $u \not\equiv 0$ and, moreover, by the concavity of $u(t)$ in I_k^+, we also have

$$u(t) \geq \frac{r}{\tau_k - \sigma_k} \min\{t - \sigma_k, \tau_k - t\}, \quad \forall\, t \in I_k^+. \tag{1.4.11}$$

In order to prove that our assumption is contradictory and hence that the topological degree is well-defined, we split our argument into three steps.

Step 1. A priori bounds for $|u'(t)|$ on I_k^+. This part of the proof follows by adapting a similar estimate obtained in Theorem 1.2.2. Notice that $h(t, u(t)) = a(t)g(u(t)) = a^+(t)g(u(t))$, for a.e. $t \in I_k^+$. Hence

$$|u''(t)| \leq \gamma_r(t) := a^+(t) \max_{0 \leq s \leq r} g(s), \quad \text{a.e. } t \in I_k^+,$$

and, therefore

$$|u'(y_1) - u'(y_2)| \leq \|\gamma_r\|_{L^1(I_k^+)}, \quad \forall\, y_1, y_2 \in I_k^+.$$

Let $\hat{t} \in I_k^+ = [\sigma_k, \tau_k]$ be such that $|u'(\hat{t})| \leq r/(\tau_k - \sigma_k)$ (otherwise we have an easy contradiction like in the proof of Theorem 1.2.2). Hence for all $t \in I_k^+$

$$|u'(t)| \leq |u'(\hat{t})| + |u'(t) - u'(\hat{t})| \leq \frac{r}{\tau_k - \sigma_k} + \|\gamma_r\|_{L^1(I_k^+)} =: M_k. \tag{1.4.12}$$

Step 2. Lower bounds for $u(t)$ on the boundary of I_k^+. Let φ_k be the positive eigenfunction on $I_k^+ = [\sigma_k, \tau_k]$ of

$$\begin{cases} \varphi'' + \lambda_1^k a^+(t)\varphi = 0 \\ \varphi|_{\partial I_k^+} = 0, \end{cases}$$

with $\|\varphi_k\|_\infty = 1$, where λ_1^k is the first eigenvalue. Then $\varphi_k(t) \geq 0$, for all $t \in I_k^+$, $\varphi_k(t) > 0$, for all $t \in]\sigma_k, \tau_k[$, and $\varphi_k'(\sigma_k) > 0 > \varphi_k'(\tau_k)$ (hence $\|\varphi_k'\|_\infty > 0$).

By (1.4.8) and $\lambda_0 \leq \lambda_1^k$, we know that

$$g(s) < (\lambda_1^k - \delta)s, \quad \forall\, 0 < s \leq r.$$

Then, by (1.4.11), we have

$$\begin{aligned}
&\|\varphi_k'\|_\infty (u(\sigma_k) + u(\tau_k)) \geq \\
&\geq u(\sigma_k)|\varphi_k'(\sigma_k)| + u(\tau_k)|\varphi_k'(\tau_k)| = u(\sigma_k)\varphi_k'(\sigma_k) - u(\tau_k)\varphi_k'(\tau_k) \\
&= \Big[u'(t)\varphi_k(t) - u(t)\varphi_k'(t)\Big]_{t=\sigma_k}^{t=\tau_k} \\
&= \int_{\sigma_k}^{\tau_k} \frac{d}{dt}\Big[u'(t)\varphi_k(t) - u(t)\varphi_k'(t)\Big]\,dt \\
&= \int_{I_k^+} \Big[u''(t)\varphi_k(t) - u(t)\varphi_k''(t)\Big]\,dt \\
&= \int_{I_k^+} \Big[-h(t,u(t))\varphi_k(t) + u(t)\lambda_1^k a^+(t)\varphi_k(t)\Big]\,dt \\
&= \int_{I_k^+} \Big[\lambda_1^k u(t) - g(u(t))\Big] a^+(t)\varphi_k(t)\,dt \\
&> \int_{I_k^+} \delta\left(\frac{r}{\tau_k - \sigma_k}\min\{t-\sigma_k, \tau_k - t\}\right) a^+(t)\varphi_k(t)\,dt \\
&= r\left[\frac{\delta}{\tau_k - \sigma_k}\int_{I_k^+} \min\{t-\sigma_k, \tau_k - t\} a^+(t)\varphi_k(t)\,dt\right].
\end{aligned}$$

Hence, from the above inequality, we conclude that there exists a constant $c_k > 0$, depending on δ, I_k^+ and $a^+(t)$, but independent of $u(t)$ and r, such that

$$u(\sigma_k) + u(\tau_k) \geq c_k r > 0.$$

As a consequence of the above inequality, we have that at least one of the two inequalities

$$0 < \frac{c_k r}{2} \leq u(\tau_k) \leq r, \qquad 0 < \frac{c_k r}{2} \leq u(\sigma_k) \leq r, \tag{1.4.13}$$

holds.

Step 3. Contradiction on an adjacent interval for μ large. Just to fix a case for the rest of the proof, suppose that the first inequality in (1.4.13) is true. In such a situation, we necessarily have $\tau_k < T$ (as $u(T) = 0$). Now we focus our attention on the right-adjacent interval $[\tau_k, \sigma_{k+1}]$, where $a(t) \leq 0$. Recall also that, by the convention we have adopted in defining the intervals

I_i^+, we have that $a(t)$ is not identically zero on all right neighborhoods of τ_k.

Since $g(s) > 0$ for all $s > 0$, we can introduce the positive constant

$$\nu_k := \min_{\frac{c_k r}{4} \leq s \leq R} g(s) > 0$$

and define

$$\delta_k^+ := \min\left\{\sigma_{k+1} - \tau_k, \frac{c_k r}{4M_k}\right\} > 0,$$

where $M_k > 0$ is the bound for $|u'|$ obtained in (1.4.12) of *Step 1.* Then, by the convexity of $u(t)$ on $[\tau_k, \sigma_{k+1}]$, we have that $u(t)$ is bounded from below by the tangent line at $(\tau_k, u(\tau_k))$, with slope $u'(\tau_k) \geq -M_k$. Therefore,

$$\frac{c_k r}{4} \leq u(t) \leq R, \quad \forall\, t \in [\tau_k, \tau_k + \delta_k^+].$$

We prove that for $\mu > 0$ sufficiently large $\max_{t\in[\tau_k,\sigma_{k+1}]} u(t) > R$ (which is a contradiction to the upper bound for $u(t)$).

Consider the interval $[\tau_k, \tau_k + \delta_k^+] \subseteq [\tau_k, \sigma_{k+1}]$. For all $t \in [\tau_k, \tau_k + \delta_k^+]$ we have

$$u'(t) = u'(\tau_k) + \int_{\tau_k}^t \mu a^-(\xi) g(u(\xi))\, d\xi \geq -M_k + \mu\nu_k \int_{\tau_k}^t a^-(\xi)\, d\xi,$$

then

$$u(t) = u(\tau_k) + \int_{\tau_k}^t u'(\xi)\, d\xi \geq \frac{c_k r}{2} - M_k(x - \tau_k) + \mu\nu_k \int_{\tau_k}^t \left(\int_{\tau_k}^s a^-(\xi)\, d\xi\right) ds.$$

Hence, for $t = \tau_k + \delta_k^+$,

$$R \geq u(\tau_k + \delta_k^+) \geq \frac{c_k r}{2} - M_k \delta_k^+ + \mu\nu_k \int_{\tau_k}^{\tau_k+\delta_k^+} \left(\int_{\tau_k}^s a^-(\xi)\, d\xi\right) ds.$$

This gives a contradiction if μ is sufficiently large, say

$$\mu > \mu_k^+ := \frac{R + M_k T}{\nu_k A_k^+},$$

where we have set

$$A_k^+ := \int_{\tau_k}^{\tau_k+\delta_k^+} \left(\int_{\tau_k}^s a^-(\xi)\, d\xi\right) ds > 0,$$

recalling that $\int_{\tau_k}^t a^-(\xi)\, d\xi > 0$ for each $t \in \,]\tau_k, \sigma_{k+1}]$.

A similar argument (with obvious modifications) applies if the second inequality in (1.4.13) is true (in such a case, we must have $\sigma_k > 0$, as

$u(0) = 0$). This time we focus our attention on the left-adjacent interval $[\tau_{k-1}, \sigma_k]$ where $a(t) \leq 0$. Recall also that, by the convention we have adopted in defining the intervals I_i^+, we have that $a(t)$ is not identically zero on all left neighborhoods of σ_k.

If we define

$$\delta_k^- := \min\left\{\sigma_k - \tau_{k-1}, \frac{c_k r}{4M_k}\right\} > 0,$$

we obtain the same contradiction for

$$\mu > \mu_k^- := \frac{R + M_k L}{\nu_k A_k^-},$$

where we have set

$$A_k^- := \int_{\sigma_k - \delta_k^-}^{\sigma_k} \left(\int_s^{\sigma_k} a^-(\xi)\, d\xi \right) ds.$$

At the end, we define

$$\mu^* := \max_{k=1,\ldots,m} \mu_k^\pm$$

and we apply Theorem 1.3.1 with $\mu > \mu^*$. The proof is completed. □

See Figure 1.6 for a numerical example concerning the multiplicity result stated in Theorem 1.4.3.

An immediate consequence of Theorem 1.4.3 is the following result which generalizes [94, Theorem 2.1].

Corollary 1.4.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $g(0) = 0$ and $g(s) > 0$ for all $s > 0$. Suppose also that*

$$\lim_{s \to 0^+} \frac{g(s)}{s} = 0 \quad \textit{and} \quad \lim_{s \to +\infty} \frac{g(s)}{s} = +\infty.$$

Let $a^\pm\colon I \to \mathbb{R}^+$ be continuous functions such that for some $0 = \tau_0 \leq \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \ldots < \sigma_m < \tau_m \leq \sigma_{m+1} = T$ it holds that

$$\begin{aligned} &a^+(t) \not\equiv 0,\ a^-(t) \equiv 0,\ \textit{on}\ [\sigma_i, \tau_i],\ i = 1, \ldots, m; \\ &a^-(t) \not\equiv 0,\ a^+(t) \equiv 0,\ \textit{on}\ [\tau_i, \sigma_{i+1}],\ i = 0, \ldots, m. \end{aligned}$$

Then there exists $\mu^* > 0$ such that, for each $\mu > \mu^*$, problem (1.4.7) has at least $2^m - 1$ positive solutions.

We conclude this section with a direct corollary of Theorem 1.4.3.

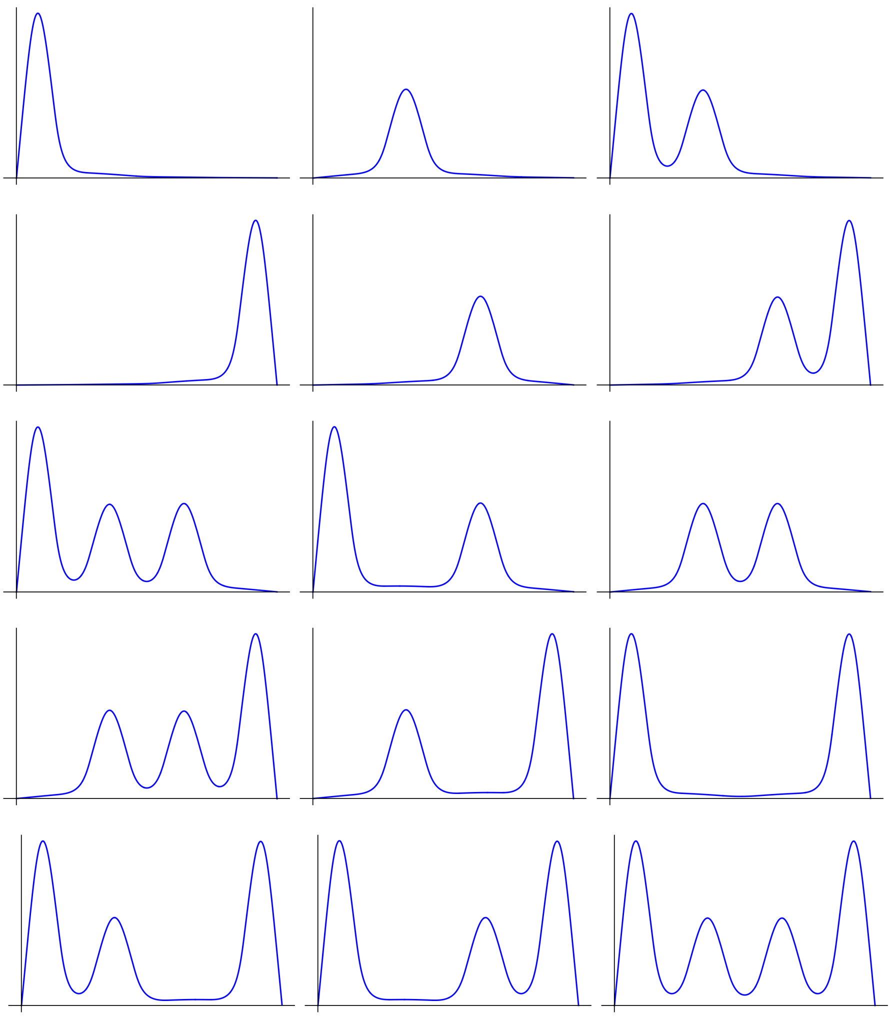

Figure 1.6: The figure shows an example of multiple positive solutions for problem (1.4.7). For this numerical simulation we have chosen $I = [0,1]$, $a(t) = \sin(7\pi t)$, $\mu = 20$ and $g(s) = \max\{0, 500\, s \arctan|s|\}$. Notice that the weight function $a(t)$ has 4 positive humps. We show the graphs of the 15 positive solutions of (1.4.7).

Corollary 1.4.3. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and let $a\colon I \to \mathbb{R}$ be a continuous function satisfying (a_*). Moreover, suppose that*

$$g_0 = 0 \qquad \textit{and} \qquad g_\infty > 0.$$

Then, there exists $\nu^* > 0$ such that, for each $\nu > \nu^*$, there exists $\mu^* = \mu^*(\nu) > 0$ so that the boundary value problem

$$\begin{cases} u'' + \big(\nu a^+(t) - \mu a^-(t)\big) g(u) = 0 \\ u(0) = u(T) = 0 \end{cases}$$

has at least $2^m - 1$ positive solutions for each $\mu > \mu^*$.

1.4.3 Radially symmetric solutions

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^N$ (for $N \geq 2$) and let

$$\Omega := B(0,R_2) \setminus B[0,R_1] = \{x \in \mathbb{R}^N : R_1 < \|x\| < R_2\}$$

be an open annular domain, with $0 < R_1 < R_2$. Let $\mathcal{A}_\mu \colon [R_1,R_2] \to \mathbb{R}$ be a continuous function defined as

$$\mathcal{A}_\mu(r) := \mathcal{A}^+(r) - \mu\mathcal{A}^-(r), \quad \mu > 0.$$

We consider the problem of existence of positive solutions for the Dirichlet boundary value problem

$$\begin{cases} -\Delta\,\mathcal{U} = \mathcal{A}_\mu(\|x\|)\, g(\mathcal{U}) & \text{in } \Omega \\ \mathcal{U} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4.14}$$

namely classical solutions such that $\mathcal{U}(x) > 0$ for all $x \in \Omega$. If we look for radially symmetric solutions of (1.4.14), we are led to the study of the two-point boundary value problem

$$v''(r) + \frac{N-1}{r}v'(r) + \mathcal{A}_\mu(r)g(v(r)) = 0, \quad v(R_1) = v(R_2) = 0. \tag{1.4.15}$$

Indeed, if $v(r)$ is any solution of (1.4.15), then $\mathcal{U}(x) := v(\|x\|)$ is a solution of (1.4.14). Using the standard change of variable

$$t = h(r) := \int_{R_1}^{r} \xi^{1-N}\, d\xi,$$

it is possible to transform (1.4.15) into the equivalent problem

$$u''(t) + r(t)^{2(N-1)}\mathcal{A}_\mu(r(t))g(u(t)) = 0, \quad u(0) = u(T) = 0, \tag{1.4.16}$$

for

$$u(t) = v(r(t)),$$

with the positions

$$T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi \quad \text{and} \quad r(t) := h^{-1}(t)$$

(see Section C.2). Clearly, problem (1.4.16) is of the same form of (1.4.7) with

$$a_\mu(t) := r(t)^{2(N-1)}\mathcal{A}_\mu(r(t)), \quad t \in [0,T].$$

Then, the following results hold, for every continuous function $g \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(0) = 0$ and $g(s) > 0$ for all $s > 0$ and satisfying

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to +\infty} \frac{g(s)}{s} = +\infty.$$

Theorem 1.4.4. *Suppose that $\mathcal{A}_\mu(r)$ changes its sign in $[R_1, R_2]$ at most a finite number of times and $\mathcal{A}^+(r) \not\equiv 0$. Then, for every $\mu \geq 0$, problem* (1.4.14) *has at least a positive radially symmetric (classical) solution.*

Theorem 1.4.5. *Suppose that for some $R_1 = \tau_0 \leq \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \ldots < \sigma_m < \tau_m \leq \sigma_{m+1} = R_2$ it holds that*

$$\mathcal{A}^+(r) \not\equiv 0,\ \mathcal{A}^-(r) \equiv 0,\ \text{on}\ [\sigma_i, \tau_i],\ i = 1, \ldots, m;$$
$$\mathcal{A}^-(r) \not\equiv 0,\ \mathcal{A}^+(r) \equiv 0,\ \text{on}\ [\tau_i, \sigma_{i+1}],\ i = 0, \ldots, m.$$

Then there exists $\mu^* > 0$ such that, for each $\mu > \mu^*$, problem (1.4.14) *has at least $2^m - 1$ positive radially symmetric (classical) solutions.*

Theorem 1.4.4 and Theorem 1.4.5 can be seen as an extension of the classical existence result of Bandle, Coffman and Marcus [13] to the case of a general sign-changing weight. It could be interesting to investigate under which supplementary assumptions the above results are sharp (that is, providing exactly one positive solution or exactly 2^m-1 positive solutions, respectively).

As a comment about the sign conditions on $\mathcal{A}_\mu(r)$, we observe that our results apply to weight functions which may vanish in some sub-intervals of $[R_1, R_2]$ (even in infinitely many sub-intervals), see Remark 1.4.1. Concerning the continuous nonlinearity $g(s)$, we notice that, besides the positivity and the conditions for $s \to 0^+$ and for $s \to +\infty$, no other assumptions (like smoothness, monotonicity or homogeneity) are required.

1.4.4 Final remarks

For the study of problem (1.4.1) we have confined ourselves to the case of a continuous weight function $a(t)$. Since the general results for problem (1.1.1) have been obtained under general Carathéodory assumptions on $f(t, s)$, we can deal with the case of $a \in L^1(I)$, too. With this respect, Theorem 1.4.1 and Theorem 1.4.2 are still valid provided that the assumption $a(t) \not\equiv 0$ on I_i^+ is meant in the sense that $a(t) \geq 0$ for a.e. $t \in I_i^+$ and $\int_{I_i^+} a(t)\, dt > 0$. Concerning the variant of Theorem 1.4.3 for $a_\mu(t) = a^+(t) - \mu a^-(t)$, with $a^\pm \in L^1(I)$ and $a^\pm \geq 0$ almost everywhere, we claim that our result still holds provided that the endpoints of the intervals are selected so that $\int_{\tau_k}^{t} a^-(\xi)\, d\xi > 0$ for all $t \in \mathopen]\tau_k, \sigma_{k+1}]$ and $\int_t^{\sigma_k} a^-(\xi)\, d\xi > 0$ for all $t \in [\tau_{k-1}, \sigma_k\mathclose[$ (for each $k = 1, \ldots, m$). In this manner, the constants $A_k^\pm$ in *Step 3* of the proof of Theorem 1.4.3 are all strictly positive. All the other parts of the proof are exactly the same.

In [93] a class of measurable weight functions which are possibly singular at the endpoints of the interval I is considered. More precisely, therein one can consider a function $a \in L^1_{\mathrm{loc}}(I)$ such that $\int_I t(T-t)|a(t)|\, dt < +\infty$. The possibility of dealing with weight functions which are not in $L^1(I)$

depends by the method of proof in [93] based on the search of fixed points for the operator associated with the Green function. Since in this chapter we follow the same approach, we can also deal with such a wider class of weight functions.

The main goal of this chapter is to present our topological approach. In order to avoid unnecessary technicalities, we have dealt with the Dirichlet problem associated with the easiest differential equation $u'' + a(t)g(u) = 0$. We underline that one can achieve the same existence and multiplicity results also for a more general equation of the form

$$u'' + c(t)u' + a(t)g(u) = 0,$$

where $c\colon [0,T] \to \mathbb{R}$ is a continuous function. Indeed, using a standard change of variable (recalled in Section C.2), we are able to transform this equation into the one that we have considered in this final section. We also refer to Chapter 2, where other generalizations of the nonlinearity $f(t,s) = a(t)g(s)$ are considered.

Our topological approach can be adapted to the study of different boundary value problems. For instance, like in [113], one can consider mixed boundary conditions like $u'(0) = u(T) = 0$ or $u(0) = u'(T) = 0$, or more in general a Sturm-Liouville boundary conditions of the form

$$\begin{cases} \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(T) - \delta u'(T) = 0, \end{cases}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ with $\gamma\beta + \alpha\gamma + \alpha\delta > 0$ (see Section 2.4 and the subsequent sections, where this type of problems are considered).

Finally, we underline that the advantage in using a topological degree approach lies also on the fact that, once we have found an open bounded set where the degree is non-zero, we know that such a result is stable under small perturbations of the operator. Thus our theorems also apply to equations which are small perturbations of the equation in (1.4.1). For example, we could even add to the equation small terms of a functional form, such as terms of (nonlocal) integral type or with a delay. Of course, in such a case, to provide positive solutions, one should look for a suitable maximum principle. In particular, the existence and multiplicity results hold for equations of the form

$$u'' + a(t)g(u) + \varepsilon h(t,u,u') = 0,$$

for $|\varepsilon|$ small enough.

Chapter 2

More general nonlinearities $f(t,s)$

This chapter is devoted to some further investigations on the nonlinear second order differential equation

$$u'' + f(t,u) = 0. \tag{2.0.1}$$

More precisely, we show how the topological approach introduced in the previous chapter can be used to obtain existence and multiplicity results for positive solutions when considering more general nonlinearities $f(t,s)$.

In the first part of this chapter (from Section 2.1 to Section 2.3) we deal with a nonlinearity of the form

$$f(t,s) = a(t)g(s),$$

where both the functions $a(t)$ and $g(s)$ are allowed to change sign. We assume that the function $g(s)$ is continuous and satisfies suitable growth conditions, including the superlinear case $g(s) = s^p$, with $p > 1$. In particular, with respect to the situation considered in Section 1.4, we still suppose that $g(s)/s$ is small near zero and large at infinity, but we do not require that $g(s)$ is non-negative in a neighborhood of zero. We give an existence result for positive solutions to the Dirichlet problem associated with (2.0.1).

In the second part (from Section 2.4 to Section 2.8) we study positive solutions of a Sturm-Liouville boundary value problem associated with (2.0.1) when the nonlinearity is of the form

$$f(t,s) := \sum_{i=1}^{m} \alpha_i a_i(t) g_i(s) - \sum_{j=0}^{m+1} \beta_j b_j(t) k_j(s),$$

where $\alpha_i, \beta_j > 0$, $a_i(t), b_j(t)$ are non-negative Lebesgue integrable functions defined in $[0,T]$, and the nonlinearities $g_i(s), k_j(s)$ are continuous, positive and satisfy suitable growth conditions, as to cover the classical superlinear equation $u'' + a(t)u^p = 0$, with $p > 1$. When the positive parameters β_j are sufficiently large, we prove the existence of at least $2^m - 1$ positive solutions.

In both parts, with *positive solution* we mean a solution in the Carathéodory sense and such that $u(t) > 0$ for every $t \in \,]0,T[$.

2.1 Sign-changing nonlinearities: introduction

In this first part we are interested in the study of positive solutions for the nonlinear two-point boundary value problem

$$\begin{cases} u'' + a(t)g(u) = 0 \\ u(0) = u(T) = 0, \end{cases} \tag{2.1.1}$$

where $a\colon [0,T] \to \mathbb{R}$ is a Lebesgue integrable function and $g\colon \mathbb{R}^+ \to \mathbb{R}$ is a continuous function. As in Section 1.4.1, we focus on the existence of at least a positive solution of (2.1.1), but, with respect to Chapter 1, we improve the conditions on the nonlinearity $g(s)$.

Our assumptions allow the weight function $a(t)$ to change its sign a finite number of times and, concerning the nonlinearity, we suppose that $g(s)$ can change its sign, even an infinite number of times, and that, roughly speaking, it has a superlinear growth at zero and at infinity. In more detail, with respect to the growth of $g(s)/s$ at zero, we assume a very general condition (cf. hypothesis (h_{III})) which depends on the sign of $g(s)$ in a right neighborhood of zero.

Our main result states that, under the conditions just presented, problem (2.1.1) has at least a positive solution. Our theorem clearly covers the case $g(s) = s^p$, with $p > 1$. Moreover, the results concerning the BVP (2.1.1) where is assumed that $a(t)g(s) \geq 0$ for a.e. $t \in [0,T]$ and for all $s \geq 0$ (see [76, 113, 146]) or that $g(s) > 0$ for all $s > 0$, when $a(t)$ is allowed to change sign (see [27, 93] and also Section 1.4), do not contain our result and, in some cases, are easy consequences of it.

Figure 2.1 and Figure 2.2 show examples of nonlinearities $g(s)$ satisfying our assumptions and which are not covered by previous results.

Now we state the hypotheses on $a(t)$ and on $g(s)$, we recall some classical results and we prove two preliminary lemmas that are then employed in Section 2.2 for the main result.

We consider the nontrivial closed interval $[0,T]$, pointing out that different choices of a nontrivial compact interval contained in $\mathbb{R}$ can be made. Let $a\colon [0,T] \to \mathbb{R}$ be an L^1-weight function. Clearly the case of a continuous function can also be treated. We assume that

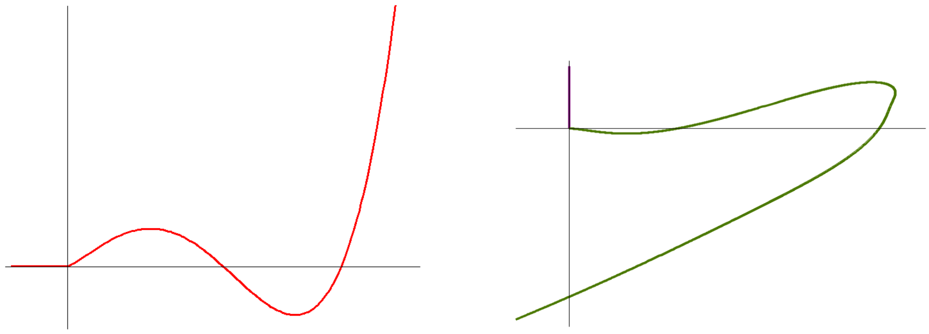

Figure 2.1: The figure shows a numerical simulation obtained by setting $a(t) = \sin(3\pi t)$ in $[0,1]$ and $g(s) = \min\{20s^{6/5} - 6s^3 + s^4, 400\, s \arctan(s)\}$, for $s \geq 0$. On the left we have shown the graph of $g(s)$. We underline that $g(s)$ changes sign and $g(s)/s \not\to +\infty$ as $s \to +\infty$. On the right we have represented the image of the segment $\{0\} \times [0,12]$ through the Poincaré map in the phase-plane (u,u'). It intersects the negative part of the u'-axis in a point, hence there is a positive initial slope at $t=0$ from which departs a solution which is positive on $]0,1[$ and vanishes at $t=1$.

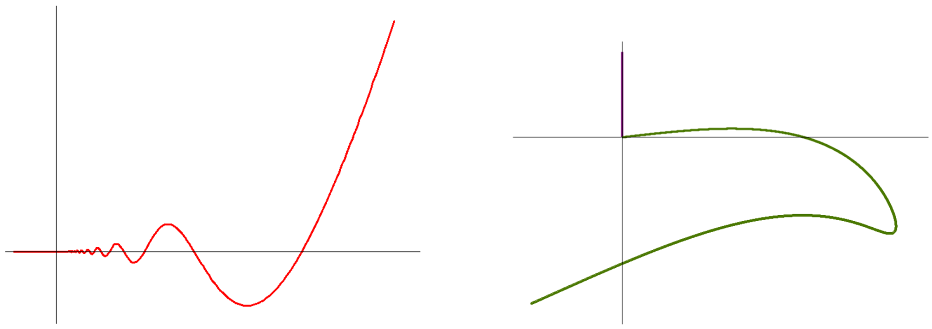

Figure 2.2: The figure shows a numerical simulation obtained by setting $a(t) = \sin(7\pi t)$ in $[0,1]$ and $g(s) = s^3 + s^2 \sin(1/s)$, for $s \geq 0$. On the left we have shown the graph of $g(s)$. The nonlinearity $g(s)$ changes sign an infinite number of times in every neighborhood of zero. On the right we have represented the image of the segment $\{0\} \times [0,16]$ through the Poincaré map in the phase-plane (u,u').

(h_I) *there exist $m \geq 1$ intervals $I_1^+, \ldots, I_m^+$, closed and pairwise disjoint, such that*

$$a(t) \geq 0, \quad \text{a.e. } t \in \bigcup_{i=1}^{m} I_i^+;$$

$$a(t) \leq 0, \quad \text{a.e. } t \in [0,T] \setminus \bigcup_{i=1}^{m} I_i^+.$$

We underline that assumption (h_I) trivially includes the case where $a(t) \geq 0$ for a.e. $t \in [0,T]$, taking $m = 1$ and $I_1^+ = [0,T]$.

Concerning the nonlinearity, we suppose that $g \colon \mathbb{R}^+ \to \mathbb{R}$ is a continuous function such that

$$(h_{II}) \qquad g(0) = 0 \quad \text{and} \quad g \not\equiv 0.$$

We set

$$g_0^{inf} := \liminf_{s\to 0^+} \frac{g(s)}{s} > -\infty, \qquad g_0^{sup} := \limsup_{s\to 0^+} \frac{g(s)}{s} < +\infty$$

and

$$g_\infty := \liminf_{s\to +\infty} \frac{g(s)}{s} > 0.$$

We emphasize that we do not suppose $g(s) \geq 0$ on $\mathbb{R}^+$ and, in particular, it is not required that $g(s) > 0$ for all $s > 0$ (as in [76, 93, 113] and also in Section 1.4). Consequently, the nonlinearity $g(s)$ could be non-negative, non-positive or it could change sign, even an infinite number of times, on a compact neighborhood of zero.

Now we show how the superlinearity of $g(s)$ is expressed at zero and at infinity. Our first step is to impose a condition on the growth of $g(s)/s$ at 0, depending on the sign of $g(s)$. Precisely we assume that

(h_{III})

- *if there exists $\delta > 0$ such that $g(s) \geq 0$, for all $s \in [0,\delta]$, it holds that*
  $$a^+(t) \not\equiv 0 \text{ on } [0,T] \quad \text{and} \quad g_0^{sup} < \lambda_0^+,$$
  where $\lambda_0^+ > 0$ is the first eigenvalue of the eigenvalue problem
  $$\varphi'' + \lambda\, a^+(t)\, \varphi = 0, \quad \varphi(0) = \varphi(T) = 0;$$
- *if there exists $\delta > 0$ such that $g(s) \leq 0$, for all $s \in [0,\delta]$, it holds that*
  $$a^-(t) \not\equiv 0 \text{ on } [0,T] \quad \text{and} \quad g_0^{inf} > -\lambda_0^-,$$
  where $\lambda_0^- > 0$ is the first eigenvalue of the eigenvalue problem
  $$\varphi'' + \lambda\, a^-(t)\, \varphi = 0, \quad \varphi(0) = \varphi(T) = 0;$$
- *if $g(s)$ changes sign an infinite number of times in every neighborhood of zero, it holds that*
  $$a(t) \not\equiv 0 \text{ on } [0,T] \quad \text{and} \quad -\lambda_0 < g_0^{inf} \leq g_0^{sup} < \lambda_0,$$
  where $\lambda_0 > 0$ is the first eigenvalue of the eigenvalue problem
  $$\varphi'' + \lambda\, |a(t)|\, \varphi = 0, \quad \varphi(0) = \varphi(T) = 0.$$

The functions $a(t)$ and $g(s)$ introduced in Figure 2.1 satisfy the first condition of hypothesis (h_{III}), while the example shown in Figure 2.2 corresponds to the third case.

As a second step we define the superlinear behavior at infinity. We suppose that

(h_{IV}) *for all* $i \in \{1,\dots,m\}$

$$a(t) \not\equiv 0 \ \textit{on}\ I_i^+ \quad \textit{and} \quad g_\infty > \lambda_1^i,$$

where $\lambda_1^i > 0$ *is the first eigenvalue of the eigenvalue problem in* I_i^+

$$\varphi'' + \lambda\, a^+(t)\,\varphi = 0, \quad \varphi|_{\partial I_i^+} = 0.$$

Now we describe the topological approach we adopt to face problem (2.1.1). Our first goal is to introduce a completely continuous operator and to define an equivalent fixed point problem.

Let $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ be the standard extension of $g(s)$ defined as

$$\tilde{g}(s) = \begin{cases} g(s), & \text{if } s \geq 0; \\ 0, & \text{if } s \leq 0. \end{cases}$$

We deal with the boundary value problem

$$\begin{cases} u'' + a(t)\tilde{g}(u) = 0 \\ u(0) = u(T) = 0. \end{cases} \tag{2.1.2}$$

From conditions (h_{II}) and (h_{III}) and by a standard maximum principle, it follows that all possible solutions of (2.1.2) are non-negative. Moreover, if these solutions are nontrivial, then they are strictly positive on $]0,T[$ and hence positive solutions of (2.1.1).

The next step is to define the classical operator $\Phi\colon \mathcal{C}([0,T]) \to \mathcal{C}([0,T])$ by

$$(\Phi u)(t) := \int_0^T G(t,\xi)a(\xi)\tilde{g}(u(\xi))\,d\xi, \quad t \in [0,T], \tag{2.1.3}$$

where $G(t,s)$ is the Green function for the differential operator $u \mapsto -u''$ with the two-point boundary condition. The map Φ is completely continuous in $\mathcal{C}([0,T])$, endowed with the sup-norm $\|\cdot\|_\infty$, and such that u is a fixed point of Φ if and only if u is a solution of (2.1.2). Therefore we have transformed problem (2.1.1) into an equivalent fixed point problem.

We close this section by proving two technical lemmas that allow us to find a nontrivial fixed point of Φ, hence a positive solution of (2.1.1). The proofs are similar to those of the analogous results proved in the previous chapter.

Using the next lemma we are able to compute the degree of $Id - \Phi$ on small balls.

Lemma 2.1.1. *There exists* $r_0 > 0$ *such that*

$$\deg_{LS}(Id - \Phi, B(0,r), 0) = 1, \quad \forall\, 0 < r \leq r_0.$$

Proof. We divide the proof in two steps.

Step 1. We prove that there exists $r_0 > 0$ such that every solution $u(t) \geq 0$ of the two-point BVP

$$\begin{cases} u'' + \vartheta a(t)g(u) = 0, & 0 \leq \vartheta \leq 1, \\ u(0) = u(T) = 0 \end{cases} \tag{2.1.4}$$

satisfying $\max_{t\in[0,T]} u(t) \leq r_0$ *is such that $u(t) = 0$, for all $t \in [0,T]$.*

The proof of this first step is given only when there exists $\delta > 0$ such that $g(s) \geq 0$, for all $s \in [0,\delta]$. The two remaining cases can be treated in an analogous way.

Using condition (h_{III}), we fix $0 < r_0 < \delta$ such that

$$\frac{g(s)}{s} < \lambda_0^+, \quad \forall\, 0 < s \leq r_0.$$

Now, suppose by contradiction that there exist $\vartheta \in [0,1]$ and a positive solution $u(t) \not\equiv 0$ of (2.1.4) such that $\max_{t\in[0,T]} u(t) = r$ for some $0 < r \leq r_0$. The choice of r_0 and the strong maximum principle imply that

$$0 \leq \vartheta g(u(t)) < \lambda_0^+ u(t), \quad \text{for all } t \in \,]0,T[.$$

Let φ be a positive eigenfunction of

$$\begin{cases} \varphi'' + \lambda_0^+ a^+(t)\varphi = 0 \\ \varphi(0) = \varphi(T) = 0. \end{cases}$$

We emphasize that $\varphi(t) > 0$, for all $t \in \,]0,T[$. Using a Sturm comparison argument, we attain

$$\begin{aligned} 0 &= \Big[u'(t)\varphi(t) - u(t)\varphi'(t)\Big]_{t=0}^{t=T} \\ &= \int_0^T \frac{d}{dt}\Big[u'(t)\varphi(t) - u(t)\varphi'(t)\Big]\,dt \\ &= \int_0^T \Big[u''(t)\varphi(t) - u(t)\varphi''(t)\Big]\,dt \\ &= \int_0^T \Big[-\vartheta a(t)g(u(t))\varphi(t) + u(t)\lambda_0^+ a^+(t)\varphi(t)\Big]\,dt \\ &\geq \int_0^T \Big[\lambda_0^+ u(t) - \vartheta g(u(t))\Big] a^+(t)\varphi(t)\,dt \\ &> 0, \end{aligned}$$

a contradiction.

Step 2. Computation of the degree. Let us fix $0 \leq \vartheta \leq 1$. As stated when we introduced the operator Φ, the maximum principle ensures that every fixed point in $\mathcal{C}([0,T])$ of the operator $\vartheta\Phi$ is non-negative and, moreover, $u \in \mathcal{C}([0,T])$ satisfies $u = \vartheta\Phi(u)$ if and only if u is a solution of the equation (2.1.4). Therefore, setting $r \in \,]0, r_0]$, *Step 1* implies that $\|u\|_\infty \neq r$ and hence

$$u \neq \vartheta\Phi(u), \quad \forall\, \vartheta \in [0,1],\ \forall\, u \in \partial B(0,r).$$

By the homotopic invariance property of the topological degree, we obtain that

$$\deg_{LS}(Id - \Phi, B(0,r), 0) = \deg_{LS}(Id, B(0,r), 0) = 1.$$

This concludes the proof of the lemma. □

Now we compute the topological degree on large balls.

Lemma 2.1.2. *There exists $R^* > 0$ such that*

$$\deg_{LS}(Id - \Phi, B(0,R), 0) = 0, \quad \forall\, R \geq R^*.$$

Proof. We divide the proof in two steps.

Step 1. A priori bounds for u on each I_i^+. For each $i \in \{1,\ldots,m\}$, we prove that there exists $R_i > 0$ such that for each L^1-Carathéodory function $h\colon [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ with

$$h(t,s) \geq a(t)g(s), \quad \text{a.e. } t \in I_i^+,\ \forall\, s \geq 0,$$

every solution $u(t) \geq 0$ of the two-point BVP

$$\begin{cases} u'' + h(t,u) = 0 \\ u(0) = u(T) = 0 \end{cases} \tag{2.1.5}$$

satisfies $\max_{t \in I_i^+} u(t) < R_i$.

We fix an index $i \in \{1,\ldots,m\}$ and set $I_i^+ := [\sigma_i, \tau_i]$. Let $0 < \varepsilon < (\tau_i - \sigma_i)/2$ be fixed such that

$$a^+(t) \not\equiv 0 \quad \text{on } I_i^{+,\varepsilon},$$

where $I_i^{+,\varepsilon} := [\sigma_i + \varepsilon, \tau_i - \varepsilon]$, and such that the first positive eigenvalue $\hat{\lambda}$ of the eigenvalue problem

$$\begin{cases} \varphi'' + \lambda a^+(t)\varphi = 0 \\ \varphi|_{\partial I_i^{+,\varepsilon}} = 0 \end{cases} \tag{2.1.6}$$

is such that

$$0 < \hat{\lambda} < g_\infty.$$

The existence of ε is ensured by the continuity of the first eigenvalue as a function of the boundary condition (see [63, 183]) and by hypothesis (h_{IV}).

From the previous inequality it follows that there exists a constant $\tilde{R} > 0$ such that

$$g(s) > \hat{\lambda} s, \quad \forall\, s \geq \tilde{R}.$$

By contradiction, suppose there is no constant $R_i > 0$ with the properties listed above. So, for each integer $n > 0$ there exists a solution $u_n \geq 0$ of (2.1.5) with $\max_{t \in I_i^+} u_n(t) =: \hat{R}_n > n$.

We claim that there exists an integer $N \geq \tilde{R}$ such that $u_n(t) > \tilde{R}$ for every $t \in I_i^{+,\varepsilon}$ and $n \geq N$. If it is not true, for every integer $n \geq \tilde{R}$ there is an integer $\hat{n} \geq n$ and $t_{\hat{n}} \in I_i^{+,\varepsilon}$ such that $u_{\hat{n}}(t_{\hat{n}}) = \tilde{R}$. We note that the solution $u_{\hat{n}}(t)$ is concave on each subinterval of I_i^+ where $u_{\hat{n}}(t) \geq \tilde{R}$, since $a(t)g(s) \geq 0$ for a.e. $t \in I_i^+$ and for all $s \geq \tilde{R}$. Then, without loss of generality, we can assume that there exists a maximum point $\hat{t}_{\hat{n}} \in I_i^+$ of $u_{\hat{n}}$ such that $u_{\hat{n}}(t) > \tilde{R}$ for all t between $t_{\hat{n}}$ and $\hat{t}_{\hat{n}}$ (if necessary, we change the choice of $t_{\hat{n}}$). From the assumptions, it follows that

$$\hat{n} < \hat{R}_{\hat{n}} = u_{\hat{n}}(\hat{t}_{\hat{n}}) = u_{\hat{n}}(t_{\hat{n}}) + \int_{t_{\hat{n}}}^{\hat{t}_{\hat{n}}} u'_{\hat{n}}(\xi)\, d\xi \leq \tilde{R} + (\tau_i - \sigma_i)|u'_{\hat{n}}(t_{\hat{n}})|. \quad (2.1.7)$$

We fix a constant $C > 0$ such that

$$C > \frac{\tilde{R}}{\varepsilon} + \|a\|_{L^1} \max_{0 \leq s \leq \tilde{R}} |g(s)|.$$

Using (2.1.7), we have that for every $n \geq (\tau_i - \sigma_i)C + \tilde{R}$ there exists $\hat{n} \geq n$ and $t_{\hat{n}} \in I_i^{+,\varepsilon}$ such that $u_{\hat{n}}(t_{\hat{n}}) = \tilde{R}$ and $|u'_{\hat{n}}(t_{\hat{n}})| > C$. We fix $n \geq (\tau_i - \sigma_i)C + \tilde{R}$, $\hat{n} \geq n$ and $t_{\hat{n}} \in I_i^{+,\varepsilon}$ with the properties just listed. Suppose that $u'_{\hat{n}}(t_{\hat{n}}) > C$ and consider the interval $[\sigma_i, t_{\hat{n}}]$. If $u'_{\hat{n}}(t_{\hat{n}}) < -C$ we proceed similarly dealing with the interval $[t_{\hat{n}}, \tau_i]$. For every $t \in [\sigma_i, t_{\hat{n}}]$

$$\begin{aligned} u'_{\hat{n}}(t) &= u'_{\hat{n}}(t_{\hat{n}}) - \int_t^{t_{\hat{n}}} u''_{\hat{n}}(\xi)\, d\xi = u'_{\hat{n}}(t_{\hat{n}}) + \int_t^{t_{\hat{n}}} h(\xi, u_{\hat{n}}(\xi))\, d\xi \\ &\geq u'_{\hat{n}}(t_{\hat{n}}) + \int_t^{t_{\hat{n}}} a(\xi) g(u_{\hat{n}}(\xi))\, d\xi, \end{aligned}$$

then

$$u'_{\hat{n}}(t) > C - \int_t^{t_{\hat{n}}} a(\xi)|g(u_{\hat{n}}(\xi)|\, d\xi.$$

From this inequality we derive that $u_{\hat{n}}(t) \leq \tilde{R}$, for all $t \in [\sigma_i, t_{\hat{n}}]$, and therefore

$$u'_{\hat{n}}(t) > \frac{\tilde{R}}{\varepsilon}, \qquad \text{for all } t \in [\sigma_i, t_{\hat{n}}].$$

Then, we obtain

$$\tilde{R} \leq \frac{\tilde{R}}{\varepsilon}(t_{\hat{n}} - \sigma_i) < \int_{\sigma_i}^{t_{\hat{n}}} u'_{\hat{n}}(\xi)\, d\xi = u_{\hat{n}}(t_{\hat{n}}) - u_{\hat{n}}(\sigma_i) \leq u_{\hat{n}}(t_{\hat{n}}) = \tilde{R},$$

a contradiction. Hence the claim is proved. So, we can fix an integer $N \geq \tilde{R}$ such that $u_n(t) > \tilde{R}$ for every $t \in I_i^{+,\varepsilon}$ and for $n \geq N$.

We denote by φ the positive eigenfunction of the eigenvalue problem (2.1.6) with $\|\varphi\|_\infty = 1$. Then $\varphi(t) > 0$, for every $t \in \,]\sigma_i + \varepsilon, \tau_i - \varepsilon[$, and $\varphi'(\sigma_i + \varepsilon) > 0 > \varphi'(\tau_i - \varepsilon)$. We emphasize that $u_n(\sigma_i + \varepsilon) > 0$ and $u_n(\tau_i - \varepsilon) > 0$, for every integer n, employing the maximum principle.

Using a Sturm comparison argument, for each $n \geq N$, we obtain

$$\begin{aligned}
0 &> u_n(\tau_i - \varepsilon)\varphi'(\tau_i - \varepsilon) - u_n(\sigma_i + \varepsilon)\varphi'(\sigma_i + \varepsilon) \\
&= \Big[u_n(t)\varphi'(t) - u_n'(t)\varphi(t)\Big]_{t=\sigma_i+\varepsilon}^{t=\tau_i-\varepsilon} \\
&= \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} \frac{d}{dt}\Big[u_n(t)\varphi'(t) - u_n'(t)\varphi(t)\Big]\,dt \\
&= \int_{I_i^{+,\varepsilon}} \Big[u_n(t)\varphi''(t) - u_n''(t)\varphi(t)\Big]\,dt \\
&= \int_{I_i^{+,\varepsilon}} \Big[-u_n(t)\hat{\lambda}a^+(t)\varphi(t) + h(t,u_n(t))\varphi(t)\Big]\,dt \\
&= \int_{I_i^{+,\varepsilon}} \Big[h(t,u_n(t)) - \hat{\lambda}a^+(t)u_n(t)\Big]\varphi(t)\,dt \\
&\geq \int_{I_i^{+,\varepsilon}} \Big[a(t)g(u_n(t)) - \hat{\lambda}a^+(t)u_n(t)\Big]\varphi(t)\,dt \\
&= \int_{I_i^{+,\varepsilon}} \Big[g(u_n(t)) - \hat{\lambda}u_n(t)\Big]a^+(t)\varphi(t)\,dt \\
&\geq 0,
\end{aligned}$$

a contradiction.

Step 2. Computation of the degree. We stress that the constant R_i, for $i \in \{1, \ldots, m\}$, does not depend on the function $h(t,s)$. Define

$$R^* := \max_{i=1,\ldots,m} R_i + \tilde{R} > 0$$

and fix a radius $R \geq R^*$.

We denote by $\mathbb{1}_A$ the indicator function of the set $A := \bigcup_{i=1}^m I_i^+$. Let us define $v(t) := \int_0^T G(t,s)\mathbb{1}_A(s)\,ds$. By the second part of Theorem A.2.1, if we show that

$$u \neq \Phi(u) + \alpha v, \quad \text{for all } u \in \partial B(0,R) \text{ and } \alpha \geq 0, \tag{2.1.8}$$

our result is proved.

Let $\alpha \geq 0$. The maximum principle ensures that any nontrivial solution $u \in \mathcal{C}([0,T])$ of $u = \Phi(u) + \alpha v$ is a non-negative solution of the equation

$u''+a(t)\tilde{g}(u)+\alpha\mathbb{1}_A(t)=0$ with $u(0)=u(T)=0$. Hence, u is a non-negative solution of (2.1.5) with

$$h(t,s)=a(t)g(s)+\alpha\mathbb{1}_A(t).$$

By definition, we have $h(t,s)\geq a(t)g(s)$, for a.e. $t\in A$ and for all $s\geq 0$, and $h(t,s)=a(t)g(s)$, for a.e. $t\in[0,T]\setminus A$ and for all $s\geq 0$. By the convexity of the solution u on the intervals of $[0,T]\setminus A$ where $u(t)\geq\tilde{R}$, we obtain

$$\|u\|_\infty=\max_{t\in[0,T]}u(t)\leq\max\Bigl\{\max_{t\in A}u(t),\tilde{R}\Bigr\}.$$

From *Step 1* and the definition of $\tilde{R}$, we deduce that $\|u\|_\infty<R^*\leq R$. Then (2.1.8) is proved and the lemma follows. □

2.2 Sign-changing nonlinearities: the main result

In this section we apply the two technical lemmas just proved to obtain the existence of a positive solution to the two-point boundary value problem (2.1.1). In more detail we use the additivity of the topological degree to provide the existence of a nontrivial fixed point of the operator Φ defined in (2.1.3).

A first immediate consequence of Lemma 2.1.1 and Lemma 2.1.2 is our main theorem.

Theorem 2.2.1. *Let $a\colon[0,T]\to\mathbb{R}$ be an L^1-function and $g\colon\mathbb{R}^+\to\mathbb{R}$ be a continuous function satisfying (h_I), (h_{II}), (h_{III}) and (h_{IV}). Then there exists at least a positive solution of the two-point boundary value problem* (2.1.1).

Proof. Let r_0 be as in Lemma 2.1.1 and R^* be as in Lemma 2.1.2. We observe that $0<r_0<R^*<+\infty$. From the additivity property and the two preliminary lemmas it follows that

$$\begin{aligned}&\deg_{LS}(Id-\Phi,B(0,R^*)\setminus B[0,r_0],0)=\\&=\deg_{LS}(Id-\Phi,B(0,R^*),0)-\deg_{LS}(Id-\Phi,B(0,r_0),0)=\\&=0-1=-1\neq 0.\end{aligned}$$

Then there exists a nontrivial fixed point of Φ and hence a corresponding positive solution of (2.1.1), as already stated. □

From Theorem 2.2.1 we easily achieve the following two results.

Corollary 2.2.1. *Let $a\colon[0,T]\to\mathbb{R}$ be an L^1-function and $g\colon\mathbb{R}^+\to\mathbb{R}$ be a continuous function satisfying (h_I) and (h_{II}). Moreover, assume that*

$$g_0:=\lim_{s\to 0^+}\frac{g(s)}{s}=0,\qquad g_\infty=\lim_{s\to+\infty}\frac{g(s)}{s}=+\infty,$$

and $a(t) \not\equiv 0$ on I_i^+, for each $i \in \{1, \dots, m\}$. Then there exists at least a positive solution of the two-point BVP (2.1.1)*.*

Corollary 2.2.2. *Let $a\colon [0,T] \to \mathbb{R}$ be an L^1-function satisfying (h_I) and such that $a(t) \not\equiv 0$ on I_i^+, for each $i \in \{1, \dots, m\}$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}$ be a continuous function satisfying (h_{II}) and such that $g_0 = 0$ and $g_\infty = \Lambda > 0$. Then there exists $\lambda^* > 0$ such that, for each $\lambda > \lambda^*$, the two-point BVP*

$$\begin{cases} u'' + \lambda a(t) g(u) = 0 \\ u(0) = u(T) = 0 \end{cases}$$

has at least a positive solution.

Although hypothesis (h_I) is more interesting when the set $[0,T] \setminus \bigcup_{i=1}^m I_i^+$ is not negligible, we can consider a weight $a(t) \geq 0$ for a.e. $t \in [0,T]$, as previously observed. In that situation Corollary 2.2.1 ensures the existence of a positive solution in the superlinear case, provided that $a \not\equiv 0$. No sign condition on the function $g(s)$ is required. Thus, with a different proof, we can extend [76, Theorem 1], which was obtained as an application of Krasnosel'skiĭ fixed point Theorem.

Remark 2.2.1. Our approach is based on the definition of a fixed point problem which is equivalent to the boundary value problem we considered. It is clear we could deal with different conditions at the boundary of $[0,T]$ like $u'(0) = u(T) = 0$ or $u(0) = u'(T) = 0$, since a suitable maximum principle and a Green function (cf. [76]) are available to define an equivalent fixed point problem and to adapt the scheme shown above (compare to Section 1.4.4 and the second part of the present chapter). $\lhd$

2.3 Sign-changing nonlinearities: radial solutions

We denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^N$ (for $N \geq 2$). Let

$$\Omega := B(0,R_2) \setminus B[0,R_1] = \bigl\{ x \in \mathbb{R}^N \colon R_1 < \|x\| < R_2 \bigr\}$$

be an open annular domain, with $0 < R_1 < R_2$. Let $a\colon [R_1,R_2] \to \mathbb{R}$ be a continuous function. In this section we consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = a(\|x\|)\, g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3.1}$$

and we are interested in the existence of positive solutions of (2.3.1), namely classical solutions such that $u(x) > 0$ for all $x \in \Omega$.

Looking for radially symmetric solutions of (2.3.1), our study can be reduced to the search of positive solutions of the two-point boundary value problem

$$w''(r) + \frac{N-1}{r} w'(r) + a(r)g(w(r)) = 0, \quad w(R_1) = w(R_2) = 0. \tag{2.3.2}$$

Indeed, if $w(r)$ is a solution of (2.3.2), then $u(x) := w(\|x\|)$ is a solution of (2.3.1). As described in Section C.2, using the standard change of variable

$$t = h(r) := \int_{R_1}^{r} \xi^{1-N}\, d\xi$$

and by defining

$$T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi, \quad r(t) := h^{-1}(t) \quad \text{and} \quad v(t) = w(r(t)),$$

we transform (2.3.2) into the equivalent problem

$$v''(t) + r(t)^{2(N-1)} a(r(t)) g(v(t)) = 0, \quad v(0) = v(T) = 0. \tag{2.3.3}$$

Consequently, the two-point boundary value problem (2.3.3) is the same as (2.1.1) considering $r(t)^{2(N-1)}a(r(t))$ as weight function.

Clearly the following result holds.

Theorem 2.3.1. *Let $a\colon [R_1,R_2] \to \mathbb{R}$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$ be continuous functions satisfying (h_I), (h_{II}), (h_{III}) and (h_{IV}). Then problem* (2.3.1) *has at least a positive radially symmetric (classical) solution.*

2.4 Conflicting nonlinearities: introduction

In this second part we study positive solutions to Sturm-Liouville boundary value problems associated with nonlinear second order ODEs. To describe our results, throughout this introductive section we focus our attention on the equation

$$u'' + a(t)g(u) - \mu\, b(t)k(u) = 0 \tag{2.4.1}$$

defined on the nontrivial compact interval $[0,T]$. We assume that $\mu > 0$ is a real parameter, $a,b\colon [0,T] \to \mathbb{R}^+$ are measurable maps and $g,k\colon \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions such that

$$(i_1) \qquad \begin{aligned} g(0) &= 0, \qquad g(s) > 0, \quad \text{for } s > 0, \\ k(0) &= 0, \qquad k(s) > 0, \quad \text{for } s > 0. \end{aligned}$$

Referring to [161], we can say that equation (2.4.1) exhibits *conflicting nonlinearities.* Moreover, we can look at (2.4.1) as an *indefinite* equation.

Our main goal is to provide multiplicity results of positive solutions to equation (2.4.1) together with the Sturm-Liouville boundary conditions, namely conditions of the form

$$\begin{cases} \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(T) + \delta u'(T) = 0, \end{cases} \tag{2.4.2}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ with $\gamma\beta + \alpha\gamma + \alpha\delta > 0$. We notice that for $\alpha = \gamma = 1$ and $\beta = \delta = 0$, we obtain the Dirichlet boundary conditions.

Starting from the Seventies, these types of problems have received a remarkable attention in the research area of nonlinear differential equations. One of the early work was due to Anderson (cf. [10]) who has proved that the equation

$$-\Delta u = u^3 - \mu u^5 - u \quad \text{in } \mathbb{R}^N$$

has a solution if $0 < \mu < 3/16$, while there are no solutions for $\mu > 3/16$.

Other two relevant contributions to the autonomous case are [8, 21]. In [21] Berestycki and Lions have analyzed the more general equation

$$-\Delta u = \nu |u|^{p-1} u - \mu |u|^{q-1} u - \lambda u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\nu, \mu, \lambda > 0$ and $1 < q < p < (N+2)/(N-2)$, and they proved existence and nonexistence results in dependence of the parameter $\mu > 0$. In [8] Ambrosetti, Brezis and Cerami proved that there is a positive solution of

$$\begin{cases} -\Delta u = \lambda u^q + u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 < q < 1 < p$, for $\lambda > 0$ small enough and no solution for λ large.

We refer to [161] for a further result in this direction and for a more complete presentation and bibliography on the subject.

A motivation for this second part of the chapter is given by the papers [4, 98], where non-autonomous differential equations on bounded domains are taken into account. The boundedness of the domain enables the authors to deal with more general equations (with respect to those considered in [8, 10, 21]) and, in particular, to consider non-negative weight functions in place of the positive coefficients in front of the nonlinearities.

In [4] Alama and Tarantello studied positive solutions of the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = \lambda u + k(x) u^q - h(x) u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ (with $N \geq 3$) is an open bounded set with smooth boundary, the functions $h, k \in L^1(\Omega)$ are non-negative and $1 < q < p$. They proved existence, nonexistence and multiplicity results depending on $\lambda \in \mathbb{R}$ and according to the properties of the ratio k^{p-1}/h^{q-1}.

In [98] Girão and Gomes dealt with nodal solutions to

$$\begin{cases} -\Delta u = a^+(x)\big(\lambda u + f(x,u)\big) - \mu a^+(x) g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ (with $N \geq 1$) is an open bounded set with smooth boundary. They proved existence of nodal solutions for $\mu > 0$ sufficiently large.

The main goal of this second part is to present a multiplicity result for positive solutions to (2.4.1)-(2.4.2) in dependence of the number of the intervals where $a(t) > 0$ and thus giving a contribution to [4, 98]. In order to explain our achievement, we now introduce it in a slightly easier framework.

Let $a, b \colon [0,T] \to \mathbb{R}^+$ be continuous functions such that

(i_2) there exist two zeros τ, σ with $0 < \tau < \sigma < T$ such that

$$\begin{aligned} &a(t) > 0 \quad \text{on }]0,\tau[\cup]\sigma,T[, && a(t) \equiv 0 \quad \text{on } [\tau,\sigma], \\ &b(t) > 0 \quad \text{on }]\tau,\sigma[, && b(t) \equiv 0 \quad \text{on } [0,\tau] \cup [\sigma,T]. \end{aligned}$$

Our main multiplicity result is the following. See Figure 2.3 for a numerical example.

Theorem 2.4.1. *Let $a, b \colon [0,T] \to \mathbb{R}^+$ be continuous functions satisfying (i_2). Let $g, k \colon \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions satisfying (i_1). Moreover, assume that*

$$\limsup_{s \to 0^+} \frac{g(s)}{s} = 0, \qquad \liminf_{s \to +\infty} \frac{g(s)}{s} = +\infty,$$

and

$$\limsup_{s \to 0^+} \frac{k(s)}{s} < +\infty.$$

Then there exists $\mu^* > 0$ such that for every $\mu > \mu^*$ the boundary value problem (2.4.1)-(2.4.2) *has at least* 3 *positive solutions.*

We notice that, under the hypotheses of Theorem 2.4.1, the function $g(s)$ is *superlinear*, thus covering the classical case $g(s) = s^p$ with $p > 1$. On the other hand, we do not impose any growth condition on $k(s)$. Hence, the case considered in [4] is clearly included in our setting (cf. Section 2.8 for other remarks in this direction).

As remarked above, Theorem 2.4.1 is a special case of the main result of this part (cf. Theorem 2.6.1), where we deal with more general (Lebesgue integrable) coefficients $a(t)$ and $b(t)$ and weaker growth conditions on $g_i(s)$. Roughly speaking, we consider a weight function $a \colon [0,T] \to \mathbb{R}^+$ (belonging to the "positive" part of the nonlinearity) which is positive on m intervals,

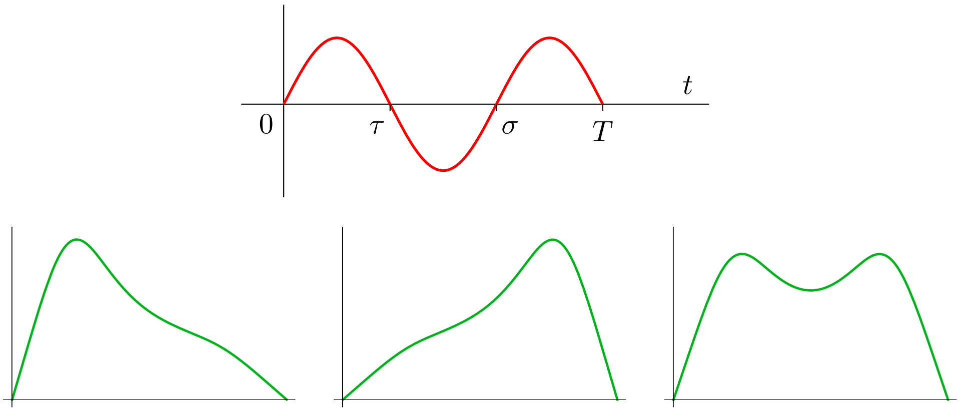

Figure 2.3: The figure shows an example of 3 positive solutions to the Dirichlet problem associated with (2.4.1) on $[0, 3\pi]$, where $\tau = \pi$, $\sigma = 2\pi$, $T = 3\pi$, $a(t) = \sin^+(t)$, $b(t) = \sin^-(t)$ (as in the upper part of the figure), $g(s) = s^2$, $k(s) = s^3$ (for $s > 0$). For $\mu = 1$, Theorem 2.4.1 ensures the existence of 3 positive solutions, whose graphs are located in the lower part of the figure.

so $a(t)$ has m positive humps. In this framework we prove the existence of $2^m - 1$ positive solutions of (2.4.1)-(2.4.2) when $b(t)$ is "sufficiently large", namely $\beta \gg 0$. We refer to the next section, where we introduce all the hypotheses on the elements involved in (2.4.1)-(2.4.2) which are assumed for the rest of the chapter.

2.5 Conflicting nonlinearities: setting and notation

In this section we present the main elements involved in the study of the positive solutions to the boundary value problem

$$\begin{cases} u'' + f(t,u) = 0 \\ \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(T) + \delta u'(T) = 0, \end{cases} \tag{2.5.1}$$

where $f\colon [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ is a function of the form

$$f(t,s) := \sum_{i=1}^{m} \alpha_i a_i(t) g_i(s) - \sum_{j=0}^{m+1} \beta_j b_j(t) k_j(s), \tag{2.5.2}$$

with $m \geq 1$, and $\alpha, \beta, \gamma, \delta \geq 0$ with $\gamma\beta + \alpha\gamma + \alpha\delta > 0$.

The following hypotheses and positions will be assumed from now on.

Let $m \geq 1$ be an integer. Let $\alpha_i > 0$, for $i = 1, \dots, m$, and $\beta_j > 0$, for $j = 0, \dots, m+1$, be real parameters.

Let $a_i\colon [0,T] \to \mathbb{R}^+$, for $i = 1,\dots,m$, and $b_j\colon [0,T] \to \mathbb{R}^+$, for $j = 0,\dots,m+1$, be (non-negative) Lebesgue integrable functions. Moreover, we assume that

(h_1) *there exist* $2m+2$ *closed and pairwise disjoint intervals* $I_1,\dots,I_m$ *and* $J_0,\dots,J_{m+1}$ *(*J_0 *and* J_{m+1} *possibly empty), such that*

$$\begin{aligned} &a_i \not\equiv 0 \;\; \textit{on } I_i, \quad a_i \equiv 0 \;\; \textit{on } [0,T]\setminus I_i, \quad i = 1,\dots,m;\\ &b_j \not\equiv 0 \;\; \textit{on } J_j, \quad b_j \equiv 0 \;\; \textit{on } [0,T]\setminus J_j, \quad j = 0,1,\dots,m+1. \end{aligned}$$

Without loss of generality, up to a relabelling of the indices, we can assume that $\max I_i \leq \min I_k$, for all $i<k$; $\max J_j \leq \min J_k$, for all $j<k$; $\max I_i \leq \min J_j$, for all $i<j$; between two intervals I_i and I_{i+1} there is an interval J_j; between two intervals J_j and J_{j+1} there is an interval I_i. Moreover, eventually extending the functions $a_i(t)$ as 0 on $([0,T]\setminus I_i)\cap[\max J_j, \min J_{j+1}]$ (with I_i between J_j and J_{j+1}), we can also suppose that

$$\bigcup_{i=1}^{m} I_i \cup \bigcup_{j=0}^{m+1} J_j = [0,T].$$

Summarizing all the conventions, it is not restrictive to label the intervals I_i and J_j following the natural order given by the standard orientation of the real line and thus determine $2m+2$ points

$$0 = \tau_0 \leq \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \ldots < \sigma_{m-1} < \tau_{m-1} < \sigma_m < \tau_m \leq \sigma_{m+1} = T,$$

so that

$$I_i := [\sigma_i, \tau_i], \quad i = 1,\dots,m, \qquad \text{and} \qquad J_j := [\tau_j, \sigma_{j+1}], \quad j = 0,\dots,m+1.$$

Finally, consistently with assumption (h_1) and without loss of generality, we select the points σ_i and τ_i in such a manner that $b_j(t) \not\equiv 0$ on all right neighborhoods of τ_j and on all left neighborhoods of σ_{j+1}. In other words, if there is an interval K contained in $[0,T]$ where $a(t)\equiv 0$, we choose the points σ_i and τ_i so that K is contained in one of the I_i or K is contained in the interior of one of the J_j.

Let $g_i\colon \mathbb{R}^+ \to \mathbb{R}^+$, for $i = 1,\dots,m$, and $k_j\colon \mathbb{R}^+ \to \mathbb{R}^+$, for $j = 0,\dots,m+1$, be continuous functions and such that

$$(h_2) \qquad \begin{aligned} &g_i(0) = 0, \qquad g_i(s) > 0, \quad \text{for } s > 0, \qquad i = 1,\dots,m;\\ &k_j(0) = 0, \qquad k_j(s) > 0, \quad \text{for } s > 0, \qquad j = 0,\dots,m+1. \end{aligned}$$

We define

$$g_0^i := \limsup_{s\to 0^+} \frac{g_i(s)}{s}, \quad g_\infty^i := \liminf_{s\to+\infty} \frac{g_i(s)}{s}, \qquad i = 1,\dots,m,$$

and

$$k_0^j := \limsup_{s\to 0^+} \frac{k_j(s)}{s}, \qquad j = 0,\dots,m+1.$$

For all $i = 1,\dots,m$ and for all $j = 0,\dots,m+1$, we suppose

$$(h_3) \qquad g_0^i < +\infty, \qquad g_\infty^i > 0, \qquad k_0^j < +\infty.$$

We denote with λ_0 the first (positive) eigenvalue of the eigenvalue problem

$$\begin{cases} \varphi'' + \lambda\bigl[\sum_{i=1}^m a_i(t)\bigr]\varphi = 0 \\ \alpha\varphi(0) - \beta\varphi'(0) = 0 \\ \gamma\varphi(T) + \delta\varphi'(T) = 0, \end{cases}$$

and, for $i = 1,\dots,m$, with λ_1^i the first eigenvalue of the eigenvalue problem in I_i

$$\begin{cases} \varphi'' + \lambda a_i(t)\varphi = 0 \\ \varphi|_{\partial I_i} = 0. \end{cases}$$

If $\tau_0 = \sigma_1 = 0$ or $\tau_m = \sigma_{m+1} = T$, we denote with λ_1^i (with $i = 1$ or $i = m$, respectively) the first eigenvalue of the eigenvalue problem

$$\begin{cases} \varphi'' + \lambda a_i(t)\varphi = 0 \\ \alpha\varphi(0) - \beta\varphi'(0) = 0 \\ \varphi(\tau_1) = 0 \end{cases} \qquad \text{or} \qquad \begin{cases} \varphi'' + \lambda a_i(t)\varphi = 0 \\ \varphi(\sigma_m) = 0 \\ \gamma\varphi(T) + \delta\varphi'(T) = 0, \end{cases}$$

respectively. Clearly, if $\beta = 0$ or $\delta = 0$, respectively, the definition of λ_1^i is the same as before. Using the assumptions on $a_i(t)$, in any case, we obtain that $\lambda_1^i > 0$ for each $i = 1,\dots,m$.

Now, we briefly review Theorem 1.3.1 in the context of a Sturm-Liouville boundary value problem. We stress that Chapter 1 (and in particular Theorem 1.3.1) concerns the Dirichlet boundary value problem (i.e. problem (2.5.1) with $\alpha = \gamma = 1$ and $\beta = \delta = 0$), but in Section 1.4.4 we observed that the approach presented therein could be adapted to the study of different boundary conditions, for example $u(0) = u'(T) = 0$ or $u'(0) = u(T) = 0$, which are clearly covered by the Sturm-Liouville ones. Since there are some difference in considering the Sturm-Liouville boundary conditions, we present the modification of Theorem 1.3.1 needed in this chapter.

Let us consider a general map $f\colon [0,T]\times\mathbb{R}^+ \to \mathbb{R}$ and suppose that $f(t,s)$ is an L^1-Carathéodory function. In order to state the multiplicity result we list the following hypotheses that will be assumed.

(f^*) $f(t,0) = 0$, for a.e. $t \in [0,T]$.

(f_0^-) There exists a function $q_- \in L^1([0,T],\mathbb{R}^+)$ such that

$$\liminf_{s\to 0^+} \frac{f(t,s)}{s} \geq -q_-(t), \quad \text{uniformly a.e. } t \in [0,T].$$

(f_0^+) There exists a function $q_0 \in L^1([0,T],\mathbb{R}^+)$ with $q_0 \not\equiv 0$ such that

$$\limsup_{s\to 0^+} \frac{f(t,s)}{s} \leq q_0(t), \quad \text{uniformly a.e. } t \in [0,T],$$

and

$$\mu_1(q_0) > 1,$$

where $\mu_1(q_0)$ is the first positive eigenvalue of the eigenvalue problem

$$\varphi'' + \mu q_0(t)\varphi = 0, \qquad \varphi(0) = \varphi(T) = 0.$$

(H) There exist $m \geq 1$ intervals $I_1, \ldots, I_m$, closed and pairwise disjoint, such that

$$f(t,s) \geq 0, \quad \text{for a.e. } t \in \bigcup_{i=1}^{m} I_i \text{ and for all } s \geq 0;$$

$$f(t,s) \leq 0, \quad \text{for a.e. } t \in [0,T] \setminus \bigcup_{i=1}^{m} I_i \text{ and for all } s \geq 0.$$

(f_∞) For all $i = 1, \ldots, m$ there exists a function $q_\infty^i \in L^1(I_i,\mathbb{R}^+)$ with $q_\infty^i \not\equiv 0$ such that

$$\liminf_{s\to +\infty} \frac{f(t,s)}{s} \geq q_\infty^i(t), \quad \text{uniformly a.e. } t \in I_i,$$

and

$$\mu_1^{I_i}(q_\infty^i) < 1,$$

where $\mu_1^{I_i}(q_\infty^i)$ is the first positive eigenvalue of the eigenvalue problem in I_i

$$\varphi'' + \mu q_\infty^i(t)\varphi = 0, \qquad \varphi|_{\partial I_i} = 0.$$

We observe that, since $f(t,s)$ satisfies condition (f_0^+), from the continuity of the eigenvalue $\mu_1(q_0)$ as a function of q_0 we can derive that there exists $r_0 > 0$ such that

(h_0) the following inequality holds

$$\frac{f(t,s)}{s} \leq q_0(t) + \varepsilon, \quad \text{a.e. } t \in [0,T],\ \forall\, 0 < s \leq r_0,$$

for every $\varepsilon > 0$ such that $\mu_1(q_0 + \varepsilon) > 1$.

Now we can state the multiplicity result for positive solutions of the boundary value problem (2.5.1).

Theorem 2.5.1. *Let* $f\colon [0,T]\times\mathbb{R}^+\to\mathbb{R}$ *be an* L^1*-Carathéodory function satisfying* (f^*), (f_0^-), (f_0^+), (H) *and* (f_∞). *Let* $r_0>0$ *satisfy* (h_0). *Suppose that*

($\bigstar$) *there exists* $r\in\,]0,r_0]$ *such that for every* $\emptyset\neq\mathcal{I}\subseteq\{1,\ldots,m\}$ *and every* L^1*-Carathéodory function* $h\colon[0,T]\times\mathbb{R}^+\to\mathbb{R}$ *satisfying*

$$\begin{aligned} h(t,s)&\geq f(t,s), \quad \text{a.e. } t\in\bigcup_{i\in\mathcal{I}} I_i,\ \forall\, s\geq 0,\\ h(t,s)&= f(t,s), \quad \text{a.e. } t\in[0,T]\setminus\bigcup_{i\in\mathcal{I}} I_i,\ \forall\, s\geq 0,\end{aligned}$$

any non-negative solution $u(t)$ *of*

$$u''+h(t,u)=0$$

satisfies $\max_{t\in I_i}u(t)\neq r$ *for every* $i\in\{1,\ldots,m\}\setminus\mathcal{I}$.

Then there exist at least 2^m-1 *positive solutions of the boundary value problem* (2.5.1).

The proof of the above result is analogous to the one presented in Section 1.3 dealing with the Dirichlet problem. We only underline that, when we reduce our study to an equivalent fixed point problem, we have to consider the Green function associated to the equation $u''+u=0$ with the Sturm-Liouville boundary conditions (cf. [75, 76]), that is

$$G(t,s):=\frac{1}{\gamma\beta+\alpha\gamma+\alpha\delta}\begin{cases}(\gamma+\delta-\gamma s)(\beta+\alpha t), & \text{if } 0\leq t\leq s\leq 1;\\ (\gamma+\delta-\gamma t)(\beta+\alpha s), & \text{if } 0\leq s\leq t\leq 1.\end{cases}$$

The computations of the degree for the map Φ, defined as in (1.1.3), are analogous. Therefore, we omit the proof.

2.6 Conflicting nonlinearities: the main result

Recalling the setting and the notation introduced in Section 2.5, now we state and prove the following main result.

Theorem 2.6.1. *Let* $m\geq 1$ *be an integer. Let* $a_i\colon[0,T]\to\mathbb{R}^+$, *for* $i=1,\ldots,m$, *and* $b_j\colon[0,T]\to\mathbb{R}^+$, *for* $j=0,\ldots,m+1$, *be Lebesgue integrable functions satisfying* (h_1). *Let* $g_i\colon\mathbb{R}^+\to\mathbb{R}^+$, *for* $i=1,\ldots,m$, *and* $k_j\colon\mathbb{R}^+\to\mathbb{R}^+$, *for* $j=0,\ldots,m+1$, *be continuous functions satisfying* (h_2) *and* (h_3). *Let* $\alpha_i>0$, *for all* $i=1,\ldots,m$. *Moreover, suppose that*

$$\alpha_i g_0^i<\lambda_0,\quad \textit{for all }\ i=1,\ldots,m, \tag{2.6.1}$$

and

$$\alpha_i g^i_\infty > \lambda^i_1, \quad \textit{for all } \; i = 1, \dots, m. \tag{2.6.2}$$

Then there exists $\beta^* > 0$ *such that, if*

$$\beta_j > \beta^*, \quad \textit{for all } \; j = 0, \dots, m+1,$$

the boundary value problem (2.5.1) *with* $f(t,s)$ *defined in* (2.5.2) *has at least* $2^m - 1$ *positive solutions.*

Proof. In order to prove the theorem, we are going to enter the setting of Theorem 2.5.1 and to check that all its hypotheses are satisfied for $\beta_j > 0$ sufficiently large.

First of all, we observe that the map $f(t,s)$ defined as in (2.5.2) is an L^1-Carathéodory function and, moreover, satisfies (f^*), due to condition (h_2). From

$$\liminf_{s\to 0^+} \frac{f(t,s)}{s} \geq \sum_{j=0}^{m+1} \beta_j b_j(t) \liminf_{s\to 0^+} -\frac{k_j(s)}{s} = -\sum_{j=0}^{m+1} \beta_j b_j(t) k^j_0,$$

for a.e. $t \in [0,T]$, and from the last assumption in (h_3), we deduce that (f^-_0) holds with $q_- \in L^1([0,T],\mathbb{R}^+)$ defined as

$$q_-(t) := \sum_{j=0}^{m+1} \beta_j b_j(t) k^j_0, \quad t \in [0,T].$$

For $i = 1, \dots, m$, by hypothesis (2.6.1) let us fix $g^i_* > 0$ such that $g^i_0 < g^i_* < \lambda_0/\alpha_i$. Next, we define

$$q_0(t) := \sum_{i=1}^{m} \alpha_i a_i(t) g^i_*, \quad t \in [0,T].$$

We observe that $q_0 \in L^1([0,T],\mathbb{R}^+)$, $q_0 \not\equiv 0$ and

$$\mu_1(q_0) \geq \mu_1\biggl(\Bigl(\max_{i=1,\dots,m} \alpha_i g^i_*\Bigr) \sum_{i=1}^{m} a_i(t)\biggr) = \frac{\lambda_0}{\displaystyle\max_{i=1,\dots,m} \alpha_i g^i_*} > 1.$$

Then condition (f^+_0) is valid. Concerning the sign of $a(t)$, we observe that hypothesis (H) directly follows from condition (h_1). Furthermore, defining

$$q^i_\infty(t) := \alpha_i a_i(t) g^i_\infty, \quad t \in I_i, \quad \text{for } \; i = 1, \dots, m,$$

and observing that $q^i_\infty \in L^1(I_i,\mathbb{R}^+)$, $q^i_\infty \not\equiv 0$ and

$$\mu^{I_i}_1(q^i_\infty) = \frac{\lambda^i_1}{\alpha_i g^i_\infty} < 1, \quad i = 1, \dots, m,$$

(by conditions (h_3) and (2.6.2)), we obtain that (f_∞) holds.

As a second step, we prove that hypothesis $(\bigstar)$ is valid. By condition (2.6.1), for all $i = 1, \ldots, m$, we can choose $\rho_i > 0$ such that $g_0^i < \lambda_0 - \rho_i$. As observed in Section 2.5, by hypothesis (f_0^+) we can take $r_0 > 0$ satisfying (h_0) (as in Theorem 2.5.1). Next, we fix $0 < r \leq r_0$ such that

$$\alpha_i \frac{g_i(s)}{s} < \lambda_0 - \rho_i, \quad \forall\, 0 < s \leq r, \tag{2.6.3}$$

(for $i = 1, \ldots, m$). We claim that $(\bigstar)$ holds for r satisfying (2.6.3) and taking the parameters β_j sufficiently large. Let us consider an arbitrary set of indices $\emptyset \neq \mathcal{I} \subseteq \{1, \ldots, m\}$ and an arbitrary L^1-Carathéodory function $h(t,s)$ as in $(\bigstar)$. Suppose by contradiction that there exists a non-negative solution $u(t)$ of $u'' + h(t,u) = 0$ such that

$$\max_{t \in I_\ell} u(t) = r, \qquad \text{for some index } \ell \in \{1, \ldots, m\} \setminus \mathcal{I}.$$

If $\mathcal{I} = \{1, \ldots, m\}$, there is nothing to prove. Then fix $\emptyset \neq \mathcal{I} \subsetneq \{1, \ldots, m\}$. By the concavity of $u(t)$ in I_ℓ, we have

$$u(t) \geq \frac{r}{\tau_\ell - \sigma_\ell} \min\{t - \sigma_\ell, \tau_\ell - t\}, \quad \forall\, t \in I_\ell = [\sigma_\ell, \tau_\ell], \tag{2.6.4}$$

(cf. [93, p. 420] for a similar estimate)

In order to prove that our assumption is contradictory, we split our argument into three steps.

Step 1. A priori bounds for $|u'(t)|$ on I_ℓ. Analogously to *Step 1* in the proof of Theorem 1.4.3, we obtain the existence of $M_\ell > 0$ such that

$$|u'(t)| \leq M_\ell, \quad \forall\, t \in I_\ell. \tag{2.6.5}$$

Step 2. Lower bounds for $u(t)$ on the boundary of I_ℓ. Let $\varphi_\ell(t)$ be the positive eigenfunction of the eigenvalue problem on I_ℓ

$$\varphi'' + \lambda_1^\ell a_\ell(t)\varphi = 0, \quad \varphi|_{\partial I_\ell} = 0,$$

with $\|\varphi_\ell\|_\infty = 1$, where $\lambda_1^\ell > 0$ is the first eigenvalue. Then $\varphi_\ell(t) \geq 0$, for all $t \in I_\ell$, $\varphi_\ell(t) > 0$, for all $t \in \,]\sigma_\ell, \tau_\ell[$, and $\varphi_\ell'(\sigma_\ell) > 0 > \varphi_\ell'(\tau_\ell)$ (hence $\|\varphi_\ell'\|_\infty > 0$).

By (2.6.3) and the fact that $\lambda_0 \leq \lambda_1^\ell$, we know that

$$\alpha_\ell g_\ell(s) < (\lambda_1^\ell - \rho_\ell)s, \quad \forall\, 0 < s \leq r.$$

Then, using (2.6.4), we have

$$
\begin{aligned}
&\|\varphi_\ell'\|_\infty (u(\sigma_\ell) + u(\tau_\ell)) \geq \\
&\geq u(\sigma_\ell)\varphi_\ell'(\sigma_\ell) + u(\tau_\ell)|\varphi_\ell'(\tau_\ell)| = u(\sigma_\ell)\varphi_\ell'(\sigma_\ell) - u(\tau_\ell)\varphi_\ell'(\tau_\ell) \\
&= \Big[u'(t)\varphi_\ell(t) - u(t)\varphi_\ell'(t)\Big]_{t=\sigma_\ell}^{t=\tau_\ell} \\
&= \int_{\sigma_\ell}^{\tau_\ell} \frac{d}{dt}\Big[u'(t)\varphi_\ell(t) - u(t)\varphi_\ell'(t)\Big]\,dt \\
&= \int_{I_\ell} \Big[u''(t)\varphi_\ell(t) - u(t)\varphi_\ell''(t)\Big]\,dt \\
&= \int_{I_\ell} \Big[-h(t, u(t))\varphi_\ell(t) + u(t)\lambda_1^\ell a_\ell(t)\varphi_\ell(t)\Big]\,dt \\
&= \int_{I_\ell} \Big[\lambda_1^\ell u(t) - \alpha_\ell g_\ell(u(t))\Big] a_\ell(t)\varphi_\ell(t)\,dt \\
&> \int_{I_\ell} \rho_\ell \left(\frac{r}{\tau_\ell - \sigma_\ell}\min\{t - \sigma_\ell, \tau_\ell - t\}\right) a_\ell(t)\varphi_\ell(t)\,dt \\
&= r\left[\frac{\rho_\ell}{\tau_\ell - \sigma_\ell}\int_{I_\ell} \min\{t - \sigma_\ell, \tau_\ell - t\} a_\ell(t)\varphi_\ell(t)\,dt\right].
\end{aligned}
$$

Hence, from the above inequality, we conclude that there exists a constant $c_\ell > 0$, depending on ρ_ℓ, I_ℓ and $a_\ell(t)$, but independent on $u(t)$ and r, such that

$$u(\sigma_\ell) + u(\tau_\ell) \geq c_\ell r > 0.$$

As a consequence of the above inequality, we have that at least one of the two inequalities

$$0 < \frac{c_\ell r}{2} \leq u(\tau_\ell) \leq r, \qquad 0 < \frac{c_\ell r}{2} \leq u(\sigma_\ell) \leq r, \tag{2.6.6}$$

holds.

The two inequalities in (2.6.6) reduce to a single one, if $\sigma_1 = 0$ and $\beta > 0$, or if $\tau_m = T$ and $\delta > 0$. Indeed, if $\sigma_1 = 0$ and $\beta > 0$, we have

$$
\begin{aligned}
&\Big[u'(t)\varphi_1(t) - u(t)\varphi_1'(t)\Big]_{t=0}^{t=\tau_1} \\
&= u'(\sigma_1)\varphi_1(\sigma_1) - u(\sigma_1)\varphi_1'(\sigma_1) - u'(\tau_1)\varphi_1(\tau_1) + u(\tau_1)\varphi_1'(\tau_1) \\
&= u'(\sigma_1)\frac{\alpha}{\beta}\varphi_1'(\sigma_1) - u(\sigma_1)\varphi_1'(\sigma_1) + u(\tau_1)\varphi_1'(\tau_1) \\
&= u(\tau_1)\varphi_1'(\tau_1) \leq \|\varphi_1'\|_\infty u(\tau_1).
\end{aligned}
$$

Analogously, if $\tau_m = T$ and $\delta > 0$, we have

$$\Big[u'(t)\varphi_1(t) - u(t)\varphi_m'(t)\Big]_{t=\sigma_m}^{t=T} \leq \|\varphi_m'\|_\infty u(\sigma_m).$$

Finally, as a consequence of the previous inequalities, we obtain that

$$0 < \frac{c_\ell r}{2} \leq u(\tau_1) \leq r \quad \text{or} \quad 0 < \frac{c_\ell r}{2} \leq u(\sigma_m) \leq r \tag{2.6.7}$$

holds, respectively.

Step 3. Contradiction on an adjacent interval for β_ℓ large. As a first case, we suppose that the first inequality in (2.6.6) is true. If $\tau_\ell = T$, then $\delta > 0$ in the boundary conditions (otherwise $u(\tau_\ell) = 0$, a contradiction) and we deal with the second inequality in (2.6.7) (see the discussion of the second case below). Consequently, whenever $\tau_\ell < T$, we can focus our attention on the right-adjacent interval $[\tau_\ell, \sigma_{\ell+1}]$, where $f(t, u(t)) = -\beta_\ell b_\ell(t) k_\ell(u(t)) \leq 0$. Recall also that, by the convention adopted in defining the intervals I_i and J_j, we have that $b_\ell(t)$ is not identically zero on all right neighborhoods of τ_ℓ.

We observe that there exists $R > r$ such that $\max_{t\in[0,T]} u(t) < R$. This is a consequence of (f^*), (f_0^-), (H) and (f_∞), as described in Lemma 1.1.3.

Since $k_\ell(s) > 0$ for all $s > 0$, we can introduce the positive constant

$$\nu_\ell := \min_{\frac{c_\ell r}{4} \leq s \leq R} k_\ell(s) > 0$$

and define

$$\delta_\ell^+ := \min\left\{\sigma_{\ell+1} - \tau_\ell, \frac{c_\ell r}{4M_\ell}\right\} > 0,$$

where $M_\ell > 0$ is the bound for $|u'(t)|$ obtained in (2.6.5) of *Step 1*. Then, by the convexity of $u(t)$ on J_ℓ, we have that $u(t)$ is bounded from below by the tangent line at $(\tau_\ell, u(\tau_\ell))$, with slope $u'(\tau_\ell) \geq -M_\ell$. Therefore,

$$\frac{c_\ell r}{4} \leq u(t) \leq R, \quad \forall\, t \in [\tau_\ell, \tau_\ell + \delta_\ell^+].$$

We are going to prove that $\max_{t\in J_\ell} u(t) > R$ for $\beta_\ell > 0$ sufficiently large (which is a contradiction with respect to the upper bound $R > 0$ for $u(t)$).

Consider the interval $[\tau_\ell, \tau_\ell + \delta_\ell^+] \subseteq J_\ell$. Proceeding as in *Step 3* in the proof of Theorem 1.4.3, we can deduce

$$R \geq u(\tau_\ell + \delta_\ell^+) \geq \frac{c_\ell r}{2} - M_\ell \delta_\ell^+ + \beta_\ell \nu_\ell \int_{\tau_\ell}^{\tau_\ell + \delta_\ell^+} \left(\int_{\tau_\ell}^{s} b_\ell(\xi)\, d\xi \right) ds.$$

This gives a contradiction if β_ℓ is sufficiently large, say

$$\beta_\ell > \beta_\ell^+ := \frac{R + M_\ell T}{\nu_\ell \int_{\tau_\ell}^{\tau_\ell+\delta_\ell^+} \int_{\tau_\ell}^{s} b_\ell(\xi)\, d\xi\, ds},$$

recalling that $\int_{\tau_\ell}^{t} b_\ell(\xi)\, d\xi > 0$ for each $t \in \,]\tau_\ell, \sigma_{k+1}]$.

A similar argument (with obvious modifications) applies if the second inequality in (2.6.6) is true. If $\sigma_\ell = 0$, then $\beta > 0$ in the boundary conditions (otherwise $u(\sigma_\ell) = 0$, a contradiction) and we deal with the first inequality in (2.6.7) (see the discussion of the first case above). Consequently, whenever $\sigma_\ell > 0$, we can focus our attention on the left-adjacent interval $J_{\ell-1}$ where $f(t,u(t)) = -\beta_{\ell-1} b_{\ell-1}(t) k_{\ell-1}(u(t)) \leq 0$. Recall also that, by the convention adopted in defining the intervals I_i and J_j, we have that $b_{\ell-1}(t)$ is not identically zero on all left neighborhoods of σ_ℓ.

If we define

$$\delta_\ell^- := \min\left\{\sigma_\ell - \tau_{\ell-1}, \frac{c_\ell r}{4M_\ell}\right\} > 0,$$

we obtain a similar contradiction for

$$\beta_\ell > \beta_\ell^- := \frac{R + M_\ell T}{\nu_\ell \int_{\sigma_\ell - \delta_\ell^-}^{\sigma_\ell} \int_s^{\sigma_\ell} b_{\ell-1}(\xi)\, d\xi\, ds}.$$

At the end, defining

$$\beta^* := \max_{k=1,\ldots,m} \beta_\ell^\pm,$$

condition (★) holds taking $\beta_j > \beta^*$, for all $j = 0, \ldots, m+1$. Finally, we can apply Theorem 2.5.1 and the proof is completed. □

From the statement of Theorem 2.6.1, one can easily notice that the parameters $\alpha_i > 0$ are involved only in hypotheses (2.6.1) and (2.6.2), therefore there is no real condition on those constants (since they can be considered as part of the functions g_i). In a moment, the role of the parameters α_i will become more clear. Indeed, investigating more on conditions (2.6.1) and (2.6.2), we can state the following corollaries (the obvious proofs are omitted).

