INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

DOCTORAL THESIS

CFT's, contact terms and anomalies

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"It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong."

Richard P. Feynman

Abstract

The first problem that is analysed in this thesis is the possibility of the existence of a parity-odd trace anomaly in $4d$ in the presence of a curved background. We show some evidence by using Feynman diagram techniques that this anomaly is present in the theory of a free chiral fermion and that it has the curious feature of having a purely imaginary coefficient in Lorentzian signature.

We also analyse the theory of a free massive fermion in $3d$. We compute two- and three-point functions of a gauge current and the energy-momentum tensor and, for instance, obtain the well-known result that in the IR limit we reconstruct the relevant CS action in the effective action. We then couple the model to higher spin currents and explicitly work out the spin 3 case. In the UV limit we obtain an effective action which was proposed many years ago as a possible generalization of spin 3 CS action. In the IR limit we derive a different higher spin action. This analysis can evidently be generalized to higher spins. We also discuss the conservation and properties of the correlators we obtain in the intermediate steps of our derivation.

I dedicate this work to my parents

Pedro and Maria,

to my brother

Daniel,

and to my love

Natalia.

Scientific production

- **Published papers**:

    1. L. Bonora, S. Giaccari, and B. Lima de Souza, *Trace anomalies in chiral theories revisited*, JHEP **1407** (2014) 117, [arXiv:1403.2606]
    2. L. Bonora, A. D. Pereira, and B. Lima de Souza, *Regularization of energy-momentum tensor correlators and parity-odd terms*, JHEP **06** (2015) 024, [arXiv:1503.03326]
    3. L. Bonora, M. Cvitan, P. D. Prester, B. Lima de Souza and I. Smolic, *Massive fermion model in 3d and higher spin currents*, JHEP **05** (2016) 072, [arXiv:1602.07178]

- **Preprints:**

    1. L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, B. Lima de Souza, T. Stemberga, *One-loop effective actions and higher spins* [arXiv: 1609.02088]

- **Conference papers**:

    1. L. Bonora, S. Giaccari and B. Lima de Souza, *Revisiting Trace Anomalies in Chiral Theories*, Springer Proc. Math. Stat. **111** (2014) 3.
    2. L. Bonora and B. Lima de Souza, *Pure contact term correlators in CFT*. Proceedings of the 18th Bled Workshop "What Comes Beyond Standard Models". [arXiv:1511.06635]

- **Software**:

    1. A Mathematica package for automated analytical computation of Feynman diagrams. The whole project is available in the github repository: *https://github.com/blimasouza/DavyFeyn*.

Contents

Part I

Introduction

Chapter 1

Introduction

1.1 Regularization of correlation functions

In recent years conformal field theories has been receiving an increasing attention. All this attention is legitimated by its vast range of applicability, namely, the AdS/CFT correspondence, description of critical phenomena, cosmological applications and etc. Apart from all these applications, CFT's are extremely interesting quantum field theories and the knowledge of exact CFT's on the top of the techniques of conformal perturbation theory will open the door to new developments on non-perturbative aspects of QFT's. All these motivations has spurred a lot of interest and activity in the theoretical aspects of conformal symmetry and conformal field theories. Recent reviews on the latter are [1, 2], older references relevant to the content of this thesis are [3, 4]. One of the most striking recently obtained results is the derivation of the general structures of conformal covariant correlators and OPE's of any kind of tensor fields in coordinate space, [5–11]. The analysis of 3-point functions of conserved currents and the energy-momentum tensor was also considered in momentum space, [12–14].

The above mentioned correlators in coordinate space are in general *unregulated* expressions, in that they have singularities at coincident points. For convenience we call them *bare*. The natural way to regularize them is provided by distribution theory. This is clear in theory, in practice it is not so simple because, except for the simplest cases, one has to do with formidable expressions. In the coordinate representation a rather natural technique is provided by the so-called *differential regularization*, [15–17]. However, in the general case, we have not been able to show that it is algorithmic and a good deal of guesswork is needed in order to obtain sensible expressions.

Regularizing correlators is not simply a procedure (legitimately) required by mathematics. Singularities in correlators usually contain useful information. For instance, in correlators of currents or energy-momentum tensors, singularities provide information about the coupling to gauge potentials and to gravity, respectively. This is the case of anomalies, which are a typical result of regularization processes, though independent of them. Regularized correlators are also necessary in the Callan-Symanzik equation, [18]. In summary, regularizing conformal correlators is the next necessary step after deriving their (unregulated or "bare") expressions.

As was said above, however, the process of regularizing higher order correlators in coordinate space representation with differential regularization does not seem to be algorithmic. For definiteness we concentrate here on the 2- and 3-point functions of the energy-momentum tensor. We show that we have a definite rule to regularize the 2-point correlators in coordinate space by means of differential regularization, but when we come to the 3-point function there is a discontinuity which does not allow us to extend the rule valid for the 2-point one. To understand the origin of the problem we resort to a model, the model of a free chiral fermion, in momentum representation. Using one-loop Feynman diagrams we can determine completely the 3-point correlator of the e.m. tensor and regularize it with standard dimensional regularization techniques. The idea is to Fourier anti-transform it in order to shed light on the regularization in the coordinate representation. For two reasons we concentrate on the parity-odd part, although the extension to the parity-even part is straightforward. The first reason is the presence of the Levi-Civita tensor which limits the number of terms to a more manageable amount, while preserving all the general features of the problem.

The second reason is more important: the appearance of the Pontryagin density in the trace anomaly of this model. This parity-odd anomaly has been recalculated explicitly in [19] after the first appearance in [20, 21], with different methods. If one uses Feynman diagram techniques the basic evaluation is that of the triangle diagram. Now, it has been proved recently (this is one of the general results mentioned above) that the parity-odd part of the 3-point function of the energy-momentum tensor in the coordinate representation vanishes identically, [7, 8]. Therefore it would seem that there is a contradiction with the existence of a parity-odd part in the trace of the e.m. tensor. Although this argument is rather naive and forgetful of the subtleties of quantum field theory, it seems to be widespread. Therefore we think it is worth clarifying it. We show that in fact there is no contradiction: a vanishing parity-odd "bare" 3-point function of the energy-momentum tensor must in fact coexist with a nonvanishing parity-odd part of the trace anomaly.

1.2 Massive theories in 3d

In the latest years, field theories, and especially conformal field theories, in 3d have become a favorite ground of research. The motivations for this are related both to gravity and to condensed matter, see for instance [22–24] and references therein, based on AdS/CFT correspondence, where 3d can feature on both sides of the duality. Also higher spin/CFT correspondence has raised interest on weakly coupled CFT in 3d, [25–27]. In this context many 3d models, disregarded in the past, are being reconsidered [28, 29]. We will be interested in the free massive fermion model in 3d coupled to various sources, not only to a gauge field and a metric, but also to higher spin fields. Unlike the free massless fermion, [30], this model has not been extensively studied, although examples of research in this direction exist, see for instance [31, 32] and also [25], and for the massless scalar model [33]. Its prominent property, as opposed to the massless one[1], is that the fermion mass parameter m breaks parity invariance, and this feature has nontrivial consequences even when $m \to 0$.

We are interested in the one-loop effective action, in particular in the local part of its UV and IR limits. These contributions are originated by contact terms of the correlators (for related aspects concerning contact terms, see [19, 35–37]). To do so we evaluate the 2-point correlators, and in some cases also the 3-point correlators, of various currents. Our method of calculation is based on Feynman diagrams and dimensional regularization. Eventually we take the limit of high and low energy compared to the mass m of the fermion. In this way we recover some well-known results, [28, 32, 38], and others which are perhaps not so well-known: in the even parity sector the correlators are those (conformal covariant) expected for the a free massless theory; in the odd parity sector the IR limit of the effective action coincides with the gauge and gravity Chern-Simons (CS) action, but also the UV limit lends itself to a similar interpretation provided we use a suitable scaling limit. We also couple the same theory to higher spin symmetric fields. The result we obtain in this case for the spin 3 current in the UV limit is a generalized CS action. We recover in this way theories proposed long ago from a completely different point of view, [39]. In the IR limit we obtain a different higher spin action.

[1] The free massless Majorana model is plagued by a sign ambiguity in the definition of the partition function, [34]. This should not be the case for the massive model.

We remark that in general the IR and UV correlators in the even sector are non-local, while the correlators in the odd-parity sector are local, i.e. made of contact terms (for related aspects, see [28]).

Apart from the final results we find other interesting things in our analysis. For instance the odd parity correlators we find as intermediate results are conformal invariant at the fixed point. However, although we obtain them by taking limits of a free field theory, these correlators cannot be obtained from any known free field theory (using the Wick theorem). Another interesting aspect is connected to the breaking of gauge or diffeomorphism symmetry in the process of taking the IR and UV limits in three-point functions. Although we use analytic regularization, when taking these limits we cannot prevent a breaking of symmetry in the correlators. They have to be "repaired" by adding suitable counterterms to the effective action.

1.3 Organization of the thesis

This thesis starts with three introductory chapters: chapters 2, 3 and 4. The chapter 2 offers a review of basic facts about the different spinor representations available in $4d$, with emphasis on the differences between Majorana and Weyl representations. In chapter 3 we present a detailed overview of the implications of symmetries in quantum field theories. We specialize to the particular cases of diffeomorphism invariance and Weyl invariance and we derive the associated Ward identities. The chapter 4 is devoted to aspects of conformal invariance in momentum space. We review the implications of spacetime symmetries to correlation functions, we specialize to the conformal symmetry and derive the conformal Ward identities (CWI's). We introduce some notation that simplifies the task of writing tensor structures for correlation functions of traceless-symmetric conserved currents.

The chapters 5, 6, 7 and 8 are the core chapters of this thesis and are heavily based on the papers [19], [37] and [40]. The chapter 5 presents the regularization of the 2-point function of energy-momentum tensors using two different methods, the differential regularization and Feynman diagrams. It is reviewed and clarified the relationship between anomalies and the regularization of short-distance singularities. In chapter 6 we focus on the theory of a chiral fermion in $4d$ and, by performing Feynman diagram computations, we show some evidence of the existence of a parity-odd trace anomaly which possess a pure imaginary coefficient in Lorentzian signature. In chapter 7 we analyse more in depth the existence of a parity-odd trace anomaly in $4d$. We start by noticing that in $4d$ there is no parity-odd contributions with support at non-coincident points in the 3-point function of energy-momentum tensors. From what was learnt in chapter 5 about the relationship between short-distance singularities and anomalies, it seems that this fact is in contradiction with the existence of the parity-odd anomaly. The remainder of the chapter is devoted to explain why there is no contradiction here and why these two facts can live together.

In chapter 8 we shift gears and we analyse correlation functions of conserved currents up to spin 3 in the theory of a free massive fermion in $3d$. We then study the UV and IR limits of these correlators.

Chapter 2

Spin-$1/2$ fields in four dimensions

In this chapter we will recall elementary facts about fermions in four dimensions. We do not have the intention to make a detailed exposition of this topic since the literature is already flooded by them. Instead, we would like to offer a minimal presentation, based on [41], with the aim of organizing ideas. Good discussions concerning fermions in arbitrary dimensions are presented in [42, 43]. For detailed discussions of discrete symmetries see for instance [44–46]. For a very complete exposition of the two-component formalism, see [47].

2.1 Spin-$1/2$ fields in four dimensions

In four dimensions there are three basic fermionic representations: Dirac, Weyl and Majorana. In what follows we are going to discuss in detail their differences.

Our starting point is the well-known Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi\,(x) = 0, \tag{2.1}$$

where the Dirac matrices γ^{μ} are 4×4 matrices which satisfy

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}, \tag{2.2}$$

$$\gamma_0\gamma_\mu\gamma_0 = \gamma_\mu^\dagger \Leftrightarrow \gamma_0^\dagger = \gamma_0,\ \ \gamma_i^\dagger = -\gamma_i. \tag{2.3}$$

It is instructive to rewrite the Dirac equation as a Schrödinger equation, namely

$$i\partial_0\psi = H\psi, \tag{2.4}$$

where the Hamiltonian H is given by

$$H = -i\gamma^0\gamma^i\partial_i + m\gamma^0. \tag{2.5}$$

The Clifford algebra requirement (2.2) comes from asking ψ to also satisfy the Klein-Gordon equation, while the second requirement (2.3) comes from asking the Hamiltonian (2.5) to be hermitian.

2.2 Reality constraint and Majorana fields

The most general solution of the Dirac equation (2.1) is the Dirac field, which is an object with four complex components. One natural question is whether it is possible to find real solutions of the Dirac equation. In order for this to be possible one should be able to find a representation of the Dirac matrices which is purely imaginary, in which case the Dirac operator (2.1) would be real, hence would admit real solutions. Indeed, such a representation for the Dirac matrices exist, it is called the *Majorana representation* and one possible realization is

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \; \gamma^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \; \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \; \gamma^3 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \tag{2.6}$$

where the σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.7}$$

In this representation we may search for solutions that satisfy the reality constraint

$$\psi = \psi^*. \tag{2.8}$$

Such a solution is called a *Majorana field* and the reality constraint is usually called the *Majorana condition*.

There are infinitely many sets of matrices that satisfy the constraints (2.2) and (2.3). Given two sets of Dirac matrices γ^μ and $\tilde{\gamma}^\mu$, there is a unitary matrix U that relate them through a similarity transformation, i.e.

$$\gamma^\mu = U\tilde{\gamma}^\mu U^\dagger. \tag{2.9}$$

If $\tilde{\psi}$ is a solution of the Dirac equation written in terms of the Dirac matrices $\tilde{\gamma}^\mu$, then $\psi = U\tilde{\psi}$ is a solution with the representation γ^μ. The Majorana condition in the form expressed in (2.8) holds in the Majorana representation (2.6). Let $\tilde{\gamma}^\mu$ be the Dirac matrices in the Majorana representation and γ^μ some other arbitrary representation. To discover what is the form of the Majorana condition (2.8) in an arbitrary representation we need to employ the unitary transformation U to rewrite it as

$$U^\dagger\psi = \left(U^\dagger\psi\right)^* \Leftrightarrow \psi = UU^T\psi^*. \tag{2.10}$$

Since U is a unitary matrix, UU^T is also an unitary matrix. It is a common practice to rewrite UU^T as γ_0 times a unitary matrix C, i.e.

$$UU^T = \gamma_0 C. \tag{2.11}$$

The right-hand side of (2.10) will be a frequent quantity in our discussion. We will refer to it as the *Lorentz-covariant conjugate*, and it will be denoted by

$$\psi^c \equiv \gamma_0 C\psi^*. \tag{2.12}$$

With these conventions, the Majorana condition becomes

$$\psi = \psi^c. \tag{2.13}$$

It can be shown that in any representation of the Dirac matrices, C is a unitary matrix that satisfies

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T, \quad \text{and} \quad C^T = -C. \tag{2.14}$$

2.3 Lorentz transformations of fermions

A conceptually important question is how do fermions transform under a Lorentz transformation. For a field Ψ living in a generic representation of the Lorentz group, under an infinitesimal Lorentz transformation $x^\mu \to x'^\mu = x^\mu + \omega^{\mu\nu} x_\nu$, it must transform as

$$\Psi(x) \to \Psi'(x') = D(\omega)\,\Psi(x)\,, \tag{2.15}$$

where $D(\omega)$ depends on the specific representation in which Ψ lives. For a Dirac field, by requiring the Dirac equation to be covariant, i.e. for its form to not depend on the reference frame, the matrix $D(\omega)$ can be fixed to be

$$D(\omega) = \exp\left(-\frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\right), \quad \Sigma_{\mu\nu} = \frac{i}{4}\left[\gamma_\mu, \gamma_\nu\right]. \tag{2.16}$$

The Majorana condition (2.13) is only physically relevant if it is independent of a choice of reference frame, in other words, any two observers must agree on the Majorana character of a fermion. Indeed, this fact holds, and to prove it one need to show that ψ^c transforms under a Lorentz transformation as ψ. To prove this fact one needs to use the relation

$$\gamma_0 C \Sigma^*_{\mu\nu} \left(\gamma_0 C\right)^{-1} = -\Sigma_{\mu\nu}. \tag{2.17}$$

Since the operation $(\cdot)^c$ does not change the behavior of a field under Lorentz transformations, it is called the *Lorentz-covariant conjugate.*

2.4 Helicity

An important quantity associated to a particle is its helicity. The helicity is defined as the projection of the spin along the direction of motion of the particle. For a spin-1/2 particle,

$$h_{\boldsymbol{p}} = \frac{2S \cdot \boldsymbol{p}}{|\boldsymbol{p}|}, \quad S^i = \frac{1}{2}\epsilon^{ijk}\Sigma_{jk}, \tag{2.18}$$

where S^i is the spin part of the total angular momentum. The possible eigenvalues of $h_{\boldsymbol{p}}$ are ± 1. An eigenstate with helicity -1 will be called left-handed, while an eigenstate with helicity $+1$ will be called right-handed.

The importance of the helicity comes from the fact that the helicity operator (2.18) commutes with the Dirac Hamiltonian, which means that it is a conserved quantity in the motion of a free spinning particle. Nevertheless, it is not a Lorentz invariant observable for massive particles and it is easy to see why is that so. Let us imagine a massive particle with a helicity $+1$ travelling in the x-direction in a given reference frame. We can always think about an observer that is moving faster than our particle, hence would see the particle traveling in the opposite direction and would infer that the helicity of the particle is -1. For massless particles the situation is different. In this case the particle is moving at the speed of light and we will not be able to find an observer moving faster than it, that is why all observers will agree on the helicity of this particle.

2.5 Chirality

The main character of this discussion is the chirality matrix γ_5, which is a matrix that anticommutes with all the Dirac matrices, i.e.

$$\{\gamma_5, \gamma_\mu\} = 0, \quad \forall\mu. \tag{2.19}$$

This matrix is given by

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \tag{2.20}$$

	Helicity	Chirality
Lorentz invariance	✗	✓
Conserved over motion	✓	✗

TABLE 2.1: Summarizes the properties of the helicity and chirality of massive spin-1/2 particles.

and the normalization is chosen such that

$$(\gamma_5)^2 = 1 \quad \text{and} \quad \gamma_5^\dagger = \gamma_5. \tag{2.21}$$

The fact that γ_5 squares to the identity matrix implies that its eigenvalues are ± 1, and we can split a solution of the Dirac equation into two pieces: one that is an eigenvector with eigenvalue $+1$ and another that is an eigenvector with eigenvalue -1. The first ones will said to be of right-chirality and the later, of left-chirality. To project over the sectors of left/right-chirality we define the projectors

$$P_L = \frac{1-\gamma_5}{2} \quad \text{and} \quad P_R = \frac{1+\gamma_5}{2}. \tag{2.22}$$

Thus, a generic solution ψ of the of the Dirac equation can be decomposed as

$$\psi = \psi_L + \psi_R, \tag{2.23}$$

where

$$\psi_L = P_L\psi \quad \text{and} \quad \psi_R = P_R\psi. \tag{2.24}$$

On one hand, chirality is a Lorentz invariant notion, meaning that all observers agree on the chirality of a particle. This derives from the fact that

$$[\gamma_5, \Sigma_{\mu\nu}] = 0, \quad \forall \mu, \nu. \tag{2.25}$$

On the other hand, chirality is not conserved over the motion of a particle, i.e. it does not commute with the Hamiltonian (2.5). Particularly, it does not commute with the mass term in the Hamiltonian, which can be easily seen by using the fact that γ_5 anticommutes with all the Dirac matrices.

The table 2.1 summarizes the facts that we have so far established concerning the notions of helicity and chirality for massive particles. In the following we are going to concentrate on the massless case.

2.6 Helicity vs. chirality for massless particles

As was already mentioned, for massless particles, both the concepts of helicity and chirality are Lorentz invariant and conserved over the motion. Furthermore, now we are going to show that these two concepts coincide in this case.

Let us consider consider a solution of the massless Dirac equation with momentum p, i.e.

$$\psi(x) = u(\boldsymbol{p})\, e^{ipx}. \tag{2.26}$$

The Dirac equation tells us that the spinor $u(\boldsymbol{p})$ is such that

$$\left(\gamma^0|\boldsymbol{p}| - \gamma \cdot \boldsymbol{p}\right) u(\boldsymbol{p}) = 0. \tag{2.27}$$

Using the fact that

$$\gamma^0\gamma^i = 2\gamma_5 S^i \tag{2.28}$$

we derive

$$\gamma_5 u(\boldsymbol{p}) = \frac{2S\cdot\boldsymbol{p}}{|\boldsymbol{p}|} u(\boldsymbol{p}), \tag{2.29}$$

which shows that helicity and chirality coincide for massless particles.

2.7 Chirality constraint and Weyl fields

In the same way that before we looked for solutions of the Dirac equation that satisfied a reality constraint, now we are going to look for solutions that satisfy a chirality constraint, i.e. solutions that are eigenvectors of the chirality matrix:

$$\begin{aligned} \text{right-handed}: &\quad \gamma_5\psi_R = +\psi_R, \\ \text{left-handed}: &\quad \gamma_5\psi_L = -\psi_L. \end{aligned} \tag{2.30}$$

Such a solution is called a *Weyl field*. It is convenient to talk about Weyl fields in a representation for the Dirac matrices in which γ_5 is diagonal, the so called chiral (or Weyl) representation. In this representation, the Dirac matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \; \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.31}$$

In the chiral representation, a generic Dirac field ψ, which is a four-component spinor, can be written as

$$\psi = \begin{pmatrix} \xi \\ \chi \end{pmatrix}, \tag{2.32}$$

where ξ and χ are two-component spinors, in agreement with (2.23). Particularly, a left-handed field has only the upper components

$$\psi_L = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \tag{2.33}$$

while a right-handed one has only the bottom components

$$\psi_R = \begin{pmatrix} 0 \\ \chi \end{pmatrix}. \tag{2.34}$$

In terms of these two-component spinors, the Dirac equation becomes

$$\begin{cases} i\bar{\sigma}^\mu\partial_\mu\xi = m\chi \\ i\sigma^\mu\partial_\mu\chi = m\xi \end{cases} \tag{2.35}$$

where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. As can be seen from the equation (2.35), the Dirac equation is a system of two coupled equations for the chiral components of the Dirac field. In the massless limit these two equations decouple and the two chiral components have independent dynamics. A left-handed Weyl field ξ is a solution of the equation

$$i\bar{\sigma}^\mu\partial_\mu\xi = 0, \tag{2.36}$$

while a right-handed Weyl field χ is a solution of

$$i\sigma^{\mu}\partial_{\mu}\chi = 0. \tag{2.37}$$

Weyl fields are irreducible representations of the Lorentz group and can be thought of as building blocks for the other representations. We already saw in (2.23) and 2.32) how to write a Dirac field in terms of Weyl fields. We have still to understand how to write a Majorana field in terms of a Weyl. In order to do that, let us compute the Lorentz-covariant conjugate of ψ_L:

$$(\psi_L)^c = \gamma_0 C \frac{1-\gamma_5^*}{2}\psi^* = P_R\psi^c \equiv (\psi^c)_R\,, \tag{2.38}$$

where it was used the hermiticity of γ_5 and the fact that

$$C^{-1}\gamma_5 C = \gamma_5^T\,. \tag{2.39}$$

Hence, the Lorentz-covariant conjugate of a left-handed field is a right-handed field. By using the properties (2.14) of the matrix C it is easy to see that $(\psi^c)^c = \psi$. Therefore, a Majorana field ψ_M can be written in terms of a Weyl field as

$$\psi_M = \psi_L + (\psi_L)^c\,, \tag{2.40}$$

where the reality condition is trivially satisfied. Of course that we could equally have used a right-handed Weyl field to write (2.40). Notice that, to write a Dirac field ψ_D in terms of only left-handed fields we would need two independent fields, namely

$$\psi_D = \psi_{1L} + (\psi_{2L})^c\,. \tag{2.41}$$

Let us see how can we write a Majorana field using the two-component notation. The Majorana condition reads

$$\psi = \begin{pmatrix} \xi \\ \chi \end{pmatrix} = \begin{pmatrix} i\sigma^2\chi^* \\ -i\sigma^2\xi^* \end{pmatrix} = \psi^c, \tag{2.42}$$

where we used the fact that in the chiral representation

$$\gamma_0 C = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}. \tag{2.43}$$

Therefore, a Majorana field, in the chiral representation, has the form

$$\psi = \begin{pmatrix} \xi \\ -i\sigma^2\xi^* \end{pmatrix} = \begin{pmatrix} i\sigma^2\chi^* \\ \chi \end{pmatrix}. \tag{2.44}$$

From (2.35) we directly see that a Majorana field satisfies

$$\bar{\sigma}^{\mu}\partial_{\mu}\xi = -m\sigma^2\xi^* \tag{2.45}$$

or equivalently

$$\sigma^{\mu}\partial_{\mu}\chi = m\sigma^2\chi^*. \tag{2.46}$$

2.8 Charge conjugation, parity and time reversal

In the quantum theory, these discrete symmetries are implemented as unitary/anti-unitary transformations that act directly on the fields though a similarity transformation and implement very concrete notions:

- a charge conjugation does the job of mapping a charged field into a field with the opposite charge;
- a parity transformation exchanges the signs of the spatial directions of spacetime, which has the consequence of inverting the sign of polar vectors and leaving axial vectors invariant. Particularly, we expect rotations to be left invariant under a parity transformation and boosts to change direction. Hence parity must map a given field into another one that transforms as the first one under rotations but transforms with the opposite sign under boosts;
- time reversal exchanges the sign of the temporal direction of spacetime, which has the consequence of inverting the signs of polar vectors and axial vectors. Analogously to parity, time reversal must map a given field into another one that transforms in the opposite way under rotations and boosts. An important distinction between time reversal and the other discrete symmetries is that it is implemented by an anti-unitary transformation, instead of an unitary one. One motivation for that is our desire to have a transformation that map positive energy states into positive energy states, which would not be achieved by a unitary transformation.

Charge conjugation The Dirac equation in the presence of an electromagnetic potential A_μ is

$$\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\psi = 0. \tag{2.47}$$

It is straightforward to see that if ψ satisfies (2.47), then ψ^c satisfies

$$\left(i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right)\psi^c = 0. \tag{2.48}$$

Thus, if ψ has charge e, ψ^c has charge $-e$. The charge conjugation must then be an operation that maps ψ into ψ^c, up to a phase, i.e.

$$\mathscr{C}\psi\left(x\right)\mathscr{C}^{-1} = \eta_C\psi^c\left(x\right), \tag{2.49}$$

where η_C is a phase.

An important remark at this point is that the charge conjugation and the Lorentz-covariant conjugate are not equivalent and their difference can be seen by applying both on a chiral field. We have already seen in (2.38) that the Lorentz-covariant conjugate of a left-handed field is a right-handed field, namely

$$\left(\psi_L\right)^c = \left(\psi^c\right)_R. \tag{2.50}$$

However, the charge conjugate of ψ_L is

$$\mathscr{C}\psi_L\mathscr{C}^{-1} = P_L\mathscr{C}\psi\mathscr{C}^{-1} = \eta_C P_L\left(\psi^c\right) = \eta_C\left(\psi^c\right)_L. \tag{2.51}$$

The important piece of information here is the fact that $\mathscr{C}$ acts directly on the fields and commute with all the Dirac matrices. Thus, while the Lorentz-covariant conjugate of ψ_L is a right-handed field, its charge-conjugate is left-handed.

The expression (2.49) written in the chiral representation has the form

$$\mathscr{C}\begin{pmatrix}\xi\\ \chi\end{pmatrix}\mathscr{C}^{-1}=\eta_C\begin{pmatrix}i\sigma^2\chi^*\\ -i\sigma^2\xi^*\end{pmatrix}, \tag{2.52}$$

from where we see that charge conjugation is only a well-defined operation for Weyl fields if we have both chiralities present in the theory.

Parity Parity is a spacetime transformation that maps $x^\mu=(t,\boldsymbol{x})$ into $x_P^\mu=(t,-\boldsymbol{x})$. As it was already mentioned, a particular consequece of this transformation is that polar vectors change sign while axial vectors remain invariant. Two examples of a couples of polar and axial vectors are boosts and rotations, and momentum and angular momentum. Under a parity transformation, momentum changes sign and angular momentum remains invariant, hence we expect parity to map particles of helicity $+1$ into particles of helicity -1 and vice-versa. Since chirality and helicity coincides for massless particles, parity must swap the chirality of particles.

Let us check how different chiralities of a Dirac field transform under a Lorentz transformation. We have seen in equations (2.15) and (2.16) how a Dirac field transforms under a Lorentz transformation. Writting the infinitesimal transformation in the chiral representation we have

$$\begin{aligned}\psi_L'(x')&=\left(1-\frac{i}{2}\theta^i\sigma_i-\frac{1}{2}\beta^i\sigma_i\right)\psi_L(x)\\ \psi_R'(x')&=\left(1-\frac{i}{2}\theta^i\sigma_i+\frac{1}{2}\beta^i\sigma_i\right)\psi_R(x)\end{aligned}, \tag{2.53}$$

where $\theta^i=\frac{1}{2}\epsilon^{ijk}\omega_{jk}$ parametrize rotations and $\beta^i=\omega^{0i}$ boosts. The crucial thing to be noticed in (2.53) is the fact that right-handed and left-handed Weyl fields transform differently under boosts. A definition of parity that respect all our expectations is

$$\mathscr{P}\psi(x)\,\mathscr{P}^{-1}=\eta_P\gamma_0\psi(x_P), \tag{2.54}$$

where η_P is a phase. In the chiral representation (2.54) becomes

$$\mathscr{P}\begin{pmatrix}\xi(x)\\ \chi(x)\end{pmatrix}\mathscr{P}^{-1}=\eta_P\begin{pmatrix}\chi(x_P)\\ \xi(x_P)\end{pmatrix}. \tag{2.55}$$

Analogously to the case of charge conjugation, we see from (2.55) that parity is only well-defined if we have the two chiralities of a Weyl field in a theory.

Time reversal Time reversal is a spacetime transformation that maps $x^\mu=(t,\boldsymbol{x})$ into $x_T^\mu=(-t,\boldsymbol{x})$. We expect time reversal to map positive energy states into positive energy states, hence e^{-iEt} must be left invariant. Since t is mapped into $-t$, our only option it to also map i into $-i$, in other words, we must require the time reversal to be implemented by an antiunitary operator.

The transformation of a Dirac field under time reversal is

$$\mathscr{T}\psi(x)\,\mathscr{T}^{-1}=\eta_T C\gamma_5\psi(x_T), \tag{2.56}$$

where η_T is a phase. Particularly, in the chiral representation we have

$$\mathscr{T}\begin{pmatrix}\xi(x)\\ \chi(x)\end{pmatrix}\mathscr{T}^{-1}=\eta_T\begin{pmatrix}i\sigma^2\xi(x_T)\\ i\sigma^2\chi(x_T)\end{pmatrix}. \tag{2.57}$$

One way to derive the factor $C\gamma_5$ in (2.56) is to require the Lagrangian for a free Dirac field to be invariant, i.e.

$$\mathscr{T}\mathcal{L}(x)\,\mathscr{T}^{-1}=\mathcal{L}(x_T). \tag{2.58}$$

Notice from (2.57) that, differently from charge conjugation and parity, time reversal is a well-defined operation even if we have only one chirality in our theory.

CP For completeness, if one considers the operation of parity followed by a charge conjugation as a new operation, this operation will have the following effect in Dirac fields written in the chiral representation

$$\mathscr{C}\mathscr{P}\begin{pmatrix}\xi(x)\\ \chi(x)\end{pmatrix}(\mathscr{C}\mathscr{P})^{-1}=\eta_{CP}\begin{pmatrix}-i\sigma^2\xi^*(x_P)\\ i\sigma^2\chi^*(x_P)\end{pmatrix}. \tag{2.59}$$

From (2.59) we see that, despite of the fact that charge conjugation and parity are not well-defined notions for single Weyl fields, CP is well-defined. As a matter of fact, let us show that the Lagrangian for a Weyl field is invariant under (2.59). The Lagrangian for a Weyl field is

$$\mathcal{L}_W(x)=i\xi^\dagger(x)\,\bar{\sigma}^\mu\partial_\mu\xi(x). \tag{2.60}$$

The CP transformation of this Lagrangian gives

$$\mathscr{C}\mathscr{P}\mathcal{L}_W(x)(\mathscr{C}\mathscr{P})^{-1}=i\mathscr{C}\mathscr{P}\xi^\dagger(x)(\mathscr{C}\mathscr{P})^{-1}\bar{\sigma}^\mu\partial_\mu\left(\mathscr{C}\mathscr{P}\xi(x)(\mathscr{C}\mathscr{P})^{-1}\right). \tag{2.61}$$

Using the transformation law (2.59), we have

$$\mathscr{C}\mathscr{P}\mathcal{L}_W(x)(\mathscr{C}\mathscr{P})^{-1}=i\left(i\xi^T(x_P)\,\sigma^2\right)\bar{\sigma}^\mu\partial_\mu\left(-i\sigma^2\xi^*(x_P)\right). \tag{2.62}$$

The fact that $\left(\sigma^2\bar{\sigma}^\mu\sigma^2\right)^T=\sigma^2$ allow us to write

$$\mathscr{C}\mathscr{P}\mathcal{L}_W(x)(\mathscr{C}\mathscr{P})^{-1}=-i\frac{\partial}{\partial x^\mu}\xi^\dagger(x_P)\,\sigma^\mu\xi(x_P)=-i\frac{\partial}{\partial x_P^\mu}\xi^\dagger(x_P)\,\bar{\sigma}^\mu\xi(x_P), \tag{2.63}$$

from which we see that, up to a total derivative, $\mathscr{C}\mathscr{P}\mathcal{L}_W(x)(\mathscr{C}\mathscr{P})^{-1}$ is equal to $\mathcal{L}_W(x_P)$.

CPT Finally, let us consider a time reversal followed by a parity transformation and then a charge conjugation. On Dirac fields this operation has the following effect:

$$\mathscr{C}\mathscr{P}\mathscr{T}\begin{pmatrix}\xi(x)\\ \chi(x)\end{pmatrix}(\mathscr{C}\mathscr{P}\mathscr{T})^{-1}=\eta_{CPT}\begin{pmatrix}\xi^*(-x)\\ -\chi^*(-x)\end{pmatrix}. \tag{2.64}$$

Analogously to the CP case, the Lagrangian for a Weyl field is invariant under CPT up to a total derivative. As a matter of fact,

$$\mathscr{C}\mathscr{P}\mathscr{T}\mathcal{L}_W(x)(\mathscr{C}\mathscr{P}\mathscr{T})^{-1}=-i\xi^T(-x)\,(\bar{\sigma}^\mu)^*\,\partial_\mu\xi^*(-x), \tag{2.65}$$

which, after a few manipulations, becomes

$$-i\frac{\partial}{\partial\left(-x\right)^{\mu}}\xi^{\dagger}\left(-x\right)\bar{\sigma}^{\mu}\xi\left(-x\right). \tag{2.66}$$

Hence, up to a total derivative,

$$\mathscr{C}\mathscr{P}\mathscr{T}\mathcal{L}_{W}\left(x\right)\left(\mathscr{C}\mathscr{P}\mathscr{T}\right)^{-1}=\mathcal{L}_{W}\left(-x\right). \tag{2.67}$$

2.9 Majorana vs. Weyl

It is common to find in the litterature the statement that a massless Majorana field is equivalent to a chargeless Weyl field. In this section we will remind differences and similarities between Majorana and Weyl fields and we will present an argument against this equivalence. Let us start by looking at their Lagrangians.

The Lagrangian for a free Majorana field is given by

$$\mathcal{L}_{M}\left(x\right)=\frac{1}{2}\bar{\psi}\left(x\right)\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi\left(x\right) \tag{2.68}$$

where ψ satisfies the Majorana condition. Using the two-components formalism, this Lagrangian becomes

$$\mathcal{L}_{M}=\frac{i}{2}\left[\xi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\xi+\xi^{T}\left(\sigma^{2}\sigma^{\mu}\sigma^{2}\right)\partial_{\mu}\xi^{*}-m\left(\xi^{T}\sigma^{2}\xi-\xi^{\dagger}\sigma^{2}\xi^{*}\right)\right], \tag{2.69}$$

where we have used the Majorana condition (2.44). This expression can be simplified by noticing that

$$\xi^{T}\left(\sigma^{2}\sigma^{\mu}\sigma^{2}\right)\partial_{\mu}\xi^{*}=-\partial_{\mu}\xi^{\dagger}\left(\sigma^{2}\sigma^{\mu}\sigma^{2}\right)^{T}\xi=-\partial_{\mu}\xi^{\dagger}\bar{\sigma}^{\mu}\xi. \tag{2.70}$$

The minus sign in the first equality comes from the grassmannian character of the spinors and in the second equality we have used the fact that

$$\left(\sigma^{2}\sigma^{\mu}\sigma^{2}\right)^{T}=\bar{\sigma}^{\mu}. \tag{2.71}$$

Thus, the Lagrangian for a massive Majorana field can be written in the form

$$\mathcal{L}_{M}=\frac{i}{2}\left[\xi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\xi-\partial_{\mu}\xi^{\dagger}\bar{\sigma}^{\mu}\xi-m\left(\xi^{T}\sigma^{2}\xi-\xi^{\dagger}\sigma^{2}\xi^{*}\right)\right]. \tag{2.72}$$

Here we have chosen to write the Majorana Lagrangian in terms of a left-handed field ξ but we could have written everything in terms of the right-handed field χ as well. We would simply have to use

$$\psi=\begin{pmatrix} i\sigma^{2}\chi^{*} \\ \chi \end{pmatrix}. \tag{2.73}$$

The Lagrangian for a Weyl field is given by

$$\mathcal{L}_{W}\left(x\right)=i\bar{\psi}_{L}\left(x\right)\gamma^{\mu}\partial_{\mu}\psi_{L}\left(x\right). \tag{2.74}$$

Using the two-components formalism we have

$$\mathcal{L}_{W}=i\xi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\xi. \tag{2.75}$$

One should notice that the Lagrangian (2.75) is hermitian up to a total derivative. Indeed,

$$\mathcal{L}_W^\dagger = -i\partial_\mu \xi^\dagger \bar{\sigma}^\mu \xi = -i\partial_\mu \left(\xi^\dagger \bar{\sigma}^\mu \xi\right) + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi. \tag{2.76}$$

Thus, up to a total derivative, we may write an hermitian Lagrangian for a Weyl field:

$$\mathcal{L}_W = \frac{i}{2}\left[\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - \partial_\mu \xi^\dagger \bar{\sigma}^\mu \xi\right]. \tag{2.77}$$

Now, by comparing (2.72) and (2.77), it is clear that the Lagrangian for a massless Majorana field is indistinguishable from the Lagrangian for a Weyl field. Yet, from the point of view of representations of the Lorentz group, since a Weyl field lives in a chiral representation, namely $({}^1\!/\!{}_2, 0)$ or $(0, {}^1\!/\!{}_2)$, and a Majorana field lives in a non-chiral representation, namely $({}^1\!/\!{}_2, 0) \oplus (0, {}^1\!/\!{}_2)$ constrained with a reality condition, they are different objects. One might think that the argument of the representations is not enough to ensure a physical distinction between Majorana and Weyl, nevertheless this difference is what entails the different behaviour of these two fields under charge conjugation and parity. As a matter of fact, these discrete symmetries only makes sense for Majorana fermions, as can be seen from our discussion in the previous section. To talk about charge conjugation and parity for Weyl fields we necessarily need both chiralities to be present. Thus, in a world where only left-handed fields exist, charge conjugation and parity are not symmetries, in contrast to the situation with a single Majorana field, which is invariant under both charge conjugation and parity.

The discussion of whether Weyl and Majorana are equivalent or not would be superfluous if there is no way to probe the difference. The only way to distinguish these two theories is to find an interaction that treats them differently. One of the results of this thesis is to show that gravity distinguishes these two theories at the quantum level. We will also present a hint that the interaction of chiral fields and gravity might lead to inconsistencies.

Chapter 3

Ward identities and trace anomalies

3.1 Symmetries of the classical theory

Let us consider a classical action S describing some matter field $\Phi(x)$ conformally coupled to of a curved background $g_{\mu\nu}$. The requirement of being conformally coupled implies that the classical action is invariant under general coordinate transformations, a.k.a. diffeomorphisms, and Weyl rescalings, i.e. local rescalings of the metric. Under a finite coordinate transformation $x \to x'(x)$, the metric transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x), \tag{3.1}$$

while under a Weyl rescaling it transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = e^{2\omega(x)} g_{\mu\nu}(x) \tag{3.2}$$

Under an infinitesimal diffeomorphism transformation

$$x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x) \tag{3.3}$$

the variation of metric is given by the Lie derivative of the metric in the direction of ϵ

$$\delta_\epsilon g_{\mu\nu} = \mathcal{L}_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu. \tag{3.4}$$

Under an infinitesimal Weyl transformation we have

$$\delta_\omega g_{\mu\nu} = 2\omega(x) g_{\mu\nu}. \tag{3.5}$$

The diffeomorphism invariance of the action assures us that, if we perform an infinitesimal coordinate transformation, the variation of the action will be zero. Thus,

$$\delta_\epsilon S = \int d^dx \frac{\delta S}{\delta g^{\mu\nu}} \delta_\epsilon g^{\mu\nu} = -\int d^dx \sqrt{g}\, T_{\mu\nu}(x) \nabla^\mu \epsilon^\nu = 0. \tag{3.6}$$

From the first to the second equality we used the fact that the energy-momentum tensor is defined by

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}, \quad T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} \tag{3.7}$$

Since the equation (3.6) must hold for an arbitrary ϵ we conclude that the energy-momentum tensor must be covariantly conserved

$$\nabla^\mu T_{\mu\nu}(x) = 0. \tag{3.8}$$

Proceeding in the same way with the Weyl invariance of the action we see that

$$\delta_\omega S = \int d^d x \frac{\delta S}{\delta g^{\mu\nu}} \delta_\omega g^{\mu\nu} = \int d^d x \sqrt{g}\, \omega(x) T^\mu_\mu(x) = 0, \tag{3.9}$$

which implies that the energy-momentum tensor is traceless

$$T^\mu_\mu(x) = 0. \tag{3.10}$$

3.2 Symmetries of the quantum theory

One fundamental object of the quantum theory is its partition function, i.e.

$$Z[g] = \int \mathcal{D}\Phi e^{iS[\Phi,g]}, \tag{3.11}$$

which is a functional of the external sources constructed from the classical action. In the expression (3.11) we are considering only a background metric as an external source, but we could consider several other sources. As we will see in the following, the addition of external sources in a quantum field theory is a very useful bookkeeping device of correlation functions of the underlying theory.

By definition, in the path-integral formalism, a correlation function of n operators $\mathcal{O}_i$, $i = 1, \dots, n$, is given by

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) e^{iS[\Phi]}. \tag{3.12}$$

The definition (3.12) refers to a correlation function in the absence of background fields. We could equally define correlators in the presence of a background metric by considering the partition function (3.11) and analogously for other sources. A particular case of interest is the 1-point function of the energy-momentum tensor in the presence of a background metric. By definition, it is given by

$$\langle T_{\mu\nu}(x) \rangle_g = \frac{1}{Z[g]} \int \mathcal{D}\Phi\, T_{\mu\nu}(x) e^{iS[\Phi,g]}, \tag{3.13}$$

where the index g on the 1-point function is there to recall that this correlator is in the presence of a background metric g, in other words, it is a function of g. Using the definition of the classical energy-momentum tensor (3.7) we may rewrite the expression (3.13) as

$$\langle T_{\mu\nu}(x) \rangle_g = \frac{1}{Z[g]} \frac{-2i}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} Z[g] \equiv \frac{2}{\sqrt{g}} \frac{\delta W[g]}{\delta g^{\mu\nu}}, \tag{3.14}$$

where we introduced the *effective action* $W[g]$, which is the object such that

$$Z[g] = e^{iW[g]} \Leftrightarrow W[g] = -i \ln Z[g]. \tag{3.15}$$

Now we are in position of deriving the implications of diffeomorphism invariance and Weyl invariance at the quantum level. If the quantum theory has these symmetries, one consequence will be that the effective action will be invariant under the infinitesimal transformations (3.4) and (3.5), namely

$$\delta_\epsilon W[g] = \int d^d x \frac{\delta W[g]}{\delta g^{\mu\nu}} \delta_\epsilon g^{\mu\nu} = - \int d^d x \sqrt{g}\, \langle T_{\mu\nu}(x) \rangle_g \nabla^\mu \epsilon^\nu = 0, \tag{3.16}$$

$$\delta_\omega W[g] = \int d^d x \frac{\delta W[g]}{\delta g^{\mu\nu}} \delta_\omega g^{\mu\nu} = \int d^d x \sqrt{g}\, \omega(x) \langle T^\mu_\mu(x) \rangle_g = 0. \tag{3.17}$$

From (3.16) we derive that the 1-point function of the energy-momentum tensor in the presence of a curved background must be covariantly conserved, i.e.

$$\nabla^\mu \langle T_{\mu\nu}(x) \rangle_g = 0, \tag{3.18}$$

while from (3.17) we derive that the 1-point function of the trace of the energy-momentum tensor in the presence of a curved background must be zero, i.e.

$$\langle T^\mu_\mu(x) \rangle_g = 0. \tag{3.19}$$

The expressions (3.18) and (3.19) are Ward identities corresponding, respectively, to diffeomorphism and Weyl invariance. It is not always possible to preserve all classical symmetries at the quantum level and when a classical symmetry is violated by quantum corrections we say that this symmetry is anomalous. A violation of the diffeomorphism invariance, i.e. of (3.18), is called a *gravitational anomaly* and it was shown in [48] that these anomalies are only possible for the spacetime dimensions $d = 4k + 2$, $k = 0, 1, \dots$. Violations of the Weyl invariance, i.e. of (3.19), are called *trace anomalies* and, if we only have a background metric as external source, they can be present in any even dimension. In the presence of other sources we may have trace anomalies in arbitrary dimensions (see [49] for a recent perspective on this matter).

3.3 The effective action as a generating function

In this section we would like to clarify the relation between the effective action W and correlation functions, and particularly the relation between 1-point functions in the presence of sources and correlation functions in the absence of sources.

Let us consider an arbitrary source A, that may live in an arbitrary representation of the Lorentz group, and an operator J that is sourced by A. Given a classical action $S[\Phi]$ for some matter field Φ, and assuming that there is a natural way of coupling this action to the source A we define $S[\Phi; A]$. By Taylor expanding $S[\Phi, A]$ around $A = 0$ we will have that

$$S[\Phi; A] = S[\Phi; 0] + \int d^d x \left(\left. \frac{\delta S}{\delta A} \right|_{A=0} \right) \cdot A + \mathcal{O}(A^2). \tag{3.20}$$

Since we have said that A sources the operator J, the equation (3.20) tell us that

$$J \equiv \left. \frac{\delta S}{\delta A} \right|_{A=0}. \tag{3.21}$$

The presence of higher-order terms in (3.20) is usually due to a symmetry principle behind the coupling of the theory to a particular source. Two examples of non-linear couplings are the couplings with a gauge field and with a metric tensor. In the first case, the higher-order terms are dictated by gauge invariance while in the second case by invariance under diffeomorphisms. In the absence of a symmetry principle to guide us, we have no reason to add extra higher-order terms, hence we will ignore them in this section.

In the quantum theory, the n-point function of the operator J is defined as

$$\langle J(x_1) \dots J(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}\Phi J(x_1) \dots J(x_n) e^{iS[\Phi]}. \tag{3.22}$$

As a reminder, in terms of Feynman diagrams, the correlator (3.22) corresponds to the sum of both connected and disconnected diagrams. Since $J = \frac{\delta S}{\delta A}$ and J do not depend on A, due to the fact that we are not considering higher-order terms in the expansion (3.20), it is easy to see that derivatives of the effective action produce the connected part of the correlator (3.22):

$$\langle J(x_1)\dots J(x_n)\rangle_c = \frac{1}{i^{n-1}}\frac{\delta^n}{\delta A(x_1)\dots\delta A(x_n)}W[A]\Big|_{A=0}. \tag{3.23}$$

The expression (3.23) shows that the effective action $W[A]$ is the generating function of correlation functions of the operator sourced by the source A, namely J. In order to be more concrete, let us consider the 2-point function of J, i.e.

$$\langle J(x_1)J(x_2)\rangle = \frac{1}{Z[A]}\int \mathcal{D}\Phi J(x_1)J(x_2)e^{iS[\Phi;A]}\Big|_{A=0}. \tag{3.24}$$

From the definition of J as derivative of S with respect to A we see that we may write the rhs of (3.24) as

$$\frac{1}{Z[A]}\int \mathcal{D}\Phi\left(\frac{1}{i}\frac{\delta}{\delta A(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta A(x_2)}\right)e^{iS[\Phi;A]}\Big|_{A=0}. \tag{3.25}$$

Writing the partition function in terms of the effective action we find

$$e^{-iW[A]}\left(\frac{1}{i}\frac{\delta}{\delta A(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta A(x_2)}\right)e^{iW[A]}\Big|_{A=0} = \left(\frac{\delta W}{\delta A(x_1)}\frac{\delta W}{\delta A(x_2)} + \frac{1}{i}\frac{\delta^2 W}{\delta A(x_1)\delta A(x_2)}\right)\Big|_{A=0}. \tag{3.26}$$

Thus, we find that

$$\frac{1}{i}\frac{\delta^2 W}{\delta A(x_1)\delta A(x_2)}\Big|_{A=0} = \langle J(x_1)J(x_2)\rangle - \langle J(x_1)\rangle\langle J(x_2)\rangle \equiv \langle J(x_1)J(x_2)\rangle_c. \tag{3.27}$$

The expression (3.23) also allow us to write $W[A]$ in terms of correlation functions in the absence of A, i.e.

$$W[A] = W[0] + \int d^dx_1\langle J(x_1)\rangle A(x_1) + \frac{i}{2!}\int d^dx_1d^dx_2\langle J(x_1)J(x_2)\rangle_c\, A(x_1)A(x_2)+\dots, \tag{3.28}$$

where $W[0] = -i\ln Z$ is a constant. For later reference, the full series is given by

$$W[A] = W[0] + \sum_{n=1}^{\infty}\frac{i^{n-1}}{n!}\int\prod_{i=1}^{n}d^dx_iA(x_i)\langle J(x_1)\cdots J(x_n)\rangle_c. \tag{3.29}$$

By definition, the 1-point function of the operator J in the presence of of the source A is given by

$$\langle J(x)\rangle_A = \frac{1}{Z[A]}\int\mathcal{D}\Phi J(x)e^{iS[\Phi;A]} = \frac{\delta W}{\delta A(x)}. \tag{3.30}$$

By using (3.29) we see that

$$\langle J(x)\rangle_A = \langle J(x)\rangle + \sum_{n=1}^{\infty}\frac{i^n}{n!}\int\prod_{i=1}^{n}d^dy_iA(y_i)\langle J(x)J(y_1)\cdots J(y_n)\rangle_c, \tag{3.31}$$

where it should be clear that $\langle J(x)\rangle_A$ denotes the 1-point function of J in the presence of the background field A, while $\langle J(x)\rangle$ denotes the 1-point function in the absence of the background field.

