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Existence and multiplicity of solutions to boundary value problems associated with nonlinear first order planar systems

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To my aunt Rosi,
who is now taking care
of me from Heaven.

Notation and terminology

- $\mathbb{N} = \{0, 1, 2, ...\}$
- $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$
- $\mathbb{R}^+_* = \{x \in \mathbb{R} \mid x > 0\}$
- $(\mathbb{R}^2, \langle\cdot|\cdot\rangle)$ = the Euclidean plane, with Euclidean norm $|\cdot|$
- $\mathbb{R}^2_*$ = the punctured plane $\mathbb{R}^2 \setminus \{0\}$
- $\mathcal{A}(r, R) = \{u \in \mathbb{R}^2 \mid r < |u| < R\}$, for $0 \leq r \leq R \leq +\infty$
- $\lfloor a \rfloor$ = the greatest integer less or equal to a, for $a \in \mathbb{R}$
- $\lceil a \rceil$ = the least integer greater or equal to a, for $a \in \mathbb{R}$
- $\chi_\Omega(x)$ = the characteristic function of $\Omega \subset \mathbb{R}^N$, i.e.,

$$\chi_\Omega(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \notin \Omega. \end{cases}$$

- J = the standard symplectic 2×2 matrix, i.e.,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For a 2×2 square matrix M:

- M^t = the transpose of M
- $\|M\|$ = the operatorial norm of M.

For a C^1-map $\Lambda : \mathcal{U} \to \mathbb{R}^2$, being $\mathcal{U} \subset \mathbb{R}^2$ an open set:

- $\Lambda'(u)$ = the Jacobian matrix of Λ, evaluated at $u \in \mathcal{U}$.

For a function $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, being $T > 0$:

- $\nabla F(t,u) = \nabla_u F(t,u)$.

We will say that a function $F : [0,T] \times \mathcal{O} \to \mathbb{R}^N$, with N a positive integer and $\mathcal{O}$ an open subset of $\mathbb{R}^N$, is L^p-Carathéodory ($p \geq 1$) if it satisfies the following properties:

- for every $u \in \mathcal{O}$, the function $t \mapsto F(t,u)$ is measurable;
- for almost every $t \in [0,T]$, the function $u \mapsto F(t,u)$ is continuous;
- for every $\mathcal{K} \subset \mathcal{O}$ compact, there exists $\eta_{\mathcal{K}} \in L^p(0,T)$ such that
$$|F(t,u)| \leq \eta_{\mathcal{K}}(t),$$
for almost every $t \in [0,T]$ and every $u \in \mathcal{K}$.

If $\mathcal{O} = \mathbb{R}^N$, the last condition will often be written as

- for every $R > 0$, there exists $\eta_R \in L^2(0,T)$ such that
$$|F(t,u)| \leq \eta_R(t),$$
for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$, with $|u| \leq R$.

According to the definition, referring to the equation

$$u' = F(t,u),$$

with $F(t,u)$ an L^1-Carathéodory function, we will mean the solutions in the generalized sense, i.e., locally absolutely continuous functions solving the differential equation almost everywhere (such a concept reduces to the one of classical solution whenever F is continuous in both the variables). By a T-periodic solution to such an equation we will mean, as usual, a solution $u(t)$ satisfying the boundary condition $u(0) = u(T)$. Indeed, whenever $F(t,u)$ is defined for every $t \in \mathbb{R}$, with $F(\cdot,u) \equiv F(\cdot + T, u)$ for every $u \in \mathbb{R}^2$, every solution to the equation defined on $[0,T]$ and satisfying $u(0) = u(T)$ can be extended, by T-periodicity, to a locally absolutely continuous T-periodic solution on the whole real line.

Introduction

This monograph is devoted to the study of the problem of existence and multiplicity of solutions to some kinds of boundary value problems associated with nonlinear first order systems in the plane. As a starting point, one can think to the T-periodic ($T > 0$) scalar boundary value problem

$$\begin{cases} x'' + g(t,x) = 0 \\ x(0) = x(T),\ x'(0) = x'(T), \end{cases} \tag{1}$$

where $g(\cdot, x)$ is T-periodic. The differential equation appearing in (1), indeed, can be written in a standard way as a first order system in the plane, being equivalent to

$$\begin{cases} x' = y \\ y' = -g(t,x). \end{cases}$$

We will be interested in the situation when the nonlinearity is linearly controlled, which in this setting means that the ratio $g(t,x)/x$ is bounded for $|x|$ large. Besides an asymptotically linear growth, we will sometimes take into account also the sublinear situation, i.e., the case when $g(t,x)/x$ converges to 0 for $|x|$ large.
One of the main issues arising in this setting is the possible occurrence of the so called phenomenon of *resonance*. Physically speaking, for an ideal vibrating system, resonance is strongly connected with the unboundedness of all its possible motions. One can think, for instance, to a spring, whose motion is forced by a time-dependent external force "constructively interacting" with the natural oscillation of the spring. More precisely, if the ratio between the natural frequency of the spring and the frequency of the forcing term is a rational number, then, at each multiple of a suitable interval of time, the amplitude of the oscillations increases. In this way, the potential occurrence of periodic oscillations is avoided.
The nonexistence of periodic motions, roughly speaking, is the main way of meaning resonance from a more mathematical point of view. In general, we will deal with a boundary value problem, most of the time periodic, whose equation will be a perturbation of a comparison one for which the existence of solutions satisfying the boundary

conditions is not guaranteed. The main aim will be to determine which hypotheses are to be additionally assumed on the nonlinearity to be ensured of the existence of a solution (for an interesting and exhaustive survey about resonance, see [109]).
Sometimes, the mathematical and the physical concepts of resonance are very similar, as it happens, for instance, in the linear case. In fact, considering the scalar equation

$$x'' + \lambda x = e(t), \tag{2}$$

together with the equivalent first order system

$$\begin{cases} x' = y \\ -y' = \lambda x - e(t), \end{cases} \tag{3}$$

with $e(t)$ a T-periodic function and λ a suitable real constant, the nonexistence of a T-periodic solution is equivalent to the unboundedness of all the solutions.
The crucial point, here, is the fact that the occurrence of resonance depends on the value of the constant λ. We have to distinguish two cases: if there exists a positive integer k such that

$$\lambda = \left(\frac{2k\pi}{T}\right)^2,$$

considering the Cauchy problem associated with (3) centered at (x_0, y_0), by direct integration it is possible to see that the solution evaluated at the time T takes the form

$$x(T) = x_0 - \frac{T^2 b_k}{4k\pi}, \quad y(T) = y_0 + \frac{T a_k}{2},$$

where

$$a_k = \frac{2}{T}\int_0^T e(s)\cos\left(\frac{2k\pi}{T}s\right) ds \quad \text{and} \quad b_k = \frac{2}{T}\int_0^T e(s)\sin\left(\frac{2k\pi}{T}s\right) ds \tag{4}$$

are the k-th Fourier coefficients of $e(t)$ with respect to the basis

$$\left\{\cos\left(\frac{2n\pi}{T}t\right), \sin\left(\frac{2n\pi}{T}t\right)\right\}_{n\in\mathbb{N}}.$$

Thus, it suffices one of a_k or b_k to be nonzero in order not to have any T-periodic solution to the linear equation (2). Indeed, in this case all the solutions are unbounded, since it can be seen that

$$x(mT) = x_0 - m\frac{T^2 b_k}{4k\pi}, \quad y(mT) = y_0 + m\frac{T a_k}{2}.$$

Incidentally, notice that, for $\lambda = 0$, the existence is not ensured: it suffices to take a T-periodic forcing term $e(t)$ with nonzero mean. For the other values of λ, straight

computations show that, on the contrary, there exists a unique T-periodic solution to (2) (for the explicit calculation and further details, see [55]).
It is thus possible to define the set

$$\Sigma_L = \left\{ \lambda_k = \left(\frac{2k\pi}{T}\right)^2 \mid k \in \mathbb{N} \right\},$$

the T-periodic *spectrum* associated with (2), whose elements take the name *eigenvalues*. It is essential to observe that only in correspondence of the eigenvalues we have existence of a nontrivial T-periodic solution to the homogeneous equation

$$x'' + \lambda x = 0, \tag{5}$$

and this could be another way to express resonance, considering equation (5). In general, there is indeed a strong link between the existence of nontrivial solutions to (5) and the solvability of (2), given by the Fredholm alternative: if $\lambda \notin \Sigma_L$, roughly speaking, the differential operator $x \mapsto x'' + \lambda x$ (acting from $C^2_T(0,T)$ to $C^0_T(0,T)$) becomes invertible, so that a T-periodic solution to (2) exists for every forcing term $e(t)$. On the other hand, in correspondence of the eigenvalues, the forcing term has to satisfy some additional requirements, otherwise the existence is not guaranteed. This parallel, which extends also to more complicated types of equation, will be crucial when considering the planar problem.

Wishing to deal with nonlinear problems in the spirit just described, the first natural step consists in perturbing the linear equation with a bounded nonlinearity, considering thus the Duffing T-periodic problem

$$\begin{cases} x'' + \lambda x + h(x) = e(t) \\ x(0) = x(T),\ x'(0) = x'(T), \end{cases} \tag{6}$$

with $e(t)$ a T-periodic function and $h(x)$ bounded. By topological methods, for instance using Schauder's theorem, it turns here out that, if $\lambda \notin \Sigma_L$, one can always find a solution to (6). Furthermore, as was already proved by Dolph in [35] using Leray-Schauder theory, the same conclusion holds for the general problem (1), whenever $g(t,x)/x$ is asymptotically far from the eigenvalues, namely there exist $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ such that

$$\lambda_k < \alpha \le \liminf_{|x|\to\infty} \frac{g(t,x)}{x} \le \limsup_{|x|\to\infty} \frac{g(t,x)}{x} \le \beta < \lambda_{k+1} \tag{7}$$

(see also [105]). In this case, we will say that we are considering a *nonresonant* problem. The history of problem (1) is quite old, starting in the first half of the 20th century, and the first results concerned the nonresonant case, which was easier to handle by means of Leray-Schauder theory or Schauder's theorem, imposing conditions on $g'(x)$ (see, for instance, [102]). On the other hand, in the case of resonance with the first eigenvalue, asymptotic sign conditions on $g(x)$ were used to ensure existence (cf. [91]).

It was the result obtained by Lazer and Leach in [93] for problem (6) which gave a new impulse to the study of problem (1), and opened the way to the huge number of generalizations obtained in the subsequent years. It was therein proved that the resonant problem

$$\begin{cases} x'' + \lambda_k x + h(x) = e(t) \\ x(0) = x(T),\, x'(0) = x'(T), \end{cases} \tag{8}$$

with $\lambda_k = \left(\frac{2k\pi}{T}\right)^2$ an arbitrary element of Σ_L and $h(x)$ bounded, has a solution if the following condition is satisfied:

$$\frac{2}{\pi}\Big(\liminf_{x\to+\infty} h(x) - \limsup_{x\to-\infty} h(x)\Big) > \sqrt{a_k^2 + b_k^2}\,,$$

where a_k and b_k are the Fourier coefficients defined in (4). Moreover, such a condition was shown to be necessary when $h(x)$ is bounded and strictly increasing in x.

The setting of problem (8) is usually referred to as *nonlinear resonance*, since we are considering a nonlinear perturbation of a linear resonant problem. With respect to the previous results obtained for (8), the one by Lazer and Leach was the first not making use of direct sign conditions on $h(x)$, but exploiting fine asymptotic information.

In 1970, the article by Landesman and Lazer [89], written for the Dirichlet boundary value problem associated with a perturbation of an elliptic PDE at resonance, provided the existence condition which is now known as the *Landesman-Lazer condition* - below, in short, (LL) - extending the Lazer-Leach result under the assumption that the kernel of the considered operator is 1-dimensional (immediate further developments were given, for example, in [23, 128]; see also [108] for further historical comments). The original form of (LL) was similar to the one by Lazer and Leach; however, referring to (1), Landesman and Lazer started considering the more general case of a time-dependent perturbation, i.e.,

$$g(t,x) = \lambda_k x + h(t,x), \tag{9}$$

with $h(t,x)$ continuous and *bounded*. Nevertheless, it was only with subsequent papers - see, e.g., the works [104] by Mawhin and [17] by Brézis and Nirenberg - that the Landesman-Lazer condition took the form in which it is known nowadays:

for every $v \neq 0$ which solves the homogeneous equation $v'' + \lambda_k v = 0$,

$$\int_{\{v>0\}} \liminf_{x\to+\infty} h(t,x)v(t)\,dt + \int_{\{v<0\}} \limsup_{x\to-\infty} h(t,x)v(t)\,dt > 0. \tag{10}$$

Here, we have used the standard notation

$$\{v > 0\} = \{t \in [0,T] \mid v(t) > 0\}, \quad \{v < 0\} = \{t \in [0,T] \mid v(t) < 0\}.$$

Notice that this condition is finer than a sign one, since it is clearly satisfied if $h(t,x)$ is positive for $x \to +\infty$ and negative for $x \to -\infty$. However, just as an intuitive idea, one can keep in mind such a shape of $h(t,x)$ as an example when (LL) is satisfied. Observe that, if $h(t,x)$ is bounded and strictly increasing in x, then (10) becomes also a necessary condition for existence: indeed, if $x(t)$ is a solution to (1), with $g(t,x)$ having the form (9), using the fact that $v'' + \lambda_k v = 0$ we get

$$\int_0^T x''(t)v(t)\,dt = -\lambda_k \int_0^T x(t)v(t)\,dt,$$

so that

$$\int_0^T h(t,x(t))\,dt = 0,$$

and (10) is obtained using the monotonicity of $h(t,x)$.
Just to say a few words about the proofs, the original arguments in [93] and [89] exploited again a clever use of Schauder's fixed point theorem. Several authors have worked in the direction of simplifying such reasonings: we only mention [81], giving an elementary proof based on a perturbation approach and finding the solution as the limit of a normalized sequence (this procedure is very similar to the one used together with coincidence degree theory), and the recent paper [78], that considered both the Lazer-Leach and the Landesman-Lazer original results, and, in particular, gave a new proof of the former, based on winding-number considerations in the phase-plane. The Landesman-Lazer condition is still a very relevant topic, with a number of citations which constantly grows in the years.
Until the middle Seventies, the study of problem (1) at resonance was mostly a matter of topological methods, but the strengthened interest for critical point theory in those years led to the alternative existence condition introduced by Ahmad, Lazer and Paul in 1976 [1], based on the study of the action functional associated with a resonant Dirichlet problem. In the spirit of this result, in [112] it was proved, by variational tools, that, assuming $g(t,x)$ to have the form (9), problem (1) has a solution if

$$\lim_{\substack{\|x\|_2 \to +\infty \\ x'' + \lambda_k x = 0}} \int_0^T \mathcal{H}(t,x(t))\,dt = +\infty,$$

where $\mathcal{H}(t,s) = \int_0^s h(t,\xi)\,d\xi$. The right variational formulation for the Ahmad-Lazer-Paul condition, which in turns gives the anticoercivity (or the coercivity, if assumed with the opposite sign) of the action functional on the kernel of the considered operator, had been provided, for the purpose, by Rabinowitz' Saddle Point Theorem [115]. In the same year 1978, Brézis and Nirenberg, in the remarkable work [17], had proposed an abstract version of Landesman-Lazer results in a Hilbert space H, introducing the concept of *recession function* for a given operator $\mathcal{N} : H \to H$ (see Section 2.2 below).

So far, the resonance assumption had been translated, concerning the shape of the nonlinearity $g(t,x)$, into the form (9), with $h(t,x)$ *bounded*. Such an assumption, which implies that

$$\lim_{|x|\to\infty} \frac{g(t,x)}{x} = \lambda_k,$$

uniformly in $t \in [0,T]$, is not really necessary, as showed first by Fabry and Fonda in [47]. One of the key tools to drop it is represented by the use of the theory of coincidence degree developed by J. Mawhin [106]. Indeed, in a spirit which reminds us of the first result by Dolph, what matters is bounding the ratio $g(t,x)/x$, from below and from above, by means of two consecutive eigenvalues. This means that, for x large, $g(t,x)$ can approach, for certain time instants, the eigenvalue λ_k, and, for other ones, the eigenvalue λ_{k+1}. Incidentally, notice that this philosophy had already been exploited in [111], introducing the so called "nonuniform nonresonance condition":

$$\lambda_k \lneqq \liminf_{|x|\to\infty} \frac{g(t,x)}{x} \leq \limsup_{|x|\to\infty} \frac{g(t,x)}{x} \lneqq \lambda_{k+1}. \tag{11}$$

In this case, no Landesman-Lazer conditions are required, since the strict inequalities holding on positive measure sets are enough to perform the desired estimates.

When, on the contrary, we only assume that

$$\lambda_k \leq \liminf_{|x|\to\infty} \frac{g(t,x)}{x} \leq \limsup_{|x|\to\infty} \frac{g(t,x)}{x} \leq \lambda_{k+1},$$

we are in the so called setting of *double resonance* and, of course, existence is no more guaranteed. It is here seen that, under some standard controls on $g(t,x)$, which are required also for the Landesman-Lazer conditions in order to make sense (cf. [47]), one can write

$$g(t,x) = \gamma(t,x)x + h(t,x), \tag{12}$$

with $\lambda_k \leq \gamma(t,x) \leq \lambda_{k+1}$ and $h(t,x)$ bounded.

The existence of solutions to (1) when the nonlinearity has this form was studied, for instance, in the quoted paper [47], and in other related papers (for example, [48]), where the results were obtained assuming one Landesman-Lazer type condition for

each of the eigenvalues λ_k and λ_{k+1}: explicitly,
for every $v \neq 0$ which solves the homogeneous equation $v'' + \lambda_k v = 0$,

$$\int_{\{v>0\}} \liminf_{x\to+\infty} h(t,x)v(t)\,dt + \int_{\{v<0\}} \limsup_{x\to-\infty} h(t,x)v(t)\,dt > 0,$$

and
for every $v \neq 0$ which solves the homogeneous equation $v'' + \lambda_{k+1} v = 0$,

$$\int_{\{v>0\}} \limsup_{x\to+\infty} h(t,x)v(t)\,dt + \int_{\{v<0\}} \liminf_{x\to-\infty} h(t,x)v(t)\,dt < 0.$$

Intuitively, this is due to the fact that the nonlinearity has to be kept sufficiently far from both λ_k and λ_{k+1}, in the right direction. However, the precise reason why the two conditions are to be assumed with opposite signs will be clear in Chapter 5, where we will analyze the consequences of the Landesman-Lazer condition on the rotational behavior of the solutions.
It has to be said that, in case of double resonance, the topological methods work better, since it would be somehow difficult to study the geometry of the action functional in the case when $g(t,x)$ is written like (12). However, on the other hand, many mentioned results exploit the possibility of using a modified type of polar coordinates, deeply relying on the planar nature of the problem, while the variational methods work in any dimension.
In the past years, there have been extensions of the mentioned results in several directions. Of course, the bibliography here given does not pretend to be complete: we cite, for instance, the works [21, 31, 38, 39, 44, 46, 47, 48, 62, 67, 81, 82, 87, 90, 92, 104, 112, 113, 134]; see also the bibliography in [55].

A possible generalization of the starting point (2), wishing to consider a nonlinearity in the operator itself, could be provided by the forced *asymmetric oscillator*, i.e., a forced oscillator which is subject to a different restoring potential according to the relative position with respect to a point fixed in advance (the oscillation center, usually the origin). In this case, equation (2) becomes piecewise linear:

$$x'' + \mu x^+ - \nu x^- = e(t), \tag{13}$$

where $\mu, \nu \in \mathbb{R}$, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. In this context, for the T-periodic problem, the linear spectrum Σ_L is replaced by the so called *Fučík spectrum* Σ_A (see [28, 67]), consisting of the couples of real numbers (μ, ν) (sometimes called *Fučík eigenvalues*) such that the homogeneous equation

$$x'' + \mu x^+ - \nu x^- = 0 \tag{14}$$

has a nontrivial T-periodic solution. Indeed, also in this case a kind of Fredholm alternative can be stated, even if with some differences; the main point, however, is that, again, the existence of T-periodic solutions to (13) may fail if and only if we have nontrivial solutions to (14). As well known, such a concept of asymmetric resonance extends the linear one (recovered for $\mu = \nu = \lambda_k$). With standard arguments regarding the time map associated with (14), it can be seen that, in the plane (μ, ν), we have

$$\Sigma_A = \{\mu = 0\} \cup \{\nu = 0\} \cup \bigcup_{k \in \mathbb{N}_0} \left\{ (\mu, \nu) \mid \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{k} \right\}.$$

Besides the axes, the other sets composing Σ_A are hyperbola-like curves in the plane, usually referred to as *Fučík* curves. In Figure 1, we have depicted the Fučík spectrum for equation (14), when considered together with π-periodic boundary conditions; the symbol $\exists$ corresponds to the connected regions of the plane (μ, ν) such that, choosing (μ, ν) therein, (13) has a π-periodic solution.
Introducing an asymmetry in the potential destroys the linearity of the operator, so that here the standard linear theory finds some difficulties in being used. However, thanks, in particular, to qualitative considerations in the phase plane, it is possible to carry out a similar study of the associated nonlinear problem as for the case of linear operators. In particular, referring to (1), this time nonresonance will mean that we are far from the Fučík curves; in this spirit, Drábek and Invernizzi [40] proved the analogous of the mentioned theorem by Dolph [35], assuming that

$$g(t, x) = \gamma_1(t, x)x^+ - \gamma_2(t, x)x^- + h(t, x),$$

with $a_i \le \gamma_i(t, x) \le b_i$, $i = 1, 2$ $(a_i, b_i \in \mathbb{R})$, and $([a_1, b_1] \times [a_2, b_2]) \cap \Sigma_A = \emptyset$. Thus, it was already clear in such a work that bounding the nonlinearity with two consecutive Fučík eigenvalues means that the ratio $g(t, x)/x$ will be subject to the controls

$$\mu_1 \le \liminf_{x \to +\infty} \frac{g(t, x)}{x} \le \limsup_{x \to +\infty} \frac{g(t, x)}{x} \le \mu_2,$$

$$\nu_1 \le \liminf_{x \to -\infty} \frac{g(t, x)}{x} \le \limsup_{x \to -\infty} \frac{g(t, x)}{x} \le \nu_2,$$

being the points $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \Sigma_A$ on two consecutive curves of the Fučík spectrum, i.e., satisfying, if we are out of the axes $\{\mu = 0\}, \{\nu = 0\}$,

$$\frac{\pi}{\sqrt{\mu_1}} + \frac{\pi}{\sqrt{\nu_1}} = \frac{T}{k}, \quad \frac{\pi}{\sqrt{\mu_2}} + \frac{\pi}{\sqrt{\nu_2}} = \frac{T}{k+1},$$

for a suitable positive integer k. Due to the change in the potential, the function $g(t, x)$ approaches thus different Fučik eigenvalues according to the sign of x.

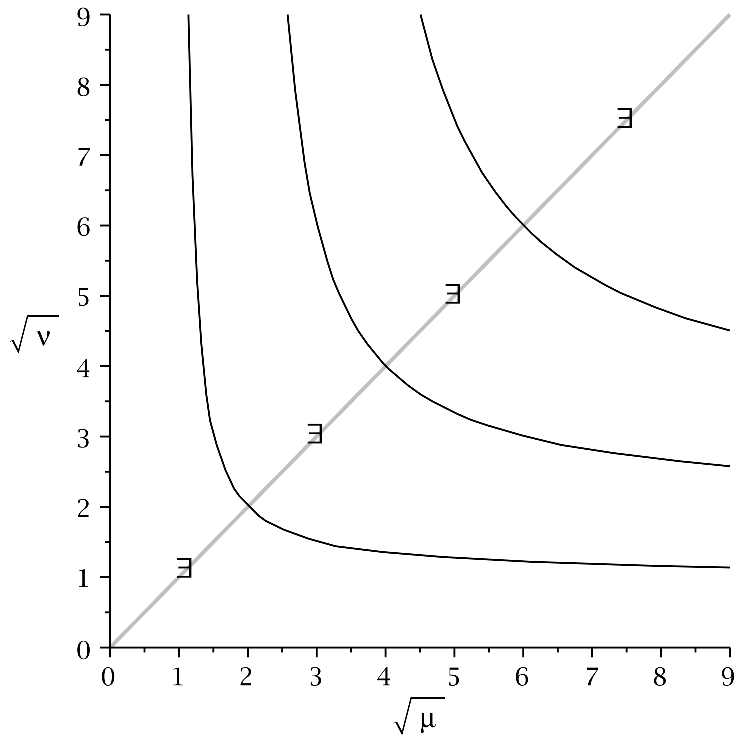

Figure 1: The T-periodic Fučík spectrum for (14), with $T = \pi$.

This line of research was opened by the already quoted paper [67], originally written for the Dirichlet problem. In the subsequent years, many authors devoted their work to this study (for instance, Dancer [28], Fučík himself [68], Fabry and Fonda [46, 49, 54], Rebelo and Zanolin [119], Capietto and Wang [22], and many others). Speaking, in particular, about Lazer-Leach and Landesman-Lazer type results for nonlinearities of the kind $g(t,x) = \mu x^+ - \nu x^- + h(t,x)$, with $(\mu, \nu) \in \Sigma_A$, we underline the significant achievements reached in the papers [28, 49]. Condition (LL) was there stated in the same way as for the linear problem, except for the fact that, as it is natural to expect, the inequality (10) had to hold for every v solving $v'' + \mu v^+ - \nu v^- = 0$.
Concerning Ahmad-Lazer-Paul conditions, the problem is more delicate, because the asymmetry of the unperturbed problem does not allow to use the linear tools usually employed to detect saddle geometry (see [125]). In this respect, some recent results were given in [9, 84]; the topic is still in progress, as we will see in Chapter 3, so we remind the reader to the discussion therein for further details.

A remarkable generalization in the framework of double resonance was given by Fabry in [46], where it was assumed that

$$g(t,x) = \gamma_1(t,x)x^+ - \gamma_2(t,x)x^- + h(t,x),$$

with $\mu_1 \le \gamma_1(t,x) \le \mu_2$, $\nu_1 \le \gamma_2(t,x) \le \nu_2$ and $h(t,x)$ bounded, being the points (μ_1, ν_1) and (μ_2, ν_2) on two consecutive curves of the Fučík spectrum (see Theorem 2.4.3 below). The existence result was obtained by adding again Landesman-Lazer conditions on both sides, similarly as for the linear case, and exploiting the theory of coincidence degree.

It is quite natural to wonder which part of these considerations is preserved when passing to consider general first order planar systems like

$$Ju' = F(t,u), \quad u \in \mathbb{R}^2, \tag{15}$$

with particular attention for the Hamiltonian case $F(t,u) = \nabla_u H(t,u)$ - we recall that we will denote by J the standard 2×2 symplectic matrix, namely

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Notice that J is put on the left-hand side since this will simplify our computations, and also because it is a more common notation when dealing with Hamiltonian systems. First of all, to proceed as for the scalar case, a suitable concept of resonance should be defined. Roughly speaking, it is quite reasonable that, if we take, as comparison terms, the eigenvalues of the T-periodic problem associated with the linear equation

$$Ju' = \lambda u,$$

a series of similar results could be obtained, more or less in the same way as for second order equations, if our nonlinearity $F(t,u)$ is suitably controlled by some symmetric 2×2 matrix $\mathbb{A}$ such that the eigenvalues of $\mathbb{A}$ are far from the spectrum of the operator $u \mapsto Ju'$ (we will not enter into details, limiting ourselves to mention the papers [3, 4, 19, 48] and the references in [14]). In this respect, in the paper [60] more general abstract results were given, applying to this particular setting; we also quote the notes [52], making an exhaustive summary of related results. This framework, which works quite well also in higher dimension, has the drawback of not including the case of the asymmetric oscillator, so that a more general environment needs to be used in order to give a unifying theory in the plane.
To this aim, as we have seen, it is not necessary that our problem is underlain by a comparison linear structure: looking also at the proofs of many previously mentioned

results, the only essential property in order to carry on similar arguments seems to be some homogeneity of the comparison equations. This consideration was clearly reflected in the work [53] by Fonda, managing to extend the framework of the problems described until now into a more general one. The main idea is that the role of the term $\lambda_k x$ (or $\mu x^+ - \nu x^-$) is now played by the gradient of a positive and positively homogeneous Hamiltonian, that is, the gradient of a C^1-function $V : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$0 < V(\lambda u) = \lambda^2 V(u), \quad \lambda > 0, \ u \in \mathbb{R}^2_*. \tag{16}$$

Indeed, writing explicitly equations (2) and (13) as first order systems,

$$\begin{cases} x' = y \\ -y' = \lambda x - e(t) \end{cases} \qquad \text{and} \qquad \begin{cases} x' = y \\ -y' = \mu x^+ - \nu x^- - e(t), \end{cases}$$

we immediately see that they have the form

$$Ju' = \nabla V(u) + E(t),$$

where $u = (x, y)$, $E(t) = (-e(t), 0)$ and, respectively,

$$V(u) = \frac{1}{2}\Big(\lambda x^2 + y^2\Big) \quad \text{or} \quad V(u) = \frac{1}{2}\Big(\mu (x^+)^2 + \nu (x^-)^2 + y^2\Big),$$

so that

$$\nabla V(u) = (\lambda x, y), \quad \text{or} \quad \nabla V(u) = (\mu x^+ - \nu x^-, y).$$

Therefore, the forced scalar second order equations previously discussed fit in this framework.

Throughout the whole monograph, we will denote by $\mathcal{P}$ the class of C^1-functions having locally Lipschitz continuous gradient and satisfying (16). The remarkable point is that if $V \in \mathcal{P}$, then every solution to the Hamiltonian system

$$Ju' = \nabla V(u) \tag{17}$$

is periodic with the same minimal period τ_V, describing a strictly star-shaped Jordan curve around the origin (in this situation, the origin is usually said to be an *isochronous center* for the system). The homogeneity, moreover, makes the set of the solutions to (17) have a precise structure, as shown in Proposition 1.1.9 below.

The fact that τ_V is a submultiple of T affects the existence of a solution when the problem is forced. Precisely, in [53, Theorem 1] it was proved that, for a T-periodic problem like

$$\begin{cases} Ju' = \nabla V(u) + E(t) \\ u(0) = u(T), \end{cases}$$

with $V \in \mathcal{P}$, if

$$\frac{T}{\tau_V} \notin \mathbb{N},$$

then there is existence for every forcing term $E : [0, T] \to \mathbb{R}^2$. On the contrary, if

$$\frac{T}{\tau_V} \in \mathbb{N},$$

in which case we will say that the Hamiltonian $V(u)$ is *resonant*, there exist forcing terms for which all the solutions are unbounded [53, Theorem 2]. It is important to remark that, in this way, we recover the concept of resonance for scalar second order equations, as it can easily be seen.
The key factor is thus to analyze the time spent by the solutions to the autonomous problem (17) to perform a whole turn around the origin, and to distinguish whether or not we could have danger of resonance through this time. Notice that this represents another way of looking at both the linear and the asymmetric problems, because it is just in correspondence of the eigenvalues (or the Fučík eigenvalues) that the solutions to (5) (or (14)) have a minimal period equal to a submultiple of T.

Our results settle exactly in this framework. In particular, this monograph is devoted to the study of nonlinear systems having the form (15), where $F(t, u)$ "interacts" with gradients of positively homogeneous Hamiltonians. We will consider the problem of existence of T-periodic solutions to system (15), mainly from a Landesman-Lazer perspective; however, also the analogue of the Ahmad-Lazer-Paul condition for this kind of equations will be taken into account, giving the possibility of better understanding condition (LL), as well. The dangerous situation will here be the one in which $F(t, u)$ is "near" the gradient of a Hamiltonian such that the associated minimal period is a submultiple of T (in the complementary case, it was essentially proved in [53] that existence is always guaranteed). In the second part, we will study some complementary problems, related, on one hand, to multiplicity of T-periodic solutions, and, on the other one, to different kinds of boundary conditions.
The techniques used are predominantly topological, exploiting degree theory, and proving existence by showing that the degree associated with the problem is equal to 1. A considerable importance is owned, as well, by qualitative arguments in the phase plane, coming from suitable changes of variables which often help in handling the considered boundary value problem. In the last chapters, we will make use, respectively, of the Poincaré-Birkhoff theorem (cf. Section 1.2) to find multiplicity of solutions and of the shooting method to deal with Sturm-Liouville type boundary conditions.

We now briefly enter into the content of each chapter.

Chapter 1 is dedicated to some preliminary notions which will be useful throughout the whole work. We will consider a slightly larger class of Hamiltonians than $\mathcal{P}$, relaxing the strict positivity assumption (considering thus the set which will be denoted by $\mathcal{P}^*$) and analyzing in details the dynamics of the associated autonomous systems (17). After that, we will recall the version of the Poincaré-Birkhoff theorem which will be useful for our purposes [34].

The material of Chapter 2 mainly comes from [56] and [70]. We will analyze in details the issue of existence for the problem

$$Ju' = F(t,u) = \gamma(t,u)\nabla V_1(u) + (1-\gamma(t,u))\nabla V_2(u) + R(t,u),$$

with $0 \leq \gamma(t,u) \leq 1$, under the assumption that $R(t,u)$ is bounded by an L^2-function (in short, is "L^2-bounded"), and $V_1 \in \mathcal{P}^*$, $V_2 \in \mathcal{P}$, with $V_1 \leq V_2$. Possibly, we will assume the Hamiltonians V_1 and V_2 to be resonant (see the beginning of the chapter for further details), so that existence is not guaranteed (immediate examples can be taken from the mentioned scalar problems). We will first take into account the case of perturbations of a fixed gradient, i.e., $\gamma(t,u) \equiv 1$ (Section 2.1), and prove existence by means of the Landesman-Lazer conditions introduced in [56, 70], generalizing the scalar ones by Fabry [46]. In Section 2.2, we will consider a real double resonance situation, assuming

$$\frac{T}{k+1} \leq \tau_{V_2} < \tau_{V_1} \leq \frac{T}{k},$$

and we will see that existence is still obtained if we impose the two right Landesman-Lazer conditions on both the Hamiltonians, in the spirit, for instance, of [46, 49].

We will then pass to consider a particular case of double resonance which turns to be quite appropriate to deal with second order equations with damping, considering systems of the type

$$\begin{cases} Ju' = \gamma(t,u)\nabla V(u) + R(t,u) \\ u(0) = u(T), \end{cases}$$

with $V \in \mathcal{P}$. We will prove some results extending part of the work by Frederickson and Lazer [66], for scalar equations of Liénard or Rayleigh type

$$x'' + p(x)x' + g(t,x) = 0, \qquad \text{or} \qquad x'' + P(x') + g(t,x) = 0,$$

where the nonlinearity also depends on the derivative of the solution x. Even if, as pointed out in [18, 73, 86, 122], this situation is qualitatively different from the one when there is no dependence on x', it is still possible to see some analogy with the result proved by Lazer and Leach. At the end, indeed, we will manage to produce a condition which includes both the Landesman-Lazer and the Frederickson-Lazer ones, even if with some restrictions on the growth of the part depending on the derivative.

The proofs of the results of the chapter use degree theory, and the degree of the associated operator is proved to be equal to 1. We anticipate here that, in order to obtain the desired a priori estimates, we exploit in several occasions the planar framework of our problem, so that some kind of polar coordinates can be used; in this setting, the Landesman-Lazer condition is needed to control the angular component of the solutions (see Chapter 5 for further developments), while the Frederickson-Lazer condition gives information on their radial part.

Chapter 3 is taken from [16], and deals with the possibility of generalizing the Ahmad-Lazer-Paul condition to the planar problem

$$\begin{cases} Ju' = \nabla V(u) + \nabla Q(t,u) \\ u(0) = u(T). \end{cases}$$

Here the topic is quite delicate from a technical point of view, and requires some care in finding a symplectic change of variable transforming the original system into a (time-dependently) forced linear one, borrowing most of the technique from [63]. After that, we will use a recent result proved in [74] to deduce existence for the new system, yielding Theorem 3.3.1 below thanks to the symplecticity of the change of variables used.

In Chapter 4, we will make a deep comparison between the Landesman-Lazer condition and the Ahmad-Lazer-Paul one. The existence of some implication between the two conditions was commonly believed to be true, but, at least to the author's knowledge, a general precise relationship was not yet established in literature. We will go through [57] and part of [16] to explore such a topic by means of a careful use of elementary analysis, and find out that the Ahmad-Lazer-Paul condition is more general than the Landesman-Lazer one. It may be of independent interest the characterization of the Landesman-Lazer condition which is provided in Propositions 4.1.3 and 4.2.2 below.

In Chapter 5, we will pass to consider the issue of multiplicity of T-periodic solutions to a Hamiltonian system like

$$Ju' = \nabla H(t,u), \tag{18}$$

assuming that the principal terms in the Taylor expansions of $\nabla H(t,u)$, at 0 and at $+\infty$, are represented by two Hamiltonians $V_0, V_\infty \in \mathcal{P}$. This assumption, in particular when V_0, V_∞ are quadratic forms, is quite standard also in higher dimension, where it is possible to apply some variational tools coming from Morse theory in presence of a certain "gap" between V_0 and V_∞ (see, for instance, [25, 97, 98, 99] and the references therein). However, in the planar setting, the Poincaré-Birkhoff fixed point theorem provides the most powerful results of multiplicity, giving a number of solutions - obtained as fixed points of the Poincaré map associated with (18) - which depends on

how large is the difference between the rotation numbers of “small” and “large” solutions to the Cauchy problems associated with (18) (see also [59]). In this spirit, to mention a very incomplete story, a quite general theorem of semiabstract type was given in [136], for second order equations which are asymptotically linear, in terms of weighted eigenvalues of the limit problems. Concerning a general planar Hamiltonian system, far fewer results are available; we mention, again for asymptotically linear systems, the paper [103], analyzing the connections between the rotation numbers and the Maslov indexes of the limit problems.
Another point which might be of interest, in Chapter 5, is represented by the possibility of giving some rotational interpretation to a number of well-known nonresonance assumptions. Indeed, we will take into account the classical nonresonance condition (7), the nonuniform nonresonance one (11) and the Landesman-Lazer condition, all in a suitable planar version (see Sections 5.1.1 and 5.1.2), and we will show that, under each of these assumptions, “large” solutions in the phase plane make a noninteger number of rotations around the origin in the time T. This is a crucial point to obtain a fine estimate of the number of solutions to our problem through the Poincaré-Birkhoff theorem.

We conclude the monograph with the study of other types of boundary value problems for first order systems in the plane, under the point of view of resonance (the material is taken from [58]). The first step consists naturally in taking into account Dirichlet or Neumann boundary conditions. The history of this kind of problems for scalar second order equations was given a great impulse by the pioneering paper by Fučík [67], dealing with the Dirichlet boundary value problem for nonlinear equations with asymmetric nonlinearities. For the model

$$\begin{cases} x'' + \mu x^+ - \nu x^- + g(x) = e(t) \\ x(0) = 0 = x(T), \end{cases} \tag{19}$$

where $\mu, \nu \in \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$, $e : [0,T] \to \mathbb{R}$ are continuous functions, he defined ([67, Lemma 2.8]) the Fučík spectrum Σ_D, whose elements, in analogy with the previous discussion, are the pairs (μ, ν) for which there is a nontrivial solution to the positively homogeneous problem

$$\begin{cases} x'' + \mu x^+ - \nu x^- = 0 \\ x(0) = 0 = x(T). \end{cases} \tag{20}$$

Assuming $g(x)$ to have a sublinear growth, Fučík proved an existence result for (19) [67, Theorem 2.11], provided that (μ, ν) belongs to some “nonresonance regions” determined by the set Σ (corresponding to the symbol $\exists$ in Figure 2, where the spectrum Σ_D for (20) has been depicted). The same kind of analysis could be performed for the Neumann problem, for which, on the contrary, the situation is similar to the one

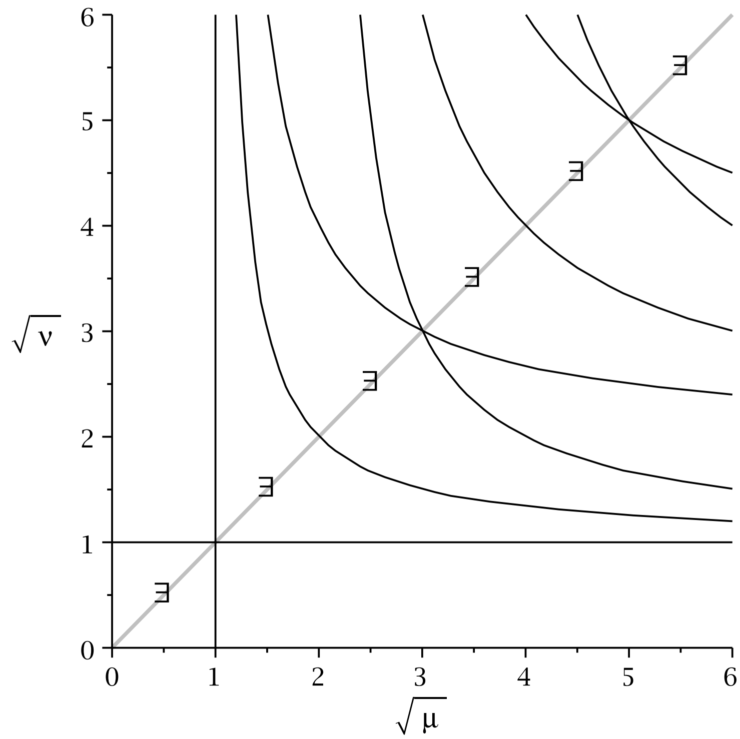

Figure 2: The Fučík spectrum for (20), with $T = \pi$.

for the periodic problem, and for Sturm-Liouville boundary conditions of the type

$$ax(0) + bx'(0) = 0 = cx(T) + dx'(T)$$

(see, e.g., [28, 36, 79, 123, 124]). In particular, the notion of Fučík spectrum naturally extends to all of these cases. Incidentally, we remember that boundary conditions of Sturm-Liouville type for planar systems have already been considered, e.g, in [127], while we refer to [110] and the references therein for nonlinear boundary conditions for second order scalar equations.

To extend these considerations to the planar setting, the starting point is the fact that, since the crucial assumption to be kept is the homogeneity with respect to the space variable, we could consider boundary conditions which are not necessarily linear, but only homogeneous. Indeed, in Chapter 6 we will consider the problem

$$\begin{cases} Ju' = \nabla V(u) + R(t,u) \\ u(0) \in \mathcal{C}_S,\ u(T) \in \mathcal{C}_A, \end{cases} \tag{21}$$

with $V \in \mathcal{P}$, where $\mathcal{C}_S$ and $\mathcal{C}_A$ are two cones in the plane. We will first use a shooting method to prove some preliminary abstract results, after having defined a suitable concept of resonance for which it will be crucial to look at the time spent by the solutions to the autonomous problem (17) to start from the cone $\mathcal{C}_S$ and arrive on $\mathcal{C}_A$. After that, we will study existence and multiplicity of solutions to (21) in some particular cases; first, we will take into account the Sturm-Liouville problem, under an assumption of nonresonance. Multiplicity can here be achieved in different ways, exploiting both a gap between zero and infinity (compare, e.g., with [4, 15, 59, 103], dealing with the periodic problem; for the Dirichlet problem, see [27, 37, 77, 120]) or, roughly speaking, a gap in the rotation numbers around some particular solution, in dependence of a real parameter (as it was done, for instance, in [76] and, for the T-periodic problem, in [32, 59] - see Subsection 6.2.2 below). In any case, roughly speaking, the number of solutions found depends again on how large is the gap between zero and infinity (see Theorem 6.1.5 below). At the end, we will examine a more general situation, which we call "polygonal problem", where we take the boundary conditions on polygonal lines, giving, also in this case, existence and multiplicity results in a nonresonant situation. In our opinion, another point of interest in this chapter might be represented by the pictures of the Fučík spectrum for Sturm-Liouville and polygonal problems associated with scalar second order equations.
The topic is very recent, and the material presented here can be considered as the first step for its study. Thus, the resonant case, in the spirit of the results achieved in the other chapters, has not been completely examined yet, being the object of a current research.

Contents

Chapter 1

Preliminaries

The chapter is divided into two parts: in the first one, we will analyze in detail the main properties of positively homogeneous C^1-Hamiltonians, and we will examine the qualitative features of the associated Hamiltonian systems. In the second part, on the other hand, we will see how the homogeneity property allows, as well, to introduce a suitable concept of modified rotation number, an important tool which will be employed in the forthcoming results.

1.1 Positively homogeneous Hamiltonian systems

In this section, we will deal with functions $V : \mathbb{R}^2 \to \mathbb{R}$ of class C^1, satisfying the following property:

$$V(\lambda u) = \lambda^2 V(u), \quad \lambda > 0,\, u \in \mathbb{R}^2. \tag{1.1}$$

We first notice that, assuming (1.1), it automatically follows that $V(0) = 0$; moreover, to determine V on the whole $\mathbb{R}^2$, it will be sufficient to know its values on each point of $\mathbb{S}^1 = \{u \in \mathbb{R}^2 \,|\, |u| = 1\}$ (or, more in general, on each point of a star-shaped curve around the origin). For this reason, for $\theta \in \mathbb{R}$ it will be useful to define the 2π-periodic C^1-function

$$\widehat{V}(\theta) = V(\cos\theta, \sin\theta).$$

We state a first proposition concerning general properties of positively homogeneous functions.

Proposition 1.1.1. *Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a C^1-function satisfying* (1.1)*. Then:*

1) *the continuous function $\nabla V : \mathbb{R}^2 \to \mathbb{R}^2$ is positively homogeneous of order* 1*;*

2) *Euler's formula holds true:*

$$\langle \nabla V(u) | u \rangle = 2V(u), \quad u \in \mathbb{R}^2.$$

Proof. To obtain 1), it suffices to differentiate formula (1.1). For what concerns point 2), let us observe that, fixed $u \in \mathbb{R}^2$, if we define $w :]0, +\infty[\to \mathbb{R}$ by

$$w(t) = \frac{V(tu)}{t^2},$$

we have that w is C^1 and

$$w'(t) = -\frac{2}{t^3} V(tu) + \frac{\langle \nabla V(tu) | u \rangle}{t^2}.$$

The thesis now follows observing that V is positively homogeneous of order 2 if and only if w is constant, i.e., $w'(t) = 0$. $\square$

As we have already remarked, we will choose our Hamiltonians to be positively homogeneous to extend to the plane most of the considerations holding for the scalar problem. In this setting, such functions will be the natural comparison terms when analyzing the possible occurrence of resonance phenomena.
Comparing with equation (5) in the Introduction, since the eigenvalues of the scalar T-periodic problem are nonnegative (so that the corresponding comparison function has the form $V(x, y) = (1/2)(y^2 + \lambda x^2)$, for a nonnegative λ), it seems natural to require our Hamiltonians to be nonnegative, leading to the following definition.

Definition 1.1.2. We set

$$\mathcal{P}^* = \left\{ V : \mathbb{R}^2 \to \mathbb{R} \;\middle|\; \begin{array}{l} V \in C^1, \nabla V \text{ is locally Lipschitz continuous} \\ 0 \le V(\lambda u) = \lambda^2 V(u),\ \lambda > 0,\ u \in \mathbb{R}^2 \end{array} \right\}.$$

Our aim is now to analyze the dynamics of the autonomous system

$$Ju' = \nabla V(u), \tag{1.2}$$

for $V \in \mathcal{P}^*$. Notice that the uniqueness and the global continuability for the Cauchy problems associated with (1.2) are ensured. Indeed, the uniqueness follows from the Lipschitz continuity of ∇V, while the global continuability is a consequence of the fact that ∇V is positively homogeneous of degree 1, and thus has an at most linear growth in u. Accordingly, we will denote by $u(t; \bar{u})$ the solution to (1.2) such that $u(0; \bar{u}) = \bar{u}$. More in general, observe that the homogeneity of V would imply uniqueness and global continuability even if ∇V was not Lipschitz continuous; we do not enter into details, referring the reader to [10, Lemma 2.1] and [118]. An example of nonnegative positively homogeneous Hamiltonian V such that ∇V is not Lipschitz continuous could be given extending by positive homogeneity the function

$$\widehat{V}(\theta) = \int_0^\theta \sqrt{|\sin s|}\, ds - \int_0^{2\pi} \sqrt{|\sin s|}\, ds.$$

Fix now a solution $u(t)$ to (1.2). Since the system is Hamiltonian, the energy V is preserved along $u(t)$:

$$\frac{d}{dt}V(u(t)) = \langle \nabla V(u(t)) | u'(t) \rangle = \langle Ju'(t) | u'(t) \rangle = 0.$$

Thus, the motion takes place on the level curves of V. Let us denote such curves by

$$\gamma_c = \{(x,y) \in \mathbb{R}^2 \mid V(x,y) = c\}, \quad c \geq 0.$$

If $c > 0$, in view of the uniqueness, $u(t)$ never reaches 0, so that it is possible to pass to polar coordinates, writing $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$. A standard computation yields then

$$-\theta'(t) = \frac{\langle Ju'(t) | u(t) \rangle}{\rho(t)^2} = \frac{2V(u(t))}{\rho(t)^2} = 2V(\cos\theta(t), \sin\theta(t)), \tag{1.3}$$

from which, since $V \geq 0$, we deduce that $u(t)$ moves clockwise on γ_c.
It turns out that the homogeneity of V deeply affects the shape of γ_c, but, as we will see, inside the class $\mathcal{P}^*$ still very different behaviors of $u(t)$ can be present, depending on the sign-definiteness of V out of the origin. For this reason, we will split our study into two cases, the first one dealing with Hamiltonians which vanish at some nonzero point in $\mathbb{R}^2$, and the other one, on the contrary, considering positive functions on $\mathbb{R}^2_*$.

1.1.1 Vanishing Hamiltonians

The first eigenvalue of the T-periodic problem for the equation

$$x'' + \lambda x = 0$$

is $\lambda = 0$, corresponding to constant eigenfunctions. Writing such an equation as a first order system, we obtain

$$Ju' = \nabla V(u),$$

with the position $V(u) = \frac{1}{2}(y^2 + \lambda x^2)$. If $\lambda = 0$, such a function vanishes along the whole axis $\{y = 0\}$, motivating the following definition.

Definition 1.1.3. We set

$$\mathcal{P}_0 = \big\{ V \in \mathcal{P}^* \mid \text{ there exists } u_0 \in \mathbb{R}^2_* \text{ with } V(u_0) = 0 \big\}.$$

Notice that, if $V \in \mathcal{P}_0$ and $u_0 \neq 0$ is such that $V(u_0) = 0$, then V vanishes on the whole half-line passing through the origin and u_0. For this reason, to determine unambiguously the set where V vanishes, it is sufficient to search for the zeros of

$\widehat{V}$. Secondly, notice that, if $V \in \mathcal{P}_0$, for every $u_0 \in \mathbb{R}^2_*$ such that $V(u_0) = 0$ we have $\nabla V(u_0) = 0$, since u_0 is a minimum point of the C^1-function V. Conversely, by Euler's formula, every zero of ∇V is a zero of V. Therefore, we can avoid the distinction between zeros of V and zeros of ∇V.
Henceforth, for a Hamiltonian $V \in \mathcal{P}_0$ we will set

$$\mathfrak{Z}_V = \{\xi \in \mathbb{S}^1 \mid V(\xi) = 0\}. \tag{1.4}$$

It is clear that

$$V(u) = 0,\, u \in \mathbb{R}^2 \quad \Longleftrightarrow \quad \text{there exist } u_0 \in \mathfrak{Z}_V \text{ and } k \geq 0 \text{ s.t. } u = ku_0.$$

As previously anticipated, we now want to examine the autonomous Hamiltonian system

$$Ju' = \nabla V(u), \quad u \in \mathbb{R}^2, \tag{1.5}$$

with $V \in \mathcal{P}_0$, keeping particular attention on the existence of periodic solutions.
Since the motion in (1.5) takes place on the level curves of V, it seems natural to start by qualitatively analyzing such curves. In particular, we state the following proposition.

Proposition 1.1.4. *Let $V \in \mathcal{P}_0$. Then:*

- *the level set γ_0 is the union of rays emanating from the origin;*
- *if $c > 0$, then γ_c is unbounded and either asymptotic or definitively parallel to γ_0 (possibly both).*

Let us first clarify what we mean by definitive parallelism of two curves. We say that γ_c is definitively parallel to γ_0 in the open angular sector $\mathcal{B}_0$ if there exists $\theta_0 \in \overline{\mathcal{B}_0}$, with $\widehat{V}(\theta_0) = 0$, such that the level curve γ_c, while lying in $\mathcal{B}_0$, is parallel to the half-line having slope equal to θ_0.
We now prove Proposition 1.1.4, referring to [70] for further details.

Proof. The first part of the statement follows directly from the homogeneity of V. For what concerns the second one, assume that $\hat{\theta}$ is such that $\widehat{V}(\hat{\theta}) > 0$. Then, by continuity, $\widehat{V}$ is positive in an angular neighborhood of $\hat{\theta}$, so that there exist θ_1, θ_2 such that $\hat{\theta} \in]\theta_1, \theta_2[$, and

$$\widehat{V}(\theta_1) = \widehat{V}(\theta_2) = 0, \quad \widehat{V}(\theta) > 0 \text{ if } \theta \in]\theta_1, \theta_2[\,.$$

Let us denote by $\mathcal{B}_{\theta_1,\theta_2}$ the open angular region from θ_1 to θ_2. By the Implicit Function Theorem, setting $A_{\theta_1,\theta_2} = \mathcal{B}_{\theta_1,\theta_2} \cap \mathbb{S}^1$, there exists a unique continuous function $\lambda_c : A_{\theta_1,\theta_2} \to \mathbb{R}$ such that, for every $u \in A_{\theta_1,\theta_2}$, it is $V(\lambda_c(u)u) = c$. Thus, the branch $\bar\gamma$ of γ_c lying in $\mathcal{B}_{\theta_1,\theta_2}$ is connected and this implies that, fixed a point $\bar u = \bar\rho(\cos\bar\theta, \sin\bar\theta) \in \bar\gamma$, we have

$$\bar\gamma = \{u(t;\bar u)\}_{t\in\mathbb{R}}.$$

Moreover, $\bar\gamma$ is unbounded, because, if it were bounded, since V is continuous and vanishes on the unbounded sets $\{k(\cos\theta_i, \sin\theta_i), k \geq 0\}, i = 1, 2$, there would exist a point $\hat u \in \mathcal{B}_{\theta_1,\theta_2}$ such that $|\hat u| > \max_{u\in\bar\gamma} |u|$ and $V(\hat u) = \hat c < c$. The contradiction would thus easily be obtained by homogeneity, being $\sqrt{c/\hat c}\,\hat u \in \bar\gamma$, and $|\sqrt{c/\hat c}\,\hat u| > \max_{u\in\bar\gamma} |u|$. It follows

$$\lim_{|t|\to+\infty} |u(t;\bar u)| = +\infty. \tag{1.6}$$

Writing the solution $u(t;\bar u)$, which never reaches the origin by uniqueness, in polar coordinates, namely

$$u(t;\bar u) = |u(t;\bar u)|(\cos\theta(t;\bar u), \sin\theta(t;\bar u)),$$

since

$$V(u(t;\bar u)) = |u(t;\bar u)|^2 V(\cos\theta(t;\bar u), \sin\theta(t;\bar u)),$$

we have, in view of the preservation of the energy and (1.6),

$$V(\cos\theta(t;\bar u), \sin\theta(t;\bar u)) \to 0 \quad \text{for } |t| \to +\infty.$$

As $\theta_1 < \bar\theta < \theta_2$, it follows that

$$\theta(t;\bar u) \to \theta_1 \ \text{ for } t \to +\infty, \quad \theta(t;\bar u) \to \theta_2 \ \text{ for } t \to -\infty, \tag{1.7}$$

agreeing with the strict monotonicity of the angle $\theta(t)$ along the level curves of V (remember that the motion is clockwise).

With these preliminaries, we now want to investigate the asymptotic behavior of the level curves of V. To this aim, let us assume that there exists a right neighborhood $\mathcal{U}_1$ of θ_1, corresponding to a certain angular region $\mathcal{B}_{\theta_1,\theta^*}$, such that the curve γ_c $(c > 0)$, written in polar coordinates (ρ,θ), is a line having slope equal to $\tan\theta_1$, for $\theta \in \mathcal{U}_1$. Then, it is of the kind

$$\rho\sin\theta = (\tan\theta_1)\rho\cos\theta + m, \tag{1.8}$$

for a suitable $m \neq 0$. Finding explicitly ρ in (1.8) and inserting it into the relation

$$V(\rho\cos\theta, \rho\sin\theta) = \rho^2 \widehat V(\theta)$$

gives, on γ_c,

$$\frac{m^2}{(\sin\theta - \tan\theta_1\cos\theta)^2}\widehat V(\theta) = c.$$

Thus, if there exists a positive constant m such that, for every $\theta \in \mathcal{U}_1$,

$$\widehat{V}(\theta) = m(\sin\theta - \tan\theta_1 \cos\theta)^2, \tag{1.9}$$

as long as the level curves of V lie in $\mathcal{B}_{\theta_1,\theta^*}$, they will be parallel to the half-line having slope θ_1. It is straightly seen that the implications in the argument hold also in the reverse direction, so that this condition is necessary and sufficient; moreover, the same argument can be repeated for a left neighborhood of θ_2. Finally, by relation (1.7), in all the other cases γ_c is asymptotic to γ_0. $\square$

An example of level curves which are definitively parallel is given, for $(x, y) \in \mathbb{R}^2$, by the Hamiltonian

$$V(x, y) = \frac{1}{2}y^2 + \frac{1}{4}(x^+)^2,$$

where $x^+ = \max\{x, 0\}$. This function vanishes along the half-line $\mathcal{L} = \{(x, y) \in \mathbb{R}^2 \mid x \le 0,\ y = 0\}$. The picture in Figure 1.1 gives an idea of the level curves of V, for three different values of c. In Figures 1.2 and 1.3, other situations are described. In the first one, we depict a situation of parallelism between the level curves, while in the second one we have drawn a mixing behavior, according to the side occupied by the considered point u with respect to the zeros of V. In this last case, the Hamiltonian V is defined as in (1.9), with $m = 1$, for $x > y$, while for $x \le y$ it is (in polar coordinates) $V(\rho, \theta) = \rho^2(1 - \cos^2 4\theta)$. In both cases, the lines colored in light grey are sets where the considered function vanishes, and have been depicted for the reader's convenience.

