International School of Advanced Studies

Mathematical Physics Sector

Academic Year 2008–09

PhD Thesis in Mathematical Physics

Duality and integrability in topological string theory

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Acknowledgements

Over the past few years I have benefited from the interaction with many people. I would like to thank first of all my supervisors: Alessandro Tanzini, for introducing me to the subject of this thesis, for his support, and for the fun we had during countless discussions, and Boris Dubrovin, from whose insightful explanations I have learned a lot. It is moreover a privilege to have Marcos Mariño and Tom Coates as referees of this thesis, and I would like to thank them for their words and active support in various occasions. Special thanks are also due to Sara Pasquetti, for her help, suggestions and inexhaustible stress-energy transfer, to Giulio Bonelli for many stimulating and inspiring discussions, and to my collaborators Renzo Cavalieri, Luca Griguolo and Domenico Seminara. I would finally like to acknowledge the many people I had discussions/correspondence with: Murad Alim, Francesco Benini, Marco Bertola, Vincent Bouchard ;-), Ugo Bruzzo, Mattia Cafasso, Hua-Lian Chang, Michele Cirafici, Tom Claeys, Andrea Collinucci, Emanuel Diaconescu, Bertrand Eynard, Jarah Evslin, Fabio Ferrari Ruffino, Barbara Fantechi, Davide Forcella, Antonio Grassi, Tamara Grava, Amar Henni, Kentaro Hori, Hiroshi Iritani, Igor Krichever, Ernst Krupnikov, Melissa Liu, Carlo Maccaferri, Cristina Manolache, Etienne Mann, Andrei Marshakov, Luca Mazzucato, Luca Mertens, Andrei Mironov, Alexei Morozov, Fabio Nironi, Alexei Oblomkov, Nicolas Orantin, Orlando Ragnisco, Andrea Raimondo, Antonio Ricco, Paolo Rossi, Yongbin Ruan, Simon Ruijsenaars, Houman Safaai, Emanuel Scheidegger, Balázs Szendrői, Nico Temme, Alessandro Tomasiello, Hsian-Hua Tseng, Stefan Vandoren, Marcel Vonk, Marlene Weiss, Katrin Wendland, Jian Zhou.

Finally, I want to thank my family, Dario, Mariacarla and Michele, and Cristina for their love and patience and for enduring my many omissions during the last few years. This thesis is dedicated to them.

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Chapter 1

Introduction

1.1 Background

The interplay between Mathematics and Physics has been for long time a most fruitful and conceptually rich arena of modern Science. On one hand, the mathematical formalization of physical models has proven to be remarkably successful in the quantitative description of natural phenomena; on the other, recent times have witnessed how surprisingly effective many ideas from particle physics (*e.g* supersymmetry) can be in the search of new mathematical structures.
The modern prototypical example of such cross–fertilization is provided by Superstring Theory. In addition to being the leading candidate for a unified theory of all forces - including a consistent quantum mechanical description of gravitational phenomena - and a surprisingly powerful tool for the study of gauge theories at strong coupling, the theory of superstrings has had a truly remarkable impact on sometimes distant areas of Mathematical Physics and Mathematics in general, both as a heuristic guiding principle and as a major unifying framework.

A central role in this respect has been played by the so-called *topological phase* of string theory (see [43, 93, 103, 118, 144, 149] for reviews). Physically, this consists of a class of two–dimensional $\mathcal{N} = 2$ superconformal field theories coupled to 2d–gravity which are characterized by an exact nilpotent fermionic symmetry Q_{BRST} [145, 149]. The most striking feature of these string theories is that fact the Q_{BRST} symmetry singles out a distinguished vector space of operators in the worldsheet σ-model, whose correlation functions do not dependent on the background metric on the Riemann surface. As the main consequence, these correlators are computed in a drastically simplified fashion as a finite-dimensional integral over on-shell, classical trajectories only.
A natural place of appearance for this theories is in the context of twisted $\mathcal{N} = (2, 2)$ σ-models, namely supersymmetric quantum field theories of maps from a compact connected Riemann surface to a Kähler manifold X, whose energy-momentum tensor is redefined by the abelian automorphisms of the $\mathcal{N} = (2, 2)$ superalgebra. When the target manifold X is the internal part of a vacuum configuration for superstrings, i.e. a Ricci-flat Kähler manifold, such topological string theories may come in two guises: the so-called A-model[1] and B-model. From the physical point of view their most attractive feature is the close relationship with the F-term sector of the parent Type IIA and IIB string theories on $\mathcal{M}_{1,3} \times X$, for which the topological invariance can be exploited to full power to give exact results. This has yielded a great deal of non-trivial information about the effective holomorphic dynamics of supersymmetric gauge and gravity theories [12, 22, 45, 143], and has provided at the same time a very useful laboratory for general ideas about black–hole dynamics [132] and dualities in

[1] In fact, conformality is not needed for the definition of the topological A-twist, which can in principle be performed on a target manifold X with a symplectic form ω, and a not necessarily integrable ω-tamed almost complex structure [17, 145]. Such level of generality will not however be needed in this thesis.

string theory and gauge theory [83].

From the mathematical viewpoint, topological string theories have perhaps an even more striking significance, as physical arguments suggest that their correlation function should capture deep and subtle aspects of the geometry of the target space. The most remarkable instance is given by the A-model on a target manifold X_A: the stationary phase reduction of the path integral in this case heuristically boils down to some sort of integration over a moduli space of holomorphic maps from the source Riemann surface to X, whose result should be invariant under deformations of the Kähler (or in general almost-Kähler) structure on X. The outcome of the integration, roughly speaking, should be related to a "count" of the number of curves in X in a fixed homology class and subject to intersect various cycles. These geometric invariants go under the name of *Gromov–Witten invariants*, and have very important applications in symplectic topology and enumerative algebraic geometry. On the other hand the same physical reasonings suggest that the B–model, which can be consistently defined only on a Calabi–Yau manifold X_B, could be interpreted as a theory of constant maps - therefore, a local theory on X_B, which computes an entirely different set of invariants related to the theory of variation of Hodge structures of X_B.
In absence of a rigorous general theory of functional integration for two dimensional quantum field theories, these path-integral motivated concepts begged for a sound mathematical foundation. In the last fifteen years, a tremendous effort has been put by the community of symplectic and algebraic geometers, somewhat in parallel, to give a rigorous description of the A–model in terms of intersection theory on moduli spaces of morphisms from marked curves to X (see Chapter 7 of [39] and references therein), with the aim to turn the physical intuition about Gromov–Witten invariants into mathematically well defined objects and relations. The success in finding a mathematical foundation for Gromov–Witten theory, however, has not stopped new ideas from Physics to be very influential and to make up an ever growing challenge for mathematicians.

Duality

A key role in this context has been played by string-inspired *dualities*. On the Physics side, it is an often recurring fact that different quantum theories might be isomorphic, meaning that their spectrum and correlators coincide in their entirety. The prime example in string theory is T-duality, namely the sign reflection of the right–moving sector of (possibly twisted) free $CFTs$. A non-linear realization of this duality relates pairs of Calabi–Yau threefolds (X_A, X_B) which are such that the topological A–model on X_A and the topological B–model on X_B are identical as physical theories. This phenomenon bears the name of *mirror symmetry*.
The consequences of such an unexpected relationship on the Mathematics side are striking. The statement of mirror symmetry builds an amazing correspondence be-

tween classically very hard enumerative problems for X_A on one side, and much simpler Hodge-theoretic computations on X_B on the other which can be efficiently performed. A paradigmatic application of mirror symmetry is the work of Candelas et al. [32] that led to startling predictions of the number of rational curves of various degrees of the quintic threefold in $\mathbb{P}^4$.
This results sparked a flurry of activity to prove rigorously the predictions of mirror symmetry [79, 81, 112], to find a proper mathematical setting for the duality [107], and to generalize it to cases of increasing difficulty. The most notable extension of the mirror symmetry program came from the work of [22], where the power of topological Ward identities was used to propose a beautiful recursive scheme for mirror symmetry calculations at higher genus.

A second, important example[2] is given by the *large N duality* between closed string theories on X_C and open strings on a background X_O with N D-branes, i.e. Dirichlet conditions for the endpoints of the string, which are constrained to live on a submanifold $M_O \subset X_O$. From the viewpoint of the gauge theory on the branes, this is a realization of 't Hooft's idea [1] that the fatgraph expansion of a gauge theory - resummed over "holes" - can be thought of as a closed string perturbation theory. In the topological A–model context, a concrete example of this duality was given [83] by relating open strings on T^*S^3 to the closed theory of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. From the mathematical point of view, this statement gets translated into a stunning mathematical connection between topological invariants of 3-dimensional real manifolds, like quantum-group theoretic invariants and knot invariants on one side, and ordinary and relative Gromov–Witten invariants of Calabi-Yau threefolds on the other. More than just an aesthetically appealing link between seemingly unrelated mathematical concepts, this relationship was exploited in full power to propose a full solution of the Gromov–Witten theory of toric Calabi–Yau threefolds, culminated in the celebrated work of [7].

Integrability

A recurring theme in the Mathematical Physics literature about topological strings is the conjecture that the high degree of solvability of the theory could be a signal of underlying *integrable structures*. The appearance of integrable systems is indeed an ubiquitous phenomenon in topological field and string theory: for example, already at genus zero and before coupling to topological gravity, the so-called associativity equations make up an integrable dynamical system [49,52,79] as do the equations that

[2] It must be noticed that this list of string dualities is far from being exhaustive and can be enlarged by including more, equally remarkable examples. For one, another kind of duality - S-duality - has proven instrumental to provide yet another instance of non-trivial mathematical correspondences on the A–model side, by relating Gromov–Witten theory with Donaldson–Thomas theory [96, 123, 131].

govern the interaction at tree–level between the chiral and the anti–chiral physical operators [34, 51]. A most important place in this context is occupied, after coupling to topological gravity, by the full partition function of the topological theory, which is conjecturally related to a τ-function of an integrable hierarchy of commuting $1+1$ PDEs. The prototypical example is given by Witten's conjecture [148], subsequently proven by Kontsevich [104], that the generating function of intersection numbers of Morita-Miller classes on the Deligne-Mumford moduli space of stable curves is a particular tau function of the KdV hierarchy.
A constructing proof of this conjecture for the A-model on a target space X (i.e., an explicit characterization of the hierarchy associated to the Gromov–Witten theory of X) would be a very far–reaching result, both in principle and computationally, and it represents possibly one of the main issues in Gromov–Witten theory. Aesthetically, it would create an *a priori* unexpected and beautiful bridge between the enumerative geometry of symplectic manifolds and the integrable dynamics of infinite–dimensional non–linear systems. Moreover, from the physics side, the integrable system would also provide a way to define non–perturbatively topological string theory on X beyond a formal power series expansion in the string coupling constant. But more in practice, knowledge of the underlying integrable flows would be a way to solve completely the intersection theoretic problem of determining the Gromov–Witten invariants of X, e.g. by establishing a set of constraints sufficiently strong to recover all the correlators recursively starting from a smaller subset of invariants. The connection with integrable systems is therefore a crucial aspect of the theory; however, despite much effort on the subject there are still only two examples (GW theory of the point and of $\mathbb{CP}^1$) where this connection is fully established mathematically, and the possibility of a systematic extension of this program to new classes of examples is to a large extent uncharted territory.

1.2 Outline of the thesis

The two aspects that we have emphasized - *duality* and *integrability* - will be the main thread of this thesis. Our focus will be on the Gromov–Witten theory of toric Calabi–Yau manifolds of complex dimension three. We will follow the following three lines of investigation:

1. in Chapter 3, we will study the variation of the A-model partition function, or Gromov–Witten potential, under a change of the Kähler moduli of the target space. By its very definition, at fixed genus the Gromov–Witten potentials take the shape of formal power series in the Novikov parameter, which is naturally interpreted from the worldsheet perspective as the exponential of the Kähler volumes of the image curves. The Gromov–Witten degree expansion is then naturally centered around the deep interior of the Kähler cone, i.e., around some sort of "large volume limit". It is natural to ask what happens if we move away from such a limit point: in particular, we would like to know how the potentials

change when we move from one patch to another of the Kähler moduli space, and also what kind of geometric information we get by considering expansions at different limiting points. We will tackle this problem by exploiting a dual description via mirror symmetry. As first advocated in [14], the α'-exactness of the B-model allows for a detailed study of the Kähler moduli space, including non-geometric ("conifold") or classically singular ("orbifold") phases. We will first propose a global B-model solution for the genus zero theory in a large class of examples, and then use the formalism of [24] to give definite predictions of the behaviour of generating functions of Gromov–Witten invariants when we move from one chamber to another of the (stringy) A-model moduli space.

2. in Chapter 4 we will study the possibility to extend the Gopakumar-Vafa duality, which relates $SU(N)$ Chern–Simons theory on S^3 to the Gromov–Witten theory of the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, to the case when the real three-sphere is replaced by a generic lens spaces $L(p,q)$. To this aim we exploit, on one hand, the matrix model representation [117] of the $SU(N)$ Chern–Simons partition function and study it in a generic flat background for the entire $L(p,q)$ family, providing a solution for its large N dynamics; on the other, we perform in full detail the construction of a family of would-be dual closed string backgrounds via conifold geometric transition from $T^*L(p,q)$. We will then explicitly prove, using mirror symmetry techniques, that the duality fails to hold true in this more general case.

3. in Chapter 5 we will tackle the problem of unveiling the integrable structures behind Gromov–Witten theory in new classes of examples and begin the study and the construction of the integrable hierarchies that govern the topological A-model on local Calabi–Yau threefolds. To this aim, we will consider the equivariant Gromov–Witten theory of Calabi–Yau rank 2-bundles over $\mathbb{P}^1$, and construct explicitly the relevant integrable hierarchies at the genus zero approximation. We will then focus on a thorough examination of the case of the resolved conifold with antidiagonal fiberwise action. We will formulate a precise conjecture about a candidate hierarchy that could encode the full theory, and test it successfully at one loop in the primary sector.

The core content of thesis is contained in Chapter 3-5. For the reader's convenience, and in order to make the exposition self–contained, we will review and collect in the next Chapter the relevant concepts that will be used throughout this thesis. This will mainly serve as a repository for the definitions of the objects that we will need in the following chapters, and has no pretension of completeness. We will first give a brief review of A-model concepts, such as Gromov–Witten invariants, their relations, and quantum co-homology; we will subsequently recall the description of the B-model on Calabi–Yau threefolds and of the mirror symmetry constructions that will be used in Chapter 3 and 4, including (local) Picard–Fuchs equations, open string mirror symmetry, and the $BKMP$ proposal.

Finally, Chapter 6 illustrates extension of results obtained in Chapter 3-5, and indicates new possible avenues of research. Two appendices have also been included, containing the B-model predictions for open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$, as well as a number of useful formulae that were used.

Chapter 2

Gromov-Witten theory of toric CY3 and mirror symmetry: a review

2.1 The A-side

In this section we will review the mathematical formalization of the topological A-model in terms of intersection theory on moduli spaces of holomorphic maps. To this aim we will first outline the properties of the relevant moduli spaces, introduce Gromov–Witten invariants in the generality we will need for this thesis, and then discuss some general properties of generating functions of Gromov–Witten invariants. We will finally conclude with the introduction of the big quantum co-homology algebra and the J-function.

2.1.1 Moduli spaces of holomorphic maps and stable compactification

Let (X, ω) be a compact Kähler manifold with Kähler form ω and let (Σ, p) be a connected, non-singular projective complex curve with n non-singular marked points $(p_1, ..., p_n)$. We will consider classes of equivalence of holomorphic maps $\phi : \Sigma \to X$, where the data (ϕ, Σ, p) and (ϕ', Σ', p') are identified if there exists a holomorphic automorphism sending $(\phi, \Sigma, p) \to (\phi', \Sigma', p')$. A holomorphic map $(\phi, \Sigma, p) \to X$ is called *stable* if for any irreducible component of Σ there is no conformal Killing vector field[1] on Σ having zeroes at the marked points. Let Σ be of arithmetic genus g and β denote the total homology class of $\phi_*[\Sigma] \in H_2(X, \mathbb{Z})$. Then a theorem from [105] states that the space of equivalence classes of stable maps $\phi : \Sigma \to X$ with given g, n, β can be closed to a compact Hausdorff topological space, called the *moduli space of stable maps* $\overline{\mathcal{M}_{g,n}}(X, \beta)$. This generalizes for $X \neq pt$ the Deligne-Mumford compactification $\mathcal{M}_{g,n}$ of moduli spaces of genus g Riemann surfaces with n marked points.

In the following we will be concerned with the case in which X is sufficiently non-negatively curved, and in particular is *semi-positive*. This means that if $\beta \in H_2(X, \mathbb{Z})$ is represented by $\phi_*([\Sigma])$ with $\Sigma \simeq S^2$, then we never have

$$\omega \cdot \beta > 0 \text{ and } 6 - \dim_{\mathbb{R}} X \leq c_1(X) \cdot \beta < 0$$

In such a case, $\overline{\mathcal{M}_{0,n}}(X, \beta)$ is a smooth compact orbifold of real dimension

$$2(\dim_{\mathbb{C}} X - 3) + 2c_1(X) \cdot \beta + 2n \tag{2.1.1}$$

and, as a quotient of a smooth manifold by a finite group, it inherits a rational fundamental cycle $[\overline{\mathcal{M}_{0,n}}(X, \beta)] \in H(\overline{\mathcal{M}_{0,n}}(X, \beta), \mathbb{Q})$. However the general case, *i.e.* $g > 0$, or non-semi-positive spaces, is considerably more involved; still, it turns out

[1]Given a metric g on Σ, $X \in \mathcal{X}(\Sigma)$ is said to be a conformal Killing vector field if

$$L_X g \propto g$$

holds pointwise.

that there exists a very rewarding approach to define nice fundamental classes and a sensible intersection theory on $\overline{\mathcal{M}_{g,n}}(X,\beta)$ in the language of algebraic geometry, namely by regarding $\overline{\mathcal{M}_{g,n}}(X,\beta)$ as a so-called "Deligne-Mumford stack". This allows for the introduction of "virtual fundamental classes" $[\overline{\mathcal{M}_{g,n}}(X,\beta)]^{vir}$, playing the role of the fundamental cycle in the general case and ensuring that all the otherwise formal expressions we will write from now on, like integration of top-degree form over the fundamental cycles, actually make sense and behave as expected. In this thesis we will take for granted that such a definition can be made, and refer the reader to [18, 106, 113] for the details of the construction.

2.1.2 Gromov-Witten invariants

Gromov-Witten theory deals with the construction of multi-linear functions $\langle \ldots \rangle_{g,n,\beta} : H^\bullet(X,\mathbb{C})^{\otimes n} \to \mathbb{C}$ via cup product of some universal cohomology classes over the fundamental cycle $[\overline{\mathcal{M}_{g,n}}(X,\beta)]^{vir}$. This is done as follows: first of all there are canonical morphisms $\mathrm{ev}_i : \overline{\mathcal{M}_{g,n}}(X,\beta) \to \overline{\mathcal{M}_{g,n-1}}(X,\beta)$, $\overline{\mathcal{M}_{g,n}}(X,\beta) \to \overline{\mathcal{M}_{g,n}}$, $\overline{\mathcal{M}_{g,n}}(X,\beta) \to X^n$ called *evalutation, forgetful*, and *contraction*, which are defined respectively by evaluating the map at the i^{th} marked point, by forgetting the map, and by forgetting one of the marked points and contracting unstable components. We will be intersted in the two following examples of characteristic classes:

1. pull-backs of cohomology classes from X^n by the evaluation maps $\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_n : \overline{\mathcal{M}_{g,n}}(X,\beta) \to X^n$ at the marked points;
2. polynomials in the curvature classes $\psi_i = c_1(\mathbb{L}_i)$ of the line orbi-bundles $\mathbb{L}_i$ over $\overline{\mathcal{M}_{g,n}}(X,\beta)$, whose fiber over the stable map $(\phi : \Sigma \to X, p)$ is the cotangent space $T^*_{p_i}\Sigma$.

This allows us to define Gromov-Witten invariants as follows.

Definition 1. *Given classes $\gamma_1, \ldots, \gamma_n \in H^\bullet(X,\mathbb{C})$, the* primary Gromov-Witten invariant $\langle\rangle_{g,n,\beta} : H^\bullet(X,\mathbb{C})^{\otimes n} \to \mathbb{C}$ *is defined by*

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,n,\beta} \quad := \quad \int_{[\overline{\mathcal{M}_{g,n}}(X,\beta)]^{vir}} \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i) \tag{2.1.2}$$

Given a collection of integers $k_i \geq 0$, we also define the descendant *(or* gravitational*)* Gromov-Witten invariant

$$\langle \tau_{k_1}\gamma_1, \ldots, \tau_{k_n}\gamma_n \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}_{g,n}}(X,\beta)]^{vir}} \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i)\psi_i^{k_i} \tag{2.1.3}$$

Remark 1. *Intuitively, the primary invariants should be related in some way to counting the number of degree-β holomorphic maps from Σ to X with n given marked points mapped to n given cycles. However it is not immediate in general to give a direct enumerative interpretaion of the invariants, as the orbifold nature of $\overline{\mathcal{M}_{g,n}}(X,\beta)$ allows them to take in general rational values.*

There is a manifold of generalizations that can be made to extend this setup to other interesting situations. We will be mostly interested on the following three (not mutually exclusive) cases:

- First of all suppose that X is acted on by an algebraic torus T; such an action then can be pulled-back to an action on $\overline{\mathcal{M}_{g,n}}(X,\beta)$. Since the evaluation, forgetful and contraction morphisms are by construction honest maps of T-spaces, one can define correlators of *equivariant* cohomology classes, which take values in $H^\bullet(BT)$

$$\begin{aligned}\langle\gamma_1,\dots,\gamma_n\rangle_{g,n,\beta}^{X\circlearrowleft T} &= \int_{[\overline{\mathcal{M}_{g,n}}(X,\beta)]^{vir}}\prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i) \\ \langle\tau_{k_1}\gamma_1,\dots,\tau_{k_n}\gamma_n\rangle_{g,n,\beta}^{X\circlearrowleft T} &= \int_{[\overline{\mathcal{M}_{g,n}}(X,\beta)]^{vir}}\prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i)\psi_i^{k_i} \qquad (2.1.4)\end{aligned}$$

  where now $\gamma_i\in H_T^\bullet(X)$. When it is implicit that we will refer to T-equivariant Gromov–Witten invariants, we will suppress the $^{\circlearrowleft T}$ apex in the notation for equivariant correlators.

- In this thesis we will actually *never* consider the case of X being projective; in fact, some of the cases of greatest interest for us will be total spaces of concave holomorphic vector bundles $X\simeq E\to B$, where B is a projective toric curve or surface. Such examples are therefore non-compact, but given that any holomorphic map of non-zero degree from a smooth projective curve Σ to X lands to the zero section of E, we have that $\overline{\mathcal{M}_{g,n}}(X,\beta)$ and $\overline{\mathcal{M}_{g,n}}(B,\beta)$ are isomorphic as topological spaces. It turns out however that their expected dimension - in gauge theory terms, the index-theoretic computation of zero-modes of the twisted Dirac operator on Σ - differs, and the two virtual cycles are related as

$$[\overline{\mathcal{M}_{g,n}}(X,\beta)]^{vir} = [\overline{\mathcal{M}_{g,n}}(B,\beta)]^{vir}\cup e(\mathcal{E}_{g,n,\beta}) \qquad (2.1.5)$$

  where e is the (possibly T-equivariant) Euler class and $\mathcal{E}_{g,n,\beta}$ is the vector bundle over $\overline{\mathcal{M}_{g,n}}(X,\beta)$ having $H^1(\Sigma,\phi^*E)$ as a fiber at a stable map $f:\Sigma\to B$. Therefore GW-invariants of X coincide with *twisted* Gromov–Witten invariants of B [36, 37].

- Another possibility we will allow is that X be a (reduced) orbifold. To this purpose, suppose X is a quotient Y/G of a smooth complex manifold Y by the possibly non-free action of a finite abelian group G by holomorphic diffeomorphisms, and denote with $X_g\subset Y$ the invariant submanifolds of the action of $g\in G$. The inertia orbifold $\mathcal{I}X$ of X is defined as the disconnected union

$$\mathcal{I}X := \bigsqcup_{g\in G} X_g \qquad (2.1.6)$$

that is, a point of $\mathcal{I}X$ is a pair (x, g) where $x \in X$ and g belongs to the isotropy group of x. The Chen–Ruan orbifold cohomology groups $H^\bullet_{CR}(X, \mathbb{C})$ are then defined as

$$H^\bullet_{CR}(X, \mathbb{C}) = H^\bullet(\mathcal{I}X; \mathbb{C})$$

As for ordinary co-homology, we have a natural grading on $H^\bullet_{CR}(X, \mathbb{C})$. To each component X_g of the inertia orbifold (2.1.6) we attach a rational number called the *age* of X_g, defined as follows: let $\mathfrak{g}$ be the representation of g inside $Aut(X)$, and consider its differential at a point $x \in X$

$$d\mathfrak{g}(x) : T_xX \to T_xX$$

Since g is abelian, T_xX splits into a direct sum

$$T_xX = \bigoplus_{0 \leq j < r} Y_j$$

where Y_j is the eigenspace of $d\mathfrak{g}$ with eigenvalues $\exp(2\pi i j/r)$. The age of X_g is defined as

$$\text{age } (X_g) = \sum_{j=0}^{r-1} \frac{j}{r} \text{dim} Y_j \tag{2.1.7}$$

We use this to turn $H^\bullet_{CR}(X, \mathbb{C})$ into a $\mathbb{Q}$-graded vector space by defining the *orbifold degree* of $\alpha \in H^k(X_g, \mathbb{C})$ as

$$\deg_{CR}(\alpha) = k + 2\text{age } (X_g) \tag{2.1.8}$$

Physically, the age-shifting is due to a shift in the fermion number of the vacuum when (quasi-periodic) twisted boundary conditions are put on the chiral fermions of a 2d CFT [152]. It turns out that a moduli space of stable maps can be defined sharing most of the desired properties of the ordinary, smooth case. Again, one considers degree d maps from n-pointed genus g orbicurves Σ to X and takes a stable compactification of the space of such maps up to isomorphism; the resulting space is again sufficiently well-behaved from the algebraic point of view to have a virtual fundamental class $[\overline{\mathcal{M}_{g,n}}(X, \beta)]^{vir}$, with the expected dimension, and with a nice notion for all the relevant morphisms of moduli spaces (and in particular evaluation maps). Taking this for granted[2], we define the *orbifold Gromov-Witten invariants of* X as

$$\langle \tau_{k_1}\gamma_1, \ldots, \tau_{k_n}\gamma_n \rangle^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}_{g,n}}(X,\beta)]^{vir}} \prod_{i=1}^n \text{ev}_i^*(\gamma_i)\psi_i^{k_i} \tag{2.1.9}$$

for orbifold cohomology classes $\gamma_i \in H^\bullet_{orb}(X)$.

[2]The intersted and algebraically-minded reader may find the details of the construction in [3]

2.1.3 Generating functions and relations between correlators

Generating functions

Gromov-Witten invariants have an obvious importance if looked at individually: as we mentioned in Remark 1, in many cases *each single* primary invariant should somehow carry information about enumerative aspects of the geometry of the target space. The main thread of this thesis, however, will be the study of the mathematical structures arising when we consider them *collectively*: that is, we will be interested in the study and characterization of the properties of *generating functions* of Gromov–Witten invariants.

To this aim, let us consider the Gromov–Witten theory of a target space X, where X is allowed to be an orbifold and/or to be acted on by a torus $T = (\mathbb{C}^*)^k$ with compact T-fixed loci. In the following we will unify notation by denoting simply with $H(X)$ the (possibly equivariant and/or orbifold) co-homology ring of X

$$H(X) = \begin{cases} H^\bullet(X, \mathbb{C}) & \text{ordinary case} \\ H^\bullet_T(X, \mathbb{C}) \otimes \mathbb{C}((\underline{\lambda})) & \text{equivariant} \\ H^\bullet_{CR}(X, \mathbb{C}) & \text{orbifold} \\ H^\bullet_{CR,T}(X, \mathbb{C}) \otimes \mathbb{C}((\underline{\lambda})) & \text{equivariant orbifold} \end{cases} \tag{2.1.10}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r)$ and $\mathbb{C}((\lambda))$ is the field of fractions of the T-equivariant cohomology of a point, $H_T(pt) \simeq \mathbb{C}[\underline{\lambda}]$. $H(X)$ is a finite dimensional vector space over a field $\mathbb{K}$, where $\mathbb{K} \simeq \mathbb{C}$ and $\mathbb{K} \simeq \mathbb{C}((\underline{\lambda}))$ in the non-equivariant and equivariant case respectively, it has a $\mathbb{Z}$- (resp. $\mathbb{Q}$-)gradation induced by the de Rham (resp. age-shifted) degree, and is endowed with a natural $\mathbb{K}$-bilinear pairing: when it is compact, or when it is acted on by a torus T with compact fixed loci F, there is an inner product $\eta : H(X) \times H(X) \to \mathbb{K}$ defined as

$$\eta(\alpha, \beta) = \begin{cases} \int_X \alpha \cup \beta & \text{ordinary case} \\ \int_F \frac{i^*(\alpha \cup \beta)}{e(\mathcal{N}_{\mathcal{I}F/\mathcal{I}X})} & \text{equivariant} \\ \int_{\mathcal{I}X} \alpha \cup I^\star \beta & \text{orbifold} \\ \int_{\mathcal{I}F} \frac{i^*(\alpha \cup I^\star \beta)}{e(\mathcal{N}_{\mathcal{I}F/\mathcal{I}X})} & \text{equivariant orbifold} \end{cases} \tag{2.1.11}$$

where in the third and fourth definition $i : \mathcal{I}F \to \mathcal{I}X$ is the inclusion of the T-fixed locus $\mathcal{I}F$ of the inertia orbifold of F into $\mathcal{I}X$, and I is the canonical involution on the inertia stack mapping X_g to $X_{g^{-1}}$.

Let $N := \dim_{\mathbb{K}} H(X)$, $r = b_2(X)$ and let us a fix a basis $\{\Phi_a\}_{a=0}^{N-1}$ of $H(X)$ such that $\deg\Phi_0 = 0$ and $\deg\Phi_i = 2$ for $0 < i \leq b_2(X)$. We will write η_{ab} the coefficient matrix of η in the basis Φ_a

$$\eta_{ab} = \eta(\Phi_a, \Phi_b)$$

Definition 2. *Let $u \in H(X)$, with $u =: \sum_a u^a \Phi_a$. The genus-*g *primary Gromov–Witten potential, or g^{th}-loop A-model free-energy of X is defined as the following* *formal power series*

$$F_g^X(u) = \sum_{n\geq 0}\sum_{\beta\in H_2(X,\mathbb{Z})} \frac{1}{n!}\langle \overbrace{u,u,\ldots,u}^{n\ times}\rangle^X_{g,n,\beta}, \qquad (2.1.12)$$

The all-genus GW potential of X is the formal power series expansion

$$F^X(u,g_s) = \sum_{g\geq 0} g_s^{2g-2}F_g^X(u) \qquad (2.1.13)$$

It is helpful also to have a generating function for correlators with insertions of ψ classes as well. Introduce the semi-infinite set of variables $t_k^a \in \mathbb{K}$, with $0\leq a\leq n$, $0\leq k<\infty$. We will denote collectively with $\mathbf{t} = \{t_k^a\}_{\substack{a=0,\ldots,n\\ k\geq 0}}$ the set of t_k^a.

Definition 3. *The genus-*g *full descendant potential of X is a formal power series in variables t_k^a, $0\leq a\leq n$, $0\leq k<\infty$ defined by*

$$\mathscr{F}_g^X(\mathbf{t}) = \sum_{n\geq 0}\sum_{0\leq k_1,\ldots,k_n<\infty}\sum_{\beta\in H_2(X,\mathbb{Z})}\frac{1}{n!}\langle \tau_{k_1}t_1\ldots\tau_{k_n}t_n\rangle^X_{g,n,\beta} \qquad (2.1.14)$$

where $H(X)\ni t_k = t_k^\alpha\Phi_\alpha$. In analogy with the case of primary invariants, we also define the all genus full descendant potential as

$$\mathscr{F}_g^X(\mathbf{t},g_s) = \sum_{g\geq 0} g_s^{2g-2}\mathcal{F}_g^X(\mathbf{t}) \qquad (2.1.15)$$

GW axioms and the worldsheet expansion

It turns out that, from the general properties of the virtual fundamental class, there exist various axioms that the correlators (2.1.3)-(2.1.2), or equally well the generating functions (2.1.12)-(2.1.14) should satisfy. Let us start by listing the following four conditions, which are valid for any genus g.

- **Degree axiom:** if γ_i are homogeneous classes, the gravitational correlator (2.1.3) can be nonzero only if the total degree is "right", in the sense that

$$\sum_{i=1}^n(2k_i+\mathrm{deg}\gamma_i) = 2(1-g)\mathrm{dim}_{\mathbb{C}}X - 2c_1(X)\cdot\beta + 2(3g-3+n) \qquad (2.1.16)$$

- **Fundamental class** (or **string**) **axiom:** let $1\in H(X)$ denote the unity class and let $2g+n\geq 3$ or $\beta\neq 0$, $n\geq 1$. Then

$$\begin{aligned}&\langle\tau_{k_1}\gamma_1,\ldots,\tau_{k_n}\gamma_n,1\rangle^X_{g,n+1,\beta}\\ =\ &\textstyle\sum_{i=1}^{n-1}\langle\tau_{k_1}\gamma_1,\ldots,\tau_{k_i-1}\gamma_i,\tau_{k_{i+1}}\gamma_{i+1},\ldots\tau_{k_n}\gamma_n\rangle_{g,n,\beta}\end{aligned} \qquad (2.1.17)$$

- **Divisor axiom:** if D is a divisor and let $2g+n \geq 3$ or $\beta \neq 0$, $n \geq 1$. Then

$$\langle \tau_{k_1}\gamma_1, \dots, \tau_{k_n}\gamma_n, D\rangle^X_{g,n+1,\beta} = (D\cdot\beta)\langle \tau_{k_1}\gamma_1, \dots, \tau_{k_n}\gamma_n\rangle^X_{g,n,\beta}$$
$$+\sum_{i=1}^{n-1}\langle \tau_{k_1}\gamma_1, \dots, \tau_{k_i-1}(D\cup\gamma_i), \tau_{k_{i+1}}\gamma_{i+1}, \dots \tau_{k_n}\gamma_n\rangle_{g,n,\beta} \qquad (2.1.18)$$

- **Dilaton axiom:** let $2g+n \geq 3$ or $\beta \neq 0$, $n \geq 1$. Then

$$\langle \tau_1\tau_{k_1}\gamma_1, \dots, \tau_{k_n}\gamma_n\rangle^X_{g,n+1,\beta} = (2g-2+n)\langle \tau_1\tau_{k_1}\gamma_1, \dots, \tau_{k_n}\gamma_n\rangle^X_{g,n,\beta} \qquad (2.1.19)$$

For $g = 0$ we also have the following

Point–splitting axiom: for the genus zero, degree zero primary invariant

$$\langle\gamma_1, \dots, \gamma_n\rangle^X_{0,n,0} = \begin{cases} (\gamma_1, \gamma_2\cup\gamma_3) & n=3 \\ 0 & \text{otherwise} \end{cases} \qquad (2.1.20)$$

With the Divisor and Point–splitting relations at hand it is instructive to rewrite the generating functions in the following way. Let us first focus on the genus zero free energy F_0^X and recall that we have picked a basis of $H(X)$ such that $\deg\Phi_0 = 0$ and $\deg\Phi_i = 2$ for $0 < i \leq b_2(X)$. Then by the Divisor and Point–splitting axiom we have

$$F_0^X(u) = \frac{1}{3!}\big(u\cup u, u\big)_X + \sum_{n\geq 0}\sum_{d_i\geq 0}\frac{e^{\beta\cdot\hat{u}}}{n!}\langle\overbrace{\tilde{u}, \tilde{u}, \dots, \tilde{u}}^{n \text{ times}}\rangle^X_{0,n,\beta} \qquad (2.1.21)$$

where $\hat{u}$ is the projection of u on $H^2(X)$ and $\tilde{u} := u - \hat{u}$. As such, the quantum ($\beta \neq 0$) part of the genus zero potential takes the shape of a formal power series expansion whose domain of convergence, if not empty, must intersect the $u_1, \dots, u_r$ hyperplane in a polydisc around $u_i \sim -\infty$. This is even more sharply evident when X is a $CY3$: by the string and degree axioms (2.1.21) becomes

$$F_0^X(u) = \frac{1}{3!}\big(u\cup u, u\big)_X + \sum_{\beta\neq 0} e^{\beta\cdot\hat{u}}\langle 1\rangle^X_{0,0,\beta} \qquad (2.1.22)$$

Remark 2. *The latter expression has boiled down to a power series expansion solely on the degree–2 parameters $\hat{u}$. In the physical picture, terms of the form $\hat{u}\cdot\beta$ arise from the non-BRST exact, purely topological part of the classical A-model action, which is simply given by the Kähler volume of the image curve represented by β in $H_2(X,\mathbb{Z})$. In the Euclidean path-integral, these give rise to worldsheet instanton contributions weighted by $e^{-\omega\cdot\beta}$, therefore identifying $\hat{u}$ with minus the Kähler form of X. As such, Gromov-Witten theory yields a formal expansion centered around the deep interior of the Kähler cone, i.e. around some sort of "large volume limit". In physical language and after properly rescaling ω in order to make it dimensionful, this is tantamount to an expansion in inverse powers of the square of the string length - they measure the increasingly* stringy *character of the corrections. On the other* *hand, the g_s-expansion of (2.1.12) takes the typical form of a topological expansion in perturbative string theory, with g_s playing the role of the string coupling constant. Higher genus contributions correspond therefore to* quantum *corrections to the space-time physics.*

WDVV and TRR

There are other non-trivial relations at genus zero, which take a particularly neat form as non-linear equations satisfied by the GW potentials. First of all, in the case of the primary invariants, a most important role is played by the so-called associativity, or Witten–Dijkgraaf–Verlinde–Verlinde ($WDVV$) equations .

Theorem 1 ([42, 43, 147])**.** *The genus zero primary Gromov–Witten potential satisfies the following system of third order non-linear equations in the form*

$$\frac{\partial^3 F_0^X}{\partial u^a \partial u^b \partial u^c}\eta^{cd}\frac{\partial^3 F_0^X}{\partial u^d \partial u^e \partial u^f} = (-1)^{\deg u^i(\deg u^j + \deg u^k)}\frac{\partial^3 F_0^X}{\partial u^a \partial u^e \partial u^c}\eta^{cd}\frac{\partial^3 F_0^X}{\partial u^d \partial u^b \partial u^f} \quad (2.1.23)$$

for any a, b, e, f.

A second, very important set of relations involves the gravitational correlators, and it allows for a recursive computation of them in terms of the primary invariants. Before we state it, we need to define the *genus g big correlators* as

$$\langle\langle \tau_{d_1}\gamma_1 \dots \tau_{d_n}\gamma_n\rangle\rangle_g(u) := \sum_{k=0}^{\infty}\langle \tau_{d_1}\gamma_1 \dots \tau_{d_n}\gamma_n, \overbrace{u, \dots, u}^{k \text{ times}}\rangle_{g,\beta,n+k} \quad (2.1.24)$$

where $u = u^a\Phi_a \in H(X)$ as before. Notice that the big correlators reduce to the usual ones (2.1.2), (2.1.3) when u is set to zero. Then the following *genus zero topological recursion relations* (TRR) hold

Theorem 2 ([44])**.** *For d_1, d_2, $d_3 \geq 0$ and $\gamma_i \in H(X)$, $i = 1, 2, 3$ we have*

$$\langle\langle \tau_{d_1}\gamma_1, \tau_{d_2}\gamma_2, \tau_{d_3}\gamma_3\rangle\rangle_0 = \langle\langle \tau_{d_1}\gamma_1, \Phi_a, \rangle\rangle_0 \langle\langle \Phi^a, \tau_{d_2}\gamma_2, \tau_{d_3}\gamma_3\rangle\rangle_0 \quad (2.1.25)$$

where Φ^a denotes the dual basis to Φ_a with respect to the cup product.

2.1.4 Quantum co-homology and the J-function

The equations (2.1.23) have a nice algebraic interpretation in terms of a deformation of the classical co-homology of the target space, corresponding physically to the notion of the chiral ring of the world–sheet TFT. This is given as follows: define *structure constants c_{ij}^k* as

$$c_{ij}^k = \frac{\partial^3 F_0^X}{\partial u^i \partial u^j \partial u^l}\eta^{lk} \quad (2.1.26)$$

Then define

$$\Phi_i * \Phi_j = c_{ij}^k \Phi_k \tag{2.1.27}$$

and extend it linearly to $H(X)$. This gives the *big quantum product* on $H(X)$; at $u = 0$, it coincides with the classical cup product in co-homology. The product is super-commutative [39], as it is its classical counterpart, it has a unity $\Phi_0 = 1$ by the String Equation and the Point–Splitting Axiom, and it is associative by the $WDVV$ equations (2.1.23). We will henceforth denote it by $QH^\bullet(X)$; when we work equivariantly with respect to a torus action T, we will often write $QH_T^\bullet(X)$.

An useful notion, which will be further explored in Sec. 5.2, is that of *flat sections* associated to the big quantum product. They are locally defined as $H(X)$ valued functions s_a from $H(X)$ such that

$$\hbar \frac{\partial s_a}{\partial u^i} = \Phi_i * s_a \tag{2.1.28}$$

where $\hbar$ is a formal parameter. The space of solutions S of (2.1.28) is N dimensional [39, 80], and we can write them as

$$S = S_{ab} \eta^{bc} \Phi_c \tag{2.1.29}$$

for a matrix valued function $S_{ab}(u)$ on $H^\bullet(X)$. It turns out that the coefficients S_{ab} have a particularly nice relationship with gravitational descendants; in particular the first row S_{0a} coincides with the so-called *big J-function of X*. The latter is a generating function of genus zero 1-point descendant Gromov–Witten invariants in the form

$$J_X(u, \hbar) := \hbar + u + \sum_{n \geq 0} \sum_{\beta \in H_2(X, \mathbb{Z})} \frac{1}{n!} \left\langle \left\langle \frac{\Phi_a}{\hbar - \psi} \right\rangle \right\rangle_{0,1,d}^X \Phi^a \tag{2.1.30}$$

We use here the shorthand notation

$$\left\langle \left\langle \tfrac{\Phi_a}{\hbar - \psi} \right\rangle \right\rangle_{0,1,d}^X \quad \text{for} \quad \textstyle\sum_{k=0}^\infty \frac{1}{\hbar^{k+1}} \langle\langle \Phi_a \psi^k \rangle\rangle_{0,1,d}^X$$

The expression above can be usefully rewritten when we restrict u to lie in $H^0(X) \oplus H^2(X)$. Writing $u = u^0 \Phi_0 + \cdots + u^{r_X} \Phi_X$, the Divisor Axiom then implies that

$$J_X(u, \hbar) := \hbar \prod_{i=0}^{r_X} e^{u^i \Phi_i / \hbar} \left(1 + \sum_{\beta \in H_2(X, \mathbb{Z})} e^{u^0 \Phi_0 / \hbar} e^{u^{r_X} \Phi_{r_X} / \hbar} \left\langle \frac{\Phi_a}{\hbar - \psi} \right\rangle_{0,1,\beta}^X \Phi^a \right) \tag{2.1.31}$$

We will refer to the latter as the *small J function* of X.

2.2 The B-side

By Q_{BRST}-localization, and by its independence on the Kähler structure of the target space, the topological B-model boils down to a theory of constant maps. Its mathematical formalization, in genus zero, is given in terms of Hodge theory and special Kähler geometry. In the following we will review the basics of Hodge theory and Picard-Fuchs equations, moving then to the higher genus theory and the holomorphic anomaly equations. We will then describe the case of toric Calabi-Yau threefolds in detail and the extensions to the open string sector. We will close the chapter with a description of the Bouchard-Klemm-Marino-Pasquetti proposal for the B-model on the mirrors of toric $CY3$.

2.2.1 A summary of the compact case

Kinematics: Hodge theory and complex moduli

Physically, the truly marginal subsector of the B-twisted $BRST$ co-homology corresponds to variations of complex structure of the target space, which are computed at tree-level via period integrals. In the following we will review the mathematical formalization of this by giving the basics of Hodge theory which will be useful for the purposes of mirror symmetry. In particular, after introducing the Hodge bundle and the Gauss-Manin connection, we will conclude this section with a classification of the types of boundary points which we will encounter in next chapters. We will follow closely the presentation of [39, 126], to which we refer the reader for details, proofs of the relevant statements, and further links to the original literature.

Let $\widehat{X}$ be a $CY3$, which in the following will be taken to be smooth and projective. Its de Rham co-homology in degree 3 with complex coefficients admits a *Hodge decomposition*

$$H^3(\widehat{X},\mathbb{C}) \simeq \bigoplus_{p+q=3} H^{p,q}(\widehat{X})$$

We have $\overline{H^{p,q}(\widehat{X})} = H^{q,p}(\widehat{X})$ relative to the real structure determined by $H^k(\widehat{X},\mathbb{R})$; the integer lattice coming from $H^k(\widehat{X},\mathbb{Z})$ gives a *Hodge structure of weight k*. It is moreover convenient to define the *Hodge filtrations*

$$F^p(\widehat{X}) = \bigoplus_{a\geq p} H^{a,k-a}(\widehat{X})$$

We will be interested in the behaviour of Hodge co-homology groups under deformations of the complex structure of $\widehat{X}$. To this purpose, let S be the full complex moduli space of $\widehat{X}$, which by the Bogomolov-Tian-Todorov theorem is a smooth manifold of dimension $h^{2,1}(\widehat{X})$. Consider a family $\pi : \mathcal{X} \to S$, with π smooth and of relative dimension 3, and denote $\widehat{X}_t := \pi^{-1}(t)$ for $t \in S$. Then [39] the subspaces $F^p(\widehat{X}_t)$ fit together to form *flat holomorhphic bundles* $\mathcal{E}^p$. There is a canonical flat connection on $\mathcal{F}^p$, called the *Gauss-Manin connection*[3]

$$\nabla : \Gamma(\mathcal{E}^0) \to \Gamma(\mathcal{E}^0) \otimes \Omega^1_S$$

which satisfies the Griffiths transversality condition $\nabla(\mathcal{E}^p) \subset \mathcal{E}^{p-1} \otimes \Omega^1_S$.

In most cases we will be interested in what happens at the *boundary* of S, and more specifically at particular points of the boundary; they will mirror the notion of specific boundary points of the Kähler moduli space of an A-model target X, like *e.g.* the classical limit (or "large radius") point of $QH^\bullet(X)$.