3.3.1 Coupling to a background metric

In our previous analysis we considered the coupling of a theory to some source A ignoring any higher-order term on A. Now we would like to consider the case of the coupling of a theory to a background metric. In this case higher-order terms on the background metric naturally arise and we would like to understand what changes.

For concreteness, let us consider the 2-point function of the energy-momentum tensor, the operator sourced by the background metric $g_{\mu\nu}$. By definition,

$$\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle = \frac{1}{Z[g]}\int \mathcal{D}\Phi T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)e^{iS[\Phi;g]}\bigg|_{g=\eta}. \tag{3.32}$$

Using the fact that $T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}$, we may write the rhs as

$$e^{-iW[g]}\left(\frac{-2i}{\sqrt{g(x_1)}}\frac{\delta}{\delta g^{\mu\nu}(x_1)}\right)\left(\frac{-2i}{\sqrt{g(x_2)}}\frac{\delta}{\delta g^{\rho\sigma}(x_2)}\right)e^{iW[g]}\bigg|_{g=\eta} + 2i\left\langle\frac{\delta T_{\rho\sigma}(x_2)}{\delta g^{\mu\nu}(x_1)}\right\rangle\bigg|_{g=\eta}. \tag{3.33}$$

Evaluating the first term of (3.33) we find

$$\begin{aligned} &-4i\,\frac{\delta^2 W}{\delta g^{\mu\nu}(x_1)\delta g^{\rho\sigma}(x_2)}\bigg|_{g=\eta} + \langle T_{\mu\nu}(x_1)\rangle\langle T_{\rho\sigma}(x_2)\rangle \\ &\qquad -\frac{i}{2}\left(\eta_{\mu\nu}\langle T_{\rho\sigma}(x_2)\rangle + \eta_{\rho\sigma}\langle T_{\mu\nu}(x_1)\rangle\right)\delta(x_1-x_2). \end{aligned} \tag{3.34}$$

Therefore,

$$\begin{aligned} \frac{4}{i}\,\frac{\delta^2 W}{\delta g^{\mu\nu}(x_1)\delta g^{\rho\sigma}(x_2)}\bigg|_{g=\eta} &= \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle - \langle T_{\mu\nu}(x_1)\rangle\langle T_{\rho\sigma}(x_2)\rangle - 2i\left\langle\frac{\delta T_{\rho\sigma}(x_2)}{\delta g^{\mu\nu}(x_1)}\right\rangle \\ &\qquad +\frac{i}{2}\left(\eta_{\mu\nu}\langle T_{\rho\sigma}(x_2)\rangle + \eta_{\rho\sigma}\langle T_{\mu\nu}(x_1)\rangle\right)\delta(x_1-x_2). \end{aligned} \tag{3.35}$$

To have an homogeneous and more compact notation we will write

$$\frac{\delta^n W}{\delta g^{\mu_1\nu_1}(x_1)\cdots\delta g^{\mu_n\nu_n}(x_n)}\bigg|_{g=\eta} = \frac{i^{n-1}}{2^n}\langle T_{\mu_1\nu_1}(x_1)\cdots T_{\mu_n\nu_n}(x_n)\rangle_{\mathrm{c}}, \tag{3.36}$$

where the c subscript still refers to the connected part of the correlation function but one should have in mind that there are extra contributions due to the non-linear coupling of the source. An important remark is that all the extra contributions are contact terms that are present to ensure covariance of the effective action. The expression for the effective action in terms of correlation functions of the energy-momentum tensor is

$$W[\eta+h] = W[\eta] + \sum_{n=1}^{\infty}\frac{i^{n-1}}{2^n n!}\int\prod_{i=1}^{n} d^d x_i h^{\mu_i\nu_i}(x_i)\langle T_{\mu_1\nu_1}(x_1)\cdots T_{\mu_n\nu_n}(x_n)\rangle_{\mathrm{c}}, \tag{3.37}$$

where we have defined $h^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$. From (3.37) we can directly read the expression for the 1-point function of the energy-momentum tensor in the presence of a curved background in terms of correlation functions in flat spacetime, namely

$$\langle T_{\mu\nu}(x)\rangle_g = \sum_{n=1}^{\infty}\frac{i^n}{2^n n!}\int\prod_{i=1}^{n}d^d y_i h^{\mu_i\nu_i}(y_i)\langle T_{\mu\nu}(x)T_{\mu_1\nu_1}(y_1)\cdots T_{\mu_n\nu_n}(y_n)\rangle_{\mathrm{c}}, \tag{3.38}$$

where we used the fact that $\langle T_{\mu\nu}(x)\rangle = 0$ due to the conformal symmetry in flat spacetime.

3.4 Implications of symmetry to correlation functions

We have already seen that the implications of diffeomorphism invariance and Weyl invariance at the quantum level are summarized by the relations (3.18) and (3.19), which we repeat here for the sake of clarity:

$$\text{Diffeomorphism} \quad \Leftrightarrow \quad \nabla^{\mu}\langle T_{\mu\nu}(x)\rangle_g = 0, \tag{3.39}$$

$$\text{Weyl} \quad \Leftrightarrow \quad \langle T^{\mu}_{\mu}(x)\rangle_g = 0. \tag{3.40}$$

Plugging the expression (3.38) in the Ward identities above and demanding the resulting expression to be zero order by order in h, we find relations between n-point functions and $(n-1)$-point functions. At order one in h we find

$$\text{Diffeomorphism} \quad \Rightarrow \quad \partial^{\mu}\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle_{\mathrm{c}} = 0, \tag{3.41}$$

$$\text{Weyl} \quad \Rightarrow \quad \langle T^{\mu}_{\mu}(x)T_{\rho\sigma}(0)\rangle_{\mathrm{c}} = 0. \tag{3.42}$$

At order two in h, diffeomorphism invariance implies that

$$\begin{aligned}
&\partial^{\mu}_{x}\left\langle T_{\mu\nu}\left(x\right)T_{\mu_1\nu_1}\left(y_1\right)T_{\mu_2\nu_2}\left(y_2\right)\right\rangle_c = \\
&2i\partial_{x(\mu_1}\left(\delta\left(x-y_1\right)\left\langle T_{\nu_1)\nu}\left(y_1\right)T_{\mu_2\nu_2}\left(y_2\right)\right\rangle_c\right) + 2i\partial_{x(\mu_2}\left(\delta\left(x-y_2\right)\left\langle T_{\nu_2)\nu}\left(x\right)T_{\mu_1\nu_1}\left(y_1\right)\right\rangle_c\right) \\
&\quad - i\eta_{\mu_1\nu_1}\partial^{\lambda}_{x}\delta\left(x-y_1\right)\left\langle T_{\lambda\nu}\left(y_1\right)T_{\mu_2\nu_2}\left(y_2\right)\right\rangle_c - i\eta_{\mu_2\nu_2}\partial^{\lambda}_{x}\delta\left(x-y_2\right)\left\langle T_{\lambda\nu}\left(y_2\right)T_{\mu_1\nu_1}\left(y_1\right)\right\rangle_c \\
&\qquad + i\left(\partial_{x,\nu}\delta\left(x-y_1\right)+\partial_{x,\nu}\delta\left(x-y_2\right)\right)\left\langle T_{\mu_1\nu_1}\left(y_1\right)T_{\mu_2\nu_2}\left(y_2\right)\right\rangle_c,
\end{aligned} \tag{3.43}$$

while Weyl invariance implies that

$$\left\langle T^{\mu}_{\mu}\left(x\right)T_{\mu_1\nu_1}\left(y_1\right)T_{\mu_2\nu_2}\left(y_2\right)\right\rangle_c = 2i\left(\delta\left(x-y_1\right)+\delta\left(x-y_2\right)\right)\left\langle T_{\mu_1\nu_1}\left(y_1\right)T_{\mu_2\nu_2}\left(y_2\right)\right\rangle_c. \tag{3.44}$$

Recall that $\langle\cdot\rangle_c$ was defined in equation (3.36) and particularly, after discarding 1-point functions,

$$\left\langle T_{\mu_1\nu_1}\left(x_1\right)T_{\mu_2\nu_2}\left(x_2\right)\right\rangle_c = \left\langle T_{\mu_1\nu_1}\left(x_1\right)T_{\mu_2\nu_2}\left(x_2\right)\right\rangle, \tag{3.45}$$

and

$$\begin{aligned}
&\left\langle T_{\mu_1\nu_1}\left(x_1\right)T_{\mu_2\nu_2}\left(x_2\right)T_{\mu_3\nu_3}\left(x_3\right)\right\rangle_c = \left\langle T_{\mu_1\nu_1}\left(x_1\right)T_{\mu_2\nu_2}\left(x_2\right)T_{\mu_3\nu_3}\left(x_3\right)\right\rangle - \\
&- 2i\left\langle T_{\mu_1\nu_1}\left(x_1\right)\frac{\delta T_{\mu_3\nu_3}\left(x_3\right)}{\delta g^{\mu_2\nu_2}\left(x_2\right)}\right\rangle - 2i\left\langle\frac{\delta T_{\mu_2\nu_2}\left(x_2\right)}{\delta g^{\mu_1\nu_1}\left(x_1\right)}T_{\mu_3\nu_3}\left(x_3\right)\right\rangle - 2i\left\langle T_{\mu_2\nu_2}\left(x_2\right)\frac{\delta T_{\mu_3\nu_3}\left(x_3\right)}{\delta g^{\mu_1\nu_1}\left(x_1\right)}\right\rangle + \\
&\quad + ig_{\mu_2\nu_2}\delta\left(x_2-x_3\right)\left\langle T_{\mu_1\nu_1}\left(x_1\right)T_{\mu_3\nu_3}\left(x_3\right)\right\rangle + ig_{\mu_1\nu_1}\delta\left(x_1-x_2\right)\left\langle T_{\mu_2\nu_2}\left(x_2\right)T_{\mu_3\nu_3}\left(x_3\right)\right\rangle + \\
&\qquad + ig_{\mu_1\nu_1}\delta\left(x_1-x_3\right)\left\langle T_{\mu_2\nu_2}\left(x_2\right)T_{\mu_3\nu_3}\left(x_3\right)\right\rangle.
\end{aligned} \tag{3.46}$$

Notice that the rhs of (3.46) is not fully symmetric under permutations of the labels, while the lrs is. Indeed, we are missing a symmetrization over the labels on the rhs, which was not written in order to not overcomplicate the expression. More details on how to derive the expressions (3.43)-(3.46) is given in the appendix A.

3.5 Trace anomalies

In this section we will concentrate in $d = 4$ and we will derive the most general trace anomaly that a theory may exhibit in the presence of a background metric.

When the Ward identity (3.19) gets an anomalous contribution, i.e.

$$\langle T^{\mu}_{\mu}(x)\rangle_g = \mathcal{A}(g), \tag{3.47}$$

we say that the theory presents a trace anomaly, where $\mathcal{A}(g)$ is a local functional of the metric. Assuming general covariance is preserved at the quantum level, we expect $\mathcal{A}(g)$ to be invariant under diffeomorphisms. On the top of that, dimensional analysis tell us that $\mathcal{A}(g)$ must be an object of dimension d, hence it must have dimension four in $4d$. In order for $\mathcal{A}(g)$ to be a diffeomorphism invariant functional of the metric it must contain the metric through Riemann tensors. Each Riemann tensor is an object of dimension two since it is given by second derivatives of the metric. Hence, to construct an object of dimension four we will need to consider squares of the Riemann tensor or the d'Alembertian of the Ricci scalar. Following these criteria, we are able to write the following terms:

$$\mathcal{A}(g) = aR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + bR_{\mu\nu}R^{\mu\nu} + cR^2 + d\Box R + e\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma\alpha\beta}. \tag{3.48}$$

As any other anomaly, the Weyl anomaly must satisfy a Wess-Zumino consistency condition, which simply reflects the abelian character of the Weyl symmetry. For the Weyl symmetry, the Wess-Zumino consistency condition reads

$$[\delta_{\omega_1}, \delta_{\omega_2}]\, W[g] = 0. \tag{3.49}$$

By imposing the condition (3.49) we discover that the coefficients of (3.48) must satisfy

$$a + b + 3c = 0, \tag{3.50}$$

which is telling us that from the three constants a, b and c, only two are independent. It is common to choose to write the trace anomaly using the Euler density and the square of the Weyl tensor, namely

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{3.51}$$

$$W^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2, \tag{3.52}$$

which are two linear combinations of the Ricci scalar, the Ricci tensor and the Riemann tensor that satisfy the condition (3.50). After imposing the Wess-Zumino consistency condition, the general form of $\mathcal{A}(g)$ reads

$$\mathcal{A}(g) = aE_4 + cW^2 + d\Box R + e\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma\alpha\beta}. \tag{3.53}$$

By definition, a true anomaly is an object that cannot be subtracted by a local counterterm. This is not the case of $\Box R$ since it can be obtained, for instance, by the Weyl variation of R^2. Therefore, the most general trace anomaly in the presence of a background metric in $4d$ is

$$\langle T^{\mu}_{\mu}(x)\rangle_g = \frac{1}{180\times 16\pi^2}\left(aE_4 + cW^2 + eP\right), \tag{3.54}$$

where

$$P = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\sigma\alpha\beta} \tag{3.55}$$

and we factorized the numerical factor of $\frac{1}{180\times 16\pi^2}$ for later convenience. The above discussion could be rephrased as a cohomology problem and the trace anomalies would be seen as non-trivial elements of a cohomology group, see for instance [50–52]. In [21] it was computed the values of a, c and e for various matter contents living in diverse representations of the Lorentz group. These numbers were computed in Euclidean signature, using the heat kernel method and are summarized in table 3.1.

TABLE 3.1: In this table we summarize the results of [21] and we translate them to our language. The values presented correspond to the contributions of the physical degrees of freedom, meaning that possible contributions of ghosts were taken into account.

	(A, B)	a	c	e
Scalar	$(0,0)$	$-\frac{1}{2}$	$\frac{3}{2}$	0
Weyl fermion	$\left(\frac{1}{2},0\right)$	$-\frac{11}{4}$	$\frac{9}{2}$	$\frac{15}{4}$
Dirac fermion	$\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$	$-\frac{11}{2}$	9	0
Gauge field	$\left(\frac{1}{2},\frac{1}{2}\right)-2(0,0)$	-31	18	0
Self-dual 2-form	$(1,0)$	$\frac{27}{2}$	$\frac{39}{2}$	30
Gravitino	$\left(1,\frac{1}{2}\right)-2\left(\frac{1}{2},0\right)$	$\frac{11}{2}$	$-\frac{255}{4}$	$-\frac{315}{4}$
Graviton	$(1,1)+(0,0)-2\left(\frac{1}{2},\frac{1}{2}\right)$	$\frac{127}{2}$	$\frac{297}{2}$	0

Appendix A

Derivation of Ward identities for 3-point functions

A.1 The "connected" 3-point correlator

By definition, the 3-point function of energy-momentum tensor is

$$\langle T_{\mu_1\nu_1}(x_1)\, T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\rangle \equiv \frac{1}{Z[g]}\int \mathscr{D}\Phi\; T_{\mu_1\nu_1}(x_1)\, T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\, e^{iS[\Phi;g]}\Big|_{g=\eta} . \quad \text{(A.1)}$$

Our aim in this section is to rewrite (A.1) in terms of derivatives of the effective action W. Let us define the operator[1]

$$\mathcal{T}_{\mu\nu} \equiv \frac{-2i}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}, \quad \text{(A.2)}$$

which is such that

$$\langle T_{\mu\nu}(x)\rangle_g = e^{-iW}\mathcal{T}_{\mu\nu}e^{iW}. \quad \text{(A.3)}$$

Using the operator (A.2) we may rewrite the correlator (A.1) as

$$\begin{aligned}\langle T_{\mu_1\nu_1}(x_1)\, T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\rangle &= e^{-iW}\mathcal{T}_{\mu_1\nu_1}\left(\int \mathscr{D}\Phi\; T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\, e^{iS[\Phi;g]}\right) - \\ &- e^{-iW}\int \mathscr{D}\Phi\; \mathcal{T}_{\mu_1\nu_1}\left(T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\right) e^{iS[\Phi;g]}. \end{aligned} \quad \text{(A.4)}$$

Notice that the term in parenthesis in the equation above may be written as

$$\mathcal{T}_{\mu_2\nu_2}\mathcal{T}_{\mu_3\nu_3}e^{iW} - \int \mathscr{D}\Phi\; \mathcal{T}_{\mu_2\nu_2}\left(T_{\mu_3\nu_3}(x_3)\right) e^{iS[\Phi;g]}. \quad \text{(A.5)}$$

Therefore,

$$\begin{aligned}\langle T_{\mu_1\nu_1}(x_1)\, T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\rangle = \;& e^{-iW}\mathcal{T}_{\mu_1\nu_1}\mathcal{T}_{\mu_2\nu_2}\mathcal{T}_{\mu_3\nu_3}e^{iW} \\ & - e^{-iW}\int \mathscr{D}\Phi\; \mathcal{T}_{\mu_1\nu_1}\mathcal{T}_{\mu_2\nu_2}\left(T_{\mu_3\nu_3}(x_3)\right) e^{iS[\Phi;g]} \\ & - e^{-iW}\int \mathscr{D}\Phi\; T_{\mu_1\nu_1}(x_1)\, \mathcal{T}_{\mu_2\nu_2}\left(T_{\mu_3\nu_3}(x_3)\right) e^{iS[\Phi;g]} \\ & - e^{-iW}\int \mathscr{D}\Phi\; \mathcal{T}_{\mu_1\nu_1}\left(T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\right) e^{iS[\Phi;g]}. \end{aligned} \quad \text{(A.6)}$$

[1] Clearly, the operator $\mathcal{T}_{\mu\nu}$ also depends on the spacetime point x, which we do not write in order to simplify the notation.

The second term on the rhs gives rise to a 1-point function which won't contribute, hence we have

$$\begin{aligned}\langle T_{\mu_1\nu_1}(x_1)\,T_{\mu_2\nu_2}(x_2)\,T_{\mu_3\nu_3}(x_3)\rangle &= e^{-iW}\mathcal{T}_{\mu_1\nu_1}\mathcal{T}_{\mu_2\nu_2}\mathcal{T}_{\mu_3\nu_3}e^{iW} - \langle T_{\mu_1\nu_1}(x_1)\,\mathcal{T}_{\mu_2\nu_2}(T_{\mu_3\nu_3}(x_3))\rangle \\ &- \langle \mathcal{T}_{\mu_1\nu_1}(T_{\mu_2\nu_2}(x_2))\,T_{\mu_3\nu_3}(x_3)\rangle - \langle T_{\mu_2\nu_2}(x_2)\,\mathcal{T}_{\mu_1\nu_1}(T_{\mu_3\nu_3}(x_3))\rangle\,. \end{aligned} \tag{A.7}$$

At this point is important to remark that the lhs of the last expression is symmetric under relabelling of the indices and points, while the rhs is not. As a matter of fact, the operators $\mathcal{T}_{\mu_i\nu_i}$ do not commute. Indeed,

$$\left[\mathcal{T}_{\mu_i\nu_i},\mathcal{T}_{\mu_j\nu_j}\right] = \delta(x_i-x_j)\left(\frac{g_{\mu_i\nu_i}}{\sqrt{g_i}}\mathcal{T}_{\mu_j\nu_j} - \frac{g_{\mu_j\nu_j}}{\sqrt{g_j}}\mathcal{T}_{\mu_i\nu_i}\right). \tag{A.8}$$

The reason why the rhs of (A.7) is not symmetric is due to the choice that we have made of factorizing first $\mathcal{T}_{\mu_1\nu_1}$ in expression (A.4). To restore the permutation symmetry we should consider

$$\begin{aligned}&\langle T_{\mu_1\nu_1}(x_1)\,T_{\mu_2\nu_2}(x_2)\,T_{\mu_3\nu_3}(x_3)\rangle \equiv \\ &\frac{1}{6}\sum_{\sigma\in S_3}\frac{1}{Z[g]}\int \mathscr{D}\Phi\; T_{\mu_{\sigma(1)}\nu_{\sigma(1)}}\left(x_{\sigma(1)}\right) T_{\mu_{\sigma(2)}\nu_{\sigma(2)}}\left(x_{\sigma(2)}\right) T_{\mu_{\sigma(3)}\nu_{\sigma(3)}}\left(x_{\sigma(3)}\right) e^{iS[\Phi;g]}\Big|_{g=\eta}\,.\end{aligned} \tag{A.9}$$

The consequence of using (A.9) instead of (A.1) is that, instead of finding (A.7) we will find its completely symmetrized version. Let us rewrite the first term of (A.7)

$$\begin{aligned}e^{-iW}\mathcal{T}_{\mu_1\nu_1}\mathcal{T}_{\mu_2\nu_2}\mathcal{T}_{\mu_3\nu_3}e^{iW} &= e^{-iW}\left(\frac{-2i}{\sqrt{g_1}}\frac{\delta}{\delta g^{\mu_1\nu_1}}\right)\left(\frac{-2i}{\sqrt{g_2}}\frac{\delta}{\delta g^{\mu_2\nu_2}}\right)\left(\frac{-2i}{\sqrt{g_3}}\frac{\delta}{\delta g^{\mu_3\nu_3}}\right)e^{iW} \\ &= (-2i)^3\,e^{-iW}\frac{1}{\sqrt{g_1}}\frac{\delta}{\delta g^{\mu_1\nu_1}}\left[\frac{1}{\sqrt{g_2}}\frac{\delta}{\delta g^{\mu_2\nu_2}}\left(\frac{1}{\sqrt{g_3}}\right)\frac{\delta}{\delta g^{\mu_3\nu_3}} + \frac{1}{\sqrt{g_2g_3}}\frac{\delta^2}{\delta g^{\mu_2\nu_2}\delta g^{\mu_3\nu_3}}\right]e^{iW}.\end{aligned}$$

The derivatives of $1/\sqrt{g}$ are given by

$$\frac{\delta}{\delta g^{\mu\nu}(x)}\left(\frac{1}{\sqrt{g(y)}}\right) = \frac{1}{2\sqrt{g(y)}}g_{\mu\nu}(x)\,\delta(x-y)\,. \tag{A.10}$$

We will ignore terms with only one free derivative because they will give rise to 1-point functions that are all zero:

$$\begin{aligned}&e^{-iW}\mathcal{T}_{\mu_1\nu_1}\mathcal{T}_{\mu_2\nu_2}\mathcal{T}_{\mu_3\nu_3}e^{iW} = \\ &\frac{(-2i)^3}{\sqrt{g_1g_2g_3}}e^{-iW}\left[\frac{1}{2}g_{\mu_2\nu_2}\delta(x_2-x_3)\frac{\delta^2}{\delta g^{\mu_1\nu_1}\delta g^{\mu_3\nu_3}} + \frac{1}{2}g_{\mu_1\nu_1}\delta(x_1-x_2)\frac{\delta^2}{\delta g^{\mu_2\nu_2}\delta g^{\mu_3\nu_3}}\right. \\ &\qquad\left. + \frac{1}{2}g_{\mu_1\nu_1}\delta(x_1-x_3)\frac{\delta^2}{\delta g^{\mu_2\nu_2}\delta g^{\mu_3\nu_3}} + \frac{\delta^3}{\delta g^{\mu_1\nu_1}\delta g^{\mu_2\nu_2}\delta g^{\mu_3\nu_3}}\right]e^{iW}\end{aligned} \tag{A.11}$$

By ignoring 1-point functions we derive that

$$e^{-iW}\frac{\delta^2}{\delta g^{\mu_1\nu_1}\delta g^{\mu_2\nu_2}}e^{iW} = i\frac{\delta^2 W}{\delta g^{\mu_1\nu_1}\delta g^{\mu_2\nu_2}} \tag{A.12}$$

and

$$e^{-iW}\frac{\delta^3}{\delta g^{\mu_1\nu_1}\delta g^{\mu_2\nu_2}\delta g^{\mu_3\nu_3}}e^{iW} = i\frac{\delta^3 W}{\delta g^{\mu_1\nu_1}\delta g^{\mu_2\nu_2}\delta g^{\mu_3\nu_3}}. \tag{A.13}$$

Putting these results together and recalling that

$$\langle T_{\mu_1\nu_1}(x_1)\, T_{\mu_2\nu_2}(x_2)\rangle_c = \frac{4}{i}\frac{\delta^2 W}{\delta g^{\mu_1\nu_1}\delta g^{\mu_2\nu_2}}, \tag{A.14}$$

$$\langle T_{\mu_1\nu_1}(x_1)\, T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\rangle_c = -8\frac{\delta^3 W}{\delta g^{\mu_1\nu_1}\delta g^{\mu_2\nu_2} g^{\mu_3\nu_3}}, \tag{A.15}$$

we get that (A.11) becomes

$$\begin{aligned} e^{-iW}\mathcal{T}_{\mu_1\nu_1}\mathcal{T}_{\mu_2\nu_2}\mathcal{T}_{\mu_3\nu_3}e^{iW}\big|_{g=\eta} &= \langle T_{\mu_1\nu_1}T_{\mu_2\nu_2}T_{\mu_3\nu_3}\rangle_c - i\left[g_{\mu_2\nu_2}\delta(x_2-x_3)\langle T_{\mu_1\nu_1}T_{\mu_3\nu_3}\rangle_c\right. \\ &\left. + g_{\mu_1\nu_1}\delta(x_1-x_2)\langle T_{\mu_2\nu_2}T_{\mu_3\nu_3}\rangle_c + g_{\mu_1\nu_1}\delta(x_1-x_3)\langle T_{\mu_2\nu_2}T_{\mu_3\nu_3}\rangle_c\right]. \end{aligned} \tag{A.16}$$

Thus, by using the result (A.16) on (A.7) we obtain

$$\begin{aligned} &\langle T_{\mu_1\nu_1}(x_1)\, T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\rangle_c = \langle T_{\mu_1\nu_1}(x_1)\, T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\rangle \\ &- 2i\left\langle T_{\mu_1\nu_1}(x_1)\frac{\delta T_{\mu_3\nu_3}(x_3)}{\delta g^{\mu_2\nu_2}(x_2)}\right\rangle - 2i\left\langle \frac{\delta T_{\mu_2\nu_2}(x_2)}{\delta g^{\mu_1\nu_1}(x_1)} T_{\mu_3\nu_3}(x_3)\right\rangle - 2i\left\langle T_{\mu_2\nu_2}(x_2)\frac{\delta T_{\mu_3\nu_3}(x_3)}{\delta g^{\mu_1\nu_1}(x_1)}\right\rangle \\ &+ i g_{\mu_2\nu_2}\delta(x_2-x_3)\langle T_{\mu_1\nu_1}(x_1)\, T_{\mu_3\nu_3}(x_3)\rangle + i g_{\mu_1\nu_1}\delta(x_1-x_2)\langle T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\rangle \\ &+ i g_{\mu_1\nu_1}\delta(x_1-x_3)\langle T_{\mu_2\nu_2}(x_2)\, T_{\mu_3\nu_3}(x_3)\rangle. \end{aligned} \tag{A.17}$$

As we mentioned earlier, the rhs of (A.17) must be symmetrized with respect to the labels.

A.2 Diffeomorphism invariance

In this section we are going to make explicit some details of the derivation of the Ward identity for diffeomorphism invariance at the level of 3-point functions of energy-momentum tensors, i.e. (3.43). As we have already seen in (3.18), diffeomorphism invariance implies that

$$\nabla^\mu\langle T_{\mu\nu}(x)\rangle_g = 0 \Leftrightarrow g^{\alpha\beta}\partial_\alpha\langle T_{\beta\nu}(x)\rangle_g - g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta}\langle T_{\lambda\nu}(x)\rangle_g - g^{\alpha\beta}\Gamma^\lambda_{\alpha\nu}\langle T_{\lambda\beta}(x)\rangle_g = 0, \tag{A.18}$$

where

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2}g^{\lambda\tau}\left(\partial_\alpha g_{\tau\beta} + \partial_\beta g_{\tau\alpha} - \partial_\tau g_{\alpha\beta}\right). \tag{A.19}$$

Considering a flat background plus a small perturbation h we may write

$$g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}, \quad g_{\alpha\beta} = \eta_{\alpha\beta} - h_{\alpha\beta} + \mathcal{O}\left(h^2\right). \tag{A.20}$$

Expanding the relevant terms up to order two in h we find

$$\Gamma^\lambda_{\alpha\beta} = -\frac{1}{2}\left(\partial_\alpha h^\lambda{}_\beta + \partial_\beta h^\lambda{}_\alpha - \partial^\lambda h_{\alpha\beta}\right) + \mathcal{O}\left(h^2\right), \tag{A.21}$$

$$g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} = -\left(\partial_\alpha h^{\lambda\alpha} - \frac{1}{2}\partial^\lambda h\right) + \mathcal{O}\left(h^2\right), \tag{A.22}$$

$$g^{\alpha\beta}\Gamma^\lambda_{\alpha\nu} = -\frac{1}{2}\left(\partial_\nu h^{\lambda\beta} + \partial^\beta h^\lambda{}_\nu - \partial^\lambda h^\beta{}_\nu\right) + \mathcal{O}\left(h^2\right). \tag{A.23}$$

We are interested on the terms of order two in h in the Ward identity, hence

$$\partial^\mu \langle T_{\mu\nu}(x)\rangle_g^{(2)} + h^{\alpha\beta}\partial_\alpha \langle T_{\beta\nu}(x)\rangle_g^{(1)} + \\ + \left(\partial_\alpha h^{\lambda\alpha} - \frac{1}{2}\partial^\lambda h\right)\langle T_{\lambda\nu}(x)\rangle_g^{(1)} + \frac{1}{2}\partial_\nu h^{\lambda\beta}\langle T_{\lambda\beta}(x)\rangle_g^{(1)} = 0, \quad \text{(A.24)}$$

where

$$\langle T_{\mu\nu}(x)\rangle_g^{(k)} = \frac{i^k}{2^k k!}\int \prod_{i=1}^{k} d^d y_i h^{\mu_i\nu_i}(y_i)\langle T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1)\cdots T_{\mu_k\nu_k}(x_k)\rangle_c . \quad \text{(A.25)}$$

Computing term by term of (A.24) we find:

$$\partial^\mu \langle T_{\mu\nu}(x)\rangle_g^{(2)} = \\ -\frac{1}{8}\int d^d y_1 d^d y_2 h^{\mu_1\nu_1}(y_1) h^{\mu_2\nu_2}(y_2)\partial_x^\mu \langle T_{\mu\nu}(x) T_{\mu_1\nu_1}(y_1) T_{\mu_2\nu_2}(y_2)\rangle_c \quad \text{(A.26)}$$

$$h^{\alpha\beta}\partial_\alpha \langle T_{\beta\nu}(x)\rangle_g^{(1)} = \\ \frac{i}{2}\int d^d y_1 d^d y_2 h^{\mu_1\nu_1}(y_1) h^{\mu_2\nu_2}(y_2)\delta(y_2 - x)\partial_{x(\mu_2}\left\langle T_{\nu_2)\nu}(x) T_{\mu_1\nu_1}(y_1)\right\rangle_c \quad \text{(A.27)}$$

$$\partial_\alpha h^{\lambda\alpha}\langle T_{\lambda\nu}(x)\rangle_g^{(1)} = \\ \frac{i}{2}\int d^d y_1 d^d y_2 h^{\mu_1\nu_1}(y_1) h^{\mu_2\nu_2}(y_2)\partial_{x(\mu_2}\delta(y_2 - x)\left\langle T_{\nu_2)\nu}(x) T_{\mu_1\nu_1}(y_1)\right\rangle_c \quad \text{(A.28)}$$

$$-\frac{1}{2}\partial^\lambda h\langle T_{\lambda\nu}(x)\rangle_g^{(1)} = \\ \frac{i}{4}\int d^d y_1 d^d y_2 h^{\mu_1\nu_1}(y_1) h^{\mu_2\nu_2}(y_2)\eta_{\mu_2\nu_2}\partial_{y_2}^\lambda \delta(y_2 - x)\langle T_{\lambda\nu}(x) T_{\mu_1\nu_1}(y_1)\rangle_c \quad \text{(A.29)}$$

$$\frac{1}{2}\partial_\nu h^{\lambda\beta}\langle T_{\lambda\beta}(x)\rangle_g^{(1)} = \\ -\frac{i}{4}\int d^d y_1 d^d y_2 h^{\mu_1\nu_1}(y_1) h^{\mu_2\nu_2}(y_2)\partial_{y_2,\nu}\delta(y_2 - x)\langle T_{\mu_2\nu_2}(x) T_{\mu_1\nu_1}(y_1)\rangle_c \quad \text{(A.30)}$$

Putting everything together we have

$$\frac{1}{2}\int d^d y_1 d^d y_2 h^{\mu_1\nu_1}(y_1) h^{\mu_2\nu_2}(y_2)\Bigg[-\frac{1}{4}\partial_x^\mu \langle T_{\mu\nu}(x) T_{\mu_1\nu_1}(y_1) T_{\mu_2\nu_2}(y_2)\rangle_c + \\ + i\partial_{x(\mu_2}\left(\delta(y_2 - x)\left\langle T_{\nu_2)\nu}(x) T_{\mu_1\nu_1}(y_1)\right\rangle_c\right) - \frac{i}{2}\eta_{\mu_2\nu_2}\partial_x^\lambda \delta(y_2 - x)\langle T_{\lambda\nu}(x) T_{\mu_1\nu_1}(y_1)\rangle_c + \\ + \frac{i}{2}\partial_{x,\nu}\delta(y_2 - x)\langle T_{\mu_2\nu_2}(x) T_{\mu_1\nu_1}(y_1)\rangle_c\Bigg] = 0. \quad \text{(A.31)}$$

Thus, the Ward identity for diffeomorphism invariance is

$$\begin{aligned}&\partial_x^\mu \left\langle T_{\mu\nu}(x)\, T_{\mu_1\nu_1}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle_c = \\ &2i\partial_{x(\mu_1}\left(\delta(x-y_1)\left\langle T_{\nu_1)\nu}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle_c\right) + 2i\partial_{x(\mu_2}\left(\delta(x-y_2)\left\langle T_{\nu_2)\nu}(x)\, T_{\mu_1\nu_1}(y_1)\right\rangle_c\right)\\ &\quad - i\eta_{\mu_1\nu_1}\partial_x^\lambda\delta(x-y_1)\left\langle T_{\lambda\nu}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle_c - i\eta_{\mu_2\nu_2}\partial_x^\lambda\delta(x-y_2)\left\langle T_{\lambda\nu}(y_2)\, T_{\mu_1\nu_1}(y_1)\right\rangle_c\\ &\qquad + i\left(\partial_{x,\nu}\delta(x-y_1) + \partial_{x,\nu}\delta(x-y_2)\right)\left\langle T_{\mu_1\nu_1}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle_c . \end{aligned} \tag{A.32}$$

A.3 Weyl invariance

In this section we are going to make explicit some details of the derivation of the Ward identity for Weyl invariance at the level of 3-point functions of energy-momentum tensors, i.e. (3.44). As we have already seen in (3.19), Weyl invariance implies that

$$\left\langle T^\mu_\mu(x)\right\rangle_g = 0 \Leftrightarrow \left(\eta^{\mu\nu} + h^{\mu\nu}\right)\left\langle T_{\mu\nu}(x)\right\rangle_g = 0. \tag{A.33}$$

We are interested in the terms of order two in h, hence

$$\eta^{\mu\nu}\left\langle T_{\mu\nu}(x)\right\rangle_g^{(2)} + h^{\mu\nu}\left\langle T_{\mu\nu}(x)\right\rangle_g^{(1)} = 0. \tag{A.34}$$

Computing each one of the terms of the expression above:

$$\eta^{\mu\nu}\left\langle T_{\mu\nu}(x)\right\rangle_g^{(2)} = -\frac{1}{8}\int d^dy_1 d^dy_2 h^{\mu_1\nu_1}(y_1)\, h^{\mu_2\nu_2}(y_2)\left\langle T^\mu_\mu(x)\, T_{\mu_1\nu_1}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle_c , \tag{A.35}$$

$$\begin{aligned}&h^{\mu\nu}\left\langle T_{\mu\nu}(x)\right\rangle_g^{(1)} = \\ &\frac{i}{4}\int d^dy_1 d^dy_2 h^{\mu_1\nu_1}(y_1)\, h^{\mu_2\nu_2}(y_2)\left(\delta(x-y_1)+\delta(x-y_2)\right)\left\langle T_{\mu_1\nu_1}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle_c .\end{aligned} \tag{A.36}$$

Putting together the results above we find

$$\begin{aligned}&-\frac{1}{8}\int d^dy_1 d^dy_2 h^{\mu_1\nu_1}(y_1)\, h^{\mu_2\nu_2}(y_2)\left[\left\langle T^\mu_\mu(x)\, T_{\mu_1\nu_1}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle - \right.\\ &\qquad\left. -2i\left(\delta(x-y_1)+\delta(x-y_2)\right)\left\langle T_{\mu_1\nu_1}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle_c\right] = 0.\end{aligned} \tag{A.37}$$

Therefore, the Ward identity for Weyl invariance is

$$\left\langle T^\mu_\mu(x)\, T_{\mu_1\nu_1}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle_c = 2i\left(\delta(x-y_1)+\delta(x-y_2)\right)\left\langle T_{\mu_1\nu_1}(y_1)\, T_{\mu_2\nu_2}(y_2)\right\rangle_c . \tag{A.38}$$

Chapter 4

Conformal invariance in momentum space

The conformal transformations are best known in configuration space, where we have a clear prescription of how to construct conformally covariant correlation functions, see for example [3–6, 11]. For several applications it is useful to have full control of correlation functions of a CFT in momentum space. For a few examples of these applications see [22, 24, 53]. Hence it would be very interesting to have a prescription to write conformally covariant correlation functions in momentum space. This problem has been addressed in the references [13, 14, 49, 54] and we are going to review some of the main ideas in this chapter.

4.1 Conformal Ward Identities

In this section we are going to review the constraints that conformal invariance implies in correlation function both in configuration space and in momentum space.

Generically, as we already saw in the previous chapter, a symmetry of the partition function Z of our theory implies Ward identities for correlators. In this chapter we would like to concentrate on the particular case in which the symmetry is a spacetime symmetry, the conformal symmetry. Consider an n-point correlation function

$$\left\langle \mathcal{O}_1\left(x_1\right)\cdots\mathcal{O}_n\left(x_n\right)\right\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\,\mathcal{O}_1\left(x_1\right)\cdots\mathcal{O}_n\left(x_n\right)e^{iS[\Phi]}. \tag{4.1}$$

Under some spacetime transformation g, $\mathcal{O}_i\left(x_i\right) \to \mathcal{O}'_i\left(x'_i\right) = \mathcal{O}_i\left(x_i\right) + i\epsilon\,\delta_g\mathcal{O}_i\left(x_i\right) + \mathcal{O}\left(\epsilon^2\right)$. If the classical action is invariant under the action of g, then

$$\left\langle \mathcal{O}'_1\left(x'_1\right)\cdots\mathcal{O}'_n\left(x'_n\right)\right\rangle = \left\langle \mathcal{O}_1\left(x_1\right)\cdots\mathcal{O}_n\left(x_n\right)\right\rangle. \tag{4.2}$$

On the other hand, the LHS of (4.2) can be written in terms of the original fields as

$$\begin{aligned}\left\langle \mathcal{O}'_1\left(x'_1\right)\cdots\mathcal{O}'_n\left(x'_n\right)\right\rangle &= \\ &= \left\langle \mathcal{O}_1\left(x_1\right)\cdots\mathcal{O}_n\left(x_n\right)\right\rangle + i\epsilon\sum_{i=1}^{n}\left\langle \mathcal{O}_1\left(x_1\right)\cdots\delta_g\mathcal{O}_i\left(x_i\right)\cdots\mathcal{O}_n\left(x_n\right)\right\rangle + O\left(\epsilon^2\right).\end{aligned} \tag{4.3}$$

Putting back this relation in (4.2) we find that at first order in ϵ we need that

$$\sum_{i=1}^{n}\left\langle \mathcal{O}_1\left(x_1\right)\cdots\delta_g\mathcal{O}_i\left(x_i\right)\cdots\mathcal{O}_n\left(x_n\right)\right\rangle = 0. \tag{4.4}$$

The conformal group is composed by Lorentz transformations, spacetime translations, dilations and special conformal transformations. The infinitesimal generators of these transformations are denoted respectively by $M_{\mu\nu}$, P_μ, D and K_μ. Of course that the infinitesimal variation of the field $\mathcal{O}(x)$ under the action of the transformation g can be written as a commutator of the infinitesimal generator $\mathfrak{g}$ of the transformation g with the field $\mathcal{O}$, i.e.

$$i\delta_g \mathcal{O}(x) \equiv [\mathfrak{g}, \mathcal{O}(x)] . \tag{4.5}$$

Example. As a simple example, consider the 2-point function of scalar operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(y)$ in a translational invariant theory. Translational invariance implies for the correlation function that

$$\langle [P_\mu, \mathcal{O}_1(x)] \mathcal{O}_2(y) \rangle + \langle \mathcal{O}_1(x) [P_\mu, \mathcal{O}_2(y)] \rangle = 0 \Leftrightarrow \left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu} \right) \langle \mathcal{O}_1(x) \mathcal{O}_2(y) \rangle = 0. \tag{4.6}$$

The solution of the differential equation

$$\left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu} \right) f(x, y) = 0 \tag{4.7}$$

is an arbitrary function of $x - y$.

4.1.1 Dilations and special conformal transformations of tensor fields

Under a general conformal transformation, a tensor field $\Phi_{\mu_1 \ldots \mu_\ell}(x)$ transforms as

$$\Phi_{\mu_1 \ldots \mu_\ell}(x) \mapsto \Phi'_{\mu_1 \ldots \mu_\ell}(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\frac{\ell - \Delta}{d}} \frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\nu_\ell}}{\partial x^{\mu_\ell}} \Phi_{\nu_1 \ldots \nu_\ell}(x') , \tag{4.8}$$

while $\Phi^{\mu_1 \ldots \mu_\ell}(x)$ transforms as[1]

$$\Phi^{\mu_1 \ldots \mu_\ell}(x) \mapsto \Phi'^{\mu_1 \ldots \mu_\ell}(x') = \left| \frac{\partial x}{\partial x'} \right|^{-\frac{\ell + \Delta}{d}} \frac{\partial x^{\mu_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\mu_\ell}}{\partial x'^{\nu_\ell}} \Phi^{\nu_1 \ldots \nu_\ell}(x') . \tag{4.9}$$

Example. Let us consider dilations, $x' = \lambda x$. Under this transformation a field $\Phi(x)$ of scaling dimension Δ becomes

$$\Phi(x) \mapsto \Phi'(x') = \lambda^\Delta \Phi(\lambda x) . \tag{4.10}$$

A field that is invariant under dilations is such that

$$\Phi'(x') = \Phi(x) \Leftrightarrow \Phi(\lambda x) = \lambda^{-\Delta} \Phi(x) . \tag{4.11}$$

From (4.10) we can derive how an infinitesimal dilation acts on a field $\Phi(x)$. For $\lambda = 1 + \epsilon$ we have

$$(1 + \epsilon)^\Delta \Phi((1 + \epsilon) x) = \Phi(x) + \epsilon (\Delta + x \cdot \partial) \Phi(x) + \mathcal{O}(\epsilon^2) . \tag{4.12}$$

Therefore

$$i\delta_D \Phi(x) \equiv [D, \Phi(x)] = i (\Delta + x \cdot \partial) \Phi(x) . \tag{4.13}$$

[1]Notice that if we have a tensor field with ℓ_1 contravariant and ℓ_2 covariant indices it will transform as

$$\Phi^{\mu_1 \ldots \mu_{\ell_1}}_{\nu_1 \ldots \nu_{\ell_2}}(x) \mapsto \Phi'^{\mu_1 \ldots \mu_{\ell_1}}_{\nu_1 \ldots \nu_{\ell_2}}(x') = \left| \frac{\partial x'}{\partial x} \right|^{\frac{(\ell_1 - \ell_2) + \Delta}{d}} \frac{\partial x^{\mu_1}}{\partial x'^{\rho_1}} \cdots \frac{\partial x^{\mu_{\ell_1}}}{\partial x'^{\rho_{\ell_1}}} \frac{\partial x'^{\sigma_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\sigma_{\ell_2}}}{\partial x^{\nu_{\ell_2}}} \Phi'^{\rho_1 \ldots \rho_{\ell_1}}_{\sigma_1 \ldots \sigma_{\ell_2}}(x') .$$

Example. Let us consider now special conformal transformations

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2}.$$

For an infinitesimal b^{μ} we have

$$x'^{\mu} = x^{\mu} + 2b \cdot x\, x^{\mu} - b^{\mu} x^2,$$

from which we may derive

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} - 2\left(b^{\mu} x_{\nu} - b_{\nu} x^{\mu}\right) + 2\left(b \cdot x\right) \delta^{\mu}_{\nu}, \qquad \left|\frac{\partial x'}{\partial x}\right| = 1 + 2d\left(b \cdot x\right).$$

Particularly, the transformations of a rank 2 tensor is given by

$$\begin{aligned} \Phi'_{\mu\nu}\left(x'\right) &= \left|\frac{\partial x'}{\partial x}\right|^{-\frac{2-\Delta}{d}} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \Phi_{\rho\sigma}\left(x'\right) \\ &= \left(1 - \left(4 - 2\Delta\right)\left(b \cdot x\right)\right) \\ &\times \left[\delta^{\rho}_{\mu} - 2\left(b^{\rho} x_{\mu} - b_{\mu} x^{\rho}\right) + 2\left(b \cdot x\right) \delta^{\rho}_{\mu}\right] \left[\delta^{\sigma}_{\nu} - 2\left(b^{\sigma} x_{\nu} - b_{\nu} x^{\sigma}\right) + 2\left(b \cdot x\right) \delta^{\sigma}_{\nu}\right] \\ &\times \left(\Phi_{\rho\sigma}\left(x\right) + \left(2\left(b \cdot x\right)\, x \cdot \partial - x^2 b \cdot \partial\right) \Phi_{\rho\sigma}\left(x\right) + \cdots\right). \end{aligned} \tag{4.14}$$

Hence

$$\Phi'_{\mu\nu}\left(x'\right) = \Phi_{\mu\nu}\left(x\right) + \left[2\Delta\left(b \cdot x\right) - 2b^{\rho} x^{\sigma} S_{\rho\sigma} + 2\left(b \cdot x\right) x \cdot \partial - x^2\left(b \cdot \partial\right)\right] \Phi_{\mu\nu}\left(x\right), \tag{4.15}$$

where

$$b^{\rho} x^{\sigma} S_{\rho\sigma} \Phi_{\mu\nu}\left(x\right) = \left(b^{\rho} x_{\mu} - b_{\mu} x^{\rho}\right) \Phi_{\rho\nu}\left(x\right) + \left(b^{\sigma} x_{\nu} - b_{\nu} x^{\sigma}\right) \Phi_{\mu\sigma}\left(x\right). \tag{4.16}$$

From this expression we can read the contributions appearing in an infinitesimal special conformal transformation:

$$i\delta_{K_{\rho}} \Phi_{\mu\nu}\left(x\right) \equiv \left[K_{\rho}, \Phi_{\mu\nu}\right] = i\left(2\Delta\, x_{\rho} - 2x^{\sigma} S_{\rho\sigma} + 2x_{\rho}\, x \cdot \partial - x^2 \partial_{\rho}\right) \Phi_{\mu\nu}\left(x\right), \tag{4.17}$$

where $S_{\rho\sigma}$ is the spin part of the generator of Lorentz transformations. Its action on a tensor field is given by

$$S_{\rho\sigma} \Phi_{\mu_1 \ldots \mu_\ell} = \sum_{i=1}^{\ell} \left(\eta_{\rho\mu_i} \delta^{\tau}_{\sigma} - \eta_{\sigma\mu_i} \delta^{\tau}_{\rho}\right) \Phi_{\mu_1 \ldots \tau \ldots \mu_\ell}, \tag{4.18}$$

i.e.,

$$x^{\sigma} S_{\rho\sigma} \Phi_{\mu_1 \ldots \mu_\ell} = \sum_{i=1}^{\ell} \left(\eta_{\rho\mu_i} x^{\tau} - x_{\mu_i} \delta^{\tau}_{\rho}\right) \Phi_{\mu_1 \ldots \tau \ldots \mu_\ell}. \tag{4.19}$$

From the equation (4.17) it is straightforward to generalize the result for an arbitrary tensor field.