Corollary 2.6.1. *Let $m \geq 1$ be an integer. Let $a_i\colon [0,T] \to \mathbb{R}^+$, for $i = 1, \ldots, m$, and $b_j\colon [0,T] \to \mathbb{R}^+$, for $j = 0, \ldots, m+1$, be Lebesgue integrable functions satisfying (h_1). Let $g_i\colon \mathbb{R}^+ \to \mathbb{R}^+$, for $i = 1, \ldots, m$, and $k_j\colon \mathbb{R}^+ \to \mathbb{R}^+$, for $j = 0, \ldots, m+1$, be continuous functions satisfying (h_2) and (h_3). Moreover, suppose that*

$$g_0^i = 0, \quad \text{for all } i = 1, \ldots, m.$$

Then there exists $\alpha^* > 0$ such that if

$$\alpha_i > \alpha^*, \quad \text{for all } i = 1, \ldots, m,$$

there exists $\beta^* = \beta^*(\alpha_1, \ldots, \alpha_m) > 0$ so that, if

$$\beta_j > \beta^*, \quad \text{for all } j = 0, \ldots, m+1,$$

then the boundary value problem (2.5.1) with $f(t,s)$ defined in (2.5.2) has at least $2^m - 1$ positive solutions.

Corollary 2.6.2. *Let $m \geq 1$ be an integer. Let $a_i \colon [0,T] \to \mathbb{R}^+$, for $i = 1,\ldots,m$, and $b_j \colon [0,T] \to \mathbb{R}^+$, for $j = 0,\ldots,m+1$, be Lebesgue integrable functions satisfying (h_1). Let $g_i \colon \mathbb{R}^+ \to \mathbb{R}^+$, for $i = 1,\ldots,m$, and $k_j \colon \mathbb{R}^+ \to \mathbb{R}^+$, for $j = 0,\ldots,m+1$, be continuous functions satisfying (h_2) and (h_3). Moreover, suppose that*

$$g_\infty^i = +\infty, \quad \textit{for all } \ i = 1,\ldots,m.$$

Then there exists $\alpha_* > 0$ such that if

$$0 < \alpha_i < \alpha_*, \quad \textit{for all } \ i = 1,\ldots,m,$$

there exists $\beta^* = \beta^*(\alpha_1,\ldots,\alpha_m) > 0$ so that, if

$$\beta_j > \beta^*, \quad \textit{for all } \ j = 0,\ldots,m+1,$$

then the boundary value problem (2.5.1) with $f(t,s)$ defined in (2.5.2) has at least $2^m - 1$ positive solutions.

2.7 Conflicting nonlinearities: radial solutions

As a consequence of Theorem 2.6.1, we can give a multiplicity result for positive radially symmetric solutions to boundary value problems associated with elliptic PDEs on an annular domain.

We briefly describe the setting, referring to the notation introduced in Section 2.5. Let $0 < R_1 < R_2$ and consider the open annulus around the origin

$$\Omega := \bigl\{ x \in \mathbb{R}^N \colon R_1 < \|x\| < R_2 \bigr\},$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^N$ (for $N \geq 2$). We define

$$\mathcal{F}(x,s) := \sum_{i=1}^{m} \alpha_i \mathcal{A}_i(x) g_i(s) - \sum_{j=0}^{m+1} \beta_j \mathcal{B}_j(x) k_j(s), \quad x \in \overline{\Omega}, \ s \in \mathbb{R}^+,$$

with $m \geq 1$. For $i = 1,\ldots,m$ and $j = 0,1,\ldots,m+1$, let $\alpha_i > 0$, $\beta_j > 0$, and moreover let $g_i \colon \mathbb{R}^+ \to \mathbb{R}^+$ and $k_j \colon \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions satisfying conditions (h_2) and (h_3). Let $\mathcal{A}_i \colon \overline{\Omega} \to \mathbb{R}^+$, for $i = 1,\ldots,m$, and $\mathcal{B}_j \colon \overline{\Omega} \to \mathbb{R}^+$, for $j = 0,1,\ldots,m+1$.

We deal with the Dirichlet boundary value problem associated with an elliptic partial differential equation

$$\begin{cases} -\Delta u = \mathcal{F}(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7.1}$$

For simplicity, we look for classical solutions to (2.7.1), namely, $u \in \mathcal{C}^2(\overline{\Omega})$. Accordingly, we assume that $\mathcal{A}_i(x)$ and $\mathcal{B}_j(x)$ are *continuous* functions.

Moreover, we suppose that $\mathcal{A}_i(x)$ and $\mathcal{B}_j(x)$ are radially symmetric function, i.e. there exist continuous functions $A_i, B_j : [R_1, R_2] \to \mathbb{R}^+$ such that

$$\mathcal{A}_i(x) = A_i(\|x\|), \quad \mathcal{B}_i(x) = B_i(\|x\|), \quad \forall\, x \in \overline{\Omega}. \tag{2.7.2}$$

In this way, we can transform the partial differential equation in (2.7.1) into a second order ordinary differential equation as the one in (2.5.1), as described in Section C.2.

Preliminarily, we introduce the function

$$F(r,s) := \sum_{i=1}^{m} \alpha_i A_i(r) g_i(s) - \sum_{j=0}^{m+1} \beta_j B_j(r) k_j(s), \quad r \in [R_1, R_2],\ s \in \mathbb{R}^+.$$

A radially symmetric (classical) solutions to (2.7.1) is a solution of the form $u(x) = \mathcal{U}(\|x\|)$, where $\mathcal{U}(r)$ is a scalar function defined on $[R_1, R_2]$. Consequently, we can convert (2.7.1) into

$$\begin{cases} \left(r^{N-1}\mathcal{U}'\right)' + r^{N-1}F(r,\mathcal{U}) = 0 \\ \mathcal{U}(R_1) = \mathcal{U}(R_2) = 0. \end{cases} \tag{2.7.3}$$

Via the change of variable

$$t = h(r) := \int_{R_1}^{r} \xi^{1-N}\, d\xi$$

and the positions

$$T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi, \quad r(t) := h^{-1}(t), \quad v(t) = \mathcal{U}(r(t)),$$

we can transform (2.7.3) into the Dirichlet problem

$$\begin{cases} v'' + f(t,v) = 0 \\ v(0) = v(T) = 0, \end{cases}$$

where

$$f(t,v) := r(t)^{2(N-1)} F(r(t), v), \quad t \in [0,T],\ v \in \mathbb{R}^+.$$

In this setting, a straightforward consequence of Theorem 2.6.1 is the following result. In the statement below, when we introduce condition (h_1^*) and the points σ_i and τ_i, we implicitly assume the convention adopted in defining the intervals I_i and J_j in Section 2.5.

Theorem 2.7.1. *Let $m \geq 1$ be an integer. Let $\mathcal{A}_i : \overline{\Omega} \to \mathbb{R}^+$, for $i = 1, \ldots, m$, and $\mathcal{B}_j : \overline{\Omega} \to \mathbb{R}^+$, for $j = 0, \ldots, m+1$, be Lebesgue integrable functions satisfying the following condition:*

(h_1^*) *there exist* $2m+2$ *points (with* $m \geq 1$*)*

$$R_1 = \tau_0 \leq \sigma_1 < \tau_1 < \sigma_2 < \ldots < \tau_{m-1} < \sigma_m < \tau_m \leq \sigma_{m+1} = R_2,$$

such that $A_i \not\equiv 0$ *on* $[\sigma_i, \tau_i]$, *for* $i = 1, \ldots, m$, *and* $B_i \not\equiv 0$ *on* $[\tau_i, \sigma_{i+1}]$, *for* $j = 1, \ldots, m+1$,

where $A_i, B_j \colon \overline{\Omega} \to \mathbb{R}^+$ *are defined as in* (2.7.2). *Let* $\alpha_i > 0$, *for all* $i = 1, \ldots, m$. *Let* $g_i \colon \mathbb{R}^+ \to \mathbb{R}^+$, *for* $i = 1, \ldots, m$, *and* $k_j \colon \mathbb{R}^+ \to \mathbb{R}^+$, *for* $j = 0, \ldots, m+1$, *be continuous functions satisfying* (h_2), (h_3), (2.6.1) *and* (2.6.2). *Then there exists* $\beta^* > 0$ *such that, if*

$$\beta_j > \beta^*, \quad \text{for all } j = 0, \ldots, m+1,$$

the Dirichlet boundary value problem (2.7.1) *has at least* $2^m - 1$ *positive radially symmetric (classical) solutions.*

Clearly, from Corollary 2.6.1 and Corollary 2.6.2 we also derive the following result.

Corollary 2.7.1. *Let* $m \geq 1$ *be an integer. Let* $\mathcal{A}_i \colon \overline{\Omega} \to \mathbb{R}^+$, *for* $i = 1, \ldots, m$, *and* $\mathcal{B}_j \colon \overline{\Omega} \to \mathbb{R}^+$, *for* $j = 0, \ldots, m+1$, *be Lebesgue integrable functions satisfying* (h_1^*). *Let* $g_i \colon \mathbb{R}^+ \to \mathbb{R}^+$, *for* $i = 1, \ldots, m$, *and* $k_j \colon \mathbb{R}^+ \to \mathbb{R}^+$, *for* $j = 0, \ldots, m+1$, *be continuous functions satisfying* (h_2) *and* (h_3).

- *If*

  $$g_0^i = 0, \quad \text{for all } i = 1, \ldots, m.$$

  Then there exists $\alpha^* > 0$ *such that if*

  $$\alpha_i > \alpha^*, \quad \text{for all } i = 1, \ldots, m,$$

  there exists $\beta^* = \beta^*(\alpha_1, \ldots, \alpha_m) > 0$ *so that, if*

  $$\beta_j > \beta^*, \quad \text{for all } j = 0, \ldots, m+1,$$

  then the Dirichlet boundary value problem (2.7.1) *has at least* $2^m - 1$ *positive radially symmetric (classical) solutions.*

- *If*

  $$g_\infty^i = +\infty, \quad \text{for all } i = 1, \ldots, m.$$

  Then there exists $\alpha_* > 0$ *such that if*

  $$0 < \alpha_i < \alpha_*, \quad \text{for all } i = 1, \ldots, m,$$

  there exists $\beta^* = \beta^*(\alpha_1, \ldots, \alpha_m) > 0$ *so that, if*

  $$\beta_j > \beta^*, \quad \text{for all } j = 0, \ldots, m+1,$$

  then the Dirichlet boundary value problem (2.7.1) *has at least* $2^m - 1$ *positive radially symmetric (classical) solutions.*

We conclude this discussion by observing that the multiplicity results given in Theorem 2.7.1 and in its corollary are also valid considering different boundary conditions of the form

$$u = 0 \ \text{ on } \big\{x \in \mathbb{R}^N \colon \|x\| = R_1\big\} \quad \text{and} \quad \frac{\partial u}{\partial r} = 0 \ \text{ on } \big\{x \in \mathbb{R}^N \colon \|x\| = R_2\big\},$$

or

$$\frac{\partial u}{\partial r} = 0 \ \text{ on } \big\{x \in \mathbb{R}^N \colon \|x\| = R_1\big\} \quad \text{and} \quad u = 0 \ \text{ on } \big\{x \in \mathbb{R}^N \colon \|x\| = R_2\big\},$$

where $r = \|x\|$ and $\partial u/\partial r$ denotes the differentiation in the radial direction (compare also to [113, 180], where an existence result for positive solutions is given for this type of conditions).

2.8 Conflicting nonlinearities: final remarks

We conclude this second part by presenting some consequences and discussions that naturally arise from our main result.

As the first point, in order to better explain our contribution to indefinite problems, we compare Theorem 2.6.1 to Theorem 1.4.3. In Chapter 1 we have presented an application of Theorem 2.5.1 (i.e. Theorem 1.3.1) to an indefinite equation of the form

$$u'' + a(t)g(u) = 0, \tag{2.8.1}$$

where $a(t) \geq 0$ on m pairwise disjoint intervals and $a(t) \leq 0$ on the complement in $[0,T]$. According to the notation of the present chapter, setting $a_i := a|_{I_i}$ and $b_j := a|_{J_j}$, one can easily see that Theorem 1.4.3 is an immediate consequence of Theorem 2.6.1. Furthermore, we observe that in the special case of (2.8.1) Theorem 2.6.1 generalizes Theorem 1.4.3. Indeed, in (2.5.2), the positive part and the negative part of the weight are associated with different nonlinearities, that is $g_i(s)$ and $k_j(s)$. This fact allows us to impose growth conditions only on the nonlinearities that have actually a role in the proof. More precisely, we assume superlinear growth conditions at zero and at infinity on the nonlinearities $g_i(s)$ (that multiply the positive part of the weight), while there are no growth conditions on the nonlinearities associated with the non-negative part. Indeed, besides the standard sign condition (h_2), we assume only that $k_0^j < +\infty$ (in (h_3)) in order to apply a standard maximum principle. In Figure 2.4 we show an example of equation which does not enter the setting of Theorem 1.4.3, while it satisfies all the hypotheses of Theorem 2.6.1.

One of the advantages in using an approach based on the topological degree is the fact that the degree is stable with respect to small perturbation of the operator and hence our multiplicity result is valid also when we

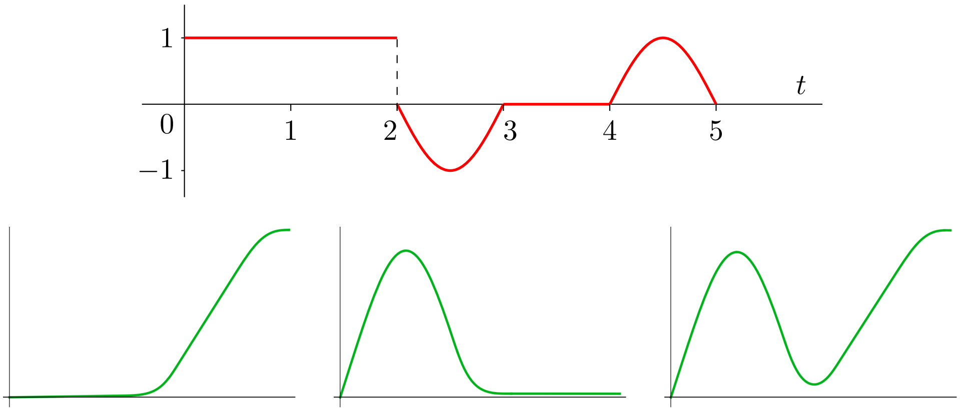

Figure 2.4: The figure shows an example of 3 positive solutions to the equation $u'' + \alpha_1 a_1(t) g_1(u) - \beta_1 b_1(t) k_1(u) + \alpha_2 a_2(t) g_2(u) = 0$ on $[0,5]$ with $u(0) = u'(5) = 0$, whose graphs are located in the lower part of the figure. For this simulation we have chosen $\alpha_1 = 10$, $\alpha_2 = 2$, $\beta_1 = 20$ and the weight functions as in the upper part of the figure, that is $a_1(t) = 1$ in $[0,2]$, $-b_1(t) = -\sin(\pi t)$ in $[2,3]$, $a_2(t) = 0$ in $[3,4]$, $a_2(t) = -\sin(\pi t)$ in $[4,5]$. Moreover, we have taken $g_1(s) = g_2(s) = s \arctan(s)$ and $k_1(s) = s/(1+s^2)$ (for $s > 0$). Notice that $k_1(s)$ has not a superlinear behavior, since $\lim_{s\to 0^+} k_1(s)/s = 1 > 0$ and $\lim_{s\to+\infty} k_1(s)/s = 0$. Then Theorem 1.4.3 does not apply, contrary to Theorem 2.6.1.

consider an equation of the form

$$u'' + \varepsilon p(t, u, u') + f(t, u) = 0,$$

for $|\varepsilon|$ sufficiently small.

From this remark we immediately obtain that we can deal with the equation

$$u'' + \lambda u + f(t, u) = 0$$

for $|\lambda|$ small enough and thus providing a contribution to [4] (compare to the discussion in Section 2.4). Moreover, we can consider the Sturm-Liouville problem associated with

$$u'' + cu' + f(t, u) = 0, \tag{2.8.2}$$

where $c \in \mathbb{R}$ is a constant, with $|c|$ small enough. The above equation has no Hamiltonian structure. An interesting question is whether Theorem 2.6.1 is still valid for an arbitrary $c \in \mathbb{R}$. With Dirichlet boundary conditions or mixed boundary conditions of the form $u'(0) = u(T) = 0$ or $u(0) = u'(T) = 0$, a standard change of variable allows to reduce equation (2.8.2) to an equation of the form as in (2.5.1); while for the general case of Sturm-Liouville boundary conditions one can adapt the approach developed in the forthcoming chapters.

In the forthcoming chapters we will also develop a technique that will enable to deal with Neumann and periodic boundary conditions and with maps that, roughly speaking, have a superlinear growth at zero and a *sublinear* growth at infinity. In this *super-sublinear* case, hypotheses (2.6.1) and (2.6.2) of Theorem 2.6.1 are replaced by

$$g_0^i = g_\infty^i = 0, \quad \text{for all} \;\; i = 1, \ldots, m.$$

We shall prove the existence of $3^m - 1$ positive solutions when α_i and β_j are sufficiently large.

Chapter 3

Neumann and periodic boundary conditions: existence results

In the present chapter we study the second order nonlinear boundary value problem

$$(\mathscr{P}) \qquad \begin{cases} u'' + a(t)g(u) = 0, \quad 0 < t < T, \\ \mathscr{B}(u,u') = \underline{0}. \end{cases}$$

As linear boundary operator we take

$$\mathscr{B}(u,u') = (u'(0), u'(T))$$

or

$$\mathscr{B}(u,u') = (u(T) - u(0), u'(T) - u'(0)),$$

so that we consider *the Neumann and the periodic boundary value problems.* The nonlinearity $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that

$$(g_*) \qquad g(0) = 0, \qquad g(s) > 0 \quad \text{for} \quad s > 0,$$

and the weight $a(t)$ is a Lebesgue integrable function defined on $[0,T]$.

A *solution* of $(\mathscr{P})$ is a continuously differentiable function $u\colon [0,T] \to \mathbb{R}$ such that its derivative $u'(t)$ is absolutely continuous and $u(t)$ satisfies $(\mathscr{P})$ for a.e. $t \in [0,T]$. We look for *positive solutions* of $(\mathscr{P})$, that is solutions u such that $u(t) > 0$ for every $t \in [0,T]$. In relation to the Neumann and the periodic boundary value problems, assumption (g_*), which requires that $g(s)$ never vanishes on $\mathbb{R}^+_0$, is essential to guarantee that the positive solutions we find are not constant.

If $u(t)$ is any positive solution to the boundary value problem $(\mathscr{P})$, then an integration on $[0,T]$ yields

$$\int_0^T a(t)g(u(t))\,dt = 0$$

and this fact, in connection with (g_*), implies that the weight function $a(t)$ (if not identically zero) must change its sign. A second relation can be derived when $g(s)$ is continuously differentiable on $\mathbb{R}_0^+$. Indeed, dividing the equation by $g(u(t))$ and integrating by parts, we obtain

$$-\int_0^T g'(u(t))\left(\frac{u'(t)}{g(u(t))}\right)^2 dt = \int_0^T a(t)\,dt$$

(cf. [15, 36]). From this relation, if $g'(s) > 0$ on $\mathbb{R}_0^+$, we find that a necessary condition for the existence of positive solutions is

$$\int_0^T a(t)\,dt < 0.$$

The above remarks suggest that, if we want to find nontrivial positive solutions for $(\mathscr{P})$ with nonlinearities which include as a particular possibility the case of $g(s)$ strictly monotone, we have to study problem $(\mathscr{P})$ considering sign-indefinite weight functions with negative mean value on $[0,T]$. This latter condition on the mean value is new with respect to the Dirichlet problems investigated in the previous chapters.

In this chapter we study the case of nonlinearities $g(s)$ which have a superlinear growth at zero and at infinity (i.e. superlinear indefinite boundary value problems) and we prove the existence of positive solutions to $(\mathscr{P})$. In more detail, necessary and sufficient conditions for the existence of nontrivial solutions are obtained.

With respect to problem $(\mathscr{P})$, the linear differential operator $u \mapsto -u''$ has a nontrivial kernel made up of the constant functions. In such a situation the operator is not invertible and we cannot proceed in the same manner as described in Chapter 1 (dealing with an equivalent fixed point problem in a suitable Banach space and applying directly some degree theoretical arguments). In this case, we have found it very useful to apply the *coincidence degree theory* developed by J. Mawhin (see Appendix B), which allows to study equations of the form $Lu = Nu$, where L is a linear operator with nontrivial kernel and N is a nonlinear one.

We remark that the existence results presented in this chapter give a solution to a problem raised by G. J. Butler in 1976 in the proof of [49, Corollary], where the author pointed out that the equation

$$u'' + w(t)|u|^{p-1}u = 0, \quad p > 1,$$

has infinitely many T-periodic solutions, assuming that $w(t)$ is a continuous T-periodic function with only isolated zeros and which is somewhere positive. It was also noted that all these solutions oscillate (have arbitrarily large zeros) if $\int_0^T w(t)\,dt \geq 0$. Since condition

$$\int_0^T w(t)\,dt < 0 \tag{3.0.1}$$

implies the existence of non-oscillatory solutions (cf. [48]), it was raised the question (see [49, p. 477]) whether *there can exist non-oscillatory periodic solutions* if (3.0.1) holds. Since in the present chapter we prove the existence of positive (i.e. non-oscillatory) T-periodic solutions under the average condition (3.0.1), we give a complete answer to Butler's question.

The plan of the chapter is the following. In Section 3.1 we apply coincidence degree theory to provide an existence theorem (see Theorem 3.1.1) for positive solutions to a general problem of the form

$$\begin{cases} u'' + f(t,u,u') = 0, \quad 0 < t < T, \\ \mathscr{B}(u,u') = \underline{0}. \end{cases}$$

The results of Section 3.1 are then employed in Section 3.2 in order to obtain two main existence theorems for problem $(\mathscr{P})$ under different conditions on the behavior of $g(s)$ near zero (see Theorem 3.2.1 and Theorem 3.2.2). Various corollaries and applications are also derived. In Section 3.3 we present two different applications where we treat separately the Neumann and the periodic problem. More precisely, in Section 3.3.1 we prove an existence result of positive radially symmetric solutions for a superlinear PDE subject to Neumann boundary conditions in annular domains, while in Section 3.3.2 we provide positive periodic solutions to a Liénard type equation. We stress that in this latter case we can give an application of our method to a non-variational setting, indeed the associated equation has not an Hamiltonian structure. Throughout the chapter we focus our study only on the existence of nontrivial solutions, while in Chapter 4 we combine the methods developed in Chapter 1 with those of the present chapter in order to achieve multiplicity results of positive solutions to the boundary value problem $(\mathscr{P})$.

3.1 An abstract existence result via coincidence degree

In this section we provide an existence result for the second order boundary value problem

$$\begin{cases} u'' + f(t,u,u') = 0, \quad 0 < t < T, \\ \mathscr{B}(u,u') = \underline{0}, \end{cases} \tag{3.1.1}$$

which includes ($\mathscr{P}$) as well as the case of more general nonlinear terms. We recall that by $\mathscr{B}(u,u')=\underline{0}$ we mean the Neumann or the periodic boundary conditions on a fixed interval $[0,T]$.

Let $X := \mathcal{C}^1([0,T])$ be the Banach space of continuously differentiable real valued functions $u(t)$ defined on $[0,T]$ endowed with the norm

$$\|u\| := \|u\|_\infty + \|u'\|_\infty$$

and let $Z := L^1([0,T])$ be the space of Lebesgue integrable functions defined on $[0,T]$ with the L^1-norm (denoted by $\|\cdot\|_{L^1}$).

We define $L\colon \operatorname{dom} L \to Z$ as

$$(Lu)(t) := -u''(t), \quad t \in [0,T],$$

and take as $\operatorname{dom} L \subseteq X$ the vector subspace

$$\operatorname{dom} L := \Big\{ u \in X \colon u' \in \mathrm{AC} \text{ and } \mathscr{B}(u,u') = \underline{0} \Big\},$$

where $u' \in \mathrm{AC}$ means that u' is absolutely continuous. In this situation, $\ker L \equiv \mathbb{R}$ is made by the constant functions and

$$\operatorname{Im} L = \left\{ w \in Z \colon \int_0^T w(t)\,dt = 0 \right\}.$$

A natural choice of the projections is given by

$$P, Q \colon u \mapsto \frac{1}{T}\int_0^T u(t)\,dt,$$

so that $\operatorname{coker} L \equiv \mathbb{R}$ and $\ker P$ is given by the continuously differentiable functions with mean value zero. With such a choice of the projection, the right inverse linear operator K_P is the map which to any $w \in L^1([0,T])$ with $\int_0^T w(t)\,dt = 0$ associates the unique solution $u(t)$ of

$$u'' + w(t) = 0, \quad \mathscr{B}(u,u') = \underline{0}, \quad \int_0^T u(t)\,dt = 0.$$

Finally, we take as a linear isomorphism $J\colon \operatorname{coker} L \to \ker L$ the identity in $\mathbb{R}$.

We are ready now to introduce the nonlinear operator $N\colon X \to Z$. First we give some assumptions on $f(t,s,\xi)$ which will be considered throughout the section.

Let $f\colon [0,T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be an L^p-Carathéodory function, for some $1 \le p \le \infty$ (cf. [104]), satisfying the following conditions

(f_1) *$f(t,0,\xi) = 0$, for a.e. $t \in [0,T]$ and for all $\xi \in \mathbb{R}$;*

(f_2) *there exists a non-negative function* $k \in L^1([0,T])$ *and a constant* $\rho > 0$ *such that*

$$|f(t,s,\xi)| \le k(t)(|s| + |\xi|),$$

for a.e. $t \in [0,T]$, *for all* $0 \le s \le \rho$ *and* $|\xi| \le \rho$.

Besides the above hypotheses, we suppose also that $f(t,s,\xi)$ satisfies a Bernstein-Nagumo type condition in order to have a priori bounds on $|u'(t)|$ whenever bounds on $u(t)$ are obtained. Typically, Bernstein-Nagumo assumptions are expressed in terms of growth restrictions on $f(t,s,\xi)$ with respect to the ξ-variable. However, depending on the given boundary value problems and on the nonlinearity, more general conditions can be considered, too. The interested reader can find in [129] a very general discussion for the periodic problem (cf. [182] for a broad list of references). See also [117] and [131] for interesting remarks and applications to different boundary value problems. For the purposes of the present chapter, we do not consider the more general situation and we confine ourselves to the classical estimate for the L^p-Carathéodory setting given in [62, § 4.4]. Accordingly, we assume that

(f_3) *for each* $\eta > 0$ *there exists a continuous function*

$$\phi = \phi_\eta \colon \mathbb{R}^+ \to \mathbb{R}^+, \quad \textit{with} \ \int^\infty \frac{\xi^{\frac{p-1}{p}}}{\phi(\xi)}\, d\xi = \infty,$$

and a function $\psi = \psi_\eta \in L^p([0,T],\mathbb{R}^+)$ *such that*

$$|f(t,s,\xi)| \le \psi(t)\phi(|\xi|), \quad \textit{for a.e. } t \in [0,T],\ \forall\, s \in [0,\eta],\ \forall\, \xi \in \mathbb{R}.$$

For technical reasons, when dealing with Nagumo functions $\phi(\xi)$ as above, we always assume further that

$$\liminf_{\xi \to +\infty} \phi(\xi) > 0.$$

This prevents the possibility of pathological examples like that in [62, pp. 46–47] and does not affect our applications.

As a first step we extend f to a Carathéodory function $\tilde{f}$ defined on $[0,T] \times \mathbb{R}^2$, by setting

$$\tilde{f}(t,s,\xi) := \begin{cases} f(t,s,\xi), & \text{if } s \ge 0; \\ -s, & \text{if } s \le 0; \end{cases}$$

and denote by $N \colon X \to Z$ the Nemytskii operator induced by $\tilde{f}$, that is

$$(Nu)(t) := \tilde{f}(t,u(t),u'(t)), \quad t \in [0,T].$$

In this setting, u is a solution of the coincidence equation

$$Lu = Nu, \quad u \in \operatorname{dom} L, \tag{3.1.2}$$

if and only if it is a solution to the boundary value problem

$$\begin{cases} u'' + \tilde{f}(t, u, u') = 0, & 0 < t < T, \\ \mathscr{B}(u, u') = \underline{0}. \end{cases} \tag{3.1.3}$$

Moreover, from the definition of $\tilde{f}$ for $s \leq 0$ and conditions (f_1) and (f_2), one can easily check by a maximum principle argument (see Lemma C.1.2 and Remark C.1.1) that if $u \not\equiv 0$, then $u(t)$ is strictly positive and hence a (positive) solution of problem (3.1.1).

Now, as an application of Lemma B.2.1 and Lemma B.2.3, we have the following result.

Theorem 3.1.1. *Let $f\colon [0,T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be an L^p-Carathéodory function (for some $1 \leq p \leq \infty$) satisfying (f_1), (f_2), (f_3). Suppose that there exist two constants $r, R > 0$, with $r \neq R$, such that the following hypotheses hold.*

(H_r) *The average condition*

$$\int_0^T f(t, r, 0)\, dt < 0$$

is satisfied. Moreover, any solution $u(t)$ of the boundary value problem

$$\begin{cases} u'' + \vartheta f(t, u, u') = 0 \\ \mathscr{B}(u, u') = \underline{0}, \end{cases} \tag{3.1.4}$$

for $0 < \vartheta \leq 1$, such that $u(t) > 0$ on $[0,T]$, satisfies $\|u\|_\infty \neq r$.

(H^R) *There exist a non-negative function $v \in L^p([0,T])$ with $v \not\equiv 0$ and a constant $\alpha_0 > 0$, such that every solution $u(t) \geq 0$ of the boundary value problem*

$$\begin{cases} u'' + f(t, u, u') + \alpha v(t) = 0 \\ \mathscr{B}(u, u') = \underline{0}, \end{cases} \tag{3.1.5}$$

for $\alpha \in [0, \alpha_0]$, satisfies $\|u\|_\infty \neq R$. Moreover, there are no solutions $u(t)$ of (3.1.5) for $\alpha = \alpha_0$ with $0 \leq u(t) \leq R$, for all $t \in [0,T]$.

Then problem (3.1.1) has at least a positive solution $u(t)$ with

$$\min\{r, R\} < \max_{t \in [0,T]} u(t) < \max\{r, R\}.$$

Proof. As we have already observed, from the choice of the spaces X, $\mathrm{dom}\, L$, Z and the operators $L\colon u \mapsto -u''$ and N (the Nemytskii operator induced by $\tilde{f}$), we have that (3.1.2) is equivalent to the boundary value problem (3.1.3). All the structural assumptions required by Mawhin's theory (that is L is Fredholm of index zero and N is L-completely continuous) are satisfied by standard facts (see [130]).

For the proof, we confine ourselves to the case

$$0 < r < R,$$

which is the interesting one for our applications. The complementary case in which $0 < R < r$ can be studied with minor changes in the proof and it will be briefly described at the end.

The coincidence equation

$$Lu = \vartheta N u, \quad u \in \mathrm{dom}\, L, \tag{3.1.6}$$

is equivalent to

$$\begin{cases} u'' + \vartheta \tilde{f}(t,u,u') = 0 \\ \mathscr{B}(u,u') = \underline{0}. \end{cases} \tag{3.1.7}$$

Let u be any solution of (3.1.6) for some $\vartheta > 0$. From the definition of $\tilde{f}$ for $s \leq 0$ and the maximum principle, we have that $u(t) \geq 0$ for every $t \in [0,T]$ and hence u is a solution of (3.1.4). Moreover, by (f_2), if $u \not\equiv 0$, then $u(t) > 0$ for all $t \in [0,T]$.

According to condition (f_3), let $\phi = \phi_r \colon \mathbb{R}^+ \to \mathbb{R}^+$ and $\psi = \psi_r \in L^p([0,T])$ be such that $|f(t,s,\xi)| \leq \psi(t)\phi(|\xi|)$, for a.e. $t \in [0,T]$, for all $s \in [0,r]$ and $\xi \in \mathbb{R}$. By Nagumo lemma (cf. [62, § 4.4, Proposition 4.7]), there exists a constant $M = M_r > 0$ (depending on r, as well as on ϕ and ψ, but not depending on $u(t)$ and $\vartheta \in \,]0,1]$) such that any solution of (3.1.7) or, equivalently, any (non-negative) solution of (3.1.4) (for some $\vartheta \in \,]0,1]$) satisfying $\|u\|_\infty \leq r$ is such that $\|u'\|_\infty < M_r$. Hence, condition (H_r) implies that, for the open and bounded set Ω_r in X defined as

$$\Omega_r := \bigl\{ u \in X \colon \|u\|_\infty < r,\ \|u'\|_\infty < M_r \bigr\},$$

it holds that

$$Lu \neq \vartheta N u, \quad \forall\, u \in \mathrm{dom}\, L \cap \partial\Omega_r,\ \forall\, \vartheta \in \,]0,1].$$

Consider now $u \in \partial\Omega_r \cap \ker L$. In this case, $u \equiv k \in \mathbb{R}$, with $|k| = r$, and

$$-JQNu = -\frac{1}{T}\int_0^T \tilde{f}(t,k,0)\, dt.$$

Notice also that $\Omega_r \cap \ker L = \,]-r,r[$.

By the definition of $\tilde{f}$ for $s \leq 0$, we have that

$$f^{\#}(s) := -\frac{1}{T}\int_0^T \tilde{f}(t,s,0)\,dt = \begin{cases} -\dfrac{1}{T}\displaystyle\int_0^T f(t,s,0)\,dt, & \text{if } s > 0; \\ s, & \text{if } s \leq 0. \end{cases}$$

Therefore, $QNu \neq 0$ for each $u \in \partial\Omega_r \cap \ker L$ and, moreover,

$$\deg_B(f^{\#},]-r, r[, 0) = 1,$$

since $f^{\#}(-r) < 0 < f^{\#}(r)$. By Lemma B.2.1 we conclude that

$$D_L(L - N, \Omega_r) = 1. \tag{3.1.8}$$

Now we study the operator equation

$$Lu = Nu + \alpha v, \quad u \in \operatorname{dom} L, \tag{3.1.9}$$

for some $\alpha \geq 0$, with v as in (H^R). This equation is equivalent to

$$\begin{cases} u'' + \tilde{f}(t,u,u') + \alpha v(t) = 0 \\ \mathscr{B}(u,u') = \underline{0}. \end{cases} \tag{3.1.10}$$

Let u be any solution of (3.1.9) for some $\alpha \geq 0$. From the definition of $\tilde{f}$ for $s \leq 0$ and the maximum principle, we have that $u(t) \geq 0$ for every $t \in [0,T]$ and hence u is a solution of (3.1.5).

According to condition (f_3), let $\phi = \phi_R \colon \mathbb{R}^+ \to \mathbb{R}^+$ and $\psi = \psi_R \in L^p([0,T])$ be such that $|f(t,s,\xi)| \leq \psi(t)\phi(|\xi|)$, for a.e. $t \in [0,T]$, for all $s \in [0,R]$ and $\xi \in \mathbb{R}$. If we take $\alpha \in [0,\alpha_0]$, we obtain that

$$|f(t,s,\xi) + \alpha v(t)| \leq \psi(t)\phi(|\xi|) + \alpha_0 v(t) \leq \tilde{\psi}(t)\tilde{\phi}(|\xi|)$$

holds for a.e. $t \in [0,T]$ and for all $s \in [0,R]$ and $\xi \in \mathbb{R}$, with

$$\tilde{\psi}(t) := \psi(t) + \alpha_0 v(t) \quad \text{and} \quad \tilde{\phi}(\xi) := \phi(\xi) + 1.$$

Observe also that $\tilde{\psi} \in L^p([0,T])$ and $\int^{\infty} \xi^{(p-1)/p}/\tilde{\phi}(\xi)\,d\xi = \infty$.

By Nagumo lemma, there exists a positive constant $M = M_R > M_r$ (depending on R, as well as on ϕ and $\tilde{\psi}$, but not depending on $u(t)$ and $\alpha \in [0,\alpha_0]$) such that any solution of (3.1.10) or, equivalently, any (non-negative) solution of (3.1.5) (for some $\alpha \in [0,\alpha_0]$) satisfying $\|u\|_\infty \leq R$ is such that $\|u'\|_\infty < M_R$. Hence, condition (H^R) implies that, for the open and bounded set Ω_R in X defined as

$$\Omega_R := \bigl\{u \in X \colon \|u\|_\infty < R,\ \|u'\|_\infty < M_R\bigr\},$$

it holds that

$$Lu \neq Nu + \alpha v, \quad \forall\, u \in \operatorname{dom} L \cap \partial\Omega_R,\ \forall\, \alpha \in [0,\alpha_0].$$

Moreover, the last hypothesis in (H^R) also implies that

$$Lu \neq Nu + \alpha_0 v, \quad \forall\, u \in \operatorname{dom} L \cap \Omega_R.$$

According to Lemma B.2.3 we have that

$$D_L(L - N, \Omega_R) = 0. \tag{3.1.11}$$

In conclusion, from (3.1.8), (3.1.11) and the additivity property of the coincidence degree, we find that

$$D_L(L - N, \Omega_R \setminus \operatorname{cl}(\Omega_r)) = -1.$$

This ensures the existence of a (nontrivial) solution $\tilde{u}$ to (3.1.2) with $\tilde{u} \in \Omega_R \setminus \operatorname{cl}(\Omega_r)$. Since $\tilde{u}$ is a nontrivial solution of (3.1.3), by the (strong) maximum principle (following from the definition of $\tilde{f}$ for $s \leq 0$, (f_1) and (f_2)), we have that $\tilde{u}$ is a solution of (3.1.1) with $\tilde{u}(t) > 0$ for all $t \in [0, T]$.

If we are in the case

$$0 < R < r,$$

we proceed in an analogous manner. With respect to the previous situation, the only relevant changes are the following. First we fix a constant $M = M_R > 0$ and, for the set Ω_R, we obtain (3.1.11). As a next step, we repeat the first part of the above proof, we fix a constant $M_r > M_R$ and, for the set Ω_r, we obtain (3.1.8). Now we have

$$D_L(L - N, \Omega_r \setminus \operatorname{cl}(\Omega_R)) = 1.$$

This ensures the existence of a (nontrivial) solution $\tilde{u}$ to (3.1.2) with $\tilde{u} \in \Omega_r \setminus \operatorname{cl}(\Omega_R)$ and then we conclude as above, showing that $\tilde{u}(t) > 0$ for all $t \in [0, T]$ (by the strong maximum principle). $\square$

Remark 3.1.1. If we consider as boundary conditions the periodic ones, namely with $\mathscr{B}(u, u') = \underline{0}$ written as

$$u(0) = u(T), \qquad u'(0) = u'(T),$$

then Theorem 3.1.1 holds true also if, in place of the differential operator $u \mapsto -u''$, we take a linear differential operator of the form $u \mapsto -u'' - cu'$, with $c \in \mathbb{R}$ a fixed constant. $\lhd$

Remark 3.1.2. The condition (f_2) is required only to assure that a non-negative solution is strictly positive. If we do not assume (f_2), with the same proof, we can provide a variant of Theorem 3.1.1 in which we obtain the existence of nontrivial non-negative solutions. In this case, condition (H_r) should be modified requiring that any $u(t) \geq 0$ satisfies $\|u\|_\infty \neq r$. $\lhd$

3.2 Existence results for problem $(\mathscr{P})$

In this section we give an application of Theorem 3.1.1 to the existence of positive solutions for problem $(\mathscr{P})$. Throughout the section, we suppose that $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that

$$(g_*) \qquad g(0) = 0, \qquad g(s) > 0 \quad \text{for} \ \ s > 0.$$

Moreover, we suppose that $g(s)$ (satisfying (g_*)) is *regularly oscillating at zero*, that is

$$\lim_{\substack{s\to 0^+\\ \omega\to 1}} \frac{g(\omega s)}{g(s)} = 1.$$

This definition is the natural transposition for $s \to 0^+$ of the usual definition of *regularly oscillating (at infinity)* considered by several authors (see [22]). Regular oscillating functions are a class of maps related to the study of Karamata regular variation theory and its many ramifications (cf. [25, 165]). They naturally appear in many different areas of real analysis like probability theory and qualitative theory of ODEs (see [71, § 1] for a brief historical survey about this subject).

The weight coefficient $a\colon [0,T] \to \mathbb{R}$ is an L^1-function such that

(a_*) *there exist $m \geq 1$ intervals $I_1^+, \ldots, I_m^+$, closed and pairwise disjoint, such that*

$$a(t) \geq 0, \quad \text{a.e. } t \in I_i^+, \ \text{with } \ a(t) \not\equiv 0 \ \text{on } I_i^+ \quad (i = 1, \ldots, m);$$
$$a(t) \leq 0, \quad \text{a.e. } t \in [0,T] \setminus \bigcup_{i=1}^m I_i^+;$$

$$(a_\#) \qquad \bar{a} := \frac{1}{T}\int_0^T a(t)\,dt < 0.$$

Let λ_1^i, $i = 1, \ldots, m$, be the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a(t)\varphi = 0, \quad \varphi|_{\partial I_i^+} = 0. \tag{3.2.1}$$

From the assumptions on $a(t)$ in I_i^+ it clearly follows that $\lambda_1^i > 0$ for each $i = 1, \ldots, m$. In the sequel, if necessary, it will be not restrictive to label the intervals I_i^+ following the natural order given by the standard orientation of the real line.

Theorem 3.2.1. *Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function satisfying (a_*) and $(a_\#)$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*). Suppose also that $g(s)$ is regularly oscillating at zero and satisfies*

$$(g_0) \qquad \lim_{s\to 0^+} \frac{g(s)}{s} = 0$$

and

$$(g_\infty) \qquad g_\infty := \liminf_{s\to+\infty} \frac{g(s)}{s} > \max_{i=1,\ldots,m} \lambda_1^i.$$

Then problem ($\mathscr{P}$) *has at least a positive solution.*

Proof. In order to enter the setting of Theorem 3.1.1 we define

$$f(t,s,\xi) = f(t,s) := a(t)g(s)$$

and observe that f is an L^1-Carathéodory function. The basic hypotheses required on $f(t,s,\xi)$ are all satisfied. In fact, (f_1) follows from $g(0) = 0$ and (f_2) is an obvious consequence of the fact that $g(s)/s$ is bounded on a right neighborhood of $s = 0$ and $a \in L^1([0,T])$. By the continuity of $g(s)$ and the integrability of $a(t)$, the Nagumo condition (f_3) is trivially satisfied since f does not depend on ξ. Indeed, we can take $p = 1$, $\phi(\xi) \equiv 1$ and $\psi(t) = |a(t)| \max_{0\leq s\leq \eta} g(s)$.

Verification of (H_r). First of all, we observe that (g_*) and $(a_\#)$ imply that

$$\int_0^T f(t,s,0)\,dt < 0, \quad \forall\, s > 0. \tag{3.2.2}$$

We claim that there exists $r_0 > 0$ such that for all $0 < r \leq r_0$ and for all $\vartheta \in \,]0,1]$ there are no solutions $u(t)$ of (3.1.4) such that $u(t) > 0$ on $[0,T]$ and $\|u\|_\infty = r$.

By contradiction, suppose the claim is not true. Then for all $n \in \mathbb{N}$ there exist $0 < r_n < 1/n$, $\vartheta_n \in \,]0,1]$ and $u_n(t)$ solution of

$$u'' + \vartheta_n a(t) g(u) = 0, \quad \mathscr{B}(u,u') = \underline{0}, \tag{3.2.3}$$

such that $u_n(t) > 0$ on $[0,T]$ and $\|u_n\|_\infty = r_n$.

Integrating on $[0,T]$ the differential equation in (3.2.3) and using the boundary conditions, we obtain

$$0 = -\int_0^T u_n''(t)\,dt = \vartheta_n \int_0^T a(t)g(u_n(t))\,dt.$$

Then

$$\int_0^T a(t)g(u_n(t))\,dt = 0 \tag{3.2.4}$$

follows. We define

$$v_n(t) := \frac{u_n(t)}{\|u_n\|_\infty}, \quad t \in [0,T],$$

and, dividing (3.2.3) by $r_n = \|u_n\|_\infty$, we get

$$v_n''(t) + \vartheta_n a(t) \frac{g(u_n(t))}{u_n(t)} v_n(t) = 0. \tag{3.2.5}$$

By (g_0), for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$0 < \frac{g(s)}{s} < \varepsilon, \quad \forall\, 0 < s < \delta_\varepsilon.$$

For $n > 1/\delta_\varepsilon$ we have $0 < u_n(t) \leq r_n < \delta_\varepsilon$ for all $t \in [0,T]$, so that

$$0 < \frac{g(u_n(t))}{u_n(t)} < \varepsilon, \quad \forall\, t \in [0,T].$$

This proves that

$$\lim_{n\to\infty} \frac{g(u_n(t))}{u_n(t)} = 0, \quad \text{uniformly on } [0,T]. \tag{3.2.6}$$

We fix $t_0 \in [0,T]$ such that $v_n'(t_0) = 0$. With the Neumann boundary condition, we choose $t_0 = 0$ (or $t_0 = T$), while, in the periodic case, the existence of such a t_0 (possibly depending on n) is ensured by Rolle's theorem. Integrating (3.2.5), we have

$$v_n'(t) = -\vartheta_n \int_{t_0}^{t} a(\xi) \frac{g(u_n(\xi))}{u_n(\xi)} v_n(\xi)\, d\xi, \quad \forall\, t \in [0,T],$$

hence

$$\|v_n'\|_\infty \leq \int_0^T |a(t)| \frac{g(u_n(t))}{u_n(t)}\, dt. \tag{3.2.7}$$

Since $a \in L^1([0,T])$, by (3.2.6) and the dominated convergence theorem, we find that $v_n'(t) \to 0$ (as $n \to \infty$) uniformly on $[0,T]$.

Since $\|v_n\|_\infty = 1$, there exists $x_1 \in [0,T]$ (possibly depending on n) such that $v_n(x_1) = 1$. From

$$v_n(t) = v_n(x_1) - \int_{x_1}^{t} v_n'(\xi)\, d\xi, \quad \forall\, t \in [0,T],$$

we conclude that

$$\lim_{n\to\infty} v_n(t) = 1, \quad \text{uniformly on } [0,T]. \tag{3.2.8}$$

Now, we write (3.2.4) as

$$0 = \int_0^T a(t) g(u_n(t))\, dt = \int_0^T \Big(a(t) g(r_n) + a(t)[g(r_n v_n(t)) - g(r_n)] \Big)\, dt.$$

Since $g(r_n) > 0$, then

$$-\frac{1}{T} \int_0^T a(t)\, dt = \frac{1}{T} \int_0^T a(t) \frac{g(r_n v_n(t)) - g(r_n)}{g(r_n)}\, dt.$$

Consequently, by $(a_{\#})$,

$$0 < -\bar{a} \leq \frac{1}{T}\|a\|_{L^1} \max_{t\in[0,T]}\left|\frac{g(r_n v_n(t))}{g(r_n)} - 1\right| = \frac{1}{T}\|a\|_{L^1}\left|\frac{g(r_n\omega_n)}{g(r_n)} - 1\right|,$$

where $\omega_n := v_n(t_n)$, for a suitable choice of $t_n \in [0,T]$, and also $\omega_n \to 1$ (as $n \to \infty$) by (3.2.8). Using the fact that $g(s)$ is regularly oscillating at zero, we obtain a contradiction as $n \to \infty$.

The claim is thus proved and, recalling also (3.2.2), we have that (H_r) holds for any $r \in\,]0, r_0]$.

Verification of (H^R). First of all, we fix a nontrivial function $v \in L^1([0,T])$, with $v(t) \not\equiv 0$ on $[0,T]$, such that

$$v(t) \geq 0, \quad \text{a.e. } t \in \bigcup_{i=1}^{m} I_i^+;$$
$$v(t) = 0, \quad \text{a.e. } t \in [0,T] \setminus \bigcup_{i=1}^{m} I_i^+.$$

For example, as $v(t)$ we can take the indicator function of the set

$$A := \bigcup_{i=1}^{m} I_i^+.$$

Secondly, we observe that $a(t) \geq 0$ on each interval I_i^+, so that

$$\liminf_{s\to+\infty} \frac{f(t,s)}{s} \geq a(t)g_\infty, \quad \text{uniformly a.e. } t \in I_i^+.$$

Moreover, condition (g_∞) implies that the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda g_\infty a(t)\varphi = 0, \quad \varphi|_{\partial I_i^+} = 0,$$

is strictly less than 1, for all $i = 1, \ldots, m$. Thus, we can apply Lemma 1.1.3 (see also Remark 1.1.1) on each interval I_i^+ (with $f(t,s) := a(t)g(s)$ and $q_\infty(t) := a(t)g_\infty$). Therefore, for each $i = 1, \ldots, m$, we obtain the existence of a constant $R_{I_i^+} > 0$ such that for each L^1-Carathéodory function $h\colon [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ with

$$h(t,s) \geq a(t)g(s), \quad \text{a.e. } t \in I_i^+,\ \forall\, s \geq 0,$$

every solution $u(t) \geq 0$ of the boundary value problem

$$\begin{cases} u'' + h(t,u) = 0 \\ \mathscr{B}(u,u') = \underline{0} \end{cases} \tag{3.2.9}$$

satisfies $\max_{t\in I_i^+} u(t) < R_{I_i^+}$.

Then, we fix a constant $R > r_0$ (with r_0 coming from the first part of the proof) such that

$$R \geq \max_{i=1,\ldots,m} R_{I_i^+} \tag{3.2.10}$$

and another constant $\alpha_0 > 0$ such that

$$\alpha_0 > \frac{\|a\|_{L^1} \max_{0\leq s\leq R} g(s)}{\|v\|_{L^1}}. \tag{3.2.11}$$

We take $\alpha \in [0, \alpha_0]$. We observe that any solution $u(t) \geq 0$ of problem (3.1.5) is a solution of (3.2.9) with

$$h(t, s) = a(t)g(s) + \alpha v(t).$$

By definition, we have that $h(t, s) \geq a(t)g(s)$ for a.e. $t \in A$ and for all $s \geq 0$, and also $h(t, s) = a(t)g(s) \leq 0$ for a.e. $t \in [0, T] \setminus A$ and for all $s \geq 0$. By the convexity of the solutions of (3.2.9) on the intervals of $[0, T] \setminus A$, we obtain

$$\max_{t\in[0,T]} u(t) = \max_{t\in A} u(t)$$

and, as an application of Lemma 1.1.3 (see also Remark 1.1.1) on each of the intervals I_i^+, we conclude that

$$\|u\|_\infty < R.$$

This proves the first part of (H^R).

It remains to verify that for $\alpha = \alpha_0$ defined in (3.2.11) there are no solutions $u(t)$ of (3.1.5) with $0 \leq u(t) \leq R$ on $[0, T]$. Indeed, if u is a solution of (3.1.5), or equivalently of

$$\begin{cases} u'' + a(t)g(u) + \alpha v(t) = 0 \\ \mathscr{B}(u, u') = \underline{0}, \end{cases}$$

with $0 \leq u(t) \leq R$, then, integrating on $[0, T]$ the differential equation and using the boundary conditions, we obtain

$$\alpha\|v\|_{L^1} = \alpha \int_0^T v(t)\, dt \leq \int_0^T |a(t)|g(u(t))\, dt \leq \|a\|_{L^1} \max_{0\leq s\leq R} g(s),$$

which leads to a contradiction with respect to the choice of α_0. Thus (H^R) is verified.

Having verified (H_r) and (H^R), the thesis follows from Theorem 3.1.1. $\square$

Remark 3.2.1. From the verification of condition (H^R) performed in the above proof, it is clear that the assumption $g_\infty > \max_i \lambda_1^i$ is employed in connection with Lemma 1.1.3 (see also Remark 1.1.1) in order to obtain the a priori bounds $R_{I_i^+}$ on the intervals I_i^+. In turn, this step in the proof is based on a Sturm comparison argument involving the eigenfunctions of (3.2.1). If among the intervals I_i^+ there is one of the form $I_1^+ = [0,\sigma]$ or one of the form $I_m^+ = [\tau, T]$ (both cases are also possible), then the choice of the eigenvalues can be made in a more refined manner in order to improve the lower bound for g_∞. More precisely, whenever such a situation occurs, we can proceed as follows.

For the Neumann problem, if $I_1^+ = [0,\sigma]$ we take as λ_1^1 the first positive eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a(t)\varphi = 0, \quad \varphi'(0) = \varphi(\sigma) = 0.$$

Similarly, if $I_m^+ = [\tau, T]$ we take as λ_1^m the first positive eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a(t)\varphi = 0, \quad \varphi(\tau) = \varphi'(T) = 0.$$

For the periodic problem, we can extend the coefficient by T-periodicity on the whole real line and, after a shift on the t-variable, we consider an equivalent problem where the weight function is negative in a neighborhood of the endpoints. We try to clarify this concept with an example. Suppose that we are interested in the search of 2π-periodic solutions of equation

$$u'' + (-k + \cos(t))g(u) = 0,$$

where $0 < k < 1$ is a fixed constant. In this case, setting the problem on the interval $[0, 2\pi]$, we should consider the eigenvalue problem (3.2.1) on the two intervals $I_1^+ = [0, \arccos k]$ and $I_2^+ = [2\pi - \arccos k, 2\pi]$, where the weight function is positive. On the other hand, since we are looking for 2π-periodic solutions, we could work on any interval of length 2π, for instance $[\pi, 3\pi]$. This is equivalent to consider the periodic boundary conditions on $[0, 2\pi]$ for the equation

$$v'' + (-k + \cos(t - \pi))g(v) = 0.$$

In this latter case, the weight is negative at the endpoints $t = 0$ and $t = 2\pi$ and there is only one interval of non-negativity, so that we have to consider the eigenvalue problem (3.2.1) only on $I_1^+ = [\pi - \arccos k, \pi + \arccos k]$. In this way we can produce a better lower bound for g_∞ by studying an equivalent problem.

The same remarks as those just made above apply in any subsequent variant of Theorem 3.2.1, for instance, we can refine the choice of the constants λ_1^i in Corollary 3.2.4 below. $\triangleleft$

The following corollaries are direct consequences of Theorem 3.2.1.

Corollary 3.2.1. *Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function satisfying (a_*) and $(a_\#)$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*). Suppose also that $g(s)$ is regularly oscillating at zero and satisfies*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \textit{and} \quad \lim_{s\to +\infty} \frac{g(s)}{s} = +\infty.$$

Then problem $(\mathscr{P})$ has at least a positive solution.

Corollary 3.2.2. *Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function satisfying (a_*) and $(a_\#)$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*). Suppose also that $g(s)$ is regularly oscillating at zero and satisfies*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \textit{and} \quad g_\infty > 0.$$

Then there exists $\nu^* > 0$ such that the boundary value problem

$$(\mathscr{P}_\nu) \qquad \begin{cases} u'' + \nu\, a(t) g(u) = 0, & 0 < t < T, \\ \mathscr{B}(u,u') = \underline{0} \end{cases}$$

has at least a positive solution for each $\nu > \nu^*$.

The following consequence of Theorem 3.2.1 provides a necessary and sufficient condition for the existence of positive solutions to problem $(\mathscr{P})$ when $g(s) = s^\gamma$ for $\gamma > 1$. It can be viewed as a version of [20, Theorem 1] for the periodic case (in [20] the authors already obtained the same result for the Neumann problem for PDEs).

Corollary 3.2.3. *The superlinear boundary value problem*

$$\begin{cases} u'' + a(t) u^\gamma = 0, & \gamma > 1, \\ \mathscr{B}(u,u') = \underline{0}, \end{cases}$$

with a integrable weight function $a\colon [0,T] \to \mathbb{R}$ satisfying (a_*), has a positive solution if and only if the average condition $(a_\#)$ holds.

Proof. The necessary part of the statement is a consequence of the fact that $g(s) := s^\gamma$, for $\gamma > 1$, has a positive derivative on $\mathbb{R}^+_0$. For the sufficient part we apply Corollary 3.2.1, observing that $g(s)$ is regularly oscillating at zero. □

As one can clearly notice from the proof, the condition $g_\infty > \max_i \lambda_1^i$ in Theorem 3.2.1 is required in order to obtain suitable a priori bounds $R_{I_i^+}$ for the maximum of the solutions on each of the intervals I_i^+, where $a \geq 0$ and $a \not\equiv 0$. Hence we can choose a constant R satisfying (3.2.10) and have (H^R)

verified. As observed in Theorem 1.4.2, if $g(s)/s$ is bounded (for s large), it is sufficient to obtain an a priori bound *only on one* of the intervals I_i^+ and the existence of a global upper bound follows from standard ODEs arguments related to the classical Gronwall's inequality. Similarly, also in the present situation, following the proof of Theorem 1.2.2, we can obtain the next result.

Corollary 3.2.4. *Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function satisfying (a_*) and $(a_\#)$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*). Suppose also that $g(s)$ is regularly oscillating at zero and satisfies*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0, \quad g_\infty > \min_{i=1,\dots,m} \lambda_1^i \quad \text{and} \quad \limsup_{s\to+\infty} \frac{g(s)}{s} < +\infty.$$

Then problem $(\mathscr{P})$ has at least a positive solution.

Actually, this result could be even improved with respect to assumption (a_*), in the sense that it would be sufficient only to find an interval $J \subseteq [0,T]$ where $a \geq 0$ and $a \not\equiv 0$ and then we can ignore completely the behavior of $a(t)$ on $[0,T]\setminus J$. If we know that g_∞ is greater than the first eigenvalue of the Dirichlet problem in J (i.e. $\varphi'' + \lambda a(t)\varphi = 0$, $\varphi|_{\partial J} = 0$), we get the upper bound R_J as in Lemma 1.1.3 (see also Remark 1.1.1) and hence a global upper bound via Gronwall's inequality. Thus we can prove the following corollary which combines Corollary 3.2.2 with Corollary 3.2.4.

Corollary 3.2.5. *Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function and assume there exists an interval $J \subseteq [0,T]$ where $a(t) \geq 0$ for a.e. $t \in J$ and also*

$$\int_0^T a(t)\,dt < 0 < \int_J a(t)\,dt.$$

Suppose that $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, regularly oscillating at zero, satisfying (g_*) and such that

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad 0 < \liminf_{s\to+\infty} \frac{g(s)}{s} \leq \limsup_{s\to+\infty} \frac{g(s)}{s} < +\infty.$$

Then there exists $\nu^* > 0$ such that the boundary value problem $(\mathscr{P}_\nu)$ has at least a positive solution for each $\nu > \nu^*$.

Remark 3.2.2. The condition of regularly oscillation at zero required on $g(s)$ is useful in order to conclude the verification of (H_r) in Theorem 3.2.1. Nevertheless, there is a disadvantage in assuming such a condition, as it does not allow to consider functions as

$$g(s) = s^\gamma \exp(-1/s) \ \text{ for } \ s > 0 \ \ (\gamma > 1), \quad g(0) = 0, \tag{3.2.12}$$

which are not regularly oscillating at zero.

With this respect we observe that, from a careful reading of the first part of the proof of Theorem 3.2.1, the key point is to demonstrate that $g(r_n\omega_n)/g(r_n) \to 1$, for $r_n \to 0^+$ and $\omega_n = v(t_n) \to 1$ (for a suitable choice of $t_n \in [0,T]$). In our proof, the sequence ω_n is not an arbitrary sequence tending to 1, since $0 < \omega_n < 1$, and, moreover, from (3.2.7) we can easily provide the estimate

$$0 \leq 1 - \omega_n \leq C \sup_{0<s\leq r_n} \frac{g(s)}{s}$$

(for $C > 0$ a suitable constant independent on r_n and ω_n). Therefore, the faster $g(r_n)/r_n$ tends to zero, the more ω_n tends to one.

Using this observation, we can apply our result also to some not regularly oscillating functions $g(s)$, provided that they tend to zero sufficiently fast. In this manner, for instance, Corollary 3.2.1 holds also for a function $g(s)$ as in (3.2.12) (the easy verification is omitted).

Another way to avoid the hypothesis of regular oscillation at the origin is described in Theorem 3.2.2. $\lhd$

A variant of Theorem 3.2.1 is the following result. Basically, we replace the computations for the verification of (H_r) given in the proof of Theorem 3.2.1 with a different argument which is essentially inspired by the approach in [40].

Theorem 3.2.2. *Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function satisfying (a_*) and $(a_\#)$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Suppose also that $g(s)$ is continuously differentiable on a right neighborhood $[0,\varepsilon_0[$ of $s = 0$. Then problem $(\mathscr{P})$ has at least a positive solution.*

Proof. As in the proof of Theorem 3.2.1, we enter the setting of Theorem 3.1.1 by defining

$$f(t,s,\xi) = f(t,s) := a(t)g(s).$$

Verification of (H_r). First of all, we observe that (g_*) and $(a_\#)$ imply that

$$\int_0^T f(t,s,0)\,dt < 0, \quad \forall\, s > 0. \tag{3.2.13}$$

We claim that there exists $r_0 \in \,]0,\varepsilon_0[$ such that for all $0 < r \leq r_0$ and for all $\vartheta \in \,]0,1]$ there are no solutions $u(t)$ of (3.1.4) such that $u(t) > 0$ on $[0,T]$ and $\|u\|_\infty = r$.

By contradiction, suppose the claim is not true. Then for all $n \in \mathbb{N}$ there exist $0 < r_n < 1/n$, $\vartheta_n \in \,]0,1]$ and $u_n(t)$ solution of (3.2.3) such that $u_n(t) > 0$ on $[0,T]$ and $\|u_n\|_\infty = r_n$. By condition (g_0), note also that

$$\lim_{n\to\infty} g'(u_n(t)) = 0, \quad \text{uniformly on } [0,T].$$

Using the identity

$$\frac{u''}{g(u)} = \frac{d}{dt}\left(\frac{u'}{g(u)}\right) + g'(u)\left(\frac{u'}{g(u)}\right)^2,$$

for $u = u_n$, and setting

$$z_n(t) := \frac{u_n'(t)}{g(u_n(t))}, \quad t \in [0, T],$$

we obtain the following relation

$$z_n'(t) + g'(u_n(t))z_n^2(t) = -\vartheta_n a(t), \quad \text{for a.e. } t \in [0, T]. \tag{3.2.14}$$

The boundary conditions (of Neumann or periodic type) on $u_n(t)$ imply that

$$z_n(0) = z_n(T) \quad \text{ and } \quad \exists\, t_n^* \in [0, T] \colon z_n(t_n^*) = 0 \tag{3.2.15}$$

(obviously, we can take $t_n^* = 0$ in the case of the Neumann boundary conditions, while the existence of such a point in the periodic case follows from Rolle's theorem).

We fix a positive constant $M > \|a\|_{L^1}$ and then a constant δ with

$$0 < \delta < \frac{M - \|a\|_{L^1}}{TM^2}. \tag{3.2.16}$$

By the continuity of $g'(s)$ on $[0, \varepsilon_0[$ and $g'(0) = 0$ (which corresponds to condition (g_0)), we find $\varepsilon \in \,]0, \varepsilon_0[$ such that

$$|g'(s)| \leq \delta, \quad \forall\, 0 \leq s \leq \varepsilon.$$

Let $n > 1/\varepsilon$. In this case, we have that $0 < u_n(t) < \varepsilon$ on $[0, T]$ and we claim that

$$\|z_n\|_\infty \leq \vartheta_n M. \tag{3.2.17}$$

Indeed, if by contradiction we suppose that (3.2.17) is not true (for some $n > 1/\varepsilon$), then, using the fact that $z_n(t)$ vanishes at some point t_n^* of $[0, T]$, we can find a maximal interval J_n of the form $[t_n^*, \tau_n]$ or $[\tau_n, t_n^*]$ such that $|z_n(t)| \leq \vartheta_n M$ for all $t \in J_n$ and $|z_n(t)| > \vartheta_n M$ for some $t \notin J_n$, or, more precisely, with $\tau_n < t \leq T$ or $0 \leq t < \tau_n$, respectively. By the maximality of the interval J_n, we also know that $|z_n(\tau_n)| = \vartheta_n M$.

Integrating (3.2.14) on J_n and passing to the absolute value, we obtain

$$\begin{aligned}
\vartheta_n M &= |z_n(\tau_n)| = |z_n(\tau_n) - z_n(t_n^*)| \\
&\leq \left| \int_{J_n} g'(u_n(t)) z_n^2(t)\, dt \right| + \vartheta_n \|a\|_{L^1} \\
&\leq \delta \vartheta_n^2 M^2 |\tau_n - t_n^*| + \vartheta_n \|a\|_{L^1} \\
&\leq \vartheta_n \big(\delta M^2 T + \|a\|_{L^1}\big)
\end{aligned}$$

(recall that $0 < \vartheta_n \leq 1$). Dividing the above inequality by $\vartheta_n > 0$, we find a contradiction with the choice of δ in (3.2.16). In this manner, we have verified that (3.2.17) is true.

Now, integrating (3.2.14) on $[0,T]$, recalling that $z_n(T) - z_n(0) = 0$ (according to (3.2.15)) and using (3.2.17), we obtain that

$$\begin{aligned} -\vartheta_n \int_0^T a(t)\,dt &= \int_0^T g'(u_n(t)) z_n^2(t)\,dt \\ &\leq T\vartheta_n^2 M^2 \max_{0\leq s\leq r_n} |g'(s)| \\ &\leq \vartheta_n T M^2 \max_{0\leq s\leq r_n} |g'(s)| \end{aligned}$$

holds for every $n > 1/\varepsilon$. From this,

$$0 < -\bar{a} \leq M^2 \max_{0\leq s\leq r_n} |g'(s)| \tag{3.2.18}$$

follows. Using the continuity of $g'(s)$ at $s = 0^+$, we get a contradiction, as $n \to \infty$.

The claim is thus proved and, recalling also (3.2.13), we have that (H_r) holds for any $r \in \,]0, r_0]$.

Verification of (H^R). This has been already checked in the second part of the proof of Theorem 3.2.1. No change is needed.

As last step, we conclude exactly as in the proof of Theorem 3.2.1, via Theorem 3.1.1. □

From Theorem 3.2.2, we can derive the same corollaries as above in which the condition of regularly oscillation at zero of $g(s)$ is systematically replaced by the smoothness of $g(s)$ on a right neighborhood $[0, \varepsilon_0[$ of zero. In particular, an obvious improvement of Corollary 3.2.3 is the following.

Corollary 3.2.6. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function satisfying (g_*), such that $g'(s) > 0$ for all $s > 0$ and*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0, \qquad \lim_{s\to +\infty} \frac{g(s)}{s} = +\infty.$$

Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function satisfying (a_*). Then problem $(\mathscr{P})$ has at least a positive solution if and only if $(a_\#)$ holds.

For the Neumann problem this result improves [40, § 3, Corollary 1] to a more general class of weight functions $a(t)$. It also extends such a result to the periodic case.

Remark 3.2.3. In Theorem 3.2.1 and Theorem 3.2.2 we have two different conditions that are required on $g(s)$ as $s \to 0^+$. It can be interesting to provide examples in which one of the two results applies, while for the other the conditions on $g(s)$ are not fulfilled. For this discussion, we confine ourselves only to the behavior of $g(s)$ on a right neighborhood $[0,1]$ of zero and we do not care about $a(t)$ or the behavior of $g(s)$ as $s \to +\infty$.

Take any function $\sigma\colon [0,1] \to \mathbb{R}^+_0$ which is continuous but not differentiable (for instance, one could even choose a nowhere differentiable function of Weierstrass type) and define

$$g(s) = \sigma(s)s^{\gamma}, \quad \gamma > 1.$$

Such a function $g(s)$ is regularly oscillating at zero (note that $\sigma(0) > 0$) and it fits for Theorem 3.2.1, but it is not suitable for Theorem 3.2.2.

As second example, we consider a function as

$$g(s) = s^{\gamma}\sin^2(1/s) + s^{\beta}, \quad \text{for} \quad s \in \left]0,1\right] \;\; (\beta > \gamma > 2), \quad g(0) = 0.$$

Such a function $g(s)$ is continuously differentiable on $[0,1]$ and it fits for Theorem 3.2.2, but it is not suitable for Theorem 3.2.1 since $g(s)$ is not regularly oscillating at zero.

See Figure 3.1 for a graphical representation of the above examples.

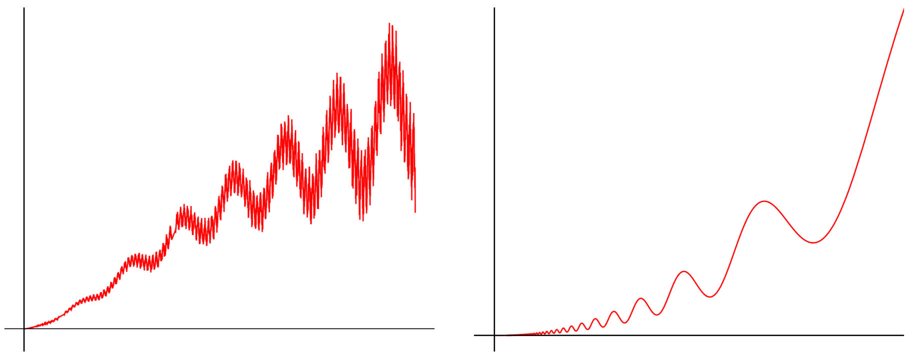

Figure 3.1: The figure on the left shows the graph of $g(s) := \sigma(s)s^{\frac{10}{9}}$ on $[0,1]$, where $\sigma(s) := 3 + \sum_{n=0}^{\infty} \sin(\pi 15^n s)/2^n$ is the *Weierstrass function.* On the right we show the graph of $g(s) := s^{2.1}\sin^2(1/s) + s^{2.2}$ on $[0,1/5]$. As explained in Remark 3.2.3, for these two functions only one of our two main existence theorems applies.

Both the above examples can be easily generalized in order to construct broad classes of nonlinearities where only one of the two existence theorems applies. $\lhd$

We end this section by presenting a variant of Corollary 3.2.5 in the smooth case and observing that the argument employed in the proof of Theorem 3.2.2 can be used to provide a *nonexistence result* for positive solutions when $g(s)$ is smooth on $\mathbb{R}^+$ and with sufficiently small derivative.

Corollary 3.2.7. *Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function and assume there exists an interval $J \subseteq [0,T]$ where $a(t) \geq 0$ for a.e. $t \in J$ and also*

$$\int_0^T a(t)\,dt < 0 < \int_J a(t)\,dt.$$

Suppose also that $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuously differentiable function satisfying (g_*) and such that

$$g'(0) = 0 \quad \text{and} \quad 0 < \liminf_{s\to+\infty} g'(s) \leq \limsup_{s\to+\infty} g'(s) < +\infty.$$

Then there exists $\nu^* > 0$ such that the boundary value problem $(\mathscr{P}_\nu)$ has at least a positive solution for each $\nu > \nu^*$.

Clearly, Corollary 3.2.7 applies also if $g(s)$ is a continuously differentiable function satisfying (g_*) and

$$g'(0) = 0 < g'(+\infty) < +\infty.$$

Indeed, using the generalized de l'Hôpital's rule, we have

$$\liminf_{s\to+\infty} g'(s) \leq \liminf_{s\to+\infty} \frac{g(s)}{s} \leq \limsup_{s\to+\infty} \frac{g(s)}{s} \leq \limsup_{s\to+\infty} g'(s).$$

Proposition 3.2.1. *Let $g\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be a continuously differentiable function with bounded derivative on $\mathbb{R}^+_0$. Let $a \in L^1([0,T])$ satisfy condition $(a_\#)$. Then there exists $\nu_* > 0$ such that the boundary value problem $(\mathscr{P}_\nu)$ has no positive solutions for each $0 < \nu < \nu_*$.*

Proof. The proof follows substantially the same argument employed in the proof of Theorem 3.2.2 from (3.2.14) to (3.2.18).

We fix two positive constants M and D such that

$$M > \|a\|_{L^1} \quad \text{and} \quad |g'(s)| \leq D, \quad \forall\, s > 0,$$

(recall that, by assumption, $g(s)$ has bounded derivative on $\mathbb{R}^+_0$) and define

$$\nu_* := \min\left\{ \frac{M - \|a\|_{L^1}}{DM^2T}, \frac{-\bar{a}}{DM^2} \right\}.$$

We shall prove that for $0 < \nu < \nu_*$ problem $(\mathscr{P}_\nu)$ has no positive solution.

Let us suppose by contradiction that $u(t) > 0$ for all $t \in [0,T]$ is a solution of problem $(\mathscr{P}_\nu)$. Setting $z(t) := u'(t)/\nu g(u(t))$, we find

$$z'(t) + \nu g'(u(t)) z^2(t) = -a(t). \tag{3.2.19}$$

As a consequence of the boundary conditions, we also have that $z(0) = z(T)$ and there exists $t^* \in [0,T]$ (with t^* depending on the solution $u(t)$) with $z(t^*) = 0$.

First of all, we claim that

$$\|z\|_\infty \leq M. \tag{3.2.20}$$

Indeed, if by contradiction we suppose that (3.2.20) is not true, then using the fact that $z(t)$ vanishes at some point of $[0,T]$, we can find a maximal interval J of the form $[t^*,\tau]$ or $[\tau,t^*]$ such that $|z(t)| \leq M$ for all $t \in J$ and $|z(t)| > M$ for some $t \notin J$. By the maximality of the interval J, we also know that $|z(\tau)| = M$. Integrating (3.2.19) on J and passing to the absolute value, we obtain

$$\begin{aligned} M = |z(\tau) - z(t^*)| &\leq \left| \int_J \nu g'(u(t)) z^2(t)\, dt \right| + \|a\|_{L^1} \\ &\leq \nu D M^2 T + \|a\|_{L^1} < M, \end{aligned}$$

a contradiction. In this manner, we have verified that (3.2.20) is true.

Now, integrating (3.2.19) on $[0,T]$, recalling that $z(T) - z(0) = 0$ and using (3.2.20), we reach

$$-\bar{a} = -\frac{1}{T}\int_0^T a(t)\, dt = \frac{1}{T}\int_0^T \nu g'(u(t)) z^2(t)\, dt \leq \nu D M^2 < -\bar{a},$$

a contradiction. This concludes the proof. □

Remark 3.2.4. The same proof as above works to prove the nonexistence of solution to problem $(\mathscr{P}_\nu)$ with range in a given open interval $]\alpha,\beta[$, for $g\colon]\alpha,\beta[\to \mathbb{R}^+_0$ a smooth function with bounded derivative. In some recent papers (see [33, 34, 36]) similar nonexistence results have been obtained under different conditions on the function $g(s)$. ◁

Remark 3.2.5. From a careful reading of the proof of Theorem 3.2.2, one can notice that in that result and in its corollaries condition (g_0) can be slightly improved to a condition of the form

$$\lim_{s\to 0^+} \frac{g(s)}{s} < \lambda_*,$$

where λ_* is a positive constant that satisfies

$$0 < \lambda_* < \frac{1}{4T\|a\|_{L^1}},$$

as can be deduced from formula (3.2.16), observing that the continuous map $M \mapsto (M - \|a\|_{L^1})/(TM^2)$ (for $M > 0$) attains its maximum for $M = 2\|a\|_{L^1}$. ◁

3.3 More general examples and applications

In Section 3.2 we have applied our abstract result Theorem 3.1.1, which deals with a general second order equation of the form

$$u'' + f(t, u, u') = 0,$$

to the simpler case given by

$$u'' + a(t)g(u) = 0.$$

In this section, we show how our result can be extended to a broader class of equations. Up to this point, by $\mathscr{B}(u, u') = \underline{0}$, we have considered together the two different boundary conditions. Now we present two different applications, one for the Neumann problem and another for periodic solutions.

For simplicity in the exposition, in our applications we will suppose that the weight function is *continuous*, in order to obtain classical positive solutions.

3.3.1 The Neumann problem: radially symmetric solutions

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^N$ (for $N \geq 2$) and let

$$\Omega := B(0, R_2) \setminus B[0, R_1] = \{x \in \mathbb{R}^N : R_1 < \|x\| < R_2\}$$

be an open annular domain, with $0 < R_1 < R_2$. Let $q\colon \overline{\Omega} \to \mathbb{R}$ be a continuous function which is radially symmetric, namely there exists a continuous scalar function $\mathcal{Q}\colon [R_1, R_2] \to \mathbb{R}$ such that

$$q(x) = \mathcal{Q}(\|x\|), \quad \forall\, x \in \overline{\Omega}.$$

In this section we consider the Neumann boundary value problem

$$\begin{cases} -\Delta u = q(x)g(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3.1}$$

and we are interested in the existence of radially symmetric positive solutions of (3.3.1), namely classical solutions such that $u(x) > 0$ for all $x \in \Omega$ and also $u(x') = u(x'')$ whenever $\|x'\| = \|x''\|$.

Since we look for radially symmetric solutions of (3.3.1), our study can be reduced to the search of positive solutions of the Neumann boundary value problem

$$w''(r) + \frac{N-1}{r} w'(r) + \mathcal{Q}(r)g(w(r)) = 0, \quad w'(R_1) = w'(R_2) = 0. \tag{3.3.2}$$

Indeed, if $w(r)$ is a solution of (3.3.2), then $u(x) := w(\|x\|)$ is a solution of (3.3.1). As illustrated in Section C.2, using the standard change of variable

$$t = h(r) := \int_{R_1}^{r} \xi^{1-N}\, d\xi$$

and defining

$$T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi, \quad r(t) := h^{-1}(t) \quad \text{and} \quad v(t) = w(r(t)),$$

we transform (3.3.2) into the equivalent problem

$$v'' + a(t)g(v) = 0, \quad v'(0) = v'(T) = 0, \tag{3.3.3}$$

with

$$a(t) := r(t)^{2(N-1)}\mathcal{Q}(r(t)), \quad t \in [0,T].$$

Consequently, the Neumann boundary value problem (3.3.3) is of the same form of $(\mathscr{P})$ and we can apply the results of Section 3.2.

Since $r(t)^{2(N-1)} > 0$ on $[0,T]$, condition (a_*) is satisfied provided that a similar condition holds for $\mathcal{Q}(r)$ on $[R_1, R_2]$. Accordingly, we assume

(q_*) *there exist $m \geq 1$ intervals $J_1^+, \ldots, J_m^+$, closed and pairwise disjoint, such that such that*

$$\mathcal{Q}(r) \geq 0, \quad \textit{for every } r \in J_i^+, \textit{ with } \max_{r \in J_i^+} \mathcal{Q}(r) > 0 \quad (i = 1, \ldots, m);$$

$$\mathcal{Q}(r) \leq 0, \quad \textit{for every } r \in [R_1, R_2] \setminus \bigcup_{i=1}^{m} J_i^+.$$

Condition $(a_\#)$ reads as

$$0 > \int_0^T r(t)^{2(N-1)}\mathcal{Q}(r(t))\, dt = \int_{R_1}^{R_2} r^{N-1}\mathcal{Q}(r)\, dr.$$

Up to a multiplicative constant, the latter integral is the integral of $q(x)$ on Ω, using the change of variable formula for radially symmetric functions (cf. [85]). Thus, $a(t)$ satisfies $(a_\#)$ if and only if

$(q_\#)$
$$\int_\Omega q(x)\, dx < 0.$$

The following theorems are easy corollaries of the results presented in Section 3.2.

Theorem 3.3.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function, regularly oscillating at zero and satisfying (g_*). Assume*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to +\infty} \frac{g(s)}{s} = +\infty.$$

Let $q(x) = \mathcal{Q}(\|x\|)$ be a continuous radially symmetric function satisfying (q_*) and $(q_\#)$. Then problem (3.3.1) *has at least a positive radially symmetric solution.*

Theorem 3.3.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function, regularly oscillating at zero and satisfying (g_*). Assume*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad \liminf_{s\to +\infty} \frac{g(s)}{s} > 0.$$

Let $q(x) = \mathcal{Q}(\|x\|)$ be a continuous radially symmetric function satisfying (q_*) and $(q_\#)$. Then there exists $\nu^* > 0$ such that problem

$$\begin{cases} -\Delta u = \nu q(x) g(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3.4}$$

has at least a positive radially symmetric solution for each $\nu > \nu^*$.

Clearly, Theorem 3.3.1 and Theorem 3.3.2 correspond to Corollary 3.2.1 and Corollary 3.2.2, respectively. The next result follows from the same argument that led to Corollary 3.2.5.

Theorem 3.3.3. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function, regularly oscillating at zero and satisfying (g_*). Assume*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad 0 < \liminf_{s\to +\infty} \frac{g(s)}{s} \le \limsup_{s\to +\infty} \frac{g(s)}{s} < +\infty.$$

Let $q(x) = \mathcal{Q}(\|x\|)$ be a continuous radially symmetric function satisfying $(q_\#)$ and such that $q(x_0) > 0$ for some $x_0 \in \Omega$. Then there exists $\nu^* > 0$ such that problem (3.3.4) *has at least a positive radially symmetric solution for each $\nu > \nu^*$.*

Note that, with respect to Theorem 3.3.2, in the above result we do not assume condition (q_*) on the weight function. In this manner, we can consider functions $\mathcal{Q}(r)$ with infinitely many changes of sign in $[R_1, R_2]$.

All the above three theorems can be stated in a version where the regularly oscillating assumption at zero is replaced with the hypothesis that $g(s)$ is continuously differentiable on a right neighborhood of zero, according to Theorem 3.2.2. For instance, the corresponding version of Theorem 3.3.1 reads as follows.

Theorem 3.3.4. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and such that $g(s)$ is continuously differentiable on a right neighborhood of $s = 0$. Assume*

$$g'(0) = 0 \quad \text{and} \quad \lim_{s\to+\infty} \frac{g(s)}{s} = +\infty.$$

Let $q(x) = \mathcal{Q}(\|x\|)$ be a continuous radially symmetric function satisfying (q_*) and $(q_\#)$. Then problem (3.3.1) has at least a positive radially symmetric solution.

If we also suppose that $g(s)$ is continuously differentiable on $\mathbb{R}_0^+$ with $g'(s) > 0$ for all $s > 0$, then condition $(q_\#)$ is also necessary for the existence of a positive solution. On the other hand, as already observed, also the fact that the weight function must change its sign is necessary for the existence of solutions. With this respect, the following corollary can be derived from the smooth version of Theorem 3.3.3 (see also Corollary 3.2.7).

Corollary 3.3.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function such that $g'(s) > 0$ for all $s > 0$. Assume*

$$g(0) = g'(0) = 0 \quad \text{and} \quad g'(+\infty) = \ell > 0.$$

Let $q(x) = \mathcal{Q}(\|x\|)$ be a continuous radially symmetric function. Then there exists $\nu^* > 0$ such that problem (3.3.4) has at least a positive radially symmetric solution for each $\nu > \nu^*$ if and only if

$$q^+(x) \not\equiv 0 \quad \text{and} \quad \int_\Omega q(x)\,dx < 0.$$

Note also that, under the assumptions of Corollary 3.3.1 there is also a constant $\nu_* > 0$ such that for each $0 < \nu < \nu_*$ problem (3.3.4) has no positive radial solutions (cf. Proposition 3.2.1).

Possible examples of functions satisfying the above conditions are

$$g(s) = Ks\arctan(s^{\gamma-1}) \quad \text{for } s \geq 0 \qquad (\gamma > 1,\ K > 0)$$

and

$$g(s) = K\frac{s^\gamma}{s^{\gamma-1} + M} \quad \text{for } s \geq 0 \qquad (\gamma > 1,\ K, M > 0).$$

Remark 3.3.1. In [19], Berestycki, Capuzzo-Dolcetta and Nirenberg obtained an existence result of positive solutions for the Neumann problem (3.3.1) in the superlinear indefinite case for Ω a bounded domain with smooth boundary. In [19, Theorem 3] the main assumptions require that $g(s)$ has a precise power-like growth at infinity, that is $g(s)/s^p \to l > 0$ (as $s \to +\infty$) for some $p \in \,]1, (N+2)/(N-1)[$, and that $\nabla q(x)$ does not vanish on the points of

$$\Gamma := \overline{\{x \in \Omega\colon q(x) > 0\}} \cap \overline{\{x \in \Omega\colon q(x) < 0\}} \subseteq \Omega.$$

Our setting is much more simplified as we consider an annular domain and a radially symmetric weight function. On the other hand, our growth condition at infinity is more general (allowing a nonlinearity which is not necessarily of power-like type) and, moreover, no condition on the zeros of $q(x)$ is required. $\lhd$

3.3.2 The periodic problem: a Liénard type equation

In this section we deal with the existence of periodic positive solutions to a *Liénard type equation*, namely positive solutions of

$$\begin{cases} u'' + h(u)u' + a(t)g(u) = 0, & 0 < t < T, \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{3.3.5}$$

where $h\colon \mathbb{R}^+ \to \mathbb{R}$ is a continuous function. As a preliminary remark, we observe that, if $u(t) > 0$ is any solution of (3.3.5), then $\int_0^T h(u(t))u'(t)\,dt = 0$ and also $\int_0^T h(u(t))u'(t)/g(u(t))\,dt = 0$. Consequently, the condition that $a(t)$ changes sign with negative average, which is necessary for $(\mathscr{P})$ (when $g'(s) > 0$), is still necessary for (3.3.5).

For simplicity, in this section we present only an extension of Corollary 3.2.1 to the Liénard equation. In particular, we do not consider the alternative approach of Theorem 3.2.2 for $g(s)$ smooth. Accordingly, applying the results in Section 3.1, we prove the following theorem.

Theorem 3.3.5. *Let $h\colon \mathbb{R}^+ \to \mathbb{R}$ be continuous and bounded. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function, regularly oscillating at zero and satisfying (g_*). Assume*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to +\infty} \frac{g(s)}{s} = +\infty. \tag{3.3.6}$$

Let $a\colon [0,T] \to \mathbb{R}$ be a continuous function satisfying (a_*) and $(a_\#)$. Then problem (3.3.5) has at least a positive solution.

Proof. We follow the same pattern as the proof of Theorem 3.2.1. In particular, we are going to show how to achieve the same main steps and formulas in that proof.

First of all, we define $f\colon [0,T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ as

$$f(t,s,\xi) = (h(s) - c)\xi + a(t)g(s), \quad \text{with } c := h(0),$$

and we remark that f is an L^1-Carathéodory function satisfying (f_1), (f_2) and (f_3). In this manner, problem (3.3.5) is of the form

$$\begin{cases} u'' + cu' + f(t,u,u') = 0, & 0 < t < T, \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases}$$

which is of the same type of (3.1.1) with $u \mapsto -(u'' + cu')$ as differential operator. The thesis will be reached using Theorem 3.1.1 with Remark 3.1.1. In order to avoid unnecessary repetitions, from now on in the proof, all the solutions that we consider satisfy the T-periodic boundary conditions.

Verification of (H_r). Observe that (3.2.2) is satisfied. We claim that there exists $r_0 > 0$ such that for all $0 < r \leq r_0$ and for all $\vartheta \in \,]0,1]$ there are no positive solutions $u(t)$ of

$$u'' + cu' + \vartheta f(t, u, u') = 0$$

such that $\|u\|_\infty = r$. By contradiction, suppose the claim is not true. Then for all $n \in \mathbb{N}$ there exist $0 < r_n < 1/n$, $\vartheta_n \in \,]0,1]$ and $u_n(t)$ positive solution of

$$u'' + cu' + \vartheta_n(h(u) - c)u' + \vartheta_n a(t)g(u) = 0 \tag{3.3.7}$$

such that $\|u_n\|_\infty = r_n$.

Integrating (3.3.7) on $[0,T]$ and using the periodic boundary conditions, we obtain again (3.2.4). We define

$$v_n(t) := \frac{u_n(t)}{\|u_n\|_\infty}, \quad t \in [0,T],$$

and, dividing (3.3.7) by $r_n = \|u_n\|_\infty$, we get

$$v_n'' + cv_n' + \vartheta_n(h(u_n(t)) - c)v_n' + \vartheta_n a(t)q(u_n(t))v_n = 0, \tag{3.3.8}$$

where $h(u_n(t)) - c \to 0$ and $q(u_n(t)) := g(u_n(t))/u_n(t) \to 0$, uniformly on $[0,T]$ as $n \to \infty$. Multiplying equation (3.3.8) by v_n and integrating on $[0,T]$, we obtain

$$\|v_n'\|^2_{L^2} \leq \|a\|_{L^1} \sup_{t\in[0,T]} |q(u_n(t))| \to 0, \qquad \text{as } n \to \infty.$$

Using this information on (3.3.8), we see that $\|v_n''\|_{L^1} \to 0$ as $n \to \infty$. From this fact and observing that v_n' must vanish at some point (by Rolle's theorem), we obtain that $v_n'(t) \to 0$ (as $n \to \infty$) uniformly on $[0,T]$ and thus (3.2.8) follows. From (3.2.4) and (3.2.8) we conclude exactly as in the verification of (H_r) in the proof of Theorem 3.2.1.

Verification of (H^R). We choose the same function $v(t)$ as in the proof of Theorem 3.2.1 and observe that the equation in (3.1.5) now reads as

$$u'' + h(u)u' + a(t)g(u) + \alpha v(t) = 0. \tag{3.3.9}$$

We also fix a constant $C > 0$ such that

$$|h(s)| \leq C, \quad \forall\, s \geq 0. \tag{3.3.10}$$

Following the proof of Theorem 3.2.1, we choose an interval J among the intervals I_i^+. We look for a bound $R_J > 0$ such that any non-negative solution $u(t)$ of (3.3.9), with $\alpha \geq 0$, satisfies $\max_{t\in J} u(t) < R_J$. For notational convenience, we set $J = [\sigma, \tau]$ and let $0 < \varepsilon < (\tau - \sigma)/2$ be fixed such that

$$a(t) \not\equiv 0 \quad \text{on } J^\varepsilon,$$

where $J^\varepsilon := [\sigma_0, \tau_0] \subseteq [\sigma, \tau]$ with $\sigma_0 - \sigma = \tau - \tau_0 = \varepsilon$.