To this aim, suppose that the smooth family $\pi : \mathcal{X} \to S$ could be completed to a flat family $\overline{\pi} : \overline{\mathcal{X}} \to \overline{S}$; here $\bar{S}$ is a compactification of S which we assume in the first place to be smooth and with normal crossings boundary divisor $D = \bar{S} - S$. A natural question is then to ask about existence and properties of extensions of $(\mathcal{E}^0, \nabla)$ on $\bar{S}$. It turns out [39] that there is a *canonical extension* $\bar{\mathcal{E}}^0$ of $\mathcal{E}^0$ on $\bar{S}$; however, the Gauss-Manin connection ∇ extends in general to a *meromorphic* connection $\bar{\nabla}$, possibling acquiring single-pole singularities on D. This implies that the sections of $\bar{\mathcal{E}}^0$ undergo non-trivial monodromy transformations $\mathcal{T}_i$ when parallel-transported around a connected component D_i of $D = \cup D_i$

$$\mathcal{T}_i : H^3(\widehat{X}_t, \mathbb{C}) \to H^3(\widehat{X}_t, \mathbb{C})$$

By the monodromy theorem [39] we have that $\mathcal{T}_i$ is quasi-unipotent, $(\mathcal{T}_i^m - I)^4 = 0$; in the cases we will deal with in this thesis, we will always have $m = 1$, and the logarithms $N_i := \log(\mathcal{T}_i)$ of the monodromy transformations will then be nilpotent of order four. In view of this, we shall classify classify boundary points as follows. Let $p \in \cap D_i$ and define a generic linear combination $N = \sum_i a_i N_i$. Then

- if $\mathrm{Res}_{D_i}\bar{\nabla} \neq 0$ for some i and $N^3 \neq 0$ for all $a_i > 0$, we will call one such p a *maximally unipotent boundary point*[4], or "large complex structure point". This will be the B-Model counterpart of the **large radius point** on the A-Model side; however, it needs not be unique;

- if $\mathrm{Res}_{D_i}\bar{\nabla} \neq 0$ for some i and $N^k = 0$ for some integer $0 < k < 4$, we will call p a **conifold point**;

- suppose now that $\mathrm{Res}_{D_i}\bar{\nabla} \neq 0$, but allow now $\bar{S}$ to be a *singular* compactification of S, with D_i having possibly orbifold-type singularities. This means that, even if the connection is non-singular, we might still have monodromy, although not of logarithmic type, due to the orbifold nature of the moduli space itself. We will call p an **orbifold point**.

[3]The Hodge bundle $\mathcal{E}^0$ is isomorphic to $R^k\pi_*\mathbb{C} \otimes \mathcal{O}_s$; ∇ is defined as the affine connection whose covariantly constant sections coincide with the local system $R^k\pi_*\mathbb{C}$

[4]Strictly speaking the definition of maximal unipotency above only applies to the case of one-dimensional moduli, and needs to be strengthened in the multi-parameter case in order to "mirror" the non-degeneracy of the Poincaré pairing.

- in all other cases, p will be called a **regular point**.

Classical dynamics: Picard-Fuchs equations

Let $r = h^{2,1}(\widehat{X})$, $z_1, \ldots, z_r$ be local holomorphic coordinates on S, and $\mathcal{D}$ be the ring of differential operators

$$\mathcal{D} = \mathbb{C}\{z_1, \ldots, z_r\} \left[\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_r}\right]$$

Denote with $\Omega(t)$ a fixed section of $\mathcal{E}^3$, that is, a choice of normalization of the holomorphic (3,0) form on $\widehat{X}$ throughout S. The Gauss-Manin connection determines an $\mathcal{O}_S$-linear map $\phi : \mathcal{D} \to \mathcal{E}^0$ through

$$\phi(X_1, \ldots, X_r) = \nabla_{X_1} \ldots \nabla_{X_r} \Omega(t)$$

for $X_i \in \mathcal{X}(S)$. We will call the ideal $I = \ker(\phi)$ the *Picard-Fuchs (*PF*) ideal* and the equations $D \cdot \Pi = 0$ for $D \in I$ the *PF equations*. They have a central role in the theory; their main properties are the following

1. the solution space of the PF system is spanned over $\mathbb{C}$ by the periods of the holomorphic (3,0) form $\Omega(t)$

$$\Pi_{\gamma_t} = \int_{\gamma_t} \Omega(t)$$

   for a locally constant homology 3-cycle γ_t

2. it is particularly interesting to look at the behavior of the solutions around one of the boundary points above, and especially around (one of) the large radius point(s). Taking local coordinates $z_1 \ldots z_r$ around such a point, we can choose a basis of solutions of $\mathcal{D} \cdot \Pi = 0$ such that their asymptotic behavior reads

$$\Pi_i(z) \sim \log {z_i}^{k_i} + \mathcal{O}(1), \qquad i = 1, \ldots, 2r+2$$

   where

$$k_i = \begin{cases} 0 & \text{for} \quad i = 1 \\ 1 & \text{for} \quad 2 \leq i \leq r+1 \\ 2 & \text{for} \quad r+1 < i < 2r+2 \\ 3 & \text{for} \quad i = 2r+2 \end{cases}$$

3. The solution with $i = 0$ is usually set to $\Pi_0 = 1$ after normalizing $\Omega(t) \to \Omega(t)/\Pi_0$. This apparently arbitrary choice enters in fact crucially in determining the *mirror map* to the A-model variables; after the normalization, these are precisely given by the set of single-logaritmic solutions $1 \leq i \leq r+1$. We will call them the *flat coordinates* around p and denote them with $t_i := \Pi_{i+1}$, $i = 1, \ldots, r$. We define analogously the *dual periods* as $\mathcal{F}_i := \Pi_{i+r+1}$.

4. the solutions with double- or triple-logarithmic singularity are not independent on the flat coordinates. There exists a complex function $\mathcal{F}_0 : \Delta \to \mathbb{C}$ defined locally on a polydisc $\Delta \ni p$, called the **prepotential** of $\widehat{X}$, such that

$$\begin{array}{rcll} \mathcal{F}_i = \Pi_{i+r+1} & = & \frac{\partial \mathcal{F}_0}{\partial t_i} & \text{for} \quad i = 1, ..., r \\ \Pi_{2r+2} & = & \sum_{i=1}^r t_i \frac{\partial \mathcal{F}_0}{\partial t_i} \mathcal{F}_0 - 2\mathcal{F}_0 & \end{array} \tag{2.2.1}$$

Therefore, knowledge of the periods of Ω, or equivalently, of a suitably normalized basis of solutions of the PF system, fixes completely the form of the prepotential up to a constant.

We have seen how to associate to $\widehat{X}$ a function $\mathcal{E}_0$, defined locally around a large complex structure point, coming from variations of Hodge structure of $\widehat{X}$. We are now in a position to give the following

Definition 4 (Tree level mirror pairs)**.** *Let X and $\widehat{X}$ be Calabi-Yau threefolds and let F_0^X and $\mathcal{F}_0^{\widehat{X}}$ be respectively the genus zero primary Gromov-Witten potential of X and the B-Model prepotential of $\widehat{X}$. Then $(X, \widehat{X})$ is said to be a* mirror pair at tree level *if*

$$F_0^X = \mathcal{F}_0^{\widehat{X}} \tag{2.2.2}$$

It is worthwhile to point out the (not just formal) analogy of such a computation with the action-angle transformation of a classical integrable system: regarding Ω as a higher-rank generalization of the Poincaré differential pdq, the flat coordinates play here the role of "action variables", and the prepotential that of the Hamilton-Jacobi function generating the canonical change of variables from z_i to t_i. We will push this analogy even further in the next section.

Quantum dynamics: the holomorphic anomaly

Thinking of the prepotential as a Hamilton-Jacobi function, we can also regard it as the eikonal limit of the (phase of) a wave-function of some quantum-mechanical system. It would seem then natural to look at it as the leading term in a formal $\hbar-$expansion,

$$\mathcal{F}_0 \to \mathcal{F}_0 + \hbar \mathcal{F}_1 + \hbar^2 \mathcal{F}_2 + \dots$$

where the higher order terms are produced by some sort of quantization of the geometric setup of the previous section and would heuristically correspond to higher string loop correction of the topological B-model $TCFT$ coupled to supersymmetric gravity. We will refer to them as the B-Model genus g free energies.

In the physics literature, it is suggested that the definition of such a deformation should be given [151] by geometric quantization of the phase space $\mathcal{F}^0$ of sec. 2.2.1. We will not go through the details of the derivation; the result of greatest interest for us is that the $\mathcal{F}_g$ obtained in this way are bound to satisfy the *holomorphic anomaly equations* of [21,22]. This means the following: first of all, there is a natural metric h on $\mathcal{F}^3$, given by $h := i\int_{\widehat{X}} \Omega \wedge \bar{\Omega}$. We will denote by $D : \Gamma(\mathcal{F}^3) \to \Gamma(\mathcal{F}^3) \otimes \Omega^1(S)$ the Levi-Civita connection associated to h. Moreover, $\log h$ defines a Kähler potential on S, whose Kähler metric, called the *Weil-Petersson* metric on S, we will write as $G_{i\bar{j}}$. Then the $\mathcal{F}_g$ are sections of $(\mathcal{E}^3)^{2-2g}$ with the following property

$$\begin{aligned} \partial_i\bar{\partial}_{\bar{j}}\mathcal{F}_1(t_i,\bar{t}_i) &= \tfrac{1}{2}C^{(0)}_{ikl}\bar{C}^{(0)kl}_{\bar{j}} - \left(\tfrac{\chi(\widehat{X})}{24}-1\right)G_{i\bar{j}}\,. && \text{for } g=1 \\ \bar{\partial}_{\bar{i}}\mathcal{F}_g &= \tfrac{1}{2}\bar{C}^{(0)jk}_{\bar{i}}\left(D_jD_kF^{(g-1)} + \textstyle\sum_{r=1}^{g-1} D_jF^{(r)}D_kF^{(g-r)}\right) && \text{for } g>1 \end{aligned} \tag{2.2.3}$$

where we defined

$$\begin{aligned} \bar{C}^{(0)kl}_{\bar{j}} &= h^2 G^{k\bar{k}}G^{l\bar{l}}\bar{C}^{(0)}_{\bar{j}\bar{k}\bar{l}} \\ \bar{C}^{(0)}_{\bar{j}\bar{k}\bar{l}} &= \partial_{\bar{j}\bar{k}\bar{l}}\bar{\mathcal{F}}_0 \end{aligned}$$

Two comments are in order:

1. The equations (2.2.3) recursively define $\mathcal{F}_g(t_i,\bar{t}_i)$ in terms of lower order $\mathcal{F}_g(t_i,\bar{t}_i)$, up to a holomorphic function of the complex moduli t_i, which has to be determined as an additional input. It will be referred to in the following as the *holomorphic ambiguity*.

2. For the purposes of mirror symmetry it is important to consider the leading term $\mathcal{F}_g(t_i)$ in a formal $\bar{t}$ expansion, which in the case of the large radius points of sec. 2.2.1 is centered around $\bar{t} \sim \infty$. We will call it the *holomorphic limit* of $\mathcal{F}_g(t_i,\bar{t}_i)$.

Suppose now to have a full solution of the holomorphic anomaly equations (2.2.3) with a prescribed choice of the holomorphic ambiguity. Then we can give the following

Definition 5 (Mirror pairs)**.** *Let X and $\widehat{X}$ be smooth Calabi-Yau threefolds and let F_g^X and $\mathcal{F}_g^{\widehat{X}}$ for $g \in \mathbb{Z}$ be respectively the genus g primary Gromov-Witten potential of X and the holomorphic limit of the B-Model genus g free energy of $\widehat{X}$. Then $(X,\widehat{X})$ is said to be a* mirror pair *if*

$$F_g^X = \mathcal{F}_g^{\widehat{X}} \quad \forall g \in \mathbb{N} \tag{2.2.4}$$

2.2.2 Closed local mirror symmetry at tree level

Establishing that X and $\widehat{X}$ are tree–level mirror pairs turns the (in general very difficult) task of computing genus 0 Gromov-Witten invariants of X into the much simpler computation of the B–model prepotential of $\widehat{X}$, which consists in either evaluating the periods of the holomorphic $(3,0)$ form Ω on a basis of $H_3(\widehat{X},\mathbb{Z})$, or finding a complete set of solutions of the PF system of $\widehat{X}$. Such a "mirror computation", although much easier, is still far from being free of difficulties. Computing explicitly the period integrals is in most cases unwieldy; on the other hand finding the solutions of a system of linear PDEs with regular singularities seems a more tractable problem, but until now we have avoided the central question "how do we construct a basis $\mathfrak{L}_i$ of the PF ideal in concrete examples?". From now on we will focus on the case in which X is a toric $CY3$. We will describe the simplifications that take place and how they allow to address both problems in practice, as well as discuss the problems due to the non–compactness of X.

The GKZ system

Let X be a toric $CY3$. Throughout this thesis, we will denote with Σ_X its fan. As a toric variety, X can be regarded as a holomorphic quotient

$$X = \frac{\mathbb{C}^{k+3} \setminus Z(\Sigma)}{(\mathbb{C}^*)^k} \tag{2.2.5}$$

where $k = b_2(X)$ and $Z(\Sigma)$ is the Stanley-Reisner subvariety of X. The algebraic k-torus $(\mathbb{C}^*)^k$ acts on $\mathbb{C}^{k+3}$ as

$$(x_1,\dots,x_{k+3}) \to (\lambda^{Q_i^1}x_1,\dots,\lambda^{Q_i^{k+3}}x_{k+3}) \qquad i=1,\dots,k \tag{2.2.6}$$

with $Q_i^j \in \mathbb{N}\ \forall\ i=1,\dots,k,\ j=1,\dots,k+3$, and moreover $\sum_j Q_i^j = 0$ by the CY condition $K_X \simeq \mathcal{O}_X$.

Let now $\{z_i\}_{i=1}^k \in \mathbb{C}^k$ and let Δ be a polydisc centered around $z_i = 0$. Introduce also a set of $k+3$ auxiliary variables $\{a_j\}_{j=1}^{k+3}$ which determine the z_i through

$$z_i = \prod_{j=1}^{k+3} a_j^{Q_i^j} \tag{2.2.7}$$

When acting on functions $f : \Delta \to \mathbb{C}$, it is immediate to verify that powers of the partial derivatives ∂_{a_i} take the form

$$\left(\frac{\partial}{\partial a_j}\right)^n = (a_j)^{(-n)} D_j^{(n)} \tag{2.2.8}$$

where $D_j^{(n)}$ is a linear differential operator in the variables $\ln z_i$ with constant coefficients. This means that the set of equations

$$\prod_{Q_i^j>0}\left(\frac{\partial f}{\partial a_j}\right)^{Q_i^j} = \prod_{Q_i^j<0}\left(\frac{\partial f}{\partial a_j}\right)^{-Q_i^j} \tag{2.2.9}$$

becomes, when f is a complex function on Δ and by using (2.2.7)-(2.2.8), a system of coupled linear PDEs in the variables z_i, having $z_i = 0$ as a regular singular point.

Definition 6. *We call the system of equations (2.2.9) the GKZ hypergeometric system associated to X. When acting on functions on Δ, we will call it the* local Picard-Fuchs system *associated to X, and the point $0 \in \Delta$ its* large complex structure point.

The GKZ system first appeared in the construction of the PF ideal of CY hypersurfaces inside toric ambient spaces [75], and its rôle in the context of local mirror symmetry was emphasized in [35]. As opposed to the flat co-ordinates t_i, we will call the z_i alternatively the *B-model moduli*, or *bare co-ordinates*.

Remark 3. *The construction of a local PF system in the mirror symmetry approach to toric $CY3$ is completely straightforward and explicit. However, there are a few drawbacks related to the non-compactness of X, which result in a sort of "incompleteness" of the solution space of (2.2.9). By this we mean the following: we have seen in section (2.2.1) that the B–model prepotential $\mathcal{F}_0$ was computed from the so-called "dual" periods, that is from the solutions of the PF system with double-logarithmic behavior. In total, we had $1 = b_0(X)$ holomorphic, $h^{2,1}(\widehat{X}) = h^{1,1}(X) = b_2(X)$ logarithimic, $b_4(X) = b_2(X)$ doubly-logarithmic, and $1 = b_6(X)$ triple-logarithmic solutions of the PF system around a large complex structure point. In the non–compact case, however, such a symmetry between the number of flat co-ordinates and the number of dual periods gets spoiled by the breakdown of Poincaré duality. In this case we have in fact*

$$b_0 = 1; \quad b_4(X) < b_2(X) = k; \quad b_6(X) = 0$$

In particular, we know less derivatives of the prepotential than the number of flat co-ordinates. This means that the resulting prepotential will have a functional ambiguity in some of the flat variables, that should be fixed by other means, either by A–model inspired considerations, or by suitably enlarging the PF system (see [68] for a proposal about this last point).

Until now, we have been discussing the local PF system quite abstractly, without reference to an underlying mirror manifold $\widehat{X}$ whose period integrals are the solutions of (2.2.9). As it turns out, $\widehat{X}$ has an equally explicit description, which we are now going to review.

The Hori-Vafa mirror and spectral curves

A general procedure for constructing mirror duals of (among others) toric CY threefolds was provided in [91]. First of all recall that since $K_X \simeq \mathcal{O}_X$, the tip of the 1-dimensional cones of its fan are all constrained to lie in some affine hyperplane in $\mathbb{R}^3$. For given X, we will denote such hyperplane as H_X. After fixing an arbitrary origin O in H_X, Pick a coordinate, integer basis $(e_1, e_2) \in GL(2, \mathbb{Z})$ of H and denote by $Q_X = H_X \cap \Sigma_X$ the polytope resulting from the intersection of the fan of X with H_X. Q_X will be called the *toric diagram* of X.

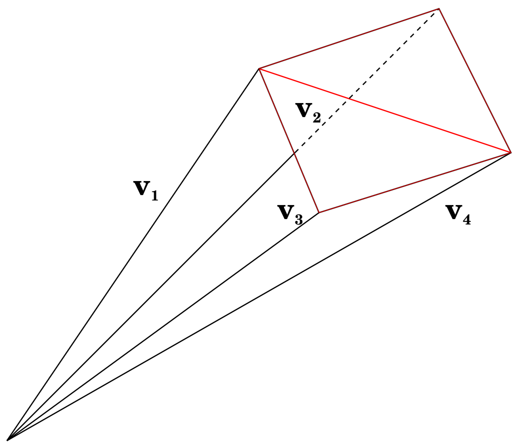

Figure 2.1: The fan of the resolved conifold, $\mathcal{O}_{\mathbb{P}}(-1) \oplus \mathcal{O}_{\mathbb{P}}(-1)$.

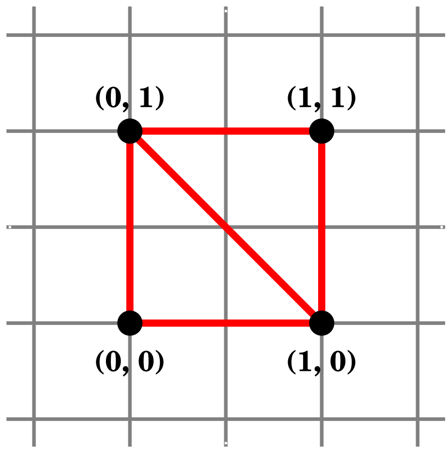

Figure 2.2: The toric diagram of the resolved conifold.

Definition 7 (Hori-Vafa mirror, [91,92]). *The B-model target space $\widehat{X}$ mirror to a toric CY three-fold X is the hypersurface in $\mathbb{C}^2(x_1, x_2) \times (\mathbb{C}^*)^2(U, V)$*

$$x_1 x_2 = P_X(U, V)$$

where $P_X(U, V)$ is the Newton polynomial associated to the polytope Q_X

$$P_X(U, V) = \sum_{p \in Q_X} a_p U^{\mathrm{pr}_1(\mathrm{p})} V^{\mathrm{pr}_2(\mathrm{p})} \tag{2.2.10}$$

and $\mathrm{pr_i}$ are the projections onto the coordinate axis e_1, e_2 of H_X.

The geometry is therefore that of a quadric fibration over the $P_X(U, V) = \lambda \in \mathbb{C}$ plane, which degenerates to a node above the punctured Riemann surface $P_X(U, V) = 0$. We will call the latter the *mirror curve* Γ_X of X. Notice that at this moment the definition of the mirror depends not only on X, but also on an arbitrary choice of basis of H - or in other words, on a choice of an automorphism of the fan Σ_X which leaves the tip of the 1–dimensional rays inside the hyperplane H_X. This turns out to be irrelevant in the computation of closed GW invariants, but it will be important in the context of open string invariants.
Since $\widehat{X}$ sits as an affine hypersurface inside $\mathbb{C}^2 \times (\mathbb{C}^*)^2$, the holomorphic $(3, 0)$ form Ω is given as the residue form on $P_X = 0$ of the holomorphic 4-form in $H^{4,0}(\mathbb{C}^2 \times (\mathbb{C}^*)^2 \setminus \widehat{X})$

$$\Omega = \mathrm{Res}_{P_X(U,V)=x_1 x_2} \left[\frac{dx_1 dx_2 dU/U dV/V}{x_1 x_2 - P_X(U, V)} \right] \tag{2.2.11}$$

In the local case under scrutiny we must actually cope with the absence of a symplectic basis for $H_3(\widehat{X}, \mathbb{Z})$. The proposal of [68, 91, 94] is then to consider non–compact cycles as well, that is we should compute

$$\int_\Gamma \Omega$$

for all $\Gamma \in H_3(\widehat{X}, \mathbb{Z}) \oplus (H_3(\widehat{X}, \mathbb{Z}))_c$, where the subscript c indicates compactly supported homology. Moreover, it was shown in [68] that the periods of Ω solve the PF system (2.2.9) if and only if those of

$$d\lambda := \mathrm{Res}_{P_X=0}\left[\frac{dUdV}{UV}\log P_X(U,V)\right] \tag{2.2.12}$$

do on a basis of $H_1(\Gamma_X, \mathbb{Z}) \oplus (H_1(\Gamma_X, \mathbb{Z}))_c$. Picking up the residue gives

$$d\lambda_X = \log V\frac{dU}{U} \tag{2.2.13}$$

The logarithmic 1-differential (2.2.13) will be called the *Hori–Vafa differential* of X. In terms of this differential the periods are computed as

$$\Pi_\gamma = \int_{\gamma\in H_1(\Gamma_X,\mathbb{Z})\oplus(H_1(\Gamma_X,\mathbb{Z}))_c} \log V\frac{dU}{U} \tag{2.2.14}$$

Proposition 3 ([68, 94]). *The periods Π_γ give the complete solution space of the (extended) local PF system of $\widehat{X}$.*

As in the compact case, a choice of polarization on Γ induces a notion of symplectically conjugated periods, and therefore that of flat co-ordinates, their dual variables, and of a prepotential. In other words, the tree-level mirror symmetry computations in the toric case are controlled, in view of Proposition 3, by the datum of a non-compact complex curve $\Gamma \subset (\mathbb{C}^*)^2$ with two marked logarithmic functions $\log U, \log V : \Gamma \to \mathbb{C}$. As we will see, this will be true much more generally.

Remark 4 (Normalizable and non–normalizable modes). *According to Remark 3, in the local CY case there is a clear dichotomy in the closed moduli sector between flat co-ordinates that have a dual doubly-logarithimic PF solution and those who don't. We will adopt the terminology of [8, 9] by calling the former normalizable and the latter non–normalizable modes of Γ. The normalizable modes correspond to flat co-ordinates that are computed as periods of the Hori–Vafa differential around non-trivial homology 1-cycles of the projectivized mirror curve $\overline{\Gamma}_X$, while non-normalizable modes are those that are computed as residues at the marked points $U = 0$, $V = 0$ of Γ_X. We add a tilde and write $\tilde{t}_i$ whenever we refer to a non–normalizable mode.*

B-model moduli and compactification

In sec. 2.2.1 we gave a partial classification of boundary points of the B–model moduli space S after compactification. An obvious question to ask is if there are natural compactifications of S, at least in some concrete examples, and what they look like. In the toric case there exists indeed such a natural compactification scheme, which we will now review. As we mentioned in Remark 3, we have $b_2(X) = k$, hence we expect to have a k-dimensional complex moduli space for $\widehat{X}$. In fact, according to (2.2.10), the mirror curves Γ, and therefore the whole mirror threefold $\widehat{X}$ come in a family parameterized by the coefficients a_p of $P_X(U,V)$; these are as many as the number of internal and external points of the polytope Q_X, namely $k+3$. The a_p are actually homogeneous coordinates for S, as an overall rescaling of them and scalings of U and V in (2.2.10) leave invariant the symplectic form

$$\frac{dU}{U} \wedge \frac{dV}{V} \tag{2.2.15}$$

in $(\mathbb{C}^*)^2$. That is, the moduli space of the mirror theory might be seen as arising from a holomorphic quotient of $\mathbb{C}^{k+3}$ by the $(\mathbb{C}^*)^3$ action

$$\begin{array}{ccccc} (\mathbb{C}^*)^3 & \times & \mathbb{C}^{k+3} & \rightarrow & \mathbb{C}^{k+3} \\ (\lambda,\mu,\nu) & & \{a_p\} & \rightarrow & \{\lambda\mu^{\mathrm{pr}_1(\mathrm{p})}\nu^{\mathrm{pr}_2(\mathrm{p})}a_p\} \end{array} \tag{2.2.16}$$

Out of these weights it is immediate to construct the (stacky) fan of a compact toric orbifold $\bar{S}$. We will call it the *compactified B–model moduli space of X*, and its fan is the *secondary fan* of X. It bears an easy relationship with the fan of X, in that its skeleton is simply given by columns of the gauged linear σ-model for X (i.e. the j_{th} ray is given by the k-tuple Q_i^j, $i = 1,\dots k$). In general it is not a toric variety, as typically the secondary fan will contain non-smooth simplicial cones, perhaps with marked points along their facets. In the latter case, this would mean that the patch parameterized by the corresponding a_i's looks like $\mathbb{C}^p/\mathbb{Z}_n$ rather than $\mathbb{C}^p$; as such, the periods of the holomorphic three-form will inherit the finite monodromy from the monodromy of the a_i themselves.

Deforming $\widehat{X}$ by moving away from a large complex structure point is the mirror process to changing the Kähler structure on X by moving from the large radius point to some small volume region. It is tempting to see what happens on the B–model side in the case in which curves or divisors in X shrink to zero size by making it (classically) singular; for example, when X is birationally isomorphic to an orbifold $\mathcal{X}$, it is natural to ask if the generating functions of X and $\mathcal{X}$ are related by some sort of analytic continuation, and how this is realized in the B–model context. Indeed, it turns out[5] that orbifold points of the compactified B–model moduli space actually are related to the Gromov–Witten expansion for $\mathcal{X}$ as follows

[5]This may be seen as following from a number of results about twisted Gromov–Witten invariants, as presented in [36, 37]

- the finite monodromy group $\mathcal{G}_{orb}$ at the orbifold point plays the same role as the parabolic monodromy at the large complex structure point: it mirrors, on the B–model side, the grading of the relevant A–model co-homology ring - in this case the age-shifted grading of $H^{\bullet}_{CR}(\mathcal{X}, \mathbb{C})$.

- therefore, in order to reflect such grading, the canonical choice of mirror co-ordinates at the orbifold is the one that *diagonalizes the monodromy.* Again, dual periods are forced by the symplectic structure of the B–model phase space to be derivatives of an *orbifold prepotential* $F_0^{\mathcal{X}}(\{t_i^{orb}\})$, and to have dual monodromy.

This procedure will then provide a set of *orbifold flat coordinates* $\{t_i^{orb}\}$, with monodromy $m \in \mathcal{G}_{orb}$, that are naturally associated to the direction in the co-homology of the m^{th} twisted sector $\mathcal{X}_m$, and an orbifold prepotential $F_0^{\mathcal{X}}(\{t_i^{orb}\})$. Quite remarkably, they are non-trivially related to the flat coordinates $\{t_i\}$ and the prepotential F_0^X at large radius: the flat co-ordinates at the orbifold point are in general obtained via a general linear transformation involving *both* the flat co-ordinates and the dual periods at large radius, and the same thing happens for the orbifold prepotential.

$$t_i^{orb} = \sum_j A_{ij} t_j + \sum_j B_{ij} \frac{\partial \mathcal{F}_0^X}{\partial t_j} \tag{2.2.17}$$

$$\frac{\partial \mathcal{F}_0^{\mathcal{X}}}{\partial t_j^{orb}} = \sum_j C_{ij} t_j + \sum_j D_{ij} \frac{\partial \mathcal{F}_0^X}{\partial t_j} \tag{2.2.18}$$

The higher genus story is more involved and will be discussed in section 2.2.4. Quite interestingly, still, the matrices A, B, C and D in (2.2.17) and (2.2.18) will completely determine how the higher genus generating functions $\mathcal{F}_g^{\mathcal{X}}(\{t_i^{orb}\})$ and $\mathcal{F}_g^X(\{t_i\})$ will be related.

2.2.3 Open local mirror symmetry

Open string generating functions

In Sec. 2.2.1 and 2.2.2 we have tried to keep our treatment of mirror pairs, either compact or non–compact, as symmetric as possible. By this we mean that every object computed on the B–model side, like the prepotential or the holomorphic limit of a solution of the holomorphic anomaly equation, has a rigorously defined counterpart on the A–model side, with a definite enumerative meaning. In this section and the following we will still introduce objects on the B–model side, and we will still hope to give them a definite enumerative meaning on the A–model side, but their rigorous construction in terms of intersection theory on some moduli space is in various cases still in progress in the literature. This objects have to do with *open string invariants*, which is what we now turn to discuss starting from the A–model.

It is desirable to generalize the setup of section 2.1.1, which consisted of maps from a compact source curve Σ to X, by now allowing Σ to have a non-trivial boundary. Let then Σ be a connected Riemann surface with $\partial\Sigma = \sqcup_{i=1}^{h} C_i$, $C_i \simeq S^1$, and let X be a $CY3$. Let L be a Lagrangian submanifold of X, and let a map $f : \Sigma \to X$ be given such that

- f is holomorphic in the interior of Σ
- $f(\partial\Sigma) \subset L$

We will always be interested in cases when $\pi_1(L) = \mathbb{Z}$. The topological type of a map $f : \Sigma_{g,h} \to X$ will be specified by a relative co-homology class $\beta \in H_2(X, L)$, as well as by winding numbers $\{w_i\}_{i=1}^{h}$, associated to the map f restricted to C_i.

We would now like to be able to compute numbers $N_{g,\beta,\vec{w}}$ which should be related to counting "holomorphic maps f from genus g curves Σ with h disconnected components C_i of the boundary to a $CY3$ X with $f_*[\Sigma] = \beta$ and $f_*[C_i] = w_i C$, where C is the generator of $H_1(L)$". If we could define and compute them, they would be worth the name of *open Gromov–Witten invariants*. Unfortunately, defining them turns out to be an even harder task than in the ordinary, closed case. The relevant moduli spaces are usually real manifolds, which forces to abandon algebraic techniques for the construction of the virtual cycle, and they have even nastier singularities which make difficult to define a sensible intersection theory [139]. Some progress has been done however in a few cases towards their rigorous definition and/or an effective way to compute them by localization techniques, which is sometimes taken as an operative definition of the $N_{g,\beta,\vec{w}}$ ([33, 101]). When this is possible, we can give the following

Definition 8. *Let (X, L) be a pair given by a Calabi–Yau three-fold X and a Lagrangian submanifold $L \subset X$ with $b_1(L) = 1$. The genus g, h−holes* open Gromov–Witten potential *is defined as the formal power series*

$$F_{g,h}^{X,L}(q, \mathcal{Z}_o) = \sum_{\beta \in H_2(X,L)} \sum_{w \in \mathbb{Z}^h} N_{g,\beta,w} q^{\beta} \mathcal{Z}_0^{w} \tag{2.2.19}$$

We will be henceforth sticking to the case in which X is toric. It turns out that there is a distinguished class of Lagrangians to consider in this case, which we are now going to describe.

Toric branes

When X is a toric $CY3$ we want to consider a particular class of Lagrangian submanifolds, which are often referred to as "*toric branes*" and were constructed in [4] generalizing [89]. The prescription of [4] relies on the realization of a toric $CY3$ as a (degenerate) $\mathbb{T}^3$ fibration, parameterized by angles θ_i, over a base B given the toric web (i.e., the image of the moment map) of X; roughly speaking, this is obtained patchwise by writing in polar form the homogeneous coordinates x_i of (2.2.6) as $x_i =: \rho_i \exp(\theta_i)$, with $\rho_i \in \mathbb{R}^+$ (the reader is referred to [23, 118] for more details on this). The authors of [4] then consider a $3-k$ real dimensional subspace W of the base

$$\sum_i q_i^\alpha \rho_i^2 = c^\alpha \qquad \alpha = 1, \dots, k; \quad q_i^\alpha \in \mathbb{Q} \tag{2.2.20}$$

for arbitrary complex numbers c^α and then specify for every $x \in W$ a distribution of tangent k-planes L_x over this subspace, defined by the condition that the vertical vectors be in the kernel of the Kähler form $\omega = \sum_i d\rho_i^2 \wedge d\theta_i$. We have the following

Proposition 4 ([4]). *The distribution of tangent planes L_x is integrable for every $x \in W$. It gives a $\mathbb{T}^k$ fibration L over the interior of W, which degenerates at the edges of the toric base. L is Lagrangian by construction and is topologically a smooth real three-cylinder $\mathbb{T}^k \times \mathbb{R}^{3-k}$; moreover, it is volume–minimizing in its homology class if and only if $\sum_i q_i^\alpha = 0$.*

In the case in which the c^α in (2.2.20) are such that W intersects the edges of the toric web, i.e. the loci where one S^1 of the toric fibration shrinks, L splits into two Lagrangians $L_\pm$ with topology $\mathbb{R}^2 \times S^1$. In the physical picture of the boundary worldsheet CFT, the open moduli $(z_o)_i$ are then given by the size of the circle, complexified with the holonomy of a $U(1)$ connection along it.

Open string mirror symmetry

On the B–model side, the mirror boundary condition for the maps from the source curve is to have $\partial\Sigma \subset \hat{Y}$, with $\hat{Y}$ a holomorphic submanifold of $\widehat{X}$. One of the most interesting features of the toric branes is the fact that the mirror symmetry construction of [91] has been extended to such open string configurations in [4]. When $k = 2$, L_+ (resp. L_-) gets mirror mapped to a curve parameterized by x_2 (resp. x_1)

$$P_X(U, V) = 0 = x_1 \quad (\text{resp.} = x_2) \tag{2.2.21}$$

The location of the mirror brane is then simply given by a point $\mathcal{Y}_o$ on the mirror curve Γ. With such a clear geometric setup at hand, it is possible to provide a physically motivated computation of B–model open string amplitudes $\mathcal{F}_{g,h}^{\widehat{X},\hat{Y}}(z, y_o)$ at genus g and h holes via the holomorphic Chern–Simons theory on $\hat{Y}$ [150]. The main results are the following [5, 24, 110]

1. the generating functions $\mathcal{F}_{g,h}^{\widehat{X},\hat{Y}}$ can be computed entirely via residue calculus on the spectral curve Γ. This will be subject of the next section;

2. $\mathcal{F}_{g,h}^{\widehat{X},\hat{Y}}$ will typically depend on a choice of basis for the fan of the A–model target X leaving fixed the affine hyperplane H, that is, they depend on an element of $SL(2,\mathbb{Z})$. Part of this ambiguity consists in specifying the location of the mirror brane;

3. the leftover freedom in the definition of $\mathcal{F}_{g,h}^{\widehat{X},\hat{Y}}$ corresponds to the subgroup of translations of $SL(2,\mathbb{Z})$ and is then parameterized by a single integer f. This is the mirror of the so–called "framing ambiguity" on the A–model side; the latter corresponds to an un-fixed weight of the torus action in the localization approach to open string invariants, or, in the physics picture, by an integer choice in the trivialization of the tangent bundle in the dual Chern-Simons theory computation.

4. as in the ordinary closed string case, the conjectural relationship of $\mathcal{F}_{g,h}^{\widehat{X},\hat{Y}}$ with open Gromov–Witten generating functions holds true only up to a change of variables relating the point on the mirror curve $\mathcal{Y}_o$ to the open modulus $\mathcal{Z}_o$ in (2.2.19). This is done via the so-called *open mirror map*: this was proposed in [5, 110] (see also [67]), who claim that at large radius $\mathcal{Y}_o$ and $\mathcal{Z}_o$ should be related as
$$\mathcal{Z}_o = \mathcal{Y}_o \prod_j \frac{z_i - q_i}{r_i^j} \tag{2.2.22}$$
where the r_i are rational numbers and q_i are exponentiated flat co-ordinates, $q_i = e^{2\pi i t_i}$. This means that the open string A–model variable is related to the B–model one by a correction involving *closed* moduli only. As discovered in [110], an extended Picard-Fuchs system may be constructed such that (2.2.22) be in its kernel and therefore to determine the r_i.

2.2.4 The BKMP formalism

A point that was avoided in the previous section was how the $\mathcal{F}_{g,h}^{\widehat{X},\hat{Y}}$ can actually be constructed from the mirror geometry. We will now review the proposal of Bouchard, Klemm, Mariño and Pasquetti (BKMP) [24,119], based on the Eynard-Orantin recursion for matrix models [62] which gives a constructive algorithm that yields explicit, closed expressions for $\mathcal{F}_{g,h}^{\widehat{X},\hat{Y}}$ via residue calculus on the mirror curve to a toric $CY3$ X. In fact, for us the acronym "BKMP" is going to mean two different things:

1. a recursive procedure for the calculation of $\mathcal{F}_{g,h}^{\widehat{X},\hat{Y}}$ for any toric $CY3$ X;

2. a prescription to give *global* predictions about Gromov–Witten invariants, including their transformation properties when they cross a wall in the extended Kähler moduli space.

The Eynard-Orantin-BKMP recursion

Let us start with the following

Definition 9. *A spectral curve $\mathcal{S}$ is a quadruplet (Γ, C, u, v) where*

1. *Γ is a family of genus g projective curves over $\mathbb{C}$;*
2. *C is a disconnected union of segments inside Γ*
3. *$u, v : \Gamma \to \mathbb{C}$ are marked analytic functions on Γ, meromorphic on $\Gamma \setminus C$ and with at most logarithmic polydromies on the real 1-dimensional locus C*

If for any p such that $du(p) = 0$ we have $dv(p) \neq 0$, the spectral curve is called *regular*. All cases considered in this thesis will satisfy this condition.

Example 1. *Let X be a toric $CY3$ and $\widehat{X}$ be its Hori–Vafa mirror. Then (Γ_X, C, $\log U$, $\log V$) make up a spectral curve, where Γ_X is the (projectivized) Hori–Vafa mirror curve, C are segments joining the marked points of Γ_X, and $\log U$, $\log V$ are the logarithms of the $\mathbb{C}^*$ co-ordinates U, V of Definition 7.*

Remark 5. *Definition 9 might be possibly generalized to include singularities for u, and v which are worse than logarithmic. This setting is however sufficient for the purposes of this thesis.*

Let $\{q_i\}$ denote the branch points of the u projection to $\mathbb{C}$. Notice that near a ramification point q_i there are two points $q, \bar{q} \in \Gamma$ with the same projection $u(q) = u(\bar{q})$. Picking a polarization $\mathcal{H} \in Sp(2g, \mathbb{Z})$ of Γ, that is a canonical basis of 1–cycles, the *Bergmann kernel* is defined as the unique meromorphic differential with a double pole at $p = q$ with no residue and no other pole, and normalized such that

$$\oint_{A_I} B(p, q) = 0, \tag{2.2.23}$$

It is useful to introduce also the 1–form

$$dE_q(p) = \frac{1}{2} \int_q^{\bar{q}} B(p, \xi), \tag{2.2.24}$$

which is defined locally near a ramification point q_i. Notice that $B(p, q)$ depends only on $(\Gamma, \mathcal{H})$ and on no additional data, like u and v.

We now define recursively an infinite sequence of *correlators* $W_h^{(g)}(p_1, \ldots, p_h)$ and *free energies* $\mathcal{F}_g$ from the spectral curve as follows:

Definition 10 (Eynard–Orantin recursion)**.** *For all $g, h \in \mathbb{Z}^+$, $h \geq 1$, a meromorphic differential $W_h^{(g)}(p_1, \ldots, p_h) \in \mathrm{Sym}^h \Omega^{(1,0)}(\Gamma)$ is defined from the following recursion*

$$
\begin{aligned}
W_1^{(0)}(p) &= 0 && (2.2.25)\\
W_2^{(0)}(p,q) &= B(p,q) && (2.2.26)\\
W_{h+1}^{(g)}(p,p_1\dots,p_h) &= \sum_{q_i} \operatorname*{Res}_{q=q_i} \frac{dE_q(p)}{\Phi(q)-\Phi(\bar{q})}\Big(W_{h+2}^{(g-1)}(q,\bar{q},p_1,\dots,p_h) \\
&+ \sum_{l=0}^{g}\sum_{J\subset H} W_{|J|+1}^{(g-l)}(q,p_J)W_{|H|-|J|+1}^{(l)}(\bar{q},p_{H\backslash J})\Big) && (2.2.27)
\end{aligned}
$$

Here we denoted $H = 1,\cdots,h$, and given any subset $J = \{i_1,\cdots,i_j\} \subset H$ we defined $p_J = \{p_{i_1},\cdots,p_{i_j}\}$. Let now $\phi(p)$ be an arbitrary anti-derivative of $\Phi(p) = v(p)du(p)$; that is, $d\phi(p) = \Phi(p)$. then for $g \geq 2$ we define the free energies

$$F_g = \frac{1}{2-2g}\sum_{q_i}\operatorname*{Res}_{q=q_i}\phi(q)W_1^{(g)}(q). \tag{2.2.28}$$

The entire set of correlators and free-energies is constructed out of the spectral curve by residue calculus on Γ. The conjecture of [24, 119] is that, when $\mathcal{S}$ is the mirror spectral curve of a toric Calabi-Yau threefold X, such quantities compute precisely the open and closed Gromov–Witten generating functions of X, for any genus g and number of holes h.

Conjecture 1 (BKMP, [24, 119])**.** *Let $\mathcal{S}$ be the mirror spectral curve to a toric CY 3–fold X, and let A_i in (2.2.23) correspond to homology 1-cycles in Γ such that the periods of the Hori–Vafa differential have logarithmic singularities at the large complex structure point. Then:*

1. *The free energies $\mathcal{F}_g$ constructed above are equal to the A-model closed topological string amplitudes on X, after plugging in the closed mirror map.*
2. *Let $\mathcal{S}_f$ be the one–integer parameter family of spectral curves obtained by sending $U \to UV^f$, $V \to V$ for $f \in \mathbb{Z}$. Then:*
   - *the integrated Hori–Vafa differential*

     $$\int d\lambda_X = \int \log V_f \frac{dU_f}{U_f} \tag{2.2.29}$$

     is equal to the framed disc generating function $F_{0,1}$ of (X, L) where L is the mirror brane to $\Gamma \subset \widehat{X}$, after plugging in the closed and open mirror maps;
   - *the integrated correlation functions $F_{g,h} = \int W_k^{(g)}(p_1,\dots,p_k)$, for $2g + h > 1$, are equal to the A-model framed open Gromov–Witten potential of (X, L) where L is the mirror brane to $\Gamma \subset \widehat{X}$, after plugging in the closed and open mirror maps.*

Conjecture 1 has had a lot of evidence supporting it. It was checked extensively in [24] (see also [25,26]), and an attempt at a physics proof was given in [46] from the Kodaira–Spencer theory of gravity of [22] dimensionally reduced to 2–dimensions. A most important hint that Conjecture 1 was the observation of [63] that there exists a canonical anti-holomorphic extension of the Bergmann kernel, called the *Schiffer kernel*. This reduces to the usual kernel in an appropriate holomorphic limit, and moreover the anti-holomorphic free energies computed with it satisfy the holomorphic anomaly equations (2.2.3) of BCOV, thus making (2.2.28) a suitable candidate for a B-model generating function. More recently, a proof that the topological vertex solution of [7, 111, 124] satisfies the recursion has been announced [65], which would correspond to proving completely Conjecture 1 at large radius.

Almost-modularity and wall-crossings

The residue computation of eq. (2.2.25)-(2.2.27) gives, in principle, $W_h^{(g)}(p_1,\dots,p_h)$ as closed functions of the open moduli $p_1,\dots p_h$ as well as of the closed moduli a_p of the Hori-Vafa curve, as appearing in (2.2.10). In fact, a remarkable property of $W_h^{(g)}(p_1,\dots,p_h)$ is that they are *almost-modular forms* of Γ. This goes as follows: define the monodromy group $\mathcal{G}$ of $\widehat{X}$ as the group generated by monodromies around each boundary point of S. The latter turns out to be a finite index subgroup of $Sp(2g,\mathbb{Z})$, where g is the genus of Γ. We have the following

Theorem 5 ([25,62]). *$W_h^{(g)}(p_1,\dots,p_h)$ is a weight zero holomorphic almost modular form of $\mathcal{G}$. More precisely, it is a polynomial*

$$W_h^{(g)}(p_1,\dots,p_h) = \sum_{n=0}^{3g-3+2h} c_n(\tau,\{\tilde{t}_i\},\{p_i\})E_2^n(\tau) \tag{2.2.30}$$

where τ is the period matrix of Γ, c_n is for all p_i and $\tilde{t}_i$ a $-2n$ modular form of $\mathcal{G}$, and E_2 is the genus-g generalization of the second Eisenstein series (see [9]).