4.1.2 Infinitesimal transformations in momentum space

Let us focus now on the transformations (4.13) and (4.17), which we repeat here

$$\left[D, \Phi\left(x\right)\right] = i\left(\Delta + x \cdot \partial\right) \Phi\left(x\right), \tag{4.20}$$

$$\left[K_{\rho}, \Phi\left(x\right)\right] = i\left[2\Delta\, x_{\rho} - 2x^{\sigma} S_{\rho\sigma} + 2x_{\rho}\, x \cdot \partial - x^2 \partial_{\rho}\right] \Phi\left(x\right). \tag{4.21}$$

Notice that we suppressed the indices of the tensor field $\Phi(x)$ but it should be understood that it could be any rank ℓ tensor field. We would like to write these expressions in momentum space. To do so we will simply replace

$$\begin{aligned} x_\mu &\rightarrow -i\frac{\partial}{\partial p^\mu}, \\ \partial_\mu &\rightarrow -ip_\mu. \end{aligned}$$

Therefore

$$\begin{aligned} \left[\hat{D},\Phi(p)\right] &= i\left((\Delta-d)-p\cdot\partial\right)\Phi(p), \qquad (4.22)\\ \left[\hat{K}_\rho\Phi(p)\right] &= \left(2(\Delta-d)\,\partial_\rho-2\partial^\sigma\hat{S}_{\rho\sigma}-2(p\cdot\partial)\,\partial_\rho+p_\rho\Box\right)\Phi(p), \qquad (4.23)\end{aligned}$$

where ∂ means here derivation with respect to p and

$$\partial^\sigma\hat{S}_{\rho\sigma}\Phi_{\mu_1\ldots\mu_\ell}(p)=\sum_{i=1}^{\ell}\left(\eta_{\rho\mu_i}\partial^\tau-\delta^\tau_\rho\partial_{\mu_i}\right)\Phi_{\mu_1\ldots\tau\ldots\mu_\ell}(p). \qquad (4.24)$$

4.2 Conformal Ward Identites in Momentum Space

From here on we will refer to the conformal Ward identities simply as the *CWI's*. Following (4.4), (4.20) and (4.21), the Ward identities for dilatations and special conformal transformations in configuration space are respectively

$$\sum_{i=1}^{n}\left(\Delta_i+x_i^\mu\frac{\partial}{\partial x_i^\mu}\right)\langle\mathcal{O}_1(\boldsymbol{x}_1)\cdots\mathcal{O}_n(\boldsymbol{x}_n)\rangle=0 \qquad (4.25)$$

and

$$\sum_{i=1}^{n}\left(\mathcal{K}_i^\kappa+\mathcal{L}_i^\kappa\right)\langle\mathcal{O}_1(\boldsymbol{x}_1)\cdots\mathcal{O}_n(\boldsymbol{x}_n)\rangle=0, \qquad (4.26)$$

where the operators $\mathcal{O}_i$ represent any operator in the CFT and $\mathcal{K}_i^\kappa$ and $\mathcal{L}_i^\kappa$ are differential operators, the first one do not depend on the tensorial structure of the correlator, while the second one does. Particularly, for scalar operators $\mathcal{L}_i^\kappa\equiv 0$. Their explicit expression are

$$\mathcal{K}_i^\kappa \equiv 2\Delta_i x_i^\kappa+2x_i^\kappa x_i^\alpha\frac{\partial}{\partial x_i^\alpha}-x_i^2\frac{\partial}{\partial x_{i\kappa}}, \qquad (4.27)$$

while

$$\begin{aligned}\mathcal{L}_i^\kappa\left\langle\mathcal{O}_1^{\mu_{11}\cdots\mu_{1r_1}}(\boldsymbol{x}_1)\cdots\mathcal{O}_n^{\mu_{n1}\cdots\mu_{nr_n}}(\boldsymbol{x}_n)\right\rangle &\equiv 2\sum_{k=1}^{r_i}\left((x_i)_{\alpha_{ik}}\delta^{\kappa\mu_{ik}}-x_i^{\mu_{ik}}\delta^\kappa_{\alpha_{ik}}\right)\times\\ &\times\left\langle\mathcal{O}_1^{\mu_{11}\cdots\mu_{1r_1}}(\boldsymbol{x}_1)\cdots\mathcal{O}_i^{\mu_{i1}\cdots\alpha_{ik}\cdots\mu_{ir_i}}(\boldsymbol{x}_i)\cdots\mathcal{O}_n^{\mu_{n1}\cdots\mu_{nr_n}}(\boldsymbol{x}_n)\right\rangle. \qquad (4.28)\end{aligned}$$

Because of translation invariance we may set $x_n = 0$, which simplifies the Ward identities leaving us with

$$\begin{aligned}\left(\Delta_t + \sum_{i=1}^{n-1} x_i^\mu \frac{\partial}{\partial x_i^\mu}\right) \langle \mathcal{O}_1(\boldsymbol{x}_1) \cdots \mathcal{O}_n(\boldsymbol{x}_n)\rangle = 0, \\ \sum_{i=1}^{n-1} \left(\mathcal{K}_i^\kappa + \mathcal{L}_i^\kappa\right) \langle \mathcal{O}_1(\boldsymbol{x}_1) \cdots \mathcal{O}_n(\boldsymbol{x}_n)\rangle = 0,\end{aligned} \tag{4.29}$$

where $\Delta_t = \sum_{i=1}^n \Delta_i$.

In momentum space, translation invariance is equivalent to momentum conservation, i.e. $\sum_{i=1}^n \boldsymbol{p}_i = 0$, and our choice $\boldsymbol{x}_n = 0$, corresponds to say that we are reexpressing $\boldsymbol{p}_n$ in terms of all the other $\boldsymbol{p}_i$, namely

$$\boldsymbol{p}_n = -\sum_{i=1}^{n-1} \boldsymbol{p}_i. \tag{4.30}$$

The CWI's in momentum space read

$$\begin{aligned}\left(\Delta_t - (n-1)\, d - \sum_{i=1}^{n-1} p_i^\mu \frac{\partial}{\partial p_i^\mu}\right) \langle\langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle\rangle = 0, \\ \sum_{i=1}^{n-1} \left(\hat{\mathcal{K}}_i^\kappa + \hat{\mathcal{L}}_i^\kappa\right) \langle\langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle\rangle = 0,\end{aligned} \tag{4.31}$$

where

$$\langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle \equiv (2\pi)^d\, \delta^d\left(\sum_{i=1}^n \boldsymbol{p}_i\right) \langle\langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle\rangle, \tag{4.32}$$

$$\hat{\mathcal{K}}_i^\kappa = 2\left(\Delta_i - d\right) \frac{\partial}{\partial p_i^\kappa} - 2 p_i^\alpha \frac{\partial}{\partial p_i^\alpha} \frac{\partial}{\partial p_i^\kappa} + (p_i)_\kappa \frac{\partial}{\partial p_i^\alpha} \frac{\partial}{\partial p_{i\alpha}} \tag{4.33}$$

and

$$\begin{aligned}\hat{\mathcal{L}}_i^\kappa \left\langle\left\langle \mathcal{O}_1^{\mu_{11}\cdots\mu_{1r_1}}(\boldsymbol{p}_1) \cdots \mathcal{O}_n^{\mu_{n1}\cdots\mu_{nr_n}}(\boldsymbol{p}_n)\right\rangle\right\rangle \equiv 2 \sum_{k=1}^{r_i} \left(\delta^{\kappa\mu_{ik}} \frac{\partial}{\partial p_i^{\alpha_{ik}}} - \delta_{\alpha_{ik}}^\kappa \frac{\partial}{\partial p_{i\mu_{ik}}}\right) \\ \left\langle\left\langle \mathcal{O}_1^{\mu_{11}\cdots\mu_{1r_1}}(\boldsymbol{p}_1) \cdots \mathcal{O}_i^{\mu_{i1}\cdots\alpha_{ik}\cdots\mu_{ir_i}}(\boldsymbol{p}_i) \cdots \mathcal{O}_n^{\mu_{n1}\cdots\mu_{nr_n}}(\boldsymbol{p}_n)\right\rangle\right\rangle.\end{aligned} \tag{4.34}$$

We call the attention of the reader to the fact that equation (4.31) is not simply the Fourier transform of (4.25) and (4.26). To get (4.31) we need to first pass throught the delta function in the definition (4.32). The details of these computations are in the appendix B.

4.2.1 CWI for 2-point function of scalar operators

A 2-point function in momentum space is a function of two momenta that sums to zero, in other words, it a function of a vector $\boldsymbol{p}$. Particularly for a scalar correlator, because of translation invariance and Lorentz invariance, it must be a function of the magnitude p of the vector $\boldsymbol{p}$. The equations (4.31) simplify considerably in the case of scalar 2-point functions.

Indeed, we find

$$\left(\Delta_1+\Delta_2-d-p\frac{d}{dp}\right)\langle\langle\mathcal{O}_1(\boldsymbol{p})\,\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle=0, \tag{4.35}$$

$$\left(\frac{d^2}{dp^2}+\frac{d+1-2\Delta_1}{p}\frac{d}{dp}\right)\langle\langle\mathcal{O}_1(\boldsymbol{p})\,\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle=0. \tag{4.36}$$

Solving the special conformal Ward identity (4.36) we obtain

$$\langle\langle\mathcal{O}_1(\boldsymbol{p})\,\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle=c_1p^{2\Delta_1-d}+c_2, \tag{4.37}$$

where c_1 and c_2 are integration constants. Inserting the solution (4.37) in the dilation Ward identity (4.35) we find

$$\Delta_1=\Delta_2\equiv\Delta,\quad c_2=0. \tag{4.38}$$

Thus, for generic Δ, the solution of the CWI's is

$$\langle\langle\mathcal{O}_1(\boldsymbol{p})\,\mathcal{O}_2(-\boldsymbol{p})\rangle\rangle=c_1\,p^{2\Delta-d}. \tag{4.39}$$

Notice that for $\Delta=\frac{d}{2}+k$, $k=0,1,2,\ldots$, (4.39) would naively become proportional to p^{2k}, which represents a contact term in configuration space, and could be subtracted by a counterterm. The story in this particular situation is more delicate and we refer to the reference [49] for the full explanation. For a taste of why there is more to this story, let us consider the 2-point function of scalar operators of scaling dimension Δ in configuration space

$$\langle\mathcal{O}(\boldsymbol{x})\,\mathcal{O}(0)\rangle=\frac{1}{x^{2\Delta}}. \tag{4.40}$$

Fourier-transforming (4.40) for generic Δ we obtain

$$\langle\langle\mathcal{O}(\boldsymbol{p})\,\mathcal{O}(-\boldsymbol{p})\rangle\rangle=\int d^dx\,e^{-ipx}\frac{1}{x^{2\Delta}}\sim\frac{\Gamma\left(\frac{d}{2}-\Delta\right)}{\Gamma(\Delta)}p^{2\Delta-d}, \tag{4.41}$$

from where we see that if $\Delta=\frac{d}{2}+k$, $k=0,1,2,\ldots$, we will hit the poles of the Gamma function $\Gamma\left(\frac{d}{2}-\Delta\right)$. Thus, we need a regularization and the renormalized correlator will have an extra logarithmic contribution, namely

$$\langle\langle\mathcal{O}(\boldsymbol{p})\,\mathcal{O}(-\boldsymbol{p})\rangle\rangle_{\text{ren.}}=p^{2k}\left(c_\Delta\ln\frac{p^2}{\mu^2}+c'_\Delta\right), \tag{4.42}$$

where c'_Δ parametrizes the scheme-dependence of the renormalized correlator.

4.2.2 CWI for 3-point function of scalar operators

A 3-point function in momentum space is a function of three momenta that sums to zero, in other words, it is function of a triangle, where the edges of the triangle are the momenta figuring in the correlator. Particularly for a scalar correlator, because of translation invariance and Lorentz invariance, the position and orientation of the triangle are not important, thus the information that we need to completly fix our triangle is only the lenght of each one of the edges or two lengths and the angle between these edges. We are going to use the first option and characterize our correlator by the norm of our momenta, i.e.

$$\langle\langle\mathcal{O}_1(\boldsymbol{p}_1)\,\mathcal{O}_2(\boldsymbol{p}_2)\,\mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle\equiv A(p_1,p_2,p_3). \tag{4.43}$$

For correlators with non-scalar operators the answer will depend also on the orientation of the triangle, i.e. the direction of each momentum is also relevant, but we are always able the decompose our correlator into scalar form factors times tensorial structures as we will see in a moment.

Since our variables now are the moduli of the momenta, we need to rewrite our momentum derivatives in terms of derivatives with respect to the moduli. In order to do so we will take $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ to be independent and $\boldsymbol{p}_3 = -(\boldsymbol{p}_1 + \boldsymbol{p}_2)$. Thus,

$$\frac{\partial}{\partial p_1^\mu} = \frac{\partial p_1}{\partial p_1^\mu}\frac{\partial}{\partial p_1} + \frac{\partial p_2}{\partial p_1^\mu}\frac{\partial}{\partial p_2} + \frac{\partial p_3}{\partial p_1^\mu}\frac{\partial}{\partial p_3}, \tag{4.44}$$

where the derivatives are given by

$$\frac{\partial p_1}{\partial p_1^\mu} = \frac{p_{1\mu}}{p_1}, \quad \frac{\partial p_2}{\partial p_1^\mu} = 0, \quad \frac{\partial p_3}{\partial p_1^\mu} = \frac{p_{1\mu}+p_{2\mu}}{p_3} = -\frac{p_{3\mu}}{p_3}. \tag{4.45}$$

Therefore

$$\frac{\partial}{\partial p_1^\mu} = \frac{p_{1\mu}}{p_1}\frac{\partial}{\partial p_1} - \frac{p_{3\mu}}{p_3}\frac{\partial}{\partial p_3}, \tag{4.46}$$

and analogously

$$\frac{\partial}{\partial p_2^\mu} = \frac{p_{2\mu}}{p_2}\frac{\partial}{\partial p_2} - \frac{p_{3\mu}}{p_3}\frac{\partial}{\partial p_3}. \tag{4.47}$$

The expressions (4.46) and (4.47) allow us to rewrite the CWI for dilatation (4.31) as

$$\left(2d - \Delta_t + \sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i}\right) A\left(p_1, p_2, p_3\right) = 0, \tag{4.48}$$

since

$$p_1^\mu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\mu} = p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} + p_3 \frac{\partial}{\partial p_3}. \tag{4.49}$$

For the expression for the Ward identity of special conformal transformation we find

$$-\sum_{j=1}^{3} p_{j\sigma}\left[\partial_j^2 + \frac{d+1-2\Delta_j}{p_j}\partial_j\right] A\left(p_1, p_2, p_3\right) = 0. \tag{4.50}$$

Since $p_{3\sigma} = -\left(p_{1\sigma} + p_{2\sigma}\right)$, (4.50) implies a system of two differential equations

$$\begin{aligned}
&\left[\left(\partial_1^2 + \frac{d+1-2\Delta_1}{p_1}\partial_1\right) - \left(\partial_3^2 + \frac{d+1-2\Delta_3}{p_3}\partial_3\right)\right] A\left(p_1, p_2, p_3\right) = 0,\\
&\left[\left(\partial_2^2 + \frac{d+1-2\Delta_2}{p_2}\partial_2\right) - \left(\partial_3^2 + \frac{d+1-2\Delta_3}{p_3}\partial_3\right)\right] A\left(p_1, p_2, p_3\right) = 0.
\end{aligned} \tag{4.51}$$

For the details of this derivation, check the appendix C. In [14, 49] it is shown that it is possible to give a very convenient integral representation, a.k.a. the triple-K representation, to the solution of (4.48) and (4.51), namely

$$A\left(p_1, p_2, p_3\right) = c_{123}\int_0^\infty dx\, x^{\frac{d}{2}-1}\prod_{j=1}^{3} p_j^{\Delta_j - \frac{d}{2}} K_{\Delta_j - \frac{d}{2}}\left(p_j x\right), \tag{4.52}$$

where $K_\nu\left(x\right)$ is the modified Bessel function of second kind.

4.3 Tensor structures in momentum space

Now we are going to consider correlation functions of tensor operators in momentum space. We will devote our analysis to completely symmetric conserved currents. The simplest examples are conserved currents J_μ associated to global symmetries and the stress-energy tensor $T_{\mu\nu}$. For free theories[2] we may also construct completely symmetric higher-spin conserved currents $J_{\mu_1\cdots\mu_s}$. We will talk more about higher-spin conserved currents in the chapter 8. For now, our only concern is to learn how to deal with the tensorial structures figuring in correlation functions of conserved currents.

4.3.1 2-point function of conserved currents

As we have seen in the chapter 3, the classical conservation laws of currents are promoted to Ward identities, i.e. relations among correlation functions at the quantum level. For 2-point functions these relations are very simple, they simply state that the 2-point function itself is conserved, i.e.

$$\partial^{\mu_1} \langle J_{\mu_1\cdots\mu_s}(x)\, J_{\nu_1\cdots\nu_s}(0)\rangle = 0. \tag{4.53}$$

In momentum space, the relation (4.53) implies

$$p^{\mu_1} \langle\langle J_{\mu_1\cdots\mu_s}(\boldsymbol{p})\, J_{\nu_1\cdots\nu_s}(-\boldsymbol{p})\rangle\rangle = 0, \tag{4.54}$$

in other words, the 2-point function $\langle\langle J_{\mu_1\cdots\mu_s}(\boldsymbol{p})\, J_{\nu_1\cdots\nu_s}(-\boldsymbol{p})\rangle\rangle$ is transverse with respect to the momentum p^μ. Let us start looking closely at the simplest case: the 2-point function of spin-1 currents. Just taking into account Lorentz invariance, the most general 2-point function of spin-1 currents is

$$\langle\langle J_\mu(\boldsymbol{p})\, J_\nu(-\boldsymbol{p})\rangle\rangle = A(p)\,\eta_{\mu\nu} + B(p)\,p_\mu p_\nu. \tag{4.55}$$

Particularly for $d = 3$ we would have a third option: $\epsilon_{\mu\nu\rho}p^\rho$. For the time being we will neglect this particular case to continue with the discussion in generic dimension. The requirement of (4.55) being transverse implies that

$$B(p) = -\frac{1}{p^2}A(p). \tag{4.56}$$

Thus,

$$\langle\langle J_\mu(\boldsymbol{p})\, J_\nu(-\boldsymbol{p})\rangle\rangle = A(p)\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right). \tag{4.57}$$

The quantity in parenthesis is a projector on the subspace transverse to p^μ and will donote it $\pi_{\mu\nu}(p)$, i.e.

$$\pi_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}. \tag{4.58}$$

The importance of the expression (4.57) is that we factorized our correlation function into a form factor times a tensor structure and the CWI's will become a system of differential equations that the form factor must satisfy.

The next example that we are going to consider is the 2-point function of stress-energy tensors. Since we now that it must be transverse, we start by attaching transverse projectors

[2] It was shown in [26] for $d = 3$ and later in [55] for $d > 3$ that a CFT possess higher-spin symmetry if and only if the CFT is free.

to every index, i.e.

$$\langle\langle T_{\mu_1\mu_2}(\boldsymbol{p})\, T_{\nu_1\nu_2}(-\boldsymbol{p})\rangle\rangle = \pi_{\mu_1}^{\alpha_1}\pi_{\mu_2}^{\alpha_2}\pi_{\nu_1}^{\beta_1}\pi_{\nu_2}^{\beta_2}\left[A(p)\left(\eta_{\alpha_1\beta_1}\eta_{\alpha_2\beta_2}+\eta_{\alpha_1\beta_2}\eta_{\alpha_2\beta_1}\right)+ \right. \\ \left. +B(p)\,\eta_{\alpha_1\alpha_2}\eta_{\beta_1\beta_2}\right]. \quad (4.59)$$

Notice that the tensors contracted to the projectors are all the available ones that could contribute. Two terms have a common form factor in order to guarantee that the RHS has the same symmetries as the LHS. The expression above may be simplified using the properties of the projector (4.58)

$$\langle\langle T_{\mu_1\mu_2}(\boldsymbol{p})\, T_{\nu_1\nu_2}(-\boldsymbol{p})\rangle\rangle = A(p)\left(\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}+\pi_{\mu_1\nu_2}\pi_{\mu_2\nu_1}\right)+B(p)\,\pi_{\mu_1\mu_2}\pi_{\nu_1\nu_2}. \quad (4.60)$$

The expression (4.60) would be the form of the 2-point functions of the stress-energy tensor in a generic QFT, but in a CFT, on the top of being transverse, this 2-point function must also be traceless. The traceless condition implies that

$$B(p) = -\frac{2}{d-1}A(p)\,, \quad (4.61)$$

and we find

$$\langle\langle T_{\mu_1\mu_2}(\boldsymbol{p})\, T_{\nu_1\nu_2}(-\boldsymbol{p})\rangle\rangle = A(p)\left(\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}+\pi_{\mu_1\nu_2}\pi_{\mu_2\nu_1}-\frac{2}{d-1}\pi_{\mu_1\mu_2}\pi_{\nu_1\nu_2}\right). \quad (4.62)$$

Again, the expression in parenthesis is a projector, this time a projector over transverse-traceless tensors and we will denote it $\Pi_{\mu\nu}^{\alpha\beta}(p)$, i.e.

$$\Pi_{\mu\nu}^{\alpha\beta} = \pi_\mu^\alpha\pi_\nu^\beta+\pi_\mu^\beta\pi_\nu^\alpha-\frac{2}{d-1}\pi_{\mu\nu}\pi^{\alpha\beta}. \quad (4.63)$$

4.3.2 Projectors and polarization vectors

A very useful idea to simplify the presentation of tensor correlators is to contract the free indices with polarization vectors n_i. We will introduce one polarization vector for each operator in the correlator. Since we are only interested in operators living in completly symmetric tensor representations, we may contract every index of an operator with the same polarization vector without losing information. On the top of that, if we want to consider objects that are transverse, we may also ask the polarization vectors to be transverse to the correspondent momentum, i.e.

$$n_i \cdot p_i = 0. \quad (4.64)$$

This feature is desirable because

$$n_i^\mu \pi_\mu^\nu(p_i) = n_i^\nu. \quad (4.65)$$

In the same way, if one wishes to describe transverse-traceless objects, it is convenient to consider

$$n_i\cdot p_i = 0 \quad \text{and} \quad n_i^2 = 0. \quad (4.66)$$

These conditions imply that

$$n_i^\mu n_i^\nu \Pi_{\mu\nu}^{\rho\sigma}(p_i) = n_i^\rho n_i^\sigma. \quad (4.67)$$

Using the polarization vectors, the 2-point function of currents (4.57) becomes

$$\langle\langle J_1(\boldsymbol{p})\, J_1(-\boldsymbol{p})\rangle\rangle = A(p)\left(n_1\cdot n_2\right), \quad (4.68)$$

where $J_1 \equiv n_i^\mu J_\mu(p_i)$ and the 1 stands for the spin of this operator. The 2-point function of stress-energy tensors in an arbitrary QFT (4.60) becomes

$$\langle\langle T(\boldsymbol{p})\,T(-\boldsymbol{p})\rangle\rangle = A(p)\,(n_1\cdot n_2)^2 + B(p)\,n_1^2 n_2^2, \tag{4.69}$$

where we absorbed a factor of 2 in the definition of the form factor $A(p)$ and $T \equiv n_i^\mu n_i^\nu T_{\mu\nu}(p_i)$. Finally, for the 2-point function of stress-energy tensors in a CFT (4.62) we have

$$\langle\langle T(\boldsymbol{p})\,T(-\boldsymbol{p})\rangle\rangle = A(p)\,(n_1\cdot n_2)^2\,. \tag{4.70}$$

With this notation it is very easy to generalize these results for higher-spin symmetric currents. If the current is transverse and traceless, we have

$$\langle\langle J_s(\boldsymbol{p})\,J_s(-\boldsymbol{p})\rangle\rangle = A(p)\,(n_1\cdot n_2)^s\,, \tag{4.71}$$

while if it is only transverse we have

$$\langle\langle J_s(\boldsymbol{p})\,J_s(-\boldsymbol{p})\rangle\rangle = \sum_{j=0}^{\lfloor \frac{s}{2}\rfloor} A_j(p)\,(n_1\cdot n_2)^{s-2j}\,n_1^{2j}n_2^{2j}. \tag{4.72}$$

As we saw, we are able to produce very compact expressions using this notation, but we need a way to come back to the representation with indices. In order to do that we need to introduce differential operators that take off the polarization vectors and produce the desired expressions. For instance, in order to recover (4.57) from (4.68) or (4.60) from (4.69) we need to use the differential operator

$$\mathcal{D}_{i,\mu} = \pi_\mu^\alpha \frac{\partial}{\partial n_1^\alpha}, \tag{4.73}$$

while if we want to recover (4.62) from (4.70) we need the operator

$$\mathcal{D}_{i,\mu_1\mu_2} = \frac{1}{2}\Pi_{\mu_1\mu_2}^{\alpha_1\alpha_2}\frac{\partial^2}{\partial n_i^{\alpha_1}\partial n_i^{\alpha_2}}. \tag{4.74}$$

In general, to recover the indices of a spin s transverse-traceless operator in a correlation function, we need the operator

$$\mathcal{D}_{i,\mu_1\cdots\mu_s} = \frac{1}{s!}\Pi_{\mu_1\cdots\mu_s}^{\alpha_1\cdots\alpha_s}\frac{\partial^s}{\partial n_i^{\alpha_1}\cdots\partial n_i^{\alpha_s}}, \tag{4.75}$$

where the explicit form of the projector $\Pi_{\mu_1\cdots\mu_s}^{\alpha_1\cdots\alpha_s}$ is derived in the appendix D.

4.4 3-point functions of conserved currents

The exact computation of 3-point functions of conserved currents is a considerably more involved task than the computation of 2-point functions. One of the main difficulties is the non-triviality of the Ward identities, as we can see in the expressions (3.43) and (3.44).

The Ward identities for a correlator prescribes which semi-local[3] terms should be present

[3]For theories invariant under spacetime translations, 2-point functions are functions of one parameter while 3-point functions depend on two parameters. For 2-point functions we have only two types of terms: either a term is a contact term (≡ local term), or it is not. In position space, contact terms are delta functions or derivatives of delta functions. In momentum space, contact terms correspond to terms that are polynomial (componentwise)

in the correlator. That being said, we can always split our correlation functions into a piece that identically satisfy the Ward identities plus a set of semi-local terms that exactly reproduces them. In this section we will be concerned only with the pieces that identically satisfy the Ward identities. It should be noted that, after the complete parametrization of a 3-point function in terms of a piece that identically satisfy the Ward identities plus the set of semi-local terms that reproduces them, one must impose the CWI's to fix the form factors. The set of semi-local terms will imply in extra constraints that the form factors must satisfy. For more details on this issue, see [14].

4.4.1 Possible tensor structures

Now we are going to extend our analysis of tensorial structures of 2-point functions to 3-point functions. The new element that we have to deal with is the fact that we have two independent momenta. Following the approach of [14], in order to simplify the analysis of permutation symmetries in correlators, we are going to use different independent momenta for different polarization vectors. If we assume that the two independent momenta are p_1 and p_2, the possible elementary tensor structures that we may have are

$$(n_1 \cdot n_2)\,, (n_1 \cdot n_3)\,, (n_2 \cdot n_3)\,, (n_1 \cdot p_2)\,, (n_2 \cdot p_1)\,, (n_3 \cdot p_1)\,, (n_3 \cdot p_2)\,, \tag{4.76}$$

where we are using the property that $(n_i \cdot p_i) = 0$. The first thing to be noticed is that $(n_3 \cdot p_1)$ and $(n_3 \cdot p_2)$ are not independent. Indeed, $(n_3 \cdot p_2) = -\,(n_3 \cdot p_1) - (n_3 \cdot p_3) = -\,(n_3 \cdot p_1)$. Notice that we also have another way of writing the tensorial structures $(n_1 \cdot p_2)$ and $(n_2 \cdot p_1)$, namely

$$(n_1 \cdot p_2) = -\,(n_1 \cdot p_3)\,, \quad (n_2 \cdot p_1) = -\,(n_2 \cdot p_3)\,. \tag{4.77}$$

By convention, we will choose the following basis of elementary tensor structures:

$$(n_1 \cdot n_2)\,, (n_1 \cdot n_3)\,, (n_2 \cdot n_3)\,, (n_1 \cdot p_2)\,, (n_2 \cdot p_3)\,, (n_3 \cdot p_1)\,. \tag{4.78}$$

The elementary tensor structures (4.78) exist in every dimension. For $d \leq 5$ we may also construct parity-odd elementary tensor structures, namely

$$\begin{aligned}
d = 3: \quad & \epsilon\,(n_1, n_2, n_3)\,, && (4.79)\\
& \epsilon\left(n_1, n_2, p_{1(2)}\right), \epsilon\left(n_1, n_3, p_{1(2)}\right), \epsilon\left(n_2, n_3, p_{1(2)}\right),\\
& \epsilon\,(n_1, p_1, p_2)\,, \epsilon\,(n_2, p_1, p_2)\,, \epsilon\,(n_3, p_1, p_2)\,,\\
d = 4: \quad & \epsilon\,(n_1, n_2, n_3, p_1)\,, \epsilon\,(n_1, n_2, n_3, p_2)\,, && (4.80)\\
& \epsilon\,(n_1, n_2, p_1, p_2)\,, \epsilon\,(n_1, n_3, p_1, p_2)\,, \epsilon\,(n_2, n_3, p_1, p_2)\,,\\
d = 5: \quad & \epsilon\,(n_1, n_2, n_3, p_1, p_2)\,. && (4.81)
\end{aligned}$$

The analysis of the parity-odd sector of correlation functions is complicated by the existence of the Schouten identity, i.e.

$$\eta_{\alpha[\beta}\,\epsilon_{\mu_1 \cdots \mu_d]} = 0. \tag{4.82}$$

on the momentum. For instance, the modulus of the momentum $|p|$ is not polynomial because $|p| \equiv \sqrt{\eta_{\mu\nu} p^\mu p^\nu}$, while any even power of $|p|$ is polynomial. For 3-point functions, since we now have two parameters, we have three types of terms: local terms, semi-local terms and non-local terms. Local terms are the ones that are local with respect to both the parameters, while semi-local are local with respect to only one of the two. Examples of local terms are any term of the form $p_1^{2n} p_2^{2m}$, with n and m positive integers, while examples of semi-local term are $p_1 p_2^2$, $\frac{p_1^2}{p_2}$ and so on.

4.4.2 Examples of 3-point functions

Now we are going to use to technology mentioned above to construct the transverse-traceless sector of a few 3-point functions, i.e. the sector that identically satisfy the Ward identities.

$\langle\langle J_1(p_1)\,\mathcal{O}_1(p_2)\,\mathcal{O}_2(p_3)\rangle\rangle$

The simplest example that we could consider is the 3-point function of a spin-1 conserved current and two scalar operators. We recall that the label 1 in the operator J_1 refers to its spin, while the labels 1 and 2 in the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are just being used to differentiate the two scalar operators. The only even tensor structure allowded in this correlator is $(n_1\cdot p_2)$, hence

$$\langle\langle J_1(p_1)\,\mathcal{O}_1(p_2)\,\mathcal{O}_2(p_3)\rangle\rangle_{\text{even}} = A_1(p_1,p_2,p_3)\,(n_1\cdot p_2)\,. \tag{4.83}$$

In $d=3$ we have one odd tensor structure allowed: $\epsilon\,(n_1,p_1,p_2)$. Thus,

$$\langle\langle J_1(p_1)\,\mathcal{O}_1(p_2)\,\mathcal{O}_2(p_3)\rangle\rangle_{\text{odd}} = B_1(p_1,p_2,p_3)\,\epsilon\,(n_1,p_1,p_2)\,. \tag{4.84}$$

Particularly, in the case that the two scalar operators are the same $\mathcal{O}_1=\mathcal{O}_2\equiv\mathcal{O}$, the LHS of (4.83) becomes symmetric under the exchange of p_2 and p_3. On the other hand,

$$(n_1\cdot p_2)\overset{p_2\leftrightarrow p_3}{\longrightarrow}(n_1\cdot p_3) = -\,(n_1\cdot p_2)\,, \tag{4.85}$$

$$\epsilon\,(n_1,p_1,p_2)\overset{p_2\leftrightarrow p_3}{\longrightarrow}\epsilon\,(n_1,p_1,p_3) = -\epsilon\,(n_1,p_1,p_2)\,, \tag{4.86}$$

hence we need

$$A_1(p_1,p_2,p_3) = -A_1(p_1,p_3,p_2)\,,\quad B_1(p_1,p_2,p_3) = -B_1(p_1,p_3,p_2)\,. \tag{4.87}$$

$\langle\langle T(p_1)\,\mathcal{O}_1(p_2)\,\mathcal{O}_2(p_3)\rangle\rangle$

A very similar case is the 3-point functions of the stress-energy tensor and two scalar operators. Again, the only even tensor structure allowded is $(n_1\cdot p_2)$, but it must come squared, since n_1 must figure twice, i.e.

$$\langle\langle T(p_1)\,\mathcal{O}_1(p_2)\,\mathcal{O}_2(p_3)\rangle\rangle_{\text{even}} = A_2(p_1,p_2,p_3)\,(n_1\cdot p_2)^2\,. \tag{4.88}$$

In $d=3$ we have one odd tensor structure allowed

$$\langle\langle T(p_1)\,\mathcal{O}_1(p_2)\,\mathcal{O}_2(p_3)\rangle\rangle_{\text{odd}} = B_2(p_1,p_2,p_3)\,(n_1\cdot p_2)\,\epsilon\,(n_1,p_1,p_2)\,. \tag{4.89}$$

If we consider the case of two identical scalar operators we conclude that

$$A_2(p_1,p_2,p_3) = A_2(p_1,p_3,p_2)\,,\quad B_2(p_1,p_2,p_3) = B_2(p_1,p_3,p_2)\,. \tag{4.90}$$

$\langle\langle J_s(p_1)\,\mathcal{O}_1(p_2)\,\mathcal{O}_2(p_3)\rangle\rangle$

It is straightforward to generalize these results to the 3-point function of a spin-s traceless and conserved current and two scalar operators. Indeed, the result is

$$\langle\langle J_s(p_1)\,\mathcal{O}_1(p_2)\,\mathcal{O}_2(p_3)\rangle\rangle_{\text{even}} = A_s(p_1,p_2,p_3)\,(n_1\cdot p_2)^s\,. \tag{4.91}$$

Particularly for $d=3$, we also have the odd contribution

$$\langle\langle J_s(p_1)\,\mathcal{O}_1(p_2)\,\mathcal{O}_2(p_3)\rangle\rangle_{\text{odd}} = B_s(p_1,p_2,p_3)\,(n_1\cdot p_2)^{s-1}\,\epsilon\,(n_1,p_1,p_2)\,. \tag{4.92}$$

For two identical scalar we need the form factor A_s to satisfy

$$A_s(p_1,p_2,p_3) = (-1)^s A_s(p_1,p_3,p_2), \quad B_s(p_1,p_2,p_3) = (-1)^s B_s(p_1,p_3,p_2). \tag{4.93}$$

$\langle\langle J_1(p_1) J_1(p_2) \mathcal{O}(p_3)\rangle\rangle$

Now we are going to consider the 3-point function of two conserved currents and a scalar operator. In this case we have two different polarization vectors available, n_1 and n_2. Hence, for the even sector we may write

$$\langle\langle J_1(p_1) J_1(p_2) \mathcal{O}(p_3)\rangle\rangle_{\text{even}} = A_1(p_1,p_2,p_3)(n_1\cdot p_2)(n_2\cdot p_3)+B_1(p_1,p_2,p_3)(n_1\cdot n_2). \tag{4.94}$$

We have an odd sector for $d=3$ and for $d=4$. For $d=3$,

$$\begin{aligned}\langle\langle J_1(p_1) J_1(p_2) \mathcal{O}(p_3)\rangle\rangle_{\text{odd}} &= C_1(p_1,p_2,p_3)\,\epsilon(n_1,n_2,p_1) - C_1(p_2,p_1,p_3)\,\epsilon(n_1,n_2,p_2)\\ &+D_1(p_1,p_2,p_3)(n_1\cdot p_2)\,\epsilon(n_2,p_1,p_2) + D_1(p_2,p_1,p_3)(n_2\cdot p_3)\,\epsilon(n_1,p_1,p_2).\end{aligned} \tag{4.95}$$

One should notice that the tensor structures present in equation (4.95) are not all independent. As a matter of fact, the Schouten identity (4.82) allow us to write

$$(n_1\cdot p_2)\,\epsilon(n_2,p_1,p_2) = -(p_1\cdot p_2)\,\epsilon(n_1,n_2,p_2) + p_2^2\epsilon(n_1,n_2,p_1), \tag{4.96}$$

$$(n_2\cdot p_3)\,\epsilon(n_1,p_1,p_2) = -p_1^2\epsilon(n_1,n_2,p_2) + (p_1\cdot p_2)\,\epsilon(n_1,n_2,p_1). \tag{4.97}$$

Using these identities in the expression (4.95) we find[4]

$$\begin{aligned}\langle\langle J_1(p_1) J_1(p_2) \mathcal{O}(p_3)\rangle\rangle_{\text{odd}} &= \left[C_1 + p_2^2 D_1 + (p_1\cdot p_2)\,D_1(1\leftrightarrow 2)\right]\epsilon(n_1,n_2,p_1)\\ &-\left[C_1(1\leftrightarrow 2) + (p_1\cdot p_2)\,D_1 + p_1^2 D_1(1\leftrightarrow 2)\right]\epsilon(n_1,n_2,p_2),\end{aligned} \tag{4.98}$$

from where we see that we may define a new form factor C_1' given by

$$\tilde{C}_1 = C_1 + p_2^2 D_1 + (p_1\cdot p_2)\,D_1(1\leftrightarrow 2), \tag{4.99}$$

in such a way that

$$\langle\langle J_1(p_1) J_1(p_2) \mathcal{O}(p_3)\rangle\rangle_{\text{odd}} = \tilde{C}_1\epsilon(n_1,n_2,p_1) - \tilde{C}_1(1\leftrightarrow 2)\,\epsilon(n_1,n_2,p_2). \tag{4.100}$$

For $d=4$,

$$\langle\langle J_1(p_1) J_1(p_2) \mathcal{O}(p_3)\rangle\rangle_{\text{odd}} = E_1(p_1,p_2,p_3)\,\epsilon(n_1,n_2,p_1,p_2). \tag{4.101}$$

The symmetry of exchanging 1 and 2 requires that

$$\begin{aligned} A_1 &= A_1(1\leftrightarrow 2),\\ B_1 &= B_1(1\leftrightarrow 2),\\ E_1 &= E_1(1\leftrightarrow 2).\end{aligned} \tag{4.102}$$

[4]To simplify the notation, we will suppress the dependence of the form factors on the momenta and the notation $A(i\leftrightarrow j)$ stands for the form factor A with the momenta p_i and p_j exchanged.

$\langle\langle T(p_1)\,T(p_2)\,\mathcal{O}(p_3)\rangle\rangle$

Our next example will be the 3-point function of two stress-energy tensors and a scalar operator. The allowed tensor structures here are

$$\langle\langle T(p_1)\,T(p_2)\,\mathcal{O}(p_3)\rangle\rangle_{\text{even}} = A_1\,(n_1\cdot n_2)^2 + A_2\,(n_1\cdot n_2)\,(n_1\cdot p_2)\,(n_2\cdot p_3) + \\ + A_3\,(n_1\cdot p_2)^2\,(n_2\cdot p_3)^2\,. \quad (4.103)$$

The LHS of (4.103) is symmetric under the exchange of 1 and 2. It is straightforward to see that all the tensorial structures present in the RHS of (4.103) are symmetric under the exchange $1\leftrightarrow 2$, therefore all the form factors need to be symmetric too.

As for the previous correlator, we have an odd sector for $d=3$ and for $d=4$. For $d=3$,

$$\langle\langle T(p_1)\,T(p_2)\,\mathcal{O}(p_3)\rangle\rangle_{\text{odd}} = \left[B_1\,(n_1\cdot n_2) + B_2\,(n_1\cdot p_2)\,(n_2\cdot p_3)\right]\epsilon\,(n_1,n_2,p_1) - \\ - \left[B_1(1\leftrightarrow 2)\,(n_1\cdot n_2) + B_2(1\leftrightarrow 2)\,(n_1\cdot p_2)\,(n_2\cdot p_3)\right]\epsilon\,(n_1,n_2,p_2)\,. \quad (4.104)$$

For $d=4$,

$$\langle\langle T(p_1)\,T(p_2)\,\mathcal{O}(p_3)\rangle\rangle_{\text{odd}} = \left[C_1\,(n_1\cdot n_2) + C_2\,(n_1\cdot p_2)\,(n_2\cdot p_3)\right]\epsilon\,(n_1,n_2,p_1,p_2)\,, \quad (4.105)$$

where the form factors C_1 and C_2 are symmetric under the exchange $1\leftrightarrow 2$.

$\langle\langle J_{s_1}(p_1)\,J_{s_2}(p_2)\,\mathcal{O}(p_3)\rangle\rangle$

From the previous cases it is easy to extrapolate to the general case of a 3-point function of two operators of arbitrary spin and a scalar operator. Without loss of generality, let us assume that $s_1 \geq s_2$, in which case we find

$$\langle\langle J_{s_1}(p_1)\,J_{s_2}(p_2)\,\mathcal{O}(p_3)\rangle\rangle_{\text{even}} = \sum_{i=0}^{s_2} A_i\,(n_1\cdot n_2)^i\,(n_1\cdot p_2)^{s_1-i}\,(n_2\cdot p_3)^{s_2-i}\,. \quad (4.106)$$

Notice that the number of independent form factors is always $\min(s_1,s_2)+1$, even in the case $s_1 = s_2$. One could expect that in the case $s_1 = s_2$ we would have less form factors because of the symmetric of exchange $1\leftrightarrow 2$. This is not the case because all the tensorial structures are symmetric by themselves in that case. For $d=3$ and $d=4$ we also have an odd sector. For $d=3$,

$$\langle\langle J_{s_1}(p_1)\,J_{s_2}(p_2)\,\mathcal{O}(p_3)\rangle\rangle_{\text{odd}} = \\ = \left(\sum_{i=0}^{s_2-1} B_i\,(n_1\cdot n_2)^i\,(n_1\cdot p_2)^{s_1-1-i}\,(n_2\cdot p_3)^{s_2-1-i}\right)\epsilon\,(n_1,n_2,p_1) - \\ - \left(\sum_{i=0}^{s_2-1} C_i\,(n_1\cdot n_2)^i\,(n_1\cdot p_2)^{s_1-1-i}\,(n_2\cdot p_3)^{s_2-1-i}\right)\epsilon\,(n_1,n_2,p_2)\,. \quad (4.107)$$

If $s_1 = s_2 = s$, the LHS of (4.107) becomes symmetric in the exchange $1 \leftrightarrow 2$, hence the RHS becomes

$$\begin{aligned}
&\langle\langle J_s(p_1)\, J_s(p_2)\, \mathcal{O}(p_3)\rangle\rangle_{\text{odd}} = \\
&\qquad = \left(\sum_{i=0}^{s-1} B_i\,(n_1\cdot n_2)^i\,(n_1\cdot p_2)^{s-1-i}\,(n_2\cdot p_3)^{s-1-i}\right)\epsilon\,(n_1,n_2,p_1) - \\
&\qquad - \left(\sum_{i=0}^{s-1} B_i(1\leftrightarrow 2)\,(n_1\cdot n_2)^i\,(n_1\cdot p_2)^{s-1-i}\,(n_2\cdot p_3)^{s-1-i}\right)\epsilon\,(n_1,n_2,p_2)\,. \qquad (4.108)
\end{aligned}$$

For $d = 4$,

$$\begin{aligned}
&\langle\langle J_{s_1}(p_1)\, J_{s_2}(p_2)\, \mathcal{O}(p_3)\rangle\rangle_{\text{odd}} = \\
&\qquad = \left(\sum_{i=0}^{s_2-1} C_i\,(n_1\cdot n_2)^i\,(n_1\cdot p_2)^{s_1-1-i}\,(n_2\cdot p_3)^{s_2-1-i}\right)\epsilon\,(n_1,n_2,p_1,p_2)\,. \qquad (4.109)
\end{aligned}$$

$\langle\langle T(p_1)\, T(p_2)\, T(p_3)\rangle\rangle$

Our next example will be the 3-point function of stress-energy tensors. There are eleven tensor structures that contribute to the even sector of this correlator, namely

$$\begin{aligned}
&\langle\langle T(p_1)\, T(p_2)\, T(p_3)\rangle\rangle_{\text{even}} = A_1\,(n_1\cdot p_2)^2\,(n_2\cdot p_3)^2\,(n_3\cdot p_1)^2 + [A_2\,(n_1\cdot n_2)\,(n_3\cdot p_1) + \\
&\quad + A_2(1\leftrightarrow 3)\,(n_2\cdot n_3)\,(n_1\cdot p_2) + A_2(2\leftrightarrow 3)\,(n_3\cdot n_1)\,(n_2\cdot p_3)]\,(n_1\cdot p_2)\,(n_2\cdot p_3)\,(n_3\cdot p_1) \\
&+ \left[A_3\,(n_1\cdot n_2)^2\,(n_3\cdot p_1)^2 + A_3(1\leftrightarrow 3)\,(n_2\cdot n_3)^2\,(n_1\cdot p_2)^2 + A_3(2\leftrightarrow 3)\,(n_3\cdot n_1)^2\,(n_2\cdot p_3)^2\right] \\
&\quad + [A_4\,(n_1\cdot n_3)\,(n_3\cdot n_2)\,(n_1\cdot p_2)\,(n_2\cdot p_3) + A_4(1\leftrightarrow 3)\,(n_2\cdot n_1)\,(n_1\cdot n_3)\,(n_2\cdot p_3)\,(n_3\cdot p_1) \\
&\quad + A_4(2\leftrightarrow 3)\,(n_1\cdot n_2)\,(n_2\cdot n_3)\,(n_1\cdot p_2)\,(n_3\cdot p_1)] + A_5\,(n_1\cdot n_2)\,(n_2\cdot n_3)\,(n_3\cdot n_1)\,. \qquad (4.110)
\end{aligned}$$

Notice that the eleven tensor structures organize themselves into five tensor structures that are symmetric under permutations of the labels. Since the LHS of (4.110) is symmetric under permutations, we need the form factors A_1 and A_5 to be completely symmetric, i.e.

$$A_i\,(p_1,p_2,p_3) = A_i\left(p_{\sigma(1)},p_{\sigma(2)},p_{\sigma(3)}\right), \quad \forall\sigma\in S_3, \quad i\in\{1,5\}, \qquad (4.111)$$

while the form factors A_i, with $i \in 2,3,4$ need to be symmetric under the exchange of 1 and 2, i.e.

$$A_i(p_1,p_2,p_3) = A_i(p_2,p_1,p_3), \quad i\in 2,3,4. \qquad (4.112)$$

$\langle\langle J_{s_1}(p_1)\, J_{s_2}(p_2)\, J_{s_3}(p_3)\rangle\rangle$

The most general 3-point function of symmetric conserved currents is given by

$$\begin{aligned}
&\langle\langle J_{s_1}(p_1)\, J_{s_2}(p_2)\, J_{s_3}(p_3)\rangle\rangle = \\
&\quad \sum_{\{i,j,k\}\in\mathcal{R}} A_{ijk}\,(n_1\cdot n_2)^i\,(n_2\cdot n_3)^j\,(n_1\cdot n_3)^k\,(n_1\cdot p_2)^{s_1-i-k}\,(n_2\cdot p_3)^{s_2-i-j}\,(n_3\cdot p_1)^{s_3-j-k}\,, \\
&\qquad (4.113)
\end{aligned}$$

where $\mathcal{R} = \left\{\{i,j,k\}\in\mathbb{N}^3 | s_1-i-k\geq 0,\ s_2-i-j\geq 0,\ s_3-j-k\geq 0\right\}$. It is possible to write down an expression in closed form for the number of terms present in (4.113), i.e. the number of elements in the set $\mathcal{R}$, namely

$$N(s_1, s_2, s_3) = \frac{(s_3+1)(s_3+2)(3s_2-s_3+3)}{6} - \frac{p(p+2)(2p+5)}{24} - \frac{1-(-1)^p}{16}, \quad (4.114)$$

where $p = \max(0, s_2+s_3-s_1)$ and we are assuming that $s_1 \geq s_2 \geq s_3$, see [56].