As for the existence of periodic solutions to (1.5), it is now clear that the only ones are given by the functions which are constantly equal to the zeros of V. Explicitly, the following proposition holds:

Proposition 1.1.5. *Let $V \in \mathcal{P}_0$. Then, the only periodic solutions to equation (1.5) are given by $u(t) \equiv ku_0$, where $u_0 \in \mathfrak{Z}_V$ and $k \ge 0$.*

Proof. Assume that $V(u(0)) > 0$. Then, $u(t; u(0))$ lies on a level curve of V corresponding to a positive value of the energy. As we have seen, the solutions lying on such curves satisfy (1.6), so that they are not periodic.
Assume, on the contrary, that $V(u(0)) = 0$. Then, $u(0) = ku_0$ for some $u_0 \in \mathfrak{Z}_V$ and $k \ge 0$, implying that $u(t; u(0)) \equiv ku_0$ by uniqueness. $\square$

Example 1.1.6. For $u = (x, y) \in \mathbb{R}^2$, consider $V(x, y) = \frac{1}{2}y^2$, which vanishes at every point of the form $(s, 0)$, for $s \in \mathbb{R}$. Such a Hamiltonian is associated with the system

$$\begin{cases} x' = y \\ y' = 0, \end{cases}$$

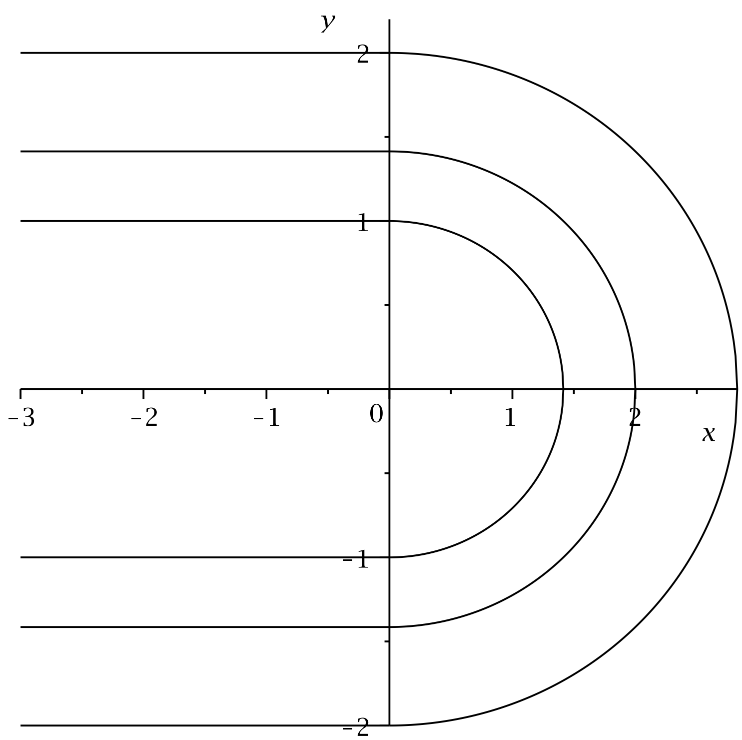

Figure 1.1: The level curves γ_c of $V(x,y) = \frac{1}{2}y^2 + \frac{1}{4}(x^+)^2$, for $c = 0.5, 1, 2$.

obviously equivalent to the scalar equation $x'' = 0$. It is well-known that the only periodic solutions to such a system are the constant ones $(x(t), y(t)) \equiv (s, 0)$. A similar discussion can be carried out for the Hamiltonian $V(x,y) = \frac{1}{2}x^2$.

1.1.2 Positive Hamiltonians

The picture arising from the definition of $\mathcal{P}_0$ changes radically when we consider, on the contrary, Hamiltonians which are always positive outside the origin. Let us start with the following definition.

Definition 1.1.7. We set

$$\mathcal{P} = \left\{ V \in \mathcal{P}^* \mid V(u) > 0 \text{ for every } u \in \mathbb{R}^2_* \right\}.$$

As a preliminary remark, notice that, if $V \in \mathcal{P}$, then $V(u) \to +\infty$ for $|u| \to +\infty$.

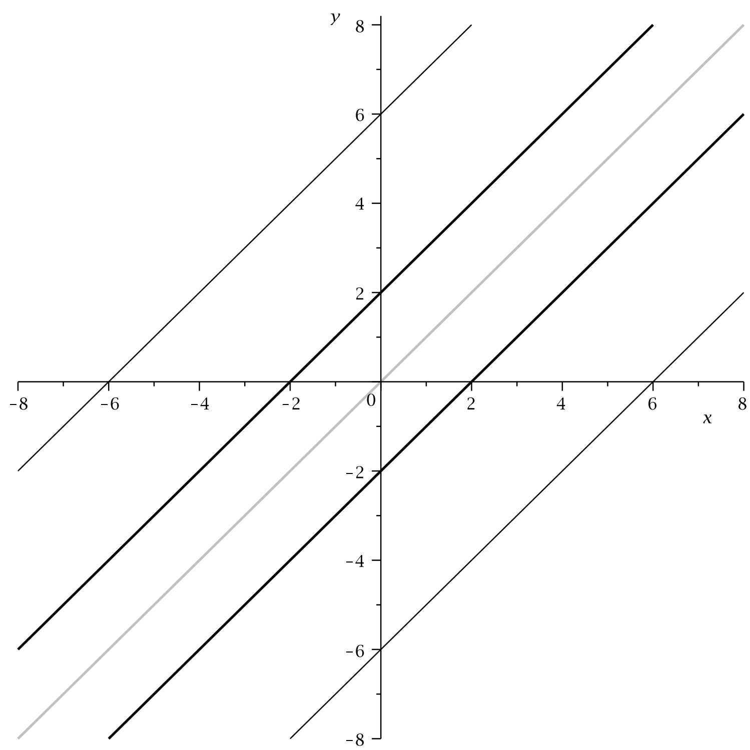

Figure 1.2: Two level curves of $V(\rho,\theta) = \rho^2(\sin\theta - \cos\theta)^2$.

Moreover, $\mathcal{P}$ is closed under addition and $\mathcal{P}_0 + \mathcal{P} \subset \mathcal{P}$. The following proposition clarifies the structure of the level curves of a Hamiltonian $V \in \mathcal{P}$.

Proposition 1.1.8. *Let $V \in \mathcal{P}$. Then, γ_c is a strictly star-shaped Jordan curve around the origin, for every $c > 0$.*

Proof. Since $V(u) > 0$ for every $u \neq 0$, by homogeneity the curve γ_c crosses every quadrant in the plane. In particular, since the map $\delta : \lambda \in \mathbb{R}^+_* \mapsto V(\lambda u)$ is strictly increasing and such that

$$\lim_{\lambda \to 0^+} \delta(\lambda) = 0, \qquad \lim_{\lambda \to +\infty} \delta(\lambda) = +\infty, \tag{1.10}$$

for every $\xi \in \mathbb{S}^1$ the ray emanating from the origin and passing through ξ intersects γ_c exactly once, so that γ_c is a star-shaped simple curve around the origin. Being, in view of Euler's formula, $\langle \nabla V(u) | u \rangle = 2V(u) \neq 0$ for every $u \in \mathbb{R}^2_*$, the curve

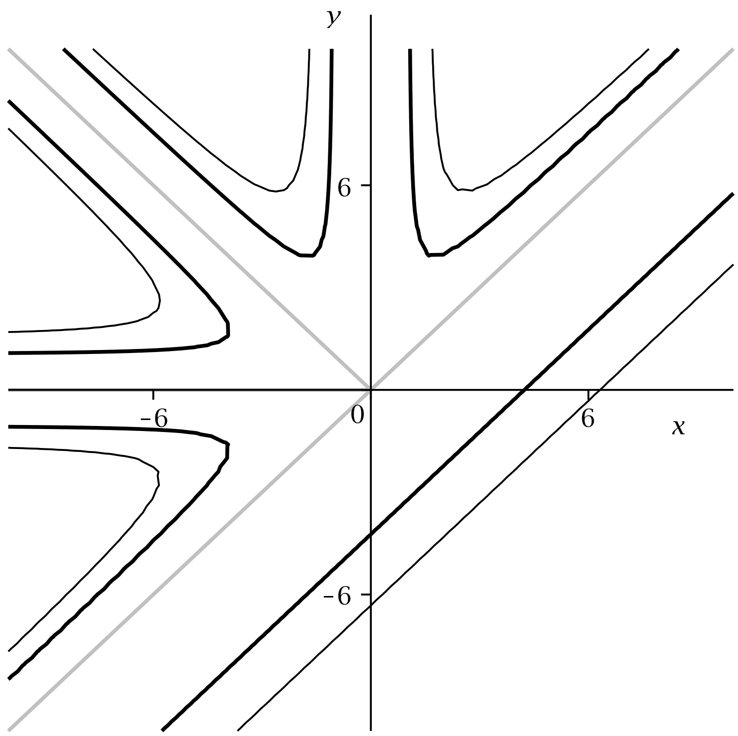

Figure 1.3: A mixing behavior: asymptotic and definitively parallel.

γ_c is strictly star-shaped, meaning that it is always transversal to the rays passing through its points. Lastly, we can use the Implicit Function Theorem, together with the compactness of $\mathbb{S}^1$, to show that γ_c is a closed curve, concluding the proof. □

We immediately see that, since the level curves of V are closed, every solution to

$$Ju' = \nabla V(u), \quad u \in \mathbb{R}^2 \tag{1.11}$$

is globally defined and periodic.
Before going on, let us remark that the homogeneity of the function V is not strictly necessary to obtain the conclusion of Proposition 1.1.8. Indeed, it suffices to assume the following three properties:

1) $V(0) = 0$ and $V(u) > 0$ for every $u \neq 0$;

2) $\nabla V(0) = 0$ and $\langle \nabla V(u) | u \rangle > 0$ for every $u \neq 0$;

3) $V(u) \to +\infty$ for $|u| \to +\infty$.

Under these assumptions, nontrivial solutions never reach the origin, in view of 1) and the preservation of $V(u)$. The star-shapedness of the level curves of V (with respect to the origin) follows from the fact that, by 2), the map $\delta : \lambda \in \mathbb{R}^+_* \mapsto V(\lambda u)$ is strictly increasing and, by 1) and 3), it satisfies (1.10). By the preservation of the energy, moreover, every solution $u(t)$ to (1.11) is globally defined and periodic, and moves clockwise by (1.3), in view of 1). In this setting, it is said that the origin is a *global center*, since it is an equilibrium point globally surrounded by closed orbits. As we will see in Chapter 3, a sufficiently regular function satisfying 1)–3), equal to the square of the Euclidean norm near the origin, can be used to construct a symplectic diffeomorphism of the plane into itself, which changes the shape of the orbits into circumferences. This will be the key point to prove the validity of an Ahmad-Lazer-Paul type result (cf. Theorem 3.3.1).

If we require V to be homogeneous, the structure of global center (in the origin) has some additional fundamental properties, which we are now going to explore. First, let us denote by ζ_θ the open half-line emanating from the origin having slope equal to θ, i.e.,

$$\zeta_\theta = \{\rho(\cos\theta, \sin\theta), \rho > 0\}.$$

If $\alpha, \beta \in \mathbb{R}$ are such that $\alpha \leq \beta < \alpha + 2\pi$, and $u(t)$ is a nontrivial solution to (1.11), the time needed by $u(t)$, starting from ζ_β, to reach ζ_α - recall that the motion is clockwise in view of (1.3) - is given by

$$\int_\alpha^\beta \frac{d\theta}{2V(\cos\theta, \sin\theta)}\,. \tag{1.12}$$

Thus, such a time is independent of the starting point on the half-line ζ_β, and this will play a very important role in our estimates. To prove such a property, it suffices to notice that, writing $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, if t_0, t_1 are such that $\theta(t_0) = \beta$, $\theta(t_1) = \alpha$, in view of (1.3) we have

$$t_1 - t_0 = \int_{t_0}^{t_1} \frac{-\theta'(t)}{2V(\cos\theta(t), \sin\theta(t))} dt = \int_\alpha^\beta \frac{d\theta}{2\widehat{V}(\theta)}.$$

Let us now choose $\theta \in [0, 2\pi[$, and set $\beta = \theta + 2\pi$, $\alpha = \theta$. In view of the 2π-periodicity of $\widehat{V}$, formula (1.12) gives that the time to perform exactly one revolution around the origin, covering the angle $\beta - \alpha = 2\pi$, is independent of θ. This shows that the minimal period of the solutions to (1.11) is always the same. Moreover, thanks to the properties of ∇V, it is possible to characterize the structure of the solution set to (1.11), as we are going to see in the following proposition.

Proposition 1.1.9. *Let $V \in \mathcal{P}$. Then, there exists $\tau_V > 0$ such that all the nonzero solutions to* (1.11) *have minimal period equal to τ_V. Fixed one of such solutions, say $\varphi_V(t)$, every other solution to* (1.11) *has the form*

$$u(t) = C\varphi_V(t+\theta),$$

for suitable constants $C \geq 0$, $\theta \in [0, \tau_V[$; moreover, it holds

$$\tau_V = \mathcal{A}_V, \tag{1.13}$$

where $\mathcal{A}_V$ is the area of the bounded region of the plane delimited by γ_1, namely

$$\mathcal{A}_V = \int_{\{V \leq 1\}} dxdy.$$

Since all the nontrivial solutions to (1.11) wind around the origin and have the same minimal period, we will say that the origin is an *isochronous center*.

Proof. Let us fix a solution $\varphi_V(t)$ to (1.11) and denote by τ_V its minimal period. Consider, for $\bar{u} \in \mathbb{R}^2_*$, the Cauchy problem

$$\begin{cases} Ju' = \nabla V(u) \\ u(0) = \bar{u}. \end{cases} \tag{1.14}$$

Since $\varphi_V(t)$ describes a star-shaped curve surrounding the origin, there exist $\hat{t} \in [0, \tau_V[$, $\hat{\lambda} > 0$ such that $\hat{\lambda}\varphi_V(\hat{t}) = \bar{u}$. Setting $u(t) = \hat{\lambda}\varphi_V(t+\hat{t})$, it is immediately seen that $u(t)$ is the unique solution to (1.14) (hence $u(t)$ has minimal period equal to τ_V), proving the first part of the statement. For the second one, choose φ_V in such a way that $V(\varphi_V(t)) = 1$ for every $t \in [0, \tau_V[$. In view of Stokes' formula, we now have

$$\begin{aligned} \mathcal{A}_V \quad &= \int_{\{V \leq 1\}} dxdy = \frac{1}{2} \int_{\partial\{V \leq 1\}^+} (xdy - ydx) = \frac{1}{2} \int_{\partial\{V \leq 1\}^+} \langle Ju | du \rangle \\ &= \frac{1}{2} \int_0^{\tau_V} \langle J\varphi_V'(t) | \varphi_V(t) \rangle dt = \tau_V, \end{aligned}$$

where for the last integral we have parametrized the boundary of the set $\{V \leq 1\}$ through $\varphi_V(t)$, and we have used Euler's formula. This concludes the proof. $\square$

As a consequence of the previous discussion, we have that, if $V \in \mathcal{P}$, the following formula holds true:

$$\tau_V = \int_0^{2\pi} \frac{d\theta}{2\widehat{V}(\theta)}. \tag{1.15}$$

1.1.3 Some remarks

In this subsection, we briefly analyze the properties of the more general class of systems having the form
$$Ju' = \zeta(t)\nabla V(u), \tag{1.16}$$
with $V \in \mathcal{P}^*$ and $\zeta : \mathbb{R} \to \mathbb{R}$ a positive continuous function. We are going to see that the arguments of the previous subsections work, with slight modifications, also in this case. For simplicity, however, in our existence results we will usually deal with the previously considered situation $\zeta(t) \equiv 1$, but several forthcoming statements will extend to this general setting, as we will see.
Concerning (1.16), observe preliminarily that also in this case we have uniqueness and global continuability for the solutions to the associated Cauchy problems, by the local Lipschitz continuity of ∇V and the at most linear growth of the right-hand side in the variable u, respectively. Moreover, the function V is preserved along the solutions:

$$\frac{d}{dt}V(u(t)) = \langle \nabla V(u(t)) | u'(t) \rangle = \zeta(t)\langle J\nabla V(u(t)) | \nabla V(u(t)) \rangle = 0.$$

Indeed, if $u(t)$ solves $Ju' = \zeta(t)\nabla V(u)$, then, setting

$$Z(t) = \int_0^t \zeta(s)\, ds, \tag{1.17}$$

by uniqueness we have
$$u(t) = v(Z(t)),$$
for some suitable function $v(t)$ solving $Jv' = \nabla V(v)$. Thus, considering the further term $\zeta(t)$ corresponds only to changing the speed on the level curves of V.
If we assume that $V \in \mathcal{P}$, we have the following proposition.

Proposition 1.1.10. *Assume that the positive continuous function $\zeta : \mathbb{R} \to \mathbb{R}$ is T-periodic, and satisfies*
$$\frac{1}{T}\int_0^T \zeta(t)\, dt = 1. \tag{1.18}$$
Then, defining $Z(t)$ as in (1.17)*, every solution to (1.16) has the form*

$$u(t) = C\varphi_V(Z(t) + \theta), \tag{1.19}$$

for suitable constants $C \geq 0$, $\theta \in [0, T[$.

Notice that assumption (1.18) can always be fulfilled, at the expense of relabeling V (thus changing the corresponding τ_V).

Proof. In view of the hypotheses, it is clear that the functions of the form (1.19) satisfy (1.16). On the other hand, fixed $\bar{u} \in \mathbb{R}^2_*$, since $Z(0) = 0$ and $\varphi_V(t)$ describes a strictly star-shaped curve around the origin in the plane, there exists a solution to (1.16) which takes the value $\bar{u}$ at $t = 0$ and has the form (1.19). The conclusion follows from the uniqueness. $\square$

If, moreover, $\zeta(t)$ is τ_V-periodic, and satisfies (1.18) with τ_V in place of T, then we can maintain the isochronicity - with the same period - for (1.16), in the sense that every solution will perform each turn around the origin in the same time τ_V. In this case, it suffices to take $\theta \in [0, \tau_V[$ in (1.19).
As a consequence of Proposition 1.1.10, anticipating some considerations about resonance, fixed $T > 0$ it is easy to see that equation (1.16) has a nontrivial T-periodic solution if and only if $Ju' = \nabla V(u)$ has a nontrivial T-periodic solution, i.e., $\frac{T}{\tau_V} \in \mathbb{N}$. In this case, observe that all the nontrivial solutions to (1.16) are T-periodic and make exactly $\frac{T}{\tau_V}$ turns around the origin in the time $[0, T]$, as it is immediately seen integrating the equality

$$\int_0^T -\frac{\theta'(t)}{2V(\cos\theta(t), \sin\theta(t))}\, dt = \int_0^T \zeta(t)\, dt.$$

1.2 Rotation numbers and the Poincaré-Birkhoff theorem

In what follows, the concept of *rotation number* of a planar curve will play an essential role. From an elementary and intuitive point of view, it simply consists in counting the number of revolutions made by the curve around a fixed point (usually, the origin). As we will see in the next chapters, a precise estimate of the rotation number of the solutions to a planar system will be one of the main tools to provide existence and multiplicity results for the periodic problem.
We first recall the following basic definition.

Definition 1.2.1. Let $t_1 < t_2$ and $u : [t_1, t_2] \to \mathbb{R}^2$ an absolutely continuous path, with $u(t) = (x(t), y(t)) \neq 0$ for every $t \in [t_1, t_2]$. The *rotation number* of $u(t)$ in the time interval $[t_1, t_2]$ is defined as

$$\mathrm{Rot}\,(u(t); [t_1, t_2]) = \frac{1}{2\pi}\int_{t_1}^{t_2} \frac{\langle Ju'(t)|u(t)\rangle}{|u(t)|^2}\, dt = \frac{1}{2\pi}\int_{t_1}^{t_2} \frac{y(t)x'(t) - x(t)y'(t)}{x(t)^2 + y(t)^2}\, dt.$$

It is well known that $\mathrm{Rot}\,(u(t); [t_1, t_2])$ counts the normalized *clockwise* angular displacement of the curve $u(t)$ around the origin, in the time interval $[t_1, t_2]$. Precisely, writing $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, with $\rho(t), \theta(t)$ absolutely continuous functions, and $\rho(t) > 0$, it holds that

$$\mathrm{Rot}\,(u(t); [t_1, t_2]) = -\frac{1}{2\pi}\int_{t_1}^{t_2} \theta'(t)\,dt = -\frac{\theta(t_2) - \theta(t_1)}{2\pi}.$$

In particular, when $u(t_1) = u(t_2)$, namely when $u(t)$ is a closed path, the number $\mathrm{Rot}\,(u(t); [t_1, t_2])$ is an integer.

Let now $u(t)$ be a solution to the linear system $Ju' = au$ (for $a > 0$). Then, since $\langle Ju'|u\rangle = a|u|^2$, the computation of the rotation number gives immediately

$$\mathrm{Rot}\,(u(t); [t_1, t_2]) = \frac{t_2 - t_1}{2\pi}a,$$

since the angular velocity of $u(t)$ is constantly equal to a.
Considering, on the other hand, the positively homogeneous system (1.11), with $V \in \mathcal{P}$ fixed, this notion of rotation number, although standard, is quite inconvenient to handle. Indeed, referring to (1.3), the angular velocity $-\theta'(t) = 2V(\cos\theta(t), \sin\theta(t))$ changes according to the values of the Hamiltonian V, and this could make our estimates more difficult. Thus, it would be preferable to have a modified version of the (angle, and thus of the) rotation number for which the angular velocity of the solutions to (1.11) is constant. Roughly speaking, as we will be able to remark repeatedly, this can be done by taking the angular coordinate on $\varphi_V(t)$, rather than on the unit circumference. Moreover, since we will be interested in T-periodic solutions, which perform an integer number of turns around the origin in the time T, this should be done possibly keeping the same value of the angular gap at least when making an integer number of turns.

To this aim, let us define the so called V-modified rotation number, as introduced in [138, p. 17].

Definition 1.2.2. Let $V \in \mathcal{P}$ and $t_1 < t_2$. Let $u : [t_1, t_2] \to \mathbb{R}^2$ be an absolutely continuous path, with $u(t) \neq 0$ for every $t \in [t_1, t_2]$. The V-modified rotation number of $u(t)$ in the time interval $[t_1, t_2]$ is defined as

$$\mathrm{Rot}_V\,(u(t); [t_1, t_2]) = \frac{1}{\tau_V}\int_{t_1}^{t_2} \frac{\langle Ju'(t)|u(t)\rangle}{2V(u(t))}\,dt.$$

In the following chapters, we will consider the T-periodic problem, for $T > 0$ fixed, and, in the case when $[t_1, t_2] = [0, T]$, we will simply write $\mathrm{Rot}_V(u(t))$ in place of $\mathrm{Rot}_V\,(u(t); [0, T])$.
The reason why we consider the expression appearing in Definition 1.2.2 can easily be understood. Indeed, take $\varphi_V(t)$ as in the previous section; since, in the time interval

$[0, \tau_V]$, it describes a strictly star-shaped curve around the origin, it can be used to introduce modified polar coordinates. Incidentally, we remark that this was originally done in [45] for the scalar problem; see also, e.g., [49, 51, 53]. In particular, let us write the path in Definition 1.2.2 as $u(t) = r(t)\varphi_V(\omega(t))$. A straightforward computation gives then

$$\omega'(t) = \frac{\langle Ju'(t)|u(t)\rangle}{2r(t)^2 V(\varphi(\omega(t)))} = \frac{\langle Ju'(t)|u(t)\rangle}{2V(u(t))},$$

so that computing Rot_V is equivalent to counting the number of turns made by the new coordinate ω (accordingly, the multiplicative factor $1/\tau_V$ comes from the fact that $\varphi_V(t)$ performs a complete turn around the origin when ω increases of τ_V). Furthermore, referring to the Hamiltonian system $Ju' = \nabla V(u)$, computing the angular velocity ω' of the solutions to the system gives the constant value

$$\omega'(t) = 1,$$

and this agrees with the previous considerations (notice that ω' is positive, since the motion is already clockwise, thanks to the direction of the motion of $\varphi_V(t)$).
We are now going to briefly discuss the relationships between the standard and the V-modified rotation number. First, it is clear that the standard (clockwise) rotation number, which, from now on, will be simply denoted by Rot, corresponds to the choice $V(u) = \frac{1}{2}|u|^2$. For a general comparison, we will need the following two lemmas, which have been proved in [10, Lemma 2.2 and Proposition 2.1]. For the reader's convenience, we briefly recall the proofs, referring to [10] for more details.

Lemma 1.2.3. *Let $V \in \mathcal{P}$. The map $\Xi_V : \mathbb{R} \to \mathbb{R}$, defined by*

$$\Xi_V(\theta) = \frac{\pi}{\tau_V}\int_0^\theta \frac{ds}{\widehat{V}(s)}, \tag{1.20}$$

is an increasing C^1-homeomorphism of $\mathbb{R}$, such that

$$\Xi_V(\theta + 2k\pi) = \Xi_V(\theta) + 2k\pi, \tag{1.21}$$

for every $\theta \in \mathbb{R}$ and every $k \in \mathbb{Z}$.

Proof. Since the integrand in (1.20) is 2π-periodic, we have $\Xi_V(\theta + 2k\pi) = \Xi_V(\theta) + k\Xi_V(2\pi)$; moreover, computing the area $\mathcal{A}_V$ using polar coordinates we have, in view of (1.15),

$$\tau_V = \mathcal{A}_V = \int_0^{2\pi} \frac{d\theta}{2\widehat{V}(\theta)} = \frac{\Xi_V(2\pi)}{2\pi}\tau_V.$$

Formula (1.21) follows immediately. On the other hand, differentiating the expression in (1.20) and observing that, for $\theta \to \pm\infty$, it is $\Xi_V(\theta) \to \pm\infty$ in view of (1.21), we have that Ξ_V is an increasing homeomorphism. □

In particular, from the statement of the lemma it follows that $\Xi_V(2k\pi) = 2k\pi$. Notice that, in the case when $V(u) = \frac{1}{2}|u|^2$, we have $\Xi_V(\theta) = \theta$, as it is natural to expect (see also (1.22) below).

Lemma 1.2.4. *Let $t_1 < t_2$ and $u : [t_1, t_2] \to \mathbb{R}^2$ be an absolutely continuous path, with $u(t) \neq 0$ for every $t \in [t_1, t_2]$. Moreover, let $\theta : [t_1, t_2] \to \mathbb{R}$ be an absolutely continuous function such that*

$$u(t) = |u(t)|(\cos\theta(t), \sin\theta(t)),$$

for every $t \in [t_1, t_2]$. Then,

$$\mathrm{Rot}_V\,(u(t); [t_1, t_2]) = \frac{\Xi_V(\theta(t_1)) - \Xi_V(\theta(t_2))}{2\pi}. \tag{1.22}$$

Proof. Defining the path

$$\Theta_V(t) = \Xi_V(\theta(t)),$$

we have

$$\Theta_V'(t) = \Xi_V'(\theta(t))\theta'(t) = -\frac{\pi}{\tau_V}\frac{1}{V(\cos\theta(t), \sin\theta(t))}\frac{\langle Ju'(t)|u(t)\rangle}{|u(t)|^2} = -\frac{2\pi}{\tau_V}\frac{\langle Ju'(t)|u(t)\rangle}{2V(u(t))},$$

whence the conclusion. □

In general, the values (computed on the same path) of Rot_V and Rot are different one from the other. Nevertheless, the two lemmas just stated allow us to prove the following fundamental property (see [10, Proposition 2.1]), which will often be employed in the following chapters.

Proposition 1.2.5. *Let $V \in \mathcal{P}$ and $t_1 < t_2$. Let $u : [t_1, t_2] \to \mathbb{R}^2$ be an absolutely continuous path, with $u(t) \neq 0$ for every $t \in [t_1, t_2]$, and $j \in \mathbb{Z}$. Then*

$$\begin{aligned} \mathrm{Rot}_V\,(u(t); [t_1, t_2]) > j \quad &\Leftrightarrow \quad \mathrm{Rot}\,(u(t); [t_1, t_2]) > j \\ \mathrm{Rot}_V\,(u(t); [t_1, t_2]) < j \quad &\Leftrightarrow \quad \mathrm{Rot}\,(u(t); [t_1, t_2]) < j. \end{aligned}$$

From the statement, it follows that

$$\mathrm{Rot}_V\,(u(t); [t_1, t_2]) = j \Leftrightarrow \mathrm{Rot}\,(u(t); [t_1, t_2]) = j,$$

namely Rot_V counts the same number of *complete* clockwise turns around the origin as Rot.

Proof. It suffices to observe that, thanks to Lemma 1.2.3,

$$\mathrm{Rot}_V\,(u(t);[t_1,t_2]) > j \Leftrightarrow \Xi_V(\theta(t_1)) > \Xi_V(\theta(t_2)) + 2j\pi = \Xi_V(\theta(t_2) + 2j\pi),$$

and, in view of the fact that $\Xi_V(\theta)$ is strictly increasing,

$$\Xi_V(\theta(t_1)) > \Xi_V(\theta(t_2) + 2j\pi) \Leftrightarrow \theta(t_1) > \theta(t_2) + 2j\pi.$$

□

What we have just seen opens some reflections about the properties of the modified rotation numbers. In the following, we will mainly deal with periodic boundary value problems, for which the solutions (if any) will perform a certain (integer) number of complete turns around the origin. It turns then out that a careful estimate of the time spent to perform an integer number of revolutions around the origin can be obtained through an accurate estimate of the modified rotation numbers, thanks to Proposition 1.2.5. However, when studying other kinds of boundary value problems, one could be interested in modified polar coordinates having different properties. For instance, wishing to apply the shooting method to ensure existence for a Dirichlet problem, it will be important to pass to new coordinates which count in the same way the number of *left and right half-turns* around the origin, while for the Neumann one it will be important to maintain the same number of *upper and lower half-turns* around the origin. Even more, for a general Sturm-Liouville (or a polygonal, see Chapter 6) problem, it will be important to preserve, with the modified coordinates, the number of "half-turns" performed in the plane starting with a certain slope, depending on the considered problem. However, we will keep this approach only for the periodic problem (see Chapter 5), but what we are going to see could be a hint for further developments regarding the topic in Chapter 6.
It is thus natural to wonder which properties have to be imposed on $V \in \mathcal{P}$ to produce a modified rotation number satisfying these different properties. We start observing that the key point in Lemma 1.2.3 is represented by formula (1.21), which is essential to ensure the validity of Proposition 1.2.5. Assume now that we want to produce a change of variables which counts all the complete half-turns around the origin as the usual polar angle; to this aim, following the previous argument, we will need a function $\Xi_V(\theta)$ such that, for every $\theta \in [0, 2\pi]$,

$$\Xi_V(\theta + k\pi) = \Xi_V(\theta) + k\pi, \tag{1.23}$$

so that $\Xi_V(k\pi) = k\pi$, for any integer k. Consequently, if a curve $u(t)$ performs an integer number j of half-turns around the origin, we will have

$$j = \frac{\Xi_V(\theta(t_1)) - \Xi_V(\theta(t_2))}{\pi}.$$

To see in which cases (1.23) is fulfilled, it suffices to observe that, differentiating (1.23), we obtain, for every $\theta \in [0, 2\pi]$,

$$\frac{1}{V(\cos(\theta + k\pi), \sin(\theta + k\pi))} = \frac{1}{V(\cos\theta, \sin\theta)},$$

and, to fulfill this last relation, $\widehat{V}(\theta)$ has to be π-periodic. This means that $V(-u) = V(u)$ for every $u \in \mathbb{R}^2$, i.e., V is even. We can thus state the following proposition.

Proposition 1.2.6. *Let $V \in \mathcal{P}$ be an* even *function, and $t_1 < t_2$. Let $u : [t_1, t_2] \to \mathbb{R}^2$ be an absolutely continuous path, with $u(t) \neq 0$ for every $t \in [t_1, t_2]$, and $j \in \mathbb{Z}$. Then*

$$\begin{aligned} \mathrm{Rot}_V\,(u(t); [t_1, t_2]) > \frac{j}{2} \quad &\Leftrightarrow \quad \mathrm{Rot}\,(u(t); [t_1, t_2]) > \frac{j}{2} \\ \mathrm{Rot}_V\,(u(t); [t_1, t_2]) < \frac{j}{2} \quad &\Leftrightarrow \quad \mathrm{Rot}\,(u(t); [t_1, t_2]) < \frac{j}{2}. \end{aligned}$$

Wishing to go on investigating this kind of issue, one could wonder when all the quarter-turns are preserved by a change of coordinates coming from a positively homogeneous Hamiltonian. In the same way as before, one easily gets that a necessary condition is represented by

$$\Xi_V\left(\theta + k\frac{\pi}{2}\right) = \Xi_V(\theta) + k\frac{\pi}{2},$$

implying $\Xi_V(k\pi/2) = k\pi/2$, for any integer k. Similarly as before, this gives

$$\frac{1}{V(\cos(\theta + k\pi/2), \sin(\theta + k\pi/2))} = \frac{1}{V(\cos\theta, \sin\theta)},$$

yielding in turns

$$V(Ju) = V(u), \qquad \text{for every } u \in \mathbb{R}^2. \tag{1.24}$$

This means that V has to be invariant under the action of the group generated by the symplectic matrix J. For the sake of completeness, we can thus state the analogous of Proposition 1.2.5 also in this case.

Proposition 1.2.7. *Let $V \in \mathcal{P}$ satisfy* (1.24)*, and $t_1 < t_2$. Let $u : [t_1, t_2] \to \mathbb{R}^2$ be an absolutely continuous path, with $u(t) \neq 0$ for every $t \in [t_1, t_2]$, and $j \in \mathbb{Z}$. Then*

$$\begin{aligned} \mathrm{Rot}_V\,(u(t); [t_1, t_2]) > \frac{j}{4} \quad &\Leftrightarrow \quad \mathrm{Rot}\,(u(t); [t_1, t_2]) > \frac{j}{4} \\ \mathrm{Rot}_V\,(u(t); [t_1, t_2]) < \frac{j}{4} \quad &\Leftrightarrow \quad \mathrm{Rot}\,(u(t); [t_1, t_2]) < \frac{j}{4}. \end{aligned}$$

Example 1.2.8. As an example, let us consider the linear case, i.e., $V(u) = \langle Bu|u\rangle$, where B is a 2×2 strictly positive definite symmetric matrix. In this situation, we have the following:

- since $V \in \mathcal{P}$, we have that Rot_V counts the same number of complete turns around the origin as Rot;
- since V is trivially even, Rot_V counts the same number of complete half-turns around the origin as Rot, as well;
- in order for V to count also the same number of quarter-turns around the origin as Rot, the following relation has to be verified:

$$\langle BJu|Ju\rangle = \langle Bu|u\rangle, \qquad \text{for every } u \in \mathbb{R}^2,$$

  implying

$$JB = BJ,$$

  so that B has to commute with the symplectic matrix J. However, the only symmetric matrices commuting with J are the multiples of the identity. If we relax the symmetry hypothesis on B (which is however essential if we want the linear system $Ju' = Bu$ to be Hamiltonian), then all the matrices having the form

$$B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

  satisfy the desired conclusion.

Dealing with the homogeneous asymmetric scalar second order equation

$$x'' + \mu x^+ - \nu x^- = 0, \tag{1.25}$$

whose associated Hamiltonian is

$$V(x,y) = \frac{1}{2}(y^2 + \mu(x^+)^2 + \nu(x^-)^2),$$

it is immediately seen that neither the hypotheses of Proposition 1.2.6 nor the ones of Proposition 1.2.7 are satisfied. However, it is often sufficient to know that only some particular half (or quarter)-turns are "preserved" when changing coordinates. For instance, with the same philosophy as before, if $V \in \mathcal{P}$ and we want the angular sector of width $A > 0$, counted clockwise starting from a slope equal to $\tan(\bar{\theta} + A)$, to be preserved when counted by means of the V-modified rotation number, it will be enough to ask that

$$\frac{\pi}{\tau_V} \int_{\bar{\theta}}^{\bar{\theta}+A} \frac{ds}{V(\cos s, \sin s)} = A,$$

as it was possible to see also by formula (1.12). If $A = 2\pi$, we thus get

$$\int_{\bar\theta}^{\bar\theta+2\pi} \frac{ds}{2V(\cos s, \sin s)} = \tau_V,$$

which is always satisfied in view of the 2π-periodicity of the integrand, meaning indeed that Rot_V preserves every complete turn around the origin (obviously, this is not the case for other values of A). Concerning (1.25), then, we have, for instance, that

$$\frac{\sqrt{\mu\nu}}{\sqrt{\mu}+\sqrt{\nu}} \int_0^\pi \frac{2\,ds}{\mu(\cos(s)^+)^2 + \nu(\cos(s)^-)^2 + \sin^2(s)} = \pi,$$

so that the upper and the lower half-turns are counted in the same way in standard and in modified polar coordinates. This can be helpful when performing certain kinds of estimates under different types of boundary conditions, and could be a good way of approaching the problems discussed in Chapter 6.

We conclude the chapter by briefly recalling the version of the Poincaré-Birkhoff fixed point theorem (see [34, 63, 95, 117]) which will be used in Chapter 5 to give multiplicity results for planar Hamiltonian systems. Indeed, the Hamiltonian structure of the considered nonlinear system is here essential, since the abstract statement of the Poincaré-Birkhoff theorem requires an area-preserving homeomorphism of two annuli in the plane. This will be reflected in a slight tightening of the class of first order systems considered in Chapter 5.
Since we are mainly interested in the applications of the Poincaré-Birkhoff theorem, we will not deepen the theoretical and the historical aspects of its statement, referring instead to [13, 26, 63] for such details.

Theorem 1.2.9. *Let $H : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a function differentiable in the second variable, with $\nabla H(t,u)$ L^1-Carathéodory, and such that*

$$\nabla H(t,0) \equiv 0.$$

Moreover, assume that the uniqueness and the global continuability (in [0,T]) for the solutions to the Cauchy problems associated with

$$Ju' = \nabla H(t,u) \tag{1.26}$$

are ensured, and denote by $u(\cdot;\bar u)$ the solution with $u(0;\bar u) = \bar u$, for $\bar u \in \mathbb{R}^2$. If there exist $0 < \rho_0 < \rho_\infty$ and two positive integers $k_0 \geq k_\infty$ such that

- $\mathrm{Rot}\,(u(t;\bar u)) > k_0\,,\quad$ *for every* $|\bar u| = \rho_0\,,$
- $\mathrm{Rot}\,(u(t;\bar u)) < k_\infty\,,\quad$ *for every* $|\bar u| = \rho_\infty\,,$

then, for every integer $k \in [k_\infty, k_0]$*, equation (1.26) has at least two (distinct)* T*-periodic solutions* $u_{1,k}(t), u_{2,k}(t)$ *such that* $u_{1,k}(0), u_{2,k}(0) \in \mathcal{A}(\rho_0, \rho_\infty)$ *and*

$$\mathrm{Rot}\,(u_{1,k}(t)) = \mathrm{Rot}\,(u_{2,k}(t)) = k.$$

Here, we used the notation $\mathcal{A}(r, R) = \{u \in \mathbb{R}^2 \mid r < |u| < R\}$, already introduced before. We only recall that Theorem 1.2.9 follows from the version of the Poincaré-Birkhoff theorem given by W. Y. Ding in [34] (see also [117]), applied to the Poincaré map

$$\Phi : \overline{\mathcal{A}(\rho_0, \rho_\infty)} \subset \mathbb{R}^2_* \to \Phi(\overline{\mathcal{A}(\rho_0, \rho_\infty)}) \subset \mathbb{R}^2_*,$$

defined by

$$\Phi(\bar{u}) = u(T; \bar{u}).$$

Indeed, such a map is a global homeomorphism and, by Liouville's theorem, it is area-preserving; moreover, observe that the condition $\nabla H(t, 0) \equiv 0$ implies that, if $\bar{u} \neq 0$, then $u(t; \bar{u}) \neq 0$ for every $t \in [0, T]$, so the rotation numbers appearing in the statement are well defined.
Lastly, we underline the fact that, in view of the previous discussions, in order to use this theorem it will be sufficient to estimate the modified rotation numbers, as we will be able to see in Chapter 5, and this will produce significant simplifications in the forthcoming computations.

Chapter 2

Existence for resonant first order systems in the plane

In this chapter, we will deal with resonant problems for first order systems in the plane. As recalled in the Introduction, in [53] the setting of positively homogeneous Hamiltonians was established as the natural generalization of the framework in which resonance appears for scalar second order equations. Thus, if $V \in \mathcal{P}$, the existence of a solution to the T-periodic forced problem

$$\begin{cases} Ju' = \nabla V(u) + E(t) \\ u(0) = u(T) \end{cases} \tag{2.1}$$

is strictly related to the existence of nontrivial solutions to the homogeneous problem

$$\begin{cases} Ju' = \nabla V(u) \\ u(0) = u(T). \end{cases} \tag{2.2}$$

As already recalled, in [53] it was proved that if $V \in \mathcal{P}$ and there exists a positive integer k such that

$$\tau_V = \frac{T}{k}, \tag{2.3}$$

then (2.2) has a nontrivial solution and (2.1) may not have any solution. On the other hand, it is clear that, if $V \in \mathcal{P}_0$, (2.1) is not necessarily solvable: as an example, one can think to the scalar equation $x'' = e(t)$ or, analogously, to the first order system $Ju' = E(t)$, where the associated Hamiltonians, referring to (2.1), are $V(x, y) = (1/2)y^2$ and $V(u) \equiv 0$, respectively. In these cases, if the forcing term does not satisfy a zero-mean condition, we clearly cannot find T-periodic solutions to the considered problem. Indeed, it is worth noticing that, while the Hamiltonians belonging to $\mathcal{P}$ are resonant with respect to the T-periodic problem only if the associated minimal period is a submultiple of T, a Hamiltonian belonging to $\mathcal{P}_0$ always gives rise to resonance. We thus want to analyze what conditions should be added to the forcing term in order to ensure the existence of a solution under the possible occurrence of resonance. In the first section, we will consider the nonlinear generalization of problem (2.1), namely when $E(t)$ is replaced by a nonlinearity $R(t,u)$, having sublinear growth in the u-variable. In the second section, we will consider the nonlinear problem associated with a more general principal term, coming from the convex combination of two resonant Hamiltonians (giving rise to a *double resonance* situation). After having examined in details the corresponding counterparts for the scalar second order analogous, at the end of the chapter we will focus on a possible relaxing of the conditions given in the previous sections.

Before going into the details of our results, we give a general property which will be useful in the following (the so called "elastic property").

Lemma 2.0.1. *Let $G : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ be an L^1-Carathéodory function such that*

$$|G(t,u)| \leq c(t)(1+|u|), \tag{2.4}$$

for almost every $t \in [0,T]$, and every $u \in \mathbb{R}^2$, being $c(t)$ a suitable function in $L^1(0,T)$. Then, for every $R_0 > 0$ there exists $R_1 \geq R_0$ such that, if $u(t)$ satisfies

$$Ju' = G(t,u), \tag{2.5}$$

and $|u(\bar{t})| \leq R_0$ for some $\bar{t} \in [0,T]$, then $|u(t)| \leq R_1$ for every $t \in [0,T]$.

Proof. Fix $R_0 > 0$; we choose $R_1 > (R_0 + \|c\|_1)e^{\|c\|_1}$, and prove that this choice makes the statement true. Indeed, otherwise, by continuity there would exist $t_0, t_1 \in [0,T]$ such that $|u(t_0)| = R_0$, $|u(t_1)| = R_1$, and

$$R_0 < |u(t)| < R_1, \qquad \text{for every } t \in]t_0, t_1[$$

(possibly with $t_1 < t_0$). It is then possible to pass to polar coordinates (ρ, θ) in (2.5), obtaining

$$|\rho'(t)| = \left| \left\langle u'(t) \,\middle|\, \frac{u(t)}{|u(t)|} \right\rangle \right| \leq |G(t,u(t))| \leq c(t)(1+\rho(t)),$$

for every $t \in [t_0, t_1]$. By Gronwall's lemma, then, the following estimate holds:

$$\rho(t) \leq (R_0 + \|c\|_1) \exp \left| \int_{t_0}^{t} c(s)\, ds \right| ,$$

for every $t \in [t_0, t_1]$. By our choice of R_1, this implies $\rho(t_1) < R_1$, hence a contradiction. □

A quite laborious proof of Lemma 2.0.1, in a more general context, can be found in [88] (proof of Theorem 6.5). As a counterpart of it, in the assumptions of the lemma, for every $R_2 > 0$ there exists $R_3 \geq R_2$ such that if $|u(\bar{t})| \geq R_3$ for some $\bar{t} \in [0,T]$, then $|u(t)| \geq R_2$ for every $t \in [0,T]$.

2.1 Fixed homogeneous principal terms

In this section, we will consider the T-periodic problem

$$\begin{cases} Ju' = G(t,u) = \nabla V(u) + R(t,u) \\ u(0) = u(T), \end{cases} \tag{2.6}$$

where $V \in \mathcal{P}^*$ and $R(t,u)$ is an L^2-Carathéodory function which is T-periodic in its first variable and such that

$$\lim_{|u|\to+\infty} \frac{R(t,u)}{|u|} = 0, \tag{2.7}$$

uniformly in $t \in [0,T]$. Accordingly, we will search for solutions which are absolutely continuous and satisfy the equation in (2.6) almost everywhere.
Two different situations can arise here: if $V \in \mathcal{P}$, then τ_V is well defined and we denote by φ_V the solution to $Ju' = \nabla V(u)$ such that $\varphi_V(0)$ lies on the positive horizontal semi-axis and

$$V(\varphi_V(t)) = 1/2, \qquad \text{for every } t \in [0,T]. \tag{2.8}$$

We recall that, in this case, every other solution to $Ju' = \nabla V(u)$ has the form $C\varphi_V(t+\theta)$, for constants $C \geq 0, \theta \in [0, \tau_V[$. On the other hand, if $V \in \mathcal{P}_0$, its level curves are unbounded and the periodic solutions to $Ju' = \nabla V(u)$ are exactly the equilibria. We recall that, in this case, we have denoted by $\mathfrak{Z}_V$ the subset of $\mathbb{S}^1$ where V vanishes (see (1.4)).
Let us introduce a further notation. We would like to distinguish among the various Hamiltonians in $\mathcal{P}$, according to their minimal period. To this aim, it seems natural to define, for a positive integer k,

$$\mathcal{P}_k = \Big\{ V \in \mathcal{P} \mid \tau_V = \frac{T}{k} \Big\}.$$

Remember that we have already defined $\mathcal{P}_0$ in Definition 1.1.3. Obviously, if $k \neq k'$, $\mathcal{P}_k \cap \mathcal{P}_{k'} = \emptyset$, while $\cup_{k\in\mathbb{N}} \mathcal{P}_k$ gives exactly the set of resonant Hamiltonians, according to (2.3) and the preliminary discussion at the beginning of the chapter.
Wishing to study problem (2.6) with the technique of topological degree, it is worth making the following digression. First of all, setting $\mathcal{D}(\mathcal{L}) = \{u \in L^2([0,T];\mathbb{R}^2) \mid u \in AC([0,T];\mathbb{R}^2), u' \in L^2([0,T];\mathbb{R}^2) \text{ and } u(0) = u(T)\}$, we can define the operators

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset L^2([0,T];\mathbb{R}^2) \to L^2([0,T];\mathbb{R}^2), \quad \mathcal{L}u = Ju'$$

and

$$\mathcal{N} : L^2([0,T];\mathbb{R}^2) \to L^2([0,T];\mathbb{R}^2), \quad (\mathcal{N}u)(t) = G(t, u(t)),$$

this last one being well defined since $G(t,u)$ satisfies (2.4). Observe that $\mathcal{L}$ is densely defined, and is a Fredholm operator of index zero. Indeed, it suffices to notice that the kernel of $\mathcal{L}$ is made up by constant functions, thus having dimension equal to 2; on the other hand, an L^2-function belongs to its rank if and only if it has zero mean, and this implies that the image of $\mathcal{L}$ is closed and has codimension equal to 2. One can thus use the theory of coincidence degree to reduce the equation

$$\mathcal{L}u = \mathcal{N}u,$$

considered on $\mathcal{D}(\mathcal{L})$, to the search for the zeros of a mapping of the form $I + K$, with K compact, through the use of the projectors on the kernel of $\mathcal{L}$ and on the complement of the image of $\mathcal{L}$ (see, for instance, [20, 106]). When the starting problem is autonomous, the remarkable result [20, Theorem 1] allows to compute the Leray-Schauder degree of such a mapping through the Brouwer degree of the right-hand side of the considered differential equation. As we will see in the proof of Theorem 2.1.1, and also afterwards, the idea is thus to exploit the homotopy invariance of the degree to connect the original problem to an autonomous one (cf. [20, Theorem 2]), whose degree can be computed and is different from zero.
Such an approach, which turns to be particularly successful for a wide range of problems, allows to prove the following theorem.

Theorem 2.1.1. *Let $V \in \mathcal{P}^*$, and assume that, for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$ with $|u| \leq 1$, and for every $\lambda \geq 1$,*

$$\langle R(t, \lambda u) | u \rangle \geq \eta(t), \tag{2.9}$$

for a suitable $\eta \in L^2(0,T)$. Then, with the previous positions, there exists a solution to problem (2.6) *provided that*

- *$V \in \mathcal{P}_0$ and the following condition is satisfied:*

  for every $\xi \in \mathfrak{Z}_V$,

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle R(t, \lambda\eta) | \eta \rangle \, dt > 0; \tag{2.10}$$

- *$V \in \mathcal{P}_k$, for an integer $k \geq 1$, and the following condition is satisfied:*

  for every $\theta \in [0,T]$,

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega) \rangle \, dt > 0. \tag{2.11}$$

Of course, (2.9) is a technical assumption, which, as we will see, is needed in order to apply Fatou's lemma, and is satisfied, in particular, when $R(t,u)$ is bounded. Conditions (2.10) and (2.11) are the Landesman-Lazer type conditions introduced in [70] and [56] (see the Introduction and Section 2.4 for their explicit formulation in the scalar case), requiring that the lower order term in the nonlinearity satisfies some integral sign conditions. Notice that, in view of the invariance of (2.10) and (2.11) with respect to the dilatations $\xi \mapsto r\xi$ and $\varphi_V(t) \mapsto r\varphi_V(t)$, respectively, for $r > 0$, instead of considering all the solutions to the resonant comparison equation it is sufficient to fix a positive number and consider only the ones having norm equal to that number (indeed, in (2.10), ξ is taken in $\mathbb{S}^1$, and in (2.11) the function φ_V is fixed).

Proof. Fix $\epsilon > 0$ and set $V_\epsilon(u) = V(u) + \frac{\epsilon}{2}|u|^2$. In any case, we have that $V_\epsilon \in \mathcal{P}$, so that τ_{V_ϵ} is well defined. By continuity and (1.13), we have that

$$\tau_{V_\epsilon} \searrow \tau_V$$

for $\epsilon \searrow 0$, thus we can choose $\bar\epsilon$ so that V_ϵ is not resonant for $0 < \epsilon \le \bar\epsilon$, namely

$$\frac{T}{\tau_{V_\epsilon}} \notin \mathbb{N} \quad \text{for every } 0 < \epsilon \le \bar\epsilon.$$

We now consider the family of problems, parametrized by $\sigma \in [0,1]$,

$$\begin{cases} Ju' = \nabla V(u) + (1-\sigma)R(t,u) + \sigma\bar\epsilon u \\ u(0) = u(T). \end{cases} \tag{2.12}$$

Since $V_\epsilon \in \mathcal{P}$ for every $0 < \epsilon \le \bar\epsilon$, for any open subset $\Omega \subset \mathbb{R}^2$ containing the origin it is

$$\deg_B(\nabla V_\epsilon, \Omega, 0) = 1,$$

where $\deg_B$ denotes the Brouwer degree (see, for instance, [88, Lemma II.6.5]). Thus, in view of [20, Theorem 2], to ensure the existence of a solution it suffices to show that we have an a priori bound in $L^\infty(0,T)$ for the solutions to (2.12), which is independent of $\sigma \in]0,1[$.

By contradiction, assume that there exists $(u_n)_n \subset L^\infty(0,T)$ and $(\sigma_n)_n \subset]0,1[$ such that u_n satisfies (2.12) for $\sigma = \sigma_n$, and $\|u_n\|_\infty \to +\infty$. We can assume that $\sigma_n \to \bar\sigma \in [0,1]$. Setting

$$v_n(t) = \frac{u_n(t)}{\|u_n\|_\infty},$$

for every n the function $v_n(t)$ satisfies

$$\begin{cases} Jv_n' = \nabla V(v_n) + (1-\sigma_n)\,\dfrac{R(t,\|u_n\|_\infty v_n)}{\|u_n\|_\infty} + \sigma_n \bar{\epsilon} v_n \\ v_n(0) = v_n(T). \end{cases} \tag{2.13}$$

In view of (2.7), it follows that v_n is bounded in $H^1(0,T)$, so that there exists $v \in H^1(0,T)$ such that, for instance, $v_n \rightharpoonup v$ in $H^1(0,T)$ and $v_n \to v$ uniformly. Moreover, since $\|v_n\|_\infty = 1$ for every n, we have that v is nonzero. Passing to the weak L^2-limit in (2.13), we thus reach, recalling (2.7),

$$Jv' = \nabla V(v) + \bar{\sigma}\bar{\epsilon} v.$$

However, this necessarily implies $\bar{\sigma} = 0$, otherwise we would have a nontrivial T-periodic solution to $Jv' = \nabla V_{\bar{\sigma}\bar{\epsilon}}(v)$, which is impossible in view of the previous positions.
We now have two cases to examine.

Case 1: $V \in \mathcal{P}_0$. In this situation, we pass to polar coordinates (thanks to the elastic property), writing $u_n(t) = \rho_n(t)(\cos\theta_n(t), \sin\theta_n(t))$. The expression of θ_n' is then given, thanks to Euler's formula, by

$$-\theta_n'(t) = 2V(\cos\theta_n(t), \sin\theta_n(t)) + (1-\sigma_n)\frac{\langle R(t, \|u_n\|_\infty v_n(t)) | u_n(t)\rangle}{\rho_n(t)^2} + \sigma_n\bar{\epsilon}. \tag{2.14}$$

Since $\|v\|_\infty = 1$, in view of Proposition 1.1.5 there exists $\xi \in \mathfrak{Z}_V$ such that $v_n \to \xi$ uniformly. Consequently, since every v_n is periodic, thus performing an integer number of turns around the origin, and $|\xi| = 1$, for n large $v_n(t)$ will turn exactly 0 times around the origin, so that integrating (2.14) from 0 to T gives 0. Since $V \geq 0$ and the sequence of positive numbers $(\sigma_n)_n$ converges to 0, for every n large it follows that

$$0 > \int_0^T \frac{\|u_n\|_\infty}{\rho_n(t)^2} \langle R(t, \|u_n\|_\infty v_n(t)) | v_n(t)\rangle \, dt,$$

and, multiplying by $\|u_n\|_\infty$,

$$0 > \int_0^T \frac{\|u_n\|_\infty^2}{\rho_n(t)^2} \langle R(t, \|u_n\|_\infty v_n(t)) | v_n(t)\rangle \, dt. \tag{2.15}$$

Using Fatou's lemma thanks to (2.9), since $\frac{\|u_n\|_\infty^2}{\rho_n(t)^2}$ converges to 1, we have

$$0 \geq \int_0^T \liminf_{n\to+\infty} \langle R(t, \|u_n\|_\infty v_n(t)) | v_n(t)\rangle \, dt;$$

in this expression, for every fixed $t \in [0,T]$ we are computing the inferior limit which appears in (2.10) along the particular subsequence $(\|u_n\|_\infty, v_n(t))$, for which $v_n(t) \to \xi$ and $\|u_n\|_\infty \to +\infty$. We deduce that

$$0 \geq \int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle R(t,\lambda\eta)|\eta\rangle \, dt,$$

which contradicts (2.10).