Let us explain more in detail what we mean by almost modularity, focusing for definiteness to the case $g=1$. Under an $Sp(2,\mathbb{Z}) = SL(2,\mathbb{Z})$ transformation

$$M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{2.2.31}$$

$$\tau \to \tilde{\tau} = (C\tau+D)^{-1}(A\tau+B) \tag{2.2.32}$$

$$\tag{2.2.33}$$

$E_2(\tau)$ transforms as

$$E_2(\tilde{\tau}) = (C\tau+D)^2E_2(\tau)+d(\tau) \tag{2.2.34}$$

where

$$d(\tau) = \frac{6}{\pi i}C(C\tau+D) \tag{2.2.35}$$

Hence, it is nearly a weight two modular form of $SL(2,\mathbb{Z})$, but for a shift linear in τ. The almost modularity of $W_h^{(g)}(p_1,\ldots,p_h)$ stems entirely from that of $E_2(\tau)$. Under a modular transformation $\tau \to \tilde{\tau}$, the expansion (2.2.30) gets transformed to

$$\begin{aligned} W_h^{(g)} &\to \tilde{W}_h^{(g)} \\ \tilde{W}_h^{(g)}(p_1,\ldots,p_h) &= \sum_{n=0}^{3g-3+2h} c_n(\tau,\{\tilde{t_i}\},\{p_i\})(E_2(\tau)+d_2(\tau))^n \end{aligned} \tag{2.2.36}$$

Eq. (2.2.36) expresses the variation of the open string generating functions under a change in the choice of polarization of the mirror curve. Recall that in Conjecture 1 a polarization was fixed by requiring the A–periods of the Hori–Vafa differential to be large radius flat co-ordinates, i.e. logarithmic solutions of the PF system around the maximally unipotent monodromy point. Changing polarization then corresponds to an $Sp(2g,\mathbb{Z})$ transformation to a different basis of solutions of the GKZ system. This acquires particular relevance when we consider the problem of studying the behaviour of Gromov–Witten invariants under variations of the Kähler structure, and in particular under *birational transformations*. As in section 2.2.2, let X be a smooth toric $CY3$ and $\mathcal{X}$ be an orbifold which is birationally isomorphic to X. Their respective bases of flat co-ordinates and dual periods will be related as in (2.2.17), (2.2.18). The second part of the $BKMP$ proposal claims the following

Conjecture 2 ([9,24])**.** *Let $W_h^{(g)}$ denote the open string correlators of X and let M be the matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.2.37}$$

representing the change of basis from the (normalizable) solutions of the PF system at large radius to those of the B–Model boundary point associated to $\mathcal{X}$. In (2.2.37), A, B, C and D are $g\times g$ matrices, where g is the genus of the mirror curve. Defining the transformed open string correlators $\tilde{W}_h^{(g)}$ of $\mathcal{X}$ as in (2.2.36), the open string generating functions $F_{g,h}^{\mathcal{X}}$ of $\mathcal{X}$ are given by the integrated correlator $\int \tilde{W}_k^{(g)}(p_1,\ldots,p_k)$, after plugging in the orbifold open and cloed mirror maps.

That is, to extract Gromov–Witten invariants of $\mathcal{X}$ starting from those of X we need to

1. transform the correlators as in (2.2.36)
2. analytically continue them from the large radius to the relevant boundary point corresponding to $\mathcal{X}$
3. expand them in powers of the appropriate local flat co-ordinates.

Remark 6. *It should be noticed that the two basis of solutions of the PF system need not be related by a simple change of polarization of the mirror curve. This will be the* *case for the orbifolds we will treat in chapter 3, where A, B, C and D in* (2.2.37) *will turn out to be complex. In that case, however, Eqs.* (2.2.35) *and* (2.2.36) *still make sense, even though they are no longer the result of the composition of* $W_h^{(g)}$ *with the modular transformation* (2.2.33). *This is why Eq.* (2.2.36) *is taken as the definition of the transformed* $\tilde{W}_h^{(g)}$ *in Conjecture 2.*

Chapter 3

Toric GW theory I: mirror symmetry and wall-crossings

3.1 Introduction

This chapter will be devoted to the study of the variation of the Gromov–Witten potential of a toric $CY3$ when we move from one chamber to another of its Kähler moduli space. In principle, assuming mirror symmetry and Conjecture 2, we have a complete recipe how to deal with this problem, whose steps were schematized at the end of the previous chapter. However, from a concrete point of view one is faced with the problem of relating different basis of solutions of the GKZ system, as in (2.2.17), (2.2.18); more general, good control on the analytic properties of such solutions as functions of the B–model moduli is required in order for the methods of Sec. 2.2.4 to work. Our aim in this chapter will be to detail a way to overcome such problems in the case of a particularly interesting 2–integer parameter family of toric $CY3$, which we will refer to as the $Y^{p,q}$ family. Our purposes will be twofold:

1. to find *global* expressions for solutions of the PF system, therefore determining their expansion in the vicinity of any boundary point;

2. to give a B-model description of wall-crossings both for open and closed Gromov–Witten invariants, and give explicit tests of Conjecture 1 and 2 for $g \leq 2$.

Point 1 is instrumental in the mirror symmetry study of wall crossings in Gromov–Witten theory of a $CY3$. Unfortunately, the GKZ system in most cases turns out to be a system of coupled $PDEs$ with irregular singular points as soon as $b_2(X) > 1$, and this jeopardizes the possibility to find solutions in closed form. The typical method adopted in the literature is the use of Frobenius' method: a power series ansatz is plugged into the PF system, which imposes a set of recursive relations on the series coefficients. Such recursions are generally not solvable in closed form[1], and knowledge of the global analytical properties of the solutions has been generally hard to obtain. We will tackle this problem by exploiting an interesting mathematical coincidence, which relates the Seiberg–Witten curves of $\mathcal{N} = 1$ gauge theories on $\mathbb{R}^4 \times S^1$ with gauge group $SU(p)$ and Chern–Simons level q with the Hori–Vafa spectral curves of the $Y^{p,q}$ family. Physically, such a coincidence is actually justified by the fact such gauge theories are "geometrically engineered" via M-theory compactifications on local Calabi-Yau geometries. We will use this as an inspiration to show how (derivatives of) the periods of the Hori–Vafa differential can be computed in closed form, therefore providing a global solution of the B–model at genus zero.

Point 2 will be studied very much in the spirit of [24]. We will focus on one particular example, namely the 2–parameter orbifold $\mathbb{C}^3/\mathbb{Z}_4$. This is slightly more complicated than the $\mathbb{C}^3/\mathbb{Z}_3$ case considered in [9, 24], but still the results about Point 1 will allow us to perform a very extensive check of Conjecture 1 and 2, up to the 6^{th} step in the recursion. We will check perturbatively that the $W_h^{(g)}$ agree

[1] It should be noticed that in the case of invariants of bundles, the methods of [36] can be used to offer a solution of such recursions in terms of finite products.

with the topological vertex results, determine explicitly the modular structure of the open string generating functions, check explicitly that the orbifold disc and closed 2-loop free energy agree with the computations from A-model localization and the holomorphic anomaly respectively, and give predictions for the remaining orbifold open and closed Gromov–Witten invariants.

3.2 The genus zero B–model on resolved $Y^{p,q}$ singularities

3.2.1 Cones over $Y^{p,q}$

The manifolds $Y^{p,q}$, with p and q integers such that $1 < q < p$, are an infinite class of real five-dimensional manifolds on which explicit Sasaki-Einstein metrics $ds^2(Y^{p,q})$ can be constructed (see [74, 122] for the details of the construction). Geometrically, they are principal $U(1)$ bundles over a base which is an (axially squashed) S^2 bundle over S^2, and p and q precisely specify the first Chern-class of the circle bundle, i.e. the monopole flux through the two 2-spheres of the base. We will not be interested in their differential–geometric characteristic, nor in the explicit knowledge of the metric: what is important for us is the fact that, since they are Sasaki-Einstein, the metric cone over $Y^{p,q}$

$$ds^2(C(Y^{p,q})) = dr^2 + r^2 ds^2(Y^{p,q})$$

is Kähler and Ricci-flat. The two extremal cases $q = 0$ and $q = p$ may be formally added to the family, corresponding to $\mathbb{Z}_p$ quotients respectively of $T^{1,1}$ (the base of the singular conifold) and of $S^5/\mathbb{Z}_2$. Given that the base has a real $\mathbb{T}^3$ of isometries, which can be lifted to effectively acting isometries of the full cone by (Hamiltonian) symplectomorphisms, $C(Y^{p,q})$ is a toric $CY3$. We have the following

Proposition 6 ([122]). *Let $C(Y^{p,q})$ denote the metric cone over $Y^{p,q}$. Then its toric diagram is given by the parallelogram spanned by*

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ p \end{pmatrix}, \quad v_4 = \begin{pmatrix} -1 \\ p-q \end{pmatrix} \tag{3.2.1}$$

$C(Y^{p,q})$ is then a Gorenstein toric variety. This means that a complete resolution r can be taken

$$\mathcal{X}_{p,q} \overset{r}{\dashrightarrow} C(Y^{p,q})$$

such that $\mathcal{X}_{p,q}$ is smooth toric and $K_{\mathcal{X}_{p,q}} = r^* K_{C(Y^{p,q})} \simeq \mathcal{O}_{K_{\mathcal{X}_{p,q}}}$. At the level of the toric diagram, this amounts [23, 85] to add the $p-1$ internal points $v_{4+j} = (0, j)$ for $j = 1, \ldots, p-1$ and declare that the set of three dimensional cones in the fan Σ be given by the simplicial cones whose projection on the affine hyperplane H yields a triangulation of the polyhedron $\{v_1, v_2, v_3, v_4\}$. If we view $\mathcal{X}_{p,q}$ as a holomorphic quotient as in Sec. 2.2.2,

$$(\mathbb{C}^{p+3} \setminus Z)/(\mathbb{C}^*)^p$$

the weights $Q_i^{(k)}$ of the $(\mathbb{C}^*)^p$ action $z_i \to \lambda^{Q_i^{(k)}} z_i$ can be chosen as

$$\begin{array}{lclcccccccccc}
Q_1 & = & (A, & -2A-B, & B, & A, & ,0, & 0, & 0, & 0, & 0, & 0, & 0) \\
Q_2 & = & (0, & 1, & 0, & 0, & -2, & 1, & 0, & 0, & \ldots, & 0, & 0) \\
Q_3 & = & (0, & 0, & 0, & 0, & 1, & -2, & 1, & 0, & \ldots, & 0, & 0) \\
Q_4 & = & (0, & 0, & 0, & 0, & 0, & 1, & -2, & 1, & \ldots, & 0, & 0) \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Q_p & = & (0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & \ldots, & 1, & -2)
\end{array} \tag{3.2.2}$$

with A and B coprime numbers solving the Diophantine equation $(p-q)A + pB = 0$ for $q < p$, while $A = 1$ and $B = 0$ for $q = p$.

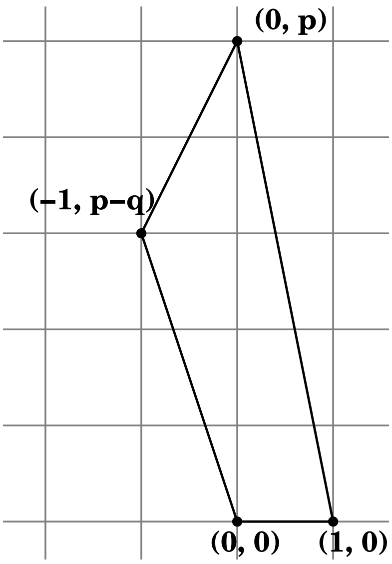

Figure 3.1: The toric diagram of $C(Y^{p,q})$ for $p = 5$, $q = 2$.

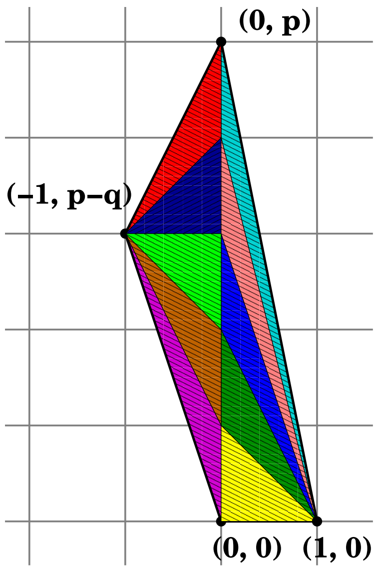

Figure 3.2: The toric diagram of $\widetilde{C(Y^{p,q})}$ for $p = 5$, $q = 2$.

From the toric data it is straightforward to extract co-homological information on $\mathcal{X}_{p,q}$, and in particular their Betti numbers. Internal vertices of the toric diagram correspond in a 1-to-1 fashion to 4-cycles; connectedness, vanishing of odd co-homologies and the area formula for the Euler characteristic [20, 71] yields

$$b_0(\mathcal{X}_{p,q}) = 1 \qquad b_2(\mathcal{X}_{p,q}) = p \qquad b_4(\mathcal{X}_{p,q}) = p - 1 \qquad b_6(\mathcal{X}_{p,q}) = 0 \tag{3.2.3}$$

p	q	$p+q$	Weights
1	0	1	$(0,0,0)$
1	1	2	$(0,1,1)$
2	0	2	$(0,1,1)$
2	1	3	$(1,1,1)$
2	2	4	$(1,1,2)$

Table 3.1: The $\mathbb{Z}_{p+q}$ orbifold points of $\mathcal{X}_{p,q}$ for the first few values of p and q. The fourth column lists the weights of the $\mathbb{Z}_{p+q}$ action on the coordinates (z_1, z_2, z_3) of $\mathbb{C}^3$.

Remark 7. *Various aspects of these geometries have been considered in the context of topological strings. First of all, for $p = 1$ we have rank 2 vector bundles on $\mathbb{P}^1$: the resolved conifold ($q = 0$) and $\mathbb{C} \times K_{\mathbb{P}^1}$ ($q = 1$). For $p = 2$, the local CY in question is the total space of the canonical line bundle $K_{\mathbb{F}_q}$ over the q^{th} Hirzebruch surface, $q = 0, 1, 2$. For higher p we have the so-called "ladder geometries" considered in [57, 90, 95, 99] in the context of geometric engineering of pure SYM theories with eight supercharges.*

The Kähler moduli space of these geometries displays a wealth of interesting phenomena which provide a natural testing ground for the A-model away from the large radius phase. It contains interesting conifold points, which in the case $q = 0$ have been related via large N duality to Chern–Simons theory on the lens space $L(p, 1)$ [6, 24, 87], as well as orbifold points of the form $\mathbb{C}^3/\mathbb{Z}_{p+q}$ (see table 3.1). We will study both types of boundary points in the examples of Sec. 3.2.4.

3.2.2 Hori–Vafa curves, integrable systems and five-dimensional gauge theories

From the toric diagram of $\mathcal{X}_{p,q}$ it is immediate to write down the Hori–Vafa mirror curve. We have the following easy

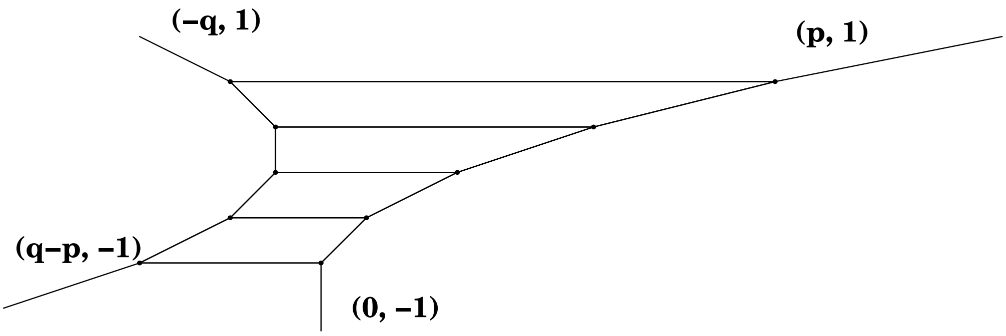

Figure 3.3: The pq-web diagram of $\widetilde{C(Y^{p,q})}$ for $p = 5$, $q = 2$.

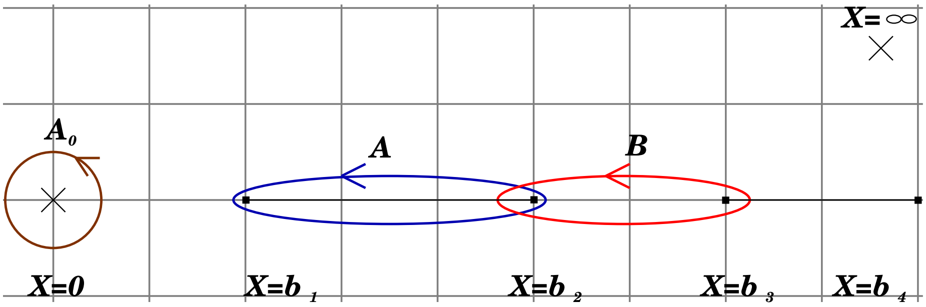

Figure 3.4: Cuts and punctures of the X plane in the genus 1 case.

Proposition 7. *The Hori–Vafa mirror curve* (2.2.10) *of* $\mathcal{X}_{p,q}$ *is given as*

$$\Gamma_{p,q} : P_{\mathcal{X}_{p,q}}(U,V) = a_1 V + \frac{a_2 U^{p-q}}{V} - \sum_{i=0}^{p} a_{i+3} U^i = 0 \tag{3.2.4}$$

We want to pose ourselves the problem of determining the periods of the Hori–Vafa differential (2.2.13) by a direct computation from (3.2.4). To this aim, let us first rewrite (3.2.4) as

$$Y^2 = P_p(X)^2 - 4a_1a_2X^{p-q} \tag{3.2.5}$$

upon setting

$$Y = a_1V - a_2U^{p-q}/V$$

$$X = U$$

$$P_p(X) = \sum_{i=0}^{p} a_{i+3}X^i \tag{3.2.6}$$

According to (3.2.5) the mirror curve $\Gamma_{p,q}$ can be seen as a two-fold covering of the X plane branched at $Y(X) = 0$, that is the locus

$$P_p(X)^2 = 4a_1a_2X^{p-q} \tag{3.2.7}$$

The resulting curve has genus $p-1$; denoting the solutions to (3.2.7) as $\{b_i\}_{i=1}^{2p}$, a basis for $H_1(\Gamma_{p,q},\mathbb{Z})$ might be taken as the homology classes of circles A_i, B_i encircling the intervals

$$I_{A_i} = [b_{2i-1}, b_{2i}] \qquad I_{B_i} = [b_{2i}, b_{2i+1}] \tag{3.2.8}$$

for $i = 1,\ldots,p-1$. We know however from the discussion of section 2.2.2 that, in order to compute a full set of periods, we should enlarge $H_1(\Gamma_{p,q},\mathbb{Z})$ by adding the non-compact cycles of $\Gamma_{p,q} \setminus \{0^\pm, \infty^\pm\}$, where we have denoted with $X^\pm$ the two inverse images of the projection onto the compactified X-plane. We will then add a circle $A_0^\pm$ around the punctures at $X = 0^\pm$ and a contour $B_0^\pm$ connecting the punctures at $0^\pm$ and $X = \infty^\pm$ on each sheet. The Hori–Vafa 1-differential $d\lambda_{p,q} = \log V dU/U$

is given, in an affine patch parameterized by X, as

$$d\lambda_{p,q}(X) = \log V(U)\frac{dU}{U} = \log\left(\frac{P_p(X) \pm \sqrt{P_p(X)^2 - 4a_1a_2X^{p-q}}}{2a_1}\right)\frac{dX}{X} \tag{3.2.9}$$

and a complete set of periods can be obtained by integrating it over the A/B-cycles

$$\Pi_{A/B} = \oint_{A/B} d\lambda_{p,q} \tag{3.2.10}$$

More explicitly,

$$\Pi_{A_i} = \int_{b_{2i-1}}^{b_{2i}} \log\left(\frac{P_p(X) + \sqrt{P_p(X)^2 - 4a_1a_2X^{p-q}}}{P_p(X) - \sqrt{P_p(X)^2 - 4a_1a_2X^{p-q}}}\right)\frac{dX}{X} \tag{3.2.11}$$

$$\Pi_{B_i} = \int_{b_{2i}}^{b_{2i+1}} \log\left(\frac{P_p(X) + \sqrt{P_p(X)^2 - 4a_1a_2X^{p-q}}}{P_p(X) - \sqrt{P_p(X)^2 - 4a_1a_2X^{p-q}}}\right)\frac{dX}{X} \tag{3.2.12}$$

$$\Pi_{A_0^\pm} = \oint_{X=0^\pm} \log\left(\frac{P_p(X) \pm \sqrt{P_p(X)^2 - 4a_1a_2X^{p-q}}}{2a_1}\right)\frac{dX}{X} \tag{3.2.13}$$

$$\Pi_{B_0^\pm} = \int_{0^\pm}^{\infty^\pm} \log\left(\frac{P_p(X) \pm \sqrt{P_p(X)^2 - 4a_1a_2X^{p-q}}}{2a_1}\right)\frac{dX}{X} \tag{3.2.14}$$

Remark 8. *This family of curves, along with the spectral data - i.e., the marked logarithmic functions $\log U$ and $\log V$ bears an interesting relationship with the Seiberg-Witten curves that encode the low energy effective actions of five dimensional gauge theories up to two derivatives [109, 128]. In the $a_1 = a_2 = (\Lambda R)^p$, $a_3 = a_p = 1$ patch the curve (3.2.5) and the differential (3.2.9) are precisely the Seiberg-Witten curve and differential [138]of $SU(p)$ $\mathcal{N} = 1$ SYM theory on $\mathbb{R}^4 \times S^1$ with a q-dependent Chern-Simons term [90, 142]. In this perspective, the field-theory limit of [99] corresponds to taking the four-dimensional $R \to 0$ limit in (3.2.5), (3.2.9). Quite interestingly, in this limit the q-dependence drops out altogether, and one gets for all q the Seiberg–Witten curves of $\mathcal{N} = 2$ $SU(p)$ SYM in four dimensions.*

Remark 9. *A further interesting identification can be made, in the case $p = q$ [128], with the spectral curve and action differential of the A_{p-1} Ruijsenaars model [137], i.e. the A_{p-1} periodic relativistic Toda chain. More precisely, setting $\zeta = \Lambda R$, (3.2.4) reads for $p = q$*

$$\Gamma_{p,p}: \quad \zeta^p\left(V + \frac{1}{V}\right) = 1 + \sum_{l=1}^{p-1} U^l S_l + U^p, \quad d\lambda_{p,p} = \log V\frac{dU}{U}$$

which can be rewritten as

$$\det(L(z) - w) = \sum_{j=0}^{p} (-w)^{p-j}\sigma_j(z) = 0 \tag{3.2.15}$$

with the Lax matrix defined as

$$\begin{aligned}
L_{ij} &= e^{Rp_i} f_i(l_{ij} + b_{ij}) \\
l_{ij} &= \delta_{i,j+1}(1 + \zeta^p z)\xi_i - \delta_{i,1}\delta_{j,p}(1 + \zeta^{-p} z^{-1})\xi_1 \\
b_{ij} &= \begin{bmatrix} -(i\zeta)^p & i \leq j-1 \\ 1 & i > j-1 \end{bmatrix} \\
f_i^2 &= (1 - \zeta^2 e^{q_{i+1}-q_i})(1 - \zeta^2 e^{q_i - q_{i-1}}) \\
\xi_i^{-1} &= 1 - \zeta^{-2} e^{q_{i-1}-q_i}
\end{aligned} \tag{3.2.16}$$

where $q_{p+1} = q_1, q_0 = q_p$, σ_j *are the elementary symmetric functions of* $L(z)$, S_j *their* z*-independent factor, and we have made the change of variables [128]*

$$-wU = 1 + \zeta^p z, \quad z = V$$

An identification of the curves for $q < p$ *as the spectral curves of some finite dimensional integrable mechanical system seems to be presently not known, and it would be interesting to understand to role of the* q *parameter in this context.*

3.2.3 Solving the PF system in the full B-model moduli space

The previous remarks establish a direct connection between the B-model on $\mathcal{X}_{p,q}$ and the Seiberg–Witten description of five dimensional theories compactified on a circle, as well as with the problem of finding action(-angle) variables for complex relativistic chains. We might wonder whether we can exploit general results from these theories to find an efficient way to compute the periods of $d\lambda_{p,q}$. Indeed, we know that in Seiberg-Witten theory, the gauge coupling matrix

$$\tau_{ij} = \frac{\partial \Pi_{B_i}}{\partial u_k} \left(\frac{\partial \Pi_{A_k}}{\partial u_j} \right)^{-1} \tag{3.2.17}$$

where u_i are Weyl-invariant functions of the scalar fields, is known to be the period matrix of the Seiberg-Witten curve, that is a ratio of periods of holomorphic differentials. We then expect that derivatives of $d\lambda_{p,q}$ with respect to suitable functions of the bare moduli be *holomorphic differentials* of $\overline{\Gamma}_{p,q}$

$$\partial_{f(a_i)} d\lambda \in \Omega^{1,0}(\Gamma_{p,q}) \tag{3.2.18}$$

From the integrable system perspective this has its origin in the fact that, for $p = q = 2$, the relativistic Toda system and the non-relativistic one share the same oscillation periods [41]; more precisely, the derivatives of the action with respect to the energy are the same (elliptic) functions of the bare parameters. This was also noticed in [128] in the study of the singularities of the moduli space of $\mathcal{N} = 1$ $SU(2)$ SYM in $d = 5$. We have the following

Proposition 8. *Let* ω_j *be defined as*

$$\omega_j := \frac{\partial d\lambda_{p,q}}{\partial a_{j+4}} \tag{3.2.19}$$

Then $\mathrm{span}_{\mathbb{C}}\left(\{\omega_j\}_{j=0}^{p-2}\right) = \Omega^{1,0}(\Gamma_{p,q})$.

Explicitly, we have

$$\frac{\partial d\lambda_{p,q}}{\partial a_{j+4}} = \frac{X^j}{\sqrt{P_p^2(X) - 4a_1 a_2 X^{p-q}}} dX \tag{3.2.20}$$

This last observation puts us in a position to give a straightforward and complete recipe for computing series expansions of solutions of the PF system (2.2.9) for $\mathcal{X}_{p,q}$ in the full B-model moduli space. The procedure is the following:

1. start with Π_{A_i/B_i} and consider its a_{j+4} derivative for $0 \leq j \leq p-2$

$$\frac{\partial \Pi_{A_i/B_i}}{\partial a_{j+4}} = \int_{e_i}^{e_{i+1}} \frac{X^j}{\sqrt{\prod_{i=1}^{2p}(X-b_i)}} dX \tag{3.2.21}$$

with $e_i = b_{2i-1}$, $e_i = b_{2i}$ for the A and the B cycles respectively. The hyperelliptic integral (3.2.21) has a closed expression given in terms of multivariate generalized hypergeometric functions of Lauricella type [59]

$$\begin{aligned} \frac{\partial \Pi_{A_i/B_i}}{\partial a_{j+4}} &= \mathrm{e}^{i\varphi}\pi \frac{(e_i)^j}{\sqrt{\prod_{k\neq i,i+1}(e_k - e_i)}} \\ &\times \; F_D^{(2p-1)}\left(\frac{1}{2}; \frac{1}{2}, \ldots, \frac{1}{2}, j; 1; x_1, \ldots, \widehat{x_i}, \widehat{x_{i+1}}, \ldots, x_{2p}, \frac{e_{i+1}-e_i}{e_i}\right) \end{aligned} \tag{3.2.22}$$

where $x_j = (e_{i+1} - e_i)/(e_j - e_i)$, $2\varphi = l\pi$, $l \in \mathbb{Z}$ is a phase depending on x_i and $F_D^{(n)}$ is the hypergeometric series

$$F_D^{(n)}(\alpha; \{\beta_i\}; \gamma; \{\delta_i\}) = \sum_{m_1 \ldots m_n = 0}^{\infty} \frac{(\alpha)_{m_+ \cdots + m_n} (\beta_1)_{m_1} \ldots (\beta_n)_{m_n} \delta_1^{m_1} \ldots \delta_n^{m_n}}{(\gamma)_{m_+ \cdots + m_n} m_1! \ldots m_n!} \tag{3.2.23}$$

which converges when $|\delta_i| < 1$ for every i. In the above formula we used the standard Pochhammer symbol $(\alpha)_m = \Gamma(\alpha+m)/\Gamma(\alpha)$. In many cases, $F_D^{(n)}$ can be reduced to a more familiar form. For instance, for $p = 2$ we have the expected complete elliptic integrals of the first kind

$$\frac{\partial \Pi_A}{\partial a_4} = \frac{2}{\sqrt{(b_1-b_3)(b_2-b_4)}} K\left[\frac{(b_1-b_2)(b_3-b_4)}{(b_1-b_3)(b_2-b_4)}\right] \qquad (3.2.24)$$

$$\frac{\partial \Pi_B}{\partial a_4} = \frac{2}{\sqrt{(b_1-b_2)(b_3-b_4)}} K\left[\frac{(b_1-b_3)(b_2-b_4)}{(b_1-b_2)(b_3-b_4)}\right] \qquad (3.2.25)$$

2. Once we have a representation for the derivatives of the periods in the form (3.2.22), (3.2.24)-(3.2.25) we can use the formulae in Appendix B.2 to analytically continue them in any given patch of the B-model moduli space and find a corresponding power series expansion in the bare moduli a_i. Integrating back with respect to a_j yields Π_{A_i} and Π_{B_i} up to a constant of integration, independent of a_j for $0 \leq j \leq p-2$. This has to be fixed either by some indirect consideration (for instance, by imposing a prescribed asymptotic behavior around a singular point) or by plugging it inside the PF system and imposing that the period be in the kernel of the GKZ operators. This operation leads to a closed ODE integrable by quadratures, which completes the solution of the problem of finding expansions for Π_{A_i/B_i} everywhere in the B-model moduli space.

3. The procedure provides us with $p-1$ flat coordinates as well as $p-1$ conjugate periods out of which to extract the prepotential. In order to find the p^{th} modulus, we pick up the residue (3.2.13)

$$\oint_{X=0_\pm} d\lambda = \begin{cases} \log\left(\pm\frac{a_3}{a_1}\right) & \text{for } q<p \\ \log\left(\frac{a_3\pm\sqrt{a_3^2-4a_1a_2}}{2a_1}\right) & \text{for } q=p \end{cases} \qquad (3.2.26)$$

which are manifestly solutions of (2.2.9). In the following, we will choose an appropriate combination of them in order to have a prescribed behavior around the expansion point under scrutiny.

Remark 10. *There are many alternative ways to express (3.2.21), for instance in terms of hyperelliptic θ functions; however, the above expression proves to be useful due to the fact that Lauricella $F_D^{(n)}$ has good analytic continuation properties outside the unit polydisc $|\delta_i| < 1$; some formulae, as well as asymptotic expansions around singular submanifolds, are collected in the Appendix, while others can be found in [59, 60]. Notice that, as opposed to the usual situation in solving PF equations by Frobenius method, we are not dealing here with hypergeometric functions of the bare moduli, but rather of the relative distance x_i between ramification points; they have singular values precisely when the latter becomes* 0, 1 *or infinity, that is when we encounter a pinching point of $\Gamma_{p,q}$. This shift in perspective is definitely an advantage compared to other expressions for hyperelliptic integrals, involving for instance the F_4 Appell function for genus 2 [102, 129]. These are simpler functions of the bare* *moduli, but have worse analytic continuation properties and are less suited for a more complete study of the moduli space, regarding for instance intersecting submanifolds of the principal discriminant locus. The above fact was already pointed out in [10], where the properties of F_D^n were exploited to study the $\mathbb{Z}_3$ point of $\mathcal{N} = 2$ $SU(3)$ SYM.*

An important advantage of the method proposed here is that, instead of integrating back patch-wise with respect to a_j, we can work directly with an Euler-type integral representation of the periods. The fact that $F_D^{(n)}$ has a single integral representation saves us most of the pain in the problem of finding the explicit analytic continuation of Π_{A_i/B_i}, which in the multi-parameter case involves the use of multi-loop Mellin-Barnes integrals. The details for the case $p = q = 2$ which will be of our interest later on for the computation of orbifold Gromov-Witten invariants are reported in Appendix B.1, where also a closed expression for the A-period can be found in terms of a generalized Kampé de Fériet hypergeometric function.

3.2.4 Examples

Let us show how the three-step recipe of the previous section allows to quickly recover some known results about mirror symmetry for local surfaces.

Local $\mathbb{F}_0$: mirror map at large radius

Local mirror symmetry for $K_{\mathbb{F}_0}$ has been studied in [6] in the check of the large N duality with Chern-Simons theory on $S^3/\mathbb{Z}_2$. The mirror curve in this case can be written as

$$a_1 V + a_2/V = a_3/U + a_4 + a_5 U \tag{3.2.27}$$

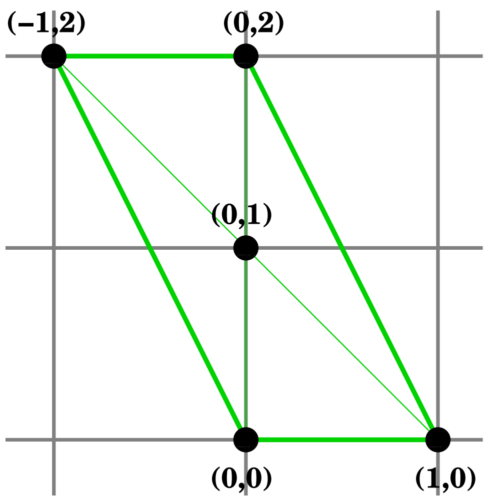

Figure 3.5: The toric diagram of local $\mathbb{F}_0$.

Good variables around the large complex structure point [35] are given by

$$z_B = \frac{a_1 a_2}{a_4^2} \qquad z_F = \frac{a_3 a_5}{a_4^2} \tag{3.2.28}$$

Let us use the scaling freedom (2.2.16) to set

$$a_3 = a_5 = 1, \qquad a_1 = a_2 \tag{3.2.29}$$

By using the change of variables (3.2.5) the curve (3.2.27) is then given by

$$Y^2 = \left(X^2 + \frac{X}{\sqrt{z_F}} + 1\right)^2 - \frac{4z_B}{z_F} X^2 \tag{3.2.30}$$

which is a double covering of the $X-$plane branched at

$$\begin{aligned} b_1 &= \frac{-1 + 2\sqrt{z_B} - \sqrt{1 - 4\sqrt{z_B} + 4z_B - 4z_F}}{2\sqrt{z_F}} && (3.2.31)\\ b_2 &= \frac{-1 - 2\sqrt{z_B} - \sqrt{1 - 4\sqrt{z_B} + 4z_B - 4z_F}}{2\sqrt{z_F}} && (3.2.32)\\ b_3 &= \frac{-1 + 2\sqrt{z_B} + \sqrt{1 - 4\sqrt{z_B} + 4z_B - 4z_F}}{2\sqrt{z_F}} && (3.2.33)\\ b_4 &= \frac{-1 - 2\sqrt{z_B} + \sqrt{1 - 4\sqrt{z_B} + 4z_B - 4z_F}}{2\sqrt{z_F}} && (3.2.34) \end{aligned}$$

We choose the $A-$cycle as the loop encircling $[b_1, b_2]$. The asymptotics of the corresponding period will indeed identify it as the flat coordinate around $z_B = z_F = 0$. By expanding (3.2.24) in (z_B, z_F) we have

$$\begin{aligned} \frac{\partial \Pi_A}{\partial a_4} &= \sqrt{z_F}(20z_B^3 + 6(30z_F + 1)z_B^2 \\ &+ 2(90z_F^2 + 12z_F + 1)z_B + 20z_F^3 + 6z_F^2 + 2z_F + 1) + \ldots \end{aligned} \tag{3.2.35}$$

which integrates to

$$\Pi_A = \log(z_F) + \frac{20z_B^3}{3} + 60z_F z_B^2 + 3z_B^2 + 60z_F^2 z_B \tag{3.2.36}$$

From (3.2.26) and (3.2.28) we can compute the remaining flat coordinate as

$$\Pi_0 = -\frac{1}{2}\log\frac{z_B}{z_F} \tag{3.2.37}$$

It is then easy to see that the combinations of periods that has the right asymptotics at large radius are given by

$$-t_B \equiv -2\Pi_0(z_B, z_F) + \Pi_A(z_B, z_F), \qquad -t_F \equiv \Pi_A(z_B, z_F) \tag{3.2.38}$$

Inversion of (3.2.36) and (3.2.37) reads, setting $Q_B = e^{-t_B}$, $Q_F = e^{-t_F}$

$$\begin{aligned} z_B &= 6Q_B^3 - 2Q_B^2 + 6Q_F^2 Q_B - 2Q_F Q_B + Q_B + \ldots \\ z_F &= 6Q_F^3 - 2Q_F^2 + 6Q_B^2 Q_F - 2Q_B Q_F + Q_F + \ldots \end{aligned} \tag{3.2.39}$$

which is the mirror map as written in [119].

Local $\mathbb{F}_0$: conifold point

Analogously, we can write down the expansion for the orbifold point [6], which corresponds to $a_1 = a_2 = a_3 = a_5 = 1$, $a_4 = 0$. Setting $a_1 = a_2 = \sqrt{1 - x1}$, $a_4 = x_1 x_2$, $a_3 = a_5 = 1$ as in [6], we have

$$\begin{aligned} s_1 &\equiv \Pi_0 = -\log(1 - x_1) \\ s_2 &\equiv \Pi_A + \Pi_B/2 = \frac{1}{61931520\pi}\Big[x_2(35(32(x_1 - 2)x_1(x_1(11x_1 - 96) + 96)E(x_1) \\ &+ x_1(x_1(x_1(x_1(105x_1 - 1856) + 8000) - 12288) + 6144)K(x_1))x_2^8 + \dots\Big] \end{aligned}$$

Upon introducing $\tilde{s}_1 = s_1$ and $\tilde{s}_2 = s_1/s_2$ we have

$$\begin{aligned} x_1(\tilde{s}_1) &= 1 - e^{-\tilde{s}_1} \\ x_2(\tilde{s}_1, \tilde{s}_2) &= \tilde{s}_2 + \frac{\tilde{s}_2}{4}\tilde{s}_1 + \left(\frac{\tilde{s}_2}{192} - \frac{\tilde{s}_2^3}{192}\right)\tilde{s}_1^2 + \left(-\frac{\tilde{s}_2}{256} - \frac{\tilde{s}_2^3}{768}\right)\tilde{s}_1^3 + \left(-\frac{49\tilde{s}_2}{737280}\right. \\ &+ \left.\frac{7\tilde{s}_2^3}{73728} - \frac{7\tilde{s}_2^5}{245760}\right)\tilde{s}_1^4 + \left(\frac{5\tilde{s}_2^3}{98304} - \frac{7\tilde{s}_2^5}{983040}\right)\tilde{s}_1^5 + \dots \end{aligned} \tag{3.2.40}$$

in perfect agreement with [6]. Needless to say, the prepotential computation can be checked exactly the same way. We have

$$\begin{aligned} F_{s_2} \equiv \Pi_A &= \frac{1}{53760}\Big[x_1 x_2\Big((75x_1^3 x_2^6 - 56x_1^2(10x_2^2 + 9)x_2^4 + 64x_1(10x_2^4 + 21x_2^2 \\ &+ 70)x_2^2 - 107520\Big)K(1 - x_1) + \dots\Big] \\ &= \log\left(\frac{x_1}{16}\right)s_2 - \frac{x_2^3}{12}x_1 + \left(\frac{x_2}{4} + \frac{x_2^3}{48}\right)x_1^2 + \left(-\frac{21}{128}x_2 + \frac{5}{768}x_2^3\right)x_1^3 \\ &+ \left(\frac{185x_2}{1536} + \frac{5x_2^3}{1024}\right)x_1^4 + \dots \end{aligned} \tag{3.2.41}$$

which reproduces the analogous formula in [6], modulo the ambiguity in the degree-zero contribution.

Local $\mathbb{F}_2$ at large radius

We might proceed along the same lines for the case of local $\mathbb{F}_2$. The curve is given by

$$a_1 V + \frac{a_2}{V} = a_3 + a_4 U + a_5 U^2 \tag{3.2.42}$$

Branch points are located at

$$U + \frac{a_4}{2a_5} = \begin{cases} \pm\frac{\sqrt{a_4^2 - 4a_3 a_5 - 8\sqrt{a_1}\sqrt{a_2}a_5}}{2a_5} \equiv \pm c_1 \\ \pm\frac{\sqrt{a_4^2 - 4a_3 a_5 + 8\sqrt{a_1}\sqrt{a_2}a_5}}{2a_5} \equiv \pm c_2 \end{cases} \tag{3.2.43}$$

and we have accordingly

$$\frac{\partial \Pi_A}{\partial a_4} = \int_{c_1}^{c_2} \frac{dX}{(X^2-c_1^2)(X^2-c_2^2)} = \frac{K\left(1-\frac{c_2^2}{c_1^2}\right)}{c_1} \tag{3.2.44}$$

$$\frac{\partial \Pi_B}{\partial a_4} = \int_{-c_1}^{c_1} \frac{dX}{(X^2-c_1^2)(X^2-c_2^2)} = \frac{2K\left(\frac{c_1^2}{c_2^2}\right)}{c_2} \tag{3.2.45}$$

In this case invariant coordinates associated to the base $\mathbb{P}^1$ and the $\mathbb{P}^1$ fiber are

$$z_B = \frac{a_1 a_2}{a_3^2}, \qquad z_F = \frac{a_3 a_5}{a_4^2} \tag{3.2.46}$$

Upon setting $a_1 = a_2$, $a_3 = a_5 = 1$, periods take the form

$$\begin{aligned} \frac{\partial t_F}{\partial z_F} &= \frac{\partial \Pi_A}{\partial z_F} = -\frac{2K\left(-\frac{16\sqrt{z_B}z_F}{-8\sqrt{z_B}z_F-4z_F+1}\right)}{\pi z_F\sqrt{1-4\left(2\sqrt{z_B}+1\right)z_F}} \\ \frac{\partial^2 \mathcal{F}}{\partial z_F \partial t_F} &= \frac{\partial \Pi_B}{\partial z_F} = -\frac{4K\left(\frac{-8\sqrt{z_B}z_F-4z_F+1}{8\sqrt{z_B}z_F-4z_F+1}\right)}{z_F\sqrt{1-4\left(1-2\sqrt{z_B}\right)z_F}} \\ t_B &= \Pi_{0_+} - \Pi_{0_-} = 2i\tan^{-1}\left(\sqrt{4z_B-1}\right) \end{aligned} \tag{3.2.47}$$

where the normalization has been chosen in order to get the right asymptotics. Integration and inversion yields the mirror map at the large radius point

$$\begin{aligned} z_B(Q_B) &= \frac{Q_B}{(Q_B+1)^2} \\ z_F(Q_B,Q_F) &= (1+Q_B)\,Q_F + \left(-2-4Q_B-2Q_B^2\right)Q_F^2 \\ &+ \left(3+3Q_B+3Q_B^2+3Q_B^3\right)Q_F^3 + \ldots \end{aligned} \tag{3.2.48}$$

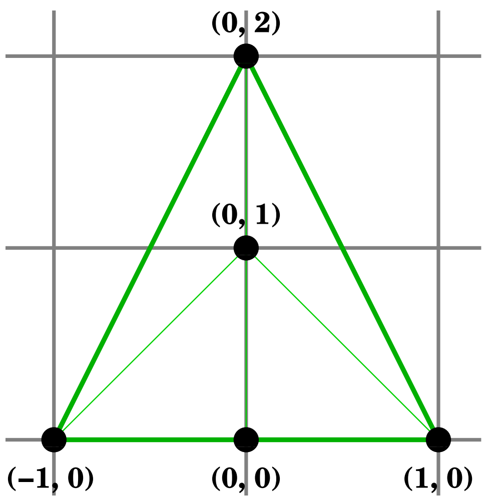

Figure 3.6: The toric diagram of local $\mathbb{F}_2$.

with $Q_B = e^{-t_B}$, $Q_F = e^{-t_F}$ and therefore

$$\begin{aligned}\partial_{t_F}\mathcal{F}(Q_B,Q_F) &= (\log(Q_F)\log(Q_BQ_F)) + (4+4Q_B)\,Q_F + (1+16Q_B + \\ &+ Q_B^2)Q_F^2 + \left(\frac{4}{9}+36Q_B+36Q_B^2+\frac{4Q_B^3}{9}\right)Q_F^3 \\ &+ \left(\frac{1}{4}+260Q_B^2+64(Q_B+Q_B^3)\right)Q_F^4+\ldots \end{aligned} \tag{3.2.49}$$

as in [35].

Local $\mathbb{F}_2$ and $\mathbb{C}^3/\mathbb{Z}_4$: genus zero orbifold invariants

We will now apply the considerations above to the study of the tip of the classical Kähler moduli space for local $\mathbb{F}_2$, where the compact divisor collapses to zero size. The resulting geometry [20] is a $\mathbb{Z}_4$ orbifold of $\mathbb{C}^3$ by the action $(\omega; z_1, z_2, z_3) \to (\omega z_1, \omega z_2, \omega^{-2} z_3)$, with $\omega \in \mathbb{Z}_4$. The inertia stack (2.1.6) takes the form

$$\mathcal{I}\mathbb{C}^3/\mathbb{Z}_4 = X_1 \cup X_i \cup X_{-1} \cup X_{-i} \tag{3.2.50}$$

where $X_0 \simeq \mathbb{C}^3$, $X_i \simeq X_{-1} \simeq \{0\}$, $X_{-1} \simeq \mathbb{C}$. The orbifold co-homology ring is spanned by classes $\mathcal{O}_{k/4}$, where $\mathrm{span}_{\mathbb{C}}\mathcal{O}_k = H^0(X_k, \mathbb{C})$, and their orbifold degree is easily computed from (2.1.7), (2.1.8) as

$$\mathrm{age}\ \mathcal{O}_1 = 0; \quad \mathrm{age}\ \mathcal{O}_i = 1; \quad \mathrm{age}\ \mathcal{O}_{-1} = 1; \quad \mathrm{age}\ \mathcal{O}_{-i} = 2 \tag{3.2.51}$$

By the Calabi–Yau condition, the orbifold Gromov–Witten potential then takes the form

$$\mathcal{F}_0^{\mathbb{C}^3/\mathbb{Z}_4}(s_i, s_{-1}) = \sum_{n,m\geq 0} \frac{1}{n!m!}\langle \mathcal{O}_i^m \mathcal{O}_{-1}^n \rangle s_i^m s_{-1}^n \tag{3.2.52}$$

To compute it, observe from figure 3.6 that Mori vectors for local $\mathbb{F}_2$ are

$$\begin{aligned} Q_1 &= (0, \quad 1, \quad 1, \quad 0, \quad -2) \\ Q_2 &= (1, \quad -2, \quad 0, \quad 1, \quad 0) \end{aligned} \tag{3.2.53}$$

and the mirror curve $\Gamma_{2,2}$ has the form (3.2.42)

$$a_1 V + \frac{a_2}{V} = a_3 + a_4 U + a_5 U^2$$

Following [40] we argue that the point we are looking for in the B-model moduli space is given by $a_3 = a_4 = 0$. This would amount to shrinking to zero size the compact divisor given, in the homogeneous coordinates x_i introduced in Sec. 2.2.2, by $x_5 = 0$. To see this, notice that the secondary fan of (3.2.53) (see Fig. 3.8) has the set of charges

$$\begin{array}{c|ccccc} & a_1 & a_3 & a_5 & a_2 & a_4 \\ \hline Q_1 & 1, & 0, & 0, & -1, & 0 \\ Q_2 & 0, & 0, & 2, & 0, & 1 \\ Q_3 & 1, & 1, & 1, & 1, & 1 \end{array} \tag{3.2.54}$$

The secondary fan of $K_{\mathbb{F}_2}$ is simplicial but with marked points, and it is therefore a toric orbifold. Its orbifold patches are, as shown in figure 3.8, a smooth $\mathbb{C}^2$ patch containing the large complex structure point, two non-smooth $\mathbb{C}^2/\mathbb{Z}_2$ cones, and finally a $\mathbb{C}^2/\mathbb{Z}_4$ patch parameterized by (a_3, a_4). The $\mathbb{Z}_4$ action on the latter reads

$$\begin{array}{cccc} \mathbb{Z}_4 \times & \mathbb{C}^2 & \to & \mathbb{C}^2 \\ \lambda & (x,y) & \to & (\lambda x, \lambda^2 y) \end{array} \tag{3.2.55}$$

$(a_3, a_4) = (0,0)$ is therefore the only point with $\mathbb{Z}_4$ monodromy in the compactified B-model moduli space. From (3.2.55) we see that good coordinates around $(a_3, a_4) = (0,0)$ are given by

$$\begin{aligned} a_3 &= \sqrt{d}e \\ a_4 &= d^{1/4} \end{aligned} \tag{3.2.56}$$

Let us then find a complete basis of solutions for the GKZ system around this point. Picard-Fuchs operators are written in this patch as

$$\begin{aligned} \mathcal{L}_1 &= a_3\partial^2_{a_4} + \frac{1}{2}\theta_{a_4}\theta_{a_3} \\ \mathcal{L}_2 &= \partial^2_{a_3} - \frac{1}{16}(\theta^2_{a_4} - 4\theta^2_{a_3}) - \frac{1}{4}\theta_{a_3}\theta_{a_4} \end{aligned} \tag{3.2.57}$$

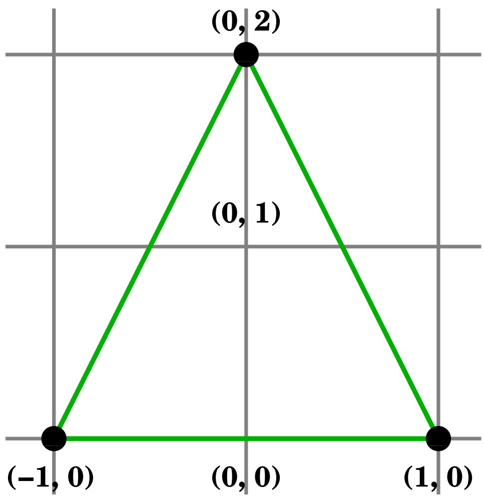

Figure 3.7: The toric diagram of $\mathbb{C}^3/\mathbb{Z}_4$.