Appendix B

Passing through the delta function

B.1 Dilatation Ward identity

In configuration space, the dilatation Ward identity is given by

$$\left(\Delta_t + \sum_{i=1}^{n} x_i^\mu \frac{\partial}{\partial x_i^\mu}\right) \langle \mathcal{O}_1(\boldsymbol{x}_1) \cdots \mathcal{O}_n(\boldsymbol{x}_n)\rangle = 0,$$

which in momentum space becomes

$$\left(\Delta_t - nd - \sum_{i=1}^{n} p_i^\mu \frac{\partial}{\partial p_i^\mu}\right) \langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle = 0.$$

Because of the translational invariance of the correlator we have that

$$\langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle = (2\pi)^d\, \delta^d \left(\boldsymbol{P} = \sum_{i=1}^{d} \boldsymbol{p}_i\right) \langle\langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle\rangle\,,$$

where, because of the delta function the function $\langle\langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle\rangle$ depends only on $n-1$ momenta. Without loss of generality we will consider that

$$\frac{\partial}{\partial p_n^\mu} \langle\langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle\rangle = 0.$$

Hence,

$$0 = \left(\Delta_t - nd - \sum_{i=1}^{n} p_i^\mu \frac{\partial}{\partial p_i^\mu}\right) \langle \mathcal{O}_1(\boldsymbol{p}_1) \cdots \mathcal{O}_n(\boldsymbol{p}_n)\rangle = \left(\Delta_t - nd - \sum_{i=1}^{n} p_i^\mu \frac{\partial}{\partial p_i^\mu}\right) \left(\delta^d(\boldsymbol{P})\, \mathcal{M}\right).$$

When the derivative acts on the delta function we will get the term

$$\mathcal{M} \sum_{i=1}^{n} p_i^\mu \frac{\partial}{\partial p_i^\mu} \delta^d(\boldsymbol{P}) = \mathcal{M} P \cdot \frac{\partial}{\partial P} \delta^d(\boldsymbol{P})\,.$$

A small computation tell us that

$$P \cdot \frac{\partial}{\partial P} \delta^d(\boldsymbol{P}) = -d\delta(\boldsymbol{P})\,.$$

To verify that this equation holds one needs to integrate both sides against a test function. Our final result is

$$\delta^d(\boldsymbol{P})\left(\Delta_t-(n-1)\,d-\sum_{i=1}^{n-1}p_i^\mu\frac{\partial}{\partial p_i^\mu}\right)\mathcal{M}=0, \tag{B.1}$$

where we also used the fact that $\frac{\partial}{\partial p_n^\mu}\mathcal{M}$.

B.2 Special conformal Ward identity

In configuration space, the special conformal Ward identity for a n-point function is given by

$$\left(\sum_{i=1}^{n}b\cdot\left(\mathcal{K}_i(\boldsymbol{x}_i)+\mathcal{L}_i(\boldsymbol{x}_i)\right)\right)\langle\mathcal{O}_1(\boldsymbol{x}_1)\cdots\mathcal{O}_n(\boldsymbol{x}_n)\rangle=0. \tag{B.2}$$

The Fourier transform of (B.2) is given by

$$\left(\sum_{i=1}^{n}b\cdot\left(\hat{\mathcal{K}}_i(\boldsymbol{p}_i)+\hat{\mathcal{L}}_i(\boldsymbol{p}_i)\right)\right)\langle\mathcal{O}_1(\boldsymbol{p}_1)\cdots\mathcal{O}_n(\boldsymbol{p}_n)\rangle=0,$$

where

$$\langle\mathcal{O}_1(\boldsymbol{p}_1)\cdots\mathcal{O}_n(\boldsymbol{p}_n)\rangle=(2\pi)^d\,\delta^d\left(\boldsymbol{P}=\sum_{i=1}^{n}\boldsymbol{p}_i\right)\langle\!\langle\mathcal{O}_1(\boldsymbol{p}_1)\cdots\mathcal{O}_n(\boldsymbol{p}_n)\rangle\!\rangle\,.$$

Now we are going to work out the action of $\hat{\mathcal{K}}_i(\boldsymbol{p}_i)$ and $\hat{\mathcal{L}}_i(\boldsymbol{p}_i)$ on $\delta^d(\boldsymbol{P})\,\mathcal{M}$, where, for simplicity, we have defined $\boldsymbol{P}=\sum_{i=1}^{n}\boldsymbol{p}_i$ and $\mathcal{M}=(2\pi)^d\,\langle\!\langle\mathcal{O}_1(\boldsymbol{p}_1)\cdots\mathcal{O}_n(\boldsymbol{p}_n)\rangle\!\rangle$.

$$\begin{aligned}
b\cdot\mathcal{K}_i(\boldsymbol{p}_i)\left[\delta^d(\boldsymbol{P})\,\mathcal{M}\right]&=\left(b\cdot p_i\frac{\partial^2}{\partial p_i^\mu\partial p_{i\mu}}-2b^\mu p_i^\nu\frac{\partial^2}{\partial p_i^\mu\partial p_i^\nu}+2\left(\Delta_i-d\right)b^\mu\frac{\partial}{\partial p_i^\mu}\right)\left[\delta^d(\boldsymbol{P})\,\mathcal{M}\right]\\
&=\mathcal{M}\left(b\cdot p_i\frac{\partial^2}{\partial P^\mu\partial P_\mu}-2b^\mu p_i^\nu\frac{\partial^2}{\partial P^\mu\partial P^\nu}+2\left(\Delta_i-d\right)b^\mu\frac{\partial}{\partial P^\mu}\right)\delta^d(\boldsymbol{P})\\
&+2b\cdot p_i\frac{\partial}{\partial P^\mu}\delta^d(\boldsymbol{P})\frac{\partial}{\partial p_{i\mu}}\mathcal{M}-2b^\mu p_i^\nu\left(\frac{\partial}{\partial P^\mu}\delta^d(\boldsymbol{P})\frac{\partial}{\partial p_i^\nu}\mathcal{M}+\frac{\partial}{\partial P^\nu}\delta^d(\boldsymbol{P})\frac{\partial}{\partial p_i^\mu}\mathcal{M}\right)\\
&+\delta^d(\boldsymbol{P})\left(b\cdot\mathcal{K}(\boldsymbol{p}_i)\right)\mathcal{M}
\end{aligned}$$

We use the facts that

$$P_\nu\frac{\partial^2}{\partial P^\mu\partial P_\mu}\delta^d(\boldsymbol{P})=-2\frac{\partial}{\partial P^\nu}\delta^d(\boldsymbol{P})\,,\quad\text{and}\quad P^\mu\frac{\partial^2}{\partial P^\mu\partial P^\nu}\delta^d(\boldsymbol{P})=-\left(d+1\right)\frac{\partial}{\partial P^\nu}\delta^d(\boldsymbol{P})$$

to rewrite our expression. After summation over i we find

$$\begin{aligned}
\sum_{i=1}^{n}b\cdot\mathcal{K}_i(\boldsymbol{p}_i)\left[\delta^d(\boldsymbol{P})\,\mathcal{M}\right]&=\mathcal{M}\left(-2b\cdot\frac{\partial}{\partial P}+2\left(d+1\right)b\cdot\frac{\partial}{\partial P}+2\left(\Delta_t-nd\right)b^\mu\frac{\partial}{\partial P^\mu}\right)\delta^d(\boldsymbol{P})\\
&-2\frac{\partial}{\partial P_\mu}\delta^d(\boldsymbol{P})\,b^\nu\sum_{i=1}^{n}\left(p_{i\mu}\frac{\partial}{\partial p_i^\nu}-p_{i\nu}\frac{\partial}{\partial p_i^\mu}\right)\mathcal{M}\\
&-2b^\mu\frac{\partial}{\partial P^\mu}\delta^d(\boldsymbol{P})\sum_{i=1}^{n}p_i\cdot\frac{\partial}{\partial p_i}\mathcal{M}+\delta^d(\boldsymbol{P})\left(\sum_{i=1}^{n}b\cdot\mathcal{K}(\boldsymbol{p}_i)\right)\mathcal{M}
\end{aligned} \tag{B.3}$$

In (B.3) we see the appearence of the term $\left(p_{i\mu}\frac{\partial}{\partial p_i^\nu} - p_{i\nu}\frac{\partial}{\partial p_i^\mu}\right)$, which is proportional to the generator of Lorentz transformations $L_{\mu\nu}$. A scalar correlator is invariant under a Lorentz transformation and we have that

$$\sum_{i=1}^{n}\left(p_{i\mu}\frac{\partial}{\partial p_i^\nu} - p_{i\nu}\frac{\partial}{\partial p_i^\mu}\right)\mathcal{M} = 0.$$

For a generic tensor correlator it is covariant under Lorentz transformations and we have

$$\sum_{i=1}^{n}\left(L_{\mu\nu}\left(\boldsymbol{p}_i\right) + S_{\mu\nu}\left(\boldsymbol{p}_i\right)\right)\mathcal{M} = 0, \tag{B.4}$$

where

$$S_{\mu\nu}\Phi_{\alpha_1\ldots\alpha_\ell} = \sum_{i=1}^{\ell}\left(\eta_{\mu\alpha_i}\delta_\nu^\tau - \eta_{\nu\alpha_i}\delta_\mu^\tau\right)\Phi_{\alpha_1\ldots\tau\ldots\alpha_\ell}.$$

One should also notice the presence of the term $\sum_{i=1}^n p_i\cdot\frac{\partial}{\partial p_i}\mathcal{M}$ in the last line of (B.3). We may rewrite this term using the fact that $\frac{\partial}{\partial p_n}\mathcal{M} = 0$ and the dilatation Ward identity (B.1). In the case of a scalar correlator we get

$$\begin{aligned}\sum_{i=1}^{n} b\cdot\mathcal{K}_i\left(\boldsymbol{p}_i\right)\left[\delta^d\left(\boldsymbol{P}\right)\mathcal{M}\right] &= 2\left(\Delta_t - (n-1)\,d\right)\mathcal{M}\left(b\cdot\frac{\partial}{\partial P}\delta^d\left(\boldsymbol{P}\right)\right)\\ &\quad - 2\left(b\cdot\frac{\partial}{\partial P}\delta^d\left(\boldsymbol{P}\right)\right)\left(\Delta_t - (n-1)\,d\right) + \delta^d\left(\boldsymbol{P}\right)\left(\sum_{i=1}^{n} b\cdot\mathcal{K}\left(\boldsymbol{p}_i\right)\right)\mathcal{M}\end{aligned}$$

i.e.

$$\sum_{i=1}^{n} b\cdot\mathcal{K}_i\left(\boldsymbol{p}_i\right)\left[\delta^d\left(\boldsymbol{P}\right)\mathcal{M}\right] = \delta^d\left(\boldsymbol{P}\right)\left(\sum_{i=1}^{n} b\cdot\mathcal{K}\left(\boldsymbol{p}_i\right)\right)\mathcal{M} = \delta^d\left(\boldsymbol{P}\right)\left(\sum_{i=1}^{n-1} b\cdot\mathcal{K}\left(\boldsymbol{p}_i\right)\right)\mathcal{M}.$$

For the generic tensor correlator we have also the term

$$-2\frac{\partial}{\partial P_\mu}\delta^d\left(\boldsymbol{P}\right)b^\nu\sum_{i=1}^{n}\left(p_{i\mu}\frac{\partial}{\partial p_i^\nu} - p_{i\nu}\frac{\partial}{\partial p_i^\mu}\right)\mathcal{M}$$

which can be written as

$$2\frac{\partial}{\partial P_\mu}\delta^d\left(\boldsymbol{P}\right)b^\nu\sum_{i=1}^{n} S_{\mu\nu}\left(\boldsymbol{p}_i\right)\mathcal{M} \tag{B.5}$$

using the Lorentz Ward identity (B.4). We also need to consider now the action of the Lorentz part of the special conformal Ward identity, i.e.

$$\sum_{i=1}^{n} b\cdot\mathcal{L}_i\left(\boldsymbol{p}_i\right)\left[\delta^d\left(\boldsymbol{P}\right)\mathcal{M}\right] = \mathcal{M}\sum_{i=1}^{n} b\cdot\mathcal{L}_i\left(\boldsymbol{p}_i\right)\delta^d\left(\boldsymbol{P}\right) + \delta^d\left(\boldsymbol{P}\right)\left(\sum_{i=1}^{n} b\cdot\mathcal{L}_i\left(\boldsymbol{p}_i\right)\mathcal{M}\right). \tag{B.6}$$

Recall that

$$b\cdot\mathcal{L}_i\left(\boldsymbol{p}_i\right) \equiv -b^\nu S_{\mu\nu}\frac{\partial}{\partial p_{i\mu}}.$$

The first term of (B.6) cancels the term (B.5) and we are left with

$$\sum_{i=1}^{n} b\cdot\left(\mathcal{K}_i\left(\boldsymbol{p}_i\right)+\mathcal{L}_i\left(\boldsymbol{p}_i\right)\right)\left[\delta^d\left(\boldsymbol{P}\right)\mathcal{M}\right]=\delta^d\left(\boldsymbol{P}\right)\sum_{i=1}^{n} b\cdot\left(\mathcal{K}_i\left(\boldsymbol{p}_i\right)+\mathcal{L}_i\left(\boldsymbol{p}_i\right)\right)\mathcal{M}.$$

Appendix C

Special conformal Ward identity in terms of p_1, p_2 and p_3

Here we are going to present some details of the derivation of (4.50). To perform these computation we will need the expressions (4.46) and (4.47). Given these quantities, let us compute

$$\begin{aligned}
\frac{\partial}{\partial p_1^\mu}\frac{\partial}{\partial p_1^\nu} &= \left(\frac{p_{1\mu}}{p_1}\frac{\partial}{\partial p_1} - \frac{p_{3\mu}}{p_3}\frac{\partial}{\partial p_3}\right)\left(\frac{p_{1\nu}}{p_1}\frac{\partial}{\partial p_1} - \frac{p_{3\nu}}{p_3}\frac{\partial}{\partial p_3}\right) \\
&= \frac{p_{1\mu}p_{1\nu}}{p_1^2}\frac{\partial^2}{\partial p_1^2} + \frac{p_{3\mu}p_{3\nu}}{p_3^2}\frac{\partial^2}{\partial p_3^2} - \frac{p_{1\mu}p_{3\nu} + p_{3\mu}p_{1\nu}}{p_1p_3}\frac{\partial^2}{\partial p_1\partial p_3} \\
&\quad + \frac{\partial}{\partial p_1^\mu}\left(\frac{p_{1\nu}}{p_1}\right)\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_1^\mu}\left(\frac{p_{3\nu}}{p_3}\right)\frac{\partial}{\partial p_3} \\
&= \frac{p_{1\mu}p_{1\nu}}{p_1^2}\frac{\partial^2}{\partial p_1^2} + \frac{p_{3\mu}p_{3\nu}}{p_3^2}\frac{\partial^2}{\partial p_3^2} - \frac{p_{1\mu}p_{3\nu} + p_{3\mu}p_{1\nu}}{p_1p_3}\frac{\partial^2}{\partial p_1\partial p_3} \\
&\quad + \left(\eta_{\mu\nu} - \frac{p_{1\mu}p_{1\nu}}{p_1^2}\right)\frac{1}{p_1}\frac{\partial}{\partial p_1} + \left(\eta_{\mu\nu} - \frac{p_{3\mu}p_{3\nu}}{p_3^2}\right)\frac{1}{p_3}\frac{\partial}{\partial p_3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial p_1^\mu}\frac{\partial}{\partial p_1^\nu} &= \frac{p_{1\mu}p_{1\nu}}{p_1^2}\frac{\partial^2}{\partial p_1^2} + \frac{p_{3\mu}p_{3\nu}}{p_3^2}\frac{\partial^2}{\partial p_3^2} - \frac{p_{1\mu}p_{3\nu} + p_{3\mu}p_{1\nu}}{p_1p_3}\frac{\partial^2}{\partial p_1\partial p_3} \\
&\quad + \left(\eta_{\mu\nu} - \frac{p_{1\mu}p_{1\nu}}{p_1^2}\right)\frac{1}{p_1}\frac{\partial}{\partial p_1} + \left(\eta_{\mu\nu} - \frac{p_{3\mu}p_{3\nu}}{p_3^2}\right)\frac{1}{p_3}\frac{\partial}{\partial p_3}.
\end{aligned}$$

Analogously for p_2

$$\begin{aligned}
\frac{\partial}{\partial p_2^\mu}\frac{\partial}{\partial p_2^\nu} &= \frac{p_{2\mu}p_{2\nu}}{p_2^2}\frac{\partial^2}{\partial p_2^2} + \frac{p_{3\mu}p_{3\nu}}{p_3^2}\frac{\partial^2}{\partial p_3^2} - \frac{p_{2\mu}p_{3\nu} + p_{3\mu}p_{2\nu}}{p_2p_3}\frac{\partial^2}{\partial p_2\partial p_3} \\
&\quad + \left(\eta_{\mu\nu} - \frac{p_{2\mu}p_{2\nu}}{p_2^2}\right)\frac{1}{p_2}\frac{\partial}{\partial p_2} + \left(\eta_{\mu\nu} - \frac{p_{3\mu}p_{3\nu}}{p_3^2}\right)\frac{1}{p_3}\frac{\partial}{\partial p_3}.
\end{aligned}$$

Using these results $\mathcal{K}_1$ becomes

$$\begin{aligned}
\mathcal{K}_{1\sigma} &= p_{1\sigma}\frac{\partial}{\partial p_1^\mu}\frac{\partial}{\partial p_{1\mu}} - 2p_1^\mu\frac{\partial}{\partial p_1^\mu}\frac{\partial}{\partial p_1^\sigma} + 2\left(\Delta_1 - d\right)\frac{\partial}{\partial p_1^\sigma} \\
&= p_{1\sigma}\left(\partial_1^2 + \partial_3^2 - 2\frac{p_1\cdot p_3}{p_1 p_3}\partial_1\partial_3 + \frac{d-1}{p_1}\partial_1 + \frac{d-1}{p_3}\partial_3\right) \\
&\quad - 2\left(p_{1\sigma}\partial_1^2 + \frac{p_1\cdot p_3}{p_3^2}p_{3\sigma}\partial_3^2 - \left(\frac{p_1}{p_3}p_{3\sigma} + \frac{p_1\cdot p_3}{p_1 p_3}p_{1\sigma}\right)\partial_1\partial_3 + \left(p_{1\sigma} - \frac{p_1\cdot p_3}{p_3^2}p_{3\sigma}\right)\frac{1}{p_3}\partial_3\right) \\
&\quad + 2\left(\Delta_1 - d\right)\left(\frac{p_{1\sigma}}{p_1}\partial_1 - \frac{p_{3\sigma}}{p_3}\partial_3\right) \\
&= -p_{1\sigma}\partial_1^2 + \left(p_{1\sigma} - 2\frac{p_1\cdot p_3}{p_3^2}p_{3\sigma}\right)\partial_3^2 + p_{3\sigma}\frac{2p_1}{p_3}\partial_1\partial_3 - p_{1\sigma}\left(\frac{d+1-2\Delta_1}{p_1}\right)\partial_1 \\
&\quad + \left(p_{1\sigma}\frac{d-1}{p_3} - p_{1\sigma}\frac{2}{p_3} + p_{3\sigma}\frac{2p_1\cdot p_3}{p_3^3} - p_{3\sigma}\frac{2\left(\Delta_1 - d\right)}{p_3}\right)\partial_3
\end{aligned}$$

and hence $\mathcal{K}_1 + \mathcal{K}_2$ is given by

$$\begin{aligned}
\mathcal{K}_{1\sigma} + \mathcal{K}_{2\sigma} &= -p_{1\sigma}\partial_1^2 - p_{2\sigma}\partial_2^2 + \left(p_{1\sigma} + p_{2\sigma} - 2\frac{(p_1+p_2)\cdot p_3 p_{3\sigma}}{p_3^2}\right)\partial_3^2 + p_{3\sigma}\frac{2p_1}{p_3}\partial_1\partial_3 + p_{3\sigma}\frac{2p_2}{p_3}\partial_2\partial_3 \\
&\quad - p_{1\sigma}\left(\frac{d+1-2\Delta_1}{p_1}\right)\partial_1 - p_{2\sigma}\left(\frac{d+1-2\Delta_2}{p_2}\right)\partial_2 \\
&\quad + \left(\left(p_{1\sigma} + p_{2\sigma}\right)\frac{d-1}{p_3} - \left(p_{1\sigma} + p_{2\sigma}\right)\frac{2}{p_3} + p_{3\sigma}\frac{2\left(p_1+p_2\right)\cdot p_3}{p_3^3} - p_{3\sigma}\frac{2\left(\Delta_1 + \Delta_2 - 2d\right)}{p_3}\right)\partial_3 \\
&= -p_{1\sigma}\partial_1^2 - p_{2\sigma}\partial_2^2 + p_{3\sigma}\partial_3^2 + p_{3\sigma}\frac{2p_1}{p_3}\partial_1\partial_3 + p_{3\sigma}\frac{2p_2}{p_3}\partial_2\partial_3 \\
&\quad - p_{1\sigma}\left(\frac{d+1-2\Delta_1}{p_1}\right)\partial_1 - p_{2\sigma}\left(\frac{d+1-2\Delta_2}{p_2}\right)\partial_2 + p_{3\sigma}\left(\frac{3d+1-2\left(\Delta_1+\Delta_2\right)}{p_3}\right)\partial_3.
\end{aligned}$$

We may use the dilatation Ward identity

$$2d - \Delta_t + \sum_{i=1}^{3} p_i\frac{\partial}{\partial p_i} = 0$$

to rewrite one of the crossed terms

$$\begin{aligned}
p_{3\sigma}\frac{2p_1}{p_3}\partial_3\partial_1 &= p_{3\sigma}\frac{2p_1}{p_3}\left(-\frac{1}{p_1}\left(2d - \Delta_t\right)\partial_3 - \frac{p_2}{p_1}\partial_2\partial_3 - \frac{p_3}{p_1}\partial_3^2 - \frac{1}{p_1}\partial_3\right) \\
&= p_{3\sigma}\left(-\frac{2}{p_3}\left(2d - \Delta_t + 1\right)\partial_3 - \frac{2p_2}{p_3}\partial_2\partial_3 - 2\partial_3^2\right).
\end{aligned}$$

Substituting this expression back in the expression for $\mathcal{K}_{1\sigma} + \mathcal{K}_{2\sigma}$ we finally find

$$\mathcal{K}_{1\sigma} + \mathcal{K}_{2\sigma} = -\sum_{j=1}^{3} p_{j\sigma}\left[\partial_j^2 + \frac{d+1-2\Delta_j}{p_j}\partial_j\right].$$

Appendix D

Higher-spin projectors

The projectors on transverse tensors

$$\pi_{\mu}^{\alpha}(p) = \delta_{\mu}^{\alpha} - \frac{p_{\mu}p^{\alpha}}{p^{2}} \tag{D.1}$$

and transverse-traceless tensors

$$\Pi_{\mu_1\mu_2}^{\alpha_1\alpha_2}(p) = \pi_{\mu_1}^{\alpha_1}\pi_{\mu_2}^{\alpha_2} + \pi_{\mu_2}^{\alpha_1}\pi_{\mu_1}^{\alpha_2} - \frac{2}{d-1}\pi_{\mu_1\mu_2}\pi^{\alpha_1\alpha_2} \tag{D.2}$$

are already well know. To construct the projector of spin 3 we start with an object that is completly symmetric and transverse but traceles with respect to 2 indices, i.e.

$$\pi_{\mu_1}^{\alpha_1}\Pi_{\mu_2\mu_3}^{\alpha_2\alpha_3} + \pi_{\mu_3}^{\alpha_1}\Pi_{\mu_1\mu_2}^{\alpha_2\alpha_3} + \pi_{\mu_2}^{\alpha_1}\Pi_{\mu_3\mu_1}^{\alpha_2\alpha_3},$$

and then subtract the non-zero traces. Taking a trace with respect to μ_1 and μ_2 we find

$$2\Pi_{\mu_3}^{\alpha_1\alpha_2\alpha_3},$$

and analogously to the other traces. Thus, substracting these traces we find

$$\begin{gathered}\pi_{\mu_1}^{\alpha_1}\Pi_{\mu_2\mu_3}^{\alpha_2\alpha_3} + \pi_{\mu_3}^{\alpha_1}\Pi_{\mu_1\mu_2}^{\alpha_2\alpha_3} + \pi_{\mu_2}^{\alpha_1}\Pi_{\mu_3\mu_1}^{\alpha_2\alpha_3} \\ -\alpha\left(\pi_{\mu_1\mu_2}\Pi_{\mu_3}^{\alpha_1\alpha_2\alpha_3} + \pi_{\mu_3\mu_1}\Pi_{\mu_2}^{\alpha_1\alpha_2\alpha_3} + \pi_{\mu_2\mu_3}\Pi_{\mu_1}^{\alpha_1\alpha_2\alpha_3}\right).\end{gathered}$$

where the coefficient α must be fixed in order for the expression to be traceless. A simple computation fix $\alpha = \frac{2}{d+1}$, hence

$$\Pi_{\mu_1\mu_2\mu_3}^{\alpha_1\alpha_2\alpha_3}(p) = \pi_{(\mu_1}^{\alpha_1}\Pi_{\mu_2\mu_3)}^{\alpha_2\alpha_3} - \frac{2}{d+1}\pi_{(\mu_1\mu_2}\Pi_{\mu_3)}^{\alpha_1\alpha_2\alpha_3} \tag{D.3}$$

Analogously to spin s we find

$$\Pi_{\mu_1\ldots\mu_s}^{\alpha_1\ldots\alpha_s}(p) = \pi_{(\mu_s}^{\alpha_s}\Pi_{\mu_1\ldots\mu_{s-1})}^{\alpha_1\ldots\alpha_{s-1}} - \frac{2}{d+2s-5}\pi_{(\mu_1\mu_2}\Pi_{\mu_3\ldots\mu_s)}^{\alpha_1\ldots\alpha_s}. \tag{D.4}$$

To fix the relative coefficient is just a question of couting the terms in the symmetrization that will constribute to the trace. In fact, there is always one term that contributes with $d-1$ and $2(s-2)$ terms that contribute with 1. The symmetrization present in equations (D.2) and (D.3) means

$$\pi_{(\mu_1}^{\alpha_1}\Pi_{\mu_2\mu_3)}^{\alpha_2\alpha_3} \equiv \pi_{\mu_1}^{\alpha_1}\Pi_{\mu_2\mu_3}^{\alpha_2\alpha_3} + \pi_{\mu_3}^{\alpha_1}\Pi_{\mu_1\mu_2}^{\alpha_2\alpha_3} + \pi_{\mu_2}^{\alpha_1}\Pi_{\mu_3\mu_1}^{\alpha_2\alpha_3}$$

while

$$\pi_{(\mu_1\mu_2}\Pi^{\alpha_1\ldots\alpha_4}_{\mu_3\mu_4)} \equiv \pi_{\mu_1\mu_2}\Pi^{\alpha_1\ldots\alpha_4}_{\mu_3\mu_4} + \pi_{\mu_1\mu_3}\Pi^{\alpha_1\ldots\alpha_4}_{\mu_2\mu_4} + \pi_{\mu_1\mu_4}\Pi^{\alpha_1\ldots\alpha_4}_{\mu_2\mu_3} + \pi_{\mu_2\mu_3}\Pi^{\alpha_1\ldots\alpha_4}_{\mu_1\mu_4} + \pi_{\mu_2\mu_4}\Pi^{\alpha_1\ldots\alpha_4}_{\mu_1\mu_3} + \pi_{\mu_3\mu_4}\Pi^{\alpha_1\ldots\alpha_4}_{\mu_1\mu_2}.$$

Notice that in $\pi^{\alpha_1}_{(\mu_1}\Pi^{\alpha_2\ldots\alpha_s}_{\mu_2\ldots\mu_s)}$ we have s terms while in $\pi_{(\mu_1\mu_2}\Pi^{\alpha_1\ldots\alpha_s}_{\mu_3\ldots\mu_s)}$ we have $\binom{s}{2} = \frac{s(s-1)}{2}$ terms.

Part II

Regularization of correlation functions, contact terms and anomalies

Chapter 5

Regularization of energy-momentum tensor correlators: 2-point functions

In this chapter we discuss the problem of regularizing correlators in conformal field theories. The only way to do it in coordinate space is to interpret them as distributions. Unfortunately except for the simplest cases we do not have tabulated mathematical results. The way out we pursue here is to go to momentum space and use Feynman diagram techniques and their regularization methods. We focus on the energy-momentum tensor correlators and, to gain insight, we compute and regularize 2-point functions in $2d$ with various techniques both in coordinate space and in momentum space, obtaining the same results. Then we do the same for 2-point functions in $4d$.

5.1 2-point function of e.m. tensors in $2d$ and trace anomaly

In this section we regularize the 2-point function of energy-momentum tensors in $2d$ using the techniques of differential regularization and we derive the very well-known $2d$ trace anomaly. The ambiguities implicit in the regularization procedure allow us to make manifest the interplay between diffeomorphism and trace anomalies.

Let us consider the 2-point function $\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle$. This 2-point function in $2d$ (i.e. the "bare" 2-point function) is very well-known and is given by[1]

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = \frac{c/2}{x^4}\left(I_{\mu\rho}(x)\, I_{\nu\sigma}(x) + I_{\nu\rho}(x)\, I_{\mu\sigma}(x) - \eta_{\mu\nu}\eta_{\rho\sigma}\right) \tag{5.1}$$

where

$$I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \tag{5.2}$$

and c is the central charge of the theory. For $x \neq 0$ this 2-point function satisfies the Ward identities

$$\partial^\mu \langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = 0, \tag{5.3}$$

$$\left\langle T^\mu_\mu(x)\, T_{\rho\sigma}(0)\right\rangle = 0. \tag{5.4}$$

The result (5.1) is obtained using the symmetry properties of the indices, dimensional analysis and eqs. (5.3) and (5.4).

The 2-point function written above are UV singular for $x \to 0$, hence this divergence has to be dealt with for the correlator to be well-defined everywhere. In this context the most convenient way to regularize this object is with the technique of *differential regularization*. The recipe of differential regularization is: given a function $f(x)$ that needs to be regularized, find the most general function $F(x)$ such that $\mathcal{D}F(x) = f(x)$, where $\mathcal{D}$ is some differential

[1] One way of deriving this expression is by using the embedding formalism, see [5], for example.

operator, and such that the Fourier transform of $\mathcal{D}F(x)$ is well-defined (alternatively $\mathcal{D}F(x)$ has integrable singularities).

In our case we have two guiding principles: the Ward identities and dimensional analysis. Differential regularization tells that our 2-point function should be some differential operator applied to a function, i.e.

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = \mathcal{D}_{\mu\nu\rho\sigma}(f(x)), \tag{5.5}$$

while conservation requires that the differential operator $\mathcal{D}_{\mu\nu\rho\sigma}$ be transverse, i.e.

$$\partial^{\mu}\mathcal{D}_{\mu\nu\rho\sigma} = \cdots = \partial^{\sigma}\mathcal{D}_{\mu\nu\rho\sigma} = 0. \tag{5.6}$$

The most general transverse operator with four derivatives, symmetric in μ,ν and in ρ,σ that one can write is

$$\mathcal{D}_{\mu\nu\rho\sigma} = \alpha\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} + \beta\mathcal{D}^{(2)}_{\mu\nu\rho\sigma}, \tag{5.7}$$

where

$$\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} = \partial_\mu\partial_\nu\partial_\rho\partial_\sigma - (\eta_{\mu\nu}\partial_\rho\partial_\sigma + \eta_{\rho\sigma}\partial_\mu\partial_\nu)\,\Box + \eta_{\mu\nu}\eta_{\rho\sigma}\Box\Box, \tag{5.8}$$

$$\begin{aligned}\mathcal{D}^{(2)}_{\mu\nu\rho\sigma} &= \partial_\mu\partial_\nu\partial_\rho\partial_\sigma - \frac{1}{2}(\eta_{\mu\rho}\partial_\nu\partial_\sigma + \eta_{\nu\rho}\partial_\mu\partial_\sigma + \eta_{\mu\sigma}\partial_\nu\partial_\rho + \eta_{\nu\sigma}\partial_\mu\partial_\rho)\,\Box \\ &\quad + \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma})\,\Box\Box.\end{aligned} \tag{5.9}$$

One important fact about these differential operators is that they may not be traceless. Indeed, by taking the trace we find

$$\eta^{\mu\nu}\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} = \eta^{\mu\nu}\mathcal{D}^{(2)}_{\mu\nu\rho\sigma} = -(\partial_\rho\partial_\sigma - \eta_{\rho\sigma}\Box)\,\Box. \tag{5.10}$$

Dimensional analysis tells us that the function $f(x)$ in (5.5) can be at most a function of $\log\mu^2x^2$ since the lhs of (5.5) scales like $1/x^4$ and this scaling is already saturated by the differential operator with four derivatives. Notice that we have introduced an arbitrary mass scale μ to make the argument of the $\log$ dimensionless. Let us write the most general ansatz for (5.5):

$$\begin{aligned}\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle \;=\; & \mathcal{D}^{(1)}_{\mu\nu\rho\sigma}\left[\alpha_1\log\mu^2x^2 + \alpha_2\left(\log\mu^2x^2\right)^2 + \cdots\right] \\ & + \mathcal{D}^{(2)}_{\mu\nu\rho\sigma}\left[\beta_1\log\mu^2x^2 + \beta_2\left(\log\mu^2x^2\right)^2 + \cdots\right].\end{aligned} \tag{5.11}$$

Now our task is to fix the coefficients α_i and β_j for (5.11) to match (5.1) for $x \neq 0$. As it turns out we only need terms up to $\log^2$ (otherwise one cannot avoid logarithmic terms for $x \neq 0$) The matching gives us

$$\alpha_1 = -\frac{c}{24} - \beta_1, \quad \alpha_2 = -\beta_2 = -\frac{c}{96},$$

thus

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = -\frac{c}{24}\mathcal{D}^{(1)}_{\mu\nu\rho\sigma}\left(\log\mu^2x^2\right) - \frac{c}{96}\left(\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} - \mathcal{D}^{(2)}_{\mu\nu\rho\sigma}\right)\left(\log\mu^2x^2\right)^2. \tag{5.12}$$

Notice that β_1 is absent in the final result. Indeed, the term with coefficient β_1 is

$$-\left(\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} - \mathcal{D}^{(2)}_{\mu\nu\rho\sigma}\right)\left(\log\mu^2x^2\right) \tag{5.13}$$

and this term identically vanishes in $2d$. If we take the trace of (5.12) we find that

$$\left\langle T_{\mu}^{\mu}\left(x\right)T_{\rho\sigma}\left(0\right)\right\rangle = -\frac{c}{48}\eta^{\mu\nu}\mathcal{D}^{(1)}_{\mu\nu\rho\sigma}\left(\log\mu^2x^2\right) = \frac{c}{48}\left(\partial_\rho\partial_\sigma - \eta_{\rho\sigma}\Box\right)\Box\log\mu^2x^2.$$

These terms have support only at $x = 0$, for in $2d$ the d'Alembertian of a $\log$ is a delta function, more precisely

$$\Box\log\mu^2x^2 = 4\pi\delta^2\left(x\right). \tag{5.14}$$

Therefore we find the anomalous Ward identity

$$\left\langle T_{\mu}^{\mu}\left(x\right)T_{\rho\sigma}\left(y\right)\right\rangle = c\frac{\pi}{12}\left(\partial_\rho\partial_\sigma - \eta_{\rho\sigma}\Box\right)\delta^2\left(x-y\right), \tag{5.15}$$

If we consider our theory in the presence of a background metric g which is a perturbation of flat spacetime, i.e. $g_{\rho\sigma}(y) = \eta_{\rho\sigma} + h_{\rho\sigma}(y) + \cdots$, eq. (5.15) gives rise to the lowest contribution to the 'full one-loop' trace of the e.m. tensor, namely

$$\langle T_{\mu}^{\mu}\rangle_g = c\frac{\pi}{12}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)h^{\mu\nu}, \tag{5.16}$$

which coincides with the lowest contribution of the expansion in h of the Ricci scalar, i.e.

$$R = (\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)h^{\mu\nu} + \mathcal{O}(h^2). \tag{5.17}$$

Covariance requires that the higher order corrections in h to the 'full one-loop' trace of the e.m. tensor in the presence of a background metric g to be such that we recover the covariant expression

$$\langle T_{\mu}^{\mu}\rangle_g = c\frac{\pi}{12}R. \tag{5.18}$$

For a free chiral fermion $c = 1/4\pi^2$, vide section 5.3 or appendix E.1. We are authorized to use the covariant expression (5.18) because the energy-momentum tensor is conserved (there are no diffemorphism anomalies).

Using the above results it is easy to verify the Callan-Symanzik equation for the 2-point function (5.12). The Callan-Symanzik differential operator reduces to the logarithmic derivative with respect to μ, because both beta functions and anomalous dimensions vanish in the case we are considering. We get

$$\mu\frac{\partial}{\partial\mu}\left\langle T_{\mu\nu}\left(x\right)T_{\rho\sigma}\left(0\right)\right\rangle \sim \left(\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} - \mathcal{D}^{(2)}_{\mu\nu\rho\sigma}\right)\left(\log\mu^2x^2\right) = 0. \tag{5.19}$$

We see that requiring that the regularized correlator satisfies conservation at $x = 0$ implies the appearance of a trace anomaly. However this is not the end of the story, since there are ambiguities in the regularization process we have so far disregarded.

5.1.1 Ambiguities

The ambiguity arises from the fact that we can add to (5.12) terms that have support only in $x = 0$. The most general modification of the parity-even part that would affect only its expression for $x = 0$ is given by

$$\begin{aligned} A_{\mu\nu\rho\sigma} \quad = \quad & A\left(\eta_{\mu\nu}\partial_\rho\partial_\sigma + \eta_{\rho\sigma}\partial_\mu\partial_\nu\right)\Box\log\mu^2x^2 \\ & + B\left(\eta_{\mu\rho}\partial_\nu\partial_\sigma + \eta_{\nu\rho}\partial_\mu\partial_\sigma + \eta_{\mu\sigma}\partial_\nu\partial_\rho + \eta_{\nu\sigma}\partial_\mu\partial_\rho\right)\Box\log\mu^2x^2 \\ & + C\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}\right)\Box\Box\log\mu^2x^2 \\ & + D\eta_{\mu\nu}\eta_{\rho\sigma}\Box\Box\log\mu^2x^2. \end{aligned} \tag{5.20}$$

We remark that this term is in general neither conserved nor traceless

$$\begin{aligned}\partial^\mu A_{\mu\nu\rho\sigma} &= 4\pi\left((A+2B)\partial_\nu\partial_\rho\partial_\sigma + (A+D)\eta_{\rho\sigma}\partial_\nu\Box\right.\\ &\quad \left.+ (B+C)\left(\eta_{\rho\nu}\partial_\sigma\Box + \eta_{\sigma\nu}\partial_\rho\Box\right)\right)\delta^{(2)}(x) \qquad (5.21)\\ A^\mu_{\mu\rho\sigma} &= 4\pi\left((2A+4B)\partial_\rho\partial_\sigma + (A+2C+2D)\eta_{\rho\sigma}\Box\right)\delta^{(2)}(x) \qquad (5.22)\end{aligned}$$

We notice that by imposing (5.21) to vanish imply that also (5.22) will vanish. We may wonder whether using this ambiguity we can cancel the trace anomaly. This can certainly be done by choosing $2A+4B = -A-2C-2D$ and adjusting the overall coefficient. But this operation gives rise to a diffeomorphism anomaly. Its form is far from appealing and not particularly illuminating, so we do not write it down (see however [51, 52]). In other words the anomaly (5.18) is a non-trivial cocycle of the overall symmetry diffeomorphisms plus Weyl transformations. As was discussed in [51, 52] it may take different forms, either as a pure diffeomorphism anomaly or a pure trace anomaly. In general both components may be nonvanishing. It is obvious that, in practice, it is more useful to preserve diffeomorphism invariance, so that the cocycle takes the form (5.18).

5.2 Parity-odd terms in $2d$

In this section we compute all possible "bare" parity-odd terms in the 2-point function of the energy-momentum tensor in $2d$. We follow three methods, the first two are general while the third is based on a specific model. Needless to say all methods give the same results up to ambiguities.

5.2.1 Using symmetries

The first method is very simple-minded, it consists in writing the most general expression $\mathcal{T}^{\rm odd}_{\mu\nu\rho\sigma}(x)$ linear in the antisymmetric tensor $\epsilon_{\alpha\beta}$ with the right dimensions which is symmetric and traceless in μ,ν and ρ,σ separately, is symmetric in the exchange $(\mu,\nu)\leftrightarrow(\rho,\sigma)$, and is conserved. The calculation is tedious but straightforward. The result is as follows. Let us define

$$T_{\mu\nu\rho\sigma} = \frac{1}{x^4}\left(I_{\mu\rho}(x)I_{\nu\sigma}(x) + I_{\mu\sigma}(x)I_{\nu\rho}(x) - \eta_{\mu\nu}\eta_{\rho\sigma}\right), \qquad (5.23)$$

and

$$\mathcal{T}^{\rm odd}_{\mu\nu\rho\sigma}(x) = \frac{\mathfrak{c}}{4}\left(\epsilon_{\mu\lambda}T^\lambda{}_{\nu\rho\sigma}(x) + \epsilon_{\nu\lambda}T_\mu{}^\lambda{}_{\rho\sigma}(x) + \epsilon_{\rho\lambda}T_{\mu\nu}{}^\lambda{}_\sigma(x) + \epsilon_{\sigma\lambda}T_{\mu\nu\rho}{}^\lambda(x)\right). \qquad (5.24)$$

where $\mathfrak{c}$ is an undetermined constant. We assume (5.24) to represent $\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle_{\rm odd}$. It satisfies all the desired properties (it is traceless and conserved). In order to make sure that it is conformal covariant, we have to check that it is chirally split. To this end we introduce the light-cone coordinates $x_\pm = x^0 \pm x^1$. It is not hard to verify that

$$\langle T_{++}(x)T_{--}(0)\rangle_{\rm odd} = 0. \qquad (5.25)$$

5.2.2 The embedding formalism

The second method is the embedding formalism [5, 6], which consists in using the fact that conformal covariance in d dimensions can be linearly realized in $d+2$. After constructing a covariant expression in $d+2$ one projects to d dimensional Minkowski space. In particular for $d=2$ the method works as follows. We write the most general parity-odd contribution to the 2-point function of a symmetric 2-tensor in $4d$ which, in addition, is transverse:

$$\langle T_{AB}(X)\,T_{CD}(Y)\rangle_{\text{odd}}=\frac{1}{(X\cdot Y)^2}\left[\epsilon_{AICJ}\frac{X^IY^J}{X\cdot Y}\left(\eta_{BD}-\frac{X_DY_B}{X\cdot Y}\right)+A\leftrightarrow B\right]+C\leftrightarrow D. \tag{5.26}$$

This term is symmetric on A, B and C, D and is transverse with respect to X_A, X_B, Y_C and Y_D. Our next step is to project this quantity to $2d$. The projected correlator is given by

$$\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(y)\rangle_{\text{odd}}=\frac{\partial X^A}{\partial x^\mu}\frac{\partial X^B}{\partial x^\nu}\frac{\partial Y^C}{\partial y^\rho}\frac{\partial Y^D}{\partial y^\sigma}\langle T_{AB}(X)\,T_{CD}(Y)\rangle_{\text{odd}}\,. \tag{5.27}$$

We recall that

$$\frac{\partial X^A}{\partial x^\mu}=\delta^A_-2x_\mu+\delta^A_\mu\equiv\left(0,2x_\mu,\delta^a_\mu\right),\quad A=+,-,a. \tag{5.28}$$

The contractions with the ϵ-tensor give rise to a determinant, namely

$$\epsilon_{AICJ}\frac{\partial X^A}{\partial x^\mu}X^I\frac{\partial Y^C}{\partial y^\rho}Y^J\equiv\begin{vmatrix}0&1&0&1\\2x_\mu&x^2&2y_\rho&y^2\\\delta^a_\mu&x^i&\delta^c_\rho&y^j\end{vmatrix}. \tag{5.29}$$

The translational invariance of the problem allows us to rewrite it in the form

$$\begin{vmatrix}0&1&0&1\\2(x-y)_\mu&(x-y)^2&0&0\\\delta^a_\mu&(x-y)^i&\delta^c_\rho&0\end{vmatrix}=-\begin{vmatrix}2(x-y)_\mu&(x-y)^2&0\\\delta^a_\mu&(x-y)^i&\delta^c_\rho\end{vmatrix}. \tag{5.30}$$

For convenience, let us relabel $x-y\to x$. This determinant is straightforward to compute and it gives us

$$-\begin{vmatrix}2x_\mu&x^2&0\\\delta^a_\mu&x^i&\delta^c_\rho\end{vmatrix}=-\left(2x_\mu\begin{vmatrix}x^i&\delta^c_\rho\end{vmatrix}-x^2\begin{vmatrix}\delta^a_\mu&\delta^c_\rho\end{vmatrix}\right)=-\left(2x_\mu\epsilon_{\alpha\rho}x^\alpha-x^2\epsilon_{\mu\rho}\right). \tag{5.31}$$

Thus, the projected correlator is given by

$$\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(0)\rangle_{\text{odd}}=\frac{e}{x^4}\left[\epsilon_{\alpha\rho}\left(\delta^\alpha_\mu-2\frac{x_\mu x^\alpha}{x^2}\right)\left(\eta_{\nu\sigma}-2\frac{x_\nu x_\sigma}{x^2}\right)+\mu\leftrightarrow\nu\right]+\rho\leftrightarrow\sigma. \tag{5.32}$$

In terms of $I_{\mu\nu}(x)$ we have

$$\begin{aligned}\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(0)\rangle_{\text{odd}}=\frac{e}{x^4}&\left[\epsilon_{\alpha\rho}\left(I^\alpha_\mu(x)\,I_{\nu\sigma}(x)+I^\alpha_\nu(x)\,I_{\mu\sigma}(x)\right)\right.\\&\left.+\epsilon_{\alpha\sigma}\left(I^\alpha_\mu(x)\,I_{\nu\rho}(x)+I^\alpha_\nu(x)\,I_{\mu\rho}(x)\right)\right].\end{aligned} \tag{5.33}$$

This correlator satisfies both tracelessness and conservation, as it can be verified by a direct computation, but it is not symmetric under the exchange of μ,ν with ρ,σ. Thus, our final expression is (5.33) symmetrized in $(\mu,\nu)\leftrightarrow(\rho,\sigma)$:

$$\begin{aligned}\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(0)\rangle_{\text{odd}}=\frac{e}{x^4}&\left[\epsilon_{\alpha\mu}\left(I^\alpha_\rho(x)\,I_{\nu\sigma}(x)+I^\alpha_\sigma(x)\,I_{\nu\rho}(x)\right)\right.\\&+\epsilon_{\alpha\nu}\left(I^\alpha_\rho(x)\,I_{\mu\sigma}(x)+I^\alpha_\sigma(x)\,I_{\mu\rho}(x)\right)\\&+\epsilon_{\alpha\rho}\left(I^\alpha_\mu(x)\,I_{\nu\sigma}(x)+I^\alpha_\nu(x)\,I_{\mu\sigma}(x)\right)\\&\left.+\epsilon_{\alpha\sigma}\left(I^\alpha_\mu(x)\,I_{\nu\rho}(x)+I^\alpha_\nu(x)\,I_{\mu\rho}(x)\right)\right].\end{aligned} \tag{5.34}$$

From (5.34) we notice a tensorial structure very similar to the parity-even part of the 2-point function of $T_{\mu\nu}$, namely

$$T_{\mu\nu\rho\sigma}(x) = \frac{1}{x^4}\left(I_{\mu\rho}(x) I_{\nu\sigma}(x) + I_{\nu\rho}(x) I_{\mu\sigma}(x) - \eta_{\mu\nu}\eta_{\rho\sigma}\right) \tag{5.35}$$

and it turns out that we may write (5.34) in terms of the partity-even part, i.e.

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle_{\text{odd}} = \frac{e}{2}\left(\epsilon_{\alpha\mu} T^{\alpha}{}_{\nu\rho\sigma}(x) + \epsilon_{\alpha\nu} T_{\mu}{}^{\alpha}{}_{\rho\sigma}(x) + \epsilon_{\alpha\rho} T_{\mu\nu}{}^{\alpha}{}_{\sigma}(x) + \epsilon_{\alpha\rho} T_{\mu\nu\rho}{}^{\alpha}(x)\right). \tag{5.36}$$

This result looks different from (5.24) but it is not hard to show that, for $x \neq 0$, they are proportional: $\mathfrak{c} = \frac{3}{4}e$.

Still another method to derive the same result is to use a free fermion model. This is deferred to appendix E.1.

5.2.3 Differential regularization of the parity-odd part

The task of regularizing the parity-odd terms is very much simplified by the fact that we are able to write them in terms of the parity-even part, see (5.36). We can therefore use the same regularization as in section 5.1. Let us start by the regularization that preserves diffeomorphisms for the parity-even part, eq. (5.12):

$$T_{\mu\nu\rho\sigma}(x) = -\frac{1}{12}\mathcal{D}^{(1)}_{\mu\nu\rho\sigma}\left(\log\mu^2 x^2\right) - \frac{1}{48}\left(\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} - \mathcal{D}^{(2)}_{\mu\nu\rho\sigma}\right)\left(\log\mu^2 x^2\right)^2. \tag{5.37}$$

Regularizing (5.36) with (5.37) leads to a trace anomaly

$$\left\langle T^{\mu}_{\mu}(x) T_{\rho\sigma}(0)\right\rangle_{\text{odd}} = \frac{\pi e}{24}\left(\epsilon_{\rho\alpha}\partial^{\alpha}\partial_{\sigma} + \epsilon_{\sigma\alpha}\partial^{\alpha}\partial_{\rho}\right)\delta^2(x), \tag{5.38}$$

and a diffeomorphism anomaly

$$\partial^{\mu}\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle_{\text{odd}} = \frac{\pi e}{24}\epsilon_{\nu\alpha}\partial^{\alpha}\left(\eta_{\rho\sigma}\Box - \partial_{\rho}\partial_{\sigma}\right)\delta^2(x). \tag{5.39}$$

In the presence of a background metric g the anomalous Ward-Identities (5.38) and (5.39) give rise to the following 'full one-loop' functions

$$\langle T^{\mu}_{\mu}(x)\rangle_g = \frac{\pi e}{24}\epsilon^{\lambda\alpha}\partial_{\alpha}\left(g^{\rho\sigma}\partial_{\lambda} g_{\rho\sigma} + g^{\rho\sigma}\partial_{\rho} g_{\lambda\sigma}\right), \tag{5.40}$$

$$\langle \nabla^{\mu} T_{\mu\nu}(x)\rangle_g = \frac{\pi e}{24}\epsilon_{\nu\alpha}\partial^{\alpha} R. \tag{5.41}$$

The second is the well-known covariant form of the diffeomorphism anomaly. The consistent form of the same anomaly is

$$\langle \nabla^{\mu} T_{\mu\nu}(x)\rangle_g \sim \epsilon^{\mu\rho}\partial_{\mu}\partial_{\alpha}\Gamma^{\alpha}_{\rho\nu}. \tag{5.42}$$

We remark however that in $2d$ the two forms (5.41) and (5.42) collapse to the same form to the lowest order, since

$$2\epsilon_{\mu\nu}\partial^{\mu}\left(\partial_{\alpha}\partial_{\beta} - \eta_{\alpha\beta}\Box\right) = \epsilon_{\mu\alpha}\left(\partial^{\mu}\partial_{\nu}\partial_{\beta} - \eta_{\nu\beta}\partial^{\mu}\Box + (\alpha \leftrightarrow \beta)\right)$$

We see that, in any case, the diffeomorphism anomaly is accompanied by the a trace anomaly.