We claim that $u'(t) \leq u(t)e^{CT}/\varepsilon$, for all $t \in [\sigma_0, \tau]$ such that $u'(t) \geq 0$, and also $|u'(t)| \leq u(t)e^{CT}/\varepsilon$, for all $t \in [\sigma, \tau_0]$ such that $u'(t) \leq 0$. The proof is based on the fact that the auxiliary function

$$\Upsilon \colon t \mapsto u'(t) \exp\biggl(\int_0^t h(u(\xi))\, d\xi\biggr)$$

is non-increasing on J. Indeed, in order to prove the first inequality, let us fix $t \in [\sigma_0, \tau]$ such that $u'(t) \geq 0$. The result is trivially true if $u'(t) = 0$. Suppose that $u'(t) > 0$. Since Υ is non-increasing on $[\sigma, t] \subseteq J$, we have

$$u'(\xi) \geq u'(t) \exp\biggl(\int_\xi^t h(u(\xi))\, d\xi\biggr) \geq u'(t)e^{-C(t-\sigma)}, \quad \forall\, \xi \in [\sigma, t].$$

Integrating on $[\sigma, t]$, we obtain

$$u(t) \geq u(t) - u(\sigma) \geq u'(t)e^{-C(t-\sigma)}(t - \sigma) \geq u'(t)e^{-CT}\varepsilon.$$

Therefore, the first inequality follows. The second one can be obtained with an analogous argument.

Let $\hat{\lambda}$ be the first (positive) eigenvalue of the eigenvalue problem

$$\begin{cases} \bigl(e^{Ct}\varphi'\bigr)' + e^{-Ct}\lambda a(t)\varphi = 0 \\ \varphi(\sigma_0) = \varphi(\tau_0) = 0. \end{cases}$$

We fix a constant $M > 0$ such that

$$M > \hat{\lambda}.$$

From (3.3.6) it follows that there exists a constant $\tilde{R} = \tilde{R}(M) > 0$ such that

$$g(s) > Ms, \quad \forall\, s \geq \tilde{R}.$$

By contradiction, suppose there is not a constant $R_J > 0$ with the properties listed above. So, for each integer $n > 0$ there exists a solution $u_n \geq 0$ of (3.3.9) with $\max_{t\in J} u_n(t) =: \hat{R}_n > n$. For each $n > \tilde{R}$ we take $\hat{t}_n \in J$ such that $u_n(\hat{t}_n) = \hat{R}_n$ and let $]\varsigma_n, \omega_n[\subseteq J$ be the intersection with $]\sigma, \tau[$

of the maximal open interval containing $\hat{t}_n$ and such that $u_n(t) > \tilde{R}$ for all $t \in]\varsigma_n, \omega_n[$. We fix an integer N such that

$$N > \tilde{R} + \frac{\tilde{R}Te^{2CT}}{\varepsilon}$$

and we claim that $]\varsigma_n, \omega_n[\supseteq [\sigma_0, \tau_0]$, for each $n \geq N$. Suppose by contradiction that $\sigma_0 \leq \varsigma_n$. In this case, we find that $u_n(\varsigma_n) = \tilde{R}$ and $u_n'(\varsigma_n) \geq 0$. Moreover, $u_n'(\varsigma_n) \leq \tilde{R}e^{CT}/\varepsilon$. Using the monotonicity of Υ, $\Upsilon(t) \leq \Upsilon(\varsigma_n)$ for every $t \in [\varsigma_n, \hat{t}_n]$ and therefore, using also (3.3.10), we find $u'(t) \leq \tilde{R}e^{2CT}/\varepsilon$ for every $t \in [\varsigma_n, \hat{t}_n]$. Finally, an integration on $[\varsigma_n, \hat{t}_n]$ yields

$$n < \hat{R}_n = u_n(\hat{t}_n) \leq \tilde{R} + \frac{\tilde{R}Te^{2CT}}{\varepsilon},$$

hence a contradiction, since $n \geq N$. A symmetric argument provides a contradiction if we suppose that $\omega_n \leq \tau_0$. This proves the claim.

So, we can fix an integer $N > \tilde{R}$ such that $u_n(t) > \tilde{R}$ for every $t \in J^\varepsilon$ and for $n \geq N$. The function $u_n(t)$, being a solution of equation (3.3.9), also satisfies

$$\begin{cases} u_n'(t) = \dfrac{y_n(t)}{p_n(t)} \\ y_n'(t) = -H_n(t, u_n(t)), \end{cases}$$

where

$$p_n(t) := \exp\left(\int_0^t h(u_n(\xi))\,d\xi\right)$$

and

$$H_n(t, u_n(t)) := \exp\left(\int_0^t h(u_n(\xi))\,d\xi\right)\Big(a(t)g(u_n(t)) + \alpha v(t)\Big).$$

Passing to the polar coordinates, via a Prüfer transformation, we consider

$$p_n(t)u_n'(t) = r_n(t)\cos\vartheta_n(t), \qquad u_n(t) = r_n(t)\sin\vartheta_n(t),$$

and obtain, for every $t \in J^\varepsilon$, that

$$\begin{aligned} \vartheta_n'(t) &= \frac{\cos^2\vartheta_n(t)}{p_n(t)} + \frac{H_n(t, u_n(t))}{u_n(t)}\sin^2\vartheta_n(t) \\ &\geq \frac{\cos^2\vartheta_n(t)}{p_n(t)} + Mp_n(t)a(t)\sin^2\vartheta_n(t). \end{aligned}$$

We also consider the linear equation

$$\big(e^{Ct}u'\big)' + e^{-Ct}Ma(t)u = 0 \tag{3.3.11}$$

and its associated angular coordinate $\vartheta(t)$ (via the Prüfer transformation), which satisfies

$$\vartheta'(t) = \frac{\cos^2\vartheta(t)}{e^{Ct}} + e^{-Ct} M a(t) \sin^2\vartheta(t).$$

Note also that the angular functions ϑ_n and ϑ are non-decreasing in J^ε. Using a classical comparison result in the frame of Sturm's theory (cf. [55, ch. 8, Theorem 1.2]), we find that

$$\vartheta_n(t) \geq \vartheta(t), \quad \forall\, t \in J^\varepsilon, \tag{3.3.12}$$

if we choose $\vartheta(\sigma_0) = \vartheta_n(\sigma_0)$. Consider now a fixed $n \geq N$. Since $u_n(t) \geq \tilde{R}$ for every $t \in J^\varepsilon$, we must have

$$\vartheta_n(t) \in \,]0,\pi[, \quad \forall\, t \in J^\varepsilon. \tag{3.3.13}$$

On the other hand, by the choice of $M > 0$, we know that any non-negative solution $u(t)$ of (3.3.11) with $u(\sigma_0) > 0$ must vanish at some point in $]\sigma_0, \tau_0[$ (see [55, ch. 8, Theorem 1.1]). Therefore, from $\vartheta(\sigma_0) = \vartheta_n(\sigma_0) \in \,]0,\pi[$, we conclude that there exists $t^* \in \,]\sigma_0,\tau_0[$ such that $\vartheta(t^*) = \pi$. By (3.3.12) we have that $\vartheta_n(t^*) \geq \pi$, which contradicts (3.3.13).

By the arbitrary choice of J among the intervals $I_1^+, \ldots, I_m^+$, for each $i = 1, \ldots, m$ we obtain the existence of a constant $R_{I_i^+} > 0$ such that any non-negative solution $u(t)$ of equation (3.3.9), with $\alpha \geq 0$, satisfies $\max_{t \in I_i^+} u(t) < R_{I_i^+}$. Finally, let us fix a constant $R > r_0$ (with r_0 coming from the first part of the proof) as in (3.2.10), so that $R \geq R_{I_i^+}$ for all $i = 1, \ldots, m$.

Consider now a (maximal) interval $\mathcal{J}$ contained in $[0,T] \setminus \bigcup_{i=1}^m I_i^+$ where $a(t) \leq 0$. For simplicity in the exposition, we suppose that $\mathcal{J}$ lies between two intervals I_i^+ where $a(t) \geq 0$, so that $\mathcal{J} = \,]\tau',\sigma'[$, with $\tau' \in I_k^+$ and $\sigma' \in I_{k+1}^+$.

Let $u(t)$ be a non-negative solution of (3.3.9). For $t \in \mathcal{J}$, equation (3.3.9) reads as

$$u'' + h(u)u' + a(t)g(u) = 0$$

and therefore the auxiliary function Υ is non-decreasing on $\mathcal{J}$. If $u'(t^*) \geq 0$, for some $t^* \in [\tau',\sigma'[$, then $u'(t) \geq 0$ for all $t \in [t^*,\sigma']$, hence $u(t^*) \leq u(\sigma') < R$ (because σ' belongs to some interval I_i^+, where $u(t)$ is bounded by R). Similarly, if $u'(t^*) \leq 0$, for some $t^* \in \,]\tau',\sigma']$, then $u'(t) \leq 0$ for all $t \in [\tau',t^*]$, hence $u(t^*) \leq u(\tau') < R$ (because τ' belongs to some interval I_i^+). Thus, we easily deduce that $u(t) < R$ for all $t \in \mathrm{cl}(\mathcal{J}) = [\tau',\sigma']$. The same argument can be easily adapted if $\mathcal{J} = [0,\sigma'[$ or $\mathcal{J} = \,]\tau',T]$ (with, respectively, $\sigma' \in I_1^+$ or $\tau' \in I_m^+$), using the T-periodic boundary conditions.

In this manner, we have found a constant $R > r_0$ such that any non-negative solution $u(t)$ of (3.3.9), with $\alpha \geq 0$, satisfies

$$\|u\|_\infty < R.$$

This shows that the first part of (H^R) is valid independently of the choice of α_0.

Now we fix α_0 as in (3.2.11). It remains to verify that for $\alpha = \alpha_0$ there are no solutions $u(t)$ of (3.3.9) with $0 \leq u(t) \leq R$ on $[0, T]$. Indeed, if there were, integrating on $[0, T]$ the differential equation and using the boundary conditions, we obtain

$$\alpha \|v\|_{L^1} = \alpha \int_0^T v(t)\, dt \leq \int_0^T |a(t)| g(u(t))\, dt \leq \|a\|_{L^1} \max_{0 \leq s \leq R} g(s),$$

which leads to a contradiction with respect to the choice of α_0. Thus (H^R) is verified.

Having verified (H_r) and (H^R), the thesis follows from Theorem 3.1.1 with Remark 3.1.1. $\square$

Chapter 4

Neumann and periodic boundary conditions: multiplicity results

In this chapter we continue the study of the Neumann and periodic boundary value problems associated with indefinite equations, introduced in Chapter 3, dealing with multiplicity results for positive solutions. Unlike the previous chapter, now we prefer to focus our attention on the periodic problem, in order to simplify our discussion and to state our theorems in a form which is suitable for the subsequent application to the search of subharmonic solutions (see Chapter 5). We stress that our results apply also to Neumann boundary conditions.

Accordingly, we study the periodic boundary value problem associated with the second order nonlinear differential equation

$$(\mathscr{E}_\mu) \qquad u'' + cu' + \bigl(a^+(t) - \mu a^-(t)\bigr)g(u) = 0,$$

where $g(u)$ is a continuous function with superlinear growth at zero and at infinity, $a(t)$ is a T-periodic locally integrable sign-changing weight, $c \in \mathbb{R}$ and $\mu > 0$ is a real parameter. Notice that for $c \neq 0$ we lose the Hamiltonian structure if we pass to the natural equivalent system in the phase-plane

$$u' = y, \qquad y' = -cy - \bigl(a^+(t) - \mu a^-(t)\bigr)g(u).$$

As main multiplicity result of this chapter, we prove the existence of $2^m - 1$ positive solutions when $a(t)$ has m positive humps separated by m negative ones (in a periodicity interval) and μ is sufficiently large. Hence, we continue the investigation on the conjecture proposed by R. Gómez-Reñasco and J. López-Gómez (for the Dirichlet boundary value problem), and finally we give a complete solution to the problem raised by G. J. Butler in 1976, too.

The technique we employ in this chapter exploits and combines the approaches introduced in Chapter 1 and in Chapter 3. Particularly, as in Chapter 3, since the linear differential operator $u \mapsto -u'' - cu'$ is not invertible, our proofs are based on the extension of Mawhin's coincidence degree defined in open and possibly unbounded sets (cf. Appendix B). In this manner, our results are stable with respect to small perturbations and, for instance, we can also extend them to equations like

$$u'' + cu' + \varepsilon u + \bigl(a^{+}(t) - \mu a^{-}(t)\bigr)g(u) = 0,$$

with $|\varepsilon| < \varepsilon_0$, where ε_0 is a sufficiently small constant depending on μ.

The plan of the chapter is as follows. In Section 4.1 we list the hypotheses on $a(t)$ and on $g(s)$ that we assume for the rest of the chapter and we introduce an useful notation. Section 4.2 is devoted to the application of coincidence degree theory to our problem. In more detail, we define an equivalent operator problem and we present three technical lemmas essential for the computation of the degree in the proof of our main result (Theorem 4.3.1), which is stated and proved in Section 4.3. In Section 4.4 we present various consequences and applications of the main theorem, including also a nonexistence result (cf. Corollary 4.4.5). Even if we focus our main attention to the study of the periodic problem, in Section 4.5 we observe that variants of our main results can be given for the Neumann problem. Therein we also provide an application to radially symmetric solutions of PDEs on annular domains.

4.1 Setting and notation

In this section we present the main elements involved in the study of the positive T-periodic solutions of the equation $(\mathscr{E}_{\mu})$. For $\mu > 0$, we set

$$a_{\mu}(t) := a^{+}(t) - \mu\, a^{-}(t), \quad t \in [0,T].$$

The hypotheses that will follow will be assumed from now on in the chapter.

Let $g \colon \mathbb{R}^{+} \to \mathbb{R}^{+}$ be a continuous function such that

$$(g_{*}) \qquad g(0) = 0, \qquad g(s) > 0 \quad \text{for} \;\; s > 0.$$

Suppose also that

$$(g_{1}) \qquad g_0 := \limsup_{s \to 0^{+}} \frac{g(s)}{s} < +\infty \quad \text{and} \quad g_{\infty} := \liminf_{s \to +\infty} \frac{g(s)}{s} > 0.$$

The weight coefficient $a \colon \mathbb{R} \to \mathbb{R}$ is a locally integrable T-periodic function such that, in a time-interval of length T, there exists a finite number of closed pairwise disjoint intervals where $a(t) \succ 0$, separated by closed intervals where $a(t) \prec 0$. In this case, thanks to the periodicity of $a(t)$, we can suitably choose an interval $[t_0, t_0 + T]$, which we identify with $[0, T]$ for notational convenience, such that the following condition (a_*) holds.

(a_*) *There exist $m \geq 2$ closed and pairwise disjoint intervals $I_1^+, \dots, I_m^+$ separated by m closed intervals $I_1^-, \dots, I_m^-$ such that*

$$a(t) \succ 0 \quad \text{on } I_i^+, \qquad a(t) \prec 0 \quad \text{on } I_i^-,$$

and, moreover,

$$\bigcup_{i=1}^{m} I_i^+ \cup \bigcup_{i=1}^{m} I_i^- = [0, T].$$

To explain this fact with an example, suppose that we take $a(t) = \cos(2t)$ as a 2π-periodic function. In this case, on $[0, 2\pi]$ we have three positive humps and two negative ones. However, in order to enter the setting of (a_*) and hence to look at the weight as a function with two positive humps separated by two negative ones in a time-interval of length 2π, we can choose $[t_0, t_0 + 2\pi]$, for $t_0 = 3\pi/4$, as interval of periodicity. When, for convenience in the exposition, we say that we work with the standard period interval $[0, 2\pi]$, we are in fact considering a shift of t_0 of the weight function, e.g. taking $\cos(2t + 2t_0)$ as effective coefficient. Clearly, this does not affect our considerations as long as we are interested in 2π-periodic solutions. In the same example, let us fix an integer $k \geq 2$ and consider the coefficient $\cos(2t + 2t_0) = \sin(2t)$ as a $2k\pi$-periodic function. In the period interval $[0, 2k\pi]$ the weight has $m = 2k$ intervals of positivity separated by $2k$ intervals of negativity. We will consider again a similar example dealing with subharmonic solutions in Chapter 5.

In the sequel, it will be not restrictive to label the intervals I_i^+ and I_i^- following the natural order given by the standard orientation of the real line and thus determine $2m + 1$ points

$$0 = \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \ldots < \sigma_{m-1} < \tau_{m-1} < \sigma_m < \tau_m < \sigma_{m+1} = T,$$

so that, for $i = 1, \dots, m$,

$$I_i^+ := [\sigma_i, \tau_i] \quad \text{and} \quad I_i^- := [\tau_i, \sigma_{i+1}].$$

Finally, as in the previous chapters, consistently with assumption (a_*) and without loss of generality, we select the points σ_i and τ_i in such a manner that $a(t) \not\equiv 0$ on all left neighborhoods of σ_i (for $i > 1$) and on all right neighborhoods of τ_i. In other words, if there is an interval J contained in $[0, T]$ where $a(t) \equiv 0$, we choose the points σ_i and τ_i so that J is contained in one of the I_i^+ or J is contained in the interior of one of the I_i^-.

We denote by λ_1^i, $i = 1, \dots, m$, the first eigenvalue of the eigenvalue problem in I_i^+

$$\varphi'' + c\varphi' + \lambda a(t)\varphi = 0, \quad \varphi|_{\partial I_i^+} = 0.$$

From the assumptions on $a(t)$ in I_i^+, it clearly follows that $\lambda_1^i > 0$ for each $i = 1, \dots, m$.

We introduce some other useful notations. Let $\mathcal{I} \subseteq \{1, \dots, m\}$ be a subset of indices (possibly empty) and let d, D be two fixed positive real numbers with $d < D$. Similarly to Section 1.3, we define two families of open unbounded sets

$$\Omega^{\mathcal{I}}_{d,D} := \left\{ u \in \mathcal{C}([0,T]) \colon \max_{t \in I_i^+} |u(t)| < D, \ i \in \mathcal{I}; \right. \\ \left. \max_{t \in I_i^+} |u(t)| < d, \ i \in \{1, \dots, m\} \setminus \mathcal{I} \right\} \tag{4.1.1}$$

and

$$\Lambda^{\mathcal{I}}_{d,D} := \left\{ u \in \mathcal{C}([0,T]) \colon d < \max_{t \in I_i^+} |u(t)| < D, \ i \in \mathcal{I}; \right. \\ \left. \max_{t \in I_i^+} |u(t)| < d, \ i \in \{1, \dots, m\} \setminus \mathcal{I} \right\}. \tag{4.1.2}$$

In the sequel, once the constants d and D are fixed, we simply use the symbols $\Omega^{\mathcal{I}}$ and $\Lambda^{\mathcal{I}}$ to denote $\Omega^{\mathcal{I}}_{d,D}$ and $\Lambda^{\mathcal{I}}_{d,D}$, respectively. See Figure 1.1 for the representations of the sets $\Omega^{\mathcal{I}}_{d,D}$ and $\Lambda^{\mathcal{I}}_{d,D}$ when $m = 2$.

4.2 The abstract setting of the coincidence degree

In this section we apply the coincidence degree theory (see Appendix B) to study the periodic problem associated with equation $(\mathscr{E}_\mu)$. We follow the same approach presented in [130].

Accordingly, let $X := \mathcal{C}([0,T])$ be the Banach space of continuous functions $u \colon [0,T] \to \mathbb{R}$, endowed with the sup-norm

$$\|u\|_\infty := \max_{t \in [0,T]} |u(t)|,$$

and let $Z := L^1([0,T])$ be the space of integrable functions $w \colon [0,T] \to \mathbb{R}$, endowed with the norm

$$\|w\|_{L^1} := \int_0^T |w(t)| \, dt.$$

We consider the linear differential operator $L \colon \operatorname{dom} L \to Z$ defined as

$$(Lu)(t) := -u''(t) - cu'(t), \quad t \in [0,T],$$

where $\operatorname{dom} L$ is determined by the functions of X which are continuously differentiable with absolutely continuous derivative and satisfying the periodic boundary condition

$$u(0) = u(T), \qquad u'(0) = u'(T). \tag{4.2.1}$$

Therefore, L is a Fredholm map of index zero, $\ker L$ and $\operatorname{coker} L$ are made up of the constant functions and

$$\operatorname{Im} L = \left\{ w \in Z \colon \int_0^T w(t)\,dt = 0 \right\}.$$

As projectors $P\colon X \to \ker L$ and $Q\colon Z \to \operatorname{coker} L$ associated with L we choose the average operators

$$Pu = Qu := \frac{1}{T}\int_0^T u(t)\,dt.$$

Notice that $\ker P$ is given by the continuous functions with mean value zero. Finally, let $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ be the right inverse of L, which is the operator that to any function $w \in Z$ with $\int_0^T w(t)\,dt = 0$ associates the unique solution u of

$$u'' + cu' + w(t) = 0, \qquad \text{with} \quad \int_0^T u(t)\,dt = 0,$$

and satisfying the boundary condition (4.2.1).

Thereafter, on $\mathbb{R}^2$ we define the L^1-Carathéodory function

$$\tilde{f}(t,s) := \begin{cases} a_\mu(t)g(s), & \text{if } s \geq 0; \\ -s, & \text{if } s \leq 0; \end{cases}$$

and observe that $\tilde{f}(t+T,s) = \tilde{f}(t,s)$ for a.e. $t \in \mathbb{R}$ and for all $s \in \mathbb{R}$. Let $N\colon X \to Z$ be the Nemytskii operator induced by $\tilde{f}$, that is

$$(Nu)(t) := \tilde{f}(t,u(t)), \quad t \in [0,T].$$

According to the above positions, if u is a T-periodic solution of

$$u'' + cu' + \tilde{f}(t,u) = 0, \tag{4.2.2}$$

then $u|_{[0,T]}$ is a solution of the coincidence equation

$$Lu = Nu, \quad u \in \operatorname{dom} L. \tag{4.2.3}$$

Conversely, any solution u of (4.2.3) can be extended by T-periodicity to a T-periodic solution of (4.2.2). Moreover, from the definition of $\tilde{f}$ and conditions (g_*) and (g_1), one can easily verify by a maximum principle argument (cf. Lemma C.1.2) that if $u \not\equiv 0$ is a solution of (4.2.3), then $u(t)$ is strictly positive and hence a positive T-periodic solution of $(\mathscr{E}_\mu)$ (once extended by T-periodicity to the whole real line).

As remarked in Appendix B, the operator equation (4.2.3) is equivalent to the fixed point problem

$$u = \Phi u := Pu + QNu + K_P(Id - Q)Nu, \quad u \in X,$$

where we have chosen the identity on $\mathbb{R}$ as linear orientation-preserving isomorphism J from $\operatorname{coker} L$ to $\ker L$ (both identified with $\mathbb{R}$).

Now we are interested in computing the coincidence degree of L and N in some open domains. For this purpose, we will consider some modifications of (4.2.2) which correspond to operator equations of the form (4.2.3) for the associated Nemytskii operators N. In the sequel we will also identify the T-periodic solutions with solutions defined on $[0,T]$ and satisfying the boundary condition (4.2.1). We also denote by L^1_T the space of locally integrable and T-periodic functions $w\colon \mathbb{R} \to \mathbb{R}$ (which can be identified with Z).

The subsequent two lemmas give conditions for the computation of the degree on some open balls and are direct applications of Lemma B.2.1 and Lemma B.2.3, respectively. We refer to the analogous result Theorem 3.1.1 for the standard proofs.

Lemma 4.2.1. *Let $\mu > 0$ be such that $\int_0^T a_\mu(t)\,dt < 0$. Assume that there exists a constant $d > 0$ such that the following property holds.*

(H_d) *If $\vartheta \in \,]0,1]$ and $u(t)$ is any non-negative T-periodic solution of*

$$u'' + cu' + \vartheta a_\mu(t) g(u) = 0, \tag{4.2.4}$$

then $\max_{t\in[0,T]} u(t) \neq d$.

Then

$$D_L(L-N, B(0,d)) = 1.$$

Lemma 4.2.2. *Assume that there exists a constant $D > 0$ such that the following property holds.*

(H^D) *There exist a non-negative function $v \in L^1_T$ with $v \not\equiv 0$ and a constant $\alpha_0 > 0$, such that every T-periodic solution $u(t) \geq 0$ of the boundary value problem*

$$u'' + cu' + a_\mu(t) g(u) + \alpha v(t) = 0, \tag{4.2.5}$$

for $\alpha \in [0,\alpha_0]$, satisfies $\|u\|_\infty \neq D$. Moreover, there are no solutions $u(t)$ of (4.2.5) for $\alpha = \alpha_0$ with $0 \leq u(t) \leq D$, for all $t \in \mathbb{R}$.

Then

$$D_L(L-N, B(0,D)) = 0.$$

In order to achieve our multiplicity result, in Section 4.3 we will fix $d, D > 0$ satisfying (H_d) and (H^D), respectively, and compute the coincidence degree in the open and unbounded sets $\Lambda^{\mathcal{I}}_{d,D}$, for $\mathcal{I} \subseteq \{1,\ldots,m\}$. To this aim the following lemma is of utmost importance (see Lemma 6.2.1 for a similar statement). In the next result we consider again equation (4.2.5) of the previous lemma.

Lemma 4.2.3. *Let $\mathcal{J} \subseteq \{1,\dots,m\}$ be a nonempty subset of indices, let $d > 0$ be a constant and $v \in L^1_T$ a non-negative nontrivial function, such that the following properties hold.*

$(A_{d,\mathcal{J}})$ *If $\alpha \geq 0$, then any non-negative T-periodic solution $u(t)$ of* (4.2.5) *satisfies $\max_{t\in I_j^+} u(t) \neq d$, for all $j \in \mathcal{J}$.*

$(B_{d,\mathcal{J}})$ *For every $\beta \geq 0$ there exists a constant $D_\beta > d$ such that if $\alpha \in [0,\beta]$ and $u(t)$ is any non-negative T-periodic solution of* (4.2.5) *with $\max_{t\in I_j^+} u(t) \leq d$, for all $j \in \mathcal{J}$, then $\max_{t\in[0,T]} u(t) \leq D_\beta$.*

$(C_{d,\mathcal{J}})$ *There exists $\alpha_0 > 0$ such that equation* (4.2.5)*, with $\alpha = \alpha_0$, does not have any non-negative T-periodic solution $u(t)$ with $\max_{t\in I_j^+} u(t) \leq d$, for all $j \in \mathcal{J}$.*

Then

$$D_L(L-N,\Gamma_{d,\mathcal{J}}) = 0,$$

where

$$\Gamma_{d,\mathcal{J}} := \left\{ u \in \mathcal{C}([0,T]) \colon \max_{t\in I_j^+} |u(t)| < d,\ j \in \mathcal{J} \right\}. \tag{4.2.6}$$

Proof. According to the setting presented in the present section, conditions $(A_{d,\mathcal{J}})$, $(B_{d,\mathcal{J}})$ and $(C_{d,\mathcal{J}})$ are equivalent to conditions (i), (ii) and (iii) of Theorem B.2.1 with respect to the open set $\Omega := \Gamma_{d,\mathcal{J}}$. Therefore, the thesis of Lemma 4.2.3 follows. □

Remark 4.2.1. From a theoretical point of view, the choice of the set of indices $\mathcal{J}$ with $\emptyset \neq \mathcal{J} \subseteq \{1,\dots,m\}$ is arbitrary. However, as we will see in the next section, in the actual applications of Lemma 4.2.3 we shall take $\mathcal{J} \subsetneq \{1,\dots,m\}$ because, in our setting, the case $\mathcal{J} = \{1,\dots,m\}$ will be discussed in the frame of Lemma 4.2.1. ◁

4.3 The main multiplicity result

In this section we use all the tools just presented in the previous sections to prove the following main result.

Theorem 4.3.1. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \text{and} \quad g_\infty > \max_{i=1,\dots,m} \lambda_1^i.$$

Then there exists $\mu^* > 0$ such that for all $\mu > \mu^*$ equation $(\mathscr{E}_\mu)$ has at least $2^m - 1$ positive T-periodic solutions.

Remark 4.3.1. The $2^m - 1$ positive T-periodic solutions are obtained as follows. Along the proof we provide two constants $0 < r < R$ (with r small and R large) such that if $\mu > \mu^*$, given any nonempty set of indices $\mathcal{I} \subseteq \{1, \ldots, m\}$, there exists at least one positive T-periodic solution $u_{\mathcal{I}} \in \Lambda^{\mathcal{I}}_{r,R}$ of $(\mathscr{E}_\mu)$. Namely, $u_{\mathcal{I}}(t)$ is small for all $t \in I^+_i$ when $i \notin \mathcal{I}$, and, on the other hand, $r < u_{\mathcal{I}}(t) < R$ for some $t \in I^+_i$ when $i \in \mathcal{I}$. We will also prove that, when μ is sufficiently large, all these solutions are small in the I^-_i intervals (see Section 4.3.5). Compare to the numerical example presented in Figure 4.1. $\lhd$

Remark 4.3.2. The assumption $g_0 = 0$ in Theorem 4.3.1 can be slightly improved to a condition of the form

$$g_0 < \lambda_*,$$

where λ_* is a positive constant which satisfies

$$0 < \lambda_* < \min_{i=1,\ldots,m} \lambda^i_1.$$

A lower bound for λ_* (although not sharp) is explicitly given by the constant $1/K_0$ provided in (4.3.15) in Section 4.3.2 (see also Remark 4.3.5 for more details). When $c = 0$ it is easy to check that $1/K_0$ is strictly less than $4/\bigl(|I^+_i| \int_{I^+_i} a^+(t)\,dt\bigr)$ (for all $i = 1, \ldots, m$), which are the constants corresponding to the application of Lyapunov inequality to each of the intervals of positivity (cf. [105, ch. XI]). $\lhd$

4.3.1 General strategy and proof of Theorem 4.3.1

In this section, we describe the main steps that define the proof of Theorem 4.3.1. The details can be found in the three following sections.

First of all, in Section 4.3.2, from $g_0 = 0$ we fix a (small) constant $r > 0$ such that

$$\eta(r) := \sup_{0<s\leq r} \frac{g(s)}{s} \tag{4.3.1}$$

is sufficiently small (cf. condition (4.3.15)). For this fixed r, we determine a constant μ_r, with

$$\mu_r > \mu^{\#} := \frac{\int_0^T a^+(t)\,dt}{\int_0^T a^-(t)\,dt}, \tag{4.3.2}$$

such that condition (H_r) of Lemma 4.2.1 is satisfied *for every* $\mu \geq \mu_r$ and therefore

$$D_L(L - N, B(0,r)) = 1. \tag{4.3.3}$$

It is important to notice that, for the validity of (4.3.3), it is necessary to take $\mu > \mu^{\#}$ in order to have $\int_0^T a_\mu(t)\,dt < 0$.

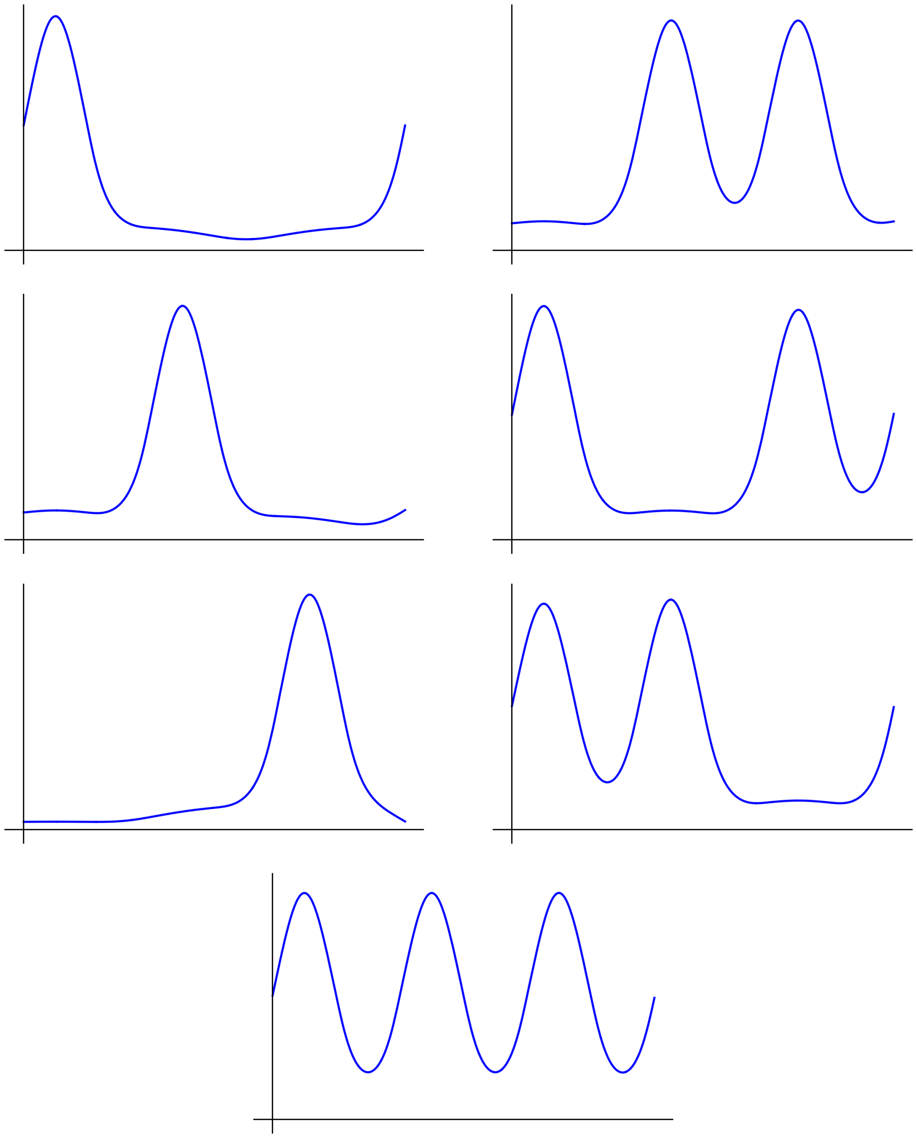

Figure 4.1: The figure shows an example of multiple positive solutions for the T-periodic boundary value problem associated with $(\mathscr{E}_{\mu})$. For this numerical simulation we have chosen $I = [0,1]$, $c = 0$, $a(t) = \sin(6\pi t)$, $\mu = 20$ and $g(s) = \max\{0, 400\, s \arctan|s|\}$. Notice that the weight function $a(t)$ has 3 positive humps. We show the graphs of the 7 positive T-periodic solutions of $(\mathscr{E}_{\mu})$.

As a second step, in Section 4.3.3, we show that there exists a constant R^*, with $0 < r < R^*$, such that, for any nontrivial function $v \in L^1([0,T])$ satisfying

$$v(t) \geq 0 \quad \text{on } \bigcup_{i=1}^{m} I_i^+, \qquad v(t) = 0 \quad \text{on } \bigcup_{i=1}^{m} I_i^-, \tag{4.3.4}$$

and for all $\alpha \geq 0$, it holds that any non-negative solution $u(t)$ of (4.2.5) is bounded by R^*, namely

$$\max_{t\in[0,T]} u(t) < R^*. \tag{4.3.5}$$

This result is proved using the lower bound of g_∞ and the constant R^* can be chosen independently on the functions $v(t)$ satisfying (4.3.4).

In this manner (for $\alpha = 0$) we obtain also a priori bound for all non-negative T-periodic solutions of $(\mathscr{E}_\mu)$. Then, we verify that condition (H^R) of Lemma 4.2.2 is satisfied for all $R \geq R^*$. Hence, we have

$$D_L(L - N, B(0, R)) = 0, \quad \forall\, R \geq R^*. \tag{4.3.6}$$

It is important to notice that, in order to prove (4.3.5) and consequently (4.3.6), we only use information about $a^+(t)$. Hence R^* can be chosen independently on $\mu > 0$.

Remark 4.3.3. Using the additivity property of the coincidence degree, from (4.3.3) and (4.3.6), we reach the following equality

$$D_L(L - N, B(0, R^*) \setminus B[0, r]) = -1.$$

Then, we obtain the existence of at least a nontrivial solution u of (4.2.3), provided that $\mu > \mu_r$. Using a standard maximum principle argument, it is easy to prove that u is a positive T-periodic solution of $(\mathscr{E}_\mu)$ (cf. Theorem 3.2.1, Theorem 3.2.2 and also Remark 4.3.6). $\lhd$

At this point, we fix a constant R with

$$0 < r < R^* \leq R$$

and, for all sets of indices $\mathcal{I} \subseteq \{1, \ldots, m\}$, we consider the open and unbounded sets

$$\Omega^{\mathcal{I}} := \Omega^{\mathcal{I}}_{r,R} \quad \text{and} \quad \Lambda^{\mathcal{I}} := \Lambda^{\mathcal{I}}_{r,R}$$

introduced in (4.1.1) and in (4.1.2), respectively.

As a third step, we will prove that

$$D_L(L - N, \Lambda^{\mathcal{I}}) \neq 0, \quad \text{for all } \mathcal{I} \subseteq \{1, \ldots, m\}. \tag{4.3.7}$$

Before the proof of (4.3.7), we make the following observation which plays a crucial role in various subsequent steps.

Remark 4.3.4. Writing equation $(\mathscr{E}_\mu)$ as

$$\left(e^{ct}u'\right)' + e^{ct}a_\mu(t)g(u) = 0,$$

we find that $(e^{ct}u'(t))' \leq 0$ for almost every $t \in I^+_i$ and $(e^{ct}u'(t))' \geq 0$ for almost every $t \in I^-_i$ (where $u(t) \geq 0$ is any solution). Then, the map

$$t \mapsto e^{ct}u'(t)$$

is non-increasing on each I_i^+ and non-decreasing on each I_i^-. This property replaces the convexity of $u(t)$ on I_i^-, which is an obvious fact when $c = 0$. For an arbitrary $c \in \mathbb{R}$ we can still preserve some convexity type properties. In particular, for all $i = 1, \ldots, m$, we have that

$$\max_{t \in I_i^-} u(t) = \max_{t \in \partial I_i^-} u(t) = \max\bigl\{u(\tau_i), u(\sigma_{i+1})\bigr\}, \tag{4.3.8}$$

which is nothing but a one-dimensional form of a maximum principle for the differential operator L. We verify now this fact since this property, although elementary, will be used several times in the sequel. Indeed, observe that if $u'(t^*) \geq 0$, for some $t^* \in [\tau_i, \sigma_{i+1}[$, then $u'(t) \geq 0$ for all $t \in [t^*, \sigma_{i+1}]$, hence $u(t^*) \leq u(\sigma_{i+1})$. Similarly, if $u'(t^*) \leq 0$, for some $t^* \in]\tau_i, \sigma_{i+1}]$, then $u'(t) \leq 0$ for all $t \in [\tau_i, t^*]$, hence $u(t^*) \leq u(\tau_i)$. From these remarks, (4.3.8) follows immediately. $\lhd$

In order to prove (4.3.7), first of all we consider $\mathcal{I} = \emptyset$. Accordingly, we have that

$$D_L(L - N, \Omega^{\emptyset}) = D_L(L - N, \Lambda^{\emptyset}) = D_L(L - N, B(0, r)) = 1. \tag{4.3.9}$$

The first identity in (4.3.9) is trivial from the definitions of the sets, since $\Omega^{\emptyset} = \Lambda^{\emptyset}$. It is also obvious that $B(0, r) \subseteq \Omega^{\emptyset}$. Conversely, let u be a T-periodic solution of (4.2.2) belonging to $\Omega^{\emptyset}$. By the maximum principle, we know that u is a (non-negative) T-periodic solution of $(\mathscr{E}_\mu)$. Moreover, $u(t) < r$ for all $t \in I_i^+$, $i = 1, \ldots, m$. Then, from (4.3.8) we have that $u(t) < r$ for all $t \in [0, T]$. (In the application of formula (4.3.8) we have considered the interval I_m^-, as an interval between I_m^+ and $I_1^+ + T$, by virtue of the T-periodicity of the solution.) Finally, by the excision property of the coincidence degree and (4.3.3), formula (4.3.9) follows.

Next, we consider a nonempty subset of indices $\mathcal{I} \subsetneq \{1, \ldots, m\}$. In Section 4.3.4, choosing $d = r$, $\mathcal{J} := \{1, \ldots, m\} \setminus \mathcal{I} \neq \emptyset$ and a nontrivial function $v \in L^1([0, T])$ such that

$$v(t) \succ 0 \quad \text{on } \bigcup_{i \in \mathcal{I}} I_i^+, \qquad v(t) = 0 \quad \text{otherwise}, \tag{4.3.10}$$

we verify that the three conditions of Lemma 4.2.3 hold, for μ sufficiently large. In more detail, we provide a lower bound $\mu^*_{\mathcal{I}} > 0$, with $\mu^*_{\mathcal{I}}$ independent on α, such that condition $(A_{r,\mathcal{J}})$ is satisfied for all $\mu > \mu^*_{\mathcal{I}}$. Then, we fix an arbitrary $\mu > \mu^*_{\mathcal{I}}$ and show that conditions $(B_{r,\mathcal{J}})$ and $(C_{r,\mathcal{J}})$ are satisfied as well.

Since R is an upper bound for all the solutions of (4.2.5) (cf. (4.3.5)), comparing the definitions (4.1.1) and (4.2.6), we see that $u \in \Omega^{\mathcal{I}}$ if and only if $u \in \Gamma_{r,\mathcal{J}}$, for each solution u. Hence, applying the excision property of the coincidence degree and Lemma 4.2.3, we obtain

$$D_L(L - N, \Omega^{\mathcal{I}}) = D_L(L - N, \Gamma_{r,\mathcal{J}}) = 0, \qquad \text{for all } \emptyset \neq \mathcal{I} \subsetneq \{1, \ldots, m\}. \tag{4.3.11}$$

Using again (4.3.8) in Remark 4.3.4 and arguing as above for r, we can check that R is an a priori bound for the solutions on the whole domain. In this manner, by (4.3.6), if $\mathcal{I} = \{1, \ldots, m\}$ we obtain

$$D_L(L - N, \Omega^{\mathcal{I}}) = D_L(L - N, B(0, R)) = 0.$$

In conclusion, putting together this latter relation with (4.3.11), we find that

$$D_L(L - N, \Omega^{\mathcal{I}}) = 0, \quad \text{for all } \emptyset \neq \mathcal{I} \subseteq \{1, \ldots, m\}. \tag{4.3.12}$$

Finally, we define

$$\mu^* := \mu_r \vee \max\bigl\{\mu^*_{\mathcal{I}} \colon \emptyset \neq \mathcal{I} \subseteq \{1, \ldots, m\}\bigr\},$$

where, as usual, "$\vee$" denotes the maximum between two numbers. As a byproduct of the proof of $(A_{\mathcal{J},r})$ in Section 4.3.4 (for $\alpha = 0$) we also have that for each $\mu > \mu^*$ the degree $D_L(L - N, \Lambda^{\mathcal{I}})$ is well-defined for all $\mathcal{I} \subseteq \{1, \ldots, m\}$ (technically, the matter is to observe that for μ sufficiently large the are no T-periodic solutions touching the level r on some intervals I^+_i).

At this point, following the same inductive argument as in proving Lemma 1.3.1, we easily obtain the following result.

Lemma 4.3.1. *Let $\mathcal{I} \subseteq \{1, \ldots, n\}$ be a set of indices. Suppose that for all $\mathcal{J} \subseteq \mathcal{I}$ the coincidence degree is defined on the sets $\Lambda^{\mathcal{J}}$ and $\Omega^{\mathcal{J}}$, with*

$$D_L(L - N, \Omega^{\emptyset}) = D_L(L - N, \Lambda^{\emptyset}) = 1$$

and

$$D_L(L - N, \Omega^{\mathcal{J}}) = 0, \quad \forall\, \emptyset \neq \mathcal{J} \subseteq \mathcal{I}.$$

Then

$$D_L(L - N, \Lambda^{\mathcal{I}}) = (-1)^{\#\mathcal{I}}.$$

Therefore, from (4.3.9) and (4.3.12), we have that

$$D_L(L - N, \Lambda^{\mathcal{I}}) = (-1)^{\#\mathcal{I}}, \quad \text{for all } \mathcal{I} \subseteq \{1, \ldots, m\},$$

holds for each $\mu > \mu^*$. In this manner, (4.3.7) is verified.

In conclusion, since the coincidence degree is non-zero in each $\Lambda^{\mathcal{I}}$, there exists a solution $u \in \Lambda^{\mathcal{I}}$ of (4.2.3), for all $\mathcal{I} \subseteq \{1, \ldots, m\}$. Notice that $0 \notin \Lambda^{\mathcal{I}}$ for all $\emptyset \neq \mathcal{I} \subseteq \{1, \ldots, m\}$. As remarked in Section 4.2, by a maximum principle argument, for $\mathcal{I} \neq \emptyset$ the solution $u \in \Lambda^{\mathcal{I}}$ of (4.2.3) is a positive T-periodic solution of $(\mathscr{E}_\mu)$. Moreover, by (4.3.8), we also deduce that $\|u\|_\infty < R$. At this moment, we can summarize what we have proved as follows.

For each nonempty set of indices $\mathcal{I} \subseteq \{1,\dots,m\}$, there exists at least a T-periodic solution $u_{\mathcal{I}}$ of $(\mathscr{E}_\mu)$ with $u_{\mathcal{I}} \in \Lambda^{\mathcal{I}}$ and such that $0 < u_{\mathcal{I}}(t) < R$ for all $t \in \mathbb{R}$.

Finally, since the number of nonempty subsets of a set with m elements is $2^m - 1$ and the sets $\Lambda^{\mathcal{I}}$ are pairwise disjoint, we conclude that there are at least 2^m-1 positive T-periodic solutions of $(\mathscr{E}_\mu)$. The thesis of Theorem 4.3.1 follows. $\square$

Having already outlined the scheme of the proof, we provide now all the missing technical details.

4.3.2 Proof of (H_r) for r small

In this section we find a sufficiently small real number $r > 0$ such that (H_r) is satisfied for all μ large enough.

Let us start by introducing some constants that are crucial for our next estimates. Define

$$K_i := \|a^+\|_{L^1(I_i^+)} e^{|c||I_i^+|}, \quad i = 1\dots,m, \tag{4.3.13}$$

and

$$K_0 := 2 \max_{i=1,\dots,m} K_i\bigl(|I_i^+| + e^{|c||I_i^-|}|I_i^-|\bigr). \tag{4.3.14}$$

By (g_*) and $g_0 = 0$, we know that $\eta(s) \to 0^+$ as $s \to 0^+$ (where $\eta(s)$ is defined in (4.3.1)). So, we fix $r > 0$ such that

$$\eta(r) < \frac{1}{K_0}. \tag{4.3.15}$$

Then, we fix a positive constant $\mu_r > \mu^{\#}$ such that

$$\mu_r > \frac{K_0\, e^{|c||I_i^-|}}{\int_{\tau_i}^{\sigma_{i+1}} \int_{\tau_i}^{t} a^-(\xi)\, d\xi\, dt} \frac{\eta(r)}{\gamma(r)}, \qquad \text{for all } i = 1,\dots,m, \tag{4.3.16}$$

where we have set

$$\gamma(r) := \min_{\frac{r}{2} \leq s \leq r} \frac{g(s)}{s}.$$

We verify that condition (H_d) of Lemma 4.2.1 is satisfied for $d = r$, chosen as in (4.3.15), and for all $\mu \geq \mu_r$. Accordingly, we claim that there is no non-negative solution $u(t)$ of (4.2.4), for some $\vartheta \in \,]0,1]$ and $\mu \geq \mu_r$, with $\|u\|_\infty = r$.

Arguing by contradiction, let us suppose that, for some ϑ and μ with $0 < \vartheta \leq 1$ and $\mu \geq \mu_r$, there exists a T-periodic solution $u(t)$ of

$$u''(t) + cu'(t) + \vartheta a_\mu(t) g(u(t)) = 0, \tag{4.3.17}$$

with $0 \leq u(t) \leq \max_{t\in[0,T]} u(t) = r$. Reasoning as in Remark 4.3.4, we observe that the solution $u(t)$ in the interval of non-positivity attains its maximum at an endpoint. Thus, there is an index $j \in \{1,\dots,m\}$ such that

$$r = \max_{t\in[0,T]} u(t) = \max_{t\in I_j^+} u(t) = u(\hat{t}_j), \qquad \text{for some } \hat{t}_j \in I_j^+ = [\sigma_j, \tau_j].$$

Next, we notice that $u'(\hat{t}_j) = 0$. Indeed, if $u'(t) \neq 0$ for all $t \in I_j^+$ such that $u(t) = r$, then $t = \sigma_j$ or $t = \tau_j$. If $t = \tau_j$, then $u'(\tau_j) > 0$ and, since the map $t \mapsto e^{ct}u'(t)$ is non-decreasing on I_j^- (cf. Remark 4.3.4), we have $u'(t) \geq u'(\tau_j)e^{c(\tau_j - t)} > 0$, for all $t \in I_j^-$. Then we obtain $r = u(\tau_j) < u(\sigma_{j+1})$, a contradiction with respect to $\|u\|_\infty = r$. If $t = \sigma_j$, one can obtain an analogous contradiction considering the interval I_{j-1}^- (if $j = 1$, we deal with $I_m^- - T$, by T-periodicity).

Writing (4.3.17) on I_j^+ as

$$\bigl(e^{ct}u'(t)\bigr)' = -\vartheta a^+(t) g(u(t)) e^{ct},$$

integrating between $\hat{t}_j$ and t and using $u'(\hat{t}_j) = 0$, we obtain

$$u'(t) = -\vartheta \int_{\hat{t}_j}^{t} a^+(\xi) g(u(\xi)) e^{c(\xi - t)}\, d\xi, \quad \forall\, t \in I_j^+.$$

Hence,

$$\|u'\|_{L^\infty(I_j^+)} \leq \vartheta \|a^+\|_{L^1(I_j^+)} \eta(r) r e^{|c||I_j^+|} = \vartheta K_j \eta(r) r. \tag{4.3.18}$$

We conclude that

$$r \geq u(\tau_j) = u(\hat{t}_j) + \int_{\hat{t}_j}^{\tau_j} u'(\xi)\, d\xi \geq r\bigl(1 - \vartheta K_j \eta(r) |I_j^+|\bigr). \tag{4.3.19}$$

Now we consider the subsequent (adjacent) interval $I_j^- = [\tau_j, \sigma_{j+1}]$ where the weight is non-positive. Since (as just remarked) the map $t \mapsto e^{ct}u'(t)$ is non-decreasing on I_j^-, we have $e^{ct}u'(t) \geq e^{c\tau_j}u'(\tau_j)$, for all $t \in I_j^-$. Therefore, recalling also (4.3.18), we get

$$u'(t) \geq e^{c(\tau_j - t)} u'(\tau_j) \geq -e^{|c||I_j^-|} \vartheta K_j \eta(r) r, \quad \forall\, t \in I_j^-.$$

Integrating on $[\tau_j, t] \subseteq I_j^-$ and using (4.3.19), we have that

$$\begin{aligned} r \geq u(t) &\geq u(\tau_j) - |I_j^-| e^{|c||I_j^-|} \vartheta K_j \eta(r) r \\ &\geq r\Bigl(1 - \vartheta K_j \bigl(|I_j^+| + |I_j^-| e^{|c||I_j^-|}\bigr)\eta(r)\Bigr) \\ &\geq r\left(1 - \vartheta \frac{K_0}{2}\eta(r)\right) \geq r\left(1 - \frac{\vartheta}{2}\right) \geq \frac{r}{2} \end{aligned} \tag{4.3.20}$$

holds for all $t \in I_j^-$. Writing (4.3.17) on I_j^- as

$$\big(e^{ct}u'(t)\big)' = \mu\vartheta a^-(t)g(u(t))e^{ct}$$

and integrating on $[\tau_j, t] \subseteq I_j^-$, we have

$$u'(t) = e^{c(\tau_j - t)}u'(\tau_j) + \mu\vartheta \int_{\tau_j}^{t} a^-(\xi)g(u(\xi))e^{c(\xi - t)}\,d\xi, \quad \forall\, t \in I_j^-.$$

Then, using (4.3.18) and recalling the definition of $\gamma(r)$, we find

$$u'(t) \geq \vartheta r\bigg(-e^{|c||I_j^-|}K_j\eta(r) + \frac{1}{2}\mu\gamma(r)e^{-|c||I_j^-|}\int_{\tau_j}^{t} a^-(\xi)\,d\xi\bigg), \quad \forall\, t \in I_j^-.$$

Finally, integrating on I_j^-, we obtain

$$\begin{aligned} u(\sigma_{j+1}) &= u(\tau_j) + \int_{\tau_j}^{\sigma_{j+1}} u'(\xi)\,d\xi \\ &\geq r\bigg(1 - \vartheta K_j\eta(r)|I_j^+| - \vartheta|I_j^-|e^{|c||I_j^-|}K_j\eta(r) \\ &\qquad + \mu\frac{\vartheta}{2}\gamma(r)e^{-|c||I_j^-|}\int_{\tau_j}^{\sigma_{j+1}}\int_{\tau_j}^{t} a^-(\xi)\,d\xi\,dt\bigg) \\ &> r, \end{aligned}$$

a contradiction with respect to the choice of $\mu \geq \mu_r$ (cf. (4.3.16)). □

Remark 4.3.5. From the proof it is clear that we do not really need that $g_0 = 0$, but we only use the fact that $r > 0$ can be chosen so that (4.3.15) is satisfied. Accordingly, our result is still valid if we assume that g_0 is sufficiently small. Clearly, some smallness condition on g_0 has to be required for the validity of our estimates. Indeed, the same argument of the proof (if applied to $g(s) = \lambda s$) shows that $1/K_0$ must be strictly less to all the first eigenvalues λ_1^i as well as to all the first eigenvalues of the Dirichlet-Neumann problems (or focal point problems) relative to the intervals I_i^+. As a consequence, we could slightly improve condition $g_0 = 0$ of Theorem 4.3.1 to an assumption of the form $g_0 < \lambda_*$, where the optimal choice for λ_* would be that of a suitable positive constant satisfying $\lambda_* < \min_i \lambda_1^i$ (as well as other similar conditions). The constant $1/K_0$ found in the proof could be improved by choosing a smaller value for K_0 in (4.3.14). Indeed, note that the factor 2 in (4.3.14) corresponds to the lower bound $u(t) \geq r/2$ in (4.3.20). We do not investigate further this aspect as it is not prominent for our results. Technical estimates related to Lyapunov type inequalities and lower bounds for the first eigenvalue of Dirichlet-Neumann problems with weights have been studied, for instance, in [45, 74, 156]. ◁

Remark 4.3.6. We stress that in the above proof we have used only the continuity of $g(s)$ (near $s = 0$), condition (g_*) and the hypothesis $g_0 = 0$. In Chapter 3, we have obtained the existence of at least a T-periodic solution of equation $(\mathscr{E}_\mu)$, proving the existence of an $r > 0$ small such that (H_d) holds for all $0 < d \leq r$, provided that the mean value of the weight is negative. In our case, such condition on the weight is equivalent to $\mu > \mu^\#$, which is a better condition than $\mu \geq \mu_r$ given here in our proof. However, in order to use a weaker assumption on the weight, in Chapter 3 we have to require a stronger hypothesis on the nonlinearity $g(s)$ near zero. In particular, we have to suppose that $g(s)$ is regularly oscillating at zero (cf. Theorem 3.2.1) or that $g(s)$ is continuously differentiable on a right neighborhood of $s = 0$ (cf. Theorem 3.2.2).

With regard to this topic, we observe that even if we have proved the existence of a sufficiently small r such that (H_r) holds, we can also verify that (H_d) holds *for all* $0 < d \leq r$ under supplementary assumptions on $g(s)$ near zero. For instance, this claim can be proved if we suppose that $g(s)$ satisfies

$$\liminf_{s \to 0^+} \frac{g(\sigma s)}{g(s)} > 0, \qquad \text{for all } \sigma > 1,$$

(cf. (4.3.16)). The above hypothesis is also called a *lower σ-condition at zero* and it is dual with respect to the more classical *Δ_2-condition at infinity* considered in the theory of Orlicz-Sobolev spaces (cf. [2, ch. VIII]). We refer to [5, 70] for a discussion about these ones and related growth assumptions at infinity, as well as for a comparison between different Karamata type conditions. $\triangleleft$

4.3.3 The a priori bound R^*

Consider an arbitrary function $v \in L^1([0,T])$ as in (4.3.4). For example, as $v(t)$ we can take the indicator function of the set

$$A := \bigcup_{i=1}^{m} I_i^+.$$

We will show that there exists $R^* > 0$ such that, for each $\alpha \geq 0$, every non-negative T-periodic solution $u(t)$ of (4.2.5) satisfies $\max_{t \in A} u(t) < R^*$.

First of all, for each $i = 1, \ldots, m$, we look for a bound $R_i > 0$ such that any non-negative T-periodic solution $u(t)$ of (4.2.5), with $\alpha \geq 0$, satisfies

$$\max_{t \in I_i^+} u(t) < R_i. \tag{4.3.21}$$

Let us fix $i \in \{1, \ldots, m\}$. Let $0 < \varepsilon < (\tau_i - \sigma_i)/2$ be fixed such that

$$a(t) \not\equiv 0 \quad \text{on } [\sigma_i + \varepsilon, \tau_i - \varepsilon] \subseteq I_i^+$$

and the first (positive) eigenvalue $\hat{\lambda}$ of the eigenvalue problem

$$\begin{cases} \varphi'' + c\varphi' + \lambda\, a(t)\varphi = 0 \\ \varphi(\sigma_i + \varepsilon) = \varphi(\tau_i - \varepsilon) = 0 \end{cases}$$

satisfies

$$g_\infty > \hat{\lambda}.$$

For notational convenience, we set $\sigma_\varepsilon := \sigma_i + \varepsilon$ and $\tau_\varepsilon := \tau_i - \varepsilon$. The existence of ε is ensured by the continuity of the first eigenvalue as a function of the boundary points and by hypothesis $g_\infty > \lambda_1^i$.

We fix a constant $M > 0$ such that

$$g_\infty > M > \hat{\lambda}.$$

It follows that there exists a constant $\tilde{R} = \tilde{R}(M) > 0$ such that

$$g(s) > Ms, \quad \forall\, s \geq \tilde{R}.$$

Arguing as in Section 3.3.2, we have that

$$\begin{aligned} u'(t) &\leq u(t)\frac{e^{|c|T}}{\varepsilon}, &&\text{for all } t \in [\sigma_\varepsilon, \tau_i] \text{ such that } u'(t) \geq 0; \\ |u'(t)| &\leq u(t)\frac{e^{|c|T}}{\varepsilon}, &&\text{for all } t \in [\sigma_i, \tau_\varepsilon] \text{ such that } u'(t) \leq 0. \end{aligned} \tag{4.3.22}$$

To prove (4.3.22), we note that the result is trivial if $u'(t) = 0$. Then, we deal separately with the cases $u'(t) > 0$ and $u'(t) < 0$. Let us fix $t \in [\sigma_\varepsilon, \tau_i]$ with $u'(t) > 0$. Since the map $\xi \mapsto e^{c\xi}u'(\xi)$ is non-increasing on I_i^+, we find that

$$u'(\xi) \geq u'(t)e^{c(t-\xi)}, \quad \forall\, \xi \in [\sigma_i, t].$$

Integrating on $[\sigma_i, t]$, we obtain

$$u(t) \geq u(t) - u(\sigma_i) \geq u'(t)e^{-|c|(t-\sigma_i)}(t - \sigma_i) \geq u'(t)e^{-|c|T}\varepsilon$$

and therefore the first inequality follows. If $t \in [\sigma_i, \tau_\varepsilon]$ and $u'(t) < 0$, we obtain the second inequality in (4.3.22), after an integration on $[t, \tau_i]$. Hence, (4.3.22) is proved.

We are ready now to prove (4.3.21). By contradiction, suppose there is not a constant $R_i > 0$ with the properties listed above. So, for each integer $n > 0$ there exists a solution $u_n \geq 0$ of (4.2.5) with $\max_{t \in I_i^+} u_n(t) =: \hat{R}_n > n$. For each $n > \tilde{R}$ we take $\hat{t}_n \in I_i^+$ such that $u_n(\hat{t}_n) = \hat{R}_n$ and let $]\varsigma_n, \omega_n[\subseteq I_i^+$ be the intersection with $]\sigma_i, \tau_i[$ of the maximal open interval containing $\hat{t}_n$ and such that $u_n(t) > \tilde{R}$ for all $t \in]\varsigma_n, \omega_n[$. We fix an integer N such that

$$N > \tilde{R} + \frac{\tilde{R}Te^{2|c|T}}{\varepsilon}$$

and we claim that $]\varsigma_n, \omega_n[\supseteq [\sigma_\varepsilon, \tau_\varepsilon]$, for each $n \geq N$. Suppose by contradiction that $\sigma_\varepsilon \leq \varsigma_n$. In this case, we find that $u_n(\varsigma_n) = \tilde{R}$ and $u'_n(\varsigma_n) \geq 0$. Moreover, $u'_n(\varsigma_n) \leq \tilde{R}e^{|c|T}/\varepsilon$. Using the monotonicity of $t \mapsto e^{ct}u'(t)$, we get $e^{ct}u'(t) \leq e^{c\varsigma_n}u'(\varsigma_n)$ for every $t \in [\varsigma_n, \hat{t}_n]$ and therefore we find $u'(t) \leq \tilde{R}e^{2|c|T}/\varepsilon$ for every $t \in [\varsigma_n, \hat{t}_n]$. Finally, an integration on $[\varsigma_n, \hat{t}_n]$ yields

$$n < \hat{R}_n = u_n(\hat{t}_n) \leq \tilde{R} + \frac{\tilde{R}Te^{2|c|T}}{\varepsilon},$$

hence a contradiction, since $n \geq N$. A symmetric argument provides a contradiction if we suppose that $\omega_n \leq \tau_\varepsilon$. This proves the claim.

So, we can fix an integer $N > \tilde{R}$ such that $u_n(t) > \tilde{R}$ for every $t \in [\sigma_\varepsilon, \tau_\varepsilon]$ and $n \geq N$. The function $u_n(t)$, being a solution of equation (4.2.5) or equivalently of

$$\big(e^{ct}u'\big)' + e^{ct}\big(a_\mu(t)g(u) + \alpha v(t)\big) = 0,$$

satisfies

$$\begin{cases} u'_n(t) = e^{-ct}y_n(t) \\ y'_n(t) = -e^{ct}\big(a_\mu(t)g(u_n(t)) + \alpha v(t)\big). \end{cases}$$

Via a Prüfer transformation, we pass to the polar coordinates

$$e^{ct}u'_n(t) = r_n(t)\cos\vartheta_n(t), \qquad u_n(t) = r_n(t)\sin\vartheta_n(t),$$

and obtain, for every $t \in [\sigma_\varepsilon, \tau_\varepsilon]$, that

$$\begin{aligned} \vartheta'_n(t) &= e^{-ct}\cos^2\vartheta_n(t) + \frac{e^{ct}\big(a^+(t)g(u_n(t)) + \alpha v(t)\big)}{u_n(t)}\sin^2\vartheta_n(t) \\ &\geq e^{-ct}\cos^2\vartheta_n(t) + e^{ct}Ma^+(t)\sin^2\vartheta_n(t). \end{aligned}$$

We also consider the linear equation

$$\big(e^{ct}u'\big)' + e^{ct}Ma^+(t)u = 0 \tag{4.3.23}$$

and its associated angular coordinate $\vartheta(t)$ (via the Prüfer transformation), which satisfies

$$\vartheta'(t) = e^{-ct}\cos^2\vartheta(t) + e^{ct}Ma^+(t)\sin^2\vartheta(t).$$

Note also that the angular functions ϑ_n and ϑ are non-decreasing in $[\sigma_\varepsilon, \tau_\varepsilon]$. Using a classical comparison result in the frame of Sturm's theory (cf. [55, ch. 8, Theorem 1.2]), we find that

$$\vartheta_n(t) \geq \vartheta(t), \quad \forall\, t \in [\sigma_\varepsilon, \tau_\varepsilon], \tag{4.3.24}$$

if we choose $\vartheta(\sigma_\varepsilon) = \vartheta_n(\sigma_\varepsilon)$. Consider now a fixed $n \geq N$. Since $u_n(t) \geq \tilde{R}$ for every $t \in [\sigma_\varepsilon, \tau_\varepsilon]$, we must have

$$\vartheta_n(t) \in\,]0, \pi[, \quad \forall\, t \in [\sigma_\varepsilon, \tau_\varepsilon]. \tag{4.3.25}$$

On the other hand, by the choice of $M > 0$, we know that any non-negative solution $u(t)$ of (4.3.23) with $u(\sigma_\varepsilon) > 0$ must vanish at some point in $]\sigma_\varepsilon, \tau_\varepsilon[$ (see [55, ch. 8, Theorem 1.1]). Therefore, from $\vartheta(\sigma_\varepsilon) = \vartheta_n(\sigma_\varepsilon) \in]0, \pi[$, we conclude that there exists $t^* \in]\sigma_\varepsilon, \tau_\varepsilon[$ such that $\vartheta(t^*) = \pi$. By (4.3.24) we have that $\vartheta_n(t^*) \geq \pi$, which contradicts (4.3.25).

We conclude that for each $i = 1, \ldots, m$ there is a constant $R_i > 0$ such that any non-negative T-periodic solution $u(t)$ of (4.2.5), with $\alpha \geq 0$, satisfies $\max_{t \in I_i^+} u(t) < R_i$.

Now we can take as R^* any constant such that $R^* > r$ (with r as in Section 4.3.2) and

$$R^* \geq R_i, \quad \text{for all } i = 1, \ldots, m. \tag{4.3.26}$$

Thus $\max_{t \in A} u(t) < R^*$ is proved. Notice that R^* does not depend on $v(t)$ and on μ, since for the constants R_i we have used only information about $a^+(t)$.

Finally, using (4.3.8) in Remark 4.3.4 and reasoning as in the proof of (4.3.9) (we just need to repeat verbatim the same argument, by replacing r with R^*), we can check that R^* is an a priori bound for the solutions on the whole domain. In this manner (4.3.5) is proved.

Remark 4.3.7. If $c = 0$ in equation (4.2.5), the existence of the upper bound R^* can be obtained in a different manner, still using a Sturm comparison argument (see the proof of Lemma 1.1.3 and Remark 1.1.1). $\lhd$

Remark 4.3.8. A careful reading of the proof of the a priori bound shows that the inequality $u(t) < R^*$ for all $t \in [0, T]$ has been proved independently on the assumption of T-periodicity of $u(t)$. Hence, the same a priori bound on $[0, T]$ is valid for any non-negative solution $u(t)$ of $(\mathscr{E}_\mu)$, with $u(t)$ defined on an interval containing $[0, T]$. We claim now that the following stronger property holds.

> *If $w(t)$ is a non-negative solution of $(\mathscr{E}_\mu)$ (not necessarily periodic), then*
> $$w(t) < R^*, \quad \forall\, t \in \mathbb{R}.$$

To check this assertion, suppose by contradiction that there exists $t^* \in \mathbb{R}$ such that $w(t^*) \geq R^*$. Let also $\ell \in \mathbb{Z}$ be such that $t^* \in [\ell T, (\ell + 1)T]$. In this case, thanks to the T-periodicity of the weight coefficient $a_\mu(t)$, the function $u(t) := w(t + \ell T)$ is still a (non-negative) solution of $(\mathscr{E}_\mu)$ with $\max_{t \in [0,T]} u(t) \geq u(t^* - \ell T) = w(t^*) \geq R^*$, a contradiction with respect to the previous established a priori bound of $u(t)$ on $[0, T]$. $\lhd$

Verification of (H^R) for $R \geq R^*$. We have found a constant $R^* > r$ such that any non-negative solution $u(t)$ of (4.2.5), with $\alpha \geq 0$, satisfies

$\|u\|_\infty < R^*$. Then, for $R \geq R^*$, the first part of (H^R) is valid independently of the choice of α_0.

Let $\alpha_0 > 0$ be fixed such that

$$\alpha_0 > \frac{\mu \|a^-\|_{L^1} \max_{0\leq s\leq R} g(s)}{\|v\|_{L^1}}.$$

We verify that for $\alpha = \alpha_0$ there are no T-periodic solutions $u(t)$ of (4.2.5) with $0 \leq u(t) \leq R$ on $[0,T]$. Indeed, if there were, integrating on $[0,T]$ the differential equation and using the boundary conditions, we obtain

$$\alpha \|v\|_{L^1} = \alpha \int_0^T v(t)\,dt = -\int_0^T a_\mu(t) g(u(t))\,dt \leq \mu \|a^-\|_{L^1} \max_{0\leq s\leq R} g(s),$$

which leads to a contradiction with respect to the choice of α_0. Thus (H^R) is verified for all $R \geq R^*$. $\square$

4.3.4 Checking the assumptions of Lemma 4.2.3 for μ large

Let $\mathcal{I} \subsetneq \{1,\ldots,m\}$ be a nonempty subset of indices and let $r > 0$ be as in Section 4.3.2, in particular r satisfies (4.3.15). Set $d = r$, $\mathcal{J} := \{1,\ldots,m\} \setminus \mathcal{I}$ and let $v \in L^1([0,T])$ be an arbitrary nontrivial function satisfying (4.3.10). For example, as $v(t)$ we can take the indicator function of the set $\bigcup_{i\in\mathcal{I}} I_i^+$.

In this section we verify that $(A_{\mathcal{J},r})$, $(B_{\mathcal{J},r})$ and $(C_{\mathcal{J},r})$ (of Lemma 4.2.3) hold for μ sufficiently large.

Verification of $(A_{\mathcal{J},r})$. Let $\alpha \geq 0$. We claim that there exists $\mu^*_{\mathcal{I}} > 0$ such that for $\mu > \mu^*_{\mathcal{I}}$ any non-negative T-periodic solution u of (4.2.5), or equivalently of

$$\big(e^{ct}u'\big)' + e^{ct}\big(a_\mu(t)g(u) + \alpha v(t)\big) = 0, \tag{4.3.27}$$

is such that $\max_{t\in I_i^+} u(t) \neq r$, for all $i \notin \mathcal{I}$.

By contradiction, suppose that there is a solution $u(t)$ of (4.2.5) with

$$\max_{t\in I_j^+} u(t) = r, \qquad \text{for some index } j \in \mathcal{J}. \tag{4.3.28}$$

Let $\hat{t}_j \in I_j^+ = [\sigma_j, \tau_j]$ be such that $u(\hat{t}_j) = r$. If $\hat{t}_j = \tau_j$, then clearly $u(\tau_j) = r$ and $u'(\tau_j) \geq 0$. If $\hat{t}_j = \sigma_j$, then $u(\sigma_j) = r$ and $u'(\sigma_j) \leq 0$. Suppose now that $\sigma_j < \hat{t}_j < \tau_j$. By conditions (a_*) and (4.3.10), the solution $u(t)$ satisfies the following initial value problem on I_j^+

$$\begin{cases} \big(e^{ct}u'\big)' + e^{ct}a^+(t)g(u) = 0 \\ u(\hat{t}_j) = r \\ u'(\hat{t}_j) = 0. \end{cases}$$

Then, we have

$$u'(t) = -\int_{\hat{t}_j}^{t} e^{c(\xi - t)} a^+(\xi) g(u(\xi))\, d\xi, \quad \forall\, t \in I_j^+,$$

and hence, recalling (4.3.13) and (4.3.15), we obtain this a priori bound for $|u'(t)|$ on I_j^+:

$$|u'(t)| \le e^{|c||I_j^+|} \|a^+\|_{L^1(I_j^+)} \eta(r) r \le K_j \frac{r}{2K_j |I_j^+|} = \frac{r}{2|I_j^+|}, \quad \forall\, t \in I_j^+.$$

Therefore, the following inequality holds

$$u(\tau_j) = u(\hat{t}_j) + \int_{\hat{t}_j}^{\tau_j} u'(t)\, dt \ge r - |I_j^+| \frac{r}{2|I_j^+|} = \frac{r}{2}.$$

Thus we have a lower bound for $u(\tau_j)$.

As a first case, suppose that $\sigma_j < \hat{t}_j \le \tau_j$. Above, we have proved that

$$u(\tau_j) \ge \frac{r}{2} \quad \text{and} \quad u'(\tau_j) \ge -\frac{r}{2|I_j^+|} \tag{4.3.29}$$

(this is also true in a trivial manner if $\hat{t}_j = \tau_j$). By the initial convention in Section 4.1 about the selection of the points σ_i and τ_i in order to separate the intervals of positivity and negativity, we know that $a(t) \prec 0$ on every right neighborhood of τ_j. Accordingly, we can fix $\delta_j^+ > 0$, with $0 < \delta_j^+ < \sigma_{j+1} - \tau_j$, such that

$$\delta_j^+ \, e^{|c|\delta_j^+} < \frac{|I_j^+|}{2} \tag{4.3.30}$$

and $a(t) \prec 0$ on $[\tau_j, \tau_j + \delta_j^+] \subseteq I_j^-$. Since $t \mapsto e^{ct} u'(t)$ is non-decreasing on I_j^-, we have

$$u'(t) \ge e^{c(\tau_j - t)} u'(\tau_j) \ge -e^{|c|\delta_j^+} \frac{r}{2|I_j^+|}, \quad \forall\, t \in [\tau_j, \tau_j + \delta_j^+],$$

then

$$u(t) = u(\tau_j) + \int_{\tau_j}^{t} u'(\xi)\, d\xi \ge \frac{r}{2} - \delta_j^+ e^{|c|\delta_j^+} \frac{r}{2|I_j^+|} > \frac{r}{4}, \quad \forall\, t \in [\tau_j, \tau_j + \delta_j^+].$$

We deduce that

$$\frac{r}{4} < u(t) \le R^* \quad \text{on } [\tau_j, \tau_j + \delta_j^+],$$

where R^* is the upper bound defined in Section 4.3.3.

Let us fix

$$\gamma := \min_{\frac{r}{4} \le s \le R^*} g(s) > 0. \tag{4.3.31}$$

We prove that for $\mu > 0$ sufficiently large $\max_{t\in[\tau_j,\tau_j+\delta_j^+]} u(t) > R^*$ (which is a contradiction to the upper bound for $u(t)$).

Note that for $t \in [\tau_j, \tau_j + \delta_j^+] \subseteq I_j^-$, equation (4.3.27) reads as

$$\big(e^{ct}u'(t)\big)' = \mu e^{ct} a^-(t) g(u(t)).$$

Hence, for all $t \in [\tau_j, \tau_j + \delta_j^+]$ we have

$$\begin{aligned} u'(t) &= e^{c(\tau_j - t)} u'(\tau_j) + \int_{\tau_j}^{t} \mu e^{c(\xi - t)} a^-(\xi) g(u(\xi))\, d\xi \\ &\geq -e^{|c|\delta_j^+} \frac{r}{2|I_j^+|} + \mu e^{-|c|\delta_j^+} \gamma \int_{\tau_j}^{t} a^-(\xi)\, d\xi, \end{aligned}$$

then

$$\begin{aligned} u(t) &= u(\tau_j) + \int_{\tau_j}^{t} u'(\xi)\, d\xi \\ &\geq \frac{r}{2} - \delta_j^+ e^{|c|\delta_j^+} \frac{r}{2|I_j^+|} + \mu e^{-|c|\delta_j^+} \gamma \int_{\tau_j}^{t} \left(\int_{\tau_j}^{s} a^-(\xi)\, d\xi \right) ds \\ &> \frac{r}{4} + \mu e^{-|c|\delta_j^+} \gamma \int_{\tau_j}^{t} \left(\int_{\tau_j}^{s} a^-(\xi)\, d\xi \right) ds. \end{aligned}$$

Therefore, for $t = \tau_j + \delta_j^+$ we get

$$R^* \geq u(\tau_j + \delta_j^+) \geq \mu e^{-|c|\delta_j^+} \gamma \int_{\tau_j}^{\tau_j + \delta_j^+} \left(\int_{\tau_j}^{s} a^-(\xi)\, d\xi \right) ds.$$

This gives a contradiction if μ is sufficiently large, say

$$\mu > \mu_j^+ := \frac{R^*\, e^{|c|\delta_j^+}}{\gamma \int_{\tau_j}^{\tau_j+\delta_j^+} \big(\int_{\tau_j}^{s} a^-(\xi)\, d\xi\big)\, ds}, \tag{4.3.32}$$

recalling that $\int_{\tau_j}^{t} a^-(\xi)\, d\xi > 0$ for each $t \in \,]\tau_j, \sigma_{j+1}]$.

As a second case, if $\hat{t}_j = \sigma_j$, we consider the interval I_{j-1}^- (if $j = 1$, we deal with $I_m^- - T$, by T-periodicity). We define δ_j^- in a similar manner, using the fact that $a(t)$ is not identically zero on all left neighborhoods of σ_j. We obtain a contradiction for

$$\mu > \mu_j^- := \frac{R^*\, e^{|c|\delta_j^-}}{\gamma \int_{\sigma_j - \delta_j^-}^{\sigma_j} \big(\int_{s}^{\sigma_j} a^-(\xi)\, d\xi\big)\, ds}. \tag{4.3.33}$$

At the end, we define

$$\mu_{\mathcal{I}}^* := \max_{j \notin \mathcal{I}} \mu_j^{\pm}$$

and we obtain a contradiction if $\mu > \mu_{\mathcal{I}}^*$. Hence condition $(A_{\mathcal{J},r})$ is verified.

Remark 4.3.9. We emphasize the fact that the constant $\mu^*_{\mathcal{I}}$ is chosen independently on the solution $u(t)$ for which we have made the estimates. In fact, the numbers $\mu_j^{\pm}$ depend only on absolute constants, like R^*, γ (depending only on $g(s)$), the constants $\delta_j^{\pm}$ defined as in (4.3.30) and, finally, the integrals of the negative part of the weight function. Observe also that, in order to have non-vanishing denominators in the definition of the $\mu_j^{\pm}$, we had to be sure that $a(t) \prec 0$ on each right neighborhood of τ_j as well as on each left neighborhood of σ_j, consistently with the choice we made at the beginning. $\lhd$

Remark 4.3.10. A careful reading of the above proof shows that the result does not involve the periodicity of the function $u(t)$, since we have only analyzed the behavior of the solution on an interval of positivity of the weight and on the adjacent intervals of negativity. Indeed, we claim that the following result holds.

> *There exists a constant $\mu^* > 0$ such that, for every $\mu > \mu^*$, any non-negative solution $w(t)$ of $(\mathscr{E}_\mu)$ (not necessarily periodic), with $w(t) < R^*$ for all $t \in \mathbb{R}$, is such that*
>
> $$\max\bigl\{w(t) \colon t \in I_i^+ + \ell T\bigr\} \neq r, \quad \forall\, i \in \{1,\dots,m\},\ \forall\, \ell \in \mathbb{Z}.$$

To check this assertion, we proceed by contradiction, assuming that there exist $i \in \{1,\dots,m\}$ and $\ell \in \mathbb{Z}$ such that $\max\bigl\{w(t) \colon t \in I_i^+ + \ell T\bigr\} = r$. Thanks to the T-periodicity of the weight coefficient $a_\mu(t)$, the function $u(t) := w(t+\ell T)$ is still a (non-negative) solution of $(\mathscr{E}_\mu)$ with $\max_{t\in I_i^+} u(t) = r$. So, we are in the same situation like at the beginning of the verification of $(A_{\mathcal{J},r})$ (cf. (4.3.28)). From now on, we proceed exactly the same as in that proof and obtain a contradiction with respect to the bound $u(t) < R^*$ (for all t) taking $\mu > \max\{\mu_i^+, \mu_i^-\}$. Hence the result is proved for μ sufficiently large, namely $\mu > \max_i \mu_i^{\pm}$. $\lhd$

Now, we fix $\mu > \mu^*_{\mathcal{I}}$ and we prove that conditions $(B_{\mathcal{J},r})$ and $(C_{\mathcal{J},r})$ hold (independently of the coefficient μ previously fixed).

Verification of $(B_{\mathcal{J},r})$. Let $u(t)$ be any non-negative T-periodic solution of (4.2.5) with $\max_{t\in I_j^+} u(t) \leq r$, for all $j \in \mathcal{J}$. Notice that R^* is an upper bound for all the solutions of (4.2.5) and R^* is independent on the functions $v \in L^1([0,T])$ satisfying (4.3.4) (and hence (4.3.10)). So condition $(B_{\mathcal{J},r})$ is verified with $D_\beta = R^*$, for every $\beta \geq 0$.

Verification of $(C_{\mathcal{J},r})$. Recalling the choice of $v(t)$ in (4.3.10), we take an index $i \in \mathcal{I} = \{1,\dots,m\} \setminus \mathcal{J}$ such that $v \not\equiv 0$ on I_i^+ and we also fix $\varepsilon > 0$ such that

$$v(t) \not\equiv 0 \quad \text{on } [\sigma_i+\varepsilon, \tau_i-\varepsilon].$$

We claim that $(C_{\mathcal{J},r})$ is satisfied for α_0 such that

$$\alpha_0 > \frac{R^*}{\int_{\sigma_i+\varepsilon}^{\tau_i+\varepsilon} v(t)\,dt}\biggl(\frac{2e^{|c|T}}{\varepsilon} + |c|\biggr).$$

To prove our assertion, first of all we observe that if $u(t)$ is any non-negative solution of (4.2.5), then

$$|u'(t)| \leq u(t)\frac{e^{|c|T}}{\varepsilon}, \quad \forall\, t \in [\sigma_i + \varepsilon, \tau_i - \varepsilon]. \tag{4.3.34}$$

Such inequality has been already proved and used in Section 4.3.3 (see the inequalities in (4.3.22)).

Let $u(t)$ be a non-negative T-periodic solution of (4.2.5), which reads as

$$\alpha v(t) = -u'' - cu' - a^+(t)g(u)$$

on the interval I_i^+. Recall also that $\|u\|_\infty < R^*$ (cf. (4.3.5)). Integrating the equation on $[\sigma_i + \varepsilon, \tau_i - \varepsilon]$, for $\alpha = \alpha_0$, we obtain

$$\begin{aligned}
&\alpha_0 \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} v(t)\,dt = \\
&= u'(\sigma_i+\varepsilon) - u'(\tau_i-\varepsilon) + c\bigl(u(\sigma_i+\varepsilon) - u(\tau_i-\varepsilon)\bigr) - \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t)g(u(t))\,dt \\
&\leq 2\frac{R^*}{\varepsilon}e^{|c|T} + |c|R^* < \alpha_0 \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} v(t)\,dt,
\end{aligned}$$

a contradiction. Hence $(C_{\mathcal{J},r})$ is verified.

Remark 4.3.11. Note that for the verification of $(B_{\mathcal{J},r})$ and $(C_{\mathcal{J},r})$ the small constant r has no played any relevant role. In fact, we used only the information about the existence of the a priori bound R^* obtained in Section 4.3.3. $\lhd$

In conclusion, all the assumptions of Lemma 4.2.3 have been verified for a fixed r and for $\mu > \mu^*_{\mathcal{I}}$. $\square$

4.3.5 A posteriori bounds

Let $\mathcal{I} \subseteq \{1, \ldots, m\}$ be a nonempty subset of indices and let r and R be fixed as explained in Section 4.3.1. Theorem 4.3.1 ensures the existence of at least a T-periodic positive solution $u(t)$ of $(\mathscr{E}_\mu)$ with $u \in \Lambda^{\mathcal{I}}_{r,R}$. In more detail, the solution $u(t)$ is such that $r < u(\hat{t}_i) < R$ for some $\hat{t}_i \in I_i^+$, if $i \in \mathcal{I}$, and $0 < u(t) < r$ for all $t \in I_i^+$, if $i \notin \mathcal{I}$.

As premised in Remark 4.3.1, in this section we prove that, for μ sufficiently large, it holds that $0 < u(t) < r$ also on the non-positivity intervals

I_i^-. First of all, by (4.3.8), we observe that the solution $u(t)$ in the interval of non-positivity attains its maximum at an endpoint. Therefore it is sufficient to show that

$$u(\sigma_i) < r \quad \text{and} \quad u(\tau_i) < r, \quad \text{for all } i = 1, \ldots, m.$$

If $i \notin \mathcal{I}$, there is nothing to prove, because $u(t) < r$ on $I_i^+ = [\sigma_i, \tau_i]$. Let us deal with the case $i \in \mathcal{I}$ and, by contradiction, suppose that

$$u(\tau_i) \geq r.$$

Proceeding as in Section 4.3.4, one can prove that

$$u'(\tau_i) \geq -K_i \max_{0 \leq s \leq R^*} g(s)$$

and hence (using estimates analogous to those following after (4.3.29)) that there exists $\delta_i^+ > 0$ such that, for $t = \tau_i + \delta_i^+ \in I_i^-$ and μ sufficiently large, we obtain

$$u(\tau_i + \delta_i^+) \geq R^*,$$

a contradiction.

A similar argument generates a contradiction also assuming $u(\sigma_i) \geq r$, for $i \in \mathcal{I}$.

Finally, repeating again the argument in Section 4.3.4, we can also check that if $u(t) = u_\mu(t)$ is a positive T-periodic solution of $(\mathscr{E}_\mu)$ (belonging to a set of the form $\Lambda^{\mathcal{I}}_{r,R}$), then, for $\mu \to +\infty$, u_μ tends uniformly to zero on the intervals I_i^-.

Remark 4.3.12. Notice that the same arguments work for any *arbitrary* non-negative solution which is upper bounded by R^* (at any effect this observation is analogous to Remark 4.3.10, since it only involves the behavior of the solution in the intervals where the weight is negative, without requiring the periodicity of the solution). Indeed, the following result holds.

> *There exists a constant $\mu^{**} > 0$ such that, for every $\mu > \mu^{**}$, any non-negative solution $w(t)$ of $(\mathscr{E}_\mu)$ (not necessarily periodic), with $w(t) < R^*$ for all $t \in \mathbb{R}$, is such that*
>
> $$\max\bigl\{w(t) \colon t \in I_i^- + \ell T\bigr\} < r, \quad \forall\, i \in \{1, \ldots, m\},\ \forall\, \ell \in \mathbb{Z}.$$

To check this assertion, we proceed by contradiction, assuming that there exist $i \in \{1, \ldots, m\}$ and $\ell \in \mathbb{Z}$ such that $\max\bigl\{w(t) \colon t \in I_i^- + \ell T\bigr\} \geq r$. Thanks to the T-periodicity of the weight coefficient $a_\mu(t)$, the function $u(t) := w(t + \ell T)$ is still a (non-negative) solution of $(\mathscr{E}_\mu)$ with $\max_{t \in I_i^-} u(t) \geq r$. This means that $u(\sigma_i) \geq r$ or $u(\tau_i) \geq r$. At this point we achieve a contradiction exactly as above. $\lhd$

4.4 Related results

In this section we deal with corollaries, variants and applications of Theorem 4.3.1. We also analyze the case of a nonlinearity $g(s)$ which is smooth in order to give a nonexistence result, too.

The following corollaries are obtained as direct applications of Theorem 4.3.1.

Corollary 4.4.1. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \textit{and} \quad g_\infty > 0.$$

Then there exists $\nu^* > 0$ such that for all $\nu > \nu^*$ there exists $\mu^* = \mu^*(\nu)$ such that for $\mu > \mu^*$ there exist at least $2^m - 1$ positive T-periodic solutions of

$$u'' + cu' + \bigl(\nu a^+(t) - \mu a^-(t)\bigr) g(u) = 0. \tag{4.4.1}$$

The constant ν^* will be chosen so that $\nu^* g_\infty > \max_i \lambda_1^i$. The lower bound for g_∞ in the main theorem is automatically satisfied when $g_\infty = +\infty$. Accordingly, we have.

Corollary 4.4.2. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \textit{and} \quad g_\infty = +\infty.$$

Then there exists $\mu^* > 0$ such that for all $\mu > \mu^*$ equation $(\mathscr{E}_\mu)$ has at least $2^m - 1$ positive T-periodic solutions.

A typical case in which the above corollary applies is for the power nonlinearity $g(s) = s^p$ (for $p > 1$), so that the next result holds.

Corollary 4.4.3. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). Then there exists $\mu^* > 0$ such that for all $\mu > \mu^*$ there exist at least $2^m - 1$ positive T-periodic solutions of*

$$u'' + cu' + \bigl(a^+(t) - \mu a^-(t)\bigr) u^p = 0, \quad p > 1.$$

Using Remark 4.3.2, we can also obtain the following result which, in some sense, is dual with respect to Corollary 4.4.1.

Corollary 4.4.4. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 > 0 \quad \textit{and} \quad g_\infty = +\infty.$$

Then there exists $\nu_* > 0$ such that for all $0 < \nu < \nu_*$ there exists $\mu^* = \mu^*(\nu)$ such that for $\mu > \mu^*$ there exist at least $2^m - 1$ positive T-periodic solutions of equation (4.4.1).

Combining Theorem 4.3.1 with Proposition 3.2.1, the following result can be obtained.

Corollary 4.4.5. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function satisfying (g_*),*

$$g_0 = 0 \quad \text{and} \quad g_\infty > 0.$$

Then for all $\mu > 0$ such that $\int_0^T a_\mu(t)\,dt < 0$ there exist two constants $0 < \omega_* \le \omega^*$ (depending on μ) such that equation

$$u'' + cu' + \nu\big(a^+(t) - \mu a^-(t)\big)g(u) = 0 \tag{4.4.2}$$

has no positive T-periodic solutions for $0 < \nu < \omega_*$ and at least one positive T-periodic solution for $\nu > \omega^*$. Moreover there exists $\nu^* > 0$ such that for all $\nu > \nu^*$ there exists $\mu^* = \mu^*(\nu)$ such that for $\mu > \mu^*$ equation (4.4.2) has at least $2^m - 1$ positive T-periodic solutions.

Equation (4.4.2) is substantially equivalent to (4.4.1). We have preferred to write it in a slightly different form for sake of convenience in stating Corollary 4.4.5.