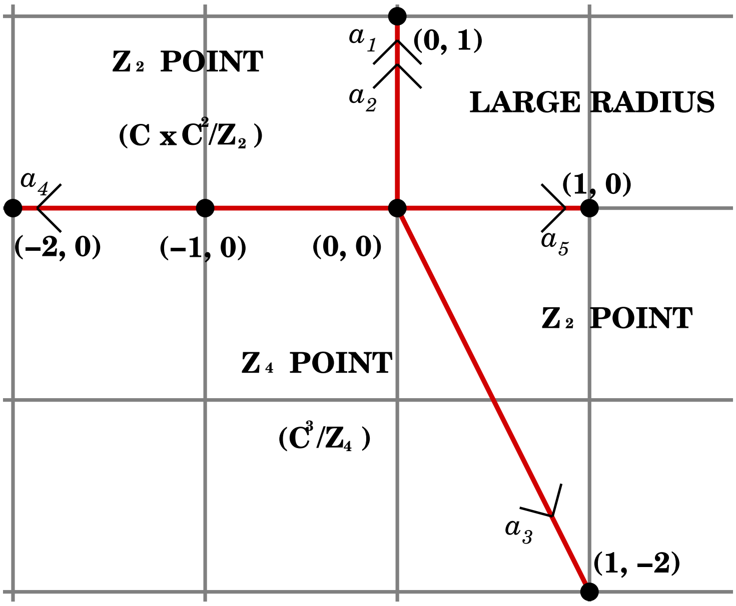

Figure 3.8: The secondary fan of $K_{\mathbb{F}_2}$.

and the branch points (3.2.43) here read

$$\begin{aligned} \pm c_1 &= \pm\frac{1}{2}\sqrt{a_4^2 - 4a_3 - 8} \\ \pm c_2 &= \pm\frac{1}{2}\sqrt{a_4^2 - 4a_3 + 8} \end{aligned} \tag{3.2.58}$$

while the period integrals (3.2.44),(3.2.45) and (3.2.26) become

$$\begin{aligned} \partial_{a_4}\Pi_A &= \frac{K\left(\frac{a_4^2-4a_3-8}{a_4^2-4a_3+8}\right)}{\sqrt{a_4^2-4a_3+8}} - \frac{K\left(\frac{a_4^2-4a_3+8}{a_4^2-4a_3-8}\right)}{\sqrt{a_4^2-4a_3-8}} \\ \partial_{a_4}\Pi_B &= 2\frac{K\left(\frac{a_4^2-4a_3-8}{a_4^2-4a_3+8}\right)}{\sqrt{a_4^2-4a_3+8}} \\ \Pi_{0\pm} &= \log\left(\frac{a_3}{2} \pm \frac{\sqrt{a_3^2-4}}{2}\right) \end{aligned} \tag{3.2.59}$$

To find solutions of the PF system (3.2.57) with prescribed monodromy around $(d, e) = (0, 0)$, let us define

$$s_i = \left[\frac{(8-8i)\pi^{1/2}}{\Gamma\left(\frac{1}{4}\right)^2}\right]\left(\frac{\Pi_B}{2} - \frac{\Pi_A}{1-i}\right) \tag{3.2.60}$$

$$s_i = \left[\frac{(4+4i)\Gamma\left(\frac{1}{4}\right)^2}{\pi^{3/2}}\right]\left(\frac{\Pi_B}{2} + \frac{\Pi_A}{1+i}\right) \tag{3.2.61}$$

$$s_{-1} = -2i\Pi_{0_-} + \pi \tag{3.2.62}$$

we then have

$$s_i(d,e) = d^{1/4}\left[1 + \left(\frac{e^2}{32} - \frac{e}{192} + \frac{1}{2560}\right)d - \frac{25e^3}{18432}d^2 + \ldots\right] \tag{3.2.63}$$

$$s_{-1}(d,e) = d^{1/2}\left[e + \frac{e^3 d}{24} + \frac{3e^5d^2}{640} + \frac{5e^7d^3}{7168} + \ldots\right] \tag{3.2.64}$$

$$s_{-i}(d,e) = d^{3/4}\left[\left(e - \frac{1}{12}\right) + \left(\frac{3e^3}{32} - \frac{3e^2}{128} + \frac{9e}{2560} - \frac{3}{14336}\right)d + \ldots\right] \tag{3.2.65}$$

The normalization of the mirror map has been fixed against the explicit I-function formulae for $\mathbb{C}^3/\mathbb{Z}_4$. Concerning the solution s_{-i}, this is identified with the derivative of the generating function $\mathcal{F}_{orb}$ in (3.2.52) with respect to s_i; in fact, the orbifold Poincaré pairing modifies this relation by a factor of 4, i.e. $s_{-i} = 4\partial_{s_i}\mathcal{F}_0^{\mathbb{C}^3/\mathbb{Z}_4}$. Taking all this into account, inversion of (3.2.63) and (3.2.64) gives the following expression for the prepotential

$$\begin{aligned} 4\frac{\partial\mathcal{F}_0^{\mathbb{C}^3/\mathbb{Z}_4}}{\partial s_i}(s_{-1},s_i) &= \left(s_{-1} + \frac{s_{-1}^3}{48} + \frac{s_{-1}^5}{960} + \frac{29s_{-1}^7}{430080} + \frac{457s_{-1}^9}{92897280} + O\left(s_{-1}^{11}\right)\right)s_i \\ &+ \left(-\frac{1}{12} - \frac{s_{-1}^2}{96} - \frac{11s_{-1}^4}{9216} - \frac{49s_{-1}^6}{368640} - \frac{601s_{-1}^8}{41287680} + O\left(s_{-1}^{10}\right)\right)s_i^3 \\ &+ \left(\frac{7s_{-1}}{3840} + \frac{s_{-1}^3}{1920} + \frac{47s_{-1}^5}{460800} + \frac{6971s_{-1}^7}{412876800} + O\left(s_{-1}^9\right)\right)s_i^5 \\ &+ \ldots \end{aligned} \tag{3.2.66}$$

As a check, the prepotential thus obtained is invariant under monodromy. The first few invariants are listed in table 3.2. Our results exactly match[2] those of [38].

[2]We are grateful to Tom Coates for sharing with us his computations and for enlightening discussions on this point.

n \ m	2	4	6	8	10
0	0	$-\frac{1}{8}$	0	$-\frac{9}{64}$	0
1	$\frac{1}{4}$	0	$\frac{7}{128}$	0	$\frac{1083}{1024}$
2	0	$-\frac{1}{32}$	0	$-\frac{143}{512}$	0
3	$\frac{1}{32}$	0	$\frac{3}{32}$	0	$\frac{85383}{16384}$
4	0	$-\frac{11}{256}$	0	$-\frac{159}{128}$	0
5	$\frac{1}{32}$	0	$\frac{47}{128}$	0	$\frac{360819}{8192}$
6	0	$-\frac{147}{1024}$	0	$-\frac{157221}{16384}$	0
7	$\frac{87}{1024}$	0	$\frac{20913}{8192}$	0	$\frac{73893099}{131072}$
8	0	$-\frac{1803}{2048}$	0	$-\frac{3719949}{32768}$	0
9	$\frac{457}{1024}$	0	$\frac{1809189}{65536}$	0	$\frac{5312434641}{524288}$
10	0	$-\frac{70271}{8192}$	0	$-\frac{498785781}{262144}$	0
11	$\frac{7859}{2048}$	0	$\frac{56072653}{131072}$	0	$\frac{254697581847}{1048576}$
12	0	$-\frac{15933327}{131072}$	0	$-\frac{11229229227}{262144}$	0
13	$\frac{801987}{16384}$	0	$\frac{2354902131}{262144}$	0	$\frac{31371782305803}{4194304}$

Table 3.2: Genus zero orbifold Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$.

3.3 A case study: the $\mathbb{C}^3/\mathbb{Z}_4$ orbifold

3.3.1 Intermezzo: the disc and torus amplitudes

The B-model disc function and open orbifold mirror map

Following the discussion of Sec. 2.2.3 we now turn on an open sector and add Lagrangian branes to $K_{\mathbb{F}_2}$. We will consider the setups I and II of figure 3.9, with a Lagrangian brane ending respectively on the upper and lower external legs of the pq-web. The choice of variables (3.2.4) we have made for the mirror curve $\Gamma_{2,2}$ corresponds to phase II. This means that V is the variable that goes to one on the brane and $X_{(II)} \equiv U$ is the good open string parameter to be taken as the independent, bare variable in (2.2.22) [24]. The transition from phase II to phase I is accomplished by the (exponentiated) $SL(2,\mathbb{Z})$ transformation

$$\begin{aligned} X_{(II)} \equiv U \quad &\rightarrow \quad \frac{1}{U} \equiv X_{(I)} \\ V \quad &\rightarrow \quad VU^2 \end{aligned} \tag{3.3.1}$$

Accordingly, the differential (3.2.9) has the form

$$d\lambda = \begin{cases} \log \frac{a_3 X_{(I)}^2 + a_4 X_{(I)} + 1 + \sqrt{(a_3 X_{(I)}^2 + a_4 X_{(I)} + 1)^2 - 4 X_{(I)}^4}}{2 X_{(I)}^4} \frac{dX_{(I)}}{X_{(I)}} & \text{phase I} \\ \log \frac{a_3 + a_4 X_{(II)} + X_{(II)}^2 + \sqrt{(a_3 + a_4 X_{(II)} + X_{(II)}^2)^2 - 4}}{2} \frac{dX_{(II)}}{X_{(II)}} & \text{phase II} \end{cases} \tag{3.3.2}$$

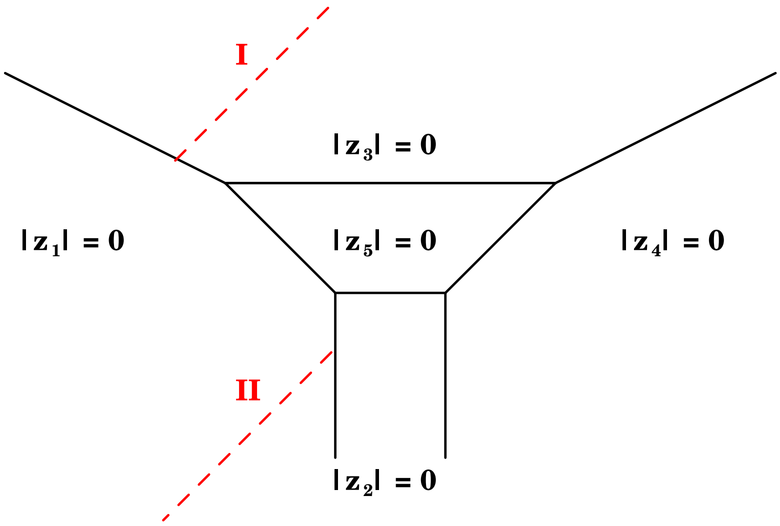

Figure 3.9: The toric web of local $\mathbb{F}_2$ with lagrangian branes on an upper (I) and lower (II) outer leg.

The unframed A-model disc amplitude for a brane in phase I can be computed as follows. First of all we need to *define* a proper open orbifold flat modulus. To this aim, we use the result of [110], where the authors show that for this outer-leg configuration the large radius open flat variable solving the extended Picard-Fuchs system is given by

$$z_o^{LR} = z + \frac{t_B}{4} + \frac{t_F}{2} + \pi i \tag{3.3.3}$$

where $z = \log X_{(I)}$. In the $(a_3,\, a_4)$ patch containing the orbifold point this becomes

$$z_o^{LR} = z + \pi i + \mathcal{O}(a_4) + \mathcal{O}(a_3) \tag{3.3.4}$$

Notice that in (3.3.3), (3.3.4), both z_{open}^{LR} and $z + \pi i$ solve the extended PF system and can then serve as a flat coordinate: z_{open}^{LR} does the job by construction, and the same is true for z because it is a difference of solutions of the Picard-Fuchs system by (3.3.3). Following [24], we will give the following operative

Definition 11. *The open orbifold mirror map for an outer brane in phase I is given, in exponentiated coordinates, by*

$$\mathcal{Z}_o^{orb} = -X_{(I)} \tag{3.3.5}$$

Indeed, we have that the difference $z_{open}^{LR} - \frac{t_B}{4} - \frac{t_F}{2} = z + \pi i$ is a global open flat variable, and moreover it is the minimal choice that yields a regular expansion at the orbifold point. Unfortunately, we have no other external input, not even from a physics-inspired BPS counting argument, to justify (3.3.5).
Having the mirror map (3.3.5) and using (2.2.29) we can now expand the chain integral and obtain the disc amplitude $\mathcal{F}_{0,1}(a_3, a_4, z)$ as a function of the bare variables,

or, using (3.2.63)-(3.2.64), of the flat variables. Notice that, since (a_3, a_4) have non-trivial $\mathbb{Z}_4$ transformations, in order to preserve the fact that the curve (3.2.42) stays invariant forces to assign weights $(1/4, 1/2)$ to (U, V) respectively, and so according to (3.3.5) $\mathcal{Z}_o^{orb}$ has weight $-1/4$. Eventually we get

$$\begin{aligned}
\mathcal{F}_{0,1}(s_i, s_{-1}, \mathcal{Z}_o^{orb}) &= \left(-\frac{s_{-1}s_i^3}{192} + \frac{s_{-1}^2 s_i}{32} - s_i\right)\mathcal{Z}_o^{orb} \\
&+ \left(\frac{s_{-1}^2 s_i^2}{64} - \frac{s_i^2}{4} + s_{-1}\left(\frac{1}{2} - \frac{s_i^4}{384}\right)\right)(\mathcal{Z}_o^{orb})^2 \\
&+ \left(\frac{7s_{-1}^2 s_i^3}{576} - \frac{s_i^3}{9} + \frac{s_{-1}s_i}{3}\right)(\mathcal{Z}_o^{orb})^3 \\
&+ \dots
\end{aligned} \tag{3.3.6}$$

which is monodromy invariant. Eq. (3.3.6) is a B-model prediction of orbifold disc invariants of $\mathbb{C}^3/\mathbb{Z}_4$.

Remark 11. *The situation for phase II appears to be more subtle. The resulting topological amplitude computed from the chain integral (2.2.29) picks up a sign flip under $\mathbb{Z}_4$. This is not completely surprising, since it is known that disc amplitudes may have non-trivial monodromy [133], and it might also be seen to be related to the more complicated geometrical structure of the $\mathbb{Z}_4$ orbifold with respect to the $\mathbb{Z}_3$ case, due to the presence of non-trivial stabilizers for the cyclic group action.*

The B-model genus 1 potential

A recipe to determine from mirror symmetry the generating function of genus 1 Gromov–Witten invariants was given in [21, 22]. In the case of toric $CY3$, we have that

$$\mathcal{F}_1 = -\frac{1}{2}\log\det\mathcal{J} - \frac{1}{12}\log\Delta \tag{3.3.7}$$

where $\mathcal{J}$ is the Jacobian matrix of the A-periods (in the appropriate polarization) with respect to the bare moduli and Δ is a rational function of the branch points, with zeroes at the discriminant locus of the curve. It is reassuring to see that we have already done most of the work in Sec. 3.2.3: the (in general difficult to compute) Jacobian $\mathcal{J}$ in (3.3.7) is precisely the main object for which we have found a closed form expression in (3.2.22). Specializing (3.3.7) to the $K_{\mathbb{F}_2}$ it is easy to obtain a closed expression for $\mathcal{F}_1$ in homogeneous (bare) coordinates at large radius. We get

$$\mathcal{F}_1^{LR}(t_F, t_B) = -\frac{1}{2}\log\left(\frac{\partial t_F}{\partial a_4}\right) + \log\left[c_1^a c_2^b (c_1 - c_2)^c (c_1 + c_2)^d\right] \tag{3.3.8}$$

where the exponents of the second term can be fixed against the topological vertex results as $a = -1/6$, $b = -1/6$, $c = -1/12$, $d = -1/12$. Then, from (3.2.43) and

(3.2.44)

$$\mathcal{F}_1^{LR}(Q_F, Q_B) = -\frac{1}{2}\frac{K\left(1-\frac{c_2^2}{c_1^2}\right)}{c_1} - \frac{1}{6}\log c_1 c_2 - \frac{1}{12}\log\left(c_1^2 - c_2^2\right) \tag{3.3.9}$$

and plugging in the mirror map (3.2.48) we can straightforwardly compute

$$\begin{aligned}\mathcal{F}_1^{LR}(Q_F, Q_B) &= \left(-\frac{\log(Q_B)}{24} - \frac{\log(Q_F)}{12}\right) - \frac{Q_F}{6} - \frac{Q_F^2}{12} - \frac{Q_F^3}{18} - \frac{Q_F^4}{24} + \\ &+ \left(-\frac{Q_F}{6} - \frac{Q_F^2}{3} - \frac{Q_F^3}{2}\right) Q_B + \left(-\frac{Q_F^2}{12} - \frac{Q_F^3}{2} + \frac{37 Q_F^4}{6}\right) Q_B^2 + \dots\end{aligned} \tag{3.3.10}$$

This agrees with the invariants computed via the topological vertex [7].

Having a closed expression for the genus 1 free energy of $K_{\mathbb{F}_2}$, we can follow [9] to obtain a prediction of elliptic orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$. We will defer it to the next section, where we display explicitly the modular properties of the open and closed generating functions and use them to bring our results for $K_{\mathbb{F}_2}$ down to the orbifold point.

3.3.2 The change of basis from large radius

In the last section we have computed the mirror map at the orbifold point by choosing solutions of the GKZ system which diagonalize the monodromy of the periods. The solutions at large radius $(1, t_B, t_F, \partial_F\mathcal{F})$, which obey a different monodromy condition at a different point, must be then be related to those at the orbifold point $(1, s_i, s_{-1}, s_{-i})$ by some general linear transformation. In particular, the restriction of this linear map to the subspace spanned by $(t_F, \partial_F\mathcal{F})$ and (s_{-1}, s_{-i}) plays a key role in the transformation rules of the open string generating functions, as well as of the higher genus closed free energies computed with the $BKMP$ method: this subsector is the one corresponding to linear combinations of periods of the Hori–Vafa differential on $H_1(\Gamma_{2,2}, \mathbb{Z})$, and it is the one that enters the shift (2.2.34) in the propagator. The methods of Sec. 3.2.3 make it much easier to study the analytic properties of the solutions, and consequently the linear change of basis. Indeed, instead of performing standard (but cumbersome) multiple Mellin-Barnes transforms, we can easily read off what S and Δ are in our case from formulae (B.1.2-B.1.3) and (3.2.59). We have the following

Proposition 9. *Define*

$$\vec{\Pi}_{LR} = \begin{pmatrix} 1 \\ t_B \\ t_F \\ \partial_F\mathcal{F} \end{pmatrix} \qquad \vec{\Pi}_{orb} = \begin{pmatrix} 1 \\ s_{-1} \\ s_i \\ s_{-i} \end{pmatrix} \tag{3.3.11}$$

and $\tilde{S} \in GL(2,\mathbb{C})$ *be such that* $\vec{\Pi}_{LR} = \tilde{S}\vec{\Pi}_{orb}$. *Then*

$$\tilde{S} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \pi & -i & 0 & 0 \\ \hline \alpha & \beta & \multicolumn{2}{c}{\multirow{2}{*}{S}} \\ \gamma & \delta & & \end{array}\right) \tag{3.3.12}$$

where

$$S = \begin{pmatrix} \frac{2\pi^{3/2}}{\Gamma\left(\frac{1}{4}\right)^2} & \frac{(1-i)\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)^2} \\ -\frac{\Gamma\left(\frac{1}{4}\right)^2}{\sqrt{\pi}} & \frac{\left(\frac{1}{2}+\frac{i}{2}\right)\Gamma\left(\frac{1}{4}\right)^2}{\pi^{3/2}} \end{pmatrix} \tag{3.3.13}$$

and α, β, γ *and* δ *are complex numbers.*

Proof. The coefficients of the submatrix S follow immediately from explicit knowledge of the a_4 derivatives of the periods, which we have obtained in the previous section, and by expanding them at the large complex structure and at the $\mathbb{Z}_4$ orbifold point. The rest of the matrix $\tilde{S}$ can be computed by using the Euler integral representation of Appendix B.1; since α, β, γ and δ do not enter the shift term of the propagator, and hence the transformation formulae of the higher genus open and closed generating functions, they will not be computed here.

3.3.3 $\Gamma(2)$ modular forms

According to (2.2.34), Eq. (3.3.13) fully determines the change of the generating functions of open and closed Gromov–Witten invariants when we move from the large radius patch of the Kähler moduli space to the $\mathbb{Z}_4$ orbifold patch. As a further step towards a complete characterization of $F^X_{g,h}$, let us show explicitly that the basic objects computed from the spectral curve data are modular forms of $\Gamma(2)$, that is the group of $SL(2,\mathbb{Z})$ matrices congruent to the identity modulo 2.

Again, we will use for inspiration the fact that our curves coincide with five-dimensional Seiberg-Witten curves, as we noticed in Remark 8, and also the fact that such curves are the same as the $R \to 0$ limit, the only thing that changes being the symplectic structure defined on the elliptic fibration, i.e., the SW differential. We have the following important

Lemma 1. *Let* c_1 *and* c_2 *denote the branch-point related variables of* (3.2.43). *They are* $\Gamma(2)$ *modular forms of* $\Gamma_{2,2}$, *and read explicitly*

$$c_1(\tau) = 2\frac{\theta_4^2(\tau)}{\theta_2^2(\tau)} \qquad c_2(\tau) = 2\frac{\theta_3^2(\tau)}{\theta_2^2(\tau)} \tag{3.3.14}$$

Proof. By formulae (3.2.5), (3.2.43) the $\Gamma_{2,2}$ family can be written as

$$Y^2 = (\hat{X}^2 - c_1^2)(\hat{X}^2 - c_2^2) \tag{3.3.15}$$

where we have shifted the X variable in (3.2.5) by $\hat{X} = X + a_4/2$. Through the $SL(2,\mathbb{C})$ automorphism of the $\hat{X}$-plane

$$\hat{X} = \frac{a\tilde{X} + b}{c\tilde{X} + d} \qquad \tilde{Y} = (c\tilde{X} + d)^2 Y$$

$$a = \frac{\sqrt{c_1 c_2 - c_2^2}}{\sqrt{4c_1 + 4c_2}}, \quad b = \frac{c_2(3c_1 + c_2)}{2\sqrt{c_2\left(c_1^2 - c_2^2\right)}}, \quad c = -\frac{\sqrt{\frac{c_1^2}{c_2} - c_2}}{2(c_1 + c_2)}, \quad d = \frac{c_1 + 3c_2}{2\sqrt{c_2\left(c_1^2 - c_2^2\right)}} \tag{3.3.16}$$

we bring (3.3.15) to the celebrated Seiberg-Witten $\Gamma(2)$-symmetric form

$$\tilde{Y}^2 = (\tilde{X}^2 - 1)(\tilde{X} - u) \tag{3.3.17}$$

where

$$u = \frac{c_1^2 + 6c_2c_1 + c_2^2}{(c_1 - c_2)^2} \tag{3.3.18}$$

With (3.3.18) at hand we can re-express the quantities computed in the previous section as $\Gamma(2)$ modular forms, whose ring is generated by the Jacobi theta functions $\theta_2(\tau)$, $\theta_3(\tau)$, $\theta_4(\tau)$, all having modular weight 1/2. This goes as follows: the Klein invariant $j(\tau)$ of the curve (3.3.17) is rationally related to u as

$$j(u) = 64\frac{(3 + u^2)^3}{(u^2 - 1)^2} \tag{3.3.19}$$

while inversion of (3.3.18) gives, writing everything for definiteness in the (a_3, a_4) patch,

$$a_4^2 - a_3 = \frac{u + 3}{\sqrt{2}\sqrt{u + 1}} \tag{3.3.20}$$

Writing the $j(\tau)$ invariant in terms of θ functions

$$j(\tau) = \frac{4}{27}\frac{(1 - \lambda(\tau) + \lambda^2(\tau))^3}{\lambda^2(\tau)(1 - \lambda(\tau))^2} \tag{3.3.21}$$

where $\lambda(\tau) = (\theta_2(\tau)/\theta_3(\tau))^4$, inverting (3.3.19) to yield $u(j)$, and plugging it into (3.3.20), we get (3.3.14).

□

Example 2. *Given (3.3.14) it is then straightforward to write the building blocks of the BCOV recursion in terms of modular forms. Let us analyze the large radius Yukawa coupling first. We have*

$$C \equiv \frac{\partial^3 \mathcal{F}}{\partial t_F^3} = \frac{4}{\pi}\left(\frac{\partial a_4}{\partial \tau}\frac{\partial t_F}{\partial a_4}\right)^{-1}$$

Using (3.2.44) we have

$$\frac{\partial t_F}{\partial a_4} = \frac{K\left(1-\frac{c_2^2}{c_1^2}\right)}{c_1} = \frac{\pi}{4}\theta_2^2(\tau) \tag{3.3.22}$$

while combining (3.3.14) and (3.2.43) yields

$$\frac{\partial a_4}{\partial \tau} = -\frac{2^6}{a_4(\tau)}\frac{\eta^{12}(\tau)}{\theta_2(\tau)^8} \tag{3.3.23}$$

where $\eta(\tau)$ *is Dedekind's function and we have used*

$$2\eta^3(\tau) = \theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \tag{3.3.24}$$

besides the modular expression of a_4 *from (3.2.43)*

$$a_4(\tau) = 2\sqrt{4\frac{\theta_4^4(\tau)}{\theta_2^4(\tau)} + a_3 + 2} \tag{3.3.25}$$

Putting it all together we arrive at

$$C(\tau) = -\frac{a_4(\tau)}{64}\frac{\theta_2^6(\tau)}{\eta^{12}(\tau)} \tag{3.3.26}$$

Example 3. *We can also complete the results of Sec. 3.3.1 by using the modular expression of* c_1, c_2 *and* $\partial_{a_4} t_F$ *to write the B-model genus one potential as an almost-modular form. From* (3.3.8) *and* (3.3.14) *we get at large radius*

$$\mathcal{F}_1^{LR}(\tau) = -\frac{1}{2}\log\eta(\tau) \tag{3.3.27}$$

Plugging in the expression for the modular parameter $q = e^{2\pi i\tau}$ *in exponentiated flat coordinates, which can be computed from (3.2.48), (3.3.19) and (3.3.20)*

$$q(Q_B, Q_F) = Q_B Q_F^2 + \left(4Q_B^2 + 4Q_B\right)Q_F^3 + \left(10Q_B^3 + 48Q_B^2 + 10Q_B\right)Q_F^4 + O\left(Q_F^5\right) \tag{3.3.28}$$

we recover precisely (3.3.10). As the logarithm of a weight $1/2$ *modular form, the genus* 1 *free energy transforms with a shift under a modular transformation*

$$\mathcal{F}_1\left(\frac{A\tau+B}{C\tau+D}\right) = \mathcal{F}_1(\tau) + \frac{1}{2}\log\frac{1}{\tau + C^{-1}D} \tag{3.3.29}$$

Writing τ *as a function of the* a_i *variables using (3.2.44), (3.2.45) and plugging the orbifold mirror map (3.2.64) we get*

$$\begin{aligned} \mathcal{F}_1^{orb}(s_i, s_{-1}) &= \sum_{n,m}\frac{N_{1,(m,n)}^{orb}}{n!m!}s_i^m s_{-1}^n \qquad (3.3.30)\\ &= -\frac{s_{14}^2 s_{12}}{384} + \frac{s_{12}^2}{192} - \frac{5s_{14}^2 s_{12}^3}{9216} + \frac{7s_{12}^4}{18432} - \frac{13s_{14}^2 s_{12}^5}{163840} + \frac{31s_{12}^6}{1105920} + \dots \end{aligned}$$

The first few GW invariants are reported in table 3.3.

n \ m	0	2	4	6	8	10
0		0	$\frac{1}{128}$	0	$\frac{441}{4096}$	0
1	0	$-\frac{1}{192}$	0	$-\frac{31}{1024}$	0	$-\frac{71291}{32768}$
2	$\frac{1}{96}$	0	$\frac{35}{3072}$	0	$\frac{235}{512}$	0
3	0	$-\frac{5}{768}$	0	$-\frac{485}{4096}$	0	$-\frac{2335165}{131072}$
4	$\frac{7}{768}$	0	$\frac{485}{12288}$	0	$\frac{458295}{131072}$	0
5	0	$-\frac{39}{2048}$	0	$-\frac{40603}{49152}$	0	$-\frac{58775443}{262144}$
6	$\frac{31}{1536}$	0	$\frac{2025}{8192}$	0	$\frac{10768885}{262144}$	0
7	0	$-\frac{2555}{24576}$	0	$-\frac{293685}{32768}$	0	$-\frac{522517275}{131072}$
8	$\frac{2219}{24576}$	0	$\frac{240085}{98304}$	0	$\frac{1437926315}{2097152}$	0
9	0	$-\frac{22523}{24576}$	0	$-\frac{73017327}{524288}$	0	$-\frac{397762755193}{4194304}$
10	$\frac{16741}{24576}$	0	$\frac{54986255}{1572864}$	0	$\frac{32280203275}{2097152}$	0
11	0	$-\frac{389975}{32768}$	0	$-\frac{18440181205}{6291456}$	0	$-\frac{12177409993695}{4194304}$
12	$\frac{1530037}{196608}$	0	$\frac{1434341595}{2097152}$	0	$\frac{7495469356455}{16777216}$	0

Table 3.3: B-model predictions for genus one orbifold Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$.

3.3.4 Bulletproofing BKMP

On the basis of the results of the previous sections, we will now perform an extensive check of the $BKMP$ proposal. We will compute the correlators of the $BKMP$ recursion up to the point where the lowest genus prediction for *closed*, ordinary GW invariants can be made. Before we do that, we will resume the formalism of kernel differentials of [24] specialized to the case of elliptic mirror curves. We will then give a proof of Theorem 5 from this formalism, and exhibit the general expression of $W_h^{(g)}$ as almost-modular forms. Finally we will exploit Conjecture 2 to predict open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$, and briefly comment on the results.

Kernel differentials

Let us specialize the recursion to the case of local $\mathbb{F}_2$. The Hori-Vafa differential (3.2.9) reads, in the (a_3, a_4) patch,

$$d\lambda_{2,2}(u) = \log\left(\frac{P_2(u) \pm Y(u)}{2}\right)\frac{du}{u} \tag{3.3.31}$$

where

$$P_2(u) = a_3 + a_4 u + u^2, \qquad Y(u) = \sqrt{P_2^2(u) - 4}$$

and the $\Gamma_{2,2}$ family can be written in the $\mathbb{Z}_2$ symmetric form (3.3.15) as a two-fold branched covering of the compactified u-plane

$$Y^2 = (u - b_1)(u - b_2)(u - b_3)(u - b_4) = (\hat{u}^2 - c_1^2)(\hat{u}^2 - c_2^2) \tag{3.3.32}$$

thanks to (3.2.43) and (3.3.15) and having defined $\hat{u} = u + a_4/2$. We have first of all that

$$d\lambda(u) - d\lambda(\bar{u}) = 2M(u)Y(u)du \tag{3.3.33}$$

where the so-called "moment function" $M(u)$ is given, after using the fact that $\log(P+Y) - \log(P-Y) = 2\tanh^{-1}(Y/P)$, as

$$M(u) = \frac{1}{uY(u)} \tanh^{-1}\left[\frac{Y(u)}{P_2(u)}\right], \tag{3.3.34}$$

Moreover, the one form $dE(p,q)$ (2.2.24) can be written as [61]

$$dE_w(u) = \frac{1}{2}\frac{Y(w)}{Y(u)}\left(\frac{1}{u-w} - LC(w)\right) du \tag{3.3.35}$$

where

$$C(w) := \frac{1}{2\pi i}\oint_A \frac{du}{Y(u)}\frac{1}{u-w}, \qquad L^{-1} := \frac{1}{2\pi i}\oint_A \frac{du}{Y(u)} \tag{3.3.36}$$

We have assumed here that w stays outside the contour A; when w lies inside the contour A, $C(w)$ in (3.3.35) should be replaced by its regularized version

$$C^{\mathrm{reg}}(w) = C(w) - \frac{1}{Y(w)} \tag{3.3.37}$$

Since $\Gamma_{2,2}$ is elliptic, it is possible to find closed form expressions for $C(u)$, $C_{reg}(u)$, $B(u,w)$ and L. We have

$$C(u) = \frac{2(b_2-b_3)}{\pi(u-b_3)(u-b_2)\sqrt{(b_1-b_3)(b_2-b_4)}}\left[\Pi(n_4,k) + \frac{u-b_2}{b_2-b_3}K(k)\right] \tag{3.3.38}$$

$$C^{\mathrm{reg}}(u) = \frac{2(b_3-b_2)}{\pi(u-b_3)(u-b_2)\sqrt{(b_1-b_3)(b_2-b_4)}}\left[\Pi(n_1,k) + \frac{u-b_3}{b_3-b_2}K(k)\right] \tag{3.3.39}$$

$$L^{-1} = \frac{2}{\sqrt{(b_1-b_3)(b_2-b_4)}}K\left[\frac{(b_1-b_2)(b_3-b_4)}{(b_1-b_3)(b_2-b_4)}\right] \tag{3.3.40}$$

$$\begin{aligned} B(u,w) &= \frac{1}{Y(u)}\left[\frac{Y^2(u)}{2Y(w)(u-w)^2} + \frac{(Y^2)'(u)}{4Y(w)(w-u)} + \frac{A(u)}{4Y(w)}\right] \\ &+ \frac{1}{2(u-w)^2} \end{aligned} \tag{3.3.41}$$

where

$$k = \frac{(b_1-b_2)(b_3-b_4)}{(b_1-b_3)(b_2-b_4)}, \qquad n_4 = \frac{(b_2-b_1)(u-b_3)}{(b_3-b_1)(u-b_2)}, \qquad n_1 = \frac{(b_4-b_3)(u-b_2)}{(b_4-b_2)(u-b_3)}, \tag{3.3.42}$$

$$A(u) = (u - b_1)(u - b_2) + (u - b_3)(u - b_4) + (b_1 - b_3)(b_2 - b_4)\frac{E(k)}{K(k)} \tag{3.3.43}$$

and $K(k)$, $E(k)$ and $\Pi(n, k)$ are the complete elliptic integrals of the first, second and third kind respectively.

With these ingredients one can compute the residues as required in (2.2.27). Given that $dE_q(p)/(d\lambda(q) - d\lambda(\bar{q}))$, as a function of q, is regular at the branch-points, all residues appearing in (2.2.27) will be linear combinations of the following *kernel differentials*

$$\begin{aligned} \chi_i^{(n)}(p) &= \mathrm{Res}_{q=x_i}\left(\frac{dE_q(p)}{d\lambda(q) - d\lambda(\bar{q})}\frac{1}{(q - x_i)^n}\right) \\ &= \frac{1}{(n-1)!}\frac{1}{Y(p)}\frac{d^{n-1}}{dq^{n-1}}\left[\frac{1}{2M(q)}\left(\frac{1}{p-q} - LC(q)\right)\right]_{q=x_i} \end{aligned} \tag{3.3.44}$$

In (3.3.44), $C(p)$ should be replaced by $C_{\mathrm{reg}}(p)$ when $i = 1, 2$.

$W_h^{(g)}$ as almost-modular forms

Let us then explicitly display the quasi-modular structure of the correlators. We have the following

Theorem 10. *The correlators* (2.2.27) *have the form*

$$W_h^{(g)}(p_1, \ldots, p_h; a_3, \tau) = \sum_{n=0}^{3g-3+2h} c_n^{(g,h)}(p_1, \ldots, p_h; a_3, \tau)E_2^n(\tau) \tag{3.3.45}$$

where $c_n^{(g,h)}(p_1, \ldots, p_h; a_3, \tau)$ *is, for every fixed* $p_1, \ldots, p_h, a_3$*, a modular form of weight* $-2n$.

Proof. Formulae (2.2.28), (2.2.27), (3.3.41) and (3.3.44) imply that $W_h^{(g)}(p_1, \ldots, p_h; a_3, \tau)$ will be a polynomial in the following five objects

$$M_i^{(n)}, \qquad \phi_i^{(n)}, \qquad A_i^{(n)}, \qquad \left(\frac{1}{Y}\right)_i^{(n)}, \qquad \mathcal{C}_i^{(n)} \tag{3.3.46}$$

where, for a function $f(x)$ with meromorphic square $f^2(x)$, we denote with $f_i^{(n)}$ the $(n+1)$-th coefficient in a Laurent expansion of $f(x)$ around b_i

$$f(x) = \sum_{n=-N_i}^{\infty} \frac{f_i^{(n+N_i)}}{(p - b_i)^{n/2}} \tag{3.3.47}$$

and have defined

$$\mathcal{C}_i^{(n)} = \begin{cases} C_{\mathrm{reg},i}^{(n)} & \text{for} \quad i = 1, 2 \\ C_i^{(n)} & \text{for} \quad i = 3, 4 \end{cases} \tag{3.3.48}$$

In fact, $\phi_i^{(n)}$ only enters the expression of the free energies $\mathcal{F}_g$. Of the five building blocks in (3.3.46), $M_i^{(n)}$ and $\phi_i^{(n)}$ are the ones which are computed most elementarily from (3.3.31) and (3.3.34), the result being in any case an algebraic function of (a_3, a_4). When re-expressed in modular form, we can actually say more about them: we have that the a_3-dependence in $M_i^{(n)}(a_3, \tau)$ and $\phi_i^{(n)}(a_3, \tau)$ is constrained to come only through $a_4(a_3, \tau)$ as written in formula (3.3.25). Indeed, from (3.2.43), (3.3.14) we have that the branch points b_i have the form

$$-\frac{a_4}{2} \pm c_1(\tau), \qquad -\frac{a_4}{2} \pm c_2(\tau) \tag{3.3.49}$$

and therefore depend on a_3 only through $a_4(a_3, \tau)$. Moreover, since $P_2(b_i) = 2$ and the derivatives of $P_2(U)$ do not depend explicitly on a_3, we have that the a_3 dependence in $W_h^{(g)}$ as obtained from the recursion may only come through $a_4(a_3, \tau)$:

$$P_2(b_i) = 2, \quad P_2'(b_i) = 2(-1)^{i+1} c_{[\frac{i-1}{2}]+1}(\tau), \quad P_2''(b_i) = -\frac{a_4}{2} + (-1)^{i+1} c_{[\frac{i-1}{2}]+1}(\tau)$$

Notice moreover that these are the only pieces bringing a dependence on the additional a_3 variable: all the others do not depend on the form of the differential (3.3.31), and are functions only of differences of branch points b_i. This means in particular that they only depend on the variables c_1 and c_2 introduced in (3.2.43) and whose modular expressions we already found in (3.3.14). This is immediate to see for $A_i^{(n)}$ and $(1/Y)_i^{(n)}$ from formulae (3.3.32) and (3.3.43). The case of $\mathcal{C}_i^{(n)}$ is just slightly more complicated: for $n = 1$, we need the first derivative of $\Pi(x, y)$ with respect to x:

$$\partial_x \Pi(x, y) = \frac{xE(y) + (y - x)K(y) + (x^2 - y)\,\Pi(x, y)}{2(x - 1)x(y - x)}$$

The above formula implies that

$$\partial_x^{(n)} \Pi(x, y) = A_n(x, y)K(y) + B_n(x, y)E(y) + C_n(x, y)\Pi(x, y) \tag{3.3.50}$$

where A_n, B_n and C_n are rational functions of x and y. From (3.3.42), to compute $\mathcal{C}_i^{(n)}$, we need to evaluate these expressions when n_1 (resp. n_4) equals either 0 or k. But using

$$\Pi(0, y) = K(y), \qquad \Pi(y, y) = \frac{E(y)}{1 - y} \tag{3.3.51}$$

we conclude that

$$\mathcal{C}_i^{(n)} = R_1^{(n)}(c_1, c_2)K(k) + R_2^{(n)}(c_1, c_2)E(k) \tag{3.3.52}$$

for two sequences of rational functions $R_i^{(n)}$. Observe now that, by (3.3.44), $\mathcal{C}_i^{(n)}$ always appears multiplied by L in the recursion. By (3.3.40)

$$L\mathcal{C}_i^{(n)} = \tilde{R}_1^{(n)}(c_1, c_2) + \tilde{R}_2^{(n)}(c_1, c_2)\frac{E(k)}{K(k)} \tag{3.3.53}$$

This last observation allows us to collect all the pieces together and state the following. By (2.2.28) and (2.2.27) we have that $W_h^{(g)}(p_1,\ldots,p_h,a_3,\tau)$ is a polynomial in $M_i^{(n)}$, $\phi_i^{(n)}$, $A_i^{(n)}$, $(1/Y)_i^{(n)}$, $\mathcal{C}_i^{(n)}$, and moreover the whole discussion above as well as formulae (3.3.43) and (3.3.53) imply that this takes the form of a polynomial in $W(\tau) := E(k)/K(k)$

$$W_h^{(g)}(a_3,\tau) = \sum_{k=0}^{n} W^k(\tau) H_{h,k}^{(g)}(a_3,\tau) \tag{3.3.54}$$

where the coefficients $H_{h,k}^{(g)}(a_3,\tau)$ are weight zero modular forms of $\Gamma(2)$, parametrically depending on a_3. To conclude the proof of (3.3.45), we can exploit the fact that [140]

$$E(k)K(k) = \left(\frac{\pi}{2}\right)^2 \frac{4E_2(2\tau) - E_2(\tau)}{3} \tag{3.3.55}$$

and that from (3.3.14) and (3.3.22)

$$K(k) = \frac{\pi}{2}\theta_3(\tau)\theta_4(\tau) \tag{3.3.56}$$

where we have used the fact that in our case

$$K(k) = \sqrt{\frac{c_2}{c_1}} K\left(1 - \frac{c_2^2}{c_1^2}\right)$$

as the reader can easily check. Moreover, the second Eisenstein series satisfies the duplication formula

$$E_2(2\tau) = \frac{E_2(\tau)}{2} + \frac{\theta_4^4(\tau) + \theta_3^4(\tau)}{4} \tag{3.3.57}$$

Therefore,

$$W(\tau) = \frac{1}{3\theta_4^2(\tau)\theta_3^2(\tau)} \left(E_2(\tau) + \theta_3^4(\tau) + \theta_4^4(\tau)\right) \tag{3.3.58}$$

This proves (3.3.45).

□

Based on (3.3.45), we can enforce Conjecture 2 by postulating the following form for the orbifold correlators.

Definition 12. *We define the orbifold correlators of* $\mathbb{C}^3/\mathbb{Z}_4$ *as*

$$\tilde{W}_h^{(g)}(p_1,\ldots,p_h;a_3,\tau) = \sum_{n=0}^{3g-3+2h} c_n^{(g,h)}(p_1,\ldots,p_h;a_3,\tau)(E_2(\tau)+d(\tau))^n \tag{3.3.59}$$

where $d(\tau)$ *is defined by* (2.2.35) *and* (3.3.13), *and* $c_n^{(g,h)}(p_1,\ldots,p_h;a_3,\tau)$ *are the modular forms computed by the recursion at large radius* (3.3.45).

Upon integration and inversion of the orbifold mirror map, the orbifold correlators will provide predictions for the generating functions of open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$.