5.2.4 Ambiguities in the parity-odd part

We know that the regularization used above is not the ultimate one, because there are ambiguities. They entail a modification of the parity-odd part given by

$$A^{\text{odd}}_{\mu\nu\rho\sigma} = \epsilon_{\alpha\mu} A^{\alpha}{}_{\nu\rho\sigma} + \epsilon_{\alpha\nu} A_{\mu}{}^{\alpha}{}_{\rho\sigma} + \epsilon_{\alpha\rho} A_{\mu\nu}{}^{\alpha}{}_{\sigma} + \epsilon_{\alpha\rho} A_{\mu\nu\rho}{}^{\alpha}, \tag{5.43}$$

where the RHS is written in terms of (5.20), which explicitly is

$$\begin{aligned} A^{\text{odd}}_{\mu\nu\rho\sigma} = \; & A\left[\eta_{\mu\nu}\left(\epsilon_{\rho\alpha}\partial^{\alpha}\partial_{\sigma} + \epsilon_{\sigma\alpha}\partial^{\alpha}\partial_{\rho}\right) + \eta_{\rho\sigma}\left(\epsilon_{\mu\alpha}\partial^{\alpha}\partial_{\nu} + \epsilon_{\nu\alpha}\partial^{\alpha}\partial_{\mu}\right)\right]\Box \log\mu^2 x^2 \\ & + B\left[\epsilon_{\mu\alpha}\left(\eta_{\nu\rho}\partial^{\alpha}\partial_{\sigma} + \eta_{\nu\sigma}\partial^{\alpha}\partial_{\rho}\right) + \epsilon_{\nu\alpha}\left(\eta_{\mu\rho}\partial^{\alpha}\partial_{\sigma} + \eta_{\mu\sigma}\partial^{\alpha}\partial_{\rho}\right)\right. \\ & \left. + \epsilon_{\rho\alpha}\left(\eta_{\sigma\mu}\partial^{\alpha}\partial_{\nu} + \eta_{\sigma\nu}\partial^{\alpha}\partial_{\mu}\right) + \epsilon_{\sigma\alpha}\left(\eta_{\rho\mu}\partial^{\alpha}\partial_{\nu} + \eta_{\rho\nu}\partial^{\alpha}\partial_{\mu}\right)\right]\Box \log\mu^2 x^2 . \end{aligned} \tag{5.44}$$

The trace and the divergence of (5.44) are given by:

$$\eta^{\mu\nu} A_{\mu\nu\rho\sigma} = 8\pi\left(A + 2B\right)\left(\epsilon_{\rho\alpha}\partial^{\alpha}\partial_{\sigma} + \epsilon_{\sigma\alpha}\partial^{\alpha}\partial_{\rho}\right)\delta^2\left(x\right), \tag{5.45}$$

$$\begin{aligned} \partial^{\mu} A_{\mu\nu\rho\sigma} = \; & 4\pi\left(B\eta_{\nu\rho}\Box + \left(A+B\right)\partial_{\nu}\partial_{\rho}\right)\epsilon_{\sigma\alpha}\partial^{\alpha}\delta^2\left(x\right) \\ & + 4\pi\left(B\eta_{\nu\sigma}\Box + \left(A+B\right)\partial_{\nu}\partial_{\sigma}\right)\epsilon_{\rho\alpha}\partial^{\alpha}\delta^2\left(x\right) \\ & + 4\pi\left(A\eta_{\rho\sigma}\Box + 2B\partial_{\rho}\partial_{\sigma}\right)\epsilon_{\nu\alpha}\partial^{\alpha}\delta^2\left(x\right). \end{aligned} \tag{5.46}$$

Using these ambiguities we can recast the expressions (5.38) and (5.39) in the form

$$\left\langle T^{\mu}_{\mu}\left(x\right) T_{\rho\sigma}\left(0\right)\right\rangle_{\text{odd}} = \left(8\pi\left(A + 2B\right) + \frac{\pi e}{24}\right)\left(\epsilon_{\rho\alpha}\partial^{\alpha}\partial_{\sigma} + \epsilon_{\sigma\alpha}\partial^{\alpha}\partial_{\rho}\right)\delta^2\left(x\right), \tag{5.47}$$

$$\begin{aligned} \partial^{\mu}\left\langle T_{\mu\nu}\left(x\right) T_{\rho\sigma}\left(0\right)\right\rangle_{\text{odd}} = \; & 4\pi\left(B\eta_{\nu\rho}\Box + \left(A+B\right)\partial_{\nu}\partial_{\rho}\right)\epsilon_{\sigma\alpha}\partial^{\alpha}\delta^2\left(x\right) \\ & + 4\pi\left(B\eta_{\nu\sigma}\Box + \left(A+B\right)\partial_{\nu}\partial_{\sigma}\right)\epsilon_{\rho\alpha}\partial^{\alpha}\delta^2\left(x\right) \\ & + \epsilon_{\nu\alpha}\partial^{\alpha}\left(\left(4\pi A + \frac{\pi e}{24}\right)\eta_{\rho\sigma}\Box + \left(8\pi B - \frac{\pi e}{24}\right)\partial_{\rho}\partial_{\sigma}\right)\delta^2\left(x\right). \end{aligned} \tag{5.48}$$

If we impose that (5.47) is zero we find

$$A = -\frac{e}{192} - 2B, \tag{5.49}$$

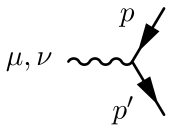
which implies that (5.48) takes the form

$$\begin{aligned} \partial^{\mu}\left\langle T_{\mu\nu}\left(x\right) T_{\rho\sigma}\left(0\right)\right\rangle_{\text{odd}} = \; & 4\pi\left[B\eta_{\nu\rho}\Box - \left(\frac{e}{192} + B\right)\partial_{\nu}\partial_{\rho}\right]\epsilon_{\sigma\alpha}\partial^{\alpha}\delta^2\left(x\right) \\ & + 4\pi\left[B\eta_{\nu\sigma}\Box - \left(\frac{e}{192} + B\right)\partial_{\nu}\partial_{\sigma}\right]\epsilon_{\rho\alpha}\partial^{\alpha}\delta^2\left(x\right) \\ & - \epsilon_{\nu\alpha}\partial^{\alpha}\left[\left(\frac{\pi e}{48} + 8\pi B\right)\eta_{\rho\sigma}\Box - \left(8\pi B - \frac{\pi e}{24}\right)\partial_{\rho}\partial_{\sigma}\right]\delta^2\left(x\right). \end{aligned} \tag{5.50}$$

The choice (5.49) allows us to eliminate the trace anomaly (5.40) but by doing so the diffeo anomaly becomes (5.50), which will not imply a covariant expression for $\langle T_{\mu\nu}\rangle_g$ for any choice of B. Thus, the most general regularization that one can write is given by the equations (5.47) and (5.48). An important point of (5.48) is that there is no choice of A and B for which it is zero, hence inevitably we will have a diffeomorphism anomaly, unless $e = 0$, which depends of course on the specific model.

5.3 The Feynman diagrams method in $2d$

It is interesting and instructive to derive the results above using Feynman diagrams. In this section we will concentrate on the theory of a free chiral fermion. There is only one non-trivial contribution that comes from the bubble diagram with one incoming and one outgoing line with momentum k and an internal momentum p (see figure 5.1). The pertinent Feynman rule is



$$= \frac{i}{8}\left[\left(p+p'\right)_{\mu}\gamma_{\nu} + \left(p+p'\right)_{\nu}\gamma_{\mu}\right]\frac{1+\gamma_{*}}{2}. \tag{5.51}$$

The relevant 2-point function is[2]

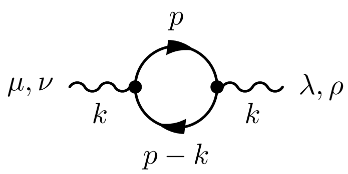


FIGURE 5.1: The relevant Feynman diagram for the computation.

$$\langle T_{\mu\nu}(x)T_{\lambda\rho}(y)\rangle = 4\int\frac{d^2k}{(2\pi)^2}e^{-ik(x-y)}\mathcal{T}_{\mu\nu\lambda\rho}(k) \tag{5.52}$$

with

$$\mathcal{T}_{\mu\nu\lambda\rho}(k) = -\frac{1}{64}\int\frac{d^2k}{(2\pi)^2}\mathrm{tr}\left(\frac{1}{\not p}(2p-k)_{\mu}\gamma_{\nu}\frac{1}{\not p-\not k}(2p-k)_{\lambda}\gamma_{\rho}\frac{1+\gamma_{*}}{2}\right) + \left\{\begin{array}{c}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{array}\right\}. \tag{5.53}$$

Taking the trace and regularizing by introducing extra components of the momentum running around the loop, $p \to p+\ell$ ($\ell = \ell_2, \ldots, \ell_{\delta+2}$), we get

$$\begin{aligned}\mathcal{T}^{\mu}{}_{\mu\lambda\rho}(k) = -\frac{1}{32}\int\frac{d^2p}{(2\pi)^2}\int\frac{d^{\delta}\ell}{(2\pi)^{\delta}}\,\mathrm{tr}\Bigg(&\frac{\not p+\not\ell}{p^2-\ell^2}\,(2\not p+2\not\ell-\not q)\\ &\frac{\not p+\not\ell-\not k}{(p-k)^2-\ell^2}(2p-k)_{\lambda}\gamma_{\rho}\frac{1+\gamma_{*}}{2}\Bigg)\end{aligned} \tag{5.54}$$

and the symmetrization $\lambda\leftrightarrow\rho$ is understood from now on. Introducing, as usual, a Feynman parametrization of the integral in (5.54) and using the results in appendix (E.2) one finally gets for the even part

$$\left(\mathcal{T}_{\text{even}}\right)^{\mu}{}_{\mu\lambda\rho}(k) = \frac{1}{192\pi}\left(\eta_{\lambda\rho}k^2 + k_{\lambda}k_{\rho}\right), \tag{5.55}$$

which corresponds to the trace anomaly

$$\langle T^{\mu}_{\mu}\rangle_g = -\frac{1}{48\pi}\left(\Box h + \partial_{\lambda}\partial_{\rho}h^{\lambda\rho}\right) + \mathcal{O}\left(h^2\right). \tag{5.56}$$

[2] The factor of 4 in (5.52) is produced by the fact that the vertex (5.51) corresponds to the insertion of $\frac{1}{2}T_{\mu\nu}$ in the correlator, not simply $T_{\mu\nu}$.

For the odd part we get instead

$$(\mathcal{T}_{\text{odd}})^{\mu}{}_{\mu\lambda\rho}(k) = -\frac{1}{192\pi}\left(\epsilon^{\sigma}{}_{\rho}k_{\sigma}k_{\lambda} + \epsilon^{\sigma}{}_{\lambda}k_{\sigma}k_{\rho}\right), \tag{5.57}$$

which corresponds to the trace anomaly

$$\langle T^{\mu}_{\mu}\rangle_g = \frac{1}{24\pi}\epsilon^{\sigma}{}_{\rho}\,\partial_{\sigma}\partial_{\lambda}h^{\lambda\rho} + \mathcal{O}\left(h^2\right). \tag{5.58}$$

The trace anomaly (5.56) is not the expected covariant one. The only possible explanation is that our regularization has broken diffeomorphism invariance. In order to check that we have to compute the divergence of the energy-momentum tensor with the same method. The relevant Feynman diagram contribution is (after regularization)

$$\begin{aligned}\mathcal{D}_{\nu\lambda\rho}(k) = -\frac{1}{64}\int\frac{d^2p}{(2\pi)^2}\int\frac{d^{\delta}\ell}{(2\pi)^{\delta}}\\ \text{tr}\Bigg(&\frac{\not p+\not\ell}{p^2-\ell^2}(2p-k)_{\mu}k^{\mu}\,\gamma_{\nu}\frac{\not p+\not\ell-\not k}{(p-k)^2-\ell^2}(2p-k)_{\lambda}\gamma_{\rho}\frac{1+\gamma_*}{2}\\ &+\frac{\not p+\not\ell}{p^2-\ell^2}(2p-k)_{\nu}\,\not k\,\frac{\not p+\not\ell-\not k}{(p-k)^2-\ell^2}(2p-k)_{\lambda}\gamma_{\rho}\frac{1+\gamma_*}{2}\Bigg).\end{aligned} \tag{5.59}$$

Explicit evaluation gives for the even part

$$(\mathcal{D}_{\text{even}})_{\nu\lambda\rho}(k) = -\frac{1}{96\pi}\eta_{\lambda\rho}k_{\nu}k^2, \tag{5.60}$$

which corresponds to the diffeomorphism anomaly

$$\nabla^{\mu}\langle T_{\mu\nu}\rangle_g = \frac{1}{12\pi}\xi^{\nu}\partial_{\nu}\Box h + \mathcal{O}\left(h^2\right). \tag{5.61}$$

For the odd part we get instead

$$(\mathcal{D}_{\text{odd}})_{\nu\lambda\rho}(k) = -\frac{1}{192\pi}k^{\sigma}\epsilon_{\sigma\rho}\left(\eta_{\nu\lambda}k^2 - k_{\nu}k_{\lambda}\right) + \{\lambda\leftrightarrow\rho\}, \tag{5.62}$$

which corresponds to the anomaly

$$\nabla^{\mu}\langle T_{\mu\nu}\rangle_g = -\frac{1}{96\pi}\epsilon^{\sigma\rho}\left(\partial_{\sigma}\partial_{\lambda}\partial_{\nu}h^{\lambda}_{\rho} - \partial_{\sigma}\Box h_{\rho\nu}\right). \tag{5.63}$$

Using the lowest order Weyl transformation

$$\delta_{\omega}h_{\mu\nu} = 2\omega\,\eta_{\mu\nu}, \tag{5.64}$$

and diffeo transformation

$$\delta_{\xi}h_{\mu\nu} = \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}, \tag{5.65}$$

it is easy to prove that the consistency relations

$$\delta_{\omega}\mathcal{A}_{\omega} = 0, \qquad \delta_{\xi}\mathcal{A}_{\omega} + \delta_{\xi}\mathcal{A}_{\omega} = 0, \qquad \delta_{\xi}\mathcal{A}_{\xi} = 0, \tag{5.66}$$

hold, where

$$\mathcal{A}_\omega = -\int d^2x\, \omega \langle T^\mu_\mu \rangle_g, \quad \text{and} \quad \mathcal{A}_\xi = \int d^2x\, \xi^\nu \nabla^\mu \langle T_{\mu\nu} \rangle_g. \tag{5.67}$$

For the even part $\mathcal{A}^{(e)}$ it is possible to add a counterterm to the action and restore covariance. The couterterm is

$$\mathcal{C} = -\frac{1}{96\pi}\int d^2x\, h\Box h. \tag{5.68}$$

After this operation the divergence of the e.m. tensor vanishes and the trace anomaly becomes

$$\mathcal{A}^{(e)}_\omega \to \mathcal{A}^{(e)}_\omega + \delta_\omega \mathcal{C} = \frac{1}{48\pi}\int d^2x\, \omega \left(\partial_\lambda \partial_\rho h^{\lambda\rho} - \Box h\right), \tag{5.69}$$

which is the expected one (see above).

Similarly the parity-odd anomalies (5.57) and (5.63) satisfy the consistency relations (5.66). One can add an odd counterterm to eliminate the odd trace anomaly but this is definitely a less interesting operation.

The results obtained in this section are well-known. The methods we have used to derive them teach us important lessons. The first concerns dimensional regularization. If not explicitly stated it is often understood in the literature that dimensional regularization of Feynman diagrams leads to covariant results. We have seen explicitly that this is not true, and a reconstruction of covariance with counterterms is inevitable. In view of the discussion on 3-points correlator of the e.m. tensor in section 7.1.4 we notice that the piece of (5.54)

$$\Delta \mathcal{T}^{\mu}_{\mu\lambda\rho}(k) = -\frac{1}{8}\int \frac{d^2p}{(2\pi)^2}\int \frac{d^\delta \ell}{(2\pi)^\delta}\, \mathrm{tr}\left(\frac{\not p + \not\ell}{p^2-\ell^2}\, 2\not\ell\, \frac{\not p + \not\ell - \not k}{(p-k)^2-\ell^2}(2p-k)_\lambda \gamma_\rho \frac{1+\gamma_*}{2}\right) \tag{5.70}$$

contributes in an essential way to both even and odd anomalies. Without this piece the result of the calculation would be inconsistent. It marks the difference between first regularizing and then taking the trace of the e.m. tensor or first taking the trace and then regularizing. From the above it is obvious that the second procedure is the correct one. In other words every irreducible Lorentz component of tensors must be regularized separately. This is the second important lesson. We will return to this point also in the final section.

5.4 2-point correlator of e.m. tensors in $4d$

In this section we are going to discuss the 2-point correlator of the e.m. tensors in $4d$. The expression in coordinate representation is well-known. We would like here to regularize it with the differential regularization method, and, later on, compare it with the expression obtained in momentum space with Feynman diagram techniques.

5.4.1 Differential regularization of the correlator

The unregulated 2-point function of e.m. tensors in arbitrary dimension d in coordinate representation is given by

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = \frac{c/2}{x^{2d}}\left(I_{\mu\rho}(x)\, I_{\nu\sigma}(x) + I_{\nu\rho}(x)\, I_{\mu\sigma}(x) - \frac{2}{d}\eta_{\mu\nu}\eta_{\rho\sigma}\right) \tag{5.71}$$

where

$$I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}. \tag{5.72}$$

As before, it can be regularized by writing down a differential operator which, acting on an integrable function, generates it for $x \neq 0$. One possibility for $d \geq 3$ is the following[3]

$$\begin{aligned}\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle &= -\frac{c/2}{2\,(d-2)^2\, d\,(d^2-1)}\mathcal{D}^{(1)}_{\mu\nu\rho\sigma}\left(\frac{1}{x^{2d-4}}\right) \\ &\quad +\frac{c/2}{2\,(d-2)^2\, d\,(d+1)}\mathcal{D}^{(2)}_{\mu\nu\rho\sigma}\left(\frac{1}{x^{2d-4}}\right), \end{aligned} \tag{5.73}$$

where

$$\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} = \partial_\mu\partial_\nu\partial_\rho\partial_\sigma - (\eta_{\mu\nu}\partial_\rho\partial_\sigma + \eta_{\rho\sigma}\partial_\mu\partial_\nu)\,\Box + \eta_{\mu\nu}\eta_{\rho\sigma}\Box\Box, \tag{5.74}$$

$$\begin{aligned}\mathcal{D}^{(2)}_{\mu\nu\rho\sigma} &= \partial_\mu\partial_\nu\partial_\rho\partial_\sigma - \frac{1}{2}\left(\eta_{\mu\rho}\partial_\nu\partial_\sigma + \eta_{\nu\rho}\partial_\mu\partial_\sigma + \eta_{\mu\sigma}\partial_\nu\partial_\rho + \eta_{\nu\sigma}\partial_\mu\partial_\rho\right)\Box \\ &\quad + \frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}\right)\Box\Box. \end{aligned} \tag{5.75}$$

Both these operators are conserved but not traceless:

$$\eta^{\mu\nu}\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} = -\,(d-1)\,(\partial_\rho\partial_\sigma - \eta_{\rho\sigma}\Box)\,\Box, \tag{5.76}$$

$$\eta^{\mu\nu}\mathcal{D}^{(2)}_{\mu\nu\rho\sigma} = -\,(\partial_\rho\partial_\sigma - \eta_{\rho\sigma}\Box)\,\Box, \tag{5.77}$$

nonetheless (5.73) is both conserved and traceless. The expression (5.73) coincides with (5.71) for $x \neq 0$, it is conserved and traceless.

There are, as usual, ambiguities in the definitions of the operators (5.74) and (5.75) for $x = 0$. Particularly, in $d = 4$ we may consider the most general modification that one could add to the expression (5.73), namely

$$\begin{aligned}\mathcal{A}_{\mu\nu\rho\sigma} = \big[&A\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\Box + B\left(\eta_{\mu\rho}\partial_\nu\partial_\sigma + \eta_{\nu\rho}\partial_\mu\partial_\sigma + \eta_{\mu\sigma}\partial_\nu\partial_\rho + \eta_{\nu\sigma}\partial_\mu\partial_\rho\right)\Box^2 \\ &+C\left(\eta_{\mu\nu}\partial_\rho\partial_\sigma + \eta_{\rho\sigma}\partial_\mu\partial_\nu\right)\Box^2 + D\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}\right)\Box^3 + E\eta_{\mu\nu}\eta_{\rho\sigma}\Box^3\big]\,\frac{1}{x^2}. \end{aligned} \tag{5.78}$$

Conservation of $\mathcal{A}$ requires

$$C = -A + 2D, \quad D = -B, \quad E = A + 2B. \tag{5.79}$$

With these conditions the trace of $\mathcal{A}$ is

$$\mathcal{A}^{\mu}{}_{\mu\rho\sigma} = -4\pi^2\,(3A+4B)\,(\eta_{\rho\sigma}\Box - \partial_\rho\partial_\sigma)\,\Box\delta(x)\,. \tag{5.80}$$

This corresponds to the trivial anomaly $\Box R$, which can be subtracted away by adding a local Weyl invariant counterterm to the action. The existence of a definition of our differential

[3]Notice that for $d > 4$, the function $1/x^{2d-4}$ is indeed integrable, while we have a function which is log divergent for $d = 4$ and linearly divergent for $d = 3$ and in both cases we need a regularization. In the spirit of differential regularization, we may use the following identities

$$d = 3: \quad \frac{1}{x^2} = \frac{1}{2}\Box\log\mu^2x^2,$$

$$d = 4: \quad \frac{1}{x^4} = -\frac{1}{4}\Box\frac{\log\mu^2x^2}{x^2},$$

where $\log\mu^2x^2$ and $\left(\log\mu^2x^2\right)/x^2$ are integrable functions in the respective dimension.

operators which do not imply in the existence of this anomaly reflects the fact that it is a trivial anomaly.

5.4.2 2-point correlator with Feynman diagrams

The computation is very similar to the one in $2d$. Again, the only diagram that contributes is the one of figure 5.1 and we have[4]

$$\langle T_{\mu\nu}(x)T_{\lambda\rho}(y)\rangle = 4\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\tilde{\mathcal{T}}_{\mu\nu\lambda\rho}(k) \tag{5.81}$$

where

$$\tilde{\mathcal{T}}_{\mu\nu\lambda\rho}(k) = -\frac{1}{64}\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left(\frac{1}{\not p}(2p-k)_\mu\gamma_\nu\frac{1}{\not p-\not k}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_5}{2}\right)+\left\{\begin{array}{c}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{array}\right\}. \tag{5.82}$$

To evaluate it we use dimensional regularization. After introducing the Feynman parameter x and shifting p as follows: $p\to p-(1-x)k$, (5.82) writes[5]

$$\begin{aligned}\tilde{\mathcal{T}}_{\mu\nu\lambda\rho}(k) \;=\; & -\frac{1}{32}\int_0^1 dx\int\frac{d^4p}{(2\pi)^4}\int\frac{d^\delta\ell}{(2\pi)^\delta}\frac{(2p+(1-2x)k)_\mu(2p+(1-2x)k)_\lambda}{(p^2+x(1-x)k^2-\ell^2)^2}\\ & \times\left[(p+(1-x)k)^\sigma(p-xk)^\tau\left(\eta_{\sigma\nu}\eta_{\tau\rho}-\eta_{\sigma\tau}\eta_{\nu\rho}+\eta_{\sigma\rho}\eta_{\nu\tau}-i\epsilon_{\sigma\nu\tau\rho}\right)-\ell^2\eta_{\nu\rho}\right]\end{aligned} \tag{5.83}$$

After the integrations (first ℓ, then p, then x) one finds[6]

$$\tilde{\mathcal{T}}_{\mu\nu\lambda\rho}(k) = \tilde{\mathcal{D}}_{\mu\nu\lambda\rho}(k)+\tilde{\mathcal{F}}_{\mu\nu\lambda\rho}(k)+\tilde{\mathcal{L}}_{\mu\nu\lambda\rho}(k) \tag{5.84}$$

where

$$\begin{aligned}\tilde{\mathcal{D}}_{\mu\nu\lambda\rho}(k) = & -\frac{i}{32(4\pi)^2}\frac{1}{15\delta}\left[8k_\mu k_\nu k_\lambda k_\rho+4k^2\left(k_\mu k_\nu\eta_{\lambda\rho}+k_\lambda k_\rho\eta_{\mu\nu}\right)\right.\\ & -6k^2\left(k_\mu k_\lambda\eta_{\nu\rho}+k_\nu k_\lambda\eta_{\mu\rho}+k_\mu k_\rho\eta_{\nu\lambda}+k_\nu k_\rho\eta_{\mu\lambda}\right)\\ & \left.-4k^4\eta_{\mu\nu}\eta_{\lambda\rho}+6k^4\left(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}\right)\right]\end{aligned} \tag{5.85}$$

which is divergent for $\delta\to 0$, but conserved and traceless,

$$\begin{aligned}\tilde{\mathcal{L}}_{\mu\nu\lambda\rho}(k) = & -\frac{i}{32(4\pi)^2}\frac{\log k^2}{30}\left[8k_\mu k_\nu k_\lambda k_\rho+4k^2\left(k_\mu k_\nu\eta_{\lambda\rho}+k_\lambda k_\rho\eta_{\mu\nu}\right)\right.\\ & -6k^2\left(k_\mu k_\lambda\eta_{\nu\rho}+k_\nu k_\lambda\eta_{\mu\rho}+k_\mu k_\rho\eta_{\nu\lambda}+k_\nu k_\rho\eta_{\mu\lambda}\right)\\ & \left.-4k^4\eta_{\mu\nu}\eta_{\lambda\rho}+6k^4\left(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}\right)\right]\end{aligned} \tag{5.86}$$

[4]For the factor of 4 in (5.81), see the footnote in section 5.3.

[5]We use the mostly minus signature for the metric.

[6]To do integration properly we have to Wick rotate the momenta and, after integration rotate them back to the Lorentzian signature. We understand this here.

which is also conserved and traceless, and

$$\begin{aligned}\tilde{\mathcal{F}}_{\mu\nu\lambda\rho}(k) = &-\frac{i}{32(4\pi)^2}\frac{1}{30}\Bigg[8\left(\gamma-\log 4\pi+\frac{31}{450}\right)k_\mu k_\nu k_\lambda k_\rho \\ &+2\left(1-\gamma+\log 4\pi+\frac{31}{150}\right)k^2\left(k_\mu k_\lambda \eta_{\nu\rho}+k_\nu k_\lambda \eta_{\mu\rho}+k_\mu k_\rho \eta_{\nu\lambda}+k_\nu k_\rho \eta_{\mu\lambda}\right) \\ &+k^4\left(\frac{10}{3}-4\gamma+4\,\log 4\pi-\frac{47}{225}\right)\eta_{\mu\nu}\eta_{\lambda\rho} \\ &-k^4\left(\frac{17}{3}-6\gamma+6\,\log 4\pi\right)\left(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}\right) \\ &-k^2\left(4-4\gamma+4\,\log 4\pi+\frac{47}{450}\right)\left(k_\mu k_\nu \eta_{\lambda\rho}+k_\lambda k_\rho \eta_{\mu\nu}\right)\Bigg]\end{aligned} \tag{5.87}$$

which is neither conserved nor traceless. Let us consider first $\tilde{\mathcal{L}}$. We recall the Fourier transform

$$\int d^4x\, e^{ikx}\frac{1}{x^2}\log \mu^2 x^2 = \frac{4\pi^2 i}{k^2}\left(\log 2-\gamma-\log\frac{k^2}{\mu^2}\right). \tag{5.88}$$

Therefore, up to the term proportional to $(\log 2-\gamma)$, by Fourier transforming (5.73) we obtain precisely (5.86) with $c = 1/\pi^4$, in agreement with the results of [3, 4]. The term proportional to $(\log 2-\gamma)$ is to be added to (5.87). Now the divergence of $\tilde{\mathcal{T}}$ contains three independent terms proportional to $k^2 k_\nu k_\lambda k_\rho$, $k^4 k_\nu \eta_{\lambda\rho}$ and $k^4(k_\lambda \eta_{\nu\rho}+k_\rho\eta_{\nu\lambda})$, respectively, while the trace contains two independent terms proportional to $k^2 k_\lambda k_\rho$ and $k^4\eta_{\lambda\rho}$. On the other hand the ambiguity (5.78) contains the same 5 independent terms with arbitrary coefficients. Therefore it is always possible to set to zero both the divergence and the trace of $\tilde{\mathcal{T}}$ by subtracting suitable counterterms. In the same way one can argue with the divergent term $\tilde{\mathcal{D}}$. This term deserves a comment: it is traceless and divergenceless, but it is infinite, so it must be subtracted away along with the $\tilde{\mathcal{F}}$ term. Both $\mathcal{F}$ and $\mathcal{D}$, the Fourier anti-transforms of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{D}}$, are contact terms and they can be written in a compact form as

$$\begin{aligned}\langle T_{\mu\nu}(x)\rangle_g \;=\;& A'\partial_\mu\partial_\nu\partial_\lambda\partial_\rho h^{\lambda\rho}(x)+B'\left(\Box\partial_\mu\partial_\lambda h^\lambda_\nu(x)+\Box\partial_\nu\partial_\lambda h^\lambda_\mu(x)\right)+C'\eta_{\mu\nu}\Box^2 h(x) \\ &+D'\,\Box^2 h_{\mu\nu}(x)+E'\left(\Box\partial_\mu\partial_\nu h(x)+\eta_{\mu\nu}\Box\partial_\lambda\partial_\rho h^{\lambda\rho}(x)\right),\end{aligned} \tag{5.89}$$

where $h = h^\lambda_\lambda$ and A', B', C', D', E' are numerical coefficients that contain also a part $\sim \frac{1}{\delta}$. The local term to be subtracted from the action is proportional to

$$\int d^4x\left(\frac{A'}{2}h^{\mu\nu}\partial_\mu\partial_\nu\partial_\lambda\partial_\rho h^{\lambda\rho} \;+\; B'h^{\mu\nu}\Box\partial_\mu\partial_\lambda h^\lambda_\nu \;+\; \frac{C'}{2}h\Box^2 h+\frac{D'}{2}h^{\mu\nu}\Box^2 h_{\mu\nu}+E'h^{\mu\nu}\Box\partial_\mu\partial_\nu h\right). \tag{5.90}$$

We can conclude that the (regularized) Feynman diagram approach to the 2-point correlator is equivalent to regularizing the 2-point function calculated with the Wick theorem approach. But we can draw also another, less pleasant, conclusion. Like in $2d$, the Feynman diagrams coupled to dimensional regularization may also produce unwelcome terms, such as the $\mathcal{D}$ and $\mathcal{F}$ terms above, which must be subtracted away by hand.

Finally we notice that, once (5.90) has been subtracted away, not only the nonvanishing trace and divergence of the em tensor disappear, but the full contact term (5.89) gets canceled. Thus the regularized 2-point correlator of the e.m. tensor coincides with the "bare" expression.

Appendix E

Details of computations

E.1 Direct computation for a chiral fermion in $2d$

Consider a free chiral fermion ψ_L in $2d$ which has the 2-point function

$$\left\langle \psi_L(x)\,\overline{\psi_L}(y)\right\rangle = \frac{i}{2\pi}\frac{\gamma\cdot(x-y)}{(x-y)^2}P_L, \quad P_L = \frac{1-\gamma_*}{2}, \tag{E.1}$$

and the e.m. tensor

$$T_{\mu\nu} = \frac{i}{4}\left(\overline{\psi_L}\gamma_\mu \overleftrightarrow{\partial}_\nu \psi_L + \mu\leftrightarrow\nu\right). \tag{E.2}$$

Before proceeding with the calculation let us recall some definitions:

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu} \Rightarrow \left(\gamma^0\right)^2 = 1, \quad \left(\gamma^i\right)^2 = -1. \tag{E.3}$$

Clearly, $\gamma^0 = \gamma_0$ and $\gamma^i = -\gamma_i$. For an arbitrary dimension D the analogous of γ_5 will be denoted γ_* and it is given by $\gamma_* = (-i)^{\frac{D}{2}+1}\gamma_0\gamma_1\ldots\gamma_{D-1}$, which for $D=2$ means $\gamma_* = -\gamma_0\gamma_1$.

It is straightforward to check that the following relations are true:

$$\gamma_\mu = \epsilon_{\mu\nu}\gamma^\nu\gamma_*, \quad \epsilon_{\mu\nu}\gamma^\nu = \gamma_\mu\gamma_*, \tag{E.4}$$

where we are using the convention where $\epsilon_{01} = 1$. It follows

$$\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_*) = -2\epsilon_{\mu\nu}. \tag{E.5}$$

Our purpose is to compute the 2-point of the em tensor in the theory (E.1). Since we are dealing with a simple free theory we can use the Wick theorem.

The non-zero part of the correlation function comes from

$$\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(y)\rangle = \frac{1}{16}\left\langle :\bar\psi_L\gamma_\mu \overleftrightarrow{\partial}_\nu \psi_L:(x) :\bar\psi_L\gamma_\rho \overleftrightarrow{\partial}_\sigma \psi_L:(y)\right\rangle + \text{sym.},$$

which is given by the full contraction of this object, namely

$$\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(y)\rangle = \frac{1}{16(2\pi)^2}\left(\mathrm{tr}\left[\gamma_\mu\partial^x_\nu\left\langle\psi_L(x)\,\bar\psi_L(y)\right\rangle\gamma_\rho\partial^y_\sigma\left\langle\psi_L(y)\,\bar\psi_L(x)\right\rangle\right]+\cdots\right) + \text{sym.}, \tag{E.6}$$

where the ellipsis stand for the three other ways of organizing the derivatives. We may use the translational invariance of this correlator to shift $x\to x-y$ and $y\to 0$. For simplicity we will relabel $x-y$ calling it simply x. Since the correlation function is simply a function of $x-y$, $\partial^y = -\partial^x$. Let us also remark that $\left\langle\psi_L(x)\,\bar\psi_L(y)\right\rangle = -\left\langle\psi_L(y)\,\bar\psi_L(x)\right\rangle$. Thus, we can exchange all the derivatives on y by derivatives on x and the correlations functions $\left\langle\psi_L\left(y\right)\bar{\psi}_L\left(x\right)\right\rangle$ by $\left\langle\psi_L\left(x\right)\bar{\psi}_L\left(y\right)\right\rangle$, which, due to translational invariance, can be written as $\left\langle\psi_L\left(x-y\right)\bar{\psi}_L\left(0\right)\right\rangle$. Therefore,

$$\begin{aligned}\langle T_{\mu\nu}\left(x\right)T_{\rho\sigma}\left(y\right)\rangle =& \frac{1}{16(2\pi)^2}\left(\mathrm{tr}\left[\gamma_\mu\partial_\nu\left\langle\psi_L\left(x\right)\bar{\psi}_L\left(0\right)\right\rangle\gamma_\rho\partial_\sigma\left\langle\psi_L\left(x\right)\bar{\psi}_L\left(0\right)\right\rangle\right]+\cdots\right)\\ &+\mathrm{sym.}\end{aligned} \tag{E.7}$$

Using the expression for the 2-point function (E.1) we have

$$\mathrm{tr}\left[\gamma_\mu\partial_\nu\left\langle\psi_L\left(x\right)\bar{\psi}_L\left(0\right)\right\rangle\gamma_\rho\partial_\sigma\left\langle\psi_L\left(x\right)\bar{\psi}_L\left(0\right)\right\rangle\right] = \frac{1}{(2\pi)^2}\partial_\nu\left(\frac{x^\alpha}{x^2}\right)\partial_\sigma\left(\frac{x^\beta}{x^2}\right)\mathrm{tr}(\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta P_L),$$

and analogously for the other terms. One should notice that

$$\mathrm{tr}(\gamma_\mu\gamma_\beta\gamma_\rho\gamma_\alpha P_L)=\mathrm{tr}(\gamma_\rho\gamma_\alpha\gamma_\mu\gamma_\beta P_L)$$

and we are able to rewrite our correlation function as

$$\begin{aligned}\langle T_{\mu\nu}\left(x\right)T_{\rho\sigma}\left(y\right)\rangle = \frac{1}{16}\frac{1}{(2\pi)^2}&\left[\partial_\nu\left(\frac{x^\alpha}{x^2}\right)\partial_\sigma\left(\frac{x^\beta}{x^2}\right)-\left(\frac{x^\alpha}{x^2}\right)\partial_\nu\partial_\sigma\left(\frac{x^\beta}{x^2}\right)\right]\\ &\times\left[\mathrm{tr}(\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta P_L)+\mu\leftrightarrow\rho\right]+\mathrm{sym.}\end{aligned} \tag{E.8}$$

Exchanging the position of γ_α and γ_ρ in $\mathrm{tr}(\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta P_L)$ we have

$$\mathrm{tr}(\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta P_L)=2\eta_{\alpha\rho}\mathrm{tr}(\gamma_\mu\gamma_\beta P_L)-\mathrm{tr}(\gamma_\mu\gamma_\rho\gamma_\alpha\gamma_\beta P_L).$$

Thus

$$\begin{aligned}\mathrm{tr}(\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta P_L)+\mu\leftrightarrow\rho &= 2\eta_{\alpha\rho}\mathrm{tr}(\gamma_\mu\gamma_\beta P_L)+2\eta_{\alpha\mu}\mathrm{tr}(\gamma_\rho\gamma_\beta P_L)-\mathrm{tr}(\{\gamma_\mu,\gamma_\rho\}\gamma_\alpha\gamma_\beta P_L)\\ &= 2\left[\eta_{\alpha\rho}\mathrm{tr}(\gamma_\mu\gamma_\beta P_L)+\eta_{\alpha\mu}\mathrm{tr}(\gamma_\rho\gamma_\beta P_L)-\eta_{\mu\rho}\mathrm{tr}(\gamma_\alpha\gamma_\beta P_L)\right].\end{aligned}$$

The trace of $\gamma_\mu\gamma_\nu P_L$ is straightforward to compute:

$$\mathrm{tr}(\gamma_\mu\gamma_\nu P_L)=\frac{1}{2}\left[\mathrm{tr}(\gamma_\mu\gamma_\nu)-\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_*)\right]=\eta_{\mu\nu}+\epsilon_{\mu\nu}.$$

Therefore

$$\mathrm{tr}(\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta P_L)+\mu\leftrightarrow\rho=2\left(\eta_{\alpha\rho}\eta_{\mu\beta}+\eta_{\alpha\mu}\eta_{\rho\beta}-\eta_{\mu\rho}\eta_{\alpha\beta}\right)+2\left(\eta_{\alpha\rho}\epsilon_{\mu\beta}+\eta_{\alpha\mu}\epsilon_{\rho\beta}-\eta_{\mu\rho}\epsilon_{\alpha\beta}\right). \tag{E.9}$$

It turns out that we are able to rewrite $\eta_{\mu\rho}\epsilon_{\alpha\beta}$ as

$$\eta_{\mu\rho}\epsilon_{\alpha\beta}=\frac{1}{2}\left(\eta_{\alpha\mu}\epsilon_{\rho\beta}-\eta_{\beta\mu}\epsilon_{\rho\alpha}+\eta_{\alpha\rho}\epsilon_{\mu\beta}-\eta_{\beta\rho}\epsilon_{\mu\alpha}\right)$$

and using this expression we may rewrite (E.9) as

$$\begin{aligned}\mathrm{tr}(\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta P_L)+\mu\leftrightarrow\rho &= 2\left(\eta_{\alpha\rho}\eta_{\mu\beta}+\eta_{\alpha\mu}\eta_{\rho\beta}-\eta_{\mu\rho}\eta_{\alpha\beta}\right)\\ &\quad+\left(\eta_{\alpha\rho}\epsilon_{\mu\beta}+\eta_{\alpha\mu}\epsilon_{\rho\beta}+\eta_{\beta\mu}\epsilon_{\rho\alpha}+\eta_{\beta\rho}\epsilon_{\mu\alpha}\right).\end{aligned} \tag{E.10}$$

Using (E.10) we can compute (E.8) and we find the parity-odd part

$$\begin{aligned} &\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle_{\rm odd} = \\ &-\frac{1}{4\pi^2}\left(\frac{\epsilon_{\mu\alpha}x^\alpha x_\nu x_\rho x_\sigma}{x^8}+\frac{\epsilon_{\nu\alpha}x^\alpha x_\mu x_\rho x_\sigma}{x^8}+\frac{\epsilon_{\rho\alpha}x^\alpha x_\mu x_\nu x_\sigma}{x^8}+\frac{\epsilon_{\sigma\alpha}x^\alpha x_\mu x_\nu x_\rho}{x^8}\right.\\ &\qquad -\frac{\epsilon_{\mu\alpha}\eta_{\rho\nu}x^\alpha x_\sigma}{4x^6}-\frac{\epsilon_{\mu\alpha}\eta_{\sigma\nu}x^\alpha x_\rho}{4x^6}-\frac{\epsilon_{\nu\alpha}\eta_{\rho\mu}x^\alpha x_\sigma}{4x^6}-\frac{\epsilon_{\nu\alpha}\eta_{\sigma\mu}x^\alpha x_\rho}{4x^6}\\ &\qquad \left.-\frac{\epsilon_{\rho\alpha}\eta_{\mu\sigma}x^\alpha x_\nu}{4x^6}-\frac{\epsilon_{\rho\alpha}\eta_{\nu\sigma}x^\alpha x_\mu}{4x^6}-\frac{\epsilon_{\sigma\alpha}\eta_{\mu\rho}x^\alpha x_\nu}{4x^6}-\frac{\epsilon_{\sigma\alpha}\eta_{\nu\rho}x^\alpha x_\mu}{4x^6}\right). \end{aligned} \tag{E.11}$$

As a matter of fact, out of this computation we find that the parity-even part matches (5.1) with $c = 1/4\pi^2$, in agreement with [3, 4]. The expression (E.11) is traceless, conserved and can be written as

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle_{\rm odd} = \frac{1}{32\pi^2}\left(\epsilon_{\alpha\mu}T^\alpha{}_{\nu\rho\sigma}+\epsilon_{\alpha\nu}T_\mu{}^\alpha{}_{\rho\sigma}+\epsilon_{\alpha\rho}T_{\mu\nu}{}^\alpha{}_\sigma+\epsilon_{\alpha\rho}T_{\mu\nu\rho}{}^\alpha\right), \tag{E.12}$$

where $T_{\mu\nu\rho\sigma}$ is given by the expression (5.23). Hence (E.11) agrees with the null cone result.

E.2 Regularization formulas in $2d$ and $4d$

In this appendix we collect the regularized integrals that are needed to evaluate the Feynman diagrams in the text both in $2d$ and $4d$. The integrals below are *Euclidean integrals*. They are an intermediate results needed in order to compute the Feynman diagrams in the text. Since the starting points and the final results are Lorentzian, it is understood that one has to do the appropriate Wick rotations in order to be able to use them.

In $2d$, after introducing δ extra dimensions in the internal momentum and a Feynman parameter u ($0 \le u \le 1$), in the limit $\delta \to 0$, we have

$$\begin{aligned} \int\frac{d^2p}{(2\pi)^2}\int\frac{d^\delta\ell}{(2\pi)^\delta}\frac{\ell^2}{(p^2+\ell^2+\Delta)^2} &= -\frac{1}{4\pi}\\ \int\frac{d^2p}{(2\pi)^2}\int\frac{d^\delta\ell}{(2\pi)^\delta}\frac{\ell^2p^2}{(p^2+\ell^2+\Delta)^2} &= \frac{1}{4\pi}\Delta \end{aligned} \tag{E.13}$$

and

$$\begin{aligned} \int\frac{d^2p}{(2\pi)^2}\int\frac{d^\delta\ell}{(2\pi)^\delta}\frac{p^2}{(p^2+\ell^2+\Delta)^2} &= \frac{1}{4\pi}\frac{1}{\Delta}\\ \int\frac{d^2p}{(2\pi)^2}\int\frac{d^\delta\ell}{(2\pi)^\delta}\frac{p^2}{(p^2+\ell^2+\Delta)^2} &= \frac{1}{4\pi}\left(-\frac{2}{\delta}-\gamma+\log(4\pi)-\log\Delta\right)\\ \int\frac{d^2p}{(2\pi)^2}\int\frac{d^\delta\ell}{(2\pi)^\delta}\frac{p^4}{(p^2+\ell^2+\Delta)^2} &= \frac{1}{2\pi}\Delta\left(\frac{2}{\delta}-1+\gamma-\log(4\pi)+\log\Delta\right) \end{aligned} \tag{E.14}$$

where $\Delta = u(1-u)k^2$.

Proceeding in the same way in $4d$, with two Feynman parameters u and v, in the limit $\delta \to 0$, beside (7.25), we find

$$\begin{aligned} \int\frac{d^4p}{(2\pi)^4}\int\frac{d^\delta\ell}{(2\pi)^\delta}\frac{p^2}{(p^2+\ell^2+\Delta)^3} &= \frac{1}{(4\pi)^2}\left(-\frac{2}{\delta}-\gamma+\log(4\pi)-\log\Delta\right)\\ \int\frac{d^4p}{(2\pi)^4}\int\frac{d^\delta\ell}{(2\pi)^\delta}\frac{p^4}{(p^2+\ell^2+\Delta)^3} &= \frac{\Delta}{2(4\pi)^2}\left(-\frac{2}{\delta}-\gamma+4+\log(4\pi)-\log\Delta\right) \end{aligned} \tag{E.15}$$

and

$$\int \frac{d^4p}{(2\pi)^4} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{\ell^2}{(p^2+\ell^2+\Delta)^3} = -\frac{1}{2(4\pi)^2}$$
$$\int \frac{d^4p}{(2\pi)^4} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{\ell^2 p^2}{(p^2+\ell^2+\Delta)^3} = \frac{1}{(4\pi)^2}\Delta \tag{E.16}$$

where $\Delta = u(1-u)k_1 + v(1-v)k_2 + 2uv\, k_1 k_2$.

Chapter 6

Trace anomalies in chiral theories

In this chapter, motivated by the search for possible CP violating terms in the trace of the energy-momentum tensor in theories coupled to gravity we revisit the problem of trace anomalies in chiral theories. We recalculate the latter and ascertain that in the trace of the energy-momentum tensor of theories with chiral fermions at one-loop the Pontryagin density appears with an imaginary coefficient. We argue that this may break unitarity, in which case the trace anomaly has to be used as a selective criterion for theories, analogous to the chiral anomalies in gauge theories. We analyze some remarkable consequences of this fact, that seem to have been overlooked in the literature.