4.5 The Neumann boundary value problem

In this section we briefly describe how to obtain the results of Section 4.3 and Section 4.4 for the Neumann boundary value problem (see also Section 3.3.1). For the sake of simplicity, we deal with the case $c = 0$. If $c \ne 0$, we can produce analogous results writing equation $(\mathscr{E}_\mu)$ as

$$\big(u'e^{ct}\big)' + \tilde{a}_\mu(t)g(u) = 0, \qquad \text{with } \tilde{a}_\mu(t) := a_\mu(t)e^{ct},$$

and entering in the setting of coincidence degree theory for the linear operator $L\colon u \mapsto -(u'e^{ct})'$. Accordingly, we consider the boundary value problem

$$\begin{cases} u'' + a_\mu(t)g(u) = 0 \\ u'(0) = u'(T) = 0, \end{cases} \tag{4.5.1}$$

where $a\colon [0,T] \to \mathbb{R}$ is an integrable function satisfying condition (a_*) and $g(s)$ fulfils the same conditions as in the previous sections. In particular, when we assume (a_*) we suppose that there exist $m \ge 2$ subintervals of $[0,T]$ where the weight is non-negative separated by $m-1$ subintervals where the weight is non-positive, namely there are $2m+2$ points

$$0 = \tau_0 \le \sigma_1 < \tau_1 < \ldots < \sigma_i < \tau_i < \ldots < \sigma_m < \tau_m \le \sigma_{m+1} = T$$

such that $a(t) \succ 0$ on $[\sigma_i, \tau_i]$ and $a(t) \prec 0$ on $[\tau_i, \sigma_{i+1}]$.

In this case, the abstract setting of Section 4.2 can be reproduced almost verbatim with $X := \mathcal{C}([0,T])$, $Z := L^1([0,T])$ and $L\colon u \mapsto -u''$, by taking in $\operatorname{dom} L$ the functions of X which are continuously differentiable with absolutely continuous derivative and such that $u'(0) = u'(T) = 0$. With the above positions $\ker L \cong \mathbb{R}$, $\operatorname{Im} L$, as well as the projectors P and Q are exactly the same as in Section 4.2. Then Theorem 4.3.1 can be restated as follows.

Theorem 4.5.1. *Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function satisfying (a_*). Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \text{and} \quad g_\infty > \max_{i=1,\dots,m} \lambda_1^i.$$

Then there exists $\mu^* > 0$ such that for all $\mu > \mu^*$ problem (4.5.1) has at least $2^m - 1$ positive solutions.

As in Theorem 4.3.1, the $2^m - 1$ positive solutions are discriminated by the fact that $\max_{t \in I_i^+} u(t) < r$ or $r < \max_{t \in I_i^+} u(t) < R$, where $I_i^+ = [\sigma_i, \tau_i]$ is the i-th interval where the weight is non-negative (cf. Remark 4.3.1). The constants λ_1^i (for $i = 1, \dots, m$) are the first eigenvalues of the eigenvalue problems in I_i^+

$$\varphi'' + \lambda a(t)\varphi = 0, \quad \varphi|_{\partial I_i^+} = 0.$$

If $\sigma_1 = \tau_0 = 0$ (that is $a(t)$ starts with a first interval of non-negativity), we can take λ_1^1 as the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a(t)\varphi = 0, \quad \varphi'(0) = \varphi(\tau_1) = 0,$$

while if $\tau_m = \sigma_{m+1} = T$ (that is $a(t)$ ends with a last interval of non-negativity), we can take λ_1^m as the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a(t)\varphi = 0, \quad \varphi(\sigma_m) = \varphi'(T) = 0.$$

Compare also to Remark 3.2.1

Clearly, for the Neumann problem (4.5.1) we can also reestablish the corollaries in Section 4.4. In particular, Corollary 4.4.2 reads as follows.

Corollary 4.5.1. *Let $a\colon [0,T] \to \mathbb{R}$ be an integrable function satisfying (a_*). Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \text{and} \quad g_\infty = +\infty.$$

Then there exists $\mu^* > 0$ such that for all $\mu > \mu^*$ problem (4.5.1) has at least $2^m - 1$ positive solutions.

In the sequel we are going to use also a variant for the Neumann problem of Corollary 4.4.5 that we do not state here explicitly.

4.5.1 Radially symmetric solutions

We show now a consequence of the above results to the study of a PDE in an annular domain. In order to simplify the exposition, we assume the continuity of the weight function. In this manner, the solutions we find are the "classical" ones (at least two times continuously differentiable). Now, proceeding in the same manner as in Section 3.3.1, we are going to complete that discussion, presenting the multiplicity results.

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^N$ (for $N \geq 2$) and let

$$\Omega := B(0,R_2) \setminus B[0,R_1] = \bigl\{x \in \mathbb{R}^N : R_1 < \|x\| < R_2\bigr\}$$

be an open annular domain, with $0 < R_1 < R_2$.

We deal with the Neumann boundary value problem

$$\begin{cases} -\Delta u = q_\mu(x) g(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5.2}$$

where $q\colon \overline{\Omega} \to \mathbb{R}$ is a continuous function which is radially symmetric, namely there exists a continuous scalar function $\mathcal{Q}\colon [R_1,R_2] \to \mathbb{R}$ such that

$$q(x) = \mathcal{Q}(\|x\|), \quad \forall\, x \in \overline{\Omega},$$

and

$$q_\mu(x) := q^+(x) - \mu q^-(x), \quad \mathcal{Q}_\mu(r) := \mathcal{Q}^+(r) - \mu \mathcal{Q}^-(r).$$

We look for existence/nonexistence and multiplicity of radially symmetric positive solutions of (4.5.2), that are classical solutions such that $u(x) > 0$ for all $x \in \Omega$ and also $u(x) = \mathcal{U}(\|x\|)$, where $\mathcal{U}$ is a scalar function defined on $[R_1,R_2]$.

Accordingly, our study can be reduced to the search of positive solutions of the Neumann boundary value problem

$$\mathcal{U}''(r) + \frac{N-1}{r}\mathcal{U}'(r) + \mathcal{Q}_\mu(r) g(\mathcal{U}(r)) = 0, \quad \mathcal{U}'(R_1) = \mathcal{U}'(R_2) = 0. \tag{4.5.3}$$

Using the standard change of variable illustrated in Section C.2

$$t = h(r) := \int_{R_1}^{r} \xi^{1-N}\, d\xi$$

and defining

$$T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi, \quad r(t) := h^{-1}(t) \quad \text{and} \quad v(t) = \mathcal{U}(r(t)),$$

we transform (4.5.3) into the equivalent problem

$$v'' + a_\mu(t) g(v) = 0, \quad v'(0) = v'(T) = 0, \tag{4.5.4}$$

with

$$a(t) := r(t)^{2(N-1)} \mathcal{Q}(r(t)), \quad t \in [0,T].$$

Consequently, the Neumann boundary value problem (4.5.4) is of the same form of (4.5.1) and we can apply the previous results.

Accordingly, suppose also that

(q_*) *there exist* $2m+2$ *points* $R_1 = \tau_0 \leq \sigma_1 < \tau_1 < \ldots < \sigma_m < \tau_m \leq \sigma_{m+1} = R_2$ *such that*

$$\mathcal{Q}(r) > 0 \ \ \textit{on} \ \ ]\sigma_i, \tau_i[, \ i = 1, \ldots, m;$$
$$\mathcal{Q}(r) < 0 \ \ \textit{on} \ \ ]\tau_i, \sigma_{i+1}[, \ i = 0, \ldots, m.$$

Notice that condition

$$\int_0^T a_\mu(t)\, dt < 0 \tag{4.5.5}$$

reads as

$$0 > \int_0^T r(t)^{2(N-1)} \mathcal{Q}_\mu(r(t))\, dt = \int_{R_1}^{R_2} r^{N-1} \mathcal{Q}_\mu(r)\, dr.$$

Up to a multiplicative constant, the latter integral is the integral of $q_\mu(x)$ on Ω, using the change of variable formula for radially symmetric functions. Thus, $\mu > 0$ satisfies (4.5.5) if and only if μ satisfies

$(q_\#)$
$$\int_\Omega q_\mu(x)\, dx < 0.$$

Similarly, the integral in (4.5.5) is sufficiently negative (depending on μ) if and only if the integral in $(q_\#)$ is negative enough (depending on μ). With these premises, Corollary 4.5.1 yields the following result.

Theorem 4.5.2. *Let $q(x)$ be a continuous (radial) weight function as above. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \textit{and} \quad g_\infty = +\infty.$$

Then there exists $\mu^* > 0$ such that for each $\mu > \mu^*$ problem (4.5.2) *has at least $2^m - 1$ positive radially symmetric solutions.*

Corollary 4.5.1 and Theorem 4.5.2 represent an extension of [28], where the same result was obtained (with a shooting type approach) for $m = 2$. Another extension of [28], for an arbitrary $m \geq 2$, has been recently achieved in [17] (using a variational approach) for a power type nonlinearity $g(s)$.

Adding the smoothness of $g(s)$, from Corollary 4.4.5 we obtain the next result.

Theorem 4.5.3. *Let $q(x)$ be a continuous (radial) weight function as above. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function satisfying (g_*),*

$$g_0 = 0 \quad \textit{and} \quad g_\infty = +\infty.$$

Then, for all $\mu > 0$ such that $(q_\#)$ holds, there exist two constants $0 < \omega_* \le \omega^*$ (depending on μ) such that the Neumann boundary value problem

$$\begin{cases} -\Delta u = \nu q_\mu(x) g(u) & \textit{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \textit{on } \partial\Omega \end{cases} \tag{4.5.6}$$

has no positive radially symmetric solutions for $0 < \nu < \omega_*$ and at least one positive radially symmetric solution for $\nu > \omega^*$. Moreover there exists $\nu^* > 0$ such that for all $\nu > \nu^*$ there exists $\mu^* = \mu^*(\nu)$ such that for $\mu > \mu^*$ problem (4.5.6) has at least $2^m - 1$ positive radially symmetric solutions.

Chapter 5

Subharmonic solutions and symbolic dynamics

In this chapter we deal with *subharmonic solutions* for superlinear indefinite equations.

In the first part of this chapter (from Section 5.1 to Section 5.3) we continue the study of the superlinear indefinite second order equation

$$(\mathscr{E}_\mu) \qquad u'' + cu' + \bigl(a^+(t) - \mu a^-(t)\bigr)g(u) = 0,$$

that we have considered in Chapter 4 (we refer to the notation introduced therein). Our goal is to apply the results therein concerning the existence and multiplicity of periodic solutions to the search of subharmonic solutions. Then, we shall use the information obtained on the subharmonics to produce bounded positive solutions which are not necessarily periodic and can reproduce an arbitrary coin-tossing sequence. More precisely, in Section 5.1 we prove the existence of infinitely many subharmonic solutions for $(\mathscr{E}_\mu)$ if the negative part of the weight is sufficiently strong (i.e. when $\mu > 0$ is large enough). This result follows from Theorem 4.3.1 applied to an interval of the form $[0, kT]$ and a careful verification that the constants needed for the proof are independent on k. Next, in Section 5.2 we discuss the number of subharmonics of a given order and in Section 5.3 we sketch how to produce bounded solutions on the real line which are not necessarily periodic.

In the second part of this chapter (from Section 5.4 to Section 5.8) we deal with a class of indefinite equations which covers the classical superlinear one

$$u'' + a(t)u^p = 0,$$

where $p > 1$ and $a(t)$ is a T-periodic sign-changing function satisfying the (sharp) mean value condition $\int_0^T a(t)\,dt < 0$. We prove that there exist positive subharmonic solutions of order k for any large integer k. The proof combines coincidence degree theory (yielding a positive harmonic solution) with the Poincaré-Birkhoff fixed point theorem (giving subharmonic solutions oscillating around it).

We conclude this introduction by recalling the definition of subharmonic solution (to the above equations) that we assume throughout the chapter. We say that $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$ is a *subharmonic solution of order* k, with $k \geq 2$ an integer number, if $u(t)$ is a kT-periodic solution of the equation which is not lT-periodic for any integer $l = 1, \dots, k-1$, that is, kT is the minimal period of $u(t)$ in the class of the integer multiples of T.

This is the most natural definition of subharmonic solution to an equation like

$$u'' + h(t, u) = 0, \tag{5.0.1}$$

when just the T-periodicity of $t \mapsto h(t, s)$ is assumed. On the lines of [137], if additional conditions on this time dependence are imposed, further information on the minimality of the period can be given. For example, if $h(t, s) = a(t)g(s)$, with $g(s) > 0$ for $s > 0$, one can notice that all the positive subharmonic solutions of order k to (5.0.1) actually have minimal period kT if we further assume that $T > 0$ is the mimimal period of $a(t)$. This is easily seen by writing the equation in the equivalent form $a(t) = -u''(t)/g(u(t))$. Let us also notice that if $u(t)$ is a subharmonic solution of order k, the $k-1$ functions $u(\cdot + lT)$, for $l = 1, \dots, k-1$, are subharmonic solutions of order k too. Then, whenever it happens that we find a subharmonic solution of order k, we also find other $k-1$ subharmonic solutions (of the same order). These solutions, though distinct, have to be considered equivalent from the point of view of the counting of subharmonics. Accordingly, given $u_1(t), u_2(t)$ subharmonic solutions of order k, we say that $u_1(t)$ and $u_2(t)$ are not in the same *periodicity class* if $u_1(\cdot) \not\equiv u_2(\cdot + lT)$ for any integer $l = 0, \dots, k-1$.

Throughout this chapter, for the sake of simplicity in the exposition, when not explicitly stated we assume that k *is an integer such that* $k \geq 2$.

5.1 Subharmonic solutions of $(\mathscr{E}_\mu)$

In Chapter 4 we have studied existence and multiplicity of T-periodic solutions to $(\mathscr{E}_\mu)$, assuming that the weight coefficient $a(t)$ is a T-period function. Since any T-periodic coefficient can be though as a kT-periodic function, with the same technique we can look for the existence of kT-periodic solutions (with k an integer).

Generally speaking, if $x(t)$ is a kT-periodic solution of a differential system $x' = f(t, x)$ in $\mathbb{R}^N$, with $f(t + T, x) = f(t, x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, the information that $x(t)$ is not lT-periodic, for any integer $l = 1, \dots, k-1$, is not enough to conclude that kT is actually the minimal positive period of the solution. However, in many significant situations, it is possible to derive such a conclusion, under suitable conditions on the vector field $f(t,x)$. For instance, in case of $(\mathscr{E}_\mu)$ and for $g(s)$ satisfying (g_*), it is easy to check that any positive subharmonic solution of order k is a solution of minimal period kT provided that T is the minimal period of the weight function. The problem of minimality of the period in the study of subharmonic solutions is a topic of considerable importance in this area of research and different approaches have been proposed depending also on the nature of the techniques adopted to obtain the solutions. See, for instance, [39, 51, 68, 137, 157] for some pertinent remarks. It may be also interesting to observe that equations of the form $(\mathscr{E}_\mu)$, with $a(t)$ a non-constant T-periodic coefficient, do not possess *exceptional solutions*, i.e. solutions having a minimal period which has an irrational ratio with T (cf. [162, ch. I, § 4]). In view of all these premises, throughout this section, Section 5.2 and Section 5.3, we suppose that the function $a(t)$ is a periodic function having $T > 0$ as a minimal period.

In order to present in a simplified manner our main multiplicity results for subharmonic solutions, we first take a class of weights of special form, namely we suppose that

> $a\colon \mathbb{R} \to \mathbb{R}$ *is a continuous periodic sign-changing function with simple zeros and with minimal period* T*, such that there exist two consecutive zeros* α, β *with* $\alpha < \beta < \alpha + T$ *so that* $a(t) > 0$ *for all* $t \in \,]\alpha, \beta[$ *and* $a(t) < 0$ *for all* $t \in \,]\beta, \alpha + T[$.

That is $a(t)$ has only one positive hump and one negative one in a period interval. In such a simplified situation, the following result holds.

Theorem 5.1.1. *Let* $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous function satisfying* (g_*),

$$g_0 = 0 \quad \text{and} \quad g_\infty = +\infty.$$

Then there exists $\mu^* > 0$ *such that, for all* $\mu > \mu^*$ *and for every integer* $k \geq 2$*, equation* $(\mathscr{E}_\mu)$ *has a subharmonic solution of order* k.

Proof. Without loss of generality (if necessary, we can make a shift by α in the time variable), we suppose that

$$a(t) > 0, \text{ for all } 0 < t < \tau := \beta - \alpha, \quad \text{and} \quad a(t) < 0, \text{ for all } \tau < t < T.$$

Let us fix an integer $k \geq 2$ and consider the T-periodic function $a(t)$ as a kT-periodic weight on the interval $[0, kT]$. In such an interval we have condition (a_*) satisfied with

$$I_i^+ = [(i-1)T, \tau + (i-1)T] \quad \text{and} \quad I_i^- = [\tau + (i-1)T, iT],$$

for $i = 1, \ldots, k$. With respect to the notation introduced in Section 4.1, we also have $m = k$, $\sigma_1 = 0$, $\sigma_{k+1} = kT$ and

$$0 < \tau_1 = \tau < \sigma_2 = T < \ldots < \sigma_k = (k-1)T < \tau_k = \tau + (k-1)T < kT.$$

In this setting we can apply Corollary 4.4.2, which ensures the existence of $2^k - 1$ positive solutions which are also kT-periodic, provided that μ is sufficiently large.

Even if we have found kT-periodic solutions, our proof is not yet complete. In fact we still have to verify that μ^* (found in the proof of Theorem 4.3.1) is independent on k and, moreover, that among the $2^k - 1$ periodic solutions there is at least one subharmonic of order k.

For the first question, we need to check how the bounds obtained in the proof of Theorem 4.3.1 depend on the weight function. First of all we underline that, by the T-periodicity of $a(t)$, the constants K_i defined in (4.3.13) are all equal for $i = 1, \ldots, k$, then K_0 does not depend on k (cf. (4.3.14)). Consequently condition (4.3.15) reads as

$$\eta(r)\, 2\|a^+\|_{L^1([0,\tau])} e^{|c|\tau} \bigl(\tau + e^{|c|(T-\tau)}(T-\tau)\bigr) < 1$$

and thus the small constant $r > 0$ is absolute and depends only on c, $\|a^+\|_{L^1([0,\tau])}$, T and τ, but it does not depend on k.

Once that we have fixed $r > 0$, using again the T-periodicity of the weight, we notice also that the lower bounds $\mu^{\#}$ and μ_r do not depend on k (cf. (4.3.2) and (4.3.16)).

The constant R^* is chosen in (4.3.26) and depends on the a priori bounds R_i, which in turn depend on the properties of $a(t)$ restricted to the interval I_i^+. In our case, by the T-periodicity of the coefficient $a(t)$, we can choose R_i as constant with respect to i. Therefore, R^* is independent on k and then also the constant γ defined in (4.3.31) does not depend on k. By the periodicity of $a(t)$, the constants δ_j^+ introduced in Section 4.3.4 (see (4.3.30)) can be also taken all equal to a common value $\delta^+ = \delta$ such that $0 < \delta < T - \tau$ and $\delta e^{|c|\delta} < \tau/2$. The same choice can be made for δ_j^- in order to have $\delta_j^- = \delta$ for all j. From these choices of the constants R^*, γ and δ, for all $j = 1, \ldots, k$ we take $\mu_j^{\pm}$, according to (4.3.32) and (4.3.33), as

$$\mu_j^+ = \mu^+ := \frac{R^*\, e^{|c|\delta}}{\gamma \int_\tau^{\tau+\delta} \bigl(\int_\tau^s a^-(\xi)\, d\xi\bigr)\, ds}$$

and

$$\mu_j^- = \mu^- := \frac{R^*\, e^{|c|\delta}}{\gamma \int_{T-\delta}^{T} \bigl(\int_s^T a^-(\xi)\, d\xi\bigr)\, ds},$$

respectively. Therefore, setting

$$\mu^* := \mu_r \vee \max\bigl\{\mu^+, \mu^-\bigr\},$$

we have found an absolute constant which is independent on k and also does not depend on the set of indices $\mathcal{I}$. This solves the first question.

To complete the proof, we show how to produce at least one subharmonic solution. It is sufficient to take $\mathcal{I} := \{1\}$. As explained in Remark 4.3.1 and also at the end of Section 4.3.1, there exists a positive kT-periodic solution $u(t)$ for $(\mathscr{E}_\mu)$ such that $u \in \Lambda^{\{1\}}_{r,R}$. This implies that there exists $\hat{t}_1 \in I^+_1 = [0,\tau]$ such that $r < u(\hat{t}_1) < R$ and, if $i \neq 1$, $0 < u(t) < r$ for all $t \in I^+_i$. Then $u(\hat{t}_1) \neq u(t)$ for all $t \in I_i$ with $i \neq 1$, and hence u is not lT-periodic for all $l = 1, \dots, k-1$. We conclude that u is a subharmonic solution of order k. $\square$

Remark 5.1.1. The fact that the weight coefficient has simple zeros has been assumed only for convenience in the exposition. The same result holds true if we suppose that there are α, β with $\alpha < \beta < \alpha + T$ such that $a(t) \succ 0$ on $[\alpha, \beta]$, $a(t) \prec 0$ on $[\beta, \alpha + T]$ and $a(t)$ is not identically zero on all left neighborhoods of α and on all right neighborhoods of β. The possibility of more changes of sign of $a(t)$ in a period can be considered as well. $\lhd$

Remark 5.1.2. We stress the fact that μ^* is chosen independent on k and also independent on the set of indices $\mathcal{I}$. This is a crucial observation if one wants to prove the existence of bounded solutions defined on the whole real line and with *any prescribed behavior* as a limit of subharmonic solutions (see Section 5.3 and [17]). $\lhd$

5.2 Counting the subharmonic solutions of $(\mathscr{E}_\mu)$

Theorem 5.1.1 guarantees the existence of at least a subharmonic solution of order k for $(\mathscr{E}_\mu)$, but, in general, there are many solutions of this kind. Even if in the statement we have not described the number of subharmonics and their behavior, this can be achieved (with the same proof) just exploiting more deeply the content of Theorem 4.3.1. In this section, given an integer $k \geq 2$, we look for an estimate on the number of subharmonic solutions of order k. To this purpose, we adapt to our setting some considerations which are typical in the area of dynamical systems, combinatorics and graph theory.

First of all, we need to introduce a notation, which is borrowed from [17]. We start with an alphabet of two symbols, conventionally indicated as $\{0,1\}$, and denote by $\{0,1\}^k$ the set of the k-tuples of $\{0,1\}$, that is the set of finite words of length k. We also denote by $0^{[k]}$ the 0-string in $\{0,1\}^k$.

For simplicity, we still consider the special weight coefficient as in the setting of Theorem 5.1.1. Recalling the definitions of $I^{\pm}_i$, for $i = 1, \dots, k$, given by

$$I^+_i = [(i-1)T, \tau + (i-1)T] \quad \text{and} \quad I^-_i = [\tau + (i-1)T, iT],$$

and reworking as in the proof of Theorem 5.1.1, we have the following result.

Theorem 5.2.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \textit{and} \quad g_\infty = +\infty.$$

Then there exist $0 < r < R$ and $\mu^{**} > 0$ such that, for all $\mu > \mu^{**}$ and for every integer $k \geq 2$, given any k-tuple $\mathcal{S}^{[k]} = (s_i)_{i=1,\ldots,k} \in \{0,1\}^k$ with $\mathcal{S}^{[k]} \neq 0^{[k]}$, there exists at least a kT-periodic positive solution of equation $(\mathscr{E}_\mu)$ such that $\|u\|_\infty < R$ and

- *$0 < u(t) < r$ on I_i^+, if $s_i = 0$;*
- *$r < u(\hat{t}_i) < R$ for some $\hat{t}_i \in I_i^+$, if $s_i = 1$;*
- *$0 < u(t) < r$ on I_i^-, for all $i = 1,\ldots,k$.*

Proof. We proceed exactly as in the proof of Theorem 5.1.1 till to the final step where we chose the set of indices $\mathcal{I}$. At this moment r, R and μ^* are determined and we are free to take any $\mu > \mu^*$. Let us consider an arbitrary integer $k \geq 2$. Observe that we took $\mathcal{I} = \{1\}$ in order to be sure to have a subharmonic, however, Theorem 4.3.1 provides the existence of a positive kT-periodic solution in $\Lambda^{\mathcal{I}}_{r,R}$ for any nonempty subset $\mathcal{I}$ of $\{1,\ldots,k\}$.

Given an arbitrary k-tuple $\mathcal{S}^{[k]} = (s_i)_{i=1,\ldots,k} \in \{0,1\}^k$ with $\mathcal{S}^{[k]} \neq 0^{[k]}$, using a typical bijection between $\{0,1\}^k$ and the power set $\mathscr{P}(\{1,\ldots,k\})$, we associate to $\mathcal{S}^{[k]}$ the set

$$\mathcal{I}_{\mathcal{S}^{[k]}} := \bigl\{ i \in \{1,\ldots,k\} \colon s_i = 1 \bigr\}.$$

Now, applying Theorem 4.3.1, we have guaranteed the existence of at least one kT-periodic solution $u(t)$ which is positive and belongs to the set $\Lambda^{\mathcal{I}_{\mathcal{S}^{[k]}}}_{r,R}$. Recalling the definition of $\Lambda^{\mathcal{I}}_{r,R}$ in (4.1.2), we find that $u(t)$ satisfies the first two conditions in the statement of the theorem. The latter condition, concerning the smallness of $u(t)$ on the intervals I_i^-, follows from the result in Section 4.3.5 provided that μ is sufficiently large, say $\mu > \mu^{**}$. Arguing as in the proof of Theorem 5.1.1, it is easy to note that μ^{**} does not depend on k. □

The above theorem provides the existence of $2^k - 1$ distinct kT-periodic solutions of $(\mathscr{E}_\mu)$ which are positive and uniformly bounded in $\mathbb{R}$. Our goal now is to detect among these solutions the "true" subharmonics of order k which do not belong to the same periodicity class. Figure 5.1 gives an explanation of what we are looking for.

In order to count the k-tuples corresponding to subharmonic solutions of order k which are not equal up to translation (geometrically distinct), we notice that the number we are looking for coincides with the number of

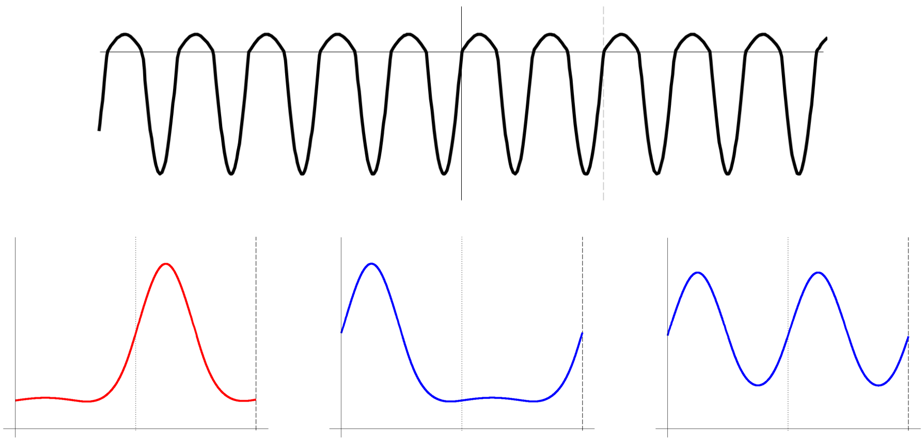

Figure 5.1: At the top we have shown the graph of the function $a_\mu(t)$, where $a(t) = \sin(2\pi t)$ and $\mu = 7$. Using a numerical simulation we have studied the subharmonic solutions of order $k = 2$ of equation $(\mathscr{E}_\mu)$ with $g(s) = \max\{0, 100\, s \arctan|s|\}$. Clearly, $T = 1$. In the lower part, the figure shows the graphs of three 2-periodic positive solutions, whose existence is consistent with Theorem 5.2.1. The first two solutions are subharmonic solutions of order 2 and the third one is a 1-periodic solution. As subharmonic solutions of order 2, we consider only the first one, since the second one is a translation by 1 of the first solution.

binary Lyndon words of length k, that is the number of binary strings inequivalent modulo rotation (cyclic permutation) of the digits and not having a period smaller than k. Usually, in each equivalence class one selects the minimal element in the lexicographic ordering. For instance, for the alphabet $\mathscr{A} = \{a, b\}$ and $k = 4$, the corresponding binary Lyndon words of length 4 are *aaab*, *aabb*, *abbb*. Note that the string *abab* is not acceptable as it represents a sequence of period 2 and the string *bbaa* is already counted as *aabb*. To give a formal definition, consider an alphabet $\mathscr{A}$ which, in our context, is a nonempty totally ordered set of $n \geq 2$ symbols. A *n-ary Lyndon word of length k* is a string of k digits of $\mathscr{A}$ which is strictly smaller in the lexicographic ordering than all of its nontrivial rotations.

The number of n-ary Lyndon words of length k is given by *Witt's formula*

$$\mathcal{L}_n(k) = \frac{1}{k} \sum_{l|k} \mu(l)\, n^{\frac{k}{l}}, \tag{5.2.1}$$

where $\mu(\cdot)$ is the Möbius function, defined on $\mathbb{N} \setminus \{0\}$ by $\mu(1) = 1$, $\mu(l) = (-1)^s$ if l is the product of s distinct primes and $\mu(l) = 0$ otherwise (cf. [119, § 5.1]). Formula (5.2.1) can be obtained by the *Möbius inversion formula*, which is strictly related with the classical *inclusion-exclusion principle*.

For instance, the values of $\mathcal{L}_2(k)$ (number of binary Lyndon words of length k) for $k = 2, \ldots, 10$ are 1, 2, 3, 6, 9, 18, 30, 56, 99.

The following proposition provides an explicit formula of $\mathcal{L}_n(k)$ (for arbitrary integers $n, k \geq 2$), depending on the *prime factorization* of k.

Proposition 5.2.1. *Let $n, k \geq 2$ be two integers. If the prime factorization of k is*

$$k = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot \ldots \cdot p_s^{\alpha_s} = \prod_{i=1}^{s} p_i^{\alpha_i},$$

where s is the number of distinct prime factors of k, then the following formula holds

$$\mathcal{L}_n(k) = \frac{1}{k} n^k + \frac{1}{k} \sum_{i=1}^{s} (-1)^i \sum_{\substack{j_d \in \{1,\ldots,s\} \\ j_1 < j_2 < \ldots < j_i}} n^{\frac{k}{p_{j_1} \cdot \ldots \cdot p_{j_i}}}.$$

Proof. First of all, we observe that the divisors l of the integer k such that $\mu(l) \neq 0$ are the square-free factors of k, hence $l = 1$ (with $\mu(1) = 1$) and the integers of the form $l = p_{j_1} \cdot \ldots \cdot p_{j_i}$ for $j_d \in \{1, \ldots, s\}$ (with $\mu(l) = (-1)^i$). The above formula immediately follows from (5.2.1). $\square$

Remark 5.2.1. Although in this context formula (5.2.1) and the more explicit one in Proposition 5.2.1 are related to the number of Lyndon words of length k in an alphabet of size n, these formulas come out in different areas of mathematics. Now we provide an overview of the several meanings of (5.2.1).

Still in combinatorics, it is not difficult to see that $\mathcal{L}_n(k)$ is also the number of *aperiodic necklaces* that can be made by arranging k beads whose color is chosen from a list of n colors (see Figure 5.2). The notions of *Lyndon words* and *necklaces* are also strictly related to *de Bruijn sequences*. We recall that a *n-ary de Bruijn sequence of order k* is a circular string of characters chosen in an alphabet of size n, for which every possible subsequence of length k appears as a substring of consecutive characters exactly once. For more details about these concepts and other aspects of the formula in the context of combinatorics on words, we refer to [119, 121] and the very interesting historical survey [23, § 4].

The number $\mathcal{L}_n(k)$ has several meanings even outside combinatorics. For instance, the integer $\mathcal{L}_2(k)$ (of binary Lyndon words of length k) corresponds to the number of periodic points with minimal period k in the iteration of the tent map $f(x) := 2\min\{x, 1-x\}$ on the unit interval (cf. [72], also for more general formulas) and to the number of distinct cycles of minimal period k in a shift dynamical system associated with a totally disconnected hyperbolic iterated function system (cf. [16, Lemma 1, p. 171]). Concerning the more general formula for $\mathcal{L}_n(k)$, we just mention two other meanings. The classical *Witt's formula* (proved in 1937), which is still widely studied in algebra, gives the dimensions of the homogeneous components of degree k of the

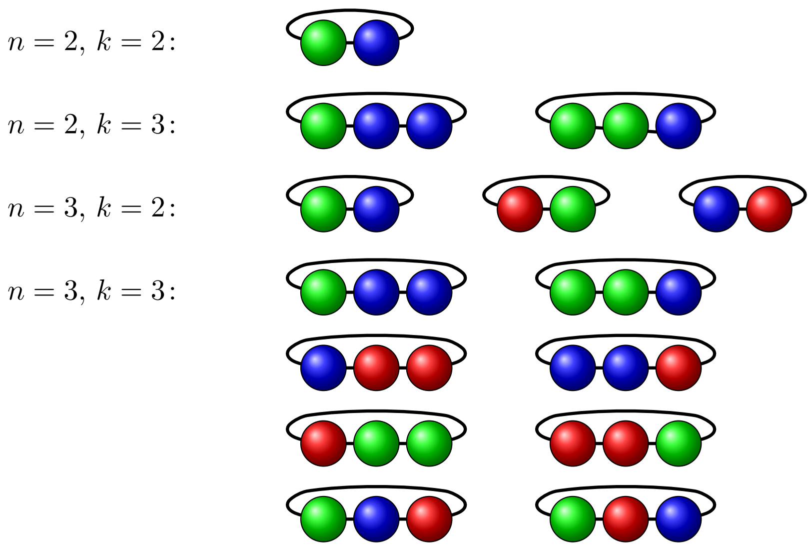

Figure 5.2: The figure shows the aperiodic necklaces made by arranging k beads whose color is chosen from a list of n colors, when $n, k \in \{2, 3\}$.

free Lie algebra over a finite set with n elements (cf. [119, Corollary 5.3.5]). Moreover, in Galois theory, $\mathcal{L}_n(k)$ is also the number of monic irreducible polynomials of degree k over the finite field $\mathbb{F}_n$, when n is a prime power (in this context (5.2.1) is also known as *Gauss formula*; we refer to [73, ch. 14, p. 588] for a possible proof).

It is not possible to mention here all the other several implications of formula (5.2.1), for example in symbolic dynamics, algebra, number theory and chaos theory. For this latter topic, we only recall the recent paper [109] where such numbers appear in connection with the study of period-doubling cascades.

Further information and references can be found in [97, 111, 170]. $\lhd$

Using the above discussion, we achieve the following result.

Theorem 5.2.2. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic continuous function with minimal period T such that there exist two consecutive zeros α, β with $\alpha < \beta < \alpha + T$ so that $a(t) > 0$ for all $t \in \,]\alpha, \beta[$ and $a(t) < 0$ for all $t \in \,]\beta, \alpha + T[$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \textit{and} \quad g_\infty = +\infty.$$

Then there exists $\mu^* > 0$ such that, for all $\mu > \mu^*$ and for every $k \geq 2$, equation $(\mathscr{E}_\mu)$ has at least $\mathcal{L}_2(k)$ positive subharmonic solutions of order k.

Proof. We have to detect the subharmonic solutions of order k among the $2^k - 1$ distinct kT-periodic positive solutions of $(\mathscr{E}_\mu)$ provided by Theorem 5.2.1. As remarked above, the number we are looking for is $\mathcal{L}_2(k)$. Therefore the thesis immediately follows. $\square$

For the sake of simplicity, above we have considered only the particular case of a continuous periodic sign-changing function $a(t)$ with minimal period T and such that it has only one positive hump and one negative one in a period interval. Moreover, we have taken a superlinear function $g(s)$. We conclude this section by stating the analogous result for more general functions $a(t)$ and $g(s)$.

Theorem 5.2.3. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*) with minimal period T. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \qquad \textit{and} \qquad g_\infty > \max_{i=1,\ldots,m} \lambda_1^i.$$

Then there exists $\mu^* > 0$ such that, for all $\mu > \mu^*$ and for every $k \geq 2$, equation $(\mathscr{E}_\mu)$ has at least $\mathcal{L}_{2^m}(k)$ positive subharmonic solutions of order k.

Proof. We only sketch the proof which is mimicked from those of Theorem 5.1.1 and of Theorem 5.2.1, using Theorem 4.3.1. To start, we need to be careful with the notation. For this reason, we call $J_1^+, \ldots, J_m^+$ the m intervals of positivity for $a(t)$ in the interval $[0,T]$ and $J_1^-, \ldots, J_m^-$ the m intervals of negativity for $a(t)$, according to assumption (a^*). Consider an arbitrary integer $k \geq 2$. The function $a(t)$ restricted to the interval $[0,kT]$ satisfies again an assumption of the form (a^*), with respect to mk intervals of positivity/negativity that we denote now with $I_1^\pm, \ldots, I_{mk}^\pm$, defined as

$$I_{j+\ell m}^\pm = J_j^\pm + \ell T, \qquad j = 1,\ldots,m, \quad \ell = 0,\ldots,k-1.$$

In other terms, in the interval $[0,kT]$ there are mk closed subintervals where $a(t) \succ 0$, separated by closed subintervals where $a(t) \prec 0$. Then we can apply Theorem 4.3.1, looking for kT-periodic solutions. In fact, by our main result, we have at least $2^{mk}-1$ positive periodic solutions of period kT (which up to now is not necessarily the minimal period for the solutions). More precisely, as in Theorem 5.2.1, there exist $0 < r < R$ and $\mu^{**} > 0$ (depending on m but not on k) such that, for all $\mu > \mu^{**}$, given any nontrivial k-tuple $\mathcal{S}^{[k]} = (s_\ell)_{\ell=0,\ldots,k-1}$ in the alphabet $\mathscr{A} := \{0,1\}^m$ of size 2^m (hence, for $\ell = 0,\ldots,k-1$, $s_\ell = (s_\ell^j)_{j=1,\ldots,m}$), there exists at least one kT-periodic positive solution

$$u(t) = u_{\mathcal{S}^{[k]}}(t)$$

of equation $(\mathscr{E}_\mu)$ such that $\|u\|_\infty < R$ and

- $0 < u(t) < r$ on $I_{j+\ell m}^+$, if $s_\ell^j = 0$;
- $r < u(\hat{t}) < R$ for some $\hat{t} \in I_{j+\ell m}^+$, if $s_\ell^j = 1$;
- $0 < u(t) < r$ on I_i^-, for all $i = 1,\ldots,mk$.

It remains to see whether, on the basis of the information we have on $u(t)$, we are able first to determine the minimality of the period and next to distinguish among solutions do not belonging to the same periodicity class. In view of the above listed properties of the solution $u(t)$, our first problem is equivalent to choosing a string $\mathcal{S}^{[k]}$ having k as a minimal period (when repeated cyclically). For the second question, given any string of this kind, we count as the same all those strings (of length k) which are equivalent by cyclic permutations. To choose exactly one string in each of these equivalence classes, we can take the minimal one in the lexicographic order. As a consequence, we can conclude that there are so many nonequivalent kT-periodic solutions which are not pT-periodic for every $p = 1, \ldots, k-1$, as many 2^m-ary Lyndon words of length k. Since we know that the equation does not possess exceptional solutions, we find that for these subharmonic solutions kT is precisely the minimal period. $\square$

We have listed before some values of $\mathcal{L}_2(k)$ which give the number of subharmonic solutions in the setting of Theorem 5.2.2. Concerning the general case addressed in Theorem 5.2.3, we observe that the number $\mathcal{L}_{2^m}(k)$, with $m \geq 2$, grows very fast with k. For instance, the values of $\mathcal{L}_{2^2}(k)$ (number of quaternary Lyndon words of length k) for $k = 2, \ldots, 10$ are 6, 20, 60, 204, 670, 2340, 8160, 29120, 104754.

5.3 Positive solutions with complex behavior

In this section we just outline a possible procedure in order to obtain the existence of solutions which follow any preassigned coding described by two symbols, say 0 and 1, that in our context will be interpreted as "small" and, respectively, "large" in the intervals where the weight is positive. In other terms we are looking for the presence of a Bernoulli shift as a factor within the set of positive and bounded solutions. Results in this direction are classical in the theory of dynamical systems (cf. [67, 141, 171]) and have been achieved in the variational setting as well (see, for instance, [41, 50, 166]). Even if the obtention of chaotic dynamics using topological degree or index theories is an established technique (see [54, 174] and the references therein), the achievement of similar results with our approach seems new in the literature.

Our proof is based on the above results about subharmonic solutions and on the following diagonal lemma, which is typical in this context. Lemma 5.3.1 is adapted from [112, Lemma 8.1] and [133, Lemma 4].

Lemma 5.3.1. *Let $f \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be an L^1-Carathéodory function. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers and $(x_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}^d$ with the following properties:*

(i) $t_n \to +\infty$ *as* $n \to \infty$*;*

(ii) for each $n \in \mathbb{N}$, $x_n(t)$ is a solution of

$$x' = f(t,x) \tag{5.3.1}$$

defined on $[-t_n, t_n]$;

(iii) for every $N \in \mathbb{N}$ there exists a bounded set $B_N \subseteq \mathbb{R}^d$ such that, for each $n \geq N$, it holds that $x_n(t) \in B_N$ for every $t \in [-t_N, t_N]$.

Then there exists a subsequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which converges uniformly on the compact subsets of $\mathbb{R}$ to a solution $\tilde{x}(t)$ of system (5.3.1)*; in particular $\tilde{x}(t)$ is defined on $\mathbb{R}$ and, for each $N \in \mathbb{N}$, it holds that $\tilde{x}(t) \in \overline{B_N}$ for all $t \in [-t_N, t_N]$.*

Proof. This result is classical and perhaps a proof is not needed. We give a sketch of the proof for the reader's convenience, following [133, Lemma 4].

First of all we observe that, by the Carathéodory assumption, for each $N \in \mathbb{N}$ there exists a measurable function $\rho_N \in L^1([-t_N, t_N], \mathbb{R}^+)$ such that

$$\|f(t,x)\| \leq \rho_N(t), \quad \text{for a.e. } t \in [-t_N, t_N] \text{ and for all } x \in B_N.$$

For every $N \in \mathbb{N}$ we also introduce the absolutely continuous function

$$\mathcal{M}_N(t) := \int_0^t \rho_N(\xi)\, d\xi, \quad t \in [-t_N, t_N].$$

By hypothesis (ii), we have that

$$x_n(t) = x_n(0) + \int_0^t f(\xi, x_n(\xi))\, d\xi, \quad \forall\, t \in [-t_n, t_n],\ \forall\, n \in \mathbb{N},$$

and, by hypothesis (iii), for every $N \in \mathbb{N}$ it follows that

$$|x_n(t') - x_n(t'')| \leq |\mathcal{M}_N(t') - \mathcal{M}_N(t'')|, \quad \forall\, t', t'' \in [-t_N, t_N],\ \forall\, n \geq N,$$

(cf. [104, p. 29]). Consequently, the sequence $(x_n)_{n\in\mathbb{N}}$ restricted to the interval $[-t_0, t_0]$ is uniformly bounded (by any constant which bounds in the Euclidean norm the set B_0) and equicontinuous. By Ascoli-Arzelà theorem, it has a subsequence $(x_n^0)_{n\in\mathbb{N}}$ which converges uniformly on $[-t_0, t_0]$ to a continuous function named $\hat{x}_0$. Similarly, the sequence $(x_n^0)_{n\geq 1}$ restricted to $[-t_1, t_1]$ is a uniformly bounded and equicontinuous sequence and has a subsequence $(x_n^1)_{n\geq 1}$ which converges uniformly on $[-t_1, t_1]$ to a continuous function $\hat{x}_1$ such that $\hat{x}_1(t) = \hat{x}_0(t)$ for all $t \in [-t_0, t_0]$. Proceeding inductively in this way, we construct a sequence of sequences $(x_n^N)_{n\geq N}$ so that $(x_n^N)_{n\geq N}$ is a subsequence of $(x_n^{N-1})_{n\geq N-1}$ and converges uniformly on $[-t_N, t_N]$ to a continuous function $\hat{x}_N$ such that $\hat{x}_N(t) = \hat{x}_{N-1}(t)$ for all $t \in [-t_{N-1}, t_{N-1}]$. By construction, we have that $\hat{x}_N(t) \in \overline{B_N}$ for all

$t \in [-t_N, t_N]$. The diagonal sequence $(\tilde{x}_n)_{n\in\mathbb{N}} := (x_n^n)_{n\in\mathbb{N}}$ converges uniformly on every compact interval to a function $\tilde{x}$ defined on $\mathbb{R}$ and such that $\tilde{x}(t) = \hat{x}_N(t)$ for all $t \in [-t_N, t_N]$ and therefore, $\tilde{x}(t) \in \overline{B_N}$ for all $t \in [-t_N, t_N]$. It remains to prove that $\tilde{x}(t)$ is a solution of (5.3.1) on $\mathbb{R}$. Indeed, let $t \in \mathbb{R}$ be arbitrary but fixed and let us fix $N \in \mathbb{N}$ such that $t \in [-t_N, t_N]$. Passing to the limit as $n \to \infty$ in the identity

$$\tilde{x}_n(t) = \tilde{x}_n(0) + \int_0^t f(\xi, \tilde{x}_n(\xi))\, d\xi, \quad \forall\, n \geq N,$$

via the dominated convergence theorem, we obtain

$$\tilde{x}(t) = \tilde{x}(0) + \int_0^t f(\xi, \tilde{x}(\xi))\, d\xi.$$

For the arbitrariness of $t \in \mathbb{R}$ and the above integral relation, we conclude that $\tilde{x}(t)$ is absolutely continuous and a solution of (5.3.1) (in the Carathéodory sense). □

If there exists a bounded set B such that $B_N \subseteq B$ for all $N \in \mathbb{N}$, then we have the stronger conclusion that $\tilde{x}(t) \in \overline{B}$ for all $t \in \mathbb{R}$ (which is precisely the result of [112, Lemma 8.1] and [133, Lemma 4]).

An application of Lemma 5.3.1 to the planar system

$$\begin{cases} u' = y \\ y' = -cy - \big(a^+(t) - \mu a^-(t)\big) g(u) \end{cases} \tag{5.3.2}$$

will produce bounded solutions with any prescribed complex behavior. In order to simplify the exposition, we suppose that the coefficient $a(t)$ is a continuous T-periodic function of minimal period T having a positive hump followed by a negative one in a period interval (these are the same assumptions for the weight coefficient as in Theorem 5.1.1). In this framework, the next result follows.

Theorem 5.3.1. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic continuous function with minimal period T such that there exist two consecutive zeros α, β with $\alpha < \beta < \alpha + T$ so that $a(t) > 0$ for all $t \in \,]\alpha, \beta[$ and $a(t) < 0$ for all $t \in \,]\beta, \alpha+T[$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \text{and} \quad g_\infty = +\infty.$$

Then there exist $0 < r < R$ and $\mu^{**} > 0$ such that, for all $\mu > \mu^{**}$, given any two-sided sequence $\mathcal{S} = (s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ which is not identically zero, there exists at least a positive solution $u(t) = u_{\mathcal{S}}(t)$ of equation $(\mathscr{E}_\mu)$ such that $\|u\|_\infty < R$ and

- *$0 < u(t) < r$ on $[\alpha + iT, \beta + iT]$, if $s_i = 0$;*

- $r < u(\hat{t}_i) < R$ *for some* $\hat{t}_i \in [\alpha + iT, \beta + iT]$, *if* $s_i = 1$;
- $0 < u(t) < r$ *on* $[\beta + iT, \alpha + (i+1)T]$, *for all* $i \in \mathbb{Z}$.

Proof. Without loss of generality, we suppose that $\alpha = 0$ and set $\tau := \beta - \alpha$, so that $a(t) > 0$ on $]0, \tau[$ and $a(t) < 0$ on $]\tau, T[$. We also introduce the intervals

$$I_i^+ := [iT, \tau + iT], \quad I_i^- := [\tau + iT, (i+1)T], \quad i \in \mathbb{Z}. \tag{5.3.3}$$

Let $0 < r < R$ and $\mu^{**} > 0$ as in Theorem 5.1.1 and Theorem 5.2.1. One more time, we wish to emphasize the fact that, once we have fixed r, R and $\mu > \mu^{**}$, we can produce kT-periodic solutions following any k-periodic sequence of two symbols, *independently on* k. Accordingly, from this moment to the end of the proof, r, R and $\mu > \mu^{**}$ are fixed.

Consider now an arbitrary sequence $\mathcal{S} = (s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ which is not identically zero. We fix a positive integer n_0 such that there is at least an index $i \in \{-n_0, \dots, n_0\}$ such that $s_i = 1$. Then, for each $n \geq n_0$ we consider the $(2n+1)$-periodic sequence $\mathcal{S}^n = (s'_i)_i \in \{0,1\}^{\mathbb{Z}}$ which is obtained by truncating $\mathcal{S}$ between $-n$ and n, and then repeating that string by periodicity. An application of Theorem 5.2.1 on the periodicity interval $[-nT, (n+1)T]$ ensures the existence of a positive periodic solution $u_n(t)$ such that $u_n(t+(2n+1)T) = u_n(t)$ for all $t \in \mathbb{R}$ and $\|u_n\|_\infty < R$. According to Theorem 5.2.1, we also know that $u_n(t) < r$ for all $t \in I_i^+$, if $s'_i = 0$, $u_n(\hat{t}) > r$ for some $\hat{t} \in I_i^+$, if $s'_i = 1$, and $\max_{t\in I_i^-} u_n(t) < r$ (for each $i \in \mathbb{Z}$).

Notice that, for $C := \max_{0\leq s\leq R} g(s)$, we have that

$$|u''_n(t)| \leq |c||u'_n(t)| + \bigl(|a^+(t)| + \mu|a^-(t)|\bigr)C, \quad \forall\, t \in \mathbb{R},$$

and hence,

$$\frac{|u''_n(t)|}{1 + |u'_n(t)|} \leq \psi_\mu(t), \quad \forall\, t \in \mathbb{R}, \tag{5.3.4}$$

where we have set $\psi_\mu(t) := |c| + (|a^+(t)| + \mu|a^-(t)|)C$.

Since the truncated string $\mathcal{S}^n$ contains at least one $s'_i = s_i = 1$, with $i \in \{-n_0, \dots, n_0\}$, we know that each periodic function $u_n(t)$ has at least a local maximum point $\hat{t}_n \in \,]-n_0T, n_0T + \tau[$ and then $u'_n(\hat{t}_n) = 0$. Suppose now that $N \geq n_0$ is fixed and define the constant

$$K_N := \exp\biggl((2N+1)\int_0^T \psi_\mu(t)\,dt\biggr).$$

We claim that

$$|u'_n(t)| \leq K_N, \quad \forall\, t \in [-NT, (N+1)T],\ \forall\, n \geq N. \tag{5.3.5}$$

Our claim follows from a Nagumo type argument as in [62, ch. I, § 4]. Suppose, by contradiction, that (5.3.5) is not true. Hence, there exist some

$n \geq N$ and a point $t_n^* \in [-NT, (N+1)T]$ such that $u_n'(t_n^*) > K_N$ or $u_n'(t_n^*) < -K_N$. In the first case there exists a maximal interval $J \subseteq [-NT, (N+1)T]$ such that one of the following two possibilities occurs:

- $J = [\xi_0, \xi_1]$ and $u_n'(\xi_0) = 0$, $u_n'(\xi_1) > K_N$ with $u_n'(t) > 0$ for all $t \in]\xi_0, \xi_1]$;
- $J = [\xi_1, \xi_0]$ and $u_n'(\xi_0) = 0$, $u_n'(\xi_1) > K_N$ with $u_n'(t) > 0$ for all $t \in [\xi_1, \xi_0[$.

Integrating $u_n''/(1+|u_n'|)$ on J and using (5.3.4), we obtain

$$\begin{aligned}\log(1+K_N) < \log(1+|u_n'(\xi_1)|) &\leq \int_J \psi_\mu(t)\,dt \\ &\leq \int_{-NT}^{(N+1)T} \psi_\mu(t)\,dt = (2N+1)\int_0^T \psi_\mu(t)\,dt = \log(K_N),\end{aligned}$$

a contradiction. We have achieved a contradiction by assuming $u_n'(t_n^*) > K_N$. A similar argument gives a contradiction if $u_n'(t_n^*) < -K_N$.

Now we write equation $(\mathscr{E}_\mu)$ as a planar system (5.3.2). From the above remarks, one can see that (up to a reparametrization of indices, counting from n_0) assumptions (i), (ii) and (iii) of Lemma 5.3.1 are satisfied, taking $t_n := nT$, $f(t,x) = (y, -cy - (a^+(t) - \mu a^-(t))g(u))$, with $x = (u,y)$, and

$$B_N := \bigl\{x \in \mathbb{R}^2 \colon 0 < x_1 < R,\ |x_2| \leq K_N\bigr\}, \quad N \in \mathbb{N},$$

as bounded set in $\mathbb{R}^2$. By Lemma 5.3.1, there is a solution $\tilde{u}(t)$ of equation $(\mathscr{E}_\mu)$ which is defined on $\mathbb{R}$ and such that $0 \leq \tilde{u}(t) \leq R$ for all $t \in [-NT, NT]$, for each $N \in \mathbb{N}$. Then $\|\tilde{u}\|_\infty \leq R$. Moreover, such a solution $\tilde{u}(t)$ is the limit of a subsequence $(\tilde{u}_n)_n$ of the sequence of the periodic solutions $u_n(t)$.

We claim that

- $0 < \tilde{u}(t) < r$ on I_i^+, if $s_i = 0$;
- $r < \tilde{u}(\hat{t}_i) < R$ for some $\hat{t}_i \in I_i^+$, if $s_i = 1$;
- $0 < \tilde{u}(t) < r$ on I_i^-, for all $i \in \mathbb{Z}$.

To prove our claim, let us fix $i \in \mathbb{Z}$ and consider the interval I_i^+ introduced in (5.3.3). For each $n \geq |i|$ (and $n \geq n_0$) the periodic solution $u_n(t)$ is defined on $\mathbb{R}$ and such that $0 < u_n(t) < r$ for all $t \in I_i^+$, if $s_i = 0$, or $\max_{t \in I_i^+} u_n(t) > r$, if $s_i = 1$. Passing to the limit on the subsequence $(\tilde{u}_n)_n$, we obtain that

$$0 \leq \tilde{u}(t) \leq r, \quad \forall\, t \in I_i^+, \quad \text{if } s_i = 0,$$

or

$$\max_{t \in I_i^+} \tilde{u}(t) \geq r, \quad \text{if } s_i = 1,$$

respectively. With the same argument we also prove that

$$0 \leq \tilde{u}(t) \leq r, \quad \forall\, t \in I_i^-, \ \forall\, i \in \mathbb{Z}.$$

By Remark 4.3.8 we get that $\tilde{u}(t) < R^* \leq R$, for all $t \in \mathbb{R}$. Moreover, since there exists at least one index $i \in \mathbb{Z}$ such that $s_i = 1$, we know that $\tilde{u}$ is not identically zero. Hence, a maximum principle argument shows that $\tilde{u}(t)$ never vanishes. In conclusion, we have proved that

$$0 < \tilde{u}(t) < R, \quad \forall\, t \in \mathbb{R}.$$

Next, we observe that

$$\max_{t \in I_i^+} \tilde{u}(t) \neq r, \quad \forall\, i \in \mathbb{Z}.$$

Indeed, this is a consequence of Remark 4.3.10, using the fact that the solution $\tilde{u}(t)$ is upper bounded by R^* and, at the beginning, μ has been chosen large enough (note also that we apply that result in the case $m = 1$ and therefore the sets $I_i^+ + \ell T$ of Remark 4.3.10 reduce, in our case, to the intervals $[0, \tau] + \ell T$). Finally, using Remark 4.3.12 we also deduce that

$$\tilde{u}(t) < r, \quad \forall\, t \in I_i^-, \ \forall\, i \in \mathbb{Z}.$$

Our claim is thus verified and this completes the proof of the theorem. □

For the equation

$$u'' + \big(a^+(t) - \mu a^-(t)\big) u^3 = 0,$$

a version of Theorem 5.3.1 has been recently obtained in [17], under the supplementary condition that in the strings of symbols the consecutive sequences of zeros are bounded in length. The proof of [17, Theorem 2.1] and ours are completely different (the former one relies on variational techniques, ours on degree theory). Our new contribution is twofold: on one side, we can deal with non Hamiltonian systems (indeed we can consider also a term of the form cu') and with a nonlinearity $g(s)$ which is not positively homogeneous; on the other hand, our approach allows to remove the condition on bounded sequences of consecutive zeros. In any case, the two results are not completely comparable since the way to associate a solution to a given string of symbols is different: the symbols 0 and 1 in our case are associated to the maximum of a solution on I_i^+, while in [17, Theorem 2.1] are associated to an integral norm on the same interval.

We remark that Theorem 5.3.1 can be generalized at the same extent like Theorem 5.2.3 generalizes Theorem 5.2.2. Indeed, combining the proofs of Theorem 5.2.3 and Theorem 5.3.1, we can obtain the following result (the proof is omitted).

Theorem 5.3.2. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*) with minimal period T. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*),*

$$g_0 = 0 \quad \text{and} \quad g_\infty > \max_{i=1,\ldots,m} \lambda_1^i.$$

Then there exist $0 < r < R$ and $\mu^{**} > 0$ such that, for all $\mu > \mu^{**}$, given any two-sided sequence $\mathcal{S} = (s_\ell)_{\ell\in\mathbb{Z}}$ in the alphabet $\mathscr{A} := \{0,1\}^m$ and not identically zero, there exists at least a positive solution $u(t) = u_{\mathcal{S}}(t)$ of equation $(\mathscr{E}_\mu)$ such that $\|u\|_\infty < R$ and the following properties hold (where we set $s_\ell = (s_\ell^i)_{i=1,\ldots,m}$, for each $\ell \in \mathbb{Z}$):

- *$0 < u(t) < r$ on $I_i^+ + \ell T$, if $s_\ell^i = 0$;*
- *$r < u(\hat{t}) < R$ for some $\hat{t} \in I_i^+ + \ell T$, if $s_\ell^i = 1$;*
- *$0 < u(t) < r$ on $I_i^- + \ell T$, for all $i \in \{1,\ldots,m\}$ and for all $\ell \in \mathbb{Z}$.*

5.4 Obtaining subharmonic solutions by means of the Poincaré-Birkhoff fixed point theorem

In this second part of the chapter, we present a different approach that provides existence of *positive subharmonic solutions* for nonlinear second order ODEs with indefinite weight. To describe our results, throughout this introductory section we focus our attention on the superlinear indefinite equation

$$u'' + a(t)u^p = 0, \tag{5.4.1}$$

with $a(t)$ a sign-changing T-periodic function and $p > 1$, which has been indeed the main motivation for our investigation.

A first crucial observation is that a mean value condition on $a(t)$ turns out to be necessary for the existence of positive kT-periodic solutions (with $k \geq 1$ an integer number); indeed, dividing equation (5.4.1) by $u(t)^p$ and integrating on $[0,kT]$, one readily obtains

$$\int_0^{kT} a(t)\,dt = -p \int_0^{kT} \left(\frac{u'(t)}{u(t)^p}\right)^2 u(t)^{p-1}\,dt,$$

so that (recalling that $a(t)$ is T-periodic)

$$\int_0^T a(t)\,dt < 0. \tag{5.4.2}$$

This fact was already observed in the introduction of Chapter 3.

In Chapter 3, a topological approach (based on Mawhin's coincidence degree) was introduced to prove that the mean value condition (5.4.2) guarantees the existence of a positive T-periodic solution for a large class of indefinite equations including (5.4.1). On one hand, this result seems to be optimal (in the sense that no more than one T-periodic solution can be expected for a general weight function with negative mean value); on the other hand, however, it is known that positive solutions to (5.4.1) can exhibit complex behavior for special choices of the weight function $a(t)$, as shown in the first part of the present chapter. Namely, whenever $a(t)$ has large negative part, equation (5.4.1) has infinitely many positive subharmonic solutions, as well as globally defined positive solutions with chaotic-like multibumb behavior.

It appears therefore a quite natural question if the sharp mean value condition (5.4.2), besides implying the existence of a positive T-periodic solution to (5.4.1), also guarantees the existence of positive subharmonic solutions. Quite unexpectedly, as a corollary of our main results, we are able to show that the answer is always affirmative .

Theorem 5.4.1. *Assume that $a\colon \mathbb{R} \to \mathbb{R}$ is a sign-changing continuous and T-periodic function, having a finite number of zeros in $[0, T[$ and satisfying the mean value condition* (5.4.2)*. Then, equation* (5.4.1) *has a positive T-periodic solution, as well as positive subharmonic solutions of order k, for any large integer number k.*

Actually, the assumptions on the weight function $a(t)$ can be considerably weakened and the conclusion about the number of subharmonic solutions obtained can be made much more precise. We refer to Section 5.6 for more general statements.

Let us emphasize that investigating the existence of subharmonic solutions for time-periodic ODEs is often a quite delicate issue, the more difficult point being the proof of the minimality of the period. In Theorem 5.4.1, kT-periodic solutions $u_k(t)$ are found (for k large enough) oscillating around a T-periodic solution $u^*(t)$ and a precise information on the number of zeros of $u_k(t) - u^*(t)$ is the key point in showing that kT is the minimal period of $u_k(t)$. This approach, based on the celebrated Poincaré-Birkhoff fixed point theorem (cf. [57, 87, 96, 114]), was introduced (and then applied to Ambrosetti-Prodi type periodic problems) in the paper [39], to which we also refer for a quite complete bibliography about the theme of subharmonic solutions. It is worth noticing, however, that the application to equation (5.4.1) of the method described in [39] is not straightforward. First, due to the superlinear character of the nonlinearity, we cannot guarantee (as needed for the application of dynamical systems techniques) the global continuability of solutions to (5.4.1) (see [47]) and some careful a priori bounds have to be performed. Second, due to the indefinite character of the equation, it seems impossible to perform explicit estimates on the solutions in order to prove the needed twist-condition of the Poincaré-Birkhoff theorem. To overcome this difficulty, we first use an idea from [35] to develop an abstract variant of the main result in [39], replacing an explicit estimate on the positive T-periodic solution $u^*(t)$ with an information about its Morse index. Using a clever trick by Brown and Hess (cf. [44]), such an information is then easily achieved. We emphasize this simple property here, since it is the crucial point for our arguments: *any positive T-periodic solution of* (5.4.1) *has non-zero Morse index.*

Let us finally recall that variational methods can be an alternative tool for the study of subharmonic solutions. In this case, information about the minimality of the period can be often achieved with careful level estimates (see, among others, [86, 167]). Maybe this technique can be successfully applied also to the superlinear indefinite equation (5.4.1); however, it has to be noticed that usually results obtained via a symplectic approach (namely, using the Poincaré-Birkhoff theorem) give sharper information (see [35, 39]).

In Section 5.5 we present, on the lines of [35, 39], an auxiliary result ensuring, for a quite broad class of nonlinearities, the existence of subharmonic solutions oscillating around a T-periodic solution with non-zero Morse index. In Section 5.6 we state our main results, dealing with equations of the type $u'' + a(t)g(u) = 0$ with $a(t)$ satisfying (5.4.2) and $g(u)$ defined on a (possibly bounded) interval of the type $[0, d[$; roughly speaking, we have that the existence of positive subharmonic solutions (oscillating around a positive T-periodic solution) is always guaranteed whenever $g(u)$ is superlinear at zero and strictly convex, with $g(u)/u$ large enough near $u = d$. Applications are given to equations superlinear at infinity (thus generalizing Theorem 5.4.1), to equations with a singularity as well as to parameter-dependent equations. In Section 5.7 we give the proof of these results; in more detail, we first prove (using a degree approach, together with the trick in [44]) the existence of a positive T-periodic solution with non-zero Morse index and we then apply the results of Section 5.5 to obtain the desired positive subharmonic solutions around it. Section 5.8 is devoted to some conclusive comments about our investigation.

5.5 Morse index, Poincaré-Birkhoff theorem, subharmonics

In this section, we present our auxiliary result for the search of subharmonic solutions to scalar second order ODEs of the type

$$u'' + h(t,u) = 0, \tag{5.5.1}$$

where $h\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function T-periodic in the first variable (for some $T > 0$). Motivated by the applications to equations like (5.4.1) with $a \in L^1([0,T])$, we set up our result in a Carathéodory setting. More precisely, we assume that the function $h(t,u)$ is measurable in the t-variable, continuously differentiable in the u-variable and satisfies the following condition: for any $r > 0$, there exists $m_r \in L^1([0,T])$ such that $|h(t,u)| + |\partial_u h(t,u)| \leq m_r(t)$ for a.e $t \in [0,T]$ and for every $u \in \mathbb{R}$ with $|u| \leq r$. Of course, in view of this assumption, solutions to (5.5.1) will be meant in the generalized sense, i.e. $W^{2,1}_{\mathrm{loc}}$-functions satisfying equation (5.5.1) for a.e. t.

Finally, we introduce the following notation. For any $q \in L^1([0,T])$, we denote by $\lambda_0(q)$ the principal eigenvalue of the linear problem

$$v'' + (\lambda + q(t))v = 0, \tag{5.5.2}$$

with T-periodic boundary conditions. As well known (see, for instance, [55, ch. 8, Theorem 2.1] and [122, Theorem 2.1]) $\lambda_0(q)$ exists and is the unique real number such that the linear equation (5.5.2) admits one-signed T-periodic solutions. Recalling that, by definition, the *Morse index* $m(q)$ of the linear equation $v'' + q(t)v = 0$ is the number of (strictly) negative T-periodic eigenvalues of (5.5.2), we immediately see that $\lambda_0(q) < 0$ if and only if $m(q) \geq 1$.

We are now in position to state the following result.

Proposition 5.5.1. *Let $h(t,u)$ be as above and assume that the global continuability for the solutions to* (5.5.1) *is guaranteed. Moreover, suppose that:*

(i) there exists a T-periodic solution $u^*(t)$ of (5.5.1) *satisfying*

$$\lambda_0(\partial_u h(t,u^*(t))) < 0; \tag{5.5.3}$$

(ii) there exists a T-periodic function $\alpha \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$ satisfying

$$\alpha''(t) + h(t,\alpha(t)) \geq 0, \quad \text{for a.e. } t \in \mathbb{R}, \tag{5.5.4}$$

and

$$\alpha(t) < u^*(t), \quad \text{for any } t \in \mathbb{R}.$$

Then there exists $k^* \geq 1$ such that for any integer $k \geq k^*$ there exists an integer $m_k \geq 1$ such that, for any integer j relatively prime with k and such that $1 \leq j \leq m_k$, equation (5.5.1) *has two subharmonic solutions $u^{(1)}_{k,j}(t)$, $u^{(2)}_{k,j}(t)$ of order k (not belonging to the same periodicity class), such that, for $i = 1,2$, $u^{(i)}_{k,j}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0,kT[$ and*

$$\alpha(t) \leq u^{(i)}_{k,j}(t), \quad \text{for any } t \in \mathbb{R}. \tag{5.5.5}$$

Incidentally, we observe that Proposition 5.5.1 in particular ensures that equation (5.5.1) has two subharmonic solutions of order k (not belonging to the same periodicity class) for any large integer k (just, take $j = 1$ in the above statement).

Remark 5.5.1. Let us recall that a T-periodic function $\alpha \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$ satisfying (5.5.4) is a *lower solution* for the T-periodic problem associated with (5.5.1) (weaker notions of lower/upper solutions could be introduced in the Carathéodory setting, see [61]). Clearly, if $\alpha(t)$ is a T-periodic solution of (5.5.1), then $\alpha(t)$ is a T-periodic lower solution; in this case, due to the uniqueness for the Cauchy problems, (5.5.5) implies that $\alpha(t) < u^{(i)}_{k,j}(t)$ for any t. $\lhd$

Proposition 5.5.1 is a variant of [39, Theorem 2.2] (actually, [39, Theorem 2.2] deals with the symmetric case assuming the existence of an upper solution $\beta(t) > u^*(t)$). However, some care is needed in comparing the two results. First, [39, Theorem 2.2] is stated for $h(t,u)$ smooth; the generalization to the Carathéodory setting in this case is not completely straightforward. Second, the assumption corresponding to (i) in [39, Theorem 2.2] reads as

$$\int_0^T \partial_u h(t, u^*(t))\, dt > 0. \tag{5.5.6}$$

The possibility of replacing this explicit condition with the abstract assumption $\lambda_0(\partial_u h(t, u^*(t))) < 0$ has been discussed in [35, Theorem 2.1] (actually, in [35, Theorem 2.1] the case $u^*(t) \equiv 0$ is taken into account, but, the two situations are equivalent via the linear change of variable given in the proof of [39, Proposition 1]). In that paper the assumption $\rho(\partial_u h(t, u^*(t))) > 0$ is used, where $\rho(q)$ is the Moser rotation number (see [142]) of the linear equation $v'' + q(t)v = 0$. However, it is very well known in the theory of the Hill's equation (see, for instance, [91, Proposition 2.1]) that $\rho(q) > 0$ if and only if $\lambda_0(q) < 0$ (that is, if and only if the equation $v'' + q(t)v = 0$ is *not disconjugate*).

Related results, yielding the existence and the multiplicity of harmonic (i.e. T-periodic) solutions according to the interaction of the nonlinearity with (non-principal) eigenvalues, can be found in [125, 126, 181].

Now, we provide just a sketch of the proof of Proposition 5.5.1, based on the Poincaré-Birkhoff fixed point theorem, referring to previous papers (in particular, to [35, 39]) for the most standard steps.

Sketch of the proof of Proposition 5.5.1. We define the truncated function

$$\tilde{h}(t,u) := \begin{cases} h(t, \alpha(t)), & \text{if } u \le \alpha(t); \\ h(t,u), & \text{if } u > \alpha(t); \end{cases}$$

and we set

$$h^*(t,v) := \tilde{h}(t, u^*(t) + v) - h(t, u^*(t)), \quad (t,v) \in \mathbb{R}^2.$$

Then, we consider the equation

$$v'' + h^*(t,v) = 0. \tag{5.5.7}$$

The following fact is easily proved, using maximum principle-type arguments (see [39, p. 95]).

($\bigstar$) If $v(t)$ is a *sign-changing* kT-period solutions of (5.5.7) (for some integer $k \geq 1$) then $v(t) \geq \alpha(t) - u^*(t)$ for any $t \in \mathbb{R}$.

Now, we observe that both uniqueness and global continuability for the solutions to the Cauchy problems associated with (5.5.7) are ensured; moreover, since $u^*(t) > \alpha(t)$, the constant function $v \equiv 0$ is a solution of (5.5.7). We can therefore transform (5.5.7) into an equivalent first order system in $\mathbb{R}^2 \setminus \{0\}$, passing to clockwise polar coordinates $v(t) = r(t)\cos\theta(t)$, $v'(t) = -r(t)\sin\theta(t)$.

We claim that:

($A1$) there exists an integer $k^* \geq 1$ such that, for any integer $k \geq k^*$, there exist an integer $m_k \geq 1$ and $r_* > 0$ such that any solution $(r(t), \theta(t))$ with $r(0) = r_*$ satisfies $\theta(kT) - \theta(0) > 2\pi m_k$;

($A2$) for any integer $k \geq k^*$ there exists $R_* > r_*$ such that any solution $(r(t), \theta(t))$ with $r(0) = R_*$ satisfies $\theta(kT) - \theta(0) < 2\pi$.

From the above facts, it follows that the Poincaré-Birkhoff theorem (in the generalized version for non-invariant annuli, see [69, 159] and also [38, § 5]) can be applied, giving, for any $k \geq k^*$ and any $1 \leq j \leq m_k$, the existence of two kT-periodic solutions $v^{(i)}_{k,j}(t)$ $(i = 1, 2)$ to equation (5.5.7) having exactly $2j$ zeros on $[0, kT[$. Using ($\bigstar$), it is then immediate to see that $u^{(i)}_{k,j}(t) := v^{(i)}_{k,j}(t) + u^*(t)$ is a kT-periodic solution of (5.5.1), satisfying (5.5.5) and such that $u^{(i)}_{k,j}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$. The fact that, for j and k relatively prime, $u^{(i)}_{k,j}(t)$ is a subharmonic solution of order k is also easily verified, while $u^{(1)}_{k,j}(t)$ and $u^{(2)}_{k,j}(t)$ are not in the same periodicity class due to a standard corollary of the Poincaré-Birkhoff theorem for the iterates of a map. For more details on the application of this method, we refer to [29, 126, 181].

To conclude the proof, we then have to verify the claims ($A1$) and ($A2$). As for the first one, it can be proved exactly as in [35, Proof of Theorem 2.1] (see also [35, Remark 2.2]). The fact that we are working in a Carathéodory setting does not cause here serious difficulties, since the dominated convergence theorem easily yields $h^*(\cdot, v)/v \to \partial_u h(\cdot, u^*(\cdot))$ in $L^1([0, T])$ for $v \to 0$, and this is enough to use continuous dependence arguments as in [35]. On the other hand, the proof of ($A2$) is more delicate (especially when dealing with Carathéodory functions) and we prefer to give some more details. We are going to use a trick based on modified polar coordinates, introduced in [77] (see also [29]). More precisely, for any $\mu > 0$, we write

$$v(t) = \frac{r_\mu(t)}{\mu}\cos\theta_\mu(t), \qquad v'(t) = -r_\mu(t)\sin\theta_\mu(t);$$

for further convenience we also compute

$$\theta_\mu'(t) = \mu \frac{v'(t)^2 - v(t)v''(t)}{\mu^2 v(t)^2 + v'(t)^2}. \tag{5.5.8}$$

The angular coordinates θ and θ_μ are in general different. However, the angular width of any quadrant of the plane is $\pi/2$ also if measured using the angle θ_μ. As a consequence, recalling (5.5.8) we can write the formula

$$\frac{1}{4} = \frac{\mu}{2\pi} \int_{t_1}^{t_2} \frac{v'(t)^2 - v(t)v''(t)}{\mu^2 v(t)^2 + v'(t)^2}\, dt, \tag{5.5.9}$$

valid whenever t_1, t_2 are such that $t_1 < t_2$, $v(t_1) = 0 = v'(t_2)$ (or viceversa) and $(v(t), v'(t))$ belongs to the same quadrant for $t \in [t_1, t_2]$. We stress that (5.5.9) holds for any $\mu > 0$.

We can now give the proof. Preliminarily, we observe that, using the Carathéodory condition together with the definition of $h^*(t, v)$, we can obtain

$$|h^*(t, v)| \leq b(t), \quad \text{a.e. } t \in [0, T],\ \forall\, v \leq 0, \tag{5.5.10}$$

where $b \in L^1([0, T])$. We now fix an integer $k \geq k^*$ and take $\mu > 0$ so small that

$$\frac{\mu k T}{2\pi} \leq \frac{1}{16}. \tag{5.5.11}$$

In view of the global continuability of the solutions, there exists $R_* > 0$ large enough such that $r(0) = R_*$ implies that

$$r_\mu(t)^2 = \mu^2 v(t)^2 + v'(t)^2 \geq \left(\frac{8k\|b\|_{L^1([0,T])}}{\pi} \right)^2, \quad \forall\, t \in [0, kT]. \tag{5.5.12}$$

At this point, assume by contradiction that $\theta(kT) - \theta(0) \geq 2\pi$ for a solution with $r(0) = R_*$. Then it is not difficult to see that there exist $t_1, t_2 \in [0, kT]$ with $t_1 < t_2$ and such that either $v(t_1) = 0 = v'(t_2)$ and $(v(t), v'(t))$ belongs to the third quadrant for $t \in [t_1, t_2]$ or $v'(t_1) = 0 = v(t_2)$ and $(v(t), v'(t))$ belongs to the fourth quadrant for $t \in [t_1, t_2]$. As a consequence, on one hand (5.5.9) holds true; on the other hand, since $v(t) \leq 0$ for $t \in [t_1, t_2]$ we can use (5.5.10) so as to obtain

$$|v(t)v''(t)| \leq b(t)|v(t)|, \quad \text{a.e. } t \in [t_1, t_2].$$

Combining these two facts, we find

$$\begin{aligned} \frac{1}{4} &\leq \frac{\mu}{2\pi} \int_{t_1}^{t_2} \frac{v'(t)^2}{\mu^2 v(t)^2 + v'(t)^2}\, dt + \frac{1}{2\pi} \int_{t_1}^{t_2} \frac{b(t) r_\mu(t) |\cos\theta_\mu(t)|}{r_\mu(t)^2}\, dt \\ &\leq \frac{\mu k T}{2\pi} + \frac{k\|b\|_{L^1([0,T])}}{2\pi} \frac{1}{\min_{t \in [0,kT]} r_\mu(t)}. \end{aligned}$$

Using (5.5.11) and (5.5.12), we finally obtain $\frac{1}{4} \leq \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$, a contradiction. □

Remark 5.5.2. We underline that, although related, conditions (5.5.3) and (5.5.6) are not equivalent. More precisely, given a general weight function $q \in L^1([0,T])$,

$$\int_0^T q(t)\,dt > 0 \quad \Longrightarrow \quad \lambda_0(q) < 0 \tag{5.5.13}$$

as an easy consequence of the variational characterization of the principal eigenvalue (see [122, Theorem 4.2])

$$\lambda_0(q) = \inf_{v \in H^1_T} \frac{\int_0^T \bigl(v'(t)^2 - q(t)v(t)^2\bigr)\,dt}{\int_0^T v(t)^2\,dt} \tag{5.5.14}$$

(just, take $v \equiv 1$ in the above formula; H^1_T denotes the Sobolev space of T-periodic H^1_{loc}-functions). Of course, (5.5.14) also implies that

$$q(t) \leq 0 \quad \Longrightarrow \quad \lambda_0(q) \geq 0$$

but there exist (sign-changing) weights $q(t)$ such that $\int_0^T q(t)\,dt \leq 0$ and $\lambda_0(q) < 0$, showing that the converse of (5.5.13) is not true. Explicit examples can be constructed, for instance, as in [29, Remark 3.5]. An even more interesting example will be given later (see Remark 5.7.3), showing that the possibility of replacing (5.5.6) with the weaker assumption (5.5.3) is crucial for our purposes. ◁

5.6 Statement of the existence results

In this section, we state our main results, dealing with positive solutions to equations of the type

$$u'' + a(t)g(u) = 0. \tag{5.6.1}$$

We always assume that $g \in \mathcal{C}^2(I)$, with $I \subseteq \mathbb{R}^+$ a right neighborhood of $s = 0$, and satisfies the following conditions:

(g_1) $$g(0) = 0$$

(g_2) $$g'(0) = 0$$

(g_3) $$g''(s) > 0, \quad \text{for every } s \in I \setminus \{0\}.$$

Hence, $g(s)$ is *superlinear at zero* and *strictly convex*. Incidentally, notice that from (g_1), (g_2) and (g_3) it follows that $g(s)$ is strictly increasing; in particular

$$g(s) > 0, \quad \text{for every } s \in I \setminus \{0\},$$

which implies that the only constant solution to (5.6.1) is the trivial one, i.e. $u \equiv 0$.

As for the weight function, we suppose that $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic and locally integrable function satisfying the following condition.

(a_*) *There exist $m \geq 1$ intervals $I_1^+, \dots, I_m^+$, closed and pairwise disjoint in the quotient space $\mathbb{R}/T\mathbb{Z}$, such that*

$$a(t) \geq 0, \quad a.e.\ t \in I_i^+, \quad a(t) \not\equiv 0 \ \ on\ I_i^+, \quad for\ i = 1, \dots, m;$$
$$a(t) \leq 0, \quad a.e.\ t \in (\mathbb{R}/T\mathbb{Z}) \setminus \bigcup_{i=1}^{m} I_i^+.$$

Moreover, motivated by the discussion in the introductive Section 5.4, we suppose that the mean value condition

$$(a_\#) \qquad \int_0^T a(t)\, dt < 0$$

holds true.

Of course, by a solution to equation (5.6.1) we mean a function $u \in W^{2,1}_{\mathrm{loc}}$, with $u(t) \in I$ for any t and solving (5.6.1) for a.e. t. Notice that, since $I \subseteq \mathbb{R}^+$, any solution is a non-negative function; as usual, we say that a solution is positive if $u(t) > 0$ for any t.

As a first result, we provide a statement generalizing Theorem 5.4.1 given for equation (5.4.1). More precisely, we show that the existence of positive subharmonic solutions to (5.6.1) is ensured for any function $g(s)$ which satisfies (g_1), (g_2), (g_3) for $I = \mathbb{R}^+$ and which is *superlinear at infinity*. Needless to say, this is the case for the model nonlinearity $g(s) = s^p$ with $p > 1$.

Theorem 5.6.1. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*) and $(a_\#)$. Let $g \in \mathcal{C}^2(\mathbb{R}^+)$ satisfy (g_1), (g_2) and (g_3), as well as*

$$(g_4) \qquad \lim_{s \to +\infty} \frac{g(s)}{s} = +\infty.$$

Then, there exists a positive T-periodic solution $u^*(t)$ of (5.6.1); moreover, there exists $k^* \geq 1$ such that for any integer $k \geq k^*$ there exists an integer $m_k \geq 1$ such that, for any integer j relatively prime with k and such that $1 \leq j \leq m_k$, equation (5.6.1) has two positive subharmonic solutions $u_{k,j}^{(i)}(t)$ ($i = 1, 2$) of order k (not belonging to the same periodicity class), such that $u_{k,j}^{(i)}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$.

In our second result, we deal with the case $I = [0, \delta[$, with $\delta > 0$ finite, assuming a *singular* behavior for $g(s)$ when $s \to \delta^-$.

Theorem 5.6.2. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*) and $(a_\#)$. Let $g \in \mathcal{C}^2([0, \delta[)$ (for some $\delta > 0$ finite) satisfy (g_1), (g_2) and (g_3), as well as*

$$(g_4') \qquad \lim_{s \to \delta^-} g(s) = +\infty.$$

Then, there exists a positive T-periodic solution $u^*(t)$ of (5.6.1)*; moreover, there exists $k^* \geq 1$ such that for any integer $k \geq k^*$ there exists an integer $m_k \geq 1$ such that, for any integer j relatively prime with k and such that $1 \leq j \leq m_k$, equation* (5.6.1) *has two positive subharmonic solutions $u^{(i)}_{k,j}(t)$ ($i = 1, 2$) of order k (not belonging to the same periodicity class), such that $u^{(i)}_{k,j}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$.*

We mention that singular equations with indefinite weight were considered in [42, 176, 177]. More precisely, these papers deal with equations like $u'' + a(t)/u^\sigma = 0$, where $\sigma > 0$. Our setting is different and Theorem 5.6.2 applies for instance to the equation

$$u'' + a(t)\,\frac{u^\gamma}{1 - u^\sigma} = 0, \tag{5.6.2}$$

for $\gamma > 1$ and $\sigma \geq 1$. To the best of our knowledge, even the mere existence of a positive T-periodic solution to (5.6.2) is a fact which has never been noticed.

Finally, we give a purely local result. More precisely, we just assume (g_1), (g_2) and (g_3) in a bounded interval $I = [0, \rho]$, with $\rho > 0$ finite; on the other hand, we deal with an equation depending on a real parameter and we manage to obtain the result by varying it.

Theorem 5.6.3. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*) and $(a_\#)$. Let $g \in \mathcal{C}^2([0, \rho])$ (for some $\rho > 0$) satisfy (g_1), (g_2) and (g_3). Then, there exists $\lambda^* > 0$ such that for any $\lambda > \lambda^*$ there exists a positive T-periodic solution $u^*(t)$ of the parameter-dependent equation*

$$u'' + \lambda a(t) g(u) = 0 \tag{5.6.3}$$

satisfying $\max_{t\in\mathbb{R}} u^*(t) < \rho$. Moreover, there exists $k^* \geq 1$ such that for any integer $k \geq k^*$ there exists an integer $m_k \geq 1$ such that, for any integer j relatively prime with k and such that $1 \leq j \leq m_k$, equation (5.6.3) *has two positive subharmonic solutions $u^{(i)}_{k,j}(t)$ ($i = 1, 2$) of order k (not belonging to the same periodicity class), with $\max_{t\in\mathbb{R}} u^{(i)}_{k,j}(t) < \rho$ and such that $u^{(i)}_{k,j}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$.*

Of course, in the above statement $g(s)$ may be defined also for $s > \rho$, but no assumptions on its behavior are made. For instance, we can apply Theorem 5.6.3 to parameter-dependent equations like

$$u'' + \lambda a(t)\,\frac{u^\gamma}{1 + u^\sigma} = 0, \tag{5.6.4}$$

with $\sigma \geq \gamma - 1 > 0$, obtaining the following: for any $\rho > 0$ small enough, there exists $\lambda^* = \lambda^*(\rho) > 0$ such that for any $\lambda > \lambda^*$ equation (5.6.4) has a positive T-periodic solution as well as positive subharmonic solutions of any large order; all these periodic solutions, moreover, have maximum less than ρ. In such a way, in the direction of proving the existence of positive subharmonics, we can complement recent results dealing with positive harmonic solutions in the asymptotically linear case $\sigma = \gamma - 1$ (see Corollary 3.2.7) and in the sublinear one $\sigma > \gamma - 1$ (see Chapter 6 and [36]). It is worth noticing that, according to Theorem 6.5.3, in this latter case a further positive T-periodic solution (having maximum greater than ρ) to (5.6.4) appears. This second solution is expected to have typically zero Morse index, and no positive subharmonic solutions oscillating around it.

5.7 Proof of the existence results

In this section we provide the proof of the results presented in Section 5.6. Actually, we are going to give and prove a further statement, which looks slightly more technical but has the advantage of unifying all the situations considered in Theorem 5.6.1, Theorem 5.6.2 and Theorem 5.6.3.

Henceforth, we deal with the equation

$$u'' + a(t)f(u) = 0, \tag{5.7.1}$$

where $f \in \mathcal{C}^2([0,\rho])$, for some $\rho > 0$ finite, and satisfies:

(f_1)
$$f(0) = 0$$

(f_2)
$$f'(0) = 0$$

(f_3)
$$f''(s) > 0, \quad \text{for every } s \in \,]0,\rho].$$

Accordingly, by a solution to equation (5.7.1) we mean a function $u \in W^{2,1}_{\mathrm{loc}}$, with $0 \leq u(t) \leq \rho$ for any t and solving (5.7.1) in the Carathéodory sense; a solution is said to be positive if $u(t) > 0$ for any t.

In this setting, the following result can be given.

Theorem 5.7.1. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*) and $(a_\#)$. Then there exist two real constants $M_1 \in \,]0,1[$ and $M_2 > 0$ such that, for any $\rho > 0$ and for any $f \in \mathcal{C}^2([0,\rho])$ satisfying (f_1), (f_2), (f_3) and*

(f_4)
$$\frac{f(M_1\rho)}{M_1\rho} > M_2,$$

the following holds true: there exists a positive T-periodic solution $u^*(t)$ of (5.7.1) with $\max_{t\in\mathbb{R}} u^*(t) < \rho$; moreover, there exists $k^* \geq 1$ such that *for any integer $k \geq k^*$ there exists an integer $m_k \geq 1$ such that, for any integer j relatively prime with k and such that $1 \leq j \leq m_k$, equation* (5.7.1) *has two positive subharmonic solutions $u^{(i)}_{k,j}(t)$ ($i = 1, 2$) of order k (not belonging to the same periodicity class), with $\max_{t\in\mathbb{R}} u^{(i)}_{k,j}(t) < \rho$ and such that $u^{(i)}_{k,j}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$.*

It is clear that all the theorems in Section 5.6 follows from Theorem 5.7.1. More precisely:

- in order to obtain Theorem 5.6.1, we take $f = g$ and $\rho > 0$ large enough: then (f_1), (f_2), (f_3) correspond to (g_1), (g_2), (g_3), while (f_4) comes from (g_4);
- in order to obtain Theorem 5.6.2, we take $f = g$ and $\rho < \delta$ with $\delta - \rho$ small enough: then (f_1), (f_2), (f_3) correspond to (g_1), (g_2), (g_3), while (f_4) comes from (g'_4);
- in order to obtain Theorem 5.6.3, we take $f = \lambda g$: then (f_1), (f_2), (f_3) correspond to (g_1), (g_2), (g_3) (independently on $\lambda > 0$), while (f_4) is certainly satisfied for $\lambda > 0$ large enough.

Now we are going to prove Theorem 5.7.1. Wishing to apply Proposition 5.5.1, we proceed as follows. First, we define an extension $\hat{f}(s)$ of $f(s)$ for $s \geq \rho$, having linear growth at infinity and thus ensuring the global continuability of the (positive) solutions of $u'' + a(t)\hat{f}(u)$; in doing this, we need to check that any periodic solution of this modified equation is actually smaller than ρ, thus solving the original equation $u'' + a(t)f(u) = 0$. This is the most technical part of the proof (producing the constants M_1, M_2 appearing in assumption (f_4)) and is developed in Section 5.7.1. Second, in Section 5.7.2, using a degree theoretic approach (and taking advantage of the a priori bound given in the previous section), we prove the existence of a positive T-periodic solution of $u'' + a(t)f(u) = 0$. Third, in Section 5.7.3 we provide the desired Morse index information. The easy conclusion of the proof is finally given in Section 5.7.4 (we just notice here that the existence of a lower solution $\alpha(t) < u^*(t)$ is straightforward, since we can take $\alpha(t) \equiv 0$).

It is worth noticing that condition (f_3) (requiring in particular that $f \in \mathcal{C}^2$) will be essential only in Section 5.7.3. For this reason, we carry out the discussion in Section 5.7.1 and Section 5.7.2 (containing results which may have some independent interests) in a slightly more general setting than the one in Theorem 5.7.1.

5.7.1 The a priori bound

In this section, we prove an a priori bound valid for periodic solutions of (5.7.1) as well as for periodic solutions of a related equation (see (5.7.3) below). This will be useful both for the application of Proposition 5.5.1 (requiring a globally defined nonlinearity) and for the degree approach discussed in the Section 5.7.2.