Remark 12. *It is interesting to remark that performing only the analytic continuation of the large radius open string generating functions to the orbifold patch, without the shift $E_2(\tau) \to E_2(\tau) + d(\tau)$, would end up in an expansion around the orbifold point with irrational (in fact transcendental) coefficients. Indeed, while the $H^{(g)}_{h,k}$ operators have an expansion with rational coefficients, the propagator $W(\tau)$ has the Taylor expansion*

$$\begin{aligned} W(\tau) &= \left(\frac{1}{2} + \frac{4\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}\right) + \left(-\frac{i}{32} - \frac{2i\pi^4}{\Gamma\left(\frac{1}{4}\right)^8}\right) s_i^2 \qquad (3.3.60)\\ &+ \left(\left(\frac{i}{8} + \frac{8i\pi^4}{\Gamma\left(\frac{1}{4}\right)^8}\right) + \left(\frac{8\pi^6}{\Gamma\left(\frac{1}{4}\right)^{12}} - \frac{\pi^2}{8\Gamma\left(\frac{1}{4}\right)^4}\right) s_i^2\right) s_{-1} \\ &+ \left(\left(-\frac{16\pi^6}{\Gamma\left(\frac{1}{4}\right)^{12}} + \frac{\pi^2}{4\Gamma\left(\frac{1}{4}\right)^4}\right) + \left(-\frac{i}{128} + \frac{24i\pi^8}{\Gamma\left(\frac{1}{4}\right)^{16}} - \frac{5i\pi^4}{8\Gamma\left(\frac{1}{4}\right)^8}\right) s_i^2\right) s_{-1}^2 + \dots \end{aligned}$$

The terms containing powers of $\Gamma(1/4)$ *are exactly cancelled by the shift in the propagator due to* $E_2(\tau) \to E_2(\tau) + d(\tau)$ *in* (3.3.59)

$$\begin{aligned} W\left(\frac{A\tau + B}{C\tau + D}\right) &= \frac{1}{2} - \frac{is_i^2}{32} - \frac{11is_i^6}{61440} + \left(\frac{i}{8} + \frac{13is_i^4}{6144} + \frac{457is_i^8}{13762560}\right) s_{-1} \\ &+ \left(-\frac{is_i^2}{128} - \frac{371is_i^6}{1474560}\right) s_{-1}^2 + \dots \qquad (3.3.61) \end{aligned}$$

Results

Let us conclude this chapter by presenting part of the explicit computation of $W_2^{(0)}$, $W_2^{(0)}$, $W_3^{(0)}$, $W_1^{(1)}$, $W_2^{(1)}$ and $W_1^{(2)}$. These correlators will also put us in a position to compute the genus 2 free energy $\mathcal{F}_2$.
The first three were computed in [24] and are given by

$$\begin{aligned} W_2^{(0)}(p_1, p_2) &= B(p_1, p_2) \\ W_3^{(0)}(p_1, p_2, p_3) &= \frac{1}{2}\sum_{i=1}^{4} M^2(b_i)(Y^2)'(b_i)\chi_i^{(1)}(p_1)\chi_i^{(1)}(p_2)\chi_i^{(1)}(p_3), \qquad (3.3.62) \\ W_1^{(1)}(p) &= \frac{1}{16}\sum_{i=1}^{4}\chi_i^{(2)}(p) + \frac{1}{8}\sum_{i=1}^{4}\left(\frac{2A(b_i)}{(Y^2)'(b_i)} - \sum_{j\neq i}\frac{1}{b_i - b_j}\right)\chi_i^{(1)}(p), \end{aligned}$$

$W_2^{(1)}$ is then given from (2.2.27) as

$$W_2^{(1)}(p, p_1) = \sum_{b_i} \operatorname*{Res}_{q=b_i} \frac{dE_q(p)}{d\lambda(q) - d\lambda(\bar{q})}\left(W_3^{(0)}(q, \bar{q}, p_1) + 2W_1^{(1)}(q)W_2^{(0)}(\bar{q}, p_1)\right) \quad (3.3.63)$$

A very lengthy, but straightforward computation leads us to

$$W_2^{(1)}(p,q) = -\frac{1}{8}\sum_{i=1}^{4}\left[A_i(q)\chi_i^{(3)}(p) + B_i(q)\chi_i^{(2)}(p) + C_i(q)\chi_i^{(1)}(p) + \sum_{j\neq i} D_{ij}(q)\chi_i^{(1)}(p)\right] \tag{3.3.64}$$

For the sake of notational brevity, we spare to the reader the very long expressions of the rational functions $A_i(q)$, $B_i(q)$, $C_i(q)$ and $D_{ij}(q)$. They involve $M_i^{(n)}$, $A_i^{(n)}$, $(1/Y)_i^{(n)}$ and $\mathcal{C}_i^{(n)}$ up to the third order in a Taylor-Laurent expansion around the branch points.

The next step is given by

$$W_1^{(2)}(p) = \sum_{b_i}\operatorname*{Res}_{q=b_i}\frac{dE_q(p)}{d\lambda(q)-d\lambda(\bar{q})}\Big(W_2^{(1)}(q,\bar{q}) + W_1^{(1)}(q)W_1^{(1)}(\bar{q})\Big) \tag{3.3.65}$$

The pole structure of $A_i(q)$, $B_i(q)$, $C_i(q)$ and $D_{ij}(q)$ dictates for $W_1^{(2)}(p)$ the following linear expression in terms of kernel differentials

$$W_1^{(2)}(p) = \sum_{n=1}^{5}\sum_{i=1}^{4} E_i^{(n)}\chi_i^{(n)}(p) \tag{3.3.66}$$

for some (very complicated) coefficients $E_i^{(n)}$.
Finally, we can compute the $g=2$ free energy by (2.2.28)

$$\mathcal{F}_2 = -\frac{1}{2}\sum_{b_i}\operatorname*{Res}_{p=b_i}\phi(p)W_1^{(2)}(p) \tag{3.3.67}$$

It is useful to collect together terms involving the same powers of $W(\tau)$. Taking the residues in (3.3.67) yields[3]

$$\sum_{n=0}^{3} h_n^{(2)}(a_3,\tau)W^n(\tau) \tag{3.3.68}$$

[3]It must be noticed that, in order to match exactly the asymptotics of the Gromov-Witten expansion at large radius, we have to subtract from (3.3.67) a constant term in τ, namely, a rational function of a_3 of the form $\frac{a_3^2-10}{1440\left(a_3^2-4\right)}$. It would be interesting to investigate the origin of this discrepancy further.

where the modular coefficients $h_n^{(2)}(a_3,\tau)$ are given as

$$
\begin{aligned}
h_3^{(2)}(a_3,\tau) &= \frac{5a_4^2(a_3,\tau)\theta_2^4(\tau)}{24576\theta_3^2(\tau)\theta_4^2(\tau)} \\
h_2^{(2)}(a_3,\tau) &= \frac{1}{1024} - \frac{a_4(a_3,\tau)^2}{49152\theta_3(\tau)^6\theta_4(\tau)^4}\Big[\theta_2(\tau)^4(15\theta_4(\tau)^6 + 16\theta_3(\tau)^2\theta_4(\tau)^4 \\
&+ \theta_2(\tau)^4\left(8\theta_3(\tau)^2 + 15\theta_4(\tau)^2\right))\Big] \\
h_1^{(2)}(a_3,\tau) &= -\frac{(\theta_2(\tau)^4 + 2\theta_4(\tau)^4 + 3\theta_3(\tau)^2\theta_4(\tau)^2)}{3072\theta_4(\tau)^2\theta_3(\tau)^2} + \frac{a_4^2(a_3,\tau)}{294192}\left[\frac{13\theta_2(\tau)^{12}}{\theta_3(\tau)^6\theta_4(\tau)^6}\right. \\
&+ \left.\frac{91\theta_2(\tau)^8}{\theta_3(\tau)^6\theta_4(\tau)^2} + \frac{48\theta_2(\tau)^8}{\theta_3(\tau)^4\theta_4(\tau)^4} + \frac{91\theta_4(\tau)^2\theta_2(\tau)^4}{\theta_3(\tau)^6} + \frac{96\theta_2(\tau)^4}{\theta_3(\tau)^4}\right] \\
h_0^{(2)}(a_3,\tau) &= \frac{1}{61440}\left(\frac{1}{a_3+2} - \frac{1}{a_3-2}\right) + \frac{\theta_2(\tau)^8 - 5\theta_3(\tau)^2\theta_2(\tau)^4 + 10\theta_3(\tau)^6}{30720\theta_3(\tau)^4\theta_4(\tau)^4} \\
&+ \frac{a_4^2(a_3,\tau)\theta_2(\tau)^4}{2949120}\left[12\left(\frac{\theta_2(\tau)^4}{\theta_3(\tau)^8} - \frac{\theta_2(\tau)^4}{\theta_4(\tau)^8}\right) - \frac{65\theta_4(\tau)^2}{\theta_3(\tau)^6} - \frac{175}{\theta_3(\tau)^2\theta_4(\tau)^2}\right. \\
&- \left.\frac{311}{2\theta_3(\tau)^4} - \frac{311}{2\theta_4(\tau)^4} - \frac{65\theta_3(\tau)^2}{\theta_4(\tau)^6}\right] + \frac{17}{46080}
\end{aligned}
\tag{3.3.69}
$$

Plugging in the expression (3.3.28) of the modular parameter q in exponentiated flat coordinates reproduces as expected the topological vertex expansion at large radius

$$
\begin{aligned}
\mathcal{F}_2^{LR}(Q_B,Q_F) &= \left(-\frac{1}{120} - \frac{Q_B}{120}\right)Q_F + \left(-\frac{1}{60} - \frac{Q_B}{60} - \frac{Q_B^2}{60}\right)Q_F^2 \\
&+ \left(-\frac{1}{40} - \frac{Q_B}{40} - \frac{Q_B^2}{40} - \frac{Q_B^3}{40}\right)Q_F^3 + \left(-\frac{1}{30} - \frac{Q_B}{30} - \frac{Q_B^2}{6} - \frac{Q_B^3}{30}\right)Q_F^4 \\
&+ \left(-\frac{1}{24} - \frac{Q_B}{24} - \frac{299Q_B^2}{24} - \frac{299Q_B^3}{24}\right)Q_F^5 + +O\left(Q_F^6\right)
\end{aligned}
\tag{3.3.70}
$$

By replacing the propagators as in Definition 12, we obtain predictions for open and closed orbifold Gromov–Witten invariants (see Table 3.4 for the genus 2 closed invariants and Tables A.1-A.21 in the Appendix for the open invariants; a `Mathematica` notebook is available on request for the detailed computation). We summarize here the main results, that make up a significant confirmation of Conjectures 1 and 2:

- *all* $W_h^{(g)}$ computed at large radius where checked to reproduce correctly the topological vertex computation of open Gromov–Witten invariants of $K_{\mathbb{F}_2}$, at the first few orders in the degree expansion;
- the closed genus 2 free energy matches *both* the genus 2 Gromov-Witten expansion at large radius, *and*, upon the GKZ change of basis and analytic continuation to the orbifold point, the orbifold genus 2 invariants predicted by the Yamaguchi-Yau polynomial method to solve the HAEs [11]. As such, this is not just a check on a finite set of numbers, but a check on the full functional form of $\mathcal{F}_2$, since we are considering Taylor expansions at *different* points.
- the disc generating function (3.3.6), at the first few orders in the open moduli expansion, agrees to *all orders* in the closed moduli with the invariants defined by localization [33], up to a sign. These results will appear in joint work with R. Cavalieri [28].

$n \backslash m$	0	2	4	6	8	10
0	$-\frac{1}{960}$	0	$-\frac{61}{30720}$	0	$-\frac{9023}{81920}$	0
1	0	$\frac{41}{46080}$	0	$\frac{6061}{245760}$	0	$\frac{36213661}{7864320}$
2	$-\frac{7}{7680}$	0	$-\frac{647}{92160}$	0	$-\frac{1066027}{1310720}$	0
3	0	$\frac{257}{92160}$	0	$\frac{168049}{983040}$	0	$\frac{887800477}{15728640}$
4	$-\frac{11}{5120}$	0	$-\frac{65819}{1474560}$	0	$-\frac{18530321}{1966080}$	0
5	0	$\frac{23227}{1474560}$	0	$\frac{43685551}{23592960}$	0	$\frac{62155559923}{62914560}$
6	$-\frac{2479}{245760}$	0	$-\frac{437953}{983040}$	0	$-\frac{9817250341}{62914560}$	0
7	0	$\frac{418609}{2949120}$	0	$\frac{452348269}{15728640}$	0	$\frac{5851085490887}{251658240}$
8	$-\frac{19343}{245760}$	0	$-\frac{303139073}{47185920}$	0	$-\frac{438364727389}{125829120}$	0
9	0	$\frac{1380551}{737280}$	0	$\frac{25384681949}{41943040}$	0	$\frac{355405937648809}{503316480}$
10	$-\frac{604199}{655360}$	0	$-\frac{2982122587}{23592960}$	0	$-\frac{16896151842371}{167772160}$	0
11	0	$\frac{200852963}{5898240}$	0	$\frac{25012290702059}{1509949440}$	0	$\frac{540498559368019561}{2013265920}$
12	$-\frac{59566853}{3932160}$	0	$-\frac{818897894611}{251658240}$	0	$-\frac{1840152188554961}{503316480}$	0

Table 3.4: The B-model prediction for genus two orbifold Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$

Chapter 4

Toric GW theory II: conifold transitions and Gopakumar-Vafa duality

4.1 Overview

After the seminal work of Witten [146], Chern-Simons (CS) theory has been deeply studied both in Mathematics and Physics. A most attractive property of this topological field theory is its large N duality with the A-model topological string as discovered by Gopakumar and Vafa in [83]. From the Physics viewpoint, this is a concrete realization of 't Hooft's intuition [1] that the large N Feynman expansion of a gauge theory with $U(N)$ structure group can be recast as a perturbative expansion of closed oriented strings in a suitable background; mathematically, such duality is a precise (and amazing!) correspondence between two seemingly very unrelated mathematical objects, namely knot invariants and (relative) Gromov-Witten invariants. This duality is realised through a particular kind of *geometric transition*, called *conifold transition*, which plays a relevant rôle in the study of the moduli space of Calabi-Yau three-folds ($CY3$) (see *e.g.* [16, 84, 135] for reviews).
Let us recall the basic features of Gopakumar-Vafa (GV) duality. We have the following

Proposition 11 (Witten, [150]). *Let M be a closed smooth 3-manifold such that T^*M is a Calabi-Yau threefold. Then the open topological A-model on T^*M, with N Lagrangian branes wrapping M, is equivalent to $U(N)$ Chern-Simons theory on M.*

Proposition 12 (Gopakumar-Vafa, [83]). *The topological open A-model on T^*S^3 with N A-branes wrapping the base S^3 is equivalent at large N to the closed topological A-model on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.*

The closed string target space $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and the open string one T^*S^3 are related by topological surgery - a birational contraction plus a complex deformation of a nodal singularity. This is what goes under the name of "*(conifold) geometric transition*". As it stands, the content of Proposition 12 is striking and somewhat mysterious: a topological invariant of a 3-manifold - the CS partition function, or Reshetikin-Turaev-Witten invariant of S^3 [134, 146] - is also a generating function of symplectic invariants of a non-compact Kähler manifold - its Gromov-Witten potential. It would be then extremely interesting to see if one could find new examples of such a duality along the same lines, replacing S^3 with a generic 3-manifold and engineering the geometric transition in such a case. Indeed, one expects on general grounds the duality to hold for Chern-Simons theories on rational homology spheres, but actually very few examples beyond S^3 are known:

Open problem. *Find when the Reshetikin-Turaev-Witten invariant of a compact smooth 3-manifold M is equal, in suitable coordinates, to the all genus Gromov-Witten potential of an algebraic threefold X_M obtained by geometric transition from T^*M: that is, X_M is given by a complex deformation of T^*M to a normal variety, followed by a birational resolution.*

This issue has been addressed in the case of $\mathbb{Z}_p \subset SU(2)$ cyclic quotients of S^3:

Proposition 13 ([87]). *Let $M = L(p, 1)$. The Hori-Vafa mirror curve and differential of the family of $CY3$ obtained by geometric transition from T^*M coincide with the spectral curve and resolvent of the $L(p, 1) = M$ Chern-Simons matrix model [117].*

This has been achieved through a detailed study of the random matrix representation for the Chern-Simons path integral, originally obtained by Mariño in [117]. An early confirmation of the above assertion was provided in [6] for $p = 2$, by matching the 't Hooft expansion of the Chern-Simons 2-matrix model with the solutions of the Picard-Fuchs system at the orbifold point of $K_{\mathbb{P}^1\times\mathbb{P}^1}$. Notice that, assuming [1] the validity of local mirror symmetry for toric $CY3$, Proposition 13 implies GV duality for $L(p, 1)$ lens spaces at least in genus zero; actually, the proposal of [24]and the announced proof of [65] would imply that the spectral data contain the full structure of the B-model on toric $CY3$ at all genera, in which case the work [87] would become automatically an all-genus proof.

However, it has been suspected [16, §7.3] that one could hardly go further along this direction for generic M. We make the following

Claim 1. *Proposition 13 is false when $M = L(p, 1)$ is replaced by the generic lens space $M = L(p, q)$ for $q > 1$.*

In this chapter we will perform a detailed analysis of the whole family $M = L(p, q)$, thus including cyclic subgroups not contained in $SU(2)$, along the lines of [87] The case $q > 1$, in which the lens space is not a $U(1)$ bundle on S^2, appears to be much harder as most of the features of the $q = 1$ case, like the mirror realization of [6] become either unclear or simply are not there. To this aim, in section 4.2 we will work out explicitly the conifold transition for the case at hand and obtain a class of would-be large N duals $\widehat{\mathcal{X}_{p,q}}$ as well as their Hori-Vafa mirror curves, correcting *en passant* a few claims in the literature [16] about the impossibility of performing such a geometric transition in the $L(p, q)$ case preserving at the same time the CY condition; in section 4.3 we will present a matrix integral representation for the Chern-Simons $U(N)$ partition function on $L(p, q)$ in a fixed flat connection background and consider its large N expansion as governed by a spectral curve and a resolvent. For $q > 1$, we will find disagreement with the B-model spectral data obtained after geometric transition from $T^*L(p, q)$.

[1]See [58, 81, 112] for rigorous mathematical proofs.

4.2 The closed string side: conifold transition for $T^*L(p,q)$

4.2.1 Geometric transition

According to Proposition 11 and 12, the GV duality for the case of the generic $L(p,q)$ lens space should be realized in two steps:

1. a complex deformation of $\widehat{\mathcal{X}_{p,q}} \equiv T^*L(p,q)$ to a normal variety $\mathcal{X}_{p,q}$ (a suitable $\mathbb{Z}_p$ quotient of the singular conifold);
2. a complete crepant resolution $\overline{\mathcal{X}_{p,q}}$ of the latter.

The first step is realized as follows: let us recall that, from [84, Theorem 1.6], the cotangent bundle to the 3-sphere T^*S^3 is diffeomorphic to a smooth hypersurface in $\mathbb{A}^4$

$$xy - zt = \mu \tag{4.2.1}$$

which is a complex structure deformation of a conifold singularity. The base S^3 is the real locus $y = \bar{x}$, $t = -\bar{z}$

$$|x|^2 + |z|^2 = \mu \tag{4.2.2}$$

Now, consider the $\mathbb{Z}_p$ action

$$\begin{array}{cccc} \mathbb{Z}_p \times & \mathbb{C}^4 & \to & \mathbb{C}^4 \\ \omega & (x,y,z,t) & \to & (\omega x, \omega^{-1} y, \omega^q z, \omega^{-q} t) \end{array} \tag{4.2.3}$$

where $\omega = e^{2\pi i/p}$, $1 \leq q < p$, $(p,q) = 1$; the orbit manifold restricted to (4.2.2) is a $L(p,q)$ lens space. At first sight, using the same coordinatization as [84], the cyclic group acts both on the fibers and on the base of T^*S^3, thus yielding something a priori different from an $\mathbb{R}^3$-bundle over $L(p,q)$. However we have the following simple

Lemma 2. *The orbit space of (4.2.3) restricted to (4.2.1) is smoothly diffeomorphic to $T^*L(p,q)$.*

Proof. Introduce the new set of variables $w_i = q_i + ip_i$

$$\begin{array}{lll} w_1 = (z_1 + z_3)/2 & & w_2 = i(-z_1 + z_3)/2 \\ w_3 = (z_2 + z_4)/2 & & w_4 = i(-z_2 + z_4)/2 \end{array} \tag{4.2.4}$$

In this coordinates, (4.2.1) is the locus in $\mathbb{R}^8$ described by

$$\sum_{j=1}^4 q_j^2 - p_j^2 = \mu, \qquad \sum_{j=1}^4 q_j p_j = 0 \tag{4.2.5}$$

Consider then the change of variables

$$\begin{array}{ll} \tilde{p}_1 = q_1 p_1 + q_2 p_2, & \tilde{p}_2 = q_1 p_2 - q_2 p_1 \\ \tilde{p}_3 = q_3 p_3 + q_4 p_4, & \tilde{p}_4 = q_3 p_4 - q_4 p_3 \end{array} \tag{4.2.6}$$

and $\tilde{q}_i = q_i$, $i = 1, \ldots, 4$. The change of variables for $\mu > 0$ is nonsingular everywhere in the set defined by (4.2.5), which is then rewritten as

$$\sum_{i=1}^{4} \tilde{q}_i^2 = \mu, \qquad \tilde{p}_1 + \tilde{p}_3 = 0 \tag{4.2.7}$$

The $\mathbb{Z}_p$ action is now represented on the tilded $\mathbb{R}^8$ in the form:

$$\begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \\ \tilde{q}_4 \end{pmatrix} \to \begin{pmatrix} \cos 2\pi/p & \sin 2\pi/p & 0 & 0 \\ -\sin 2\pi/p & \cos 2\pi/p & 0 & 0 \\ 0 & 0 & \cos 2\pi q/p & \sin 2\pi q/p \\ 0 & 0 & -\sin 2\pi q/p & \cos 2\pi q/p \end{pmatrix} \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \\ \tilde{q}_4 \end{pmatrix} \tag{4.2.8}$$

$$\begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} \to \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} \tag{4.2.9}$$

realizing therefore the $\mathbb{Z}_p$ quotient (4.2.3) of the deformed conifold as a trivial $\mathbb{R}^3$-bundle over $L(p,q)$. By Stiefel's theorem [141], the latter being an orientable three-manifold, there exists a (strong) C^∞ bundle isomorphism mapping $\mathbb{R}^3 \times L(p,q)$ to $T^*L(p,q) =: \widehat{\mathcal{X}_{p,q}}$.

□

With the algebraic realization (4.2.1), (4.2.3) of $\widehat{\mathcal{X}_{p,q}}$ at hand it is straightforward to perform the second step of the transition. As in the S^3 case, the μ parameter measures the size of the lens space and sending μ to zero amounts to deforming $\widehat{\mathcal{X}_{p,q}}$ to the singular variety $\mathcal{X}_{p,q}$, where the Lagrangian null section $L(p,q)$ has shrunk to zero size. We have the following

Theorem 14. *The singular variety $\mathcal{X}_{p,q}$, obtained as the orbit space of (4.2.3) inside (4.2.1) with $\mu = 0$, is a toric variety with trivial canonical sheaf, $K_{\mathcal{X}_{p,q}} \simeq \mathcal{O}_{\mathcal{X}_{p,q}}$.*

Proof. For $q = 1$ the theorem was proven in [87], where the authors exploited the fact that $\mathcal{X}_{p,1}$ is obtained from the resolved conifold (a rank 2 bundle over S^2 as in proposition 12) by quotienting a fiberwise-acting $\mathbb{Z}_p$ group and "blowing-down" the base S^2. For $q > 1$, though, the $\mathbb{Z}_p$ group does no longer act fiberwise and we have to deal with it in a different way.

By definition , we have to prove that $\mathcal{X}_{p,q}$ contains an algebraic three-torus as an open subset effectively acting through an extension of its obvious action on itself. This is identified as follows: the singular conifold $\mathcal{X}$, as an affine variety

$$\mathcal{X} := \mathrm{Spec}\frac{\mathbb{C}[x,y,z,t]}{\{xy - zt\}}$$

is toric with torus action given by

$$\begin{array}{rcl} (\mathbb{C}^*)^3 & \stackrel{j}{\hookrightarrow} & \mathcal{X} \\ (t_1,t_2,t_3) & \to & (t_1,t_2,t_3,t_1t_2t_3^{-1}) \end{array} \tag{4.2.10}$$

This action descends to an action on the orbifolded conifold $\mathcal{X}_{p,q}$ by (4.2.3)

$$(\mathbb{C}^*)^3/\mathbb{Z}_p \stackrel{\tilde{j}}{\hookrightarrow} \mathcal{X}_{p,q} \tag{4.2.11}$$

Proving that $\mathcal{X}_{p,q}$ is toric therefore amounts to find explicitly an isomorphism π : $(\mathbb{C}^*)^3/\mathbb{Z}_p \mapsto (\mathbb{C}^*)^3$

$$0 \to \mathbb{Z}_p \xrightarrow{i} (\mathbb{C}^*)^3 \xrightarrow{\pi} (\mathbb{C}^*)^3 \to 0 \tag{4.2.12}$$

where the injection i is dictated by (4.2.3) to be

$$\begin{array}{rrcl} i: & \mathbb{Z}_p & \hookrightarrow & (\mathbb{C}^*)^3 \\ & \omega & \mapsto & (\omega,\omega^{-1},\omega^q) \end{array} \tag{4.2.13}$$

and by (4.2.12) we can write for π

$$\begin{array}{rrcl} \pi: & (\mathbb{C}^*)^3 & \hookrightarrow & (\mathbb{C}^*)^3 \\ & (t_1,t_2,t_3) & \mapsto & (t_1^p, t_1t_2, t_1^q t_3^{-1}) \end{array} \tag{4.2.14}$$

The three-torus inside the quotient of the conifold by the action (4.2.3) is then identified by

$$\begin{array}{rcl} (\mathbb{C}^*)^3 & \stackrel{\tilde{j}\circ\pi^{-1}}{\hookrightarrow} & \mathcal{X}_{p,q} \\ (t_1,t_2,t_3) & \to & (t_1^{1/p}, t_1^{-1/p}t_2, t_1^{q/p}t_3^{-1}, t_1^{-q/p}t_2t_3) \end{array} \tag{4.2.15}$$

From (4.2.15) we can read off the dual cone as the real tetrahedron spanned by

$$a_1 = \begin{pmatrix} 1/p \\ 0 \\ 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} -1/p \\ 1 \\ 0 \end{pmatrix} \quad a_3 = \begin{pmatrix} q/p \\ 0 \\ -1 \end{pmatrix} \quad a_4 = \begin{pmatrix} -q/p \\ 1 \\ 1 \end{pmatrix} \tag{4.2.16}$$

The fan is then obtained by taking the inward pointing normal to each facet, normalized in such a way to hit the first point on the $\mathbb{Z}^3$ lattice. Modulo an automorphism of the lattice, we thus get that the rays of the fan of $\mathcal{X}_{p,q}$ are given by

$$b_1 = \begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix} \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ q+1 \end{pmatrix} \quad b_3 = \begin{pmatrix} p \\ 1 \\ 1 \end{pmatrix} \quad b_4 = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} \tag{4.2.17}$$

and the fan consists of a single cone generated by b_i, $i = 1 \dots 4$. Notice that the tip of the rays all lie in an an affine hyperplane, namely $y = 1$, thus implying triviality of the canonical class [71]. The theorem is proved.

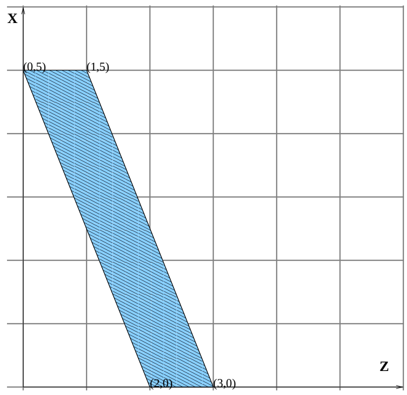

Figure 4.1: The toric diagram of $\mathcal{X}_{p,q}$ for $p = 5$, $q = 2$.

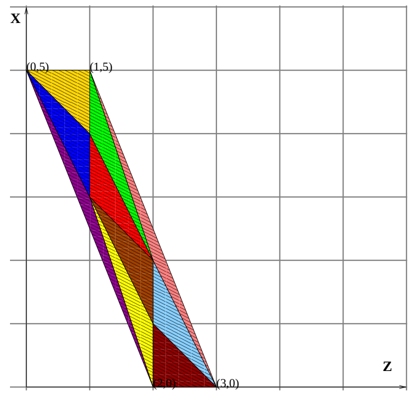

Figure 4.2: The toric diagram of $\overline{\mathcal{X}_{p,q}}$ for $p = 5$, $q = 2$

$\square$

Remark 13. *It is instructive to point out an interesting new geometrical fact in the $1 < q < p - 1$ case. Let us consider the orbifold of the resolved conifold geometry $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by the $\mathbb{Z}_p$ action (4.2.3), which corresponds to a partial resolution of $\mathcal{X}_{p,q}$. This can be described as an orbi-bundle fibration of a Hirzebruch-Jung singularity over a $\mathbb{P}^1$ with two marked points with $\mathbb{Z}_p$-monodromy. One way to do that is to realize the projectivization of the resolved conifold as a subspace $\{[z_0, z_1, z_2], [z_3, z_4, z_5], [r, s] \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 | z_1 r = z_2 s, z_3 r = z_4 s\}$. The $\mathbb{Z}_p$ action on the above variables is inferred from (4.2.3) via the identification $(z_1, z_2, z_3, z_4) = (x, -z, t, -y)$ and imposing invariance of the relations, which gives $(r, s) \to (\omega^{q-1} r, s)$. The fiber over the north pole $r = 0, s = 1$ is parametrized by (z_1, z_3), which describe precisely a Hirzebruch-Jung singularity. The analogue of the above is valid for the fiber over the south pole.*

All we are left to do to complete step 2 is to take a complete resolution $\overline{\mathcal{X}_{p,q}}$ of $\mathcal{X}_{p,q}$

$$\overline{\mathcal{X}_{p,q}} \overset{r}{\dashrightarrow} \mathcal{X}_{p,q}$$

Since $\mathcal{X}_{p,q}$ is Gorenstein, the birational morphism r can be taken to preserve the condition of $\overline{\mathcal{X}_{p,q}}$ being both toric and Calabi-Yau, i.e. to be a crepant toric resolution. We can realize this diagrammatically [71] by adding all the interior lattice vectors inside the tetrahedron spanned by b_i, and declaring that the (top-dimensional part of the) fan of $\overline{\mathcal{X}_{p,q}}$ is made by the cones constructed above the 2-simplices which triangulate the projection of the fan onto the $y = 1$ plane. Notice that the latter is a parallelogram with shorter sides of length 1 (see figure 4.1). This means that the intersection of the parallelogram with the horizontal lines $x = 1, \dots, p$ contains either one point of the lattice in the interior or two points on the diagonal edges, the latter possibility being excluded by the coprimality condition $(p, q) = 1$. Thus the number of points of the lattice (apart from the 4 external vertices) in the interior of the parallelogram is precisely $p-1$. As is clear from the picture, these points have the form (on the plane) $([q+1-jq/p],j) = (q-[jq/p],j)$, where square brackets denote the integer part of the argument.

It is straightforward to get a complete crepant resolution of the orbifold by taking a triangulation of the $p+3$ points

$$v_1 \equiv \begin{pmatrix} q+1 \\ 0 \end{pmatrix} \quad v_2 \equiv \begin{pmatrix} q \\ 0 \end{pmatrix} \quad v_{p+3} \equiv \begin{pmatrix} 1 \\ p \end{pmatrix}$$

$$v_{j+2} \equiv \begin{pmatrix} q-[jq/p] \\ j \end{pmatrix}, \qquad j=1,\ldots,p \tag{4.2.18}$$

Definition 13. *We will call $\overline{\mathcal{X}_{p,q}}$ the toric variety defined (modulo flops) by a fan supported by the rays*

$$b_i \equiv \begin{pmatrix} v_i \\ 1 \end{pmatrix} \tag{4.2.19}$$

and whose 3-dimensional cones are defined by having their intersection with the $z=1$ hyperlane coincide with the simplices of a complete triangulation of the convex hull of (4.2.19).

By construction $\overline{\mathcal{X}_{p,q}}$ is a simplicial, smooth[2] toric CY three-fold which is birationally isomorphic to $\mathcal{X}_{p,q}$. Step 2 is completed.

□

The toric data (4.2.19) allows us to extract some useful information on the geometry of $\overline{\mathcal{X}_{p,q}}$. First of all, since internal vertices are in 1-to-1 correspondence with linear equivalence classes of (compact) divisors of $\overline{\mathcal{X}_{p,q}}$, we have that the fourth Betti number is

$$b_4(\overline{\mathcal{X}_{p,q}}) = p-1$$

for every q. Moreover, given that the Euler characteristic $\chi(\overline{\mathcal{X}_{p,q}})$ is simply given by twice the area of the base of the tetrahedron and that odd Betti numbers vanish, we can easily compute the dimension of the second cohomology group as

$$b_2(\overline{\mathcal{X}_{p,q}}) = \chi(\overline{\mathcal{X}_{p,q}}) - b_0(\overline{\mathcal{X}_{p,q}}) - b_4(\overline{\mathcal{X}_{p,q}}) = 2p-1-(p-1) = p \tag{4.2.20}$$

[2]A triangulation of the $p+3$ points (4.2.18) realizes the projection of our cone onto the plane $y=1$ as the disjoint union of precisely $2p$ triangles. The fact that the number of triangles is $2p$ is a consequence of Euler's formula: denoting with m the number of triangles in a triangulation of (4.2.18), since each triangle has 3 edges and the convex hull has 4, the number of edges is $(3m+4)/2$, due to the fact that each edge is incident to exactly two faces. Plugging all the ingredients (number of points and edges) into Euler's formula, it follows that the number of triangles is exactly $2p$. Now, each triangle has half-integer area since each vertex is a site of the lattice, and the area is then given by half the determinant of an integer matrix. But p is the area of the whole parallelogram, so having $2p$ triangles implies that each triangle must have area $1/2$. The cones which project onto those triangles are then simplicial and smooth (i.e. each triple of vectors spanning a cone in the fan of the resolution is an integer basis of the lattice), which is precisely the non-singularity condition for a toric variety.

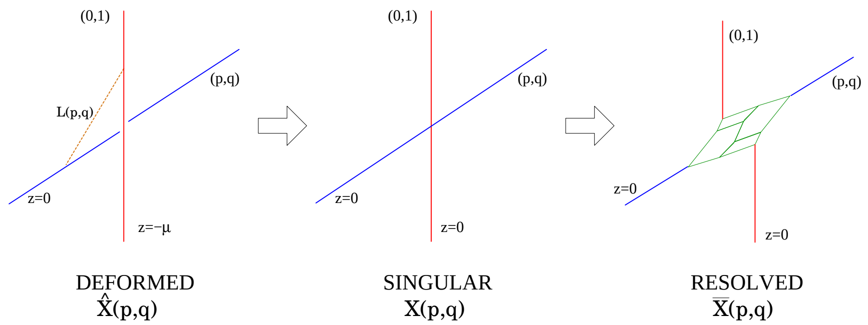

Figure 4.3: A pictorial representation of the geometric transition for Lens spaces $L(p,q)$ as $\mathbb{T}^2$ fibrations.

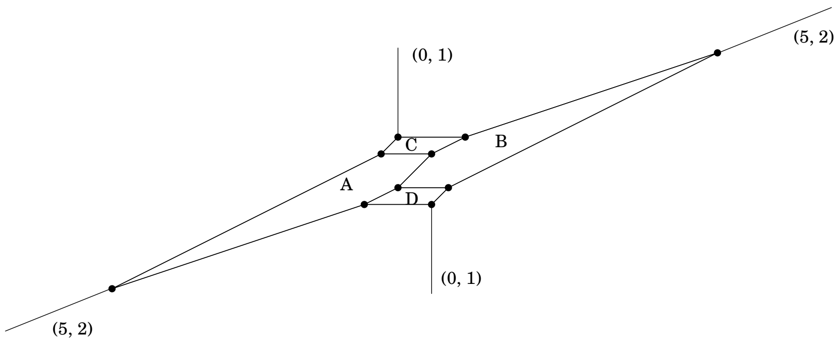

Figure 4.4: The pq-web diagram for the resolution of the orbifold $p=5$, $q=2$. A, B and C, D represent two dP_2 and two $\mathbb{F}_1$ surfaces respectively.

for every q. This is expected: the dimension of the Kähler moduli space of $\overline{\mathcal{X}_{p,q}}$ should match the number of inequivalent flat connections of the CS $SU(N)$ theory on $L(p,q)$, which is $\pi_1(L(p,q)) = p$.
It is however remarkable, as it is also apparent from figure 4.2 and 4.4, that the intersection structure of $\overline{\mathcal{X}_{p,q}}$ for $q>1$ is significantly more complicated than the simple case $q=1$. Instead of the simple ladder diagrams describing the pq-webs of the A_{p-1} geometries, which were built out of a single tower of (nef) Hirzebruch surfaces, the compact divisors here are generic toric Fano surfaces and intersect in a wildly more intricate way, due to the fact that the vertices of the rays of the fan are no-longer tetravalent and vertically aligned.

4.2.2 Mirror symmetry

From the toric data (4.2.19) we can straightforwardly write down the Hori-Vafa mirror curve to $\overline{\mathcal{X}_{p,q}}$. We have the following

Proposition 15. *The B-model target space mirror to a toric CY three-fold X is the* *hypersurface in* $\mathbb{C}^2(x_1,x_2)\times(\mathbb{C}^*)^2(U,V)$

$$x_1x_2 = P_X(U,V)$$

where

$$P_{p,q}(U,V) = (U^pV^q-1)(V-1) + d_p + \sum_{j=1}^{p-1} d_j U^j V^{q-[(p-j)q/p]} \tag{4.2.21}$$

When $d_j=0$, corresponding to the singular $\mathcal{X}_{p,q}$, this form for the mirror curve had already been suggested by [6]. Notice, from (4.2.18), that there is no $GL(3,\mathbb{Z})$ transformation sending the points in $\mathcal{F}_{\overline{\mathcal{X}_{p,q}}}$ into a strip of horizontal width less than 3 for $1<q<p-1$. Moreover, by (4.2.3), the fan of $\overline{\mathcal{X}_{p,q}}$ and $\overline{\mathcal{X}_{p,p-q}}$ are related by an automorphism of the lattice, thus yielding isomorphic toric varieties. Collecting it all together we have proven the following

Proposition 16. *The Hori-Vafa mirror curve and* 1*–differential are given by*

$$\Gamma^{HV}_{p,q} : P_{p,q}(e^u,e^v)=0, \qquad d\lambda_{p,q} = u dv$$

where $P_{p,q}$ *is given by (4.2.21) and* $u=\log U$,$v=\log V\in\mathbb{R}\times S^1$. *The curve has 4 punctures and genus* $p-1$ *for all* q*, and its periods have a symmetry given by* $q\to p-q$. *For* $1<q<p-1$, $\Gamma^{HV}_{p,q}$ *is not hyperelliptic.*

□

4.3 The open string side: CS theory on $L(p,q)$ and matrix models

The goal of this section is to provide a suitable matrix model representation of the partition function of Chern-Simons theory on a $L(p,q)$ lens space in a given vacuum. This case has been already considered in [117], where a general matrix integral representation for the partition function of Chern-Simons theory on Seifert homology spheres has been derived (see also [72] and [48]). Here we find a slightly different, but equivalent, representation more useful for our purposes.

More precisely, let us consider $U(N)$ Chern-Simons theory at level $k\in\mathbb{Z}$ on a $L(p,q)$ lens space ($1\le q<p$, p and q coprime) and index with $\mathbf{m}\in\mathbb{Z}_p^N$ the set of $U(N)$-flat connections on $L(p,q)$. We will denote the corresponding partition function, or Reshetikin-Turaev-Witten invariant, as $Z^{L(p,q)}_{U(N)}\big(k,\mathbf{m}\big)$. We have the following

Theorem 17 (Hansen-Takata, [88])**.** *The Chern-Simons partition function for a* $L(p,q)$ *lens space, gauge group* $U(N)$*, level* k *and a fixed choice of flat connection* $\mathbf{m}$ *is given by*

$$Z^{L(p,q)}_{U(N)}\big(k,\mathbf{m}\big) = C_N(p,q;g_s)\, e^{-\frac{4\pi^2 q}{g_s^2 p}\mathbf{m}^2} \sum_{\tilde\omega,\omega\in S_N} \varepsilon(\omega)\, e^{\frac{g_s^2}{2p}\omega(\rho)\cdot\rho}\, e^{\frac{2\pi\imath}{p}\tilde\omega(\mathbf{m})\cdot(q\rho+\omega(\rho))} \tag{4.3.1}$$

where $g_s^2 = \frac{4\pi i}{k+N} = \frac{4\pi i}{\hat{k}}$, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the Weyl vector of $SU(N)$, S_N is the permutation group of N elements and $C_N(p,q;g_s)$ is a fixed overall factor, not depending on the particular flat connection (the exact expression of $C_N(p,q;g_s)$ is given in [88] and does not play any role here).

To obtain a matrix model representation it is useful to observe that this expression, up to an overall normalization factor, can be also written as

$$Z_{U(N)}^{L(p,q)}\big(k,\mathbf{m}\big) = \sum_{\tilde{\omega},\omega\in S_N} \varepsilon(\omega)\; e^{\frac{1}{4g_s^2 p}\left(g_s^2(\omega(\rho)+\rho)+4i\pi\tilde{\omega}(\mathbf{m})\right)^2+2\pi i\frac{(q-1)}{p}\tilde{\omega}(\mathbf{m})\cdot\rho}. \quad (4.3.2)$$

By exploiting a trivial integral representation of the gaussian function, we can rewrite the above partition function as an integral

$$\begin{aligned} Z_{U(N)}^{L(p,q)}\big(k,\mathbf{m}\big) &= \int_{-\infty}^{\infty} d^N x \sum_{\tilde{\omega},\omega\in S_N}\varepsilon(\omega)\; e^{i\hat{k}p\pi(x\cdot x)+2\pi\left(\omega(\rho)+\rho+\hat{k}\;\tilde{\omega}(\mathbf{m})\right)\cdot x+2\pi i\frac{(q-1)}{p}\tilde{\omega}(m)\cdot\rho} \\ &= \int_{-\infty}^{\infty} d^N x \sum_{\tilde{\omega},\omega\in S_N}\varepsilon(\omega)\; e^{i\hat{k}p\pi(x\cdot x)+2\pi\left(\tilde{\omega}^{-1}(\omega(\rho))+\tilde{\omega}^{-1}(\rho)+\hat{k}\;\mathbf{m}\right)\cdot\tilde{\omega}^{-1}(x)+2\pi i\frac{(q-1)}{p}\mathbf{m}\cdot\tilde{\omega}^{-1}(\rho)}. \end{aligned} \quad (4.3.3)$$

Since the measure of integration and $(x\cdot x)$ are symmetric under permutations, the partition function can be rearranged as

$$Z_{U(N)}^{L(p,q)}\big(k,\mathbf{m}\big) = \int_{-\infty}^{\infty} d^N x \sum_{\tilde{\omega},\omega\in S_N}\varepsilon(\omega)\varepsilon(\omega')\; e^{i\hat{k}p\pi(x\cdot x)+2\pi\left(\omega(\rho)+\tilde{\omega}'(\rho)+\hat{k}\;\mathbf{m}\right)\cdot x+2\pi i\frac{(q-1)}{p}\mathbf{m}\cdot\tilde{\omega}'(\rho)}. \quad (4.3.4)$$

To perform the sum over ω and ω', it is sufficient to recall the Weyl-formula

$$\sum_{\omega\in S_N}\varepsilon(\omega)e^{i(\phi\cdot\omega(\rho))} = \prod_{\alpha>0} 2\sin\left(\frac{\alpha\cdot\phi}{2}\right), \quad (4.3.5)$$

and we thus get, restoring $g_s = \frac{4\pi i}{\hat{k}}$,

$$\begin{aligned} &Z_{U(N)}^{L(p,q)}\big(k,\mathbf{m}\big) = \\ &= \int_{-\infty}^{\infty} d^N x\;\; e^{i\hat{k}p\pi(x\cdot x)+2\pi\hat{k}\;\mathbf{m}\cdot x}\prod_{\alpha>0}\sinh\left(\pi\alpha\cdot x\right)\sinh\left(\pi\alpha\cdot\left(x+i\frac{(q-1)}{p}\mathbf{m}\right)\right) \\ &= \int_{-\infty}^{\infty} d^N x\;\; e^{-g_s p(x\cdot x)+4\pi i\;\mathbf{m}\cdot x}\prod_{i<j}\sinh\left(\Delta_{ij}+\frac{i\pi(q-1)}{p}(\mathbf{m}_i-\mathbf{m}_j)\right)\sinh\left(\Delta_{ij}\right), \end{aligned} \quad (4.3.6)$$

where $\Delta_{ij} \equiv \frac{g_s}{2}(x_i - x_j)$. All the equalities hold up to irrelevant multiplicative constant factors and we have

Theorem 18. *The partition function of Chern-Simons theory on a $L(p,q)$ lens space for a choice* $\mathbf{m}$ *of flat connection can be written as a multi-eigenvalue integral as*

$$\begin{aligned} Z^{L(p,q)}_{U(N)}(k,\mathbf{m}) &= \int \prod_{I=1}^{p} d^{N_I} u_k^{(I)} e^{-\sum_{j=1}^{N} u_j^2 \frac{p}{2g_s}} \prod_{i<j} \sinh\left(\hat\Delta_{ij}^{(I)}\right) \sinh\left(\hat\Delta_{ij}^{(I)}\right) \\ & \prod_{I<J}\prod_{i<j} \sinh\left(\hat\Delta_{ij}^{(IJ)} + \frac{\pi i(I-J)}{p}\right) \sinh\left(\hat\Delta_{ij}^{(IJ)} + q\frac{\pi i(I-J)}{p}\right) \end{aligned} \tag{4.3.7}$$

where $u_i^I \in \mathbb{R}$, $I = 1,\ldots,p$, $i = 1,\ldots,N_I$ *and we have defined* $\hat\Delta_{ij}^{(I)} \equiv \frac{1}{2}\left(u_i^{(I)} - u_j^{(I)}\right)$, $\hat\Delta_{ij}^{(IJ)} \equiv \frac{1}{2}\left(u_i^{(I)} - u_j^{(J)}\right)$.

In (4.3.7) we have eventually rescaled g_s by a factor of two in order to make contact with the notation of [6, 87], to which it reduces in the case $q = 1$ and discarded a constant in front of the final matrix integral. This representation is of course equivalent to the one found in [117], up to an overall multiplicative constant.

Remark 14. *At this stage we can already spot a few signals of the fact that GV duality could break down for $q > 1$. Indeed, two $L(p,q)$ and $L(p',q')$ lens spaces are homeomorphic if and only if $p = p'$ and $q = \pm q' \pmod p$ or $qq' = \pm 1 \pmod p$: the related partition functions are topological invariants and should thus be equal. This can be verified explicitly when the sum over the flat connections is performed and the Chern-Simons level is correctly quantized [88], but the same property does not seem to show up for the partition function in the background of a fixed flat connection: as one can easily check by explicit examples, different flat connection sectors are mixed under the relevant transformations. On the other hand, as pointed out in Proposition 16, $q = \pm q' \pmod p$ is instead a symmetry of the closed string background described in the the previous section. Therefore it is expected that the spectral data (4.2.21) will be different from what we will extract from the large N analysis of (4.3.7).*

4.3.1 Large N limit of the CS matrix model

We now would like to prove that, as in the case of hermitian matrix models, the eigenvalue integral (4.3.7) is governed by a pair $(\Gamma^{CS}_{p,q}, dR_{p,q})$ made up of a spectral curve $\Gamma^{CS}_{p,q}$ and a resolvent $dR_{p,q}$, out of which the genus zero free energy is extracted by the usual relations of special geometry. Proving spectral equivalence as in proposition 13 amounts then to finding an isomorphism of curves ϕ such that

$$\begin{array}{rccl} \phi: & \Gamma^{HV}_{p,q} & \mapsto & \Gamma^{CS}_{p,q} \quad \text{isomorphism} \\ & (\phi^{-1})^* d\lambda_{p,q} & = & dR_{p,q} \end{array} \tag{4.3.8}$$

Let us first introduce some basic objects in the discussion of the large N limit.