Case 2: $V \in \mathcal{P}$. This time, we introduce modified polar coordinates, by writing $u_n(t) = r_n(t)\varphi_V(t+\omega_n(t))$, with $w_n(0) \in [0,\tau_V[$ for every n. In view of (2.12), Euler's formula and the properties of $\varphi_V(t)$, we have

$$\omega_n'(t) = \sigma_n\bar{\epsilon}|\varphi_V(t+\omega_n(t))|^2 + (1-\sigma_n)\,\frac{\langle R(t, r_n(t)\varphi_V(t+\omega_n(t)))|\varphi_V(t+\omega_n(t))\rangle}{r_n(t)}.$$

Being $T/\tau_V = k$, $v(t)$ performs k turns around the origin in the time T; moreover, in view of Proposition 1.1.9, it will be $v(t) = r_v\varphi_V(t+\omega_v)$, for suitable constants $r_v > 0$, $\omega_v \in [0,\tau_V[$. Since the sequence of T-periodic functions v_n converges to v uniformly, for n sufficiently large every v_n (and so every u_n) performs exactly k turns around the origin. As a consequence, for such n we have $\omega_n(0) = \omega_n(T)$, thus integrating the expression of ω_n' gives 0. It follows that

$$0 > \int_0^T \frac{\langle R(t, r_n(t)\varphi_V(t+\omega_n(t)))|\varphi_V(t+\omega_n(t))\rangle}{r_n(t)}\, dt,$$

from which, setting $r_n^V(t) = r_n(t)/\|u_n\|_\infty$, we obtain

$$0 > \int_0^T \frac{\langle R(t, r_n(t)\varphi_V(t+\omega_n(t)))|\varphi_V(t+\omega_n(t))\rangle}{r_n^V(t)}\, dt.$$

Hypothesis (2.9) now allows us to apply Fatou's lemma, which gives

$$0 \geq \int_0^T \liminf_{n\to+\infty} \frac{\langle R(t, r_n(t)\varphi_V(t+\omega_n(t)))|\varphi_V(t+\omega_n(t))\rangle}{r_n^V(t)}\, dt.$$

We now observe that the relation

$$v_n(t) = r_n^V(t)\varphi_V(t+\omega_n(t)) \to v(t) = r_v\varphi_V(t+\omega_v), \qquad \text{uniformly in } t \in [0,T],$$

implies that $r_n^V \to r_v$ uniformly (it suffices to apply V to both members). In view of standard properties of inferior limits, this yields, since $r_v > 0$,

$$0 \geq \int_0^T \liminf_{n\to+\infty} \langle R(t, r_n(t)\varphi_V(t+\omega_n(t)))|\varphi_V(t+\omega_n(t))\rangle \, dt. \tag{2.16}$$

We now want to show that it is possible to choose ω_n in such a way that $\omega_n \to \omega_v$ uniformly. Observe preliminarily that adding any integer multiple of τ_V to ω_n does not change either the value of $\varphi_V(t+\omega_n(t))$ or the one of ω_n'. Now, consider the sequence $\omega_n(0)$: since it takes value in the compact set $[0,\tau_V]$, either it converges or it has more than one cluster point belonging to $[0,\tau_V]$. Assume this second case, and suppose, for the moment, that $\omega_n(0)$ has two distinct cluster points $\omega_1, \omega_2 \in [0,\tau_V]$. Thus, there exist two subsequences ω_{n_j}, ω_{n_h} of ω_n such that $\omega_{n_j}(0) \to \omega_1$ and $\omega_{n_h}(0) \to \omega_2$. Since

$$\varphi_V(t+\omega_n(t)) \to \varphi_V(t+\omega_v) \quad \text{uniformly in } t \in [0,T], \tag{2.17}$$

setting $t = 0$ we see that $\varphi_V(\omega_1) = \varphi_V(\omega_2) = \varphi_V(\omega_v)$, so that it will necessarily be $\omega_1 = \omega_v + i_1\tau_V$ and $\omega_2 = \omega_v + i_2\tau_V$, for suitable integers i_1, i_2. However, since $\omega_1, \omega_2 \in [0,\tau_V]$, the only possibility is that $\omega_1 = 0$ (and $\omega_2 = \tau_V$), or $\omega_1 = \tau_V$ (and $\omega_2 = 0$); this also shows that $\omega_n(0)$ cannot have more than two distinct cluster points, and that $\omega_v = 0$. Now, assume for instance that $\omega_1 = 0$: changing the original subsequence ω_{n_h} into $\omega_{n_h} - \tau_V$, we will have the convergence of the entire sequence:

$$\omega_n(0) \to \omega_1 = 0. \tag{2.18}$$

Thus, without loss of generality we can assume that $\omega_n(0)$ converges to ω_v. Our aim is now to show that $\omega_n \to \omega_v$ uniformly in $[0,T]$. To this aim, assume by contradiction that there exist $\bar\epsilon > 0$ and two sequences $(n_j)_j \subset \mathbb{N}, (t_j)_j \subset [0,\tau_V]$ such that

$$|\omega_{n_j}(t_j) - \omega_v| > \bar\epsilon. \tag{2.19}$$

By continuity, (2.19) implies that there exists a sequence $(s_j)_j \subset [0,\tau_V]$ such that $\omega_{n_j}(s_j) = \omega_v + \bar\epsilon$; we can assume that $s_j \to \bar s$, for a suitable $\bar s \in [0,\tau_V]$. We thus have

$$\varphi_V(s_j + \omega_{n_j}(s_j)) = \varphi_V(s_j + \omega_v + \bar\epsilon)$$

and, in view of (2.17), we deduce

$$\varphi_V(\bar s + \omega_v) = \varphi_V(\bar s + \omega_v + \bar\epsilon),$$

a contradiction (notice that (2.18) allows to exclude that $\bar\epsilon$ is a multiple of τ_V).
We can now finish the proof. Referring to (2.16), for every fixed $t \in [0,T]$ we are computing the inferior limit which appears in (2.11) along the particular subsequence $(r_n(t), \omega_n(t))$, for which $\omega_n(t) \to \omega_v$ and $r_n(t) = \|u_n\|_\infty \, r_n^V(t) \to +\infty$. We deduce that

$$0 \geq \int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\omega_v)} \langle R(t, \lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle \, dt,$$

which contradicts (2.11). $\square$

Remark 2.1.2. To perform the same proof as before, it suffices that, in place of (2.10), the following weaker condition is satisfied:

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle G(t,\lambda\eta)|\eta\rangle\, dt > 0. \tag{2.20}$$

However, for the sake of uniformity we have preferred to write a similar condition for both the Hamiltonians involved.
Furthermore, referring to (2.15), to achieve the same conclusion we could also propose a kind of sign condition which is weaker than (2.10) and can be formulated as follows: for every $\xi \in \mathfrak{Z}_V$, there exist λ_ξ, $\delta_\xi > 0$ and $h_\xi \in L^2(0,T)$ such that, for almost every $t \in [0,T]$, for every η such that $|\eta - \xi| \le \delta_\xi$ and every $\lambda \ge \lambda_\xi$, it holds

$$\frac{\langle G(t,\lambda\eta)|\eta\rangle}{|\eta|^2} \ge h_\xi(t), \tag{2.21}$$

with

$$\int_0^T h_\xi(t)\, dt \ge 0.$$

This requirement is similar to hypothesis (H) in [46, Theorem 1'] (for the scalar problem), which, however, does not seem to imply it.
Let us show that (2.20) implies (2.21). To this aim, let $\xi \in \mathfrak{Z}_V$ and set

$$l(t) = \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle G(t,\lambda\eta)|\eta\rangle;$$

by (2.20), choose $\epsilon > 0$ such that $\int_0^T (l(t) - \epsilon)\, dt > 0$. Accordingly, it will be possible to find $\lambda_0, \delta_0 > 0$ such that, for almost every $t \in [0,T]$, every $\lambda \ge \lambda_0$, and every η with $|\eta - \xi| \le \delta_0$,

$$\frac{\langle G(t,\lambda\eta)|\eta\rangle}{|\eta|^2} \ge \frac{l^+(t) - l^-(t) - \epsilon}{|\eta|^2},$$

where, as usual, $l^+(t) = \max\{l(t), 0\}$, and $l^-(t) = \max\{-l(t), 0\}$. Set

$$h_\xi(t) = \frac{l^+(t)}{\max_{|\eta-\xi|\le\delta_0} |\eta|^2} - \frac{l^-(t) + \epsilon}{\min_{|\eta-\xi|\le\delta_0} |\eta|^2},$$

i.e.,

$$h_\xi(t) = \frac{l^+(t)}{(1+\delta_0)^2} - \frac{l^-(t) + \epsilon}{(1-\delta_0)^2}.$$

Shrinking δ_0, if necessary, it is not difficult to see that (2.21) holds and $\int_0^T h_\xi(t)\, dt > 0$.

Remark 2.1.3. It is interesting to notice that, when the function $R(t,u)$ does not depend on u, condition (2.10) can provide a sufficient condition of existence for a forced autonomous system like

$$Ju' = \nabla V(u) + E(t),$$

with $V \in \mathcal{P}_0$. In particular, in this case (2.10) reads as

$$\int_0^T \langle E(t)|\xi\rangle \, dt > 0,$$

for every $\xi \in \mathfrak{Z}_V$.

Remark 2.1.4. In particular, Theorem 2.1.1 allows to consider the case when $V(u) \equiv 0$, so that a system of the type

$$Ju' = R(t,u),$$

with $R : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ an L^2-bounded function, is included in our framework, as well. This kind of simple resonance seems to be different from the usual concept of resonance for scalar second order equations. However, since in this case (2.10) has to be verified for every point of $\mathbb{S}^1$, it turns into a sign condition on the product $\langle R(t,\xi)|\xi\rangle$ for every $\xi \in \mathbb{R}^2$ with $|\xi|$ sufficiently large.

Remark 2.1.5. It is possible to give an alternative proof of Theorem 2.1.1, using the Poincaré-Bohl fixed point theorem. Indeed, anticipating what we will see in Chapter 5 concerning the rotational interpretation of the Landesman-Lazer conditions, the V-rotation number of "large" solutions to the Cauchy problems associated with (2.6) turns out not to be an integer, if we assume (2.10) or (2.11). In this case, notice that, in order to find solutions when $R(t,u)$ is not locally Lipschitz continuous, one can use an approximation procedure similar to the one exploited in Section 6.1. We are currently trying to understand how to extend this reasoning to the case of double resonance (cf. Remark 2.2.6); also for this reason, we have chosen to give the above proof, which works well both in the simple and in the double resonance setting. Let us remark, moreover, that, under the Landesman-Lazer condition which will be discussed in Chapter 5, the Poincaré-Bohl theorem could be used, as well, to give existence results for systems like $Ju' = \zeta(t)\nabla V(u) + R(t,u)$ (cf. Subsection 1.1.3 and Proposition 5.1.9).

Before concluding the section, let us observe that, changing the homotopy used to prove Theorem 2.1.1, we could prove the following counterpart of it.

Theorem 2.1.6. *Let $V \in \mathcal{P}_k$, for some integer $k \geq 1$, and assume that, for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$ with $|u| \leq 1$, and for every $\lambda \geq 1$,*

$$\langle R(t,\lambda u)|u\rangle \leq \eta(t),$$

for a suitable $\eta \in L^2(0,T)$. If, moreover, the following condition holds:
for every $\theta \in [0,T]$,

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle\, dt < 0,$$

then there exists a solution to problem (2.6)*.*

Indeed, it suffices to replace V_ϵ with the Hamiltonian $V(u) - \frac{\epsilon}{2}|u|^2$, for ϵ sufficiently small.
We conclude the section with an example.

Example 2.1.7. Theorem 2.1.1 ensures, for instance, the existence of a T-periodic solution to the system

$$Ju' = \nabla V(u) + \frac{u}{|u|^\alpha} + E(t),$$

being $E(t)$ a T-periodic function, $V \in \mathcal{P}^*$ and $0 < \alpha < 1$. This is clear since the integral appearing in (2.10) is equal to

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} [\lambda^{1-\alpha}|\eta|^{2-\alpha} + \langle E(t)|\eta\rangle]\, dt = +\infty,$$

while, on the other hand, the one appearing in (2.11) is given by

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} [\lambda^{1-\alpha}|\varphi_V(t+\omega)|^{2-\alpha} + \langle E(t)|\varphi_V(t+\omega)\rangle]\, dt = +\infty.$$

Setting, for simplicity, $T = 2\pi$, one can easily construct some variants like, e.g.,

$$R(t,u) = |1+\sin(t)|\frac{u}{|u|^\alpha} + E(t),$$

with α as above.

2.2 The case of double resonance

In this section, we will investigate the existence of a solution to the boundary value problem

$$\begin{cases} Ju' = F(t,u) \\ u(0) = u(T), \end{cases} \tag{2.22}$$

with $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ having the form

$$F(t,u) = \gamma(t,u)\nabla V_1(u) + (1-\gamma(t,u))\nabla V_2(u) + R(t,u), \tag{2.23}$$

where $\gamma(t,u)$ and $R(t,u)$ are L^2-Carathéodory functions such that $0 \le \gamma(t,u) \le 1$, and $R(t,u)$ satisfies the sublinear growth condition (2.7). We will assume

$$V_1 \in \mathcal{P}^*, \quad V_2 \in \mathcal{P},$$

with

$$V_1(u) \le V_2(u), \qquad \text{for every } u \in \mathbb{R}^2. \tag{2.24}$$

This situation was brought to the attention, first for second order equations and then in more general cases, in the works by Fabry and Fonda [46, 47]. Referring to the scalar case, since $V_1 \in \mathcal{P}^*$ we are also considering the case when double resonance involves, on one side, the first eigenvalue of the linear problem (or, for the asymmetric equation, the positive semi-axes in the plane where the Fučík spectrum is usually represented).

The following lemma is the key point to extend the considerations of the previous section to the case of double resonance.

Lemma 2.2.1. *Let $V_1 \in \mathcal{P}_k$ and $V_2 \in \mathcal{P}_{k+1}$, for a* nonnegative *integer k. Moreover, let $\alpha \in L^2(0,T)$ be such that, for almost every $t \in [0,T]$, $0 \le \alpha(t) \le 1$. Then, if $u(t)$ is a nontrivial solution to*

$$\begin{cases} Ju' = \alpha(t)\nabla V_1(u) + (1-\alpha(t))\nabla V_2(u) \\ u(0) = u(T), \end{cases} \tag{2.25}$$

then $u(t)$ solves either $Ju' = \nabla V_1(u)$, or $Ju' = \nabla V_2(u)$.

Proof. First of all, we observe that a nontrivial solution to (2.25) never reaches the origin. Indeed, if $v(t)$ solves (2.25) then also $sv(t)$ does, for every $s > 0$, thanks to the homogeneity of the right-hand side; moreover, since the right-hand side grows at most linearly in v, Lemma 2.0.1 holds. It follows that, if $v(\bar{t}) \neq 0$ for some $\bar{t} \in [0,T]$, then $v(t) \neq 0$ for every $t \in [0,T]$.

Consequently, the usual system of polar coordinates (ρ, θ) is well defined for system (2.25), so that we can write $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$. In view of Euler's formula, we get

$$-\theta'(t) = 2\alpha(t)V_1(\cos\theta(t), \sin\theta(t)) + 2(1-\alpha(t))V_2(\cos\theta(t), \sin\theta(t)). \tag{2.26}$$

It follows that $\theta'(t) \le 0$ for every $t \in [0,T]$; moreover, since $V_2 \in \mathcal{P}$ and (2.24) holds, we have

$$-\frac{\theta'(t)}{2V_2(\cos\theta(t), \sin\theta(t))} \le 1, \tag{2.27}$$

for almost every $t \in [0,T]$. Since $u(t)$ is T-periodic, it will perform an integer number of clockwise turns around the origin, say m. We now have two cases:

1) if $k = 0$, then, integrating (2.27) from 0 to T, relation (1.12) implies (since $V_2 \in \mathcal{P}_1$) that

$$\int_0^T \frac{-\theta'(t)\,dt}{2V_2(\cos\theta(t), \sin\theta(t))} = m\tau_{V_2} \leq T \;\Rightarrow\; m \leq 1.$$

On the other hand, since $u(t)$ moves clockwise, it will perform a nonnegative number of clockwise turns around the origin in the time T. This suffices to infer $m = 0$ or $m = 1$.
If $m = 0$, by (2.26) and the fact that $\theta'(t) \leq 0$ for almost every $t \in [0, T]$, we have

$$2\alpha(t)V_1(\cos\theta(t), \sin\theta(t)) + 2(1 - \alpha(t))V_2(\cos\theta(t), \sin\theta(t)) = 0$$

for almost every $t \in [0, T]$, from which it immediately follows that $\alpha(t) = 1$ almost everywhere (since $V_1 \geq 0$, $V_2 > 0$), and

$$-\theta'(t) = 2\alpha(t)V_1(\cos\theta(t), \sin\theta(t)) = 2V_1(\cos\theta(t), \sin\theta(t)) = 0 \tag{2.28}$$

for almost every $t \in [0, T]$. Thus, $u(t)$ is a solution to $Ju' = \nabla V_1(u)$ and V_1 is preserved along $u(t)$. Consequently, if $V_1(u(0)) = 0$, then $u(t) \equiv u(0)$ (and thus $u(t)$ satisfies $Ju' = \nabla V_1(u)$); on the other hand, it is not possible that $V_1(u(0)) > 0$, otherwise it would be $V_1(u(t)) > 0$ for every $t \in [0, T]$, contradicting (2.28).
If $m = 1$, we pass to generalized polar coordinates by writing $u(t) = r(t)\varphi_{V_2}(t + \omega(t))$, and get the equations for $r'(t)$ and $\omega'(t)$:

$$r'(t) = -\alpha(t)r(t)\langle \nabla V_1(\varphi_{V_2}(t + \omega(t))) | \varphi'_{V_2}(t + \omega(t))\rangle, \tag{2.29}$$

and

$$\omega'(t) = 2\alpha(t)V_1(\varphi_{V_2}(t + \omega(t))) - \alpha(t). \tag{2.30}$$

In view of (2.24) and the fact that $V_2(\varphi_{V_2}(t)) = \frac{1}{2}$ for every $t \in [0, T]$, $\omega'(t) \leq 0$ for almost every $t \in [0, T]$. Since, on the other hand, $\omega(0) = \omega(T)$ (being $m = 1$), it follows that $\omega'(t) = 0$ for almost every $t \in [0, T]$, so that $\omega(t) \equiv \hat{\omega}$, for a suitable $\hat{\omega} \in [0, T]$. Concerning (2.29), it follows that

$$r'(t) = -\alpha(t)r(t)\langle \nabla V_1(\varphi_{V_2}(t + \hat{\omega}) | \varphi'_{V_2}(t + \hat{\omega})\rangle,$$

while (2.30) now gives

$$\omega'(t) = \alpha(t)(2V_1(\varphi_{V_2}(t + \hat{\omega})) - 1) = 0 \tag{2.31}$$

We now want to prove that $r'(t) = 0$ for almost every $t \in [0, T]$. In view of (2.31), for almost every $t \in [0, T]$ we have that either $\alpha(t) = 0$, or $V_1(\varphi_{V_2}(t + \hat{\omega})) = \frac{1}{2}$. Consider thus $\bar{t} \in [0, T]$; if $\alpha(\bar{t}) = 0$, then $r'(\bar{t}) = 0$, as desired. On the contrary, if $\alpha(\bar{t}) > 0$, since $V_2(\varphi_{V_2}(t+\hat{\omega})) = \frac{1}{2}$ for every $t \in [0, T]$, we have that $\bar{t}$ is a zero of the function $t \mapsto V_1(\varphi_{V_2}(t + \hat{\omega})) - V_2(\varphi_{V_2}(t + \hat{\omega}))$, which is of class C^1 and nonpositive. Necessarily $\bar{t}$ is then a maximum of this function, and so

$$\frac{d}{dt} V_1(\varphi_{V_2}(t + \hat{\omega}))\Big|_{t=\bar{t}} = \frac{d}{dt} V_2(\varphi_{V_2}(t + \hat{\omega}))\Big|_{t=\bar{t}} = 0,$$

as V_2 is preserved along φ_{V_2}. It follows that $\langle \nabla V_1(\varphi_{V_2}(\bar{t}+\hat{\omega}))|\varphi'_{V_2}(\bar{t}+\hat{\omega})\rangle = 0$, so that $r'(\bar{t}) = 0$. Since $r'(t) = 0$ for almost every $t \in [0, T]$, and $r(t)$ is absolutely continuous, this implies that $r(t)$ is constant. It follows that $u(t) = \hat{r}\varphi_{V_2}(t + \hat{\omega})$ for some nonnegative constant $\hat{r}$, so that $u(t)$ is a solution to

$$Ju' = \nabla V_1(u).$$

2) if $k > 0$, the reasoning is very similar. In particular, since $V_1 \leq V_2$ and in view of (2.26), one immediately gets (recall that, in this case, V_1 is strictly positive)

$$-\frac{\theta'(t)}{2V_2(\cos\theta(t), \sin\theta(t))} \leq 1 \leq -\frac{\theta'(t)}{2V_1(\cos\theta(t), \sin\theta(t))}. \tag{2.32}$$

Integrating (2.32) from 0 to T, taking into account that, in view of (1.15),

$$\int_0^{2\pi} \frac{d\theta}{2V_1(\cos\theta, \sin\theta)} = \tau_{V_1}\,, \quad \int_0^{2\pi} \frac{d\theta}{2V_2(\cos\theta, \sin\theta)} = \tau_{V_2}\,,$$

we get $m\tau_{V_2} \leq T \leq m\tau_{V_1}$, for a suitable integer $m \geq 1$. Hence, since $k\tau_{V_1} = T = (k+1)\tau_{V_2}$, and m is integer, it follows $m = k$ or $m = k+1$. We only sketch the proof for $m = k$, the other case being analogous. Similarly as before, we write $u(t) = r(t)\varphi_{V_1}(t + \omega(t))$, and get the equations for $r'(t)$ and $\omega'(t)$:

$$r'(t) = -(1 - \alpha(t))r(t)\langle \nabla V_2(\varphi_{V_1}(t + \omega(t)))|\varphi'_{V_1}(t + \omega(t))\rangle,$$

and

$$\omega'(t) = 2(1-\alpha(t))V_2(\varphi_{V_1}(t+\omega(t)))+\alpha(t)-1 = (1-\alpha(t))(2V_2(\varphi_{V_1}(t+\omega(t)))-1).$$

Now we get $\omega'(t) \equiv 0$ since $\omega'(t) \leq 0$ and $\omega(0) = \omega(T)$. This implies, as before, that $r'(t) \equiv 0$, so that $u(t)$ satisfies $Ju' = \nabla V_1(u)$.

□

Remark 2.2.2. We notice that, if (2.25) has a nontrivial solution, it is thus not possible to say that $\alpha(t) = 0$ or $\alpha(t) = 1$ almost everywhere: this is a priori true only if $\widehat{V}_1(\theta) < \widehat{V}_2(\theta)$ for every $\theta \in [0, 2\pi]$. For instance, if $V_1(x, y) = \frac{1}{2}((x^+)^2 + a_-(x^-)^2 + y^2)$ and $V_2(x, y) = \frac{1}{2}((x^+)^2 + b_-(x^-)^2 + y^2)$, with $0 < a_- < b_-$, then $\alpha(t)$ does not affect the orbit of the solutions in the half-plane $\{x > 0\}$.

Remark 2.2.3. Actually, it has been proved in [70, Lemma 3.1] that Lemma 2.2.1 holds true in the slightly more general case of equation

$$Ju' = \alpha(t)\nabla V_1(u) + \beta(t)\nabla V_2(u),$$

where $\alpha, \beta \in L^2(0, T)$ are such that, for almost every $t \in [0, T]$, $\alpha(t) \geq 0$, $\beta(t) \geq 0$, $\alpha(t) + \beta(t) \leq 1$, and $\alpha(t) + \beta(t) > 0$ for t belonging to a subset of $[0, T]$ having positive measure. The proof is very similar, and we refer the reader to [70] for the details.

In the previous section, we have seen that, when the principal part of the right-hand side in the considered equation is a fixed resonant Hamiltonian, a Landesman-Lazer condition is sufficient to ensure existence. Wishing to consider problem (2.22), on the other hand, the interaction of the nonlinearity takes place with two different Hamiltonians, so that it is expected that only one Landesman-Lazer condition is not enough to ensure the existence of a solution. The reason why will be more evident after Chapter 5, where we will place us in a rotational perspective (think also to Remark 2.1.5 about the possibility of proving existence through the Poincaré-Bohl theorem).
Before proving the main result of the section, let us introduce some notation for the version of conditions (2.10) and (2.11) which will be needed for problem (2.22).
If $V \in \mathcal{P}_0$, we set

- $(\mathrm{LL})_0$: *for every* $\xi \in \mathfrak{Z}_V$,

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle F(t, \lambda\eta)|\eta\rangle \, dt > 0.$$

On the other hand, if $V \in \mathcal{P}_k$, where $k \geq 1$ is an integer, we define the conditions $(\mathrm{LL}+)_k$ and $(\mathrm{LL}\text{-})_k$ as follows:

- $(\mathrm{LL}+)_k$: *for every* $\theta \in [0, T]$,

$$\begin{aligned}&\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle F(t, \lambda\varphi_V(t+\omega)) - \nabla V(\lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle \, dt \\ &= \int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} [\langle F(t, \lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle - \lambda] \, dt > 0. \qquad (2.33)\end{aligned}$$

- $(\mathrm{LL}\text{-})_k$: *for every* $\theta \in [0,T]$,

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle F(t,\lambda\varphi_V(t+\omega)) - \nabla V(\lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle \, dt$$
$$= \int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} [\langle F(t,\lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle - \lambda] \, dt < 0, \tag{2.34}$$

where φ_V has been defined in the previous section (satisfying, in particular, (2.8)). Obviously, we have used Euler's formula and the homogeneity of ∇V to simplify the way of writing (2.33) and (2.34), but we have preferred to maintain explicitly also the first expression to underline the importance of the Hamiltonian V as a comparison term for $F(t,u)$, in order to write a condition ensuring existence. Wishing to make a comparison between $(\mathrm{LL})_0$ and $(\mathrm{LL}+)_k$, it is not strange that no correction terms are added to $F(t,\lambda\eta)$ under the integral sign in $(\mathrm{LL})_0$, since in this case V vanishes along the T-periodic solutions to $Ju' = \nabla V(u)$. On the other hand, since when dealing with Hamiltonians in $\mathcal{P}_0$ we do not have a natural star-shaped curve around the origin providing a system of modified polar coordinates, when approaching to the limit problem in $(\mathrm{LL})_0$ we need to control the behavior of the solutions separately in each direction, thus considering a "triple" inferior limit, over the three-dimensional variable $(\lambda,\eta) \in \mathbb{R}\times\mathbb{R}^2$.
In the remaining part of the section, for V_1, V_2 as in (2.23), we will briefly write τ_1, τ_2 in place of τ_{V_1}, τ_{V_2} (if such times are well defined), and, analogously, φ_1, φ_2 instead of $\varphi_{V_1}, \varphi_{V_2}$.

Theorem 2.2.4. *Let $V_1 \in \mathcal{P}_k$, $V_2 \in \mathcal{P}_{k+1}$, for a nonnegative integer k. In the previous setting (assuming, in particular, (2.23)), suppose that, for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$, with $|u| \le 1$, and for every $\lambda \ge 1$,*

$$\langle F(t,\lambda u)|u\rangle - 2\lambda V_1(u) \ge -\eta(t), \tag{2.35}$$

and

$$2\lambda V_2(u) - \langle F(t,\lambda u)|u\rangle \ge -\eta(t), \tag{2.36}$$

being η a suitable function in $L^2(0,T)$. Then, there exists a solution to problem (2.22) provided that

- $$V_1 \in \mathcal{P}_0 \quad \text{and} \quad (\mathrm{LL})_0 \text{ is satisfied for } V = V_1,$$

  and

  $$V_2 \in \mathcal{P}_1 \quad \text{and} \quad (\mathrm{LL})_1 \text{ is satisfied for } V = V_2;$$

or

- *for a suitable integer $k \geq 1$,*

$$V_1 \in \mathcal{P}_k \quad \text{and} \quad (\text{LL+})_k \text{ is satisfied for } V = V_1;$$

and

$$V_2 \in \mathcal{P}_{k+1} \quad \text{and} \quad (\text{LL-})_{k+1} \text{ is satisfied for } V = V_2.$$

Proof. Let us first observe that the problem

$$\begin{cases} Ju' = \dfrac{1}{2}(\nabla V_1(u) + \nabla V_2(u)) \\ u(0) = u(T) \end{cases}$$

has only the trivial solution, thanks to Lemma 2.2.1 and to the fact that $\widehat{V}_1(\theta) < \widehat{V}_2(\theta)$ (remember the notation introduced in Chapter 1) for θ belonging to a subset of $[0, 2\pi[$ having positive measure. Since $V_1 + V_2 \in \mathcal{P}$, moreover, for any open subset $\Omega \subset \mathbb{R}^2$ containing 0 it is

$$\deg_B(J\nabla V_1 + J\nabla V_2, \Omega, 0) = \deg_B(\nabla V_1 + \nabla V_2, \Omega, 0) = 1,$$

where $\deg_B$ denotes the Brouwer degree (cf. [88, Lemma II.6.5]). In view of [20, Theorem 2], it is thus sufficient to prove that the solutions to the family of problems, parametrized by $\sigma \in]0, 1[$, given by

$$\begin{cases} Ju' = \sigma F(t, u) + \dfrac{1-\sigma}{2}\,(\nabla V_1(u) + \nabla V_2(u)) \\ u(0) = u(T), \end{cases} \tag{2.37}$$

are a priori L^∞-bounded, the bound not depending on the homotopy parameter σ. Therefore, by contradiction we assume that there exist $(u_n)_n \subset L^\infty(0, T)$ and $(\sigma_n)_n \subset]0, 1[$ such that $u_n(t)$ satisfies

$$\begin{cases} Ju_n' = \sigma_n(\gamma(t, u_n)\nabla V_1(u_n) + (1 - \gamma(t, u_n))\nabla V_2(u_n) + R(t, u_n)) \\ \qquad\qquad + \dfrac{1-\sigma_n}{2}(\nabla V_1(u_n) + \nabla V_2(u_n)) \\ u_n(0) = u_n(T), \end{cases}$$

with $\|u_n\|_\infty \to +\infty$ for $n \to +\infty$. We can assume that $\sigma_n \to \bar{\sigma} \in [0, 1]$. Setting $v_n(t) = u_n(t)/\|u_n\|_\infty$, for every n the function $v_n(t)$ satisfies

$$\begin{cases} Jv_n' = \sigma_n(\gamma(t, \|u_n\|_\infty v_n)\nabla V_1(v_n) + (1 - \gamma(t, \|u_n\|_\infty v_n))\nabla V_2(v_n) \\ \qquad\qquad + \dfrac{R(t, \|u_n\|_\infty v_n))}{\|u_n\|_\infty} + \dfrac{1-\sigma_n}{2}(\nabla V_1(v_n) + \nabla V_2(v_n)) \\ v_n(0) = v_n(T). \end{cases} \tag{2.38}$$

Since $(v_n)_n$ is bounded in $L^2(0,T)$ and in view of (2.7), the sequence $(v_n)_n$ is bounded in $H^1(0,T)$, so there exists a T-periodic function $v \in H^1(0,T)$ such that (up to subsequences) $v_n \to v$ uniformly and $v_n \rightharpoonup v$ weakly in $H^1(0,T)$. Moreover, being $\|v_n\|_\infty = 1$ for every n, it is $v \neq 0$. On the other hand, the sequence $(\gamma(\cdot, \|u_n\|_\infty v_n(\cdot)))_n$ is bounded in $L^2(0,T)$, so there exists $\Gamma \in L^2(0,T)$ such that $\gamma(\cdot, \|u_n\|_\infty v_n(\cdot)) \rightharpoonup \Gamma(t)$ in $L^2(0,T)$ (extracting a new subsequence, if necessary). As $\{w \in L^2(0,T) \mid 0 \leq w(t) \leq 1$ for almost every $t \in [0,T]\}$ is a convex and closed subset of $L^2(0,T)$, it is weakly closed and this implies $0 \leq \Gamma(t) \leq 1$ for almost every $t \in [0,T]$. Passing to the weak limit in (2.38), noticing that the term containing $R(t,u)$ vanishes thanks to condition (2.7), we then get

$$\begin{cases} Jv' = \left(\dfrac{1-\bar{\sigma}}{2} + \bar{\sigma}\Gamma(t)\right)\nabla V_1(v) + \left(\dfrac{1+\bar{\sigma}}{2} - \bar{\sigma}\Gamma(t)\right)\nabla V_2(v) \\ v(0) = v(T). \end{cases} \tag{2.39}$$

Notice that this excludes the case $\bar{\sigma} = 0$, since in this case v (which is nonzero) would be a solution to the periodic problem

$$\begin{cases} Jv' = \frac{1}{2}(\nabla V_1(v) + \nabla V_2(v)) \\ v(0) = v(T), \end{cases}$$

which, as already remarked, has only the trivial solution. Since $0 \leq \Gamma(t) \leq 1$ for almost every $t \in [0,T]$, the right-hand side of the differential equation in (2.39) is a convex combination of $\nabla V_1(v)$ and $\nabla V_2(v)$, so that we can use Lemma 2.2.1 to infer that the T-periodic function $v(t)$ solves either

$$Jv' = \nabla V_1(v),$$

or

$$Jv' = \nabla V_2(v).$$

The proof now is a bit different, according as $V_1 \in \mathcal{P}_0$ or $V_1 \in \mathcal{P}$.

Case 1: $V_1 \in \mathcal{P}_0$. In this situation, we only take into account the case when $Jv' = \nabla V_1(v)$, leaving the complete discussion to Case 2.

Since $\|v\|_\infty = 1$, by Proposition 1.1.5 we have that there exists $\xi \in \mathfrak{Z}_{V_1}$ such that $v(t) \equiv \xi$. In this situation, it is $\frac{1-\bar{\sigma}}{2} + \bar{\sigma}\Gamma(t) = 1$, which in turns implies $\bar{\sigma} = 1$ and $\Gamma(t) \equiv 1$. Thanks to the elastic property (recall that $\|u_n\|_\infty \to +\infty$), for n large we can pass to polar coordinates, writing $u_n(t) = \rho_n(t)(\cos\theta_n(t), \sin\theta_n(t))$. The expression of θ_n' is then given by

$$-\theta_n'(t) = \frac{\sigma_n \langle F(t, u_n(t)) | u_n(t)\rangle}{\rho_n(t)^2} + \frac{1-\sigma_n}{\rho_n(t)^2}(V_1(u_n(t)) + V_2(u_n(t))). \tag{2.40}$$

However, since $v_n \to \xi \neq 0$ uniformly, for every n large $v_n(t)$ will not be able to turn around the origin, so that integrating (2.40) from 0 to T gives 0. Recalling that $V_1 \geq 0$ and $V_2 > 0$, for every n large it follows that

$$0 > \int_0^T \sigma_n \frac{\|u_n\|_\infty}{\rho_n(t)^2} \langle F(t, \|u_n\|_\infty v_n(t)) | v_n(t) \rangle \, dt, \tag{2.41}$$

and, multiplying by $\|u_n\|_\infty$,

$$0 > \int_0^T \sigma_n \frac{\|u_n\|_\infty^2}{\rho_n(t)^2} \langle F(t, \|u_n\|_\infty v_n(t)) | v_n(t) \rangle \, dt. \tag{2.42}$$

Using Fatou's lemma thanks to (2.35), since $\frac{\|u_n\|_\infty^2}{\rho_n(t)^2}$ converges to 1 and $\sigma_n \to 1$, we have

$$0 \geq \int_0^T \liminf_{n \to +\infty} \langle F(t, \|u_n\|_\infty v_n(t)) | v_n(t) \rangle \, dt,$$

contradicting $(\mathrm{LL})_0$.

Case 2: $V_1 \in \mathcal{P}_k$. Let us assume that $Jv' = \nabla V_1(v)$. Thus, for suitable $r_v > 0$, $\omega_v \in [0, \tau_1[$, it will be $v(t) = r_v \varphi_1(t + \omega_v)$. Writing, in generalized polar coordinates, $u_n(t) = r_n(t)\varphi_1(t + \omega_n(t))$, with $\omega_n(0) \in [0, \tau_1[$ for every n, (2.37) gives

$$\begin{aligned} \omega_n'(t) &= \sigma_n \frac{\langle F(t, r_n(t)\varphi_1(t+\omega_n(t))) | \varphi_1(t+\omega_n(t)) \rangle}{r_n(t)} + \\ &\quad + (1-\sigma_n)(V_1(\varphi_1(t+\omega_n(t))) + V_2(\varphi_1(t+\omega_n(t)))) - 1. \end{aligned} \tag{2.43}$$

Since v performs k turns around the origin in the time T, and the sequence of T-periodic functions v_n converges to v uniformly, for n sufficiently large every v_n performs k turns around the origin, and so every u_n. As a consequence, for such n it is $\omega_n(0) = \omega_n(T)$, thus integrating (2.43) from 0 to T gives 0. In view of Euler's formula and (2.8), it follows that

$$0 > \int_0^T \sigma_n \frac{\langle F(t, r_n(t)\varphi_1(t+\omega_n(t))) | \varphi_1(t+\omega_n(t)) \rangle - r_n(t)}{r_n(t)} \, dt,$$

from which we obtain, for n large,

$$0 > \int_0^T \frac{\langle F(t, r_n(t)\varphi_1(t+\omega_n(t))) | \varphi_1(t+\omega_n(t)) \rangle - r_n(t)}{r_n^V(t)} \, dt, \tag{2.44}$$

where $r_n^V(t) = r_n(t)/\|u_n\|_\infty$. Hypotheses (2.23) and (2.35) now allow us to apply Fatou's lemma, which gives

$$0 \geq \int_0^T \liminf_{n \to +\infty} \frac{\langle F(t, r_n(t)\varphi_1(t+\omega_n(t))) | \varphi_1(t+\omega_n(t)) \rangle - r_n(t)}{r_n^V(t)} \, dt;$$

using standard properties of the inferior limit, taking into account that, since $v_n \to v$ uniformly, also $r_n^V \to r_v$ uniformly, this yields

$$0 \geq \int_0^T \liminf_{n\to+\infty} [\langle F(t, r_n(t)\varphi_1(t+\omega_n(t)))|\varphi_1(t+\omega_n(t))\rangle - r_n(t)]\, dt.$$

Moreover, using again the fact that $v_n \to v$ uniformly, we can argue as in the proof of Theorem 2.1.1 and assume, without loss of generality, that $\omega_n(t) \to \omega_v$ uniformly (passing, if necessary, to a further subsequence). Thus, recalling (2.8), we deduce that

$$0 \geq \int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\omega_v)} [\langle F(t, \lambda\varphi_1(t+\omega))|\varphi_1(t+\omega)\rangle - \lambda]\, dt,$$

which contradicts $(\mathrm{LL}+)_k$.
On the other hand, if $Jv' = \nabla V_2(v)$, then it will be $v(t) = r_v\varphi_2(t+\omega_v)$, for suitable $r_v > 0$, $\omega_v \in [0, \tau_2[$, and we pass again to generalized polar coordinates writing, this time, $u_n(t) = r_n(t)\varphi_2(t+\omega_n(t))$. A computation similar as before gives

$$\begin{aligned} \omega_n'(t) &= \sigma_n \frac{\langle F(t, r_n(t)\varphi_2(t+\omega_n(t)))|\varphi_2(t+\omega_n(t))\rangle}{r_n(t)} + \\ &\quad +(1-\sigma_n)(V_1(\varphi_2(t+\omega_n(t))) + V_2(\varphi_2(t+\omega_n(t)))) - 1, \end{aligned}$$

and, in view of the fact that v performs $k+1$ turns in the time T, we reach analogously

$$0 < \int_0^T \sigma_n \frac{\langle F(t, r_n(t)\varphi_2(t+\omega_n(t)))|\varphi_2(t+\omega_n(t))\rangle - r_n(t)}{r_n^V(t)}\, dt.$$

Thanks to (2.36), we can apply Fatou's lemma to infer that

$$0 \leq \int_0^T \limsup_{n\to+\infty} [\langle F(t, r_n(t)\varphi_2(t+\omega_n(t)))|\varphi_2(t+\omega_n(t))\rangle - r_n(t)]\, dt,$$

and the conclusion is obtained exactly as before, in view of the produced contradiction

$$0 \leq \int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\omega_v)} [\langle F(t, \lambda\varphi_2(t+\omega))|\varphi_2(t+\omega)\rangle - \lambda]\, dt.$$

□

Remark 2.2.5. In our results, we were able to prove existence by showing that the coincidence degree associated with the considered problem is equal to 1. We recall that, when $V_1 = V_2$, different assumptions generalizing the Landesman-Lazer condition [89, 93] have been proposed in [21, 22, 49, 50, 54, 61] (the same is true for conditions generalizing the existence result in [66]). The main point in these papers, however, is that the associated degree can also be an arbitrary negative number, and can sometimes take large positive values, as well. The possibility of obtaining this kind of results in the case of double resonance with two different Hamiltonians is still to be investigated.

Remark 2.2.6. It seems more difficult here to estimate the rotation numbers of large solutions to our system, since there are two different Hamiltonians involved, so it is not clear which modified rotation number is convenient to compute. This point is strictly connected to the possibility of giving, in Chapter 5, a rotational interpretation to the Landesman-Lazer conditions when the starting system is at double resonance. We are actually trying to understand how to overcome this problem. This could be useful to deal, for instance, with the more general situation

$$\begin{cases} Ju' = \gamma(t,u)\zeta_1(t)\nabla V_1(u) + (1-\gamma(t,u))\zeta_2(t)\nabla V_2(u) + R(t,u) \\ u(0) = u(T). \end{cases}$$

Remark 2.2.7. Reasoning similarly as in Chapter 5 (see [15, Section 4]), it is possible to show that Theorem 2.2.4 holds, in the same way, in the L^1-Carathéodory setting, thanks to an application of the Dunford-Pettis theorem. In general, however, the L^2-Carathéodory framework is more typical when dealing with Landesman-Lazer conditions.

We now state the following corollary of Theorem 2.2.4.

Corollary 2.2.8. *Assume* (2.23) *and suppose that for almost every* $t \in [0,T]$ *and every* $u \in \mathbb{R}^2$*, with* $|u| \leq 1$*, and for every* $\lambda \geq 1$*,*

$$\langle R(t, \lambda u) | u \rangle \geq \eta(t),$$

for a suitable $\eta \in L^2(0,T)$*. Then, problem* (2.22) *has a solution provided that* $R(t,u)$ *satisfies*

1) for every $\theta \in [0,T]$*,*

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_1(t+\omega)) | \varphi_1(t+\omega) \rangle \, dt > 0, \tag{2.45}$$

if $V_1 \in \mathcal{P}_k$*, for* $k > 0$*, or*

1') for every $\xi \in \mathfrak{Z}_V$*,*

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle R(t, \lambda\eta) | \eta \rangle \, dt > 0, \tag{2.46}$$

if $V_1 \in \mathcal{P}_0$*,*

and

2) for every $\theta \in [0,T]$,

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_2(t+\omega))|\varphi_2(t+\omega)\rangle \, dt < 0. \tag{2.47}$$

The sufficiency of assumptions 1) and 2), as stronger requirements with respect to $(\text{LL}+)_k$ and $(\text{LL-})_{k+1}$, has been discussed in [56, Corollary 2.7]. On the other hand, writing explicitly the expression of $F(t,u)$ in (2.40) and (2.41), we immediately see that 1') - which just corresponds to condition (2.10) - is sufficient to perform the same proof as above in the case when $V_1 \in \mathcal{P}_0$.
The corollary can be useful in the applications: from a practical point of view, indeed, we can first check if the part which has lower order satisfies the hypotheses of the theorem.

We conclude the section by analyzing the relationships between our Landesman- Lazer conditions and the ones introduced by Brézis and Nirenberg in [17], where the following definition was given.

Definition 2.2.9. Let H be a Hilbert space, and $\mathcal{N} : [0,T] \times H \to H$. The *recession function* $\mathcal{J}_{\mathcal{N}} : H \to \mathbb{R}$ is defined as

$$\mathcal{J}_{\mathcal{N}}(z) = \liminf_{(\lambda,w)\to(+\infty,z)} (\mathcal{N}(t,\lambda w)|w)_H,$$

where $(\cdot|\cdot)_H$ denotes the scalar product in H.

With this definition, when one of the comparison Hamiltonians belongs to $\mathcal{P}_0$, namely referring, e.g., to case 1') of Corollary 2.2.8, the Landesman-Lazer condition (2.46) requires that, for every $\xi \in \mathfrak{Z}_V$, it holds

$$\int_0^T \mathcal{J}_R(t,\xi)\, dt > 0,$$

where the recession function $\mathcal{J}_R$ is meant to be defined in $\mathbb{R}^2$, i.e., here $H = \mathbb{R}^2$. In [17, Chapter III], sign conditions involving the recession function were used to give existence results for abstract equations in a domain Ω and for systems. In particular, in [17, Section III.4], condition (2.46) was exploited, for instance, to prove existence for systems of the kind $Au + g(t,u) = f$, asking its validity for every ξ in the kernel of the linear operator A. Considering the T-periodic problem associated with the system

$$Ju' = \nabla V(u) + R(t,u),$$

with $V \in \mathcal{P}_0$, if ∇V is linear (i.e., $V(u) = \frac{1}{2}\langle Bu|u\rangle$, with B a positive semidefinite square matrix having nontrivial kernel), then our result coincides with the one by Brézis and Nirenberg. On the other hand, if ∇V is only homogeneous, case which was not investigated in [17], wishing to use [17, condition (3.41)] we would have to ask the validity of (2.46) for every $\xi \in \mathbb{R}^2$, while we have seen that it is sufficient to impose it only for the zeros of V. However, it is worth mentioning that the results in [17] were stated in a general abstract setting, and applicable in arbitrary dimension.

Speaking about positive Hamiltonians, the situation is slightly different, and in this case the Landesman-Lazer results provided in [17] rely on the abstract assumption that

$$\mathcal{J}_{\mathcal{N}}(v) > 0$$

for every $v \neq 0$ belonging to the kernel of the linear operator A appearing in the considered equation [17, Theorem III.1]. Referring to the Introduction, in the particular case of problem (1), with $g(t,x)$ as in (9), taking $H = L^2(0,T)$ and denoting by $\mathcal{N}$ the Nemytzkii operator associated with $h(t,x)$, Brézis and Nirenberg showed that

$$\mathcal{J}_{\mathcal{N}}(v) \geq \int_{\{v>0\}} \liminf_{x\to+\infty} h(t,x)v(t)\,dt + \int_{\{v<0\}} \limsup_{x\to-\infty} h(t,x)v(t)\,dt,$$

for every $v \neq 0$ satisfying $v'' + \lambda_k v = 0$, and they were able to recover the existence result in [89].

It seems difficult to apply the Brézis-Nirenberg approach to our type of situation, for the lack of an underlying linear structure and for the presence of double resonance. However, our Landesman-Lazer conditions seem to involve a sort of finite dimensional concept of recession function also in the case of positive Hamiltonians. More precisely, speaking about double resonance, denoting by $\mathcal{N}_1$ the Nemytzkii operator associated with the function $F - \nabla V_1$, and by $\mathcal{N}_2$ the Nemytzkii operator associated with $\nabla V_2 - F$, the recession functions defined in [17] would be given by

$$\mathcal{J}_{\mathcal{N}_1}(\theta) = \liminf_{(\lambda,w)\to(+\infty,z)} \int_0^T [\langle F(t,\lambda w(t))|w(t)\rangle - 2\lambda V_1(w(t))]\,dt,$$

and

$$\mathcal{J}_{\mathcal{N}_2}(\theta) = \liminf_{(\lambda,w)\to(+\infty,z)} \int_0^T [2\lambda V_2(w(t)) - \langle F(t,\lambda w(t))|w(t)\rangle]\,dt,$$

where $w \to z$ in $L^2([0,T];\mathbb{R}^2)$. On the other hand, setting

$$\tilde{\mathcal{J}}_1(t;\theta) = \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle \mathcal{N}_1(\lambda\varphi_1(t+\omega))|\varphi_1(t+\omega)\rangle,$$

and

$$\tilde{\mathcal{J}}_2(t;\theta) = \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle \mathcal{N}_2(\lambda\varphi_2(t+\omega))|\varphi_2(t+\omega)\rangle,$$

our Landesman-Lazer conditions read as

$$\int_0^T \tilde{\mathcal{J}}_1(t;\theta)\,dt > 0, \quad \text{and} \quad \int_0^T \tilde{\mathcal{J}}_2(t;\theta)\,dt > 0,$$

for every $\theta \in [0,T]$. In some sense, $\tilde{\mathcal{J}}_1(t;\theta)$ and $\tilde{\mathcal{J}}_2(t;\theta)$ can then be thought as finite dimensional recession functions (living in $\mathbb{R}^2$ instead of L^2), depending on t (and thus still to be integrated in order to write a Landesman-Lazer type condition). From our point of view, this approach gives the advantage of providing conditions which are easier to handle.

2.3 Simple resonance and nonresonance

In this brief section, we take into account some cases which are complementary with respect to the ones analyzed in Sections 2.1 and 2.2. The forthcoming results can be proved using the same technique exploited in the proof of Theorem 2.2.4; however, concerning, in particular, the nonresonant case, we think that it is interesting to perform explicitly the computations, to see which kind of estimates on the angular coordinate can be shown to hold.
First, we consider a nonlinearity satisfying (2.23), but interacting with only one resonant Hamiltonian: wishing not to cross other resonant functions in $\mathcal{P}$, we will have to impose some controls on the minimal periods of the comparison Hamiltonians (given by (2.48) below).

Corollary 2.3.1. *Assume that $V_1 \in \mathcal{P} \setminus \cup_{k\in\mathbb{N}}\mathcal{P}_k$, $V_2 \in \mathcal{P}_{k+1}$, with $V_1 \le V_2$, satisfy*

$$\frac{T}{k+1} = \tau_2 \le \tau_1 < \frac{T}{k}. \tag{2.48}$$

If, moreover, $(\text{LL-})_{k+1}$ *holds with $V = V_2$, then problem* (2.22) *has a solution.*

Proof. The result can be obtained following the lines of the proof of Theorem 2.2.4, performing a homotopy of the type

$$Ju' = \sigma(\gamma(t,u)\nabla V_1(u) + (1-\gamma(t,u))\nabla V_2(u) + R(t,u)) + (1-\sigma)\nabla V_1(u),$$

for $\sigma \in [0,1]$. In this case, the normalized sequence v_n will necessarily converge to a solution to $Jv' = \nabla V_2(v)$. We omit the details for briefness. $\square$

Clearly, we have a similar statement if we assume $V_1 \in \mathcal{P}_k$ and replace (2.48) by

$$\frac{T}{k+1} < \tau_2 \le \tau_1 = \frac{T}{k}\,.$$

In this case, we will need condition $(\mathrm{LL+})_k$ instead of $(\mathrm{LL-})_{k+1}$. In an analogous way, it could also be possible to consider $V_1 \in \mathcal{P}_0$ assuming $(\mathrm{LL})_0$, with the same final outcome.
On the other hand, if we want to investigate the case when neither V_1 nor V_2 are resonant, it is even possible to drop some of the hypotheses of Theorem 2.2.4, still performing a similar proof, as we are going to show. However, we must require the same control for the minimal periods of V_1, V_2, to ensure that no other resonant Hamiltonians are "included" between them two.

Theorem 2.3.2. *Assume that $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ grows at most linearly in the second variable, i.e.*

$$|F(t,u)| \le c(t)(1 + |u|),$$

with $c \in L^2(0,T)$, and that (2.35), (2.36) hold. If $V_1, V_2 \in \mathcal{P}$, with $V_1 \le V_2$, and there exists a positive integer k such that

$$\frac{T}{k+1} < \tau_2 \le \tau_1 < \frac{T}{k}, \tag{2.49}$$

then problem (2.22) *has a solution.*

Proof. As in the proof of Theorem 2.2.4, we argue by contradiction, assuming that (2.37) holds for an unbounded (in L^∞-norm) sequence $(u_n)_n$, i.e.

$$\begin{cases} Ju_n' = \sigma_n F(t,u_n) + \frac{1-\sigma_n}{2}\left(\nabla V_1(u_n) + \nabla V_2(u_n)\right) \\ u_n(0) = u_n(T). \end{cases}$$

By the elastic property, $\min |u_n(t)| \to \infty$ for $n \to \infty$. Consequently, it is possible to introduce polar coordinates, writing $u_n(t) = \rho_n(t)(\cos\theta_n(t), \sin\theta_n(t))$, and we know that u_n will perform an integer number m_n of rotations around the origin in the time T. A direct computation of θ_n', together with the use of (2.35), (2.36) gives

$$\frac{\theta_n'(t)}{2V_2(\cos\theta_n(t), \sin\theta_n(t))} \ge \frac{-\eta(t)}{\rho_n^2(t) 2V_2(\cos\theta_n(t), \sin\theta_n(t))} - 1, \tag{2.50}$$

and

$$\frac{\theta_n'(t)}{2V_1(\cos\theta_n(t), \sin\theta_n(t))} \le \frac{\eta(t)}{\rho_n^2(t) 2V_1(\cos\theta_n(t), \sin\theta_n(t))} - 1. \tag{2.51}$$

Since, as we have already recalled,

$$\int_0^{2\pi} \frac{d\theta}{2V_1(\cos\theta, \sin\theta)} = \tau_1\,, \quad \int_0^{2\pi} \frac{d\theta}{2V_2(\cos\theta, \sin\theta)} = \tau_2\,,$$

integrating in (2.50) and (2.51) from 0 to T yields

$$T \geq m_n \tau_2 + \int_0^T \frac{-\eta(t)}{\rho_n^2(t) 2V_2(\cos\theta_n(t), \sin\theta_n(t))}\, dt,$$

and

$$T \leq m_n \tau_1 + \int_0^T \frac{\eta(t)}{\rho_n^2(t) 2V_1(\cos\theta_n(t), \sin\theta_n(t))}\, dt.$$

However, since $\rho_n \to \infty$ uniformly, the contribution of the two terms

$$\int_0^T \frac{-\eta(t)}{\rho_n^2(t) 2V_2(\cos\theta_n(t), \sin\theta_n(t))}\, dt \quad \text{and} \quad \int_0^T \frac{\eta(t)}{\rho_n^2(t) 2V_1(\cos\theta_n(t), \sin\theta_n(t))}\, dt$$

vanishes for $n \to \infty$. As a consequence, in view of (2.49) we will have, for a suitable $\epsilon > 0$, to be chosen sufficiently small,

$$k < \frac{T}{\tau_1} - \epsilon \leq m_n \leq \frac{T}{\tau_2} + \epsilon < k+1,$$

for n sufficiently large. Since m_n is integer, this is a contradiction. □

Notice that here (2.23) is not needed, since (2.35) and (2.36) already imply that $F(t,u)$ is, in some sense, included between V_1 and V_2; moreover, neither the Landesman-Lazer conditions $(\text{LL+})_k$ and $(\text{LL-})_{k+1}$ are needed, in view of the nonresonance hypothesis (2.49).

2.4 The scalar case

We now examine the consequences of Theorem 2.2.4 for the scalar problem

$$\begin{cases} x'' + f(t,x) = 0 \\ x(0) = x(T),\ x'(0) = x'(T), \end{cases} \tag{2.52}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^2-Carathéodory function. To begin with, assume

$$f(t,x) = \mu x^+ - \nu x^- + r(t,x),$$

where μ and ν are nonnegative constants such that the pair (μ,ν) belongs to the T-periodic Fučík spectrum (cf. [28, 67]). The equation can then be written as

$$\begin{cases} x' = y \\ -y' = \mu x^+ - \nu x^- + r(t,x), \end{cases}$$

and setting

$$u = (x, y), \quad R(t, u) = \begin{pmatrix} r(t, x) \\ 0 \end{pmatrix}, \quad V(x, y) = \frac{1}{2}(\mu(x^+)^2 + \nu(x^-)^2 + y^2),$$

it becomes equivalent to the first order planar system

$$Ju' = \nabla V(u) + R(t, u).$$

It is clear that, depending on the values of μ, ν, the Hamiltonian V will belong to $\mathcal{P}_0$ or to $\mathcal{P}$. We are going to see that in both cases, with the slight changes due to the different formulations of conditions $(\mathrm{LL})_0$ and $(\mathrm{LL}+)_k$ (or $(\mathrm{LL}\text{-})_k$), the scalar version of the Landesman-Lazer condition is equivalent to the planar one given in the previous sections (see [56, 69]).

We will assume the following hypothesis on $r(t, x)$:

(ll_1) for every $v \neq 0$ satisfying the homogeneous equation

$$x'' + \mu x^+ - \nu x^- = 0, \tag{2.53}$$

the following inequality holds:

$$\int_{\{v>0\}} \liminf_{x\to+\infty} r(t, x)v(t)\, dt + \int_{\{v<0\}} \limsup_{x\to-\infty} r(t, x)v(t)\, dt > 0.$$

The well known Landesman-Lazer condition (ll_1), in the particular case when $\mu = \nu = 0$, reduces to

$$\int_0^T \limsup_{x\to-\infty} r(t, x)\, dt < 0 < \int_0^T \liminf_{x\to+\infty} r(t, x)\, dt, \tag{2.54}$$

(see, for instance, [104]) while, if only $\mu = 0$ (resp. only $\nu = 0$), it reads as

$$\int_0^T \liminf_{x\to+\infty} r(t, x)\, dt > 0, \quad \Big(\text{resp. } \int_0^T \limsup_{x\to-\infty} r(t, x)\, dt < 0\Big). \tag{2.55}$$

Before stating the main proposition of this section, we recall that, if $v(t)$ solves (2.53), then also $Cv(t + \theta)$ does, for every $C \geq 0$ and $\theta \in [0, T[$.

Proposition 2.4.1. *Hypothesis (ll_1) is equivalent to the following:*

- *if $\mu \neq 0$ and $\nu \neq 0$:*
  for every $\theta \in [0, T]$,

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_V(t + \omega))|\varphi_V(t + \omega)\rangle\, dt > 0;$$

- *if either* $\mu = 0$ *or* $\nu = 0$*:*
  for every $\xi \in \mathfrak{Z}_V$*,*

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle R(t,\lambda\eta)|\eta\rangle\, dt > 0.$$

Proof. Assume that $\mu \neq 0$, $\nu \neq 0$. Since equation (2.53) is equivalent to the planar system $Ju' = \nabla V(u)$, it will be $\varphi_V = (v, v')$, for a suitable v solving (2.53). Let $\theta \in [0,T]$ be fixed; we can write

$$[0,T] = \{t \in [0,T] \mid v(t+\theta) > 0\} \cup \{t \in [0,T] \mid v(t+\theta) < 0\} \cup Z_0,$$

where $Z_0 = \{t \in [0,T] \mid v(t+\theta) = 0\}$ has Lebesgue measure equal to 0 (Z_0 is made up by a finite number of points, as it can be easily seen by computing explicitly $v(t+\theta)$). Let us fix t such that $v(t+\theta) > 0$ and consider

$$\liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle = \liminf_{(\lambda,\omega)\to(+\infty,\theta)} r(t,\lambda v(t+\omega))v(t+\omega).$$

Since $\lim_{\omega\to\theta} v(t+\omega) = v(t+\theta) > 0$, we have, by standard properties of the inferior limits,

$$\liminf_{(\lambda,\omega)\to(+\infty,\theta)} r(t,\lambda v(t+\omega))v(t+\omega) = \liminf_{(\lambda,\omega)\to(+\infty,\theta)} r(t,\lambda v(t+\omega))\, v(t+\theta).$$

Now, observe that

$$\liminf_{(\lambda,\omega)\to(+\infty,\theta)} r(t,\lambda v(t+\omega)) \geq \liminf_{x\to+\infty} r(t,x),$$

and

$$\liminf_{(\lambda,\omega)\to(+\infty,\theta)} r(t,\lambda v(t+\omega)) \leq \liminf_{\lambda\to+\infty} r(t,\lambda v(t+\theta));$$

since, being θ fixed,

$$\liminf_{\lambda\to+\infty} r(t,\lambda v(t+\theta)) = \liminf_{x\to+\infty} r(t,x),$$

we then deduce

$$\liminf_{(\lambda,\omega)\to(+\infty,\theta)} r(t,\lambda v(t+\omega))v(t+\omega) = \liminf_{x\to+\infty} r(t,x)v(t+\theta).$$

On the other hand, if t is such that $v(t+\theta) < 0$, noticing that, for ω close to θ, the sign of $v(t+\omega)$ will now be negative, we have

$$\begin{aligned} \liminf_{(\lambda,\omega)\to(+\infty,\theta)} r(t,\lambda v(t+\omega))v(t+\omega) &= \liminf_{(\lambda,\omega)\to(+\infty,\theta)} (-r(t,\lambda v(t+\omega)))(-v(t+\omega)) \\ = \limsup_{(\lambda,\omega)\to(+\infty,\theta)} r(t,\lambda v(t+\omega))v(t+\theta) &= \limsup_{x\to-\infty} r(t,x)\, v(t+\theta). \end{aligned}$$

So,

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle\,dt =$$

$$\int_{\{v(t+\theta)>0\}} \liminf_{x\to+\infty} r(t,x)v(t+\theta)\,dt + \int_{\{v(t+\theta)<0\}} \limsup_{x\to-\infty} r(t,x)v(t+\theta)\,dt > 0,$$

and the conclusion follows.
On the other hand, if $\mu=0$ or $\nu=0$, then $v(t)$ is constant, so there exists $c\in\mathbb{R}\setminus\{0\}$ such that $v(t)\equiv c$. We now set $\eta=(\eta_x,\eta_y)$ and $\xi=(c,0)$ and observe that

$$\liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle R(t,\lambda\eta)|\eta\rangle = \liminf_{(\lambda,\eta)\to(+\infty,\xi)} r(t,\lambda\eta_x)\eta_x = \liminf_{(\lambda,\eta_x)\to(+\infty,c)} r(t,\lambda\eta_x)\eta_x.$$

Now, if $c>0$, we have that

$$\liminf_{(\lambda,\eta_x)\to(+\infty,c)} r(t,\lambda\eta_x)\eta_x = \liminf_{(\lambda,\eta_x)\to(+\infty,c)} r(t,\lambda\eta_x)c = \liminf_{x\to+\infty} r(t,x)c,$$

where the last equality is justified by standard properties of the inferior limit, in the same way as before. If, on the contrary, $c<0$, we reach, in a similar way,

$$\liminf_{(\lambda,\eta_x)\to(+\infty,c)} r(t,\lambda\eta_x)\eta_x = \limsup_{(\lambda,\eta_x)\to(+\infty,c)} r(t,\lambda\eta_x)c = \limsup_{x\to+\infty} r(t,x)c.$$

Thus, in view of (ll_1) we get

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle R(t,\lambda\eta)|\eta\rangle\,dt$$

$$= \int_{\{v>0\}} \liminf_{x\to+\infty} r(t,x)v(t)\,dt \quad + \int_{\{v<0\}} \limsup_{x\to-\infty} r(t,x)v(t)\,dt > 0,$$

for every v - constant - which solves (2.53), concluding the proof. □

As a counterpart, let $\mu,\nu>0$ and consider the following assumption on h:

(ll_2) for every $v\neq 0$ satisfying the homogeneous equation (2.53), the following inequality holds:

$$\int_{\{v>0\}} \limsup_{x\to+\infty} r(t,x)v(t)\,dt + \int_{\{v<0\}} \liminf_{x\to-\infty} r(t,x)v(t)\,dt < 0.$$

The following proposition can be proved in the same way as Proposition 2.4.1.