6.1 One-loop trace anomaly in chiral theories

The model we will consider is the simplest possible one: a right-handed spinor coupled to external gravity in $4d$. The action is

$$S = \int d^4x \sqrt{|g|}\, i\overline{\psi_R}\gamma^m \left(\nabla_m + \frac{1}{2}\omega_m\right)\psi_R, \tag{6.1}$$

where $\gamma^m = e_a^m\gamma^a$, ∇ ($m, n, \dots$ are world indices, $a, b, \dots$ are flat indices) is the covariant derivative with respect to the world indices and ω_m is the spin connection:

$$\omega_m = \omega_m^{ab}\Sigma_{ab}$$

where $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ are the Lorentz generators. Finally $\psi_R = \frac{1+\gamma_5}{2}\psi$. Classically the energy-momentum tensor

$$T_{\mu\nu} = \frac{i}{4}\overline{\psi_R}\gamma_\mu \overset{\leftrightarrow}{\nabla}_\nu \psi_R + (\mu \leftrightarrow \nu) \tag{6.2}$$

is both conserved and traceless on-shell. At one-loop to make sense of the calculations one must introduce regulators. The latter generally breaks both diffeomorphism and conformal invariance. A careful choice of the regularization procedure may preserve diff invariance, but anyhow breaks conformal invariance, so that the trace of the e.m. tensor takes the form (3.54), with specific non-vanishing coefficients a, c and e. There are various techniques to calculate the latter: cutoff, point splitting, Pauli-Villars, dimensional regularization and etc. Here we would like to briefly recall the heat kernel method utilized in [21] and in references cited therein. Denoting by D the relevant Dirac operator in (6.1) one can show that

$$\delta W = -\int d^4x \sqrt{|g|}\sigma \langle T_\mu^\mu(x)\rangle_g = -\frac{1}{16\pi^2}\int d^4x \sqrt{|g|}\sigma b_4\left(x, x; D^\dagger D\right).$$

Thus

$$\langle T^{\mu}_{\mu}(x)\rangle_g = \frac{1}{16\pi^2} b_4\left(x,x;D^\dagger D\right) \tag{6.3}$$

The coefficient $b_4\left(x,x;D^\dagger D\right)$ appears in the heat kernel. The latter has the general form

$$K\left(t,x,y;\mathcal{D}\right) \sim \frac{1}{\left(4\pi t\right)^2} e^{-\frac{\sigma(x,y)}{2t}} \left(1 + t b_2\left(x,y;\mathcal{D}\right) + t^2 b_4\left(x,y;\mathcal{D}\right) + \cdots\right),$$

where $\mathcal{D} = D^\dagger D$ and $\sigma\left(x,y\right)$ is the half square length of the geodesic connecting x and y, so that $\sigma\left(x,x\right) = 0$. For coincident points we therefore have

$$K\left(t,x,x;\mathcal{D}\right) \sim \frac{1}{16\pi^2}\left(\frac{1}{t^2} + \frac{1}{t} b_2\left(x,x;\mathcal{D}\right) + b_4\left(x,x;\mathcal{D}\right) + \cdots\right). \tag{6.4}$$

This expression is divergent for $t \to 0$ and needs to be regularized. This can be done in various ways. The finite part, which we are interested in, has been calculated first by DeWitt, [57], and then by others with different methods. The results are reported in [21]. For a spin $\frac{1}{2}$ right-handed spinor as in our example one gets

$$b_4\left(x,x;D^\dagger D\right) = \frac{1}{180}\left(a\,E_4 + c\,W^2 + e\,P\right), \tag{6.5}$$

with

$$a = -\frac{11}{4}, \qquad c = \frac{9}{2}, \qquad e = \frac{15}{4}. \tag{6.6}$$

This result was obtained with an entirely Euclidean calculation. Coming back to Lorentzian signature the e.m. trace at one-loop is

$$\langle T^{\mu}_{\mu}\rangle_g = \frac{1}{180\times 16\pi^2}\left(-\frac{11}{4}\,E_4 + \frac{9}{2}\,W^2 + i\frac{15}{4}\,P\right). \tag{6.7}$$

As pointed out above the important aspect of (6.7) is the i appearing in front of the Pontryagin density. The origin of this imaginary coupling is easy to trace. It comes from the trace of gamma matrices including a γ_5 factor. In $4d$, while the trace of an even number of gamma matrices, which give rise to first two terms in the RHS of (6.7), is a real number, the trace of an even number of gamma's multiplied by γ_5 is always imaginary. The Pontryagin term comes precisely from the latter type of traces. It follows that, as a one loop effect, the energy momentum tensor becomes complex, and, in particular, since T^0_0 is the Hamiltonian density, we must conclude that unitarity may not be preserved in this type of theories. It is legitimate to ask whether, much like chiral gauge theories with non-vanishing chiral gauge anomalies are rejected as sick theories, also chiral models with complex trace anomalies are not acceptable theories. We will return to this point later on.

6.2 Other derivations of the Pontryagin trace anomaly

The derivation of the results in the previous section are essentially based on the method invented by DeWitt, [57], which is a point splitting method, the splitting distance being geodesic. As such, it guarantees covariance of the anomaly expression. To our surprise we have found that, while for the CP preserving part of the trace anomaly various methods of calculation are available in the literature, no other method is met to calculate the coefficient of the Pontryagin density. Given the important consequences of such a (imaginary) coefficient, we have decided to recalculate the results of the previous section with a different method, based on Feynman diagram techniques. We will use it in conjunction with dimensional regularization.

To start with from (6.1) we have to extract the Feynman rules.[1] More explicitly the action (6.1) can be written as

$$S = \int d^4x \sqrt{|g|} \left[\frac{i}{2} \overline{\psi_R} \gamma^\mu \overleftrightarrow{\partial}_\mu \psi_R - \frac{1}{4} \epsilon^{\mu abc} \omega_{\mu ab} \overline{\psi_R} \gamma_c \gamma_5 \psi_R \right] \tag{6.8}$$

where it is understood that the derivative applies to ψ_R and $\overline{\psi_R}$ only. We have used the relation $\{\gamma^a, \Sigma^{bc}\} = i\,\epsilon^{abcd}\gamma_d\gamma_5$. Now we expand

$$e^a_\mu = \delta^a_\mu + \chi^a_\mu + \cdots, \quad e^\mu_a = \delta^\mu_a + \hat{\chi}^\mu_a + \cdots, \quad \text{and} \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \cdots \tag{6.9}$$

Inserting these expansions in the defining relations $e^a_\mu e^\mu_b = \delta^a_b$, $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, we find

$$\hat{\chi}^\mu_\nu = -\chi^\mu_\nu \quad \text{and} \quad h_{\mu\nu} = 2\,\chi_{\mu\nu}. \tag{6.10}$$

From now on we will use both χ^a_μ and $h_{\mu\nu}$, since we are interested in the lowest order contribution, we will raise and lower the indices *only* with δ. We will not need to pay attention to the distinction between flat and world indices. Let us expand accordingly the spin connection. Using

$$\omega_{\mu ab} = e_{\nu a}(\partial_\mu e^\nu_b + e^\sigma{}_b \Gamma_\sigma{}^\nu{}_\mu) \quad \text{and} \quad \Gamma_\sigma{}^\nu{}_\mu = \frac{1}{2}\eta^{\nu\lambda}(\partial_\sigma h_{\lambda\mu} + \partial_\mu h_{\lambda\sigma} - \partial_\lambda h_{\sigma\mu}) + \cdots,$$

after some algebra we get

$$\omega_{\mu ab}\,\epsilon^{\mu abc} = -\epsilon^{\mu abc}\,\partial_\mu \chi_{a\lambda}\,\chi^\lambda_b + \cdots. \tag{6.11}$$

For later use let us quote the following approximation for the Pontryagin density

$$\epsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau} = 8\epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma\chi^a_\nu\,\partial_\lambda\partial_a\chi^\sigma_\rho - \partial_\mu\partial_\sigma\chi^a_\nu\,\partial_\lambda\partial^\sigma\chi_{a\rho}\right) + \cdots \tag{6.12}$$

Therefore, up to second order the action can be written (by incorporating $\sqrt{|g|}$ in a redefinition of the ψ field [2])

$$S \approx \int d^4x \left[\frac{i}{2}(\delta^\mu_a - \chi^\mu_a)\overline{\psi_R}\gamma^a \overleftrightarrow{\partial}_\mu \psi_R + \frac{1}{4}\epsilon^{\mu abc}\,\partial_\mu\chi_{a\lambda}\,\chi^\lambda_b\,\overline{\psi}_R\gamma_c\gamma_5\psi_R \right]$$

The free action is

$$S_{free} = \int d^4x\, \frac{i}{2}\overline{\psi_R}\gamma^a \overleftrightarrow{\partial}_a \psi_R \tag{6.13}$$

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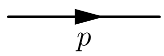
[1] We follow closely the derivation of the chiral anomaly in [58, 59], although with a different regularization. For other derivations of this anomaly see also [60, 61].

[2] This is the simplest way to deal with $\sqrt{|g|}$. Alternatively one can keep it explicitly in the action and approximate it as $1 + \frac{1}{2}h^\mu_\mu$; this would produce two additional vertices, which however do not contribute to our final result.

and the lowest interaction terms are

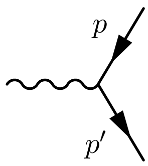
$$
\begin{aligned}
S_{int} &= \int d^4x \left[-\frac{i}{2}\chi_a^\mu \overline{\psi_R}\gamma^a \overset{\leftrightarrow}{\partial}_\mu \psi_R + \frac{1}{4}\epsilon^{\mu abc}\,\partial_\mu \chi_{a\lambda}\,\chi_b^\lambda\,\overline{\psi}_R\gamma_c\gamma_5\psi_R \right] \\
&= \int d^4x \left[-\frac{i}{4}h_a^\mu \overline{\psi_R}\gamma^a \overset{\leftrightarrow}{\partial}_\mu \psi_R + \frac{1}{16}\epsilon^{\mu abc}\,\partial_\mu h_{a\lambda}\,h_b^\lambda\,\overline{\psi}_R\gamma_c\gamma_5\psi_R \right]
\end{aligned}
\tag{6.14}
$$

As a consequence of (6.13) and (6.14) the Feynman rules are as follows (the external gravitational field is assumed to be $h_{\mu\nu}$). The fermion propagator is



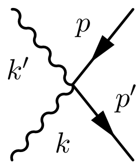
$$
= \frac{i}{\not p + i\epsilon}. \tag{6.15}
$$

The two-fermion-one-graviton vertex (V_{ffg}) is



$$
= \frac{i}{8}\left[(p+p')_\mu \gamma_\nu + (p+p')_\nu \gamma_\mu \right] \frac{1+\gamma_5}{2}. \tag{6.16}
$$

The two-fermion-two-graviton vertex (V_{ffgg}) is



$$
= \frac{1}{64} t_{\mu\nu\mu'\nu'\kappa\lambda}\,(k-k')^\lambda\,\gamma^\kappa \frac{1+\gamma_5}{2}, \tag{6.17}
$$

where the momenta of the gravitons are ingoing and

$$
t_{\mu\nu\mu'\nu'\kappa\lambda} = \eta_{\mu\mu'}\epsilon_{\nu\nu'\kappa\lambda} + \eta_{\nu\nu'}\epsilon_{\mu\mu'\kappa\lambda} + \eta_{\mu\nu'}\epsilon_{\nu\mu'\kappa\lambda} + \eta_{\nu\mu'}\epsilon_{\mu\nu'\kappa\lambda}. \tag{6.18}
$$

Due to the non-polynomial character of the action the diagrams contributing to the trace anomaly are infinitely many. Fortunately, using diffeomorphism invariance, it is enough to determine the lowest order contributions and consistency takes care of the rest. There are two potential lowest order diagrams (see figures F.1 and F.2 in the appendices F.1 and F.2) that may contribute. The first contribution, the bubble graph, turns out to vanish, see appendix F.1. It remains for us to calculate the triangle graph. To limit the size and number of formulas in the sequel *we will be concerned only with the contribution of the diagrams to the Pontryagin density.*

6.2.1 The fermion triangle diagram

It is constructed by joining three vertices V_{ffg} with three fermion lines. The external momenta are q (ingoing) with labels σ and τ, and k_1, k_2 (outgoing), with labels μ, ν and μ', ν' respectively. Of course $q = k_1 + k_2$. The internal momenta are p, $p - k_1$ and $p - k_1 - k_2$, respectively. After contracting σ and τ the total contribution is

$$-\frac{1}{256}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\left(\frac{1}{\not p}\left((2p-k_1)_\mu\gamma_\nu+(\mu\leftrightarrow\nu)\right)\frac{1}{\not p-\not k_1}\right.\right.$$
$$\left.\left.\times\left((2p-2k_1-k_2)_{\mu'}\gamma_{\nu'}+(\mu'\leftrightarrow\nu')\right)\frac{1}{\not p-\not k_1-\not k_2}(2\not p-\not k_1-\not k_2)\right)\frac{1+\gamma_5}{2}\right] \quad (6.19)$$

to which we have to add the cross diagram in which k_1,μ,ν is exchanged with k_2,μ',ν'. This integral is divergent. To regularize it we use dimensional regularization. To this end we introduce additional components of the momentum running on the loop (for details see, for instance, [62]): $p\to p+\ell$, $\ell=(\ell_4,\ldots,\ell_{n-4})$

$$T_{\mu\nu\mu'\nu'}(k_1,k_2)=-\frac{1}{256}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^{n-4}\ell}{(2\pi)^{n-4}}\,\mathrm{tr}\left(\frac{\not p+\not\ell}{p^2-\ell^2}(2p+2\ell-k_1)_\mu\gamma_\nu\right.$$
$$\left.\times\frac{\not p+\not\ell-\not k_1}{(p-k_1)^2-\ell^2}(2p+2\ell-2k_1-k_2)_{\mu'}\gamma_{\nu'}\frac{\not p+\not\ell-\not q}{(p-q)^2-\ell^2}(2\not p+2\not\ell-\not q)\frac{1+\gamma_5}{2}\right) \quad (6.20)$$

where the symmetrization over μ,ν and μ',ν' has been understood.[3] After some manipulations this becomes

$$T_{\mu\nu\mu'\nu'}(k_1,k_2)=T^{(1)}_{\mu\nu\mu'\nu'}(k_1,k_2)+T^{(2)}_{\mu\nu\mu'\nu'}(k_1,k_2)$$
$$-\frac{1}{256}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^{n-4}\ell}{(2\pi)^{n-4}}\mathrm{tr}\left(\frac{\not p+\not\ell}{p^2-\ell^2}(2p+2\ell-k_1)_\mu\gamma_\nu\right.$$
$$\left.\times\frac{\not p+\not\ell-\not k_1}{(p-k_1)^2-\ell^2}(2p+2\ell-2k_1-k_2)_{\mu'}\gamma_{\nu'}\frac{\not p+\not\ell-\not q}{(p-q)^2-\ell^2}\not\ell\gamma_5\right) \quad (6.21)$$

The terms $T^{(1)},T^{(2)}$ turn out to vanish. The rest, after a Wick rotation (see appendix F.2), gives

$$T_{\mu\nu\mu'\nu'}(k_1,k_2)=\frac{1}{6144\pi^2}\left(k_1\cdot k_2\,t_{\mu\nu\mu'\nu'\lambda\rho}-t^{(21)}_{\mu\nu\mu'\nu'\lambda\rho}\right)k_1^\lambda k_2^\rho \quad (6.22)$$

where

$$t^{(21)}_{\mu\nu\mu'\nu'\kappa\lambda}=k_{2\mu}k_{1\mu'}\epsilon_{\nu\nu'\kappa\lambda}+k_{2\nu}k_{1\nu'}\epsilon_{\mu\mu'\kappa\lambda}+k_{2\mu}k_{1\nu'}\epsilon_{\nu\mu'\kappa\lambda}+k_{2\nu}k_{1\mu'}\epsilon_{\mu\nu'\kappa\lambda} \quad (6.23)$$

Finally we have to add the cross graph contribution, obtained by $k_1,\mu,\nu\leftrightarrow k_2,\mu',\nu'$. Under this exchange the t tensors transform as follows:

$$t\leftrightarrow -t,\qquad t^{(21)}\leftrightarrow -t^{(21)},\quad i\neq j \quad (6.24)$$

Therefore the cross graph gives the same contribution as (6.22). So for the triangle diagram we get

$$T^{(tot)}_{\mu\nu\mu'\nu'}(k_1,k_2)=\frac{1}{3072\pi^2}\left(k_1\cdot k_2\,t_{\mu\nu\mu'\nu'\lambda\rho}-t^{(21)}_{\mu\nu\mu'\nu'\lambda\rho}\right)k_1^\lambda k_2^\rho \quad (6.25)$$

[3] Some attention has to be paid in introducing the additional momentum components ℓ. Due to the chiral projectors in the V_{ffg} vertex it would seem that $\not\ell$ should not be present in the first and third terms in (6.20) (because $[\not\ell,\gamma_5]=0$); however this regularization prescription would give a wrong result for the CP even part of the anomaly. The right prescription is (6.20).

To obtain the above results we have set the external lines on-shell. This deserves a comment.

6.2.2 On-shell conditions

Putting the external lines on-shell means that the corresponding fields have to satisfy the EOM of gravity $R_{\mu\nu} = 0$. In the linearized form this means

$$\Box\chi_{\mu\nu} = \partial_\mu\partial_\lambda\chi^\lambda_\nu + \partial_\nu\partial_\lambda\chi^\lambda_\mu - \partial_\mu\partial_\nu\chi^\lambda_\lambda \tag{6.26}$$

We also choose the de Donder gauge

$$\Gamma^\lambda_{\mu\nu}g^{\mu\nu} = 0 \tag{6.27}$$

which at the linearized level becomes

$$2\partial_\mu\chi^\mu_\lambda - \partial_\lambda\chi^\mu_\mu = 0 \tag{6.28}$$

In this gauge (6.26) becomes

$$\Box\chi_{\mu\nu} = 0 \tag{6.29}$$

In momentum space this implies that $k_1^2 = k_2^2 = 0$. Since we know that the final result is covariant this simplification does not jeopardize it.

6.2.3 Overall contribution

The overall one-loop contribution to the trace anomaly in momentum space, *as far as the CP violating part is concerned*, is given by (6.25). After returning to the Minkowski metric and Fourier-antitransforming this we can extract the local expression of the trace anomaly, as explained in appendix F.3. The saturation with $h^{\mu\nu}, h^{\mu'\nu'}$ brings a multiplication by 4 of the anomaly coefficient. The result is, to lowest order,

$$\langle T^\mu_\mu(x)\rangle_g = \frac{i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma h^\tau_\nu\,\partial_\lambda\partial_\tau h^\sigma_\rho - \partial_\mu\partial_\sigma h^\tau_\nu\,\partial_\lambda\partial^\sigma h_{\tau\rho}\right) \tag{6.30}$$

Comparing with (7.50) we deduce the covariant expression of the CP violating part of the trace anomaly

$$\langle T^\mu_\mu(x)\rangle_g = \frac{i}{768\pi^2}\,\frac{1}{2}\,\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau} \tag{6.31}$$

which is the same as (6.7).

6.3 Consequences of the Pontryagin trace anomaly in chiral theories

In this section we would like to expand on the consequences of a non-vanishing Pontryagin term in the trace anomaly. To start with let us spend a few words on a misconception we sometime come across: the gravitational charge of matter is its mass and, as a consequence, gravity interacts with matter via its mass. This would imply in particular that massless particles do not feel gravity, which is clearly false (e.g., the photon). The point is that gravity interacts with matter via its energy-momentum tensor. In particular, for what concerns us here, the e.m. tensor is different for left-handed and right-handed massless matter, and this is the origin of a different trace anomaly for them.

As we have already noticed in 6.1, in theories with a chiral unbalance, as a consequence of the Pontryagin trace anomaly, the energy momentum tensor becomes complex, and, in particular, unitarity is not preserved. This raises a question: much like chiral gauge theories with non-vanishing chiral gauge anomalies are rejected as unfit theories, should we conclude also that chiral models with complex trace anomalies are not acceptable theories? To answer this question it is important to put it in the right framework. To start with let us consider the example of the standard model. In its pre-neutrino-mass-discovery period its spectrum was usually written as follows:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \widehat{u_R}, \quad \widehat{d_R}, \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \widehat{e_R} \tag{6.32}$$

together with two analogous families (here and in the sequel, for any fermion field ψ, $\hat{\psi} = \gamma^0 C\psi^*$, where C is the charge conjugation matrix, i.e. $\hat{\psi}$ represents the Lorentz covariant conjugate field). All the fields are Weyl spinors and a hat represents CP conjugation. If a field is right-handed its CP conjugate is left-handed. Thus all the fields in (6.32) are left-handed. This is the well-known chiral formulation of the SM. So we could represent the entire family as a unique left-handed spinor ψ_L and write the kinetic part of the action as in (6.1). However the coupling to gravity of a CP conjugate field is better described as follows (see, for instance, [41]). First, for a generic spinor field ψ, let us define (with $L = \frac{1-\gamma_5}{2}$, $R = \frac{1+\gamma_5}{2}$, and $\psi_L = L\psi, \psi_R = R\psi$)

$$\widehat{\psi_R} = \gamma^0 C\psi_R^* = \gamma^0 C R^* \psi^* = L\gamma^0 C\psi^* = L\hat{\psi} = \hat{\psi}_L \tag{6.33}$$

where we have used the properties of the gamma matrices and the charge conjugation matrix C:

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T, \qquad CC^\dagger = 1, \qquad CC^* = -1, \qquad C^T = -C$$

and in particular $C^{-1}\gamma_5 C = \gamma_5^T$. Let us stress in (6.33) the difference implied by the use of $\widehat{\ }$ and $\hat{\ }$, respectively.

With the help of these properties one can easily show that

$$\sqrt{|g|}\,\overline{\hat{\psi}_L}\,\gamma^m\left(\nabla_m + \frac{1}{2}\omega_m\right)\hat{\psi}_L = \sqrt{|g|}\,\overline{\widehat{\psi_R}}\,\gamma^m\left(\nabla_m + \frac{1}{2}\omega_m\right)\widehat{\psi_R}$$
$$= \sqrt{|g|}\,\psi_R^T C^{-1}\gamma^m\gamma^0\left(\nabla_m + \frac{1}{2}\omega_m\right)C\psi_L^*$$

which, after a partial integration and an overall transposition, becomes

$$\sqrt{|g|}\,\overline{\psi_R}\,\gamma^m\left(\nabla_m + \frac{1}{2}\omega_m\right)\psi_R \tag{6.34}$$

i.e. the right-handed companion of the initial left-handed action. This follows in particular from the property $C^{-1}\Sigma_{ab}C = -\Sigma_{ab}^T$.

From the above we see that in the multiplet (6.32) there is a balance between the left-handed and right-handed field components except for the left-handed field ν_e. Therefore the multiplet (6.32) when weakly coupled to gravity, will produce an overall non-vanishing (imaginary) coefficient e for the Pontryagin density in the trace anomaly and, in general, a breakdown of unitarity (this argument must be seen in the context of the discussion in the following section). This breakdown is naturally avoided if we add to the SM multiplet a right-handed neutrino field, because in that case the balance of chirality is perfect. Another possibility is that the unique neutrino field in the multiplet be Majorana, because a Majorana fermion can be viewed as a superposition of a left-handed and a right-handed Weyl spinor, with the additional condition of reality, and, therefore its contribution to the Pontryagin density is null. In both cases the neutrino can have mass.

In hindsight this could have been an argument in favor of massive neutrinos.

From a certain point of view what we have just said may sound puzzling because it is often stated that in 4D massless Weyl and Majorana fermions are physically indistinguishable: they have the same number of components and we can define a one-to-one correspondence between the latter. A theory of Majorana fermions cannot have the kind of (chiral) anomaly we have found. So where does our anomaly comes from? It is therefore necessary to spend some time recalling the crucial difference between Weyl and Majorana fields in 4D. To start with, the map between Majorana and Weyl fields mentioned above is not representable by means of a linear invertible operator and this fact radically changes the way they transform under Lorentz transformations. Majorana fields transform as real representations and Weyl fields as complex representations of the Lorentz group. As a consequence, the relevant Dirac operators are different. Now, when we compute anomalies using the path integral we have to integrate over fields, not over particles. Therefore anomalies are determined by the field content of the theory and by the appropriate Dirac operator. On the other hand anomalies like our Pontryagin anomaly (and many others) are not physical objects, but defects of the theory. Thus what we are saying is: if we want to formulate a theory with a different number of left-handed and right-handed Weyl fields, we are bound to find a dangerous anomaly in the trace of the em tensor. This does not prevent us from constructing a theory with the same physical content in an another way, which is anomaly-free, by using Majorana fields. But the path integrals of the two theories are not coincident. This, in turn, is connected with a related question: it is well known that, by means of mere algebraic manipulations, we can rewrite the kinetic action term of a Weyl field as the kinetic term of the corresponding Majorana field. So at first sight that seems to be no difference between the two. But this conclusion would forget that the transformation from Weyl to Majorana fields is not linear and invertible, so that one must take into account the Jacobian in the path integral. This is hard to compute directly, but what we have stressed in this paper is that it manifests itself (at least) in the Pontryagin anomaly.

6.4 Discussion and conclusion

The main point of this paper is a reassessment of the role of trace anomalies in theories with chiral matter coupled to gravity. In particular we have explicitly calculated the trace anomaly for a chiral fermion. The result is the expected one on the basis of the existing literature, except for the fact that, in our opinion, it had never been explicitly stated before (save for a footnote in [63, 64]), and, especially, its consequences had never been seriously considered. As we have seen, for chiral matter the trace anomaly at one-loop contains the Pontryagin density P with an imaginary coefficient. This implies, in particular, that the Hamiltonian density becomes complex and breaks unitarity. This poses the problem of whether this anomaly is on the same footing as chiral gauge anomalies in a chiral theory, which, when present, spoil its consistency. It is rightly stressed that the standard model is free of any chiral anomaly, including the gravitational ones. But in the case of ordinary chiral gauge anomalies the gauge fields propagate and drag the inconsistency in the internal loops, while in gravitational anomalies (including our trace anomaly) gravity is treated as a background field. So, do the latter have the same status as chiral gauge anomalies?

Let us analyse the question by asking: are there cases in which the Pontryagin density vanishes identically? The answer is: yes, there are background geometries where the Pontryagin density vanishes. They include for instance the FRW and Schwarzschild [65]. Therefore, in such backgrounds the problem of unitarity simply does not exist. But the previous ones are very special 'macroscopic' geometries. For a generic geometry the Pontryagin density does not vanish. For instance in a cosmological framework, we can imagine to go up to higher energies where gravity inevitably back-reacts. In this case it does not seem to be possible to avoid the conclusion that the Pontryagin density does not vanish and unitarity is affected due to the trace anomaly, the more so because gravitational loops cannot cancel it. Thus, seen in this more general context, the breakdown of unitarity due to a chirality unbalance in an asymptotically free matter theory should be seriously taken into account.

Returning now to the problem we started with in the introduction, that is the appearance of a CP violating Pontryagin density in the trace of the energy-momentum tensor, we conclude that unitarity seems to prevent it at one-loop, and we cannot imagine a mechanism that may produce it at higher loops. In [63, 64] a holographic model was presented which yields a Pontryagin density in the trace of the e.m. tensor, but again with a unitarity problem [64]. Anyhow it would be helpful to understand its (very likely, non-perturbative) origin in the boundary theory. This mechanism for CP violation is very interesting and, above, we have seen another attractive aspect of it: its effect evaporates automatically while the universe evolves towards 'simpler' geometries.

A final comment about supersymmetry. In a previous paper, [66], the compatibility between the appearance of the Pontryagin term in the trace anomaly and supersymmetry was considered and evidence was produced that they are not compatible. Altogether this and the results of this paper point towards the need for a theory which is neither chiral nor supersymmetric, if we wish to see the Pontryagin density with a real coefficient appear in the trace of the energy-momentum tensor. How this may actually be realized, as suggested in [63, 64], is still an open and intriguing problem.

Appendix F

Details of computations

F.1 Calculation details: the bubble diagram

In this appendix we give a few details of the calculations in section 6.2. Let us consider first the bubble graph (see figure F.1). It is obtained by joining two vertices, V_{ffg} (on the left)

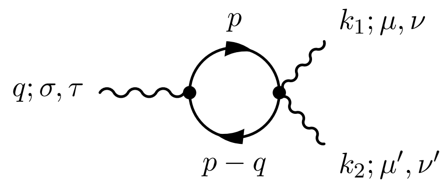


FIGURE F.1: Bubble diagram with ingoing momentum q and outgoing k_1 and k_2.

and V_{ffgg} (on the right) with two fermion propagators. The ingoing graviton in V_{ffg} has momentum q and Lorentz labels σ, τ and the two outgoing gravitons in V_{ffgg} are specified by k_1, μ, ν and k_2, μ', ν', respectively. Of course $q = k_1 + k_2$. The two fermion propagators form a loop. The running momentum is clockwise oriented. We denote the momentum in the upper branch of the loop by p and in the lower branch by $p - q$. This diagram is

$$2 \times \frac{i}{512} \int \frac{d^4 p}{(2\pi)^4} \operatorname{tr} \left[\frac{1}{\not p} t_{\mu\nu\mu'\nu'\lambda\rho} (k_2 - k_1)^\lambda \gamma^\rho \frac{1}{\not p - \not q} \left((2p^\sigma - q^\sigma) \gamma^\tau + (\sigma \leftrightarrow \tau) \right) \frac{1+\gamma_5}{2} \right] \tag{F.1}$$

The factor of two in front of it comes from the combinatorics of diagrams: this one must contributes twice. Its possible contribution to the trace anomaly comes from contracting the indices σ and τ with a Kronecker delta (in principle we should consider contracting also the other couple of indices μ, ν and μ', ν', but this gives zero due to the symmetry properties of the t tensor). The integral is divergent and needs to be regularized. We use dimensional regularization. To this end we introduce additional components of the momentum running on the loop: $p \to p + \ell$, $\ell = (\ell_4, \ldots, \ell_{n-4})$. The relevant integral becomes

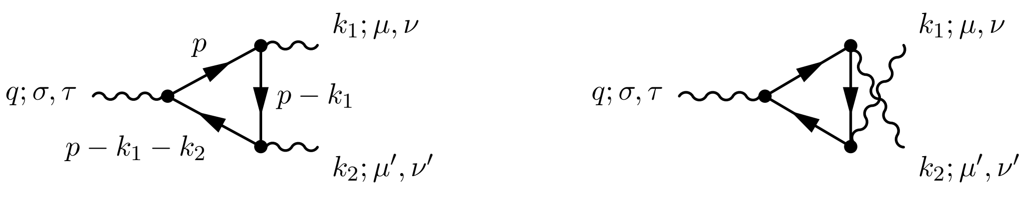
$$\begin{aligned} D_{\mu\nu\mu'\nu'}(k_1, k_2) = & \frac{i}{256} \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^{n-4}\ell}{(2\pi)^{n-4}} t_{\mu\nu\mu'\nu'\lambda\rho} (k_2 - k_1)^\lambda \\ & \times \operatorname{tr} \left(\frac{\not p + \not \ell}{p^2 - \ell^2} \gamma^\rho \frac{\not p - \not q + \not \ell}{(p-q)^2 - \ell^2} (2\not p + 2\not \ell - \not q) \right) \end{aligned} \tag{F.2}$$

After some algebra and introducing a parametric representation for the denominators, one finally gets

$$D_{\mu\nu\mu'\nu'}(k_1,k_2) = -\frac{i}{64}\, t_{\mu\nu\mu'\nu'\lambda\rho}(k_2-k_1)^{\lambda}\int_0^1 dx \int \frac{d^4p}{(2\pi)^4}\int \frac{d^{n-4}\ell}{(2\pi)^{n-4}}$$
$$\times\left[\left(\frac{3}{2}(2x-1)p^2 + x(x-1)(2x-1)q^2 - (2x-1)\ell^2\right) q^{\rho}\right]\frac{1}{(p^2+x(1-x)q^2-\ell^2)^2} \qquad \text{(F.3)}$$

This vanishes because of the x integration.

F.2 Calculation details: the triangle diagram



(A) Triangle diagram. (B) Crossed triangle diagram.

FIGURE F.2: In both these diagrams the momentum q is ingoing while the momenta k_1 and k_2 is outgoing.

As for the triangle diagram (see figure F.2), with reference to eq.(6.21), we have

$$T^{(1)}_{\mu\nu\mu'\nu'}(k_1,k_2) = -\frac{1}{256}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^{n-4}\ell}{(2\pi)^{n-4}}\mathrm{tr}\left(\frac{\not p+\not\ell}{p^2-\ell^2}(2p+2\ell-k_1)_{\mu}\gamma_{\nu}\right.$$
$$\left.\times\,\frac{\not p+\not\ell-\not k_1}{(p-k_1)^2-\ell^2}\,(2p+2\ell-2k_1-k_2)_{\mu'}\gamma_{\nu'}\frac{\gamma_5}{2}\right)$$
$$= -\frac{i}{256}\int_0^1 dx\int \frac{d^4p}{(2\pi)^4}\int \frac{d^{n-4}\ell}{(2\pi)^{n-4}}\epsilon_{\nu\nu'\lambda\rho}k_1^{\rho}\frac{p^2}{(p^2+x(1-x)k_1^2-\ell^2)^2}$$
$$\times\left(\delta^{\lambda}_{\mu}(2\ell-2xk_1-k_2)_{\mu'}+\delta^{\lambda}_{\mu'}(2\ell-2xk_1-k_2)_{\mu}\right), \qquad \text{(F.4)}$$

which evidently vanishes when we symmetrize μ with ν and μ' with ν'. $T^{(2)}$ is similar to $T^{(1)}$ and vanishes for the same reason. Setting $k_1^2=k_2^2=0$, the remaining term in (6.21) can be written

$$T_{\mu\nu\mu'\nu'}(k_1,k_2) = \frac{i}{32}\epsilon_{\nu\nu'\lambda\rho}k_1^{\lambda}k_2^{\rho}\int_0^1 dx\int_0^{1-x}dy\int \frac{d^4p}{(2\pi)^4}\int\frac{d^{n-4}\ell}{(2\pi)^{n-4}}$$
$$\times\,\ell^2\,\frac{p^2\eta_{\mu\mu'}+((2x+2y-1)k_1+2yk_2)_{\mu}(2(x+y-1)k_1+(2y-1)k_2)_{\mu'}}{(p^2-\ell^2+2y(1-x-y)k_1\cdot k_2)^3}. \qquad \text{(F.5)}$$

It involves two integrals over the momenta

$$\int \frac{d^4p}{(2\pi)^4} \int \frac{d^{n-4}\ell}{(2\pi)^{n-4}} \frac{\ell^2}{(p^2+x(1-x)q^2-\ell^2)^3} \overset{n\to 4}{=} \frac{i}{32\pi^2} \tag{F.6}$$

and

$$\int \frac{d^4p}{(2\pi)^4} \int \frac{d^{n-4}\ell}{(2\pi)^{n-4}} \frac{p^2\ell^2}{(p^2+x(1-x)q^2-\ell^2)^3} \overset{n\to 4}{=} \frac{i}{16\pi^2} 2y\,(1-x-y)\, k_1\!\cdot\! k_2. \tag{F.7}$$

Integration over x and y is elementary and one gets (6.22). Both RHS's are obtained by Wick-rotating all the momenta.

F.3 Local expression of the trace anomaly

In this paper we focus on the amplitude

$$\langle T^\sigma_\sigma(q)\, T_{\mu\nu}(k_1)\, T_{\mu'\nu'}(k_2)\rangle_c = \int d^4x\, d^4y\, d^4z\, e^{i(k_1x+k_2y-qz)} \langle T^\sigma_\sigma(z)\, T_{\mu\nu}(x)\, T_{\mu'\nu'}(y)\rangle_c \tag{F.8}$$

at one-loop order. On the basis of the previous discussion, the local expression of the anomaly is obtained by Fourier-antitransforming (6.25) and inserting it into the trace of (3.38). One relevant contribution is $t_{\mu\nu\mu'\nu'\lambda\rho}\, k_1^\lambda k_2^\rho\; k_1\cdot k_2\, \delta(q-k_1-k_2)$, from which

$$\begin{aligned} t_{\mu\nu\mu'\nu'\lambda\rho} \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, e^{-i(k_1x+k_2y-qz)}\, k_1^\lambda k_2^\rho\; k_1\cdot k_2\, \delta(q-k_1-k_2) \\ = t_{\mu\nu\mu'\nu'\lambda\rho}\partial_x^\lambda \partial_x^\tau \delta(x-z)\, \partial_y^\rho \partial_{y\tau}\delta(y-z) \end{aligned} \tag{F.9}$$

Inserting this into the trace of (3.38) we get

$$\begin{aligned} \langle T^\mu_\mu(z)\rangle_g^{(1)} &= t_{\mu\nu\mu'\nu'\lambda\rho}\int d^4x d^4y\, h^{\mu\nu}(x) h^{\mu'\nu'}(y) \partial_x^\lambda \partial_x^\tau \delta(x-z)\, \partial_y^\rho \partial_{y\tau}\delta(y-z) \\ &= 4\,\epsilon_{\nu\nu'\lambda\rho}\partial^\lambda\partial^\tau h^{\mu\nu}\, \partial^\rho\partial_\tau h^{\nu'}_\mu \end{aligned} \tag{F.10}$$

Another relevant contribution is given by (it comes from the term containing $t^{(21)}$)

$$\begin{aligned} k_{2\nu}k_{1\nu'}\, \epsilon_{\mu\mu'\lambda\rho}\, k_1^\lambda k_2^\rho\; \delta(q-k_1-k_2) \\ = \epsilon_{\mu\mu'\lambda\rho} \int d^4x d^4y d^4z\, e^{i(k_1x+k_2y-qz)}\, \partial_x^\lambda \partial_{x\nu'}\delta(x-z)\, \partial_y^\rho \partial_{y\nu}\delta(y-z) \end{aligned} \tag{F.11}$$

Inserting it into the trace of (3.38) we get

$$\langle T^\mu_\mu(z)\rangle^{(2)} = 4\,\epsilon_{\mu\mu'\lambda\rho}\,\partial^\lambda\partial_\tau h^{\mu\nu}\, \partial^\rho\partial_\nu h^{\mu'\tau} \tag{F.12}$$

This result is still Euclidean.

Chapter 7

Regularization of energy-momentum tensor correlators: 3-point functions

In this chapter we finally turn to the 3-point function of e.m. tensors in 4d, and concentrate on its parity-odd part. We derive in particular the regularized trace and divergence of the energy-momentum tensor in a chiral fermion model. We discuss the problems related to the parity-odd trace anomaly.

7.1 The 3-point correlator

The calculation of the 3-point correlator brings new elements into the game. First and foremost new (nontrivial) anomalies, but also an enormous complexity as compared to the 2-point correlator. In this section we first show that generically at non-coincident points the 3-point function of e.m. tensors in $4d$ does not possess a parity-odd contribution due to the permutation symmetry of the correlator. Then we compute the “bare” 3-point correlator by means of the Wick theorem in the same specific chiral fermionic model considered above, disregarding regularization. We find that, as expected, the parity-odd part identically vanishes. Subsequently we compute the same amplitude using Feynman diagrams and regularize it. It turns out that not only the parity-even but also the parity-odd trace of the e.m. tensor is nonvanishing. We will explain this apparent paradox in section 7.2.

7.1.1 No-go for parity-odd contributions

In this subsection we will review the fact that in four dimensions there are no parity-odd “bare” contributions to the 3-point function of energy-momentum tensors, which has already been emphasized in [7, 8].

A very powerful tool to analyse which tensorial structures can exist in a given correlation function in a CFT is the embedding formalism as it was formulated in [6]. In their language, to construct conformally covariant tensorial structures becomes a game of putting together building blocks respecting the tensorial requirements of your correlator. Particularly for the 3-point function of e.m. tensors we have seven building blocks. These building blocks are written in terms of points P_i of the six-dimensional embedding space and lightlike polarization vectors Z_i. Three of them depend on two points, namely

$$H_{12} = -2\left[(Z_1\cdot Z_2)\,(P_1\cdot P_2)-(Z_1\cdot P_2)\,(Z_2\cdot P_1)\right], \tag{7.1}$$
$$H_{23} = -2\left[(Z_2\cdot Z_3)\,(P_2\cdot P_3)-(Z_2\cdot P_3)\,(Z_3\cdot P_2)\right], \tag{7.2}$$
$$H_{13} = -2\left[(Z_1\cdot Z_3)\,(P_1\cdot P_3)-(Z_1\cdot P_3)\,(Z_3\cdot P_1)\right]. \tag{7.3}$$

Four of them depend on three points, three being parity-even, namely

$$V_1 = \frac{(Z_1 \cdot P_2)(P_1 \cdot P_3) - (Z_1 \cdot P_3)(P_1 \cdot P_2)}{P_2 \cdot P_3}, \tag{7.4}$$

$$V_2 = \frac{(Z_2 \cdot P_3)(P_2 \cdot P_1) - (Z_2 \cdot P_1)(P_2 \cdot P_3)}{P_3 \cdot P_1}, \tag{7.5}$$

$$V_3 = \frac{(Z_3 \cdot P_1)(P_3 \cdot P_2) - (Z_3 \cdot P_2)(P_3 \cdot P_1)}{P_1 \cdot P_2}, \tag{7.6}$$

while the last one is parity-odd, being the only object that one may construct with an epsilon tensor, i.e.

$$O_{123} = \epsilon(Z_1, Z_2, Z_3, P_1, P_2, P_3). \tag{7.7}$$

Our job now is to put together these objects to form a conformally covariant object with the tensorial structure of the 3-point function of e.m. tensors. Particularly, the objects that we will construct must present twice each polarization vector Z_i, since each Z_i is associated with one index of the i-th e.m. tensor. Since we are interested on parity-odd terms we will necessarily have the building block O_{123} which already takes care of one factor of each Z_i, thus it is clear that our only options are

$$T_1 = O_{123} V_1 V_2 V_3, \tag{7.8}$$

$$T_2 = O_{123}(V_1 H_{23} + V_2 H_{13} + V_3 H_{12}). \tag{7.9}$$

In the following we will show that both T_1 and T_2 are antisymmetric under the permutation of 1 and 2 for example, which forbids them to be present in the 3-point function of e.m. tensors. By inspection of the expressions (7.1)-(7.7) we see that under the exchange of 1 and 2 our building blocks change as follows:

$$\begin{aligned} H_{12} &\to H_{12}, \\ H_{23} &\to H_{13}, \\ H_{13} &\to H_{23}, \\ V_1 &\to -V_2, \\ V_2 &\to -V_1, \\ V_3 &\to -V_3, \\ O_{123} &\to O_{123}. \end{aligned}$$

From these rules it is clear that both T_1 and T_2 are antisymmetric under the exchange of 1 and 2. Of course the same result holds for the exchanges $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$.

7.1.2 The "bare" parity-odd 3-point correlator

Consider a free chiral fermion ψ_L in four dimensions which has the 2-point function[1]

$$\langle \psi_L(x) \overline{\psi_L}(y) \rangle = \frac{i}{2\pi^2} \frac{\gamma \cdot (x-y)}{(x-y)^4} P_L, \quad P_L = \frac{1-\gamma_5}{2}, \tag{7.10}$$

and the e.m. tensor

$$T_{\mu\nu} = \frac{i}{4}\left(\overline{\psi_L}\gamma_\mu \overset{\leftrightarrow}{\partial}_\nu \psi_L + \mu \leftrightarrow \nu\right), \text{ where } \overset{\leftrightarrow}{\partial}_\nu \equiv \partial_\nu - \overset{\leftarrow}{\partial}_\nu . \tag{7.11}$$

[1] The factor of $\frac{1}{2\pi^2}$ in the propagator of a fermion in $4d$ is needed in order for its Fourier-transform to give the usual propagator, namely $\frac{i}{\not p}$.

Since we are dealing with a free theory we are able to compute the 3-point function of e.m. tensors by applying the Wick theorem. Using the explicit form of the e.m. tensor (7.11) we write

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(y)\, T_{\alpha\beta}(z)\rangle = -\frac{i}{64}\left\langle :\overline{\psi}_L\gamma_\mu \overleftrightarrow{\partial}_\nu \psi_L:(x):\overline{\psi}_L\gamma_\rho \overleftrightarrow{\partial}_\sigma \psi_L:(y):\overline{\psi}_L\gamma_\alpha \overleftrightarrow{\partial}_\beta \psi_L:(z)\right\rangle$$
$$+\text{ symmetrization.} \tag{7.12}$$

There are two ways to fully contract these fields, as shown in equation (7.12). Each of the contractions is composed by a certain tensor with six indices $f^{(i)}_{\nu a\sigma b\beta c}$ contracted with a trace of six gamma matrices and a projector P_L, namely

$$f^{(1)}_{\nu a\sigma b\beta c}\mathrm{tr}\Big(\gamma_\mu\gamma^a\gamma_\rho\gamma^b\gamma_\alpha\gamma^c P_L\Big) \text{ and } f^{(2)}_{\nu a\sigma b\beta c}\mathrm{tr}\Big(\gamma_\mu\gamma^a\gamma_\alpha\gamma^b\gamma_\rho\gamma^c P_L\Big), \tag{7.13}$$

where the upper index of f is 1 for the first way of contracting and 2 for the second way. The ordering of the free indices in the trace are given by the two ways of performing the full contraction. The functions $f^{(i)}_{\nu a\sigma b\beta c}$ are composed by eight terms which are the eight forms of distributing the derivatives in the right hand side of (7.12). We will show that in reality $f^{(1)}$ and $f^{(2)}$ are the same object. To see this we will only need to exchange a with c in the expression for the second way of contracting, i.e.

$$f^{(2)}_{\nu a\sigma b\beta c}\mathrm{tr}\Big(\gamma_\mu\gamma^a\gamma_\alpha\gamma^b\gamma_\rho\gamma^c P_L\Big) = f^{(1)}_{\nu a\sigma b\beta c}\mathrm{tr}\Big(\gamma_\mu\gamma^c\gamma_\alpha\gamma^b\gamma_\rho\gamma^a P_L\Big). \tag{7.14}$$

Hence, the sum of the two ways of contracting will simplify to

$$f^{(1)}_{\nu a\sigma b\beta c}\left[\mathrm{tr}\Big(\gamma_\mu\gamma^a\gamma_\rho\gamma^b\gamma_\alpha\gamma^c P_L\Big) + \mathrm{tr}\Big(\gamma_\mu\gamma^c\gamma_\alpha\gamma^b\gamma_\rho\gamma^a P_L\Big)\right]. \tag{7.15}$$

It is possible to put the second trace in the form $\mathrm{tr}\big(\gamma_\rho\gamma^a\gamma_\mu\gamma^c\gamma_\alpha\gamma^b P_L\big)$, which reduces our final expression to

$$f^{(1)}_{\nu a\sigma b\beta c}\left[\mathrm{tr}\Big(\gamma_\mu\gamma^a\gamma_\rho\gamma^b\gamma_\alpha\gamma^c P_L\Big) + \mathrm{tr}\Big(\gamma_\rho\gamma^a\gamma_\mu\gamma^c\gamma_\alpha\gamma^b P_L\Big)\right]. \tag{7.16}$$

The trace of six gamma matrices and a gamma five is given by

$$\begin{aligned}\mathrm{tr}(\gamma_\mu\gamma_a\gamma_\rho\gamma_b\gamma_\alpha\gamma_c\gamma_5) = 4i\,(&\eta_{\mu a}\epsilon_{\rho b\alpha c} - \eta_{\mu\rho}\epsilon_{ab\alpha c} + \eta_{\rho a}\epsilon_{\mu b\alpha c}\\ &+\eta_{\alpha c}\epsilon_{\mu a\rho b} - \eta_{bc}\epsilon_{\mu a\rho\alpha} + \eta_{\alpha b}\epsilon_{\mu a\rho c})\,.\end{aligned} \tag{7.17}$$

As one can easily check, the trace (7.17) is antisymmetric under the exchange $(\mu \leftrightarrow \rho, b \leftrightarrow c)$, thus the odd part of the correlation function is zero.

Now we will work out what are the functions $f^{(i)}$ and show the relation between $f^{(1)}$ and $f^{(2)}$ mentioned above. From the first way of contracting we derive the expression

$$\mathrm{tr}\left[\gamma_\mu\partial_\nu\left(\gamma^a\partial_a\frac{1}{(x-y)^2}P_L\right)\gamma_\rho\partial_\sigma\left(\gamma^b\partial_b\frac{1}{(y-z)^2}P_L\right)\gamma_\alpha\partial_\beta\left(\gamma^c\partial_c\frac{1}{(z-x)^2}P_L\right)\right]+\cdots, \tag{7.18}$$

where the ellipsis stand for the seven other ways of organizing the derivatives ∂_ν, ∂_σ and ∂_β. From (7.18) we see that we will have some expression that we call $f^{(1)}$ contracted with

$\mathrm{tr}\big(\gamma_\mu\gamma^a\gamma_\rho\gamma^b\gamma_\alpha\gamma^c P_L\big)$. The expression for $f^{(1)}$ can be read off from (7.18):

$$\begin{aligned}
f^{(1)}_{\nu a\sigma b\beta c} = \;& \partial^x_\nu\partial^x_a\frac{1}{(x-y)^2}\partial^y_\sigma\partial^y_b\frac{1}{(y-z)^2}\partial^z_\beta\partial^z_c\frac{1}{(z-x)^2} - \partial^y_\sigma\partial^x_a\frac{1}{(x-y)^2}\partial^z_\beta\partial^y_b\frac{1}{(y-z)^2}\partial^x_\nu\partial^z_c\frac{1}{(z-x)^2}\\
&-\partial^y_\sigma\partial^x_\nu\partial^x_a\frac{1}{(x-y)^2}\left[\partial^y_b\frac{1}{(y-z)^2}\partial^z_\beta\partial^z_c\frac{1}{(z-x)^2} - \partial^z_\beta\partial^y_b\frac{1}{(y-z)^2}\partial^z_c\frac{1}{(z-x)^2}\right]\\
&-\partial^z_\beta\partial^y_\sigma\partial^y_b\frac{1}{(y-z)^2}\left[\partial^x_\nu\partial^x_a\frac{1}{(x-y)^2}\partial^z_c\frac{1}{(z-x)^2} - \partial^x_a\frac{1}{(x-y)^2}\partial^x_\nu\partial^z_c\frac{1}{(z-x)^2}\right]\\
&-\partial^x_\nu\partial^z_\beta\partial^z_c\frac{1}{(z-x)^2}\left[\partial^x_a\frac{1}{(x-y)^2}\partial^y_\sigma\partial^y_b\frac{1}{(y-z)^2} - \partial^y_\sigma\partial^x_a\frac{1}{(x-y)^2}\partial^y_b\frac{1}{(y-z)^2}\right]. \qquad (7.19)
\end{aligned}$$

The second way of contracting give us the expression

$$\mathrm{tr}\left[\gamma_\mu\partial_\nu\left(\gamma^a\partial_a\frac{1}{(x-z)^2}P_L\right)\gamma_\alpha\partial_\beta\left(\gamma^b\partial_b\frac{1}{(z-y)^2}P_L\right)\gamma_\alpha\partial_\beta\left(\gamma^c\partial_c\frac{1}{(y-x)^2}P_L\right)\right]+\cdots, \qquad (7.20)$$

from where we may read off the expression for $f^{(2)}$:

$$\begin{aligned}
f^{(2)}_{\nu a\sigma b\beta c} = \;& \partial^x_\nu\partial^x_a\frac{1}{(x-z)^2}\partial^z_\beta\partial^z_b\frac{1}{(z-y)^2}\partial^y_\sigma\partial^y_c\frac{1}{(y-x)^2} - \partial^z_\beta\partial^x_a\frac{1}{(x-z)^2}\partial^y_\sigma\partial^z_b\frac{1}{(z-y)^2}\partial^x_\nu\partial^y_c\frac{1}{(y-x)^2}\\
&-\partial^x_\nu\partial^y_\sigma\partial^y_c\frac{1}{(y-x)^2}\left[\partial^x_a\frac{1}{(x-z)^2}\partial^z_\beta\partial^z_b\frac{1}{(z-y)^2} - \partial^z_\beta\partial^x_a\frac{1}{(x-z)^2}\partial^z_b\frac{1}{(z-y)^2}\right]\\
&-\partial^y_\sigma\partial^z_\beta\partial^z_b\frac{1}{(z-y)^2}\left[\partial^x_\nu\partial^x_a\frac{1}{(x-z)^2}\partial^y_c\frac{1}{(y-x)^2} - \partial^x_a\frac{1}{(x-z)^2}\partial^x_\nu\partial^y_c\frac{1}{(y-x)^2}\right]\\
&-\partial^z_\beta\partial^x_\nu\partial^x_a\frac{1}{(x-z)^2}\left[\partial^z_b\frac{1}{(z-y)^2}\partial^y_\sigma\partial^y_c\frac{1}{(y-x)^2} - \partial^y_\sigma\partial^z_b\frac{1}{(z-y)^2}\partial^y_c\frac{1}{(y-x)^2}\right]. \qquad (7.21)
\end{aligned}$$

It is now a straightforward exercise to check that if one exchanges a with c in the expression of $f^{(2)}_{\nu a\sigma b\beta c}$ one gets $f^{(1)}_{\nu a\sigma b\beta c}$, i.e.

$$f^{(2)}_{\nu c\sigma b\beta a} = f^{(1)}_{\nu a\sigma b\beta c}. \qquad (7.22)$$

We remind the reader that in this computation we have ignored coincident point singularities. The next task will be to take them into account, which will be done in momentum space.