Definition 14. *Let $N \in \mathbb{N}_0$, $I = 1,\ldots,p$ and $x \in \mathbb{R}$. For every I, the sequence of tempered distributions $\rho_N^{(I)} \in S'(\mathbb{R})$*

$$\rho_N^{(I)}(x) := \frac{1}{N}\sum_{i=1}^{N} \delta(x - u_i^{(I)})$$

will be called I$^{\text{th}}$ eigenvalue density at rank N. *Their integral on the real line gives the relative fraction of eigenvalues* (**filling fraction**) *in the I^{th} group*

$$\int_{\mathbb{R}} \rho_N(x)dx = \frac{N_I}{N} \tag{4.3.9}$$

We then make the following basic

Assumption. *We* **assume** *that the $N \to \infty$ distributional limit*

$$\rho^{(I)}(x) := \lim_{N\to\infty} \rho_N^{(I)}(x)$$

is a compactly supported continous function on the real line, $\rho^{(I)} \in \mathcal{C}_c^0(\mathbb{R})$, $\forall I$.

This assumption is motivated by the analogous situation for hermitian matrix ensembles as well as for the CS matrix models of [87], and we will prove that it is self-consistent. It will be useful in the following to denote with $t \equiv g_s N$ the total 't Hooft coupling and with S_I the large N limit of the filling fractions

$$S_I := t \lim_{N\to\infty} \frac{N_I}{N}$$

normalized so that $\sum_I S_I = t$.

We will now construct explicitly the spectral curve $\Gamma_{p,q}^{CS}$ and differential $dR_{p,q}$ emerging from the large N study of (4.3.7). As usual in random matrix theory, this will be accomplished by finding an implicit algebraic expression $P_{p,q}(u,v)$ for the force $v(u)$ on a probe eigenvalue u at large N, in terms of which we will define

$$\begin{aligned} \Gamma_{p,q}^{CS} &:= \{(u,v) \in (\mathbb{R}\times S^1)\times(\mathbb{R}\times S^1) | P_{p,q}(u,v) = 0\} \\ dR_{p,q} &:= v(u)du \end{aligned} \tag{4.3.10}$$

At large N, $v(u)$ will have cuts in the complex plane whose discontinuity yields the individual eigenvalue densities ρ^I; its regularized integral from infinity to the I^{th} cut will instead measure, by construction, the variation of the (leading order) free energy with respect to the I^{th} filling fraction. This is summarised by the *special geometry* relations

$$\oint_{A_I} dR_{p,q} = S_I \qquad \oint_{B_I} dR_{p,q} = \frac{\partial \mathcal{F}}{\partial S_I} \tag{4.3.11}$$

We will now show, from the explicit form of $(\Gamma_{p,q}^{CS}, dR_{p,q})$, that no such a ϕ as in (4.3.8) does in fact exist for $1 < q < p-1$.

Proof of Claim 1. As is customary for CS multi-matrix models, the steepest descent (saddle-point) method to evaluate (4.3.7) at large N yields a singular integral equation for $\rho^{(I)}$ with q-dependent hyperbolic kernels. From (4.3.7) we can straightforwardly write the saddle point equation as

$$\begin{aligned} p\lambda_I &= t \fint \coth\left(\frac{\lambda_I - \lambda_I'}{2}\right)\rho_I(\lambda_I')d\lambda_I' \\ &+ \frac{t}{2}\sum_{J\neq I}\int_{\mathbb{R}}\left[\coth\left(\frac{\lambda_I-\lambda_J}{2}+d_{IJ}\right)+\coth\left(\frac{\lambda_I-\lambda_J}{2}+qd_{IJ}\right)\right]\rho_J(\lambda_J)d\lambda_J \end{aligned} \tag{4.3.12}$$

where $d_{IJ} := i\pi(I-J)/p$ and the slashed integral indicates the Cauchy principal value ("improper") integral.
Now let us define the following set of *resolvents*

$$\omega_I(z) \equiv t\int_{\mathbb{R}}\coth\left(\frac{z-\lambda_I}{2}\right)\rho_I(\lambda_I)d\lambda_I, \tag{4.3.13}$$

$$\omega(z) = \frac{1}{2}\sum_{I=1}^{p}\left[\omega_I\left(z-2\pi i\frac{I}{p}\right)+\omega_I\left(z-2\pi i\frac{qI}{p}\right)\right], \tag{4.3.14}$$

We will need the following easy generalization of the Sokhotski-Plemelij lemma

Lemma 3. *Define the following limits in $S'(\mathbb{R})$*

$$\coth_{\pm}(z) := \lim_{\epsilon\to 0}\coth\left(z\pm i\epsilon\right).$$

Then the following identities in $S'(\mathbb{R})$ hold true:

$$\coth_+ + \coth_- = 2\mathrm{pv}(\coth), \tag{4.3.15}$$

$$\coth_+ - \coth_- = -2\pi i\delta. \tag{4.3.16}$$

For notational purposes, we define accordingly $\omega_{\pm}(z) \equiv \lim_{\epsilon\to 0}\omega(z+i\epsilon)$.

Given that the eigenvalue densities are supported on the real axis, we conclude immediately from (4.3.13) and (4.3.16) that the individual resolvents $\omega_I(z)$ have branch cuts which coincide with[3] $\mathrm{supp}(\rho_i) = [-a_I, a_I]$ for some $a_I \in \mathbb{R}$. This implies that $\omega(z)$, as a function from the cylinder $0 \le \Im m z < 2\pi$ to the Riemann sphere, has p cuts centered at $z = 2\pi i I/p$, whose width as usual depends on a choice of filling fractions S_I. Explicitly, from (4.3.14) we have for $J = 0, \dots, p-1$

$$2\omega\left(z+\frac{2\pi i J}{p}\right) = \omega_J\left(z\right)+\omega_{\hat{J}}(z)+\sum_{I\neq J}\omega_I\left(z-\frac{2\pi i(I-J)}{p}\right)+\sum_{I\neq\hat{J}}\omega_I\left(z-\frac{2\pi i(qI-J)}{p}\right) \tag{4.3.17}$$

[3]The eigenvalue integral is parity invariant, which therefore implies a $\mathbb{Z}_2$ symmetry in the location of the branch points.

where $\hat{J}$ is defined by $q\hat{J} = J \bmod p$. When $I = 0$ we have $\hat{I} = 0$, and for $x \in [-a_0, a_0]$ we get that

$$\begin{array}{rcl} \frac{\omega_+(x)+\omega_-(x)}{2} & = & t \fint \coth\left(\frac{x-\lambda'_I}{2}\right)\rho_0(x')dx' \\ & + & \frac{t}{2}\sum_{J\neq 0}\int_{\mathbb{R}}\left[\coth\left(\frac{x-x'}{2}+d_{pJ}\right)+\coth\left(\frac{x-x'}{2}+qd_{pJ}\right)\right]\rho_J(x')dx' \\ & = & px \end{array} \tag{4.3.18}$$

due to (4.3.12) and lemma 3. However, a quick inspection shows that for no other $0 < I < p$ it is possible to find a closed expression for the average of the resolvent on the I^{th} cut. Indeed, since $I \neq \hat{I}$ for $0 < I < p$, different individual resolvents become singular at $x + 2\pi i I/p$ inside the total sum (4.3.14), namely ω_I and $\omega_{\hat{I}} \neq \omega_I$, and it appears to be very intricate to infer the structure of $\omega(z)$ from (4.3.12). However, let us restrict ourselves for the moment to the special case in which

$$\rho_I = \rho_1 \qquad 1 < I < p \tag{4.3.19}$$

This corresponds to a particular symmetric choice of filling fractions, that is one in which a fraction of S_0 eigenvalues have been put on the cut on the real axis, corresponding to the trivial Chern-Simons connection, and $S_I = (t - S_0)/(p-1)$ for $I > 0$ are democratically distributed between the non-trivial flat connections. This would amount to explore a peculiar codimension $p-2$ subspace in the space of 't Hooft parameters, for which the large N data can be described in complete detail. In particular, this would give a 2-parameter closed subset of our sought-for p-dimensional family $(\Gamma^{CS}_{p,q}, dR_{p,q})$.
Under the constraints (4.3.19) we now have that, for $x \in [-a_I, a_I]$,

$$\frac{\omega_+\left(x+\frac{2\pi i I}{p}\right)+\omega_-\left(x+\frac{2\pi i I}{p}\right)}{2} = px \qquad \forall I = 0,\ldots,p-1 \tag{4.3.20}$$

Now, let's map conformally the cylinder of width 2π to the punctured complex plane via

$$\begin{array}{rccc} Z: & \mathbb{R}\times S^1 & \mapsto & \mathbb{C}^* \\ & z & \mapsto & e^z. \end{array} \tag{4.3.21}$$

Exponentiating (4.3.20) yields

$$Z^p = e^{\omega_+/2}e^{\omega_-/2} \tag{4.3.22}$$

Introduce now

$$g(z) = e^{\omega/2} + Z^p e^{-\omega/2} \tag{4.3.23}$$

This is a function which is single-valued on the whole strip $0 \leq \Im m z < 2\pi$, because for all $x \in [-a_I, a_I]$ we have that

$$\begin{array}{rcl} g_+(x+2\pi i I/p) & = & e^{\omega_+/2} + e^{p(x+2\pi i I/p)}e^{-\omega_+/2} = e^{\omega_+/2} + e^{px} \\ & \times & e^{-\omega_+/2} = e^{px}e^{-\omega_-/2} + e^{\omega_-/2} = g_-(x+2\pi i I/p). \end{array} \tag{4.3.24}$$

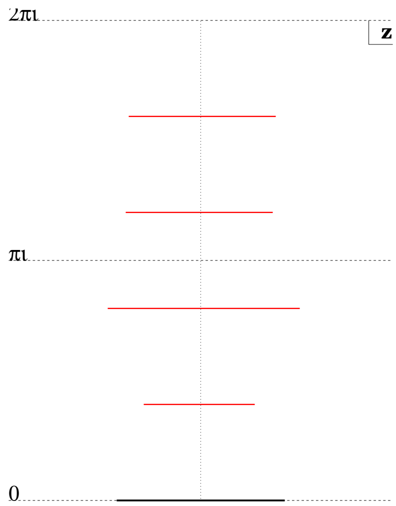

Figure 4.5: Cuts of the resolvent for $p = 5$. Cuts relative to non-trivial flat connections are drawn in red.

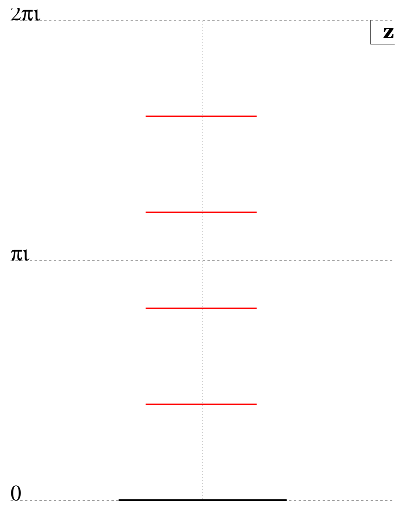

Figure 4.6: Cuts of the resolvent for $p = 5$, imposing the constraint (4.3.19). Cuts relative to non-trivial flat connections are drawn in red.

and it is regular everywhere, except perhaps at infinity. This implies that $g(Z)$ is an entire analytic function in Z with algebraic growth

$$g(Z) = \sum_{n=0}^{p} d_n Z^n \tag{4.3.25}$$

where the d_n's are (still unknown) functions of the two filling fractions S_0, $S_I = S_1 = (t - S_0)/(p-1)$. The resolvent $\omega(z)$ is then determined by solving the quadratic equation (4.3.23) with the appropriate boundary condition at infinity, which yields

$$\omega(z) = \log\left[\frac{1}{2}\left(g(z) - \sqrt{g^2(z) - 4e^{pz}}\right)\right]. \tag{4.3.26}$$

Defining $u \equiv z$, $v \equiv (t - 2\omega)$ we arrive at the following form for $\Gamma^{CS}_{p,q}$ under the constraint[4] (4.3.19)

$$\Gamma^{CS}_{p,q}: \qquad e^{t-2v} - e^{t/2-v}e^{-t/2}\left(e^{pu} + \sum_{n=1}^{p-1} d_n e^{nu} + 1\right) + e^{pu} = 0 \Rightarrow$$

$$e^t - e^v\left(e^{pu} + \sum_{n=1}^{p-1} d_n e^{nu} + 1\right) + e^{pu+2v} = (e^v - 1)(e^{pu+v} - 1) + e^t - 1 + e^v \sum_{n=1}^{p-1} d_n e^{nu} = 0, \tag{4.3.27}$$

which coincides with the Hori-Vafa mirror curve (4.2.21) for $q = 1$ and a proper identification of the complex structure parameters. We have then proven the following

[4]We also set $d_1 = d_p = 1$ with a redefinition u and dividing by an overall factor.

Proposition 19. *Let $\Gamma^{CS}_{p,q}$ and $dR_{p,q} \equiv vdu$ be the 2-parameter family (4.3.27) of large N spectral curves and differentials of the $L(p,q)$ Chern-Simons matrix model (4.3.7) under the constraint (4.3.19). Then they are q-independent and they make up a closed subset of the family of Hori-Vafa mirror curves (4.2.21) with $q = 1$.*

This concludes the proof of Claim 1 for the following reason. Notice that this restricted class of $L(p,q)$ large N curves consists of hyperlliptic Riemann surfaces, since they coincide with the large N curves of the $q = 1$ case. Moreover, they are generically smooth and have topological genus $p-1$. But for $1 < q < p-1$, there is *no* such a subfamily inside the Hori-Vafa family of mirror curves (4.2.21), as follows from the discussion preceding Proposition 16.[5]

□

[5]Indeed, the very definition of the Hori-Vafa map and the fact that the toric diagram is not contained in a strip of width at most two for $1 < q < p-1$ imply that hyperellipticity can be obtained only imposing a vanishing condition on a coefficient multiplying a monomial associated to an external point in the toric diagram. This amounts to discard a 1-dimensional ray in the fan and all the three-dimensional cones in which it is contained, as is familiar from the degenerate limit in which local surfaces reduce to local curves. But this fact will automatically lower the number of internal points and thus the genus of the mirror curve. Hence there can be no hyperelliptic *and* genus $p-1$ subfamily of curves inside (4.2.21).

Chapter 5

Toric GW theory III: local curves and integrable hierarchies

5.1 Topological string theory and integrable systems

Topological string theory can often be solved very effectively, as we have witnessed in chapter 3 and 4 in the toric $CY3$ case. It has been conjectured for long that "solvability" could be seen as a consequence of an underlying *integrability*. It is a general fact that integrable systems appear in topological field theories, as we reviewed in the Introduction; many more characters appear in the play when we look at the mirror symmetry description of the A–Model on a toric $CY3$:

1. first of all, at fixed moduli, the Jacobian of the mirror spectral curve can often be identified with the Liouville torus of a complex integrable system, and the Hori–Vafa differential with its Poincaré 1–form pdq [26, 109];

2. secondarily, and more importantly, the Kodaira–Spencer theory of gravity [22], dimensionally reduced on the mirror curve, provides [8] a free–fermion description of the integrable hierarchies that govern the "slow motion" over the moduli space of spectral curves [50, 108]. In addition, this lays the basis for a dispersive deformation of the hierarchy in terms of a $\mathcal{D}$-module quantization of the spectral data [47];

3. finally, the Eynard–Orantin recursion itself dictates [62] for genus zero spectral curves that the all–genus partition function should be a τ–function of some underlying integrable hierarchy[1].

A distinguished place in this context, and partly related to point 2 and 3 above, has been historically occupied by the conjectural existence of an integrable hierarchy that governs the *full* Gromov–Witten theory of a target space X, including the observables that come from the gravity sector of the topological A–model: the gravitational descendants. Let X be a smooth Kähler manifold or a reduced orbifold, possibly acted on by an algebraic torus T with compact fixed loci, and consider its all–genus, full descendant Gromov–Witten potential (2.1.15)

$$\mathscr{F}^X(\lambda, \mathbf{t}) = \sum_{g\geq 0} g_s^{2g-2} \sum_{n\geq 0} \sum_{0\leq k_1,\dots,k_n<\infty} \sum_{\beta\in H_2(X,\mathbb{Z})} \frac{1}{n!} \langle \tau_{k_1} t_1 \dots \tau_{k_n} t_n \rangle^X_{g,n,\beta} \tag{5.1.1}$$

$\mathscr{F}^X(\lambda, \mathbf{t})$ is by construction the object of our desire: it is defined in such a way as to package the entire set of correlators of the topological A–model on X in a single generating function. However, moving from single intersection numbers to generating functions is more than just a bookkeeping device: we learn from examples

[1] Quite interestingly, the generalization to $g > 0$ spectral curves is intimately related to the proposal of a non–perturbative definition of the model [64]

[104, 130, 148] and have general evidence from conjectural properties of Gromov–Witten theory [53, 56] that the following statement holds true:

Conjecture 3. *There exists an integrable hierarchy of evolutionary* $1+1$ *PDEs such that* $\mathscr{F}^X(\lambda, \mathbf{t})$ *is the logarithm of a* τ*–function associated to one of its solutions.*

As we reviewed in the Introduction, proving this conjecture would be of crucial importance both conceptually and computationally. Unfortunately, to find out whether such an integrable structure can be found and effectively described is a very tough problem, and it should be no surprise that the catalogue of answers to this question is restricted to a discouragingly low number of examples. Indeed, we have only two cases where Conjecture 3 has been rigorously proven

1. $X = pt$, that is, the intersection theory on the Deligne–Mumford compactification of the moduli space of curves. The Witten–Kontsevich theorem states [104, 148] that the KdV hierarchy is the relevant integrable system in this case;

2. $X = \mathbb{P}^1$, in which case the associated system is the extended Toda hierarchy [53, 55, 76, 125, 130]

A constructive proof of Conjecture 3 for a generic target space X appears to be out of reach at the moment. In fact, even adding new examples to the above list seems to be a very challenging problem: the next–to–simplest case of the complex projective plane $\mathbb{P}^2$ is already very hard to tackle, and it is as of today unsolved. For this reason, it would be a very valuable step forward if one could find more examples, or perhaps families of examples, of target spaces for which we can give an explicit construction of the relevant integrable model.

In this chapter, we will try to address this problem by beginning the study of the integrable structures that govern the equivariant Gromov-Witten theory of Calabi-Yau rank 2 bundles over the projective line, building on known results on the local Gromov–Witten theory of curves due to Bryan and Pandharipande [30]. This would constitute a new, one–integer parameter family of examples for which a relationship can be found with (possibly new) integrable models. In spirit, we will be very close to the perturbative philosophy of [49, 53], where the whole hierarchy is constructed according to the following two–step process:

1. find a closed form description of its genus zero approximation;

2. find a reconstruction procedure to incorporate the higher genus corrections.

In the following sections we will thus first describe the general recipe of [49] that associates a dispersionless integrable hierarchy to the (possibly equivariant) big quantum cohomology ring of a target space X; we will then review the results of [30, 31] on the primary, equivariant potential for a class of rank 2 bundles over

$\mathbb{CP}^1$ with a torus action rotating the fibers, and complete the construction of the prepotential; finally, we will consider the application of Dubrovin's construction to such target spaces, deriving the general structure of the dispersionless limit for the whole family and discussing in detail the case of the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. This example will be the main focus of this chapter: by adapting results of [54], we will relate its genus zero approximation to the so–called long–wave limit of the Ablowitz–Ladik lattice, and we will formulate a precise conjecture for the all–genus theory, propose a series of tests to verify it, and perform the simplest of them.

5.2 Dispersionless hierarchies from associativity equations

In the following we will review some general facts about associativity equations and their associated Principal Hierarchies. This is basically a review of known material; the interested reader is referred to the original literature [49, 52] for more details.

5.2.1 WDVV and the Gauss-Manin connection

The discussion here is mostly adapted to the case we will have in mind in section 5.3. Let $\mathbb{K}$ be a field; we will always have in mind either the complex numbers $\mathbb{K} = \mathbb{C}$ or a field of fractions of a polynomial ring over the complex numbers $\mathbb{K} = \mathbb{C}(\lambda_1, \ldots, \lambda_n)$.

Definition 15. *A Frobenius manifold is a pair $(\mathcal{N}, F_0)$ where*

- *$\mathcal{N}$ is a finite dimensional $\mathbb{K}$-vector space,* $\dim_{\mathbb{K}} N = N \in \mathbb{N}$. *We will also fix a set of basis vectors Φ_α, $\alpha = 0, \ldots, N-1$; a generic point in $\mathcal{N}$ will be denoted as $u = \sum_\alpha u^\alpha \Phi_\alpha$ with $u^\alpha \in \mathbb{K}$.*

- *$F_0 : \mathcal{N} \to \mathbb{K}$ is an analytic function. Here this means, in absence (possibly) of a metric structure on $\mathcal{N}$ and an analytic theory of functions, that F is a formal power series*
$$F_0(u) = \sum_j \sum_{i_1 \ldots i_j} c_{i_1 \ldots i_j} (u^1)^{i_1} \ldots (u^j)^{i_j}_j \tag{5.2.1}$$
Analytic operations such as integration and derivation are defined algebraically.

- *the direction spanned by Φ_0 is marked in the following sense: for every $u \in \mathcal{N}$, $\partial_0 \partial^2_{\alpha\beta} F_0(u) =: \eta_{\alpha\beta}$ is a nondegenerate, constant symmetric matrix.*

- *for every $u \in \mathcal{N}$, $c^\gamma_{\alpha\beta}(u) := \partial^3_{\alpha\beta\delta} F_0(u) \eta^{\delta\gamma}$ are point-dependent structure constants of an* associative *algebra (with a unity, because of the previous item).*

The last statement is tantamount to imposing a set of third order, non-linear PDEs on F_0, called the *WDVV equations*

$$\partial^3_{\alpha\beta\gamma}F_0\eta^{\gamma\delta}\partial^3_{\delta\varepsilon\zeta}F_0 = \partial^3_{\alpha\varepsilon\gamma}F_0\eta^{\gamma\delta}\partial^3_{\delta\beta\zeta}F_0 \tag{5.2.2}$$

The matrix η will be often referred to in the following as the "metric" or the "topological[2] metric", although it need not be positive definite. It will be implicit that the operation of lowering and raising indices of tensor fields on $\mathcal{N}$ will be performed with η.

Remark 15. *These are actually two thirds of the usual definition of a Frobenius manifold [52], which also includes a grading condition on N. In fact, we do not impose here any quasi-homogeneity condition on F_0. The construction of the Principal Hierarchy - see below - will go through unaffected; however, this will leave an ambiguity to the choice of basis of conserved currents, which in the enumerative context it will be crucial to fix - let alone the consequences on the Dubrovin-Zhang perturbative reconstruction of the higher genus theory.*

Example 4. *Let X be a n–dimensional compact Kähler manifold (respectively orbifold) with vanishing odd co-homologies and let $\mathcal{N} = QH^\bullet(X)$ Then the arguments of section 2.1.3 $\mathcal{N}$ is a Frobenius manifold over $\mathbb{K} = \mathbb{C}$: the marked direction Φ_0 corresponds to the unity class in co-homology, and the metric η coincides with the Poincaré (resp. orbifold Poincaré) pairing (2.1.11). The $WDVV$ equations are precisely the associativity condition (2.1.23). Notice that the assumption $b_{2k+1}(X) = 0$ for all k is needed in order to ensure that $\mathcal{N}$ be a commutative ring.*

Example 5. *Let X be a possibly non–compact n–dimensional Kähler manifold (respectively orbifold) with vanishing odd co-homologies and endowed with a holomorphic $T \simeq (\mathbb{C}^*)^k$ action ($k \leq n$) with compact fixed locus F. Let $\mathbb{C}[\lambda]$ denote the T-equivariant co-homology of a point and $\mathbb{K} := \mathbb{C}(\lambda)$ be its field of fractions. Then $\mathcal{N} = QH^\bullet_T(X)$ is a Frobenius manifold over $\mathbb{K}$: the marked direction Φ_0 corresponds to the unity class in equivariant co-homology, and the metric η coincides with the equivariant Poincaré (resp. equivariant orbifold Poincaré) pairing (2.1.11). Again, the $WDVV$ equations are precisely the associativity condition (2.1.23)*

Remark 16. *To have a well-defined metric η it was important in Example 5 to require that the fixed locus F be compact, in order to ensure non–degeneracy of the Poincaré pairing. When F is non–compact, and more particularly when X is non–compact and we consider the non–equivariant theory, the metric becomes degenerate. Indeed, the very definition of the degree zero term in the Gromov–Witten potential becomes more subtle in this case [35, 69] and in no case does it lead to a Frobenius manifold, not in even in the relaxed meaning we have adopted here. They still lead to solutions of $WDVV$ though, as happens in the case of the prepotentials coming from Seiberg–Witten theory or the large N limit of matrix models, but the following discussion cannot apply to such cases.*

[2]The adjective here might be needed to distinguish it from the Zamolodchikov, or tt^* metric.

One nice implication of (5.2.2) is the following. Let us define $\mathcal{M} := T^*\mathcal{N}$ and the following 1-parameter family of connections on $\mathcal{M}$

$$D_z := d + \Gamma \tag{5.2.3}$$

where the Christoffel symbol $\Gamma^\gamma_{\alpha\beta} = zc^\gamma_{\alpha\beta}$ and $z \in \mathbb{K}$. We want to find the horizontal sections of D_z, i.e.

$$D_z\omega = 0, \quad \omega \in \mathcal{M} \tag{5.2.4}$$

Now, notice that because of integrability of $c_{\alpha\beta\gamma}$ and WDVV, we have that

$$D_z^2 = 0 \quad \forall z \tag{5.2.5}$$

that is, the connection is flat. This means that all ω satisfying (5.2.4) are closed 1-forms; the trivial cohomology of $\mathcal{N}$ and (5.2.4) then imply that $\omega = df$ and also

$$\partial^2_{\alpha\beta} f = zc^\gamma_{\alpha\beta}\partial_\gamma f \tag{5.2.6}$$

The system of PDEs (5.2.6) has a $N = \dim_{\mathbb{K}}\mathcal{N}$ dimensional space of solutions $h^\delta(u, z)$. We will call them the *flat functions* of $\mathcal{N}$. Their duals $h_\alpha(u, z) = \eta_{\alpha\beta}h^\beta(u, z)$ can be normalized such that

$$\begin{aligned}
h_\alpha(u, 0) &= w_\alpha = \eta_{\alpha\beta}u^\beta && (5.2.7)\\
\partial_\gamma h_\alpha(u, z)\eta^{\gamma\delta}\partial_\delta h_\beta(u, -z) &= \eta_{\alpha\beta} && (5.2.8)\\
\partial_0 h_\alpha(u, z) &= zh_\alpha(u, z) + \eta_{0\alpha} && (5.2.9)
\end{aligned}$$

In the following section, such a normalization will always be implicitly assumed, unless otherwise stated.

5.2.2 The Principal Hierarchy

The flat functions $h^\delta(\tau; z)$ are closely related to the one–point, "big" correlators (2.1.24). Taylor–expanding $h_\delta(\tau; z)$ with respect to z

$$h_\alpha(u, z) =: \sum_{z=0}^{\infty} h_{\alpha,p}(u)z^p \tag{5.2.10}$$

we have [52]

$$h_{\alpha,p} = \langle\langle \tau_p \Phi_\alpha \rangle\rangle \tag{5.2.11}$$

It was first conjectured by Witten [148] that the one–point "big" correlators could be seen as densities of a Hamiltonian integrable system on $\mathcal{N}$, whose flow would allow to reconstruct the full descendant, genus zero potential starting from the primary one. A complete formalization in the context of a generic Frobenius manifold was made by Dubrovin in [49], and we will now schematically review it here. Consider the loop space $L\mathcal{N}$ of $\mathcal{N}$, $L\mathcal{N} = u : S^1 \to N$; elements of $L\mathcal{N}$ will be written as $u^\alpha(x)$.

$L\mathcal{N}$ can be turned into an infinite dimensional Poisson manifold by introducing the hydrodynamic Poisson bracket

$$\{u^\alpha(x), u^\beta(y)\} = \eta^{\alpha\beta}\delta'(x-y) \tag{5.2.12}$$

We then define the following hierarchy of quasilinear PDEs

$$\partial_{t^{\alpha,p}} u^\beta = \{u^\beta, H_{\alpha,p}\} \tag{5.2.13}$$

where

$$H_{\alpha,p}(u) \;=\; \int h_{\alpha,p+1}(u(x))dx \tag{5.2.14}$$

Theorem 20 (Dubrovin)**.** *The set of Hamiltonians* (5.2.13) *mutually Poisson–commute with respect to the Poisson bracket* (5.2.12)*. Let then* $u_\alpha(x,\mathbf{t})$ *solve the system (5.2.13) with boundary condition*

$$u_\alpha(x,\{t_k=0\}) = \partial^2_{x,u^\alpha}F_0(u^0+x, u^1, \ldots\ldots)|_{u^i=0}$$

and define for all times

$$\begin{aligned} \partial^2_{x,u^\alpha}\mathscr{F}_0(x+t^{0,0}, t^{1,0}, \ldots, t^{\gamma,k}, \ldots) &:= u^\alpha(\mathbf{t}) & (5.2.15)\\ \langle\langle\phi_{\alpha,p}\phi_{\beta,q}\ldots\rangle\rangle &:= \partial_{t^{\alpha,p}}\partial_{t^{\beta,q}}\ldots\partial_{\ldots}\mathscr{F}_0(\mathbf{t}) & (5.2.16)\end{aligned}$$

Then $\mathscr{F}_0$ *- the logarithm of a* τ *function for the hierarchy (5.2.13) - satisfies*

$$\begin{array}{rcll} \mathscr{F}_0|_{t^{a,p}=0 \text{ for } p>0} & = & F_0(t^{a,0}) & \text{(reduction to primaries)}\\ \partial_x\mathscr{F}_0 & = & \sum t^{\alpha,p}\partial_{t^{\alpha,p-1}}\mathscr{F}_0 + \frac{1}{2}\eta_{\alpha\beta}t^{\alpha,0}t^{\beta,0} & \text{(string equation)}\\ \langle\langle\phi_{\alpha,p}\phi_{\beta,q}\phi_{\gamma,r}\rangle\rangle & = & \langle\langle\phi_{\alpha,p}\phi_{\delta,0}\rangle\rangle\eta^{\delta\epsilon}\langle\langle\phi_{\epsilon,0}\phi_{\beta,q}\phi_{\gamma,r}\rangle\rangle & \text{(TRRs)} \end{array} \tag{5.2.17}$$

This is particularly interesting in the case in which $(\mathcal{N}, F_0)$ is the (possibly equivariant) big quantum cohomology algebra of a Kähler manifold or orbifold X, $N = QH^\bullet_T(X)$, where $BT = \mathbb{K}$. In this case, the genus zero TRRs (5.2.17) and the string equation allow to determine the descendant correlators, starting from the primary invariants. The statement of Theorem 20 is that such a procedure is given through a dispersionless flow, generated by $h_{\alpha,p}$. In particular we have in this case

Corollary 1.

$$\langle\langle\phi_{\alpha_1,i_1}\ldots\phi_{\alpha_n,i_n}\rangle\rangle = \langle\langle\tau_{i_1}\Phi_{\alpha_1}\ldots\tau_{i_n}\Phi_{\alpha_n}\rangle\rangle_0 \tag{5.2.18}$$

where $i_k \geq 0$, $0 \leq \alpha_k < N$.

Remark 17. *We are deliberately hiding some subtleties here. Indeed, the normalization conditions* (5.2.7)*–*(5.2.9) *do not single out a unique basis of Hamiltonians: in fact, there is a leftover functional ambiguity in* z*. In the quasi-homogeneous case, this is mostly ruled out by the presence of the Euler vector field; however, in more general cases the problem will have to be dealt with otherwise. We will see how to overcome this difficulty in more detail in the example of the resolved conifold in section 5.4.2.*

Let us summarize the situation. We have a full construction of an integrable hierarchy, associated to the (possibly equivariant and/or orbifold) big quantum cohomology of a target space, which is valid as long as a metric can be sensibly defined. This has a number of good, as well as weak points, which we can schematize as follows

- the good point is that the construction is both *general*[3] and *(almost) complete*. The gravitational invariants of X, i.e. $\mathscr{F}_0$, is entirely determined by the primary structure, i.e. F_0, on whose form no extra constraining assumption is made besides the fact that it should give rise to a Frobenius structure on $T_u\mathcal{N}$ for $u \in \mathcal{N}$. Moreover, this is achieved through an integrable flow on $L\mathcal{N}$ which solves a hierarchy of differential equations of hydrodynamic type. *Almost everything* is dictated by the functional form of F_0 and the flatness condition (5.2.6) for the Hamiltonian densities. Such a procedure therefore proves, in this generality, Conjecture 3, at the leading order in the topological expansion parameter g_s.
- the weak point is that we need F_0 in *closed* form - no implicit, recursive or up-to-inversion-of-the-mirror-map form will do the job. Any polynomial truncation of (5.2.1) affects dramatically the form of the three-point couplings c_{ijk} and therefore the flat functions. In other words, we must know beforehand *all* the coefficients $c_{i_1\dots i_j}$ in (5.2.1). Moreover, if we want the recursion to have geometric meaning - i.e., the derivatives of the deformed flat coordinates to reproduce the J function of the target space (see section 5.4.2), we will need to pick a canonical normalization of the $h_\alpha(w; z)$. In absence of an Euler vector field, this will need to be done using some external input.

The good news are actually beautiful news: there is an integrable system governing the intersection theory on the spaces of holomorphic maps from the Riemann sphere to X. But the bad ones are, in the Gromov-Witten case, extremely constraining in practice. It is an overly fortunate case to actually have the full Gromov-Witten potential F_0 in closed form. This is the case of the Gromov-Witten theory of the point

$$F_0(u) = \frac{1}{3!}u^3$$

for which (5.2.13) is the dispersionless limit of KdV, and of the complex projective line

$$F_0(u) = \frac{1}{2}(u^{(0)})^2 u^{(2)} + e^{u^{(2)}}$$

in which case (5.2.13) becomes the long-wave limit of the Toda lattice. The fact that this list coincides with the only two known cases for which Conjecture 3 was proven is indeed not an accident: already for $\mathbb{CP}^2$ the full form of the potential is only available in a recursive, non-closed form. Such a construction becomes then impracticable in most of the cases, and the form of the integrable system has to be argued by different means.

[3] Again, as long as $b_{2k+1}(X) = 0$, as we stressed at the end of Example 4.

Besides, a second problem arises when one wonders what happens with the higher genus theory. This is the case that is of more interest on the geometry side, where in general no recursive relations like 2.1.25 are known, and for which having an integrable system at hand could become the key to solve the Gromov–Witten theory of X completely. Adding higher order corrections in λ takes the shape, on the integrable system side, of a dispersive deformation of the Hamiltonians 5.2.14 in the form of a *long–wave expansion* [53]

$$\partial_{t^{\alpha,p}} u^\beta = \{u^\beta, H_{\alpha,p,g_s}\} = A(u)u_x + g_s^2 B(u, u_x, u_{xx}) + g_s^4 C(u, u_x, u_{xx}, u_{xxx}) + \dots \tag{5.2.19}$$

Addressing fully the statement of Conjecture 3 would require to find a way to determine all the higher order coefficients in the genus expansion. In the quasi–homogeneous case and assuming the Virasoro conjecture, a reconstruction theorem was given in the monumental work of [53]. On the other hand, nothing is known to date about its generalization to the non-quasihomogeneous case, which is the one relevant for equivariant Gromov–Witten theory.

5.3 The local Gromov–Witten theory of curves

In this section, we will review the findings of [30,31] about the local Gromov–Witten theory of curves. We will just give the general frame and quote the results we are going to need; the interested reader is referred to the original papers for the details.

We want to consider a one-integer-parameter family of target spaces, isomorphic (differentially) to neighbourhoods of a rational curve inside a Calabi-Yau threefold. More precisely, we are going to consider a family of Calabi-Yau complex rank 2 bundles over the Riemann sphere. By Grothendieck's theorem, such bundles split (in the holomorphic category) into a sum of line bundles: $\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1}(n_2)$, $n_i \in \mathbb{Z}$; by the Calabi-Yau condition, we must have that $k := n_1 = -n_2 - 2$. We will denote with X_k the total spaces of these bundles. Moreover, we will consider a $T \simeq \mathbb{C}^*$ algebraic torus acting fiberwise with *opposite* characters on each $\mathbb{C}$ fiber - that is, it will rescale the fibers with equivariant parameter λ and $-\lambda$ respectively.

As in section 2.1.3, let $H(X_k) := H_T^\bullet(X_k, \mathbb{C}) \otimes \mathbb{C}(\lambda)$ denote the localized T-equivariant cohomology of X_k and $F_k \simeq \mathbb{P}^1$ be the fixed locus of the T-action, that is, the null section of $X_k \to \mathbb{P}^1$. Let $(1, p)$ be a canonical basis for $H(X_k)$ (regarded as a free $\mathbb{C}(\lambda)$-module), where 1 and p denote respectively the identity and the Kähler class, and let $u =: v + wp$ with $u_i \in \mathbb{C}(\lambda)$. Since X_k is a $CY3$, the prepotential is given by (2.1.22) as

$$F_0^{X_k}(\tau) = F_{0,cl}^{X_k}(u) + F_{0,qu}^{X_k}(u) \tag{5.3.1}$$

where

$$
\begin{aligned}
F^{X_k}_{0,cl}(u) &= \frac{1}{3!}(u \cup u, u) \\
F^{X_k}_{0,qu}(u) &= \sum_{d>0} e^{dw} N_{0,d} \\
N_{g,d} &= \int_{[(X_k)_{g,0,d}]^{vir}} 1
\end{aligned}
\tag{5.3.2}
$$

This allows us to separate the classical, $d = 0$ part from the quantum, worldsheet instanton corrected one given by degree > 0 maps to X_k.

We have seen that one of the weak points of Theorem 20 is the fact that we are supposed to know beforehand *all* the genus zero primary invariants to construct explicitly the hierarchy, and that this can be hardly attained in general. However, the case of X_k is very special: their big quantum co-homology shares most of the desirable features of the ordinary Fano case (e.g., it is semisimple), yet, being a toric $CY3$ with an equivariantly CY action, it borrows most of the characteristics of its non-equivariant limit, like the possibility to compute Gromov–Witten invariants via the topological vertex [7, 111, 124] or via mirror symmetry: either way, the fact that $b_4(X_k) = 0$ guarantees then the possibility to give a closed formula for the prepotential.

5.3.1 Calculating $F^{X_k}_{0,qu}(u)$

In the mathematical literature, it was indeed shown that localization techniques allow to compute $N_{g,d}$ for *arbitrary* rank 2 bundles over *arbitrary* genus g projective curves with *arbitrary* $(\mathbb{C}^*)^2$ action on the fibers [30]. Here's a corollary of their main theorem, which is just a specialization of the latter to our $X_k^{\circlearrowleft T}$ case.

Theorem 21 (Bryan-Pandharipande)**.** *The* fixed-degree $d > 0$, all-genus *Gromov - Witten free energies of $X_k^{\circlearrowleft T}$ are given by the following sum over partitions*

$$
\sum_{g \geq 0} g_s^{2g-2} N_{g,d} = (-1)^{d(k-1)} \sum_{\rho} \left(\frac{\dim_Q \rho}{d!} \right)^2 Q^{c_\rho (1-k)}
\tag{5.3.3}
$$

In (5.3.3), ρ is a Young diagram (a partition of length $l(\rho)$), c_ρ is its total content, $Q = e^{ig_s}$, $h(\Box)$ is the hooklength of a box in ρ and

$$
\frac{\dim_Q \rho}{d!} = \prod_{\Box \in \rho} \left(2 \sin \frac{h(\Box) g_s}{2} \right)^{-1}
$$

Theorem (21) can be regarded as a specialization of the topological vertex method of [7] to X_k.

This gives a great deal of information on X_k, but still not exactly what we need. In fact, what we will eventually want is a closed form expression for the big quantum cohomology $N = QH^\bullet_T(X_k)$, i.e., a closed form expression for its *prepotential* $F^0_{X_k}$. In a sense, this means that we want to sum up things the other way round with respect to (5.3.3): we will need the *all-degree, fixed genus* (=0) free energies of X_k. The problem how to extract them from (5.3.3) was addressed in [31], where the authors perform a steepest descent analysis of the sum over partitions with a *chirality* assumption, which enables them to find closed form expressions for $F^0_{X_k}$. Their results match perfectly with known results for the case $k = 1$ [15]; additionally, a confirmation was given mathematically via mirror symmetry (Coates-Givental+Bikhoff factorization) in [70] and via symplectic field theory methods in [136]. The result is the following:

Proposition 22 (CGMSP)**.** *The genus zero, quantum corrected tail $F^{X_k}_{0,qu}(u)$ of the A–model prepotential of X_k is given by*

$$\begin{aligned}
F^{X_1}_{0,qu}(\tau) &= Li_3(e^w) && (5.3.4)\\
F^{X_2}_{0,qu}(\tau) &= -Li_3(e^w) && (5.3.5)\\
F^{X_k}_{0,qu}(\tau) &= (-)^{k-1}e^{-w}{}_{n+3}F_{n+2}\Bigg[1,1,1,1,\frac{1}{n},\frac{2}{n},\dots,1-\frac{1}{n};2,2,2,2,\frac{1}{n-1},\\
&\qquad \dots,1-\frac{1}{n-1};(-1)^k\left(\frac{n}{n-1}\right)^{n-1} n\exp(w)\Bigg] \qquad (k>2)\\
& && (5.3.6)
\end{aligned}$$

where $n = (k-1)^2$.

5.3.2 Calculating $F^{X_k}_{0,cl}(u)$

The only missing ingredient for the complete calculation of the prepotential is the trivial computation of the classical (in α') term $F^{X_k}_{0,cl}(\tau)$. From (5.3.1), this is given by the triple intersection "number"

$$F^{X_k}_{0,cl}(\tau) = \frac{1}{3!}\int_{[\mathbb{P}^1]}\frac{u\cup u\cup u}{e(\mathcal{N}_{X_k^T/X_k})} \qquad (5.3.7)$$

Remember that $u =: v + wp$, $X_k = \mathcal{O}_{\mathbb{P}^1}(-k)\oplus\mathcal{O}_{\mathbb{P}^1}(k-2)$ and T acts antidiagonally on the fibers. That is,

$$\begin{aligned}
(e(\mathcal{N}_{X_k^T/X_k}))^{-1} &= [(-kp+\lambda)((k-2)p-\lambda)]^{-1} = -\frac{1}{\lambda^2}\frac{1}{1-\frac{kp}{\lambda}-\frac{(k-2)p}{\lambda}}\\
&= -\frac{1}{\lambda^2}\left[1+2\frac{(k-1)p}{\lambda}\right] \qquad (5.3.8)
\end{aligned}$$

Eq. (5.3.7) then becomes

$$\begin{aligned} F^{X_k}_{0,cl}(\tau) &= -\frac{1}{3!\lambda^2}\int_{[\mathbb{P}^1]}(v+wp)^3\left[1+2\frac{(k-1)p}{\lambda}\right] \\ &= \frac{1}{3!}\left(-\frac{1}{\lambda^2}\right)\left(\frac{2k-2}{\lambda}\right)v^3-\frac{1}{2}v^2w\frac{1}{\lambda^2} \end{aligned} \tag{5.3.9}$$

Therefore,

$$F^0_{X_k}(\tau)=\frac{2-2k}{3!}\left(\frac{v}{\lambda}\right)^3-\frac{1}{2}w\left(\frac{v}{\lambda}\right)^2+F^{X_k}_{0,qu}(w) \tag{5.3.10}$$

We now have all the ingredients to compute the structure constants of $QH^\bullet_T(X_k)$. First of all the metric, after rescaling $v\to v\lambda$, is given as

$$\eta_{ij}=\partial^3_{0ij}F^0_{X_k}(\tau)=\begin{pmatrix} \frac{2-2k}{\lambda^3} & -\frac{1}{\lambda^2} \\ -\frac{1}{\lambda^2} & 0 \end{pmatrix} \tag{5.3.11}$$

with inverse

$$\eta^{ij}:=(\eta^{-1})_{ij}=\begin{pmatrix} 0 & -\lambda^2 \\ -\lambda^2 & -(2-2k)\lambda \end{pmatrix} \tag{5.3.12}$$

while the only other non-trivial Yukawa coupling $Y_k(w):=\partial^3_{w^3}F^0_{X_k}(\tau)$ is given by

$$Y_k(w)=\begin{cases} \frac{\exp(w)}{1-\exp(w)} & k=1 \\ \frac{\exp(w)}{\exp(w)-1} & k=2 \\ \frac{1}{n}-\frac{1}{n}{}_{n-1}F_{n-2}\left[\frac{1}{n},\dots,\frac{n-1}{n};\frac{1}{n-1},\dots,\frac{n-2}{n-1};(-1)^k\frac{n^n}{(n-1)^{n-1}}\exp(-w)\right] & k>2 \end{cases} \tag{5.3.13}$$

5.3.3 Warming up: principal hierarchies in the equivariantly CY case

The exact form of the prepotential is the main ingredient to determine the form of the Principal Hierarchy. As a warm–up, let us determine the first few flows here in the general case.