Proposition 2.4.2. *Hypothesis (ll_2) is equivalent to the following:*

- *for every* $\theta \in [0, T]$,

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle \, dt < 0.$$

We are now ready to show that Theorem 2.2.4 includes the main result proved by Fabry in [46], which we now state for the reader's convenience.

Theorem 2.4.3. *Let* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function such that the following conditions hold:*

$$a_+ x - \eta(t) \le f(t, x) \le b_+ x + \eta(t) \quad \textit{for every } x \ge 0, \textit{ and a.e. } t \in [0, T], \tag{2.56}$$

and

$$b_- x - \eta(t) \le f(t, x) \le a_- x + \eta(t) \quad \textit{for every } x \le 0, \textit{ and a.e. } t \in [0, T], \tag{2.57}$$

for suitable constants $a_-, a_+ \ge 0$ *and* $b_-, b_+ > 0$ *satisfying, for a suitable positive integer* k*, the equality*

$$\frac{\pi}{\sqrt{b_+}} + \frac{\pi}{\sqrt{b_-}} = \frac{T}{k}\,. \tag{2.58}$$

Moreover, if $a_-, a_+ > 0$*, we assume that* $k > 1$ *and*

$$\frac{\pi}{\sqrt{a_+}} + \frac{\pi}{\sqrt{a_-}} = \frac{T}{k-1}\,. \tag{2.59}$$

Finally, assume that for every nontrivial solutions ϕ, ψ *to*

$$\phi'' + a_+\phi^+ - a_-\phi^- = 0, \quad \textit{and} \quad \psi'' + b_+\psi^+ - b_-\psi^- = 0,$$

respectively, the following conditions are satisfied:

$$\int_{\{\phi>0\}} \liminf_{x\to+\infty} (f(t, x) - a_+ x)\, \phi(t)\, dt + \int_{\{\phi<0\}} \limsup_{x\to-\infty} (f(t, x) - a_- x)\, \phi(t)\, dt > 0,$$

and

$$\int_{\{\psi>0\}} \limsup_{x\to+\infty} (f(t, x) - b_+ x)\, \psi(t)\, dt + \int_{\{\psi<0\}} \liminf_{x\to-\infty} (f(t, x) - b_- x)\, \psi(t)\, dt < 0.$$

Then, problem (2.52) has a solution.

Proof. It can be shown (see e.g. [46, Lemma 1]) that, under conditions (2.56) and (2.57), one can write

$$f(t,x) = \gamma_1(t,x)x^+ - \gamma_2(t,x)x^- + r(t,x), \tag{2.60}$$

where $r(t,x)$ is L^2-bounded and

$$a_+ \le \gamma_1(t,x) \le b_+, \quad a_- \le \gamma_2(t,x) \le b_-, \tag{2.61}$$

for almost every $t \in [0,T]$, and every $x \in \mathbb{R}$. Defining

$$V_1(u) = \frac{1}{2}(a_+(x^+)^2 + a_-(x^-)^2 + y^2), \quad V_2(u) = \frac{1}{2}(b_+(x^+)^2 + b_-(x^-)^2 + y^2),$$

$$R(t,u) = \begin{pmatrix} r(t,x) \\ 0 \end{pmatrix}, \quad F(t,u) = \begin{pmatrix} f(t,x) \\ 0 \end{pmatrix},$$

we now show that Theorem 2.2.4 can be applied. Indeed, (2.58) and (2.59) imply that $V_1 \in \mathcal{P}_{k-1}$, $V_2 \in \mathcal{P}_k$, while (2.23), (2.35) and (2.36) hold thanks to (2.60) and (2.61). Now, condition (LL+)$_{k-1}$ follows from Proposition 2.4.1, with $\mu = a_+$ and $\nu = a_-$, applied to $r(t,x) = f(t,x) - a_+x^+ + a_-x^-$, and condition (LL-)$_k$ follows from Proposition 2.4.2, with $\mu = b_+$ and $\nu = b_-$, applied to $r(t,x) = f(t,x) - b_+x^+ + b_-x^-$. □

Observe that Theorem 2.4.3 allows to consider also the situation of resonance with the first eigenvalue, namely

$$f(t,x) = \gamma_1(t,x)x + r(t,x),$$

with $0 \le \gamma_1(t,x) \le \lambda_1$, where $\lambda_1 = \left(\frac{2\pi}{T}\right)^2$.

Remark 2.4.4. Let us observe that, under the hypothesis that $r(t,x)$ is bounded and *strictly increasing* in x, Lazer and Leach proved in [93] that condition (ll$_1$) is *necessary and sufficient* for the existence of a T-periodic solution. Hence, in this case, the original Landesman-Lazer condition (ll$_1$), the Brézis-Nirenberg condition and our condition (LL+)$_k$ are all equivalent one with the other.

Remark 2.4.5. When $a_- = a_+ = 0$ in Theorem 2.4.3, we use the Landesman-Lazer condition (2.54) in order to be far from resonance at the first eigenvalue; however, as shown in [46, Theorem 1'], such an assumption could also be weakened (by means of hypothesis (H) therein, see Remark 2.1.2). On the other hand, as already remarked, when only one of a_-, a_+ is zero, (2.55) states that only a one-sided Landesman-Lazer condition is needed. This perfectly agrees with the visual representation of the Fučík spectrum in the plane (μ,ν), since in this case only one edge of the rectangle $[0,b_+] \times [a_-,b_-]$ (resp. $[a_+,b_+] \times [0,b_-]$) lies on an axis (and thus on a “dangerous region” with respect to resonance).

Remark 2.4.6. A further comment about resonance with the axes of the Fučík spectrum is in order. In particular, consider for instance the special case of equation

$$x'' - \nu_0 x^- + r(t,x) = 0, \quad (\text{resp. } x'' + \mu_0 x^+ + r(t,x) = 0),$$

with $\nu_0 > 0$ (resp. $\mu_0 > 0$), and $r : [0,T] \times \mathbb{R} \to \mathbb{R}$ bounded. This equation describes a nonlinear oscillator where a one-sided restoring force acts (see, for instance, [121, 126]). In the plane (μ, ν) where the Fučík spectrum is usually represented, here the occurrence of resonance can be “visualized” on the vertical (respectively, horizontal) semi-axis, because the parameter μ (respectively, ν) is equal to 0.
In this situation, to apply Theorem 2.2.4 we need the validity of condition (2.55), which is the one-sided analogous of (2.54). For example, in the particular situation of equation

$$x'' - \nu_0 x^- = e(t),$$

such a Landesman-Lazer condition reads as

$$\int_0^T e(t)\, dt < 0,$$

recovering a well-known classical result of existence (we refer again, in a more general setting, to [121, 126]).

2.5 An example

In this section we will show an application of Theorem 2.2.4 to a class of planar systems which are asymptotically controlled by piecewise linear functions. For the sake of simplicity, we will only search for conditions which allow us to apply Corollary 2.2.8. Moreover, we will take into account the only case of double resonance with positive Hamiltonians, but a similar statement could be obtained when one of the comparison functions belongs to $\mathcal{P}_0$.
We will consider a perturbation of the so called “bi-asymmetric” oscillator. Precisely, for $u = (x, y) \in \mathbb{R}^2$, let us write $u^+ = (x^+, y^+)$ and $u^- = (x^-, y^-)$ and study the problem

$$\begin{cases} Ju' = \gamma(t,u)[\mathbb{A}_+ u^+ - \mathbb{A}_- u^-] + (1-\gamma(t,u))[\mathbb{B}_+ u^+ - \mathbb{B}_- u^-] + R(t,u) \\ u(0) = u(T), \end{cases} \tag{2.62}$$

where $\gamma(t,u)$ and $R(t,u)$ are L^2-Carathéodory functions such that $0 \le \gamma(t,u) \le 1$ for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$, and $R(t,u)$ is L^2-bounded. Moreover, we assume that

$$R(t,u) = \begin{pmatrix} r_{1,1}(t,x) + r_{1,2}(t,y) \\ r_{2,1}(t,x) + r_{2,2}(t,y) \end{pmatrix}, \tag{2.63}$$

and

$$\mathbb{A}_+ = \begin{pmatrix} a_+ & c \\ c & A_+ \end{pmatrix}, \mathbb{B}_+ = \begin{pmatrix} b_+ & c \\ c & B_+ \end{pmatrix}, \mathbb{A}_- = \begin{pmatrix} a_- & c \\ c & A_- \end{pmatrix}, \mathbb{B}_- = \begin{pmatrix} b_- & c \\ c & B_- \end{pmatrix},$$

for positive numbers $a_\pm, A_\pm, b_\pm, B_\pm$ satisfying $a_\pm \leq b_\pm$, $A_\pm \leq B_\pm$, with at least one of these inequalities strict, and $c \in \mathbb{R}$ such that

$$c^2 < \min\{a_+A_+, a_+A_-, a_-A_+, a_-A_-\},$$

in order to ensure that the two Hamiltonians

$$V_1(u) = \frac{1}{2}\left(a_+(x^+)^2 + a_-(x^-)^2 + A_+(y^+)^2 + A_-(y^-)^2 + cxy\right),$$

$$V_2(u) = \frac{1}{2}\left(b_+(x^+)^2 + b_-(x^-)^2 + B_+(y^+)^2 + B_-(y^-)^2 + cxy\right)$$

are positive. The particular form of the system is due to the fact that we want the right-hand side of the differential equation in (2.62) to be (up to $R(t,u)$) a convex combination of the gradients of the two comparison Hamiltonians.
It is immediately seen that condition (2.23) holds. Concerning the Landesman-Lazer conditions, fix $\varphi_1 = (\phi^{(1)}, \phi^{(2)})$ such that $J\varphi_1' = \nabla V_1(\varphi_1)$, and $\varphi_2 = (\psi^{(1)}, \psi^{(2)})$ such that $J\varphi_2' = \nabla V_2(\varphi_2)$. We will ask a Landesman-Lazer condition which is slightly stronger than the ones introduced in Theorem 2.2.4 and in Corollary 2.2.8, but has the advantage of being more understandable. Making use of the notation $\phi_\theta^{(j)}(t) = \phi^{(j)}(t+\theta)$, and the same for $\psi_\theta^{(j)}$, $j = 1,2$, define, for $i,j = 1,2$,

$$L_{i,j}(\theta) = \int_{\{\phi_\theta^{(j)}>0\}} \liminf_{s\to+\infty} r_{i,j}(t,s)\phi_\theta^{(i)}(t)\,dt + \int_{\{\phi_\theta^{(j)}<0\}} \limsup_{s\to-\infty} r_{i,j}(t,s)\phi_\theta^{(i)}(t)\,dt,$$

and

$$U_{i,j}(\theta) = \int_{\{\psi_\theta^{(j)}>0\}} \limsup_{s\to+\infty} r_{i,j}(t,s)\psi_\theta^{(i)}(t)\,dt + \int_{\{\psi_\theta^{(j)}<0\}} \liminf_{s\to-\infty} r_{i,j}(t,s)\psi_\theta^{(i)}(t)\,dt.$$

Setting

$$\tilde{\Gamma}_1^-(\theta) = L_{1,1}(\theta) + L_{1,2}(\theta) + L_{2,1}(\theta) + L_{2,2}(\theta), \tag{2.64}$$

and

$$\tilde{\Gamma}_1^+(\theta) = U_{1,1}(\theta) + U_{1,2}(\theta) + U_{2,1}(\theta) + U_{2,2}(\theta), \tag{2.65}$$

to fulfill conditions (2.45) and (2.47) we will then ask

$$\tilde{\Gamma}_1^-(\theta) > 0 > \tilde{\Gamma}_1^+(\theta), \tag{2.66}$$

for every $\theta \in [0, T]$. For the computation of the periods of the solutions to the comparison systems

$$\begin{cases} -y' = a_+ x^+ - a_- x^- + cy \\ x' = cx + A_+ y^+ - A_- y^-, \end{cases} \quad \text{and} \quad \begin{cases} -y' = b_+ x^+ - b_- x^- + cy \\ x' = cx + B_+ y^+ - B_- y^-, \end{cases}$$

we refer to [61, Section 4]. In the particular case $c = 0$, they have the following simple expressions:

$$\tau_1 = \frac{\pi}{2}\left[\frac{1}{\sqrt{a_+ A_+}} + \frac{1}{\sqrt{a_- A_+}} + \frac{1}{\sqrt{a_- A_-}} + \frac{1}{\sqrt{a_+ A_-}}\right],$$

and

$$\tau_2 = \frac{\pi}{2}\left[\frac{1}{\sqrt{b_+ B_+}} + \frac{1}{\sqrt{b_- B_+}} + \frac{1}{\sqrt{b_- B_-}} + \frac{1}{\sqrt{b_+ B_-}}\right],$$

respectively. Summing up, we infer:

Corollary 2.5.1. *Assume that $V_1 \in \mathcal{P}_k$, $V_2 \in \mathcal{P}_{k+1}$, being k a positive integer. Moreover, let $V_1 \leq V_2$, and suppose that (2.66) holds. Then problem (2.62) has a solution.*

We have thus proved a double resonance existence result, which, in the scalar case without damping, corresponding to $r_{1,2} \equiv r_{2,1} \equiv r_{2,2} \equiv 0$, $A_\pm = B_\pm = 1$, and $c = 0$, is strongly related to Fabry's one in [46]. As particular cases of system (2.62), one can also consider scalar second order equations of Liénard or Rayleigh type (see [61] for details). In the next sections, we will see that, using some estimates on the radial coordinate, we are able to give some finer results of existence also for these equations.

2.6 A possible relaxing of the double resonance conditions

From now on, we will focus on an intermediate case of double resonance which leads to more refined results. We will consider the situation in which the two Hamiltonians involved are both multiples of the same element of $\mathcal{P}$, and such that their minimal periods are two consecutive submultiples of T. The main idea consists in taking into account also the radial component when performing the a priori estimates. It is worth noticing that, in this way, it becomes more difficult to handle the case when one of the two considered Hamiltonians belongs to $\mathcal{P}_0$, since here the lack of a modified system of polar coordinates results in problems in estimating the contribution of the principal term to the derivative of the radius (see Remark 2.6.3 below). For this reason, we will only consider the case of two Hamiltonians in $\mathcal{P}$. Furthermore, we underline that, with

this approach, we could also deal with a more general problem at double resonance like the one studied in Section 2.2, but this would lead to existence conditions which are too involved, so that we prefer to consider a slightly easier situation.
Precisely, let $V \in \mathcal{P}$ and let α, β be two positive constants such that $\alpha < \beta$. It is clear that, defining as usual φ_V as the solution to $J\varphi_V' = \nabla V(\varphi_V)$ such that $\varphi_V(0)$ is on the positive horizontal semi-axis and $V(\varphi_V(t)) = 1/2$ for every t, the functions $\varphi_V(\alpha t)$ and $\varphi_V(\beta t)$ solve, respectively, the equations $Ju' = \alpha\nabla V(u)$ and $Ju' = \beta\nabla V(u)$. Moreover, writing

$$\tau_\alpha = \frac{\tau_V}{\alpha}, \quad \tau_\beta = \frac{\tau_V}{\beta},$$

we notice that τ_α, τ_β are, respectively, the minimal periods of $\varphi_V(\alpha t)$ and $\varphi_V(\beta t)$.
Our crucial assumption will be the following:

$$\frac{T}{k+1} \leq \tau_\beta < \tau_\alpha \leq \frac{T}{k}\,, \tag{2.67}$$

for some positive integer k. Under this requirement, we consider the problem

$$\begin{cases} Ju' = \hat{\gamma}(t,u)\nabla V(u) + R(t,u) \\ u(0) = u(T), \end{cases} \tag{2.68}$$

being $\alpha \leq \hat{\gamma}(t,u) \leq \beta$ for almost every $t \in [0,T]$, and every $u \in \mathbb{R}^2$, and $R(t,u)$ an L^2-bounded function. Defining $V_1(u) = \alpha V(u)$ and $V_2(u) = \beta V(u)$ and setting $\varphi_1(t) = \varphi_V(\alpha t)$ and $\varphi_2(t) = \varphi_V(\beta t)$, if we assume that V_1 satisfies the Landesman-Lazer condition (LL+)$_k$ and V_2 fulfills (LL-)$_{k+1}$, it is possible to apply Theorem 2.2.4. Indeed, conditions (2.23) and (2.7) are plainly satisfied, in view of the possibility of writing $\hat{\gamma}(t,u)\nabla V(u)$ as a convex combination of the gradients of the Hamiltonians V_1 and V_2.

However, it is possible to prove a better result which includes this one, as we are going to show. Referring to (LL+)$_k$, (LL-)$_{k+1}$, we introduce some notation, setting, for every $\theta \in [0,T]$,

$$\Gamma_1^-(\theta) = \int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(\alpha(t+\omega)))|\varphi_V(\alpha(t+\omega))\rangle\, dt$$

$$\Gamma_1^+(\theta) = \int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(\beta(t+\omega)))|\varphi_V(\beta(t+\omega))\rangle\, dt.$$

Notice that conditions (2.45) and (2.47) can be written, in this particular setting, as $\Gamma_1^-(\theta) > 0$ and $\Gamma_1^+(\theta) < 0$ for every $\theta \in [0,T]$, respectively. Moreover, we define the functions $\Gamma_2^\pm$ and $\Gamma_3^\pm$ by

$$\Gamma_2^-(\theta) = \int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(\alpha(t+\omega)))|\varphi_V'(\alpha(t+\omega))\rangle\, dt,$$

$$\Gamma_2^+(\theta) = \int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_V(\beta(t+\omega))) | \varphi_V'(\beta(t+\omega))\rangle \, dt,$$

and

$$\Gamma_3^-(\theta) = \int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_V(\alpha(t+\omega))) | \varphi_V'(\alpha(t+\omega))\rangle \, dt,$$

$$\Gamma_3^+(\theta) = \int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_V(\beta(t+\omega))) | \varphi_V'(\beta(t+\omega))\rangle \, dt.$$

We will prove the following statement.

Theorem 2.6.1. *Suppose that (2.67) holds. Moreover, assume that, for every $\theta \in [0, T]$,*

$$\Gamma_1^-(\theta) > 0 \quad \text{or} \quad \Gamma_2^-(\theta) > 0 \quad \text{or} \quad \Gamma_3^-(\theta) < 0, \tag{2.69}$$

and

$$\Gamma_1^+(\theta) < 0 \quad \text{or} \quad \Gamma_2^+(\theta) < 0 \quad \text{or} \quad \Gamma_3^+(\theta) > 0. \tag{2.70}$$

Then problem (2.68) has a solution.

Proof. Following the lines of the proof of Theorem 2.2.4, we proceed by performing a suitable homotopy. Assume by contradiction that an unbounded (in L^∞-norm) sequence $(u_n)_n$ satisfies

$$\begin{cases} Ju_n' = \sigma_n(\hat\gamma(t,u_n)\nabla V(u_n) + R(t,u_n)) + (1-\sigma_n)\delta\nabla V(u_n) \\ u_n(0) = u_n(T), \end{cases} \tag{2.71}$$

where $\sigma_n \in]0,1[$, and $\delta \in \mathbb{R}$ is a fixed number such that $\alpha < \delta < \beta$ (for example, $\delta = \frac{1}{2}(\alpha+\beta)$); without loss of generality, we can suppose $\sigma_n \to \bar\sigma \in [0,1]$. We can show that $\bar\sigma \neq 0$ exactly as in the proof of Theorem 2.2.4. Moreover, setting $v_n = \frac{u_n}{\|u_n\|_\infty}$, v_n converges uniformly, up to subsequences, to a function v which has the form $v(t) = r_v\varphi_V(\alpha(t+\omega_v))$ or $v(t) = r_v\varphi_V(\beta(t+\omega_v))$, for suitable constants $r_v > 0$, $\omega_v \in [0, \tau_\alpha[$ or $\omega_v \in [0, \tau_\beta[$. For example, suppose that this second situation occurs; we pass to generalized polar coordinates in (2.71), writing $u_n(t) = r_n(t)\varphi_V(\beta(t+\omega_n(t)))$, with $\omega_n(0) \in [0, \tau_\beta[$ for every n. For a subsequence, we have that $\omega_n(t) \to \omega_v$ uniformly. We have already seen, in the proof of Theorem 2.2.4, that the result holds if $\Gamma_1^+(\omega_v) < 0$. Assume now $\Gamma_1^+(\omega_v) \geq 0$. We have

$$-r_n'(t) = \sigma_n\langle R(t, r_n(t)\varphi_V(\beta(t+\omega_n(t)))) | \varphi_V'(\beta(t+\omega_n(t)))\rangle, \tag{2.72}$$

which, in view of the T-periodicity of u_n, gives

$$0 = \int_0^T \sigma_n\langle R(t, r_n(t)\varphi_V(\beta(t+\omega_n(t)))) | \varphi_V'(\beta(t+\omega_n(t)))\rangle \, dt.$$

By a straight use of Fatou's lemma, since $R(t,u)$ is L^2-bounded (notice that $\bar{\sigma} \neq 0$), it follows that

$$0 \geq \int_0^T \liminf_{n\to+\infty} \langle R(t, r_n(t)\varphi_V(\beta(t+\omega_n(t))))|\varphi_V'(\beta(t+\omega_n(t)))\rangle \, dt,$$

and

$$0 \leq \int_0^T \limsup_{n\to+\infty} \langle R(t, r_n(t)\varphi_V(\beta(t+\omega_n(t))))|\varphi_V'(\beta(t+\omega_n(t)))\rangle \, dt,$$

whence $\Gamma_3^+(\omega_v) \leq 0 \leq \Gamma_2^+(\omega_v)$, in contradiction with the hypothesis. □

Remark 2.6.2. To all intents and purposes, the considered problem is at double resonance, even if one may think to a simple resonance situation since there is really only one Hamiltonian involved. This is reflected in the difficulties mentioned in Remark 2.2.6, wishing to treat, for example, the equation $Ju' = \gamma(t,u)\nabla V(u) + R(t,u)$, with $\zeta_1(t) \leq \gamma(t,u) \leq \zeta_2(t)$. In this case, one of ζ_1 and ζ_2 will necessarily have mean different from 1, so that relabeling V leads to consider, in fact, a situation with two different Hamiltonians like the one in Section 2.2.

Remark 2.6.3. Wishing to consider $V \in \mathcal{P}_0$, one could perform a homotopy using the positive Hamiltonian $\epsilon(V(u)+|u|^2)$, for a sufficiently small ϵ such that this Hamiltonian is nonresonant. The point is that it is then difficult to have a neat expression for the derivative of the radius like (2.72), since with standard polar coordinates one is led to estimate the quantity $\langle \nabla V(\cos\theta, \sin\theta)|(-\sin\theta, \cos\theta)\rangle$, which, in general, is nonzero: it suffices to take $u = (x,y)$ and $V(u) = \frac{1}{2}y^2$ to obtain the result $\cos\theta\sin\theta$, which is not sign-definite. This contribution makes the derivative of the radius more difficult to estimate.

Remark 2.6.4. Let us give a geometrical interpretation of conditions (2.69) and (2.70). Defining the two curves $\Gamma^\pm : [0,T] \to \mathbb{R}^3$ as

$$\Gamma^-(\theta) = (\Gamma_1^-(\theta), \Gamma_2^-(\theta), \Gamma_3^-(\theta)), \quad \Gamma^+(\theta) = (\Gamma_1^+(\theta), \Gamma_2^+(\theta), \Gamma_3^+(\theta)),$$

condition (2.69) requires that $\Gamma^-(\theta)$ never enters the sector $\{(x,y,z) \in \mathbb{R}^3 \mid x \leq 0,\ y \leq 0,\ z \geq 0\}$, while condition (2.70) imposes that $\Gamma^+(\theta)$ never enters the sector $\{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ z \leq 0\}$. Recall that, in Theorem 2.2.4, we assumed, in a more restrictive way, that $\Gamma^-(\theta)$ always had to remain in the half-space $\{x > 0\}$ and $\Gamma^+(\theta)$ in $\{x < 0\}$.

Remark 2.6.5. It could happen that $\Gamma_2^-(\theta) = \Gamma_3^-(\theta)$ for every $\theta \in [0,T]$ (or $\Gamma_2^+(\theta) = \Gamma_3^+(\theta)$ for every $\theta \in [0,T]$). In this case, there is no need to define the curve Γ^- in $\mathbb{R}^3$, and one could define, instead, $\Gamma^- : [0,T] \to \mathbb{R}^2$ as $\Gamma^-(\theta) = (\Gamma_1^-(\theta), \Gamma_2^-(\theta))$.

Then, condition (2.69) requires that $\Gamma^-(\theta)$ never touches the half-line $\{(x,y) \in \mathbb{R}^2 \mid x \le 0, y = 0\}$. Clearly, in such a situation, the winding number of the curve Γ^-, with respect to the origin, is equal to 0. In the case of simple resonance, it was shown in [54] that this winding number $\mathrm{rot}(\Gamma^-, 0)$ is related to the topological degree associated with the considered periodic problem. Different examples were given (see [21, 22, 49, 50, 54, 61]) where $\mathrm{rot}(\Gamma^-, 0) \neq 0$. It can indeed be proved that the degree associated with the problem is equal to $1 - \mathrm{rot}(\Gamma^-, 0)$, see [54]. This agrees with the fact that, in our situation, the degree is equal to 1.

As in Section 2.5, we now show some possible applications to a particular class of planar systems. Consider the T-periodic problem

$$\begin{cases} Ju' = \hat{\gamma}(t,u)[\mathbb{A}_+ u^+ - \mathbb{A}_- u^-] + R(t,u) \\ u(0) = u(T), \end{cases} \tag{2.73}$$

being $\alpha \le \hat{\gamma}(t,u) \le \beta$ for some positive constants $\alpha < \beta$, for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$, and $R(t,u)$ an L^2-bounded function of the form (2.63). Moreover, we assume that

$$\mathbb{A}_+ = \begin{pmatrix} a_+ & c \\ c & A_+ \end{pmatrix}, \qquad \text{and} \qquad \mathbb{A}_- = \begin{pmatrix} a_- & c \\ c & A_- \end{pmatrix},$$

for positive constants $a_\pm$, $A_\pm$, and $c \in \mathbb{R}$ such that

$$c^2 < \alpha^2 \min\{a_+A_+, a_+A_-, a_-A_+, a_-A_-\}.$$

Notice that, without loss of generality, we can assume $\alpha = 1$. Hence, we are dealing with a particular case of the systems treated in Section 2.5, with $\mathbb{B}_+ = \beta\mathbb{A}_+$, and $\mathbb{B}_- = \beta\mathbb{A}_-$; as a consequence, the functions $\tilde{\Gamma}_1^-$ and $\tilde{\Gamma}_1^+$ can be explicitly written as in (2.64) and (2.65). However, in view of Theorem 2.6.1, it is possible to improve Corollary 2.5.1. Being $\varphi_1 = (\phi^{(1)}, \phi^{(2)})$, $\varphi_2 = (\psi^{(1)}, \psi^{(2)})$, $\tilde{\Gamma}_1^-(\theta)$ and $\tilde{\Gamma}_1^+(\theta)$ as in Section 2.5, and using the same notation therein, we define, for $i,j = 1,2$,

$$\mathcal{L}_{i,j}(\theta) = \int_{\{[\phi_\theta^{(j)}]'>0\}} \liminf_{s\to+\infty} r_{i,j}(t,s)[\phi_\theta^{(i)}]'(t)\,dt + \int_{\{[\phi_\theta^{(j)}]'<0\}} \limsup_{s\to-\infty} r_{i,j}(t,s)[\phi_\theta^{(i)}]'(t)\,dt,$$

$$\mathcal{U}_{i,j}(\theta) = \int_{\{[\psi_\theta^{(j)}]'>0\}} \limsup_{s\to+\infty} r_{i,j}(t,s)[\psi_\theta^{(i)}]'(t)\,dt + \int_{\{[\psi_\theta^{(j)}]'<0\}} \liminf_{s\to-\infty} r_{i,j}(t,s)[\psi_\theta^{(i)}]'(t)\,dt,$$

and

$$\mathcal{M}_{i,j}(\theta) = \int_{\{[\phi_\theta^{(j)}]'>0\}} \limsup_{s\to+\infty} r_{i,j}(t,s)[\phi_\theta^{(i)}]'(t)\,dt + \int_{\{[\phi_\theta^{(j)}]'<0\}} \liminf_{s\to-\infty} r_{i,j}(t,s)[\phi_\theta^{(i)}]'(t)\,dt,$$

$$\mathcal{V}_{i,j}(\theta) = \int_{\{[\psi_\theta^{(j)}]'>0\}} \liminf_{s\to+\infty} r_{i,j}(t,s)[\psi_\theta^{(i)}]'(t)\,dt + \int_{\{[\psi_\theta^{(j)}]'<0\}} \limsup_{s\to-\infty} r_{i,j}(t,s)[\psi_\theta^{(i)}]'(t)\,dt.$$

Moreover, we set

$$\tilde\Gamma_2^-(\theta) = \mathcal{L}_{1,1}(\theta) + \mathcal{L}_{1,2}(\theta) + \mathcal{L}_{2,1}(\theta) + \mathcal{L}_{2,2}(\theta),$$
$$\tilde\Gamma_2^+(\theta) = \mathcal{U}_{1,1}(\theta) + \mathcal{U}_{1,2}(\theta) + \mathcal{U}_{2,1}(\theta) + \mathcal{U}_{2,2}(\theta),$$

and

$$\tilde\Gamma_3^-(\theta) = \mathcal{M}_{1,1}(\theta) + \mathcal{M}_{1,2}(\theta) + \mathcal{M}_{2,1}(\theta) + \mathcal{M}_{2,2}(\theta),$$
$$\tilde\Gamma_3^+(\theta) = \mathcal{V}_{1,1}(\theta) + \mathcal{V}_{1,2}(\theta) + \mathcal{V}_{2,1}(\theta) + \mathcal{V}_{2,2}(\theta).$$

To satisfy (2.69) and (2.70), we will then ask that, for every $\theta \in [0, T]$, it holds

$$\tilde\Gamma_1^-(\theta) > 0 \quad \text{or} \quad \tilde\Gamma_2^-(\theta) > 0 \quad \text{or} \quad \tilde\Gamma_3^-(\theta) < 0, \tag{2.74}$$

and

$$\tilde\Gamma_1^+(\theta) < 0 \quad \text{or} \quad \tilde\Gamma_2^+(\theta) < 0 \quad \text{or} \quad \tilde\Gamma_3^+(\theta) > 0. \tag{2.75}$$

For the computation of the periods of the comparison Hamiltonian systems, we refer again to [61]. With a direct application of Theorem 2.6.1, we now obtain, in this particular framework, the following improvement of Corollary 2.5.1.

Corollary 2.6.6. *Assume that conditions (2.67), (2.74) and (2.75) hold. Then problem (2.73) has a solution.*

We conclude the section by remarking that, if $\alpha = \beta$, the considered problem takes the form studied in Section 2.1 and it is even easier to give an existence result like Theorem 2.6.1 (see Corollary 2.7.1 below). We are going to see that this can be successfully exploited when studying scalar equations with damping.

2.7 Scalar equations with damping in a case of simple resonance

In this section, we will examine an application of a particular case of Theorem 2.6.1 to scalar equations with damping which fit in the framework of system (2.73). Preliminarily, it is interesting to notice that in 1969, the same year of publication of the Lazer-Leach result, Frederickson and Lazer introduced in [66] a rather similar condition for second order equations of Liénard or Rayleigh type, where the nonlinearity depends on the derivative of the solution x. For instance, considering the Rayleigh T-periodic problem

$$\begin{cases} x'' + P(x') + \lambda_k x = e(t), \\ x(0) = x(T),\ x'(0) = x'(T), \end{cases} \tag{2.76}$$

with the assumption that $P(x)$ is strictly increasing, and that

$$\frac{2}{\pi}\Big(\lim_{x\to+\infty}P(x)-\lim_{x\to-\infty}P(x)\Big)>\sqrt{a_k^2+b_k^2}\,,$$

being a_k and b_k defined by

$$a_k=\frac{2}{T}\int_0^T e(s)\cos\left(\frac{2k\pi}{T}s\right)\,ds \quad\text{and}\quad b_k=\frac{2}{T}\int_0^T e(s)\sin\left(\frac{2k\pi}{T}s\right)\,ds,$$

they proved that (2.76) has a solution.
As we are going to see, the Landesman-Lazer condition is not suitable to deal with this kind of problems, since the angular coordinate is more difficult to control. However, thanks to the results of the previous section, the planar framework seems to be quite convenient to give a unique statement including also this situation. It has to be said that, for equations with damping, especially the Liénard one, a suitable definition of spectrum was given in [65], and it would be interesting to investigate the possibility of giving existence results when considering resonance with respect to this concept of eigenvalue, as the paper [72] tried to do. We will limit ourselves to other kinds of considerations, giving results where, in the spirit of the previous section, the idea is to control the radial component of the solutions, rather than the angular one. This will allow us to recover results of Frederickson-Lazer type.
For simplicity, referring to (2.73), we will consider only the symmetric case, namely $\mathbb{A}_+=\mathbb{A}_-$, assuming

$$\alpha=\beta=1,$$

and $T=2\pi$. The same arguments would apply to the asymmetric case, as well.
Let us first state the following immediate consequence of Theorem 2.6.1, as mentioned at the end of the previous section. We will use the notation introduced in Section 2.6.

Corollary 2.7.1. *Let $V\in\mathcal{P}_k$, for a positive integer k, and consider the problem*

$$\begin{cases} Ju'=\nabla V(u)+R(t,u) \\ u(0)=u(T), \end{cases} \tag{2.77}$$

being $R(t,u)$ an L^2-bounded function. Moreover, assume that, for every $\theta\in[0,T]$,

$$\Gamma_1^-(\theta)>0,\qquad\text{or}\qquad \Gamma_2^-(\theta)>0,\qquad\text{or}\qquad \Gamma_3^-(\theta)<0. \tag{2.78}$$

Then problem (2.77) has a solution.

Clearly, assumption (2.78) can be replaced by the following one:

$$\Gamma_1^+(\theta)<0,\qquad\text{or}\qquad \Gamma_2^+(\theta)<0,\qquad\text{or}\qquad \Gamma_3^+(\theta)>0. \tag{2.79}$$

Notice that, in this situation, since $\varphi_V(\alpha t)$ and $\varphi_V(\beta t)$ coincide and they both solve $Ju' = \nabla V(u)$, we have that

$$\Gamma_1^-(\theta) \leq \Gamma_1^+(\theta), \quad \text{and} \quad \Gamma_2^-(\theta) = \Gamma_3^+(\theta) \leq \Gamma_3^-(\theta) = \Gamma_2^+(\theta),$$

for every $\theta \in [0, T]$.
We can now take in consideration the following two problems:

$$\begin{cases} x'' + p(x)x' + k^2 x = e(t) \\ x(0) = x(2\pi),\ x'(0) = x'(2\pi), \end{cases} \tag{2.80}$$

and

$$\begin{cases} x'' + P(x') + k^2 x = e(t), \\ x(0) = x(2\pi),\ x'(0) = x'(2\pi), \end{cases} \tag{2.81}$$

being k a positive integer, and $e(t)$ continuous and 2π-periodic. Clearly, similar considerations would hold for the T-periodic problem, with k^2 replaced by the corresponding λ_k.
The differential equations in (2.80) and (2.81) are equivalent to the systems

$$\begin{cases} x' = y - P(x) \\ y' = -k^2 x + e(t), \end{cases} \tag{2.82}$$

and

$$\begin{cases} x' = y \\ y' = -k^2 x - P(y) + e(t), \end{cases}$$

respectively, where in (2.82) we have set $P(x) = \int_0^x p(s)\, ds$. They are thus included in our framework, with

$$V(x, y) = \frac{1}{2}(k^2 x^2 + y^2).$$

For simplicity, in the following we will deal only with the Liénard problem (2.80), and hence with (2.82). The 2π-periodic problem associated with the Rayleigh equation can be treated in the same way, yielding similar results. As a structural hypothesis, we assume that P is a *bounded* function.
Let us observe that, as a matter of fact, Theorem 2.2.4 is not suitable to deal with this kind of systems. Considering (2.82), if we assume that $P(x)$ is strictly increasing, there always exists $\theta \in [0, 2\pi]$ such that neither condition $(\text{LL+})_k$ nor $(\text{LL-})_k$ is satisfied (with V as above). To see this, for instance for what concerns $(\text{LL+})_k$, set

$$\phi(t) = \frac{1}{k}\cos(kt),$$

and $\varphi_V(t) = (\phi(t), \phi'(t))$; after some computations we see that, if $(\mathrm{LL}+)_k$ holds, the quantity

$$\int_0^{2\pi} \Big(\liminf_{(\lambda,\omega)\to(+\infty,\theta)} [-P(\lambda\phi(t+\omega))\phi'(t+\omega)] - e(t)\phi(t+\theta) \Big)\, dt$$

has to be strictly positive for every $\theta \in [0, 2\pi]$. Noticing that, since P has finite limits at $\pm\infty$, the inferior limit which appears under the integral sign is indeed a finite limit, this is true if and only if

$$\int_0^{2\pi} e(t)\phi(t+\theta)\, dt < 0, \qquad \text{for every } \theta \in [0, 2\pi].$$

Such a condition, however, is never satisfied, due to the form of $\phi(t+\theta)$: explicitly, it should be

$$\cos(k\theta) \int_0^{2\pi} e(t)\cos(kt)\, dt - \sin(k\theta) \int_0^{2\pi} e(t)\sin(kt)\, dt < 0,$$

for every $\theta \in [0, 2\pi]$, which is clearly impossible. In the same way, we see that also $(\mathrm{LL}\text{-})_k$ fails.
We now show how it is possible to overcome this problem using Corollary 2.7.1. Consider system (2.82): setting

$$P_-(+\infty) = \liminf_{x\to+\infty} P(x), \quad P_+(+\infty) = \limsup_{x\to+\infty} P(x),$$

$$P_-(-\infty) = \liminf_{x\to-\infty} P(x), \quad P_+(-\infty) = \limsup_{x\to-\infty} P(x),$$

and

$$\Delta P(+\infty) = P_+(+\infty) - P_-(+\infty), \quad \Delta P(-\infty) = P_+(-\infty) - P_-(-\infty),$$

the following result holds true:

Corollary 2.7.2. *Assume that, for every $\theta \in [0, 2\pi]$,*

$$\begin{aligned} &\int_0^{2\pi} e(t)\phi(t+\theta)\, dt < -\frac{1}{k}(\Delta P(+\infty) + \Delta P(-\infty)), \quad \text{or} \\ &\int_0^{2\pi} e(t)\phi'(t+\theta)\, dt < 2(P_-(+\infty) - P_+(-\infty)), \ \ \text{or} \\ &\int_0^{2\pi} e(t)\phi'(t+\theta)\, dt > 2(P_+(+\infty) - P_-(-\infty)). \end{aligned} \tag{2.83}$$

Then problem (2.80) has a solution.

Notice that the statement follows from Corollary 2.7.1, since (2.83) implies (2.78). A symmetric result can be stated assuming, for every $\theta \in [0, 2\pi]$,

$$\begin{aligned}
&\int_0^{2\pi} e(t)\phi(t+\theta)\,dt > \frac{1}{k}(\Delta P(+\infty) + \Delta P(-\infty)), \quad \text{or} \\
&\int_0^{2\pi} e(t)\phi'(t+\theta)\,dt < 2(P_-(+\infty) - P_+(-\infty)), \ \ \text{or} \\
&\int_0^{2\pi} e(t)\phi'(t+\theta)\,dt > 2(P_+(+\infty) - P_-(-\infty)),
\end{aligned} \tag{2.84}$$

since (2.84) implies (2.79).

The last part of this section will be dedicated to compare Corollary 2.7.2, and its symmetric version with (2.84) instead of (2.83), with the following result proved by Frederickson and Lazer in [66], in the particular case $k = 1$.

Theorem 2.7.3. *Assume that $k = 1$ and that $P(x)$ is strictly increasing. Then, setting*

$$P(+\infty) = \lim_{x\to+\infty} P(x), \qquad \text{and} \qquad P(-\infty) = \lim_{x\to-\infty} P(x),$$

the condition

$$\left| \int_0^{2\pi} e(t)e^{-it}\,dt \right| < 2(P(+\infty) - P(-\infty))$$

is both necessary and sufficient for the existence of a solution to (2.80).

We thus consider, in our framework, the case when P is increasing, so that $P(+\infty) - P(-\infty) > 0$. Since P has finite limits at $\pm\infty$ (recall that we are assuming P to be bounded), the inferior limits appearing under the integral sign in our hypotheses are finite limits. So, by Corollary 2.7.2, if for every $\theta \in [0, 2\pi]$ we have

$$\int_0^{2\pi} e(t)\phi(t+\theta)\,dt < 0 \qquad \text{or} \qquad \int_0^{2\pi} e(t)\phi'(t+\theta)\,dt \neq 2(P(+\infty) - P(-\infty)), \tag{2.85}$$

being $\phi(t) = \cos t$, then problem (2.80) has a solution. It is straightly seen that this hypothesis follows from the Frederickson-Lazer condition, which implies indeed

$$\int_0^{2\pi} e(t)\phi'(t+\theta)\,dt < 2(P(+\infty) - P(-\infty)),$$

for every $\theta \in [0, 2\pi]$. Apparently, however, (2.85) seems to be more general, which looks strange, as the Frederickson-Lazer condition is also necessary for the existence, in the setting of the theorem. We now show that the two statements are indeed equivalent. Suppose, for simplicity, $e(t) = \cos t$, and assume that (2.85) holds, namely

$$\cos\theta < 0, \qquad \text{or} \qquad -\pi\sin\theta \neq 2(P(+\infty) - P(-\infty)),$$

for every $\theta \in [0, T]$. We claim that, necessarily, it will be

$$2(P(+\infty) - P(-\infty)) > \pi; \tag{2.86}$$

otherwise, we could always find $\theta_0 \in [0, 2\pi]$ such that

$$2(P(+\infty) - P(-\infty)) = -\pi\sin\theta_0, \qquad \text{and} \qquad \cos\theta_0 \geq 0$$

hold at the same time, making (2.85) fail. Being

$$\left|\int_0^{2\pi} (\cos t)e^{-it}\, dt\right| = \pi,$$

we have that (2.86) implies the Frederickson-Lazer condition, so we are done in the particular case $e(t) = \cos t$. The reasoning works, in the same way, for $e(t) = \cos jt$ and $e(t) = \sin jt$, for every $j \in \mathbb{N}$. Using the fact that $\{\cos jt, \sin jt\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(0, 2\pi)$, the previous considerations can be extended to every continuous forcing term $e(t)$. Summing up, if P is bounded and increasing, Corollary 2.7.2 generalizes Frederickson and Lazer's result.

Remark 2.7.4. By the above discussion, we can conclude that Corollary 2.7.1 generalizes, for the periodic problem, both the Lazer and Leach existence result and the Frederickson and Lazer one, in the case when P is bounded (see also [21, 22, 61, 92]). Notice, however, that, in [66], P was not assumed to be bounded, and the almost periodic problem was also considered, obtaining a similar existence result.

Remark 2.7.5. The above arguments can be adapted to the case when P is not bounded, but has sublinear growth, provided that the functions $\Gamma_1^\pm$, $\Gamma_2^\pm$ and $\Gamma_3^\pm$ are well defined. Even in this case, if P is increasing, we have that the Frederickson-Lazer condition and ours turn out to be equivalent.

2.8 Higher-order Landesman-Lazer conditions

In this last section, we are interested in the case when the inferior and superior limits which appear in the Landesman-Lazer conditions are equal to 0, and so conditions $(\text{LL}+)_k$ and $(\text{LL-})_{k+1}$ do not hold. This problem has already been studied in the scalar setting, see, e.g., [46] and [114]. We propose here a possible generalization of

[46, Theorem 2], based on Theorem 2.2.4, and consisting in refining (LL+)$_k$, (LL-)$_{k+1}$. We will use again the notation introduced there. Moreover, we will also assume as hypotheses the corresponding refinements of conditions (2.35) and (2.36) (the idea is that $|R(t,u)|$ has to be controlled by some negative power of $|u|$).

Theorem 2.8.1. *Let $k \in \mathbb{N}$ and assume that $V_1 \in \mathcal{P}_k$, $V_2 \in \mathcal{P}_{k+1}$ are such that (2.23) holds, together with (2.7). Moreover, assume that there exists $j \geq 0$ such that, for a positive function $\eta \in L^2(0,T)$, it is*

$$\lambda^j(\langle F(t,\lambda w)|w\rangle - 2\lambda V_1(w)) \geq -\eta(t), \tag{2.87}$$

and

$$\lambda^j(2\lambda V_2(w) - \langle F(t,\lambda w)|w\rangle) \geq -\eta(t),$$

for almost every $t \in [0,T]$, every $w \in \mathbb{R}^2$ with $|w| \leq 1$ and every $\lambda \geq 1$. Then, there exists a solution to problem (2.22), provided that

- *$V_1 \in \mathcal{P}_0$ and, for every $\xi \in \mathfrak{Z}_{V_1}$,*

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \lambda^j \langle F(t,\lambda\eta)|\eta\rangle \, dt > 0, \tag{2.88}$$

or

$V_1 \in \mathcal{P}$ and, for every $\theta \in [0,T]$,

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \lambda^j [\langle F(t,\lambda\varphi_1(t+\omega))|\varphi_1(t+\omega)\rangle - 2\lambda V_1(\varphi_1(t))] \, dt > 0, \tag{2.89}$$

and

- *for every $\theta \in [0,T]$,*

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \lambda^j [\langle F(t,\lambda\varphi_2(t+\omega))|\varphi_2(t+\omega)\rangle - 2\lambda V_2(\varphi_2(t))] \, dt < 0. \tag{2.90}$$

Proof. It is sufficient to follow the lines of the proof of Theorem 2.2.4. Anyway, for the sake of clarity, we will consider separately the case $V_1 \in \mathcal{P}_0$. In this situation, we have that (2.42) yields, after multiplication by $\|u_n\|_\infty^j$,

$$0 > \int_0^T \|u_n\|_\infty^j \frac{\langle F(t,\|u_n\|_\infty v_n(t))|v_n(t)\rangle}{(\rho_n^V(t))^2} \, dt,$$

where $\rho_n^V(t) = \rho_n(t)/\|u_n\|_\infty$. Since $v_n = u_n/\|u_n\|_\infty$ converges uniformly to a point $\xi \in \mathbb{S}^1$, it follows that $\rho_n^V(t) \to 1$ uniformly, so that, using Fatou's lemma thanks to (2.87), we obtain

$$0 \geq \int_0^T \liminf_{n\to+\infty} \|u_n\|_\infty^j \langle F(t, \|u_n\|_\infty v_n(t))|v_n(t)\rangle \, dt.$$

Since $\|u_n\|_\infty \to +\infty$, this contradicts (2.88).
The case when both the Hamiltonians V_1 and V_2 belong to $\mathcal{P}$ is similar: focusing, for instance, on (2.89), we have that (2.44) gives

$$0 > \int_0^T r_n(t)^j \, \frac{\langle F(t, r_n(t)\varphi_1(t+\omega_n(t)))|\varphi_1(t+\omega_n(t))\rangle - r_n(t)}{(r_n^V(t))^{j+1}} \, dt.$$

Again by Fatou's lemma, this implies that

$$0 \geq \int_0^T \liminf_{n\to+\infty} r_n(t)^j \left[\langle F(t, r_n(t)\varphi_1(t+\omega_n(t)))|\varphi_1(t+\omega_n(t))\rangle - r_n(t)\right] dt,$$

contradicting (2.89). □

Notice that Theorem 2.2.4 is a particular case of this result (for $j = 0$). In a similar way, moreover, it is possible to obtain, also in this framework, results analogous to the ones proved in Sections 2.5, 2.6 and 2.7.

Chapter 3

A variational approach: the Ahmad-Lazer-Paul condition

In this chapter, we will discuss the possibility of proving existence for a planar system at resonance by means of an Ahmad-Lazer-Paul condition.
Just to mention a bit of history, let us start our discussion with the T-periodic boundary value problem associated with the scalar second order differential equation

$$x'' + \lambda_k x + r(t, x) = 0, \qquad x \in \mathbb{R}, \tag{3.1}$$

being $r : [0, T] \times \mathbb{R} \to \mathbb{R}$ a continuous and bounded function and $\lambda_k = \left(\frac{2k\pi}{T}\right)^2$ ($k = 0, 1, \ldots$), so that we are considering a resonant situation. In [112, Theorem 4.8 and Exercise 4.9], an existence result for (3.1) was proved - by variational tools - under the assumption that

$$\lim_{\substack{\|x\|_\infty \to +\infty \\ x'' + \lambda_k x = 0}} \int_0^T \mathcal{R}(t, x(t))\, dt = +\infty, \tag{3.2}$$

being $\mathcal{R}(t, x) = \int_0^x r(t, \xi)\, d\xi$. Condition (3.2) is usually referred to as the *Ahmad-Lazer-Paul condition*, since it is the version, for the T-periodic problem, of the assumption introduced in [1], dealing with the Dirichlet problem for a partial differential equation at resonance. Qualitatively, from the point of view introduced by Rabinowitz in [115], (3.2) expresses the anticoercivity of the Lagrange functional associated with (3.1) on the eigenspace relative to λ_k, so that a T-periodic solution can be provided as a critical point of saddle type. Incidentally, notice that the result in [112] holds as well for second order systems of gradient type

$$x'' + \lambda_k x + \nabla_x \mathcal{R}(t, x) = 0, \qquad x \in \mathbb{R}^N;$$

further developments along this direction were obtained, among the others, in [7, 75, 130].

Later, related results [30, 74, 85, 96, 129] were given for general Hamiltonian systems of the type

$$Ju' = A(t)u + \nabla_u Q(t,u), \qquad u \in \mathbb{R}^{2N}, \tag{3.3}$$

where $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard symplectic matrix, $A(t)$, $t \in [0,T]$, is a continuous path of $2N \times 2N$ symmetric matrices and $Q : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ is a regular function with bounded gradient. Thus, the principal term here is of linear type, and resonance is meant in the sense that the linear problem $Ju' = A(t)u$ has nontrivial T-periodic solutions. In this case, more sophisticated techniques from critical point theory are needed, since the natural variational formulation of (3.3) leads to a strongly indefinite functional (i.e., its quadratic part is unbounded both from below and from above).

Considering, on the other hand, the asymmetric equation

$$x'' + \mu x^+ - \nu x^- + r(t,x) = 0, \qquad x \in \mathbb{R}, \tag{3.4}$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, and (μ, ν) belongs to the T-periodic Fučík spectrum, the existence of T-periodic solutions to (3.4) via Ahmad-Lazer-Paul type conditions is a more subtle problem, because the asymmetry of the unperturbed problem avoids the use of the linear tools usually employed to detect saddle geometry (see [125]). In this connection, some results were given in [9, 84].

We now want to extend part of the mentioned results to the more general situation of system

$$Ju' = \nabla V(u) + \nabla Q(t,u), \tag{3.5}$$

where $V \in \mathcal{P}$, and $Q : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ has bounded gradient, so to include both the results for the linear and for the asymmetric case. As we know, in this setting resonance appears if and only if $V \in \mathcal{P}_k$, for a nonnegative integer k (in this chapter, we will only take into account the case $k > 0$). Our aim is to provide an Ahmad-Lazer-Paul condition (see (3.22) below) to ensure existence in this resonant setting. Actually, as we have already seen in the previous chapter, $\nabla_u Q(t,u)$ does not need to be bounded (cf. Remark 3.3.2), but is allowed to grow at infinity as a sublinear power in the u-variable (see [30, 74, 130]).
Throughout the chapter, after having recalled the variational setting associated with a planar Hamiltonian system with linear principal part, we will study the properties of a suitable symplectic change of variables, which will be crucial to prove, in Section 3.3, the desired Ahmad-Lazer-Paul existence result.

3.1 The variational setting

We now briefly recall the variational setting to study (3.5) when the principal term is linear, i.e.,

$$Jv' = B(t)v + \nabla Q(t,v), \tag{3.6}$$

being $B(t)$, $t \in [0,T]$, a continuous path of symmetric 2×2 matrices and $Q : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ a C^1-function. As we have remarked before, the existence of T-periodic solutions to such a kind of systems has been extensively studied by a large number of authors, providing several results with many different approaches.
Thanks to the variational structure of the problem, one possibility is here to search for solutions as critical points of a suitable functional associated with (3.6). To this aim, we define

$$E = \Big\{ v \in L^2([0,T]\,;\mathbb{R}^2) \mid \sum_{j\in\mathbb{Z}} (1+|j|)|v_j|^2 < +\infty \Big\},$$

being $\sum_{j\in\mathbb{Z}} e^{\frac{2j\pi}{T}tJ} v_j$, with $v_j \in \mathbb{R}^2$, the Fourier expansion of $v(t)$. The space E is a fractional Sobolev space (usually denoted also by $H_T^{1/2}$) and has a structure of Hilbert space, endowed with the scalar product

$$\langle v|w\rangle_E = \sum_{j\in\mathbb{Z}} (1+|j|)\langle v_j|w_j\rangle.$$

Since, for v,w smooth, the bilinear map

$$(v,w) \mapsto \int_0^T \langle Jv'(t)|w(t)\rangle\,dt$$

is continuous with respect to the norm of E, by density and the Riesz representation theorem there exists a unique linear bounded operator $\mathcal{L} : E \to E$ such that, for v,w smooth,

$$\langle \mathcal{L}v|w\rangle_E = \int_0^T \langle Jv'(t)|w(t)\rangle\,dt.$$

It is now possible to define the functional

$$\mathcal{I}(v) = \frac{1}{2}\langle \mathcal{L}v|v\rangle_E - \frac{1}{2}\int_0^T \langle B(t)v(t)|v(t)\rangle\,dt - \int_0^T Q(t,v(t))\,dt, \qquad v \in E.$$

We notice that the first integral is well-defined in view of the embedding $E \hookrightarrow L^2([0,T]\,;\mathbb{R}^2)$, whereas the second one is just formal, since $v(t)$ may not be continuous. However, we have the following proposition (see [6, 116]).

Proposition 3.1.1. *Assume that there exist $m > 0, s \in]2, +\infty[$ such that*

$$|\nabla Q(t, v)| \leq m(1 + |v|^{s-1}), \quad \text{for every } t \in [0, T],\ v \in \mathbb{R}^2. \tag{3.7}$$

Then, $\mathcal{I} : E \to \mathbb{R}$ is of class C^1 and its critical points are (classical) T-periodic solutions to (3.6)*.*

Thus, searching for T-periodic solutions to (3.6) will be equivalent to searching for critical points of $\mathcal{I}$. Keeping in mind the theory of resonance recalled in Chapter 2, we are especially interested in the case when, once the perturbation is excluded, the remaining autonomous system has nontrivial solutions.
The following result deals just with this situation, when (3.6) is the perturbation of a linear problem at resonance (see [74, Theorem 1.1] and [84, Remark, p. 1225]).

Theorem 3.1.2. *Denote by $\mathfrak{S}$ the set of the T-periodic solutions to $Jv' = B(t)v$ and assume $\mathfrak{S} \neq \{0\}$. Moreover, suppose that there exists $\widetilde{M} > 0$ and $\alpha \in [0, 1[$ such that*

$$|\nabla Q(t, v)| \leq \widetilde{M}(1 + |v|^{\alpha}), \quad \text{for every } t \in [0, T],\ v \in \mathbb{R}^2. \tag{3.8}$$

If

$$\lim_{\substack{\|v\|_E \to +\infty \\ v \in \mathfrak{S}}} \frac{1}{\|v\|_E^{2\alpha}} \int_0^T Q(t, v(t))\, dt = +\infty, \tag{3.9}$$

then system (3.6) *has a T-periodic solution.*

Observe that $\mathfrak{S}$ is a linear subspace having finite dimension, so that any other norm on $\mathfrak{S}$ could be used in (3.9). In the following, for simplicity, we will use the L^∞-norm.
For what concerns the proof of Theorem 3.1.2, observe first that (3.8) implies (3.7), so that (3.6) can be studied in the previously introduced variational setting. The assumptions (3.8) and (3.9) are then used to ensure the validity of the Palais-Smale condition and a saddle type geometry associated with an orthogonal decomposition $E = E_1 \oplus E_2$. However, since both E_1 and E_2 are infinite dimensional, a finer version of Rabinowitz saddle point Theorem needs to be used [116, Theorem 5.29 and Example 5.22].