7.1.3 Relevant Fourier transforms

In the next subsection, in order to compute the 3-point amplitude of the e.m. tensor, with the Feynman diagram technique we will use (momentum space) Feynman diagrams. Although essentially equivalent to the Wick theorem they lend themselves more naturally to regularization. The two techniques are related by Fourier transform. Hereby we collect a series of Fourier transforms of distributions that are used in our calculations. The source is [67]. The notation is as follows

$$\mathcal{F}[\phi(x)](k) \equiv \tilde{\phi}(k) = \int d^4x\, e^{ikx}\phi(x), \qquad \phi(x) = \int \frac{d^4k}{(2\pi)^4}e^{-ikx}\tilde{\phi}(k)$$

In particular

$$\int d^4x\, e^{ikx}\, \frac{1}{x^2} = \frac{4\pi^2 i}{k^2}, \tag{7.23}$$

$$\int d^4x\, e^{ikx}\, \frac{\log x^2\mu^2}{x^2} = -\frac{4\pi^2 i}{k^2} \log\left(\frac{-k^2}{\bar{\mu}^2}\right), \tag{7.24}$$

where $\bar{\mu}^2 \equiv 2\mu^2 e^{-\gamma}$, $\gamma = 0.57721\ldots$ being the Euler constant. As we have seen this is essentially what one needs to compute the Fourier transform of the 2-point correlator. The novel feature in the calculation of the 3-point correlator is the appearance of products of similar expressions in different points, a prototype being

$$\frac{1}{(x-y)^2(x-z)^2(y-z)^2}. \tag{7.25}$$

This is singular at coincident points and has a non-integrable singularity at $x = y = z = 0$. Ignoring this let us proceed to Fourier-transforming it

$$\int d^4x\, d^4y\, d^4z\, \frac{e^{i(k_1x+k_2y-qz)}}{(x-y)^2(x-z)^2(y-z)^2} = \int d^4x\, d^4y\, d^4z\, \frac{e^{i(k_1x+k_2y+(k_1+k_2-q)z)}}{(x-y)^2x^2y^2}$$
$$= (2\pi)^4\delta(q-k_1-k_2) \int d^4x\, d^4y\, \frac{e^{i(k_1x+k_2y)}}{(x-y)^2x^2y^2}. \tag{7.26}$$

Let us set $f(x,y) = \frac{1}{(x-y)^2x^2y^2}$. Then, using the convolution theorem, the Fourier transform of f with respect to x is

$$\begin{aligned} \mathcal{F}_x[f(x,y)](k_1) &= \int d^4x\, e^{ik_1x} f(x,y) = \frac{1}{y^2}\int d^4x\, \frac{e^{ik_1x}}{x^2(x-y)^2} \\ &= \frac{1}{y^2}\int \frac{d^4p}{(2\pi)^4}\, \mathcal{F}_x\left[\frac{1}{x^2}\right](k_1-p)\, \mathcal{F}_x\left[\frac{1}{(x-y)^2}\right](p) \\ &= -\frac{1}{y^2}\int d^4p\, \frac{e^{ipy}}{p^2(p-k_1)^2}. \end{aligned} \tag{7.27}$$

Therefore

$$\begin{aligned} \int d^4x\, d^4y\, \frac{e^{i(k_1x+k_2y)}}{(x-y)^2x^2y^2} &= \int d^4y\, e^{ik_2y}\, \mathcal{F}_x[f(x,y)](k_1) \\ &= -i(2\pi)^6 \int \frac{d^4p}{(2\pi)^4}\, \frac{1}{p^2(p-k_1)^2(p+k_2)^2}. \end{aligned} \tag{7.28}$$

We can now compute the RHS of (7.28) in the usual way by introducing a Feynman parametrization in terms of two parameters u, v:

$$\int \frac{d^4p}{(2\pi)^4}\, \frac{1}{p^2(p-k_1)^2(p+k_2)^2} = \int_0^1 du \int_0^{1-u} dv \int \frac{d^4p'}{(2\pi)^4}\, \frac{1}{(p'^2-\ell^2+\Delta)^3} \tag{7.29}$$

where $p' = p - uk_1 + vk_2$ and $\Delta = u(1-u)k_1^2 + v(1-v)k_2^2 + 2uv\, k_1k_2$. Performing the p' integral one gets

$$\int_0^1 du \int_0^{1-u} dv \int \frac{d^4p}{(2\pi)^4} \int \frac{d^\delta\ell}{(2\pi)^\delta}\, \frac{1}{(p^2-\ell^2+\Delta)^3} = \frac{i}{2(4\pi)^2}\int_0^1 du \int_0^{1-u} dv \frac{1}{\Delta} \tag{7.30}$$

Our attitude will be to define the regularization of (7.25) as the Fourier anti-transform of the (7.30).

In general, however, the expressions we have to do with are not as simple as (7.30) and the integrals as simple as (7.28). The typical integral of the type (7.28) contains a polynomial of p, k_1, k_2 in the numerator of the integrand. In this case we have two ways to proceed: either we extend the running momentum p to extra dimensions (dimensional regularization), as we have done in 2d, carry out the integration and Fourier-anti-transform the final result, or we reduce the calculations to a differential operator applied to the Fourier-anti-transform of (7.30) (differential regularization). Usually the former procedure is more convenient, while in many cases the latter is problematic.

Other analogous expressions are obtained in appendix G.1.

7.1.4 The parity-odd 3-point correlator with Feynman diagrams

This section is devoted to the same calculation as in subsection (7.1.2), but with Feynman diagram techniques. In order to compute the 3-point function of the energy-momentum tensor for a chiral fermion, it is very convenient to couple it minimally to gravity and extract from the corresponding action the Feynman rules, as presented in chapter 6 and in [19, 59]. Due to the non polynomial character of the action the diagrams contributing to the trace anomaly are infinitely many. Fortunately, using diffeomorphism invariance, it is enough to determine the lowest order contributions and consistency takes care of the rest. There are two potential lowest order diagrams that may contribute. The first contribution, the bubble graph, turns out to give a vanishing contribution. The important term comes from the triangle graph. This has an incoming line with momentum $q = k_1 + k_2$ with Lorentz indices μ, ν. The two outgoing lines have momenta k_1, k_2 with Lorentz labels λ, ρ and α, β, respectively. The contribution is formally written as

$$\mathcal{T}^{(1)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2) = -\frac{1}{512}\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\left(\frac{1}{\not p}\big((2p-k_1)_\lambda\gamma_\rho + (\lambda\leftrightarrow\rho)\big)\,\frac{1}{\not p - \not k_1}\right.\right.$$
$$\left.\left.\times\,\big((2p-2k_1-k_2)_\alpha\gamma_\beta + (\alpha\leftrightarrow\beta)\big)\,\frac{1}{\not p-\not q}\big((2p-q)_\mu\gamma_\nu + (\mu\leftrightarrow\nu)\big)\right)\frac{1+\gamma_5}{2}\right] \tag{7.31}$$

to which the cross graph contribution $\mathcal{T}^{(2)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2) = \mathcal{T}^{(1)}_{\mu\nu\lambda\rho\alpha\beta}(k_2,k_1)$ has to be added. We regularize (7.31) as usual by introducing extra component of the momentum running around the loop $p \to p+\ell$, $\ell = \ell_4,\dots,\ell_{\delta+4}$:

$$\mathcal{T}^{(1)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2) = -\frac{1}{512}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^\delta\ell}{(2\pi)^\delta}\,\mathrm{tr}\left(\frac{\not p+\not\ell}{p^2-\ell^2}(2p-k_1)_\lambda\gamma_\rho\right.$$
$$\left.\times\,\frac{\not p+\not\ell-\not k_1}{(p-k_1)^2-\ell^2}\,(2p-2k_1-k_2)_\alpha\gamma_\beta\,\frac{\not p+\not\ell-\not q}{(p-q)^2-\ell^2}\,(2p-q)_\mu\gamma_\nu\frac{1+\gamma_5}{2}\right) \tag{7.32}$$

where the symmetrization with respect to $\alpha\leftrightarrow\beta$, $\lambda\leftrightarrow\rho$ and $\mu\leftrightarrow\nu$ is understood. We should now proceed to the explicit calculation. However one quickly realizes that this involves a huge number of terms. To find an orientation among the latter it is very useful to first compute the trace and the divergence of the e.m. tensor in the above formulas. They are connected to the trace and divergence of the full one-loop e.m. tensor by the general formulas of chapter 3.

7.1.4.1 The trace

The trace of (7.32) is

$$\mathcal{T}^{(1a)\mu}{}_{\mu\alpha\beta\lambda\rho}(k_1,k_2) = -\frac{1}{256}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^\delta \ell}{(2\pi)^\delta}\,\mathrm{tr}\left(\frac{\not p+\not\ell}{p^2-\ell^2}(2p-k_1)_\lambda\gamma_\rho \right.$$
$$\left.\times\frac{\not p+\not\ell-\not k_1}{(p-k_1)^2-\ell^2}(2p-2k_1-k_2)_\alpha\gamma_\beta\frac{\not p+\not\ell-\not q}{(p-q)^2-\ell^2}(2\not p-\not q)\frac{1+\gamma_5}{2}\right). \tag{7.33}$$

On the other hand if we first take the trace of (7.31) and then regularize it, we get

$$\mathcal{T}^{(1b)\mu}{}_{\mu\alpha\beta\lambda\rho}(k_1,k_2) = -\frac{1}{256}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^\delta \ell}{(2\pi)^\delta}\,\mathrm{tr}\left(\frac{\not p+\not\ell}{p^2-\ell^2}(2p-k_1)_\lambda\gamma_\rho \right.$$
$$\left.\times\frac{\not p+\not\ell-\not k_1}{(p-k_1)^2-\ell^2}(2p-2k_1-k_2)_\alpha\gamma_\beta\frac{\not p+\not\ell-\not q}{(p-q)^2-\ell^2}\left(2\not p+2\not\ell-\not q\right)\frac{1+\gamma_5}{2}\right). \tag{7.34}$$

The difference between the two is[2]

$$\Delta\mathcal{T}^{(1)\mu}{}_{\mu\alpha\beta\lambda\rho}(k_1,k_2) = -\frac{1}{128}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^\delta \ell}{(2\pi)^\delta}\,\mathrm{tr}\left(\frac{\not p+\not\ell}{p^2-\ell^2}(2p-k_1)_\lambda\gamma_\rho \right.$$
$$\left.\times\frac{\not p+\not\ell-\not k_1}{(p-k_1)^2-\ell^2}(2p-2k_1-k_2)_\alpha\gamma_\beta\frac{\not p+\not\ell-\not q}{(p-q)^2-\ell^2}\not\ell\,\frac{1+\gamma_5}{2}\right). \tag{7.35}$$

Similar expressions hold for $\mathcal{T}^{(2)}$. Now it is easy to show that (7.33) vanishes along with the analogous expression for $\mathcal{T}^{(2)}$, while (7.34) does not, and in fact the odd-parity part of (7.35) is precisely the anomalous term computed in [19], which, together with the cross term coming form $\mathcal{T}^{(2)}$, gives rise to the Pontryagin anomaly. More precisely, the two terms yield

$$\mathcal{T}^{\mu}_{\mu\alpha\beta\lambda\rho}(k_1,k_2) = \frac{1}{192(4\pi)^2}k_1^\sigma k_2^\tau\left(t^{(21)}_{\lambda\rho\alpha\beta\sigma\tau} - t_{\lambda\rho\alpha\beta\sigma\tau}(k_1^2+k_2^2+k_1k_2)\right) \tag{7.36}$$

The tensors t and $t^{(21)}$ were defined in [19]. In [19] the external lines were put on shell (in the de Donder gauge): $k_1^2 = k_2^2 = 0$. This is the right thing to do, as we shall see, but it is important to clarify the role of the off-shell terms too. Therefore let us consider nonvanishing external square momenta. While the remaining terms, when inserted into the reconstruction formula (3.38), reproduce the Pontryagin density to order h^2,

$$\sim \epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma h^\tau_\nu\,\partial_\lambda\partial_\tau h^\sigma_\rho - \partial_\mu\partial_\sigma h^\tau_\nu\,\partial_\lambda\partial^\sigma h_{\tau\rho}\right) + \mathcal{O}(h^3), \tag{7.37}$$

the term proportional to $k_1^2+k_2^2$ in (7.36) leads to a term proportional to

$$\epsilon^{\mu\nu\lambda\rho}\partial_\mu\Box h^\alpha_\nu\partial_\lambda h_{\rho\alpha}. \tag{7.38}$$

They are both invariant under the Weyl rescaling $\delta h_{\mu\nu} = 2\omega\,\eta_{\mu\nu}$. Thus the corresponding anomalous terms obtained by integrating (7.37) and (7.38) multiplied by the Weyl parameter ω are consistent. But while the first gives rise to a true anomaly, the second one must be

[2] Eqs.(3.38) and (3.44) suggest that the right prescription is (7.34), not (7.33). This has been fully confirmed by the calculations in $2d$. The anomaly is determined by the n-point functions where the entries are one trace of the e.m. tensor and $n-1$ e.m. tensors. We have quoted the 'wrong' formula (7.33) on purpose in order to stress this point.

trivial because there is no covariant cocycle containing the ϵ tensor beside the Pontryagin one. In fact it is easy to guess the counterterm that cancels it: it is proportional to

$$\int d^4x\, h\, \epsilon^{\mu\nu\lambda\rho}\partial_\mu \Box h^\alpha_\nu \partial_\lambda h_{\rho\alpha} \tag{7.39}$$

where $h = h^\mu_\mu$. But this counterterm breaks invariance under general coordinate transformations, which to lowest order take the form $\delta_\xi h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ (with $\delta_\xi\omega = 0$). Thus we must expect that off-shell terms break the e.m. tensor conservation. This does not mean that there are true diff anomalies, but simply that we have to subtract counterterms (actually, a lot of them, see below) in order to recover a covariant regularization. In other words taking into account off-shell terms is a very effective way to complicate one's own life, while disregarding them does not spoil the result if our aim is to find a covariant expression of the anomaly. The reason for this is that the equation of motion of gravity in vacuum

$$\Box h_{\mu\nu} - \partial_\mu\partial_\lambda h^\lambda_\nu - \partial_\nu\partial_\lambda h^\lambda_\mu + \partial_\mu\partial_\nu h^\lambda_\lambda = 0 \tag{7.40}$$

is covariant. If we impose the De Donder gauge

$$2\partial_\mu h^\mu_\lambda - \partial_\lambda h^\mu_\mu = 0 \tag{7.41}$$

the last three terms in the RHS of (7.40) vanish and the latter reduces to $\Box h_{\mu\nu} = 0$. Therefore choosing this gauge and putting the external legs on shell (as we have just done) does not break covariance and considerably simplifies the calculations[3].

7.1.4.2 The divergence

The discussion in the previous subsection raises a problem. For not only can we subtract (7.38) via the counterterm (7.39), but also (7.37) can be subtracted away by means of the counterterm

$$\sim \int d^4x\, h\, \epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma h^\tau_\nu\, \partial_\lambda\partial_\tau h^\sigma_\rho - \partial_\mu\partial_\sigma h^\tau_\nu\, \partial_\lambda\partial^\sigma h_{\tau\rho}\right) + \mathcal{O}(h^3), \tag{7.42}$$

as it is easy to verify. This of course generates new terms in the divergence of the e.m. tensor. Choosing the on-shell option to simplify the problem, they corresponds, in the momentum notation, to the terms

$$\sim \epsilon_{\beta\rho\sigma\tau}\, k_{1\nu} k_1^\sigma k_2^\tau \left(k_{1\lambda}k_{2\alpha} - \eta_{\alpha\lambda}k_1\cdot k_2\right) + \{\lambda \leftrightarrow \rho\} + \{\alpha \leftrightarrow \beta\} + \{1 \leftrightarrow 2\} \tag{7.43}$$

where the subscript ν, in coordinate representation, is saturated with the diffeomorphism parameter ξ^ν.

Let us remark that, when we refer to the lowest order in h, any anomaly appears to be trivial and can be subtracted (see what we have done above in $2d$). This is true also for the even parity anomalies, but it is an accident of the approximation. What is decisive about triviality or not of the anomalies is their diff partner. We must arrive at a configuration in which the diff partner of the trace anomaly vanishes. In this case we can conclude that a nonvanishing trace anomaly is nontrivial even if it is expressed at the lowest order in h. This expression will be the lowest order expansion of a covariant expression (much as (7.37) is). In conclusion we expect that subtracting away (7.37) by means of (7.42) is a forbidden

[3]Sometimes it oversimplifies them, for instance in $2d$ or in $4d$ for the 2-point correlator. In such cases there is no way but doing the calculations in full, as we have done above.

operation (it breaks covariance). But it is important to verify it by a direct calculation. This is what we intend to do in the sequel.

The relevant lowest order contribution to $\langle \nabla^\mu T_{\mu\nu}\rangle_g$, see (3.43), comes from the 3-point function $\langle 0|\mathcal{T}\{\partial^\mu T_{\mu\nu}(x)T_{\lambda\rho}(y)T_{\alpha\beta}(z)\}|0\rangle$. The latter corresponds to two graphs, the bubble and the triangle ones (see [19]). The bubble graph contribution vanishes. The triangle contribution is given by

$$q^\mu \mathcal{T}^{(1)}_{\mu\nu\lambda\rho\alpha\beta}(k_1,k_2) = -\frac{1}{512}\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\left(\frac{1}{\not p}((2p-k_1)_\lambda\gamma_\rho + (\lambda\leftrightarrow\rho))\;\frac{1}{\not p-\not k_1}\right.\right. \\ \left.\left.\times((2p-2k_1-k_2)_\alpha\gamma_\beta + (\alpha\leftrightarrow\beta))\;\frac{1}{\not p-\not q}((2p-q)\cdot q\,\gamma_\nu + (2p-q)_\nu \not q)\right)\frac{1+\gamma_5}{2}\right] \tag{7.44}$$

to which the cross contribution $q^\mu\mathcal{T}^{(1)}_{\mu\nu\alpha\beta\lambda\rho}(k_2,k_1)$ has to be added. We regulate the integral as usual with an extra dimensional momentum ℓ and introduce Feynman parameters as needed. After a rather lengthy algebra, in particular with explicit use of the identity

$$\eta_{\mu\nu}\epsilon_{\lambda\rho\sigma\tau} - \eta_{\mu\lambda}\epsilon_{\nu\rho\sigma\tau} + \eta_{\mu\rho}\epsilon_{\nu\lambda\sigma\tau} - \eta_{\mu\sigma}\epsilon_{\nu\lambda\rho\tau} + \eta_{\mu\tau}\epsilon_{\nu\lambda\rho\sigma} = 0, \tag{7.45}$$

the regularized (7.44) can be recast into the form

$$\mathcal{D}^{(1)}_{\nu\lambda\rho\alpha\beta}(k_1,k_2) \equiv q^\mu\left(\mathcal{T}^{(1)}_{\mu\nu\lambda\rho\alpha\beta}(k_1,k_2) + \mathcal{T}^{(1)}_{\mu\nu\alpha\beta\lambda\rho}(k_2,k_1)\right) \\ = \frac{i}{256}\int_0^1 dx\int_0^{1-x}dy\int\frac{d^4p}{(2\pi)^4}\int\frac{d^\delta\ell}{(2\pi)^\delta}\Big[-\epsilon_{\nu\beta\sigma\tau}\left(p_\rho k_1^\sigma k_2^\tau + (k_{1\rho}k_2^\tau + k_{2\rho}k_1^\tau + 2k_2^\tau p_\rho)p^\sigma\right) \\ +\epsilon_{\nu\beta\rho\tau}\left(p^2(k_1+k_2-p)^\tau + p\cdot k_1\,k_2^\tau - p\cdot k_2\,k_1^\tau + (k_2-2p)\cdot k_1 p^\tau\right) + \epsilon_{\nu\sigma\tau\kappa}\eta_{\beta\rho}\,p^\sigma k_1^\tau k_2^\kappa \\ +\epsilon_{\nu\rho\sigma\tau}\left(p_\beta k_1^\tau k_2^\sigma + p^\sigma(k_{1\beta}k_2^\tau + k_{2\beta}k_1^\tau - 2p_\beta k_1^\tau)\right) + \ell^2\epsilon_{\nu\beta\rho\tau}(p+k_1-k_2)^\tau\Big] \\ \times\frac{2p\cdot(k_1+k_2)(2p+k_1)_\lambda(2p-k_2)_\alpha}{[(p+xk_1-yk_2)^2 + 2xy\,k_1\cdot k_2 - \ell^2]^3}. \tag{7.46}$$

This expression does not contain any of the terms (7.43), but of course this is not enough. We have to prove that all the terms in (7.46) either vanish or are trivial in the sense that they can be canceled by counterterms that are Weyl invariant. This analysis is carried out in appendix G.2, where counterterms are constructed which cancel all the onvanishing terms in (7.46) without altering the result of the trace anomaly calculation. Thus the lowest order expression (7.37) cannot be canceled (except at the price of breaking diffeomorphism invariance) and is a genuine covariant expression. It represents the lowest order approximation of the Pontryagin density.

7.1.4.3 (Partial) conclusion

The results obtained in this section fully confirm those of [19–21]. The apparent contradiction inherent in the fact that the "bare" parity-odd correlator of three energy-momentum tensors vanishes will be explained in the next section. Here we would like to draw some conclusion on the regularized e.m. tensor 3-point function. We have seen that the trace and the traceless part of the correlator must be regularized separately. The traceless part of the correlator can be regularized starting from (7.32). We would like to be able to conclude that the regularized traceless part coincides with the "bare" part, i.e. it vanishes, but in order to justify this conclusion the calculations are very challenging, because it is not enough to regularize and compute (7.32), but we must also take into account all the counterterms (with the exact coefficients) that we have subtracted in order to guarantee covariance, see appendix G.2. This can realistically be done only with a computer algebra program. For the time being, although we believe the regularized traceless part of the correlator vanishes, we leave its proof as an open problem.

Finally a comment on the parity-even part of the 3-point e.m. tensor correlator. The calculation of the trace and divergence involves many more terms than in the parity-odd part, but it does not differ in any essential way from it. Also in the parity-even part it is necessary to introduce counterterms in order to guarantee covariance and the correct final expression for the trace anomaly. On the other hand this is pretty clear already in the $2d$ case, as we have shown above. Since the results for the parity-even part of the 3-point function, both "bare" and regularized, [3, 4], and relevant even-parity anomalies are well-known, see [68], we dispense with an explicit calculation.

7.2 The ugly duckling anomaly

The title is due to the non-overwhelming consideration met so far by the Pontryagin trace anomaly. Needless to say its presence in the free chiral fermion model is at first sight surprising. The basic ingredient to evaluate this anomaly in the Feynman diagram approach is traditionally the triangle diagram, which can be seen as the lowest order approximation of the 3-point correlator, whose entries are one e.m. trace and two e.m. tensors. On the other hand, since the "bare" parity-odd part of the 3-point correlator of the e.m. tensor vanishes on the basis of very general considerations of symmetry, it would seem that even the triangle diagram contributions should vanish, because the regularization of zero should be zero.

The remark made in connection with formulas (7.33), (7.34) and (7.35) may seem to add strength to this argument because it leaves the impression that the Pontryagin anomaly is something we can do without. After all its existence in the 3-point correlators is related to the order in which we regularize. One might argue that if we regularize in a specific order the anomaly disappears, but this is not the case. First of all we remark that what one does in all kind of anomalies is to regularize the divergence of a current or of the e.m. tensor, or the trace of the latter, rather than regularizing the current or the e.m. tensor and then taking the divergence or the trace thereof. In other words the regularization should be done independently for each irreducible component that enters into play. But, even forgetting this, in order to make a decision about such an ambiguous occurrence one must resort to some consistency argument, and this is what we will do below.

In fact the apparent contradiction is based on a misunderstanding, which consists in assuming that the (unregulated) 3-point correlator in the coordinate representation is the sole ingredient of the anomaly. This is not true[4]. The 3-point correlator of the energy-momentum tensor is one of the possible markers of the trace anomaly, but, as we shall see, there are infinite many of them and consistency demands that they all agree (the more so if the correlator is unregulated). Let us start with by clarifying this point.

In chapter 3 we have shown how to reconstruct the full one-loop e.m. tensor starting from the one-loop correlators of the e.m. tensors, see (3.38). What matters here is that the full one-loop e.m. tensor contains the information about the e.m. tensor correlators with any number of entries. The first non-trivial one corresponds of course to $n = 2$.

Now let us apply the reconstruction formula (3.38) to a single chiral fermion theory. Classically the energy-momentum tensor for a left-handed fermion is

$$T^{(L)}_{\mu\nu} = \frac{i}{4}\overline{\psi_L}\gamma_\mu \overset{\leftrightarrow}{\partial}_\nu \psi_L + \{\mu \leftrightarrow \nu\} \tag{7.47}$$

[4]We remark that the parity-even 3-point correlator of the e.m. trace and two e.m. tensors also vanishes for non-coincident points, but this does not prevent the even parity anomaly from being nonvanishing.

which is both conserved and traceless on shell. An analogous expression holds for a right-handed fermion. It has been proved in general (and we have shown it above) that the (unregulated) parity-odd 3-point function in the coordinate representation vanishes. Thus let us ask ourselves what would happen if parity-odd amplitudes

$$\langle 0|\mathcal{T}\{T^{(L)}_{\mu_1\nu_1}(x_1)\dots T^{(L)}_{\mu_n\nu_n}(x_n)\}|0\rangle_{\text{odd}}$$

to all orders were to vanish. We would have the same also for the right handed counterpart, while the even-parity amplitudes are equal. Therefore the difference

$$\langle T^{(L)}_{\mu\nu}(x)\rangle_g - \langle T^{(R)}_{\mu\nu}(x)\rangle_g = 0. \tag{7.48}$$

This would imply that the quantum analog of $\overline{\psi}\gamma_\mu\gamma_5\overset{\leftrightarrow}{\partial}_\nu\psi + \{\mu \leftrightarrow \nu\}$ would vanish identically. This is nonsense, and means that the vanishing of the parity-odd 3-point function is an accidental occurrence and that the ("bare") parity-odd amplitudes will generically be non-vanishing [5]. Inserting now these results in the reconstruction formula (3.38) and resumming the series we would reconstruct the parity-odd anomaly. Let us apply this to the trace of the quantum energy-momentum tensor. Since the parity-odd amplitudes are generically non-vanishing we would obtain a nonvanishing trace anomaly. Now the only possible covariant parity-odd anomaly is the Pontryagin density

$$P = \frac{1}{2}\left(\epsilon^{nmlk}\mathcal{R}_{nmpq}\mathcal{R}_{lk}{}^{pq}\right) \tag{7.49}$$

whose first nonvanishing contribution is quadratic in $h_{\mu\nu}$

$$\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau} = 2\epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma h^\tau_\nu\,\partial_\lambda\partial_\tau h^\sigma_\rho - \partial_\mu\partial_\sigma h^\tau_\nu\,\partial_\lambda\partial^\sigma h_{\tau\rho}\right) + \dots, \tag{7.50}$$

and can come only from the parity-odd 3-point correlator. But, if the latter vanishes, we would get an incomplete, and therefore non-covariant, expression for this anomaly.

The conclusion of this argument is: covariance (and consistency) requires that, even if the (unregulated) parity-odd 3-point function in the coordinate representation vanishes, the corresponding regularized counterpart must be non-vanishing. This is precisely what was found in [19] with (regularized) Feynman diagram techniques.

The existence of the Pontryagin anomaly is confirmed also by other methods of calculation: the heat kernel method, see [19, 21] and references therein, and the mass regularization of [59], although the latter method have not been applied with the same accuracy as the dimensional regularization in the present paper. We should mention also the dispersive method which uses unitarity as an input. Of course we do not expect this method to reproduce this anomaly, which violates unitarity, [19]. In fact using such a method would be a reversal of the burden of proof. The dispersive argument is very elegant and powerful, [12, 62, 69], but it assumes unitarity. Unitarity is normally given for granted and assumed by default. But the case presented in this paper is precisely an example in which this cannot be done.

Finally we would like to notice that the so-called Delbourgo-Salam anomaly, [59], i.e. the anomaly in the divergence of the chiral current $j_{\mu 5} = i\bar{\psi}\gamma_\mu\gamma_5\psi$, is determined by a term (7.33) in which the factor $(2\not{p}-\not{q})$ is replaced by $\not{q}$. If, in such a term, we rewrite $\not{q}$ as $2\not{p}-(2\not{p}-\not{q})$, we see that the second part reproduces the Pontryagin anomaly we have computed, while the term containing $2\not{p}$, once regularized, is easily seen to vanish. In other words the Pontryagin trace anomaly and the Delbourgo-Salam chiral anomaly come from the same term.

[5]The analogue of the parity-odd 3-point correlator vanishing theorem does not exist for generic amplitudes.

7.3 Conclusions

In conclusion, let us summarize what was reviewed and what was shown in this paper. Our paradigm is always the theory of a free chiral fermion, thus every time that we refer to Feynman diagram techniques or Wick theorem, we are making reference to these techniques applied to this specific model.

We started in sections 5.1, 5.2 and 5.3 by reviewing the regularization of the 2-point function of e.m. tensors in $2d$, using both differential regularization and dimensional regularization of the expression obtained with Feynman diagrams. Demanding the correlator to satisfy the Ward identity for diffeomorphism invariance we obtain a violation of the Ward identity for conformal invariance and we recover the known result of the $2d$ trace anomaly. In section 5.4 the analogous result was shown also in $4d$ where the situation is different because we are able to regularize the correlator in such a way that both Ward identities are satisfied.

In section 7.1, moving to the 3-point function of e.m. tensors in $4d$, we first noted a discrepancy between the computations in momentum space through Feynman diagrams and the computation in coordinate space using the Wick theorem. The direct computation through Wick theorem tells us that there is no (unregulated) parity-odd contribution in the 3-point correlator of e.m. tensors for the free chiral fermion. This result is indeed in agreement with the general fact that in $4d$ there are no parity-odd contribution in the correlation function of three e.m. tensors which was reviewed in section 7.1.1. With this fact in hand one could try to regularize this correlator with the techniques of differential regularization and would be obliged to conclude that there is no parity-odd trace anomaly simply because there is no parity-odd contribution to be regularized. On the other hand, by doing the computation in momentum space with Feynman diagram techniques we do find a parity-odd trace anomaly. Is this result forced to be wrong?

We argued in section 7.2 that these results can perfectly coexist and the result in coordinate space by no means is a no-go for the existence of the Pontryagin anomaly.

Appendix G

Details of computations

G.1 Fourier transforms

In this appendix we expand on the results of section (7.1.3). Let us start from the following formal transformations:

$$\begin{aligned}
&-i(2\pi)^6 \int \frac{d^4p}{(2\pi)^4} \frac{k_{1\mu}}{p^2(p-k_1)^2(p-q)^2} = i\int d^4x\, d^4y\, e^{i(k_1x+k_2y)} \frac{\partial}{\partial x^\mu}\left(\frac{1}{(x-y)^2x^2y^2}\right) \\
&-i(2\pi)^6 \int \frac{d^4p}{(2\pi)^4} \frac{k_{2\mu}}{p^2(p-k_1)^2(p-q)^2} = i\int d^4x\, d^4y\, e^{i(k_1x+k_2y)} \frac{\partial}{\partial y^\mu}\left(\frac{1}{(x-y)^2x^2y^2}\right)
\end{aligned} \tag{G.1}$$

According to the procedure outlined in section (7.1.3), the LHS's of these equations will be defined by means of (7.30) and, via Fourier anti-transform, will define the corresponding regularized rational function in the RHS's. The generalization to multiple powers of the momenta k_1, k_2 in the numerator is straightforward. The (G.1) formulas and the like define a *differential regularization*.

In the main body of the paper we have to do with similar integrals in which, however, the numerator of the integrand contains polynomials of p beside k_1 and k_2. In this case we do not know a straightforward way to differentially regularize them and resort instead to *dimensional regularization*, in which case other Fourier transforms are needed. For instance

$$\begin{aligned}
&\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{e^{i(k_1(x-z)+k_2(y-z))}}{(k_1+k_2)^2} \\
&= \frac{1}{16}\left(\int \frac{d^4\tilde{k}_1}{(2\pi)^4}\frac{e^{i\tilde{k}_1\left(\frac{(x-z)+(y-z)}{2}\right)}}{\tilde{k}_1^2}\right)\left(\int \frac{d^4\tilde{k}_2}{(2\pi)^4} e^{i\tilde{k}_2\left(\frac{x-y}{2}\right)}\right) = \frac{1}{16\pi^2}\frac{1}{(x-z)^2}\delta^{(4)}(x-y)
\end{aligned} \tag{G.2}$$

where we set $\tilde{k}_1 = k_1 + k_2$ and $\tilde{k}_2 = k_1 - k_2$. Proceeding in the same way,

$$\begin{aligned}
&\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{e^{i(k_1(x-z)+k_2(y-z))}}{(k_1+k_2)^2}\log(k_1+k_2)^2 = \\
&\qquad = \frac{1}{4\pi^2}\delta^{(4)}(x-y)\frac{1}{(x-z)^2}\log\frac{(x-z)^2}{4},
\end{aligned} \tag{G.3}$$

and it is understood that

$$\begin{aligned}
&\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} e^{i(k_1(x-z)+k_2(y-z))}\log(k_1+k_2)^2 = \\
&\qquad = -(\partial_x+\partial_y)^2 \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{e^{i(k_1(x-z)+k_2(y-z))}}{(k_1+k_2)^2}\log(k_1+k_2)^2.
\end{aligned}$$

G.2 Conservation of the e.m. tensor

In this appendix we complete the proof of section 7.1.4.2.

To start with we write down the structure of the various terms in (7.46) in momentum representation and in coordinate space after applying (3.38)

$$\begin{aligned}
\epsilon_{\nu\beta\rho\tau}k_1\cdot k_2 k_{1\lambda}k_{1\alpha}k_1^\tau \to &\quad \int \xi^\nu \epsilon_{\nu\beta\rho\tau}\partial_\sigma\partial_\lambda\partial_\alpha\partial^\tau h^{\lambda\rho}\partial^\sigma h^{\alpha\beta} = 0,\\
\epsilon_{\nu\beta\rho\tau}k_1\cdot k_2 k_{1\lambda}k_{1\alpha}k_2^\tau \to &\quad \int \xi^\nu \epsilon_{\nu\beta\rho\tau}\partial_\sigma\partial_\lambda\partial_\alpha h^{\lambda\rho}\,\partial^\tau\partial^\sigma h^{\alpha\beta} \qquad \text{(G.4)}\\
&= \frac{1}{2}\int \xi^\nu \epsilon_{\nu\beta\rho\tau}\partial_\sigma\partial^\rho\partial_\alpha h\,\partial^\tau\partial^\sigma h^{\alpha\beta}, \qquad \text{(G.5)}\\
\epsilon_{\nu\beta\rho\tau}k_1\cdot k_2 k_{1\lambda}k_{2\alpha}k_2^\tau \to &\quad \int \xi^\nu \epsilon_{\nu\beta\rho\tau}\partial_\sigma\partial_\alpha h^{\lambda\rho}\,\partial_\lambda\partial^\tau\partial^\sigma h^{\alpha\beta}, \qquad \text{(G.6)}\\
\epsilon_{\nu\beta\rho\tau}k_1\cdot k_2 k_{1\lambda}k_{2\alpha}k_1^\tau \to &\quad \int \xi^\nu \epsilon_{\nu\beta\rho\tau}\partial_\sigma\partial_\alpha\partial^\tau h^{\lambda\rho}\,\partial_\lambda\partial^\sigma h^{\alpha\beta}, \qquad \text{(G.7)}
\end{aligned}$$

and other similar terms obtained by exchanging 1 and 2. (G.6) is the opposite of (G.7). In addition we have the term

$$\eta_{\alpha\lambda}\epsilon_{\nu\beta\rho\tau}(k_1\cdot k_2)^2 k_1^\tau \to \quad \int \xi^\nu \epsilon_{\nu\beta\rho\tau}\partial_\sigma\partial_\kappa\partial^\tau h^{\lambda\rho}\partial^\kappa\partial^\sigma h^{\alpha\beta} \qquad \text{(G.8)}$$

and the opposite one obtained by exchanging 1 and 2. All these terms appear with (nonvanishing) coefficients which are rational numbers or rational numbers multiplied by

$$\frac{2}{\delta} + \gamma - \log 4\pi + \log 2k_1\cdot k_2 \qquad \text{(G.9)}$$

in the limit $\delta \to 0$. The terms proportional to $\log 2k_1\cdot k_2$ will be disregarded here because, due to the results in appendix G.1, they corresponds to the 2-point terms of eq.(3.43). All the other terms have to be canceled by subtracting counterterms from the action. The important point is that such counterterm must be Weyl invariant to the appropriate order in h, otherwise they would modify the trace of the e.m. tensor. We show next that this is in fact true for all the above terms.

The terms (G.4) and (G.8) are trivial, for we have

$$\begin{aligned}
&\delta_\omega \int h\,\epsilon_{\mu\nu\lambda\rho}\partial^\mu h^{\tau\nu}\,\partial^\lambda\Box h^\rho_\tau = 0,\\
&\delta_\xi \int h\,\epsilon_{\mu\nu\lambda\rho}\partial^\mu h^{\tau\nu}\,\partial^\lambda\Box h^\rho_\tau = \int \xi^\nu \epsilon_{\nu\sigma\rho\lambda}\partial_\tau\partial^\lambda\partial^\kappa h\,\partial^\sigma\partial_\kappa h^{\rho\tau}, \qquad \text{(G.10)}
\end{aligned}$$

and

$$\begin{aligned}
&\delta_\omega \int \epsilon_{\mu\nu\lambda\rho}\,h^{\mu\sigma}\,\partial^\tau\partial^\lambda h^\rho_\sigma\,\Box h^\nu_\tau = 0,\\
&\delta_\xi \int \epsilon_{\mu\nu\lambda\rho}\,h^{\mu\sigma}\,\partial^\tau\partial^\lambda h^\rho_\sigma\,\Box h^\nu_\tau = -\int \xi^\nu \epsilon_{\nu\tau\lambda\rho}\partial^\tau\partial^\kappa h^{\mu\sigma}\,\partial_\mu\partial_\kappa\partial^\lambda h^\rho_\sigma, \qquad \text{(G.11)}\\
&+2\int \xi^\nu \epsilon_{\nu\tau\lambda\rho}\partial^\kappa\partial^\alpha h^{\tau\sigma}\partial_\kappa\partial_\alpha\partial^\lambda h^\rho_\sigma + \frac{1}{2}\int \xi^\nu \epsilon_{\nu\mu\tau\lambda}\partial^\tau\partial^\kappa h^{\mu\sigma}\partial_\kappa\partial^\lambda\partial_\sigma h.
\end{aligned}$$

Similarly

$$\begin{aligned}\delta_\omega \int \epsilon_{\nu\beta\rho\tau}\, h^{\nu\alpha}\partial_\kappa h^{\sigma\rho}\,\partial_\sigma\partial^\tau\partial^\kappa h_\alpha^\beta &= 0,\\ \delta_\xi \int \epsilon_{\nu\beta\rho\tau}\, h^{\nu\alpha}\partial_\kappa h^{\sigma\rho}\,\partial_\sigma\partial^\tau\partial^\kappa h_\alpha^\beta &= \int \xi^\nu \epsilon_{\nu\rho\beta\tau}\,\partial_\sigma\partial_\kappa h^{\alpha\rho}\,\partial^\sigma\partial^\kappa\partial^\tau h_\alpha^\beta\\ &\quad -\frac{1}{2}\int \xi^\nu \epsilon_{\nu\rho\beta\tau}\,\partial^\rho\partial_\kappa h^{\sigma\tau}\,\partial_\sigma\partial^\kappa\partial^\beta h,\end{aligned} \tag{G.12}$$

and

$$\begin{aligned}&\delta_\omega \int \epsilon_{\nu\beta\rho\tau}\, h^{\nu\sigma}\partial_\kappa h^{\alpha\rho}\,\partial^\tau\partial_\sigma\partial^\kappa h_\alpha^\beta = 0,\\ &\delta_\xi \int \epsilon_{\nu\beta\rho\tau}\, h^{\nu\sigma}\partial_\kappa h^{\alpha\rho}\,\partial^\tau\partial_\sigma\partial^\kappa h_\alpha^\beta = -\int \xi^\nu \Big(2\epsilon_{\nu\beta\rho\tau}\,\partial_\kappa\partial^\lambda h^{\alpha\rho}\,\partial_\alpha\partial^\kappa\partial^\tau h_\alpha^\beta\\ &+2\epsilon_{\nu\rho\beta\tau}\,\partial_\kappa\partial^\rho h^{\alpha\sigma}\,\partial_\sigma\partial^\kappa\partial^\tau h_\alpha^\beta + 2\epsilon_{\nu\beta\rho\tau}\,\partial^\kappa\partial^\tau\partial_\sigma h_\alpha^\beta\,\partial^\alpha\partial_\kappa h^{\rho\sigma} + \epsilon_{\nu\beta\rho\tau}\,\partial_\kappa\partial^\rho h^{\beta\sigma}\,\partial^\kappa\partial^\tau\partial_\sigma h\Big),\end{aligned} \tag{G.13}$$

as well as

$$\begin{aligned}&\delta_\omega \int \epsilon_{\nu\beta\rho\tau}\, h^{\nu\sigma}\Box h^{\alpha\rho}\,\partial^\tau\partial_\sigma h_\alpha^\beta = 0,\\ &\delta_\xi \int \epsilon_{\nu\beta\rho\tau}\, h^{\nu\sigma}\Box h^{\alpha\rho}\,\partial^\tau\partial_\sigma h_\alpha^\beta = \int \xi^\nu \Big(\frac{1}{2}\epsilon_{\nu\beta\rho\tau}\,\partial_\kappa\partial^\rho h^{\beta\sigma}\,\partial_\sigma\partial^\kappa\partial^\tau h\\ &\quad -2\epsilon_{\nu\rho\beta\tau}\,\partial_\kappa\partial^\alpha h^{\rho\sigma}\,\partial_\sigma\partial^\kappa\partial^\tau h_\alpha^\beta - \epsilon_{\nu\beta\rho\tau}\,\partial_\kappa\partial^\rho h^{\alpha\sigma}\,\partial^\kappa\partial^\tau\partial_\sigma h_\alpha^\beta\Big),\end{aligned} \tag{G.14}$$

and other similar ones. Using combinations of these relations it is easy to see that all the terms listed above, which appear in (7.46), see (G.4), (G.6) and (G.7), are in fact trivial. They can be reabsorbed in a redefinition of the action without altering the already calculated trace anomaly.

Chapter 8

Massive fermion model in 3d

In this chapter we analyze the 3d free massive fermion theory coupled to external sources. The presence of a mass explicitly breaks parity invariance. We calculate two- and three-point functions of a gauge current and the energy momentum tensor and, for instance, obtain the well-known result that in the IR limit (but also in the UV one) we reconstruct the relevant CS action. We then couple the model to higher spin currents and explicitly work out the spin 3 case. In the UV limit we obtain an effective action which was proposed many years ago as a possible generalization of spin 3 CS action. In the IR limit we derive a different higher spin action. This analysis can evidently be generalized to higher spins. We also discuss the conservation and properties of the correlators we obtain in the intermediate steps of our derivation.

8.1 The 3d massive fermion model coupled to external sources

The simplest model is that of a Dirac fermion[1] coupled to a gauge field. The action is

$$S[A] = \int d^3x \, \left[i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi \right], \quad D_\mu = \partial_\mu + A_\mu, \tag{8.1}$$

where $A_\mu = A_\mu^a(x)T^a$ and T^a are the generators of a gauge algebra in a given representation determined by ψ. We will use the antihermitean convention, so $[T^a, T^b] = f^{abc}T^c$, and the normalization $\mathrm{tr}(T^aT^b) = \delta^{ab}$. The current

$$J_\mu^a(x) = \bar{\psi}\gamma_\mu T^a \psi \tag{8.2}$$

is (classically) covariantly conserved on shell as a consequence of the gauge invariance of (8.1)

$$(DJ)^a = (\partial^\mu \delta^{ac} + f^{abc}A^{b\mu})J_\mu^c = 0. \tag{8.3}$$

The next example involves the coupling to gravity

$$S[g] = \int d^3x \, e \left[i\bar{\psi}E_a^\mu \gamma^a \nabla_\mu \psi - m\bar{\psi}\psi \right], \quad \nabla_\mu = \partial_\mu + \frac{1}{2}\omega_{\mu bc}\Sigma^{bc}, \quad \Sigma^{bc} = \frac{1}{4}\left[\gamma^b, \gamma^c\right]. \tag{8.4}$$

The corresponding energy momentum tensor

$$T_{\mu\nu} = \frac{i}{4}\bar{\psi}\left(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu\right)\psi \tag{8.5}$$

[1]The minimal representation of the Lorentz group in 3d is a real Majorana fermion. A Dirac fermion is a complex combination of two Majorana fermions. The action for a Majorana fermion is $\frac{1}{2}$ of (8.1).

is covariantly conserved on shell as a consequence of the diffeomorphism invariance of the action,

$$\nabla^\mu T_{\mu\nu}(x) = 0. \tag{8.6}$$

However we can couple the fermions to more general fields. Consider the free action

$$S = \int d^3x \,\left[i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi\right], \tag{8.7}$$

and the spin three conserved current

$$\begin{aligned} J_{\mu_1\mu_2\mu_3} &= \frac{1}{2}\bar{\psi}\gamma_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3)}\psi + \frac{1}{2}\partial_{(\mu_1}\partial_{\mu_2}\bar{\psi}\gamma_{\mu_3)}\psi - \frac{5}{3}\partial_{(\mu_1}\bar{\psi}\gamma_{\mu_2}\partial_{\mu_3)}\psi \\ &+\frac{1}{3}\eta_{(\mu_1\mu_2}\partial^\sigma\bar{\psi}\gamma_{\mu_3)}\partial_\sigma\psi - \frac{m^2}{3}\eta_{(\mu_1\mu_2}\bar{\psi}\gamma_{\mu_3)}\psi. \end{aligned} \tag{8.8}$$

Using the equation of motion one can prove that

$$\partial^\mu J_{\mu\nu\lambda} = 0, \tag{8.9}$$

$$J_\mu{}^\mu{}_\lambda = \frac{4}{9}m\left(-i\partial_\lambda\bar{\psi}\psi + i\bar{\psi}\partial_\lambda\psi + 2\bar{\psi}\gamma_\lambda\psi\right). \tag{8.10}$$

Therefore, the spin three current (8.8) is conserved on shell and its tracelessness is softly broken by the mass term. Similarly to the gauge field and the metric, we can couple the fermion ψ to a new external source $b_{\mu\nu\lambda}$ by adding to (8.7) the term

$$\int d^3x J_{\mu\nu\lambda}b^{\mu\nu\lambda}. \tag{8.11}$$

Notice that this requires b to have canonical dimension -1. Due to the (on shell) current conservation this coupling is invariant under the (infinitesimal) gauge transformations

$$\delta b_{\mu\nu\lambda} = \partial_{(\mu}\Lambda_{\nu\lambda)}, \tag{8.12}$$

where round brackets stand for symmetrization. In the limit $m \to 0$ we have also invariance under the local transformations

$$\delta b_{\mu\nu\lambda} = \Lambda_{(\mu}\eta_{\nu\lambda)}, \tag{8.13}$$

which are usually referred to as (generalized) Weyl transformations and which induce the tracelessness of $J_{\mu\nu\lambda}$ in any couple of indices.