The non-trivial structure constant is the one relative to the product $\partial_w\cdot\partial_w$, namely $c^\gamma_{ww}=c_{ww\delta}\eta^{\delta\gamma}$. We have

$$c^v_{ww}=c_{www}\eta^{wv}=c_{www}=-\lambda^2Y_k(w) \tag{5.3.14}$$

$$c^w_{ww}=c_{www}\eta^{ww}=(2k-2)\lambda Y_k(w) \tag{5.3.15}$$

The flat functions must then satisfy the system of PDEs

$$\partial^2_v f = z\partial_v f \tag{5.3.16}$$

$$\partial^2_{vw} f = z\partial_w f \tag{5.3.17}$$

$$\partial^2_w f = -z\lambda Y_k(w)(\lambda\partial_v f+(2k-2)\partial_w f) \tag{5.3.18}$$

The general integral of the first equation is

$$f(v,w;z) = A(w,z,\lambda)\frac{e^{zv}}{z} + B(w,z) \tag{5.3.19}$$

Replacing it in the second equation imposes

$$B(w,z) = B(z) \tag{5.3.20}$$

and the third equation reduces to a linear ODE for A

$$A''(w) = \left[z^2 A(w) + 2zA'(w)\right] Y_k(w) \tag{5.3.21}$$

When $k = 1, 2$ this is a Fuchsian ODE which can be integrated in closed form; there is however no general recipe to solve it exactly for $k > 2$. However, let us solve it perturbatively in z. First of all, notice that the normalization condition (5.2.9)for the Hamiltonian densities

$$\begin{aligned} h_v^k(v,w;z) &= A_v^k(w,z)\frac{e^{zv}}{z} + B_v^k(z) \\ h_w^k(v,w;z) &= A_w^k(w,z)\frac{e^{zv}}{z} + B_w^k(z) \end{aligned} \tag{5.3.22}$$

now implies that

$$B_v^k(z) = \frac{2k-2}{z}, \qquad B_w^k(z) = -\frac{1}{z} \tag{5.3.23}$$

while, upon defining

$$\begin{aligned} A_v^k(w,z) &=: A_{0,0}^k(w) + zA_{0,1}^k(w) + \frac{z}{2!}A_{0,1}^k(w) + \dots \\ A_w^k(w,z) &=: A_{2,0}^k(w) + zA_{2,1}^k(w) + \frac{z}{2!}A_{2,1}^k(w) + \dots \end{aligned}$$

Eq. (5.2.7) yields

$$A_{0,0}^k(w) = 2-2k, \quad A_{0,1}^k(w) = w, \quad A_{2,0}^k(w) = 1, \quad A_{2,1}^k(w) = 0 \tag{5.3.24}$$

while the flatness condition (5.3.18) imposes that

$$\partial_w^2 A_{0,2}^k(w) = 0, \qquad \partial_w^2 A_{2,2}^k(w) = 2Y_k(w) \tag{5.3.25}$$

This implies that the densities have the expansion

$$h_v^k(v,w;z) = (2-2k)v + w + \frac{1}{2}\left((2-2k)v^2 + 2vw + A_{0,2}^k(w)\right)z + \mathcal{O}(z^2) \tag{5.3.26}$$

$$h_w^k(v,w;z) = v + \frac{1}{2}\left(v^2 + A_{2,2}^k(w)\right)z \tag{5.3.27}$$

Let us focus on the $t^{2,0}$ flow, compute the Poisson bracket of the coordinates with the $H_{w,0}$ Hamiltonian, and eliminate v. We have

$$\partial_t v(x,t) = \{v(x,t), H_{w,0}\} = \frac{1}{2}\partial_w^2 A_{2,2}^k(w)(w)_x = Y_k(w)_x \tag{5.3.28}$$
$$\partial_t w(x,t) = \{w(x,t), H_{w,0}\} = (v)_x + (2k-2)Y_k(w)_x \tag{5.3.29}$$

which reduces to the following non–linear wave equation

$$(w)_{tt} = \left(Y_k(w)(w)_x\right)_x + 2\left(Y_k(w)(w)_x\right)_t \tag{5.3.30}$$

This sort of "hypergeometric hierarchy", given by (5.3.30) and its conservation laws generated by $H_{\alpha,p}$ for $p > 0$, seems to be unknown in the literature. Let us mention, as a curiosity, that the hypergeometric Yukawa $Y_k(w)$ might have an expression in terms of elementary (in fact algebraic) functions of e^w (see appendix B.5 for the details of the case $k = 3$).

5.4 A case study: the resolved conifold and the Ablowitz–Ladik hierarchy

5.4.1 Solving the flatness condition

The case $k = 1$ corresponds to the special case of a rigid $\mathbb{P}^1$ inside a $CY3$, for which concavity of the normal bundle and the Calabi–Yau condition imply the latter to be of the form $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. This case was treated, with one minor difference, in [54], where it is shown that a prepotential of the form

$$F(u,v) = \frac{1}{2}uv^2 - Li_3(e^{-u}) \tag{5.4.1}$$

was related to the long-wave limit of the Ablowitz-Ladik lattice[4].

Let us compute explicitly the flat functions $h_v(\tau, z)$, $h_w(\tau, z)$. In this case the linear ODE (5.3.21) becomes

$$A''(w,y) = -y^2\frac{A(w)\exp(w)}{1-\exp(w)} \tag{5.4.2}$$

where we defined $y := z\lambda$. A basis of solutions is given by

$$f_1(w,y) = {}_2F_1\left(-y, y; 1; e^w\right); \qquad f_2(w,y) = (1-e^w)\,{}_2F_1\left(1-y, y+1; 2; 1-e^w\right) \tag{5.4.3}$$

[4]Finding the correct geometric interpretation for the prepotential of [54] in the context of Gromov–Witten theory was indeed the first motivation of this chapter.

Therefore, Hamiltonian densities for the hierarchy (5.2.13) have the form

$$\begin{aligned} h_v(v,w;z) &= A_v(w,z\lambda)\frac{e^{zv}}{z} + B_v(z) \\ h_w(v,w;z) &= A_w(w,z\lambda)\frac{e^{zv}}{z} + B_w(z) \end{aligned} \tag{5.4.4}$$

where

$$A_v(w,y) = c_1^{(v)}(y)\, {}_2F_1\left(-y,y;1;e^w\right) + \left(1-e^w\right) c_2^{(v)}(y)\, {}_2F_1\left(1-y,y+1;2;1-e^w\right) \tag{5.4.5}$$

$$A_w(w,y) = c_1^{(w)}(y)\, {}_2F_1\left(-y,y;1;e^w\right) + \left(1-e^w\right) c_2^{(w)}(y)\, {}_2F_1\left(1-y,y+1;2;1-e^w\right) \tag{5.4.6}$$

for some constants of integration $c_j^{(i)}(y)$. As we stressed in Remark 17, the set of conditions (5.2.7)–(5.2.9) is insufficient to uniquely fix them without any more external information. In fact, such extra data are available, as we will now turn to discuss.

5.4.2 Normalizing the deformed flat coordinates

The extra information that we need comes from a result of Coates–Givental [36], which allows in particular to compute the twisted Gromov–Witten invariants of the total space of a concave vector bundle $E \to B$, where B is a compact orbifold, in terms of the ordinary invariants of B. For the sake of brevity we will just mention the result that we need without justifying it[5]. We have

Proposition 23 ([36])**.** *The J–function of the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$ with anti–diagonal action is given by*

$$J(q,z) = e^{zp\log q}\sum_{d\geq 0}\frac{\prod_{m=-d+1}^{0}\left(-p+m/z+\lambda\right)\left(-p+m/z-\lambda\right)}{\prod_{m=1}^{d}\left(p+m/z\right)^2}q^d \tag{5.4.7}$$

Here, we have redefined $\hbar = 1/z$ in eq. (2.1.31).

By the discussion of section 2.1.4, the J function is very closely related to the flat functions (5.2.6). Comparing (5.2.6) and (2.1.28), we see that the basis of solutions h^α of (5.2.6) is related to the fundamental solution (2.1.29) as

$$\partial_\alpha h_\beta = S_{\alpha\beta}$$

[5] The result comes from knowledge of the J function of $\mathbb{P}^1$, the Coates–Givental hypergeometric modification [36] defining the twisted I function of the conifold, and uniqueness properties of the J function.

and the J function is then given by

$$J = \partial_0 h_\beta \Phi^b \tag{5.4.8}$$

By this relation, we can use our knowledge of the J function of the conifold (5.4.7) to completely fix the unknown coefficients $c_j^{(i)}(y)$. In particular, it allows us to prove the following

Theorem 24. *The dispersionless integrable hierarchy associated to the genus zero equivariant Gromov–Witten theory of the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with anti–diagonal $\mathbb{C}^*$ action on the fibers is given by*

$$\partial_{t^{\alpha,p}} u^\beta = \left\{ u^\beta, \int h_{\alpha,p} \right\} \tag{5.4.9}$$

where $\{,\}$ *denotes the Poisson bracket* (5.2.12)*,* η *is the topological metric* (5.3.11) *with* $k=1$*, and the Hamiltonian densities* $h_{\alpha,p}$ (5.2.10) *are given by*

$$\begin{aligned} h_v(v,w,z,\lambda) &= \frac{e^{vz}}{\lambda^2}\Big[(H_{-z\lambda} + H_{z\lambda})\, {}_2F_1(-z\lambda, z\lambda; 1; e^w) + (-1+e^w) \\ &\times \pi z\lambda \csc(\pi z\lambda)\, {}_2F_1(1-z\lambda, z\lambda+1; 2; 1-e^w)\Big] \qquad (5.4.10) \\ h_w(v,w,z,\lambda) &= \frac{1 - e^{vz}\, {}_2F_1(-z\lambda, z\lambda; 1; e^w)}{z\lambda^2} \qquad (5.4.11) \end{aligned}$$

In (5.4.10)*,* $\psi^{(0)}(z)$ *is the polygamma function*

$$\psi^{(0)}(z) = \frac{d \log \Gamma(z)}{dz}$$

while H_z *is the harmonic number function*

$$H_z = \psi^{(0)}(z+1) + \gamma$$

where γ *is the Euler–Mascheroni constant.*

Proof. The $\mathcal{O}(z)$ term of the expansion of J is the statement that the mirror map is *trivial* in this case

$$\log q = w \pmod{2\pi i} \tag{5.4.12}$$

Let us examine the summand in (5.4.7) above more closely, starting from the numerator. It simply reads

$$\frac{m^2}{z^2} - \frac{2pm}{z} - \lambda^2$$

and performing the product (remember that $p^2 = 0$) yields

$$-\frac{y\left(\frac{\lambda^2}{y^2}\right)^d \Gamma(d-y)\Gamma(d+y)\sin(\pi y)}{\pi}+$$

$$\frac{y^2\left(\frac{\lambda^2}{y^2}\right)^d \Gamma(d-y)\Gamma(d+y)\left(-\psi^{(0)}(d-y)+\psi^{(0)}(-y)+\psi^{(0)}(y)-\psi^{(0)}(d+y)\right)\sin(\pi y)}{\pi\lambda}p$$

while for the inverse of the denominator we obtain simply

$$\frac{(z^2)^d}{\Gamma(d+1)^2}-\frac{2z\,(z^2)^d\,H_d}{\Gamma(d+1)^2}p$$

By (5.4.8), this means that the deformed flat coordinate of the unity is given as

$$h^0(0,w,z,\lambda)=\sum_{d\geq 0}-\frac{y\left(\frac{y^2}{\lambda^2}\right)^d\left(\frac{\lambda^2}{y^2}\right)^d\Gamma(d-y)\Gamma(d+y)\sin(\pi y)}{\pi\Gamma(d+1)^2}e^{dw}=f_1(w,y) \tag{5.4.13}$$

This simply sets

$$c_1^{(v)}(y)=1, \qquad c_2^{(v)}(y)=0 \tag{5.4.14}$$

On the other hand, the term proportional to the volume form is a series whose general term looks as follows

$$\frac{1}{\pi\Gamma(d+1)^2}z^2\lambda\Gamma(d-z\lambda)\Gamma(d+z\lambda)\Big(2H_d+\psi^{(0)}(-z\lambda)$$

$$+\psi^{(0)}(z\lambda)-\psi^{(0)}(d-z\lambda)-\psi^{(0)}(d+z\lambda)\Big)\sin(\pi z\lambda)$$

and has moreover a zw term, coming from the $e^{zp\log q}$ prefactor of the I function, which multiplies $f_1(w,y)$. Let us then fix the coefficients $c_i^{(w)}(z)$ by Taylor expanding the sum at $q=\exp w=0$. We get an expansion of the form $a\log q+b+o(1)$

$$A_2(w,y)=\left(c_1^{(w)}(y)-\frac{c_2^{(w)}(y)\left(H_{-y}+H_y+\log(q)\right)\sin(\pi y)}{\pi y}\right)+O\left(q^1\right)$$

while from the explicit form of the I function we get

$$A_2(w,y)=z\log q+\mathcal{O}(q)$$

Matching the logarithmic coefficient gives

$$c_2^{(w)}(y)=-z\frac{\pi y}{\sin(\pi y)} \tag{5.4.15}$$

while the $\mathcal{O}(1)$ term gives

$$c_1^{(w)}(y) = -\frac{\pi y \cot(\pi y)}{\lambda} - \frac{2yH_y}{\lambda} + \frac{1}{\lambda} \tag{5.4.16}$$

that completely fixes the form of the deformed flat coordinates. It is straightforward to check that the normalization condition (5.2.7)-(5.2.9) are satisfied.

□

5.4.3 The dispersionless Ablowitz–Ladik hierarchy

Let us work out the z-expansion of (5.4.10), (5.4.11) explicitly. Using (B.4.1), (B.4.2) we can write

$$\begin{aligned}
h_v(v,w,z,\lambda) &= -\frac{w}{\lambda^2} - \frac{vwz}{\lambda^2} + \Bigg[\frac{1}{6}w\left(-\frac{3v^2}{\lambda^2} - 6w\log\left(1-e^w\right) + \pi^2\right) \\
&- w\mathrm{Li}_2\left(1-e^w\right) - 2\mathrm{Li}_3\left(e^w\right)\Bigg] z^2 + \Bigg[\frac{1}{6}vw\Big(-\frac{v^2}{\lambda^2} - 6w\log\left(1-e^w\right) \\
&+ \pi^2\Big) - vw\mathrm{Li}_2\left(1-e^w\right) - 2v\mathrm{Li}_3\left(e^w\right)\Bigg] z^3 + O\left(z^4\right) \qquad (5.4.17) \\
h_w(v,w,z,\lambda) &= -\frac{v}{\lambda^2} + \left(\mathrm{Li}_2\left(e^w\right) - \frac{v^2}{2\lambda^2}\right) z + \left(v\mathrm{Li}_2\left(e^w\right) - \frac{v^3}{6\lambda^2}\right) z^2 \\
&+ \left(-\frac{v^4}{24\lambda^2} + \frac{1}{2}\mathrm{Li}_2\left(e^w\right)\left(v^2 - \lambda^2\mathrm{Li}_2\left(e^w\right)\right) + 2\lambda^2 S_{2,2}\left(e^w\right)\right) z^3 + O\left(z^4\right)
\end{aligned}$$
(5.4.18)

where $S_{\nu,p}(z)$ denotes the Nielsen polylogarithm

$$S_{\nu,p}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+p} k^{-\nu} z^k S_k^{(p)}}{k!}$$

and $S_l^{(p)}$ are the Stirling numbers of the second kind.

Let us focus on the first two non-trivial flows, which are generated by $H_{\alpha,0}$. $H_{v,0}$ generates space translation, $\partial_{t^{v,0}} = \partial_x$; on the other hand the $t := t^{w,0}$ flow is given by

$$\begin{aligned}
\partial_t v(x,t) &= \{v(x,t), H_{w,0}\} = \frac{\lambda^2}{e^{-w}-1}(w)_x &(5.4.19)\\
\partial_t w(x,t) &= \{w(x,t), H_{w,0}\} = (v)_x &(5.4.20)
\end{aligned}$$

Eliminating v we get

$$(w)_{tt} = \lambda^2 \left(\frac{1}{e^{-w}-1}(w)_x\right)_x \tag{5.4.21}$$

Equations (5.4.19) were recognized in [54] to be related to the dispersionless limit of an integrable system called the *Ablowitz–Ladik lattice* [2]; we will here review, almost verbatim, the arguments of [54] relating the solution of $WDVV$ (5.4.1) to such an integrable lattice. The latter is defined as

$$\begin{aligned} i\,\dot{a}_n &= -\frac{1}{2}(1-a_nb_n)(a_{n-1}+a_{n+1})+a_n \\ i\,\dot{b}_n &= \frac{1}{2}(1-a_nb_n)(b_{n-1}+b_{n+1})-b_n. \end{aligned} \tag{5.4.22}$$

Introducing new variables

$$\begin{aligned} u_n &= -\log(1-a_nb_n) \\ y_n &= \frac{1}{2i}\left(\log\frac{a_n}{a_{n-1}}-\log\frac{b_n}{b_{n-1}}\right). \end{aligned} \tag{5.4.23}$$

the evolution (5.4.22) can be written as a Hamiltonian flow generated by

$$H_{AL}=\sum_n\sqrt{(1-e^{-u_n})\,(1-e^{-u_{n-1}})}\,\cos y_n \tag{5.4.24}$$

with the Poisson bracket

$$\{u_n,y_m\}=\delta_{n,m-1}-\delta_{n,m},\quad \{u_n,u_m\}=\{y_n,y_m\}=0. \tag{5.4.25}$$

Taking the long-wave expansion means that we interpolate the space variable and rescale the time parameter

$$u_n=u(g_sn,g_st),\quad y_n=y(g_sn,g_st)$$

This leads, at leading order in g_s, to the dispersionless system

$$\begin{aligned} u_t &= \partial_X\left[(e^u-1)\,\sin y\right] \\ y_t &= \partial_X\left[e^{-u}\cos y\right]. \end{aligned} \tag{5.4.26}$$

In order to make contact with the Principal Hierarchy of the resolved conifold, we will follow the argument of [54] replacing $v(X)$,$u(X)$ by

$$x := i\lambda X \tag{5.4.27}$$

$$v(x) := iy(x)\lambda \tag{5.4.28}$$

$$w(x) := \frac{g_s\lambda\partial_x}{e^{g_s\lambda\partial_x}-1}\,u(x) \tag{5.4.29}$$

In this way, the Poisson brackets of w and v take the standard form (5.2.12), and the Hamiltonian (5.4.24) becomes upon interpolation

$$\begin{aligned} H_{\mathrm{AL}} &= \int h_{\mathrm{AL}}\, dx \\ &= \int \sqrt{\left(1-\exp\left\{\frac{1-e^{g_s\lambda\partial_x}}{g_s\lambda\partial_x}w\right\}\right)\left(1-\exp\left\{\frac{e^{-g_s\lambda\partial_x}-1}{g_s\lambda\partial_x}w\right\}\right)}\cosh\left(\frac{v}{\lambda}\right)dx \end{aligned} \tag{5.4.30}$$

$$h_{\mathrm{AL}} = (-1+e^w)\cosh\left(\frac{v}{\lambda}\right)+\frac{\left(e^w\lambda^2\cosh\left(\frac{v}{\lambda}\right)\left(4\left(-1+e^w\right)w_{xx}-3(w_{xx})^2\right)\right)g_s^2}{24\left(-1+e^w\right)}+O\left(g_s^4\right) \tag{5.4.31}$$

The long–wave limit of the AL system admits an infinite set of conserved currents, and a direct computation shows [54] that such densities coincide precisely with the coefficients of the z-expansion of the Hamiltonian densities of the Principal Hierarchy, thereby justifying the identification of the latter with the dispersionless Ablowitz–Ladik hierarchy.

5.4.4 The equivariant Gromov–Witten theory of the conifold and the dispersionful Ablowitz–Ladik hierarchy: the Main Conjecture

The Main Conjecture

Our results about the resolved conifold can be summarized in the following two statements: on one hand, we have written down the Principal Hierarchy associated to the genus zero subsector and singled out the correct normalization of the flows; on the other, we have borrowed and adapted the results of [54], that identify such a system with the dispersionless limit of the Ablowitz–Ladik hierarchy.

For the purposes of solving the full Gromov–Witten theory of X, the first result was a necessary, non–trivial, but still unsatisfactory step. From the point of view of geometry, in fact, knowing the Principal Hierarchy adds little to the complete knowledge we already have about gravitational correlators in genus zero: the latter are fully recovered, via the Dijkgraaf–Witten TRRs (2.1.25), by the primary invariants. This fortunate situation is however restricted to low genus only: for $g = 1$ a complete set of topological recursion relations were found by Getzler [77], and additional universal relations were established for $g = 2$ by [19, 78, 127], but apart from that very little (see *e.g.* [73]) has been established in general, without appealing to extra working hypothesis - like the Virasoro conjecture. This could be a, if not the main, domain of application of an integrable hierarchy associated to the Gromov–Witten theory of some target space, since it provides in a specific case a powerful tool to solve the full theory completely. For these reasons it is of great interest to establish a precise connection, however conjectural, to an integrable system of the PDEs in the form

(5.2.19) in order to obtain extract higher genus enumerative data on the geometry side. The discussion of the previous section pushes us to formulate the following

Conjecture 4. *The dispersionful Ablowitz–Ladik hierarchy is the integrable hierarchy associated to the all–genus equivariant Gromov–Witten theory of the resolved conifold* $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

The reader might feel that such a proposal is driven by some sort of psychological bias, due to the fact that all we happened to lay our hands on is not more than one acceptable candidate; in fact, lacking more information this would not rule out at all - and in fact tells us nothing about - that the right answer might be given by a different system, that shares with AL the same dispersionless limit. However, postulating that the dispersive corrections should really take the form of the Ablowitz–Ladik Hamiltonians is less naive than it might seem. First of all, it was shown in [54] that not every dispersive deformation of a two-component dispersionless limit preserves integrability: in our case, blindly discretizing space–derivatives would not result in an integrable system already at the leading order in the g_s–expansion. Therefore, if we insist to look for an *integrable* hierarchy governing the all–genus theory, it is already fortunate to have one such system at our disposal. Secondarily, and most importantly, Conjecture 4 can be effectively tested, at least perturbatively in g_s, which is what we now turn to do.

Dispersive deformation and D-operator formalism

To check Conjecture 4, it would be desirable to find explicitly the form of the dispersive corrections (5.2.19) to the Hamiltonians of the Principal Hierarchy (5.4.10), (5.4.11). Without a quasi–homogeneous prepotential we cannot avail ourselves of the Reconstruction Theorem of [53]; however, assuming (5.4.30) as the full dispersive completion of the dispersionless density

$$h_{AL}^{(0)} = (1 - e^w) \cosh v/\lambda \tag{5.4.32}$$

is sufficient to determine the dispersive deformation of *all* flows of the Principal Hierarchy. A method to do it was devised in [54] and relies on the construction of a so–called D–operator, which we here review.

Let H be one of the Hamiltonians of a dispersionless Hamiltonian system (5.2.13), and let $\hat{H}$ be a dispersive deformation of the form

$$\hat{H} = H + g_s H^{(1)} + g_s^2 H^{(2)} + \dots \tag{5.4.33}$$

The dispersionless hierarchy (5.2.13) has an infinite set of conserved Hamiltonians $\{H_{\alpha,p}\}$, which Poisson commute with $\hat{H}$ at leading order in g_s

$$\left\{ H_{\alpha,p}, \hat{H} \right\} = \mathcal{O}(g_s) \quad \forall \alpha, p$$

What we would like to find is a dispersive deformation of each dispersionless Hamiltonian $H_{\alpha,p}$ in the form (5.4.33)

$$\hat{H}_{\alpha,p} = H_{\alpha,p} + g_s\, H^{(1)}_{\alpha,p} + g_s^2 H^{(2)}_{\alpha,p} + \dots \tag{5.4.34}$$

which be such that

$$\left\{\hat{H}_{\alpha,p}, \hat{H}\right\} = 0 \quad \forall \alpha, p \tag{5.4.35}$$

A key object in the determination of the dispersive corrections (5.4.34) is a so-called D–operator. We recall its definition from [54], adapting it to the case of the Ablowitz–Ladik system with dependent variables v, w (5.4.27).

Definition 16. *The linear differential operator*

$$D = D^{[0]} + g_s\, D^{[1]} + g_s^2 D^{[2]} + \dots$$

$$D^{[0]} = \mathrm{id}, \quad D^{[k]} = \sum b^{[k]}_{i_1,i_2}(w, w_x, \dots, v, v_x, \dots) \frac{\partial^{m(k)}}{\partial v^{i_1} \dots \partial w^{i_2}} \tag{5.4.36}$$

*is said to be a D-*operator *for the perturbations* (5.4.33), (5.4.34) *if for any dispersionless Hamiltonian density $h_{\alpha,p}$ the Hamiltonian*

$$\hat{H}_{\alpha,p} = \int D h_{\alpha,p}\, dx \tag{5.4.37}$$

satisfies the involutivity condition (5.4.35).

In (5.4.36), the coefficients $b^{[k]}_{i_1,i_2}$ are smooth functions of v, w in some domain $\mathcal{D} \subset \mathbb{R}^2$, and depend polynomially on the jet coordinates w_x, w_{xx}, v_x, v_{xx}, ... We have denoted $m(k) = \left[\frac{3k}{2}\right]$, and i_1, i_2 are non-negative integers such that $i_1 + i_2 = m(k)$.

A D-operator need not exist for an arbitrary perturbation (5.4.33); as we have already mentioned, not every dispersive perturbation preserves integrability. Moreover, picking $H = h^{(0)}_{AL} = (1 - e^w)\cosh(v/\lambda)$, (5.4.35) implies that the tree–level densities $h_{\alpha,p}$ satisfy

$$\partial_w^2 h_{\alpha,p} - \frac{\lambda^2}{e^{-w} - 1}\partial_v^2 h_{\alpha,p} = 0 \tag{5.4.38}$$

Therefore a normal form for this operator should be chosen, for example by keeping derivatives of order at most 1 with respect to u. Apart from this ambiguity, it was shown in [54] that if a D-operator (5.4.36) exists, it is unique. We can rephrase the last statement as follows: the involutivity condition (5.4.35) implies that knowing *one* perturbed Hamiltonian allows to reconstruct the dispersive completion of *all* the Hamiltonians of the Principal Hierarchy, order by order in g_s. In our case, postulating that (5.4.30) be the perturbation of (5.4.32) allows to reconstruct the flows related to higher genus descendant invariants.

A check at genus one

We can now apply the above arguments to determine explicitly the D–operator of the AL hierarchy at one–loop. This was already computed in [54]; the result here differs slightly because of the minor differences between the prepotential (5.4.1) considered in [54] and the genus zero Gromov–Witten potential of the resolved conifold (5.3.4), (5.3.10). Imposing the involution condition (5.4.35), we get

$$\begin{aligned} D_{AL} f &= f - g_s^2 \left[\frac{e^{w(x)}\left(-1+2e^{w(x)}\right) w'(x)^2 f_{vv} \lambda^4}{24\left(-1+e^{w(x)}\right)^2} + \frac{e^{w(x)} w'(x)^2 f_{vvw} \lambda^4}{12\left(-1+e^{w(x)}\right)} \right. \\ &+ \left. \frac{e^{w(x)} w'(x) v'(x) f_{vvv} \lambda^4}{6\left(-1+e^{w(x)}\right)} + \frac{v'(x)^2 f_{vv} \lambda^2}{-12+12e^{-w(x)}} + \frac{1}{12} v'(x)^2 f_{vvw} \lambda^2 \right] + \mathcal{O}(g_s^4) \end{aligned} \tag{5.4.39}$$

Let us apply it to the first non–trivial Hamiltonian flow associated to the Kähler class $[\mathbb{P}^1]$, i.e. $H_{w,0}$. At $\mathcal{O}(g_s^2)$ we get

$$H_{w,0} = \int \left[-\frac{v^2}{2\lambda^2} + \mathrm{Li}_2(e^w) + g_s^2 \left(\frac{e^{w(x)}\left(-1+2e^{w(x)}\right)\lambda^2 w'(x)^2}{24\left(-1+e^{w(x)}\right)^2} + \frac{v'(x)^2}{-12+12e^{-w(x)}} \right) \right] dx \tag{5.4.40}$$

With the perturbed Hamiltonian (5.4.40) at hand, we can perform a first check of Conjecture 4. Recall [52] that the reduction to primaries is obtained by setting

$$u_x \Big|_{t^{\alpha,p}=0 \quad p>0} = \Phi_0 \tag{5.4.41}$$

Moreover, the Hamiltonian densities are related (see (5.2.16)) to second derivatives of the logarithm of the topological τ–function, which in turn, according to Conjecture 4, should be related to "big" 2–point correlators (2.1.24)

$$h_{\beta,p}\Big|_{s.p.s.} = \partial_x \partial_{t^{\beta,p}} \ln \tau \Big|_{s.p.s.} = \langle\langle \tau_p \Phi_\beta v \Phi_0 \rangle\rangle = \langle\langle \tau_{p-1} \Phi_\beta \rangle\rangle \tag{5.4.42}$$

where by $|_{s.p.s.}$ we have denoted the reduction to small phase space (5.4.41). Combining (5.4.40), (5.4.41) and (5.4.42) with $p=1$ we obtain the following

Corollary 2. *Assuming Conjecture 4, the genus 1 primary Gromov–Witten potential F_1 of the resolved conifold with anti–diagonal action is given by*

$$\langle\langle \tau_0 \Phi_\beta \rangle\rangle = \frac{\partial F_1}{\partial w} = \frac{1}{12} \mathrm{Li}_0(e^w) \tag{5.4.43}$$

This result agrees perfectly with the known answer in Gromov–Witten theory, for example from the localization computation of [30] (see also [100, 116])

$$F_g(w) = \chi(\mathcal{M}_g) \mathrm{Li}_{3-2g}(e^w) \tag{5.4.44}$$

for $g=1$.

Chapter 6

Conclusions

The results of this thesis open the way for various avenues of research. We will mention here a few of them, subdivided into the three main directions that we have followed.

Mirror symmetry and wall-crossings

In Chapter 3 we have proposed an exact solution for the B-model on a 2–parameter family of mirrors of toric $CY3$ which applies to the full B-model moduli space, including orbifold and conifold divisors. Our focus was mostly on local Hirzebruch surfaces, that is, $Y^{p,q}$ geometries with $p = 2$; extending it to the more complicated $p > 2$ case is just technically more involved, and would allow for a considerably simplified study of a number of new singular points, including Argyres–Douglas like conifold points [13] and a variety of orbifold points of the form $\mathbb{C}^3/\mathbb{Z}_n$. It is also noteworthy to see that our methods can be extended beyond the case of $Y^{p,q}$: it is in fact straightforward to show that holomorphicity of (derivatives of) the differential can be proven whenever the mirror curve is hyperelliptic[1], like for example for local Del Pezzo surfaces.
Another interesting aspect is the relationship with integrable systems. The identification of the mirror geometry in the $q = p$ case as a fibration over the spectral curve of the relativistic A_{p-1} Toda chain pushes to give a meaning to the q parameter as an integrable deformation of the Toda chain, perhaps leading to new classes of algebraically integrable systems.
Finally, it would be worthwile to back the large number of B-model computations of of Sec 3.3.4 with further checks, especially at the orbifold point. We already mentioned the result, which will appear in joint work with R. Cavalieri [28], that the A-model orbifold disc function computed via localization [33] agrees, at the first few order in the open moduli expansion, to all orders in the closed moduli with the predictions from mirror symmetry, up to signs. It would be very interesting, first of all, to show a complete identity between the A-model and the B-model generating functions, therefore giving a full proof of the mirror symmetry computation in this case, and secondarily to perform an explicit A-model check of the predictions for the annulus function, which, by the non-trivial almost modular structure of the Bergmann kernel would constitute a remarkable verification of Conjecture 2 in an open string case. We plan to report on this in the near future [28].

Geometric transitions and Gopakumar–Vafa duality

Even though our proof of Claim 1 in Chapter 4 might appear to be an obstruction to the program of extending the duality of [83] to more general backgrounds, let us outline two possible avenues of further investigation which we think might lead to the solution of the puzzle for the case under scrutiny.

[1] At a pictorial level, this class coincides with those toric CY whose toric diagram is contained into a vertical strip of width 2, modulo $SL(2,\mathbb{Z})$ transformations.

One possible way out is to regard GV duality as an identity between the full CS partition on a 3-manifold M and some suitable *non-perturbative* definition of the $A-$model on the $CY3$ obtained through conifold geometric transition from T^*M. Indeed, as first advocated[2] in [120], a proper non-perturbative definition of the $A-$model on toric target spaces with a dual matrix integral description should be given in terms of a filling fraction independent sum over multi-instanton sectors. This would be dual to the proper definition of the Reshetikin-Turaev-Witten invariant as a sum over flat connections.
A second possibility, hinted at by the geometric picture arising in the discussion of section 2, might consist in a refinement of the notion of "orbifold of the GV duality for S^3" in order to properly encompass the case of the generic lens space. Indeed, for $1 < q < p-1$ the cyclic group does no longer act fiberwise on the resolved conifold, giving rise to an orbibundle over a rational curve with marked points (see Remark 13). This new feature with respect to the $q = 1$ case definitely begs for further understanding, in order to clarify the correct formulation of GV duality in this case as well as its possible relation with a (suitably twisted) Gromov-Witten theory of orbicurves.

The local Gromov–Witten theory of curves and integrable hierarchies

The results of Chapter 5 raise a large number of new problems and directions of research. Let us list a few of them:

- the first, impelling problem is a thorough verification of Conjecture 4. From a computational point of view, the reconstruction of the D-operator is feasible, albeit quite lengthy, for genus 2. This would immediately yield an extension of Corollary 2 to genus 2. The real testing ground however is the correct computation of descendant invariants from the perturbative hierarchy, at least for $g \leq 2$. In particular, if it could be proven that the AL flows imply the genus 1 [77] and the genus 2 [19, 78, 127] topological recursion relations, which in the semi–simple case exhaust the whole set of universal relations between gravitational Gromov–Witten invariants, then this would be the smoking gun of the validity of Conjecture 4 at higher genus. A key tool for a proof of these statements could be given by a perturbative reconstruction of the dispersive τ-function, possibly via a quasi–triviality transformation as in the ordinary Frobenius manifold case [53]. These topics will be addressed in a forthcoming paper [29].

- it would also be nice to see how the results for the resolved conifold with antidiagonal action could be adapted to more general cases, where the quantum part of the prepotential is still dictated by the Aspinwall–Morrison formula.

[2]See also [64] for related work on background independence and [121, §6.3], [64, §5.2] for a discussion precisely about the case of topological strings with a $L(p, 1)$ Chern-Simons matrix model representation.

Examples are given by the $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k-2)$ bundle with diagonal action and the $k = 2$ case with generic torus action. It would also be fascinating, moreover, to study in the integrable systems setting the $k \to \infty$ limit, which is related to Hurwitz theory and the ordinary Toda hierarchy. A further extension could be given by the study of multi–parameter cases, such as the local "chains of $\mathbb{P}^1$" as well as their orbifold points of the form $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$.

- on a more conceptual level, it would be of great interest to re-interpret topics in Gromov–Witten theory in the framework of integrable hierarchies. One example is given by the Crepant Resolution Conjecture, which in the cases mentioned above, like $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$, could perhaps be regarded as the same kind of transition that relates, in the Gelfand-Dikii case, topological $(p, 1)$ minimal models to (p, q) minimal models, or ordinary to topological $2d$ gravity - that is, a flow of the *same* solution at a different time. Another topic is the re-interpretation of the relationship between descendant invariants and open string invariants, as suggested for example by localization on orbifolds of $\mathbb{C}^3$ [33]. Moreover, it would be of sure interest to find a mirror symmetry description of equivariant Gromov–Witten theory in terms of spectral curves, and therefore find a connection with the Eynard–Orantin recursion. Finally, the explicit construction of the hierarchies might also be a playground for the study of higher genus relations in equivariant Gromov–Witten theory. A natural question is whether the Virasoro conjecture should still be expected to be true, or if some other, more general kind of symmetries, like $\mathcal{W}$-symmetries take over in this general setting.

- to conclude, there are a number of applications to the physics of type II/M theory compactifications that beg for further understanding. It would be interesting to find a clear interpretation of the equivariant deformation and to find a meaning for the gravitational correlators from a target–space point of view, perhaps due to couplings to suitable background superfields. An important issue is the relationship of the AL flows with UV deformations of $SU(2)$ Seiberg–Witten theory on $\mathbb{R}^5$, which is geometrically engineered by M-theory compactification on local curves. This would generalize to the non-abelian (yet purely perturbative, from a four-dimensional point of view) case the results of [114, 115] for abelian extended Seiberg-Witten theories.

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Appendix A

The $\mathbb{C}^3/\mathbb{Z}_4$ computation

A.1 Predictions for open orbifold Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$

A.1.1 $g = 0$, $h = 1$

n \ m	1	3	5	7	9
0	1	0	$-\frac{3}{64}$	0	$-\frac{1491}{4096}$
1	0	$\frac{1}{32}$	0	$\frac{207}{2048}$	0
2	$-\frac{1}{16}$	0	$-\frac{37}{1024}$	0	$-\frac{79869}{65536}$
3	0	$\frac{9}{512}$	0	$\frac{9963}{32768}$	0
4	$-\frac{3}{256}$	0	$-\frac{1551}{16384}$	0	$-\frac{8292567}{1048576}$
5	0	$\frac{321}{8192}$	0	$\frac{940047}{524288}$	0
6	$-\frac{101}{4096}$	0	$-\frac{130737}{262144}$	0	$-\frac{1385156769}{16777216}$
7	0	$\frac{23689}{131072}$	0	$\frac{144472923}{8388608}$	0
8	$-\frac{6343}{65536}$	0	$-\frac{18265531}{4194304}$	0	$-\frac{337142625627}{268435456}$
9	0	$\frac{2963841}{2097152}$	0	$\frac{32673216687}{134217728}$	0
10	$-\frac{696201}{1048576}$	0	$-\frac{3803089437}{67108864}$	0	$-\frac{112241155641669}{4294967296}$

Table A.1: Predictions for $g = 0$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 1.

n \ m	0	2	4	6	8	10
0	0	$-\frac{1}{2}$	0	$\frac{9}{64}$	0	$\frac{861}{512}$
1	$\frac{1}{2}$	0	$-\frac{1}{16}$	0	$-\frac{93}{256}$	0
2	0	$\frac{1}{16}$	0	$\frac{23}{256}$	0	$\frac{44133}{8192}$
3	$-\frac{1}{8}$	0	$-\frac{3}{128}$	0	$-\frac{1047}{1024}$	0
4	0	0	0	$\frac{225}{1024}$	0	$\frac{548475}{16384}$
5	$\frac{1}{32}$	0	$-\frac{27}{512}$	0	$-\frac{47061}{8192}$	0
6	0	$\frac{7}{512}$	0	$\frac{2259}{2048}$	0	$\frac{44197653}{131072}$
7	$-\frac{1}{128}$	0	$-\frac{239}{1024}$	0	$-\frac{1741719}{32768}$	0
8	0	$\frac{25}{512}$	0	$\frac{304325}{32768}$	0	$\frac{1305131775}{262144}$
9	$\frac{1}{512}$	0	$-\frac{7251}{4096}$	0	$-\frac{95530011}{131072}$	0
10	0	$\frac{339}{1024}$	0	$\frac{15386001}{131072}$	0	$\frac{211795044723}{2097152}$
11	$-\frac{1}{2048}$	0	$-\frac{1340031}{65536}$	0	$-\frac{7258371969}{524288}$	0

Table A.2: Predictions for $g = 0$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 2.

n \ m	1	3	5	7	9
0	0	$\frac{2}{3}$	0	$-\frac{21}{32}$	0
1	$-\frac{1}{3}$	0	$\frac{43}{192}$	0	$\frac{8417}{4096}$
2	0	$-\frac{7}{48}$	0	$-\frac{1229}{3072}$	0
3	$\frac{7}{48}$	0	$\frac{73}{1024}$	0	$\frac{412041}{65536}$
4	0	$\frac{1}{96}$	0	$-\frac{4697}{4096}$	0
5	$-\frac{41}{768}$	0	$\frac{11963}{49152}$	0	$\frac{41384697}{1048576}$
6	0	$-\frac{299}{4096}$	0	$-\frac{1783953}{262144}$	0
7	$\frac{229}{4096}$	0	$\frac{362393}{262144}$	0	$\frac{6936052081}{16777216}$
8	0	$-\frac{11747}{32768}$	0	$-\frac{142831999}{2097152}$	0
9	$\frac{26639}{196608}$	0	$\frac{56047641}{4194304}$	0	$\frac{1730826138737}{268435456}$
10	0	$-\frac{10669787}{3145728}$	0	$-\frac{68424562163}{67108864}$	0
11	$\frac{4170167}{3145728}$	0	$\frac{38784663739}{201326592}$	0	$\frac{598285614606521}{4294967296}$
12	0	$-\frac{196698881}{4194304}$	0	$-\frac{2844990148951}{134217728}$	0

Table A.3: Predictions for $g = 0$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 3.

A.1.2 $g = 0$, $h = 2$

n \ m	0	2	4	6	8	10
0	0	$\frac{1}{8}$	0	$\frac{3}{128}$	0	$\frac{4893}{4096}$
1	$-\frac{1}{4}$	0	$-\frac{1}{128}$	0	$-\frac{33}{128}$	0
2	0	0	0	$\frac{35}{512}$	0	$\frac{55935}{8192}$
3	$\frac{1}{32}$	0	$-\frac{3}{128}$	0	$-\frac{22407}{16384}$	0
4	0	$\frac{3}{256}$	0	$\frac{21}{64}$	0	$\frac{4220337}{65536}$
5	$-\frac{1}{64}$	0	$-\frac{201}{2048}$	0	$-\frac{195789}{16384}$	0
6	0	$\frac{5}{128}$	0	$\frac{42915}{16384}$	0	$\frac{118015665}{131072}$
7	$-\frac{23}{1024}$	0	$-\frac{2881}{4096}$	0	$-\frac{20423991}{131072}$	0
8	0	$\frac{63}{256}$	0	$\frac{2067209}{65536}$	0	$\frac{18287528607}{1048576}$
9	$-\frac{131}{1024}$	0	$-\frac{506487}{65536}$	0	$-\frac{185694621}{65536}$	0
10	0	$\frac{9945}{4096}$	0	$\frac{140110485}{262144}$	0	$\frac{58498600995}{131072}$
11	$-\frac{9049}{8192}$	0	$-\frac{1981089}{16384}$	0	$-\frac{287144765007}{4194304}$	0
12	0	$\frac{4542901}{131072}$	0	$\frac{3177309831}{262144}$	0	$\frac{3818835874293}{262144}$

Table A.4: Predictions for $g = 0$, $h = 2$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(1, 1)$.

n \ m	1	3	5	7	9
0	0	$-\frac{3}{16}$	0	$-\frac{21}{1024}$	0
1	$\frac{1}{8}$	0	$-\frac{3}{512}$	0	$\frac{27309}{32768}$
2	0	$\frac{5}{256}$	0	$-\frac{2505}{16384}$	0
3	$-\frac{5}{128}$	0	$\frac{255}{8192}$	0	$\frac{2302815}{524288}$
4	0	$-\frac{35}{4096}$	0	$-\frac{189045}{262144}$	0
5	$\frac{21}{2048}$	0	$\frac{15737}{131072}$	0	$\frac{312154689}{8388608}$
6	0	$-\frac{795}{65536}$	0	$-\frac{23293305}{4194304}$	0
7	$-\frac{465}{32768}$	0	$\frac{1732635}{2097152}$	0	$\frac{63058278675}{134217728}$
8	0	$-\frac{80835}{1048576}$	0	$-\frac{4329443445}{67108864}$	0
9	$-\frac{25879}{524288}$	0	$\frac{291969237}{33554432}$	0	$\frac{17804785464789}{2147483648}$
10	0	$-\frac{11668795}{16777216}$	0	$-\frac{1135730450505}{1073741824}$	0
11	$-\frac{3887325}{8388608}$	0	$\frac{70313936215}{536870912}$	0	$\frac{6698323708804935}{34359738368}$
12	0	$-\frac{2459089635}{268435456}$	0	$-\frac{400027626445845}{17179869184}$	0

Table A.5: Predictions for $g = 0$, $h = 2$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(2, 1)$.

n \ m	0	2	4	6	8	10
0	$\frac{1}{2}$	0	$\frac{3}{8}$	0	$-\frac{21}{128}$	0
1	0	$-\frac{1}{8}$	0	$\frac{3}{32}$	0	$-\frac{11463}{4096}$
2	0	0	$-\frac{5}{64}$	0	$\frac{45}{128}$	0
3	0	$\frac{1}{16}$	0	$-\frac{3}{64}$	0	$-\frac{122187}{8192}$
4	0	0	$\frac{5}{256}$	0	$\frac{6645}{4096}$	0
5	0	$-\frac{7}{256}$	0	$-\frac{233}{2048}$	0	$-\frac{4019679}{32768}$
6	0	0	$-\frac{15}{512}$	0	$\frac{98805}{8192}$	0
7	0	$\frac{17}{512}$	0	$-\frac{12633}{16384}$	0	$-\frac{196225827}{131072}$
8	0	0	$-\frac{235}{2048}$	0	$\frac{8814945}{65536}$	0
9	0	$\frac{31}{512}$	0	$-\frac{246663}{32768}$	0	$-\frac{6705257547}{262144}$
10	0	0	$-\frac{39985}{32768}$	0	$\frac{17375355}{8192}$	0
11	0	$\frac{2641}{4096}$	0	$-\frac{1721123}{16384}$	0	$-\frac{612160163421}{1048576}$
12	0	0	$-\frac{2340015}{131072}$	0	$\frac{23610629685}{524288}$	0

Table A.6: Predictions for $g = 0$, $h = 2$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(2, 2)$.