Therefore, we have here an existence result based on a nonresonance condition of Ahmad-Lazer-Paul type. We want to establish a similar statement for a perturbation of a positively homogeneous system like

$$Ju' = \nabla V(u) + \nabla Q(t, u), \qquad u \in \mathbb{R}^2, \tag{3.10}$$

where $V \in \mathcal{P}_k$, $k \geq 1$, $Q : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and, as usual, we write $\nabla Q(t, u)$ in place of $\nabla_u Q(t, u)$. Since the principal term ∇V is not necessarily linear, we cannot apply directly the results which have just been mentioned; on the other hand, treating directly the problem writing the associated action functional would lead to some difficulties in the estimates. To achieve a result in the spirit of the theory of resonance, the idea is thus to perform a suitable change of variables transforming the original problem into a perturbation of a linear one. However, this task is quite subtle, since we want it to preserve the structure of the starting equation and the form of the Ahmad-Lazer-Paul condition (3.9). For this reason, we will spend the next section of the chapter constructing explicitly the change of variables, on the lines of [63]; in the second section, we will state and prove our existence result.

3.2 A symplectic change of variables

Recall that, for an open set $\mathcal{U} \subset \mathbb{R}^2$, a C^1-map $\Lambda : \mathcal{U} \to \mathbb{R}^2$ is called *symplectic* if

$$\Lambda'(u)^t J \Lambda'(u) = J, \qquad \text{for every } u \in \mathcal{U}. \tag{3.11}$$

It is well known that, given a C^1-function $H : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and a symplectic C^1-diffeomorphism Λ of the plane onto itself, the Hamiltonian system $Ju' = \nabla H(t,u)$ is changed, via the change of variables $v = \Lambda(u)$, into the system $Jv' = \nabla \widetilde{H}(t,v)$, being $\widetilde{H}(t,v) = H(t, \Lambda^{-1}(v))$. This means that the transformed system is still Hamiltonian, and the associated Hamiltonian is just the "old" one, evaluated on the "new" variable v.

We are going to construct a symplectic C^1-diffeomorphism of the plane, in association with a nonnegative function $W : \mathbb{R}^2 \to \mathbb{R}$ satisfying the following hypotheses:

(W0) $W \in C^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2_*)$;

(W1) $W(0) = 0$ and $W(u) > 0$ for every $u \neq 0$;

(W2) $\nabla W(0) = 0$ and $\langle \nabla W(u) | u \rangle > 0$ for every $u \neq 0$;

(W3) $W(u) \to +\infty$ for $|u| \to +\infty$.

Notice that, in view of Euler's formula, if $W \in C^2(\mathbb{R}^2_*) \cap \mathcal{P}$, then it satisfies (W0)–(W3).

The dynamics of the planar autonomous Hamiltonian system

$$Ju' = \nabla W(u), \tag{3.12}$$

when $W(u)$ fulfills (W0)–(W3), has already been discussed after Proposition 1.1.8. We know that, in this situation, the origin is a global center for system (3.12). Consequently, denoting by $\zeta(t;u)$ ($u \neq 0$) the solution to (3.12) such that $\zeta(0;u) = u$, we can define $\tau(u) \in \mathbb{R}^+_*$ as the minimal period of $\zeta(t;u)$.

The following proposition holds.

Proposition 3.2.1. *The map* $\tau : \mathbb{R}^2_* \to \mathbb{R}^+_*$ *is of class* C^1.

As a notation, for functions depending on $u \in \mathbb{R}^2$, we will write ∂_i, $i = 1, 2$, to denote the partial derivative with respect to the i-th component of u. Observe that the map

$$\mathbb{R} \times \mathbb{R}^2_* \ni (t, u) \mapsto \zeta(t; u) = (\zeta_1(t; u), \zeta_2(t; u)) \in \mathbb{R}^2_*$$

is of class C^1.

Proof. For $c \in \mathbb{R}^+_*$, let $\xi(c)$ be the unique positive number such that $W(\xi(c), 0) = c$. The map $c \mapsto \xi(c)$ is continuous and the Implicit Function Theorem ensures that it is of class C^1. Indeed,

$$\frac{\partial}{\partial d}\Big(W(d, 0) - c\Big)\Big|_{d=\xi(c)} = \partial_1 W(\xi(c), 0) = \frac{1}{\xi(c)}\langle \nabla W(\xi(c), 0) | (\xi(c), 0)\rangle \neq 0.$$

Next, for $x \in \mathbb{R}^+_*$, let $\Pi(x)$ be the second strictly positive real number such that $\zeta_2(\Pi(x); (x, 0)) = 0$. Since all the nontrivial solutions to $Ju' = \nabla W(u)$ are periodic and describe, in the clockwise sense, a strictly star-shaped Jordan curve, $\Pi(x)$ is the period of the orbit passing through $(x, 0)$. As $\nabla W(u) \neq 0$ for every $u \neq 0$, [80, (v), p. 83] ensures that $\Pi(x)$ is continuous. We claim that it is actually of class C^1. Indeed, using again the Implicit Function Theorem,

$$\begin{aligned}\frac{\partial}{\partial t}\zeta_2(t; (x, 0))\Big|_{t=\Pi(x)} &= -\partial_1 W(\zeta(\Pi(x); (x, 0))) \\ &= -\frac{\langle \nabla W(\zeta(\Pi(x); (x, 0))) | \zeta(\Pi(x); (x, 0))\rangle}{\zeta_1(\Pi(x); (x, 0))} \neq 0.\end{aligned}$$

Since $\tau(u) = \Pi(\xi(W(u)))$, for $u \in \mathbb{R}^2_*$, we conclude. □

Assume now that the origin is an *isochronous* center, i.e., there exists $\tau > 0$ such that

$$\tau(u) = \tau, \qquad \text{for every } u \neq 0. \tag{3.13}$$

In this case, we define $\theta(u) \in [0, 2\pi[$ as the minimum time for which

$$\zeta\left(-\frac{\tau}{2\pi}\theta(u); u\right) \in \mathbb{R}^+_* \times \{0\}.$$

Let us first briefly analyze some regularity issues about the function $\theta(u)$.

Proposition 3.2.2. *The following facts hold:*

(i) the map $\theta : \mathbb{R}^2_* \setminus (\mathbb{R}^+_* \times \{0\}) \to \mathbb{R}$ *is of class* C^1 *and* $\nabla\theta(u)$ *extends to a continuous function on the whole* $\mathbb{R}^2_*$ *(which we still denote by* $\nabla\theta(u)$*); moreover, if* $u(t)$ *is a solution to* (3.12)*, it holds that*

$$\Big(\langle J\nabla\theta(u(t)) | \nabla W(u(t)) \rangle = \Big) \frac{d}{dt}\theta(u(t)) = \frac{2\pi}{\tau}; \tag{3.14}$$

(ii) the map $\mathbb{R}^2_* \ni u \mapsto (\cos\theta(u), -\sin\theta(u))$ *is of class* C^1*.*

Proof. The two assertions basically follow from the geometrical interpretation of $\theta(u)$ and the regularity of $\zeta(t;u)$, together with the Implicit Function Theorem. However, we will briefly examine each case.

Concerning (i), let us first fix $u^* \in \mathbb{R}^2_* \setminus (\mathbb{R}^+_* \times \{0\})$, and observe that the continuity of $\theta(u)$ in u^* follows from the continuity of $\zeta(t;u)$. We are going to show that $\theta(u)$ is of class C^1 in a neighborhood of u^*. By the Implicit Function Theorem, repeating a similar argument as before, we deduce that there exist:

- a neighborhood $\mathcal{U}$ of u^* and a neighborhood $\mathcal{T}$ of $\frac{\tau}{2\pi}\theta(u^*)$;
- a C^1-map $t : \mathcal{U} \to \mathcal{T}$,

such that

$$\zeta_2(-t;u) = 0, \quad (t,u) \in \mathcal{T} \times \mathcal{U} \quad \Longleftrightarrow \quad t = t(u). \tag{3.15}$$

Since $\theta(u)$ is continuous at u^*, we get $t(u) = \frac{\tau}{2\pi}\theta(u)$ for u in a neighborhood of u^*, getting the desired conclusion.
On the other hand, if $u_0 \in \mathbb{R}^+_* \times \{0\}$, one can construct locally the function $t(u)$ satisfying (3.15) as before. By the definition of $\theta(u) \in [0, 2\pi[$, it is then possible to infer that, for $u = (u_1, u_2)$ in a neighborhood of u_0,

$$\theta(u) = \begin{cases} \dfrac{2\pi}{\tau} t(u) & \text{for} \quad u_2 < 0 \\[2ex] \dfrac{2\pi}{\tau}(t(u) + \tau) & \text{for} \quad u_2 \geq 0. \end{cases} \tag{3.16}$$

From (3.16), we deduce both the fact that $\nabla\theta(u)$ extends to a continuous function on the whole $\mathbb{R}^2_*$ and the fact that the map $\mathbb{R}^2_* \ni u \mapsto (\cos\theta(u), -\sin\theta(u))$ is continuous.

We now discuss point (ii) concerning the differentiability of $u \mapsto (\cos\theta(u), -\sin\theta(u))$. Of course, we only need to focus on a point $u_0 \in \mathbb{R}^+_* \times \{0\}$; for simplicity, moreover, we just consider the map $u \mapsto \cos\theta(u)$.
The existence in u_0 and the continuity in a neighborhood of u_0 are ensured for

$\partial_2 \cos(\theta(u))$, since $\nabla\theta(u)$ exists out of $\mathbb{R}^+_* \times \{0\}$ and can be extended to the whole $\mathbb{R}^2_*$. For what concerns $\partial_1 \cos\theta(u)$, we have, since $\theta(u) \equiv 0$ on $\mathbb{R}^+_* \times \{0\}$,

$$\partial_1 \cos\theta(u)\big|_{u=u_0} = \lim_{\delta\to 0} \frac{\cos\theta(u_0 + (\delta, 0)) - \cos\theta(u_0)}{\delta} = 0,$$

so that the existence in u_0 is guaranteed. As for the continuity in a neighborhood of u_0, observe that, for $u \notin \mathbb{R}^+_* \times \{0\}$, one has

$$\begin{aligned}\lim_{u\to u_0} \partial_1 \cos\theta(u) &= -\lim_{u\to u_0} \sin\theta(u)\partial_1\theta(u) \\ &= -\frac{2\pi}{\tau} \lim_{u\to u_0} \sin\theta(u)\partial_1 t(u) = -\frac{2\pi}{\tau} \sin\theta(u_0)\partial_1 t(u_0).\end{aligned}$$

The conclusion follows from the fact that, since $t(u) \equiv 0$ on $\mathbb{R}^+_* \times \{0\}$, it holds $\langle \nabla t(u_0) | u_0 \rangle = 0$. $\square$

After these preliminary considerations, let us state the following proposition, dealing with the existence of a symplectic diffeomorphism of $\mathbb{R}^2$ into itself, changing the original system into a linear one, in the case when the origin is an isochronous center.

Proposition 3.2.3. *Let $W(u)$ satisfy (W0)–(W3). Assume that there exist $\tau, r > 0$ such that* (3.13) *holds and*

$$W(u) = \frac{\pi}{\tau}|u|^2, \quad \text{for every } |u| < r. \tag{3.17}$$

Then, there exists a symplectic C^1-diffeomorphism Λ_W of $\mathbb{R}^2$ onto itself such that

$$W(\Lambda_W^{-1}(v)) = \frac{\pi}{\tau}|v|^2, \quad \text{for every } v \in \mathbb{R}^2. \tag{3.18}$$

Let us observe that, if (3.17) holds, all the solutions $u(t)$ to (3.12) such that $|u(t)| < r$ are of the type $\lambda(\cos(\frac{2\pi}{\tau}(t+\theta)), -\sin(\frac{2\pi}{\tau}(t+\theta)))$, for suitable constants $\lambda, \theta > 0$, so that they are periodic with minimal period equal to τ. Hence, (3.13) and (3.17) are not contradictory.
Geometrically, (3.18) means that the level curves of $W(u)$ are transformed, through Λ_W, into circumferences around the origin, so that the nonlinear system (3.12) is changed into the linear one

$$Jv' = \frac{2\pi}{\tau} v.$$

We now prove Proposition 3.2.3.

Proof. Let us define the map $\Lambda_W : \mathbb{R}^2_* \to \mathbb{R}^2$ as

$$\Lambda_W(u) = \sqrt{\frac{\tau}{\pi} W(u)}(\cos\theta(u), -\sin\theta(u)). \tag{3.19}$$

In view of Proposition 3.2.2, Λ_W is of class C^1; moreover, using (3.17), it is easy to see that $\Lambda_W(u) = u$ for every $|u| < r$, so that Λ_W extends (setting $\Lambda_W(0) = 0$) to a C^1 function on the whole $\mathbb{R}^2$, still denoted in the same way. Second, a simple calculation yields, in view of (3.14),

$$\det \Lambda'_W(u) = \frac{\tau}{2\pi}\langle J\nabla\theta(u)|\nabla W(u)\rangle = 1, \tag{3.20}$$

which, by a direct computation, implies (3.11). Third, the fact that Λ_W is a C^1-diffeomorphism follows from the Hadamard-Caccioppoli global inversion Theorem. Indeed, (3.20) implies that Λ_W is C^1-locally invertible; moreover, Λ_W is also a proper map (i.e., the preimage of compact sets is compact), since $|\Lambda_W(u)| \to +\infty$ for $|u| \to +\infty$, in view of (W3).
Finally, relation (3.18) follows from (3.19) - just taking the modulus and setting $u = \Lambda_W^{-1}(v)$. □

Remark 3.2.4. We stress that assumption (3.17) is crucial, since it is needed to guarantee that Λ_W is of class C^1 up to the origin; this is not the case for a general function $W(u)$ satisfying (3.13). To show this, consider, for example, a function $W \in \mathcal{P}$. In this case, it can be seen - noticing that $\theta(u)$ is positively homogeneous of degree 0 - that Λ_W is positively homogeneous of degree 1, so that Λ'_W is constant on every ray emanating from the origin. Accordingly, Λ'_W is not continuous, except when Λ'_W is constant, i.e., Λ_W linear. However, $\Lambda_W(u) = Au$ for a square matrix A implies, using (3.19),

$$W(u) = \frac{\pi}{\tau}|\Lambda_W(u)|^2 = \frac{\pi}{\tau}\langle A^t Au|u\rangle,$$

namely $W(u)$ is a (positive definite) quadratic form, which is not in general the case. We finally remark that, when only (3.13) is assumed, Λ_W is a symplectic diffeomorphism of $\mathbb{R}^2_*$ onto itself, as it can be seen by slightly different arguments (see [63, 84] for a guideline).

3.3 The planar version of the Ahmad-Lazer-Paul condition

Here is the statement of the main result of the chapter.

Theorem 3.3.1. *Let $V \in \mathcal{P}_k \cap C^2(\mathbb{R}^2_*)$, $k \geq 1$, and let $Q \in C^1([0,T] \times \mathbb{R}^2)$ fulfill, for suitable constants $M > 0$, $\alpha \in [0,1[$, the growth condition*

$$|\nabla Q(t,u)| \leq M(1 + |u|^\alpha), \qquad \text{for every } t \in [0,T],\ u \in \mathbb{R}^2. \tag{3.21}$$

Moreover, suppose that

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda^{2\alpha}} \int_0^T Q(t, \lambda\varphi_V(t+\theta))\, dt = +\infty, \qquad \text{uniformly in } \theta \in [0, \tau_V[\ , \tag{3.22}$$

where $\varphi_V(t)$ denotes the solution to $Ju' = \nabla V(u)$ such that $\varphi_V(0) = (1,0)$, Then, system (3.10) has a T-periodic solution.

Recall that, by Proposition 1.1.9, the family

$$\{\lambda\varphi_V(\cdot + \theta) \mid \lambda > 0,\ \theta \in [0, \tau_V[\,\}$$

gives exactly the set of nontrivial solutions to $Ju' = \nabla V(u)$.

Proof. The main ingredient of the proof of Theorem 3.3.1 consists in transforming system (3.10), via the symplectic change of variables described in Section 3.2, into a perturbation of a *linear* one. However, since $V(u)$ does not satisfy (3.17) - unless $V(u) = \frac{\pi}{\tau_V}|u|^2$ for every $u \in \mathbb{R}^2$ - we need the following preliminary trick.

Fix $0 < r_1 < r_2$, and $\epsilon > 0$ such that

$$\epsilon|u|^2 \leq V(u), \qquad \text{for every } r_1 \leq |u| \leq r_2. \tag{3.23}$$

Moreover, choose a regular nondecreasing function $\beta : [0, +\infty[\to \mathbb{R}$ such that

- $\beta(x) = 0$ for every $x \leq r_1^2$ and $\beta(x) = 1$ for every $x \geq r_2^2$;
- $0 < \beta(x) < 1$ for every $r_1^2 < x < r_2^2$.

Now, define $W^* : \mathbb{R}^2 \to \mathbb{R}$ as

$$W^*(u) = (1 - \beta(|u|^2))\epsilon|u|^2 + \beta(|u|^2)V(u).$$

We claim that $W^*(u)$ satisfies (W0)–(W3). Indeed, (W0), (W1) and (W3) are straightly proved, while (W2) follows, in view of (3.23), from the fact that

$$\begin{aligned}\langle \nabla W^*(u)|u\rangle &= 2\beta'(|u|^2)|u|^2[V(u) - \epsilon|u|^2] + 2\epsilon(1 - \beta(|u|^2))|u|^2 \\ &\quad + \beta(|u|^2)\langle \nabla V(u)|u\rangle.\end{aligned}$$

Hence, in view of the discussion after Proposition 1.1.8, the origin is a center for $Ju' = \nabla W^*(u)$, and we denote by $\tau_{W^*}(u)$ the minimal period of the solutions to such a system which pass through u. Since, in general, $\tau_{W^*}(u)$ is not constant, in order to apply Proposition 3.2.3 we need a further modification.

As in the proof of Proposition 3.2.1, for $c \in \mathbb{R}^+_*$ we define $\xi(c)$ as the unique positive number such that $W^*(\xi(c), 0) = c$, i.e. $(\xi(c), 0)$ is the intersection between the level curve $\{u \in \mathbb{R}^2 \mid W^*(u) = c\}$ and the positive x-semiaxis; the map $c \mapsto \xi(c)$ is clearly continuous. We define

$$W(u) = \frac{1}{\tau_V} \int_0^{W^*(u)} \tau_{W^*}(\xi(c), 0)\, dc;$$

by construction we have

$$\nabla W(u) = \frac{\tau_{W^*}(\xi(W^*(u)), 0)}{\tau_V} \nabla W^*(u) = \frac{\tau_{W^*}(u)}{\tau_V} \nabla W^*(u).$$

Since, as remarked before Proposition 3.2.3, $u \mapsto \tau_{W^*}(u)$ is of class C^1 on $\mathbb{R}^2_*$, we have that $W(u)$ satisfies (W0). On the other hand, (W1)–(W3) are easily proved and, by construction, all the nontrivial solutions to $Ju' = \nabla W(u)$ have minimal period τ_V (cf. (3.13)). Moreover, for $|u| < r_1$, we see that

$$W(u) = \frac{1}{\tau_V} \int_0^{\epsilon |u|^2} \tau_{W^*}\left(\sqrt{\frac{c}{\epsilon}}, 0\right) dc = \frac{1}{\tau_V} \int_0^{\epsilon |u|^2} \frac{\pi}{\epsilon}\, dc = \frac{\pi}{\tau_V} |u|^2,$$

so that $W(u)$ fulfills (3.17) with $r = r_1$, as well. Accordingly, we can apply Proposition 3.2.3 to produce the symplectic diffeomorphism Λ_W of the plane onto itself.

For further convenience, observe that $W(u) = V(u)$ for $|u| > r_2$, so that there exists $r^* > 0$ such that $\Lambda_W^{-1}(v) = \Lambda_V^{-1}(v)$ for $|v| \geq r^*$. Indeed, it suffices to take $\varphi^*(t)$ solving $Ju' = \nabla W(u)$, with $|\varphi^*(t)| \geq r_2$ for every $t \in [0, \tau_V]$ (thus solving also $Ju' = \nabla V(u)$) and set $r^* = \sqrt{\frac{\pi}{\tau_V} W(\varphi^*(t))}$. This fact has two important consequences:

- it holds
$$\Lambda_W^{-1}(\lambda v) = \lambda \Lambda_W^{-1}(v), \quad \text{for every } |v| \geq r^*,\ \lambda > 1; \tag{3.24}$$

- we have
$$\|(\Lambda_W^{-1})'(v)\| \leq L, \quad \text{for every } v \in \mathbb{R}^2, \tag{3.25}$$
for a suitable constant $L > 0$.

Both claims follow from the fact that $\Lambda_V(u)$ is positively homogeneous of degree 1, since $\theta(u)$ is homogeneous of degree 0 (cf. Remark 3.2.4).

For $u \in \mathbb{R}^2$, we now set $S(u) = V(u) - W(u)$; moreover, we define, for $v \in \mathbb{R}^2$ and $t \in [0, T]$,

$$\widetilde{W}(v) = W(\Lambda_W^{-1}(v)), \quad \widetilde{S}(v) = S(\Lambda_W^{-1}(v)), \quad \widetilde{Q}(t, v) = Q(t, \Lambda_W^{-1}(v)).$$

With these positions, system (3.10) is changed, via Λ_W, into

$$Jv' = \nabla\widetilde{W}(v) + \nabla\widetilde{S}(v) + \nabla\widetilde{Q}(t,v),$$

that is, using (3.18) and the fact that $V \in \mathcal{P}_k$,

$$Jv' = \frac{2k\pi}{T}v + \nabla\widetilde{S}(v) + \nabla\widetilde{Q}(t,v).$$

We now claim that we are in the setting of Theorem 3.1.2; precisely,

- there exists $\widetilde{M} > 0$ such that, for every $t \in [0,T]$ and every $v \in \mathbb{R}^2$,
$$|\nabla\widetilde{Q}(t,v) + \nabla\widetilde{S}(v)| \le \widetilde{M}(1 + |v|^\alpha); \tag{3.26}$$
- denoting by $\mathfrak{S}$ the set of the T-periodic solutions to $Jv' = \frac{2k\pi}{T}v$, it holds that
$$\lim_{\substack{\|v\|_\infty\to+\infty\\ v\in\mathfrak{S}}} \frac{\int_0^T [\widetilde{Q}(t,v(t)) + \widetilde{S}(v(t))]\,dt}{\|v\|_\infty^{2\alpha}} = +\infty.$$

For the first claim, we observe preliminarily that, applying the Mean Value Theorem for C^1-maps on convex subsets of $\mathbb{R}^2$ to Λ_W^{-1}, (3.25) and $\Lambda_W^{-1}(0) = 0$ imply

$$|\Lambda_W^{-1}(v)| \le L|v|, \qquad \text{for every } v \in \mathbb{R}^2. \tag{3.27}$$

From (3.21), (3.25) and (3.27), we obtain

$$\begin{aligned}|\nabla\widetilde{Q}(t,v)| &= |\,[(\Lambda_W^{-1})'(v)]^t \nabla Q(t,\Lambda_W^{-1}(v))| \le L|\nabla Q(t,\Lambda_W^{-1}(v))| \\ &\le LM(1 + |\Lambda_W^{-1}(v)|^\alpha) \le LM(1 + L^\alpha|v|^\alpha),\end{aligned}$$

proving the claim since $\nabla\widetilde{S}(v)$ is bounded (indeed, $S(u) = 0$ for $|u| > r_2$).
We now prove the second claim. Again in view of the boundedness of $\widetilde{S}(v)$, it is equivalent to show that

$$\lim_{\substack{\|v\|_\infty\to+\infty\\ v\in\mathfrak{S}}} \frac{\int_0^T \widetilde{Q}(t,v(t))\,dt}{\|v\|_\infty^{2\alpha}} = +\infty. \tag{3.28}$$

To this aim, notice that

$$v(t) \in \mathfrak{S} \iff v(t) = \widetilde{\lambda}\psi(t + \widetilde{\theta}),$$

for suitable positive constants $\widetilde{\lambda} > 0$, $\widetilde{\theta} \in [0, T/k[$, where

$$\psi(t) = \left(\cos\left(\frac{2k\pi}{T}t\right), -\sin\left(\frac{2k\pi}{T}t\right)\right).$$

In particular, it turns out that $\|v\|_\infty = \widetilde{\lambda}$, so that (3.28) is equivalent to

$$\lim_{\widetilde{\lambda}\to+\infty} \frac{1}{\widetilde{\lambda}^{2\alpha}} \int_0^T \widetilde{Q}(t, \widetilde{\lambda}\psi(t+\widetilde{\theta}))\,dt = +\infty, \qquad \text{uniformly in } \widetilde{\theta} \in [0, \tau_V[.$$

We now observe that, in view of (3.24) and the position of r^*, for $\widetilde{\lambda} \geq r^*$ we have

$$\begin{aligned} \Lambda_W^{-1}(\widetilde{\lambda}\psi(t+\widetilde{\theta})) &= \Lambda_W^{-1}\left(\frac{\widetilde{\lambda}}{r^*} r^* \psi(t+\widetilde{\theta})\right) \\ &= \frac{\widetilde{\lambda}}{r^*}\Lambda_W^{-1}(r^*\psi(t+\widetilde{\theta})) = \frac{\widetilde{\lambda}}{r^*}\varphi^*(t+\theta^*(\widetilde{\theta})). \end{aligned}$$

Since, as remarked before, $\varphi^*(t)$ solves $Ju' = \nabla V(u)$, there exist $\rho > 0$ and $\theta \in [0, \tau_V[$ (depending on $\varphi^*(t)$ and $\theta^*(\widetilde{\theta})$) such that

$$\varphi^*(t+\theta^*(\widetilde{\theta})) = \rho\varphi_V(t+\theta).$$

Summing up, we have

$$\begin{aligned} \frac{1}{\widetilde{\lambda}^{2\alpha}} \int_0^T \widetilde{Q}(t, \widetilde{\lambda}\psi(t+\widetilde{\theta}))\,dt &= \frac{1}{\widetilde{\lambda}^{2\alpha}} \int_0^T Q(t, \Lambda_H^{-1}(\widetilde{\lambda}\psi(t+\widetilde{\theta})))\,dt \\ &= \frac{1}{\widetilde{\lambda}^{2\alpha}} \int_0^T Q\Big(t, \widetilde{\lambda}\frac{\rho}{r^*}\varphi_V(t+\theta)\Big)\,dt, \end{aligned}$$

so that we conclude in view of (3.22). $\square$

A couple of remarks about Theorem 3.3.1 are now in order.

Remark 3.3.2. When $\alpha = 0$ in (3.21), i.e., when $\nabla Q(t,u)$ is bounded, (3.22) reads as

$$\lim_{\lambda\to+\infty} \int_0^T Q(t, \lambda\varphi_V(t+\theta))\,dt = +\infty, \qquad \text{uniformly in } \theta \in [0, \tau_V[\,. \tag{3.29}$$

Bounded perturbations of resonant problems represent the setting where the Ahmad-Lazer-Paul condition was originally introduced [1, 112].
We also point out that, according to [74, Theorem 1.1], the conclusion of Theorem 3.3.1 still holds true if (3.22) is replaced by

$$\lim_{\lambda\to+\infty} \frac{1}{\lambda^{2\alpha}} \int_0^T Q(t, \lambda\varphi_V(t+\theta))\,dt = -\infty, \qquad \text{uniformly in } \theta \in [0, \tau_V[\,.$$

Remark 3.3.3. Our choice to consider a positively homogeneous Hamiltonian of degree 2 in Theorem 3.3.1 is mainly motivated by the fact that, in this setting, (3.25) holds true. As a consequence, one easily gets relation (3.26), assuming the corresponding bound (3.21) for the growth of $\nabla Q(t,u)$.
However, some generalizations to perturbations of other isochronous centers are possible. For instance, as in [84], one can consider the scalar p-Laplacian equation, $p>1$,

$$(|x'|^{p-2}x')' + \mu|x|^{p-2}x^+ - \nu|x|^{p-2}x^- + r(t,x) = 0,$$

whose associated Hamiltonian $V(x,y) = \frac{1}{p}(\mu(x^+)^p + \nu(x^-)^p) + \frac{1}{q}|y|^q$ (with q given by $\frac{1}{p}+\frac{1}{q}=1$) is not positively homogeneous for $p\neq 2$, but gives birth to an isochronous center. In this case, even if (3.25) is not fulfilled, a suitable growth assumption on $r(t,x)$, depending on p, ensures the validity of (3.26), with $\alpha=0$.

Remark 3.3.4. It would be interesting to consider the case when $V\in\mathcal{P}_0$, as well. However, in this situation there is lack of suitable coordinates allowing to transform the problem into a linear one, since it is not possible to produce the analogous of Proposition 3.2.3. We are currently trying to understand if this situation can be studied in other ways, to reach a statement similar to the one of Theorem 3.3.1.

3.4 Some examples

Theorem 3.3.1 contains some results previously achieved in literature, dealing with the scalar second order equation

$$x'' + \mu x^+ - \nu x^- + r(t,x) = 0, \tag{3.30}$$

where the couple of positive parameters (μ,ν) belongs to the T-periodic Fučík spectrum, i.e., it satisfies, for some positive integer k, the equality

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{k}. \tag{3.31}$$

In particular, we have the following corollary, improving [9] (where $r(t,x)$ is bounded and (μ,ν) is "not too far" from the diagonal) and [84] (where $r(t,x)$ is bounded; cf. Remark 3.3.3).

Corollary 3.4.1. *Let $\mu,\nu>0$ as above and let $r\in C([0,T]\times\mathbb{R})$ fulfill, for suitable constants $M>0$, $\alpha\in[0,1[$, the growth condition*

$$|r(t,x)| \leq M(1+|x|^\alpha), \qquad \text{for every } t\in[0,T],\ x\in\mathbb{R}. \tag{3.32}$$

Moreover, setting

$$\phi(t) = \begin{cases} \dfrac{1}{\sqrt{\mu}} \sin(\sqrt{\mu} t) & \textit{if} \quad t \in \left[0, \frac{\pi}{\sqrt{\mu}}\right] \\ \dfrac{1}{\sqrt{\nu}} \sin\left(\sqrt{\nu}\Big(\dfrac{\pi}{\sqrt{\mu}} - t\Big)\right) & \textit{if} \quad t \in \left[\frac{\pi}{\sqrt{\mu}}, \frac{T}{k}\right], \end{cases}$$

and still denoting by $\phi(t)$ its $\frac{T}{k}$-periodic extension, suppose that

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda^{2\alpha}} \int_0^T \mathcal{R}(t, \lambda\phi(t+\theta))\, dt = +\infty, \qquad \textit{unif. in } \theta \in \left[0, \frac{T}{k}\right[, \tag{3.33}$$

where $\mathcal{R}(t,x) = \int_0^x r(t,\xi)\, d\xi$. Then, equation (3.30) has a T-periodic solution.

Proof. Set $u = (x, y)$,

$$V(u) = \frac{1}{2}(\mu(x^+)^2 + \nu(x^-)^2 + y^2), \quad Q(t,u) = \mathcal{R}(t,x),$$

and

$$\varphi_V(t) = \sqrt{\mu}\left(\phi\Big(t + \frac{\pi}{2\sqrt{\mu}}\Big), \phi'\Big(t + \frac{\pi}{2\sqrt{\mu}}\Big)\right).$$

The thesis follows plainly from Theorem 3.3.1, observing that (3.31) implies that $V \in \mathcal{P}_k$, (3.32) implies (3.21) and (3.33) implies (3.22). □

This situation can be extended to the perturbed "bi-asymmetric oscillator"

$$\begin{cases} x' &= \mu_1 y^+ - \nu_1 y^- + r_1(t,y) \\ y' &= -\mu_2 x^+ + \nu_2 x^- - r_2(t,x), \end{cases} \tag{3.34}$$

being

- $\mu_1, \mu_2, \nu_1, \nu_2 > 0$ such that, for a positive integer k,

$$\frac{\pi}{2}\left(\frac{1}{\sqrt{\mu_1\mu_2}} + \frac{1}{\sqrt{\mu_1\nu_2}} + \frac{1}{\sqrt{\nu_1\nu_2}} + \frac{1}{\sqrt{\mu_2\nu_1}}\right) = \frac{T}{k}; \tag{3.35}$$

- $r_1, r_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ continuous functions satisfying, for suitable constants $M > 0$, $\alpha \in [0, 1[$,

$$|r_i(t,x)| \le M(1 + |x|^\alpha), \qquad \text{for every } t \in [0,T],\ x \in \mathbb{R},\ i = 1,2.$$

System (3.34) (cf. system (2.62) in Section 2.5) was already considered, for instance, in [135], including the asymmetric equation (3.4) (for $\mu_1 = \nu_1 = 1$, $q_1(t,y) \equiv 0$). When (3.35) is fulfilled, every nontrivial solution to the autonomous system

$$\begin{cases} x' &= \mu_1 y^+ - \nu_1 y^- \\ y' &= -\mu_2 x^+ + \nu_2 x^- \end{cases}$$

is T-periodic; fixed the solution $\varphi_V(t) = (\varphi_1(t), \varphi_2(t))$ such that $\varphi_V(0) = (1,0)$, every other (nontrivial) one has the form $\lambda\varphi_V(t+\theta)$ for suitable constants $\lambda > 0$, $\theta \in [0, \frac{T}{k}[$. Hence, according to Theorem 3.3.1, the planar system (3.34) has a T-periodic solution if the following condition is fulfilled:

uniformly in $\theta \in [0, \frac{T}{k}[$, *it holds*

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda^{2\alpha}} \int_0^T \big[\mathcal{R}_1(t, \lambda\varphi_1(t+\theta)) + \mathcal{R}_2(t, \lambda\varphi_2(t+\theta))\big]\, dt = +\infty,$$

being $\mathcal{R}_i(t,z) = \int_0^z r_i(t,\xi)\, d\xi$, $i = 1, 2$.

Chapter 4

Comparing Landesman-Lazer and Ahmad-Lazer-Paul conditions

In this chapter, we will analyze in details the relationships between the Landesman-Lazer condition and the Ahmad-Lazer-Paul one. The existence of a strong connection between these two conditions was already pointed out in the original work by Ahmad, Lazer and Paul (see also [107]), who considered the semilinear problem

$$Lu = g(x, u), \quad x \in \Omega.$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary, the operator $L : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ is a linear self-adjoint operator with compact resolvent, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^2-bounded function, i.e., there exists $\eta \in L^2(\Omega)$ such that

$$|g(x, s)| \leq \eta(x),$$

for almost every $x \in \Omega$ and every $s \in \mathbb{R}$. The Landesman-Lazer condition reads then as follows:

(LL) for every $v \in \ker L \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s\to+\infty} g(x, s)v(x)\, dx + \int_{\{v<0\}} \limsup_{s\to-\infty} g(x, s)v(x)\, dx > 0,$$

while, denoting by G a primitive of g in the second variable, i.e.,

$$G(x, s) = \int_0^s g(x, \xi)\, d\xi,$$

the Ahmad-Lazer-Paul condition can be written as

(ALP) as long as $v \in \ker L$,

$$\lim_{\|v\|_2 \to +\infty} \int_\Omega G(x, v(x))\, dx = +\infty.$$

In [1], it was shown that, in the case when g depends only on s, the limits $\lim_{s\to\pm\infty} g(s)$ exist and $\dim \ker L = 1$, then (LL) implies (ALP). Under these assumptions, indeed, writing $\ker L = \langle \phi_L \rangle$, we can use Fatou's Lemma to obtain

$$\liminf_{s\to+\infty} \frac{1}{s} \int_\Omega G(s\phi_L(x))\, dx \geq \int_\Omega \liminf_{s\to+\infty} \frac{G(s\phi_L(x))}{s}\, dx =$$

$$= \int_{\{\phi_L>0\}} g(+\infty)\phi_L(x)\, dx + \int_{\{\phi_L<0\}} g(-\infty)\phi_L(x)\, dx > 0,$$

thanks to (LL). It follows, for $v \in \ker L$,

$$\lim_{\|v\|_2\to+\infty} \int_\Omega G(v(x))\, dx = \lim_{s\to+\infty} \int_\Omega G(s\phi_L(x))\, dx = +\infty.$$

The general case is more subtle, and requires some careful use of elementary analysis. As a first step, we will provide a characterization of both the scalar and the planar Landesman-Lazer conditions, which will then be used to prove the desired implication. However, we think that Propositions 4.1.3 and 4.2.2 below could be independently of some interest, particularly when required to check the validity of the Landesman-Lazer condition.

We will split our discussion into two sections. The first one, essentially taken from [57], deals with the scalar case, in an abstract setting. We will work in a general measure space, without any topological requirement, and avoid considering any differential problem, focusing only on the nonresonance conditions. For this reason, we will not need a linear subspace as the kernel of L, but just a cone Σ, with some compactness properties. As possible examples, the nonlinear asymmetric oscillator, the Dirichlet (or Neumann) problem for an elliptic equation like $\Delta u + \lambda u = g(x,u)$, where λ is an eigenvalue of the differential operator $-\Delta$, or more general equations involving the p-Laplacian will be included in our setting. However, some care could be recommended in this last case, since the spectral properties of the p-Laplacian are not completely established yet (see, e.g., [8, 41, 42], and the references therein). In principle, boundary value problems associated to hyperbolic equations fit in our framework, as well. However, in this case we do not know about existence results under these general assumptions (see, however, [5, 115]).

On the other hand, the second section is devoted to the analysis of the above connection in the setting of planar systems, providing as well a characterization which is slightly different from the one for the scalar framework, and will be briefly compared with it at the end of the chapter.

4.1 The general implication for scalar equations

Let (Ω, μ) be a σ-finite measure space (in the applications, Ω is usually an open subset of $\mathbb{R}^n$ with the standard Lebesgue measure). We will briefly write "measurable" in place of μ-measurable, and $L^q(\Omega)$ instead of $L^q(\Omega, d\mu)$. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function; for the reader's convenience, we recall that this means that

- $x \mapsto g(x, s)$ is measurable for every $s \in \mathbb{R}$;
- $s \mapsto g(x, s)$ is continuous for almost every $x \in \Omega$;
- for every $R > 0$, there exists $\eta_R \in L^1(\Omega)$ such that, for almost every $x \in \Omega$, and every $s \in \mathbb{R}$ with $|s| \le R$,
$$|g(x, s)| \le \eta_R(x).$$

Moreover, let $p, q \in [1, +\infty]$ be conjugate exponents, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and assume that there exist $d > 0$ and a nonnegative function $h \in L^q(\Omega)$ such that, for almost every $x \in \Omega$,

$$|s| \ge d \quad \Rightarrow \quad \operatorname{sgn}(s) g(x, s) \ge -h(x). \tag{4.1}$$

Let $\Sigma \subset L^p(\Omega)$ satisfy the following properties:

- if $u \in \Sigma$, and $\lambda > 0$, then $\lambda u \in \Sigma$;
- $\Sigma \cap S_1$ is compact in $L^p(\Omega)$, where $S_1 = \{u \in L^p(\Omega) \mid \|u\|_p = 1\}$.

Lastly, set $G(x, s) = \int_0^s g(x, \xi)\, d\xi$. We consider the following two conditions:

(LL) for every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s \to +\infty} g(x, s) v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s \to -\infty} g(x, s) v(x)\, d\mu > 0;$$

(ALP) as long as $v \in \Sigma$,

$$\lim_{\|v\|_p \to +\infty} \int_\Omega G(x, v(x))\, d\mu = +\infty.$$

We are now going to prove the following statement:

Theorem 4.1.1. *(LL) implies (ALP).*

A couple of remarks are in order. Notice that, if λ is the first eigenvalue, our assumptions are known to be fulfilled, since $\Sigma \cap S_1$ is a finite set. In the above statement, moreover, we do not need any growth assumption on g - which, however, are usually necessary to prove existence results - other than (4.1), and, concerning the applications, Ω has not necessarily to be a bounded subset of $\mathbb{R}^n$. Hence, we can also deal with problems on unbounded domains, topic which has been studied by several authors in the recent years, mainly using variational methods, yielding existence results by means of both Landesman-Lazer (see for instance [2, 101]) and Ahmad-Lazer-Paul conditions (see, e.g., [83, 100]). Our theorem could be useful in these cases, since it seems easier to check if (LL) holds, rather than (ALP).
Lastly, notice that a symmetric result with respect to Theorem 4.1.1 can be stated if we take into account the following two conditions:

(LL′) for every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \limsup_{s\to+\infty} g(x,s)v(x)\,d\mu + \int_{\{v<0\}} \liminf_{s\to-\infty} g(x,s)v(x)\,d\mu < 0,$$

(ALP′) as long as $v \in \Sigma$,

$$\lim_{\|v\|_p\to+\infty} \int_\Omega G(x,v(x))\,d\mu = -\infty.$$

Let us give some preliminaries for the proof of Theorem 4.1.1. First of all, notice that condition (4.1) guarantees that the integrals appearing in (LL) are both well defined, with values in $\mathbb{R} \cup \{+\infty\}$. Along the proof of Theorem 4.1.1, we will show that, in this setting, the same is true for the integral appearing in (ALP). On the other hand, since Ω is σ-finite, there exists a family $\{K_m\}_{m\in\mathbb{N}}$ of measurable subsets of Ω such that

- $\mu(K_m) < +\infty$, for every $m \in \mathbb{N}$;
- $K_m \subset K_{m+1}$, for every $m \in \mathbb{N}$;
- $\cup_{m\in\mathbb{N}} K_m = \Omega$.

Thus, for every $m \in \mathbb{N}$ we can define the truncation function $\zeta_m : \Omega \to \mathbb{R}$ by

$$\zeta_m(x) = \begin{cases} m & \text{if } x \in K_m \\ 0 & \text{if } x \in \Omega \setminus K_m; \end{cases}$$

it is worth noticing that, for every $m \in \mathbb{N}$, ζ_m belongs to $L^q(\Omega)$, for every $q \geq 1$.
The following lemma says that, if the Landesman-Lazer condition is satisfied by g, then it is satisfied also by some suitable truncation of g.

Lemma 4.1.2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy condition (LL). Then, setting*

$$g_m(x,s) = \begin{cases} \min\{g(x,s), \zeta_m(x)\} & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ \max\{g(x,s), -\zeta_m(x)\} & \text{if } s < 0, \end{cases}$$

there exists $\bar{m} \in \mathbb{N}$ such that, for every $m \geq \bar{m}$ and every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s\to+\infty} g_m(x,s)\, v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g_m(x,s) v(x)\, d\mu > 0. \tag{4.2}$$

Proof. It suffices to prove the statement for every $v \in \Sigma \cap S_1$, since the left-hand side in (4.2) is positively homogeneous of degree 1 with respect to v. Consequently, we will assume $\|v\|_p = 1$.
Since $g(x,s) = \lim_{m\to+\infty} g_m(x,s)$ for almost every $x \in \Omega$ and every $s \in \mathbb{R} \setminus \{0\}$, this limit being monotone (increasing for $s > 0$, decreasing for $s < 0$), we can rewrite condition (LL) as

$$\int_{\{v>0\}} \Big(\liminf_{s\to+\infty} \lim_{m\to+\infty} g_m(x,s)\Big) v(x)\, d\mu + \\ + \int_{\{v<0\}} \Big(\limsup_{s\to-\infty} \lim_{m\to+\infty} g_m(x,s)\Big) v(x)\, d\mu > 0. \tag{4.3}$$

We show that it is possible to exchange the inferior limit and the limit under the first integral. First of all, since $g_m(x,s) \leq g(x,s)$ for almost every $x \in \Omega$ and every $s > 0$, it follows easily that

$$\lim_{m\to+\infty} \liminf_{s\to+\infty} g_m(x,s) \leq \liminf_{s\to+\infty} \lim_{m\to+\infty} g_m(x,s).$$

On the other hand, after having observed that $\liminf_{s\to+\infty} g(x,s) > -\infty$ for almost every $x \in \Omega$, thanks to (4.1), we have to consider the two following cases.

- if $x \in \Omega$ is such that $\liminf_{s\to+\infty} g(x,s) = +\infty$, then, fixed $K > 0$, there exists s_K such that, if $s \geq s_K$, then $g(x,s) \geq K$. Moreover, there exists $m_x \in \mathbb{N}$ such that $x \in K_m$ for every $m \geq m_x$. For every $m \geq \max\{K, m_x\}$, then, it will be $g_m(x,s) \geq K$, for every $s \geq s_K$, from which $\liminf_{s\to+\infty} g_m(x,s) \geq K$, so that

$$\lim_{m\to+\infty} \liminf_{s\to+\infty} g_m(x,s) = +\infty.$$

- if, on the contrary, $x \in \Omega$ is such that $\liminf_{s\to+\infty} g(x,s) = l \in \mathbb{R}$, then, fixed $\epsilon > 0$, there exists s_ϵ such that $g(x,s) \geq l - \epsilon$ for $s \geq s_\epsilon$. Moreover, there exists $m_x \in \mathbb{N}$ such that $x \in K_m$ for every $m \geq m_x$. For every $m \geq \max\{l, m_x\}$, then, we have $g_m(x,s) \geq l-\epsilon$, for every $s \geq s_\epsilon$, so that $\liminf_{s\to+\infty} g_m(x,s) \geq l-\epsilon$, from which we deduce that

$$\lim_{m\to+\infty} \liminf_{s\to+\infty} g_m(x,s) \geq l.$$

With the same computations, it is possible to show that the superior limit and the limit under the second integral can be exchanged.
According to (4.3), then,

$$\int_{\{v>0\}} \Big(\lim_{m\to+\infty} \liminf_{s\to+\infty} g_m(x,s)\Big) v(x)\,d\mu + \\ + \int_{\{v<0\}} \Big(\lim_{m\to+\infty} \limsup_{s\to-\infty} g_m(x,s)\Big) v(x)\,d\mu > 0.$$

The two sequences $(\liminf_{s\to+\infty} g_m(x,s)v(x))_m$ and $(\limsup_{s\to-\infty} g_m(x,s)v(x))_m$, considered on their domains of integration $\{v>0\}$ and $\{v<0\}$, respectively, are monotone increasing. Moreover, they are bounded from below by the L^1-functions $-h(x)v(x)$ and $h(x)v(x)$, respectively. By the monotone convergence theorem, then,

$$\lim_{m\to+\infty} \left(\int_{\{v>0\}} \liminf_{s\to+\infty} g_m(x,s)v(x)\,d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g_m(x,s)v(x)\,d\mu \right) > 0.$$

Hence, there exists $M_v \in \mathbb{N}$ such that, defining

$$I_m(v) = \int_{\{v>0\}} \liminf_{s\to+\infty} g_m(x,s)v(x)\,d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g_m(x,s)v(x)\,d\mu,$$

one has $I_m(v) > 0$ for every $m \geq M_v$. Choose $M \geq M_v$ and set

$$g_+(x) = \liminf_{s\to+\infty} g_M(x,s), \quad g_-(x) = \limsup_{s\to-\infty} g_M(x,s).$$

Observe that g_+ and g_- belong to $L^q(\Omega)$: similarly as before, indeed, for almost every $x \in \Omega$,

$$-h(x) \leq g_+(x) \leq \zeta_M(x), \quad -\zeta_M(x) \leq g_-(x) \leq h(x).$$

We now claim that $I_M : L^p(\Omega) \to \mathbb{R}$ is continuous at v. To show it, let $v_j \to v$ in $L^p(\Omega)$, and fix the following notation:

$$A_j^+ = \{v_j \geq 0\}, \quad A_j^- = \{v_j < 0\},$$

$$A^+ = \{v \geq 0\}, \quad A^- = \{v < 0\}.$$

We have

$$I_M(v_j) - I_M(v) = \Upsilon_{1,j} + \Upsilon_{2,j} + \Upsilon_{3,j} + \Upsilon_{4,j}\,,$$

where

$$\begin{aligned}
\Upsilon_{1,j} &= \int_{A_j^+ \cap A^+} g_+(x)(v_j(x) - v(x))\, d\mu,\\
\Upsilon_{2,j} &= \int_{A_j^- \cap A^-} g_-(x)(v_j(x) - v(x))\, d\mu,\\
\Upsilon_{3,j} &= \int_{A_j^- \cap A^+} (g_-(x)v_j(x) - g_+(x)v(x))\, d\mu,\\
\Upsilon_{4,j} &= \int_{A_j^+ \cap A^-} (g_+(x)v_j(x) - g_-(x)v(x))\, d\mu.
\end{aligned}$$

As $j \to +\infty$, $\Upsilon_{1,j}$ and $\Upsilon_{2,j}$ vanish thanks to the Hölder inequality, since $v_j \to v$ in $L^p(\Omega)$. Concerning $\Upsilon_{3,j}$, for every subsequence of $(v_j)_j$ we can find a further subsequence, still denoted by $(v_j)_j$, such that $v_j(x) \to v(x)$ for almost every $x \in \Omega$. Hence, by the Lebesgue dominated convergence Theorem,

$$\int_{A_j^- \cap A^+} g_+(x)v(x)\, d\mu \to 0,$$

since $\mu((A_j^- \cap A^+) \setminus \{v = 0\}) \to 0$. On the other hand, writing

$$\int_{A_j^- \cap A^+} g_-(x)v_j(x)\, d\mu = \int_{A_j^- \cap A^+} g_-(x)(v_j(x) - v(x))\, d\mu + \int_{A_j^- \cap A^+} g_-(x)v(x)\, d\mu,$$

arguing similarly we see that

$$\int_{A_j^- \cap A^+} g_-(x)v_j(x)\, d\mu \to 0.$$

This shows that $\Upsilon_{3,j} \to 0$ as $j \to +\infty$. With the same reasonings, we see that $\Upsilon_{4,j}$ vanishes, as well. The continuity of I_M is thus proved.

It follows that there exists $\delta_v > 0$ such that $I_M(w) > 0$ for $\|w - v\|_p < \delta_v$, namely

$$\int_{\{w>0\}} \liminf_{s\to+\infty} g_M(x,s)w(x)\, d\mu + \int_{\{w<0\}} \limsup_{s\to-\infty} g_M(x,s)w(x)\, d\mu > 0.$$

Since, thanks to our hypotheses, $\Sigma \cap S_1$ is compact in $L^p(\Omega)$, it will be possible to find $\bar{m} \in \mathbb{N}$ such that, for every $v \in \Sigma \cap S_1$,

$$I_{\bar{m}}(v) = \int_{\{v>0\}} \liminf_{s\to+\infty} g_{\bar{m}}(x,s)v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g_{\bar{m}}(x,s)v(x)\, d\mu > 0.$$

The fact that (4.2) holds for every $m \geq \bar{m}$ is a simple consequence of the monotonicity of the integrands with respect to m. □

We now give a characterization of the Landesman-Lazer condition.

Proposition 4.1.3. *The following conditions are equivalent:*

1) $g(x,s)$ satisfies (LL);

2) there exist $\bar{\eta} > 0$, $R \geq d$ and $\psi_-, \psi_+ \in L^q(\Omega)$ such that

- *$g(x,s) \geq \psi_+(x)$ for a.e. $x \in \Omega$, and every $s \geq R$;*
- *$g(x,s) \leq \psi_-(x)$ for a.e. $x \in \Omega$, and every $s \leq -R$;*
- *for every $v \in \Sigma$,*

$$\int_{\{v>0\}} \psi_+(x)v(x)\,d\mu + \int_{\{v<0\}} \psi_-(x)v(x)\,d\mu \geq \bar{\eta}\|v\|_p\,. \tag{4.4}$$

Moreover, there exists $M > 0$ such that

$$-h(x) \leq \psi_+(x) \leq M, \quad -M \leq \psi_-(x) \leq h(x),$$

for almost every $x \in \Omega$, and, if $x \in \Omega \setminus K_M$, then $\psi_+(x) \leq 0$ and $\psi_-(x) \geq 0$.

Proof. In view of the positive homogeneity of both sides of (4.4) with respect to v, it is not restrictive to assume $\|v\|_p = 1$. We will only prove that 1) implies 2), since the other implication is straightforward. Suppose that (LL) holds: by Lemma 4.1.2, using the same notation, there exists $\bar{m} \in \mathbb{N}$ such that, for every $m \geq \bar{m}$ and every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s\to+\infty} g_m(x,s)v(x)\,d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g_m(x,s)v(x)\,d\mu > 0,$$

i.e.,

$$\int_{\{v>0\}} \Big(\lim_{n\to+\infty} \inf_{s\geq n} g_m(x,s)\Big)v(x)\,d\mu + \int_{\{v<0\}} \Big(\lim_{n\to+\infty} \sup_{s\leq -n} g_m(x,s)\Big)v(x)\,d\mu > 0.$$

Fix $M \geq \bar{m}$ and set

$$\gamma_n^+(x) = \inf_{s\geq n} g_M(x,s), \quad \gamma_n^-(x) = \sup_{s\leq -n} g_M(x,s).$$

Observe that, for every $n \geq d$, γ_n^+ and γ_n^- belong to $L^q(\Omega)$, since, for almost every $x \in \Omega$,

$$-h(x) \leq \gamma_n^+(x) \leq M, \quad -M \leq \gamma_n^-(x) \leq h(x).$$

On their domains of integration $\{v > 0\}$ and $\{v < 0\}$, respectively, the sequences of L^1-functions $(\gamma_n^+ v)_{n\geq d}$ and $(\gamma_n^- v)_{n\geq d}$ are both monotone increasing, and bounded from below by the L^1-functions $-hv$ and hv respectively. By the monotone convergence theorem, for every $v \in \Sigma \cap S_1$,

$$\begin{aligned}
\lim_{n\to+\infty} &\left(\int_{\{v>0\}} \gamma_n^+(x)v(x)\,d\mu + \int_{\{v<0\}} \gamma_n^-(x)v(x)\,d\mu \right) = \\
&= \int_{\{v>0\}} \lim_{n\to+\infty} \gamma_n^+(x)v(x)\,d\mu + \int_{\{v<0\}} \lim_{n\to+\infty} \gamma_n^-(x)v(x)\,d\mu \\
&= \int_{\{v>0\}} \liminf_{s\to+\infty} g_M(x,s)v(x)\,d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g_M(x,s)v(x)\,d\mu > 0.
\end{aligned}$$

Then, there exist $\eta_v > 0$ and $N_v \in \mathbb{N}$, with $N_v \geq d$, such that, defining

$$J_n(v) = \int_{\{v>0\}} \gamma_n^+(x)v(x)\,d\mu + \int_{\{v<0\}} \gamma_n^-(x)v(x)\,d\mu,$$

one has $J_n(v) \geq \eta_v$ for every $n \geq N_v$. Choose $N \geq N_v$: with the same reasonings as in the proof of Lemma 4.1.2, we can show that $J_N : L^p(\Omega) \to \mathbb{R}$ is continuous at v. Hence, there exists $\delta_v > 0$ such that, if $\|w - v\|_p \leq \delta_v$,

$$\int_{\{w>0\}} \gamma_N^+(x)w(x)\,d\mu + \int_{\{w<0\}} \gamma_N^-(x)w(x)\,d\mu \geq \frac{\eta_v}{2}\,.$$

By the compactness of $\Sigma \cap S_1$ it is possible to find $\bar{n} \in \mathbb{N}$, with $\bar{n} \geq d$, and $\bar{\eta} > 0$ such that, for every $v \in \Sigma \cap S_1$,

$$\int_{\{v>0\}} \gamma_{\bar{n}}^+(x)v(x)\,d\mu + \int_{\{v<0\}} \gamma_{\bar{n}}^-(x)v(x)\,d\mu \geq \bar{\eta}.$$

Setting

$$\psi_+(x) = \gamma_{\bar{n}}^+(x), \quad \psi_-(x) = \gamma_{\bar{n}}^-(x),$$

the proof is easily completed, taking $R = \bar{n}$. □

Remark 4.1.4. In the study of an elliptic boundary value problem at resonance with the first eigenvalue, Gossez and Omari characterized the Landesman-Lazer condition, as well (see [71, Proposition 4.1]). In their particular case, the eigenspace is 1-dimensional and generated by a positive eigenfunction (see also, in a different context, [64, Lemma 1]).

We are now ready to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Let $v \in \Sigma \setminus \{0\}$ and set

$$\Omega_v^+ = \{x \in \Omega \mid v(x) > R\},$$

$$\Omega_v^- = \{x \in \Omega \mid v(x) < -R\},$$

$$\Omega_v^0 = \{x \in \Omega \mid -R \leq v(x) \leq R\},$$

where $R > 0$ is given by Proposition 4.1.3. Writing

$$\int_\Omega G(x, v(x))\, d\mu = \int_{\Omega_v^+} G(x, v(x))\, d\mu + \int_{\Omega_v^-} G(x, v(x))\, d\mu + \int_{\Omega_v^0} G(x, v(x))\, d\mu,$$

we are led to consider each term separately. For what concerns the first one, notice that, using the notation of Proposition 4.1.3, we have $g(x, s) \geq \psi_+(x)$ for $s > R$, and $|g(x, s)| \leq \eta_R(x)$ for $|s| \leq R$, thanks to the Carathéodory assumptions. Moreover, recalling that $\psi_+(x) \leq M$ for almost every $x \in \Omega$ and $\psi_+(x) \leq 0$ for almost every $x \in \Omega \setminus K_M$,

$$\begin{aligned}
G(x, v(x)) &= \int_0^R g(x, \xi)\, d\xi + \int_R^{v(x)} g(x, \xi)\, d\xi \\
&\geq -R\eta_R(x) + (v(x) - R)\psi_+(x) \\
&\geq -R\eta_R(x) + v(x)\psi_+(x) - R\psi_+(x)\chi_{K_M}(x) \\
&\geq -R\eta_R(x) + v(x)\psi_+(x) - RM\chi_{K_M}(x),
\end{aligned}$$

for almost every $x \in \Omega_v^+$. Hence,

$$\begin{aligned}
&\int_{\Omega_v^+} G(x, v(x))\, d\mu \geq -R\|\eta_R\|_1 + \int_{\Omega_v^+} \psi_+(x)v(x)\, d\mu - RM\mu(K_M) \\
&= -R(\|\eta_R\|_1 + M\mu(K_M)) + \int_{\{v>0\}} \psi_+(x)v(x)\, d\mu - \int_{\{0<v(x)\leq R\}} \psi_+(x)v(x)\, d\mu \\
&\geq -R(\|\eta_R\|_1 + M\mu(K_M)) + \int_{\{v>0\}} \psi_+(x)v(x)\, d\mu - \int_{\{0<v(x)\leq R\}\cap K_M} \psi_+(x)v(x)\, d\mu \\
&\geq -R(\|\eta_R\|_1 + M\mu(K_M)) + \int_{\{v>0\}} \psi_+(x)v(x)\, d\mu - \int_{\{0<v(x)\leq R\}\cap K_M} Mv(x)\, d\mu \\
&\geq -R(\|\eta_R\|_1 + 2M\mu(K_M)) + \int_{\{v>0\}} \psi_+(x)v(x)\, d\mu.
\end{aligned}$$

A similar computation yields

$$\int_{\Omega_v^-} G(x, v(x))\, d\mu \geq -R(\|\eta_R\|_1 + 2M\mu(K_M)) + \int_{\{v<0\}} \psi_-(x)v(x)\, d\mu,$$

while we have

$$\int_{\Omega_v^0} G(x, v(x))\, d\mu = \int_{\Omega_v^0} \int_0^{v(x)} g(x, \xi)\, d\xi\, d\mu \geq -R\|\eta_R\|_1\,. \tag{4.5}$$

Summing up, using (4.4), we have

$$\int_\Omega G(x, v(x))\, d\mu \geq \bar{\eta}\|v\|_p - R\big[3\|\eta_R\|_1 + 4M\mu(K_M)\big], \tag{4.6}$$

where $\bar{\eta} > 0$ is given by Proposition 4.1.3. This concludes the proof. □

We have just proved that the Ahmad-Lazer-Paul condition is more general than the Landesman-Lazer one. However, in the setting of the theorem, adding some monotonicity assumption on g (with respect to s) makes the two conditions equivalent, as shown in the following proposition.