As already anticipated, in this section we do not assume all the conditions on $f(s)$ required in Theorem 5.7.1. More precisely, we are going to deal with continuously differentiable functions $f\colon [0,\rho] \to \mathbb{R}^+$ satisfying (f_1) and the following condition

$$(f_*) \qquad f(s) > 0, \quad \text{for every } s \in \,]0,\rho].$$

Moreover, instead of (f_3) we just suppose that $f(s)$ is a *convex function*, namely

$$f(\vartheta s_1 + (1-\vartheta)s_2) \leq \vartheta f(s_1) + (1-\vartheta) f(s_2), \quad \forall\, s_1, s_2 \in [0,\rho],\ \forall\, \vartheta \in [0,1].$$

For further convenience, we observe that from the above conditions it follows that $f(s)$ is non-decreasing and such that $s \mapsto f(s)/s$ is a non-decreasing map in $]0,\rho]$. Indeed, let $0 < s_1 < s_2$ and let $\vartheta \in [0,1]$ be such that $s_1 = \vartheta s_2$. Then, we have

$$f(s_1) = f(\vartheta s_2 + (1-\vartheta)0) \leq \vartheta f(s_2) + (1-\vartheta) f(0) = \frac{s_1}{s_2} f(s_2)$$

and thus the map $s \mapsto f(s)/s$ is non-decreasing in $]0,\rho]$. Consequently, we immediately obtain that $s \mapsto f(s)$ is a non-decreasing map in $[0,\rho]$, since it is the product of two non-decreasing positive maps in $]0,\rho]$.

We also recall that a function $f \in \mathcal{C}^1([0,\rho])$ is convex if and only if $f(s)$ lies above all of its tangents, hence

$$f(s_1) \geq f(s_2) + f'(s_2)(s_1 - s_2), \quad \forall\, s_1, s_2 \in [0,\rho].$$

Using (f_1), (f_*) and the above inequality (with $s_1 = 0$ and $s_2 = \rho$), we immediately obtain

$$f'(\rho) \geq \frac{f(\rho)}{\rho} > 0.$$

With this in mind, we introduce the extension $\hat{f}\colon \mathbb{R}^+ \to \mathbb{R}$ defined as

$$\hat{f}(s) := \begin{cases} f(s), & \text{if } s \in [0,\rho]; \\ f(\rho) + f'(\rho)(s-\rho), & \text{if } s \in \,]\rho,+\infty[. \end{cases} \tag{5.7.2}$$

It is easily seen that the map $\hat{f}(s)$ is continuously differentiable, convex, non-decreasing and such that $\hat{f}(s) > 0$ for all $s > 0$. Then, arguing as above, we immediately obtain that the map $s \mapsto \hat{f}(s)/s$ is non-decreasing as well.

We are now in a position to state our technical result (whose proof benefits from some arguments developed in [93, p. 421] and in Lemma 7.4.1) giving a priori bounds for periodic solutions of the equation

$$u'' + a(t)\hat{f}(u) + \nu \mathbb{1}_{\bigcup_{i=1}^{m} I_i^+}(t) = 0, \tag{5.7.3}$$

where $\nu \geq 0$ and $\mathbb{1}_{\bigcup_{i=1}^{m} I_i^+}$ denotes the indicator function of the set $\bigcup_{i=1}^{m} I_i^+$. Incidentally, notice that neither the mean value condition $(a_{\#})$ nor the superlinearity assumption at zero (f_2) are required in the statement below.

Lemma 5.7.1. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). Then there exist two real constants $M_1 \in \,]0,1[$ and $M_2 > 0$ such that, for every $\rho > 0$, for every convex function $f \in \mathcal{C}^1([0,\rho])$ satisfying (f_1), (f_*) and (f_4), for every $\nu \geq 0$ and for every integer $k \geq 1$, any kT-periodic solution $u(t)$ to* (5.7.3) *satisfies* $\max_{t\in\mathbb{R}} u(t) < \rho$.

Proof. According to condition (a_*), we can find $2m+1$ points

$$\sigma_1 < \tau_1 < \ldots < \sigma_i < \tau_i < \ldots < \sigma_m < \tau_m < \sigma_{m+1}, \quad \text{with } \sigma_{m+1} - \sigma_1 = T,$$

such that

$$I_i^+ = [\sigma_i, \tau_i], \quad i = 1, \ldots, m.$$

We then fix $\varepsilon > 0$ such that

$$\varepsilon < \frac{|I_i^+|}{2} \quad \text{and} \quad \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t)\,dt > 0, \qquad \text{for all } i \in \{1, \ldots, m\};$$

so that the constant

$$\eta_\varepsilon := \min_{i=1,\ldots,m} \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t)\,dt$$

is well-defined and positive. Next, we define the constants

$$M_1 := \frac{\varepsilon}{\max\limits_{i=1,\ldots,m} |I_i^+|} \quad \text{and} \quad M_2 := \frac{2}{M_1 \varepsilon \eta_\varepsilon}.$$

Notice that $M_1 \in \,]0,1[$, since $\varepsilon < |I_i^+|$, for all $i = 1, \ldots, m$. We stress that M_1 and M_2 depend only on the weight function $a(t)$.

Let us consider an arbitrary $\nu \geq 0$ and an arbitrary convex function $f \in \mathcal{C}^1([0,\rho])$ satisfying (f_1), (f_*) and (f_4). By contradiction, we suppose that $u(t)$ is a kT-periodic solution of (5.7.3) such that

$$\max_{t\in\mathbb{R}} u(t) =: \rho^* \geq \rho.$$

Setting $I_{i,\ell}^+ := I_i^+ + \ell T$ (for $i = 1, \ldots, m$ and $\ell \in \mathbb{Z}$), the convexity of $u(t)$ on $\mathbb{R} \setminus \bigcup_{i,\ell} I_{i,\ell}^+$ ensures that the maximum is attained in some $I_{i,\ell}^+$. Accordingly, we can suppose that there is an index $i \in \{1, \ldots, m\}$ and $\ell \in \{0, \ldots, k-1\}$ such that

$$\max_{t \in I^+_{i,\ell}} u(t) = \rho^*.$$

Up to a relabeling of the intervals $I^+_{i,\ell}$, we can also suppose $\ell = 0$ (notice that the constants M_1 and M_2 do not change since $a(t)$ is T-periodic). From now on, we therefore assume that

$$\max_{t \in I^+_{i}} u(t) = \rho^*.$$

From this fact, together with the concavity of $u(t)$ on I^+_i, we obtain

$$u(t) \geq \frac{\rho^*}{|I^+_i|} \min\{t - \sigma_i, \tau_i - t\}, \quad \forall\, t \in I^+_i,$$

(cf. [93, p. 420] for a similar estimate) and hence

$$u(t) \geq \frac{\varepsilon \rho^*}{\max\limits_{i=1,\ldots,m} |I^+_i|} = M_1 \rho^*, \quad \forall\, t \in I^+_i. \tag{5.7.4}$$

On the other hand, arguing as in Section 3.3.2 and Section 4.3.3 (with $c = 0$), from

$$|u'(t)| \leq \frac{u(t)}{\varepsilon}, \quad \forall\, t \in [\sigma_i + \varepsilon, \tau_i - \varepsilon],$$

we deduce

$$|u'(t)| \leq \frac{\rho^*}{\varepsilon}, \quad \forall\, t \in [\sigma_i + \varepsilon, \tau_i - \varepsilon]. \tag{5.7.5}$$

Integrating equation (5.7.3) on $[\sigma_i + \varepsilon, \tau_i - \varepsilon]$ and using (5.7.4), (5.7.5) and the monotonicity of $s \mapsto \hat{f}(s)$, we have

$$\begin{aligned} \hat{f}(M_1\rho^*) \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t)\, dt &\leq \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t) \hat{f}(u(t))\, dt = \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} \bigl(-u''(t) - \nu\bigr)\, dt \\ &= u'(\sigma_i + \varepsilon) - u'(\tau_i - \varepsilon) - \nu\, (\tau_i - \sigma_i - 2\varepsilon) \leq \frac{2\rho^*}{\varepsilon}. \end{aligned}$$

Dividing by $M_1 \rho^* \eta_\varepsilon$ the above inequality and using the monotonicity of the map $s \mapsto \hat{f}(s)/s$, we obtain

$$\frac{f(M_1\rho)}{M_1\rho} = \frac{\hat{f}(M_1\rho)}{M_1\rho} \leq \frac{\hat{f}(M_1\rho^*)}{M_1\rho^*} \leq \frac{2}{M_1 \varepsilon \eta_\varepsilon} = M_2,$$

a contradiction with respect to hypothesis (f_4). $\square$

Remark 5.7.1. It is worth noticing that, in the above proof, the fact that $u(t)$ is kT-periodic is used only to ensure, via a convexity argument, that its maximum is achieved in some positivity interval I^+_i. Accordingly, it is easily seen that the conclusion of Lemma 5.7.1 still holds true for any globally defined bounded solution of (5.7.3), as well as for solutions defined on compact intervals and satisfying Dirichlet/Neumann conditions at the boundary. $\lhd$

5.7.2 Existence of T-periodic solutions: a degree approach

In this section, using the topological degree approach introduced in the previous chapters, we prove the existence of a positive T-periodic solution of (5.7.1).

Since we are going to take advantage of the a priori bound developed in Lemma 5.7.1, we assume again that $f(s)$ is a convex function satisfying (f_1), (f_*) and (f_4); moreover, now also the superlinearity at zero condition (f_2) and the mean value assumption $(a_\#)$ play a crucial role.

Proposition 5.7.1. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*) and $(a_\#)$. Let $f \in \mathcal{C}^1([0,\rho])$ be a convex function satisfying (f_1), (f_2), (f_*) and (f_4) with $M_1 \in \,]0,1[$ and $M_2 > 0$ the constants given in Lemma 5.7.1. Then there exists a positive T-periodic solution $u^*(t)$ of* (5.7.1) *such that* $\max_{t\in\mathbb{R}} u^*(t) < \rho$.

Proof. Since this proposition is not a direct consequence of the existence results of Chapter 3, we prefer to sketch the proof (which is based on Mawhin's coincidence degree theory).

First of all, taking into account condition (f_1), we introduce the L^1-Carathéodory function

$$\tilde{f}(t,s) := \begin{cases} -s, & \text{if } s \leq 0; \\ a(t)f(s), & \text{if } 0 \leq s \leq \rho; \\ a(t)f(\rho), & \text{if } s \geq \rho; \end{cases}$$

and we consider the T-periodic problem associated with

$$u'' + \tilde{f}(t,u) = 0. \tag{5.7.6}$$

A standard maximum principle ensures that every T-periodic solution of (5.7.6) is non-negative; moreover, in view of (f_2), if $u(t)$ is a T-periodic solution of (5.7.6) with $u \not\equiv 0$, then $u(t) > 0$ for all t (see Lemma C.1.2).

Next, we write the T-periodic problem associated with (5.7.6) as a *coincidence equation*

$$Lu = Nu, \quad u \in \operatorname{dom} L. \tag{5.7.7}$$

As a first observation, let us recall that finding a T-periodic solution of (5.7.6) is equivalent to solving equation (5.7.6) on $[0,T]$ together with the periodic boundary condition

$$u(0) = u(T), \qquad u'(0) = u'(T).$$

Accordingly, let $X := \mathcal{C}([0,T])$ be the Banach space of continuous functions $u\colon [0,T] \to \mathbb{R}$, endowed with the sup-norm $\|u\|_\infty := \max_{t\in[0,T]} |u(t)|$, and let $Z := L^1([0,T])$ be the Banach space of Lebesgue integrable functions

$v\colon [0,T] \to \mathbb{R}$, endowed with the norm $\|v\|_{L^1} := \int_0^T |v(t)|\,dt$. Next we consider the differential operator

$$L\colon u \mapsto -u'',$$

defined on

$$\operatorname{dom} L := \bigl\{u \in W^{2,1}([0,T])\colon u(0) = u(T),\ u'(0) = u'(T)\bigr\} \subseteq X.$$

It is easy to prove that L is a linear Fredholm map of index zero. Moreover, in order to enter the coincidence degree setting, we have to define the projectors $P\colon X \to \ker L \cong \mathbb{R}$, $Q\colon Z \to \operatorname{coker} L \cong Z/\operatorname{Im} L \cong \mathbb{R}$, the right inverse $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ of L, and the linear (orientation-preserving) isomorphism $J\colon \operatorname{coker} L \to \ker L$. For the standard positions we refer to Section 3.1 or to Section 4.2. Finally, let us denote by $N\colon X \to Z$ the Nemytskii operator induced by the function $\tilde{f}(t,s)$, that is

$$(Nu)(t) := \tilde{f}(t,u(t)), \quad t \in [0,T].$$

With this position, now we prove that there exists an open (bounded) set $\Omega \subseteq X$, with

$$\Omega \subseteq B(0,\rho) \setminus \{0\}, \tag{5.7.8}$$

such that the coincidence degree $D_L(L-N,\Omega)$ of L and N in Ω is defined and different from zero. In this manner, by the existence property of the degree, there exists at least a nontrivial solution u^* of (5.7.7) with $\|u^*\|_\infty < \rho$. Hence, $u^*(t)$ is a T-periodic solution of (5.7.6). As a consequence of the maximum principle, as already noticed, this solution is positive; moreover, being $u^*(t) < \rho$ for any t, it solves the original equation (5.7.1).

We split our argument into three steps. In the following, when referring to a solution $u(t)$ of (5.7.9) and (5.7.10) we implicitly assume that $0 \leq u(t) \leq \rho$, for all $t \in \mathbb{R}$, since $f(s)$ is defined on $[0,\rho]$.

Step 1. *There exists a constant $r \in \,]0,\rho[$ such that any T-periodic solution $u(t)$ of*

$$u'' + \vartheta a(t) f(u) = 0, \tag{5.7.9}$$

for $0 < \vartheta \leq 1$, satisfies $\|u\|_\infty \neq r$.

Indeed, since condition (f_2) can be written in the equivalent form

$$\lim_{s\to 0^+} \frac{f(s)}{s} = 0,$$

we can proceed (by contradiction) exactly as in the proof of Theorem 3.2.2 (thus, we omit the proof).

Step 2. *There exists a constant $\nu_0 > 0$ such that any T-periodic solution $u(t)$ of*

$$u'' + a(t)f(u) + \nu \mathbb{1}_{\bigcup_{i=1}^m I_i^+}(t) = 0, \tag{5.7.10}$$

for $\nu \in [0,\nu_0]$*, satisfies* $\|u\|_\infty \neq \rho$*. Moreover, there are no* T*-periodic solutions* $u(t)$ *of* (5.7.10) *for* $\nu = \nu_0$*.*

From Lemma 5.7.1 we deduce that, for any $\nu \geq 0$, every T-periodic solution $u(t)$ of (5.7.10) satisfies $\|u\|_\infty \neq \rho$ (notice that the definition of the extension $\tilde{f}(t,s)$ for $s \geq \rho$ has no role in this proof). Next, we fix a constant $\nu_0 > 0$ such that

$$\nu_0 > \frac{\|a\|_{L^1} \max_{0\leq s\leq\rho} f(s)}{\sum_{i=1}^m |I_i^+|}.$$

We have only to verify that, for $\nu = \nu_0$, there are no T-periodic solutions $u(t)$ of (5.7.10). Indeed, if $u(t)$ is a T-periodic solution of (5.7.10) then, integrating (5.7.10) on $[0,T]$, we obtain

$$\nu \sum_{i=1}^m |I_i^+| = \nu \int_0^T \mathbb{1}_{\bigcup_{i=1}^m I_i^+}(t)\, dt \leq \int_0^T |a(t)| f(u(t))\, dt \leq \|a\|_{L^1} \max_{0\leq s\leq\rho} f(s),$$

a contradiction with respect to the choice of ν_0.

Step 3. Computation of the degree. First of all, we compute the coincidence degree on $B(0,r)$. From [132, Theorem 2.4] and *Step 1*, we obtain that

$$D_L(L-N, B(0,r)) = \deg_B\left(-\frac{1}{T}\int_0^T \tilde{f}(t,\cdot)\, dt, \left]-r,r\right[, 0\right) = 1. \tag{5.7.11}$$

For the details, we refer to the proof of Theorem 3.1.1.

Secondly, we compute the coincidence degree on $B(0,\rho)$. From the homotopy invariance of the degree and *Step 2*, we obtain that

$$D_L(L-N, B(0,\rho)) = 0. \tag{5.7.12}$$

For the details, we refer to the proof of Theorem 3.1.1.

In conclusion, from (5.7.11), (5.7.12) and the additivity property of the coincidence degree, we find that

$$D_L(L-N, B(0,\rho)\setminus B[0,r]) = -1.$$

This ensures the existence of a nontrivial solution u^* to (5.7.7) with

$$u^* \in \Omega := B(0,\rho)\setminus B[0,r].$$

Recalling (5.7.8) and the argument explained therein, the proof is concluded. $\square$

Remark 5.7.2. The above existence of a positive T-periodic solution to (5.7.1) could likely be proved under less restrictive assumptions of $f(s)$. In particular, as shown in Chapter 3, the conclusion in *Step 1* is still valid when $f(s)$ is only continuous and regularly oscillating at zero; on the other hand, we expect that *Step 2* can be proved (with slightly different arguments) under alternative assumptions not requiring the convexity of $f(s)$. We have chosen however to take advantage of the a priori bound developed in Lemma 5.7.1, therefore giving the proof in this simplified setting, since a convexity assumption will be in any case essential in the subsequent Section 5.7.3. $\lhd$

5.7.3 The Morse index computation

In this section we present the (crucial) Morse index computation. As remarked in Section 5.4, it is based on an algebraic trick already employed in the proof of [44, Theorem 1], exploiting in an essential way the strict convexity assumption (f_3) (together with the sign condition (f_*)). Notice that all the other assumptions on $f(s)$ and $a(t)$ are not required in Lemma 5.7.2 below, which is indeed an a priori Morse index estimate, valid for positive T-periodic solutions of (5.7.1) independently of their existence.

Lemma 5.7.2. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function. Let $f \in \mathcal{C}^2([0,\rho])$ satisfy (f_*) and (f_3). If $u(t)$ is a positive T-periodic solution of* (5.7.1)*, then*

$$\lambda_0\bigl(a(t)f'(u(t))\bigr) < 0,$$

where $\lambda_0\bigl(a(t)f'(u(t))\bigr)$ denotes (as in Section 5.5) the principal eigenvalue of the T-periodic problem associated with $v'' + (\lambda + a(t)f'(u(t)))v = 0$.

Proof. Let $u(t)$ be a positive T-periodic solution of (5.7.1) and let $v(t)$ be a *positive* eigenfunction associated to the principal eigenvalue $\lambda_0 = \lambda_0(a(t)f'(u(t)))$. Then, $v(t)$ satisfies

$$v'' + \bigl(\lambda_0 + a(t)f'(u(t))\bigr)v = 0, \tag{5.7.13}$$

is T-periodic and $v(t) > 0$ for all $t \in \mathbb{R}$ (cf. [55]).

By multiplying (5.7.1) by $f'(u)v$ we obtain

$$u''f'(u)v + a(t)f(u)f'(u)v = 0$$

and, respectively, by multiplying (5.7.13) by $f(u)$ we have

$$v''f(u) + \bigl(\lambda_0 + a(t)f'(u)\bigr)vf(u) = 0.$$

From the above equalities, we therefore immediately deduce

$$\begin{aligned} \lambda_0 v(t)f(u(t)) &= -a(t)f(u(t))f'(u(t))v(t) - v''(t)f(u(t)) \\ &= u''(t)f'(u(t))v(t) - v''(t)f(u(t)), \quad \forall\, t \in \mathbb{R}. \end{aligned}$$

Integrating by parts this equality, we obtain

$$\lambda_0 \int_0^T v(t)f(u(t))\,dt = \int_0^T \big(u''(t)f'(u(t))v(t) - v''(t)f(u(t))\big)\,dt$$
$$= \Big[-v'(t)f(u(t))\Big]_{t=0}^{t=T} + \int_0^T \big(u''(t)f'(u(t))v(t) + v'(t)f'(u(t))u'(t)\big)\,dt$$
$$= \int_0^T \big(u''(t)f'(u(t))v(t) + v'(t)f'(u(t))u'(t)\big)\,dt.$$

Via a further integration by parts, we find

$$\int_0^T \big(v(t)f'(u(t))u''(t) + v'(t)f'(u(t))u'(t)\big)\,dt$$
$$= \Big[v(t)f'(u(t))u'(t)\Big]_{t=0}^{t=T} - \int_0^T v(t)f''(u(t))u'(t)^2\,dt$$
$$= -\int_0^T v(t)f''(u(t))u'(t)^2\,dt.$$

In conclusion,

$$\lambda_0 \int_0^T v(t)f(u(t))\,dt = -\int_0^T v(t)f''(u(t))u'(t)^2\,dt.$$

Observing now that both the above integrals are positive, since $v(t) > 0$ for all $t \in \mathbb{R}$ and $f(s)$ satisfies (f_3) and (f_*) (notice that $u'(t) \not\equiv 0$, again in view of (f_*)), we immediately deduce that

$$\lambda_0 = \lambda_0\big(a(t)f'(u(t))\big) < 0.$$

The lemma is thus proved. $\square$

Remark 5.7.3. We observe that Lemma 5.7.2 in particular applies to the function $f(s) = s^p$ with $p > 1$, implying that, whenever $u(t)$ is a positive T-periodic solution of $u'' + a(t)u^p = 0$, the Morse index of the linear equation

$$v'' + p\,a(t)u(t)^{p-1}v = 0$$

is non-zero. On the other hand, it is worth noticing that

$$\int_0^T a(t)u(t)^{p-1}\,dt < 0,$$

as it can be easily seen by writing $a(t)u(t)^{p-1} = u''(t)/(pu(t))$ and integrating by parts (compare with the computation leading to (5.4.2) in Section 5.4). This provides an elegant proof of the claim made in Remark 5.5.2, that is, for a linear equation $v'' + q(t)v = 0$, with $q(t)$ sign-changing, the mean value condition $\int_0^T q(t)\,dt \le 0$ does not imply that the Morse index is zero. Furthermore, this fact shows that the main result in [39] is not applicable to the equation $u'' + a(t)u^p = 0$, emphasizing the essential role of its abstract variant given in Proposition 5.5.1 (see again the discussion in Remark 5.5.2). $\lhd$

Remark 5.7.4. Recalling that the Morse index of a positive T-periodic solution $u(t)$ of equation (5.7.1) is the Morse index of the linear equation $v'' + a(t)f'(u(t))v = 0$, Lemma 5.7.2 asserts that any positive T-periodic solution of (5.7.1) has non-zero Morse index. From a variational point of view, this implies that $u(t)$, as a critical point of the action functional

$$J(u) = \int_0^T \left(\frac{1}{2}u'(t)^2 - a(t)F(u(t))\right) dt, \quad \text{where } F(u) := \int_0^u f(\xi)\,d\xi,$$

is not a local minimum. We stress again that this is an a priori information, valid for *any* positive T-periodic solution of (5.7.1); on the other hand, it requires the global convexity assumption (f_3), which is usually not needed for existence results (see Remark 5.7.2). It appears therefore a natural question if it is possible to prove, using variational arguments of mountain pass type (on the lines of [3, 20]), the existence of *at least one* positive T-periodic solution with non-zero Morse index, under less restrictive assumptions on $f(s)$. This however does not seem to be an easy task, since (thought the local topological structure of a functional near a min-max critical point can be analyzed) estimates from below for the Morse index are usually possible only under non-denegeracy assumptions (see, for instance, [95, 108]). $\lhd$

5.7.4 Conclusion of the proof

We are now in a position to easily complete the proof of Theorem 5.7.1. Of course, we assume henceforth that (f_1), (f_2), (f_3) and (f_4) are satisfied, with M_1, M_2 the constants given by Lemma 5.7.1. As a consequence, $f(s)$ is (strictly) convex and the sign condition (f_*) holds true, so that all the results in Section 5.7.1, Section 5.7.2 and Section 5.7.3 can be used.

Let us define, for $(t,s) \in \mathbb{R}^2$,

$$h(t,s) := \begin{cases} 0, & \text{if } s \le 0; \\ a(t)\hat{f}(s), & \text{if } s \ge 0; \end{cases}$$

where $\hat{f}(s)$ is given by (5.7.2). Using (f_1) and (f_2), it is easy to see that the function $h(t,s)$ satisfies the smoothness conditions required in Proposition 5.5.1. Moreover, since $\hat{f}(s)$ has linear growth at infinity, the global continuability for the solutions of

$$u'' + h(t,u) = 0 \tag{5.7.14}$$

is guaranteed. We now claim that both the assumptions (i) and (ii) of Proposition 5.5.1 are satisfied.

Indeed, Proposition 5.7.1 implies the existence of a T-periodic function $u^*(t)$, solving (5.7.1) and such that $0 < u^*(t) < \rho$ for any $t \in \mathbb{R}$; moreover, from Lemma 5.7.2 we know that

$$\lambda_0\big(a(t)f'(u^*(t))\big) < 0.$$

Since $h(t,s) = a(t)f(s)$ for $0 \leq s \leq \rho$, we have thus obtained a T-periodic solution of (5.7.14) satisfying (5.5.3). Hence, condition (i) is fulfilled.

As for condition (ii), we simply take $\alpha(t) \equiv 0$, due to the fact that $\alpha(t)$ is a (trivial) solution of (5.7.14), and $0 = \alpha(t) < u^*(t)$ for any $t \in \mathbb{R}$.

Proposition 5.5.1 thus ensures that there exists $k^* \geq 1$ such that, for any integer $k \geq k^*$, there exists an integer $m_k \geq 1$ such that, for any integer j relatively prime with k and such that $1 \leq j \leq m_k$, equation (5.7.14) has two subharmonic solutions $u^{(i)}_{k,j}(t)$ $(i = 1, 2)$ of order k (not belonging to the same periodicity class), such that $u^{(i)}_{k,j}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$. From the fact that (5.5.5) holds true (with $\alpha(t) \equiv 0$) together with Remark 5.5.1, we obtain that $u^{(i)}_{k,j}(t) > 0$ for any $t \in \mathbb{R}$. We finally use Lemma 5.7.1 (with $\nu = 0$) to ensure that $u^{(i)}_{k,j}(t) < \rho$ for any $t \in \mathbb{R}$. Thus $u^{(i)}_{k,j}(t)$ is a positive subharmonic solutions of equation (5.7.1) and the proof is concluded. $\square$

Remark 5.7.5. Reading more carefully the proof of Lemma 5.7.2, one can notice that we do not use the fact that the interval $[0, \rho]$ is a right neighborhood of zero. Indeed, the same conclusion holds true by taking an interval $J \subseteq \mathbb{R}$ in place of $[0, \rho]$. Accordingly, we can state the following result.

> *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function. Let $J \subseteq \mathbb{R}$ be an interval. Let $f \in \mathcal{C}^2(J)$ satisfy $f(s) > 0$ and $f''(s) > 0$ for any $s \in J$. If $u(t)$ is a T-periodic solution of $u'' + a(t)f(u) = 0$ (thus, in particular, $u(t) \in J$ for any t), then $\lambda_0\big(a(t)f'(u(t))\big) < 0$.*

Although less general, Lemma 5.7.2 is the version more suitable to be subsequently applied to the search of positive subharmonics with range in $]0, \rho[$. However, a natural question arises. Suppose to consider a nonlinearity $f(s)$ as above, namely a $\mathcal{C}^2$-function which is positive and strictly convex in an interval $J \subseteq \mathbb{R}$. Given a T-periodic solution $u^*(t)$ to $u'' + a(t)f(u) = 0$ (with $u^*(t) \in J$ for any t), which additional conditions on $f(s)$ guarantee the applicability of the method adopted in this second part of the chapter to find subharmonics of order k? $\lhd$

5.8 Final remarks

We conclude this investigation on subharmonic solutions with a brief discussion about some natural questions which our result may suggest, if compared to the existing literature. As in the introductive Section 5.4, we focus our attention on the model superlinear equation

$$u'' + a(t)u^p = 0, \tag{5.8.1}$$

with $a(t)$ satisfying (a_*) and $(a_\#)$, and $p > 1$.

In the first part of the present chapter and in [17] it was shown that, assuming to deal with a parameter-dependent weight

$$a_\mu(t) := q^+(t) - \mu q^-(t),$$

equation (5.8.1) with $a(t) = a_\mu(t)$ has positive subharmonic solutions (of *any* order) whenever $\mu \gg 0$ (i.e. when the weight function $a(t)$ has "large" negative part). Such a result, which may be interpreted in the context of singular perturbation problems, provides indeed positive subharmonic solutions which can be characterized by the fact of being either "small" or "large" on the intervals of positivity of the weight function (according to a chaotic-like multibump behavior). A careful comparison between this result and Theorem 5.4.1 could deserve some interest.

In a similar spirit, it is worth recalling that, again according to Chapter 4 and [17], whenever (a_*) holds with $m \geq 2$, equation (5.8.1) with $a(t) = a_\mu(t)$ and $\mu \gg 0$ has at least $2^m - 1$ distinct positive T-periodic solutions, say $u_i^*(t)$ for $i = 1, \ldots, 2^m - 1$. Since Lemma 5.7.2 implies that any of these periodic solutions has non-zero Morse index, Proposition 5.5.1 can be in principle applied $2^m - 1$ times to obtain positive subharmonic solutions oscillating around each $u_i^*(t)$. It seems however a quite delicate question to understand if these subharmonic solutions are actually distinct or not.

Finally, we observe that it appears very natural to consider the damped version of (5.8.1), namely

$$u'' + cu' + a(t)u^p = 0, \tag{5.8.2}$$

where $c \in \mathbb{R}$ is an arbitrary constant. Indeed, it was shown in Chapter 3 that conditions (a_*) and $(a_\#)$ also guarantee the existence of a positive T-periodic solution to (5.8.2). What about positive suhharmonic solutions? It is generally expected that the periodic solutions provided by the Poincaré-Birkhoff fixed point theorem disappear for (even small) perturbations destroying the Hamiltonian structure, but maybe this is not the case for the positive subharmonic solutions to (5.8.2). Let us observe, for instance, that the multibump subharmonics constructed in the first part of this chapter via degree theory for $a(t) = a_\mu(t)$ (and μ large) still exist for $c \neq 0$. Since both the symplectic approach and the variational one are useless in a non-Hamiltonian setting, investigating the general case of an arbitrary weight function with a negative mean value seems to be a difficult problem.

Part II

Super-sublinear indefinite problems

Chapter 6

Existence results

In this chapter we deal with boundary value problems associated with the nonlinear second order ordinary differential equation

$$(\mathscr{E}_\lambda) \qquad u'' + cu' + \lambda a(t) g(u) = 0,$$

where $c \in \mathbb{R}$, $\lambda > 0$ is a real parameter, $a(t)$ is a sign-changing weight and $g(s)$ has *superlinear growth at zero* and *sublinear growth at infinity*. We refer to $(\mathscr{E}_\lambda)$ as a *super-sublinear indefinite equation*. The main contribution of the present chapter is to provide an existence result for pairs of positive solutions to equation $(\mathscr{E}_\lambda)$ in the possibly non-variational setting (when $c \neq 0$).

As in Chapter 4, we focus our attention on the periodic problem associated with $(\mathscr{E}_\lambda)$ and then we discuss other boundary conditions. For the periodic problem, we prove the existence of two positive T-periodic solutions when $\int_0^T a(t)\,dt < 0$ and $\lambda > 0$ is sufficiently large. Our approach is based on Mawhin's coincidence degree theory and index computations.

The plan of the chapter is the following. In Section 6.1 we list the hypotheses on $a(t)$ and on $g(s)$ that we assume for the rest of the chapter and we present our main result (Theorem 6.1.1). In Section 6.2 we state two lemmas for the computation of the coincidence degree (see Lemma 6.2.1 and Lemma 6.2.2), that are then employed in the proof of Theorem 6.1.1, as explained in Section 6.3 where we provide the main steps of the proof. The tecnical details are performed in Section 6.4. We present in Section 6.5 some consequences and variants of the main theorem (including existence of small/large solutions using only conditions for $g(s)$ near zero/near infinity, respectively). In the same section we also deal with the smooth case and give a nonexistence result. Section 6.6 is devoted to a brief description of how all the results can be adapted to the Dirichlet and Neumann problems, including a final application to radially symmetric solutions on annular domains.

6.1 The main existence result

In this section we present our main existence results for *positive* T-*periodic* solutions to $(\mathscr{E}_\lambda)$, namely functions $u(t)$ satisfying $(\mathscr{E}_\lambda)$ in the Carathéodory sense and such that $u(t+T)=u(t)>0$ for all $t\in\mathbb{R}$.

We suppose that $a\colon \mathbb{R}\to\mathbb{R}$ is a locally integrable T-periodic function and the nonlinear map $g\colon \mathbb{R}^+\to\mathbb{R}^+$ is continuous and such that

$$(g_*) \qquad g(0)=0, \qquad g(s)>0 \quad \text{for} \;\; s>0.$$

The real constant c is arbitrary and results will be given depending on the parameter $\lambda>0$.

As main assumptions on the nonlinearity we require that $g(s)$ tends to zero for $s\to 0^+$ faster than linearly and it has a sublinear growth at infinity, that is

$$(g_0) \qquad \lim_{s\to 0^+}\frac{g(s)}{s}=0$$

and

$$(g_\infty) \qquad \lim_{s\to +\infty}\frac{g(s)}{s}=0.$$

When $g(s)$ is continuously differentiable with $g'(s)>0$ for all $s>0$, using the same argument as in Chapter 3 (when $c=0$), we deduce that condition

$$(a_\#) \qquad \int_0^T a(t)\,dt<0.$$

is necessary for the existence of positive T-periodic solutions to $(\mathscr{E}_\lambda)$ with an arbitrary $c\in\mathbb{R}$.

Before stating our main result, we recall that a continuous function $h\colon \mathbb{R}^+_0\to\mathbb{R}^+_0$ is said to be *regularly oscillating at zero* if

$$\lim_{\substack{s\to 0^+\\ \omega\to 1}}\frac{h(\omega s)}{h(s)}=1.$$

Analogously, we say that h is *regularly oscillating at infinity* if

$$\lim_{\substack{s\to +\infty\\ \omega\to 1}}\frac{h(\omega s)}{h(s)}=1.$$

We refer to Chapter 3 for a discussion on regularly oscillating functions.

Now we are in position to state our main result.

Theorem 6.1.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*). Suppose also that g is regularly oscillating at zero and at infinity and satisfies (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition $(a_\#)$. Furthermore, suppose that there exists an interval $I \subseteq [0,T]$ such that $a(t) \geq 0$ for a.e. $t \in I$ and $\int_I a(t)\,dt > 0$. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ equation $(\mathscr{E}_\lambda)$ has at least two positive T-periodic solutions.*

As will become clear from the proof, the constant λ^* can be chosen depending (besides on c and $g(s)$) only on the behavior of $a(t)$ on the interval I. This remark allows to obtain the following corollary for the related two-parameter equation

$$u'' + cu' + \bigl(\lambda a^+(t) - \mu a^-(t)\bigr) g(u) = 0, \tag{6.1.1}$$

with $\lambda, \mu > 0$.

Corollary 6.1.1. *Let $g(s)$ be as above and let $a(t)$ be a T-periodic function with $a^\pm \in L^1([0,T])$ and $a^- \not\equiv 0$. Suppose also that there exists an interval $I \subseteq [0,T]$ such that*

$$\int_I a^-(t)\,dt = 0 < \int_I a^+(t)\,dt.$$

Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for each

$$\mu > \lambda \,\frac{\int_0^T a^+(t)\,dt}{\int_0^T a^-(t)\,dt}$$

equation (6.1.1) *has at least two positive T-periodic solutions.*

Our results are sharp in the sense that there are examples of functions $g(s)$ satisfying all the assumptions of Theorem 6.1.1 or of Corollary 6.1.1 and such that there are no positive T-periodic solutions if $\lambda > 0$ is small or if $(a_\#)$ is not satisfied. For this remark see [36, § 2], where the assertions were proved in the case $c = 0$. One can easily check that those results can be extended to the case of an arbitrary $c \in \mathbb{R}$ (see also Section 6.5.4).

Another sharp result can be given when $g(s)$ is smooth. Indeed, first of all we produce a variant of Theorem 6.1.1 by replacing the hypothesis of regular oscillation of g at zero or at infinity with the condition of continuous differentiability of $g(s)$ in a neighborhood of $s = 0$ or, respectively, near infinity (see Theorem 6.5.3). Next, in the smooth case and further assuming that $|g'(s)|$ is bounded on $\mathbb{R}^+_0$, we can also provide a nonexistence result for $\lambda > 0$ small (see Theorem 6.5.4). As a consequence of these results, the following variant of Theorem 6.1.1 can be stated. We denote by $g'(\infty) = \lim_{s\to+\infty} g'(s)$.

Theorem 6.1.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function satisfying (g_*) and such that $g'(0) = 0$ and $g'(\infty) = 0$. Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition $(a_{\#})$. Furthermore, suppose that there exists an interval $I \subseteq [0,T]$ such that $a(t) \geq 0$ for a.e. $t \in I$ and $\int_I a(t)\,dt > 0$. Then there exists $\lambda_* > 0$ such that for each $0 < \lambda < \lambda_*$ equation $(\mathscr{E}_\lambda)$ has no positive T-periodic solution. Moreover, there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ equation $(\mathscr{E}_\lambda)$ has at least two positive T-periodic solutions. Condition $(a_{\#})$ is also necessary if $g'(s) > 0$ for $s > 0$.*

To show a simple example of applicability of Theorem 6.1.2, we consider the T-periodic boundary value problem

$$\begin{cases} u'' + cu' + \lambda(\sin(t) + \delta)g(u) = 0 \\ u(2\pi) - u(0) = u'(2\pi) - u'(0) = 0, \end{cases} \tag{6.1.2}$$

where $\delta \in \mathbb{R}$ and

$$g(s) = \arctan(s^\alpha), \quad \text{with } \alpha > 1,$$

(other examples of functions $g(s)$ can be easily produced). Since $g'(s) > 0$ for all $s > 0$, we know that there are positive T-periodic solutions only if $-1 < \delta < 0$. Moreover, for any fixed $\delta \in\,]-1,0[$ there exist two constants $0 < \lambda_{*,\delta} \leq \lambda^{*,\delta}$ such that for $0 < \lambda < \lambda_{*,\delta}$ there are no positive solutions for problem (6.1.2), while for $\lambda > \lambda^{*,\delta}$ there are at least two positive solutions. Estimates for $\lambda_{*,\delta}$ and $\lambda^{*,\delta}$ can be given for any specific equation.

Figure 6.1 below illustrates another example of existence of two positive T-periodic solutions to $(\mathscr{E}_\lambda)$, when the weight function possesses a positive hump separated by a negative one.

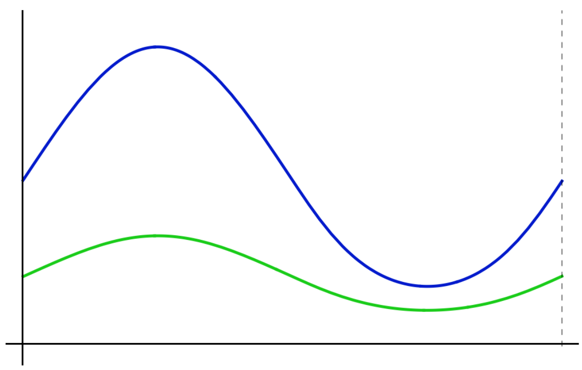

Figure 6.1: The figure shows an example of pairs of positive T-periodic solutions to $(\mathscr{E}_\lambda)$, with $c = 0$. For this simulation we have chosen the interval $[0,T]$ with $T = 2\pi$, $\lambda = 1$, $a(t) := \sin^+(t) - 5\sin^-(t)$, $g(s) := s^2/(1+s^2)$ for $s > 0$.

6.2 The abstract setting of the coincidence degree

Let $X := \mathcal{C}_T$ be the Banach space of continuous and T-periodic functions $u\colon \mathbb{R} \to \mathbb{R}$, endowed with the norm

$$\|u\|_\infty := \max_{t\in[0,T]} |u(t)| = \max_{t\in\mathbb{R}} |u(t)|,$$

and let $Z := L^1_T$ be the Banach space of measurable and T-periodic functions $v\colon \mathbb{R} \to \mathbb{R}$ which are integrable on $[0,T]$, endowed with the norm

$$\|v\|_{L^1_T} := \int_0^T |v(t)|\,dt.$$

The linear differential operator

$$L\colon u \mapsto -u'' - cu'$$

is a (linear) Fredholm map of index zero defined on $\operatorname{dom} L := W^{2,1}_T \subseteq X$, with range

$$\operatorname{Im} L = \left\{ v \in Z\colon \int_0^T v(t)\,dt = 0 \right\}.$$

Associated with L we have the projectors

$$P\colon X \to \ker L \cong \mathbb{R}, \qquad Q\colon Z \to \operatorname{coker} L \cong Z/\operatorname{Im} L \cong \mathbb{R},$$

that, in our situation, can be chosen as the average operators

$$Pu = Qu := \frac{1}{T}\int_0^T u(t)\,dt.$$

Finally, let

$$K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$$

be the right inverse of L, which is the operator that to any function $v \in L^1_T$ with $\int_0^T v(t)\,dt = 0$ associates the unique T-periodic solution u of

$$u'' + cu' + v(t) = 0, \qquad \text{with} \quad \int_0^T u(t)\,dt = 0.$$

Next, we define the L^1-Carathéodory function

$$f_\lambda(t,s) := \begin{cases} -s, & \text{if } s \leq 0; \\ \lambda a(t) g(s), & \text{if } s \geq 0; \end{cases}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic and locally integrable function, $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $g(0) = 0$ and $\lambda > 0$ is a fixed parameter. Let us denote by $N_\lambda\colon X \to Z$ the Nemytskii operator induced by the function f_λ, that is

$$(N_\lambda u)(t) := f_\lambda(t, u(t)), \quad t \in \mathbb{R}.$$

By coincidence degree theory we know that the equation

$$Lu = N_\lambda u, \quad u \in \operatorname{dom} L, \tag{6.2.1}$$

is equivalent to the fixed point problem

$$u = \Phi_\lambda u := Pu + QN_\lambda u + K_P(Id - Q)N_\lambda u, \quad u \in X.$$

Technically, the term $QN_\lambda u$ in the above formula should be more correctly written as $JQN_\lambda u$, where J is a linear (orientation-preserving) isomorphism from $\operatorname{coker} L$ to $\ker L$. However, in our situation, we can take as J the identity on $\mathbb{R}$, having identified $\operatorname{coker} L$, as well as $\ker L$, with $\mathbb{R}$. It is standard to verify that $\Phi_\lambda\colon X \to X$ is a completely continuous operator. In such a situation, we usually say that N_λ is *L-completely continuous* (see [130], where the treatment has been given for the most general cases).

Within the framework introduced above, we present now two auxiliary semi-abstract results which are useful for the computation of the coincidence degree (see also Theorem 3.1.1 and Lemma 4.2.3). For Lemma 6.2.1 and Lemma 6.2.2 we do not require all the assumptions on $a(t)$ and $g(s)$ stated in Theorem 6.1.1. In this way we hope that the two results may have an independent interest beyond that of providing a proof of Theorem 6.1.1.

Lemma 6.2.1. *Let $\lambda > 0$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $g(0) = 0$. Suppose $a \in L^1_T$. Assume that there exists a constant $d > 0$ and a compact interval $\mathcal{I} \subseteq [0, T]$ such that the following properties hold.*

$(A_{d,\mathcal{I}})$ *If $\alpha \geq 0$, then any non-negative T-periodic solution $u(t)$ of*

$$u'' + cu' + \lambda a(t)g(u) + \alpha = 0 \tag{6.2.2}$$

satisfies $\max_{t\in\mathcal{I}} u(t) \neq d$.

$(B_{d,\mathcal{I}})$ *For every $\beta \geq 0$ there exists a constant $D_\beta \geq d$ such that if $\alpha \in [0, \beta]$ and $u(t)$ is any non-negative T-periodic solution of equation (6.2.2) with $\max_{t\in\mathcal{I}} u(t) \leq d$, then $\max_{t\in[0,T]} u(t) \leq D_\beta$.*

$(C_{d,\mathcal{I}})$ *There exists $\alpha^* \geq 0$ such that equation (6.2.2), with $\alpha = \alpha^*$, does not have any non-negative T-periodic solution $u(t)$ with $\max_{t\in\mathcal{I}} u(t) \leq d$.*

Then

$$D_L(L - N_\lambda, \Omega_{d,\mathcal{I}}) = 0,$$

where

$$\Omega_{d,\mathcal{I}} := \Big\{ u \in X \colon \max_{t\in\mathcal{I}} |u(t)| < d \Big\}.$$

Notice that $\Omega_{d,\mathcal{I}}$ is open but not bounded (unless $\mathcal{I} = [0,T]$).

Proof. For a fixed constant $d > 0$ and a compact interval $\mathcal{I} \subseteq [0,T]$ as in the statement, let us consider the open set $\Omega_{d,\mathcal{I}}$ defined above. We study the equation

$$u'' + cu' + f_\lambda(t,u) + \alpha = 0, \tag{6.2.3}$$

for $\alpha \geq 0$, which can be written as a coincidence equation in the space X

$$Lu = N_\lambda u + \alpha \mathbf{1}, \quad u \in \operatorname{dom} L,$$

where $\mathbf{1} \in X$ is the constant function $\mathbf{1}(t) \equiv 1$.

As a first step, we check that $D_L(L - N_\lambda - \alpha\mathbf{1}, \Omega_{d,\mathcal{I}})$ is well-defined for any $\alpha \geq 0$. To this aim, suppose that $\alpha \geq 0$ is fixed and consider the set

$$\begin{aligned}\mathcal{R}_\alpha &:= \bigl\{u \in \operatorname{cl}(\Omega_{d,\mathcal{I}}) \cap \operatorname{dom} L \colon Lu = N_\lambda u + \alpha\mathbf{1}\bigr\} \\ &= \bigl\{u \in \operatorname{cl}(\Omega_{d,\mathcal{I}}) \colon u = \Phi_\lambda u + \alpha\mathbf{1}\bigr\}.\end{aligned}$$

We have that $u \in \mathcal{R}_\alpha$ if and only if $u(t)$ is a T-periodic solution of (6.2.3) such that $|u(t)| \leq d$ for every $t \in \mathcal{I}$. By a standard application of the maximum principle, we find that $u(t) \geq 0$ for all $t \in \mathbb{R}$ and, indeed, $u(t)$ solves (6.2.2), with $\max_{t\in\mathcal{I}} u(t) \leq d$. Condition $(B_{d,\mathcal{I}})$ gives a constant D_α such that $\|u\|_\infty \leq D_\alpha$ and so $\mathcal{R}_\alpha$ is bounded. The complete continuity of the operator Φ_λ ensures the compactness of $\mathcal{R}_\alpha$. Moreover, condition $(A_{d,\mathcal{I}})$ guarantees that $|u(t)| < d$ for all $t \in \mathcal{I}$ and then we conclude that $\mathcal{R}_\alpha \subseteq \Omega_{d,\mathcal{I}}$. In this manner we have proved that the coincidence degree $D_L(L - N_\lambda - \alpha\mathbf{1}, \Omega_{d,\mathcal{I}})$ is well-defined for any $\alpha \geq 0$.

Now, condition $(C_{d,\mathcal{I}})$, together with the property of existence of solutions when the degree D_L is non-zero, implies that there exists $\alpha^* \geq 0$ such that

$$D_L(L - N_\lambda - \alpha^*\mathbf{1}, \Omega_{d,\mathcal{I}}) = 0.$$

On the other hand, from condition $(B_{d,\mathcal{I}})$ applied on the interval $[0,\beta] := [0,\alpha^*]$, by repeating the same argument as in the first step above, we find that the set

$$\begin{aligned}\mathcal{S} := \bigcup_{\alpha\in[0,\alpha^*]} \mathcal{R}_\alpha &= \bigcup_{\alpha\in[0,\alpha^*]} \bigl\{u \in \operatorname{cl}(\Omega_{d,\mathcal{I}}) \cap \operatorname{dom} L \colon Lu = N_\lambda u + \alpha\mathbf{1}\bigr\} \\ &= \bigcup_{\alpha\in[0,\alpha^*]} \bigl\{u \in \operatorname{cl}(\Omega_{d,\mathcal{I}}) \colon u = \Phi_\lambda u + \alpha\mathbf{1}\bigr\}\end{aligned}$$

is a compact subset of $\Omega_{d,\mathcal{I}}$. Hence, by the homotopic invariance of the coincidence degree, we have that

$$D_L(L - N_\lambda, \Omega_{d,\mathcal{I}}) = D_L(L - N_\lambda - \alpha^*\mathbf{1}, \Omega_{d,\mathcal{I}}) = 0.$$

This concludes the proof. $\square$

Lemma 6.2.2. *Let $\lambda > 0$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $g(0) = 0$. Suppose $a \in L^1_T$ with $\int_0^T a(t)\,dt < 0$. Assume that there exists a constant $d > 0$ such that $g(d) > 0$ and the following property holds.*

(H_d) *If $\vartheta \in \,]0,1]$ and $u(t)$ is any non-negative T-periodic solution of*

$$u'' + cu' + \vartheta\lambda a(t)g(u) = 0, \tag{6.2.4}$$

then $\max_{t\in[0,T]} u(t) \neq d$.

Then

$$D_L(L - N_\lambda, B(0,d)) = 1.$$

Proof. The proof follows substantially the same argument employed in the proof of Theorem 3.1.1. First of all, we claim that there are no solutions to the parameterized coincidence equation

$$Lu = \vartheta N_\lambda u, \quad u \in \partial B(0,d) \cap \operatorname{dom} L, \ \ 0 < \vartheta \leq 1.$$

Indeed, if any such a solution u exists, it is a T-periodic solution of

$$u'' + cu' + \vartheta f_\lambda(t,u) = 0,$$

with $\|u\|_\infty = d$. By the definition of $f_\lambda(t,s)$ and a standard application of the maximum principle, we easily get that $u(t) \geq 0$ for every $t \in \mathbb{R}$. Therefore, $u(t)$ is a non-negative T-periodic solution of (6.2.4) with $\max_{t\in[0,T]} u(t) = d$. This contradicts property (H_d) and the claim is thus proved.

As a second step, we consider $QN_\lambda u$ for $u \in \ker L$. Since $\ker L \cong \mathbb{R}$, we have

$$QN_\lambda u = \frac{1}{T}\int_0^T f_\lambda(t,s)\,dt, \qquad \text{for } \ u \equiv \text{constant} = s \in \mathbb{R}.$$

For notational convenience, we set

$$f^{\#}_\lambda(s) := \frac{1}{T}\int_0^T f_\lambda(t,s)\,dt = \begin{cases} -s, & \text{if } s \leq 0; \\ \lambda\Big(\dfrac{1}{T}\displaystyle\int_0^T a(t)\,dt\Big)g(s), & \text{if } s \geq 0. \end{cases}$$

Note that $sf^{\#}_\lambda(s) < 0$ for each $s \neq 0$. As a consequence, we find that $QN_\lambda u \neq 0$ for each $u \in \partial B(0,d) \cap \ker L$.

An important result from Mawhin's continuation theorem (see [132, Theorem 2.4] and also [127], where the result was previously given in the context of the periodic problem for ODEs) guarantees that

$$\begin{aligned} D_L(L - N_\lambda, B(0,d)) &= \deg_B(-QN_\lambda|_{\ker L}, B(0,d) \cap \ker L, 0) \\ &= \deg_B(-f^{\#}_\lambda,]-d,d[, 0). \end{aligned}$$

This latter degree is clearly equal to 1 as

$$-f^{\#}_\lambda(-d) = -d < 0 < \lambda\Big(-\frac{1}{T}\int_0^T a(t)\,dt\Big)g(d) = -f^{\#}_\lambda(d).$$

This concludes the proof. □

6.3 Proof of Theorem 6.1.1: the general strategy

With the aid of the two lemmas proved in the Section 6.2, we can prove Theorem 6.1.1. Omitting the technical details and estimates (presented in Section 6.4), in this section we show the general strategy of the proof.

From now on, all the assumptions on $a(t)$ and $g(s)$ in Theorem 6.1.1 are implicitly assumed.

We fix a constant $\rho > 0$ and consider, for $\mathcal{I} := I$, the open set

$$\Omega_{\rho,I} := \Big\{ u \in X \colon \max_{t \in I} |u(t)| < \rho \Big\}.$$

First of all, we show that condition $(A_{\rho,I})$ is satisfied provided that $\lambda > 0$ is sufficiently large, say $\lambda > \lambda^* := \lambda^*_{\rho,I}$. Such lower bound for λ does not depend on α. Then, we fix an arbitrary $\lambda > \lambda^*$ and show that conditions $(B_{\rho,I})$ and $(C_{\rho,I})$ are satisfied as well. In particular, for $\beta = 0$, we find a constant $D_0 = D_0(\rho, I, \lambda) \geq \rho$ such that any possible solution of

$$Lu = N_\lambda u, \quad u \in \mathrm{cl}(\Omega_{\rho,I}) \cap \mathrm{dom}\, L,$$

satisfies

$$\|u\|_\infty \leq D_0.$$

In this manner, we have that

$$B(0,\rho) \subseteq \Omega_{\rho,I} \quad \text{and} \quad \mathrm{Fix}\,(\Phi_\lambda, \Omega_{\rho,I}) \subseteq B(0,R), \;\; \forall\, R > D_0.$$

Moreover,

$$D_L(L - N_\lambda, \Omega_{\rho,I}) = D_L(L - N_\lambda, \Omega_{\rho,I} \cap B(0,R)) = 0, \quad \forall\, R > D_0.$$

As a next step, using (g_0) and the regular oscillation of $g(s)$ at zero, we find a positive constant $r_0 < \rho$ such that for each $r \in \,]0, r_0]$ condition (H_r) (of Lemma 6.2.2) is satisfied and therefore

$$D_L(L - N_\lambda, B(0,r)) = 1, \quad \forall\, 0 < r \leq r_0.$$

With a similar argument, using (g_∞) and the regular oscillation of $g(s)$ at infinity, we find a positive constant $R_0 > D_0$ such that for each $R \geq R_0$ condition (H_R) is satisfied too and therefore

$$D_L(L - N_\lambda, B(0,R)) = 1, \quad \forall\, R \geq R_0.$$

By the additivity property of the coincidence degree we obtain

$$D_L\big(L - N_\lambda, \Omega_{\rho,I} \setminus B[0,r]\big) = -1, \quad \forall\, 0 < r \leq r_0, \tag{6.3.1}$$

and

$$D_L\big(L - N_\lambda, B(0,R) \setminus \mathrm{cl}(\Omega_{\rho,I} \cap B(0,R_0))\big) = 1, \quad \forall\, R > R_0. \tag{6.3.2}$$

Thus, in conclusion, we find a first solution $\underline{u}$ of (6.2.1) with $\underline{u} \in \Omega_{\rho,I} \setminus B[0,r]$ (using (6.3.1) for a fixed $r \in]0, r_0]$) and a second solution $\overline{u}$ of (6.2.1) with $\overline{u} \in B(0,R) \setminus \mathrm{cl}(\Omega_{\rho,I} \cap B(0,R_0))$ (using (6.3.2) for a fixed $R > R_0$). Both $\underline{u}(t)$ and $\overline{u}(t)$ are nontrivial T-periodic solutions of

$$u'' + cu' + f_\lambda(t,u) = 0$$

and, by the maximum principle, they are actually *non-negative* solutions of $(\mathscr{E}_\lambda)$. Finally, since by condition (g_0) we know that $a(t)g(s)/s$ is L^1-bounded in a right neighborhood of $s = 0$, it is immediate to prove (by an elementary form of the strong maximum principle) that such solutions are in fact *strictly positive* (cf. Lemma C.1.2).

6.4 Proof of Theorem 6.1.1: the technical details

In this section we give a proof of Theorem 6.1.1 by following the steps described in Section 6.3. To this aim, it is sufficient to check separately the validity of the assumptions in Lemma 6.2.1, for $\mathcal{I} := I$ and $d = \rho > 0$ a fixed number, and the ones in Lemma 6.2.2, for $d = r > 0$ small ($0 < r \leq r_0$) and for $d = R > 0$ large ($R \geq R_0$). Notice that r_0 and R_0 are chosen after both ρ and $\lambda > 0$ have been fixed.

Throughout the section, for the sake of simplicity, we suppose the validity of all the assumptions in Theorem 6.1.1. However, from a careful checking of the proofs below, one can see that not all of them are needed for the verification of each single lemma.

6.4.1 Checking the assumptions of Lemma 6.2.1 for λ large

Let $\rho > 0$ be fixed. Let $I := [\sigma, \tau] \subseteq [0,T]$ be such that $a(t) \geq 0$ for a.e. $t \in I$ and $\int_I a(t)\,dt > 0$. We fix $\varepsilon > 0$ such that for

$$\sigma_0 := \sigma + \varepsilon < \tau - \varepsilon =: \tau_0$$

it holds that

$$\int_{\sigma_0}^{\tau_0} a(t)\,dt > 0.$$

Let us consider the non-negative solutions of equation (6.2.2) for $t \in I$. Such an equation takes the form

$$u'' + cu' + h(t,u) = 0, \tag{6.4.1}$$

where we have set (for notational convenience)

$$h(t,s) = h_{\lambda,\alpha}(t,s) := \lambda a(t)g(s) + \alpha,$$

where $\lambda > 0$ and $\alpha \geq 0$. Note that $h(t,s) \geq 0$ for a.e. $t \in I$ and for all $s \geq 0$.

Writing equation (6.4.1) as

$$\big(e^{ct}u'\big)' + e^{ct}h(t,u) = 0,$$

we find that $(e^{ct}u'(t))' \leq 0$ for a.e. $t \in I$, so that the map $t \mapsto e^{ct}u'(t)$ is non-increasing on I.

We split the proof into different steps.

Step 1. A general estimate. For every non-negative solution $u(t)$ of (6.4.1) the following estimate holds:

$$|u'(t)| \leq \frac{u(t)}{\varepsilon}\, e^{|c|T}, \quad \forall\, t \in [\sigma_0, \tau_0]. \tag{6.4.2}$$

Such inequality has been already proved and used in Chapter 4. The proof is based on the fact that the map $\xi \mapsto e^{c\xi}u'(\xi)$ is non-increasing on I_i^+ and we refer to Section 4.3.3 for the details (cf. equations (4.3.22) and (4.3.34)). Observe that only a condition on the sign of $h(t,s)$ is used and, therefore, the estimate is valid independently on $\lambda > 0$ and $\alpha \geq 0$.

Step 2. Verification of $(A_{\rho,I})$ for $\lambda > \lambda^*$, with λ^* depending on ρ and I but not on α. Suppose that $u(t)$ is a non-negative T-periodic solution of (6.2.2) with

$$\max_{t\in I} u(t) = \rho.$$

Let $t_0 \in I$ be such that $u(t_0) = \rho$ and observe that $u'(t_0) = 0$, if $\sigma < t_0 < \tau$, while $u'(t_0) \leq 0$, if $t_0 = \sigma$, and $u'(t_0) \geq 0$, if $t_0 = \tau$.

First of all, we prove the existence of a constant $\delta \in \,]0,1[$ such that

$$\min_{t\in[\sigma_0,\tau_0]} u(t) \geq \delta\rho. \tag{6.4.3}$$

This follows from the estimate (6.4.2). Indeed, if $t_* \in [\sigma_0, \tau_0]$ is such that $u(t_*) = \min_{t\in[\sigma_0,\tau_0]} u(t)$, we obtain that

$$|u'(t_*)| \leq \frac{u(t_*)}{\varepsilon}\, e^{|c|T}. \tag{6.4.4}$$

On the other hand, by the monotonicity of the function $t \mapsto e^{ct}u'(t)$ in $[\sigma,\tau]$,

$$u'(\xi)e^{c\xi} \geq u'(t_*)e^{ct_*}, \quad \forall\, \xi \in [\sigma, t_*], \tag{6.4.5}$$

and

$$u'(\xi)e^{c\xi} \leq u'(t_*)e^{ct_*}, \quad \forall\, \xi \in [t_*, \tau]. \tag{6.4.6}$$

From the properties about $u'(t_0)$ listed above, we deduce that if $t_0 > t_*$, then $u'(t_0) \geq 0$ and, therefore, we must have $u'(t_*) \geq 0$. Similarly, if $t_0 < t_*$, then $u'(t_0) \leq 0$ and, therefore, we must have $u'(t_*) \leq 0$. The case in which

$t_* = t_0$ can be handled in a trivial way and we do not consider it. In this manner, we have that one of the two following situations occurs: either

$$\sigma \le t_0 < t_* \in [\sigma_0, \tau_0], \quad u(t_0) = \rho, \quad u'(\xi) \le 0, \ \forall\, \xi \in [t_0, t_*], \tag{6.4.7}$$

or

$$\tau \ge t_0 > t_* \in [\sigma_0, \tau_0], \quad u(t_0) = \rho, \quad u'(\xi) \ge 0, \ \forall\, \xi \in [t_*, t_0]. \tag{6.4.8}$$

Suppose that (6.4.7) holds. In this situation, from (6.4.5) we have $-u'(\xi) \le -u'(t_*)e^{c(t_*-\xi)}$ for all $\xi \in [t_0, t_*]$ and thus, integrating on $[t_0, t_*]$ and using (6.4.4), we obtain

$$\rho - u(t_*) \le |u'(t_*)|\, e^{|c|T}(t_* - t_0) \le \frac{u(t_*)}{\varepsilon}\, e^{2|c|T}T.$$

This gives (6.4.3) for

$$\delta := \frac{\varepsilon}{\varepsilon + e^{2|c|T}T}.$$

We get exactly the same estimate in case of (6.4.8), by using (6.4.6) and then integrating on $[t_*, t_0]$. Observe that the constant $\delta \in \,]0,1[$ does not depend on λ and α.

Having found the constant δ, we now define

$$\eta = \eta(\rho) := \min\bigl\{g(s) \colon s \in [\delta\rho, \rho]\bigr\}.$$

Then, integrating equation (6.2.2) on $[\sigma_0, \tau_0]$ and using (6.4.2) (for $t = \sigma_0$ and $t = \tau_0$), we obtain

$$\begin{aligned}\lambda\eta \int_{\sigma_0}^{\tau_0} a(t)\,dt &\le \lambda \int_{\sigma_0}^{\tau_0} a(t)g(u(t))\,dt \\ &= u'(\sigma_0) - u'(\tau_0) + c\bigl(u(\sigma_0) - u(\tau_0)\bigr) - \alpha\,(\tau_0 - \sigma_0) \\ &\le 2\frac{\rho}{\varepsilon}e^{|c|T} + 2|c|\rho.\end{aligned}$$

Now, we define

$$\lambda^* := \frac{2\rho\bigl(\varepsilon|c| + e^{|c|T}\bigr)}{\varepsilon\eta\int_{\sigma_0}^{\tau_0} a(t)\,dt}. \tag{6.4.9}$$

Arguing by contradiction, we immediately conclude that there are no (non-negative) T-periodic solutions $u(t)$ of (6.2.2) with $\max_{t\in I} u(t) = \rho$ if $\lambda > \lambda^*$. Thus condition $(A_{\rho,I})$ is proved.

Step 3. Verification of $(B_{\rho,I})$. Let $u(t)$ be any non-negative T-periodic solution of (6.2.2) with $\max_{t\in I} u(t) \le \rho$. Let us fix an instant $\hat{t} \in [\sigma_0, \tau_0]$. By (6.4.2), we know that

$$|u'(\hat{t})| \le \frac{\rho}{\varepsilon}\, e^{|c|T}.$$

Using the fact that

$$|h(t,s)| \leq M(t)|s| + N(t), \quad \text{for a.e. } t \in [0,T],\ \forall\, s \in \mathbb{R},\ \forall\, \alpha \in [0,\beta],$$

with suitable $M, N \in L^1_T$ (depending on β), from a standard application of the (generalized) Gronwall's inequality (cf. [104]), we find a constant $D_\beta = D_\beta(\rho,\lambda)$ such that

$$\max_{t\in[0,T]} \bigl(|u(t)| + |u'(t)|\bigr) \leq D_\beta.$$

So condition $(B_{\rho,I})$ is verified.

Step 4. Verification of $(C_{\rho,I})$. Let $u(t)$ be an arbitrary non-negative T-periodic solution of (6.2.2) with $\max_{t\in I} u(t) \leq \rho$. Integrating (6.2.2) on $[\sigma_0,\tau_0]$ and using (6.4.2) (for $t=\sigma_0$ and $t=\tau_0$), we obtain

$$\begin{aligned}\alpha\,(\tau_0-\sigma_0) &= u'(\sigma_0) - u'(\tau_0) + c\bigl(u(\sigma_0)-u(\tau_0)\bigr) - \lambda\biggl(\int_{\sigma_0}^{\tau_0} a(t)g(u(t))\,dt\biggr)\\ &\leq 2\frac{\rho}{\varepsilon}e^{|c|T} + 2|c|\rho =: K = K(\rho,\varepsilon).\end{aligned}$$

This yields a contradiction if $\alpha > 0$ is sufficiently large. Hence $(C_{\rho,I})$ is verified, by taking $\alpha^* > K/(\tau_0-\sigma_0)$.

In conclusion, all the assumptions of Lemma 6.2.1 have been verified for a fixed $\rho > 0$ and for $\lambda > \lambda^*$. □

Remark 6.4.1. Notice that, among the assumptions of Theorem 6.1.1, in this part of the proof we have used only the following ones: $g(s) > 0$ for all $s \in\,]0,\rho]$, $\limsup_{s\to+\infty} |g(s)|/s < +\infty$, $a \in L^1_T$ and $a(t) \geq 0$ for a.e. $t \in I$, with $\int_I a(t)\,dt > 0$. ◁

6.4.2 Checking the assumptions of Lemma 6.2.2 for r small

We prove that condition (H_d) of Lemma 6.2.2 is satisfied for $d = r$ sufficiently small. We proceed as in the proof of Theorem 3.2.1. Indeed, we claim that there exists $r_0 > 0$ such that there is no non-negative T-periodic solution $u(t)$ of (6.2.4) for some $\vartheta \in\,]0,1]$ with $\|u\|_\infty = r \in\,]0,r_0]$. Arguing by contradiction, we suppose that there exists a sequence of T-periodic functions $u_n(t)$ with $u_n(t) \geq 0$ for all $t \in \mathbb{R}$ and such that

$$u_n''(t) + cu_n'(t) + \vartheta_n\lambda a(t)g(u_n(t)) = 0, \tag{6.4.10}$$

for a.e. $t \in \mathbb{R}$ with $\vartheta_n \in\,]0,1]$, and also such that $\|u_n\|_\infty = r_n \to 0^+$. Let $t_n^* \in [0,T]$ be such that $u_n(t_n^*) = r_n$.

We define

$$v_n(t) := \frac{u_n(t)}{\|u_n\|_\infty} = \frac{u_n(t)}{r_n}, \quad t \in \mathbb{R},$$

and observe that (6.4.10) can be equivalently written as

$$v_n''(t) + cv_n'(t) + \vartheta_n \lambda a(t) q(u_n(t)) v_n(t) = 0, \tag{6.4.11}$$

where the map $q\colon \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $q(s) := g(s)/s$ for $s > 0$ and $q(0) = 0$. Notice that q is continuous on $\mathbb{R}^+$ (by condition (g_0)). Moreover, $q(u_n(t)) \to 0$ uniformly in $\mathbb{R}$, as a consequence of $\|u_n\|_\infty \to 0$. Multiplying equation (6.4.11) by v_n and integrating on $[0,T]$, we find

$$\|v_n'\|^2_{L^2_T} = \int_0^T v_n'(t)^2 \, dt \leq \lambda \|a\|_{L^1_T} \sup_{t\in[0,T]} |q(u_n(t))| \to 0, \quad \text{as } n \to \infty.$$

As an easy consequence $\|v_n - 1\|_\infty \to 0$, as $n \to \infty$.

Integrating (6.4.10) on $[0,T]$ and using the periodic boundary conditions, we have

$$0 = \int_0^T a(t) g(u_n(t)) \, dt = \int_0^T a(t) g(r_n) \, dt + \int_0^T a(t) \big(g(r_n v_n(t)) - g(r_n) \big) \, dt$$

and hence, dividing by $g(r_n) > 0$, we obtain

$$0 < -\int_0^T a(t) \, dt \leq \|a\|_{L^1_T} \sup_{t\in[0,T]} \left| \frac{g(r_n v_n(t))}{g(r_n)} - 1 \right|.$$

Using the fact that $g(s)$ is regularly oscillating at zero and $v_n(t) \to 1$ uniformly as $n \to \infty$, we find that the right-hand side of the above inequality tends to zero and thus we achieve a contradiction. $\square$

Remark 6.4.2. Notice that, among the assumptions of Theorem 6.1.1, in this part of the proof we have used only the following ones (for verifying (H_r)): $g(s) > 0$ for all s in a right neighborhood of $s = 0$, $g(s)$ regularly oscillating at zero and satisfying (g_0), $a \in L^1_T$ with $\int_0^T a(t)\, dt < 0$. $\lhd$

6.4.3 Checking the assumptions of Lemma 6.2.2 for R large

We are going to check that condition (H_d) of Lemma 6.2.2 is satisfied for $d = R$ sufficiently large. In other words, we claim that there exists $R_0 > 0$ such that there is no non-negative T-periodic solution $u(t)$ of (6.2.4) for some $\vartheta \in \,]0,1]$ with $\|u\|_\infty = R \geq R_0$. Arguing by contradiction, we suppose that there exists a sequence of T-periodic functions $u_n(t)$ with $u_n(t) \geq 0$ for all $t \in \mathbb{R}$ and such that

$$u_n''(t) + cu_n'(t) + \vartheta_n \lambda a(t) g(u_n(t)) = 0, \tag{6.4.12}$$

for a.e. $t \in \mathbb{R}$ with $\vartheta_n \in \,]0,1]$, and also such that $\|u_n\|_\infty = R_n \to +\infty$. Let $t_n^* \in [0,T]$ be such that $u_n(t_n^*) = R_n$.

First of all, we claim that $u_n(t) \to +\infty$ uniformly in t (as $n \to \infty$). Indeed, to be more precise, we have that $u_n(t) \geq R_n/2$ for all t. To prove this assertion, let us suppose, by contradiction, that $\min u_n(t) < R_n/2$. In this case, we can take a maximal compact interval $[\alpha_n, \beta_n]$ containing t_n^* and such that $u_n(t) \geq R_n/2$ for all $t \in [\alpha_n, \beta_n]$. By the maximality of the interval, we also have that $u_n(\alpha_n) = u_n(\beta_n) = R_n/2$ with $u_n'(\alpha_n) \geq 0 \geq u_n'(\beta_n)$.

We set

$$w_n(t) := u_n(t) - \frac{R_n}{2}, \quad t \in \mathbb{R},$$

and observe that $0 \leq w_n(t) \leq R_n/2$ for all $t \in [\alpha_n, \beta_n]$. Equation (6.4.12) reads equivalently as

$$-w_n''(t) - cw_n'(t) = \vartheta_n \lambda a(t) g(u_n(t)).$$

Multiplying this equation by $w_n(t)$ and integrating on $[\alpha_n, \beta_n]$, we obtain

$$\int_{\alpha_n}^{\beta_n} w_n'(t)^2 \, dt \leq \lambda \|a\|_{L^1_T} \frac{R_n}{2} \sup_{\frac{R_n}{2} \leq s \leq R_n} |g(s)|.$$

From condition (g_∞), for any fixed $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that $|g(s)| \leq \varepsilon s$, for all $s \geq L_\varepsilon$. Thus, for n sufficiently large so that $R_n \geq 2L_\varepsilon$, we find

$$\int_{\alpha_n}^{\beta_n} w_n'(t)^2 \, dt \leq \frac{1}{2} \lambda \varepsilon R_n^2 \|a\|_{L^1_T}.$$

By an elementary form of the Poincaré-Sobolev inequality, we conclude that

$$\frac{R_n^2}{4} = \max_{t \in [\alpha_n, \beta_n]} |w_n(t)|^2 \leq T \int_{\alpha_n}^{\beta_n} w_n'(t)^2 \, dt \leq \frac{1}{2} \lambda \varepsilon T R_n^2 \|a\|_{L^1_T}$$

and a contradiction is achieved if we take ε sufficiently small.

Consider now the auxiliary function

$$v_n(t) := \frac{u_n(t)}{\|u_n\|_\infty} = \frac{u_n(t)}{R_n}, \quad t \in \mathbb{R},$$

and divide equation (6.4.12) by R_n. In this manner we obtain again (6.4.11). By (g_∞) and the fact that $u_n(t) \to +\infty$ uniformly in t, we conclude that $q(u_n(t)) = g(u_n(t))/u_n(t) \to 0$ uniformly (as $n \to \infty$). Hence, we are exactly in the same situation as in the case we have already discussed above in Section 6.4.2 for r small and we can end the proof in a similar way. More precisely, $\|v_n'\|_{L^2_T} \to 0$ as $n \to \infty$ (this follows by multiplying equation (6.4.11) by $v_n(t)$ and integrating on $[0, T]$) so that $\|v_n - 1\|_\infty \to 0$, as $n \to \infty$. Then, integrating equation (6.4.12) on $[0, T]$ and dividing by $g(R_n) > 0$, we obtain

$$0 < -\int_0^T a(t) \, dt \leq \|a\|_{L^1_T} \sup_{t \in [0,T]} \left| \frac{g(R_n v_n(t))}{g(R_n)} - 1 \right|.$$

Using the fact that $g(s)$ is regularly oscillating at infinity and $v_n(t) \to 1$ uniformly as $n \to \infty$, we find that the right-hand side of the above inequality tends to zero and thus we achieve a contradiction. $\square$

Remark 6.4.3. Notice that, among the assumptions of Theorem 6.1.1, in this part of the proof we have used only the following ones (for verifying (H_R)): $g(s) > 0$ for all s in a neighborhood of infinity, $g(s)$ regularly oscillating at infinity and satisfying (g_∞), $a \in L^1_T$ with $\int_0^T a(t)\,dt < 0$. $\lhd$

6.5 Related results

In this section we present some consequences and variants obtained from Theorem 6.1.1. We also examine the cases of nonexistence of solutions when the parameter λ is small.

6.5.1 Proof of Corollary 6.1.1

In order to deduce Corollary 6.1.1 from Theorem 6.1.1, we stress the fact that the constant $\lambda^* > 0$ (defined in (6.4.9)) is produced along the proof of Lemma 6.2.1 in dependence of an interval $I \subseteq [0,T]$ where $a(t) \geq 0$ and $\int_I a(t)\,dt > 0$. For this step in the proof we do not need any information about the weight function on $[0,T] \setminus I$. As a consequence, when we apply our result to equation (6.1.1), we have that λ^* can be chosen independently on μ. On the other hand, for Lemma 6.2.2 with r small as well as with R large, we do not need any special condition on λ (except that λ in (6.4.10) or in (6.4.12) is fixed) and we use only the fact that $\int_0^T a(t)\,dt < 0$ (without requiring any other information on the sign of $a(t)$). Accordingly, once that $\lambda > \lambda^*$ is fixed, to obtain a pair of positive T-periodic solutions we only need to check that the integral of the weight function on $[0,T]$ is negative. For equation (6.1.1) this condition is equivalent to

$$\frac{\mu}{\lambda} > \frac{\int_0^T a^+(t)\,dt}{\int_0^T a^-(t)\,dt}.$$

By the above remarks, we deduce immediately Corollary 6.1.1 from Theorem 6.1.1. $\square$

6.5.2 Existence of small/large solutions

Theorem 6.1.1 guarantees the existence of at least two positive T-periodic solutions of equation $(\mathscr{E}_\lambda)$. In more detail, we have found a first solution in $\Omega_{\rho,I} \setminus B[0,r]$ and a second one in $B(0,R) \setminus \mathrm{cl}(\Omega_{\rho,I} \cap B(0,R_0))$, verifying that the coincidence degree is non-zero in these sets (see (6.3.1) and (6.3.2)). The positivity of both the solutions follows from maximum principle arguments.

A careful reading of the proof (cf. Section 6.4) shows that weaker conditions on $g(s)$ are sufficient to repeat some of the steps in Section 6.3 in order to prove (6.3.1) (or (6.3.2)) and thus obtain the existence of a small (or large, respectively) positive T-periodic solution of $(\mathscr{E}_\lambda)$.

More precisely, taking into account Remark 6.4.1 and Remark 6.4.2 we can state the following theorem, ensuring the existence of a small positive T-periodic solution.

Theorem 6.5.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and*

$$\limsup_{s\to+\infty} \frac{g(s)}{s} < +\infty. \tag{6.5.1}$$

Suppose also that g is regularly oscillating at zero and satisfies (g_0). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition $(a_\#)$. Furthermore, suppose that there exists an interval $I \subseteq [0,T]$ such that $a(t) \geq 0$ for a.e. $t \in I$ and $\int_I a(t)\,dt > 0$. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ equation $(\mathscr{E}_\lambda)$ has at least a positive T-periodic solution.

On the other hand, in view of Remark 6.4.1 and Remark 6.4.3 we have the following result giving the existence of a large positive T-periodic solution.

Theorem 6.5.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and*

$$\limsup_{s\to0^+} \frac{g(s)}{s} < +\infty. \tag{6.5.2}$$

Suppose also that g is regularly oscillating at infinity and satisfies (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition $(a_\#)$. Furthermore, suppose that there exists an interval $I \subseteq [0,T]$ such that $a(t) \geq 0$ for a.e. $t \in I$ and $\int_I a(t)\,dt > 0$. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ equation $(\mathscr{E}_\lambda)$ has at least a positive T-periodic solution.

Notice that the possibility of applying a strong maximum principle (in order to obtain positive solutions) is ensured by (g_0) in Theorem 6.5.1, while it follows by (6.5.2) in Theorem 6.5.2. The dual condition (6.5.1) in Theorem 6.5.1 is, on the other hand, needed to apply Gronwall's inequality (checking the assumptions of Lemma 6.2.1).

6.5.3 Smoothness versus regular oscillation

It can be observed that the assumptions of regular oscillation of $g(s)$ at zero or, respectively, at infinity can be replaced by suitable smoothness assumptions. Indeed, we can provide an alternative manner to check the assumptions of Lemma 6.2.2 for r small or R large, by assuming that $g(s)$

is smooth in a neighborhood of zero or, respectively, in a neighborhood of infinity. For this purpose, we present some preliminary considerations (cf. Proposition 3.2.1).

Let $u(t)$ be a positive and T-periodic solution of

$$u'' + cu' + \nu a(t) g(u) = 0, \tag{6.5.3}$$

where $\nu > 0$ is a given parameter (in the following, we will take $\nu = \lambda$ or $\nu = \vartheta\lambda$). Suppose that the map $g(s)$ is continuously differentiable on an interval containing the range of $u(t)$. In such a situation, we can perform the change of variable

$$z(t) := \frac{u'(t)}{\nu g(u(t))}, \quad t \in \mathbb{R}, \tag{6.5.4}$$

and observe that $z(t)$ satisfies

$$z' + cz = -\nu g'(u(t)) z^2 - a(t). \tag{6.5.5}$$

The function $z(t)$ is absolutely continuous, T-periodic with $\int_0^T z(t)\,dt = 0$ and, moreover, there exists a $t^* \in [0,T]$ such that $z(t^*) = 0$.

This change of variable (recently considered also in [39]) is used to provide a nonexistence result as well as a priori bounds for the solutions. We premise the following result.

Lemma 6.5.1. *Let $J \subseteq \mathbb{R}$ be an interval. Let $g\colon J \to \mathbb{R}^+_0$ be a continuously differentiable function with bounded derivative (on J). Let $a \in L^1_T$ satisfy $(a_{\#})$. Then there exists $\omega_* > 0$ such that, if*

$$\nu \sup_{s \in J} |g'(s)| < \omega_*,$$

there are no T-periodic solutions of (6.5.3) *with $u(t) \in J$, for all $t \in \mathbb{R}$.*

Proof. For notational convenience, let us set

$$D := \sup_{s \in J} |g'(s)|.$$

First of all, we fix a positive constant $M > e^{|c|T} \|a\|_{L^1_T}$ and define

$$\omega_* := \min\left\{ \frac{M - e^{|c|T}\|a\|_{L^1_T}}{M^2 T e^{|c|T}}, \frac{-\int_0^T a(t)\,dt}{M^2 T} \right\}.$$

Note that ω_* does not depend on ν, J and D. We shall prove that if

$$0 < \nu D < \omega_*$$

equation (6.5.3) has no T-periodic solution $u(t)$ with range in J.

By contradiction we suppose that $u(t)$ is a solution of (6.5.3) with $u(t) \in J$, for all $t \in \mathbb{R}$. Setting $z(t)$ as in (6.5.4), we claim that

$$\|z\|_\infty \le M. \tag{6.5.6}$$

Indeed, if by contradiction we suppose that (6.5.6) is not true, then using the fact that $z(t)$ vanishes at some point of $[0,T]$, we can find a maximal interval $\mathcal{I}$ of the form $[t^*,\tau]$ or $[\tau,t^*]$ such that $|z(t)| \le M$ for all $t \in \mathcal{I}$ and $|z(t)| > M$ for some $t \notin \mathcal{I}$. By the maximality of the interval $\mathcal{I}$, we also know that $|z(\tau)| = M$. Multiplying equation (6.5.5) by $e^{c(t-\tau)}$ yields

$$\big(z(t)e^{c(t-\tau)}\big)' = \big(-\nu g'(u(t))z^2(t) - a(t)\big)e^{c(t-\tau)}.$$

Then, integrating on $\mathcal{I}$ and passing to the absolute value, we obtain

$$\begin{aligned} M = |z(\tau)| &= \left| z(\tau) - z(t^*)e^{c(t^*-\tau)} \right| \\ &\le \left| \int_{\mathcal{I}} \nu g'(u(t))z^2(t)\,dt \right| e^{|c|T} + \|a\|_{L^1_T} e^{|c|T} \\ &\le \nu D M^2 T e^{|c|T} + \|a\|_{L^1_T} e^{|c|T} \\ &< \omega_* M^2 T e^{|c|T} + \|a\|_{L^1_T} e^{|c|T} \le M, \end{aligned}$$

a contradiction. In this manner, we have verified that (6.5.6) is true.

Now, integrating (6.5.5) on $[0,T]$ and using (6.5.6), we reach

$$0 < -\int_0^T a(t)\,dt = \int_0^T \nu g'(u(t))z^2(t)\,dt < \omega_* M^2 T \le -\int_0^T a(t)\,dt,$$

a contradiction. This concludes the proof. $\square$

The same change of variable is employed to provide the following variant of Theorem 6.1.1.

Theorem 6.5.3. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and such that $g(s)$ is continuously differentiable on a right neighborhood of $s = 0$ and on a neighborhood of infinity. Suppose also that (g_0) and*

$$(g'_\infty) \qquad g'(\infty) := \lim_{s\to+\infty} g'(s) = 0$$

hold. Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition $(a_\#)$. Furthermore, suppose that there exists an interval $I \subseteq [0,T]$ such that $a(t) \ge 0$ for a.e. $t \in I$ and $\int_I a(t)\,dt > 0$. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ equation $(\mathscr{E}_\lambda)$ has at least two positive T-periodic solutions.

Proof. We follow the scheme described in Section 6.3. The verification of the assumptions of Lemma 6.2.1 for λ large is exactly the same as in Section 6.4.1. We just describe the changes with respect to Section 6.4.2 and Section 6.4.3. It is important to emphasize that $\lambda > \lambda^*$ is fixed from now on.

Verification of the assumption of Lemma 6.2.2 for r small. Let $[0, \varepsilon_0[$ be a right neighborhood of 0 where g is continuously differentiable. We claim that there exists $r_0 \in \,]0, \varepsilon_0[$ such that for all $0 < r \leq r_0$ and for all $\vartheta \in \,]0, 1]$ there are no non-negative T-periodic solutions $u(t)$ of (6.2.4) such that $\|u\|_\infty = r$.

First of all, we observe that any non-negative T-periodic solution $u(t)$ of (6.2.4), with $\|u\|_\infty = r$, is positive. This follows either by the uniqueness of the trivial solution (due to the smoothness of $g(s)$ in $[0, \varepsilon_0[$), or by an elementary form of the strong maximum principle. Thus we have to prove that there are no T-periodic solutions $u(t)$ of (6.2.4) with range in the interval $]0, r]$ (for all $0 < r \leq r_0$).

We apply Lemma 6.5.1 to the present situation with $\nu = \vartheta\lambda$ and $J = \,]0, r]$. There exists a constant $\omega_* > 0$ (independent on r) such that there are no T-periodic solutions with range in $]0, r]$ if

$$\sup_{0<s\leq r} |g'(s)| = \max_{0\leq s\leq r} |g'(s)| < \frac{\omega_*}{\lambda}$$

(recall that $0 < \vartheta \leq 1$). This latter condition is clearly satisfied for every $r \in \,]0, r_0]$, with $r_0 > 0$ suitably chosen using the continuity of $g'(s)$ at $s = 0^+$.

Verification of the assumption of Lemma 6.2.2 for R large. Let $]N, +\infty[$ be a neighborhood of infinity where g is continuously differentiable. As in Section 6.4.3, we argue by contradiction. Suppose that there exists a sequence of non-negative T-periodic functions $u_n(t)$ satisfying (6.4.12) and such that $\|u_n\|_\infty = R_n \to +\infty$. By the same argument as previously developed therein, we find that $u_n(t) \geq R_n/2$, for all $t \in \mathbb{R}$ (for n sufficiently large). Notice that for this part of the proof we require condition (g_∞), but we do not need the hypothesis of regular oscillation at infinity. Clearly, (g_∞) is implied by (g'_∞).

For n sufficiently large (such that $R_n > 2N$), we apply Lemma 6.5.1 to the present situation with $\nu = \nu_n := \vartheta_n\lambda$ and $J = J_n := [R_n/2, R_n]$. There exists a constant $\omega_* > 0$ (independent on n) such that there are no T-periodic solutions with range in J_n if

$$\max_{\frac{R_n}{2}\leq s\leq R_n} |g'(s)| < \frac{\omega_*}{\lambda}$$

(recall that $0 < \vartheta_n \leq 1$). This latter condition is clearly satisfied for every n sufficiently large as a consequence of condition (g'_∞). The desired contradiction is thus achieved. □

Remark 6.5.1. Clearly one can easily produce two further theorems, by combining the assumptions of regular oscillation at zero (at infinity) with the smoothness condition at infinity (at zero, respectively). $\lhd$

6.5.4 Nonexistence results

In the proof of Theorem 6.5.3 we have applied Lemma 6.5.1 to intervals of the form $]0, r]$ or, respectively, $[R_n/2, R_n]$ in order to check the assumptions of Lemma 6.2.2. Clearly, one could apply such a lemma to the whole interval $\mathbb{R}^+_0$ of positive real numbers. In this manner, we can easily provide a nonexistence result of positive T-periodic solutions to $(\mathscr{E}_\lambda)$ when $g'(s)$ is bounded in $\mathbb{R}^+_0$ and λ is small. With this respect, the following result holds.

Theorem 6.5.4. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function satisfying (g_*), (g_0) and (g'_∞). Let $a \in L^1_T$ satisfy $(a_\#)$. Then there exists $\lambda_* > 0$ such that for each $0 < \lambda < \lambda_*$ equation $(\mathscr{E}_\lambda)$ has no positive T-periodic solution.*

Proof. First of all, we observe that g' is bounded on $\mathbb{R}^+_0$ (since $g(s)$ is continuously differentiable in $\mathbb{R}^+$ with $g'(0) = g'(\infty) = 0$). Accordingly, let us set

$$D := \max_{s \geq 0} |g'(s)|.$$

We apply now Lemma 6.5.1 to equation $(\mathscr{E}_\lambda)$ for $J = \mathbb{R}^+_0$. This lemma guarantees the existence of a constant $\omega_* > 0$ such that, if $0 < \lambda < \omega_*/D$, $(\mathscr{E}_\lambda)$ has no positive T-periodic solution. This ensures the existence of a suitable constant $\lambda_* \geq \omega_*/D$, as claimed in the statement of the theorem. $\square$

At this point, Theorem 6.1.2 is a straightforward consequence of Theorem 6.5.3 and Theorem 6.5.4.

6.6 Dirichlet and Neumann boundary conditions

In this final section we briefly describe how to obtain the preceding results for the Dirichlet and Neumann boundary value problems. For the sake of simplicity, we deal with the case $c = 0$. If $c \neq 0$, we can write equation $(\mathscr{E}_\lambda)$ as

$$\big(u' e^{ct}\big)' + \lambda \tilde{a}(t) g(u) = 0, \qquad \text{with } \tilde{a}(t) := a(t) e^{ct},$$

and enter the setting of coincidence degree theory for the linear operator $L\colon u \mapsto -(u' e^{ct})'$.

Accordingly, we consider the boundary value problems

$$\begin{cases} u'' + \lambda a(t) g(u) = 0 \\ u(0) = u(T) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} u'' + \lambda a(t) g(u) = 0 \\ u'(0) = u'(T) = 0, \end{cases} \tag{6.6.1}$$

where $a\colon [0,T] \to \mathbb{R}$ and $g(s)$ satisfy the same conditions as in the previous sections. The abstract setting of Section 6.2 can be reproduced almost verbatim with $X := \mathcal{C}([0,T])$, $Z := L^1([0,T])$ and $L\colon u \mapsto -u''$, by taking

$$\operatorname{dom} L := \bigl\{u \in W^{2,1}([0,T]) \colon u(0) = u(T) = 0\bigr\}$$

with Dirichlet boundary conditions, and

$$\operatorname{dom} L := \bigl\{u \in W^{2,1}([0,T]) \colon u'(0) = u'(T) = 0\bigr\}$$

with Neumann boundary conditions, respectively. We stress that, concerning the Dirichlet problem, the differential operator L is invertible (indeed it can be expressed by means of the Green's function), so that the coincidence degree theory reduces to the classical Leray-Schauder one for the locally compact operators as $\Phi_\lambda = L^{-1}N_\lambda$.

All the results till Section 6.5 can be now restated for problems (6.6.1). In particular, we obtain again Theorem 6.1.1, Theorem 6.5.3 and Theorem 6.5.4, as well as their corollaries for equation $(\mathscr{E}_\lambda)$ (with $c = 0$ and the Dirichlet/Neumann boundary conditions).