A.1.3 $g = 0$, $h = 3$

n \ m	1	3	5	7	9
0	0	$-\frac{3}{64}$	0	$\frac{189}{4096}$	0
1	$\frac{1}{32}$	0	$-\frac{33}{2048}$	0	$-\frac{132711}{131072}$
2	0	$\frac{11}{1024}$	0	$\frac{14547}{65536}$	0
3	$-\frac{7}{512}$	0	$-\frac{1989}{32768}$	0	$-\frac{18108903}{2097152}$
4	0	$\frac{353}{16384}$	0	$\frac{1809801}{1048576}$	0
5	$-\frac{79}{8192}$	0	$-\frac{218993}{524288}$	0	$-\frac{3593874231}{33554432}$
6	0	$\frac{33711}{262144}$	0	$\frac{330787647}{16777216}$	0
7	$-\frac{7287}{131072}$	0	$-\frac{36190149}{8388608}$	0	$-\frac{990617610423}{536870912}$
8	0	$\frac{4907493}{4194304}$	0	$\frac{84814988181}{268435456}$	0
9	$-\frac{889439}{2097152}$	0	$-\frac{8528369313}{134217728}$	0	$-\frac{363568459048071}{8589934592}$
10	0	$\frac{1045989811}{67108864}$	0	$\frac{29188217357547}{4294967296}$	0
11	$-\frac{167510567}{33554432}$	0	$-\frac{2728134070309}{2147483648}$	0	$-\frac{171647294174135943}{137438953472}$
12	0	$\frac{307481197833}{1073741824}$	0	$\frac{13004327932052961}{68719476736}$	0

Table A.7: Predictions for $g = 0$, $h = 3$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(1, 1, 1)$.

n \ m	0	2	4	6	8	10
0	$\frac{1}{2}$	0	$\frac{9}{64}$	0	$-\frac{21}{128}$	0
1	0	$-\frac{1}{16}$	0	$\frac{3}{64}$	0	$\frac{28137}{8192}$
2	$\frac{1}{16}$	0	$-\frac{7}{256}$	0	$-\frac{4377}{8192}$	0
3	0	$\frac{3}{128}$	0	$\frac{87}{1024}$	0	$\frac{464121}{16384}$
4	$-\frac{1}{32}$	0	$-\frac{3}{512}$	0	$-\frac{64593}{16384}$	0
5	0	$-\frac{7}{512}$	0	$\frac{4509}{8192}$	0	$\frac{11087523}{32768}$
6	$\frac{17}{512}$	0	$-\frac{111}{2048}$	0	$-\frac{5600217}{131072}$	0
7	0	$-\frac{1}{64}$	0	$\frac{170919}{32768}$	0	$\frac{1472070201}{262144}$
8	$\frac{9}{512}$	0	$-\frac{11867}{32768}$	0	$-\frac{85512669}{131072}$	0
9	0	$-\frac{423}{2048}$	0	$\frac{4679277}{65536}$	0	$\frac{16303071447}{131072}$
10	$\frac{1091}{4096}$	0	$-\frac{493299}{131072}$	0	$-\frac{14083706541}{1048576}$	0
11	0	$-\frac{176659}{65536}$	0	$\frac{701236689}{524288}$	0	$\frac{29811430249887}{8388608}$
12	$\frac{22219}{8192}$	0	$-\frac{7019643}{131072}$	0	$-\frac{753800096679}{2097152}$	0

Table A.8: Predictions for $g = 0$, $h = 3$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(2, 1, 1)$.

n \ m	1	3	5	7	9
0	$-\frac{3}{4}$	0	$-\frac{111}{256}$	0	$\frac{19593}{16384}$
1	0	$\frac{15}{128}$	0	$-\frac{2595}{8192}$	0
2	$-\frac{1}{64}$	0	$\frac{603}{4096}$	0	$\frac{595491}{262144}$
3	0	$-\frac{177}{2048}$	0	$-\frac{40479}{131072}$	0
4	$\frac{65}{1024}$	0	$\frac{1245}{65536}$	0	$\frac{63319245}{4194304}$
5	0	$\frac{895}{32768}$	0	$-\frac{4055235}{2097152}$	0
6	$-\frac{541}{16384}$	0	$\frac{343623}{1048576}$	0	$\frac{10240884231}{67108864}$
7	0	$-\frac{68737}{524288}$	0	$-\frac{568999599}{33554432}$	0
8	$\frac{34245}{262144}$	0	$\frac{39364105}{16777216}$	0	$\frac{2384870136465}{1073741824}$
9	0	$-\frac{5665425}{8388608}$	0	$-\frac{119917956675}{536870912}$	0
10	$\frac{1984519}{4194304}$	0	$\frac{7678005843}{268435456}$	0	$\frac{757515211867371}{17179869184}$
11	0	$-\frac{1112041297}{134217728}$	0	$-\frac{35129545858719}{8589934592}$	0
12	$\frac{386924425}{67108864}$	0	$\frac{2109027490965}{4294967296}$	0	$\frac{315113944321865685}{274877906944}$

Table A.9: Predictions for $g = 0$, $h = 3$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(2, 2, 1)$.

n \ m	1	3	5	7	9
0	$-\frac{2}{3}$	0	$-\frac{19}{32}$	0	$\frac{3017}{2048}$
1	0	$\frac{35}{192}$	0	$-\frac{1655}{4096}$	0
2	$-\frac{1}{16}$	0	$\frac{599}{3072}$	0	$\frac{216111}{65536}$
3	0	$-\frac{127}{1024}$	0	$-\frac{34639}{65536}$	0
4	$\frac{7}{64}$	0	$\frac{71}{1024}$	0	$\frac{7515233}{262144}$
5	0	$\frac{385}{16384}$	0	$-\frac{4900175}{1048576}$	0
6	$-\frac{733}{12288}$	0	$\frac{263963}{262144}$	0	$\frac{6306454721}{16777216}$
7	0	$-\frac{280541}{786432}$	0	$-\frac{972023479}{16777216}$	0
8	$\frac{8599}{32768}$	0	$\frac{71107289}{6291456}$	0	$\frac{935960475531}{134217728}$
9	0	$-\frac{12976415}{4194304}$	0	$-\frac{276764938375}{268435456}$	0
10	$\frac{1372579}{1048576}$	0	$\frac{12755007193}{67108864}$	0	$\frac{744181335698931}{4294967296}$
11	0	$-\frac{3203261567}{67108864}$	0	$-\frac{105795705480319}{4294967296}$	0
12	$\frac{28966097}{1572864}$	0	$\frac{1153922108479}{268435456}$	0	$\frac{47596772308516649}{8589934592}$

Table A.10: Predictions for $g = 0$, $h = 3$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(3, 1, 1)$.

A.1.4 $g = 1$, $h = 1$

n \ m	1	3	5	7	9
0	$\frac{1}{48}$	0	$\frac{9}{1024}$	0	$\frac{31003}{65536}$
1	0	$-\frac{5}{1536}$	0	$-\frac{3375}{32768}$	0
2	$\frac{1}{768}$	0	$\frac{1367}{49152}$	0	$\frac{3206563}{1048576}$
3	0	$-\frac{81}{8192}$	0	$-\frac{326497}{524288}$	0
4	$\frac{65}{12288}$	0	$\frac{40345}{262144}$	0	$\frac{534647035}{16777216}$
5	0	$-\frac{19145}{393216}$	0	$-\frac{50714835}{8388608}$	0
6	$\frac{4321}{196608}$	0	$\frac{5760669}{4194304}$	0	$\frac{129857120323}{268435456}$
7	0	$-\frac{2469623}{6291456}$	0	$-\frac{11529490917}{134217728}$	0
8	$\frac{490945}{3145728}$	0	$\frac{3642090395}{201326592}$	0	$\frac{43135367843675}{4294967296}$
9	0	$-\frac{158835215}{33554432}$	0	$-\frac{3604297162935}{2147483648}$	0
10	$\frac{85184641}{50331648}$	0	$\frac{354513303549}{1073741824}$	0	$\frac{18747883132922083}{68719476736}$
11	0	$-\frac{128738647003}{1610612736}$	0	$-\frac{1481653476327337}{34359738368}$	0
12	$\frac{21004177025}{805306368}$	0	$\frac{137005640391385}{17179869184}$	0	$\frac{10315653154790585915}{1099511627776}$

Table A.11: Predictions for $g = 1$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 1.

n \ m	0	2	4	6	8	10
0	0	$-\frac{1}{32}$	0	$-\frac{3}{512}$	0	$-\frac{30093}{16384}$
1	$\frac{1}{48}$	0	$-\frac{1}{512}$	0	$\frac{19}{64}$	0
2	0	$\frac{1}{192}$	0	$-\frac{27}{512}$	0	$-\frac{200327}{16384}$
3	$-\frac{5}{384}$	0	$\frac{5}{512}$	0	$\frac{121865}{65536}$	0
4	0	$-\frac{5}{3072}$	0	$-\frac{315}{1024}$	0	$-\frac{33148025}{262144}$
5	$\frac{1}{768}$	0	$\frac{1277}{24576}$	0	$\frac{1175231}{65536}$	0
6	0	$-\frac{11}{2048}$	0	$-\frac{89893}{32768}$	0	$-\frac{247637721}{131072}$
7	$-\frac{85}{12288}$	0	$\frac{6995}{16384}$	0	$\frac{131562305}{524288}$	0
8	0	$-\frac{275}{6144}$	0	$-\frac{9362985}{262144}$	0	$-\frac{161674997735}{4194304}$
9	$-\frac{389}{12288}$	0	$\frac{1350073}{262144}$	0	$\frac{631910777}{131072}$	0
10	0	$-\frac{25459}{49152}$	0	$-\frac{337003153}{524288}$	0	$-\frac{8635006486289}{8388608}$
11	$-\frac{27155}{98304}$	0	$\frac{4250765}{49152}$	0	$\frac{2044307220305}{16777216}$	0
12	0	$-\frac{4343505}{524288}$	0	$-\frac{16051763495}{1048576}$	0	$-\frac{584421859946805}{16777216}$

Table A.12: Predictions for $g = 1$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 2.

n \ m	1	3	5	7	9
0	0	$\frac{1}{12}$	0	$-\frac{21}{256}$	0
1	$-\frac{5}{144}$	0	$\frac{115}{3072}$	0	$-\frac{258695}{196608}$
2	0	$-\frac{59}{2304}$	0	$\frac{9679}{49152}$	0
3	$\frac{53}{2304}$	0	$-\frac{1903}{49152}$	0	$-\frac{34606561}{3145728}$
4	0	$\frac{305}{18432}$	0	$\frac{219465}{131072}$	0
5	$-\frac{235}{12288}$	0	$-\frac{716555}{2359296}$	0	$-\frac{6451124795}{50331648}$
6	0	$\frac{4819}{65536}$	0	$\frac{240631049}{12582912}$	0
7	$-\frac{16007}{589824}$	0	$-\frac{14352681}{4194304}$	0	$-\frac{567229333447}{268435456}$
8	0	$\frac{962165}{1179648}$	0	$\frac{7715651635}{25165824}$	0
9	$-\frac{2979965}{9437184}$	0	$-\frac{3575613975}{67108864}$	0	$-\frac{608448940883375}{12884901888}$
10	0	$\frac{1821378401}{150994944}$	0	$\frac{21402084232819}{3221225472}$	0
11	$-\frac{207837889}{50331648}$	0	$-\frac{10779639149749}{9663676416}$	0	$-\frac{94491992113090307}{68719476736}$
12	0	$\frac{32205472535}{134217728}$	0	$\frac{1609762782468295}{8589934592}$	0

Table A.13: Predictions for $g = 1$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 3.

n \ m	0	2	4	6	8	10
0	$\frac{1}{3}$	0	$-\frac{5}{16}$	0	$\frac{315}{256}$	0
1	0	$\frac{3}{32}$	0	$-\frac{181}{512}$	0	$\frac{106299}{16384}$
2	$-\frac{1}{24}$	0	$\frac{39}{256}$	0	$-\frac{109}{128}$	0
3	0	$-\frac{11}{128}$	0	$\frac{387}{2048}$	0	$\frac{5026167}{65536}$
4	$\frac{7}{96}$	0	$-\frac{3}{32}$	0	$-\frac{162943}{16384}$	0
5	0	$\frac{233}{3072}$	0	$\frac{11689}{8192}$	0	$\frac{132120547}{131072}$
6	$-\frac{31}{384}$	0	$-\frac{403}{2048}$	0	$-\frac{8297873}{65536}$	0
7	0	$-\frac{331}{12288}$	0	$\frac{286821}{16384}$	0	$\frac{2328425659}{131072}$
8	$\frac{187}{1536}$	0	$-\frac{31927}{12288}$	0	$-\frac{70090611}{32768}$	0
9	0	$\frac{1389}{4096}$	0	$\frac{73463763}{262144}$	0	$\frac{868198249737}{2097152}$
10	$\frac{389}{6144}$	0	$-\frac{5027977}{131072}$	0	$-\frac{25016794729}{524288}$	0
11	0	$\frac{193951}{49152}$	0	$\frac{6224145569}{1048576}$	0	$\frac{12994166797747}{1048576}$
12	$\frac{47767}{24576}$	0	$-\frac{99929913}{131072}$	0	$-\frac{2869421365529}{2097152}$	0

Table A.14: Predictions for $g = 1$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 4.

A.1.5 $g = 1,\ h = 2$

$n \backslash m$	0	2	4	6	8	10
0	0	$\frac{1}{192}$	0	$-\frac{83}{2048}$	0	$-\frac{179179}{32768}$
1	$-\frac{1}{96}$	0	$\frac{19}{1536}$	0	$\frac{16183}{16384}$	0
2	0	$-\frac{3}{512}$	0	$-\frac{1313}{6144}$	0	$-\frac{3412701}{65536}$
3	$\frac{1}{128}$	0	$\frac{59}{1024}$	0	$\frac{581143}{65536}$	0
4	0	$-\frac{127}{6144}$	0	$-\frac{116319}{65536}$	0	$-\frac{383135601}{524288}$
5	$\frac{11}{1024}$	0	$\frac{10619}{24576}$	0	$\frac{15337479}{131072}$	0
6	0	$-\frac{3329}{24576}$	0	$-\frac{1426777}{65536}$	0	$-\frac{7424556503}{524288}$
7	$\frac{761}{12288}$	0	$\frac{238597}{49152}$	0	$\frac{1121762999}{524288}$	0
8	0	$-\frac{5593}{4096}$	0	$-\frac{1171872737}{3145728}$	0	$-\frac{3036747101441}{8388608}$
9	$\frac{4423}{8192}$	0	$\frac{20167831}{262144}$	0	$\frac{217372440663}{4194304}$	0
10	0	$-\frac{15545773}{786432}$	0	$-\frac{4454455581}{524288}$	0	$-\frac{395787223226897}{33554432}$
11	$\frac{228811}{32768}$	0	$\frac{2578302709}{1572864}$	0	$\frac{26951369828703}{16777216}$	0
12	0	$-\frac{1225078949}{3145728}$	0	$-\frac{4181947560489}{16777216}$	0	$-\frac{1000188285379879}{2097152}$

Table A.15: Predictions for $g = 1$, $h = 2$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(1,1)$.

$n \backslash m$	1	3	5	7	9
0	0	$-\frac{3}{128}$	0	$\frac{749}{8192}$	0
1	$\frac{1}{64}$	0	$-\frac{73}{4096}$	0	$-\frac{806391}{262144}$
2	0	$\frac{11}{2048}$	0	$\frac{59347}{131072}$	0
3	$-\frac{13}{3072}$	0	$-\frac{4277}{65536}$	0	$-\frac{114835519}{4194304}$
4	0	$\frac{323}{98304}$	0	$\frac{7763577}{2097152}$	0
5	$\frac{161}{16384}$	0	$-\frac{1512059}{3145728}$	0	$-\frac{23730473751}{67108864}$
6	0	$\frac{14031}{524288}$	0	$\frac{1484285887}{33554432}$	0
7	$\frac{28147}{786432}$	0	$-\frac{87714437}{16777216}$	0	$-\frac{6775943551839}{1073741824}$
8	0	$\frac{6282863}{25165824}$	0	$\frac{395292407237}{536870912}$	0
9	$\frac{4059523}{12582912}$	0	$-\frac{21529218793}{268435456}$	0	$-\frac{2563997898607031}{17179869184}$
10	0	$\frac{1355800153}{402653184}$	0	$\frac{140469708824427}{8589934592}$	0
11	$\frac{277853969}{67108864}$	0	$-\frac{21346688578591}{12884901888}$	0	$-\frac{1243192010808991359}{274877906944}$
12	0	$\frac{132430149801}{2147483648}$	0	$\frac{64325487060690897}{137438953472}$	0

Table A.16: Predictions for $g = 1$, $h = 2$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(2,1)$.

n \ m	0	2	4	6	8	10
0	$-\frac{1}{4}$	0	$\frac{7}{128}$	0	$-\frac{63}{256}$	0
1	0	$-\frac{1}{96}$	0	$\frac{19}{512}$	0	$\frac{164059}{16384}$
2	$-\frac{1}{32}$	0	$-\frac{19}{1536}$	0	$-\frac{15543}{16384}$	0
3	0	$\frac{5}{768}$	0	$\frac{85}{2048}$	0	$\frac{2917315}{32768}$
4	$\frac{1}{64}$	0	$\frac{73}{3072}$	0	$-\frac{254719}{32768}$	0
5	0	$-\frac{47}{3072}$	0	$\frac{2213}{6144}$	0	$\frac{73876031}{65536}$
6	$-\frac{17}{1024}$	0	$\frac{391}{4096}$	0	$-\frac{23761463}{262144}$	0
7	0	$-\frac{25}{768}$	0	$\frac{246515}{65536}$	0	$\frac{10287089645}{524288}$
8	$-\frac{9}{1024}$	0	$\frac{187937}{196608}$	0	$-\frac{383755867}{262144}$	0
9	0	$-\frac{4013}{12288}$	0	$\frac{14299413}{262144}$	0	$\frac{474325562521}{1048576}$
10	$-\frac{1091}{8192}$	0	$\frac{10594097}{786432}$	0	$-\frac{66113744979}{2097152}$	0
11	0	$-\frac{1594085}{393216}$	0	$\frac{3320919305}{3145728}$	0	$\frac{224405506520675}{16777216}$
12	$-\frac{22219}{16384}$	0	$\frac{68613211}{262144}$	0	$-\frac{3672961920137}{4194304}$	0

Table A.17: Predictions for $g = 1$, $h = 2$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(2, 2)$.

n \ m	0	2	4	6	8	10
0	$-\frac{4}{9}$	0	$\frac{1}{8}$	0	$-\frac{161}{384}$	0
1	0	$-\frac{29}{576}$	0	$\frac{189}{2048}$	0	$\frac{1521551}{98304}$
2	$\frac{5}{144}$	0	$-\frac{95}{2304}$	0	$-\frac{48635}{24576}$	0
3	0	$\frac{17}{512}$	0	$\frac{6733}{24576}$	0	$\frac{22458213}{131072}$
4	$-\frac{1}{18}$	0	$-\frac{59}{1536}$	0	$-\frac{135383}{6144}$	0
5	0	$-\frac{13}{18432}$	0	$\frac{215983}{65536}$	0	$\frac{1055600059}{393216}$
6	$\frac{115}{4608}$	0	$-\frac{1335}{2048}$	0	$-\frac{134219405}{393216}$	0
7	0	$\frac{16493}{73728}$	0	$\frac{13351407}{262144}$	0	$\frac{44963141923}{786432}$
8	$-\frac{433}{2304}$	0	$-\frac{179953}{18432}$	0	$-\frac{87659531}{12288}$	0
9	0	$\frac{8527}{3072}$	0	$\frac{1095436053}{1048576}$	0	$\frac{6682846176571}{4194304}$
10	$-\frac{50965}{36864}$	0	$-\frac{25251695}{131072}$	0	$-\frac{610917761875}{3145728}$	0
11	0	$\frac{117348941}{2359296}$	0	$\frac{348308890001}{12582912}$	0	$\frac{5664941001390979}{100663296}$
12	$-\frac{387979}{18432}$	0	$-\frac{3841015873}{786432}$	0	$-\frac{10511662024063}{1572864}$	0

Table A.18: Predictions for $g = 1$, $h = 2$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number $(3, 1)$.

A.1.6 $g = 2$, $h = 1$

n \ m	1	3	5	7	9
0	$\frac{1}{3840}$	0	$-\frac{851}{81920}$	0	$-\frac{7860797}{5242880}$
1	0	$\frac{391}{122880}$	0	$\frac{716029}{2621440}$	0
2	$-\frac{91}{61440}$	0	$-\frac{235817}{3932160}$	0	$-\frac{1298196793}{83886080}$
3	0	$\frac{32689}{1966080}$	0	$\frac{112050491}{41943040}$	0
4	$-\frac{6263}{983040}$	0	$-\frac{11465707}{20971520}$	0	$-\frac{313635847269}{1342177280}$
5	0	$\frac{1451137}{10485760}$	0	$\frac{76688664827}{2013265920}$	0
6	$-\frac{739891}{15728640}$	0	$-\frac{7332916417}{1006632960}$	0	$-\frac{103745095940193}{21474836480}$
7	0	$\frac{855699469}{503316480}$	0	$\frac{24010004494133}{32212254720}$	0
8	$-\frac{394660109}{754974720}$	0	$-\frac{2156215517801}{16106127360}$	0	$-\frac{134782837320297767}{1030792151040}$
9	0	$\frac{233999737631}{8053063680}$	0	$\frac{3291576679832789}{171798691840}$	0
10	$-\frac{32994415691}{4026531840}$	0	$-\frac{279022560888339}{85899345920}$	0	$-\frac{2464123523429889593}{5497558138880}$
11	0	$\frac{85088983138249}{128849018880}$	0	$\frac{5161051100360148793}{8246337208320}$	0
12	$-\frac{3705845271181}{21474836480}$	0	$-\frac{414824442483351281}{4123168604160}$	0	$-\frac{16682185968014850733109}{87960930222080}$

Table A.19: Predictions for $g = 2$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 1.

n \ m	0	2	4	6	8	10
0	0	$-\frac{23}{3840}$	0	$\frac{397}{20480}$	0	$\frac{3708677}{655360}$
1	$\frac{37}{5760}$	0	$-\frac{9}{2560}$	0	$-\frac{752263}{983040}$	0
2	0	$\frac{47}{46080}$	0	$\frac{6763}{61440}$	0	$\frac{468854467}{7864320}$
3	$-\frac{37}{23040}$	0	$-\frac{1837}{122880}$	0	$-\frac{30020327}{3932160}$	0
4	0	$\frac{11}{73728}$	0	$\frac{404921}{393216}$	0	$\frac{5658808193}{6291456}$
5	$\frac{599}{184320}$	0	$-\frac{97087}{737280}$	0	$-\frac{856150369}{7864320}$	0
6	0	$\frac{2443}{491520}$	0	$\frac{108494989}{7864320}$	0	$\frac{773230153649}{41943040}$
7	$\frac{10501}{737280}$	0	$-\frac{2175833}{1310720}$	0	$-\frac{66501755201}{31457280}$	0
8	0	$\frac{22249}{294912}$	0	$\frac{1589644841}{6291456}$	0	$\frac{49653533899283}{100663296}$
9	$\frac{191287}{1474560}$	0	$-\frac{9401217}{327680}$	0	$-\frac{13531886494963}{251658240}$	0
10	0	$\frac{517601}{368640}$	0	$\frac{15956291063}{2621440}$	0	$\frac{33622369415282627}{2013265920}$
11	$\frac{10024913}{5898240}$	0	$-\frac{123550208597}{188743680}$	0	$-\frac{1747956403728227}{1006632960}$	0
12	0	$\frac{414232039}{12582912}$	0	$\frac{9407394255163}{50331648}$	0	$\frac{93672968896625789}{134217728}$

Table A.20: Predictions for $g = 2$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 2.

n \ m	1	3	5	7	9
0	0	$\frac{221}{5760}$	0	$-\frac{12691}{122880}$	0
1	$-\frac{643}{34560}$	0	$\frac{19633}{737280}$	0	$\frac{175714631}{47185920}$
2	0	$-\frac{7651}{552960}$	0	$-\frac{1942183}{3932160}$	0
3	$\frac{7231}{552960}$	0	$\frac{907759}{11796480}$	0	$\frac{35888259253}{754974720}$
4	0	$-\frac{16787}{1105920}$	0	$-\frac{300043261}{47185920}$	0
5	$\frac{24737}{8847360}$	0	$\frac{583154159}{566231040}$	0	$\frac{10087645073971}{12079595520}$
6	0	$-\frac{10955107}{47185920}$	0	$-\frac{334185141679}{3019898880}$	0
7	$\frac{12947971}{141557760}$	0	$\frac{53326890019}{3019898880}$	0	$\frac{37577538677641473}{193273528320}$
8	0	$-\frac{4233264583}{1132462080}$	0	$-\frac{61018503742387}{24159191040}$	0
9	$\frac{2754790277}{2264924160}$	0	$\frac{2105326035057}{5368709120}$	0	$\frac{1799638347922850591}{3092376453120}$
10	0	$-\frac{2843659084991}{36238786560}$	0	$-\frac{6332775558829201}{85899345920}$	0
11	$\frac{825581791111}{36238786560}$	0	$\frac{25692970234497637}{2319282339840}$	0	$\frac{107897653642130395489 3}{49478023249920}$
12	0	$-\frac{101535236275903}{48318382080}$	0	$-\frac{415476048776607671 3}{1546188226560}$	0

Table A.21: Predictions for $g = 2$, $h = 1$ open orbifold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_4$ at winding number 3.

Appendix B

Useful formulae

B.1 Euler integral representations, analytic continuation and generalized hypergeometric functions

As we pointed out in Sec. 2.2.2, an important feature of our formalism for solving the genus zero B-model is the fact that we can work directly with an Euler-type integral representation for the periods. We will focus here in the case $p = q = 2$, but the strategy is completely general and computationally feasible as long as x_i is algebraically related to a_i.
For $p = q = 2$, the derivatives of the periods have the simple form (3.2.44), (3.2.45). Using the standard Euler integral representation for the complete elliptic integral $K(x)$

$$2K(x) = \int_0^1 \frac{d\theta}{\sqrt{\theta}\sqrt{1-\theta}} \frac{1}{\sqrt{1-x\theta}} \tag{B.1.1}$$

we can integrate back a_4 and get

$$\Pi_A(a_i) = \int_0^1 \frac{2a_5 d\theta}{\sqrt{\theta}\sqrt{1-\theta}} \log\left[a_4 + \sqrt{c_1^2 + (c_2^2 - c_1^2)(1-\theta)}\right] \tag{B.1.2}$$

$$\Pi_B(a_i) = 4\int_0^1 \frac{2a_5 d\theta}{\sqrt{1-\theta^2}\theta} \log\left[\frac{1}{\sqrt{c_1^2 - c_2^2}}\left(a_4\theta + \sqrt{(c_1^2 - c_2^2) + c_2\theta^2}\right)\right] \tag{B.1.3}$$

where the constant factors in a_4 are introduced as a constant of integration in order to satisfy (2.2.9). Formulae (B.1.2), (B.1.3) then yield simple and globally valid expressions for the periods and significantly ease the task of finding their analytic continuation from patch to patch. For small a_4, we can simply expand the integrand and integrate term by term. For large a_4 Π_A has the following asymptotic behavior

$$\Pi_A = 2a_5 \log(2a_4) - 2\left(a_3 a_5^2\right)\left(\frac{1}{a_4}\right)^2 + \left(-3a_3^2 a_5^3 - 6a_1 a_2 a_5^3\right)\left(\frac{1}{a_4}\right)^4 + O\left(\frac{1}{a_4}\right)^5 \tag{B.1.4}$$

but an expansion for Π_B is much harder to find. The leading order term can still be extracted, for example in the $a_2 = a_3 = a_5 = 1$ patch using

$$\int_0^1 \log\left[\theta a + \sqrt{1 + \left(b + \frac{a^2}{4}\right)\theta^2}\right] \frac{d\theta}{\sqrt{1-\theta^2}\theta} = 2Li_2(-1-a) + O(\log a) \tag{B.1.5}$$

which gives

$$\Pi_B = 4\left(\log\left(\frac{1}{2\sqrt[4]{a_1}}\right) - \log\left(\frac{1}{a_4}\right)\right)^2 + O\left(\log a_4\right) \tag{B.1.6}$$

Single and double logarithmic behaviors as in (B.1.4, B.1.6) are characteristic of the large radius patch in the moduli space, which as we will see will be given precisely by $a_4 \to \infty$ (and $a_1 \to 0$).

Lastly, a nice fact to notice is that the periods for this particular case take the form of known generalized hypergeometric functions of two variables. For example we have that, modulo a_4 independent terms, the A period can be written as

$$\Pi_A = \frac{\pi}{4}\log c_1 + \frac{\pi}{4}\left(\frac{a_4^2}{c_1} - c_1\right) F^{1,2,2}_{1,1,1}\left[\begin{array}{c|cc|cc|c} 1 & \frac{3}{2}, & 1 & \frac{1}{2}, & \frac{1}{2} & \\ 2 & & 2 & & 1 & \end{array} c_1\left(1-\frac{a_4^2}{c_1^2}\right), c_1\left(1-\frac{c_2^2}{c_1^2}\right)\right] \tag{B.1.7}$$

in terms of the Kampé de Fériet[1] hypergeometric function of two variables.

[1]See Eric Weinstein, *"Kampé de Fériet Function"*, `http://mathworld.wolfram.com/KampedeFerietFunction.html` or [59, 60] for a more detailed account on such functions.

B.2 Lauricella functions

We collect here a number of properties and useful formulae for Lauricella's $F_D^{(n)}$ functions. The interested reader might want to look at [59] for a detailed discussion of this topic.

B.2.1 Definition

The usual power series definition of Lauricella $F_D^{(n)}$ of n complex variables is

$$\begin{aligned} & F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) \\ = & \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m1+\cdots+m_n} m_1!\cdots m_n!} x_1^{m_1}\cdots x_n^{m_n}, \end{aligned} \tag{B.2.1}$$

whenever $|x_1|, \ldots, |x_n| < 1$. For $n = 1$ this is nothing but Gauss' hypergeometric function ${}_2F_1(a, b; c; x)$; for $n = 2$ is boils down to Appell's $F_1(a, b, c; d; x, y)$. It also satisfies the following system of PDE's, which generalizes the $n = 1$ hypergeometric equation

$$\begin{aligned} ab_j F_D &= x_j(1 - x_j)\frac{\partial^2 F_D}{\partial x_j^2} + (1 - x_j)\sum_{k\neq j} x_k \frac{\partial^2 F_D}{\partial x_k \partial x_j} + [c - (a + b_j + a)x_j]\frac{\partial F_D}{\partial x_j} \\ &- b_j \sum_{\neq j} x_k \frac{\partial F_D}{\partial x_k} \qquad j = 1, \ldots, n \end{aligned} \tag{B.2.2}$$

The system (B.2.2) has regular singular points when

$$x_i = 0, 1, \infty \quad \text{and} \quad x_i = x_j \qquad i = 1, \ldots, n, j \neq i \tag{B.2.3}$$

The number of intersecting singular submanifolds in correspondence of the generic singular point

$$(x_1, \ldots, x_n) = (\underbrace{0, \ldots, 0}_{p}, \underbrace{1 \ldots, 1}_{q}, \underbrace{\infty, \ldots, \infty}_{n-p-q}) \tag{B.2.4}$$

is

$$\binom{p+1}{2}\binom{q+1}{2}\binom{n-p-q+1}{2}$$

In contrast with the well-known $n = 1$ case, typically the Lauricella system does not close under analytic continuation around a singular point. As explained in [59], a complete set of solutions of the F_D^n system (B.2.2) away from the region of convergence $|x_i| < 1$ involves a larger set of functions, namely Exton's C_n^k and $D_{(n)}^{p;q}$. We will report here a number of analytic continuation formulae valid for generic n, and refer to [59] for further results in this direction. See also [66] for further developments in finding asymptotic expressions for large values of the parameters.

B.2.2 Analytic continuation formulae for Lauricella F_D

In the following, results on analytic continuation for F_D will be expressed in terms of Exton's C and D functions

$$\begin{aligned}&C_n^{(k)}(\{b_i\}, a, a'; \{x_i\})\\&= \sum_{m_1,\ldots m_n} \prod_i (b_i)_{m_i} (a)_{\sum_{i=k+1}^n m_i - \sum_{i=1}^k m_i} (a')_{-\sum_{i=k+1}^n m_i + \sum_{i=1}^k m_i} \prod_i \frac{x_i^{m_i}}{m_i!}\end{aligned} \tag{B.2.5}$$

$$\begin{aligned}&D_{(n)}^{p,q}(a, b_1, \ldots, b_n; c, c'; x_1, \ldots, x_n)\\&= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_{p+1}+\cdots+m_n-m1-\cdots-m_p}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_{q+1}+\cdots+m_n-m_1-\cdots-m_p} c'_{m_{p+1}+\cdots+m_q} m_1!\cdots m_n!} x_1^{m_1} \cdots x_n^{m_n},\end{aligned} \tag{B.2.6}$$

- **Continuation around** $(0, 0, \ldots, 0, \infty)$

$$\begin{aligned}&F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) =\\&\Gamma\left[\begin{array}{cc} c, & b_n - a \\ b_n, & c-a \end{array}\right] (-x_n)^{-a} F_D^{(n)}(a, b_1, \ldots, b_{n-1}, 1-c+a; 1-b_n+a; \tfrac{x_1}{x_n}, \ldots, \tfrac{x_{n-1}}{x_n}, \tfrac{1}{x_n})\\&+\Gamma\left[\begin{array}{cc} c, & -b_n + a \\ a, & c - b_n \end{array}\right] (-x_n)^{-b} C_n^{(n-1)}(b_1, \ldots, b_n, 1-c+b_n; a-b_n; -x_1, \ldots, -x_{n-1}, \tfrac{1}{x_n})\end{aligned} \tag{B.2.7}$$

- **Continuation around** $(0, 0, \ldots, 0, 1)$

$$\begin{aligned}&F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \Gamma\left[\begin{array}{cc} c, & c-b_n-a \\ c-a, & c-b_n \end{array}\right] (1-x_1)^{-b_1} \ldots (1-x_{n-1})^{-b_{n-1}}\\&\times x_n^{-b_n} C_n^{(n-1)}(b_1, \ldots, b_n, 1+b_n-c; c-a-b_n; \tfrac{x_1}{1-x_1}, \ldots, \tfrac{x_{n-1}}{1-x_{n-1}}, \tfrac{1-x_n}{x_n})\\&+\Gamma\left[\begin{array}{cc} c, & b_n+a-c \\ a, & b_n \end{array}\right] (1-x_1)^{-b_1} \ldots (1-x_{n-1})^{-b_{n-1}} (1-x_n)^{c-a-b_n}\\&\times F_D^{(n)}(c-a, b_1, \ldots, b_{n-1}; c-a-b_n+1; \tfrac{1-x_n}{1-x_1}, \ldots, \tfrac{1-x_n}{1-x_{n-1}}, 1-x_n)\end{aligned} \tag{B.2.8}$$

- **Continuation around** $(0, 0, \ldots, \infty, 1)$

$$\begin{aligned}&F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \Gamma\left[\begin{array}{cc} c, & b_n+a-c \\ a, & b_n \end{array}\right] (1-x_n)^{c-a-b_n} \prod_{i=1}^{n-1} (1-x_i)^{-b_i}\\&\times F_D^{(n)}(c-a, b_1, \ldots, b_{n-1}; c-b_1-\cdots-b_n; c-a-b_n+1; \tfrac{1-x_n}{1-x_1}, \ldots, \tfrac{1-x_n}{1-x_{n-1}}, 1-x_n)\\&+\Gamma\left[\begin{array}{ccc} c, & c-a-b_n, a-b_{n-1} & \\ c-a, & c-b_{n-1}-b_n, & a \end{array}\right] (1-x_1)^{-b_1} \ldots (1-x_{n-1})^{-b_{n-1}} x_n^{-b_n}\\&\times D_{(n)}^{1,2}(c-a-b_n, b_n, \ldots, b_1; c-b_{n-1}-b_n; b_{n-1}-a+1; \tfrac{x_n-1}{x_n}, \tfrac{1}{1-x_{n-1}}, \tfrac{x_{n-2}}{1-x_{n-2}} \ldots, \tfrac{x_1}{1-x_1})\\&+\Gamma\left[\begin{array}{cc} c, & b_{n-1}-a \\ c-a, & b_{n-1} \end{array}\right] (1-x_{n-1})^{-a}\\&\times F_D^{(n)}(a, b_1, \ldots, b_{n-2}; c-\textstyle\sum_{i=1}^n b_i; b_n, a-b_{n-1}+1; \tfrac{1-x_1}{1-x_{n-1}}, \ldots, \tfrac{1-x_{n-2}}{1-x_{n-1}}, \tfrac{1}{1-x_{n-1}}, \tfrac{1-x_n}{1-x_{n-1}})\end{aligned} \tag{B.2.9}$$

Notice that the formulae above are valid only for generic values of the parameters b_i, a and c. Should one be confronted with singular cases, it would be necessary to take a suitable regularization (such as $b_i \to b_i + \epsilon$) and after analytic continuation take the $\epsilon \to 0$ limit. See Appendix B in [10] for more details; suffice it here to report as an example the case $b_n = a$:

$$\begin{aligned}
&F_D^{(n)}(a, b_1, \ldots, b_{n-1}, a; c; x_1, \ldots, x_n) \\
&= \Gamma\left[\begin{array}{c} c \\ a, c-a \end{array}\right](-x_n)^{-a} \sum_M \sum_{m_n=0}^{\infty} \Gamma\left[\begin{array}{c} c-a-|M|| \\ c-a+|M| \end{array}\right] \tfrac{(a)_{|M|+m_n}(1-c+a)_{2|M|+m_n}}{(|M|+m_n)!m_n!} \prod_{i=1}^{n-1} \tfrac{(b_i)_{m_i}}{m_i!} \times \\
&\times (\log(-x_n) + h_{m_n}) \left(\tfrac{x_1}{x_n}\right)^{m_1} \cdots \left(\tfrac{x_{n-1}}{x_n}\right)^{m_n-1} \left(\tfrac{1}{x_n}\right)^{m_n} \\
&+\Gamma\left[\begin{array}{c} c, c-a \\ a \end{array}\right](-x_n)^{-a} \sum_M \sum_{m_n=0}^{|M|-1} \tfrac{(a)_{m_n}\Gamma(|M|-m_n)}{m_n!(c-a)_{|M|-m_n}} \prod_{i=1}^{n-1} \tfrac{(b_i)_{m_i}}{m_i!} x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \left(\tfrac{1}{x_n}\right)^{m_n},
\end{aligned} \tag{B.2.10}$$

with

$$h_{m_n} = \psi(1+|M|+m_n) + \psi(1+m_n) - \psi(a+|M|+m_n) - \psi(c-a-m_n), \tag{B.2.11}$$

and $M = (m_1, \ldots, m_n)$ is a multindex (so that $|M| \equiv \sum_{i=1}^{m_n} m_i$).

B.3 CSMM as a unitary matrix model

An alternative matrix model realization can be given, in the trivial vacuum $m = 0$, as an integral over unitary matrices. Starting from (4.3.1) and reasoning along the lines of [117] we have that the CS partition function can be written as

$$Z_{U(N)}^{L(p,q)}(k, \mathbf{0}) = \int d^N x e^{-\frac{p(x\cdot x)}{g_s q}} \prod_{i<j} \sinh\left(\frac{x_i - x_j}{2q}\right) \sinh\left(\frac{x_i - x_j}{2}\right), \tag{B.3.1}$$

and by means of the identity

$$\prod_{i<j} \sinh(a(x_i - x_j)) = \frac{e^{-a(N-1)\sum_i x_i}}{2^{N(N-1)/2}} \Delta(e^{2ax_i}), \tag{B.3.2}$$

we can write (B.3.1) as follows

$$Z_{CS} = \int_{-\infty}^{+\infty} \frac{d^N x}{2^{N(N-1)/2}} \exp\left[\frac{-p(x\cdot x)}{g_s q} - \frac{(N-1)(q+1)}{2q} \sum_i x^i\right] \Delta(x_i)^2 \frac{\Delta(e^{x_i/q})\Delta(e^{x_i})}{\Delta(x_i)^2}. \tag{B.3.3}$$

Now, with the help of the Itzykson-Zuber formula [97]

$$\frac{\det(e^{jx_i/q})}{\Delta(i^j)\Delta(x_i^j)} = \frac{1}{\prod_{p=0}^{N-1} p!} \left(\frac{1}{q}\right)^{\frac{N(N-1)}{2}} \int dU_1 e^{\frac{1}{q}\mathrm{Tr}(U_1 A_D U_1^\dagger X_d)} \tag{B.3.4}$$

with $A_D = \mathrm{diag}\,(1, \ldots, N)$ and using that

$$\int_{\mathfrak{u}(N)} f(X)dX = \Omega_N \int_{\mathbb{R}^N} d^N x \Delta^2(x) f(\mathrm{diag}(x_i)) \tag{B.3.5}$$

for any Ad-invariant $f : \mathfrak{u}(N) \to \mathbb{C}$, where $\Omega_N = (2\pi)^{N(N-1)/2} / \prod_{j=1}^N j!$, we can turn (4.3.6) for $\mathbf{m} = 0$ into a HUU 3-matrix integral

$$\begin{aligned} Z_{CS} &= \left(\frac{N!}{(4\pi q)^{N(N-1)/2}} \right) \int dX e^{\frac{-p}{g_s q} \mathrm{Tr} X^2 - \frac{(N-1)(q+1)}{2q} \mathrm{Tr} X} \\ &\times \int dU_1 dU_2 e^{\frac{1}{q} \mathrm{Tr}(U_1 A_D U_1^\dagger X) + \mathrm{Tr}(U_2 A_D U_2^\dagger X)} \end{aligned} \tag{B.3.6}$$

Defining $\hat{X} \equiv U_1^\dagger X U_1$, $U \equiv U_1^\dagger U_2$ and exploiting the translation invariance of the Haar measure on $U(N)$

$$\begin{aligned} Z_{CS} &= \left(\frac{N!}{(4\pi q)^{N(N-1)/2}} \right) \int d\hat{X} e^{\frac{-p}{g_s q} \mathrm{Tr} \hat{X}^2 - \frac{(N-1)(q+1)}{2q} \mathrm{Tr} \hat{X}} \\ &\times \int dU e^{\frac{1}{q} \mathrm{Tr}(A_D \hat{X}) + \mathrm{Tr}(U A_D U^\dagger X \hat{)}} \end{aligned} \tag{B.3.7}$$

The gaussian integral over $\hat{X}$ gives

$$\begin{aligned} Z_{CS} &= \left(\frac{N!}{(4\pi q)^{N(N-1)/2}} \right) \int dU e^{\frac{g_s}{2p} \mathrm{Tr} U A U^\dagger A} \\ &\exp \left\{ \frac{g_s}{8pq} \left[\left(\frac{N-1}{4} \right)^2 N(q+1)^2 + (q^2+1) \mathrm{Tr} A^2 - (N-1)(q+1)^2 \mathrm{Tr} A \right] \right\} \end{aligned} \tag{B.3.8}$$

Notice that this last integral can be explicitly evaluated by means of the Itzykson-Zuber formula [97]. The result is in agreement with [48], where the same expression was computed by exploiting the biorthogonal polynomials.

B.4 Expansion formulae for Gauss' hypergeometric function around integer parameters

We give here some useful expansion formulae [98] for the expansion of Gauss' hypergeometric function ${}_2F_1(a, b, c; x)$ around integer values of a, b, and c. By hypergeometric recursions, this can be reduced to the following cases:

$$
{}_2F_1\left(\begin{array}{c} 1+a_1\varepsilon, 1+a_2\varepsilon \\ 2+c\varepsilon \end{array} \middle| z\right) = \frac{1+c\varepsilon}{z}\Bigg(-\ln(1-z) - \varepsilon\Bigg\{\frac{c-a_1-a_2}{2}\ln^2(1-z)
$$
$$
+c\mathrm{Li}_2(z)\Bigg\} + \varepsilon^2\Bigg\{\left[(a_1+a_2)c - c^2 - 2a_1a_2\right]\mathrm{S}_{1,2}(z) + \left[(a_1+a_2)c - c^2 - a_1a_2\right]
$$
$$
\ln(1-z)\mathrm{Li}_2(z) + c^2\mathrm{Li}_3(z) - \frac{1}{6}(c-a_1-a_2)^2\ln^3(1-z)\Bigg\}
$$
$$
-\varepsilon^3\Bigg\{c\left[(a_1+a_2)c - c^2 - 2a_1a_2\right]\mathrm{S}_{2,2}(z) + c\left[(a_1+a_2)c - c^2 - a_1a_2\right]
$$
$$
\ln(1-z)\mathrm{Li}_3(z) + (c-a_1)(c-a_2)(c-a_1-a_2)\Bigg[\ln(1-z)\mathrm{S}_{1,2}(z)
$$
$$
+\frac{1}{2}\ln^2(1-z)\mathrm{Li}_2(z)\Bigg] + \frac{1}{24}(c-a_1-a_2)^3\ln^4(1-z)
$$
$$
+c(c-a_1-a_2)^2\mathrm{S}_{1,3}(z) + c^3\mathrm{Li}_4(z)\Bigg\} + \mathcal{O}(\varepsilon^4)\Bigg)\,, \tag{B.4.1}
$$

$$
{}_2F_1\left(\begin{array}{c} a_1\varepsilon, a_2\varepsilon \\ 1+c\varepsilon \end{array} \middle| z\right) = 1 + a_1a_2\varepsilon^2\Bigg(\mathrm{Li}_2(z) - \varepsilon\Bigg\{(c-a_1-a_2)\mathrm{S}_{1,2}(z) + c\mathrm{Li}_3(z)\Bigg\}
$$
$$
+\varepsilon^2\Bigg\{c^2\mathrm{Li}_4(z) + (c-a_1-a_2)^2\mathrm{S}_{1,3}(z) + \frac{1}{2}\left[c(c-a_1-a_2) + a_1a_2\right]\left[\mathrm{Li}_2(z)\right]^2
$$
$$
-\left[c(c-a_1-a_2) + 2a_1a_2\right]\mathrm{S}_{2,2}(z)\Bigg\} + \mathcal{O}(\varepsilon^3)\Bigg)\,. \tag{B.4.2}
$$

B.5 Hypergeometric Yukawas as algebraic functions

For $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ we have

$$
Y_3(\tau_2) = \frac{1}{4}{}_3F_2\left[\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{1}{3},\frac{2}{3};-\frac{4^4}{3^3}\exp(-\tau_2)\right] - \frac{1}{4} \tag{B.5.1}
$$

By the remarkable formula [82]

$$
{}_{n-1}F_{n-2}\left[\frac{1}{n},\ldots,\frac{n-1}{n};\frac{2}{n-1},\ldots,\frac{n-2}{n-2},\frac{n}{n-1};\frac{x(1-x^{n-1})^{n-1}}{f_n}\right] = \frac{1}{1-x^{n-1}} \tag{B.5.2}
$$

where $f_n = (n-1)n^{-n/(n-1)}$, and the fact that

$$ {}_3F_2\left[\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{1}{3},\frac{2}{3};x^3\right] = \frac{d}{dx}\left\{\frac{1}{x}{}_3F_2\left[\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{2}{3},\frac{4}{3};x^3\right]\right\} $$

we get, upon defining

$$ x^3 := -\frac{4^4}{3^3}\exp(-\tau_2) \tag{B.5.3} $$

$$ t(x) := \frac{\sqrt{1-x^3}-1}{2} \tag{B.5.4} $$

$$ y(x) := \frac{1}{2}\sqrt{\frac{x}{\sqrt[3]{4t}}+\sqrt[3]{t}-\frac{2\sqrt[3]{2}}{\sqrt{-\frac{x+(2t)^{2/3}}{\sqrt[3]{t}}}}}-\frac{\sqrt{-\frac{x+(2t)^{2/3}}{\sqrt[3]{t}}}}{2\sqrt[3]{2}} \tag{B.5.5} $$

that

$$ Y_3(x) = \frac{-y(x)^3+3xy'(x)y(x)^2+1}{\left(y(x)^3-1\right)^2} \tag{B.5.6} $$

Unfortunately, this does not seem to simplify the task of finding a closed form in z for the flat functions.