Proposition 4.1.5. *Assume that $g(x, s)$ is nondecreasing in s, for almost every $x \in \Omega$. Then (LL) and (ALP) are equivalent.*

Proof. Assume that (LL) is not satisfied. Hence, there exists $v \in \Sigma \setminus \{0\}$ such that

$$\int_{\{v>0\}} \liminf_{s\to+\infty} g(x, s)v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g(x, s)v(x)\, d\mu \leq 0;$$

setting $g_+(x) = \lim_{s\to+\infty} g(x, s)$, and $g_-(x) = \lim_{s\to-\infty} g(x, s)$, this reads as

$$\int_{\{v>0\}} g_+(x)v(x)\, d\mu + \int_{\{v<0\}} g_-(x)v(x)\, d\mu \leq 0.$$

Let us show that the function $\mathcal{G} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\mathcal{G}(\lambda) = \int_\Omega G(x, \lambda v(x))\, d\mu = \int_\Omega \int_0^{\lambda v(x)} g(x, \xi)\, d\xi\, d\mu,$$

is nonpositive for $\lambda > 0$. Indeed, since, for almost every $x \in \Omega$, and every $s \in \mathbb{R}$,

$$g_-(x) \leq g(x, s) \leq g_+(x),$$

we have

$$\begin{aligned}\mathcal{G}(\lambda) &= \int_{\{v>0\}} \int_0^{\lambda v(x)} g(x, \xi)\, d\xi\, d\mu + \int_{\{v<0\}} \int_0^{\lambda v(x)} g(x, \xi)\, d\xi\, d\mu \\ &\leq \lambda \int_{\{v>0\}} g_+(x)v(x)\, d\mu + \lambda \int_{\{v<0\}} g_-(x)v(x)\, d\mu \leq 0.\end{aligned}$$

Consequently, $\limsup_{\lambda\to+\infty} \mathcal{G}(\lambda) \leq 0$, so that (ALP) does not hold. □

Remark 4.1.6. As shown in (4.6), condition (LL) implies that, for every $\beta \in [0,1[$,

$$\lim_{\|v\|_p \to +\infty} \frac{1}{\|v\|_p^\beta} \int_\Omega G(x, v(x))\, d\mu = +\infty,$$

as long as $v \in \Sigma$. Conditions of this type (compare with (3.9)) were considered, e.g., in [30, 75, 130, 131].

Remark 4.1.7. As already pointed out in [9], it is possible to compare conditions (LL) and (ALP) with another existence condition introduced by Tomiczek in [132] and [133], the so called *potential Landesman-Lazer condition.* In our abstract framework, with the same notation as before, such a condition can be written as follows:

(p-LL) for every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s\to+\infty} \frac{G(x,s)}{s} v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} \frac{G(x,s)}{s} v(x)\, d\mu > 0.$$

Let us first show that (LL) implies (p-LL). Using Proposition 4.1.3, with the notation therein, for almost every $x \in \Omega$ we have

$$\begin{aligned}\liminf_{s\to+\infty} \frac{G(x,s)}{s} &= \liminf_{s\to+\infty} \frac{G(x,s) - G(x,R)}{s} \\ &= \liminf_{s\to+\infty} \frac{1}{s} \int_R^s g(x,\xi)\, d\xi \\ &\geq \liminf_{s\to+\infty} \frac{s-R}{s} \psi_+(x) = \psi_+(x).\end{aligned}$$

By analogous computations, we see that

$$\limsup_{s\to-\infty} \frac{G(x,s)}{s} \leq \psi_-(x),$$

and the statement follows then from (4.4).

We now show that (p-LL) implies (ALP). To this aim, define the function

$$f(x,s) = \begin{cases} \dfrac{1}{s}(G(x,s) - G(x,d)) & \text{if } s \geq d \\ 0 & \text{if } -d < s < d \\ \dfrac{1}{s}(G(x,s) - G(x,-d)) & \text{if } s \leq -d \end{cases}$$

and notice that (p-LL) implies

$$\int_{\{v>0\}} \liminf_{s\to+\infty} f(x,s)v(x)\,d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} f(x,s)v(x)\,d\mu > 0,$$

i.e., f satisfies (LL). Moreover, it is easily seen that the function f satisfies the same L^1-Carathéodory conditions as g, and

$$|s| \geq d \quad \Rightarrow \quad \operatorname{sgn}(s)f(x,s) \geq -h(x),$$

as well. Consequently, assuming (p-LL), Lemma 4.1.2 and Proposition 4.1.3 can be applied, with g replaced by f, yielding the existence of $\bar\eta > 0$, $R \geq d$, and Ψ_+, Ψ_- belonging to $L^q(\Omega)$, such that $f(x,s) \geq \Psi_+(x)$ for $s \geq R$, $f(x,s) \leq \Psi_-(x)$ for $s \leq -R$, and

$$\int_{\{v>0\}} \Psi_+(x)v(x)\,d\mu + \int_{\{v<0\}} \Psi_-(x)v(x)\,d\mu \geq \bar\eta\|v\|_p\,, \tag{4.7}$$

for every $v \in \Sigma$. Moreover, there exists $M > 0$ such that

$$-h(x) \leq \Psi_+(x) \leq M, \quad -M \leq \Psi_-(x) \leq h(x),$$

for almost every $x \in \Omega$, and, if $x \in \Omega \setminus K_M$, then $\Psi_+(x) \leq 0$ and $\Psi_-(x) \geq 0$. Letting $v \in \Sigma \setminus \{0\}$ and

$$\begin{aligned}
\Omega_v^+ &= \{x \in \Omega \mid v(x) > R\},\\
\Omega_v^- &= \{x \in \Omega \mid v(x) < -R\},\\
\Omega_v^0 &= \{x \in \Omega \mid -R \leq v(x) \leq R\},
\end{aligned}$$

we write

$$\int_\Omega G(x,v(x))\,d\mu = \int_{\Omega_v^+} G(x,v(x))\,d\mu + \int_{\Omega_v^-} G(x,v(x))\,d\mu + \int_{\Omega_v^0} G(x,v(x))\,d\mu.$$

For what concerns the first term, since, for almost every $x \in \Omega_v^+$,

$$\begin{aligned}
G(x,v(x)) &= \int_0^d g(x,\xi)\,d\xi + \int_d^{v(x)} g(x,\xi)\,d\xi\\
&\geq -d\eta_d(x) + f(x,v(x))v(x)\\
&\geq -d\eta_d(x) + \Psi_+(x)v(x),
\end{aligned}$$

with similar computations as in the proof of Theorem 4.1.1 we obtain

$$\begin{aligned}
\int_{\Omega_v^+} G(x,v(x))\,d\mu &\geq -d\|\eta_d\|_1 + \int_{\Omega_v^+} \Psi_+(x)v(x)\,d\mu\\
&\geq -d\|\eta_d\|_1 - RM\mu(K_M) + \int_{\{v>0\}} \Psi_+(x)v(x)\,d\mu.
\end{aligned}$$

Similarly,

$$\int_{\Omega_v^-} G(x, v(x))\, d\mu \geq -d\|\eta_d\|_1 - RM\mu(K_M) + \int_{\{v<0\}} \Psi_-(x)v(x)\, d\mu,$$

while the integral on Ω_v^0 can be estimated as in (4.5). Hence, by (4.7),

$$\int_\Omega G(x, v(x))\, d\mu \geq \bar{\eta}\|v\|_p - \big[R\|\eta_R\|_1 + 2d\|\eta_d\|_1 + 2RM\mu(K_M)\big].$$

It follows that the Ahmad-Lazer-Paul condition is fulfilled.

Clearly, as a consequence of Proposition 4.1.5, condition (p-LL) is equivalent to both (LL) and (ALP) when g is nondecreasing with respect to its second variable.

4.2 The planar case

We now compare the planar version of the Ahmad-Lazer-Paul condition (3.22) with the planar version of the Landesman-Lazer one, thoroughly discussed in Chapter 2. For simplicity, we will limit ourselves to the equation

$$Ju' = \nabla V(u) + \nabla Q(t, u),$$

with $V \in \mathcal{P}$, and $\nabla Q(t, u)$ a *bounded* function. Accordingly, we are interested in condition (3.29), written in the equivalent way

$$\lim_{\lambda\to+\infty} \int_0^T Q(t, \lambda\varphi_V(t+\theta))\, dt = +\infty, \qquad \text{uniformly in } \theta \in [0, T], \tag{4.8}$$

being $\varphi_V(t)$ fixed as in Section 3.3. We recall that, in this setting, the planar Landesman-Lazer condition [56] reads as follows:
for every $\theta \in [0, T]$, it holds

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle \nabla Q(t, \lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle\, dt > 0. \tag{4.9}$$

The analogue of Theorem 4.1.1 is then represented by the following result.

Theorem 4.2.1. *Condition* (4.9) *implies* (4.8).

To prove the theorem, we state the following preliminary proposition [16, Lemma 4.2].

Proposition 4.2.2. *Assume* (4.9). *Then, there exist* $\lambda_0 > 0$*,* $\theta_1, \ldots, \theta_j \in [0,T]$*,* $\delta_1, \ldots, \delta_j > 0$ *and* $h_1, \ldots, h_j \in L^1(0,T)$*, with* $\int_0^T h_i(t)\,dt > 0$ *for every* $i = 1, \ldots, j$ *(where* j *is a suitable integer), such that*

$$\bigcup_{i=1}^{j} [\theta_i - \delta_i, \theta_i + \delta_i] \supset [0,T], \tag{4.10}$$

and, for every $i = 1, \ldots, j$ *and every* $t \in [0,T]$*,*

$$\langle \nabla Q(t, \lambda \varphi_V(t+\omega)) | \varphi_V(t+\omega) \rangle \geq h_i(t), \qquad \text{if } |\omega - \theta_i| \leq \delta_i,\ \lambda \geq \lambda_0. \tag{4.11}$$

Proof. For simplicity, we set $l_\kappa(t,\omega) = \langle \nabla Q(t, \kappa \varphi_V(t+\omega)) | \varphi_V(t+\omega) \rangle$.
Fix $\hat{\theta} \in [0,T]$. In view of the definition of inferior limit and Fatou's lemma, (4.9) gives

$$\liminf_{(\lambda,\delta) \to (+\infty, 0)} \int_0^T \inf_{\kappa \geq \lambda,\, |\omega - \hat{\theta}| \leq \delta} l_\kappa(t,\omega)\,dt > 0.$$

Therefore, there exist $\lambda_0 = \lambda_0(\hat{\theta}) \geq 1$, $\delta_0 = \delta_0(\hat{\theta}) > 0$ such that

$$\int_0^T \inf_{\kappa \geq \lambda_0,\, |\omega - \hat{\theta}| \leq \delta_0} l_\kappa(t,\omega)\,dt > 0.$$

We set

$$h(t, \hat{\theta}) = \inf_{\kappa \geq \lambda_0,\, |\omega - \hat{\theta}| \leq \delta_0} l_\kappa(t,\omega);$$

of course, $\int_0^T h(t,\hat{\theta})\,dt > 0$ and, by the definition, $l_\kappa(t,\omega) \geq h(t,\hat{\theta})$ for every $t \in [0,T]$, $\kappa \geq \lambda_0(\hat{\theta})$ and $|\omega - \hat{\theta}| \leq \delta_0(\hat{\theta})$. Repeating the argument for every $\hat{\theta} \in [0,T]$ and using the compactness of $[0,T]$, there exist $\theta_1, \ldots, \theta_j \in [0,T]$, $\delta_1(\theta_1), \ldots, \delta_j(\theta_j) > 0$ such that (4.10) holds true, with an analogous control on $l_\kappa(t,\omega)$ in the corresponding interval. Setting

$$\lambda_0 = \max_{i=1,\ldots,j} \{\lambda_0(\theta_i)\}, \quad h_i(t) = h(t,\theta_i), \qquad i = 1, \ldots, j,$$

we finally get (4.11). □

Since the converse statement is easily seen to hold true, Proposition 4.2.2 can be viewed as a characterization of condition (4.9), just as Proposition 4.1.3. It is thus natural to wonder which kind of relationship exists between these two propositions. To this aim, we will only consider, for simplicity, the case of an asymmetric equation like

$$x'' + \mu x^+ - \nu x^- + r(t,x) = 0, \tag{4.12}$$

with (μ,ν) belonging to the T-periodic Fučík spectrum, assuming that $r(t,x)$ fulfills the usual Landesman-Lazer condition:
for every nontrivial solution $v(t)$ to $v''+\mu v^+-\nu v^-=0$,

$$\int_{\{v>0\}} \liminf_{x\to+\infty} r(t,x)v(t)\,dt + \int_{\{v<0\}} \limsup_{x\to-\infty} r(t,x)v(t)\,dt > 0. \tag{4.13}$$

We will still denote by ψ_+,ψ_- the functions provided by Proposition 4.1.3, such that, for every $v(t)$ as in (4.13),

$$\int_{\{v>0\}} \psi_+(t)v(t)\,dt + \int_{\{v<0\}} \psi_-(t)v(t)\,dt \geq \bar\eta\|v\|_2 > 0. \tag{4.14}$$

Once written, as usual, equation (4.12) as a first order system, being in particular $V(x,y)=\frac{1}{2}(\mu(x^+)^2+\nu(x^-)^2+y^2)$ and $Q(t,x,y)=\int_0^x r(t,\xi)\,d\xi$, we want to determine explicitly the functions h_i appearing in Proposition 4.2.2. For this purpose, notice first that it is possible to write $\varphi_V(t)=(v(t),v'(t))$, for a suitable $v(t)$ solving $v''+\mu v^+-\nu v^-=0$. As a consequence, fixed $\theta\in[0,T]$, we have (compare with the computations in Section 2.4)

$$\langle\nabla Q(t,\lambda\varphi_V(t+\theta))|\varphi_V(t+\theta)\rangle = r(t,\lambda v(t+\theta))v(t+\theta).$$

We now have two cases: if $t\in\{v(t+\theta)>0\}$, then we can find a closed neighborhood $\mathcal{U}_\theta$ of θ and $\lambda_0>0$ such that, for every $\omega\in\mathcal{U}_\theta$ and every $\lambda\geq\lambda_0$, it holds

$$r(t,\lambda v(t+\omega))v(t+\omega) \geq \psi_+(t)v(t+\omega),$$

since $v(t+\omega)$ can be chosen to have positive minimum over $\mathcal{U}_\theta$. Similarly, if $t\in\{v(t+\theta)<0\}$, we can find another closed neighborhood $\mathcal{U}'_\theta$ of θ and $\lambda'_0>0$ such that, for every $\omega\in\mathcal{U}'_\theta$ and every $\lambda\geq\lambda'_0$, it holds

$$r(t,\lambda v(t+\omega))v(t+\omega) \geq \psi_-(t)v(t+\omega).$$

We now set

$$h_\theta(t)=\Big(\min_{\omega\in\mathcal{U}_\theta\cap\mathcal{U}'_\theta}\psi_+(t)v(t+\omega)\Big)\chi_{\{v(t+\theta)>0\}}(t)+\Big(\min_{\omega\in\mathcal{U}_\theta\cap\mathcal{U}'_\theta}\psi_-(t)v(t+\omega)\Big)\chi_{\{v(t+\theta)<0\}}(t).$$

It is immediately seen that, whatever the choice of $t\in[0,T]$ (up to a null measure set), we have

$$\langle\nabla Q(t,\lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle \geq h_\theta(t),$$

for every $\omega \in \mathcal{U}_\theta \cap \mathcal{U}'_\theta$. The point is now to prove that the integral of $h_\theta(t)$ is positive. However, in view of Lebesgue dominated convergence theorem we have

$$\begin{aligned}\int_0^T h_\theta(t)\,dt &= \int_{\{v(t+\theta)>0\}} \min_{\omega\in\mathcal{U}_\theta\cap\mathcal{U}'_\theta} \psi_+(t)v_\omega(t)\,dt + \int_{\{v(t+\theta)<0\}} \min_{\omega\in\mathcal{U}_\theta\cap\mathcal{U}'_\theta} \psi_-(t)v_\omega(t)\,dt\\ &= \min_{\omega\in\mathcal{U}_\theta\cap\mathcal{U}'_\theta} \int_{\{v(t+\theta)>0\}} \psi_+(t)v_\omega(t)\,dt + \min_{\omega\in\mathcal{U}_\theta\cap\mathcal{U}'_\theta} \int_{\{v(t+\theta)<0\}} \psi_-(t)v_\omega(t)\,dt,\end{aligned}$$

where we have used the notation $v_\omega(t) = v(t+\omega)$. Since (4.14) holds, and both the summands in its left-hand side are continuous in v, replacing there $v(t)$ with $v(t+\omega)$ we obtain a continuous expression in ω (cf. the proofs of Lemma 4.1.2 and Proposition 4.1.3), so that we can find a (smaller, if necessary) neighborhood of θ, which we denote by $\mathcal{U}$, such that $\mathcal{U} \subset \mathcal{U}_\theta \cap \mathcal{U}'_\theta$ and, taking the minimum for $\omega \in \mathcal{U}$, the last expression above is positive. Hence, $h_\theta(t)$ is the desired function for every $\omega \in \mathcal{U}$; reasoning by compactness, one can then reconstruct a finite number of functions $h_i(t)$ as in the statement of Proposition 4.2.2.

We now conclude the chapter by proving that the Landesman-Lazer condition implies the Ahmad-Lazer-Paul one also in the planar setting.

Proof of Theorem 4.2.1. Without loss of generality, we can assume $Q(t,0) \equiv 0$; moreover, as in the proof of Proposition 4.2.2, we set

$$l_\kappa(t,\theta) = \langle \nabla Q(t, \kappa\varphi_V(t+\theta)) | \varphi_V(t+\theta)\rangle.$$

For every $t \in [0,T]$, $\lambda \geq \lambda_0$ and $\theta \in [0,T]$, we have

$$\begin{aligned}Q(t,\lambda\varphi_V(t+\theta)) &= \int_0^1 \frac{d}{d\kappa} Q(t,\kappa\lambda\varphi_V(t+\theta))\,d\kappa\\ &= \lambda \int_0^1 \langle \nabla Q(t,\kappa\lambda\varphi_V(t+\theta)) | \varphi_V(t+\theta)\rangle\,d\kappa = \int_0^\lambda l_\kappa(t,\theta)\,d\kappa\\ &= \int_0^{\lambda_0} l_\kappa(t,\theta)\,d\kappa + \int_{\lambda_0}^\lambda l_\kappa(t,\theta)\,d\kappa\\ &\geq -\lambda_0 \max_{t,\theta\in[0,T],\kappa\in[0,\lambda_0]} |l_\kappa(t,\theta)| + \int_{\lambda_0}^\lambda h_i(t)\,d\kappa,\end{aligned}$$

being the index i such that $\theta \in [\theta_i - \delta_i, \theta_i + \delta_i]$ (keeping the notation of Proposition 4.2.2). Integrating on $[0,T]$ we obtain

$$\begin{aligned}\int_0^T Q(t,\lambda\varphi_V(t+\theta))\,dt &\geq -C_1 + (\lambda - \lambda_0)\int_0^T h_i(t)\,dt\\ &\geq -C_1 + (\lambda-\lambda_0) \min_{i=1,\dots,j} \int_0^T h_i(t)\,dt, \qquad (4.15)\end{aligned}$$

being $C_1 = T\lambda_0 \max_{t,\theta\in[0,T],\kappa\in[0,\lambda_0]} |l_\kappa(t,\theta)|$. The conclusion follows, since, for $i = 1,\dots,j$, $\int_0^T h_i(t)\,dt > 0$ in view of Proposition 4.2.2. □

Remark 4.2.3. In view of (4.15), the same argument show that, when (3.21) is satisfied for $\alpha \in [0,1/2[$, condition (4.9) still implies (3.22). In such a case, however, in order for the integral in (4.9) to make sense, one has to assume that, for a suitable $\eta \in L^1(0,T)$, it holds

$$\langle \nabla Q(t,\lambda u)|u\rangle \geq \eta(t), \qquad \text{for every } t\in[0,T],\ |u|\leq 1,\ \lambda\geq 1.$$

Chapter 5

The rotational approach: multiple solutions in the Hamiltonian case

In this chapter, we will shift our attention to the problem of multiplicity of periodic solutions for a first order planar system. Our approach will be topological, searching for solutions as fixed points of the Poincaré map associated with the considered problem. In this context, the use of the Poincaré-Birkhoff fixed point theorem (in the formulation given in Theorem 1.2.9) will be extremely convenient, once it is possible to give some estimates on the rotational behavior of the system around the origin, which is assumed to be an equilibrium. Indeed, whenever it is possible to exhibit a gap in the rotations of “small” and “large” solutions to the Cauchy problems associated with the considered equation, the Poincaré-Birkhoff theorem provides fixed points of the Poincaré map which are distinguished through their number of rotations around the origin, in a number which depends on the amplitude of such a gap. For this reason, it will be important to examine the influence of some well-known nonresonance conditions on the number of turns of the solutions to some reference equation around the origin, topic to which the first section is devoted. Afterwards, we will use the Poincaré-Birkhoff fixed point theorem to find multiple solutions, showing how the estimates can be performed when we compare our problem with positively homogeneous ones (actually, we will take into account as comparison terms the systems discussed in Subsection 1.1.3, so that we will consider a slightly more general version of our nonresonance conditions). Lastly, we highlight that, in order to obtain the multiplicity results, the role of resonance is secondary (indeed, it suffices the mentioned gap to be large enough): the main problem is rather the fact that, in association with a resonant Hamiltonian, the estimate of the modified rotation numbers is generally rougher, in absence of other assumptions on the nonlinearity.

5.1 A rotational interpretation of nonresonance

Our discussion will be split into two subsections: in the first one, we will analyze the nonresonance and the nonuniform nonresonance conditions from a rotational viewpoint, while the second one will be dedicated to such a discussion for the Landesman-Lazer condition.

5.1.1 Conditions of nonresonance and nonuniform nonresonance

In this section, we will consider a general first order system like

$$Ju' = F(t,u), \tag{5.1}$$

where $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is an L^1-Carathéodory function. We will perform some estimates of the rotation numbers of both "small" and "large" solutions, essentially referring to a nonresonance situation.

We start with a very simple observation, which extends to the plane, even if still with a quite unpractical formulation, the nonuniform nonresonance condition by Mawhin and Ward and the nonresonance conditions introduced in [45] (see also Proposition 5.1.2 below).

Lemma 5.1.1. *Let $V \in \mathcal{P}$, $0 \leq r < R \leq +\infty$ and $\beta : [0,T] \times \mathcal{A}(r,R) \to \mathbb{R}$ an L^1-Carathéodory function - we recall that $\mathcal{A}(r,R)$ denotes the open annulus of radii r and R. The following statements hold true.*

- *Assume:*
    - *for almost every $t \in [0,T]$ and for every $u \in \mathcal{A}(r,R)$,*

    $$\frac{\langle F(t,u)|u\rangle}{2V(u)} \geq \beta(t,u), \tag{5.2}$$

    - *for every $u : [0,T] \to \mathbb{R}^2$ solving (5.1), with $u(t) \in \mathcal{A}(r,R)$ for every $t \in [0,T]$, it holds*

    $$\frac{1}{T}\int_0^T \beta(t,u(t))\,dt > 1. \tag{5.3}$$

  Then, for every $u : [0,T] \to \mathbb{R}^2$ solving (5.1), with $u(t) \in \mathcal{A}(r,R)$ for every $t \in [0,T]$, it holds

  $$\operatorname{Rot}(u(t)) > \left\lfloor \frac{T}{\tau_V} \right\rfloor.$$

- *Assume:*
  - *for almost every* $t \in [0,T]$ *and for every* $u \in \mathcal{A}(r,R)$,
$$\frac{\langle F(t,u)|u\rangle}{2V(u)} \le \beta(t,u), \tag{5.4}$$
  - *for every* $u : [0,T] \to \mathbb{R}^2$ *solving (5.1), with* $u(t) \in \mathcal{A}(r,R)$ *for every* $t \in [0,T]$, *it holds*
$$\frac{1}{T}\int_0^T \beta(t,u(t))\,dt < 1. \tag{5.5}$$

  Then, for every $u : [0,T] \to \mathbb{R}^2$ *solving (5.1), with* $u(t) \in \mathcal{A}(r,R)$ *for every* $t \in [0,T]$, *it holds*
$$\mathrm{Rot}\,(u(t)) < \left\lceil \frac{T}{\tau_V} \right\rceil.$$

Proof. We only prove the first statement. For $u : [0,T] \to \mathbb{R}^2$ solving (5.1), with $u(t) \in \mathcal{A}(r,R)$ for every $t \in [0,T]$, we have, in view of (5.2), (5.3),

$$\begin{aligned}\mathrm{Rot}_V\,(u(t)) &= \frac{1}{\tau_V}\int_0^T \frac{\langle Ju'(t)|u(t)\rangle}{2V(u(t))}\,dt = \frac{1}{\tau_V}\int_0^T \frac{\langle F(t,u(t))|u(t)\rangle}{2V(u(t))}\,dt \\ &\ge \frac{1}{\tau_V}\int_0^T \beta(t,u(t))\,dt > \frac{T}{\tau_V} \ge \left\lfloor \frac{T}{\tau_V} \right\rfloor.\end{aligned}$$

The proof of the second statement is analogous, using (5.4) and (5.5). □

As a first consequence, we give a corollary which can be applied when (5.1) can be compared, in a suitable weak sense, with a system of the form $Ju' = \zeta(t)\nabla V(u)$ (with $V \in \mathcal{P}$), either at 0 or at $+\infty$. In order to give a unifying statement, we will denote by the symbol $\odot$ either 0 or $+\infty$.

Proposition 5.1.2. *Let* $V \in \mathcal{P}$ *and* $\zeta \in L^1(0,T)$, *with*
$$\frac{1}{T}\int_0^T \zeta(t)\,dt = 1. \tag{5.6}$$

Moreover, if $\odot = 0$, *assume that* $F(t,0) \equiv 0$. *The following statements hold true.*

- *If*
$$\liminf_{|u|\to\odot} \frac{\langle F(t,u)|u\rangle}{2V(u)} \ge \zeta(t), \quad \textit{uniformly for a.e. } t \in [0,T], \tag{5.7}$$

then there exists $\rho > 0$ such that, for every $u : [0,T] \to \mathbb{R}^2$ solving (5.1) and satisfying, for every $t \in [0,T]$, $u(t) \in \mathcal{A}(0,\rho)$ if $\odot = 0$, or $u(t) \in \mathcal{A}(\rho,+\infty)$ if $\odot = +\infty$, it holds

$$\begin{cases} \operatorname{Rot}(u(t)) > \left\lfloor \dfrac{T}{\tau_V} \right\rfloor & \text{if } \dfrac{T}{\tau_V} \notin \mathbb{N}_0, \\ \operatorname{Rot}(u(t)) > \dfrac{T}{\tau_V} - 1 & \text{if } \dfrac{T}{\tau_V} \in \mathbb{N}_0. \end{cases}$$

- *If*

$$\limsup_{|u|\to\odot} \frac{\langle F(t,u)|u\rangle}{2V(u)} \le \zeta(t), \quad \textit{uniformly for a.e. } t \in [0,T], \tag{5.8}$$

then there exists $\rho > 0$ such that, for every $u : [0,T] \to \mathbb{R}^2$ solving (5.1) and satisfying, for every $t \in [0,T]$, $u(t) \in \mathcal{A}(0,\rho)$ if $\odot = 0$, or $u(t) \in \mathcal{A}(\rho,+\infty)$ if $\odot = +\infty$, it holds

$$\begin{cases} \operatorname{Rot}(u(t)) < \left\lceil \dfrac{T}{\tau_V} \right\rceil & \text{if } \dfrac{T}{\tau_V} \notin \mathbb{N}_0, \\ \operatorname{Rot}(u(t)) < \dfrac{T}{\tau_V} + 1 & \text{if } \dfrac{T}{\tau_V} \in \mathbb{N}_0. \end{cases}$$

Proof. We will only prove the first statement for $\odot = 0$, the other cases being analogous. Define, for $\delta > 0$ small, the function $V_\delta \in \mathcal{P}$ as

$$V_\delta(u) = (1-\delta)V(u), \quad u \in \mathbb{R}^2.$$

Clearly, uniformly for almost every $t \in [0,T]$,

$$\liminf_{|u|\to 0} \frac{\langle F(t,u)|u\rangle}{2V_\delta(u)} \ge \frac{1}{1-\delta}\,\zeta(t). \tag{5.9}$$

Since $\frac{1}{T}\int_0^T \frac{1}{1-\delta}\,\zeta(t)\,dt > 1$, we can fix $\epsilon > 0$ such that $\frac{1}{T}\int_0^T \frac{1}{1-\delta}\,(\zeta(t)-\epsilon)\,dt > 1$. In view of (5.9), there exists $\rho > 0$ such that, for almost every t and every $u \in \mathcal{A}(0,\rho)$,

$$\frac{\langle F(t,u)|u\rangle}{2V_\delta(u)} \ge \frac{1}{1-\delta}\Big(\zeta(t)-\epsilon\Big),$$

so that, from Lemma 5.1.1, we infer that, for every solution $u(t)$ satisfying $u(t) \in \mathcal{A}(0,\rho)$,

$$\operatorname{Rot}(u(t)) > \left\lfloor \frac{T}{\tau_{V_\delta}} \right\rfloor.$$

Since

$$\tau_{V_\delta} = \frac{1}{1-\delta}\tau_V \searrow \tau_V$$

for $\delta \to 0$, the conclusion follows. $\square$

Remark 5.1.3. Notice that the equality involving $\zeta(t)$ assumed in (5.6) is just a matter of normalization, provided that $\int_0^T \zeta(t)\,dt > 0$, since (5.7) and (5.8) are invariant under the dilatation

$$\zeta(t) \mapsto \lambda\zeta(t), \quad V(u) \mapsto \frac{1}{\lambda}V(u),$$

for $\lambda > 0$.

Remark 5.1.4. Observe that hypotheses (5.7) and (5.8) can be weakened to hold in an L^1-sense. Precisely, focusing for instance on (5.7) with $\odot = 0$, we can require the following:

for every $\epsilon > 0$ there exist $r_\epsilon > 0$ and $\eta_\epsilon \in L^1(0,T)$, with $\int_0^T |\eta_\epsilon(t)|\,dt \leq \epsilon$, such that, for almost every $t \in [0,T]$, and every $u \in \mathcal{A}(0, r_\epsilon)$,

$$\frac{\langle F(t,u)|u\rangle}{2V(u)} \geq \zeta(t) - \eta_\epsilon(t).$$

We now turn our attention to possible corollaries of Lemma 5.1.1 for scalar second order equations like

$$x'' + f(t,x) = 0, \tag{5.10}$$

with $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ an L^1-Carathéodory function.
The first one concerns a nonresonant case. Again, $\odot$ will denote either 0 or $+\infty$. We postpone our comments after the statement.

Proposition 5.1.5. *If $\odot = 0$, assume $f(t,0) \equiv 0$. The following statements hold true.*

- *Assume that there exists $p \in L^1(0,T)$ such that*

$$\liminf_{|x|\to\odot} \frac{f(t,x)}{x} \geq p(t), \quad \textit{uniformly for a.e. } t \in [0,T]. \tag{5.11}$$

If there exists $k \in \mathbb{N}_0$ such that

$$\sqrt{\lambda_k} < \sup_{\xi>0} \frac{\frac{1}{T}\int_0^T \min\{p(t),\xi\}\,dt}{\sqrt{\xi}}, \tag{5.12}$$

then there exists $\rho > 0$ such that, for every $x : [0,T] \to \mathbb{R}$ solving (5.10) and satisfying, for every $t \in [0,T]$, $0 < x(t)^2 + x'(t)^2 < \rho^2$ if $\odot = 0$, or $x(t)^2 + x'(t)^2 > \rho^2$ if $\odot = +\infty$, it holds

$$\mathrm{Rot}\,((x(t), x'(t))) > k.$$

- *Assume that there exists $q \in L^1(0,T)$ such that*

$$\limsup_{|x|\to\odot} \frac{f(t,x)}{x} \le q(t), \quad \textit{uniformly for a.e. } t \in [0,T].$$

If there exists $k \in \mathbb{N}_0$ such that

$$\inf_{\zeta>0} \frac{\frac{1}{T}\int_0^T \max\{q(t),\zeta\}\,dt}{\sqrt{\zeta}} < \sqrt{\lambda_k}, \tag{5.13}$$

then there exists $\rho > 0$ such that, for every $x : [0,T] \to \mathbb{R}$ solving (5.10) and satisfying, for every $t \in [0,T]$, $0 < x(t)^2 + x'(t)^2 < \rho^2$ if $\odot = 0$, or $x(t)^2 + x'(t)^2 > \rho^2$ if $\odot = +\infty$, it holds

$$\mathrm{Rot}\,((x(t), x'(t))) < k.$$

Proof. We will only prove the first statement, the second being similar. Anyway, the cases $\odot = 0$ and $\odot = +\infty$ are slightly different, so we will prove them both. Set $u = (x,y)$, $F(t,u) = (f(t,x), y)$ and choose $\xi, \epsilon > 0$ such that, in view of (5.12),

$$\sqrt{\lambda_k} < \frac{\frac{1}{T}\int_0^T \min\{p(t)-\epsilon,\xi\}\,dt}{\sqrt{\xi}}. \tag{5.14}$$

Set $V(x,y) = \frac{1}{2}\sqrt{\lambda_k \xi}\left(x^2 + \frac{1}{\xi}y^2\right)$; computing τ_V, it is easy to see that $\frac{T}{\tau_V} = k$. Assume $\odot = 0$. Setting

$$\beta(t) = \frac{\min\{p(t)-\epsilon,\xi\}}{\sqrt{\lambda_k \xi}},$$

we have, in view of (5.14), that $\frac{1}{T}\int_0^T \beta(t)\,dt > 1$. On the other hand, by (5.11), there exists $\rho > 0$ such that, for almost every $t \in [0,T]$ and every $|x| < \rho$,

$$f(t,x)x \ge (p(t)-\epsilon)x^2.$$

For $u = (x,y) \in \mathcal{A}(0,\rho)$, it follows that

$$\begin{aligned}
\frac{\langle F(t,u)|u\rangle}{2V(u)} &= \frac{f(t,x)x + y^2}{\sqrt{\lambda_k\xi}\left(x^2 + \frac{1}{\xi}y^2\right)} \\
&\ge \frac{\min\{p(t)-\epsilon,\xi\}x^2 + \frac{1}{\xi}\min\{p(t)-\epsilon,\xi\}y^2}{\sqrt{\lambda_k\xi}\left(x^2 + \frac{1}{\xi}y^2\right)} \\
&\ge \frac{\min\{p(t)-\epsilon,\xi\}}{\sqrt{\lambda_k\xi}} = \beta(t),
\end{aligned}$$

whence the conclusion by Lemma 5.1.1 in the case $\odot = 0$.

Assume now $\odot = +\infty$. Using (5.11) and the fact that $f(t,x)$ is L^1-Carathéodory, there exists $z \in L^1(0,T)$ such that, for almost every $t \in [0,T]$ and every $x \in \mathbb{R}$,

$$f(t,x)x \geq (p(t) - \epsilon)x^2 - z(t).$$

Setting, for $t \in [0,T]$ and $(x,y) \in \mathcal{A}(0,+\infty)$,

$$\beta(t,x,y) = \frac{\min\{p(t)-\epsilon,\xi\}}{\sqrt{\lambda_k \xi}} - \frac{z(t)}{\sqrt{\lambda_k \xi}\Big(x^2 + \frac{1}{\xi}y^2\Big)},$$

we see, in view of (5.14), that it is possible to choose $\rho > 0$ sufficiently large such that

$$\frac{1}{T}\int_0^T \beta(t,x(t),x'(t))\,dt > 1,$$

for every $x : [0,T] \to \mathbb{R}$ solving (5.10) such that $x(t)^2 + x'(t)^2 > \rho^2$. On the other hand, with computations similar as before, we have, for almost every $t \in [0,T]$ and every $u = (x,y) \in \mathbb{R}^2$, that

$$\frac{\langle F(t,u)|u\rangle}{2V(u)} \geq \beta(t,u),$$

whence the conclusion by Lemma 5.1.1. □

Remark 5.1.6. Conditions (5.12) and (5.13), this last one with λ_k replaced by λ_{k+1}, were introduced by Fabry in [45], for a nonlinearity satisfying

$$p(t) \leq \liminf_{|x|\to+\infty} \frac{f(t,x)}{x} \leq \limsup_{|x|\to+\infty} \frac{f(t,x)}{x} \leq q(t), \tag{5.15}$$

in order to ensure the solvability of (5.10). Here, we have seen a separate interpretation of each of the inequalities in (5.15) in terms of the rotation number, and we have shown that the estimates can be carried out in an analogous way both at 0 and at $+\infty$. We highlight the two following special cases of conditions (5.12) and (5.13):

1) if

$$\operatorname{ess\,inf}_{[0,T]} p(t) > 0 \quad \text{and} \quad \sqrt{\lambda_k} < \sqrt{\operatorname{ess\,inf}_{[0,T]} p(t)}, \tag{5.16}$$

then (5.12) is satisfied. Indeed, we have

$$\sqrt{\operatorname{ess\,inf}_{[0,T]} p(t)} \leq \sup_{\xi>0} \frac{\frac{1}{T}\int_0^T \min\{p(t),\xi\}\,dt}{\sqrt{\xi}},$$

as it can be seen taking $\xi = \operatorname{ess\,inf}_{[0,T]} p(t)$ in the right-hand side. Analogously, if

$$\operatorname{ess\,sup}_{[0,T]} q(t) > 0 \quad \text{and} \quad \sqrt{\operatorname{ess\,sup}_{[0,T]} q(t)} < \sqrt{\lambda_k}, \tag{5.17}$$

then (5.13) is satisfied. Conditions (5.16) and (5.17) are standard nonresonance conditions with respect to λ_k;

2) if

$$\operatorname{ess\,sup}_{[0,T]} p(t) > 0 \quad \text{and} \quad \sqrt{\lambda_k} < \frac{\frac{1}{T}\int_0^T p(t)\,dt}{\sqrt{\operatorname{ess\,sup}_{[0,T]} p(t)}},$$

then (5.12) is satisfied. Indeed, we have

$$\frac{\frac{1}{T}\int_0^T p(t)\,dt}{\sqrt{\operatorname{ess\,sup}_{[0,T]} p(t)}} \le \sup_{\xi>0} \frac{\frac{1}{T}\int_0^T \min\{p(t),\xi\}\,dt}{\sqrt{\xi}},$$

as it can be seen taking $\xi = \operatorname{ess\,sup}_{[0,T]} p(t)$ in the right-hand side. Analogously, if

$$\operatorname{ess\,inf}_{[0,T]} q(t) > 0 \quad \text{and} \quad \frac{\frac{1}{T}\int_0^T q(t)\,dt}{\sqrt{\operatorname{ess\,inf}_{[0,T]} q(t)}} < \sqrt{\lambda_k},$$

then (5.13) is satisfied. This shows, in particular, that conditions (5.12) and (5.13) allow $\frac{f(t,x)}{x}$ to cross an arbitrary number of eigenvalues.

As a second corollary of Lemma 5.1.1, we give another sufficient condition to achieve the same conclusion, which is independent of the previous one. Again, we postpone our comments after the statement.

Proposition 5.1.7. *If $\odot = 0$, assume $f(t,0) \equiv 0$. Assume that there exist $p, q \in L^1(0,T)$ such that*

$$p(t) \le \liminf_{|x|\to\odot} \frac{f(t,x)}{x} \le \limsup_{|x|\to\odot} \frac{f(t,x)}{x} \le q(t), \quad \textit{uniformly for a.e. } t \in [0,T]. \tag{5.18}$$

The following statements hold true.

- *If there exists $k \in \mathbb{N}_0$ such that*

$$\lambda_k \le p(t), \quad \textit{with} \quad \lambda_k < \frac{1}{T}\int_0^T p(t)\,dt, \tag{5.19}$$

then there exists $\rho > 0$ such that, for every $x : [0,T] \to \mathbb{R}$ solving (5.10) and satisfying, for every $t \in [0,T]$, $0 < x(t)^2 + x'(t)^2 < \rho^2$ if $\odot = 0$, or $x(t)^2 + x'(t)^2 > \rho^2$ if $\odot = +\infty$, it holds

$$\operatorname{Rot}\left((x(t), x'(t))\right) > k.$$

- *If there exists $k \in \mathbb{N}_0$ such that*

$$q(t) \leq \lambda_k, \quad \textit{with} \quad \frac{1}{T}\int_0^T q(t)\,dt < \lambda_k, \tag{5.20}$$

then there exists $\rho > 0$ such that, for every $x : [0,T] \to \mathbb{R}$ solving (5.10) and satisfying, for every $t \in [0,T]$, $0 < x(t)^2 + x'(t)^2 < \rho^2$ if $\odot = 0$, or $x(t)^2 + x'(t)^2 > \rho^2$ if $\odot = +\infty$, it holds

$$\operatorname{Rot}\left((x(t), x'(t))\right) < k.$$

Proof. We will only prove the first statement, the second being similar. Anyway, the cases $\odot = 0$ and $\odot = +\infty$ are slightly different, so we will prove them both. Set $u = (x,y)$, $F(t,u) = (f(t,x), y)$ and $V(x,y) = \frac{1}{2}(\lambda_k x^2 + y^2)$; it is easy to see that $\frac{T}{\tau_V} = k$. Moreover, write $p(t) = \lambda_k + \eta(t)$, so that $\eta(t) \geq 0$ and $\frac{1}{T}\int_0^T \eta(t)\,dt > 0$. Assume first $\odot = 0$.

<u>Claim.</u> *There exist $\sigma, r > 0$ such that, for every $x : [0,T] \to \mathbb{R}$ solving (5.10) with $0 < x(t)^2 + x'(t)^2 < r^2$ for every $t \in [0,T]$, it holds*

$$\int_0^T \frac{\eta(t)x(t)^2}{\lambda_k x(t)^2 + x'(t)^2}\,dt \geq \sigma.$$

By contradiction, assume that there exists a sequence of functions $x_n(t)$ solving (5.10), with $0 < x_n(t)^2 + x_n'(t)^2 < \frac{1}{n}$ for every $t \in [0,T]$, such that, defining

$$\mathcal{I}(x_n) = \int_0^T \frac{\eta(t)x_n(t)^2}{\lambda_k x_n(t)^2 + x_n'(t)^2}\,dt,$$

it is $\mathcal{I}(x_n) \to 0$ for $n \to +\infty$. Set

$$v_n(t) = \frac{x_n(t)}{\|x_n\|_{C^1}},$$

and observe that, for every n,

$$v_n''(t) + \hat{f}(t, x_n(t))v_n(t) = 0, \tag{5.21}$$

with

$$\hat{f}(t,x) = \begin{cases} \dfrac{f(t,x)}{x} & x \neq 0 \\[2ex] 0 & x = 0. \end{cases}$$

Since $x_n(t)$ has at most a finite number of zeros (otherwise it would have a double zero, which is impossible since $x_n(t)^2 + x_n'(t)^2 > 0$ for every $t \in [0,T]$), it is easy to see that $\hat{f}(t, x_n(t))$ is measurable, for every n. Moreover, hypothesis (5.18) ensures that there exists $\alpha \in L^1(0,T)$ such that, for almost every $t \in [0,T]$ and every $|x| < 1$,

$$|\hat{f}(t,x)| \leq \alpha(t). \tag{5.22}$$

As $|x_n(t)| < 1$, then, by the Dunford-Pettis theorem (see [43]) there exists $a \in L^1(0,T)$ such that, up to subsequences, $\hat{f}(t, x_n(t)) \rightharpoonup a(t)$ in $L^1(0,T)$. On the other hand, a standard argument based on (5.22), on the Dunford-Pettis theorem and on Ascoli's theorem ensures the existence of a nonzero $v \in W^{2,1}(0,T)$ such that, up to subsequences, $v_n \to v$ strongly in $C^1([0,T])$ and weakly in $W^{2,1}(0,T)$.
For the reader's convenience, we just give a sketch of this argument. First of all, Ascoli's theorem implies that, up to subsequences, $v_n \to v$ in $C([0,T])$ for a suitable $v \in C([0,T])$. Secondly, since $\|v_n\|_{C^1} = 1$, and in view of (5.22),

$$|v_n''(t)| \leq \alpha(t), \tag{5.23}$$

so that the Dunford-Pettis theorem applies again, giving the existence of $w \in L^1(0,T)$ such that (up to subsequences) $v_n'' \rightharpoonup w$ in $L^1(0,T)$. Moreover, (5.23) gives the equicontinuity of the family $\{v_n'(t)\}_n$, so that, by Ascoli's theorem, $v \in C^1([0,T])$ and $v_n \to v$ in $C^1([0,T])$, implying, in particular, that $v(t)$ is nonzero. It is now easy to see that $v \in W^{2,1}(0,T)$, with $v'' = w$, as desired.
Passing to the weak L^1-limit in (5.21), it follows that $v(t)$ satisfies the linear equation

$$v'' + a(t)v = 0,$$

so that, by the uniqueness of the associated Cauchy problem, $v(t)$ has a finite number of zeros, all of them simple. Consequently, $\mathcal{I}(v)$ is well defined, and, since $\eta(t) \geq 0$ and $\frac{1}{T}\int_0^T \eta(t)\,dt > 0$, we deduce that

$$\mathcal{I}(v) = \int_0^T \frac{\eta(t)v(t)^2}{v(t)^2 + v'(t)^2}\,dt > 0.$$

On the other hand, since $v_n \to v$ in $C^1([0,T])$,

$$\mathcal{I}(x_n) = \mathcal{I}(v_n) \to \mathcal{I}(v) = 0,$$

a contradiction which proves the claim. $\square$
We now show that Lemma 5.1.1 implies the conclusion. Fix $0 < \epsilon < \lambda_k \sigma$: by assumption (5.18), there exists $0 < \rho < r$ such that, for almost every $t \in [0,T]$ and for every $|x| < \rho$,

$$f(t,x)x \geq (\lambda_k + \eta(t) - \epsilon)x^2.$$

Hence, defining, for $(x, y) \in \mathcal{A}(0, \rho)$,

$$\beta(t, x, y) = 1 + \frac{(\eta(t) - \epsilon)x^2}{\lambda_k x^2 + y^2},$$

the hypotheses of Lemma 5.1.1 are satisfied, whence the conclusion in the case $\odot = 0$.

Assume now $\odot = +\infty$.

<u>Claim.</u> *There exist $\sigma, R > 0$ such that, for every $x : [0, T] \to \mathbb{R}$ solving (5.10) with $x(t)^2 + x'(t)^2 > R^2$ for every $t \in [0, T]$, it holds*

$$\int_0^T \frac{\eta(t)x(t)^2}{\lambda_k x(t)^2 + x'(t)^2}\, dt \geq \sigma.$$

By contradiction, assume that there exists a sequence of functions $x_n(t)$ solving (5.10), with $x_n(t)^2 + x_n'(t)^2 > n$ for every $t \in [0, T]$, such that, defining

$$\mathcal{I}(x_n) = \int_0^T \frac{\eta(t)x_n(t)^2}{\lambda_k x_n(t)^2 + x_n'(t)^2}\, dt,$$

it is $\mathcal{I}(x_n) \to 0$ for $n \to +\infty$. Set

$$v_n(t) = \frac{x_n(t)}{\|x_n\|_{C^1}},$$

and observe that, for every n,

$$v_n''(t) + \hat{f}(t, x_n(t))v_n(t) + \frac{r(t, x_n(t))}{\|x_n\|_{C^1}} = 0,$$

with

$$\hat{f}(t, x) = \begin{cases} \dfrac{f(t, x)}{x} & |x| \geq 1 \\[2ex] \dfrac{(x+1)}{2} f(t, 1) + \dfrac{(x-1)}{2} f(t, -1) & |x| \leq 1, \end{cases}$$

and $r(t, x) = f(t, x) - \hat{f}(t, x)x$. Since $\hat{f}(t, x)$ and $r(t, x)$ are L^1-Carathéodory, in particular $\hat{f}(t, x_n(t))$ and $r(t, x_n(t))$ are measurable for every n. Moreover, by (5.18), there exists $\alpha \in L^1(0, T)$ such that, for almost every $t \in [0, T]$ and every $x \in \mathbb{R}$,

$$|\hat{f}(t, x)| \leq \alpha(t), \quad |r(t, x)| \leq \alpha(t).$$

Similarly as in the case $\odot = 0$, we conclude the validity of the claim. □
Fix now $0 < \epsilon < \frac{\lambda_k \sigma}{2}$: by assumption (5.18), and the fact that $f(t,x)$ is L^1-Carathéodory, there exists $z \in L^1(0,T)$ such that, for almost every $t \in [0,T]$ and every $x \in \mathbb{R}$,

$$f(t,x)x \geq (\lambda_k + \eta(t) - \epsilon)x^2 - z(t).$$

Now, choose $\rho \geq R$ such that, if $x^2 + y^2 > \rho^2$,

$$\frac{1}{\lambda_k x^2 + y^2} \int_0^T |z(t)|\, dt < \frac{\sigma}{2}.$$

Defining, for $(x,y) \in \mathcal{A}(\rho, +\infty)$,

$$\beta(t,x,y) = 1 + \frac{(\eta(t) - \epsilon)x^2 - z(t)}{\lambda_k x^2 + y^2},$$

the hypotheses of Lemma 5.1.1 are satisfied, so that the conclusion follows. □

Remark 5.1.8. Conditions (5.19) and (5.20), this last with λ_k replaced by λ_{k+1}, are the nonuniform nonresonance conditions first introduced by Mawhin and Ward in [111], for a nonlinearity satisfying

$$p(t) \leq \liminf_{|x| \to +\infty} \frac{f(t,x)}{x} \leq \limsup_{|x| \to +\infty} \frac{f(t,x)}{x} \leq q(t),$$

ensuring the solvability of (5.10). Again, here we have seen an interpretation of each of such inequalities in terms of the rotation number, and we have shown that the estimates can be carried out in an analogous way both at 0 and at $+\infty$.

5.1.2 The Landesman-Lazer condition from a rotation number viewpoint

In this subsection, we will consider again the system

$$Ju' = F(t,u), \tag{5.24}$$

with $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ an L^1-Carathéodory function. We will examine a resonant situation, namely when (5.24) is asymptotic, at infinity, to a resonant system of the type $Ju' = \zeta(t)\nabla V(u)$, with $V \in \mathcal{P}$. Our attention will be focused on the role played by the Landesman-Lazer condition in the estimate of the rotation number of large solutions. As we have already observed before, we will consider a slightly more general assumption, which includes the Landesman-Lazer conditions largely discussed in the first sections of Chapter 2.

Proposition 5.1.9. *Let $V \in \mathcal{P}$, with*

$$\frac{T}{\tau_V} \in \mathbb{N}_0.$$

Moreover, assume that there exist a continuous function $\zeta : [0,T] \to \mathbb{R}$ such that $\zeta(t) > 0$ for every $t \in [0,T]$ and

$$\frac{1}{T}\int_0^T \zeta(t)\,dt = 1, \tag{5.25}$$

and an L^1-Carathéodory function $R : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$, with

$$\lim_{|u| \to +\infty} \frac{R(t,u)}{|u|} = 0, \quad \textit{uniformly for a.e. } t \in [0,T], \tag{5.26}$$

such that

$$F(t,u) = \zeta(t)\nabla V(u) + R(t,u).$$

Set $Z(t) = \int_0^t \zeta(s)\,ds$. The following statements hold true.

- *Assume:*
  - *for almost every $t \in [0,T]$, for every $u \in \mathbb{R}^2$ with $|u| \le 1$ and for every $\lambda > 1$,*
  $$\langle R(t,\lambda u) | u \rangle \ge \eta(t), \tag{5.27}$$
  for a suitable $\eta \in L^1(0,T)$,
  - *for every $\theta \in [0,T]$,*
  $$\int_0^T \liminf_{(\lambda,\omega) \to (+\infty,\theta)} \langle R(t, \lambda\varphi_V(Z(t)+\omega)) | \varphi_V(Z(t)+\omega) \rangle > 0. \tag{5.28}$$

  Then, there exists $\rho > 0$ such that, for every $u : [0,T] \to \mathbb{R}^2$ solving (5.1), with $u(t) \in \mathcal{A}(\rho, +\infty)$ for every $t \in [0,T]$, it holds

  $$\mathrm{Rot}\,(u(t)) > \frac{T}{\tau_V}.$$

- *Assume:*
  - *for almost every $t \in [0,T]$, for every $u \in \mathbb{R}^2$ with $|u| \le 1$ and for every $\lambda > 1$,*
  $$\langle R(t,\lambda u) | u \rangle \le \eta(t), \tag{5.29}$$
  for a suitable $\eta \in L^1(0,T)$,

- *for every* $\theta \in [0, T]$,

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_V(Z(t)+\omega)) | \varphi_V(Z(t)+\omega)\rangle < 0. \tag{5.30}$$

Then, there exists $\rho > 0$ *such that for every* $u : [0, T] \to \mathbb{R}^2$ *solving (5.1), with* $u(t) \in \mathcal{A}(\rho, +\infty)$ *for every* $t \in [0, T]$*, it holds*

$$\mathrm{Rot}\,(u(t)) < \frac{T}{\tau_V}.$$

In particular, with the notation of Chapter 2, this implies that, if $V \in \mathcal{P}_k$ and $(\mathrm{LL}+)_k$ is satisfied, then large solutions to $Ju' = \nabla V(u) + R(t, u)$ have rotation number strictly larger than k. As mentioned in Remark 2.1.5, condition (5.28) (or (5.30)) could ensure existence for systems of the kind $Ju' = \zeta(t)\nabla V(u) + R(t, u)$, through the Poincaré-Bohl theorem. For briefness, we do not discuss the details.

Proof. We prove the first statement. For $u : [0, T] \to \mathbb{R}^2$ solving (5.1) and such that $u(t) \neq 0$ for every $t \in [0, T]$, we have, in view of Euler's formula and (5.25),

$$\begin{aligned}
\mathrm{Rot}_V\,(u(t)) &= \frac{1}{\tau_V}\int_0^T \frac{\langle Ju'(t)|u(t)\rangle}{2V(u(t))}\,dt = \frac{1}{\tau_V}\int_0^T \frac{\langle F(t,u(t))|u(t)\rangle}{2V(u(t))}\,dt \\
&= \frac{1}{\tau_V}\int_0^T \zeta(t)\frac{\langle \nabla V(u(t)|u(t)\rangle}{2V(u(t))}\,dt + \frac{1}{\tau_V}\int_0^T \frac{\langle R(t,u(t))|u(t)\rangle}{2V(u(t))}\,dt \\
&= \frac{T}{\tau_V} + \frac{1}{\tau_V}\int_0^T \frac{\langle R(t,u(t))|u(t)\rangle}{2V(u(t))}\,dt.
\end{aligned}$$

Hence, to conclude the proof it is sufficient to show that there exists $\rho > 0$ such that

$$\int_0^T \frac{\langle R(t,u(t))|u(t)\rangle}{2V(u(t))}\,dt > 0,$$

for every $u : [0, T] \to \mathbb{R}^2$ solving (5.1) and such that $u(t) \in \mathcal{A}(\rho, +\infty)$ for every $t \in [0, T]$. Let us suppose by contradiction that there exists a sequence of functions $u_n(t)$ solving (5.1) such that $|u_n(t)| \to +\infty$ uniformly, with

$$\int_0^T \frac{\langle R(t,u_n(t))|u_n(t)\rangle}{2V(u_n(t))}\,dt \leq 0. \tag{5.31}$$

We set $v_n(t) = \frac{u_n(t)}{\|u_n\|_{L^\infty}}$; since

$$Jv_n' = \zeta(t)\nabla V(v_n) + \frac{R(t,u_n)}{\|u_n\|_{L^\infty}}, \tag{5.32}$$

it follows from (5.26) that there exists $\alpha \in L^1(0,T)$ such that, for almost every $t \in [0,T]$,

$$|v_n'(t)| \leq \alpha(t). \tag{5.33}$$

This implies the equicontinuity of the family $\{v_n(t)\}_n$, so that, by Ascoli's theorem, there exists a nonzero $v \in C([0,T];\mathbb{R}^2)$ such that, up to subsequences, $v_n(t) \to v(t)$ uniformly. On the other hand, in view of (5.33), the Dunford-Pettis theorem can be applied, yielding the existence of $w \in L^1([0,T]\,;\mathbb{R}^2)$ such that (up to subsequences) $v_n' \rightharpoonup w$ in $L^1([0,T]\,;\mathbb{R}^2)$. It is now easy to see that $v \in W^{1,1}(]0,T[\,;\mathbb{R}^2)$, with $v' = w$. In view of (5.26), passing to the weak L^1-limit in (5.32), we then get $Jv' = \zeta(t)\nabla V(v)$, implying, by Lemma 1.1.10, that

$$v(t) = r_v \varphi_V(Z(t) + \omega_v),$$

for suitable constants $r_v > 0$ and $\omega_v \in [0, \tau_V[$. Performing the change of variables

$$u_n(t) = r_n(t)\varphi_V(Z(t) + \omega_n(t)),$$

with $\omega_n(0) \in [0, \tau_V[$ for every n, it follows that

$$\frac{r_n(t)}{\|u_n\|_{L^\infty}} \to r_v, \quad \text{uniformly in } t \in [0,T], \tag{5.34}$$

and it can be seen, as already done in the proof of Theorem 2.1.1, that

$$\omega_n(t) \to \omega_v, \quad \text{uniformly in } t \in [0,T].$$

Multiplying (5.31) by $\|u_n\|_{L^\infty}$, we get

$$\int_0^T \frac{\langle R(t, r_n(t)\varphi_V(Z(t)+\omega_n(t)))|\varphi_V(Z(t)+\omega_n(t))\rangle}{2\frac{r_n(t)}{\|u_n\|_{L^\infty}} V(\varphi_V(Z(t)+\omega_n(t)))}\, dt \leq 0.$$

Using Fatou's lemma, thanks to (5.27), and noticing that $V(\varphi_V(Z(t)+\omega_n(t))) \equiv \frac{1}{2}$, we get

$$\int_0^T \liminf_{n\to+\infty} \frac{\langle R(t, r_n(t)\varphi_V(Z(t)+\omega_n(t)))|\varphi_V(Z(t)+\omega_n(t))\rangle}{\frac{r_n(t)}{\|u_n\|_{L^\infty}}}\, dt \leq 0.$$

In view of (5.34) and using standard properties of the inferior limit, we infer that

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\omega_v)} \langle R(t, \lambda\varphi_V(Z(t)+\omega))|\varphi_V(Z(t)+\omega)\rangle\, dt \leq 0,$$

a contradiction. $\square$

Remark 5.1.10. Notice that, similarly as in Remark 5.1.4, hypothesis (5.26) can be weakened into the following L^1-condition:

for every $\epsilon > 0$, there exists $b_\epsilon \in L^1(0,T)$ such that, for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$,

$$|R(t,u)| \leq \epsilon|u| + b_\epsilon(t).$$

Indeed, this is enough to carry out the same proof, in particular to pass to the limit in (5.32).