We present now a consequence of these results to the study of a PDE in an annular domain. In order to simplify the exposition of the next results, we assume the continuity of the weight function. In this manner, the solutions we find are the "classical" ones (at least two times continuously differentiable).

6.6.1 Radially symmetric solutions

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^N$ (for $N \geq 2$) and let

$$\Omega := B(0,R_2) \setminus B[0,R_1] = \bigl\{x \in \mathbb{R}^N \colon R_1 < \|x\| < R_2\bigr\}$$

be an open annular domain, with $0 < R_1 < R_2$.

We deal with the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = \lambda q(x) g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6.6.2}$$

and with the Neumann boundary value problem

$$\begin{cases} -\Delta u = \lambda q(x) g(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.6.3}$$

We assume that $q\colon \overline{\Omega} \to \mathbb{R}$ is a continuous function which is radially symmetric, namely there exists a continuous scalar function $\mathcal{Q}\colon [R_1,R_2] \to \mathbb{R}$ such that

$$q(x) = \mathcal{Q}(\|x\|), \quad \forall\, x \in \overline{\Omega}.$$

We look for existence/nonexistence and multiplicity of radially symmetric positive solutions of (6.6.2) and of (6.6.3), that are classical solutions such that $u(x) > 0$ for all $x \in \Omega$ and also $u(x) = \mathcal{U}(\|x\|)$, where $\mathcal{U}$ is a scalar function defined on $[R_1, R_2]$.

Accordingly, our study can be reduced to the search of positive solutions of the Dirichlet/Neumann boundary value problem on $[R_1, R_2]$ associated with

$$\mathcal{U}''(r) + \frac{N-1}{r}\mathcal{U}'(r) + \lambda \mathcal{Q}(r) g(\mathcal{U}(r)) = 0. \tag{6.6.4}$$

As explained in Section C.2, using the standard change of variable

$$t = h(r) := \int_{R_1}^{r} \xi^{1-N}\, d\xi$$

and defining

$$T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi, \quad r(t) := h^{-1}(t) \quad \text{and} \quad v(t) = \mathcal{U}(r(t)),$$

we transform (6.6.4) into the equivalent equation

$$v'' + \lambda a(t) g(v) = 0 \tag{6.6.5}$$

with

$$a(t) := r(t)^{2(N-1)} \mathcal{Q}(r(t)).$$

Moreover, the boundary conditions becomes

$$v(0) = v(T) = 0 \quad \text{and} \quad v'(0) = v'(T) = 0,$$

respectively. Consequently, the Dirichlet/Neumann boundary value problems associated with (6.6.5) are of the same form of (6.6.1) and we can apply the previous results.

Notice that condition $(a_{\#})$ reads as

$$0 > \int_0^T r(t)^{2(N-1)} \mathcal{Q}(r(t))\, dt = \int_{R_1}^{R_2} r^{N-1} \mathcal{Q}(r)\, dr.$$

Up to a multiplicative constant, the latter integral is the integral of $q(x)$ on Ω, using the change of variable formula for radially symmetric functions. Thus, $a(t)$ satisfies $(a_{\#})$ if and only if

$$(q_{\#}) \qquad \int_{\Omega} q(x)\, dx < 0.$$

The analogue of Theorem 6.1.1 for problems (6.6.2) and (6.6.3) now becomes the following.

Theorem 6.6.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*). Suppose also that g is regularly oscillating at zero and at infinity and satisfies (g_0) and (g_∞). Let $q(x)$ be a continuous (radial) weight function as above satisfying $(q_\#)$ and such that $q(x_0) > 0$ for some $x_0 \in \Omega$. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ problem* (6.6.2) *((6.6.3), respectively) has at least two positive radially symmetric solutions.*

Analogously, if we replace the regularly oscillating conditions with the smoothness assumptions, from Theorem 6.5.3 and Theorem 6.5.4, we obtain the next result.

Theorem 6.6.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function satisfying (g_*), (g_0) and (g'_∞). Let $q(x)$ be a continuous (radial) weight function as above satisfying $(q_\#)$ and such that $q(x_0) > 0$ for some $x_0 \in \Omega$. Then there exist two positive constant $\lambda_* \leq \lambda^*$ such that for each $0 < \lambda < \lambda_*$ there are no positive radially symmetric solutions for problem* (6.6.2) *((6.6.3), respectively), while for each $\lambda > \lambda^*$ there exist at least two positive radially symmetric solutions. Moreover, if $g'(s) > 0$ for all $s > 0$, then condition $(q_\#)$ is also necessary.*

Chapter 7

High multiplicity results

In this chapter we continue the investigation initiated in Chapter 6 with the aim to provide some multiplicity results for positive solutions to Dirichlet, Neumann and periodic boundary value problems associated with the second order nonlinear differential equation

$$(\mathscr{E}_{\lambda,\mu}) \qquad u''+\bigl(\lambda a^{+}(t)-\mu a^{-}(t)\bigr)g(u)=0,$$

where $a(t)$ is a sign-changing weight function and $g(s)$ is a function with superlinear growth at zero, sublinear growth at infinity and positive on $\mathbb{R}^{+}_{0}$. As in the previous chapter, we focus our attention on the periodic problem; however, for simplicity in the exposition, we prefer to treat the case without the additive term cu' (see also the discussion in Section 7.7.1).

For λ,μ positive and large, we prove the existence of $3^{m}-1$ positive T-periodic solutions when the weight function $a(t)$ has m positive humps separated by m negative ones (in a T-periodicity interval). The proof is based on coincidence degree theory for locally compact operators on open unbounded sets and also applies to Neumann and Dirichlet boundary conditions.

The plan of the chapter is the following. In Section 7.1 we list the hypothesis on $a(t)$ and on $g(s)$ that we assume for the rest of the chapter and we present our main result (Theorem 7.1.1). In Section 7.2 we introduce the functional analytic framework to deal with the periodic problem associated with $(\mathscr{E}_{\lambda,\mu})$ in the setting of the Mawhin's coincidence degree theory. In Section 7.3 we define the open and unbounded sets $\Lambda^{\mathcal{I},\mathcal{J}}$ and describe the general strategy for the proof of the degree formula

$$\mathrm{deg}_{LS}(Id-\Phi_{\lambda,\mu},\Lambda^{\mathcal{I},\mathcal{J}},0)\neq 0. \tag{7.0.1}$$

In more detail, we first introduce some auxiliary open and unbounded sets $\Omega^{\mathcal{I},\mathcal{J}}$ and we then present two lemmas (Lemma 7.3.1 and Lemma 7.3.2) for the computation of

$$\deg_{LS}(Id - \Phi_{\lambda,\mu}, \Omega^{\mathcal{I},\mathcal{J}}, 0). \tag{7.0.2}$$

The obtention of (7.0.1) from the evaluation of the degrees in (7.0.2) is justified in Section 7.5 using a purely combinatorial argument. In Section 7.4 we actually show, by means of some careful estimates on the solutions of some parameterized equations related to $(\mathscr{E}_{\lambda,\mu})$, that the above lemmas and the general strategy can be applied for λ and μ large, thus concluding the proof of Theorem 7.1.1. In Section 7.6 we present some general properties of (not necessarily periodic) positive solutions of $(\mathscr{E}_{\lambda,\mu})$ defined on the whole real line and we discuss the limit behavior for $\mu \to +\infty$. The chapter ends with Section 7.7, where we discuss variants and extensions of Theorem 7.1.1 and we also present an application to radially symmetric solutions for some elliptic PDEs.

7.1 The main multiplicity result

In this section we present our main existence results for positive T-periodic solutions to $(\mathscr{E}_{\lambda,\mu})$ in dependence of the real parameters $\lambda, \mu > 0$.

Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying the sign hypothesis

(g_*) $$g(0) = 0, \qquad g(s) > 0 \quad \text{for} \quad s > 0,$$

as well as the conditions of superlinear growth at zero

(g_0) $$\lim_{s \to 0^+} \frac{g(s)}{s} = 0$$

and sublinear growth at infinity

(g_∞) $$\lim_{s \to +\infty} \frac{g(s)}{s} = 0.$$

Concerning the weight

$$a_{\lambda,\mu}(t) := \lambda a^+(t) - \mu a^-(t), \quad t \in \mathbb{R},$$

we assume that $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic locally integrable sign-changing function, that is

$$\int_0^T a^+(t)\,dt \neq 0 \neq \int_0^T a^-(t)\,dt.$$

More precisely, we suppose that

(a_*) *there exist* $2m+1$ *points (with* $m \geq 1$*)*

$$\sigma_1 < \tau_1 < \ldots < \sigma_i < \tau_i < \ldots < \sigma_m < \tau_m < \sigma_{m+1},$$

with $\sigma_{m+1} - \sigma_1 = T$*, such that, for* $i = 1, \ldots, m$*,* $a(t) \succ 0$ *on* $[\sigma_i, \tau_i]$ *and* $a(t) \prec 0$ *on* $[\tau_i, \sigma_{i+1}]$*,* Without loss of generality, due to the T-periodicity of the function $a(t)$, in the sequel we assume that $\sigma_1 = 0$ and $\sigma_{m+1} = T$. For $i = 1, \ldots, m$, we also set

$$I_i^+ := [\sigma_i, \tau_i] \quad \text{and} \quad I_i^- := [\tau_i, \sigma_{i+1}]. \tag{7.1.1}$$

As already observed in the previous chapters, whenever $g'(s) > 0$ for any $s > 0$, a necessary condition for the existence of positive Neumann/periodic solutions to $(\mathscr{E}_{\lambda,\mu})$ turns out to be

$$\int_0^T a_{\lambda,\mu}(t)\,dt < 0,$$

which equivalently reads as

$$\mu > \mu^{\#}(\lambda) := \lambda \frac{\int_0^T a^+(t)\,dt}{\int_0^T a^-(t)\,dt}. \tag{7.1.2}$$

Then, the following result holds true.

Theorem 7.1.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ there exists $\mu^*(\lambda) > 0$ such that for each $\mu > \mu^*(\lambda)$ equation $(\mathscr{E}_{\lambda,\mu})$ has at least $3^m - 1$ positive T-periodic solutions.*

More precisely, fixed an arbitrary constant $\rho > 0$ there exists $\lambda^* = \lambda^*(\rho) > 0$ such that for each $\lambda > \lambda^*$ there exist two constants r, R with $0 < r < \rho < R$ and $\mu^*(\lambda) = \mu^*(\lambda, r, R) > 0$ such that for any $\mu > \mu^*(\lambda)$ and any finite string $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_m) \in \{0,1,2\}^m$, with $\mathcal{S} \neq (0, \ldots, 0)$, there exists a positive T-periodic solution $u(t)$ of $(\mathscr{E}_{\lambda,\mu})$ such that

- $\max_{t \in I_i^+} u(t) < r$, *if $\mathcal{S}_i = 0$;*
- $r < \max_{t \in I_i^+} u(t) < \rho$, *if $\mathcal{S}_i = 1$;*
- $\rho < \max_{t \in I_i^+} u(t) < R$, *if $\mathcal{S}_i = 2$.*

Remark 7.1.1. As already anticipated, the same multiplicity result holds true for the Neumann as well as for the Dirichlet problems associated with $(\mathscr{E}_{\lambda,\mu})$ on the interval $[0,T]$. Dealing with these boundary value problems, the weight function $a(t)$ is allowed to be negative on a right neighborhood of 0 and/or positive on a left neighborhood of T. Indeed, what is crucial to obtain $3^m - 1$ positive solutions is the fact that there are m positive humps of the weight function which are separated by negative ones. Accordingly, if we study the Neumann or the Dirichlet problems on $[0,T]$ it will be sufficient to suppose that there are $m-1$ intervals where $a(t) \prec 0$ separating m intervals where $a(t) \succ 0$. On the other hand, the nature of periodic boundary conditions requires that the positive humps of the weight coefficient are separated by negative humps on $[0,T]/\{0,T\} \simeq \mathbb{R}/T\mathbb{Z} \simeq S^1$. This is the reason for which condition (a_*) for the periodic problem is conventionally expressed assuming that, in an interval of length T, the weight function starts positive and ends negative. For a more detailed discussion, see Section 7.7.2. $\lhd$

Let us now make some comments about Theorem 7.1.1, trying at first to explain its meaning in an informal way. The existence of $3^m - 1$ positive solutions comes from the possibility of prescribing, for a positive T-periodic solution of $(\mathscr{E}_{\lambda,\mu})$, the behavior in each interval of positivity of the weight function $a(t)$ among three possible ones: either the solution is "very small" on I_i^+ (if $\mathcal{S}_i = 0$), or it is "small" (if $\mathcal{S}_i = 1$) or it is "large" (if $\mathcal{S}_i = 2$). This is related to the fact that three non-negative solutions for the Dirichlet problem associated with $u'' + \lambda a^+(t)g(u) = 0$ on I_i^+ are always available, when $g(s)$ is super-sublinear, for $\lambda > 0$ large enough: the trivial one, and two positive solutions given by Rabinowitz's theorem (cf. [158]). This point of view can be made completely rigorous by showing that the solutions constructed in Theorem 7.1.1 converge, for $\mu \to +\infty$, to solutions of the Dirichlet problem associated with $u'' + \lambda a^+(t)g(u) = 0$ on each I_i^+ and to zero on $\bigcup_i I_i^-$ (see the second part of Section 7.6 for a detailed discussion). With this is mind, one can interpret Theorem 7.1.1 as a singular perturbation result from the limit case $\mu = +\infty$. Indeed, by taking into account all the possibilities for the non-negative solutions of the Dirichlet problem associated with $u'' + \lambda a^+(t)g(u) = 0$ on each I_i^+, one finds 3^m limit profiles for positive solutions to $(\mathscr{E}_{\lambda,\mu})$. Among them, $3^m - 1$ are nontrivial and give rise, for $\mu \gg 0$, to $3^m - 1$ positive T-periodic solutions to $(\mathscr{E}_{\lambda,\mu})$, while the trivial limit profile still persists as the trivial solution to $(\mathscr{E}_{\lambda,\mu})$ for any $\mu > 0$.

What may appear as a relevant aspect of our result is the fact that a minimal set of assumptions on the nonlinearity $g(s)$ is required. Indeed, only positivity, continuity and the hypotheses on the limits $g(s)/s$ for $s \to 0^+$ and $s \to +\infty$ are required. In particular, no supplementary power-type growth conditions at zero or at infinity are needed. In Chapter 6 we obtain the existence of at least two positive T-periodic solutions under the sharp condition (7.1.2) on the coefficient μ; on the other hand, some extra (although mild) assumptions on $g(s)$ are imposed. It is interesting to observe that increasing the value of μ yields both abundance of solutions and no-extra assumptions on $g(s)$.

7.2 The abstract setting of the coincidence degree

Dealing with boundary value problems, it is often convenient to choose spaces of functions defined on compact domains. Therefore, for the T-periodic problem, as usual, we shall restrict ourselves to functions $u(t)$ defined on $[0,T]$ and such that

$$u(0) = u(T), \qquad u'(0) = u'(T). \tag{7.2.1}$$

In the sequel, solutions of a given second order differential equation satisfying the boundary condition (7.2.1) will be referred to as *T-periodic solutions*.

Let $X := \mathcal{C}([0,T])$ be the space of continuous functions $u\colon [0,T] \to \mathbb{R}$, endowed with the norm

$$\|u\|_\infty := \max_{t\in[0,T]} |u(t)|,$$

and let $Z := L^1([0,T])$ be the space of integrable functions $v\colon [0,T] \to \mathbb{R}$, endowed with the norm

$$\|v\|_{L^1} := \int_0^T |v(t)|\,dt.$$

As well known, the differential operator

$$L\colon u \mapsto -u'',$$

defined on

$$\operatorname{dom} L := \bigl\{u \in W^{2,1}([0,T])\colon u(0) = u(T),\ u'(0) = u'(T)\bigr\} \subseteq X,$$

is a linear Fredholm map of index zero with range

$$\operatorname{Im} L = \biggl\{v \in Z\colon \int_0^T v(t)\,dt = 0\biggr\}.$$

Moreover, as usual, we can define the projectors $P\colon X \to \ker L \cong \mathbb{R}$ and $Q\colon Z \to \operatorname{coker} L \cong Z/\operatorname{Im} L \cong \mathbb{R}$, as the average operators

$$Pu = Qu := \frac{1}{T}\int_0^T u(t)\,dt.$$

Finally, let $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ be the right inverse of L, that is the operator that to any function $v \in L^1([0,T])$ with $\int_0^T v(t)\,dt = 0$ associates the unique T-periodic solution u of

$$u'' + v(t) = 0, \qquad \text{with} \quad \int_0^T u(t)\,dt = 0.$$

Next, we introduce the L^1-Carathéodory function

$$f_{\lambda,\mu}(t,s) := \begin{cases} -s, & \text{if } s \leq 0; \\ a_{\lambda,\mu}(t)g(s), & \text{if } s \geq 0; \end{cases}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a locally integrable T-periodic function, $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $g(0) = 0$ and $\lambda, \mu > 0$ are fixed parameters. Let us denote by $N_{\lambda,\mu}\colon X \to Z$ the Nemytskii operator induced by the function $f_{\lambda,\mu}$, that is

$$(N_{\lambda,\mu}u)(t) := f_{\lambda,\mu}(t, u(t)), \quad t \in [0,T].$$

By coincidence degree theory, the operator equation

$$Lu = N_{\lambda,\mu}u, \quad u \in \operatorname{dom} L,$$

is equivalent to the fixed point problem

$$u = \Phi_{\lambda,\mu}u := Pu + QN_{\lambda,\mu}u + K_P(Id - Q)N_{\lambda,\mu}u, \quad u \in X.$$

Notice that the term $QN_{\lambda,\mu}u$ in the above formula should be more correctly written as $JQN_{\lambda,\mu}u$, where J is a linear (orientation-preserving) isomorphism from $\operatorname{coker} L$ to $\ker L$. However, in our situation, both $\operatorname{coker} L$, as well as $\ker L$, can be identified with $\mathbb{R}$, so that we can take as J the identity on $\mathbb{R}$. It is standard to verify that $\Phi_{\lambda,\mu}\colon X \to X$ is a completely continuous operator and thus we say that $N_{\lambda,\mu}$ is *L-completely continuous*.

In the sequel we will apply this general setting in the following manner. We consider a L-completely continuous operator $\mathcal{N}$ and an open (not necessarily bounded) set $\mathcal{A}$ such that the solution set $\{u \in \overline{\mathcal{A}} \cap \operatorname{dom} L\colon Lu = \mathcal{N}u\}$ is compact and disjoint from $\partial\mathcal{A}$. Therefore $D_L(L - \mathcal{N}, \mathcal{A})$ is well-defined. We will proceed analogously when dealing with homotopies.

We notice that, by the existence theorem, if $D_L(L - N_{\lambda,\mu}, \Omega) \neq 0$ for some open set $\Omega \subseteq X$, then equation

$$u'' + f_{\lambda,\mu}(t,u) = 0 \tag{7.2.2}$$

has at least one solution in Ω satisfying the boundary condition (7.2.1). If we denote by $u(t)$ such a solution, we have that $u(t)$ can be extended by T-periodicity to a T-periodic solution of (7.2.2) defined on the whole real line. Moreover, a standard application of the maximum principle ensures that $u(t) \geq 0$ for all $t \in \mathbb{R}$. Finally, if $g(s)/s$ is bounded in a right neighborhood of $s = 0$ (a situation which always occurs if we assume (g_0)), then either $u \equiv 0$ or $u(t) > 0$ for all $t \in \mathbb{R}$.

Remark 7.2.1. As already observed in the introduction and in Section 7.1, our main attention is devoted to the investigation of the periodic problem for $Lu = -u''$, while, for Neumann and Dirichlet boundary conditions, as well as for other operators, we only underline which modifications are needed.

If we study the Neumann problem, we just modify the domain of L as

$$\operatorname{dom} L := \bigl\{u \in W^{2,1}([0,T])\colon u'(0) = u'(T) = 0\bigr\} \subseteq X$$

and all the rest is basically the same with elementary modifications. Obviously, the right inverse of L now is the operator which associates to any function $v \in L^1([0,T])$ satisfying $\int_0^T v(t)\,dt = 0$ the unique solution of $u'' + v(t) = 0$ with $u'(0) = u'(T) = 0$ and $\int_0^T u(t)\,dt = 0$.

In the case of the Dirichlet problem, the domain of L is

$$\operatorname{dom} L := W_0^{2,1}([0,T]) = \bigl\{u \in W^{2,1}([0,T]) \colon u(0) = u(T) = 0\bigr\} \subseteq X,$$

but now the differential operator L is invertible (indeed it can be expressed by means of the Green's function), so that $\Phi_{\lambda,\mu} = L^{-1}N_{\lambda,\mu}$. In this situation, coincidence degree theory reduces to the classical Leray-Schauder one for locally compact operators.

Finally, we observe that the above framework remains substantially unchanged for other classes of linear differential operators. In the periodic case, exactly the same considerations as above are valid if we take the operator

$$L \colon u \mapsto -u'' - cu',$$

where $c \in \mathbb{R}$ is an arbitrary but fixed constant (recall that the maximum principle is still valid in this setting, see Appendix C). This, in principle, allows us to insert a dissipation term in equation $(\mathscr{E}_{\lambda,\mu})$ (see Section 7.7.1 for a more detailed discussion).

Concerning the Neumann and the Dirichlet problems, we can easily deal with self-adjoint differential operators of the form

$$L \colon u \mapsto -(p(t)u')',$$

with $p(t) > 0$ for all $t \in [0,T]$. We do not insist further on these aspects; however, we will present later a special example of $p(t)$ which naturally arises in the study of radially symmetric solutions of elliptic PDEs (see Section 7.7.3). $\lhd$

7.3 Proof of Theorem 7.1.1: an outline

The proof of Theorem 7.1.1 and its variants is based on the abstract setting described in the previous section but it also requires some careful estimates on the solutions of $(\mathscr{E}_{\lambda,\mu})$ and of some related equations. In this section we first introduce some special open sets of the Banach space X where the coincidence degree will be computed and next we present the main steps which are required for these computations. In this manner we can skip for a moment all the technical estimates (which are developed in Section 7.4) and focus ourselves on the general strategy of the proof.

From now on, all the assumptions on $a(t)$ and $g(s)$ in Theorem 7.1.1 will be implicitly assumed.

7.3.1 General strategy

Let us fix an arbitrary constant $\rho > 0$. Depending on ρ, we determine a value $\lambda^* = \lambda^*(\rho) > 0$ such that, for $\lambda > \lambda^*$, any non-negative solution to

$$u'' + \lambda a^+(t) g(u) = 0,$$

with $\max_{t \in I_i^+} u(t) = \rho$, must vanish on I_i^+ (whatever the index $i = 1 \ldots, m$). This fact is expressed in a more formal way in Lemma 7.4.1 (where we also consider a more general equation). From now on, both ρ and $\lambda > \lambda^*$ are fixed.

Next, given any constants r, R with $0 < r < \rho < R$ and for any pair of subsets of indices $\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, m\}$ (possibly empty) with $\mathcal{I} \cap \mathcal{J} = \emptyset$, we define the open and unbounded set

$$\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)} := \left\{ u \in X \colon \begin{array}{l} \max_{I_i^+} |u| < r, \ i \in \{1, \ldots, m\} \setminus (\mathcal{I} \cup \mathcal{J}) \\ \max_{I_i^+} |u| < \rho, \ i \in \mathcal{I} \\ \max_{I_i^+} |u| < R, \ i \in \mathcal{J} \end{array} \right\}. \tag{7.3.1}$$

See Figure 7.1 for the representations of the sets $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ when $m = 2$.

Then, in Section 7.4.2 we determine two specific constants r, R with $0 < r < \rho < R$ such that, for any choice of $\mathcal{I}, \mathcal{J}$ as above, the coincidence degree

$$D_L\big(L - N_{\lambda,\mu}, \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\big)$$

is defined, provided that μ is sufficiently large (say $\mu > \mu^*(\lambda, r, R)$). Along this process, in Section 7.4.3 and Section 7.4.4 we also prove Theorem 7.3.1 below.

Theorem 7.3.1. *In the above setting, it holds that*

$$D_L\big(L - N_{\lambda,\mu}, \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\big) = \begin{cases} 0, & \textit{if } \mathcal{I} \neq \emptyset; \\ 1, & \textit{if } \mathcal{I} = \emptyset. \end{cases} \tag{7.3.2}$$

Then, having fixed ρ, λ, r, R, μ as above, we further introduce the open and unbounded sets

$$\Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)} := \left\{ u \in X \colon \begin{array}{l} \max_{I_i^+} |u| < r, \ i \in \{1, \ldots, m\} \setminus (\mathcal{I} \cup \mathcal{J}) \\ r < \max_{I_i^+} |u| < \rho, \ i \in \mathcal{I} \\ \rho < \max_{I_i^+} |u| < R, \ i \in \mathcal{J} \end{array} \right\}. \tag{7.3.3}$$

See Figure 7.2 for the representations of the sets $\Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ when $m = 2$.

From Theorem 7.3.1 and the combinatorial argument presented in Section 7.5, we can prove the following.

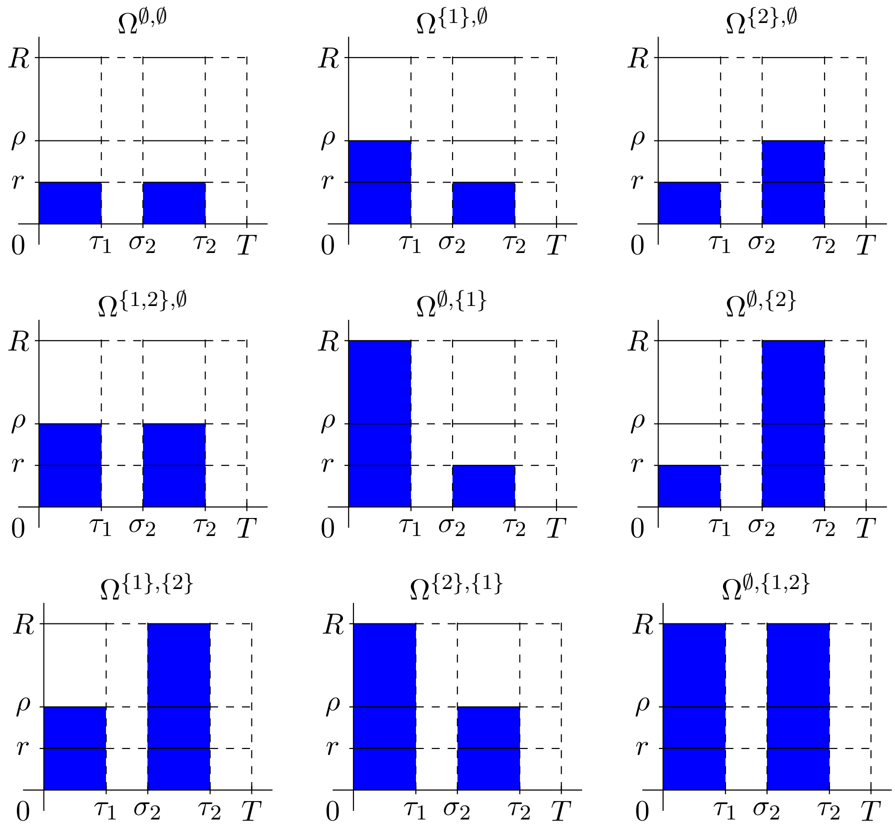

Figure 7.1: The figure represents the family of sets $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$, when $m = 2$ and the subintervals of positivity $I_1^+ := [0, \tau_1]$ and $I_2^+ := [\sigma_2, \tau_2]$ are arranged as in the figure. The sets $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ are made up of the continuous functions on $[0, T]$ which are in the blue area on the intervals I_1^+ and I_2^+, while in the remaining intervals there are no constraints. We consider only the non-negative function in $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$.

Theorem 7.3.2. *In the above setting, it holds that*

$$D_L\big(L - N_{\lambda,\mu}, \Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\big) = (-1)^{\#\mathcal{I}}. \tag{7.3.4}$$

As a consequence of the existence property for the coincidence degree, we thus obtain the existence of a T-periodic solution of (7.2.2) in each of these 3^m sets $\Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ (taking into account all the possible cases for $\mathcal{I},\mathcal{J}$). Notice that $\Lambda^{\emptyset,\emptyset}(r,\rho,R)$ contains the trivial solution. In all the other $3^m - 1$ sets the solution must be nontrivial and hence, by the maximum principle argument recalled in the previous section, a *positive* solution of $(\mathscr{E}_{\lambda,\mu})$. In this manner we can conclude that, for each choice of $\mathcal{I},\mathcal{J}$ with $\mathcal{I} \cup \mathcal{J} \neq \emptyset$, there exists at least one positive T-periodic solution $u(t)$ of $(\mathscr{E}_{\lambda,\mu})$ such that

- $0 < \max_{t \in I_i^+} u(t) < r$, for $i \notin \mathcal{I} \cup \mathcal{J}$;

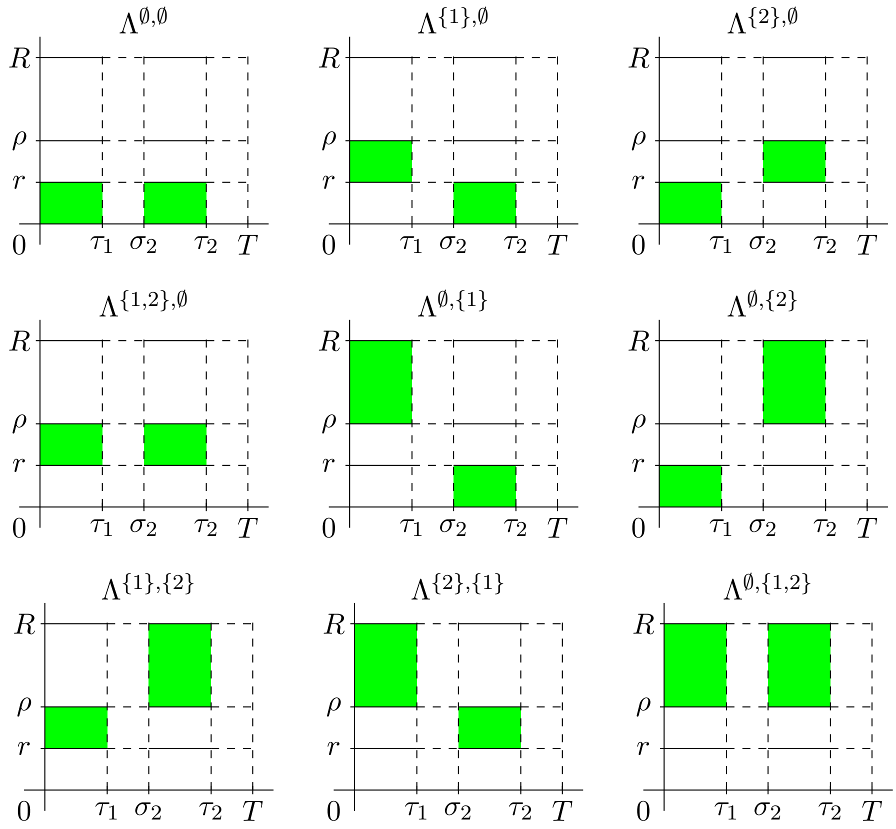

Figure 7.2: The figure represents the family of sets $\Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$, when $m = 2$ and the subintervals of positivity $I^+_1 := [0, \tau_1]$ and $I^+_2 := [\sigma_2, \tau_2]$ are arranged as in the figure. The sets $\Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ are made up of the functions in $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ such that the maximum on I^+_i $(i = 1, 2)$ is in the green area. We consider only the non-negative function in $\Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$.

- $r < \max_{t\in I^+_i} u(t) < \rho$, for all $i \in \mathcal{I}$;

- $\rho < \max_{t\in I^+_i} u(t) < R$, for all $i \in \mathcal{J}$.

Finally, in order to achieve the conclusion of Theorem 7.1.1, we just observe that, given any finite string $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_m) \in \{0, 1, 2\}^m$, with $\mathcal{S} \neq (0, \ldots, 0)$, we can associate to $\mathcal{S}$ the sets

$$\mathcal{I} := \bigl\{i \in \{1, \ldots, m\} : \mathcal{S}_i = 1\bigr\}, \quad \mathcal{J} := \bigl\{i \in \{1, \ldots, m\} : \mathcal{S}_i = 2\bigr\},$$

so that $\mathcal{S}_i = 0$ when $i \notin \mathcal{I} \cup \mathcal{J}$. This fact completes the proof of Theorem 7.1.1. $\square$

7.3.2 Degree lemmas

For the proof of Theorem 7.3.1, we need to compute the topological degrees in formula (7.3.2). To this end, we will use the following results.

Lemma 7.3.1. *Let $\mathcal{I} \neq \emptyset$ and $\lambda, \mu > 0$. Assume that there exists $v \in L^1([0,T])$, with $v(t) \succ 0$ on $[0,T]$ and $v \equiv 0$ on $\bigcup_i I_i^-$, such that the following properties hold.*

(H_1) *If $\alpha \geq 0$, then any T-periodic solution $u(t)$ of*

$$u'' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) + \alpha v(t) = 0, \tag{7.3.5}$$

with $0 \leq u(t) \leq R$ for all $t \in [0,T]$, satisfies

- $\max_{t \in I_i^+} u(t) \neq r$, *if $i \notin \mathcal{I} \cup \mathcal{J}$;*
- $\max_{t \in I_i^+} u(t) \neq \rho$, *if $i \in \mathcal{I}$;*
- $\max_{t \in I_i^+} u(t) \neq R$, *if $i \in \mathcal{J}$.*

(H_2) *There exists $\alpha^* \geq 0$ such that equation (7.3.5), with $\alpha = \alpha^*$, does not possess any non-negative T-periodic solution $u(t)$ with*

$$u(t) \leq \rho, \quad \forall\, t \in \bigcup_{i \in \mathcal{I}} I_i^+.$$

Then it holds that

$$D_L\big(L - N_{\lambda,\mu}, \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\big) = 0.$$

Proof. We adapt to our situation an argument from Lemma 6.2.1 (cf. also Lemma 4.2.3). We first write the equation

$$u'' + f_{\lambda,\mu}(t,u) + \alpha v(t) = 0 \tag{7.3.6}$$

as a coincidence equation in the space X

$$Lu = N_{\lambda,\mu} u + \alpha v, \quad u \in \operatorname{dom} L,$$

and we check that the coincidence degree $D_L\big(L - N_{\lambda,\mu} - \alpha v, \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\big)$ is well-defined for any $\alpha \geq 0$. To this end, for $\alpha \geq 0$, we consider the solution set

$$\mathcal{R}_\alpha := \Big\{ u \in \operatorname{cl}\big(\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\big) \cap \operatorname{dom} L \colon Lu = N_{\lambda,\mu} u + \alpha v \Big\}.$$

We have that $u \in \mathcal{R}_\alpha$ if and only if $u(t)$ is a T-periodic solution of (7.3.6) with $|u(t)| \leq r$ for all $t \in I_i^+$ if $i \notin \mathcal{I} \cup \mathcal{J}$, $|u(t)| \leq \rho$ for all $t \in I_i^+$ if $i \in \mathcal{I}$, and $|u(t)| \leq R$ for all $t \in I_i^+$ if $i \in \mathcal{J}$. By a maximum principle argument, we find $u(t) \geq 0$ for any t. Moreover, taking into account that $v(t) \succ 0$ on $[0,T]$ and $v \equiv 0$ on $\bigcup_i I_i^-$, we have that $u(t)$ is concave in each

I_i^+ and convex in each I_i^-. As a consequence, $u(t) \leq R$ for any t. Hence, $\mathcal{R}_\alpha \subseteq B[0,R] := \{u \in X \colon \|u\|_\infty \leq R\}$ and the complete continuity of $\Phi_{\lambda,\mu}$ implies that $\mathcal{R}_\alpha$ is compact. Furthermore, condition (H_1) guarantees that $\max_{I_i^+} u < r$ if $i \notin \mathcal{I} \cup \mathcal{J}$, $\max_{I_i^+} u < \rho$ if $i \in \mathcal{I}$, and $\max_{I_i^+} u < R$ if $i \in \mathcal{J}$. Thus, $\mathcal{R}_\alpha \subseteq \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$. In this way we conclude that the coincidence degree $D_L\bigl(L - N_{\lambda,\mu} - \alpha v, \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\bigr)$ is well-defined for any $\alpha \geq 0$.

Now, using α as homotopy parameter and using the homotopic invariance of the degree (with the same argument as above, we can see that $\bigcup_{\alpha\in[0,\alpha^*]} \mathcal{R}_\alpha$ is a compact subset of $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$), we have that

$$D_L\bigl(L - N_{\lambda,\mu}, \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\bigr) = D_L\bigl(L - N_{\lambda,\mu} - \alpha^* v, \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\bigr).$$

If, by contradiction, this degree is non-null, then there exists at least one T-periodic solution $u \in \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ of (7.3.6) with $\alpha = \alpha^*$. Again by the maximum principle, we then have a non-negative T-periodic solution of (7.3.5) with $\alpha = \alpha^*$ and, since $u \in \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ with $\mathcal{I} \neq \emptyset$, it holds that $\max_{I_i^+} u \leq \rho$ if $i \in \mathcal{I}$. This contradicts assumption (H_2) and the proof is completed. $\square$

The next result uses a duality theorem by Mawhin which relates the coincidence degree with the (finite dimensional) Brouwer degree. We recall also the definition of $\mu^{\#}(\lambda)$ given in (7.1.2).

Lemma 7.3.2. *Let $\mathcal{I} = \emptyset$, $\lambda > 0$ and $\mu > \mu^{\#}(\lambda)$. Assume the following property.*

(H_3) *If $\vartheta \in \,]0,1]$, then any T-periodic solution $u(t)$ of*

$$u'' + \vartheta\bigl(\lambda a^+(t) - \mu a^-(t)\bigr)g(u) = 0, \tag{7.3.7}$$

with $0 \leq u(t) \leq R$ for all $t \in [0,T]$, satisfies

- $\max_{t\in I_i^+} u(t) \neq r$, *if $i \notin \mathcal{J}$;*
- $\max_{t\in I_i^+} u(t) \neq R$, *if $i \in \mathcal{J}$.*

Then it holds that

$$D_L\bigl(L - N_{\lambda,\mu}, \Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}\bigr) = 1.$$

Proof. We argue similarly as in Lemma 6.2.2. We consider the parameterized equation

$$u = \Psi_\vartheta(u) := Pu + QN_{\lambda,\mu}u + \vartheta K_P(Id - Q)N_{\lambda,\mu}u, \quad u \in X,\ \vartheta \in [0,1].$$

Let also

$$\mathcal{S} := \bigcup_{\vartheta\in[0,1]} \Bigl\{u \in \mathrm{cl}\bigl(\Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}\bigr) \colon u = \Psi_\vartheta(u)\Bigr\}.$$

Suppose that $0 < \vartheta \leq 1$. In this situation, $u = \Psi_\vartheta(u)$ if and only if

$$Lu = \vartheta N_{\lambda,\mu} u, \quad u \in \operatorname{dom} L,$$

or, equivalently, $u(t)$ is a T-periodic solution of

$$u'' + \vartheta f_{\lambda,\mu}(t,u) = 0.$$

If $u \in \operatorname{cl}\bigl(\Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}\bigr)$, we know that $\max_{I_i^+} |u| \leq r$ if $i \notin \mathcal{J}$ and $\max_{I_i^+} |u| \leq R$ if $i \in \mathcal{J}$. Hence, by a maximum principle, $u(t)$ is a non-negative T-periodic solution of (7.3.7) and, by a convexity argument, $u(t) \leq R$ for any t. Moreover, by (H_3), $\max_{I_i^+} u < r$ if $i \notin \mathcal{J}$ and $\max_{I_i^+} u < R$ if $i \in \mathcal{J}$.

On the other hand, if $\vartheta = 0$, u is a solution of $u = \Psi_0(u)$ if and only if $u = Pu + QN_{\lambda,\mu}u$, that is, $u \in \ker L$ and $QN_{\lambda,\mu}u = 0$. Since $\ker L \cong \mathbb{R}$ and

$$QN_{\lambda,\mu}u = \frac{1}{T}\int_0^T f_{\lambda,\mu}(t,s)\,dt, \qquad \text{for } u \equiv \text{constant} = s \in \mathbb{R},$$

we conclude that $u \equiv s \in \mathbb{R}$ is a solution of $u = \Psi_0(u)$ with $u \in \operatorname{cl}\bigl(\Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}\bigr)$ if and only if $|s| \leq r$ if $\mathcal{J} \neq \{1,\ldots,m\}$ and $|s| \leq R$ if $\mathcal{J} = \{1,\ldots,m\}$ and, moreover, $f^{\#}_{\lambda,\mu}(s) = 0$, where we have set

$$f^{\#}_{\lambda,\mu}(s) := \frac{1}{T}\int_0^T f_{\lambda,\mu}(t,s)\,dt = \begin{cases} -s, & \text{if } s \leq 0; \\ \displaystyle\biggl(\frac{1}{T}\int_0^T a_{\lambda,\mu}(t)\,dt\biggr) g(s), & \text{if } s \geq 0. \end{cases}$$

If $\mu > \mu^{\#}(\lambda)$, we have that $f^{\#}_{\lambda,\mu}$ satisfies $f^{\#}_{\lambda,\mu}(s)s < 0$ for $s \neq 0$. Hence $u \equiv 0$.

We conclude that the set $\mathcal{S}$ is compact and contained in $\Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}$. By the homotopic invariance of the coincidence degree, we have that

$$\begin{aligned} D_L\bigl(L - N_{\lambda,\mu}, \Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}\bigr) &= \deg_{LS}\bigl(Id - \Psi_1, \Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}, 0\bigr) \\ &= \deg_{LS}\bigl(Id - \Psi_0, \Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}, 0\bigr) \\ &= \deg_B\bigl(-QN_{\lambda,\mu}|_{\ker L}, \Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)} \cap \ker L, 0\bigr) \\ &= \deg_B\bigl(-f^{\#}_{\lambda,\mu}|_{\ker L},]-d,d[, 0\bigr) = 1, \end{aligned}$$

where $d = r$ or $d = R$ according to whether $\mathcal{J} \neq \{1,\ldots,m\}$ or $\mathcal{J} = \{1,\ldots,m\}$. This concludes the proof. □

Remark 7.3.1. When dealing with other differential operators L or with Neumann and Dirichlet boundary conditions, some changes are required.

First of all we notice that Lemma 7.3.1 and Lemma 7.3.2 hold exactly the same for the T-periodic problem and the differential operator $u \mapsto -u'' - cu'$.

The same is true for Neumann boundary conditions: we have only to assume for equation (7.3.5) and (7.3.7) that $u(t)$ is a solution satisfying $u'(0) = u'(T) = 0$. For these cases, no relevant changes are needed in the proofs.

Concerning the Dirichlet problem the following modifications are in order. First, in all the degree formulas the terms $D_L(L - N_{\lambda,\mu}, \cdot)$ have to be replaced by $\deg_{LS}(Id - L^{-1}N_{\lambda,\mu}, \cdot, 0)$. Secondly, in equations (7.3.5) and (7.3.7) we have to suppose that $u(t)$ is a solution satisfying $u(0) = u(T) = 0$. Finally, we strongly simplify the argument in the proof of Lemma 7.3.2 since, when $\vartheta = 0$, we directly reduce to the trivial equation $u = 0$. Therefore the homotopic invariance of the Leray-Schauder degree (with respect to the parameter $\vartheta \in [0, 1]$) yields

$$D_L(L - N_{\lambda,\mu}, \Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}) = \deg_{LS}(Id, \Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}, 0) = 1,$$

because $0 \in \Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}$. In this case the condition $\mu > \mu^{\#}(\lambda)$ is not required in Lemma 7.3.2. However, the largeness of μ will be in any case needed later in subsequent technical estimates. $\lhd$

7.4 Proof of Theorem 7.1.1: the details

In view of the general strategy for the proof described in Section 7.3, we are going to prove that the assumptions (H_1), (H_2) of Lemma 7.3.1 and (H_3) of Lemma 7.3.2 are satisfied for suitable choices of r, ρ, R and λ, μ large enough. These proofs are given in the second part of this section (see Section 7.4.3 and Section 7.4.4). Lemma 7.3.1 and Lemma 7.3.2 involve the study of the solutions of (7.3.5) and (7.3.7), respectively. These equations, although different, present common features and, for this reason, we premise some technical estimates on the solutions which will help and simplify our subsequent proofs.

Keeping in mind that all the assumptions on $a(t)$ and $g(s)$ in Theorem 7.1.1 are assumed, we introduce now the following notation. For any constant $d > 0$, we set

$$\zeta(d) := \max_{\frac{d}{2} \leq s \leq d} \frac{g(s)}{s}, \qquad \gamma(d) := \min_{\frac{d}{2} \leq s \leq d} \frac{g(s)}{s}. \tag{7.4.1}$$

Moreover, we also define

$$g^*(d) := \max_{0 \leq s \leq d} g(s), \qquad g_*(d, D) := \min_{d \leq s \leq D} g(s),$$

where $D > d$ is another arbitrary constant. Furthermore, recalling (a_*) and the positions in (7.1.1), for all $i = 1, \ldots, m$, we set

$$\|a\|_{\pm,i} := \int_{I_i^{\pm}} a^{\pm}(t)\, dt$$

and

$$A_i(t) := \int_{\tau_i}^{t} a^-(\xi)\, d\xi, \quad t \in I_i^-, \qquad \|A_i\| := \int_{I_i^-} A_i(t)\, dt,$$
$$B_i(t) := \int_{t}^{\sigma_{i+1}} a^-(\xi)\, d\xi, \quad t \in I_i^-, \qquad \|B_i\| := \int_{I_i^-} B_i(t)\, dt.$$

Notice that, in general, $\|A_i\|$ and $\|B_i\|$ may be different.

7.4.1 Technical estimates

We present now some preliminary technical lemmas. We stress the fact that all the results in this subsection concern the properties of solutions of given equations without any reference to the boundary conditions.

Lemma 7.4.1. *For any $\rho > 0$, there exists $\lambda^* = \lambda^*(\rho) > 0$ such that, for any $\lambda > \lambda^*$, $\alpha \geq 0$ and $i \in \{1, \ldots, m\}$, there are no non-negative solutions $u(t)$ to*

$$u'' + \lambda a^+(t) g(u) + \alpha = 0, \tag{7.4.2}$$

with $u(t)$ defined for all $t \in I_i^+$, and such that $\max_{t \in I_i^+} u(t) = \rho$.

Proof. We fix $\varepsilon > 0$ such that, for each $i \in \{1, \ldots, m\}$, $\varepsilon < (\tau_i - \sigma_i)/2$ and, moreover, $\int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t)\, dt > 0$. In this manner, the quantity

$$\nu_\varepsilon := \min_{i=1,\ldots,m} \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t)\, dt$$

is well-defined and positive.

Let $\rho > 0$ be fixed and consider $\alpha \geq 0$ and $i \in \{1, \ldots, m\}$. Suppose that $u(t)$ is a non-negative solution of (7.4.2) defined on I_i^+ and such that

$$\max_{t \in I_i^+} u(t) = \rho.$$

Using the concavity of $u(t)$ on I_i^+ and proceeding as in Section 5.7 (see also Section 3.3.2, Section 4.3 and Section 6.4.1), we have

$$|u'(t)| \leq \frac{u(t)}{\varepsilon}, \quad \forall\, t \in [\sigma_i + \varepsilon, \tau_i - \varepsilon].$$

As a consequence,

$$|u'(t)| \leq \frac{\rho}{\varepsilon}, \quad \forall\, t \in [\sigma_i + \varepsilon, \tau_i - \varepsilon]. \tag{7.4.3}$$

On the other hand, the concavity of $u(t)$ on I_i^+ ensures that

$$u(t) \geq \frac{\rho}{|I_i^+|} \min\{t - \sigma_i, \tau_i - t\}, \quad \forall\, t \in I_i^+. \tag{7.4.4}$$

We introduce now the positive constant

$$\eta_{\varepsilon,\rho} := \min\left\{ g(s) \colon \frac{\varepsilon\rho}{\max_{i=1,\ldots,m} |I_i^+|} \leq s \leq \rho \right\}.$$

Integrating equation (7.4.2) on $[\sigma_i + \varepsilon, \tau_i - \varepsilon]$ and using (7.4.3) and (7.4.4), we obtain

$$\begin{aligned}\lambda \eta_{\varepsilon,\rho} \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t)\,dt &\leq \lambda \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t) g(u(t))\,dt = \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} (-u''(t) - \alpha)\,dt \\ &= u'(\sigma_i+\varepsilon) - u'(\tau_i-\varepsilon) - \alpha\,(\tau_i - \sigma_i - 2\varepsilon) \leq \frac{2\rho}{\varepsilon}.\end{aligned}$$

Now, we set

$$\lambda^* = \lambda^*(\rho) := \frac{2\rho}{\varepsilon \nu_\varepsilon \eta_{\varepsilon,\rho}}.$$

Arguing by contradiction, from the last inequality we immediately conclude that there are no non-negative solutions $u(t)$ of (7.4.2) with $\max_{t\in I_i^+} u(t) = \rho$, if $\lambda > \lambda^*$. $\square$

Lemma 7.4.2. *Let $\lambda, \mu > 0$. Let $d > 0$ be such that*

$$\zeta(d) < \frac{1}{2\lambda \max_{i=1,\ldots,m} (|I_i^+| + |I_i^-|)\|a\|_{+,i}}. \tag{7.4.5}$$

Suppose that $u(t)$ is a non-negative solution of

$$u'' + \vartheta\big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0, \quad \vartheta \in \,]0,1],$$

defined on $I_i^+ \cup I_i^-$ for some $i \in \{1,\ldots,m\}$ and such that

$$\max_{t\in I_i^+} u(t) = d \quad \textit{and} \quad u'(\sigma_i) \geq 0.$$

Then it holds that

$$u(\sigma_{i+1}) \geq d\left[1 + \frac{\vartheta}{2}\Big(\mu\gamma(d)\|A_i\| - 1\Big)\right]$$

and

$$u'(\sigma_{i+1}) \geq \vartheta d\left(\mu\frac{\gamma(d)}{2}\|a\|_{-,i} - \lambda\|a\|_{+,i}\zeta(d)\right).$$

Proof. The proof is split into two parts. In the first one we provide some estimates for $u(\tau_i)$ and $u'(\tau_i)$, while in the second part we obtain the desired inequality on $u(\sigma_{i+1})$ and $u'(\sigma_{i+1})$.

Let $\hat{t}_i \in I_i^+$ be such that

$$\max_{t \in I_i^+} u(t) = d = u(\hat{t}_i).$$

Observe that $u'(\hat{t}_i) = 0$, if $\sigma_i \leq \hat{t}_i < \tau_i$ (since $u'(\sigma_i) \geq 0$), while $u'(\hat{t}_i) \geq 0$, if $\hat{t}_i = \tau_i$. As a first instance, suppose that

$$u'(\hat{t}_i) = 0.$$

Let $[s_1, s_2] \subseteq I_i^+$ be the maximal closed interval containing $\hat{t}_i$ and such that $u(t) \geq d/2$ for all $t \in [s_1, s_2]$. We claim that $[s_1, s_2] = I_i^+$. From

$$u''(t) = -\vartheta\lambda a^+(t) g(u(t)), \quad t \in I_i^+,$$

and

$$u'(t) = u'(\hat{t}_i) + \int_{\hat{t}_i}^{t} u''(\xi)\, d\xi, \quad \forall\, t \in I_i^+,$$

it follows that

$$|u'(t)| \leq \vartheta\lambda \|a\|_{+,i} \zeta(d) d, \quad \forall\, t \in [s_1, s_2].$$

Then, in view of (7.4.5),

$$u(t) = u(\hat{t}_i) + \int_{\hat{t}_i}^{t} u'(\xi)\, d\xi \geq d - \vartheta\lambda |I_i^+| \|a\|_{+,i} \zeta(d) d > \frac{d}{2}, \quad \forall\, t \in [s_1, s_2].$$

This inequality, together with the maximality of the interval $[s_1, s_2]$, implies that $[s_1, s_2] = I_i^+$. Hence

$$u'(t) \geq -\vartheta\lambda \|a\|_{+,i} \zeta(d) d, \quad \forall\, t \in I_i^+, \tag{7.4.6}$$

and, a fortiori,

$$u'(\tau_i) \geq -\vartheta\lambda \|a\|_{+,i} \zeta(d) d. \tag{7.4.7}$$

Moreover, after an integration of (7.4.6) on $[\hat{t}_i, \tau_i]$, we obtain

$$u(\tau_i) \geq d\big(1 - \vartheta\lambda |I_i^+| \|a\|_{+,i} \zeta(d)\big). \tag{7.4.8}$$

On the other hand, if we suppose that $\hat{t}_i = \tau_i$ and $u'(\hat{t}_i) > 0$, we immediately have

$$u(\tau_i) = d \geq d\big(1 - \vartheta\lambda |I_i^+| \|a\|_{+,i} \zeta(d)\big) \quad \text{and} \quad u'(\tau_i) > 0 \geq -\vartheta\lambda \|a\|_{+,i} \zeta(d) d.$$

Thus, in any case, (7.4.7) and (7.4.8) hold. Having produced some estimates on $u(\tau_i)$ and $u'(\tau_i)$ we are in position now to proceed with the second part of the proof.

We consider the subsequent (adjacent) interval $I_i^- = [\tau_i, \sigma_{i+1}]$ where the weight is non-positive. Since $u'(t)$ is non-decreasing, from (7.4.7) we get

$$u'(t) \geq -\vartheta\lambda\|a\|_{+,i}\zeta(d)d, \quad \forall\, t \in I_i^-.$$

Therefore, integrating on $[\tau_i, t]$ and using (7.4.8), we have

$$\begin{aligned} u(t) = u(\tau_i) + \int_{\tau_i}^{t} u'(\xi)\,d\xi &\geq d\big(1 - \vartheta\lambda|I_i^+|\|a\|_{+,i}\zeta(d) - \vartheta\lambda|I_i^-|\|a\|_{+,i}\zeta(d)\big) \\ &\geq d\big(1 - \lambda(|I_i^+| + |I_i^-|)\|a\|_{+,i}\zeta(d)\big) > \frac{d}{2}, \quad \forall\, t \in I_i^-, \end{aligned} \tag{7.4.9}$$

where the last inequality follows from (7.4.5). On the other hand, integrating

$$u''(t) = \vartheta\mu a^-(t)g(u(t)), \quad t \in I_i^-,$$

on $[\tau_i, t]$ and using (7.4.7) and (7.4.9), we find

$$\begin{aligned} u'(t) &= u'(\tau_i) + \int_{\tau_i}^{t} \vartheta\mu a^-(\xi)g(u(\xi))\,d\xi \\ &\geq -\vartheta\lambda\|a\|_{+,i}\zeta(d)d + \vartheta\frac{d}{2}\mu\gamma(d)A_i(t), \quad \forall\, t \in I_i^-. \end{aligned}$$

In particular,

$$u'(\sigma_{i+1}) \geq \vartheta d\left(\mu\frac{\gamma(d)}{2}\|a\|_{-,i} - \lambda\|a\|_{+,i}\zeta(d)\right).$$

Finally, a further integration and condition (7.4.5) yield

$$\begin{aligned} u(\sigma_{i+1}) &= u(\tau_i) + \int_{\tau_i}^{\sigma_{i+1}} u'(t)\,dt \\ &\geq d - \vartheta\lambda(|I_i^+| + |I_i^-|)\|a\|_{+,i}\zeta(d)d + \vartheta\frac{d}{2}\mu\gamma(d)\|A_i\| \\ &\geq d\left[1 + \vartheta\left(\mu\frac{\gamma(d)}{2}\|A_i\| - \lambda(|I_i^+| + |I_i^-|)\|a\|_{+,i}\zeta(d)\right)\right] \\ &\geq d\left[1 + \frac{\vartheta}{2}\Big(\mu\gamma(d)\|A_i\| - 1\Big)\right]. \end{aligned}$$

This concludes the proof. □

Symmetrically, we have the following.

Lemma 7.4.3. *Let $\lambda, \mu > 0$. Let $d > 0$ be such that*

$$\zeta(d) < \frac{1}{2\lambda \max_{i=1,\dots,m}(|I_{i-1}^-| + |I_i^+|)\|a\|_{+,i}}.$$

Suppose that $u(t)$ is a non-negative solution of

$$u'' + \vartheta\big(\lambda a^{+}(t) - \mu a^{-}(t)\big)g(u) = 0, \quad \vartheta \in \,]0,1],$$

defined on $I^{-}_{i-1} \cup I^{+}_{i}$ for some $i \in \{1,\ldots,m\}$ and such that

$$\max_{t\in I^{+}_{i}} u(t) = d \quad \text{ and } \quad u'(\tau_i) \leq 0.$$

Then it holds that

$$u(\tau_{i-1}) \geq d\left[1 + \frac{\vartheta}{2}\Big(\mu\gamma(d)\|B_{i-1}\| - 1\Big)\right]$$

and

$$u'(\tau_{i-1}) \leq -\vartheta d\bigg(\mu\frac{\gamma(d)}{2}\|a\|_{-,i-1} - \lambda\|a\|_{+,i}\zeta(d)\bigg).$$

Remark 7.4.1. In the sequel, when dealing with the periodic problem, we observe that the solutions we consider are defined on $[0,T]$ and satisfy T-periodic boundary conditions $u(T) - u(0) = u'(T) - u'(0) = 0$. Hence it is convenient to count the intervals cyclically. Accordingly, in the special case in which $i = 1$, we apply Lemma 7.4.3 with the agreement $I^{-}_{0} = I^{-}_{m}$. This makes sense because, if we extend the solution by T-periodicity on the whole real line, we can consider the interval $I^{-}_{m} - T$ as adjacent on the left to I^{+}_{1}. ◁

Lemma 7.4.4. *Let $\lambda > 0$ and $0 < d < D$. For any $i \in \{1,\ldots,m\}$ there exists a constant*

$$\mu^{*,+}_{i} = \mu^{*,+}_{i}(I^{-}_{i}, I^{+}_{i+1}) > 0$$

such that for all $\mu > \mu^{*,+}_{i}$ any non-negative solution $u(t)$ of

$$u'' + \vartheta\big(\lambda a^{+}(t) - \mu a^{-}(t)\big)g(u) = 0, \quad \vartheta \in \,]0,1],$$

defined on $I^{-}_{i} \cup I^{+}_{i+1}$ and such that

$$\|u\|_{\infty} \leq D, \quad u(\tau_i) > d \quad \text{ and } \quad u'(\tau_i) > 0,$$

satisfies

$$u(t) > d, \quad u'(t) > 0, \quad \forall\, t \in I^{-}_{i} \cup I^{+}_{i+1}.$$

Proof. Clearly, by the convexity of $u(t)$ on I^{-}_{i}, we have

$$u(t) > d, \quad u'(t) > 0, \quad \forall\, t \in I^{-}_{i}.$$

Integrating

$$u''(t) = \vartheta\mu a^{-}(t)g(u(t)) \geq \vartheta\mu a^{-}(t)g_{*}(d,D), \quad t \in I^{-}_{i},$$

on $[\tau_i, t] \subseteq I_i^-$ we find

$$u'(t) = u'(\tau_i) + \int_{\tau_i}^{t} u''(\xi)\, d\xi > \vartheta\mu A_i(t) g_*(d, D), \quad \forall\, t \in I_i^-,$$

so that

$$u'(\sigma_{i+1}) > \vartheta\mu A_i(\sigma_{i+1}) g_*(d, D) = \vartheta\mu \|a\|_{-,i}\, g_*(d, D).$$

On the other hand, integrating

$$u''(t) = -\vartheta\lambda a^+(t) g(u(t)) \geq -\vartheta\lambda a^+(t) g^*(D), \quad t \in I_{i+1}^+,$$

on $[\sigma_{i+1}, t] \subseteq I_{i+1}^+$ we find

$$\begin{aligned} u'(t) &= u'(\sigma_{i+1}) + \int_{\sigma_{i+1}}^{t} u''(\xi)\, d\xi \\ &> \vartheta\Big(\mu\|a\|_{-,i} g_*(d, D) - \lambda\|a\|_{+,i+1} g^*(D)\Big) > 0, \quad \forall\, t \in I_{i+1}^+, \end{aligned}$$

where the last inequality holds provided that

$$\mu > \mu_i^{*,+} = \mu_i^{*,+}(I_i^-, I_{i+1}^+) := \frac{\lambda\|a\|_{+,i+1} g^*(D)}{\|a\|_{-,i} g_*(d, D)}.$$

Then the solution $u(t)$ is increasing in $I_{i+1}^+ = [\sigma_{i+1}, \tau_{i+1}]$ and hence

$$u(t) > u(\sigma_{i+1}) > d, \quad \forall\, t \in I_{i+1}^+.$$

The proof is thus completed. □

Symmetrically, we have the following.

Lemma 7.4.5. *Let $\lambda > 0$ and $0 < d < D$. For any $i \in \{1, \ldots, m\}$ there exists a constant*

$$\mu_i^{*,-} = \mu_i^{*,-}(I_{i-1}^+, I_{i-1}^-) > 0$$

such that for all $\mu > \mu_i^{*,-}$ any non-negative solution $u(t)$ of

$$u'' + \vartheta\big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0, \quad \vartheta \in \,]0, 1],$$

defined on $I_{i-1}^+ \cup I_{i-1}^-$ and such that

$$\|u\|_\infty \leq D, \quad u(\sigma_i) > d \quad \text{and} \quad u'(\sigma_i) < 0,$$

satisfies

$$u(t) > d, \quad u'(t) < 0, \quad \forall\, t \in I_{i-1}^+ \cup I_{i-1}^-.$$

Remark 7.4.2. Similarly as in Remark 7.4.1, in order to make the statements of Lemma 7.4.4 and Lemma 7.4.5 meaningful for each possible choice of the index $i \in \{1, \ldots, m\}$, when dealing with the periodic problem we shall use the cyclic agreement $I_0^- = I_m^-$ (as above) and, moreover, $I_{m+1}^+ = I_1^+$, $I_0^+ = I_m^+$. ◁

7.4.2 Fixing the constants ρ, λ, r and R

First of all, we arbitrarily choose a constant $\rho > 0$. Then, we determine the constant $\lambda^* = \lambda^*(\rho) > 0$ according to Lemma 7.4.1 and we take an arbitrary $\lambda > \lambda^*$. Next, we fix two positive constants r, R with

$$0 < r < \rho < R$$

and such that

$$\zeta(s) < \frac{1}{2\lambda \max\limits_{i=1,\dots,m} (|I^-_{i-1}| + |I^+_i| + |I^-_i|)\|a\|_{+,i}}, \quad \forall\, 0 < s \leq r,\ \forall\, s \geq R, \tag{7.4.10}$$

where $\zeta(s)$ is defined in (7.4.1). In the above formula, we use again the cyclic agreement $I^-_0 = I^-_m$. The existence of r and R with the above property is guaranteed by the fact that $g(s)/s \to 0^+$ for $s \to 0^+$ and for $s \to +\infty$, namely conditions (g_0) and (g_∞).

With this choice of r, ρ and R, we consider the sets $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ defined in (7.3.1). We are ready now to prove Theorem 7.3.1, by checking that Lemma 7.3.1 and Lemma 7.3.2 can be applied for $\mu > 0$ sufficiently large (say $\mu > \mu^*(\lambda, r, R)$).

In the proofs of the next two subsections we deal with solutions satisfying T-periodic boundary conditions. Accordingly, we apply Lemma 7.4.2, Lemma 7.4.3, Lemma 7.4.4 and Lemma 7.4.5 with the cyclic convention about the labelling of the intervals described in Remark 7.4.1 and Remark 7.4.2.

7.4.3 Checking the assumptions of Lemma 7.3.1 for μ large

In this section we are going to prove the first part of Theorem 7.3.1, that is

$$D_L\big(L - N_{\lambda,\mu}, \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\big) = 0, \quad \text{if } \mathcal{I} \neq \emptyset. \tag{7.4.11}$$

As usual, we implicitly suppose that $\mathcal{I}, \mathcal{J} \subseteq \{1, \dots, m\}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$.

Given $\mathcal{I}, \mathcal{J}$ as above, with $\mathcal{I} \neq \emptyset$, it is sufficient to check that the assumptions of Lemma 7.3.1 are satisfied, taking as $v(t)$ the indicator function of the set $\bigcup_{i\in\mathcal{I}} I^+_i$.

Verification of (H_1). Let $\alpha \geq 0$. By contradiction, suppose that there exists a non-negative T-periodic solution $u(t)$ of (7.3.5) with $\|u\|_\infty \leq R$ such that at least one of the following conditions holds:

(a_1) there is an index $i \notin \mathcal{I} \cup \mathcal{J}$ such that $\max_{t\in I^+_i} u(t) = r$;

(a_2) there is an index $i \in \mathcal{I}$ such that $\max_{t\in I^+_i} u(t) = \rho$;

(a_3) there is an index $i \in \mathcal{J}$ such that $\max_{t\in I^+_i} u(t) = R$.

Suppose that (a_1) holds. On the interval $I_i^+ \cup I_i^-$ (with $i \notin \mathcal{I} \cup \mathcal{J}$) equation (7.3.5) reads as

$$u'' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0.$$

Consider at first the case $u'(\sigma_i) \geq 0$. By Lemma 7.4.2 (with $\vartheta = 1$ and $d = r$), we have that

$$u(\sigma_{i+1}) \geq r \left(1 + \mu \frac{\gamma(r)}{2} \|A_i\| - \frac{1}{2}\right) \geq \mu\, r \frac{\gamma(r)}{2} \|A_i\|.$$

Thus, taking

$$\mu > \hat{\mu}_i^{\mathrm{right}} := \frac{2R}{r\gamma(r)\|A_i\|}, \tag{7.4.12}$$

we obtain

$$u(\sigma_{i+1}) > R,$$

a contradiction. On the other hand, if $u'(\sigma_i) < 0$, by the concavity of $u(t)$ in I_i^+ we have that $u'(\tau_i) < 0$. In this case we reach the contradiction

$$u(\tau_{i-1}) > R$$

using Lemma 7.4.3 (with $\vartheta = 1$ and $d = r$) and taking

$$\mu > \hat{\mu}_i^{\mathrm{left}} := \frac{2R}{r\gamma(r)\|B_{i-1}\|}. \tag{7.4.13}$$

Suppose that (a_2) holds. This fact contradicts Lemma 7.4.1 in view of our choice of $\lambda > \lambda^*$. In this case no assumption on $\mu > 0$ is needed.

Finally, if (a_3) holds, we obtain again a contradiction arguing as in case (a_1) and using Lemma 7.4.2 (with $\vartheta = 1$ and $d = R$). Indeed, $u'(\sigma_i)$ cannot be negative, otherwise $u(\sigma_i) = R$ and we get a contradiction with $\max_{t \in I_i^+} u(t) = R = \|u\|_\infty$. Hence, only the instance $u'(\sigma_i) \geq 0$ may occur and we have a contradiction for

$$\mu > \check{\mu}_i := \frac{1}{\gamma(R)\|A_i\|}. \tag{7.4.14}$$

We conclude that (H_1) holds true for

$$\mu > \mu^{(H_1)} := \max_{i=1,\ldots,m} \big\{ \hat{\mu}_i^{\mathrm{right}}, \hat{\mu}_i^{\mathrm{left}}, \check{\mu}_i \big\}.$$

Verification of (H_2). Let $u(t)$ be an arbitrary non-negative T-periodic solution of (7.3.5) (with $\alpha \geq 0$) such that $u(t) \leq \rho$ for every $t \in \bigcup_{i \in \mathcal{I}} I_i^+$.

We fix an index $j \in \mathcal{I}$ and observe that on the interval I_j^+ equation (7.3.5) reads as

$$u'' + \lambda a^+(t) g(u) + \alpha = 0.$$

Now, we choose a constant $\varepsilon \in \,]0,(\tau_j-\sigma_j)/2[$ and we notice that the inequality

$$|u'(t)| \leq \frac{|u(t)|}{\varepsilon}, \quad \forall\, t \in [\sigma_j+\varepsilon, \tau_j-\varepsilon],$$

used in the proof of Lemma 7.4.1 is still valid. Integrating the differential equation on $[\sigma_j+\varepsilon,\tau_j-\varepsilon]$ and using the above inequality, we obtain

$$\alpha\,(\tau_i-\sigma_i-2\varepsilon) = u'(\sigma_i+\varepsilon) - u'(\tau_i-\varepsilon) - \lambda \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} a^+(t) g(u(t))\,dt \leq \frac{2\rho}{\varepsilon}.$$

This yields a contradiction if $\alpha > 0$ is sufficiently large. Hence (H_2) is verified (with $\alpha^* > 2\rho/\varepsilon(\tau_i-\sigma_i-2\varepsilon)$). Notice that for the validity of (H_2) we do not impose any condition on $\mu > 0$.

Summing up, we can apply Lemma 7.3.1 for $\mu > \mu^{(H_1)}$ and therefore formula (7.4.11) is verified. □

7.4.4 Checking the assumptions of Lemma 7.3.2 for μ large

In this section we are going to prove the second part of Theorem 7.3.1, that is

$$D_L\big(L - N_{\lambda,\mu}, \Omega^{\emptyset,\mathcal{J}}_{(r,\rho,R)}\big) = 1, \tag{7.4.15}$$

where $\mathcal{J} \subseteq \{1,\ldots,m\}$.

Given an arbitrary $\mathcal{J} \subseteq \{1,\ldots,m\}$, it is sufficient to check that the assumption of Lemma 7.3.2 is satisfied.

Verification of (H_3). Let $\vartheta \in \,]0,1]$. By contradiction, suppose that there exists a non-negative T-periodic solution $u(t)$ of (7.3.7) with $\|u\|_\infty \leq R$ such that at least one of the following conditions holds:

(b_1) there is an index $i \notin \mathcal{J}$ such that $\max_{t\in I_i^+} u(t) = r$;

(b_2) there is an index $i \in \mathcal{J}$ such that $\max_{t\in I_i^+} u(t) = R$.

Suppose that (b_1) holds. Consider at first the case $u'(\sigma_i) \geq 0$. Applying Lemma 7.4.2 (with $d = r$), we obtain

$$u(\sigma_{i+1}) \geq r\left[1 + \frac{\vartheta}{2}\Big(\mu\gamma(r)\|A_i\| - 1\Big)\right]$$

and

$$u'(\sigma_{i+1}) \geq \vartheta r \left(\mu \frac{\gamma(r)}{2}\|a\|_{-,i} - \lambda \|a\|_{+,i}\zeta(r)\right).$$

Notice that if

$$\mu > \hat{\mu}_i^{\mathrm{right}}, \tag{7.4.16}$$

with $\hat{\mu}_i^{\mathrm{right}}$ defined in (7.4.12), then $\mu > 1/(\gamma(r)\|A_i\|)$, and hence $u(\sigma_{i+1}) > r$ (as $\vartheta > 0$).

On the interval I^+_{i+1} equation (7.3.7) yields

$$u''(t) = -\vartheta\lambda a^+(t)g(u(t)) \geq -\vartheta\lambda a^+(t)g^*(R).$$

Then, integrating on $[\sigma_{i+1}, t] \subseteq I^+_{i+1}$ and using the above lower estimate on $u'(\sigma_{i+1})$, we obtain

$$\begin{aligned} u'(t) &= u'(\sigma_{i+1}) + \int_{\sigma_{i+1}}^{t} u''(\xi)\,d\xi \geq u'(\sigma_{i+1}) - \vartheta\lambda\|a\|_{+,i+1}g^*(R) \\ &\geq \vartheta r\biggl(\mu\frac{\gamma(r)}{2}\|a\|_{-,i} - \lambda\|a\|_{+,i}\zeta(r) - \lambda\|a\|_{+,i+1}\frac{g^*(R)}{r}\biggr), \quad \forall\, t \in I^+_{i+1}. \end{aligned}$$

Taking μ sufficiently large, precisely

$$\mu > \tilde{\mu}_i^{\mathrm{right}} := \frac{2\lambda\bigl(\|a\|_{+,i}r\zeta(r) + \|a\|_{+,i+1}g^*(R)\bigr)}{\gamma(r)r\|a\|_{-,i}}, \tag{7.4.17}$$

we obtain that

$$u'(t) > 0, \quad \forall\, t \in I^+_{i+1}.$$

Consequently

$$u(t) = u(\sigma_{i+1}) + \int_{\sigma_{i+1}}^{t} u'(\xi)\,d\xi \geq u(\sigma_{i+1}) > r, \quad \forall\, t \in I^+_{i+1}.$$

We conclude that for

$$\mu > \max\bigl\{\hat{\mu}_i^{\mathrm{right}}, \tilde{\mu}_i^{\mathrm{right}}\bigr\},$$

we have that

$$u(t) > r, \quad u'(t) > 0, \quad \forall\, t \in I^+_{i+1},$$

and, in particular,

$$u(\tau_{i+1}) > r \quad \text{ and } \quad u'(\tau_{i+1}) > 0.$$

Now we can apply Lemma 7.4.4 (with $d = r$ and $D = R$) on the interval $I^-_{i+1} \cup I^+_{i+2}$, which ensures that

$$u(t) > r, \quad u'(t) > 0, \quad \forall\, t \in I^-_{i+1} \cup I^+_{i+2},$$

provided that

$$\mu > \mu^{*,+}_{i+1} = \mu^{*,+}(I^-_{i+1}, I^+_{i+2}) = \frac{\lambda\|a\|_{+,i+2}g^*(R)}{\|a\|_{-,i+1}g_*(r,R)}. \tag{7.4.18}$$

Repeating inductively the same argument $m-1$ times we cover an interval T-periodicity with intervals (of the form $I^-_j \cup I^+_{j+1}$) where the function $u(t)$

is strictly increasing, provided that μ is sufficiently large. More precisely, for

$$\mu > \max_{i=1,\dots,m} \mu_i^{*,+}$$

it holds that

$$u(t) > r, \quad u'(t) > 0, \quad \forall\, t \in [0,T].$$

This clearly contradicts the T-periodicity of $u(t)$.

Consider now the case $u'(\sigma_i) < 0$, which implies (by the concavity of $u(t)$ in I_i^+) that $u'(\tau_i) < 0$. The same proof as above leads to a contradiction, proceeding backward and using at first Lemma 7.4.3 (with $d = r$) and then Lemma 7.4.5 (with $d = r$ and $D = R$), inductively. Conditions (7.4.16), (7.4.17) and (7.4.18) will be replaced by the analogous inequalities

$$\mu > \hat{\mu}_i^{\mathrm{left}},$$

with $\hat{\mu}_i^{\mathrm{left}}$ defined in (7.4.13),

$$\mu > \tilde{\mu}_i^{\mathrm{left}} := \frac{2\lambda\big(\|a\|_{+,i} r\zeta(r) + \|a\|_{+,i-1} g^*(R)\big)}{\gamma(r) r \|a\|_{-,i-1}},$$

and

$$\mu > \mu_{i-1}^{*,-} = \mu^{*,-}(I_{i-2}^+, I_{i-2}^-) = \frac{\lambda \|a\|_{+,i-2} g^*(R)}{\|a\|_{-,i-2} g_*(r,R)},$$

respectively. Thus a contradiction comes for

$$\mu > \max_{i=1,\dots,m} \mu_i^{*,-},$$

by showing that $u'(t) < 0$ for all $t \in [0,T]$.

Taking into account all the possible situations we conclude that the case (b_1) never occurs if

$$\mu > \mu_1^{(H_3)} := \max_{i=1,\dots,m} \big\{\hat{\mu}_i^{\mathrm{right}}, \hat{\mu}_i^{\mathrm{left}}, \tilde{\mu}_i^{\mathrm{right}}, \tilde{\mu}_i^{\mathrm{left}}, \mu_i^{*,+}, \mu_i^{*,-}\big\}.$$

To conclude the proof, suppose now that (b_2) holds. As observed in the previous proof, the fact that $\max_{t \in I_i^+} u(t) = R = \|u\|_\infty$ prevents the possibility that $u'(\sigma_i) < 0$. Hence only the instance $u'(\sigma_i) \geq 0$ may occur. Applying Lemma 7.4.2 (with $d = R$), we obtain

$$u(\sigma_{i+1}) \geq R \left[1 + \frac{\vartheta}{2}\Big(\mu\gamma(R)\|A_i\| - 1\Big)\right].$$

Hence, if

$$\mu > \check{\mu}_i = \frac{1}{\gamma(R)\|A_i\|}$$

(already defined in (7.4.14)) we get $u(\sigma_{i+1}) > R$ and thus a contradiction with $\|u\|_\infty \leq R$. We conclude that the case (b_2) never occurs if

$$\mu > \mu_2^{(H_3)} := \max_{i=1,\dots,m} \check{\mu}_i.$$

Summing up, we can apply Lemma 7.3.2 for

$$\mu > \mu^{(H_3)} := \max\Big\{\mu_1^{(H_3)}, \mu_2^{(H_3)}, \mu^{\#}(\lambda)\Big\}$$

and therefore formula (7.4.15) is verified. □

7.4.5 Completing the proof of Theorem 7.1.1

With reference to Section 7.3 we summarize what we have proved until now and we give the final details of the proof of our main theorem.

First, we have fixed an arbitrary constant $\rho > 0$ and determined a constant $\lambda^* = \lambda^*(\rho) > 0$ via Lemma 7.4.1. We stress the fact that λ^* depends only on $g(s)$ for $s \in [0,\rho]$ and on the behavior of $a(t)$ in each of the intervals I_i^+.

Next, for $\lambda > \lambda^*$, we have found two constants (a small one r and a large one R) with $0 < r < \rho < R$ such that condition (7.4.10) holds. To choose r and R we only require conditions on the smallness of $g(s)/s$ for s near zero and near infinity, which is an obvious consequence of (g_0) and (g_∞). We notice also that condition (7.4.10) depends on the behavior of $a(t)$ in each of the intervals I_i^+ as well as on the lengths of pairs of consecutive intervals.

As a further step, we have established that both Lemma 7.3.1 and Lemma 7.3.2 can be applied provided that

$$\mu > \mu^*(\lambda) = \mu^*(\lambda, r, R) := \max\Big\{\mu^{(H_1)}, \mu^{(H_3)}\Big\}.$$

Checking carefully the estimates leading to $\mu^{(H_1)}$ and $\mu^{(H_3)}$ one realizes that again only local conditions about the behavior of $a(t)$ on the intervals $I_i^\pm$ are involved.

As a consequence, for all $\mu > \mu^*(\lambda)$, formula (7.3.2) in Theorem 7.3.1 holds. From this latter result, via a purely combinatorial argument (independent on the particular equation under consideration), we achieve formula (7.3.4) in Theorem 7.3.2 and the existence of $3^m - 1$ positive T-periodic solutions to $(\mathscr{E}_{\lambda,\mu})$ is guaranteed, as already explained at the end of Section 7.3.1. □

7.5 Combinatorial argument

In this section we present the combinatorial argument needed in the proof of Theorem 7.3.2. In more detail, recalling the definitions of $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$

and $\Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ given in (7.3.1) and (7.3.3) respectively, from formula (7.3.2) (concerning the degrees on $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$), we prove that, for any pair of subset of indices $\mathcal{I},\mathcal{J} \subseteq \{1,\dots,m\}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$, we have

$$D_L\big(L - N_{\lambda,\mu}, \Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\big) = (-1)^{\#\mathcal{I}}.$$

We offer two independent proofs since we believe that both possess some peculiar aspects which might be also adapted to different situations.

7.5.1 First argument

In this first part we present a combinatorial argument which is related to the concept of *valuation*, as introduced in [110].

Let $m \in \mathbb{N}$ be a positive integer. We denote by

$$\mathbb{A} := \big\{A_1 \times A_2 \times \ldots \times A_m \colon A_i \in \mathscr{P}(\{0,1,2\})\big\}$$

the set of the 8^m Cartesian products of m subsets of $\{0,1,2\}$.

Let

$$\mathcal{A} := A_1 \times A_2 \times \ldots \times A_m \tag{7.5.1}$$

be an element of $\mathbb{A}$, let $i \in \{1,\dots,m\}$ be a fixed index and let also $B_i \in \mathscr{P}(\{0,1,2\})$. We introduce the following notation

$$\mathcal{A}[i : B_i] := A_1 \times \ldots \times A_{i-1} \times B_i \times A_{i+1} \times \ldots \times A_m.$$

Note that for any fixed $\mathcal{A}$ as above and for any $i \in \{1,\dots,m\}$ it holds that $\mathcal{A} = \mathcal{A}[i : A_i]$.

We consider a function

$$d \colon \mathbb{A} \to \mathbb{Z}$$

which satisfies the following property.

> *Additivity property.* Let $i \in \{1,\dots,m\}$ and $B_i \in \mathscr{P}(\{0,1,2\})$. Suppose that $B_i', B_i'' \subseteq B_i$ are disjoint (possibly empty) and such that
>
> $$B_i = B_i' \cup B_i''.$$
>
> Then, for all $\mathcal{A} \in \mathbb{A}$, it holds that
>
> $$d(\mathcal{A}[i : B_i]) = d(\mathcal{A}[i : B_i']) + d(\mathcal{A}[i : B_i'']).$$

From the additivity property (applied in the case $B_i = B_i' = B_i'' = \emptyset$) we immediately obtain that, if there exists an index $i \in \{1,\dots,m\}$ such that $A_i = \emptyset$, then $d(A_1 \times \ldots \times A_m) = 0$.

Moreover, we assume that d satisfies the following rules.

$(R1)$ If there exists an index $i \in \{1, \ldots, m\}$ such that $A_i = \{0,1\}$ and $A_j \in \{\{0\}, \{0,1,2\}\}$ for all $j \in \{1, \ldots, m\}$ such that $A_j \neq \{0,1\}$, then

$$d(A_1 \times \ldots \times A_m) = 0.$$

$(R2)$ If $A_i \in \{\{0\}, \{0,1,2\}\}$, for all $i = 1, \ldots, m$, then

$$d(A_1 \times \ldots \times A_m) = 1.$$

Our goal is to compute the value of $d(A_1 \times \ldots \times A_m)$ when $A_i \in \{\{0\}, \{1\}, \{2\}, \{0,1,2\}\}$, for all $i = 1, \ldots, m$.

As a first step we prove a generalization of rule $(R1)$.

Lemma 7.5.1. *If there exists an index $i \in \{1, \ldots, m\}$ such that $A_i = \{0,1\}$ and $A_j \in \{\{0\}, \{2\}, \{0,1,2\}\}$ for all $j \in \{1, \ldots, m\}$ such that $A_j \neq \{0,1\}$, then*

$$d(A_1 \times \ldots \times A_m) = 0.$$

Proof. We prove the statement by induction on the non-negative integer

$$k := \#\big\{j \in \{1, \ldots, m\} \colon A_j = \{2\}\big\}.$$

Case $k = 0$. If there is no $j \in \{1, \ldots, m\}$ such that $A_j = \{2\}$, the thesis follows by rule $(R1)$.

Case $k = 1$. Suppose that there is exactly one index $j \in \{1, \ldots, m\}$ such that $A_j = \{2\}$. Recalling the definition of $\mathcal{A}$ in (7.5.1), it is easy to see that

$$\mathcal{A}[j : \{0,1,2\}] = \mathcal{A} \cup \mathcal{A}[j : \{0,1\}].$$

Then, by the additivity property of d and rule $(R1)$, we obtain

$$d(\mathcal{A}) = d(\mathcal{A}[j : \{0,1,2\}]) - d(\mathcal{A}[j : \{0,1\}]) = 0 - 0 = 0.$$

Inductive step. Suppose that the statement holds for k. We prove it for $k+1$. Let $j \in \{1, \ldots, m\}$ be such that $A_j = \{2\}$. As above, from

$$\mathcal{A}[j : \{0,1,2\}] = \mathcal{A} \cup \mathcal{A}[j : \{0,1\}],$$

we obtain

$$d(\mathcal{A}) = d(\mathcal{A}[j : \{0,1,2\}]) - d(\mathcal{A}[j : \{0,1\}]).$$

By the inductive hypothesis, we know that $d(\mathcal{A}[j : \{0,1,2\}]) = 0$ and that $d(\mathcal{A}[j : \{0,1\}]) = 0$ (since $\mathcal{A}[j : \{0,1,2\}]$ and $\mathcal{A}[j : \{0,1\}]$ both have exactly k indices i such that $A_i = \{2\}$). The thesis immediately follows. $\square$

Now we provide a generalization of rule $(R2)$.

Lemma 7.5.2. *If $A_i \in \{\{0\},\{2\},\{0,1,2\}\}$, for all $i = 1,\dots,m$, then*

$$d(A_1 \times \ldots \times A_m) = 1.$$

Proof. We prove the statement by induction on the non-negative integer

$$k := \#\big\{j \in \{1,\dots,m\} \colon A_j = \{2\}\big\}.$$

Case $k = 0$. If there is no $j \in \{1,\dots,m\}$ such that $A_j = \{2\}$, the thesis follows by rule $(R2)$.

Case $k = 1$. Suppose that there is exactly one index $j \in \{1,\dots,m\}$ such that $A_j = \{2\}$. Recalling the definition of $\mathcal{A}$ in (7.5.1), it is easy to see that

$$\mathcal{A}[j:\{0,1,2\}] = \mathcal{A} \cup \mathcal{A}[j:\{0,1\}].$$

Then, by the additivity property of d and rules $(R1)$ and $(R2)$, we obtain

$$d(\mathcal{A}) = d(\mathcal{A}[j:\{0,1,2\}]) - d(\mathcal{A}[j:\{0,1\}]) = 1 - 0 = 1.$$

Inductive step. Suppose that the statement holds for k. We prove it for $k+1$. Let $j \in \{1,\dots,m\}$ be such that $A_j = \{2\}$. As above, from

$$\mathcal{A}[j:\{0,1,2\}] = \mathcal{A} \cup \mathcal{A}[j:\{0,1\}],$$

we obtain

$$d(\mathcal{A}) = d(\mathcal{A}[j:\{0,1,2\}]) - d(\mathcal{A}[j:\{0,1\}]).$$

By the inductive hypothesis, we obtain that $d(\mathcal{A}[j:\{0,1,2\}]) = 1$ (since $\mathcal{A}[j:\{0,1,2\}]$ has exactly k indices i such that $A_i = \{2\}$). By Lemma 7.5.1, we have that $d(\mathcal{A}[j:\{0,1\}]) = 0$. The thesis immediately follows. □

Finally, using the rules presented above, we obtain the final lemma.

Lemma 7.5.3. *If $A_i \in \{\{0\},\{1\},\{2\},\{0,1,2\}\}$, for all $i = 1,\dots,m$, then*

$$d(A_1 \times \ldots \times A_m) = (-1)^{\#\mathcal{I}},$$

where $\mathcal{I} := \big\{i \in \{1,\dots,m\} \colon A_i = \{1\}\big\}$.

Proof. We prove the statement by induction on the non-negative integer $k := \#\mathcal{I}$.

Case $k = 0$. If there is no $i \in \{1,\dots,m\}$ such that $A_i = \{1\}$, the thesis follows by Lemma 7.5.2.

Case $k = 1$. Suppose that there is exactly one index $i \in \{1,\dots,m\}$ such that $A_i = \{1\}$. Recalling the definition of $\mathcal{A}$ in (7.5.1), it is easy to see that

$$\mathcal{A}[i:\{0,1,2\}] = \mathcal{A}[i:\{0\}] \cup \mathcal{A} \cup \mathcal{A}[i:\{2\}].$$

Then, by the additivity property of d and Lemma 7.5.2, we obtain

$$\begin{aligned} d(\mathcal{A}) &= d(\mathcal{A}[i:\{0,1,2\}]) - d(\mathcal{A}[i:\{0\}]) - d(\mathcal{A}[i:\{2\}]) \\ &= 1 - 1 - 1 = -1 = (-1)^{\#\mathcal{I}}. \end{aligned}$$

Inductive step. Suppose that the statement holds when the set $\mathcal{I}$ has k elements. We prove it for $\#\mathcal{I} = k+1$. Let $i \in \{1,\ldots,m\}$ be such that $A_i = \{1\}$. By assumption there are $k+1$ indices with such a property. As above, from

$$\mathcal{A}[i:\{0,1,2\}] = \mathcal{A}[i:\{0\}] \cup \mathcal{A} \cup \mathcal{A}[i:\{2\}],$$

we obtain

$$d(\mathcal{A}) = d(\mathcal{A}[i:\{0,1,2\}]) - d(\mathcal{A}[i:\{0\}]) - d(\mathcal{A}[i:\{2\}]).$$

Now, all the sets $\mathcal{A}[i:\{0,1,2\}]$, $\mathcal{A}[i:\{0\}]$ and $\mathcal{A}[i:\{2\}]$ have precisely k indices j such that $A_j = \{1\}$. Then, by the inductive hypothesis, we obtain that

$$d(\mathcal{A}[i:\{0,1,2\}]) = d(\mathcal{A}[i:\{0\}]) = d(\mathcal{A}[i:\{2\}]) = (-1)^k$$

and hence

$$d(\mathcal{A}) = -(-1)^k = (-1)^{k+1} = (-1)^{\#\mathcal{I}}.$$

The thesis immediately follows. $\square$

We conclude this first part by showing how to apply this approach to obtain formula (7.3.4).

To any element $\mathcal{A} \in \mathbb{A}$ we associate an open set $\Omega_{\mathcal{A}}$ made up of the continuous functions $u\colon [0,T] \to \mathbb{R}$ which, for all $i = 1,\ldots,m$, satisfy

- $\max_{t\in I_i^+} |u(t)| < r$, if $A_i = \{0\}$;
- $r < \max_{t\in I_i^+} |u(t)| < \rho$, if $A_i = \{1\}$;
- $\rho < \max_{t\in I_i^+} |u(t)| < R$, if $A_i = \{2\}$;
- $\max_{t\in I_i^+} |u(t)| < \rho$, if $A_i = \{0,1\}$;
- either $\max_{t\in I_i^+} |u(t)| < r$ or $\rho < \max_{t\in I_i^+} |u(t)| < R$, if $A_i = \{0,2\}$;
- $r < \max_{t\in I_i^+} |u(t)| < R$, if $A_i = \{1,2\}$;
- $\max_{t\in I_i^+} |u(t)| < R$, if $A_i = \{0,1,2\}$.

By convention, we also set $\Omega_{\mathcal{A}} = \emptyset$ if there is an index $i \in \{1, \dots, m\}$ such that $A_i = \emptyset$. In this manner the set $\Omega_{\mathcal{A}}$ is well-defined for every $\mathcal{A} \in \mathbb{A}$.

Having fixed ρ, $\lambda > \lambda^*$, $r < \rho < R$ and $\mu > \mu^*(\lambda)$ as in Section 7.4, we have that the coincidence degree $D_L(L - N_{\lambda,\mu}, \Omega_{\mathcal{A}})$ is well-defined for every $\mathcal{A} \in \mathbb{A}$. Hence we set

$$d(\mathcal{A}) := D_L\big(L - N_{\lambda,\mu}, \Omega_{\mathcal{A}}\big).$$

Notice that the sets $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ introduced in (7.3.1) are of the form $\Omega_{\mathcal{A}}$ for $\mathcal{A}$ with $A_i = \{0\}$ for any $i \in \{1, \dots, m\} \setminus (\mathcal{I} \cup \mathcal{J})$, $A_i = \{0, 1\}$ for any $i \in \mathcal{I}$ and $A_i = \{0, 1, 2\}$ for any $i \in \mathcal{J}$. Similarly, the sets $\Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ introduced in (7.3.3) are of the form $\Omega_{\mathcal{A}}$ for $\mathcal{A}$ with $A_i = \{0\}$ for any $i \in \{1, \dots, m\} \setminus (\mathcal{I} \cup \mathcal{J})$, $A_i = \{1\}$ for any $i \in \mathcal{I}$ and $A_i = \{2\}$ for any $i \in \mathcal{J}$.

With these positions, the *additivity property* of the valuation d follows from the additivity property of the coincidence degree. Moreover, rules $(R1)$ and $(R2)$ are satisfied since they correspond to formula (7.3.2). Then, all the above lemmas on the valuation d apply and, in particular, Lemma 7.5.3 gives precisely formula (7.3.4). This completes the proof of Theorem 7.3.2.

7.5.2 Second argument

In this second part we present a different combinatorial argument, in the same spirit of the one adopted in the proof of Lemma 1.3.1 (see also Lemma 4.3.1).

Let r, ρ, R be three positive real numbers such that $0 < r < \rho < R$ and let $m \geq 1$ be an integer. Recalling the definitions of $\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ and $\Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}$ given in (7.3.1) and (7.3.3) respectively, we note that, for any pair of subset of indices $\mathcal{I}, \mathcal{J} \subseteq \{1, \dots, m\}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$, we have

$$\Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)} = \bigcup_{\substack{\mathcal{I}' \subseteq \mathcal{I} \cup \mathcal{J} \\ \mathcal{J}' \subseteq \mathcal{J} \\ \mathcal{I}' \cap \mathcal{J}' = \emptyset}} \Lambda^{\mathcal{I}',\mathcal{J}'}_{(r,\rho,R)} \cup \Upsilon^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}, \tag{7.5.2}$$

where

$$\Upsilon^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)} := \bigcup_{i \in \mathcal{I} \cup \mathcal{J}} \Big\{ u \in \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)} \colon \max_{I_i^+} |u| = r \Big\} \\ \cup \bigcup_{i \in \mathcal{J}} \Big\{ u \in \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)} \colon \max_{I_i^+} |u| = \rho \Big\}.$$

We notice that the union in (7.5.2) is disjoint, since $\Lambda^{\mathcal{I}',\mathcal{J}'}_{(r,\rho,R)} \cap \Lambda^{\mathcal{I}'',\mathcal{J}''}_{(r,\rho,R)} = \emptyset$, for $\mathcal{I}' \neq \mathcal{I}''$ or for $\mathcal{J}' \neq \mathcal{J}''$.

In the next lemma we observe that the set made up of the pairs $(\mathcal{I}, \mathcal{J})$ with $\mathcal{I}, \mathcal{J} \subseteq \{1, \dots, m\}$ such that $\mathcal{I} \cap \mathcal{J} = \emptyset$ has cardinality equal to 3^m.

Lemma 7.5.4. *Let S be a finite set with cardinality $\#S = m$. Then*

$$\#\{(A,B) \subseteq S \times S \colon A \cap B = \emptyset\} = 3^m.$$

Proof. Let us set $\mathcal{Q}(S) := \{(A,B) \subseteq S \times S \colon A \cap B = \emptyset\}$. We prove the statement by induction on $m = \#S$. If $m = 0$, clearly $S = \emptyset$ and thus $\mathcal{Q}(S) = \{(\emptyset,\emptyset)\}$. Then obviously $\#(\mathcal{Q}(S)) = 1 = 3^0$. Next, suppose that the formula holds for $m \in \mathbb{N}$, that is, if S' is a set of cardinality m, then $\#(\mathcal{Q}(S')) = 3^m$. We prove the formula for $m+1$.

Suppose $\#S = m+1$. Then $S \neq \emptyset$. Let $s_0 \in S$. For every couple $(A,B) \in \mathcal{Q}(S)$ one of the following three possibilities holds: $s_0 \notin A \cup B$; $s_0 \in A$ (so $s_0 \notin B$); $s_0 \in B$ (so $s_0 \notin A$). We observe that:

- the couples $(A,B) \in \mathcal{Q}(S)$ not containing s_0 are precisely the couples in $\mathcal{Q}(S \setminus \{s_0\})$, and so they are 3^m (by the inductive hypothesis);
- the couples $(A,B) \in \mathcal{Q}(S)$ such that $s_0 \in A$ are the couples of the form $(A' \cup \{s_0\}, B)$ with $(A',B) \in \mathcal{Q}(S \setminus \{s_0\})$, and so they are 3^m (by the inductive hypothesis);
- the couples $(A,B) \in \mathcal{Q}(S)$ such that $s_0 \in B$ are the couples of the form $(A, B' \cup \{s_0\})$ with $(A,B') \in \mathcal{Q}(S \setminus \{s_0\})$, and so they are 3^m (by the inductive hypothesis).

Then, we deduce that $\#(\mathcal{Q}(S)) = 3^m + 3^m + 3^m = 3^{m+1}$. The lemma is thus proved. □

Now we are in position to present the following result.

Lemma 7.5.5. *Let $\mathcal{I}, \mathcal{J} \subseteq \{1,\ldots,m\}$ be two subsets of indices (possibly empty) such that $\mathcal{I} \cap \mathcal{J} = \emptyset$. Suppose that the coincidence degrees $D_L\big(L - N_{\lambda,\mu}, \Omega^{\mathcal{I}',\mathcal{J}'}_{(r,\rho,R)}\big)$ and $D_L\big(L - N_{\lambda,\mu}, \Lambda^{\mathcal{I}',\mathcal{J}'}_{(r,\rho,R)}\big)$ are well-defined for all $\mathcal{I}' \subseteq \mathcal{I} \cup \mathcal{J}$ and for all $\mathcal{J}' \subseteq \mathcal{J}$ with $\mathcal{I}' \cap \mathcal{J}' = \emptyset$. Assume also*

$$D_L\big(L - N_{\lambda,\mu}, \Omega^{\mathcal{I}',\mathcal{J}'}_{(r,\rho,R)}\big) = 1, \quad \textit{if } \mathcal{I}' = \emptyset, \tag{7.5.3}$$

and

$$D_L\big(L - N_{\lambda,\mu}, \Omega^{\mathcal{I}',\mathcal{J}'}_{(r,\rho,R)}\big) = 0, \quad \textit{if } \mathcal{I}' \neq \emptyset. \tag{7.5.4}$$

Then

$$D_L\big(L - N_{\lambda,\mu}, \Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}\big) = (-1)^{\#\mathcal{I}}. \tag{7.5.5}$$

Proof. For simplicity of notation, in this proof we set

$$\Omega^{\mathcal{I},\mathcal{J}} = \Omega^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)} \quad \text{and} \quad \Lambda^{\mathcal{I},\mathcal{J}} = \Lambda^{\mathcal{I},\mathcal{J}}_{(r,\rho,R)}.$$

First of all, we underline that $\Omega^{\emptyset,\emptyset} = \Lambda^{\emptyset,\emptyset}$ and, in view of (7.5.3), we have that

$$D_L\big(L - N_{\lambda,\mu}, \Omega^{\emptyset,\emptyset}\big) = D_L\big(L - N_{\lambda,\mu}, \Lambda^{\emptyset,\emptyset}\big) = 1. \tag{7.5.6}$$

Hence the conclusion is trivially satisfied when $\mathcal{I} = \mathcal{J} = \emptyset$.

Now we consider two arbitrary subsets of indices (possibly empty) such that $\mathcal{I} \cup \mathcal{J} \neq \emptyset$ and $\mathcal{I} \cap \mathcal{J} = \emptyset$. We are going to prove formula (7.5.5) by using an inductive argument. Instead of a double induction on $\#\mathcal{I}$ and on $\#\mathcal{J}$, it seems more convenient to introduce the bijection

$$(i, j) \leftrightarrow i + (m+1)j$$

from the set of couples $(i, j) \in \{0, 1, \dots, m\}^2$ and the integers $0 \le n \le m(m+2)$, in order to reduce our argument to a single induction. More precisely, we define

$$n := \#\mathcal{I} + (m+1)\#\mathcal{J} \ge 1$$

and, for every integer k with $0 \le k \le n$, we introduce the property $\mathscr{P}(k)$ which reads as follows.

$\mathscr{P}(k)$: *The formula*

$$D_L(L - N_{\lambda,\mu}, \Lambda^{\mathcal{I}',\mathcal{J}'}) = (-1)^{\#\mathcal{I}'}$$

holds for each $\mathcal{I}' \subseteq \mathcal{I} \cup \mathcal{J}$ *and for each* $\mathcal{J}' \subseteq \mathcal{J}$ *such that* $\mathcal{I}' \cap \mathcal{J}' = \emptyset$ *and* $\#\mathcal{I}' + (m+1)\#\mathcal{J}' \le k$.

In this manner, if we are able to prove $\mathscr{P}(n)$, then (7.5.5) immediately follows.

Verification of $\mathscr{P}(0)$. See (7.5.6).

Verification of $\mathscr{P}(1)$. For $\mathcal{I}' = \mathcal{J}' = \emptyset$ the result is already proved in (7.5.6). If $\mathcal{I}' = \{i\}$, with $i \in \mathcal{I} \cup \mathcal{J}$, and $\mathcal{J}' = \emptyset$, by the additivity property of the coincidence degree and hypothesis (7.5.4), we have

$$\begin{aligned} D_L(L - N_{\lambda,\mu}, \Lambda^{\mathcal{I}',\mathcal{J}'}) &= D_L(L - N_{\lambda,\mu}, \Lambda^{\{i\},\emptyset}) \\ &= D_L(L - N_{\lambda,\mu}, \Omega^{\{i\},\emptyset} \setminus \Lambda^{\emptyset,\emptyset}) \\ &= D_L(L - N_{\lambda,\mu}, \Omega^{\{i\},\emptyset}) - D_L(L - N_{\lambda,\mu}, \Lambda^{\emptyset,\emptyset}) \\ &= 0 - 1 = -1 = (-1)^{\#\mathcal{I}'}. \end{aligned}$$

There are no other possible choices of $\mathcal{I}'$ and $\mathcal{J}'$ with $\#\mathcal{I}' + (m+1)\#\mathcal{J}' \le 1$ (since $m \ge 1$).

Verification of $\mathscr{P}(k-1) \Rightarrow \mathscr{P}(k)$, *for* $1 \le k \le n$. Assuming the validity of $\mathscr{P}(k-1)$ we have that the formula is true for every $\mathcal{I}' \subseteq \mathcal{I} \cup \mathcal{J}$ and for every $\mathcal{J}' \subseteq \mathcal{J}$ such that $\mathcal{I}' \cap \mathcal{J}' = \emptyset$ and $\#\mathcal{I}' + (m+1)\#\mathcal{J}' \le k - 1$. Therefore, in order to prove $\mathscr{P}(k)$, we have only to check that the formula is true for any possible choice of $\mathcal{I}' \subseteq \mathcal{I} \cup \mathcal{J}$ and $\mathcal{J}' \subseteq \mathcal{J}$ with $\mathcal{I}' \cap \mathcal{J}' = \emptyset$ and such that

$$\#\mathcal{I}' + (m+1)\#\mathcal{J}' = k. \tag{7.5.7}$$

We distinguish two cases: either $\mathcal{I}' = \emptyset$ or $\mathcal{I}' \neq \emptyset$. As a first instance, let $\mathcal{I}' = \emptyset$ and, in view of (7.5.7), suppose $\mathcal{J}' \neq \emptyset$ and $\#\mathcal{J}' = k/(m+1)$.

By formula (7.5.2), $\Omega^{\emptyset,\mathcal{J}'}$ can be written as the disjoint union

$$\Omega^{\emptyset,\mathcal{J}'} = \bigcup_{\substack{\mathcal{L}\subseteq\mathcal{J}'\\ \mathcal{K}\subseteq\mathcal{J}'\\ \mathcal{L}\cap\mathcal{K}=\emptyset}} \Lambda^{\mathcal{L},\mathcal{K}} \cup \Upsilon^{\emptyset,\mathcal{J}'} = \Lambda^{\emptyset,\mathcal{J}'} \cup \bigcup_{\substack{\mathcal{L}\subseteq\mathcal{J}'\\ \mathcal{K}\subsetneq\mathcal{J}'\\ \mathcal{L}\cap\mathcal{K}=\emptyset}} \Lambda^{\mathcal{L},\mathcal{K}} \cup \Upsilon^{\emptyset,\mathcal{J}'}.$$

We observe that there is no solution of $Lu = N_{\lambda,\mu}u$ with $u \in \Upsilon^{\emptyset,\mathcal{J}'}$, due to the fact that the degree is well-defined on the sets $\Lambda^{\mathcal{L},\mathcal{K}}$. Consequently, since $\#\mathcal{L} + (m+1)\#\mathcal{K} \leq k-1$ if $\mathcal{K} \subsetneq \mathcal{J}'$, by (7.5.3) and by the inductive hypothesis, we obtain

$$\begin{aligned} D_L(L - N_{\lambda,\mu}, \Lambda^{\emptyset,\mathcal{J}'}) &= D_L(L - N_{\lambda,\mu}, \Omega^{\emptyset,\mathcal{J}'}) - \sum_{\substack{\mathcal{L}\subseteq\mathcal{J}'\\ \mathcal{K}\subsetneq\mathcal{J}'\\ \mathcal{L}\cap\mathcal{K}=\emptyset}} D_L(L - N_{\lambda,\mu}, \Lambda^{\mathcal{L},\mathcal{K}}) \\ &= 1 - \sum_{\substack{\mathcal{L}\subseteq\mathcal{J}'\\ \mathcal{K}\subsetneq\mathcal{J}'\\ \mathcal{L}\cap\mathcal{K}=\emptyset}} (-1)^{\#\mathcal{L}}. \end{aligned}$$

Now we observe that

$$\sum_{\substack{\mathcal{L}\subseteq\mathcal{J}'\\ \mathcal{K}\subsetneq\mathcal{J}'\\ \mathcal{L}\cap\mathcal{K}=\emptyset}} (-1)^{\#\mathcal{L}} = \sum_{\mathcal{K}\subsetneq\mathcal{J}'} \sum_{\mathcal{L}\subseteq\mathcal{J}'\setminus\mathcal{K}} (-1)^{\#\mathcal{L}} = 0,$$

due to the fact that in a finite set there are so many subsets with even cardinality as there are with odd cardinality. Thus we conclude that

$$D_L(L - N_{\lambda,\mu}, \Lambda^{\emptyset,\mathcal{J}'}) = 1 = (-1)^{\#\mathcal{I}'}.$$

As a second instance, let $\mathcal{I}' \neq \emptyset$. Using (7.5.2), we can write $\Omega^{\mathcal{I}',\mathcal{J}'}$ as the disjoint union

$$\Omega^{\mathcal{I}',\mathcal{J}'} = \bigcup_{\substack{\mathcal{L}\subseteq\mathcal{I}'\cup\mathcal{J}'\\ \mathcal{K}\subseteq\mathcal{J}'\\ \mathcal{L}\cap\mathcal{K}=\emptyset}} \Lambda^{\mathcal{L},\mathcal{K}} \cup \Upsilon^{\mathcal{I}',\mathcal{J}'} = \Lambda^{\mathcal{I}',\mathcal{J}'} \cup \bigcup_{\substack{\mathcal{L}\subseteq\mathcal{I}'\cup\mathcal{J}'\\ \mathcal{K}\subseteq\mathcal{J}'\\ \mathcal{L}\cap\mathcal{K}=\emptyset\\ (\mathcal{L},\mathcal{K})\neq(\mathcal{I}',\mathcal{J}')}} \Lambda^{\mathcal{L},\mathcal{K}} \cup \Upsilon^{\mathcal{I}',\mathcal{J}'}.$$

We observe that there is no solution of $Lu = N_{\lambda,\mu}u$ with $u \in \Upsilon^{\mathcal{I}',\mathcal{J}'}$, due to the fact that the degree is well-defined on the sets $\Lambda^{\mathcal{L},\mathcal{K}}$. Consequently, since $\#\mathcal{L} + (m+1)\#\mathcal{K} \leq k-1$, if $\mathcal{K} \subsetneq \mathcal{J}'$ or if $\mathcal{K} = \mathcal{J}'$ and $\mathcal{L} \subsetneq \mathcal{I}'$, by

(7.5.4) and by the inductive hypothesis, we obtain

$$\begin{aligned}
&D_L(L - N_{\lambda,\mu}, \Lambda^{\mathcal{I}',\mathcal{J}'}) = \\
&= D_L(L - N_{\lambda,\mu}, \Omega^{\mathcal{I}',\mathcal{J}'}) - \sum_{\substack{\mathcal{L}\subseteq\mathcal{I}'\cup\mathcal{J}' \\ \mathcal{K}\subseteq\mathcal{J}' \\ \mathcal{L}\cap\mathcal{K}=\emptyset \\ (\mathcal{L},\mathcal{K})\neq(\mathcal{I}',\mathcal{J}')}} D_L(L - N_{\lambda,\mu}, \Lambda^{\mathcal{L},\mathcal{K}}) \\
&= 0 - \sum_{\substack{\mathcal{L}\subseteq\mathcal{I}'\cup\mathcal{J}' \\ \mathcal{K}\subseteq\mathcal{J}' \\ \mathcal{L}\cap\mathcal{K}=\emptyset \\ (\mathcal{L},\mathcal{K})\neq(\mathcal{I}',\mathcal{J}')}} (-1)^{\#\mathcal{L}} = (-1)^{\#\mathcal{I}'} - \sum_{\substack{\mathcal{L}\subseteq\mathcal{I}'\cup\mathcal{J}' \\ \mathcal{K}\subseteq\mathcal{J}' \\ \mathcal{L}\cap\mathcal{K}=\emptyset}} (-1)^{\#\mathcal{L}} = (-1)^{\#\mathcal{I}'},
\end{aligned}$$

observing, as above, that

$$\sum_{\substack{\mathcal{L}\subseteq\mathcal{I}'\cup\mathcal{J}' \\ \mathcal{K}\subseteq\mathcal{J}' \\ \mathcal{L}\cap\mathcal{K}=\emptyset}} (-1)^{\#\mathcal{L}} = \sum_{\mathcal{K}\subseteq\mathcal{J}'} \sum_{\mathcal{L}\subseteq\mathcal{I}'\cup(\mathcal{J}'\setminus\mathcal{K})} (-1)^{\#\mathcal{L}} = 0.$$

Then $\mathscr{P}(k)$ is proved and the lemma follows. $\square$

Now, since (7.5.5) is exactly formula (7.3.4), in order to complete the proof of Theorem 7.3.2 we have only to check that the degrees are well-defined and assumptions (7.5.3) and (7.5.4) in the above combinatorial lemma are satisfied. All these requests are obviously guaranteed by the discussion in Section 7.3.1 and by formula (7.3.2). Then Lemma 7.5.5 applies and this completes the proof of Theorem 7.3.2.

7.6 General properties for globally defined solutions and some a posteriori bounds

In this section we focus our attention on non-negative solutions of $(\mathscr{E}_{\lambda,\mu})$ which are defined for all $t \in \mathbb{R}$. On one hand, we show how some computations in the proofs of the technical lemmas in Section 7.4 are still valid in this setting. This will be useful in view of further applications of Theorem 7.1.1 described in Chapter 8. On the other hand, we provide some additional information for the solutions when $\mu \to +\infty$.

In order to avoid repetitions, throughout this section we assume that the constants $\rho > 0$, $\lambda > \lambda^*$, $0 < r < \rho < R$ and $\mu > \mu^*(\lambda)$ are all fixed as in Section 7.4.2 and Section 7.4.5. We stress the fact that even if these constants have been determined with respect to the T-periodic problem, all the results below are valid for arbitrary globally defined non-negative solutions.

The first result concerns the behavior of the solutions with respect to the constant R.

Proposition 7.6.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). If $w(t)$ is any non-negative solution of $(\mathscr{E}_{\lambda,\mu})$ with $\sup_{t\in\mathbb{R}} w(t) \leq R$, then $w(t) < R$ for all $t \in \mathbb{R}$.*

Proof. Suppose by contradiction that there exists a point $t^* \in \mathbb{R}$ such that $w(t^*) = \max_{t\in\mathbb{R}} w(t) = R$. Let also $\ell \in \mathbb{Z}$ be such that $t^* \in [\ell T, (\ell+1)T]$. In this case, thanks to the T-periodicity of the weight coefficient $a_{\lambda,\mu}(t)$, the function $u(t) := w(t + \ell T)$ is still a (non-negative) solution of $(\mathscr{E}_{\lambda,\mu})$ with $\max_{t\in[0,T]} u(t) = u(t^* - \ell T) = w(t^*) = R$. From now on, the proof uses exactly the same argument as for the discussion of the case (a_3) in the verification of (H_1) in Section 7.4.3 (for $\alpha = 0$) and the same contradiction can be achieved. $\square$

A straightforward application of Lemma 7.4.1 gives the following result (the obvious proof is omitted).

Proposition 7.6.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). If $w(t)$ is any non-negative solution of $(\mathscr{E}_{\lambda,\mu})$ and $I^+_{i,\ell} := I^+_i + \ell T$ is any interval of the real line where $a(t) \succ 0$, then $\max_{t\in I^+_{i,\ell}} w(t) \neq \rho$.*

The next result concerns the behavior of the solutions with respect to the constant r.

Proposition 7.6.3. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). If $w(t)$ is any non-negative solution of $(\mathscr{E}_{\lambda,\mu})$ with $\sup_{t\in\mathbb{R}} w(t) \leq R$ and $I^+_{i,\ell} := I^+_i + \ell T$ is any interval of the real line where $a(t) \succ 0$, then $\max_{t\in I^+_{i,\ell}} w(t) \neq r$.*

Proof. We follow the same scheme as for Proposition 7.6.1. Suppose by contradiction that there exists $t^* \in I^+_{i,\ell}$ such that $w(t^*) = \max_{t\in I^+_{i,\ell}} w(t) = r$. The function $u(t) := w(t + \ell T)$ is a non-negative solution of $(\mathscr{E}_{\lambda,\mu})$ with $\max_{t\in I^+_i} u(t) = w(t^*) = r$. From now on, the proof uses exactly the same argument as for the discussion of the case (a_1) in the verification of (H_1) in Section 7.4.3 (for $\alpha = 0$) and the same contradiction can be achieved, in the sense that we find a point where $w(t) > R$. $\square$

We now focus on some properties of globally defined non-negative solutions of $(\mathscr{E}_{\lambda,\mu})$ when $\mu \to +\infty$. The first result in this direction concerns the behavior on the intervals where $a(t) \succ 0$: roughly speaking, any "very small" solution becomes arbitrarily small as $\mu \to +\infty$.

Proposition 7.6.4. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function satisfying (a_*). Then for every ε with $0 < \varepsilon \leq r$ there exists $\mu^\star_\varepsilon \geq \mu^*(\lambda)$ such that for any fixed $\mu > \mu^\star_\varepsilon$ the following holds: if $w(t)$ is any non-negative solution of $(\mathscr{E}_{\lambda,\mu})$ with $\sup_{t\in\mathbb{R}} w(t) \leq R$ and $\max_{t\in I^+_{i,\ell}} w(t) \leq r$, where $I^+_{i,\ell} := I^+_i + \ell T$ is any interval of the real line where $a(t) \succ 0$, then $\max_{t\in I^+_{i,\ell}} w(t) < \varepsilon$.*

Proof. Repeating the same approach as in the proof of the previous propositions and using the T-periodicity of the weight, without loss of generality, we can restrict ourselves to the analysis of the non-negative solution $w(t)$ on an interval I^+_i, for $i = 1, \ldots, m$.

The proof uses exactly the same argument as for the discussion of the case (a_1) in the verification of (H_1) in Section 7.4.3 (for $\alpha = 0$). Let $\varepsilon \in \,]0, r]$. By contradiction, suppose that there exists a non-negative solution $w(t)$ of $(\mathscr{E}_{\lambda,\mu})$ such that $\sup_{t\in\mathbb{R}} w(t) \leq R$ and $\max_{t\in I^+_i} w(t) = \varepsilon_0 \in [\varepsilon, r]$. Consider at first the case $w'(\sigma_i) \geq 0$. Recalling condition (7.4.10), by Lemma 7.4.2 (with $\vartheta = 1$ and $d = \varepsilon_0$), we have that

$$w(\sigma_{i+1}) \geq \mu\, \varepsilon_0 \frac{\gamma(\varepsilon_0)}{2} \|A_i\|.$$

Observing that

$$\gamma(\varepsilon_0) = \min_{\frac{\varepsilon_0}{2} \leq s \leq \varepsilon_0} \frac{g(s)}{s} \geq \min_{\frac{\varepsilon}{2} \leq s \leq r} \frac{g(s)}{s} =: \gamma^*(\varepsilon, r) > 0$$

and thus taking

$$\mu > \mu^{\star,+}_i(\varepsilon) := \frac{2R}{\varepsilon \gamma^*(\varepsilon, r) \|A_i\|}$$

we obtain $w(\sigma_{i+1}) > R$, a contradiction. On the other hand, if $w'(\sigma_i) < 0$, by the concavity of $w(t)$ in I^+_i we have that $w'(\tau_i) < 0$. In this case we reach the contradiction $w(\tau_{i-1}) > R$ using Lemma 7.4.3 (with $\vartheta = 1$ and $d = \varepsilon_0$) and taking

$$\mu > \mu^{\star,-}_i(\varepsilon) := \frac{2R}{\varepsilon \gamma^*(\varepsilon, r) \|B_{i-1}\|}$$

(if $i = 1$, we count cyclically and consider the interval I^-_0 as I^+_m). In conclusion, taking

$$\mu > \mu^\star_\varepsilon := \max_{i=1,\ldots,m} \bigl\{ \mu^{\star,+}_i(\varepsilon), \mu^{\star,-}_i(\varepsilon), \mu^*(\lambda) \bigr\}$$

the proposition follows. □

Our final result in this section concerns the behavior of non-negative solutions to $(\mathscr{E}_{\lambda,\mu})$ on the intervals where $a(t) \prec 0$. With reference to condition (a_*), for technical reasons we further suppose that $a(t) \not\equiv 0$ *in each right* *neighborhood of* τ_i *and in each left neighborhood of* σ_{i+1}. Such an assumption does not require any new constraint on the weight function, but just a more careful selection of the points τ_i and σ_{i+1}. What we mean is that for a weight function $a(t)$ satisfying (a_*) the way to select the intervals I_i^+ and I_i^- may be not univocal. Indeed, we could have an interval J where $a(t) \equiv 0$ between an interval of positivity and an interval of negativity for the weight. Up to now the decision whether incorporate such an interval J as a part of I_i^+ or I_i^- was completely arbitrary. On the contrary, for the next result, we prefer to consider an interval as J as a part of I_i^+. In any case, we can allow a closed interval where $a(t) \equiv 0$ to lie in the interior of one of the I_i^-. With this in mind, we can now present our next result.

Proposition 7.6.5. *Let* $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous function satisfying* (g_*)*,* (g_0) *and* (g_∞)*. Let* $a\colon \mathbb{R} \to \mathbb{R}$ *be a* T*-periodic locally integrable function satisfying* (a_*)*. Then for every* ε *with* $0 < \varepsilon \leq r$ *there exists* $\mu^{\star\star}_\varepsilon \geq \mu^*(\lambda)$ *such that for any fixed* $\mu > \mu^{\star\star}_\varepsilon$ *the following holds: if* $w(t)$ *is any non-negative solution of* $(\mathscr{E}_{\lambda,\mu})$ *with* $\sup_{t\in\mathbb{R}} w(t) \leq R$ *and* $I^-_{i,\ell} := I_i^- + \ell T$ *is any interval of the real line where* $a(t) \prec 0$*, then* $\max_{t\in I^-_{i,\ell}} w(t) < \varepsilon$*.*

Proof. Without loss of generality, we can restrict ourselves to the analysis of the non-negative solution $w(t)$ on an interval I_i^-, for $i = 1, \ldots, m$.

Given $\varepsilon \in \,]0, r]$, we consider the values of the solution $w(t)$ at the boundary of the interval I_i^-, for an arbitrary but fixed index $i \in \{1, \ldots, m\}$. If $w(\tau_i) < \varepsilon$ and $w(\sigma_{i+i}) < \varepsilon$, then, by convexity, $w(t) < \varepsilon$ for all $t \in I_i^-$ and we have nothing to prove. Therefore, we discuss only the cases when $w(\tau_i) \geq \varepsilon$ or $w(\sigma_{i+1}) \geq \varepsilon$. We are going to show that this cannot occur if μ is sufficiently large. Accordingly, suppose that $w(\tau_i) \geq \varepsilon$. Knowing that $w(t) \leq R$ on the whole real line, in particular in the interval I_i^+, we easily find that there is at least a point $t_0 \in I_i^+$ such that $|w'(t_0)| \leq R/|I_i^+|$. On the other hand, equation $(\mathscr{E}_{\lambda,\mu})$ on I_i^+ reads as $w'' = -\lambda a^+(t) g(w)$, so that an integration on $[t_0, \tau_i]$ yields

$$w'(\tau_i) = w'(t_0) - \lambda \int_{t_0}^{\tau_i} a^+(t) g(w(t))\, dt \geq -\frac{R}{|I_i^+|} - \lambda \|a\|_{+,i}\, g^*(R) =: -\kappa_i,$$

where the constants $\|a\|_{+,i}$ and $g^*(R)$ are those defined at the beginning of Section 7.4. The convexity of $w(t)$ in I_i^- guarantees that $w'(t) \geq -\kappa_i$ for all $t \in I_i^-$. Hence, if we fix a constant $\delta_i > 0$ with $\tau_i + \delta_i < \sigma_{i+1}$ and such that $\delta_i < \varepsilon/(2\kappa_i)$, it is clear that $w(t) \geq \varepsilon/2$ for all $t \in [\tau_i, \tau_i + \delta_i]$. On the interval I_i^- equation $(\mathscr{E}_{\lambda,\mu})$ reads as $w'' = \mu a^-(t) g(w)$, so that an integration on $[\tau_i, t] \subseteq [\tau_i, \tau_i + \delta_i]$ yields

$$w'(t) = w'(\tau_i) + \mu \int_{\tau_i}^{t} a^-(\xi) g(w(\xi))\, d\xi \geq -\kappa_i + \mu A_i(t)\, g_*(\varepsilon/2, R),$$

where the function $A_i(t)$ and the constant $g_*(\varepsilon/2, R)$ are defined at the beginning of Section 7.4. Since we have supposed that $a^-(t)$ is not identically zero in each right neighborhood of τ_i, we know that the function $A_i(t)$ is strictly positive for each $t \in]\tau_i, \sigma_{i+1}]$. Then, integrating the above inequality on $[\tau_i, \tau_i + \delta_i]$, we obtain

$$w(\tau_i + \delta_i) = w(\tau_i) + \int_{\tau_i}^{\tau_i+\delta_i} w'(t)\, dt \geq \varepsilon - \kappa_i \delta_i + \mu\, g_*(\varepsilon/2, R) \int_{\tau_i}^{\tau_i+\delta_i} A_i(t)\, dt.$$

This latter inequality implies $w(\tau_i + \delta_i) > R$ (and hence a contradiction) for

$$\mu > \mu_i^{\text{left}}(\varepsilon) := \frac{R + \kappa_i \delta_i}{g_*(\varepsilon/2, R) \int_{\tau_i}^{\tau_i+\delta_i} A_i(t)\, dt}.$$

On the other hand, if we suppose that $w(\sigma_{i+1}) \geq \varepsilon$, then by the same argument we have

$$w'(\sigma_{i+1}) \leq \kappa_{i+i} := \frac{R}{|I_{i+1}^+|} + \lambda \|a\|_{+,i+1} g^*(R)$$

(if $i = m$, we count cyclically and consider the interval I_{m+1}^+ as I_1^+). As before, we fix a constant $\delta_{i+1} > 0$ with $\sigma_{i+1} - \delta_{i+1} > \tau_i$ and such that $\delta_{i+1} < \varepsilon/(2\kappa_{i+1})$, so that $u(t) \geq \varepsilon/2$ for all $t \in [\sigma_{i+1} - \delta_{i+1}, \sigma_{i+1}]$. An integration of the equation on $[t, \sigma_{i+1}]$ yields

$$w'(t) \leq \kappa_{i+1} - \mu B_i(t)\, g_*(\varepsilon/2, R).$$

Since we have supposed that $a^-(t)$ is not identically zero in each left neighborhood of σ_{i+1}, we know that the function $B_i(t)$ is strictly positive for each $t \in [\tau_i, \sigma_{i+1}[$. Then, integrating the above inequality on $[\sigma_{i+1} - \delta_{i+1}, \sigma_{i+1}]$, we obtain

$$w(\sigma_{i+1} - \delta_{i+1}) \geq \varepsilon - \kappa_{i+1}\delta_{i+1} + \mu g_*(\varepsilon/2, R) \int_{\sigma_{i+1}-\delta_{i+1}}^{\sigma_{i+1}} B_i(t)\, dt.$$

This latter inequality implies $w(\sigma_{i+1} - \delta_{i+1}) > R$ (and hence a contradiction) for

$$\mu > \mu_i^{\text{right}}(\varepsilon) := \frac{R + \kappa_{i+1}\delta_{i+1}}{g_*(\varepsilon/2, R) \int_{\sigma_{i+1}-\delta_{i+1}}^{\sigma_{i+1}} B_i(t)\, dt}.$$

In conclusion, for

$$\mu > \mu_\varepsilon^{\star\star} := \max_{i=1,\ldots,m} \bigl\{ \mu_i^{\text{left}}(\varepsilon), \mu_i^{\text{right}}(\varepsilon), \mu^*(\lambda) \bigr\} \tag{7.6.1}$$

our result is proved. □

We conclude this section by briefly describing, as typical in singular perturbation problems, the limit behavior of positive solutions of $(\mathscr{E}_{\lambda,\mu})$ for $\mu \to +\infty$ (compare with [17], where a similar discussion was performed in the superlinear case). We focus our attention on the solutions found in Theorem 7.1.1 for the T-periodic problem; however, similar considerations are valid for Dirichlet and Neumann boundary conditions, as well as for globally defined positive solutions.

Let us fix a non-null string $\mathcal{S} \in \{0,1,2\}^m$. Theorem 7.1.1 ensures the existence (in general, not the uniqueness) of a positive T-periodic solution of $(\mathscr{E}_{\lambda,\mu})$ associated with it, if $\lambda > \lambda^*$ and $\mu > \mu^*(\lambda)$; in order to emphasize its dependence on the parameter μ, we will denote it by $u_\mu(t)$. Then, as a direct consequence of Proposition 7.6.4 and Proposition 7.6.5, we have that $u_\mu(t)$ converges uniformly to zero both in the intervals I_i^+ with $\mathcal{S}_i = 0$ as well as in the intervals I_i^-, for $\mu \to +\infty$. As for the behavior of $u_\mu(t)$ on the intervals I_i^+ such that $\mathcal{S}_i \in \{1,2\}$, with a standard compactness argument (based on the facts that $0 \leq u_\mu(t) \leq R$ and that equation $(\mathscr{E}_{\lambda,\mu})$ is independent on the parameter μ in the intervals I_i^+), we can prove that the family $\{u_\mu|_{I_i^+}\}_{\mu>\mu^*(\lambda)}$ is relatively compact in $\mathcal{C}(I_i^+)$ and that each of its cluster points $u_\infty(t)$ has to be a non-negative solution of $u''+\lambda a^+(t)g(u) = 0$ on I_i^+. We claim that $u_\infty(t)$ is actually a positive solution, satisfies Dirichlet boundary condition on I_i^+ and is "small" if $\mathcal{S}_i = 1$ and "large" if $\mathcal{S}_i = 2$. Indeed, the first assertion follows from the fact that, passing to the limit, $r \leq \max_{t\in I_i^+} u_\infty(t) \leq \rho$ if $\mathcal{S}_i = 1$ and $\rho \leq \max_{t\in I_i^+} u_\infty(t) \leq R$ if $\mathcal{S}_i = 2$. As for Dirichlet boundary condition on I_i^+, this is a consequence of $u_\mu(t) \to 0$ on every interval of negativity. Finally, using Lemma 7.4.1, we infer $r \leq \max_{t\in I_i^+} u_\infty(t) < \rho$ if $\mathcal{S}_i = 1$ (that is, $u_\infty(t)$ is "small") and $\rho < \max_{t\in I_i^+} u_\infty(t) \leq R$ if $\mathcal{S}_i = 2$ (that is, $u_\infty(t)$ is "large").

In conclusion, up to subsequences, we have that $u_\mu(t) \to u_\infty(t)$ uniformly for $\mu \to +\infty$, with $u_\infty(t)$ a function made up of "null", "small" and "large" solutions of Dirichlet problems in the intervals I_i^+ (depending on $\mathcal{S}_i = 0,1,2$ respectively) connected by null functions in I_i^-. See Figure 7.3 for a numerical simulation. Notice that this discussion is simplified whenever we are able to prove that each Dirichlet problem associated with $u'' + \lambda a^+(t)g(u) = 0$ on I_i^+ has *exactly* two positive solutions; indeed, in this case every string $\mathcal{S} \in \{0,1,2\}^m$ uniquely determines a limit profile $u_\infty(t)$ and $u_\mu(t) \to u_\infty(t)$ uniformly, without the need of taking subsequences (even if $u_\mu(t)$ could be not unique in the class of positive solutions to $(\mathscr{E}_{\lambda,\mu})$ associated with $\mathcal{S}$).

7.7 Related results

In this final section we briefly describe some results which can be obtained by minor modifications of the arguments developed along this chapter.

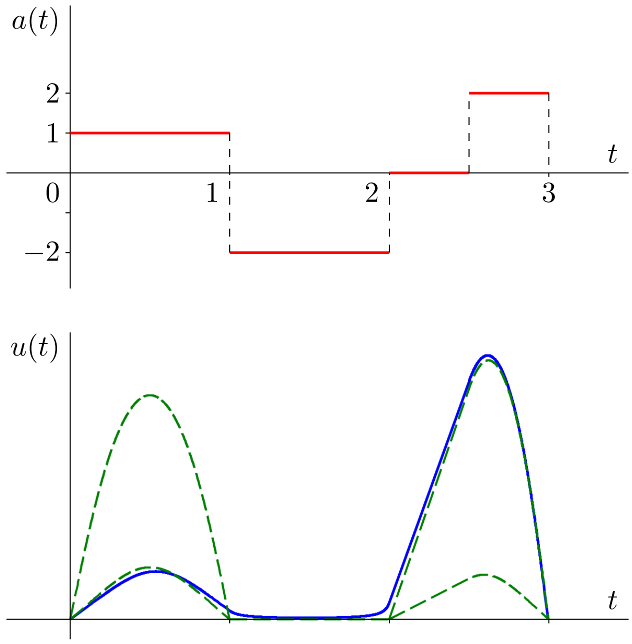

Figure 7.3: The lower part of the figure shows a positive solution of equation ($\mathscr{E}_{\lambda,\mu}$) for the super-sublinear nonlinearity $g(s) = \arctan(s^3)$, for $s \geq 0$, and Dirichlet boundary conditions. For this simulation we have chosen the interval $[0, T]$ with $T = 3$ and the weight function $a_{\lambda,\mu}(t)$ with $a(t)$ having a stepwise graph as represented in the upper part of the figure. First, with a dashed line we have drawn the Dirichlet solutions ("small" and "large") on the intervals $[0, 1]$ and $[2, 3]$. Then, for $\lambda = 20$ and $\mu = 10000$, we have exhibited a solution of the form "small" in the first interval of positivity $[0, 1]$ and "large" in the second interval of positivity $[2, 3]$. Such a solution is very close to the limit profile for the class of solutions associated with the string $(1, 2)$, which is made by a "small" solution of the Dirichlet problem in $[0, 1]$ and a "large" solution of the Dirichlet problem in $[2, 3]$ connected by the null solution in $[1, 2]$. Notice that, for the given weight function which is identically zero on the interval $[2, 2.5]$ separating the negative and the positive hump, the solution is very small (and the limit profile is zero) only in the interval $[1, 2]$ where the weight is negative. This is in complete accordance with Proposition 7.6.5 and the choice of the endpoints of the intervals $I_i^{\pm}$.

7.7.1 The non-Hamiltonian case

One of the advantages in obtaining results of existence/multiplicity with a topological degree technique lies in the fact that the degree is stable with respect to small perturbations of the operator. Such a remark, when applied to equation ($\mathscr{E}_{\lambda,\mu}$), allows us to establish the same result for the equation

$$u'' + cu' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0, \tag{7.7.1}$$

where $c \in \mathbb{R}$ and $c \neq 0$. More precisely, in the same setting of Theorem 7.1.1, once $\lambda > \lambda^*$ and $\mu > \mu^*(\lambda)$ are fixed, there exists a constant $\varepsilon = \varepsilon(\lambda, \mu) > 0$

such that the statement of the theorem is still true for any $c \in \mathbb{R}$ with $|c| < \varepsilon$.

A possibly interesting question which naturally arises is whether these multiplicity results are still valid for an arbitrary $c \in \mathbb{R}$. In the superlinear indefinite case, besides the results presented in the first part of the present thesis, Capietto, Dambrosio and Papini in [52] produced such kind of results for sign-changing (oscillatory) solutions. Concerning our super-sublinear setting, all the abstract approach and the strategy for the proof work exactly the same for the linear differential operator $u \mapsto -u'' - cu'$ for an arbitrary $c \in \mathbb{R}$ (see Remark 7.2.1 and Remark 7.3.1). Thus, the only problem in extending all our results of the previous sections to equation (7.7.1) comes from some additional difficulties related to the technical estimates. In particular, we have often exploited the convexity of the solutions in the intervals I^{-}_{i} and their concavity in the intervals I^{+}_{i}. In Chapter 6 we have proved the existence of two positive T-periodic solutions to equation (7.7.1) by effectively replacing the convexity/concavity properties with suitable monotonicity properties for the map $t \mapsto e^{ct}u'(t)$. Similar tricks have been successfully applied in Chapter 4 to obtain multiplicity results for equation (7.7.1) with a superlinear $g(s)$. It is therefore quite reasonable that these arguments can be adapted to our case. However, due to the lengthy and complex technical details required in Section 7.4, we prefer to skip further investigations in this direction.

7.7.2 Neumann and Dirichlet boundary conditions

As anticipated, versions of Theorem 7.1.1 for both Neumann and Dirichlet boundary conditions can be given. In these cases, we can consider a slightly more general sign condition for the measurable weight function $a\colon [0,T] \to \mathbb{R}$, which reads as follows:

(a_{**}) *there exist* $2m+2$ *points (with* $m \geq 1$*)*

$$0 = \tau_0 \leq \sigma_1 < \tau_1 < \ldots < \sigma_i < \tau_i < \ldots < \sigma_m < \tau_m \leq \sigma_{m+1} = T$$

such that $a(t) \succ 0$ *on* $[\sigma_i, \tau_i]$, *for* $i = 1, \ldots, m$, *and* $a(t) \prec 0$ *on* $[\tau_i, \sigma_{i+1}]$, *for* $i = 0, \ldots, m$.

This means that $a(t)$ has m positive humps $[\sigma_i, \tau_i]$ $(i = 1, \ldots, m)$ separated by $m-1$ negative ones $[\tau_i, \sigma_{i+1}]$ $(i = 1, \ldots, m-1)$; in addition, $a(t)$ might have one/two further negativity intervals, precisely an initial one $[\tau_0, \sigma_1] = [0, \sigma_1]$ or/and a final one $[\tau_m, \sigma_{m+1}] = [\tau_m, T]$ (compare with Remark 7.1.1). In this setting, the following result holds true.

Theorem 7.7.1. *Let* $g\colon \mathbb{R}^{+} \to \mathbb{R}^{+}$ *be a continuous function satisfying* (g_*), (g_0) *and* (g_∞). *Let* $a\colon [0,T] \to \mathbb{R}$ *be an integrable function satisfying* (a_{**}). *Then there exists* $\lambda^* > 0$ *such that for each* $\lambda > \lambda^*$ *there exists* $\mu^*(\lambda) > 0$

such that for each $\mu > \mu^*(\lambda)$ *the Neumann problem*

$$\begin{cases} u'' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0 \\ u'(0) = u'(T) = 0 \end{cases}$$

has at least $3^m - 1$ *positive solutions. The same result holds for the Dirichlet problem*

$$\begin{cases} u'' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0 \\ u(0) = u(T) = 0. \end{cases}$$

Of course, such solutions can again be coded via a non-null string $\mathcal{S} \in \{0,1,2\}^m$ as described in Theorem 7.1.1. We also remark that, as usual, a positive solution of the Dirichlet problem is a function $u(t)$ solving the equation and such that $u(0) = u(T) = 0$ and $u(t) > 0$ for any $t \in \,]0,T[$.

For the proof of Theorem 7.7.1, we rely on the abstract setting of Section 7.2 (with the changes underlined in Remark 7.2.1) and on the general strategy presented in Section 7.3.1. The key point is then the verification of the assumptions of Lemma 7.3.1 and Lemma 7.3.2 (in the slightly modified versions described in Remark 7.3.1). To this end, we can take advantage of the technical estimates developed in Section 7.4.1 (which indeed are independent of the boundary conditions) and we can prove the result with minor modifications of the arguments in the remaining part of Section 7.4.

Finally, we observe that the same result can be obtained for positive solutions of equation $(\mathscr{E}_{\lambda,\mu})$ satisfying the mixed boundary conditions $u(0) = u'(T) = 0$ or $u'(0) = u(T) = 0$ (compare with Section 1.4.4 and Section 2.8).

7.7.3 Radially symmetric solutions

As a standard consequence of Theorem 7.7.1, we can produce multiplicity results for radially symmetric positive solutions to elliptic BVPs on an annulus.

More precisely, let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^N$ (for $N \geq 2$) and let

$$\Omega := \big\{ x \in \mathbb{R}^N \colon R_1 < \|x\| < R_2 \big\}$$

be an open annular domain, with $0 < R_1 < R_2$. We deal with the elliptic partial differential equation

$$-\Delta u = \big(\lambda q^+(x) - \mu q^-(x)\big) g(u) \quad \text{in } \Omega \tag{7.7.2}$$

together with Neumann boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega \tag{7.7.3}$$

or Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega. \tag{7.7.4}$$

For simplicity, we look for classical solutions to (7.7.2) (namely, $u \in \mathcal{C}^2(\overline{\Omega})$) and, accordingly, we assume that $q\colon \overline{\Omega} \to \mathbb{R}$ is a *continuous* function. Moreover, in order to transform the partial differential equation (7.7.2) into a second order ordinary differential equation of the form $(\mathscr{E}_{\lambda,\mu})$ so as to apply Theorem 7.7.1, we also require that $q(x)$ is a radially symmetric function, i.e. there exists a continuous function $\mathcal{Q}\colon [R_1, R_2] \to \mathbb{R}$ such that

$$q(x) = \mathcal{Q}(\|x\|), \quad \forall\, x \in \overline{\Omega}. \tag{7.7.5}$$

We also set

$$\mathcal{Q}_{\lambda,\mu}(r) := \lambda \mathcal{Q}^+(r) - \mu \mathcal{Q}^-(r), \quad r \in [R_1, R_2],$$

where, as usual, $\lambda, \mu > 0$.

Looking for radially symmetric (classical) solutions to (7.7.2), i.e. solutions of the form $u(x) = \mathcal{U}(\|x\|)$ where $\mathcal{U}(r)$ is a scalar function defined on $[R_1, R_2]$, we transform equation (7.7.2) into

$$\big(r^{N-1}\mathcal{U}'\big)' + r^{N-1}\mathcal{Q}_{\lambda,\mu}(r) g(\mathcal{U}) = 0. \tag{7.7.6}$$

Moreover, the boundary conditions (7.7.3) and (7.7.4) become

$$\mathcal{U}(R_1) = \mathcal{U}(R_2) = 0 \quad \text{and} \quad \mathcal{U}'(R_1) = \mathcal{U}'(R_2) = 0,$$

respectively. Via the change of variable described in Section C.2

$$t = h(r) := \int_{R_1}^{r} \xi^{1-N}\, d\xi$$

and the positions

$$T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi, \quad r(t) := h^{-1}(t) \quad \text{and} \quad v(t) = \mathcal{U}(r(t)),$$

we can further convert (7.7.6) and the corresponding boundary conditions into the Neumann and Dirichlet problems

$$\begin{cases} v'' + a_{\lambda,\mu}(t) g(v) = 0 \\ v'(0) = v'(T) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} v'' + a_{\lambda,\mu}(t) g(v) = 0 \\ v(0) = v(T) = 0, \end{cases}$$

respectively, where

$$a(t) := r(t)^{2(N-1)} \mathcal{Q}(r(t)), \quad t \in [0, T].$$

In this setting, Theorem 7.7.1 gives the following result. The straightforward proof is omitted.

Theorem 7.7.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $\mathcal{Q}\colon [R_1, R_2] \to \mathbb{R}$ be a continuous function satisfying*

$(\mathcal{Q}_{**})$ *there exist* $2m+2$ *points (with* $m \geq 1$*)*

$$R_1 = \tau_0 \leq \sigma_1 < \tau_1 < \ldots < \sigma_i < \tau_i < \ldots < \sigma_m < \tau_m \leq \sigma_{m+1} = R_2$$

such that $\mathcal{Q}(r) \succ 0$ *on* $[\sigma_i, \tau_i]$, *for* $i = 1, \ldots, m$, *and* $\mathcal{Q}(r) \prec 0$ *on* $[\tau_i, \sigma_{i+1}]$, *for* $i = 0, \ldots, m$,

and let $q \colon \overline{\Omega} \to \mathbb{R}$ *be defined as in* (7.7.5). *Then there exists* $\lambda^* > 0$ *such that for each* $\lambda > \lambda^*$ *there exists* $\mu^*(\lambda) > 0$ *such that for each* $\mu > \mu^*(\lambda)$ *the Neumann problem associated with* (7.7.2) *has at least* $3^m - 1$ *radially symmetric positive (classical) solutions. The same result holds for the Dirichlet problem associated with* (7.7.2).

Chapter 8

Subharmonic solutions and symbolic dynamics

In this chapter we continue the discussion of the previous chapter for the super-sublinear indefinite equation

$$(\mathscr{E}_{\lambda,\mu}) \qquad u''+\bigl(\lambda a^{+}(t)-\mu a^{-}(t)\bigr)g(u)=0,$$

dealing with subharmonic solutions and symbolic dynamics, following the line of research initiated in Chapter 5 for the superlinear indefinite case.

Throughout the chapter we implicitly assume all the hypotheses of Theorem 7.1.1; in particular we suppose that $g\colon \mathbb{R}^{+}\to\mathbb{R}^{+}$ is a continuous function satisfying (g_{*}) as well as (g_{0}) and (g_{∞}) and $a\colon\mathbb{R}\to\mathbb{R}$ is a T-periodic locally integrable function satisfying (a_{*}). For convenience, we also suppose that $T>0$ is the *minimal period* of $a(t)$. Moreover, we recall the notation

$$I^{\pm}_{i,\ell}:=I^{\pm}_{i}+\ell T, \quad \text{for } \; i=1,\ldots,m \; \text{ and } \; \ell\in\mathbb{Z}. \tag{8.0.1}$$

Taking advantage of the approach developed in Chapter 7 , we are going to present an application of our main theorem concerning the multiplicity of positive T-periodic solutions of $(\mathscr{E}_{\lambda,\mu})$ (i.e. Theorem 7.1.1). In fact, an important feature of Theorem 7.1.1 is that all the constants appearing in the statement (precisely λ^{*}, r, R and $\mu^{*}(\lambda)$) can be explicitly estimated (depending on $g(s)$, $a(t)$, as well as on the arbitrary choice of ρ) and, as remarked in Section 7.4.5, these estimates are of local nature. In particular, it turns out that, whenever Theorem 7.1.1 is applied to an interval of the form $[0,kT]$, with $k\geq 1$ an integer number, these constants can be chosen independently on k. This implies that, for any fixed $\lambda>\lambda^{*}$ and for any $\mu>\mu^{*}(\lambda)$, equation $(\mathscr{E}_{\lambda,\mu})$ has positive T-periodic solutions as well as positive kT-periodic solutions for any $k\geq 2$. Such solutions can of course be coded similarly as the T-periodic ones, by prescribing their behavior on the intervals $I^+_{i,\ell}$, for $i = 1, \dots, m$ and $\ell \in \mathbb{Z}$, according to a non-null bi-infinite km-periodic string $\mathcal{S}$ in the alphabet $\mathscr{A} := \{0, 1, 2\}$ of 3 symbols (see Theorem 8.1.1). This information can be used to prove that many of these positive kT-periodic solutions have kT as minimal period, namely they are *subharmonic solutions* of order k (see Theorem 8.1.2, where a lower bound based on the combinatorial concept of *Lyndon words* is given). Next, in Section 8.2, using an approximation argument of Krasnosel'skiĭ-Mawhin type (cf. [112, 133]) for $k \to \infty$, it is possible to construct globally defined bounded (not necessarily periodic) positive solutions to $(\mathscr{E}_{\lambda,\mu})$, whose behavior on each $I^+_{i,\ell}$ can be prescribed a priori with a nontrivial bi-infinite string $\mathcal{S} \in \mathscr{A}^{\mathbb{Z}}$ and thus exhibiting *chaotic-like dynamics* (see Theorem 8.2.2). In this way we can improve the main result in [37], where arguments from topological horseshoes theory were used to construct a symbolic dynamics on two symbols (1 and 2, according to the notation of the present chapter). This is a hint of complex dynamics and indeed we conclude the chapter by describing some dynamical consequences of our results. More precisely, in Section 8.3, in a dynamical system perspective, we also prove the presence of a Bernoulli shift as a factor within the set of positive bounded solutions of $(\mathscr{E}_{\lambda,\mu})$.

Recalling the discussion in Section 7.7.1, we underline that we can prove the existence of infinitely many positive subharmonic solutions as well as the presence of chaotic dynamics on 3^m symbols also for the equation

$$u'' + cu' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0,$$

namely when we lose the Hamiltonin structure of equation $(\mathscr{E}_{\lambda,\mu})$ adding the friction term cu' (compare also to Chapter 5).

8.1 Positive subharmonic solutions

In this section we investigate the existence and multiplicity of positive subharmonic solutions to equation $(\mathscr{E}_{\lambda,\mu})$. Let $k \geq 2$ be a fixed integer. Following a standard definition, we recall that a *subharmonic solution of order* k is a kT-periodic solution which is not lT-periodic for any integer $l = 1, \dots, k-1$. As observed in Chapter 5 (as a consequence of (g_*) and the fact that $T > 0$ is the minimal period of $a(t)$) any positive subharmonic solution of order k has actually kT as minimal period. Moreover, we underline that if we find a (positive) subharmonic solution $u(t)$ of order k, we also obtain altogether a family of k (positive) subharmonic solutions $v_\ell(t) := u(t + \ell T)$ (for $0 \leq \ell \leq k-1$) of the same order. These solutions, even if formally distinct, will be considered as belonging to the same periodicity class.

We split the search of subharmonic solutions to $(\mathscr{E}_{\lambda,\mu})$ into two steps. In the first one we present a theorem of existence and multiplicity of positive kT-periodic solutions which is a direct application of Theorem 7.1.1 for the interval $[0,kT]$. As a second step, we show how the code "very small/small/large" allows us to prove the minimality of the period for some of such kT-periodic solutions and determine a lower bound for the number of k-th order subharmonics.

First of all, in order to apply Theorem 7.1.1 to the interval $[0,kT]$, we need to observe that now $a(t)$ is treated as a kT-periodic function (even if it has T as minimal period). Recalling the notation in (8.0.1), in the "new" periodicity interval $[0,kT]$ the weight $a(t)$ turns out to be a function with km positive humps $I^{+}_{i,\ell}$ separated by km negative ones $I^{-}_{i,\ell}$ (for $i=1,\dots,m$ and $\ell=0,\dots,k-1$).

In this setting, Theorem 7.1.1 reads as follows.

Theorem 8.1.1. *Let $g\colon \mathbb{R}^{+}\to\mathbb{R}^{+}$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $a\colon\mathbb{R}\to\mathbb{R}$ be a locally integrable periodic function of minimal period $T>0$ satisfying (a_*). Then there exists $\lambda^*>0$ such that for each $\lambda>\lambda^*$ there exists $\mu^*(\lambda)>0$ such that, for each $\mu>\mu^*(\lambda)$ and each integer $k\geq 2$, equation $(\mathscr{E}_{\lambda,\mu})$ has at least $3^{km}-1$ positive kT-periodic solutions.*

More precisely, fixed an arbitrary constant $\rho>0$ there exists $\lambda^*=\lambda^*(\rho)>0$ such that for each $\lambda>\lambda^*$ there exist two constants r,R with $0<r<\rho<R$ and $\mu^*(\lambda)=\mu^*(\lambda,r,R)>0$ such that, for any $\mu>\mu^*(\lambda)$ and for any integer $k\geq 2$, the following holds: given any finite string $\mathcal{S}=(\mathcal{S}_1,\dots,\mathcal{S}_{km})\in\{0,1,2\}^{km}$, with $\mathcal{S}\neq(0,\dots,0)$, there exists a positive kT-periodic solution $u(t)$ of $(\mathscr{E}_{\lambda,\mu})$ such that

- $\max_{t\in I^{+}_{i,\ell}} u(t)<r$, *if $\mathcal{S}_j=0$ for $j=i+\ell m$;*
- $r<\max_{t\in I^{+}_{i,\ell}} u(t)<\rho$, *if $\mathcal{S}_j=1$ for $j=i+\ell m$;*
- $\rho<\max_{t\in I^{+}_{i,\ell}} u(t)<R$, *if $\mathcal{S}_j=2$ for $j=i+\ell m$.*

Proof. This statement follows from Theorem 7.1.1 (for the search of positive kT-periodic solutions and the weight $a(t)$ considered as a kT-periodic function), after having checked that the constants λ^*, r, R and $\mu^*(\lambda)$ can be chosen independently on k. This is a consequence of the fact that, for the part in which they depend on $a(t)$, these constants involve either integrals of $a^{\pm}(t)$ on $I^{\pm}_i$ or interval lengths of the form $|I^{\pm}_i|$, with $i=1,\dots,m$ (compare with the discussion in Section 7.4.5), and of the fact that the "new" intervals $I^{\pm}_{i,\ell}$ (for $i=1,\dots,m$ and $\ell=0,\dots,k-1$) are just ℓT-translations of the original $I^{\pm}_i$ (with $a(t)$ T-periodic). □

Remark 8.1.1. As a further information, up to selecting the intervals $I^{\pm}_i$ so that $a(t)\not\equiv 0$ on each right neighborhood of τ_i and on each left neighborhood of σ_{i+1}, among the properties of the positive kT-periodic solutions listed in Theorem 8.1.1, we can add the following one (if μ is sufficiently large):

- *$0 < u(t) < r$ on $I^{-}_{i,\ell}$, for all $i = 1,\dots,m$ and $\ell = 0,\dots,k-1$.*

This assertion is justified by Proposition 7.6.5 taking $\mu > \mu^{\star\star}_{r}$ defined in (7.6.1) for $\varepsilon = r$, and observing also that the constants $\mu^{\mathrm{left}}_{i}(r)$ and $\mu^{\mathrm{right}}_{i}(r)$ depend on $a(t)$ on a T-periodicity interval and do not depend on k. ◁

From now on, we can use Theorem 8.1.1 to produce subharmonics. The trick is that of selecting strings which are minimal in some sense, in order to obtain the minimality of the period. On the other hand, in counting the subharmonic solutions we wish to avoid duplications, in the sense that we count only once subharmonics belonging to the same periodicity class. To this end, we can take advantage of some combinatorial results related to the concept of Lyndon words. We recall that a *n-ary Lyndon word of length k* is a string of k digits of an alphabet $\mathscr{B}$ with n symbols which is strictly smaller in the lexicographic ordering than all of its nontrivial rotations. As in Chapter 5, we denote by $\mathcal{L}_n(k)$ the number of n-ary Lyndon words of length k. According to formula (5.2.1) and Proposition 5.2.1, for instance, the values of $\mathcal{L}_3(k)$ (number of ternary Lyndon words of length k) for $k = 2,\dots,10$ are 3, 8, 18, 48, 116, 312, 810, 2184, 5880. See Chapter 5 for more details and remarks (cf. also Figure 5.2).

In this setting we can now provide the following consequence of Theorem 8.1.1.

Theorem 8.1.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable periodic function of minimal period $T > 0$ satisfying (a_*). Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ there exists $\mu^*(\lambda) > 0$ such that for each $\mu > \mu^*(\lambda)$ and each integer $k \geq 2$, equation $(\mathscr{E}_{\lambda,\mu})$ has at least $\mathcal{L}_{3^m}(k)$ positive subharmonic solutions of order k.*

Proof. We consider an alphabet $\mathscr{B}$ made by 3^m symbols and defined as

$$\mathscr{B} := \{0,1,2\}^m.$$

Let us fix a non-null k-tuple $\mathcal{T}^{[k]} := (\mathcal{T}_\ell)_{\ell=0,\dots,k-1}$ in the alphabet $\mathscr{B}$. We have that for each $\ell = 0,\dots,k-1$, the element $\mathcal{T}_\ell \in \mathscr{B}$ can be written as $\mathcal{T}_\ell = (\mathcal{T}^i_\ell)_{i=1,\dots,m}$, where $\mathcal{T}^i_\ell \in \{0,1,2\}$ for $i = 1,\dots,m$ and $\ell = 0,\dots,k-1$. By Theorem 8.1.1, there exists at least one positive kT-periodic solution $u(t)$ of equation $(\mathscr{E}_{\lambda,\mu})$ such that

- $\max_{t\in I^{+}_{i,\ell}} u(t) < r$, if $\mathcal{T}^i_\ell = 0$;

- $r < \max_{t\in I^{+}_{i,\ell}} u(t) < \rho$, if $\mathcal{T}^i_\ell = 1$;

- $\rho < \max_{t\in I^{+}_{i,\ell}} u(t) < R$, if $\mathcal{T}^i_\ell = 2$.

In fact, the k-tuple $\mathcal{T}^{[k]}$ determines the string $\mathcal{S}$ of length km with

$$\mathcal{S}_j := \mathcal{T}^i_\ell, \quad \text{for } j = i + \ell m.$$

It remains to see whether, on the basis of the information we have on $u(t)$, we are able first to prove the minimality of the period and next to distinguish among solutions not belonging to the same periodicity class. In view of the above listed properties of the solution $u(t)$, the minimality of the period is guaranteed when the string $\mathcal{T}^{[k]}$ has k as a minimal period (when repeated cyclically). For the second question, given any string of this kind, we count as the same all those strings (of length k) which are equivalent by cyclic permutations. To choose exactly one string in each of these equivalence classes, we can take the minimal one in the lexicographic order, namely a Lyndon word. As a consequence, we find that each 3^m-ary Lyndon word of length k determines at least one kT-periodic solution which is not pT-periodic for every $p = 1, \ldots, k-1$. This solution has indeed kT as minimal period. Moreover, by definition, solutions associated with different Lyndon words are not in the same periodicity class. □

8.2 Positive solutions with complex behavior

Having shown the existence of a mechanism producing subharmonic solutions of arbitrary order, letting $k \to \infty$ we can provide positive (not necessarily periodic) bounded solutions coded by a non-null bi-infinite string of three symbols. A similar procedure has been performed in [17] and in Chapter 5 for the superlinear case.

Our proof is based on the following diagonal lemma borrowed from [112, Lemma 8.1] and [133, Lemma 4].

Lemma 8.2.1. *Let $f\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be an L^1-Carathéodory function. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers and $(x_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}^d$ with the following properties:*

(i) *$t_n \to +\infty$ as $n \to \infty$;*

(ii) *for each $n \in \mathbb{N}$, $x_n(t)$ is a solution of*

$$x' = f(t, x) \tag{8.2.1}$$

defined on $[-t_n, t_n]$;

(iii) *there exists a closed and bounded set $B \subseteq \mathbb{R}^d$ such that, for each $n \in \mathbb{N}$, $x_n(t) \in B$ for every $t \in [-t_n, t_n]$.*

Then there exists a subsequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which converges uniformly on the compact subsets of $\mathbb{R}$ to a solution $\tilde{x}(t)$ of system (8.2.1)*; in particular $\tilde{x}(t)$ is defined on $\mathbb{R}$ and $\tilde{x}(t) \in B$ for all $t \in \mathbb{R}$.*

In order to simplify the exposition we suppose that the coefficient $a(t)$ has a positive hump followed by a negative one in a period interval (i.e. $m = 1$ in hypothesis (a_*)). In this framework, the next result follows.

Theorem 8.2.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function such that there exist α, β with $\alpha < \beta < \alpha + T$ so that $a(t) \succ 0$ on $[\alpha, \beta]$ and $a(t) \prec 0$ on $[\beta, \alpha + T]$. Then, fixed an arbitrary constant $\rho > 0$ there exists $\lambda^* = \lambda^*(\rho) > 0$ such that for each $\lambda > \lambda^*$ there exist two constants r, R with $0 < r < \rho < R$ and $\mu^*(\lambda) = \mu^*(\lambda, r, R) > 0$ such that for any $\mu > \mu^*(\lambda)$ the following holds: given any two-sided sequence $\mathcal{S} = (\mathcal{S}_j)_{j\in\mathbb{Z}} \in \{0,1,2\}^{\mathbb{Z}}$ which is not identically zero, there exists at least a positive solution $u(t)$ of $(\mathscr{E}_{\lambda,\mu})$ such that*

- $\max_{t\in[\alpha+jT,\beta+jT]} u(t) < r$, *if* $\mathcal{S}_j = 0$;
- $r < \max_{t\in[\alpha+jT,\beta+jT]} u(t) < \rho$, *if* $\mathcal{S}_j = 1$;
- $\rho < \max_{t\in[\alpha+jT,\beta+jT]} u(t) < R$, *if* $\mathcal{S}_j = 2$.

Proof. Without loss of generality, we suppose that $\alpha = 0$ and set $\tau := \beta - \alpha$, so that $a(t) \succ 0$ on $[0, \tau]$ and $a(t) \prec 0$ on $[\tau, T]$. We also introduce the intervals

$$J_j^+ := [jT, \tau + jT], \quad J_j^- := [\tau + jT, (j+1)T], \quad j \in \mathbb{Z}. \tag{8.2.2}$$

Let ρ, $\lambda > \lambda^*$, r, R and $\mu^*(\lambda)$ be fixed as in Section 7.4.2 and Section 7.4.5 for $m = 1$. Once more, we emphasize that all our constants can be chosen independently on k. Thus, having fixed all these constants and taken $\mu > \mu^*(\lambda)$, we can produce kT-periodic solutions following any k-periodic two-sided sequence of three symbols, as in Theorem 8.1.1.

Consider now an arbitrary sequence $\mathcal{S} = (\mathcal{S}_j)_{j\in\mathbb{Z}} \in \{0,1,2\}^{\mathbb{Z}}$ which is not identically zero. We fix a positive integer n_0 such that there is at least an index $j \in \{-n_0, \ldots, n_0\}$ such that $\mathcal{S}_j \neq 0$. Then, for each $n \geq n_0$ we consider the $(2n+1)$-periodic sequence $\mathcal{S}^n = (\mathcal{S}'_j)_j \in \{0,1,2\}^{\mathbb{Z}}$ which is obtained by truncating $\mathcal{S}$ between $-n$ and n, and then repeating that string by periodicity. We apply Theorem 8.1.1, with $m = 1$, on the periodicity interval $[-nT, (n+1)T]$ and find a *positive* periodic solution $u_n(t)$ such that $u_n(t + (2n+1)T) = u_n(t)$ for all $t \in \mathbb{R}$ and $\|u_n\|_\infty < R$ (by the concavity of the solutions in the intervals J_j^- where $a(t) \prec 0$). Moreover, we also know that

- $\max_{t\in J_j^+} u_n(t) < r$, if $\mathcal{S}'_j = 0$;
- $r < \max_{t\in J_j^+} u_n(t) < \rho$, if $\mathcal{S}'_j = 1$;
- $\rho < \max_{t\in J_j^+} u_n(t) < R$, if $\mathcal{S}'_j = 2$.

In each interval J_j^+ (of length τ) the positive solution $u_n(t)$ is bounded by R and therefore there exists at least a point $t_{n,j} \in J_j^+$ such that $|u_n'(t_{n,j})| \leq R/\tau$. Hence, for each $t \in J_j^+$ and every $n \geq n_0$, it holds that

$$\begin{aligned} |u_n'(t)| &= \left| u_n'(t_{n,j}) + \int_{t_{n,j}}^t u_n''(\xi)\, d\xi \right| \leq \frac{R}{\tau} + \lambda \int_{J_j^+} a^+(\xi) g(u_n(\xi))\, d\xi \\ &\leq \frac{R}{\tau} + \lambda \|a\|_{+,1} g^*(R) =: K, \end{aligned} \tag{8.2.3}$$

where the constants $\|a\|_{+,1}$ and $g^*(R)$ are those defined at the beginning of Section 7.4. Notice that K is independent on j and this provides a uniform estimate for all the intervals where the weight is positive. On the other hand, using the convexity of $u_n(t)$ in the intervals J_j^-, we know that

$$|u_n'(t)| \leq \max_{\xi \in \partial J_j^-} |u_n'(\xi)| \leq \max_{\xi \in J_j^+ \cup J_{j+1}^+} |u_n'(\xi)| \leq K, \quad \forall\, t \in J_j^-,\ \forall\, n \geq n_0,$$

and thus we are able to find the global uniform estimate

$$|u_n'(t)| \leq K, \quad \forall\, t \in \mathbb{R},\ \forall\, n \geq n_0.$$

Now we write equation $(\mathscr{E}_{\lambda,\mu})$ as the planar system

$$\begin{cases} u' = y \\ y' = -\big(\lambda a^+(t) - \mu a^-(t)\big) g(u). \end{cases}$$

From the above estimates, one can see that (up to a reparametrization of indices, counting from n_0) assumptions (i), (ii) and (iii) of Lemma 8.2.1 are satisfied, taking $t_n := nT$, $f(t,x) = (y, -(\lambda a^+(t) - \mu a^-(t)) g(u))$, with $x = (u,y)$, and

$$B := \big\{ x = (x_1, x_2) \in \mathbb{R}^2 \colon 0 \leq x_1 \leq R,\ |x_2| \leq K \big\},$$

which is a closed and bounded set in $\mathbb{R}^2$. By Lemma 8.2.1, there is a solution $\tilde{u}(t)$ of equation (8.2.1) which is defined on $\mathbb{R}$ and such that $0 \leq \tilde{u}(t) \leq R$ for all $t \in \mathbb{R}$. Moreover, such a solution $\tilde{u}(t)$ is the limit of a subsequence $(\tilde{u}_n)_n$ of the sequence of the periodic solutions $u_n(t)$.

We claim that

- $\max_{t \in J_j^+} \tilde{u}(t) < r$, if $\mathcal{S}_j = 0$;
- $r < \max_{t \in J_j^+} \tilde{u}(t) < \rho$, if $\mathcal{S}_j = 1$;
- $\rho < \max_{t \in J_j^+} \tilde{u}(t) < R$, if $\mathcal{S}_j = 2$.

To prove our claim, let us fix $j \in \mathbb{Z}$ and consider the interval J_j^+ introduced in (8.2.2). For each $n \geq |j|$ (and $n \geq n_0$) the periodic solution $u_n(t)$ is defined on $\mathbb{R}$ and such that $\max_{J_j^+} u_n < r$ if $\mathcal{S}_j = 0$, $r < \max_{J_j^+} u_n < \rho$ if $\mathcal{S}_j = 1$, $\rho < \max_{J_j^+} u_n < R$ if $\mathcal{S}_j = 2$. Passing to the limit on the subsequence $(\tilde{u}_n)_n$, we obtain that

- $\max_{t \in J_j^+} \tilde{u}(t) \leq r$, if $\mathcal{S}_j = 0$;
- $r \leq \max_{t \in J_j^+} \tilde{u}(t) \leq \rho$, if $\mathcal{S}_j = 1$;
- $\rho \leq \max_{t \in J_j^+} \tilde{u}(t) \leq R$, if $\mathcal{S}_j = 2$.

By Proposition 7.6.1 we get that $\tilde{u}(t) < R$, for all $t \in \mathbb{R}$. Moreover, since there exists at least one index $j \in \mathbb{Z}$ such that $\mathcal{S}_j \neq 0$, we know that $\tilde{u}(t)$ is not identically zero. Hence, a maximum principle argument shows that $\tilde{u}(t)$ never vanishes. In conclusion, we have proved that

$$0 < \tilde{u}(t) < R, \quad \forall\, t \in \mathbb{R}.$$

Next, using this fact, by Proposition 7.6.2 we observe that

$$\max_{t \in J_j^+} \tilde{u}(t) \neq \rho, \quad \forall\, j \in \mathbb{Z},$$

and by Proposition 7.6.3 we have

$$\max_{t \in J_j^+} \tilde{u}(t) \neq r, \quad \forall\, j \in \mathbb{Z},$$

since, at the beginning, μ has been chosen large enough (note also that we apply those propositions in the case $m = 1$ and so the sets $I_{i,\ell}^+$ reduce to the intervals $[0,\tau] + \ell T$). Our claim is thus verified and this completes the proof of the theorem. □

Theorem 8.2.1 can be compared with the main result in [37], providing (under a few technical conditions on $a(t)$ and $g(s)$) globally defined positive solutions to $(\mathscr{E}_{\lambda,\mu})$ according to a symbolic dynamics on *two* symbols. More precisely, using a dynamical systems technique it was shown in [37, Theorem 2.3] the existence of two disjoint compact sets $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathbb{R}^2$ such that for any two-sided sequence $\mathcal{S} = (\mathcal{S}_j)_{j \in \mathbb{Z}} \in \{1,2\}^{\mathbb{Z}}$ there is a positive solution $u(t)$ to $(\mathscr{E}_{\lambda,\mu})$ satisfying $(u(\alpha + jT), u'(\alpha + jT)) \in \mathcal{K}_{\mathcal{S}_j}$ for all $j \in \mathbb{Z}$. Even if this conclusion is not directly comparable with the one of Theorem 8.2.1 (in which solutions are distinguished in dependence of the value $\max_{t \in [\alpha + jT, \beta + jT]} u(t)$), a careful reading of the arguments in [37] should convince us that the solutions obtained therein correspond to solutions which are "small" or "large" according to the code of the present chapter. From this point of view, Theorem 8.2.1 can thus be seen as an improvement of [37, Theorem 2.3], providing in addition solutions which are "very small" on some intervals of positivity of the weight function and thus leading to a symbolic dynamics on *three* symbols. It has to be noticed, however, that in [37] some further information for the Poincaré map associated with $(\mathscr{E}_{\lambda,\mu})$ were obtained; we will comment again on this point in Section 8.3.

Theorem 8.2.1 can be extended to the case of a weight function with more than one positive hump in the interval $[0,T]$, as described in hypothesis (a_*). The corresponding more general result is given in the next theorem.

Theorem 8.2.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable periodic function of minimal period $T > 0$ satisfying (a_*). Then, fixed an arbitrary constant $\rho > 0$ there exists $\lambda^* = \lambda^*(\rho) > 0$ such that for each $\lambda > \lambda^*$ there exist two constants r, R with $0 < r < \rho < R$ and $\mu^*(\lambda) = \mu^*(\lambda, r, R) > 0$ such that for any $\mu > \mu^*(\lambda)$ the following holds: given any two-sided sequence $\mathcal{S} = (\mathcal{S}_j)_{j\in\mathbb{Z}}$ in the alphabet $\mathscr{A} := \{0,1,2\}$ which is not identically zero, there exists at least a positive solution $u(t)$ of $(\mathscr{E}_{\lambda,\mu})$ such that*

- $\max_{t\in I^+_{i,\ell}} u(t) < r$, *if $\mathcal{S}_j = 0$ for $j = i + \ell m$;*
- $r < \max_{t\in I^+_{i,\ell}} u(t) < \rho$, *if $\mathcal{S}_j = 1$ for $j = i + \ell m$;*
- $\rho < \max_{t\in I^+_{i,\ell}} u(t) < R$, *if $\mathcal{S}_j = 2$ for $j = i + \ell m$.*

Proof. The proof requires only minor modifications in the argument applied for Theorem 8.2.1 and thus the details are omitted. We only observe that the uniform bound K for $|u_n'(t)|$ is now achieved by working separately on each interval $I^+_{i,\ell}$. When arguing like in (8.2.3) one obtains

$$|u_n'(t)| \leq \frac{R}{|I^+_i|} + \lambda \|a\|_{+,i} g^*(R) =: K_i, \quad \forall\, t \in I^+_{i,\ell},\ \forall\, n \geq n_0.$$

Now all the rest works fine for

$$K := \max_{i=1,\dots,m} K_i.$$

The same final arguments allow us to obtain the theorem. □

Remark 8.2.1. As a further information, up to selecting the intervals $I^\pm_i$ so that $a(t) \not\equiv 0$ on each right neighborhood of τ_i and on each left neighborhood of σ_{i+1}, among the properties of the positive solutions listed in Theorem 8.2.1 and Theorem 8.2.2, we can add the following one (if μ is sufficiently large):

- $0 < u(t) < r$ *on $I^-_{i,\ell}$, for all $i \in \{1,\dots,m\}$ and for all $\ell \in \mathbb{Z}$.*

This assertion is justified by Proposition 7.6.5 taking $\mu > \mu_r^{\star\star}$ defined in (7.6.1) for $\varepsilon = r$, and observing also that the constants $\mu_i^{\mathrm{left}}(r)$ and $\mu_i^{\mathrm{right}}(r)$ depend on $a(t)$ on a T-periodicity interval. ◁

8.3 A dynamical systems perspective

In the two previous sections we have proved the presence of chaotic-like dynamics which is highlighted by the coexistence of infinitely many subharmonic solutions together with non-periodic bounded solutions which can be coded by sequences of three symbols. Our next goal is to show that our results allow us to enter a classical framework for complex dynamical systems, namely the semiconjugation with the Bernoulli shift.

We start with some formal definitions. Let $n \geq 2$. Let $\mathscr{B}$ be a finite set of n elements (called *symbols*), conventionally denoted as $\mathscr{B} := \{b_1, \dots, b_n\}$, which is endowed with the discrete topology. Let $\Sigma_n := \mathscr{B}^{\mathbb{Z}}$ be the set of all two-sided sequences $\mathcal{T} = (\mathcal{T}_\ell)_{\ell \in \mathbb{Z}}$ where, for each $\ell \in \mathbb{Z}$, the element $\mathcal{T}_\ell$ is a symbol of the alphabet $\mathscr{B}$. The set $\Sigma_n = \prod_{\ell \in \mathbb{Z}} \mathscr{B}$, endowed with the product topology, turns out to be a compact metrizable space. As a suitable distance on Σ_n we take

$$d(\mathcal{T}', \mathcal{T}'') := \sum_{\ell \in \mathbb{Z}} \frac{\delta(\mathcal{T}'_\ell, \mathcal{T}''_\ell)}{2^{|\ell|}}, \quad \mathcal{T}', \mathcal{T}'' \in \Sigma_n,$$

where δ is the discrete distance on $\mathscr{B}$, that is $\delta(s', s'') = 0$ if $s' = s''$ and $\delta(s', s'') = 1$ if $s' \neq s''$. We introduce a map $\sigma\colon \Sigma_n \to \Sigma_n$ called the *shift automorphism* (cf. [171, p. 770]) or *Bernoulli shift* (cf. [178]) and defined as

$$\sigma(\mathcal{T}) = \mathcal{T}', \quad \text{with } \mathcal{T}'_\ell := \mathcal{T}_{\ell+1}, \ \forall\, \ell \in \mathbb{Z}.$$

The map σ is a bijective continuous map (a homeomorphism) of Σ_n which possesses all the features usually associated with the concept of chaos, such as transitivity, density of the set of periodic points and positive topological entropy (which is $\log(n)$ for an alphabet of n symbols).

Given a topological space X and a continuous map $\psi\colon X \to X$, a typical way to prove that ψ is "chaotic" consists into verifying that ψ has the shift map as a *factor*, namely that there exist a compact set $Y \subseteq X$ which is invariant for ψ (i.e. $\psi(Y) = Y$) and a *continuous and surjective* map $\pi\colon Y \to \Sigma_n$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"] \\ \Sigma_n \arrow[r, "\sigma"'] & \Sigma_n \end{tikzcd}$$

commutes, that is

$$\pi \circ \psi = \sigma \circ \pi. \tag{8.3.1}$$

If we are in this situation we say that the map $\psi|_Y$ is *semiconjugate* with the shift on n symbols. Usually the best form of chaos occurs when the map

$\pi\colon Y \to \Sigma_n$ is a homeomorphism. In this latter case the map $\psi|_Y$ is said to be *conjugate* with the shift σ. This, for instance, occurs for the classical Smale horseshoe (see [141, 171]). In many concrete examples of differential equations, the conjugation with the shift map is not feasible and many investigations have been addressed toward the proof of a semiconjugation with the Bernoulli shift, possibly accompanied by some further information, such as density of periodic points, in order to provide a description of chaotic dynamics which is still interesting for the applications. Quoting Block and Coppel from [26, Introduction],

> " ... there is no generally accepted definition of chaos. It is our view that any definition for more general spaces should agree with ours in the case of an interval. ... we show that a map is chaotic if and only if some iterate has the shift map as a *factor*, and we propose this as a general definition."

Indeed, the semiconjugation of an iterate of a map ψ with the Bernoulli shift is defined as *B/C-chaos* in [12].

We plan to prove the existence of a strong form of B/C-chaos coming from Theorem 8.1.1 and Theorem 8.2.2, namely the existence of a compact invariant set Y for a continuous homeomorphism ψ such that $\psi|_Y$ satisfies (8.3.1) and such that to any periodic sequence of symbols corresponds a periodic solution of $(\mathscr{E}_{\lambda,\mu})$. Such a stronger form of chaos has been produced by several authors using dynamical systems techniques (see, for instance, [37, 54, 138, 139, 173, 184, 185]). The obtention of this kind of results with the coincidence degree approach appears new in the literature.

Let us start by defining a suitable metric space and a homeomorphism on it. Let X be the set of the continuous functions $z = (x, y)\colon \mathbb{R} \to \mathbb{R}^2$. For each $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in X$, we define

$$\vartheta_N(z_1, z_2) := \max_{t\in[-N,N]} \bigl(|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|\bigr), \quad N \in \mathbb{N} \setminus \{0\},$$

and we set

$$\mathrm{dist}\,(z_1, z_2) := \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{\vartheta_N(z_1, z_2)}{1 + \vartheta_N(z_1, z_2)}.$$

It is a standard task to check that (X, dist) is a complete metric space. Moreover, given a sequence of functions $(z_k)_k$ in X and a function $\hat{z} \in X$, we have that $z_k \to \hat{z}$ with respect to the distance of X if and only if $z_k(t)$ converges uniformly to $\hat{z}(t)$ in each compact interval of $\mathbb{R}$ (cf. [24, ch. 1], [164, ch. III] and [168, § 20]). We also recall that a family of functions $\mathcal{M} \subseteq X$ is relatively compact if and only if for every compact interval J the set of restrictions to J of the functions belonging to $\mathcal{M}$ is relatively compact in $\mathcal{C}(J, \mathbb{R}^2)$ (cf. [56, p. 2]). Next, recalling that $T > 0$ is the minimal period of the weight function $a(t)$, we introduce the shift map $\psi\colon X \to X$ defined by

$$(\psi u)(t) := u(t+T), \quad t \in \mathbb{R},$$

which is a homeomorphism of X onto itself. The discrete dynamical system induced by ψ is usually referred to as a *Bebutov dynamical system* on X.

For the next results we assume the standard hypotheses on the nonlinearity $g(s)$ and on the coefficient $a(t)$, that is, $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying (g_*), (g_0), (g_∞), $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic locally integrable function satisfying (a_*) with minimal period T. We suppose also that all the positive constants ρ, $\lambda > \lambda^*$, r, R and $\mu^*(\lambda)$ are fixed as in Section 7.4.2 and Section 7.4.5. Let also $\mu > 0$.

We consider the first order differential system

$$\begin{cases} x' = y \\ y' = -\bigl(\lambda a^+(t) - \mu a^-(t)\bigr) g(x) \end{cases} \tag{8.3.2}$$

associated with $(\mathscr{E}_{\lambda,\mu})$. Even if all our results concern non-negative solutions of $(\mathscr{E}_{\lambda,\mu})$, in dealing with system (8.3.2) it would be convenient to have the vector field (i.e. the right-hand side of the system) defined for all $t \in \mathbb{R}$ and $(x,y) \in \mathbb{R}^2$. For this reason, we extend $g(s)$ to the whole real line, for instance by setting $g(s) = 0$ for $s \leq 0$ (any extension we choose will have no effect in what follows). As usual the solutions of (8.3.2) are meant in the Carathéodory sense.

Next, we denote by Y_0 the subset of X made up of the globally defined solutions $(x(t), y(t))$ of (8.3.2) such that $0 \leq x(t) \leq R$, for all $t \in \mathbb{R}$. Observe that $(0,0) \in Y_0$ (as $u(t) \equiv 0$ is the trivial solution of $(\mathscr{E}_{\lambda,\mu})$). On the other hand, if $(x,y) \in Y_0$ with $x \not\equiv 0$, then $x(t) > 0$ for all $t \in \mathbb{R}$.

Lemma 8.3.1. *There exists a constant $K > 0$ such that for each $(x,y) \in Y_0$ it holds that*

$$|y(t)| \leq K, \quad \forall\, t \in \mathbb{R}. \tag{8.3.3}$$

Moreover, Y_0 is a compact subset of X which is invariant for the map ψ.

Proof. The estimates needed to prove this result have been already obtained along the proof of Theorem 8.2.1. We briefly repeat the argument since the context here is slightly different. Let $(x,y) \in Y_0$. Since $0 \leq x(t) \leq R$ for all $t \in \mathbb{R}$, we have that, for all $i \in \{1,\ldots,m\}$ and $\ell \in \mathbb{Z}$, there exists at least a point $\hat{t}_{i,\ell} \in I^+_{i,\ell}$ such that $|y(\hat{t}_{i,\ell})| \leq R/|I^+_i|$ (recall the definition of $I^+_{i,\ell}$ in (8.0.1)). Hence, for each $t \in I^+_{i,\ell}$, it holds that

$$\begin{aligned} |y(t)| &= \left| y(\hat{t}_{i,\ell}) + \int_{\hat{t}_{i,\ell}}^t y'(\xi)\, d\xi \right| \leq \frac{R}{|I^+_i|} + \lambda \int_{I^+_{i,\ell}} a^+(\xi) g(x(\xi))\, d\xi \\ &\leq \frac{R}{|I^+_i|} + \lambda \|a\|_{+,i}\, g^*(R) =: K_i. \end{aligned}$$

Note that the constant K_i does not depend on the index ℓ. Therefore, setting

$$K := \max_{i=1,\dots,m} K_i,$$

we get

$$|y(t)| \leq K, \quad \forall\, t \in I^+_{i,\ell},\ \forall\, i = 1,\dots,m,\ \forall\, \ell \in \mathbb{Z}.$$

On the other hand, using the convexity of $x(t)$ in the intervals $I^-_{i,\ell}$ we know that

$$|y(t)| = |x'(t)| \leq \max_{\xi\in\partial I^-_{i,\ell}} |x'(\xi)| \leq K, \quad \forall\, t \in I^+_{i,\ell},\ \forall\, i = 1,\dots,m,\ \forall\, \ell \in \mathbb{Z}.$$

This proves inequality (8.3.3).

From system (8.3.2), we know that the absolutely continuous vector function $(x,y) \in Y_0$ satisfies

$$|x'(t)| + |y'(t)| \leq K + \big(\lambda a^+(t) + \mu a^-(t)\big) g^*(R), \quad \text{for a.e. } t \in \mathbb{R}.$$

Therefore, Ascoli-Arzelà theorem implies that the set of restrictions of the functions in Y_0 to any compact interval is relatively compact in the uniform norm. Thus we conclude that the closed set Y_0 is a compact subset of X.

Finally, we observe that the invariance of Y_0 under the map ψ follows from the T-periodicity of the coefficients in system (8.3.2), which in turn implies that $(x(t),y(t))$ is a solution of (8.3.2) if and only if $(x(t+T),y(t+T))$ is a solution of the same system. $\square$

The next result summarizes the properties obtained in Proposition 7.6.1, Proposition 7.6.2 and Proposition 7.6.3.

Lemma 8.3.2. *Suppose that $\mu > \mu^*(\lambda)$. Then, given any $(x,y) \in Y_0$, for each $i \in \{1,\dots,m\}$ and $\ell \in \mathbb{Z}$ we have that one of the following alternatives holds: $\max_{t\in I^+_{i,\ell}} x(t) < r$, $r < \max_{t\in I^+_{i,\ell}} x(t) < \rho$ or $\rho < \max_{t\in I^+_{i,\ell}} x(t) < R$.*

Let

$$\mathscr{B} := \{0,1,2\}^m$$

be the alphabet of the 3^m elements of the form $(\omega_1,\dots,\omega_m)$, where $\omega_i \in \{0,1,2\}$ for each $i = 1,\dots,m$.