Let us now focus on the second order case. Consider again the equation

$$x'' + f(t,x) = 0, \tag{5.35}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory. As a corollary of Proposition 5.1.9, we have the following result.

Corollary 5.1.11. *Assume that there exist $k \in \mathbb{N}_0$ and $\mu, \nu > 0$, with*

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{k}, \tag{5.36}$$

such that, uniformly for almost every $t \in [0,T]$,

$$\lim_{x \to +\infty} \frac{f(t,x)}{x} = \mu, \quad \lim_{x \to -\infty} \frac{f(t,x)}{x} = \nu. \tag{5.37}$$

The following statements hold true.

- *Assume:*
  - *for almost every $t \in [0,T]$, and every $x \in \mathbb{R}$,*

    $$\operatorname{sgn}(x)(f(t,x) - \mu x^+ + \nu x^-) \geq \eta(t), \tag{5.38}$$

    for a suitable $\eta \in L^1(0,T)$,
  - *for every $\phi(t)$ nontrivial solution (defined on $[0,T]$) to $x'' + \mu x^+ - \nu x^- = 0$,*

    $$\int_{\{\phi>0\}} \liminf_{x \to +\infty}(f(t,x) - \mu x)\phi(t)\,dt + \int_{\{\phi<0\}} \limsup_{x \to -\infty}(f(t,x) - \nu x)\phi(t)\,dt > 0. \tag{5.39}$$

  Then, there exists $\rho > 0$ such that for every $x : [0,T] \to \mathbb{R}$ solving (5.35), with $x(t)^2 + x'(t)^2 > \rho^2$ for every $t \in [0,T]$, it holds

  $$\operatorname{Rot}((x(t), x'(t))) > k.$$

- *Assume:*
  - *for almost every* $t \in [0,T]$, *and every* $x \in \mathbb{R}$,
$$\operatorname{sgn}(x)(f(t,x) - \mu x^+ + \nu x^-) \leq \eta(t), \tag{5.40}$$
for a suitable $\eta \in L^1(0,T)$,
  - *for every* $\phi(t)$ *nontrivial solution (defined on* $[0,T]$*) to* $x'' + \mu x^+ - \nu x^- = 0$,
$$\int_{\{\phi>0\}} \limsup_{x\to+\infty}(f(t,x) - \mu x)\phi(t)\,dt + \int_{\{\phi<0\}} \liminf_{x\to-\infty}(f(t,x) - \nu x)\phi(t)\,dt < 0. \tag{5.41}$$

Then, there exists $\rho > 0$ *such that for every* $x : [0,T] \to \mathbb{R}$ *solving (5.35), with* $x(t)^2 + x'(t)^2 > \rho^2$ *for every* $t \in [0,T]$, *it holds*
$$\operatorname{Rot}((x(t), x'(t))) < k.$$

Proof. Setting $u = (x,y)$, $F(t,u) = (f(t,x),y)$, $V(x,y) = \frac{1}{2}(\mu(x^+)^2 + \nu(x^-)^2 + y^2)$ and $\zeta(t) \equiv 1$, we briefly show that the hypotheses of Proposition 5.1.9 (actually, in the weaker version of Remark 5.1.10) are satisfied. Using (5.36), it is easy to see that $\frac{T}{\tau_V} = k$. Moreover, (5.37) and the Carathéodory assumption imply that, fixed $\epsilon > 0$, there exists $b_\epsilon \in L^1(0,T)$ such that
$$|f(t,x) - \mu x^+ + \nu x^-| \leq \epsilon|x| + b_\epsilon(t),$$
from which the condition of Remark 5.1.10 follows. Finally, (5.38) implies (5.27), and (5.40) implies (5.29), while for the proof that (5.39) and (5.41) imply (5.28) and (5.30), respectively, we refer to Section 2.4. □

Remark 5.1.12. Observe that, in this case, it is considered the more general situation of resonance with respect to the T-periodic Fučík spectrum (see [68]). Indeed, when (5.36) holds, all the nontrivial solutions to $x'' + \mu x^+ - \nu x^- = 0$ are T-periodic, and have $2k$ zeros in the interval $[0,T[$. When $\mu = \nu$, (5.36) implies $\mu = \lambda_k$, so that we recover the well known linear theory.

Remark 5.1.13. We do not know if a condition of Landesman-Lazer type can be formulated at zero, to control the behavior of "small" solutions (clearly, the natural one obtained by the mere replacement of ∞ with 0 is senseless). To this aim, to the best of our knowledge, rougher sign conditions are usually employed.

5.2 Multiplicity results for unforced planar Hamiltonian systems and second order equations

In this section, we will apply the estimates developed in Section 5.1, together with the Poincaré-Birkhoff theorem, in order to give some multiplicity results for the (unforced) planar Hamiltonian system $Ju' = \nabla H(t,u)$ and the (unforced undamped) second order equation $x'' + f(t,x) = 0$. Notice that, even if the results of such sections are valid in the general context of planar systems, now we need the Hamiltonian structure in order to apply the Poincaré-Birkhoff fixed point theorem. In particular, this is the case for undamped second order equations, with $H(t,x,y) = (1/2)y^2 + \int_0^x f(t,\xi)\,d\xi$. All the forthcoming statements will exploit both an assumption at zero and an assumption at infinity among the ones previously introduced (so that, for related comments, we refer the reader back to the corresponding subsections), in order to show that a twist condition is satisfied. For the sake of brevity, we will limit ourselves to a few combinations of them, systematically considering a lim inf-inequality at zero and a lim sup-inequality at infinity, but it will be clear that several other statements could be obtained, in the same way. We postpone a sketch of the (standard) proofs after the statements.

The first two results, in particular, concern the planar Hamiltonian system

$$Ju' = \nabla H(t,u), \tag{5.42}$$

where $H : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is differentiable in the second variable and $\nabla H(t,u)$ is L^1-Carathéodory.
We consider first a nonresonance situation.

Theorem 5.2.1. *Assume the uniqueness and the global continuability for the solutions to the Cauchy problems associated with (5.42). Moreover, assume:*

(H_0) $\nabla H(t,0) \equiv 0$ *and there exist* $V_0 \in \mathcal{P}$, $k_0 \in \mathbb{N}_0$ *and* $\zeta_0 \in L^1(0,T)$, *with*

$$\frac{T}{\tau_{V_0}} > k_0, \qquad \frac{1}{T}\int_0^T \zeta_0(t)\,dt = 1,$$

such that

$$\liminf_{|u|\to 0} \frac{\langle \nabla H(t,u) | u \rangle}{2V_0(u)} \geq \zeta_0(t), \quad \textit{uniformly for a.e. } t \in [0,T];$$

(H_∞) there exist $V_\infty \in \mathcal{P}$, $k_\infty \in \mathbb{N}_0$ *and* $\zeta_\infty \in L^1(0,T)$, *with*

$$\frac{T}{\tau_{V_\infty}} < k_\infty, \qquad \frac{1}{T}\int_0^T \zeta_\infty(t)\,dt = 1,$$

such that

$$\limsup_{|u|\to\infty} \frac{\langle \nabla H(t,u)|u\rangle}{2V_\infty(u)} \leq \zeta_\infty(t), \quad \textit{uniformly for a.e. } t \in [0,T].$$

Then, for every integer $k \in [k_\infty, k_0]$ *(if any), there exist two* T*-periodic solutions* $u_{1,k}(t), u_{2,k}(t)$ *to (5.42) such that*

$$\mathrm{Rot}\,(u_{1,k}(t)) = \mathrm{Rot}\,(u_{2,k}(t)) = k.$$

Remark 5.2.2. Clearly, a symmetric statement can be obtained when we assume a lim sup-inequality in (H_0) and a lim inf-inequality in (H_∞), giving existence of solutions for any integer $k \in [k_0, k_\infty]$ (if any). As a consequence, when

$$\lim_{|u|\to 0} \frac{\langle \nabla H(t,u)|u\rangle}{2V_0(u)} = \zeta_0(t), \qquad \lim_{|u|\to\infty} \frac{\langle \nabla H(t,u)|u\rangle}{2V_\infty(u)} = \zeta_\infty(t),$$

and

$$\frac{T}{\tau_{V_0}} \notin \mathbb{N}, \quad \frac{T}{\tau_{V_\infty}} \notin \mathbb{N},$$

the statement provides the existence of

$$2\left|\left\lfloor \frac{T}{\tau_{V_0}} \right\rfloor - \left\lfloor \frac{T}{\tau_{V_\infty}} \right\rfloor\right|$$

T-periodic solutions. Notice that this, in particular, holds in the case when

$$\nabla H(t,u) = \zeta_0(t)\nabla V_0(u) + o(|u|), \qquad |u| \to 0,$$

$$\nabla H(t,u) = \zeta_\infty(t)\nabla V_\infty(u) + o(|u|), \qquad |u| \to \infty.$$

The spirit of this kind of results is similar to the ones of [53, Theorem 6], [59, Theorem 1].

Now we pass to consider a resonant (at infinity) situation, i.e., when, with the previous notation,

$$\frac{T}{\tau_{V_\infty}} = k_\infty \in \mathbb{N}_0,$$

or, equivalently, $V_\infty \in \mathcal{P}_{k_\infty}$. Of course, in this case, Theorem 5.2.1 can still be applied, giving T-periodic solutions with rotation number equal to k for every $k \in [k_\infty + 1, k_0]$ (if any). However, the existence of solutions making exactly k_∞ revolutions is no longer guaranteed. To recover it, we will add a Landesman-Lazer condition.

Theorem 5.2.3. *Assume the uniqueness for the solutions to the Cauchy problems associated with (5.42). Moreover, assume:*

(H_0) $\nabla H(t,0) \equiv 0$ *and there exist* $V_0 \in \mathcal{P}$, $k_0 \in \mathbb{N}_0$ *and* $\zeta_0 \in L^1(0,T)$, *with*

$$\frac{T}{\tau_{V_0}} > k_0, \qquad \frac{1}{T}\int_0^T \zeta_0(t)\,dt = 1,$$

such that

$$\liminf_{|u|\to 0} \frac{\langle \nabla H(t,u)|u\rangle}{2V_0(u)} \geq \zeta_0(t), \quad \text{uniformly for a.e. } t \in [0,T];$$

(H_∞) there exist $V_\infty \in \mathcal{P}$, $k_\infty \in \mathbb{N}_0$, *and a continuous function* $\zeta_\infty : [0,T] \to \mathbb{R}$, *with* $\zeta_\infty(t) > 0$ *for every* $t \in [0,T]$, *such that*

$$\frac{T}{\tau_{V_\infty}} = k_\infty, \qquad \frac{1}{T}\int_0^T \zeta_\infty(t)\,dt = 1,$$

and an L^1*-Carathéodory function* $R_\infty : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$, *with*

$$\lim_{|u|\to+\infty} \frac{R_\infty(t,u)}{|u|} = 0, \quad \text{uniformly for a.e. } t \in [0,T],$$

such that

$$\nabla H(t,u) = \zeta_\infty(t)\nabla V_\infty(u) + R_\infty(t,u).$$

Moreover, setting $Z_\infty(t) = \int_0^t \zeta_\infty(s)\,ds$, *suppose that*

- *for almost every* $t \in [0,T]$, *for every* $u \in \mathbb{R}^2$ *with* $|u| \leq 1$ *and for every* $\lambda > 1$,

$$\langle R_\infty(t,\lambda u)|u\rangle \leq \eta(t),$$

for a suitable $\eta \in L^1(0,T)$,

- *for every* $\theta \in [0,T]$,

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R_\infty(Z_\infty(t), \lambda\varphi_{V_\infty}(Z_\infty(t)+\omega))|\varphi_{V_\infty}(Z_\infty(t)+\omega)\rangle < 0.$$

Then, for every integer $k \in [k_\infty, k_0]$ *(if any), there exist two* T*-periodic solutions* $u_{1,k}(t), u_{2,k}(t)$ *to (5.42) such that*

$$\mathrm{Rot}\,(u_{1,k}(t)) = \mathrm{Rot}\,(u_{2,k}(t)) = k.$$

We now deal with the second order equation

$$x'' + f(t,x) = 0, \tag{5.43}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, as usual, is an L^1-Carathéodory function. The first result concerns a nonresonant situation, and can be seen as a more applicable version of the results in [136]. Observe, however, that neither at zero nor at infinity do we require the existence of the limits for the ratio $\frac{f(t,x)}{x}$.

Theorem 5.2.4. *Assume the uniqueness and the global continuability for the solutions to the Cauchy problems associated with (5.43). Moreover, assume:*

(f_0) $f(t,0) \equiv 0$ *and there exist* $k_0 \in \mathbb{N}_0$ *and* $p \in L^1(0,T)$ *with*

$$\sup_{\xi>0} \frac{\frac{1}{T}\int_0^T \{\min p(t), \xi\}\, dt}{\sqrt{\xi}} > \sqrt{\lambda_{k_0}},$$

such that

$$\liminf_{x\to 0} \frac{f(t,x)}{x} \geq p(t), \quad \textit{uniformly for a.e. } t \in [0,T];$$

(f_∞) there exist $k_\infty \in \mathbb{N}_0$ *and* $q \in L^1(0,T)$ *with*

$$\inf_{\zeta>0} \frac{\frac{1}{T}\int_0^T \max\{q(t), \zeta\}\, dt}{\sqrt{\zeta}} < \sqrt{\lambda_{k_\infty}},$$

such that

$$\limsup_{|x|\to+\infty} \frac{f(t,x)}{x} \leq q(t), \quad \textit{uniformly for a.e. } t \in [0,T].$$

Then, for every integer $k \in [k_\infty, k_0]$ *(if any), there exist two* T*-periodic solutions* $x_{1,k}(t), x_{2,k}(t)$ *to (5.43) having* $2k$ *zeroes in* $[0,T[$.

Finally, as in Theorem 5.2.3, we consider the situation when the nonlinearity interacts, at infinity, with the Fučík spectrum, adding again a Landesman-Lazer condition (in the Fabry sense [46]). Even if assumption (f_0) of Theorem 5.2.4 is still suitable, this time we propose a nonuniform nonresonance condition.

Theorem 5.2.5. *Assume the uniqueness for the Cauchy problems associated with (5.43). Moreover, assume:*

(f_0) $f(t,0) \equiv 0$ and there exist $k_0 \in \mathbb{N}_0$ and $p, q \in L^1(0,T)$, with

$$p(t) \geq \lambda_{k_0} \quad \text{and} \quad \frac{1}{T}\int_0^T p(t)\,dt > \lambda_{k_0},$$

such that

$$p(t) \leq \liminf_{x\to 0} \frac{f(t,x)}{x} \leq \limsup_{x\to 0} \frac{f(t,x)}{x} \leq q(t), \quad \textit{uniformly for a.e. } t \in [0,T],$$

(f_∞) there exist $k_\infty \in \mathbb{N}_0$ and $\mu, \nu > 0$ with

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{k_\infty},$$

and $\eta \in L^1(0,T)$ such that:

- *uniformly for almost every $t \in [0,T]$,*

$$\lim_{x\to+\infty} \frac{f(t,x)}{x} = \mu, \quad \lim_{x\to-\infty} \frac{f(t,x)}{x} = \nu,$$

- *for almost every $t \in [0,T]$, and every $x \in \mathbb{R}$,*

$$\operatorname{sgn}(x)(f(t,x) - \mu x^+ + \nu x^-) \geq \eta(t),$$

- *for every $\phi(t)$ nontrivial solution (defined on $[0,T]$) to $x'' + \mu x^+ - \nu x^- = 0$,*

$$\int_{\{\phi>0\}} \limsup_{x\to+\infty}(f(t,x) - \mu x)\phi(t)\,dt + \int_{\{\phi<0\}} \liminf_{x\to-\infty}(f(t,x) - \nu x)\phi(t)\,dt < 0.$$

Then, for every integer $k \in [k_\infty, k_0]$ (if any), there exist two T-periodic solutions $x_{1,k}(t), x_{2,k}(t)$ to (5.43) having $2k$ zeroes in $[0,T[$.

We now sketch the proofs of Theorems 5.2.1, 5.2.3, 5.2.4 and 5.2.5. First of all, notice that the uniqueness is always assumed; on the other hand, the global continuability is assumed in Theorem 5.2.1 and in Theorem 5.2.4, while it follows from (H_∞) in Theorem 5.2.3 and (f_∞) in Theorem 5.2.5. As it is well known, these facts, together with $\nabla H(t,0) \equiv 0$, imply the following elastic properties (compare with Lemma 2.0.1):

(E_0) for every $r > 0$, there exists $0 < \rho_0 < r$ such that, if $\bar{u} \in \overline{\mathcal{A}(0,\rho_0)}$, then $u(t;\bar{u}) \in \mathcal{A}(0,r)$ for every $t \in [0,T]$;

(E_∞) for every $R > 0$, there exists $\rho_\infty > R$ such that, if $\bar{u} \in \overline{\mathcal{A}(\rho_\infty, +\infty)}$, then $u(t; \bar{u}) \in \mathcal{A}(R, \infty)$ for every $t \in [0, T]$.

We can now conclude by Theorem 1.2.9, in view of Proposition 5.1.2 for Theorem 5.2.1, Propositions 5.1.2 and 5.1.9 for Theorem 5.2.3, Proposition 5.1.5 for Theorem 5.2.4 and Proposition 5.1.7 and Corollary 5.1.11 for Theorem 5.2.5.

Remark 5.2.6. We remark that, in Theorems 5.2.1 and 5.2.3, $\zeta_0(t)$ and $\zeta_\infty(t)$ are essentially positive functions. It would be possible to consider negative functions, as well, possibly giving the existence of T-periodic solutions rotating counterclockwise. For the sake of brevity, however, since this is not possible for the second order case in the standard phase plane setting, we have chosen to present our results from a simpler point of view.

5.3 Final remarks

So far, we have examined the situation of positive Hamiltonians, i.e., functions belonging to the class $\mathcal{P}$. It is now natural to wonder if it is possible to give a rotational interpretation of condition (2.10) introduced in Chapter 2, and if it is useful in order to refine some multiplicity results for planar systems. We will only briefly explain some features of this situation, referring, for what follows, to [11] for the complete discussion.

One can thus imagine to deal with the system

$$Ju' = \nabla H(t, u), \tag{5.44}$$

where we assume $\nabla H(t, 0) \equiv 0$ and it is possible to linearize $\nabla H(t, u)$ at infinity as follows:

$$\nabla H(t, u) = \nabla V_\infty(u) + R_\infty(t, u), \quad |u| \to +\infty,$$

with $V_\infty \in \mathcal{P}_0$. On the other hand, since we want to produce a gap between 0 and infinity and large solutions move clockwise, a possibility, roughly speaking, is to assume that near 0 some solutions of the linearized system rotate counterclockwise around the origin, or, in alternative, that the origin is a saddle equilibrium point. Next step consists in checking if the rotation number of large solutions to (5.44), which is difficult to control since V_∞ is resonant, is positively affected by the Landesman-Lazer condition (2.10). An affirmative answer is given by the following proposition [11, Theorem 4.2], which is in the spirit of Proposition 5.1.9.

Proposition 5.3.1. *Let $V_\infty \in \mathcal{P}_0$ and assume that $R_\infty : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies* (5.26)*,* (5.27)*, and condition* (2.10)*. Then, there exists $\rho > 0$ such that, for every $u : [0, T] \to \mathbb{R}^2$ solving*

$$Ju' = \nabla V_\infty(u) + R_\infty(t, u),$$

such that $u(t) \in \mathcal{A}(\rho, +\infty)$ *for every* $t \in [0, T]$*, it holds*

$$\mathrm{Rot}(u(t)) > 0.$$

The proof can be performed by contradiction, making use of the Lebesgue dominated convergence theorem.
Thus, referring to (5.44), (2.10) implies that large-norm solutions necessarily turn clockwise strictly more than 0 times in the time T; the gap between the rotations of small and large solutions is then obtained thanks to the hypothesis near 0. Indeed, a classical gap construction as in the previous sections of this chapter would not be possible, since, with the assumptions therein, small solutions would turn clockwise, in any case (a positively homogeneous principal term gives a contribution in this direction). In fact, the classical Poincaré-Birkhoff theorem is not suitable to deal with this particular case, but it is needed a modified version in which, roughly speaking, only the rotations of solutions starting at certain points of the plane need to be controlled. In this way, under the previous assumptions, Proposition 5.3.1 turns to be sufficient to provide the desired gap but only one solution, with rotation number equal to 0, is found. For the details and the precise statements, we refer to [11, Theorem 2.1, Corollary 2.1 and Theorem 4.2].

To conclude the chapter, since we have dealt with the Poincaré-Birkhoff theorem, a last remark is deserved by the search for subharmonic solutions to the planar Hamiltonian system (5.42). In this case, of course, we mean that $H(t, u)$ is defined on $\mathbb{R} \times \mathbb{R}^2$, and T-periodic in the first variable. Accordingly, by a subharmonic solution of order $m \in \mathbb{N}_0$, $m \geq 2$, we mean an mT-periodic solution (in the usual sense) which is not lT-periodic for any integer $l < m$, namely mT is the minimal period inside the class of the integer multiples of T. Notice that, if $u(t)$ is a mT-periodic solution with

$$\mathrm{Rot}\,(u(t); [0, mT]) = k,$$

then $u(t)$ is a subharmonic of order m whenever m and k are relatively prime integers, i.e., their greatest common divisor is 1.
Therefore, since the multiplicity results obtained via the Poincaré-Birkhoff theorem produce solutions with prescribed rotation numbers, it is seen that they are also suitable, in principle, to get subharmonic solutions (just by replacing T with mT, finding solutions with rotation number equal to k when m and k are relatively prime). The situation when, keeping assumption (H_0) in Theorem 5.2.1, the Hamiltonian satisfies a subquadraticity-like condition at infinity has been considered in [10, Theorem 3.1], in order to produce subharmonics of order m, for every m large enough, with a sharp nodal characterization. On the other hand, we incidentally notice that the case of superquadratic growth at infinity has been recently studied in [12] (proving existence

of infinitely many periodic solutions, harmonic and subharmonic).
We now show that, whenever *any* kind of gap between 0 and ∞ is assumed, once again independently of the fact that the considered Hamiltonians are resonant or not, it is still possible, exploiting a number theory argument developed in [33], to prove the existence of subharmonics of order m, for every m large enough (though with a weaker nodal characterization with respect to [10]).

Theorem 5.3.2. *Assume the uniqueness and the global continuability for the solutions to the Cauchy problems associated with (5.42). Moreover, suppose:*

(H_0) $\nabla H(t,0) \equiv 0$ *and there exist* $V_0 \in \mathcal{P}$ *and* $\zeta_0 \in L^1(0,T)$*, with*

$$\frac{1}{T}\int_0^T \zeta_0(t)\,dt = 1,$$

such that

$$\liminf_{|u|\to 0} \frac{\langle \nabla H(t,u)|u\rangle}{2V_0(u)} \geq \zeta_0(t), \quad \textit{uniformly for a.e. } t \in [0,T];$$

(H_∞) there exist $V_\infty \in \mathcal{P}$ *and* $\zeta_\infty \in L^1(0,T)$ *with*

$$\frac{1}{T}\int_0^T \zeta_\infty(t)\,dt = 1,$$

such that

$$\limsup_{|u|\to \infty} \frac{\langle \nabla H(t,u)|u\rangle}{2V_\infty(u)} \leq \zeta_\infty(t), \quad \textit{uniformly for a.e. } t \in [0,T].$$

Finally, assume that

$$\frac{T}{\tau_{V_\infty}} < \frac{T}{\tau_{V_0}}. \tag{5.45}$$

Then, for every $r \in \mathbb{N}_0$*, there exists an integer* $m^*(r)$ *such that, for every* $m \geq m^*(r)$*, there exist* $2r$ *subharmonics of order* m *solving (5.42).*

We remark that condition (5.45) is not sufficient to obtain the existence of a T-periodic solution, since there could be no integers in the interval $\left]\frac{T}{\tau_{V_\infty}}, \frac{T}{\tau_{V_0}}\right[$. Moreover, observe that, fixed $m \in \mathbb{N}_0$, we are not showing the existence of infinitely many subharmonics of order m; nevertheless, we have subharmonics for any m large enough, and such a number increases with m.

Proof. We give a brief sketch of the proof. It has to be shown that, for every $r \in \mathbb{N}_0$, there exists $m^*(r)$ such that, for every $m \geq m^*(r)$, there exist r integers $k_1^{(m)}, \ldots, k_r^{(m)}$, each one relatively prime with m, and such that, for every $i = 1, \ldots, r$,

$$\frac{T}{\tau_{V_0}} < \frac{k_i^{(m)}}{m} < \frac{T}{\tau_{V_\infty}}.$$

In view of Theorem 5.2.1, this gives, for every $i = 1, \ldots, r$, the existence of two mT-periodic solutions making $k_i^{(m)}$ turns around the origin, whence the conclusion. Dividing the interval $\left]\frac{T}{\tau_{V_\infty}}, \frac{T}{\tau_{V_0}}\right[$ into r subintervals, without loss of generality it is possible to assume $r = 1$. In this case, the argument to achieve the conclusion is based on some properties of prime numbers, and can be found in the proof of [33, Theorem 2.3]. □

Remark 5.3.3. We observe that, as pointed out in the proof of [119, Theorem 5], it is possible to show that, for each $i = 1, \ldots, r$, the two subharmonics of order m making $k_i^{(m)}$ turns around the origin do not belong to the same periodicity class, i.e., each of them is not a time translation of the other one by lT, with $l = 1, \ldots, m-1$.

Of course, Theorem 5.3.2 describes just a model situation, and the more complicated cases previously considered could also be taken into account.

Chapter 6

General positively homogeneous boundary conditions

In this chapter, we leave the periodic problem and take into account different boundary conditions from the point of view of resonance. Starting from the simplest case, represented by the Dirichlet boundary conditions, we notice that the associated situation is substantially different from the periodic case. For the Dirichlet problem, as already noticed in [29, Proposition 1], it is not sufficient that the pair (μ, ν) $(\mu, \nu > 0)$ does not belong to the Fučík spectrum to guarantee, for every forcing term $e(t)$, the existence of a solution to $x'' + \mu x^+ - \nu x^- = e(t)$ satisfying the boundary conditions, but some regions between the Fučík curves must also be avoided.
Passing then to consider a planar system of the type

$$Ju' = \nabla V(u) + R(t, u), \tag{6.1}$$

with $V \in \mathcal{P}$ and $R : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ continuous, together with some boundary conditions including the Sturm-Liouville ones, it is not strange that some stronger nonresonance assumptions is needed to ensure existence.
Throughout the chapter, we will consider equation (6.1), together with the following boundary conditions: fixed a "starting" cone $\mathcal{C}_S$ and an "arrival" cone $\mathcal{C}_A$, we will ask

$$u(0) \in \mathcal{C}_S, \quad u(T) \in \mathcal{C}_A. \tag{6.2}$$

This choice is very natural, in view of the positive homogeneity of $V(u)$. Notice that, in this way, we can include in our study the Dirichlet problem (also known as "Bolza problem" in this framework), as well as the Neumann or the Sturm-Liouville ones. We will generalize to this setting the nonresonance conditions introduced by Fučík in [67], following the scheme proposed in [53] for the periodic problem, and then discuss the solvability of (6.1)–(6.2) under these assumptions. Notice that we assume $V \in \mathcal{P}$

since we need to define some comparison times to cover certain angular regions, but it would be interesting to consider the case $V \in \mathcal{P}_0$, as well, also in view of the fact that this time a vanishing Hamiltonian does not necessarily give rise to resonance.

6.1 Abstract existence and multiplicity results

We start by preliminarily specifying what we mean by an *admissible cone* in the plane.

Definition 6.1.1. A nonempty closed subset $\mathcal{C}$ of $\mathbb{R}^2$ is a *cone* if

$$[u \in \mathcal{C} \text{ and } \kappa \geq 0] \implies \kappa u \in \mathcal{C}.$$

We say that a cone $\mathcal{C}$ is *admissible* if $\mathbb{R}^2 \setminus \mathcal{C}$ is disconnected.

For every $\bar{u} \in \mathbb{R}^2$, let us denote by $u(\cdot\,;\bar{u})$ the solution to

$$\begin{cases} Ju' = \nabla V(u) \\ u(0) = \bar{u}, \end{cases}$$

which is unique and globally defined, since $V \in \mathcal{P}$. We define the continuous function $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathcal{F}(\bar{u}) = u(T;\bar{u}). \tag{6.3}$$

It is clear that, if $\mathcal{C}$ is a cone, then $\mathcal{F}(\mathcal{C})$ is also a cone, since, by homogeneity, $u(\cdot\,;\kappa\bar{u}) = \kappa u(\cdot\,;\bar{u})$, for every $\kappa \geq 0$.

We now fix two cones $\mathcal{C}_S$ and $\mathcal{C}_A$ (the "starting" and the "arrival" cones) and consider the boundary value problem

$$\begin{cases} Ju' = \nabla V(u) + R(t,u) \\ u(0) \in \mathcal{C}_S,\ u(T) \in \mathcal{C}_A. \end{cases} \tag{6.4}$$

Let us state our main abstract result.

Theorem 6.1.2. *Let the following assumption hold:*

(A) *The cone $\mathcal{C}_A$ is admissible and $\mathcal{F}(\mathcal{C}_S)$ has a nonempty intersection with at least two different connected components of $\mathbb{R}^2 \setminus \mathcal{C}_A$.*

If, moreover,

$$\lim_{|u|\to+\infty} \frac{R(t,u)}{|u|} = 0, \qquad \text{uniformly in } t \in [0,T], \tag{6.5}$$

then, problem (6.4) *is solvable.*

Proof. In view of assumption (A), there exist $\bar{u}_1, \bar{u}_2 \in \mathcal{C}_S \setminus \{0\}$ such that their images under the map $\mathcal{F}$ defined in (6.3) belong to two different connected components $\mathcal{W}_1, \mathcal{W}_2$ of $\mathbb{R}^2 \setminus \mathcal{C}_A$. Let $(R_n)_n$ be a sequence of locally Lipschitz continuous functions $R_n : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $R_n(t,u) \to R(t,u)$ uniformly for $(t,u) \in [0,T] \times \mathbb{R}^2$. We consider, for $\lambda \geq 1$, the Cauchy problems

$$\begin{cases} Ju' = \nabla V(u) + R_n(t,u) \\ u(0) = \lambda \bar{u}_1 \,. \end{cases} \tag{6.6}$$

Setting $v(t) = \frac{1}{\lambda} u(t)$, system (6.6) is equivalent to the following:

$$\begin{cases} Jv' = \nabla V(v) + \dfrac{R_n(t,\lambda v)}{\lambda} \\ v(0) = \bar{u}_1 \,. \end{cases} \tag{6.7}$$

We denote by $v_{\lambda,n}(\cdot\,; \bar{u}_1)$ the solution to (6.7). By (6.5) and the uniform convergence of $R_n(t,u)$ to $R(t,u)$, we have that

$$\frac{R_n(t,\lambda v)}{\lambda} \to 0, \qquad \text{as } \lambda \to +\infty \text{ and } n \to +\infty,$$

uniformly in $t \in [0,T]$ and v in any compact subset of $\mathbb{R}^2$. By continuous dependence, cf. [24] , for every fixed $\eta > 0$, there are λ and n sufficiently large such that

$$|v_{\lambda,n}(t; \bar{u}_1) - u(t; \bar{u}_1)| \leq \eta,$$

for every $t \in [0,T]$. Since $\mathcal{W}_1$ is open and $u(T; \bar{u}_1) \in \mathcal{W}_1$, taking η sufficiently small there exist some sufficiently large $\lambda^* \geq 1$ and $n^* \geq 1$ such that $v_{\lambda,n}(T; \bar{u}_1)$ belongs to $\mathcal{W}_1$ for every $\lambda \geq \lambda^*$ and $n \geq n^*$. Analogously, enlarging λ^* and n^* if necessary, we will have that $v_{\lambda,n}(T; \bar{u}_2) \in \mathcal{W}_2$ for every $\lambda \geq \lambda^*$ and $n \geq n^*$, with the analogous convention in the notation.

We now fix $\lambda = \lambda^*$ and use the notation $v_n(t; \bar{u}) = v_{\lambda^*,n}(t, \bar{u})$. We consider the continuous path $\gamma : [0,1] \to \mathcal{C}_S$ defined by

$$\gamma(s) = \begin{cases} (1-2s)\bar{u}_1 & \text{if} \quad s \in \left[0, \frac{1}{2}\right] \\ (2s-1)\bar{u}_2 & \text{if} \quad s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

By continuous dependence on the initial data, the map $s \mapsto v_n(T; \gamma(s))$ is continuous, for every $n \geq n^*$; moreover, $v_n(T; \gamma(0)) \in \mathcal{W}_1$ and $v_n(T; \gamma(1)) \in \mathcal{W}_2$. Since $\mathcal{C}_A$ separates $\mathcal{W}_1$ and $\mathcal{W}_2$, by continuity this implies that, for every $n \geq n^*$, there exists s_n^* such that $v_n(T; \gamma(s_n^*)) \in \mathcal{C}_A$.

Let $u_n(t) = \lambda^* v_n(t; \gamma(s_n^*))$. By the above arguments, $(u_n)_n$ is uniformly bounded.

Passing to a subsequence, we can assume that $s_n^* \to s^*$ for some $s^* \in [0,1]$. On the other hand, as $u_n(t)$ solves the differential equation in (6.6), we have that $(u_n')_n$ is also uniformly bounded, so that, by Ascoli-Arzelà Theorem, there is a subsequence $(u_{n_k})_k$ which uniformly converges to some continuous function $\hat{u}(t)$. Then, $\hat{u}(0) = \gamma(s^*) \in \mathcal{C}_S$ and, passing to the limit in

$$u_{n_k}(t) = u_{n_k}(0) - J \int_0^t (\nabla V(u_{n_k}(r)) + R_{n_k}(r, u_{n_k}(r)))\, dr,$$

we have that $\hat{u}(t)$ is a solution to the differential equation in (6.4). Since $\mathcal{C}_A$ is closed and $u_n(T; \gamma(s_n^*)) \in \mathcal{C}_A$, being

$$\hat{u}(T) = \lim_{n \to +\infty} u_n(T; \gamma(s_n^*)),$$

we have that $\hat{u}(T) \in \mathcal{C}_A$. The proof is thus completed. $\square$

As an immediate corollary, under assumption (A) we have existence in the case when the function $R(t,u)$ appearing in (6.4) does not depend on u. This fact reminds a classical feature of nonresonance for the forced system

$$\begin{cases} Ju' = \nabla V(u) + E(t) \\ u(0) \in \mathcal{C}_S,\ u(T) \in \mathcal{C}_A. \end{cases}$$

Remark 6.1.3. The approximation process used above could have been avoided using the shooting approach without uniqueness developed in [27, Section 2].

We now turn to the issue of multiplicity for the boundary value problem

$$\begin{cases} Ju' = F(t,u) \\ u(0) \in \mathcal{C}_S,\ u(T) \in \mathcal{C}_A, \end{cases} \tag{6.8}$$

being $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ continuous, locally Lipschitz continuous in its second variable, and satisfying

$$F(t,0) \equiv 0.$$

Hence, $u(t) \equiv 0$ trivially satisfies (6.8). Moreover, we will assume that there exist $V_0, V_\infty \in \mathcal{P}$ such that

$$\lim_{|u| \to 0} \frac{F(t,u) - \nabla V_0(u)}{|u|} = 0, \tag{6.9}$$

$$\lim_{|u| \to +\infty} \frac{F(t,u) - \nabla V_\infty(u)}{|u|} = 0. \tag{6.10}$$

Given $\bar{u} \in \mathbb{R}^2$, let us denote by $u_0(t;\bar{u})$ and $u_\infty(t;\bar{u})$, respectively, the solutions to the Cauchy problems

$$\begin{cases} Ju' = \nabla V_0(u) \\ u(0) = \bar{u}, \end{cases} \qquad \begin{cases} Ju' = \nabla V_\infty(u) \\ u(0) = \bar{u}. \end{cases} \tag{6.11}$$

We will write the starting and the arrival cones as union of half-lines: precisely,

$$\mathcal{C}_S = \bigcup_{\alpha \in \mathcal{I}_S} \eta_S^\alpha, \quad \mathcal{C}_A = \bigcup_{\beta \in \mathcal{I}_A} \eta_A^\beta,$$

where $\mathcal{I}_S$ and $\mathcal{I}_A$ are sets of indexes, possibly infinite and uncountable, η_S^α and η_A^β are half-lines emanating from the origin and the above unions are disjoint (except for the origin). Moreover, we define the nonnegative integers $n_0^{\alpha,\beta}, n_\infty^{\alpha,\beta}$ as follows: denoting by $\hat{u}^\alpha$ the only point in η_S^α with $|\hat{u}^\alpha| = 1$,

$$n_0^{\alpha,\beta} = \#\{t \in]0,T[\mid u_0(t,\hat{u}^\alpha) \in \eta_A^\beta\}, \tag{6.12}$$

and

$$n_\infty^{\alpha,\beta} = \#\{t \in]0,T[\mid u_\infty(t,\hat{u}^\alpha) \in \eta_A^\beta\}. \tag{6.13}$$

The numbers $n_0^{\alpha,\beta}, n_\infty^{\alpha,\beta}$ just count the intersections of the solutions to the autonomous systems (6.11), starting on the half-line η_S^α, with the half-line η_A^β.

We can now state the following.

Lemma 6.1.4. *For $\alpha \in \mathcal{I}_S$, $\beta \in \mathcal{I}_A$ fixed, there exist at least $|n_\infty^{\alpha,\beta} - n_0^{\alpha,\beta}|$ nontrivial solutions to*

$$\begin{cases} Ju' = F(t,u) \\ u(0) \in \eta_S^\alpha,\ u(T) \in \eta_A^\beta. \end{cases} \tag{6.14}$$

Proof. We consider the Cauchy problem

$$\begin{cases} Ju' = F(t,u) \\ u(0) = \bar{u}^\alpha, \end{cases}$$

with $\bar{u}_\alpha \in \mathbb{R}^2_*$, and denote by $u(t;\bar{u}^\alpha)$ its solution. Since $F(t,0) \equiv 0$, in view of the uniqueness it is possible to write $u(t;\bar{u}^\alpha) = \rho(t;\bar{u}^\alpha)(\cos\theta(t;\bar{u}^\alpha), \sin\theta(t;\bar{u}^\alpha))$, from which, since $V_0 \in \mathcal{P}$ and in view of Euler's identity, we have the equation for the angle $\theta = \theta(t;\bar{u}^\alpha)$:

$$-\theta' = \frac{\langle F(t,u)|u\rangle}{|u|^2} = 2V_0(\cos\theta, \sin\theta) + \frac{\langle R_0(t,u)|u\rangle}{|u|^2},$$

where $R_0(t,u) = F(t,u) - \nabla V_0(u)$. By continuous dependence, from (6.9) we deduce that, for $|\bar{u}^\alpha|$ small, the last term gives a negligible contribution, so that we can infer that the number of intersections of $u(t;\bar{u}^\alpha)$ with η_A^β for $t \in]0,T[$ is equal to $n_0^{\alpha,\beta}$. Similarly, by the elastic property (cf. Lemma 2.0.1), since $F(t,u)$ has an at most linear growth, we have that if $|\bar{u}^\alpha|$ is sufficiently large, then $u(t;\bar{u}^\alpha)$ remains sufficiently far from the origin for every $t \in [0,T]$, and the number of intersections of $u(t;\bar{u}^\alpha)$ with η_A^β for $t \in]0,T[$ is equal to $n_\infty^{\alpha,\beta}$.
We now exploit the continuous dependence of $\theta(t;\bar{u}^\alpha)$ on the initial datum $\bar{u}^\alpha$, to infer that, moving $\bar{u}^\alpha$ on η_S^α, we will find $|n_\infty^{\alpha,\beta} - n_0^{\alpha,\beta}|$ points $\bar{u}_i^\alpha \in \eta_S^\alpha$ such that $u(T;\bar{u}_i^\alpha) \in \eta_A^\beta$, giving the desired conclusion. □

With these preliminaries, we can now state the following result.

Theorem 6.1.5. *Problem* (6.8) *has at least*

$$\sum_{\alpha \in \mathcal{I}_S} \sum_{\beta \in \mathcal{I}_A} |n_\infty^{\alpha,\beta} - n_0^{\alpha,\beta}| \tag{6.15}$$

nontrivial solutions.

Proof. It suffices to repeat the reasoning in the proof of Lemma 6.1.4 for every couple of half-lines η_S^α, η_A^β. Since, for every $\alpha \in \mathcal{I}_S$, $\beta \in \mathcal{I}_A$, we find $|n_\infty^{\alpha,\beta} - n_0^{\alpha,\beta}|$ initial conditions in η_S^α yielding to a solution to problem (6.14), the thesis follows. □

Observe that the sum appearing in (6.15) is well defined, since it is a sum of positive integers. Notice moreover that the number of solutions found through Theorem 6.1.5 could be infinite.

Remark 6.1.6. The statement of Theorem 6.1.5 holds the same if we weaken conditions (6.9) and (6.10) into the following ones, respectively:

$$\lim_{|u|\to 0} \left[\frac{\langle F(t,u)|u\rangle}{|u|^2} - 2V_0\left(\frac{u}{|u|}\right)\right] = 0,$$

$$\lim_{|u|\to +\infty} \left[\frac{\langle F(t,u)|u\rangle}{|u|^2} - 2V_\infty\left(\frac{u}{|u|}\right)\right] = 0,$$

up to requiring that $F(t,u)$ has an at most linear growth in the variable u.

Remark 6.1.7. We could extend Theorem 6.1.5 assuming a more general control on $F(t,u)$, namely

$$F(t,u) = \gamma_0(t,u)\nabla V_0^{(1)}(u) + (1-\gamma_0(t,u))\nabla V_0^{(2)}(u) + R_0(t,u)$$

in a neighborhood of 0, and

$$F(t,u) = \gamma_\infty(t,u)\nabla V_\infty^{(1)}(u) + (1-\gamma_\infty(t,u))\nabla V_\infty^{(2)}(u) + R_\infty(t,u)$$

at infinity, where $V_0^{(i)}, V_\infty^{(i)} \in \mathcal{P}$, $i = 1,2$, satisfy $V_0^{(1)} \leq V_0^{(2)}$, $V_\infty^{(1)} \leq V_\infty^{(2)}$, the functions $\gamma_0(t,u)$, $\gamma_\infty(t,u)$ are continuous, taking values between 0 and 1, and the functions $R_0(t,u)$, $R_\infty(t,u)$ are negligible. The multiplicity result then comes from similar estimates on the gap of the angular speeds at zero and infinity (see, e.g., the arguments in Chapter 5). For briefness, we do not enter into the details.

6.2 Sturm-Liouville boundary conditions

We now want to examine how the results of the previous section can be rephrased in a more concrete way, when taking the boundary points on two straight lines in the plane (so that we consider Sturm-Liouville boundary conditions). Moreover, we will study the possibility of proving multiplicity of solutions in dependence of a real parameter, in the spirit of the work by Hart, Lazer and McKenna [76], concerning the scalar second order equation. Using a suitable change of variable, we will reformulate the problem so as to obtain some kind of gap between the rotation numbers of "small" and "large" solutions, finding then multiple solutions for large values of the real parameter. We remark that, for the periodic problem, results of this kind were obtained, for instance, in [32, 59, 94, 137].

6.2.1 Existence and multiplicity with a gap between zero and infinity

Let us fix two lines passing through the origin, say l_S and l_A. We are interested in the following Sturm-Liouville boundary value problem:

$$\begin{cases} Ju' = \nabla V(u) + R(t,u) \\ u(0) \in l_S,\, u(T) \in l_A, \end{cases} \tag{6.16}$$

where $V \in \mathcal{P}$ and $R(t,u)$ is a continuous function.
Before stating our existence result, we need the following preliminary digression. Let us follow a nontrivial solution $u(t)$ to the equation $Ju' = \nabla V(u)$, for which it will be $u(t) \neq 0$ for every $t \in \mathbb{R}$ in view of the uniqueness. Starting from the vertical positive semi-axis and moving clockwise, at some nonnegative time instant t_0 such a solution will arrive at a point u_1 in l_S (see Figure 6.1). We denote by τ_1 the least positive time needed by $u(t)$ to arrive at a point u_2 in l_A, starting from u_1, and, correspondingly, we denote by θ_1 the angular width covered in the time τ_1. Continuing in covering the orbit described by $u(t)$, we define σ_1 as the least nonnegative time needed to encounter again l_S, starting from u_2, and, accordingly, we denote by θ_2 the

angular width spanned in the time τ_2. In the same way, we define τ_2 as the positive time needed to arrive once more on l_A and σ_2 as the remaining nonnegative time to complete a whole revolution (see Figure 6.1 to visualize such definitions). In this way, $\tau_V = \tau_1 + \sigma_1 + \tau_2 + \sigma_2$ (and $2\theta_1 + 2\theta_2 = 2\pi$). It is important to underline that, in view of (1.12), the times $\tau_1, \tau_2, \sigma_1, \sigma_2$ are well-defined and independent of the choice of $u(t)$; moreover, if the lines l_S and l_A coincide, then $\sigma_1 = \sigma_2 = 0$.

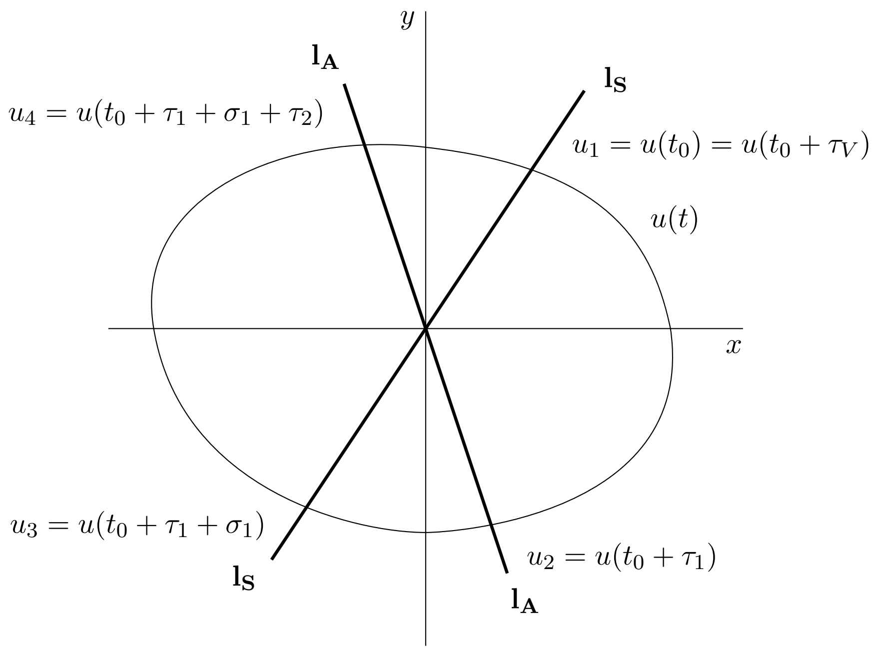

Figure 6.1: Following a solution $u(t)$ to define the times $\tau_1, \sigma_1, \tau_2, \sigma_2$.

We are now able to state the following theorem.

Theorem 6.2.1. *Let $R(t,u)$ satisfy the sublinear growth condition* (6.5)*, and assume that there exists a nonnegative integer k such that one of the following nonresonance assumptions holds: either*

$$(k-1)\tau_V + \tau_1 + \tau_2 + \max\{\sigma_1, \sigma_2\} < T < k\tau_V + \min\{\tau_1, \tau_2\}, \tag{6.17}$$

or

$$k\tau_V + \max\{\tau_1, \tau_2\} < T < k\tau_V + \tau_1 + \tau_2 + \min\{\sigma_1, \sigma_2\}. \tag{6.18}$$

Then, problem (6.16) *has a solution.*

Proof of Theorem 6.2.1. Setting $\mathcal{C}_S = l_S$ and $\mathcal{C}_A = l_A$, we have to prove that assumption (A) holds.

We focus on the case when condition (6.17) is assumed. Consider a nontrivial solution $u(t) = u(t;\bar{u})$ to the Cauchy problem

$$\begin{cases} Ju' = \nabla V(u) \\ u(0) = \bar{u} \in l_S, \end{cases}$$

and the corresponding function $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined in (6.3). Using polar coordinates, we can write $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, for some continuously differentiable functions $\rho(t) > 0$, $\theta(t) \in \mathbb{R}$. Hence,

$$-\theta' = 2V(\cos\theta, \sin\theta),$$

yielding

$$\int_{\theta(0)}^{\theta(T)} \frac{d\theta}{2V(\cos\theta, \sin\theta)} = T.$$

By (6.17) and the definition of θ_1 and θ_2, we get

$$2(k-1)\pi + \theta_1 + \theta_2 + \theta_1 < \theta(0) - \theta(T) < 2k\pi + \theta_1,$$

so that, since $\theta_1 + \theta_2 = \pi$,

$$\pi < \theta(0) - \theta(T) - 2(k-1)\pi - \theta_1 < 2\pi.$$

Hence, following the solution $u(t)$ when t varies from 0 to T, we cover an angular width greater than $(2k-1)\pi + \theta_1$ and smaller than $2k\pi + \theta_1$, where θ_1 has been defined above. It follows that the points $\mathcal{F}(\bar{u})$ and $\mathcal{F}(-\bar{u})$ lie in two different connected components of $\mathbb{R}^2 \setminus l_A$.
In the case when (6.18) is assumed, we analogously get

$$0 < \theta(0) - \theta(T) - 2k\pi - \theta_1 < \pi,$$

giving the conclusion with a similar argument. □

In order to clarify the assumptions of Theorem 6.2.1, let us make some considerations about the autonomous problem

$$\begin{cases} Ju' = \nabla V(u) \\ u(0) \in l_S, \ u(T) \in l_A, \end{cases} \tag{6.19}$$

for $V \in \mathcal{P}$. It follows directly from the above definitions that such a problem has a nontrivial solution if and only if, for some nonnegative integer k,

$$T - k\tau_V \in \{\tau_1, \tau_2, \tau_1 + \sigma_1 + \tau_2, \tau_2 + \sigma_2 + \tau_1\}. \tag{6.20}$$

The set of Hamiltonians belonging to $\mathcal{P}$ and satisfying (6.20) generalizes to the plane the classical notion of Fučík spectrum, originally introduced, as already remarked, for the equation

$$x'' + \mu x^+ - \nu x^- = 0, \tag{6.21}$$

being μ, ν positive parameters. In this case,

$$\tau_V = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\,, \tag{6.22}$$

and the Fučík spectrum is defined as the set Σ of the couples (μ, ν) such that the Sturm-Liouville boundary value problem (6.19), with $u = (x, x')$ and

$$V(x,y) = \frac{1}{2}(y^2 + \mu(x^+)^2 + \nu(x^-)^2), \tag{6.23}$$

has nontrivial solutions. For instance, in the particular case of the Dirichlet problem, where $l_S = l_A = \{u = (x,y) \in \mathbb{R}^2 \mid x = 0\}$, we have $\sigma_1 = \sigma_2 = 0$, and

$$\tau_1 = \frac{\pi}{\sqrt{\mu}}\,, \quad \tau_2 = \frac{\pi}{\sqrt{\nu}}\,,$$

so that the Fučík spectrum Σ can be easily computed (see [67]), giving rise to the sequence of curves which has been depicted in Figure 2 below. On the other hand, in the case of the Neumann problem, where l_S and l_A both coincide with the horizontal axis, we have $\sigma_1 = \sigma_2 = 0$, and

$$\tau_1 = \tau_2 = \frac{1}{2}\left(\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\right) = \frac{\tau_V}{2}\,.$$

The corresponding Fučík spectrum can be easily determined in this case, as well. These two examples carry a substantial difference: while for Neumann boundary conditions it is enough to assume that $V \in \mathcal{P}$ is nonresonant (i.e., it does not satisfy (6.20)) in order to apply Theorem 6.2.1, this is not the case for the Dirichlet problem, as shown in [29, Proposition 1].

In particular, for the Dirichlet problem, we need the two stronger conditions (6.17) and (6.18), namely either

$$k\tau_V < T < k\tau_V + \min\{\tau_1, \tau_2\},$$

or

$$k\tau_V + \max\{\tau_1, \tau_2\} < T < (k+1)\tau_V.$$

Notice that these two assumptions also avoid the existence of T-periodic solutions to (6.19), case which would give rise to resonance, as well, since $l_S = l_A$. As a consequence of Theorem 6.2.1, we thus obtain Fučík's original result [67, Theorem 2.11].

We now give two pictures of the Fučík spectra for the asymmetric equation (6.21), referring, respectively, to the problems

$$\begin{cases} x'' + \mu x^+ - \nu x^- = 0 \\ x(0) = 0,\, x(T) + x'(T) = 0, \end{cases} \qquad \begin{cases} x'' + \mu x^+ - \nu x^- = 0 \\ x(0) + x'(0) = 0,\, x'(T) = 0. \end{cases}$$

Using (1.12), it is readily seen that, in the first case, we have

$$\tau_1 = \frac{\pi}{2\sqrt{\mu}} + \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}\,, \quad \sigma_1 = \frac{\pi}{2\sqrt{\mu}} - \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}\,,$$

$$\tau_2 = \frac{\pi}{2\sqrt{\nu}} + \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}\,, \quad \sigma_2 = \frac{\pi}{2\sqrt{\nu}} - \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}\,.$$

In the second one, it is

$$\tau_1 = \frac{\tau_V}{2} - \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}\,, \quad \sigma_1 = \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}\,,$$

$$\tau_2 = \frac{\tau_V}{2} - \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}\,, \quad \sigma_2 = \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}\,.$$

Thus, in these two cases, condition (6.20) is quite easy to verify. Moreover, let us observe the following qualitative difference: while in the first case (Figure 6.2) the first curves of the spectrum are two straight lines, parallel to the coordinate axes, like for Dirichlet boundary conditions, in the second one such straight lines disappear (see Figure 6.3). This is due to the fact that any solution to (6.19), with $V(u)$ given by (6.23), has to cross both the half-plane where x is positive and the one where x is negative. Observe that, as in the classical case introduced by Fučík, the regions for which there exists a solution are the connected components of $\mathbb{R}^2 \setminus \Sigma$ which have nonempty intersection with the diagonal.

Concerning multiplicity, we now apply Theorem 6.1.5 to give a few corollaries concerning the Sturm-Liouville problem.

Corollary 6.2.2. *Let l_S, l_A be two lines through the origin. Moreover, let $k > m$ be two positive integers, and $V_0, V_\infty \in \mathcal{P}$. Denoting by $\tau_V^0, \tau_1^0, \sigma_1^0, \tau_2^0, \sigma_2^0$ the times defined before for the Sturm-Liouville problem, relative to the system $Ju' = \nabla V_0(u)$, and using the same convention for V_∞, assume that*

$$(k-1)\tau_V^0 + \tau_1^0 + \tau_2^0 + \max\{\sigma_1^0, \sigma_2^0\} < T < k\tau_V^0 + \min\{\tau_1^0, \tau_2^0\},$$

and

$$m\tau_V^\infty + \max\{\tau_1^\infty, \tau_2^\infty\} < T < m\tau_V^\infty + \tau_1^\infty + \tau_2^\infty + \min\{\sigma_1^\infty, \sigma_2^\infty\}.$$

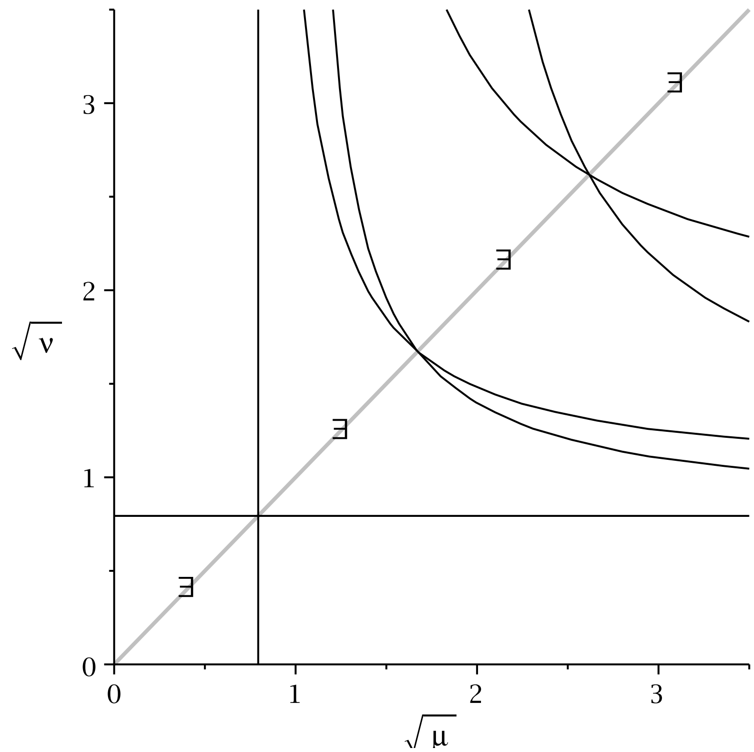

Figure 6.2: The Fučík spectrum for (6.21), with $x(0) = 0$, $x(\pi) + x'(\pi) = 0$.