We define a semiconjugation π between Y_0 and the set Σ_{3^m} associated with $\mathscr{B}$ as follows. Suppose that $\mu > \mu^*(\lambda)$. To each element $z = (x,y) \in Y_0$ the map π associates a sequence $\pi(z) = \mathcal{T} = (\mathcal{T}_\ell)_{\ell\in\mathbb{Z}} \in \Sigma_{3^m}$ defined as

$$\mathcal{T}_\ell = (\mathcal{T}^1_\ell,\dots,\mathcal{T}^m_\ell) \in \mathscr{B}, \quad \ell \in \mathbb{Z},$$

where, for $i = 1,\dots,m$,

- $\mathcal{T}^i_\ell = 0$, if $\max_{t\in I^+_{i,\ell}} x(t) < r$;

- $\mathcal{T}_\ell^i = 1$, if $r < \max_{t \in I_{i,\ell}^+} x(t) < \rho$;

- $\mathcal{T}_\ell^i = 2$, if $\rho < \max_{t \in I_{i,\ell}^+} x(t) < R$.

Lemma 8.3.2 guarantees that the above map is well-defined.

Now we are in position to state the main result of this section.

Theorem 8.3.1. *Suppose that $\mu > \mu^*(\lambda)$. Then the map $\pi \colon Y_0 \to \Sigma_{3^m}$ is continuous, surjective and such that the diagram*

$$\begin{tikzcd} Y_0 \arrow[r, "\psi"] \arrow[d, "\pi"'] & Y_0 \arrow[d, "\pi"] \\ \Sigma_{3^m} \arrow[r, "\sigma"'] & \Sigma_{3^m} \end{tikzcd}$$

commutes. Furthermore, for every integer $k \geq 1$, the counterimage of any k-periodic sequence in Σ_{3^m} contains at least a point $(u, y) \in Y_0$ such that $u(t)$ is a kT-periodic solution of $(\mathscr{E}_{\lambda,\mu})$.

Proof. Part of the statement follows immediately from our previous results. The surjectivity of the map π is a consequence of Theorem 8.2.2. Indeed, if $\mathcal{T} \in \Sigma_{3^m}$ is the null sequence then it is the image of the trivial solution $(0,0) \in Y_0$. On the other hand, given any non-null sequence $\mathcal{T} = (\mathcal{T}_\ell)_{\ell \in \mathbb{Z}}$, with $\mathcal{T}_\ell = (\mathcal{T}_\ell^1, \ldots, \mathcal{T}_\ell^m)$ for each $\ell \in \mathbb{Z}$, there exists at least one globally defined positive solution $u(t)$ to equation $(\mathscr{E}_{\lambda,\mu})$ such that

- $\max_{t \in I_{i,\ell}^+} u(t) < r$, if $\mathcal{T}_\ell^i = 0$;

- $r < \max_{t \in I_{i,\ell}^+} u(t) < \rho$, if $\mathcal{T}_\ell^i = 1$;

- $\rho < \max_{t \in I_{i,\ell}^+} u(t) < R$, if $\mathcal{T}_\ell^i = 2$.

Then π maps $(u(t), u'(t)) = (x(t), y(t)) \in Y_0$ to $\mathcal{T}$. In a similar way, Theorem 8.1.1 ensures that, for any integer $k \geq 1$, the counterimage of a k-periodic sequence in Σ_{3^m} can be chosen as a kT-periodic solution of (8.3.2).

The commutativity of the diagram follows from the fact that, whenever $(x(t), y(t))$ is a solution of (8.3.2), then $(x(t+T), y(t+T))$ is also a solution of the same system and, moreover, if $(\mathcal{T}_\ell)_{\ell \in \mathbb{Z}}$ is the sequence of symbols associated with $(x(t), y(t))$, then the sequence corresponding to $(x(t+T), y(t+T))$ must be $(\mathcal{T}_{\ell+1})_{\ell \in \mathbb{Z}}$. This proves (8.3.1).

Thus we have only to check the continuity of π. Let $\tilde{z} = (\tilde{x}, \tilde{y}) \in Y_0$ and $\tilde{\mathcal{T}} = \pi(\tilde{z})$. Let $z_n = (x_n, y_n) \in Y_0$ be a sequence such that $z_n \to \tilde{z}$ in Y_0. This means that $(x_n(t), y_n(t))$ converges uniformly to $(\tilde{x}(t), \tilde{y}(t))$ on any compact interval $[-NT, NT]$ of the real line. For any interval $I_{i,\ell}^+ \subseteq [-NT, NT]$, we have that either $\max_{I_{i,\ell}^+} \tilde{x} < r$ or $r < \max_{I_{i,\ell}^+} \tilde{x} < \rho$ or $\rho < \max_{I_{i,\ell}^+} \tilde{x} < R$. By the uniform convergence of the sequence of solutions on $I^+_{i,\ell}$, there exists an index $n^*_{i,\ell}$ such that, for each $n \geq n^*_{i,\ell}$, the solution $x_n(t)$ satisfies the same inequalities as $\tilde{x}(t)$ on the interval $I^+_{i,\ell}$. Hence, for any fixed N, there is an index

$$n^*_N := \max\bigl\{n^*_{i,\ell} \colon i = 1, \ldots, m,\ \ell = -N, \ldots, N-1\bigr\}$$

such that, setting $\mathcal{T}^n = \pi(z_n)$, it holds that $\mathcal{T}^n_\ell = \tilde{\mathcal{T}}_\ell$ for all $n \geq n^*_N$ and $\ell = -N, \ldots, N-1$. By the topology of Σ_{3^m}, this means that $\mathcal{T}^n$ converges to $\tilde{\mathcal{T}}$. This concludes the proof. □

From Theorem 8.3.1 many consequences can be produced. For instance, we can refine the set Y_0 in order to obtain an invariant set with dense periodic trajectories of any period. This follows via a standard procedure that we describe below for the reader's convenience.

Let Y_{per} be the set of all the pairs $(x, y) \in Y_0$ which are kT-periodic solutions of (8.3.2) for some integer $k \geq 1$ and let

$$Y := \mathrm{cl}(Y_{\mathrm{per}}) \subseteq Y_0,$$

where the closure is taken with respect to the distance in the space X. Clearly, the set Y is compact, invariant for the map ψ and Y_{per} is dense in Y. Then, from Theorem 8.3.1 we immediately have that for $\mu > \mu^*(\lambda)$ the map $\psi|_Y \colon Y \to Y$ is semiconjugate (via the surjection $\pi|_Y$) with the shift σ on Σ_{3^m} and, moreover, for every integer $k \geq 1$, the counterimage by π of any k-periodic sequence in Σ_{3^m} contains at least a point $(u, y) \in Y$ such that $u(t)$ is a kT-periodic solution of $(\mathscr{E}_{\lambda,\mu})$.

As a last step, we want to express our results in terms of the Poincaré map associated with system (8.3.2). To this end, we further suppose that the nonlinearity $g(s)$ is locally Lipschitz continuous on $\mathbb{R}^+$. This, in turn, implies the uniqueness of the solutions for the initial value problems associated with (8.3.2). We recall that the Poincaré map associated with system (8.3.2) is defined as

$$\Psi_T \colon \mathrm{dom}\,\Psi_T (\subseteq \mathbb{R}^2) \to \mathbb{R}^2, \quad z_0 = (x_0, y_0) \mapsto z(T, z_0),$$

where $z(t, z_0) = (x(t, z_0), y(t, z_0))$ is the solution of system (8.3.2) such that $x(0) = x_0$ and $y(0) = y_0$. The map Ψ_T is defined provided that the solutions can be extended to the interval $[0, T]$. In general the domain of Ψ_T is an open subset of $\mathbb{R}^2$ and Ψ_T is a homeomorphism of $\mathrm{dom}\,\Psi_T$ onto its image. In our case, due to the sublinear growth at infinity (g_∞), we have that $\mathrm{dom}\,\Psi_T = \mathbb{R}^2$ and Ψ_T is a homeomorphism of $\mathbb{R}^2$ onto itself.

Let

$$\mathcal{W}_0 := \bigl\{(x(0), y(0)) \in [0, R] \times [-K, K] \colon (x, y) \in Y_0\bigr\}$$

and define $\Pi \colon \mathcal{W}_0 \to \Sigma_{3^m}$ as

$$\Pi(z_0) := \pi(z(\cdot, z_0)), \quad z_0 \in \mathcal{W}_0.$$

Notice that the map Π is well-defined; indeed, if $z_0 \in \mathcal{W}_0$, then $z(\cdot, z_0) \in Y_0$.

The next result is an equivalent version of Theorem 8.3.1 where *chaotic dynamics* are described in terms of the Poincaré map.

Theorem 8.3.2. *Suppose that $\mu > \mu^*(\lambda)$. Then the map $\Pi\colon \mathcal{W}_0 \to \Sigma_{3^m}$ is continuous, surjective and such that the diagram*

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r, "\Psi_T"] \arrow[d, "\Pi"'] & \mathcal{W}_0 \arrow[d, "\Pi"] \\ \Sigma_{3^m} \arrow[r, "\sigma"'] & \Sigma_{3^m} \end{tikzcd}$$

commutes. Furthermore, for every integer $k \geq 1$, the counterimage of any k-periodic sequence in Σ_{3^m} contains at least a point $w \in \mathcal{W}_0$ which is a k-periodic point of the Poincaré map and so that the solution $u(t)$ of $(\mathscr{E}_{\lambda,\mu})$, with $(u(0), u'(0)) = w$, is a kT-periodic solution of $(\mathscr{E}_{\lambda,\mu})$.

Proof. Let $\zeta : \mathcal{W}_0 \to Y_0$ be the map which associates to any initial point z_0 the solution $z(\cdot, z_0)$ of (8.3.2) with $(x(0), y(0)) = z_0$. We consider the diagram

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r, "\Psi_T"] \arrow[d, "\zeta"'] & \mathcal{W}_0 \arrow[d, "\zeta"] \\ Y_0 \arrow[r, "\psi"'] & Y_0 \end{tikzcd}$$

and observe that the map ζ is bijective, continuous and with continuous inverse. Indeed, if $z_n \to z_0$ in $\mathbb{R}^2$, then $z(t, z_n)$ converges uniformly to $z(t, z_0)$ on the compact subsets of $\mathbb{R}$. The above diagram is also commutative because (by the uniqueness of the solutions to the initial value problems) the solution of (8.3.2) starting at the point $z(T, z_0)$ coincides with $z(t + T, z_0)$. From these remarks and the commutativity of the diagram in Theorem 8.3.1 we easily conclude. $\square$

We conclude this section with a final remark concerning a dynamical consequence of Theorem 8.3.2. Consider again the alphabet $\mathscr{B}$ of 3^m elements of the form $\omega = (\omega_1, \ldots, \omega_m)$, where $\omega_i \in \{0, 1, 2\}$ for each $i = 1, \ldots, m$. To each element $\omega \in \mathcal{B}$ we associate the set

$$\mathcal{K}_\omega := \left\{ w \in \mathcal{W}_0 \colon \begin{array}{c} \max_{t \in I_i^+} x(t, w) < r, \text{ if } \omega_i = 0 \\ r < \max_{t \in I_i^+} x(t, w) < \rho, \text{ if } \omega_i = 1 \\ \rho < \max_{t \in I_i^+} x(t, w) < R, \text{ if } \omega_i = 2 \end{array} \right\},$$

which is compact, as an easy consequence of Lemma 8.3.2. By definition, the sets $\mathcal{K}_\omega$ for $\omega \in \mathscr{B}$ are pairwise disjoint subsets of $[0, R] \times [-K, K]$. Hence, another way to describe our results is the following.

For each two-sided sequence $(\mathcal{T}_\ell)_{\ell\in\mathbb{Z}}$ *there exists a corresponding sequence* $(w_\ell)_{\ell\in\mathbb{Z}} \in (\mathcal{W}_0)^{\mathbb{Z}}$ *such that, for all* $\ell \in \mathbb{Z}$*,*

$$w_{\ell+1} = \Psi_T(w_\ell) \quad \text{and} \quad w_\ell \in \mathcal{K}_{\mathcal{T}_\ell}; \tag{8.3.4}$$

moreover, whenever $(\mathcal{T}_\ell)_{\ell\in\mathbb{Z}}$ *is a* k*-periodic sequence for some integer* $k \geq 1$*, there exists a* k*-periodic sequence* $(w_\ell)_{\ell\in\mathbb{Z}} \in (\mathcal{W}_0)^{\mathbb{Z}}$ *satisfying condition* (8.3.4)*.*

In this manner, we enter a setting of coin-tossing type dynamics widely explored in the literature. As a consequence, in the case $m = 1$, we obtain a dynamics on three symbols, described as itineraries for the Poincaré map jumping among three compact mutually disjoint sets $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2$. A previous result in this direction, but involving only two symbols, was obtained in [37] with a completely different approach.

Part III

Appendices

Appendix A

Leray-Schauder degree for locally compact operators on open possibly unbounded sets

In this appendix we present a general version of the *Leray-Schauder topological degree* for locally compact operators on open possibly unbounded sets in a normed linear space. Actually, we do not introduce the degree in the most general version, that is the one set in metric absolute neighborhood retracts (ANR) which is due to R. Nussbaum (cf. [147, 148]). We prefer to display the topological degree in the version best suited to the applications presented in this thesis.

We propose an axiomatic treatement and we omitt the proofs. For more details we refer to [134, 147, 148] and the references therein. Moreover, concerning the classical Brouwer degree and the classical Leray-Schauder degree we refer to the well known books about those theories (see, for instance, [46, 66, 103, 116]).

A.1 Definition, axioms and properties

Let X be a normed linear space, $\Omega \subseteq X$ an open (possibly unbounded) subset and $z \in X$. Consider a continuous map $\phi\colon \Omega \to X$ such that

$$\mathcal{S}_z := \big\{ x \in \Omega \colon x - \phi(x) = z \big\}$$

is a compact set (possibly empty) and such that there exists an open neighborhood V of $\mathcal{S}_z$ with $\overline{V} \subseteq \Omega$ such that $\phi|_{\overline{V}}$ is compact. If all the previous assumptions are satisfied, the triplet $(Id - \phi, \Omega, z)$ is called *admissible* (where $Id = Id_X$ is the identity map in X).

To the admissible triplet $(Id - \phi, \Omega, z)$ we associate the integer

$$\deg_{LS}(Id - \phi, \Omega, z),$$

called the *Leray-Schauder degree of* $Id-\phi$ *on* Ω *in* z, satisfying the following three axioms.

- *Additivity.* If Ω_1, Ω_2 are open and disjoint subsets of Ω such that $\mathcal{S}_z \subseteq \Omega_1 \cup \Omega_2$, then

$$\deg_{LS}(Id - \phi, \Omega, z) = \deg_{LS}(Id - \phi, \Omega_1, z) + \deg_{LS}(Id - \phi, \Omega_2, z).$$

- *Homotopic invariance.* Let $U \subseteq X \times [a, b]$ be an open subset (typically $U = \Omega \times [a, b]$, with $\Omega \subseteq X$). Let $h\colon U \to X$ be a continuous map. Define $h_\lambda(x) := h(x, \lambda)$ and $U_\lambda := \{x \in X \colon (x, \lambda) \in U\}$. Suppose that the set
$$\Sigma := \big\{(x, \lambda) \in U \colon x - h_\lambda(x) = z\big\}$$
is compact (possibly empty) and that there exists an open neighborhood W of Σ such that $h|_{\overline{W}}$ is a compact map. Then
$$\deg_{LS}(Id - h_\lambda, U_\lambda, z)$$
is constant with respect to $\lambda \in [a, b]$.

- *Normalization.* It holds that
$$\deg_{LS}(Id, \Omega, z) := \begin{cases} 1, & \text{if } z \in \Omega; \\ 0, & \text{if } z \in X \setminus \overline{\Omega}. \end{cases}$$

Dealing with the special case of an open and bounded set Ω in a real Banach space X and a completely continuous map $\phi\colon \overline{\Omega} \to X$ such that

$$x - \phi(x) \neq z, \quad x \in \partial\Omega,$$

clearly the triplet $(Id-\phi, \Omega, z)$ is admissible and it is easy to verify that the above definition of the Leray-Schauder degree reduces to the classical one.

From the *Additivity* of the topological degree, one can easily prove that $\deg_{LS}(Id - \phi, \emptyset, z) = 0$ and that the following properties hold (where we implicitly assume that $(Id - \phi, \Omega, z)$ is an admissible triplet).

- *Excision.* If Ω_0 is an open subset of Ω such that $\mathcal{S}_z \subseteq \Omega_0$, then

$$\deg_{LS}(Id - \phi, \Omega, z) = \deg_{LS}(Id - \phi, \Omega_0, z)$$

- *Existence theorem.* If $\deg_{LS}(Id-\phi, \Omega, z) \neq 0$, then $\mathcal{S}_z \neq \emptyset$, and hence there exists $\hat{x} \in \Omega$ such that $\hat{x} - \phi(\hat{x}) = z$.

We conclude this section by stating some additional properties of the Leray-Schauder degree which are relevant for our applications. For simplicity we take $z = 0$.

Theorem A.1.1 (Commutativity property)**.** *Let X_1, X_2 be normed linear spaces. Let $\Omega_i \subseteq X_i$, $i = 1, 2$, be an open (possibly unbounded) set. Let $\psi_1 \colon \Omega_1 \to X_2$ and $\psi_2 \colon \Omega_2 \to X_1$ be continuous maps. Consider the maps*

$$\psi_2 \circ \psi_1 \colon \psi_1^{-1}(\Omega_2)(\subseteq X_1) \to X_1 \quad \textit{and} \quad \psi_1 \circ \psi_2 \colon \psi_2^{-1}(\Omega_1)(\subseteq X_2) \to X_2$$

and the sets

$$\mathcal{S}_1 := \big\{x \in \psi_1^{-1}(\Omega_2) \colon x - \psi_2(\psi_1(x)) = 0\big\}$$

and

$$\mathcal{S}_2 := \big\{x \in \psi_2^{-1}(\Omega_1) \colon x - \psi_1(\psi_2(x))) = 0\big\}.$$

Assume that $\mathcal{S}_1$ (or $\mathcal{S}_2$) is compact (possibly empty) and that ψ_1 is compact on some open neighborhood of $\mathcal{S}_1$ (or ψ_2 is compact on some open neighborhood of $\mathcal{S}_2$, respectively). Then

$$\deg_{LS}(Id - \psi_2 \circ \psi_1, \psi_1^{-1}(\Omega_2), 0) = \deg_{LS}(Id - \psi_1 \circ \psi_2, \psi_2^{-1}(\Omega_1), 0).$$

In particular $\mathcal{S}_1 = \psi_2(\mathcal{S}_2)$ and $\mathcal{S}_2 = \psi_1(\mathcal{S}_1)$ are compact and the Leray-Schauder degrees in the above formula are defined.

As a direct corollary of the *Commutativity property*, one can deduce the *Reduction formula.*

Corollary A.1.1 (Reduction formula)**.** *Let X be a normed linear space. Let $\Omega \subseteq X$ be an open (possibly unbounded) set. Let $\phi \colon \Omega \to X$ be a continuous map such that the degree $\deg_{LS}(Id - \phi, \Omega, z)$ is defined. Let $Y \subseteq X$ be a subspace such that $\phi(\Omega) \subseteq Y$. Then*

$$\deg_{LS}(Id - \phi, \Omega, 0) = \deg_{LS}(Id_Y - \phi|_Y, \Omega \cap Y, 0).$$

In the statement of Corollary A.1.1, we implicitly identify ϕ with $j \circ \phi$, where $j \colon Y \to X$ is the (continuous) inclusion.

Finally we present the *Multiplicativity property.*

Theorem A.1.2 (Multiplicativity property)**.** *Let X be a normed linear space. Let $\Omega_1 \subseteq X_1$ and $\Omega_2 \subseteq X_2$ be open (possibly unbounded) sets. Let $\phi_1 \colon \Omega_1 \to X_1$ and $\phi_2 \colon \Omega_2 \to X_2$ be continuous maps such that the degrees $\deg_{LS}(Id_{X_i} - \phi_i, \Omega_i, 0)$, for $i = 1, 2$, are defined. Let $\phi \colon \Omega_1 \times \Omega_2 \to X_1 \times X_2$ be defined as $\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$. Then*

$$\begin{aligned} &\deg_{LS}(Id_{X_1 \times X_2} - \phi, \Omega_1 \times \Omega_2, 0) = \\ &= \deg_{LS}(Id_{X_1} - \phi_1, \Omega_1, 0) \deg_{LS}(Id_{X_2} - \phi_2, \Omega_2, 0). \end{aligned}$$

In the framework of this thesis, in most cases we need a simpler version of the general topological degree described above. Namely, in our applications we often deal with a completely continuous operator $\phi\colon X \to X$ and an open set $\Omega \subseteq X$. Since we focus on the existence of fixed points of a map ϕ, we take $z = 0$ and we are interested in studying the integer

$$\deg_{LS}(Id - \phi, \Omega, 0).$$

To prove the admissibility of $(Id - \phi, \Omega, 0)$ it is sufficient either to establish that

$$\mathcal{S}_0 = \bigl\{x \in \Omega\colon x - \phi(x) = 0\bigr\}$$

is compact or, equivalently, to show that the set of all possible fixed points of ϕ in the whole space X is contained in an open and bounded set W satisfying $x - \phi(x) \neq 0$, for all $x \in \partial(\Omega \cap W)$.

A.2 Computation of the degree: a useful theorem

In this section we present a theorem which is of crucial importance for our applications. First of all, we recall a result for the computation of the degree on open and bounded sets.

Theorem A.2.1. *Let X be a normed linear space and $\Omega \subseteq X$ be an open and bounded set. Let $\phi\colon \overline{\Omega} \to X$ be a compact map. If $F\colon \overline{\Omega} \times [0,+\infty[\to X$ is a compact map such that*

(i) $F(x,0) = \phi(x)$, for all $x \in \partial\Omega$;

(ii) $F(x,\alpha) \neq x$, for all $x \in \partial\Omega$ and $\alpha \geq 0$;

(iii) there exists $\alpha_0 \geq 0$ such that $F(x,\alpha) \neq x$, for all $x \in \overline{\Omega}$ and $\alpha \geq \alpha_0$;

then

$$\deg_{LS}(Id - \phi, \Omega, 0) = 0.$$

Moreover, if there exists $v \in X \setminus \{0\}$ such that $x \neq \phi(x) + \alpha v$, for all $x \in \partial\Omega$ and $\alpha \geq 0$, then conditions (i), (ii), (iii) are satisfied.

As we are going to state and prove a generalization of Theorem A.2.1, we omit the proof. In [63, pp. 67–68] the author proved the statement for an open ball (see also [145, Lemma 1.1]).

Now we consider open and possibly unbounded sets, as in the context of our applications.

Theorem A.2.2. *Let X be a normed linear space and $\Omega \subseteq X$ be an open set. Let $\phi\colon X \to X$ be a continuous map and $F\colon X \times [0,+\infty[\to X$ a completely continuous map. Suppose that* (*i*) $F(x, 0) = \phi(x)$, *for all* $x \in X$;

(*ii*) *for all* $\alpha \geq 0$ *there exists* $R_\alpha > 0$ *such that if there exist* $x \in \overline{\Omega}$ *and* $\zeta \in [0, \alpha]$ *such that* $x = F(x, \zeta)$, *then* $\|x\| \leq R_\alpha$ *and* $x \in \Omega$;

(*iii*) *there exists* $\alpha_0 \geq 0$ *such that* $x \neq F(x, \alpha)$, *for all* $x \in \overline{\Omega}$ *and* $\alpha \geq \alpha_0$.

Then the triplet $(Id - \phi, \Omega, 0)$ *is admissible and*

$$\deg_{LS}(Id - \phi, \Omega, 0) = 0.$$

Proof. Without loss of generality, we can assume that $R_{\alpha'} < R_{\alpha''}$, if $\alpha' < \alpha''$. The set $A := B(0, R_{\alpha_0+1}) \cap \Omega$ is open and bounded, and, by conditions (ii) and (iii), it contains all possible fixed points of $F(\cdot, \alpha)$ in $\overline{\Omega}$. Using also (i), we have that

$$\deg_{LS}(Id - \phi, \Omega, 0) = \deg_{LS}(Id - F(\cdot, 0), \Omega, 0) = \deg_{LS}(Id - F(\cdot, 0), A, 0).$$

Taking $h_\alpha := F(\cdot, \alpha)$, $\alpha \in [0, \alpha_0]$, as admissible homotopy, by (ii) and the homotopic invariance of the degree we obtain that

$$\deg_{LS}(Id - F(\cdot, \alpha), A, 0) = const., \qquad 0 \leq \alpha \leq \alpha_0.$$

By (iii), we conclude that

$$\deg_{LS}(Id - \phi, \Omega, 0) = \deg_{LS}(Id - F(\cdot, \alpha_0), A, 0) = 0.$$

This proves the theorem. □

Appendix B

Mawhin's coincidence degree

This appendix is devoted to the *coincidence degree*. First we recall the classical coincidence degree theory introduced by J. Mawhin for open and bounded sets in a normed linear space. Subsequently we present a more general version for locally compact operators on open and possibly unbounded sets. This latter version, which is the best suited to the applications given in the thesis, is based on the topological degree exhibited in Appendix A (see also [147, 148]). For more details, omitted proofs and applications, we refer to the classical books [89, 130, 132] (and the references therein) and to [169].

B.1 Definition and axioms

Let X and Z be normed linear spaces and let

$$L\colon \operatorname{dom} L (\subseteq X) \to Z$$

be a linear Fredholm mapping of index zero, i.e. $\operatorname{Im} L$ is a closed subspace of Z and $\dim(\ker L) = \operatorname{codim}(\operatorname{Im} L)$ are finite. We denote by $\ker L = L^{-1}(0) (\subseteq X)$ the *kernel* of L, by $\operatorname{Im} L \subseteq Z$ the *range* or *image* of L and by $\operatorname{coker} L = Z/\operatorname{Im} L$ the quotient space of Z under the equivalence relation $w_1 \sim w_2$ if and only if $w_1 - w_2 \in \operatorname{Im} L$. Thus $\operatorname{coker} L$ is a complementary subspace of $\operatorname{Im} L$ in Z.

From basic results of linear functional analysis, due to the fact that L is a Fredholm mapping, there exist linear continuous projections

$$P\colon X \to \ker L, \qquad Q\colon Z \to \operatorname{coker} L$$

so that

$$X = \ker L \oplus \ker P, \qquad Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

We denote by

$$K\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$$

the *right inverse* of L, i.e. $LK(w) = w$ for each $w \in \operatorname{Im} L$. Since $\ker L$ and $\operatorname{coker} L$ are finite dimensional vector spaces of the same dimension, once an orientation on both spaces is fixed, we choose a linear orientation-preserving isomorphism

$$J\colon \operatorname{coker} L \to \ker L.$$

Let

$$N\colon \operatorname{dom} N(\subseteq X) \to Z$$

be a (possibly nonlinear) *L-completely continuous* operator, namely N and $K(Id_Z - Q)N$ are continuous, and also $QN(B)$ and $K(Id_Z - Q)N(B)$ are relatively compact sets, for each bounded set $B \subseteq \operatorname{dom} N$. For example, N is L-completely continuous when N is continuous, maps bounded sets to bounded sets and K is a compact linear operator.

Now we define the *coincidence equation*

$$Lu = Nu, \quad u \in \operatorname{dom} L \cap \operatorname{dom} N. \tag{B.1.1}$$

One can prove that equation (B.1.1) is equivalent to the fixed point problem

$$u = \Phi(u) := Pu + JQNu + K_P(Id_Z - Q)Nu, \quad u \in \operatorname{dom} N. \tag{B.1.2}$$

Moreover, since N is L-completely continuous, we notice that the operator Φ is completely continuous.

Let $\mathcal{O} \subseteq \operatorname{dom} N$ be an open and *bounded* set such that

$$Lu \neq Nu, \quad \forall\, u \in \partial\mathcal{O} \cap \operatorname{dom} L.$$

The *coincidence degree of L and N in $\mathcal{O}$* is defined as

$$D_L(L - N, \mathcal{O}) := \deg_{LS}(Id - \Phi, \mathcal{O}, 0).$$

A remarkable result from coincidence degree theory guarantees that D_L is independent on the choice of the projectors P and Q. Moreover, it is also independent of the choice of the linear isomorphism J, provided that we have fixed an orientation on $\ker L$ and $\operatorname{coker} L$ and considered for J only orientation-preserving isomorphisms. Furthermore, this generalized degree has all the usual properties of Brouwer and Leray-Schauder degree, like additivity, excision and homotopic invariance (see [130, ch. II]). In particular, equation (B.1.1) has at least one solution in $\mathcal{O}$ if $D_L(L - N, \mathcal{O}) \neq 0$. We will list later the main properties of the coincidence degree in a more general setting.

In our applications we need to consider a slight extension of the coincidence degree to open *possibly unbounded* sets. To this purpose, we just follow the standard approach used in the theory of fixed point index to define the Leray-Schauder degree for locally compact maps on arbitrary open sets (cf. [102, 134, 147, 148] and Appendix A). We underline that extensions of coincidence degree to the case of general open sets have been already considered, for instance, in [53, 136, 140].

Let $\Omega \subseteq \operatorname{dom} N$ be an open set and suppose that the solution set

$$\operatorname{Fix}(\Phi,\Omega) := \bigl\{u \in \Omega \colon u = \Phi u\bigr\} = \bigl\{u \in \Omega \cap \operatorname{dom} L \colon Lu = Nu\bigr\}$$

is compact. The extension of the Leray-Schauder degree in Appendix A allows to define

$$\deg_{LS}(Id - \Phi, \Omega, 0) := \deg_{LS}(Id - \Phi, \mathcal{V}, 0),$$

where $\mathcal{V}$ is an open and bounded set with

$$\operatorname{Fix}(\Phi,\Omega) \subseteq \mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \Omega.$$

One can check that the definition is independent of the choice of $\mathcal{V}$. Accordingly, we define the *coincidence degree of L and N in* Ω as

$$D_L(L-N,\Omega) := D_L(L-N,\mathcal{V}) = \deg_{LS}(Id - \Phi, \mathcal{V}, 0),$$

with $\mathcal{V}$ as above. Using the excision property of the Leray-Schauder degree, it is easy to check that if Ω is an open and bounded set satisfying $Lu \neq Nu$, for all $u \in \partial\Omega \cap \operatorname{dom} L$, this definition is exactly the usual definition of coincidence degree described above.

Combining the properties of coincidence degree from [130, ch. II] with the theory of fixed point index for locally compact operators, it is possible to derive the following versions of the main properties of the degree.

- *Additivity.* Let Ω_1, Ω_2 be open and disjoint subsets of Ω such that $\operatorname{Fix}(\Phi,\Omega) \subseteq \Omega_1 \cup \Omega_2$. Then

$$D_L(L-N,\Omega) = D_L(L-N,\Omega_1) + D_L(L-N,\Omega_2).$$

- *Excision.* Let Ω_0 be an open subset of Ω such that $\operatorname{Fix}(\Phi,\Omega) \subseteq \Omega_0$. Then

$$D_L(L-N,\Omega) = D_L(L-N,\Omega_0).$$

- *Existence theorem.* If $D_L(L-N,\Omega) \neq 0$, then $\operatorname{Fix}(\Phi,\Omega) \neq \emptyset$, hence there exists $u \in \Omega \cap \operatorname{dom} L$ such that $Lu = Nu$.

- *Homotopic invariance.* Let $H \colon [0,1] \times \Omega \to X$, $H_\vartheta(u) := H(\vartheta,u)$, be a continuous homotopy such that

$$\mathcal{S} := \bigcup_{\vartheta \in [0,1]} \{u \in \Omega \cap \operatorname{dom} L \colon Lu = H_\vartheta u\}$$

is a compact set and there exists an open neighborhood $\mathcal{W}$ of $\mathcal{S}$ such that $\overline{\mathcal{W}} \subseteq \Omega$ and $(K_P(Id_Z - Q)H)|_{[0,1]\times\overline{\mathcal{W}}}$ is a compact map. Then the map $\vartheta \mapsto D_L(L - H_\vartheta, \Omega)$ is constant on $[0, 1]$.

In the present thesis, we apply this general setting in the following manner. Usually, we deal with an L-completely continuous operator $\mathcal{N}\colon X \to Z$ and an open set $\mathcal{A}$ such that the solution set $\{u \in \overline{\mathcal{A}} \cap \operatorname{dom} L\colon Lu = \mathcal{N}u\}$ is compact and disjoint from $\partial\mathcal{A}$. Therefore $D_L(L - \mathcal{N}, \mathcal{A})$ is well-defined. We proceed analogously when dealing with homotopies.

B.2 Computation of the degree: useful results

A typical degree theoretic approach in order to prove the existence of nontrivial solutions to the coincidence equation (B.1.1) consists into showing that the degree changes from small balls to large balls, so that the additivity property of the degree guarantees the existence of a solution in the annular domain. From this point of view, results ensuring that the degree is zero on some open sets may be useful for the applications.

In order to present our results in the version best suited to the applications of the thesis, from now on we suppose that $\operatorname{dom} N = X$. Analogous results are valid even when dealing with an arbitrary $\operatorname{dom} N \subseteq X$.

The following lemma is of crucial importance in order to compute the coincidence degree in open and bounded sets. Using a reduction property, it relates the coincidence degree to the finite dimensional Brouwer degree of the operator N projected into $\ker L$ (see [135] for an interesting discussion on the reduction formula in the context of coincidence degree). This result was exhibited in [128] in its abstract form and, previously, in [127] in the context of periodic problems for ODEs. We give only a sketch of the proof and we refer to [89, Theorem IV.1] and [132, Theorem 2.4] for the missing details.

Lemma B.2.1 (Mawhin, 1969-1972). *Let L and N be as in Section B.1 and let $\Omega \subseteq X$ be an open and bounded set. Suppose that*

$$Lu \neq \vartheta Nu, \quad \forall\, u \in \partial\Omega \cap \operatorname{dom} L,\ \forall\, \vartheta \in \,]0, 1],$$

and

$$QN(u) \neq 0, \quad \forall\, u \in \partial\Omega \cap \ker L.$$

Then

$$D_L(L - N, \Omega) = \deg_B(-JQN|_{\ker L}, \Omega \cap \ker L, 0).$$

Proof. Consider the operator Φ_ϑ defined as

$$\Phi_\vartheta(u) := Pu + JQNu + \vartheta K_P(Id_Z - Q)Nu, \quad \text{for } \ \vartheta \in [0, 1],$$

and observe that $\Phi_1 = \Phi$ and Φ_0 has finite dimensional range in $\ker L$. The assumptions of the lemma imply that $u \neq \Phi_\vartheta u$, for all $u \in \partial\Omega$ and $\vartheta \in [0,1]$. The homotopic invariance and the reduction property of the Leray-Schauder degree then give

$$\begin{aligned} D_L(L-N,\Omega) &= \deg_{LS}(Id - \Phi_1, \Omega, 0) = \deg_{LS}(Id - \Phi_0, \Omega, 0) \\ &= \deg_B(-JQN|_{\ker L}, \Omega \cap \ker L, 0). \end{aligned}$$

Hence the lemma is proved. □

The next result is a simple adaptation to our setting of a well know lemma (see [145] and also Theorem A.2.1).

Lemma B.2.2. *Let L and N be as in Section B.1 and let $\Omega \subseteq X$ be an open and bounded set. Suppose that $v \neq 0$ is a vector such that*

$$Lu \neq Nu + \alpha v, \quad \forall\, u \in \partial\Omega \cap \operatorname{dom} L,\ \forall\, \alpha \geq 0.$$

Then

$$D_L(L-N,\Omega) = 0.$$

Proof. First of all, we observe that $u \in \operatorname{dom} L$ is a solution of the equation $Lu = Nu + \alpha v$ if and only if $u \in X$ is a solution of

$$u = \Phi u + \alpha v^*, \quad \text{with } v^* := JQv + K_P(Id_Z - Q)v, \tag{B.2.1}$$

where Φ is the operator defined in (B.1.2). We claim that $v^* \neq 0$. Indeed, if $v^* = 0$, then $Qv = 0$ and also $K_P v = 0$. Hence, $v \in \operatorname{Im} L$ and therefore $v = LK_P v = 0$, a contradiction. Thus the claim is proved.

Since Φ is compact on the bounded set $\overline{\Omega}$, we have that

$$M := \sup_{u \in \overline{\Omega}} \|u - \Phi u\| < \infty.$$

We conclude that, if we fix any number

$$\alpha_0 > \frac{M}{\|v^*\|},$$

then (B.2.1) has no solutions on $\overline{\Omega}$ for all $\alpha = \alpha_0$ (furthermore, there are no solutions also for $\alpha \geq \alpha_0$).

By the homotopic invariance of the coincidence degree (using $\alpha \in [0,\alpha_0]$ as a parameter), we find

$$D_L(L-N,\Omega) = \deg_{LS}(Id - \Phi, \Omega, 0) = \deg_{LS}(Id - \Phi - \alpha_0 v^*, \Omega, 0) = 0.$$

Hence the result is proved. □

From the proof of Lemma B.2.2 it is clear that the following variant holds.

Lemma B.2.3. *Let L and N be as in Section B.1 and let $\Omega \subseteq X$ be an open and bounded set. Suppose that there exist a vector $v \neq 0$ and a constant $\alpha_0 > 0$ such that*

$$Lu \neq Nu + \alpha v, \quad \forall\, u \in \partial\Omega \cap \operatorname{dom} L,\ \forall\, \alpha \in [0, \alpha_0],$$

and

$$Lu \neq Nu + \alpha_0 v, \quad \forall\, u \in \operatorname{dom} L \cap \Omega.$$

Then

$$D_L(L - N, \Omega) = 0.$$

Finally, we state and prove a key theorem for the computation of the degree in open (possibly unbounded) sets. This result is a more general version of Lemma B.2.3 (cf. Theorem A.2.2).

Theorem B.2.1. *Let L and N be as above and let $\Omega \subseteq X$ be an open set. Suppose that there exist a vector $v \neq 0$ and a constant $\alpha_0 > 0$ such that*

- (i) *$Lu \neq Nu + \alpha v$, for all $u \in \partial\Omega \cap \operatorname{dom} L$ and for all $\alpha \geq 0$;*
- (ii) *for all $\beta \geq 0$ there exists $R_\beta > 0$ such that if there exist $u \in \overline{\Omega} \cap \operatorname{dom} L$ and $\alpha \in [0, \beta]$ with $Lu = Nu + \alpha v$, then $\|u\|_X \leq R_\beta$;*
- (iii) *there exists $\alpha_0 > 0$ such that $Lu \neq Nu + \alpha v$, for all $u \in \Omega \cap \operatorname{dom} L$ and $\alpha \geq \alpha_0$.*

Then

$$D_L(L - N, \Omega) = 0.$$

Proof. For $\alpha \geq 0$, let us consider the set

$$\mathcal{R}_\alpha := \bigl\{u \in \overline{\Omega} \cap \operatorname{dom} L \colon Lu = Nu + \alpha v\bigr\} = \bigl\{u \in \overline{\Omega} \colon u = \Phi u + \alpha v^*\bigr\},$$

where $v^* := JQv + K_P(Id_Z - Q)v$. Without loss of generality, we assume that $R_{\alpha'} < R_{\alpha''}$ for $\alpha' < \alpha''$. By conditions (i), for all $\alpha \geq 0$, the solution set $\mathcal{R}_\alpha$ is disjoint from $\partial\Omega$. Moreover, by conditions (ii) and (iii), $\mathcal{R}_\alpha$ is contained in $\Omega \cap B(0, R_{\alpha_0+1})$. So $\mathcal{R}_\alpha$ is bounded, and hence compact. In this manner we have proved that the coincidence degree $D_L(L - N - \alpha v, \Omega)$ is well-defined for any $\alpha \geq 0$.

Now, condition (iii), together with the property of existence of solutions when the degree D_L is non-zero, implies that there exists $\alpha_0 \geq 0$ such that

$$D_L(L - N - \alpha_0 v, \Omega) = 0.$$

On the other hand, from condition (ii) applied to $\beta = \alpha_0$, repeating the same argument as above, we find that the set

$$\mathcal{S} := \bigcup_{\alpha\in[0,\alpha_0]} \mathcal{R}_\alpha = \bigcup_{\alpha\in[0,\alpha_0]} \bigl\{u \in \overline{\Omega} \cap \operatorname{dom} L \colon Lu = Nu + \alpha v\bigr\}$$

is a compact subset of Ω. Hence, by the homotopic invariance of the coincidence degree, we have that

$$D_L(L - N, \Omega) = D_L(L - N - \alpha_0 v, \Omega) = 0.$$

This concludes the proof. $\square$

Appendix C

Maximum principles and a change of variable

This appendix is devoted to some technical results which constitute important tools for the discussion in the present thesis. More precisely, in Section C.1 we show two *maximum principles* which guarantee that the solutions of the considered boundary value problems are actually non-negative or positive (in the sense explained in the first chapters of the thesis). Subsequently, in Section C.2 we introduce a *change of variable* that allows us to transform an equation of the form

$$u'' + c(t)u' + f(t,u) = 0$$

into the differential equation

$$u'' + h(t,u) = 0.$$

In this manner, we will notice that, when Dirichlet or Neumann conditions are taken into account, we can reduce our discussion to an equation of a simpler form.

C.1 Maximum principles

In this section we deal with the second order nonlinear differential equation

$$u'' + h(t,u) = 0, \tag{C.1.1}$$

where $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. We will present some *maximum principles* that ensure the non-negativity or the positivity of the solutions to the Dirichlet/Neumann/periodic boundary value problem associated with (C.1.1).

The first result concerns the solutions of the Dirichlet boundary value problem

$$\begin{cases} u'' + h(t,u) = 0 \\ u(0) = u(T) = 0. \end{cases} \tag{C.1.2}$$

We omit the standard proof (see, for instance, [62, 124]), since in the sequel we will prove an analogous result (cf. Lemma C.1.2).

Lemma C.1.1. *Let $h\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be an L^1-Carathéodory function.*

(*i*) *If*

$$h(t,s) \geq 0, \quad \text{a.e. } t\in[0,T], \text{ for all } s\leq 0,$$

then any solution of (C.1.2) *is non-negative on* $[0,T]$.

(*ii*) *If $h(t,0)\equiv 0$ and there exist $k_1,k_2\in L^1([0,T],\mathbb{R}^+)$ such that*

$$\liminf_{s\to 0^+}\frac{h(t,s)}{s} \geq -k_1(t), \quad \text{uniformly a.e. } t\in[0,T];$$
$$\limsup_{s\to 0^+}\frac{h(t,s)}{s} \leq k_2(t), \quad \text{uniformly a.e. } t\in[0,T];$$

then every nontrivial non-negative solution $u(t)$ of (C.1.2) *satisfies $u(t)>0$, for all $t\in\,]0,T[$ and, moreover, $u'(0)>0>u'(T)$.*

We stress that the same result is valid also when considering Sturm-Liouville boundary conditions of the form

$$\begin{cases} \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(T) - \delta u'(T) = 0, \end{cases} \tag{C.1.3}$$

where $\alpha,\beta,\gamma,\delta\geq 0$ with $\gamma\beta+\alpha\gamma+\alpha\delta>0$ (cf. Section 2.4 and the subsequent sections).

Lemma C.1.1 is stated in a form which is useful for the applications presented in this thesis. We notice that, for example, assertion (*ii*) can be equivalently expressed in a simpler manner: in effect if we only suppose that there exists $k\in L^1([0,T],\mathbb{R}^+)$ such that

$$\limsup_{s\to 0^+}\frac{|h(t,s)|}{s} \leq k(t), \quad \text{uniformly a.e. } t\in[0,T],$$

the conclusion remains valid.

Now, we consider the boundary value problem

$$\begin{cases} u'' + h(t,u) = 0 \\ \mathscr{B}(u,u') = \underline{0}, \end{cases} \tag{C.1.4}$$

where $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. Recalling the notation introduced in Chapter 3, by $\mathscr{B}(u,u') = \underline{0}$ we mean the Neumann or the periodic boundary conditions on $[0,T]$.

In this framework, the following result holds.

Lemma C.1.2. *Let $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function.*

(*i*) *If*

$$h(t,s) > 0, \quad \text{a.e. } x \in [0,T], \text{ for all } s < 0,$$

then any solution of (C.1.4) *is non-negative on* $[0,T]$.

(*ii*) *If $h(t,0) \equiv 0$ and there exists $k \in L^1([0,T],\mathbb{R}^+)$ such that*

$$\limsup_{s\to 0^+} \frac{|h(t,s)|}{s} \leq k(t), \quad \text{uniformly a.e. } t \in [0,T],$$

then every nontrivial non-negative solution $u(t)$ of (C.1.4) *satisfies $u(t) > 0$, for all $t \in [0,T]$.*

Proof. (*i*). By contradiction, suppose that there exists a solution $u(t)$ of (C.1.4) and $\hat{t} \in [0,T]$ such that $u(\hat{t}) < 0$. Let $]t_1,t_2[\subseteq]0,T[$ be the maximal open interval containing $\hat{t}$ with $u(t) < 0$, for all $t_1 < t < t_2$. Since $u''(t) < 0$ for a.e. $t \in [t_1,t_2]$, an elementary convexity argument shows that $0 < t_1 < t_2 < T$ is not possible. Similarly, also $u(t) < 0$ for all $t \in]0,T[$ can be excluded, otherwise, $0 > \int_0^T u''(t)\,dt = u'(T) - u'(0)$, contradicting the boundary conditions. Hence, there are only two possibilities: either $t_1 = 0$ and $t_2 < T$, or $0 < t_1$ and $t_2 = T$. Suppose $t_1 = 0$ (the other case can be treated in a similar manner). In this case, $u(0) \leq 0$ and moreover $u'(0) > 0$ (otherwise, by concavity, one has $u(t) < 0$ for all $t \in]0,T]$, a situation previously excluded). This already gives a contradiction with the Neumann boundary condition at $t = 0$. On the other hand, if we consider the periodic boundary condition, we have that $u(T) = u(0) \leq 0$ and $u'(T) = u'(0) > 0$. Hence, by the concavity of u on the intervals where $u(t) < 0$, we obtain that $u(t) < 0$ for every $t \in [0,T[$, a contradiction.

(*ii*). By contradiction, suppose that there exists a solution $u(t) \geq 0$ of (C.1.4) and $t^* \in [0,T]$ such that $u(t^*) = 0$ (so, $u'(t^*) = 0$).

We claim that there exists a real number $\varepsilon > 0$ such that $u(t) = 0$, for all $t \in [t^* - \varepsilon, t^* + \varepsilon]$. So that $u \equiv 0$ on $[0,T]$, a contradiction.

From the hypotheses, we obtain that there exists $\delta > 0$ such that

$$|h(t,s)| \leq k_*(t)s, \quad \text{a.e. } t \in [0,T],\ \forall\, 0 \leq s \leq \delta,$$

where $k_*(t) := k(t) + 1$. Using the continuity of $u(t)$, we fix $\varepsilon > 0$ such that $0 \leq u(t) \leq \delta$, for all $t \in [t^* - \varepsilon, t^* + \varepsilon]$.

We employ $\|(\xi_1,\xi_2)\| = |\xi_1| + |\xi_2|$ as a standard norm in $\mathbb{R}^2$. For all $t \in \,]t^*, t^*+\varepsilon]$ we have

$$\begin{aligned}
0 \leq \|(u(t),u'(t))\| = |u(t)| + |u'(t)| &= u(t) + |u'(t)| = \\
&= u(t^*) + \int_{t^*}^{t} u'(\xi)\,d\xi + \left| u'(t^*) + \int_{t^*}^{t} -h(\xi,u(\xi))\,d\xi \right| \\
&\leq \int_{t^*}^{t} |u'(\xi)|\,d\xi + \int_{x^*}^{t} |h(\xi,u(\xi))|\,d\xi \\
&\leq \int_{t^*}^{t} \Big[k_*(\xi)|u(\xi)| + |u'(\xi)| \Big]\,d\xi \\
&\leq \int_{t^*}^{t} (k_*(\xi)+1)(|u(\xi)| + |u'(\xi)|)\,d\xi.
\end{aligned}$$

Using the classical Gronwall's inequality, we attain

$$0 \leq u(t) \leq \|(u(t),u'(t))\| = 0, \quad \forall\, t \in \,]t^*, t^*+\varepsilon].$$

With an analogous computation one can prove that $u(t) = 0$ for all $t \in [t^*-\varepsilon, t^*[$. Hence the claim and (ii) are proved. $\square$

Remark C.1.1. The maximum principle just presented can be also stated for the more general boundary value problem

$$\begin{cases} u'' + \tilde{f}(t,u,u') = 0, & 0 < t < T, \\ \mathscr{B}(u,u') = \underline{0}, \end{cases}$$

where $\tilde{f}\colon [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is an L^p-Carathéodory function as in Section 3.1, hence equal to $-s$ for $s \leq 0$ and satisfying conditions (f_1) and (f_2). The proof of this result is the same as that just viewed with minor changes. See also [62]. $\triangleleft$

We finally underline that the maximum principles presented in this section can be also stated in the case of a more general differential operator of the form

$$u \mapsto -u'' - cu',$$

where $c \in \mathbb{R}$ is a constant. The proofs of this more general results are analogous to the one presented above. In this case the convexity argument largely employed in the proof above is replaced by the fact that the map $t \mapsto e^{ct}u'(t)$ is non-increasing in the intervals where $h(t,s) \geq 0$ and non-decreasing in the intervals where $h(t,s) \leq 0$ (compare to the discussion in Remark 4.3.4).

C.2 A change of variable

In this section we exhibit a standard change of variable that allows us to transform the second order differential equation

$$v'' + c(x)v' + f(x,v) = 0, \quad x_1 < x < x_2, \tag{C.2.1}$$

into the equation

$$u'' + h(t,u) = 0, \quad 0 < t < T. \tag{C.2.2}$$

We will remark that a similar transformation is used in this thesis to reduce an elliptic equation with a radially symmetric weight in an annular domain to an ordinary differential equation in an interval (cf. Section 1.4.3, where this technique has been used for the first time in the thesis). We also refer to [13, 28] for analogous changes of variable.

We start from equation (C.2.1). Let $f\colon [x_1,x_2]\times\mathbb{R}\to\mathbb{R}$ be an L^1-Carathéodory function and let $c\colon [x_1,x_2]\to\mathbb{R}$ be an L^1-function. Now, we define the change of variable. Let

$$t = \psi(x) := \int_{x_1}^{x} e^{-\int_{x_1}^{\xi} c(\zeta)\,d\zeta}\,d\xi,$$

where $\psi\colon [x_1,x_2]\to[0,T]$ is an increasing $\mathcal{C}^2$-diffeomorphism with

$$T := \int_{x_1}^{x_2} e^{-\int_{x_1}^{\xi} c(\zeta)\,d\zeta}\,d\xi.$$

Consequently,

$$x = \varphi(t) := \psi^{-1}(t).$$

From now on, we implicitly suppose that $t\in[0,T]$ and $x\in[x_1,x_2]$. Next, we define

$$u(t) := v(\varphi(t)) = v(x).$$

Hence, we have

$$u'(t) = v'(\varphi(t))\varphi'(t), \qquad u''(t) = v''(\varphi(t))(\varphi'(t))^2 + v'(\varphi(t))\varphi''(t).$$

Using equation (C.2.1), we obtain

$$\begin{aligned} u''(t) &= -(\varphi'(t))^2\big[c(\varphi(t))v'(\varphi(t)) + f(\varphi(t), v(\varphi(t)))\big] + v'(\varphi(t))\varphi''(t) \\ &= -(\varphi'(t))^2 f(\varphi(t),u(t)) + v'(\varphi(t))\big[\varphi''(t) - (\varphi'(t))^2 c(\varphi(t))\big]. \end{aligned}$$

We claim that

$$\varphi''(t) - (\varphi'(t))^2 c(\varphi(t)) = 0.$$

First, we observe that

$$\psi'(x) = e^{-\int_{x_1}^{x} c(\zeta)\,d\zeta} > 0$$

and moreover that

$$1 = (\psi(\varphi(t)))' = \psi'(\varphi(t))\varphi'(t) = e^{-\int_{x_1}^{x} c(\zeta)\,d\zeta}\varphi'(t).$$

Therefore, we have

$$\begin{aligned}\varphi''(t) &= \frac{d}{dt}\varphi'(t) = \frac{d}{dt}\frac{1}{\psi'(\varphi(t))} = \varphi'(t)\frac{d}{dx}\frac{1}{\psi'(x)} \\ &= \varphi'(t)c(x)e^{\int_{x_1}^{x} c(\zeta)\,d\zeta} = (\varphi'(t))^2 c(x).\end{aligned}$$

The claim is thus proved. Consequenlty, we find that

$$u''(t) + (\varphi'(t))^2 f(\varphi(t), u(t)) = 0.$$

Defining

$$h(t,s) := (\varphi'(t))^2 f(\varphi(t), s), \quad t \in [0,T],\ s \in \mathbb{R},$$

we have that $h\colon [x_1, x_2] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. We conclude that the change of variable $t = \psi(x)$ transforms (C.2.1) into (C.2.2).

As an alternative, we notice that also the change of variable $t = K\psi(x)$, for a constant $K > 0$, is suitable for our purpose.

We now show an application of the change of variable illustrated above. In the present thesis, when dealing with Dirichlet and Neumann boundary value problems associated to a second order ordinary differential equation of the form (C.2.2), our existence and multiplicity results for positive solutions induce analogous results for positive radially symmetric solutions to Dirichlet and Neumann problems associated with an elliptic partial differential equation of the form

$$-\Delta\phi = q(x)g(\phi) \quad \text{in } \Omega, \tag{C.2.3}$$

where $\Omega := \{x \in \mathbb{R}^N : R_1 < \|x\| < R_2\}$ is an open annular domain (with $0 < R_1 < R_2$) and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^N$ (for $N \geq 2$). In order to study (classical) radial solutions, we suppose that $q\colon \overline{\Omega} \to \mathbb{R}$ and there exists a continuous function $b\colon [R_1, R_2] \to \mathbb{R}$ such that

$$q(x) = b(\|x\|), \quad \forall\, x \in \overline{\Omega}.$$

Accordingly, looking for solutions of the form $\phi(x) = v(\|x\|)$ where $v(r)$ is a scalar function defined on $[R_1, R_2]$, we can write equation (C.2.3) as

$$v'' + \frac{N-1}{r}v' + b(r)g(v) = 0, \quad R_1 < r < R_2. \tag{C.2.4}$$

Equation (C.2.4) is of the same form as equation (C.2.1), setting

$$c(r) := \frac{N-1}{r}, \quad f(r,v) := b(r)g(v), \quad x_1 := R_1, \quad x_2 := R_2.$$

By performing the change of variable described above

$$t = \psi(r) := (R_1)^{1-N} \int_{R_1}^{r} e^{-\int_{R_1}^{\xi} c(\zeta)\, d\zeta}\, d\xi = \int_{R_1}^{r} \xi^{1-N}\, d\xi,$$

and by defining

$$T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi, \quad r(t) := \varphi(t) = \psi^{-1}(t), \quad u(t) := v(r(t)),$$

we convert (C.2.4) into

$$u'' + a(t)g(u) = 0, \quad 0 < t < T, \tag{C.2.5}$$

where

$$a(t) := r(t)^{2(N-1)} b(r(t)), \quad t \in [0, T],$$

since

$$r'(t) = \varphi'(t) = \frac{1}{\psi'(r(t))} \quad \text{and} \quad \psi'(r(t)) = r(t)^{1-N}.$$

Therefore, we have a correspondence between solutions to (C.2.5) and radial solutions to (C.2.3) (cf. also [13, 28])

We conclude this section, with a discussion about boundary conditions. We are going to illustrate how the different boundary conditions transform under the change of variable described above. First, we observe that the change of variable $t = \psi(x)$ gives

$$u(t) = v(x)$$

and

$$u'(t) = v'(x)\varphi'(t) = \frac{v'(x)}{\psi'(x)} = v'(x) e^{\int_{x_1}^{x} c(\zeta)\, d\zeta}.$$

Then, we have

$$u(0) = v(x_1), \quad u(T) = v(x_2), \quad u'(0) = 0, \quad u'(T) = v'(x_2) e^{\int_{x_1}^{x_2} c(\zeta)\, d\zeta}.$$

Consequently, we deduce that Dirichlet and Neumann boundary value problems in $[x_1, x_2]$ associated with (C.2.1) become Dirichlet and Neumann boundary value problems in $[0, T]$ associated with (C.2.2), respectively.

Concerning Sturm-Liouville boundary conditions (namely (C.1.3)), only the case

$$u'(0) = u(T) = 0 \quad \text{or} \quad u(0) = u'(T) = 0$$

(i.e. when $\alpha = \delta = 0$ or $\beta = \gamma = 0$, respectively) keep the same form under our change of variable. Finally, it is obvious to note that the periodic boundary conditions are not preserved under the change of variable displayed in this section.

Bibliography

[1] N. Ackermann, Long-time dynamics in semilinear parabolic problems with autocatalysis, in: Recent progress on reaction-diffusion systems and viscosity solutions, World Sci. Publ., Hackensack, NJ, 2009, pp. 1–30.

[2] R. A. Adams, Sobolev spaces, Pure and Applied Mathematics, Vol. 65, Academic Press, New York-London, 1975.

[3] S. Alama, G. Tarantello, On semilinear elliptic equations with indefinite nonlinearities, Calc. Var. Partial Differential Equations 1 (1993) 439–475.

[4] S. Alama, G. Tarantello, Elliptic problems with nonlinearities indefinite in sign, J. Funct. Anal. 141 (1996) 159–215.

[5] S. Aljančić, D. Arandelović, $\mathscr{O}$-regularly varying functions, Publ. Inst. Math. (Beograd) (N.S.) 22 (36) (1977) 5–22.

[6] H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, J. Functional Analysis 11 (1972) 346–384.

[7] H. Amann, J. López-Gómez, A priori bounds and multiple solutions for superlinear indefinite elliptic problems, J. Differential Equations 146 (1998) 336–374.

[8] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519–543.

[9] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Functional Analysis 14 (1973) 349–381.

[10] D. L. T. Anderson, Stability of time-dependent particlelike solutions in nonlinear field theories. II, J. Math. Phys. 12 (1971) 945–952.

[11] F. V. Atkinson, W. N. Everitt, K. S. Ong, On the m-coefficient of Weyl for a differential equation with an indefinite weight function, Proc. London Math. Soc. 29 (1974) 368–384.

[12] B. Aulbach, B. Kieninger, On three definitions of chaos, Nonlinear Dyn. Syst. Theory 1 (2001) 23–37.

[13] C. Bandle, C. V. Coffman, M. Marcus, Nonlinear elliptic problems in annular domains, J. Differential Equations 69 (1987) 322–345.

[14] C. Bandle, M. A. Pozio, A. Tesei, The asymptotic behavior of the solutions of degenerate parabolic equations, Trans. Amer. Math. Soc. 303 (1987) 487–501.

[15] C. Bandle, M. A. Pozio, A. Tesei, Existence and uniqueness of solutions of nonlinear Neumann problems, Math. Z. 199 (1988) 257–278.

[16] M. Barnsley, Fractals everywhere, Academic Press, Inc., Boston, MA, 1988.

[17] V. L. Barutello, A. Boscaggin, G. Verzini, Positive solutions with a complex behavior for superlinear indefinite ODEs on the real line, J. Differential Equations 259 (2015) 3448–3489.

[18] A. K. Ben-Naoum, C. De Coster, On the existence and multiplicity of positive solutions of the p-Laplacian separated boundary value problem, Differential Integral Equations 10 (1997) 1093–1112.

[19] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, Superlinear indefinite elliptic problems and nonlinear Liouville theorems, Topol. Methods Nonlinear Anal. 4 (1994) 59–78.

[20] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, Variational methods for indefinite superlinear homogeneous elliptic problems, NoDEA Nonlinear Differential Equations Appl. 2 (1995) 553–572.

[21] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Rational Mech. Anal. 82 (1983) 313–345.

[22] S. M. Berman, High level sojourns of a diffusion process on a long interval, Z. Wahrsch. Verw. Gebiete 62 (1983) 185–199.

[23] J. Berstel, D. Perrin, The origins of combinatorics on words, European J. Combin. 28 (2007) 996–1022.

[24] N. P. Bhatia, G. P. Szegö, Stability theory of dynamical systems, Die Grundlehren der mathematischen Wissenschaften, Band 161, Springer-Verlag, New York-Berlin, 1970.

[25] N. H. Bingham, C. M. Goldie, J. L. Teugels, Regular variation, vol. 27 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1987.

[26] L. S. Block, W. A. Coppel, Dynamics in one dimension, vol. 1513 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1992.

[27] D. Bonheure, J. M. Gomes, P. Habets, Multiple positive solutions of superlinear elliptic problems with sign-changing weight, J. Differential Equations 214 (2005) 36–64.

[28] A. Boscaggin, A note on a superlinear indefinite Neumann problem with multiple positive solutions, J. Math. Anal. Appl. 377 (2011) 259–268.

[29] A. Boscaggin, Subharmonic solutions of planar Hamiltonian systems: a rotation number approach, Adv. Nonlinear Stud. 11 (2011) 77–103.

[30] A. Boscaggin, G. Feltrin, Positive subharmonic solutions to nonlinear ODEs with indefinite weight, submitted.

[31] A. Boscaggin, G. Feltrin, F. Zanolin, Pairs of positive periodic solutions of nonlinear ODEs with indefinite weight: a topological degree approach for the super-sublinear case, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016) 449–474.

[32] A. Boscaggin, G. Feltrin, F. Zanolin, Positive solutions for super-sublinear indefinite problems: high multiplicity results via coincidence degree, Trans. Amer. Math. Soc., to appear.

[33] A. Boscaggin, M. Garrione, Multiple solutions to Neumann problems with indefinite weight and bounded nonlinearities, J. Dynam. Differential Equations 28 (2016) 167–187.

[34] A. Boscaggin, M. Garrione, Positive solutions to indefinite Neumann problems when the weight has positive average, Discrete Contin. Dyn. Syst. 36 (2016) 5231–5244.

[35] A. Boscaggin, R. Ortega, F. Zanolin, Subharmonic solutions of the forced pendulum equation: a symplectic approach, Arch. Math. (Basel) 102 (2014) 459–468.

[36] A. Boscaggin, F. Zanolin, Pairs of positive periodic solutions of second order nonlinear equations with indefinite weight, J. Differential Equations 252 (2012) 2900–2921.

[37] A. Boscaggin, F. Zanolin, Positive periodic solutions of second order nonlinear equations with indefinite weight: multiplicity results and complex dynamics, J. Differential Equations 252 (2012) 2922–2950.

[38] A. Boscaggin, F. Zanolin, Pairs of nodal solutions for a class of nonlinear problems with one-sided growth conditions, Adv. Nonlinear Stud. 13 (2013) 13–53.

[39] A. Boscaggin, F. Zanolin, Subharmonic solutions for nonlinear second order equations in presence of lower and upper solutions, Discrete Contin. Dyn. Syst. 33 (2013) 89–110.

[40] A. Boscaggin, F. Zanolin, Second-order ordinary differential equations with indefinite weight: the Neumann boundary value problem, Ann. Mat. Pura Appl. 194 (2015) 451–478.

[41] E. Bosetto, E. Serra, S. Terracini, Generic-type results for chaotic dynamics in equations with periodic forcing terms, J. Differential Equations 180 (2002) 99–124.

[42] J. L. Bravo, P. J. Torres, Periodic solutions of a singular equation with indefinite weight, Adv. Nonlinear Stud. 10 (2010) 927–938.

[43] H. Brézis, R. E. L. Turner, On a class of superlinear elliptic problems, Comm. Partial Differential Equations 2 (1977) 601–614.

[44] K. J. Brown, P. Hess, Stability and uniqueness of positive solutions for a semi-linear elliptic boundary value problem, Differential Integral Equations 3 (1990) 201–207.

[45] R. C. Brown, D. B. Hinton, Norm eigenvalue bounds for some weighted Sturm-Liouville problems, in: General inequalities, 7 (Oberwolfach, 1995), vol. 123 of Internat. Ser. Numer. Math., Birkhäuser, Basel, 1997, pp. 129–144.

[46] R. F. Brown, A topological introduction to nonlinear analysis, Birkhäuser Boston, Inc., Boston, MA, 1993.

[47] T. Burton, R. Grimmer, On continuability of solutions of second order differential equations, Proc. Amer. Math. Soc. 29 (1971) 277–283.

[48] G. J. Butler, Oscillation theorems for a nonlinear analogue of Hill's equation, Quart. J. Math. Oxford Ser. 27 (1976) 159–171.

[49] G. J. Butler, Rapid oscillation, nonextendability, and the existence of periodic solutions to second order nonlinear ordinary differential equations, J. Differential Equations 22 (1976) 467–477.

[50] J. Byeon, P. H. Rabinowitz, On a phase transition model, Calc. Var. Partial Differential Equations 47 (2013) 1–23.

[51] N. P. Các, A. C. Lazer, On second order, periodic, symmetric, differential systems having subharmonics of all sufficiently large orders, J. Differential Equations 127 (1996) 426–438.

[52] A. Capietto, W. Dambrosio, D. Papini, Superlinear indefinite equations on the real line and chaotic dynamics, J. Differential Equations 181 (2002) 419–438.

[53] A. Capietto, M. Henrard, J. Mawhin, F. Zanolin, A continuation approach to some forced superlinear Sturm-Liouville boundary value problems, Topol. Methods Nonlinear Anal. 3 (1994) 81–100.

[54] M. C. Carbinatto, J. Kwapisz, K. Mischaikow, Horseshoes and the Conley index spectrum, Ergodic Theory Dynam. Systems 20 (2000) 365–377.

[55] E. A. Coddington, N. Levinson, Theory of ordinary differential equations, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[56] C. Corduneanu, Integral equations and stability of feedback systems, vol. 104 of Mathematics in Science and Engineering, Academic Press, New York-London, 1973.

[57] F. Dalbono, C. Rebelo, Poincaré-Birkhoff fixed point theorem and periodic solutions of asymptotically linear planar Hamiltonian systems, Rend. Sem. Mat. Univ. Politec. Torino 60 (2002) 233–263.

[58] F. Dalbono, F. Zanolin, Multiplicity results for asymptotically linear equations, using the rotation number approach, Mediterr. J. Math. 4 (2007) 127–149.

[59] E. N. Dancer, The effect of domain shape on the number of positive solutions of certain nonlinear equations, J. Differential Equations 74 (1988) 120–156.

[60] E. N. Dancer, The effect of domain shape on the number of positive solutions of certain nonlinear equations. II, J. Differential Equations 87 (1990) 316–339.

[61] C. De Coster, P. Habets, Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results, in: Nonlinear analysis and boundary value problems for ordinary differential equations (Udine), vol. 371 of CISM Courses and Lectures, Springer, Vienna, 1996, pp. 1–78.

[62] C. De Coster, P. Habets, Two-point boundary value problems: lower and upper solutions, vol. 205 of Mathematics in Science and Engineering, Elsevier B. V., Amsterdam, 2006.

[63] D. G. De Figueiredo, Positive solutions of semilinear elliptic problems, in: Differential equations (São Paulo, 1981), vol. 957 of Lecture Notes in Mathematics, Springer, Berlin-New York, 1982, pp. 34–87.

[64] D. G. De Figueiredo, J.-P. Gossez, P. Ubilla, Local superlinearity and sublinearity for indefinite semilinear elliptic problems, J. Funct. Anal. 199 (2003) 452–467.

[65] D. G. De Figueiredo, P.-L. Lions, R. D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pures Appl. 61 (1982) 41–63.

[66] K. Deimling, Nonlinear functional analysis, Springer-Verlag, Berlin, 1985.

[67] R. L. Devaney, An introduction to chaotic dynamical systems, Addison-Wesley Studies in Nonlinearity, 2nd ed., Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.

[68] T. R. Ding, F. Zanolin, Subharmonic solutions of second order nonlinear equations: a time-map approach, Nonlinear Anal. 20 (1993) 509–532.

[69] W. Y. Ding, Fixed points of twist mappings and periodic solutions of ordinary differential equations, Acta Math. Sinica 25 (1982) 227–235.

[70] D. Djurčić, $\mathscr{O}$-regularly varying functions and strong asymptotic equivalence, J. Math. Anal. Appl. 220 (1998) 451–461.

[71] D. Djurčić, A. Torgašev, Strong asymptotic equivalence and inversion of functions in the class K_c, J. Math. Anal. Appl. 255 (2001) 383–390.

[72] B. S. Du, The minimal number of periodic orbits of periods guaranteed in Sharkovskiĭ's theorem, Bull. Austral. Math. Soc. 31 (1985) 89–103.

[73] D. S. Dummit, R. M. Foote, Abstract algebra, 3rd ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[74] S. B. Eliason, Lyapunov inequalities and bounds on solutions of certain second order equations, Canad. Math. Bull. 17 (1974) 499–504.

[75] L. H. Erbe, S. C. Hu, H. Wang, Multiple positive solutions of some boundary value problems, J. Math. Anal. Appl. 184 (1994) 640–648.

[76] L. H. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc. 120 (1994) 743–748.

[77] C. Fabry, Periodic solutions of the equation $x'' + f(t,x) = 0$, Séminaire de Mathématique, No. 117, Louvain-la-Neuve (1987).

[78] G. Feltrin, Multiple positive solutions of a Sturm-Liouville boundary value problem with conflicting nonlinearities, submitted.

[79] G. Feltrin, A note on a fixed point theorem on topological cylinders, submitted.

[80] G. Feltrin, Existence of positive solutions of a superlinear boundary value problem with indefinite weight, Discrete Contin. Dyn. Syst. (Dynamical systems, differential equations and applications. 10th AIMS Conference. Suppl.) (2015) 436–445.

[81] G. Feltrin, F. Zanolin, An application of coincidence degree theory to cyclic feedback type systems associated with nonlinear differential operators, submitted.

[82] G. Feltrin, F. Zanolin, Multiplicity of positive periodic solutions in the superlinear indefinite case via coincidence degree, submitted.

[83] G. Feltrin, F. Zanolin, Existence of positive solutions in the superlinear case via coincidence degree: the Neumann and the periodic boundary value problems, Adv. Differential Equations 20 (2015) 937–982.

[84] G. Feltrin, F. Zanolin, Multiple positive solutions for a superlinear problem: a topological approach, J. Differential Equations 259 (2015) 925–963.

[85] G. B. Folland, Real analysis, Modern techniques and their applications, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1984.

[86] A. Fonda, M. Ramos, Large-amplitude subharmonic oscillations for scalar second-order differential equations with asymmetric nonlinearities, J. Differential Equations 109 (1994) 354–372.

[87] A. Fonda, M. Sabatini, F. Zanolin, Periodic solutions of perturbed Hamiltonian systems in the plane by the use of the Poincaré-Birkhoff theorem, Topol. Methods Nonlinear Anal. 40 (2012) 29–52.

[88] B. Franchi, E. Lanconelli, J. Serrin, Existence and uniqueness of nonnegative solutions of quasilinear equations in $\mathbb{R}^n$, Adv. Math. 118 (1996) 177–243.

[89] R. E. Gaines, J. Mawhin, Coincidence degree, and nonlinear differential equations, vol. 568 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1977.

[90] R. E. Gaines, J. Santanilla M., A coincidence theorem in convex sets with applications to periodic solutions of ordinary differential equations, Rocky Mountain J. Math. 12 (1982) 669–678.

[91] S. Gan, M. Zhang, Resonance pockets of Hill's equations with two-step potentials, SIAM J. Math. Anal. 32 (2000) 651–664.

[92] M. Gaudenzi, P. Habets, F. Zanolin, An example of a superlinear problem with multiple positive solutions, Atti Sem. Mat. Fis. Univ. Modena 51 (2003) 259–272.

[93] M. Gaudenzi, P. Habets, F. Zanolin, Positive solutions of superlinear boundary value problems with singular indefinite weight, Commun. Pure Appl. Anal. 2 (2003) 411–423.

[94] M. Gaudenzi, P. Habets, F. Zanolin, A seven-positive-solutions theorem for a superlinear problem, Adv. Nonlinear Stud. 4 (2004) 149–164.

[95] N. Ghoussoub, Location, multiplicity and Morse indices of min-max critical points, J. Reine Angew. Math. 417 (1991) 27–76.

[96] P. Gidoni, Two explorations in dynamical systems and mechanics. Part I: Avoiding cones conditions and higher dimensional twist, Ph.D. thesis, SISSA, Trieste, 2016.

[97] E. N. Gilbert, J. Riordan, Symmetry types of periodic sequences, Illinois J. Math. 5 (1961) 657–665.

[98] P. M. Girão, J. M. Gomes, Multi-bump nodal solutions for an indefinite non-homogeneous elliptic problem, Proc. Roy. Soc. Edinburgh Sect. A 139 (2009) 797–817.

[99] P. M. Girão, J. M. Gomes, Multibump nodal solutions for an indefinite superlinear elliptic problem, J. Differential Equations 247 (2009) 1001–1012.

[100] R. Gómez-Reñasco, J. López-Gómez, The effect of varying coefficients on the dynamics of a class of superlinear indefinite reaction-diffusion equations, J. Differential Equations 167 (2000) 36–72.

[101] J. R. Graef, L. Kong, H. Wang, Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem, J. Differential Equations 245 (2008) 1185–1197.

[102] A. Granas, The Leray-Schauder index and the fixed point theory for arbitrary ANRs, Bull. Soc. Math. France 100 (1972) 209–228.

[103] A. Granas, J. Dugundji, Fixed point theory, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[104] J. K. Hale, Ordinary differential equations, 2nd ed., Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980.

[105] P. Hartman, Ordinary differential equations, 2nd ed., Birkhäuser, Boston, Mass., 1982.

[106] D. Henry, Geometric theory of semilinear parabolic equations, vol. 840 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1981.

[107] P. Hess, T. Kato, On some linear and nonlinear eigenvalue problems with an indefinite weight function, Comm. Partial Differential Equations 5 (1980) 999–1030.

[108] H. Hofer, A note on the topological degree at a critical point of mountainpass-type, Proc. Amer. Math. Soc. 90 (1984) 309–315.

[109] M. R. Joglekar, E. Sander, J. A. Yorke, Fixed points indices and period-doubling cascades, J. Fixed Point Theory Appl. 8 (2010) 151–176.

[110] D. A. Klain, G.-C. Rota, Introduction to geometric probability, Lezioni Lincee [Lincei Lectures], Cambridge University Press, Cambridge, 1997.

[111] T. Kociumaka, J. Radoszewski, W. Rytter, Computing k-th Lyndon word and decoding lexicographically minimal de Bruijn sequence, in: Combinatorial pattern matching, vol. 8486 of Lecture Notes in Computer Science, Springer, Cham, 2014, pp. 202–211.

[112] M. A. Krasnosel'skiĭ, The operator of translation along the trajectories of differential equations, Translations of Mathematical Monographs, Vol. 19, American Mathematical Society, Providence, R.I., 1968.

[113] K. Lan, J. R. L. Webb, Positive solutions of semilinear differential equations with singularities, J. Differential Equations 148 (1998) 407–421.

[114] P. Le Calvez, About Poincaré-Birkhoff theorem, Publ. Mat. Urug. 13 (2011) 61–98.

[115] P.-L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (1982) 441–467.

[116] N. G. Lloyd, Degree theory, Cambridge Tracts in Mathematics, No. 73, Cambridge University Press, Cambridge-New York-Melbourne, 1978.

[117] N. H. Loc, K. Schmitt, Bernstein-Nagumo conditions and solutions to nonlinear differential inequalities, Nonlinear Anal. 75 (2012) 4664–4671.

[118] J. López-Gómez, Varying bifurcation diagrams of positive solutions for a class of indefinite superlinear boundary value problems, Trans. Amer. Math. Soc. 352 (2000) 1825–1858.

[119] M. Lothaire, Combinatorics on words, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1997.

[120] Y. Lou, W.-M. Ni, L. Su, An indefinite nonlinear diffusion problem in population genetics. II. Stability and multiplicity, Discrete Contin. Dyn. Syst. 27 (2010) 643–655.

[121] P. A. MacMahon, Applications of a theory of permutations in circular procession to the theory of numbers, Proc. London Math. Soc. S1-23 (1891/92) 305–313.

[122] W. Magnus, S. Winkler, Hill's equation, Interscience Tracts in Pure and Applied Mathematics, No. 20, Interscience Publishers John Wiley & Sons, 1966.

[123] R. Manásevich, J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian-like operators, J. Differential Equations 145 (1998) 367–393.

[124] R. Manásevich, F. I. Njoku, F. Zanolin, Positive solutions for the one-dimensional p-Laplacian, Differential Integral Equations 8 (1995) 213–222.

[125] A. Margheri, C. Rebelo, P. J. Torres, On the use of Morse index and rotation numbers for multiplicity results of resonant BVPs, J. Math. Anal. Appl. 413 (2014) 660–667.

[126] A. Margheri, C. Rebelo, F. Zanolin, Maslov index, Poincaré-Birkhoff theorem and periodic solutions of asymptotically linear planar Hamiltonian systems, J. Differential Equations 183 (2002) 342–367.

[127] J. Mawhin, Équations intégrales et solutions périodiques des systèmes différentiels non linéaires, Acad. Roy. Belg. Bull. Cl. Sci. 55 (1969) 934–947.

[128] J. Mawhin, Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, J. Differential Equations 12 (1972) 610–636.

[129] J. Mawhin, Boundary value problems for nonlinear second-order vector differential equations, J. Differential Equations 16 (1974) 257–269.

[130] J. Mawhin, Topological degree methods in nonlinear boundary value problems, vol. 40 of CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, R.I., 1979.

[131] J. Mawhin, The Bernstein-Nagumo problem and two-point boundary value problems for ordinary differential equations, in: Qualitative theory of differential equations, Vol. I, II (Szeged, 1979), vol. 30 of Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam-New York, 1981, pp. 709–740.

[132] J. Mawhin, Topological degree and boundary value problems for nonlinear differential equations, in: Topological methods for ordinary differential equations (Montecatini Terme, 1991), vol. 1537 of Lecture Notes in Mathematics, Springer, Berlin, 1993, pp. 74–142.

[133] J. Mawhin, Bounded solutions of nonlinear ordinary differential equations, in: Non-linear analysis and boundary value problems for ordinary differential equations (Udine), vol. 371 of CISM Courses and Lectures, Springer, Vienna, 1996, pp. 121–147.

[134] J. Mawhin, Leray-Schauder degree: a half century of extensions and applications, Topol. Methods Nonlinear Anal. 14 (1999) 195–228.

[135] J. Mawhin, Reduction and continuation theorems for Brouwer degree and applications to nonlinear difference equations, Opuscula Math. 28 (2008) 541–560.

[136] J. Mawhin, C. Rebelo, F. Zanolin, Continuation theorems for Ambrosetti-Prodi type periodic problems, Commun. Contemp. Math. 2 (2000) 87–126.

[137] R. Michalek, G. Tarantello, Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems, J. Differential Equations 72 (1988) 28–55.

[138] K. Mischaikow, M. Mrozek, Chaos in the Lorenz equations: a computer-assisted proof, Bull. Amer. Math. Soc. (N.S.) 32 (1995) 66–72.

[139] K. Mischaikow, M. Mrozek, Isolating neighborhoods and chaos, Japan J. Indust. Appl. Math. 12 (1995) 205–236.

[140] P. Morassi, A note on the construction of coincidence degree, Boll. Un. Mat. Ital. A 10 (1996) 421–433.

[141] J. Moser, Stable and random motions in dynamical systems, With special emphasis on celestial mechanics, Hermann Weyl Lectures, the Institute for Advanced Study, Princeton, N.J., Annals of Mathematics Studies, No. 77, Princeton University Press, Princeton, N.J., 1973.

[142] J. Moser, An example of a Schroedinger equation with almost periodic potential and nowhere dense spectrum, Comment. Math. Helv. 56 (1981) 198–224.

[143] K. Nakashima, W.-M. Ni, L. Su, An indefinite nonlinear diffusion problem in population genetics. I. Existence and limiting profiles, Discrete Contin. Dyn. Syst. 27 (2010) 617–641.

[144] F. I. Njoku, F. Zanolin, Positive solutions for two-point BVPs: existence and multiplicity results, Nonlinear Anal. 13 (1989) 1329–1338.

[145] R. D. Nussbaum, Periodic solutions of some nonlinear, autonomous functional differential equations. II, J. Differential Equations 14 (1973) 360–394.

[146] R. D. Nussbaum, Positive solutions of nonlinear elliptic boundary value problems, J. Math. Anal. Appl. 51 (1975) 461–482.

[147] R. D. Nussbaum, The fixed point index and some applications, vol. 94 of Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], Presses de l'Université de Montréal, Montreal, QC, 1985.

[148] R. D. Nussbaum, The fixed point index and fixed point theorems, in: Topological methods for ordinary differential equations (Montecatini Terme, 1991), vol. 1537 of Lecture Notes in Mathematics, Springer, Berlin, 1993, pp. 143–205.

[149] F. Obersnel, P. Omari, Positive solutions of elliptic problems with locally oscillating nonlinearities, J. Math. Anal. Appl. 323 (2006) 913–929.

[150] T. Ouyang, J. Shi, Exact multiplicity of positive solutions for a class of semilinear problems, J. Differential Equations 146 (1998) 121–156.

[151] T. Ouyang, J. Shi, Exact multiplicity of positive solutions for a class of semilinear problem. II, J. Differential Equations 158 (1999) 94–151.

[152] D. Papini, Prescribing the nodal behaviour of periodic solutions of a superlinear equation with indefinite weight, Atti Sem. Mat. Fis. Univ. Modena 51 (2003) 43–63.

[153] D. Papini, F. Zanolin, Periodic points and chaotic-like dynamics of planar maps associated to nonlinear Hill's equations with indefinite weight, Georgian Math. J. 9 (2002) 339–366.

[154] D. Papini, F. Zanolin, On the periodic boundary value problem and chaotic-like dynamics for nonlinear Hill's equations, Adv. Nonlinear Stud. 4 (2004) 71–91.

[155] D. Papini, F. Zanolin, Some results on periodic points and chaotic dynamics arising from the study of the nonlinear Hill equations, Rend. Semin. Mat. Univ. Politec. Torino 65 (2007) 115–157.

[156] J. P. Pinasco, Lyapunov-type inequalities, Springer Briefs in Mathematics, Springer, New York, 2013.

[157] V. A. Pliss, Nonlocal problems of the theory of oscillations, Academic Press, New York-London, 1966.

[158] P. H. Rabinowitz, Pairs of positive solutions of nonlinear elliptic partial differential equations, Indiana Univ. Math. J. 23 (1973/74) 173–186.

[159] C. Rebelo, A note on the Poincaré-Birkhoff fixed point theorem and periodic solutions of planar systems, Nonlinear Anal. 29 (1997) 291–311.

[160] W. E. Ricker, Stock and recruitment, J. Fish. Res. Board Can. 11 (1954) 559–623.

[161] H.-J. Ruppen, Multiplicity results for a semilinear, elliptic differential equation with conflicting nonlinearities, J. Differential Equations 147 (1998) 79–122.

[162] G. Sansone, R. Conti, Non-linear differential equations, Revised edition. International Series of Monographs in Pure and Applied Mathematics, Vol. 67, A Pergamon Press Book. The Macmillan Co., New York, 1964.

[163] K. Schmitt, Fixed point and coincidence theorems with applications to nonlinear differential and integral equations, Séminaires de Mathématique Appliquée et Mécanique, Rapport No. 97, Université catholique de Louvain, Vander, Louvain-la-Neuve, 1976.

[164] G. R. Sell, Topological dynamics and ordinary differential equations, Van Nostrand Reinhold Mathematical Studies, No. 33, Van Nostrand Reinhold Co., London, 1971.

[165] E. Seneta, Regularly varying functions, vol. 508 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1976.

[166] É. Séré, Looking for the Bernoulli shift, Ann. Inst. H. Poincaré Anal. Non Linéaire 10 (1993) 561–590.

[167] E. Serra, M. Tarallo, S. Terracini, Subharmonic solutions to second-order differential equations with periodic nonlinearities, Nonlinear Anal. 41 (2000) 649–667.

[168] K. S. Sibirsky, Introduction to topological dynamics, Noordhoff International Publishing, Leiden, 1975.

[169] A. Sırma, S. Ṣevgin, A note on coincidence degree theory, Abstr. Appl. Anal. (2012) Art. ID 370946, 18.

[170] N. J. A. Sloane, The on-line encyclopedia of integer sequences, published electronically at `http://oeis.org`, 2010, Sequence A001037.

[171] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73 (1967) 747–817.

[172] E. Sovrano, F. Zanolin, Remarks on Dirichlet problems with sublinear growth at infinity, Rend. Istit. Mat. Univ. Trieste 47 (2015) 265–305.

[173] R. Srzednicki, K. Wójcik, A geometric method for detecting chaotic dynamics, J. Differential Equations 135 (1997) 66–82.

[174] R. Srzednicki, K. Wójcik, P. Zgliczyński, Fixed point results based on the Ważewski method, in: Handbook of topological fixed point theory, Springer, Dordrecht, 2005, pp. 905–943.

[175] S. Terracini, G. Verzini, Oscillating solutions to second-order ODEs with indefinite superlinear nonlinearities, Nonlinearity 13 (2000) 1501–1514.

[176] A. J. Ureña, A counterexample for singular equations with indefinite weight, submitted.

[177] A. J. Ureña, Periodic solutions of singular equations, Topol. Methods Nonlinear Anal. 47 (2016) 55–72.

[178] P. Walters, An introduction to ergodic theory, vol. 79 of Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1982.

[179] P. Waltman, An oscillation criterion for a nonlinear second order equation, J. Math. Anal. Appl. 10 (1965) 439–441.

[180] H. Wang, On the existence of positive solutions for semilinear elliptic equations in the annulus, J. Differential Equations 109 (1994) 1–7.

[181] C. Zanini, Rotation numbers, eigenvalues, and the Poincaré-Birkhoff theorem, J. Math. Anal. Appl. 279 (2003) 290–307.

[182] F. Zanolin, On the periodic boundary value problem for forced nonlinear second order vector differential equations, Riv. Mat. Pura Appl. (1987) 105–124.

[183] A. Zettl, Sturm-Liouville theory, vol. 121 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2005.

[184] P. Zgliczyński, Fixed point index for iterations of maps, topological horseshoe and chaos, Topol. Methods Nonlinear Anal. 8 (1996) 169–177.

[185] P. Zgliczyński, M. Gidea, Covering relations for multidimensional dynamical systems, J. Differential Equations 202 (2004) 32–58.

Acknowledgments

I am extremely grateful to my advisor and mentor Prof. Fabio Zanolin for his dedicated supervision, for proposing very interesting topics, for continuously encouraging my research, for always teaching me something new at every meeting and, despite the many commitments, for always making time for me during the last four years. I will always be indebted to Fabio for his priceless advice on my research and on my career, for his kindness, for his human generosity, for his patience to follow and support me, for his immense knowledge and deep research experience shared with me, and for the passion for Mathematics that he has passed down to me.

During my Ph.D. studies, I had the chance and the pleasure to know and work with Alberto Boscaggin, a collaborator and a friend. I remember with delight our many mathematical discussions, the conferences we attended together, his visits at SISSA, his professional generosity and the way he thinks as a researcher. I am indebted to him for every precious advice he gave me. Working with Alberto has been a great experience from which I learnt a lot and improved as a mathematician. I hope to work again with him.

My Ph.D. would never have been such an amazing experience without meeting my mathematical colleagues, especially Stefano Almi, Paolo Gidoni, Matteo Rizzi and Elisa Sovrano. I am glad to have shared with them this wonderful time.

I wish to thank Stefano for showing me sincere friendship, for the many chats and for the good time we had together. It was a great pleasure sharing with him the long days at SISSA and even taking breaks from Math! Thank you so much!

I know Paolo since we were at university, but our relationship became closer during our doctorate. We spent many great times together, especially during the several mathematical trips. I thank him also for solving my several doubts concerning LaTeX. I wish I had worked more with him, but I am sure there will be opportunities in the future.

I thank Matteo for the great liveliness and the friendship he shows me. I remember all the curious and funny stories of his adventures and definitely his passion for traveling will allow us to find a meeting point in the forthcoming years.

I am glad to have known Elisa. I thank her for the time spent together, both working on one of the many research interests which we have in common and during the conferences. Although we did not achieve our research project, certainly in the coming years we will work together again.

Next, special thanks to Francesco Boarotto, Andrea Carioli, Vito Crismale, Stefano Modena and all my other colleagues (Ph.D. students and post-docs, at SISSA and at University of Udine), whose friendship accompanied and supported me during these four years.

Being a Ph.D. student at SISSA gave me the possibility to follow lectures and seminars of excellent professors, which was a unique and precious opportunity. Therefore, I wish to thank all the professors of Mathematics Area.

A special acknowledgment to the research group in differential equations: Alberto Boscaggin, Chiara Corsato, Tobia Dondè, Maurizio Garrione, Paolo Gidoni, Andrea Sfecci, Elisa Sovrano, and Professors Alessandro Fonda, Franco Obersnel, Pierpaolo Omari, Duccio Papini, Fabio Zanolin. A sincere thanks for the nice mathematical meetings and mostly for the pleasant time spent together, also during the international conferences (above all the ones in Lisbon, Madrid, Toruń, Orlando and Rio de Janeiro). I also thank Prof. Anna Capietto for her kindness and helpfulness.

Furthermore, I remember with pleasure all the people I have met during this journey, especially Andrea Tellini (it is always a pleasure to spend time with him) and the SISSA Library staff.

Next, I wish to thank those who financially supported me and thus enabled me to attend several conferences, schools and workshops: SISSA, University of Udine (especially Prof. Fabio Zanolin), Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), University of Torino (especially Prof. Susanna Terracini), Prof. Pierluigi Benevieri and Prof. Alessandro Michelangeli (also for his remarkable generosity and kindness).

Finally, I would like to express my deep gratitude to my parents, my brother Ludovico, my grandmother Luciana and all my family for the continuous support during my Ph.D. studies.

September 30, 2016