Assume that $F(t, u)$ *satisfies* (6.9) *and* (6.10), *and* $F(t, 0) \equiv 0$. *Then, the problem*

$$\begin{cases} Ju' = F(t, u) \\ u(0) \in l_S, \ u(T) \in l_A, \end{cases}$$

has at least $4(k - m) - 2$ *nontrivial solutions.*

Proof. It suffices to notice that, writing $l_S = \eta_S^1 \cup \eta_S^2$, $l_A = \eta_A^1 \cup \eta_A^2$, where η_S^i, η_A^i, $i = 1, 2$, are half-lines emanating from the origin, in such a way that the first half-line encountered starting on η_S^1 and moving clockwise is η_A^1, one has

$$n_0^{1,1} = n_0^{1,2} = n_0^{2,1} = n_0^{2,2} = k,$$

$$n_\infty^{1,1} = n_\infty^{2,2} = m + 1, \quad n_\infty^{1,2} = n_\infty^{2,1} = m.$$

The conclusion follows. $\square$

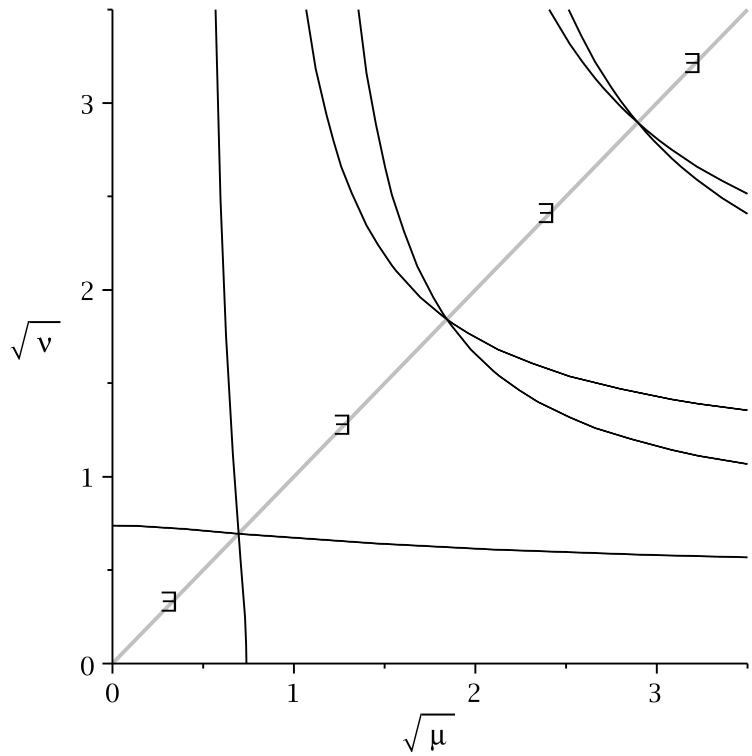

Figure 6.3: The Fučík spectrum for (6.21), with $x(0) + x'(0) = 0$, $x'(\pi) = 0$.

It is clear that several combinations of the conditions previously introduced are possible, giving various different results of multiplicity. For example, we could have the following.

Corollary 6.2.3. *Let l_S, l_A be two lines through the origin. Moreover, let $k > m$ be two positive integers, $V_0, V_\infty \in \mathcal{P}$. Using the same notation as in Corollary 6.2.2, assume that*

$$k\tau_V^0 + \max\{\tau_1^0, \tau_2^0\} < T < k\tau_V^0 + \tau_1^0 + \tau_2^0 + \min\{\sigma_1^0, \sigma_2^0\},$$

and

$$m\tau_V^\infty + \max\{\tau_1^\infty, \tau_2^\infty\} < T < m\tau_V^\infty + \tau_1^\infty + \tau_2^\infty + \min\{\sigma_1^\infty, \sigma_2^\infty\}.$$

Assume that $F(t,u)$ satisfies (6.9) *and* (6.10)*, and $F(t,0) \equiv 0$. Then, the problem*

$$\begin{cases} Ju' = F(t,u) \\ u(0) \in l_S,\ u(T) \in l_A, \end{cases}$$

has at least $4(k-m)$ nontrivial solutions.

Remark 6.2.4. Referring to Theorem 6.1.5 for the Sturm-Liouville problem, we point out that other kinds of controls on the angular speed could be considered. For instance, in the case of a problem like

$$\begin{cases} x'' + f(t,x) = 0 \\ x(0) = 0 = x(T), \end{cases} \tag{6.24}$$

following [27] we could assume that $f(t,0) \equiv 0$,

$$a_0(t) \le \liminf_{|x|\to 0} \frac{f(t,x)}{x} \le \limsup_{|x|\to 0} \frac{f(t,x)}{x} \le b_0(t),$$

and

$$a_\infty(t) \le \liminf_{|x|\to +\infty} \frac{f(t,x)}{x} \le \limsup_{|x|\to +\infty} \frac{f(t,x)}{x} \le b_\infty(t),$$

for suitable functions $a_0(t), a_\infty(t), b_0(t), b_\infty(t)$. Denoting by $\lambda_n(\gamma)$ the n-th eigenvalue of the Dirichlet problem

$$\begin{cases} x'' + \lambda\gamma(t)x = 0 \\ x(0) = 0 = x(T), \end{cases}$$

it was proved in [27, Theorem 1.1] that, if there exist two integers $m \le n$ such that

$$\lambda_n(a_0) < 1 < \lambda_m(b_\infty), \tag{6.25}$$

then there are $2(n-m+1)$ solutions to (6.24).

Comparing with our previous results, in the case when the above functions a_0, a_∞, b_0, b_∞ are constant, let

$$V_0^{(1)}(x,y) = \frac{1}{2}(y^2 + \widehat{\lambda}_n x^2), \quad V_\infty^{(2)}(x,y) = \frac{1}{2}(y^2 + \widehat{\lambda}_m x^2),$$

where $\widehat{\lambda}_1 < \widehat{\lambda}_2 < \dots$ are the usual eigenvalues of the Dirichlet problem on $[0,T]$. We observe that (6.25) is then equivalent to

$$b_\infty < \widehat{\lambda}_m \le \widehat{\lambda}_n < a_0. \tag{6.26}$$

Let us denote by $\eta_S^1 = \eta_A^1$ the positive vertical semi-axis, and by $\eta_S^2 = \eta_A^2$ the negative one. Using the same notation as in (6.12), (6.13), from (6.26) we deduce that

$$n_0^{1,1} \geq \left\lfloor \frac{n}{2} \right\rfloor, \quad n_0^{1,2} \geq \left\lceil \frac{n}{2} \right\rceil, \quad n_0^{2,1} \geq \left\lceil \frac{n}{2} \right\rceil, \quad n_0^{2,2} \geq \left\lfloor \frac{n}{2} \right\rfloor,$$

and

$$n_\infty^{1,1} \leq \left\lfloor \frac{m-1}{2} \right\rfloor, \quad n_\infty^{1,2} \leq \left\lceil \frac{m-1}{2} \right\rceil, \quad n_\infty^{2,1} \leq \left\lceil \frac{m-1}{2} \right\rceil, \quad n_\infty^{2,2} \leq \left\lfloor \frac{m-1}{2} \right\rfloor.$$

We recall that, for a real number a, the symbol $\lfloor a \rfloor$ denotes the largest integer less than or equal to a, while $\lceil a \rceil$ denotes the least integer greater than or equal to a. Arguing as in the proofs of Lemma 6.1.4 and Theorem 6.1.5, we then find at least

$$2\left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{m-1}{2} \right\rceil \right)$$

nontrivial solutions. This number can be checked to be exactly equal to $2(n-m+1)$, thus agreeing with [27, Theorem 1.1].

We conclude this subsection by observing that, as in [27], we could characterize the nontrivial solutions obtained by their nodal properties. For briefness, we will not enter into details.

6.2.2 Multiplicity in dependence of a real parameter

We now consider the possibility of giving multiplicity results for the Sturm-Liouville boundary value problem, but in dependence of the value of a real parameter. Let l_S, l_A be two fixed lines passing through the origin. We will deal with the problem

$$\begin{cases} Ju' = \nabla V(u) + R(t,u) + sv^*(t) \\ u(0) \in l_S, \ u(T) \in l_A, \end{cases} \tag{6.27}$$

with s a positive parameter and $v^*(t)$ a fixed continuous function. Moreover, we will suppose that $R(t,u)$ fulfills the sublinear growth assumption (6.5), and $V(u)$ satisfies some nonresonance hypothesis.

We will still denote by τ_i, σ_i, with $i = 1, 2$, the times introduced in Subsection 6.2.1. Moreover, $\tau_{\mathbb{A}}, \tau_{i,\mathbb{A}}, \sigma_{i,\mathbb{A}}$, with $i = 1, 2$, will refer to the times, defined as in Subsection 6.2.1, associated with the linear problem

$$\begin{cases} Ju' = \mathbb{A}u \\ u(0) \in l_S, \ u(T) \in l_A, \end{cases}$$

where $\mathbb{A}$ is a symmetric 2×2 matrix.
We can now state the main result of this section.

Theorem 6.2.5. *Let $V \in \mathcal{P}$ satisfy, for a suitable nonnegative integer k,*

$$(k-1)\tau_V + \tau_1 + \tau_2 + \max\{\sigma_1, \sigma_2\} < T < k\tau_V + \min\{\tau_1, \tau_2\}. \tag{6.28}$$

Moreover, assume that there exist a function $w^* : \mathbb{R} \to \mathbb{R}^2$ solving

$$\begin{cases} Jw' = \nabla V(w) + v^*(t) \\ w(0) \in l_S, \ w(T) \in l_A, \end{cases}$$

such that $0 \notin \mathcal{W}^* = \{w^*(t) : t \in [0,T]\}$, and two positive definite symmetric matrices $\mathbb{A} \leq \mathbb{B}$ satisfying

$$\langle \mathbb{A}(u-v)|u-v\rangle \leq \langle \nabla V(u) - \nabla V(v)|u-v\rangle \leq \langle \mathbb{B}(u-v)|u-v\rangle, \tag{6.29}$$

for every $u, v \in \mathcal{W}^*$. Assume also that $\mathbb{A}$, $\mathbb{B}$ fulfill, for a suitable nonnegative integer m,

$$m\tau_{\mathbb{A}} + \max\{\tau_{1,\mathbb{A}}, \tau_{2,\mathbb{A}}\} < T < m\tau_{\mathbb{B}} + \tau_{1,\mathbb{B}} + \tau_{2,\mathbb{B}} + \min\{\sigma_{1,\mathbb{B}}, \sigma_{2,\mathbb{B}}\}. \tag{6.30}$$

Lastly, suppose that $R(t,u)$ satisfies the sublinear growth condition (6.5)*. Then, there exists $s^* > 0$ such that, for every $s \geq s^*$, problem* (6.27) *has at least*

$$2|2(m-k)+1|+1$$

solutions.

The proof is similar to the one for the T-periodic problem given in [59, Theorem 1.1]. First, we change variables in (6.27), setting

$$\lambda = \frac{1}{s}, \quad y = \lambda u - w^*.$$

In this way, for $\lambda \in]0,+\infty[$, problem (6.27) is equivalent to

$$\begin{cases} Jy' = \nabla V(y + w^*(t)) - \nabla V(w^*(t)) + f(t,y;\lambda) \\ y(0) \in l_S, \ y(T) \in l_A, \end{cases} \tag{6.31}$$

where

$$f(t,y;\lambda) = \begin{cases} \lambda R\Big(t, \dfrac{1}{\lambda}(y + w^*(t))\Big) & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0. \end{cases}$$

In view of (6.5), we have

$$\lim_{\lambda \to 0^+} f(t,y;\lambda) = 0, \tag{6.32}$$

uniformly in $t \in [0,T]$ and $y \in B(0,r)$, being $B(0,r)$ the open ball centered at 0 with radius $r > 0$. Thus, $f(t,y;\lambda)$ is continuous up to $\lambda = 0$.

The next two statements are crucial in order to find a first solution to (6.27) for λ small enough, via topological methods. In the following, we will denote by $B_\infty(0,r)$ the open ball in $L^\infty(0,T)$, centered in 0 and with radius $r>0$, and by $\overline{B}_\infty(0,r)$ its closure.

Lemma 6.2.6. *There exists $r^*>0$ such that, if $y(t)$ solves*

$$\begin{cases} Jy' = \nabla V(y+w^*(t)) - \nabla V(w^*(t)) \\ y(0)\in l_S,\ y(T)\in l_A, \end{cases} \tag{6.33}$$

and $y\in \overline{B}_\infty(0,r^*)$, then $y(t)\equiv 0$.

Proof. If there were a sequence $(y_n)_n \in \overline{B}_\infty(0,1/n)$ of nontrivial solutions to (6.33), by uniqueness it would be $y_n(t)\neq 0$ for every $t\in[0,T]$, and we could write

$$y_n(t) = \rho_n(t)(\cos\theta_n(t), \sin\theta_n(t)),$$

so that

$$-\theta_n'(t) = \frac{\langle \nabla V(y_n + w^*(t)) - \nabla V(w^*(t)) | y_n\rangle}{|y_n|^2}\,. \tag{6.34}$$

Fix $\bar n$ sufficiently large; in view of the strict inequalities in (6.30), we are able to find two matrices $\widehat{\mathbb{A}}, \widehat{\mathbb{B}}$, with $0<\widehat{\mathbb{A}}\le \mathbb{A}\le\mathbb{B}\le\widehat{\mathbb{B}}$, such that, replacing $\mathbb{A}$ with $\widehat{\mathbb{A}}$ and $\mathbb{B}$ with $\widehat{\mathbb{B}}$, (6.30) is still satisfied and (6.29) holds for every $u,v\in\{w^*(t)+x : t\in[0,T], |x|\le 1/\bar n\}$. Therefore, since $y_n\in B_\infty(0,1/\bar n)$ for $n\ge\bar n$, from (6.34) we deduce

$$\int_{\theta_n(0)}^{\theta_n(T)} \frac{d\theta}{\langle \widehat{\mathbb{B}}(\cos\theta,\sin\theta)|(\cos\theta,\sin\theta)\rangle} \le T \le \int_{\theta_n(0)}^{\theta_n(T)} \frac{d\theta}{\langle \widehat{\mathbb{A}}(\cos\theta,\sin\theta)|(\cos\theta,\sin\theta)\rangle}\,.$$

Hence, in view of (6.30), it follows that

$$2m\pi + \theta_1 < \theta_n(0) - \theta_n(T) < (2m+1)\pi + \theta_1, \tag{6.35}$$

where θ_1 is as in Subsection 6.2.1. This implies that it is not possible that $y_{\bar n}(0)\in l_S$ and $y_{\bar n}(T)\in l_A$ at the same time, a contradiction. □

Lemma 6.2.7. *For every $\delta>0$, there exists $\lambda^*=\lambda^*(\delta)$ such that, for every $\lambda\in[0,\lambda^*(\delta)]$, there is a solution y_λ to (6.31), satisfying*

$$\|y_\lambda\|_\infty \le \delta.$$

Proof. In view of Lemma 6.2.6, it turns out that, for $\lambda=0$, $y_0(t)\equiv 0$. We would like to continue such a solution in a neighborhood of $\lambda=0$. Let $L: D(L)\subset C^0([0,T])\to C^0([0,T])$, with $D(L)=\{u\in C^1([0,T]) \mid u(0)\in l_S, u(T)\in l_A\}$, be defined by

$Lu = Ju'$, and let N_λ be the Nemytzkii operator associated with the right-hand side of the equation in (6.31). If α does not belong to the spectrum of L, we can define $\Phi : C([0,T]) \times [0,1] \to C([0,T])$ by

$$\Phi(y,\lambda) = (L - \alpha I)^{-1}(N_\lambda y - \sigma y).$$

In this way, (6.31) is equivalent to the fixed point problem

$$\Phi(y,\lambda) = y.$$

Moreover, in view of (6.32), we have that

$$\lim_{\lambda \to 0^+} \Phi(y;\lambda) = \Phi(y;0),$$

uniformly in $y \in \overline{B}_\infty(0,r^*)$, where $r^* > 0$ is as above.
We are now going to compute the Leray-Schauder degree

$$\deg\left(\Phi(\cdot\,;\lambda) - I, B_\infty(0,r^*)\right),$$

showing that it is different from 0. To this aim, we first notice that, with the same proof as in [59, Lemma 2.2], we can deduce that there exists $\lambda^* > 0$ such that there are no solutions to (6.31) on the boundary of $B_\infty(0,r^*)$, for $\lambda \in [0,\lambda^*]$. We then pass to consider the problem

$$\begin{cases} Jy' = \sigma(\nabla V(y + w^*(t)) - \nabla V(w^*(t))) + \dfrac{(1-\sigma)}{2}(\mathbb{A} + \mathbb{B})y \\ y(0) \in l_S,\ y(T) \in l_A. \end{cases}$$

Since $\mathbb{A} \le \mathbb{B}$, we can use the same argument as the one to obtain (6.35), to deduce that this problem has only the trivial solution in $\overline{B}_\infty(0,r^*)$. By the homotopy invariance of the topological degree and the previous considerations, it follows that

$$\begin{aligned} \deg\left(\Phi(\cdot\,;\lambda) - I, B_\infty(0,r^*)\right) &= \deg\left(\Phi(\cdot\,;0) - I, B_\infty(0,r^*)\right) \\ &= \deg\left((L-\alpha I)^{-1}\Big(\frac{\mathbb{A}+\mathbb{B}}{2} - \alpha I\Big) - I, B_\infty(0,r^*)\right) \\ &\neq 0, \end{aligned}$$

since the operator involved in the last degree is linear and invertible.

We are now ready to conclude the proof of the lemma. So far, for every $\lambda \in [0,\lambda^*]$, we have found a solution y_λ to (6.31), belonging to $B_\infty(0,r^*)$. We want to prove that

$$\lim_{\lambda \to 0^+} \|y_\lambda\|_\infty = 0. \tag{6.36}$$

By contradiction, assume that there exist $\epsilon > 0$, $(t_n)_n$ in $[0, T]$ and $(\lambda_n)_n$ in $[0, 1]$, with $\lambda_n \to 0$, such that, for every n,

$$|y_{\lambda_n}(t_n)| \geq \epsilon.$$

Since $(y_{\lambda_n})_n$ is bounded in $L^\infty(0, T)$, being $y_{\lambda_n} \in B_\infty(0, r^*)$, the sequence $y_{\lambda_n}(t_n)$ is bounded, so there exists $\bar{y}$ such that, up to subsequences,

$$y_{\lambda_n}(t_n) \to \bar{y}; \tag{6.37}$$

obviously, $|\bar{y}| \geq \epsilon$. Moreover, we can assume, for a subsequence, that $t_n \to \bar{t} \in [0, T]$. We now consider, for every n, the Cauchy problem

$$\begin{cases} Jy' = \nabla V(y + w^*(t)) - \nabla V(w^*(t)) + f(t, y; \lambda) \\ y(t_n) = y_{\lambda_n}(t_n). \end{cases} \tag{6.38}$$

By uniqueness, (6.38) is solved by $y_{\lambda_n}(t)$; moreover, in view of (6.37), we can infer, by continuous dependence, that

$$\lim_{n \to +\infty} y_{\lambda_n}(t) = \hat{y}(t),$$

uniformly in $t \in [0, T]$, where $\hat{y}(t)$ solves

$$\begin{cases} Jy' = \nabla V(y + w^*(t)) - \nabla V(w^*(t)) \\ y(\bar{t}) = \bar{y}. \end{cases}$$

It follows that $\hat{y} \in \overline{B}_\infty(0, r^*)$, $\hat{y}(\bar{t}) = \bar{y} \neq 0$, $\hat{y}(0) \in l_S$, $\hat{y}(T) \in l_A$, so that $\hat{y}$ is a nontrivial solution to (6.33). This contradicts Lemma 6.2.6. □

We now change variables, by setting

$$z = y - y_\lambda,$$

which transforms problem (6.31) into the following one:

$$\begin{cases} Jz' = g(t, z; \lambda) \\ z(0) \in l_S,\ z(T) \in l_A, \end{cases} \tag{6.39}$$

where

$$g(t, z; \lambda) = \nabla V(z + y_\lambda(t) + w^*(t)) - \nabla V(y_\lambda(t) + w^*(t)) + f(t, z + y_\lambda(t); \lambda) - f(t, y_\lambda(t); \lambda).$$

With the goal of applying a shooting method to prove Theorem 6.2.5, we are now going to consider the Cauchy problem

$$\begin{cases} Jz' = g(t, z; \lambda) \\ z(0) = \bar{z} \in l_S. \end{cases} \tag{6.40}$$

We will denote by $z(t, \bar{z}; \lambda)$ the solution to such a problem. We will use the same philosophy as for Theorem 6.1.5, counting the number of intersections of such a solution with the arrival line l_A; precisely, we define

$$n(z(t, \bar{z}; \lambda)) = \#\{t \in]0, T[\mid z(t, \bar{z}; \lambda) \in l_A\}.$$

We first state a lemma concerning the limit case $\lambda = 0$. In view of (6.32), (6.36) and the Lipschitz continuity of ∇V, we have

$$\lim_{\lambda \to 0^+} g(t, z; \lambda) = g(t, z; 0) = \nabla V(z + w^*(t)) - \nabla V(w^*(t)), \tag{6.41}$$

uniformly for every $t \in [0, T]$ and every $z \in B(0, r)$, with $r > 0$.

Lemma 6.2.8. *Let $r^* > 0$ be as in Lemma 6.2.6. There exist two positive constants $\hat{r}$, $\bar{r}$, with $4\hat{r} < \bar{r} < r^*/4$, such that, if $\bar{z} \in \mathbb{R}^2$ satisfies*

$$|\bar{z}| = \bar{r},$$

then the solution $z(t)$ to the Cauchy problem

$$\begin{cases} Jz' = \nabla V(z + w^*(t)) - \nabla V(w^*(t)) \\ z(0) = \bar{z} \end{cases} \tag{6.42}$$

satisfies, for every $t \in [0, T]$,

$$4\hat{r} \le |z(t)| \le \frac{r^*}{4}.$$

The proof can be found in [59, Lemma 2.4] and is essentially based on Gronwall's Lemma, which can be used thanks to the Lipschitz continuity of $\nabla V(u)$. In the following, without ambiguity, we will denote by $z(t, \bar{z}; 0)$ the solution to (6.42).

We are now going to display the gap between "small" and "large" solutions to (6.40), in order to apply the shooting method and find multiple solutions to the original problem (6.27).

Lemma 6.2.9. *Let $\bar{r} > 0$ be as in the previous lemma. Then, there exists $\lambda_1 \in]0, \lambda^*]$ such that, if $|\bar{z}| = \bar{r}$, every solution $z(t, \bar{z}; \lambda)$ to (6.40), with $\lambda \in [0, \lambda_1]$, satisfies*

$$n(z(t, \bar{z}; \lambda)) = 2m + 1.$$

Proof. We first focus on (6.42), to show that $n(z(t, \bar{z}; 0)) = 2m+1$. In view of Lemma 6.2.8, if $|\bar{z}| = \bar{r}$, then $z(\cdot\,, \bar{z}; 0)$ belongs to $\overline{B}_\infty(0, r^*/4)$, so that, reasoning on (6.34) as in Lemma 6.2.6, we can use (6.30) to argue that $z(t, \bar{z}; 0)$ meets l_A exactly $2m + 1$ times in the time interval $]0, T[$, in view of (6.35).

We now turn our attention to the solution $z(t, \bar{z}; \lambda)$ to (6.40), with a *fixed* $\bar{z}$ such that $|\bar{z}| = \bar{r}$. By continuous dependence, in view of (6.41), we have that $z(t, \bar{z}; \lambda)$ will stay near $z(t, \bar{z}; 0)$, for $\lambda > 0$ sufficiently small. Using Lemma 6.2.8, there exists $\lambda_0 > 0$ such that, for every $t \in [0, T]$ and $\lambda \in]0, \lambda_0]$, we have

$$|z(t, \bar{z}; \lambda)| \leq \frac{r^*}{2}\,.$$

Moreover, by (6.36), we can assume that $\|y_\lambda\|_\infty \leq r^*/2$ for every $\lambda \in]0, \lambda_0]$. Thus, we can control the angle as in the proof of Lemma 6.2.6 and, since the inequalities in (6.35) are strict, we deduce

$$n(z(t, \bar{z}; \lambda)) = 2m + 1,$$

for every $\lambda \in]0, \lambda_0]$.
Finally, in view of the continuous dependence and the compactness of $\partial B(0, \bar{r}) \subset \mathbb{R}^2$, we can find $\lambda_1 > 0$ as in the statement. □

We now fix $\lambda \in]0, \lambda_1]$. The following lemma gives an estimate for the number of intersections of "large" solutions to (6.40) with the line l_A.

Lemma 6.2.10. *There exists $\bar{R}_\lambda > \bar{r}$ such that, if $|\bar{z}| = \bar{R}_\lambda$, the solution $z(t, \bar{z}; \lambda)$ to* (6.40) *satisfies*

$$n(z(t, \bar{z}; \lambda)) = 2k.$$

Proof. Let us take $\bar{z} \in \mathbb{R}^2$ sufficiently far from the origin. By uniqueness, the usual system of polar coordinates is well defined for $z(t, \bar{z}; \lambda)$. Writing

$$z(t, \bar{z}; \lambda) = \rho(t, \bar{z}; \lambda)(\cos\theta(t, \bar{z}; \lambda), \sin\theta(t, \bar{z}; \lambda)),$$

we are led to the usual equation for $\theta'(t) = \theta'(t, \bar{z}; \lambda)$:

$$\begin{aligned} -\theta'(t) \;\; = \;\; & \frac{\langle \nabla V(z + y_\lambda(t) + w^*(t)) | z \rangle}{|z|^2} - \frac{\langle \nabla V(y_\lambda(t) + w^*(t)) | z \rangle}{|z|^2} + \\ & + \frac{\langle f(t, z + y_\lambda(t); \lambda) | z \rangle}{|z|^2} - \frac{\langle f(t, y_\lambda(t); \lambda) | z \rangle}{|z|^2}\,. \end{aligned} \tag{6.43}$$

We notice that, for $|z| \to +\infty$, the second and the fourth term in the right-hand side vanish, since $\|y_\lambda + w^*\|_\infty$ is bounded. For what concerns the third summand, writing explicitly its expression we have

$$\frac{\langle f(t, z + y_\lambda(t); \lambda) | z \rangle}{|z|^2} = \lambda \frac{\langle R(t, \frac{1}{\lambda}(z + y_\lambda(t) + w^*(t))) | z \rangle}{|z|^2}\,.$$

In view of (6.5), fixed $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $|R(t,u)| \leq C_\epsilon + \epsilon|u|$ for every $t \in [0,T]$ and every $u \in \mathbb{R}^2$. Hence, the third term in (6.43) goes to 0 when $|z| \to +\infty$, as well. To estimate the remaining part, we write it as

$$\frac{\langle \nabla V(z + y_\lambda(t) + w^*(t))|z\rangle}{|z|^2} = \frac{\langle \nabla V(z + y_\lambda(t) + w^*(t)) - \nabla V(z)|z\rangle}{|z|^2} + \frac{\langle \nabla V(z)|z\rangle}{|z|^2} \tag{6.44}$$

and observe that the Lipschitz continuity of $\nabla V(u)$ gives $|\nabla V(z + y_\lambda(t) + w^*(t)) - \nabla V(z)| \leq L|y_\lambda(t) + w^*(t)|$, for a suitable constant $L > 0$, so that the first term of the right-hand side in (6.44) vanishes for $|z| \to +\infty$. By Euler's identity, this implies that

$$-\theta'(t) = 2V(\cos\theta(t), \sin\theta(t)) + h(t, z(t)), \tag{6.45}$$

being h a function which satisfies $h(t, z(t)) \to 0$, uniformly in $t \in [0,T]$, when $\min_{[0,T]} |z(t)| \to +\infty$. As a consequence, there exists a number $M > 0$ such that, if $|z(t, \bar{z}; \lambda)| > M$ for every $t \in [0,T]$, then, in view of (6.45) and the strict inequalities in (6.28), $z(t, \bar{z}; \lambda)$ encounters exactly $2k$ times the line l_A. It is now possible to find $\bar{R}_\lambda > M$ through the elastic property, which ensures that, if we start with $|\bar{z}| = \bar{R}_\lambda$, it will be $|z(t, \bar{z}; \lambda)| > M$ for every $t \in [0,T]$. $\square$

We are now ready to conclude the proof of Theorem 6.2.5. Let l_S^1 be one of the two half-lines of l_S, starting from the origin. In view of Lemmas 6.2.9 and 6.2.10 and the continuous dependence on the initial datum, there will be $|2(m-k)+1|$ distinct points $\bar{z}_{i,\lambda} \in l_S^1$ such that the solution $z(t, \bar{z}_{i,\lambda}; \lambda)$ to (6.40) satisfies $z(T, \bar{z}_{i,\lambda}; \lambda) \in l_A$, thus solving (6.39). Notice that, for $\lambda = 0$, the points $\bar{z}_{i,0}$ do not coincide with the origin. Returning to the original variable u through the inverse change of variable

$$z(t) = \lambda u(t) - y_\lambda(t) - w^*(t),$$

we find the corresponding (all distinct) starting points $\bar{u}_{i,\lambda} = \bar{z}_{i,\lambda} + y_\lambda(0) + w^*(0)$ yielding to a solution to (6.27). In particular, since $z(0) \in l_S^1, z(T) \in l_A$, it will be $u(0, \bar{u}_{i,\lambda}; \lambda) \in l_S$, $u(T, \bar{u}_{i,\lambda}; \lambda) \in l_A$. The same reasoning could be done on the other half-line of l_S. Taking into account the further solution $y_\lambda(t)$ found in Lemma 6.2.7 which, we recall, for $\lambda = 0$ is identically 0, we see that, for λ sufficiently small, $y_\lambda(t)$ does not coincide with any of the other solutions found. The proof of Theorem 6.2.5 is thus complete.

Remark 6.2.11. Notice that it has been essential to deal with linear boundary conditions, otherwise problems (6.31) and (6.39) would not have been equivalent to our original boundary value problem, and we could not have used (at least, in a plain way) reasonings involving the topological degree.

Remark 6.2.12. We briefly compare Theorem 6.2.5 with a result by Hart, Lazer and McKenna [76, Theorem 1], concerning the scalar Dirichlet problem

$$\begin{cases} x'' + g(x) = e(t) + s\sin(t) \\ x(0) = 0 = x(\pi), \end{cases} \tag{6.46}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a C^1-function. Using the notation therein, we set

$$a = \lim_{\xi \to -\infty} g'(\xi) \quad \text{and} \quad b = \lim_{\xi \to +\infty} g'(\xi),$$

and assume $a < b$. Moreover, we fix $l = l_S = l_A = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$. Writing the equation as a first order system, we have that (6.46) is equivalent to the problem

$$\begin{cases} Ju' = \nabla V(u) + R(t, u) + sv^*(t) \\ u(0) \in l,\ u(\pi) \in l, \end{cases}$$

where $u = (x, x')$, $V(u) = \frac{1}{2}(b(x^+)^2 + a(x^-)^2 + (x')^2)$, $v^*(t) = (-\sin t, 0)$ and $R(t, u)$ is a bounded function which can be computed explicitly. Thus, it turns out that the choice

$$\mathbb{A} = \mathbb{B} = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$$

makes the control (6.29) true in the whole set $\{w^*(t) \mid t \in [0, \pi]\}$, where

$$w^*(t) = \frac{1}{b-1}(\sin t, \cos t)$$

(notice that $\sin(t)/(b-1)$ solves the equation $x'' + bx^+ - ax^- = \sin t$, for $t \in [0, \pi]$). We are going to show that Theorem 6.2.5 agrees with [76, Theorem 1]; not to confuse the notation, we will write the integer numbers appearing therein with their original letters, but in Gothic style. Thus, $\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$) will be the number of zeros of a solution to $x'' + bx^+ - ax^- = 0$, with $x(0) = 0, x'(0) > 0$ (resp. $x(0) = 0, x'(0) < 0$), in $]0, \pi[$. Moreover, it is assumed in [76] that there exists a positive integer $\mathfrak{n}$ such that

$$\mathfrak{n}^2 < b < (\mathfrak{n} + 1)^2, \quad \mathfrak{n} \in \mathbb{N}.$$

Observe that assumption (6.28) becomes here

$$k\tau_V < \pi < k\tau_V + \min\{\tau_1, \tau_2\},$$

so that $\mathfrak{m}^+ = \mathfrak{m}^- = 2k$. On the other hand, a comparison with condition (6.30) yields $\mathfrak{n} = 2m+1$. Thus, since the assumption $a < b$ implies $2\mathfrak{n} > \mathfrak{m}^+ + \mathfrak{m}^-$ and thus $m \geq k$, we find the same number of solutions than in [76], i.e.,

$$2\mathfrak{n} - (\mathfrak{m}^+ + \mathfrak{m}^-) + 1 = 2(2m + 1) - 4k + 1 = 4(m - k) + 3.$$

Finally, notice that, in [76], also the case of a negative parameter s has been considered. This situation can be recovered by means of the change of variable $\tilde{x}(t) = -x(t)$ in (6.46).

In the statement of Theorem 6.2.5, we have presented a particular result of multiplicity, relying on conditions (6.28) and (6.30). For the sake of completeness, we now combine the nonresonance conditions given in Subsection 6.2.1 in different ways, and state the corresponding multiplicity results.

Theorem 6.2.13. *Assume that $V \in \mathcal{P}$ satisfies, for a suitable nonnegative integer k,*

$$k\tau_V + \max\{\tau_1, \tau_2\} < T < k\tau_V + \tau_1 + \tau_2 + \min\{\sigma_1, \sigma_2\} \tag{6.47}$$

instead of (6.28)*. Then, under all the other assumptions of Theorem 6.2.5, problem* (6.27) *has at least*

$$4|m-k|+1$$

solutions.

It is interesting to observe that, if $m = k$, Theorem 6.2.5 provides at least three solutions, while in Theorem 6.2.13 we only find a single solution, the one given by the topological degree argument. This can be explained by the fact that, roughly speaking, assuming together (6.28) and (6.30) implies that a gap between "small" and "large" solutions is already present even if k, m are equal, since "large" solutions starting on a fixed half-line of l_S intersect the arrival line l_A a number of times equal to $2k$, while "small" ones intersect it $2m+1$ times. In this last theorem, on the contrary, the number of intersections of "large" solutions starting on a fixed half-line of l_S, with l_A, is equal to $2k+1$.

Acting on condition (6.30), on the other hand, we have the following counterparts of Theorems 6.2.5 and 6.2.13.

Theorem 6.2.14. *Assume that $V \in \mathcal{P}$ satisfies* (6.28)*, and that $\mathbb{A}$, $\mathbb{B}$ fulfill, for a nonnegative integer m,*

$$(m-1)\tau_{\mathbb{A}} + \tau_{1,\mathbb{A}} + \tau_{2,\mathbb{A}} + \max\{\sigma_{1,\mathbb{A}}, \sigma_{2,\mathbb{A}}\} < T < m\tau_{\mathbb{B}} + \min\{\tau_{1,\mathbb{B}}, \tau_{2,\mathbb{B}}\}, \tag{6.48}$$

instead of (6.30)*. Then, under all the other assumptions of Theorem 6.2.5, problem* (6.27) *has at least*

$$4|m-k|+1$$

solutions.

Theorem 6.2.15. *Assume that $V \in \mathcal{P}$ satisfies* (6.47) *and $\mathbb{A}, \mathbb{B}$ fulfill* (6.48). *Then, under all the other assumptions of Theorem 6.2.5, problem* (6.27) *has at least*

$$2|2(m-k)-1|+1$$

solutions.

Remark 6.2.16. Comparing with the periodic boundary value problem, the number of solutions found, e.g., in [15, 32, 59, 94], is given in term of the gap between the behavior at 0 and at $+\infty$, similarly as in Corollary 6.2.3, or with a similar interpretation, after a change of variables involving a real parameter, as shown in the proof of Theorem 6.2.5. Indeed, every complete turn around the origin makes the number $n(z(t, \bar{z}; \lambda))$ defined above increase of two unities, so that the final number of solutions found, e.g., in Theorems 6.2.13 and 6.2.14 corresponds to the gap between the rotation numbers of "small" and "large" solutions.

6.3 The polygonal boundary value problem

Our general setting allows to consider boundary conditions which are not necessarily linear; indeed, as already remarked, the only important feature is homogeneity. In this section, we thus consider more general boundary conditions fitting in the setting of Theorem 6.1.2, choosing, as the starting and the arrival cones, two polygonal (piecewise linear) lines p_S and p_A which are the union of two half-lines emanating from the origin. For simplicity, we will assume that 0 is the only point of intersection of p_S and p_A. Obviously, each of these polygonal lines divide the plane into two connected regions. We will provide, similarly as in the previous section, existence and multiplicity results by the use of Theorem 6.1.2 and Theorem 6.1.5.

Precisely, we will deal with the boundary value problem

$$\begin{cases} Ju' = \nabla V(u) + R(t,u) \\ u(0) \in p_S, \ u(T) \in p_A. \end{cases} \tag{6.49}$$

We need to consider two cases, depending on the mutual position of p_S and p_A.

<u>Case 1</u>: the polygonal line p_S crosses both the connected regions of the plane separated by p_A. This situation is similar to the one for the Sturm-Liouville problem. As before, we follow a nontrivial solution $u(t)$ to the equation $Ju' = \nabla V(u)$, starting again from the vertical positive semi-axis and moving clockwise. In this way, at some nonnegative time instant t_0, $u(t)$ will arrive at a point u_1 in p_S, and we denote by τ_1 the least time needed by $u(t)$ to arrive at a point u_2 in p_A, starting from u_1. Continuing in covering the orbit described by $u(t)$, we then define σ_1 as the least time needed to encounter

again p_S, starting from u_2, and, similarly, we define τ_2 and σ_2 as the times needed to arrive once more on p_A and p_S. The only difference with the Sturm-Liouville problem lies in the fact that the four angles determined by the intersection of p_S and p_A will all be different, in general (see Figure 6.4). Aside from such a difference, this case can

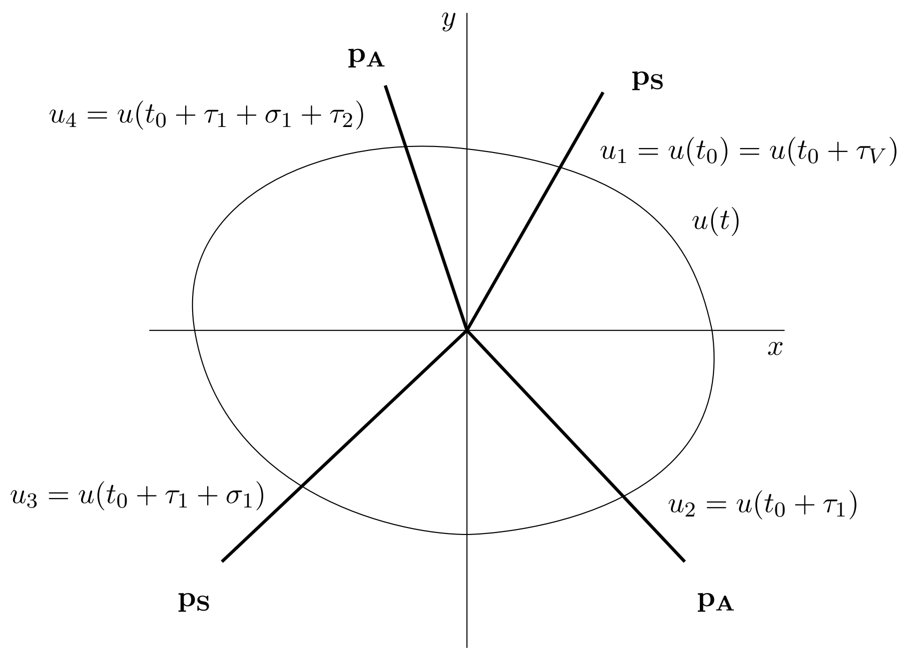

Figure 6.4: The situation described in Case 1.

be treated exactly as the previous one, so that we have the following result.

Theorem 6.3.1. *In the above configuration, the statement of Theorem 6.2.1 holds the same for problem* (6.49)*.*

The proof can be done as for Theorem 6.2.1, except for the fact that, instead of the antipodal points $\bar{u}$ and $-\bar{u}$, one has to take two points on the two different half-lines of p_S. As one can expect, the picture concerning the Fučík spectrum, in this case, can be similar to the one of the Sturm-Liouville boundary value problem. However, if the polygonal lines are chosen so as to mix the two situations briefly described before Corollary 6.2.2, some curious phenomena can appear. To give an idea, let us consider the scalar asymmetric equation (6.21), with the boundary conditions

$$\{x'(0) = 0,\, x(0) \geq 0\} \;\text{ or }\; \{x(0) + x'(0) = 0,\, x(0) \leq 0\}, \tag{6.50}$$

and

$$\{x(T) - x'(T) = 0,\, x'(T) \geq 0\} \;\text{ or }\; \{x(T) + x'(T) = 0,\, x'(T) \leq 0\}. \tag{6.51}$$

We clarify such boundary conditions in Figure 6.5.

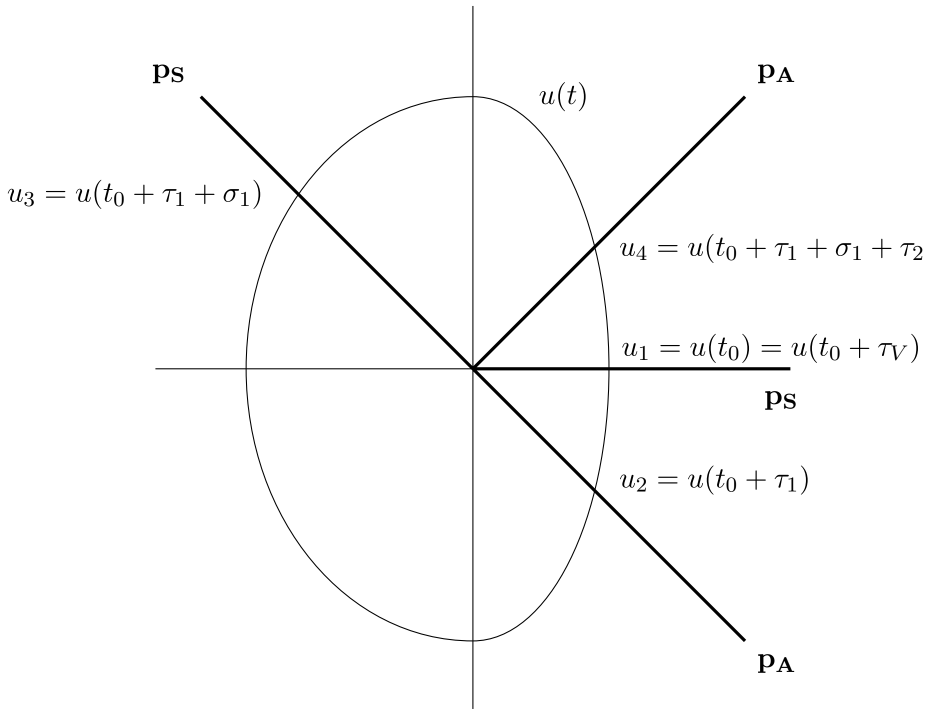

Figure 6.5: A "snapshot" of the boundary conditions (6.50), (6.51) for eq. (6.21).

The Fučík spectrum is defined exactly as in the previous section. Recalling (6.22), a direct computation gives, in this case,

$$\tau_1 = \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}\,, \quad \sigma_1 = \frac{\tau_V}{2} - \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}\,,$$

and

$$\tau_2 = \frac{\tau_V}{2} - \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}} - \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}\,, \quad \sigma_2 = \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}\,.$$

We thus obtain, for the Fučík spectrum Σ, the curves depicted in Figure 6.6; notice the asymmetry coming from the fact that the four times $\tau_1, \sigma_1, \tau_2, \sigma_2$ are generally not obtainable one from the other by simply exchanging μ and ν (as it is the case for the Sturm-Liouville boundary value problem). The regions for which there exists a solution are not so intuitively clear as in the classical case. Referring to Figure 6.6, starting from the origin and proceeding along the diagonal, in $\mathbb{R}^2 \setminus \Sigma$ one enters the existence regions alternatively, being them the first, the third, the fifth, ...

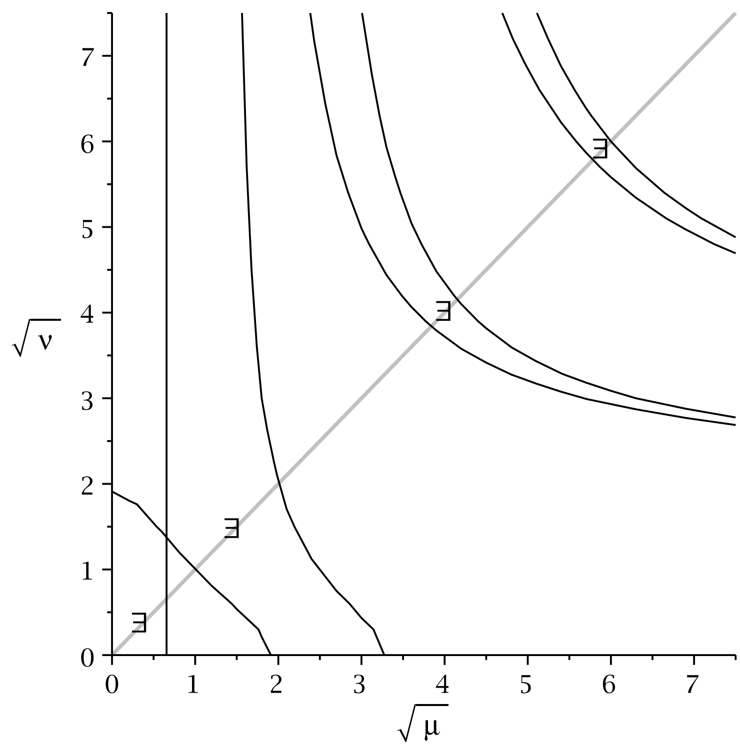

Figure 6.6: The Fučík spectrum for (6.21), with (6.50), (6.51), for $T = \frac{\pi}{2}$.

<u>Case 2</u>: the polygonal line p_S is all contained (except for the origin) into only one of the two connected regions of the plane separated by p_A. Once again, we follow a nontrivial solution $u(t)$ to $Ju' = \nabla V(u)$ starting from the vertical positive semi-axis, but, to simplify the notation, it is convenient to proceed in a slightly different way. We define $\hat{\tau}_1$ as the least time needed by $u(t)$ to start from p_S and arrive at p_A moving clockwise. Assume that this has been done starting from $u_1 \in p_S$ and arriving at $u_2 \in p_A$, covering an angular width $\hat{\theta}_1$. From there on, we resume our path along the orbit, defining $\hat{\tau}_2$ as the positive time to arrive again on p_A (at some point u_3) starting from u_2; we denote by $\hat{\theta}_2$ the amplitude of the corresponding angular region. We further define $\hat{\tau}_3$ as the time to reach p_S in a point u_4, starting at u_3 (and $\hat{\theta}_3$ as the angle covered in such a time), and $\hat{\tau}_4$ as the time to reach again p_S, starting from u_4 (and $\hat{\theta}_4$ correspondingly). In view of the mutual position of p_S and p_A, it is guaranteed that $\hat{\tau}_1 + \hat{\tau}_2 + \hat{\tau}_3 + \hat{\tau}_4 = \tau_V$; on the other hand, $\hat{\theta}_1 + \hat{\theta}_2 + \hat{\theta}_3 + \hat{\theta}_4 = 2\pi$. We

depict this situation in Figure 6.7.

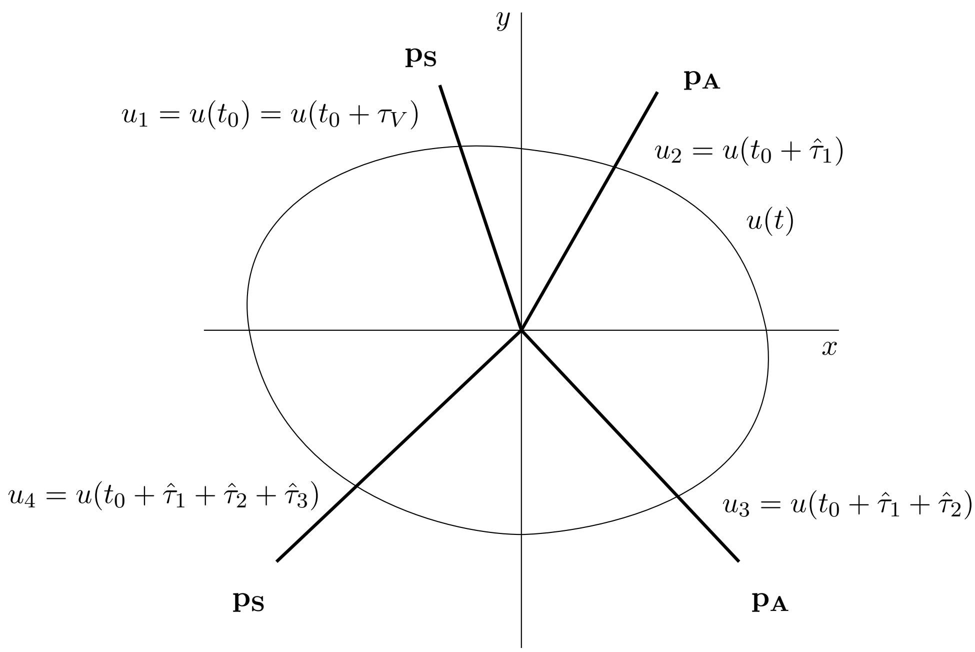

Figure 6.7: The situation described in Case 2.

We now state the following result.

Theorem 6.3.2. *Let $R(t,u)$ satisfy the sublinear growth condition* (6.5)*, and assume that there exists a nonnegative integer k such that one of the following nonresonance assumptions holds:*

$$k\tau_V + \hat{\tau}_1 + \max\{\hat{\tau}_2, \hat{\tau}_4\} < T < k\tau_V + \hat{\tau}_1 + \hat{\tau}_2 + \hat{\tau}_4, \tag{6.52}$$

or

$$k\tau_V + \hat{\tau}_1 < T < k\tau_V + \hat{\tau}_1 + \min\{\hat{\tau}_2, \hat{\tau}_4\}. \tag{6.53}$$

Then, problem (6.49) *has a solution.*

Proof. We consider a solution $u(t)$ to the Cauchy problem

$$\begin{cases} Ju' = \nabla V(u) \\ u(0) = \bar{u} \in p_S, \end{cases}$$

and the corresponding function $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined in (6.3). Passing to polar coordinates, it is possible to write $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$.

Similarly as in the proof of Theorem 6.3.1, it is possible to see that (6.52) implies

$$\max\{\hat{\theta}_2, \hat{\theta}_4\} < \theta(0) - \theta(T) - 2k\pi - \hat{\theta}_1 < \hat{\theta}_2 + \hat{\theta}_4.$$

This means that the points belonging to different half-lines of p_S are mapped, through the map $\mathcal{F}$, into different connected components of $\mathbb{R}^2 \backslash p_A$. In particular, $\mathcal{F}(u_1)$ (where u_1 is as above) will lie in the region - delimited by p_A - which contains p_S, while $\mathcal{F}(u_4)$ will belong to the interior of its complementary.
When (6.53) is assumed, we obtain

$$0 < \theta(0) - \theta(T) - 2k\pi - \hat{\theta}_1 < \min\{\hat{\theta}_2, \hat{\theta}_4\},$$

giving rise to the opposite situation.
In both cases, assumption (A) is thus satisfied and we conclude in view of Theorem 6.1.2. □

We point out that, in this situation, the resonance phenomenon is quite different. In particular, as a counterpart of conditions (6.52) and (6.53), it is readily seen that the problem

$$\begin{cases} Ju' = \nabla V(u) \\ u(0) \in p_S,\, u(T) \in p_A \end{cases}$$

has a (nontrivial) solution if and only if, for some integer k,

$$T - k\tau_V \in \{\hat{\tau}_1, \hat{\tau}_1 + \hat{\tau}_2, \hat{\tau}_4 + \hat{\tau}_1, \hat{\tau}_4 + \hat{\tau}_1 + \hat{\tau}_2\}.$$

As an example, let us consider the scalar second order equation (6.21), with the boundary conditions

$$\{x'(0) = 0,\, x(0) \geq 0\} \;\text{ or }\; \{x(0) = 0,\, x'(0) \geq 0\}, \tag{6.54}$$

and

$$\{x(T) = 0,\, x'(T) \leq 0\} \;\text{ or }\; \{x'(T) = 0,\, x(T) \leq 0\}. \tag{6.55}$$

This means that a solution to such a problem will start on one positive semi-axis and arrive on a negative one, no matter which. In this case, we will have

$$\hat{\tau}_1 = \hat{\tau}_4 = \frac{\pi}{2\sqrt{\mu}}\,, \quad \hat{\tau}_2 = \hat{\tau}_3 = \frac{\pi}{2\sqrt{\nu}}\,,$$

and the Fučík curves overlap in a quite curious way, as shown in Figure 6.8.
Notice that, as already visible in Figure 6.6, due to the nonlinear boundary conditions, the existence regions do not correspond to those having nonempty intersections with the diagonal.

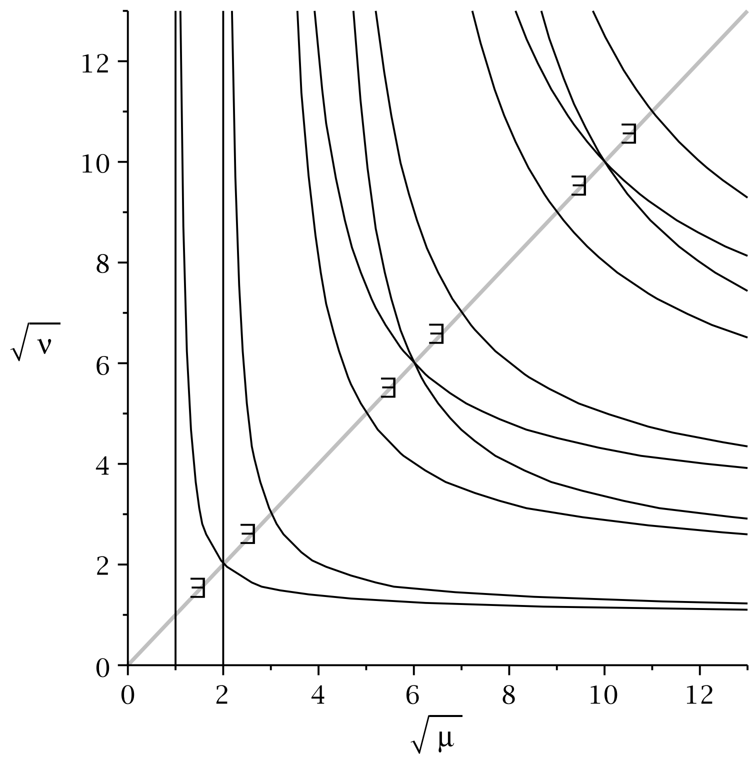

Figure 6.8: The Fučík spectrum for (6.21), with (6.54), (6.55), for $T = \frac{\pi}{2}$.

For what concerns multiplicity results, for the sake of brevity we will limit ourselves to the following example; in particular, considered two polygonal lines p_S and p_A, whose mutual position is as in Case 2 before, we give the following statement, keeping the same notation therein.

Corollary 6.3.3. *Let p_S, p_A be two polygonal lines through the origin as in Case 2. Moreover, let $k > m$ be two positive integers, $V_0, V_\infty \in \mathcal{P}$. With an analogous convention for the notation as in the previous corollaries, suppose that*

$$k\tau_V^0 + \hat{\tau}_1^0 + \max\{\hat{\tau}_2^0, \hat{\tau}_4^0\} < T < k\tau_V^0 + \hat{\tau}_1^0 + \hat{\tau}_2^0 + \hat{\tau}_4^0,$$

and

$$m\tau_V^\infty + \hat{\tau}_1^\infty < T < m\tau_V^\infty + \hat{\tau}_1^\infty + \min\{\hat{\tau}_2^\infty, \hat{\tau}_4^\infty\}.$$

Assume that $F(t,u)$ *satisfies* (6.9) *and* (6.10), *and* $F(t,0) \equiv 0$. *Then, the problem*

$$\begin{cases} Ju' = F(t,u) \\ u(0) \in p_S, \; u(T) \in p_A, \end{cases}$$

has at least $4(k-m)+2$ *nontrivial solutions.*

Proof. In this case, setting $p_S = p_S^1 \cup p_S^2$, where p_S^1 is the half-line of p_S which is closer to p_A with respect to the clockwise motion, and $p_A = p_A^1 \cup p_A^2$, where p_A^1 is the first half-line of p_A which is encountered by p_S^1 after a clockwise rotation, one has

$$n_0^{1,1} = n_0^{1,2} = n_0^{2,1} = k+1, \quad n_0^{2,2} = k, \quad n_\infty^{1,1} = m+1, \quad n_\infty^{1,2} = n_\infty^{2,1} = n_\infty^{2,2} = m,$$

yielding the desired conclusion. □

In the multiplicity results of the present section and of Subsection 6.2.1, we always assumed conditions of nonresonance type. Clearly, other types of corollaries of Theorem 6.1.5 can easily be obtained, without this restriction, at the expense of finding a lower number of solutions (see, e.g., [15, 27]). This fits exactly in the philosophy which is characteristic also of the multiplicity results for the T-periodic problem through the Poincaré-Birkhoff theorem (see Chapter 5): the essential requirement consists in producing a gap between small and large solutions, independently of whether or not we are at resonance.

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