The construction of conserved currents can be generalized as follows, see [29, 30]. There is a generating function for $J^{(n)}$. Introduce the following symbols

$$u_\mu = \overrightarrow{\partial}_\mu, \qquad v_\mu = \overleftarrow{\partial}_\mu, \qquad \langle uv\rangle = u^\mu v_\mu, \qquad \langle uz\rangle = u^\mu z_\mu, \qquad \langle\gamma z\rangle = \gamma^\mu z_\mu, \qquad \text{etc},$$

where z^μ are external parameters. Now define

$$J(x;z) = \sum_n J^{(n)}_{\mu_1\ldots\mu_n}z^{\mu_1}\ldots z^{\mu_n} = \bar{\psi}\langle\gamma z\rangle F(u,v,z)\psi, \tag{8.14}$$

where

$$F(u,v,z) = e^{(\langle uz\rangle - \langle vz\rangle)}\, f(X), \qquad f(X) = \frac{\sinh\sqrt{X}}{\sqrt{X}}, \qquad X = 2\langle uv\rangle\langle zz\rangle - 4\langle uz\rangle\langle vz\rangle. \tag{8.15}$$

Defining next the operator $\mathcal{D} = \langle (u+v)\frac{\partial}{\partial z}\rangle$, it is easy to prove that, using the free equation of motion,

$$\mathcal{D}J(x;z) = 0. \tag{8.16}$$

Therefore all the homogeneous terms in z in $J(x;z)$ are conserved if $m=0$. If $m \neq 0$ one has to replace X with $Y = X - 2m^2\langle zz\rangle$. Then we define

$$J_m(x;z) = \sum_n J^{(n)}_{\mu_1\dots\mu_n} z^{\mu_1}\dots z^{\mu_n} = \bar\psi\langle\gamma z\rangle e^{(\langle uz\rangle - \langle vz\rangle)}\, f(Y)\psi \tag{8.17}$$

and one can prove that

$$\mathcal{D}J_m(x;z) = 0, \tag{8.18}$$

with $m \neq 0$. The case $J^{(3)}$ in (8.17) coincides with the third order current introduced before.

For any conserved current $J^{(n)}_{\mu_1\dots\mu_n}$ we can introduce an associated source field $b^{\mu_1\dots\mu_n}$ similar to the rank three one introduced above, with a transformation law that generalizes (8.12). However, in this regard, a remark is in order. In fact, (8.12) has to be understood as the transformation of the fluctuating field $b_{\mu\nu\lambda}$, which is the lowest order term in the expansion of a field $B_{\mu\nu\lambda} = b_{\mu\nu\lambda} + \dots$ whose background value is 0. $b_{\mu\nu\lambda}$ plays a role similar to $h_{\mu\nu}$ in the expansion of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \dots$ (see also Appendix B). In order to implement full invariance we should introduce in the free action the analog of the spin connection for $B_{\mu\nu\lambda}$ and a full covariant conservation law would require introducing in (8.9) the analog of the Christoffel symbols.

8.1.1 Generating function for effective actions

The generating function of the effective action of (8.1) is

$$W[A] = \sum_{n=1}^{\infty}\frac{i^{n+1}}{n!}\int\prod_{i=1}^{n} d^3x_i A^{a_1\mu_1}(x_1)\dots A^{a_n\mu_n}(x_n)\langle 0|\mathcal{T}J^{a_1}_{\mu_1}(x_1)\dots J^{a_n}_{\mu_n}(x_n)|0\rangle, \tag{8.19}$$

where the time ordered correlators are understood to be those obtained with the Feynman rules. The full one-loop 1-point correlator for J^a_μ is

$$\begin{aligned}\langle\!\langle J^a_\mu(x)\rangle\!\rangle &= \frac{\delta W[A]}{\delta A^{a\mu}(x)} \\ &= -\sum_{n=1}^{\infty}\frac{i^n}{n!}\int\prod_{i=1}^{n} d^3x_i A^{a_1\mu_1}(x_1)\dots A^{a_n\mu_n}(x_n)\langle 0|\mathcal{T}J^a_\mu(x)J^{a_1}_{\mu_1}(x_1)\dots J^{a_n}_{\mu_n}(x_n)|0\rangle.\end{aligned} \tag{8.20}$$

Later on we will need also the one-loop conservation

$$(D^\mu\langle\!\langle J_\mu(x)\rangle\!\rangle)^a = \partial^\mu\langle\!\langle J^a_\mu(x)\rangle\!\rangle + f^{abc}A^b_\mu(x)\langle\!\langle J^{\mu c}(x)\rangle\!\rangle = 0. \tag{8.21}$$

We can easily generalize this to the case of higher tensor currents $J^{(p)}$. The generating function is

$$\begin{aligned}W^{(p)}[a] = \sum_{n=1}^{\infty}\frac{i^{n+1}}{n!}\int\prod_{i=1}^{n} d^3x_i a^{\mu_{11}\dots\mu_{1p}}(x_1)\dots a^{\mu_{n1}\dots\mu_{np}}(x_n) \\ \times\langle 0|\mathcal{T}J^{(p)}_{\mu_{11}\dots\mu_{1p}}(x_1)\dots J^{(p)}_{\mu_{n1}\dots\mu_{np}}(x_n)|0\rangle.\end{aligned} \tag{8.22}$$

In particular $a_{\mu\nu} = h_{\mu\nu}$ and $J^{(2)}_{\mu\nu} = T_{\mu\nu}$, and $a_{\mu\nu\lambda} = b_{\mu\nu\lambda}$. The full one-loop 1-pt correlator for J^a_μ is

$$\langle\langle J^{(p)}_{\mu_1\ldots\mu_p}(x)\rangle\rangle = \frac{\delta W[a,p]}{\delta a^{\mu_1\ldots\mu_p}(x)} = -\sum_{n=1}^{\infty}\frac{i^n}{n!}\int\prod_{i=1}^{n} d^3x_i a^{\mu_{11}\ldots\mu_{1p}}(x_1)\ldots a^{\mu_{n1}\ldots\mu_{np}}(x_n)$$
$$\times\langle 0|\mathcal{T} J^{(p)}_{\mu_1\ldots\mu_p}(x) J^{(p)}_{\mu_{11}\ldots\mu_{1p}}(x_1)\ldots J^{(p)}_{\mu_{n1}\ldots\mu_{np}}(x_n)|0\rangle. \quad (8.23)$$

The full one-loop conservation law for the energy-momentum tensor is

$$\nabla^\mu\langle\langle T_{\mu\nu}(x)\rangle\rangle = 0. \quad (8.24)$$

A similar covariant conservation should be written also for the other currents, but in this paper for $p > 2$ we will content ourselves with the lowest nontrivial order in which the conservation law reduces to

$$\partial^{\mu_1}\langle\langle J^{(p)}_{\mu_1\ldots\mu_p}(x)\rangle\rangle = 0. \quad (8.25)$$

Warning. One must be careful when applying the previous formulas for generating functions. If the expression $\langle 0|\mathcal{T} J^{(p)}_{\mu_{11}\ldots\mu_{1p}}(x_1)\ldots J^{(p)}_{\mu_{n1}\ldots\mu_{np}}(x_n)|0\rangle$ in (8.22) is meant to denote the n-th point-function calculated by using Feynman diagrams, a factor i^n is already included in the diagram themselves and so it should be dropped in (8.22). When the current is the energy-momentum tensor an additional precaution is necessary: the factor $\frac{i^{n+1}}{n!}$ must be replaced by $\frac{i}{2^n n!}$. The factor $\frac{1}{2^n}$ is motivated by the fact that when we expand the action

$$S[\eta + h] = S[\eta] + \int d^dx \frac{\delta S}{\delta g^{\mu\nu}}\Big|_{g=\eta} h^{\mu\nu} + \cdots,$$

the factor $\frac{\delta S}{\delta g^{\mu\nu}}\Big|_{g=\eta} = \frac{1}{2}T_{\mu\nu}$. Another consequence of this fact will be that the presence of vertices with one graviton in Feynman diagrams will correspond to insertions of the operator $\frac{1}{2}T_{\mu\nu}$ in correlation functions.

8.1.2 General structure of 2-point functions of currents

In order to compute the generating function (effective action) W we will proceed in the next section to evaluate 2-point and 3-point correlators using the Feynman diagram approach. It is however possible to derive their general structure on the basis of covariance. In this subsection we will analyze the general form of 2-point correlators.

As long as 2-point correlators of currents are involved the conservation law is simply represented by the vanishing of the correlator divergence:

$$\partial^{\mu_1}\langle 0|\mathcal{T} J^{(p)}_{\mu_1\ldots\mu_p}(x) J^{(p)}_{\nu_1\ldots\nu_p}(y)|0\rangle = 0. \quad (8.26)$$

Using Poincaré covariance and this equation we can obtain the general form of the correlators in momentum space in terms of distinct tensorial structures and form factors. Denoting by

$$\tilde{J}_{\mu_1\ldots\mu_p,\nu_1\ldots\nu_p}(k) = \langle\tilde{J}^{(p)}_{\mu_1\ldots\mu_p}(k)\tilde{J}^{(p)}_{\nu_1\ldots\nu_p}(-k)\rangle \quad (8.27)$$

the Fourier transform of the 2-point function, the conservation is simply represented by the contraction of $\tilde{F}_{\mu\dots}$ with k^μ:

$$k^{\mu_1}\tilde{J}_{\mu_1\dots\mu_p,\nu_1\dots\nu_p}(k) = 0. \tag{8.28}$$

The result is as follows. For 1-currents we have

$$\tilde{J}^{ab}_{\mu\nu}(k) = \langle \tilde{J}^a_\mu(k)\tilde{J}^b_\nu(-k)\rangle = \delta^{ab}\left[\tau\left(\frac{k^2}{m^2}\right)\frac{k_\mu k_\nu - k^2\eta_{\mu\nu}}{16|k|} + \kappa\left(\frac{k^2}{m^2}\right)\frac{k^\tau\epsilon_{\tau\mu\nu}}{2\pi}\right]. \tag{8.29}$$

where $|k| = \sqrt{k^2}$ and τ, κ are model dependent form factors.

The most general 2-point function for the energy-momentum tensor has the form

$$\begin{aligned}\left\langle \tilde{T}_{\mu\nu}(k)\,\tilde{T}_{\rho\sigma}(-k)\right\rangle \;=\;& \frac{\tau_g\left(k^2/m^2\right)}{|k|}\left(k_\mu k_\nu - \eta_{\mu\nu}k^2\right)\left(k_\rho k_\sigma - \eta_{\rho\sigma}k^2\right)\\ &+\frac{\tau_g'\left(k^2/m^2\right)}{|k|}\left[\left(k_\mu k_\rho - \eta_{\mu\rho}k^2\right)\left(k_\nu k_\sigma - \eta_{\nu\sigma}k^2\right) + \mu\leftrightarrow\nu\right] \\ &+\frac{\kappa_g\left(k^2/m^2\right)}{192\pi}\left[\left(\epsilon_{\mu\rho\tau}k^\tau\left(k_\nu k_\sigma - \eta_{\nu\sigma}k^2\right) + \rho\leftrightarrow\sigma\right) + \mu\leftrightarrow\nu\right].\end{aligned} \tag{8.30}$$

where τ_g, τ_g' and κ_g are model-dependent form-factors. Vanishing of traces over $(\mu\nu)$ or $(\rho\sigma)$ requires $\tau_g + \tau_g' = 0$. Both here and in the previous case, the notation, the signs and the numerical factors are made to match our definition with the ones used in [28]. [2]

As for the order 3 tensor currents the most general form of the 2-point function in momentum representation is

$$\begin{aligned}\langle \tilde{J}_{\mu_1\mu_2\mu_3}(k)\tilde{J}_{\nu_1\nu_2\nu_3}(-k)\rangle = \tau_b\left(\frac{k^2}{m^2}\right)|k|^5\pi_{\mu_1\mu_2}\pi_{\mu_3\nu_1}\pi_{\nu_2\nu_3} + \tau_b'\left(\frac{k^2}{m^2}\right)|k|^5\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\\ + k^4\epsilon_{\mu_1\nu_1\sigma}k^\sigma\left[\kappa_b\left(\frac{k^2}{m^2}\right)\pi_{\mu_2\mu_3}\pi_{\nu_2\nu_3} + \kappa_b'\left(\frac{k^2}{m^2}\right)\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\right],\end{aligned} \tag{8.31}$$

where complete symmetrisation of the indices (μ_1, μ_2, μ_3) and (ν_1, ν_2, ν_3) is implicit[3] and

$$\pi_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \tag{8.32}$$

is the transverse projector. This expression is, by construction, conserved but not traceless. Vanishing of traces requires

$$4\tau_b + 3\tau_b' = 0, \qquad 4\kappa_b + \kappa_b' = 0. \tag{8.33}$$

8.2 Two-point functions

In this section we compute the the 2-point function of spin 1, 2 and 3 currents using Feynman diagrams with finite mass m. Then we take the limit $m \to 0$ or $m \to \infty$ with respect to the total energy of the process, i.e. the UV and IR limit of the 2-point functions, respectively.

[2] Except that we work in spacetime with Lorentzian signature $(+--)$.

[3] When we say that the complete symmetrisation is implicit it means that one should understand, for instance

$$\pi_{\mu_1\mu_2}\pi_{\mu_3\nu_1}\pi_{\nu_2\nu_3} \to \frac{1}{9}\left[\pi_{\mu_1\mu_2}\pi_{\mu_3\nu_1}\pi_{\nu_2\nu_3} + \pi_{\mu_1\mu_3}\pi_{\mu_2\nu_1}\pi_{\nu_2\nu_3} + \dots\right].$$

These are expected to correspond to 2-point functions of conformal field theories at the relevant fixed points. We will be mostly interested in the odd parity part of the correlators, because in the UV and IR limit they give rise to local effective actions, but occasionally we will also consider the even parity part.

8.2.1 Two-point function of the current $J^a_\mu(x)$

This case has been treated in [35], therefore we will be brief. The only contribution comes from the bubble diagram with external momentum k and momentum p in the fermion loop. In momentum representation we have

$$\begin{aligned}\tilde{J}^{ab}_{\mu\nu}(k) = -\int \frac{d^3p}{(2\pi)^3}\mathrm{Tr}\left(\gamma_\mu T^a \frac{1}{\not p - m}\gamma_\nu T^b \frac{1}{\not p - \not k - m}\right) = -2\delta^{ab} \\ \times \int \frac{d^3p}{(2\pi)^3}\frac{p_\nu(p-k)_\mu - p\cdot(p-k)\eta_{\mu\nu} + p_\mu(p-k)_\nu + im\epsilon_{\mu\nu\sigma}k^\sigma + m^2\eta_{\mu\nu}}{(p^2-m^2)((p-k)^2-m^2)}\end{aligned} \tag{8.34}$$

For the even parity part we get

$$\tilde{J}^{ab(even)}_{\mu\nu}(k) = \frac{2i}{\pi}\delta^{ab}\left[\left(1+\frac{4m^2}{k^2}\right)\mathrm{arctanh}\left(\frac{|k|}{2|m|}\right) - \frac{2|m|}{|k|}\right]\frac{k_\mu k_\nu - k^2\eta_{\mu\nu}}{16|k|}, \tag{8.35}$$

while for the odd parity part we get

$$\tilde{J}^{ab(odd)}_{\mu\nu}(k) = \frac{1}{2\pi}\delta^{ab}\epsilon_{\mu\nu\sigma}k^\sigma \frac{m}{|k|}\mathrm{arctanh}\left(\frac{|k|}{2|m|}\right) \tag{8.36}$$

where $|k| = \sqrt{k^2}$. The conservation law (8.28) is readily seen to be satisfied. In the following we are going to consider the IR and UV limit of the expressions (8.35) and (8.36) and it is important to remark that we have two possibilities here: we may consider a timelike momentum ($k^2 > 0$) or a spacelike one ($k^2 < 0$). In the first case, we must notice that the function $\mathrm{arctanh}\left(\frac{|k|}{2|m|}\right)$ has branch-cuts on the real axis for $\frac{|k|}{2|m|} > 1$ and it acquires an imaginary part. On the other hand, if we consider spacelike momenta, we will have $\mathrm{arctanh}\left(\frac{i|k|}{2|m|}\right) = i\arctan\left(\frac{|k|}{2|m|}\right)$ and $\arctan\left(\frac{|k|}{2|m|}\right)$ is real on the real axis. The region of spacelike momenta reproduces the Euclidean correlators. Throughout this paper we will always consider UV and IR limit as being respectively the limits of very large or very small spacelike momentum with respect to the mass scale m. In these two limits we get

$$\tilde{J}^{ab(even)}_{\mu\nu}(k) = \frac{i}{8\pi}\delta^{ab}\frac{k_\mu k_\nu - k^2\eta_{\mu\nu}}{|k|}\begin{cases}\frac{2|k|}{3|m|} & \mathrm{IR} \\ \frac{\pi}{2} & \mathrm{UV}\end{cases}, \tag{8.37}$$

$$\tilde{J}^{ab(odd)}_{\mu\nu}(k) = \frac{1}{2\pi}\delta^{ab}\epsilon_{\mu\nu\sigma}k^\sigma\begin{cases}\frac{1}{2}\frac{m}{|m|} & \mathrm{IR} \\ \frac{\pi}{2}\frac{m}{|k|} & \mathrm{UV}\end{cases}. \tag{8.38}$$

The UV limit is actually vanishing in the odd case (this is also the case for all the 2-point functions we will meet in the following). However we can consider a model made of N identical copies of free fermions coupled to the same gauge field. Then the result (8.38) would be

$$\tilde{J}^{ab(odd)}_{\mu\nu}(k) = \frac{N}{4}\delta^{ab}\epsilon_{\mu\nu\sigma}k^\sigma\frac{m}{|k|}. \tag{8.39}$$

In this case we can consider the scaling limit $\frac{m}{|k|} \to 0$ and $N \to \infty$ in such a way that $N\frac{m}{|k|}$ is fixed. Then the UV limit (8.39) becomes nonvanishing.

Fourier transforming (8.38) and inserting the result in the generating function (8.19) we get the first (lowest order) term of the CS action

$$\begin{aligned} CS &= \frac{\kappa}{4\pi}\int d^3x \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \\ &= \frac{\kappa}{4\pi}\int d^3x \epsilon^{\mu\nu\lambda}\left(A^a_\mu\partial_\nu A^a_\lambda + \frac{1}{3}f^{abc}A^a_\mu A^b_\nu A^c_\lambda\right). \end{aligned} \tag{8.40}$$

In particular, from (8.38) we see that in the IR limit $\kappa = \pm\frac{1}{2}$. The CS action (8.40) is invariant not only under the infinitesimal gauge transformations

$$\delta A = d\lambda + [A,\lambda], \qquad \lambda = \lambda^a(x)T^a, \tag{8.41}$$

but also under large gauge transformations when $\kappa \in \mathbb{Z}$. From (8.38) follows that $\kappa_{UV} = 0$ and $\kappa_{IR} = \pm 1/2$, which suggests that the gauge symmetry is broken unless there is an even number of fermions. A further discussion of this phenomenon can be found in [28].

8.2.2 Two-point function of the e.m. tensor

The lowest term of the effective action in an expansion in $h_{\mu\nu}$ come from the two-point function of the e.m. tensor. So we now set out to compute the latter. The correlators of the e.m. tensor will be denoted with the letter $\tilde{T}$ instead of $\tilde{J}$. The Feynman propagator and vertices are given in Appendix H.2. For simplicity from now on we assume $m > 0$.

The bubble diagram (one graviton entering and one graviton exiting with momentum k, one fermionic loop) contribution to the e.m. two-point function is given in momentum space by

$$\tilde{T}_{\mu\nu\lambda\rho}(k) = -\frac{1}{64}\int\frac{d^3p}{(2\pi)^3}\mathrm{Tr}\left(\frac{1}{\not p - m}(2p-k)_\mu\gamma_\nu\frac{1}{\not p - \not k - m}(2p-k)_\lambda\gamma_\rho\right), \tag{8.42}$$

where symmetrization over (μ,ν) and (λ,ρ) will be always implicit.

8.2.2.1 The odd parity part

The odd-parity part of (8.42) is

$$\tilde{T}^{(odd)}_{\mu\nu\lambda\rho}(k) = \frac{im}{32}\int_0^1 dx\int\frac{d^3p}{(2\pi)^3}\epsilon_{\sigma\nu\rho}k^\sigma\,\frac{(2p+(2x-1)k)_\mu\,(2p+(2x-1)k)_\lambda}{[p^2-m^2+x(1-x)k^2]^2}. \tag{8.43}$$

The evaluation of this integral is described in detail in Appendix H.4. The result is

$$\begin{aligned} \tilde{T}^{(odd)}_{\mu\nu\lambda\rho}(k) = -\frac{3m}{4|k|}\left[\left(1-\frac{4m^2}{k^2}\right)\mathrm{arctanh}\left(\frac{|k|}{2|m|}\right)+\frac{2|m|}{|k|}\right]\frac{\epsilon_{\mu\lambda\sigma}k^\sigma\left(k_\nu k_\rho-\eta_{\nu\rho}k^2\right)}{192\pi} - \\ -\frac{\mathrm{sign}(m)|m|^2}{64\pi}\epsilon_{\mu\lambda\sigma}k^\sigma\eta_{\nu\rho}. \end{aligned} \tag{8.44}$$

A surprising feature of (8.44) is that if we contract it with k^μ we do not get zero. Let us look closer into this problem.

8.2.2.2 The divergence of the e.m. tensor: odd-parity part

To see whether the expression of the one-loop effective action is the legitimate one, one must verify that the procedure to obtain it does not break diffeomorphism invariance. The bubble diagram contribution to the divergence of the e.m. tensor is

$$\begin{aligned} k^\mu \tilde{T}_{\mu\nu\lambda\rho}(k) &= -\frac{1}{64}\int \frac{d^3p}{(2\pi)^3}\left[\mathrm{Tr}\left(\frac{1}{\not p - m}(2p-k)\cdot k\,\gamma_\nu \frac{1}{\not p - \not k - m}(2p-k)_\lambda \gamma_\rho\right)\right. \\ &\left. + \,\mathrm{Tr}\left(\frac{1}{\not p - m}(2p-k)_\nu\, \not k \frac{1}{\not p - \not k - m}(2p-k)_\lambda \gamma_\rho\right)\right] + (\lambda \leftrightarrow \rho)\,. \end{aligned} \tag{8.45}$$

Repeating the same calculation as above one finally finds

$$k^\mu \tilde{T}^{(odd)}_{\mu\nu\lambda\rho}(k) = -\frac{\mathrm{sign}(m)|m|^2}{64\pi}\epsilon_{\sigma\nu\rho}\, k^\sigma k_\lambda + (\lambda \leftrightarrow \rho)\,. \tag{8.46}$$

This is a local expression. It corresponds to the anomaly

$$\Delta_\xi = -\frac{\mathrm{sign}(m)|m|^2}{32\pi}\int \epsilon_{\sigma\nu\rho}\xi^\nu \partial^\sigma \partial_\lambda h^{\lambda\rho}. \tag{8.47}$$

The counterterm to cancel it is

$$\mathcal{C} = \frac{\mathrm{sign}(m)|m|^2}{64\pi}\int \epsilon_{\sigma\nu\rho} h^\nu_\lambda \partial^\sigma h^{\lambda\rho}. \tag{8.48}$$

Once this is done the final result is

$$\langle T_{\mu\nu}(k)\, T_{\lambda\rho}(-k)\rangle_{odd} = \frac{\kappa_g(k^2/m^2)}{192\pi}\epsilon_{\sigma\nu\rho}\, k^\sigma \left(k_\mu k_\lambda - k^2 \eta_{\mu\lambda}\right) + \begin{pmatrix} \mu \leftrightarrow \nu \\ \lambda \leftrightarrow \rho \end{pmatrix} \tag{8.49}$$

with

$$\kappa_g(k^2/m^2) = -\frac{3m}{|k|}\left[\left(1 - \frac{4m^2}{k^2}\right)\mathrm{arctanh}\left(\frac{|k|}{2|m|}\right) + \frac{2|m|}{|k|}\right]. \tag{8.50}$$

Now (8.49) is conserved and traceless. To obtain (8.50) we have to recall that

$$\tilde{T}_{\mu\nu\lambda\rho}(k) = \frac{1}{4}\langle T_{\mu\nu}(k)\, T_{\lambda\rho}(-k)\rangle, \tag{8.51}$$

which was explained in the warning of section 8.1.1. To complete the discussion we should also take into account a tadpole graph which might contribute to the two-point function. With the vertex V_{ggff} it is in fact possible to construct such a graph. It yields the contribution

$$\frac{3}{32\pi}\mathrm{sign}(m)|m|^2\, t_{\mu\nu\lambda\rho\sigma}\, k^\sigma. \tag{8.52}$$

This term violates conservation, just as the previous (8.46), but it has a different coefficient. So it must be subtracted in the same way.

8.2.2.3 The UV and IR limit

Let us set $\lim_{|\frac{m}{k}|\to 0}\kappa_g = \kappa_{UV}$, and $\lim_{|\frac{k}{m}|\to 0}\kappa_g = \kappa_{IR}$. We get

$$\kappa_{IR} = \frac{m}{|m|}, \qquad \kappa_{UV} = \frac{3}{2}\pi\frac{m}{|k|} = 0 + \mathcal{O}\left(\left|\frac{m}{k}\right|\right). \tag{8.53}$$

As before for the gauge case, in the UV limit we can get a finite result by considering a system of N identical fermions. Then the above 2-point function gets multiplied by N. In the UV limit, $\left|\frac{m}{k}\right| \to 0$, we can consider the scaling limit $N \to \infty, \left|\frac{m}{k}\right| \to 0$ such that

$$\lambda = N\frac{m}{|k|} \tag{8.54}$$

is fixed and finite. In this limit

$$\lim_{N\to\infty, \left|\frac{m}{k}\right|\to 0} N\kappa_g(k) = \frac{3\pi}{2}\frac{m}{|m|}\lambda. \tag{8.55}$$

For a unified treatment let us call both UV and IR limits of κ_g simply κ. In such limits contribution to the parity odd part of the effective action can be easily reconstructed by

$$S_{\text{eff}}^{(odd)} = \frac{\kappa}{192\pi}\int d^3x\, \epsilon_{\sigma\nu\rho}\, h^{\mu\nu}\, \partial^\sigma (\partial_\mu\partial_\lambda - \eta_{\mu\lambda}\Box)\, h^{\lambda\rho}. \tag{8.56}$$

This exactly corresponds to a gravitational CS term in 3d, for which at the quadratic order in $h_{\mu\nu}$ we have

$$\begin{aligned} CS &= -\frac{\kappa}{96\pi}\int d^3x\, \epsilon^{\mu\nu\lambda}\left(\partial_\mu\omega_\nu^{ab}\omega_{\lambda ba} + \frac{2}{3}\omega_{\mu a}{}^{b}\omega_{\nu b}{}^{c}\omega_{\lambda c}{}^{a}\right) \\ &= \frac{\kappa}{192\pi}\int d^3x\, \epsilon_{\sigma\nu\rho}\, h^{\lambda\rho}\left(\partial^\sigma\partial_\lambda\partial_b h^{b\nu} - \partial^\sigma\Box h_\lambda^\nu\right) + \dots \end{aligned} \tag{8.57}$$

Once again we note that the topological arguments combined with path integral quantization force κ to be an integer ($\kappa \in \mathbb{Z}$). From $\kappa_{IR} = \pm 1$ we see that the quantum contribution to the parity-odd part of the effective action in the IR is given by the local gravitational CS term, with the minimal (positive or negative unit) coupling constant. The constant $\frac{3\pi}{2}\lambda$ in the UV has of course to be integer in order for the action to be well defined also for large gauge transformations. Finally we recall that the CS Lagrangian is diffeomorphism and Weyl invariant up to a total derivative. However, note that for the Majorana fermion one would obtain half of the result as for the Dirac fermion, i.e. $\kappa_{IR} = \pm 1/2$.

8.2.2.4 Two-point function: even parity part

Although in this paper we are mostly interested in the odd-parity amplitudes, for completeness in the following we calculate also the even parity part of the e.m. tensor 2-point correlator.

The even parity part of the two-point function comes from the bubble diagram alone, eq.(8.42). Proceeding in the same way as above one gets

$$\begin{aligned} \tilde{T}^{(even)}_{\mu\nu\lambda\rho}(k) &= \frac{1}{4}\tau_g\left(\frac{k^2}{m^2}\right)\frac{1}{|k|}\left(k_\mu k_\nu - \eta_{\mu\nu}k^2\right)\left(k_\lambda k_\rho - \eta_{\lambda\rho}k^2\right) \\ &+ \frac{1}{4}\tau_g'\left(\frac{k^2}{m^2}\right)\frac{1}{|k|}\left[\left(k_\mu k_\lambda - \eta_{\mu\lambda}k^2\right)\left(k_\nu k_\rho - \eta_{\nu\rho}k^2\right) + \mu \leftrightarrow \nu\right] \\ &- \frac{im^3}{48\pi}\left(\eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda} + 2\eta_{\mu\nu}\eta_{\lambda\rho}\right), \end{aligned} \tag{8.58}$$

where

$$\tau_g\left(\frac{k^2}{m^2}\right)=\frac{i}{64\pi|k|^3}\left[|m|k^2+4|m|^3-\frac{\left(k^2-4m^2\right)^2}{2|k|}\text{arctanh}\left(\frac{|k|}{2|m|}\right)\right], \tag{8.59}$$

$$\tau_g'\left(\frac{k^2}{m^2}\right)=\frac{i}{64\pi|k|^3}\left[4|m|^3-|m|k^2+\frac{\left(k^4-16m^4\right)}{2|k|}\text{arctanh}\left(\frac{|k|}{2|m|}\right)\right]. \tag{8.60}$$

Saturating (8.58) with k^μ we find

$$k^\mu\tilde{T}^{(even)}_{\mu\nu\lambda\rho}(k)=-\frac{im^3}{48\pi}\left(k_\lambda\eta_{\nu\rho}+k_\rho\eta_{\nu\lambda}+2k_\nu\eta_{\lambda\rho}\right). \tag{8.61}$$

The same result can be obtained directly from the even part of (8.45). The term (8.61) is local and corresponds to an anomaly proportional to

$$\mathcal{A}_\xi=\int\xi^\nu\left(\partial_\lambda h^\lambda_\nu+\partial_\nu h\right). \tag{8.62}$$

This can be eliminated by subtracting the counterterm

$$\mathcal{C}=-\frac{1}{2}\int(h^{\lambda\nu}h_{\lambda\nu}+h^2). \tag{8.63}$$

After this we can write

$$\begin{aligned}\langle T_{\mu\nu}(k)\,T_{\lambda\rho}(-k)\rangle_{even}=&\,\tau_g\left(\frac{k^2}{m^2}\right)\frac{1}{|k|}\left(k_\mu k_\nu-\eta_{\mu\nu}k^2\right)\left(k_\lambda k_\rho-\eta_{\lambda\rho}k^2\right)\\&+\tau_g'\left(\frac{k^2}{m^2}\right)\frac{1}{|k|}\left[\left(k_\mu k_\lambda-\eta_{\mu\lambda}k^2\right)\left(k_\nu k_\rho-\eta_{\nu\rho}k^2\right)+\mu\leftrightarrow\nu\right].\end{aligned} \tag{8.64}$$

The UV limit gives

$$\lim_{|\frac{k}{m}|\to\infty}\tau_g=-\lim_{|\frac{k}{m}|\to\infty}\tau_g'=\frac{1}{256}, \tag{8.65}$$

so that in this limit

$$\begin{aligned}\langle T_{\mu\nu}(k)\,T_{\lambda\rho}(-k)\rangle^{UV}_{even}=&-\frac{i}{256}\frac{1}{|k|}\Big(\left(k_\mu k_\nu-\eta_{\mu\nu}k^2\right)\left(k_\rho k_\lambda-\eta_{\rho\lambda}k^2\right)\\&-\left[\left(k_\mu k_\rho-\eta_{\mu\rho}k^2\right)\left(k_\nu k_\lambda-\eta_{\nu\lambda}k^2\right)+\mu\leftrightarrow\nu\right]\Big).\end{aligned} \tag{8.66}$$

This represents the two-point function of a CFT in 3d, which is a free theory, the massless limit of the massive fermion theory we are studying.

The IR limit of the form factors (8.59) and (8.60) is

$$\tau_g=\frac{1}{24\pi}\left|\frac{m}{k}\right|+\mathcal{O}\left(\left|\frac{k}{m}\right|\right), \tag{8.67}$$

$$\tau_g'=-\frac{1}{48\pi}\left|\frac{m}{k}\right|+\mathcal{O}\left(\left|\frac{k}{m}\right|\right). \tag{8.68}$$

In this limit we have

$$\langle T_{\mu\nu}(k)T_{\lambda\rho}(-k)\rangle^{IR}_{even} = \frac{i|m|}{96\pi}\left[\frac{1}{2}\left((k_\mu k_\lambda \eta_{\nu\rho} + \lambda \leftrightarrow \rho) + \mu \leftrightarrow \nu\right) - \right.$$
$$\left. -\left(k_\mu k_\nu \eta_{\lambda\rho} + k_\lambda k_\rho \eta_{\mu\nu}\right) - \frac{k^2}{2}\left(\eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda}\right) + k^2\eta_{\mu\nu}\eta_{\lambda\rho}\right]. \quad (8.69)$$

The expression (8.69) is transverse but not traceless because $\tau_g + \tau_g' \overset{IR}{\neq} 0$. To have a well-behaved IR limit we may add local counterterms to cancel the whole IR expression. That may be accomplished by simply performing the shifts

$$\tau_g \to \tau_g - \frac{i}{24\pi}\left|\frac{m}{k}\right|, \quad \tau_g' \to \tau_g' + \frac{i}{48\pi}\left|\frac{m}{k}\right|. \quad (8.70)$$

These shifts correspond to the addition of a set of local counterterms in the expression (8.64) and they do not change the UV behavior since they go to zero in that limit.

8.2.3 Two-point function of the spin 3 current

Let us recall that we have postulated for the spin 3 current an action term of the form

$$S_{int} \sim \int d^3x\, J_{\mu\nu\lambda} b^{\mu\nu\lambda}, \quad (8.71)$$

where b is a completely symmetric 3rd order tensor (in this subsection we assume $h_{\mu\nu} = 0$ for simplicity). This interaction term gives rise to the following b-field-fermion-fermion vertex V_{bff}

$$\frac{1}{2}\left(\gamma_{(\mu_1} q_{2\mu_2} q_{2\mu_3)} + q_{1(\mu_1} q_{1\mu_2}\gamma_{\mu_3)}\right) - \frac{5}{3} q_{1(\mu_1}\gamma_{\mu_2} q_{2\mu_3)} + \frac{1}{3}\eta_{(\mu_1\mu_2}\gamma_{\mu_3)}\left(q_1 \cdot q_2 + m^2\right), \quad (8.72)$$

where q_1 and q_2 are the incoming momenta of the two fermions. For a spin n current, the analogous vertex can be obtained from the formula

$$V_{bff} : \langle \gamma z\rangle\, e^{i\langle (q_1-q_2)z\rangle} f\left(-2\langle q_1 q_2\rangle\langle zz\rangle + 4\langle q_1 z\rangle\langle q_2 z\rangle - 2m^2\langle zz\rangle\right) \quad (8.73)$$

by differentiating with respect to z the right number of times (and setting $z = 0$).

As usual the contribution from the 2-point function comes from the bubble diagram with incoming and outgoing momentum k_μ. Using the V_{bff} vertex the bubble diagram gives

$$\begin{aligned}\tilde{J}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k) = &\int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\left(\frac{1}{\not p - m}\left[\frac{1}{2}\left(\gamma_{(\nu_1}(p-k)_{\nu_2}(p-k)_{\nu_3)} + p_{(\nu_1} p_{\nu_2}\gamma_{\nu_3)}\right)\right.\right.\\ &\left. + \frac{5}{3} p_{(\nu_1}\gamma_{\nu_2}(p-k)_{\nu_3)} - \frac{1}{3}\eta_{(\nu_1\nu_2}\gamma_{\nu_3)}\left(p\cdot(p-k) - m^2\right)\right]\frac{1}{\not p - \not k - m}\\ &\cdot\left[\frac{1}{2}\left(\gamma_{(\mu_1}(p-k)_{\mu_2}(p-k)_{\mu_3)} + p_{(\mu_1} p_{\mu_2}\gamma_{\mu_3)}\right)\right.\\ &\left.\left. + \frac{5}{3} p_{(\mu_1}\gamma_{\mu_2}(p-k)_{\mu_3)} - \frac{1}{3}\eta_{(\mu_1\mu_2}\gamma_{\mu_3)}\left(p\cdot(p-k) - m^2\right)\right]\right). \end{aligned} \quad (8.74)$$

The parity-even part of the final result is given by

$$\tilde{J}^{(even)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k) = \tau_b\left(\frac{k^2}{m^2}\right)|k|^5\pi_{\mu_1\mu_2}\pi_{\mu_3\nu_1}\pi_{\nu_2\nu_3} + \tau_b'\left(\frac{k^2}{m^2}\right)|k|^5\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3} + \mathcal{A}^{(even)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}, \tag{8.75}$$

where

$$\tau_b = \frac{i}{288\pi k^6}\Big[6|k||m|\left(k^4+8k^2m^2-32m^4\right) - \left(3\left(k^2-4m^2\right)^3+8m^2\left(k^2-6m^2\right)\left(k^2+4m^2\right)\right)\operatorname{arctanh}\left(\frac{|k|}{2|m|}\right)\Big], \tag{8.76}$$

$$\tau_b' = \frac{i}{216\pi k^6}\Big[-6|k||m|\left(k^4-\frac{8}{3}k^2m^2+16m^4\right) + +3\left(k^2-4m^2\right)^2\left(k^2+4m^2\right)\operatorname{arctanh}\left(\frac{|k|}{2|m|}\right)\Big] \tag{8.77}$$

and $\mathcal{A}^{(even)}$ corresponds to a set of contact terms that are not transverse but may be subtracted by local counterterms. It is given by

$$\mathcal{A}^{(even)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} = \frac{im^3}{9\pi}\Big[\frac{3}{4}k_{\mu_1}k_{\nu_1}\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3} + \frac{7}{8}\left(k_{\mu_1}k_{\mu_2}\eta_{\nu_1\nu_2}\eta_{\mu_3\nu_3} + k_{\nu_1}k_{\nu_2}\eta_{\mu_1\mu_2}\eta_{\mu_3\nu_3}\right) + \frac{32}{15}m^2\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3} + \frac{52}{15}m^2\eta_{\mu_1\nu_1}\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3} - \frac{3}{4}k^2\eta_{\mu_1\nu_1}\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3}\Big]. \tag{8.78}$$

The parity-odd part is given by

$$\tilde{J}^{(odd)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k) = k^4\epsilon_{\mu_1\nu_1\sigma}k^\sigma\left[\kappa_b\left(\frac{k^2}{m^2}\right)\pi_{\mu_2\mu_3}\pi_{\nu_2\nu_3} + \kappa_b'\left(\frac{k^2}{m^2}\right)\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\right] + \mathcal{A}^{(odd)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}, \tag{8.79}$$

where

$$\kappa_b = \frac{m}{72\pi|k|^5}\left[-20|k|^3|m| + 16|k||m|^3 + \left(k^4-32m^4\right)\operatorname{arctanh}\left(\frac{|k|}{2|m|}\right)\right], \tag{8.80}$$

$$\kappa_b' = \frac{m}{18\pi|k|^5}\left[2|k|^3|m| + 8|k||m|^3 - \left(k^2-4m^2\right)^2\operatorname{arctanh}\left(\frac{|k|}{2|m|}\right)\right], \tag{8.81}$$

and, as before, $\mathcal{A}^{(odd)}$ corresponds to a set of contact terms that are not transverse but may be subtracted by local counterterms. It is given by

$$\mathcal{A}^{(odd)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} = -\frac{\operatorname{sign}(m)|m|^2}{16\pi}\epsilon_{\mu_1\nu_1\sigma}k^\sigma\Big[\left(k_{\mu_2}k_{\mu_3}\eta_{\nu_2\nu_3} + k_{\nu_2}k_{\nu_3}\eta_{\mu_2\mu_3}\right) + \frac{128}{27}m^2\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3} + \frac{32}{27}m^2\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3} - k^2\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3}\Big]. \tag{8.82}$$

8.2.3.1 Even parity UV and IR limits

In the UV limit, i.e. $\left|\frac{m}{k}\right| \to 0$, we find

$$\lim_{\left|\frac{m}{k}\right|\to 0} \tau_b = -\frac{3}{4} \lim_{\left|\frac{m}{k}\right|\to 0} \tau_b' = \frac{1}{192}. \tag{8.83}$$

In the IR limit, i.e. $\left|\frac{k}{m}\right| \to 0$, we find

$$\tau_b = \frac{8}{135\pi}\left|\frac{m}{k}\right| + \mathcal{O}\left(\left|\frac{k}{m}\right|\right), \quad \tau_b' = -\frac{4}{135\pi}\left|\frac{m}{k}\right| + \mathcal{O}\left(\left|\frac{k}{m}\right|\right). \tag{8.84}$$

As in the case of the IR limit of the 2-point function of the stress-energy tensor, these leading divergent contributions of the form factors give rise to a set of contact terms in the IR that are all proportional to $|m|$. To add counter terms to make the IR well-behaved is equivalent to perform the following shift in the form factors τ_b and τ_b':

$$\tau_b \to \tau_b - \frac{8i}{135\pi}\left|\frac{m}{k}\right|, \quad \tau_b' \to \tau_b' + \frac{4i}{135\pi}\left|\frac{m}{k}\right|. \tag{8.85}$$

8.2.3.2 Odd parity UV

In the UV limit we find

$$\kappa_b = \frac{1}{144}\frac{m}{|k|} + \mathcal{O}\left(\left|\frac{m}{k}\right|^2\right), \quad \kappa_b' = -\frac{1}{36}\frac{m}{|k|} + \mathcal{O}\left(\left|\frac{m}{k}\right|^2\right). \tag{8.86}$$

As in the previous cases the UV is specified by the leading term in $\frac{m}{|k|}$. We get (after Wick rotation)

$$\tilde{J}^{(odd,UV)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k) = \frac{1}{4}\frac{m}{|k|}\,\epsilon_{\mu_1\nu_1\sigma}k^\sigma\Big[\frac{1}{12}k_{\mu_2}k_{\mu_3}k_{\nu_2}k_{\nu_3} - \frac{2}{9}k^2 k_{\mu_3}k_{\nu_3}\eta_{\mu_2\nu_2}$$
$$+\frac{k^2}{36}\left(k_{\nu_2}k_{\nu_3}\eta_{\mu_2\mu_3} + k_{\mu_2}k_{\mu_3}\eta_{\nu_2\nu_3}\right) + \frac{1}{9}k^4\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3} - \frac{1}{36}k^4\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3}\Big]. \tag{8.87}$$

From now on in this section we understand symmetrization among μ_1, μ_2, μ_3 and among ν_1, ν_2, ν_3. The anti-Wick rotation does not yield any change. We can contract (8.87) with any k^{μ_i} and any two indexes μ_i and find zero. Therefore (8.87) is conserved and traceless (it satisfies eq.(8.33)).

We have obtained the same result (8.87) with the method illustrated in Appendix H.5.

8.2.3.3 Odd parity IR

In the IR limit we find

$$\kappa_b = \frac{8}{27\pi}\frac{m^2}{k^2} + \frac{1}{240\pi} + \mathcal{O}\left(\left|\frac{k}{m}\right|^2\right), \quad \kappa_b' = -\frac{8}{27\pi}\frac{m^2}{k^2} - \frac{2}{135\pi} + \mathcal{O}\left(\left|\frac{m}{k}\right|^2\right). \tag{8.88}$$

Once again the IR limit contain divergent contributions that can be treated by adding local counter terms, which is equivalent to perform the following shifts on the form factors:

$$\kappa_b \to \kappa_b + \frac{8}{27\pi}\frac{m^2}{k^2}, \quad \kappa_b' \to \kappa_b' - \frac{8}{27\pi}\frac{m^2}{k^2}. \tag{8.89}$$

The final result in Lorentzian metric (obtained with the two different methods above) is

$$\tilde{J}^{(odd,IR)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k) = \frac{1}{4\pi}\epsilon_{\mu_1\nu_1\sigma}k^\sigma\Big[\frac{1}{60}k^4\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3} - \frac{8}{135}k^4\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}$$
$$- \frac{1}{60}k^2\left(k_{\nu_2}k_{\nu_3}\eta_{\mu_2\mu_3} + k_{\mu_2}k_{\mu_3}\eta_{\nu_2\nu_3}\right) + \frac{16}{135}k^2 k_{\mu_2}k_{\nu_2}\eta_{\mu_3\nu_3} - \frac{23}{540}k_{\mu_2}k_{\mu_3}k_{\nu_2}k_{\nu_3}\Big]. \quad (8.90)$$

The trace of (8.90) does not vanish. However at this point we must avoid a semantic trap. A nonvanishing trace of this kind does not contradict the fact that it represents a fixed point of the renormalization group. An RG fixed point is expected to be conformal, but this means vanishing of the e.m. trace, not necessarily of the trace of the spin three current.

8.2.3.4 The lowest order effective action for the field B

The odd 2-point correlator in a scaling UV limit similar to (8.55), Fourier anti-transformed and inserted in (8.22), gives rise to the action term

$$S^{(UV)} \sim \int d^3x \quad \epsilon_{\mu_1\nu_1\sigma}\Big[3\partial^\sigma B^{\mu_1\mu_2\mu_3}\partial_{\mu_2}\partial_{\mu_3}\partial_{\nu_2}\partial_{\nu_3}B^{\nu_1\nu_2\nu_3} - 8\partial^\sigma B^{\mu_1\mu_2\mu_3}\Box\partial_{\mu_3}\partial_{\nu_3}B^{\nu_1\nu_3}{}_{\mu_2}$$
$$+2\partial^\sigma B^{\mu_1\lambda}{}_\lambda\Box\partial_{\nu_2}\partial_{\nu_3}B^{\nu_1\nu_2\nu_3} + 4\partial^\sigma B^{\mu_1\mu_2\mu_3}\Box^2 B^{\nu_1}{}_{\mu_2\mu_3}$$
$$-\partial^\sigma B^{\mu_1\lambda}{}_\lambda\Box^2 B^{\nu_1\rho}{}_\rho\Big], \quad (8.91)$$

where $B_{\mu\nu\lambda} = b_{\mu\nu\lambda} + \ldots$. This is the lowest order term of the analog of the CS action for the field B. This theory is extremely constrained. The field B has 10 independent components. The gauge freedom counts 6 independent functions, the conservation equations are 3. The generalized Weyl (g-Weyl) invariance implies two additional degrees of freedom. So altogether the constraints are more than the degrees of freedom. The question is whether such CS actions contain nontrivial (i.e. non pure gauge) solutions.

In a similar way (8.90) gives rise to the action

$$S^{(IR)} = \frac{1}{32\pi}\frac{1}{540}\int d^3x \quad \epsilon_{\mu_1\nu_1\sigma}\Big[-23\partial^\sigma B^{\mu_1\mu_2\mu_3}\partial_{\mu_2}\partial_{\mu_3}\partial_{\nu_2}\partial_{\nu_3}B^{\nu_1\nu_2\nu_3}$$
$$+64\partial^\sigma B^{\mu_1\mu_2\mu_3}\Box\partial_{\mu_3}\partial_{\nu_3}B^{\nu_1\nu_3}{}_{\mu_2} - 18\partial^\sigma B^{\mu_1\lambda}{}_\lambda\Box\partial_{\nu_2}\partial_{\nu_3}B^{\nu_1\nu_2\nu_3}$$
$$-32\partial^\sigma B^{\mu_1\mu_2\mu_3}\Box^2 B^{\nu_1}{}_{\mu_2\mu_3} + 9\partial^\sigma B^{\mu_1\lambda}{}_\lambda\Box^2 B^{\nu_1\rho}{}_\rho\Big]. \quad (8.92)$$

This action is invariant under (8.12), but not under (8.13).

Remark The action (8.91) is similar to eq.(30) of [39]. The latter is written in terms of spinor labels, therefore the relation is not immediately evident. After turning to the ordinary notation, eq.(30) of [39] becomes

$$\sim \int d^3x \quad \epsilon_{\mu_1\nu_1\sigma}\Big[\frac{3}{2}\partial^\sigma h^{\mu_1\mu_2\mu_3}\partial_{\mu_2}\partial_{\mu_3}\partial_{\nu_2}\partial_{\nu_3}h^{\nu_1\nu_2\nu_3} - 4\partial^\sigma h^{\mu_1\mu_2\mu_3}\Box\partial_{\mu_3}\partial_{\nu_3}h^{\nu_1\nu_3}{}_{\mu_2}$$
$$+2\partial^\sigma h^{\mu_1\mu_2\mu_3}\Box^2 h^{\nu_1}{}_{\mu_2\mu_3}\Big] \quad (8.93)$$

and one can see that they are equal if we set $B^{\mu\lambda}{}_\lambda = 0$ in (8.91). The reason of the difference is that in [39] the field $h^{\mu\nu\lambda}$ is traceless, while in (8.91) the field $B_{\mu\nu\lambda}$ is not. The presence of the trace part modifies the conservation law and thus the action.

8.3 Chern-Simons effective actions

In the previous section we have seen that the odd parity 2-point correlators of the massive fermion model, either in the IR or UV limit, are local and give rise to action terms which coincide with the lowest (second) order of the gauge CS action and gravity CS action for the 2-point function of the gauge current and the e.m. tensor, respectively; and to the lowest order of a CS-like action for the rank 3 tensor field B. It is natural to expect that the n-th order terms of such CS actions will originate in a similar way from the n-point functions of the relevant currents. In particular the next to leading (third order) term in the CS actions is expected to be determined by the 3-point functions of the relevant currents. This is indeed so, but in a quite nontrivial way, with complications due both to the regularization and to the way we take the IR and UV limit.

The purpose of this section is to elaborate on the properties of the gauge and gravity CS actions, (8.40) and (8.57), respectively, in order to prepare the ground for the following discussion. The point we want to stress here is that in order to harmonize the formalism with the perturbative expansion in quantum field theory we need perturbative cohomology. The latter is explained in detail in Appendix H.3. It consists of a sequence of coboundary operators which approximate the full cohomology: in the case of a gauge theory the sequence reduces to two elements, in the case of gravity or higher tensor theories the sequence is infinite.

8.3.1 CS term for the gauge field

Let us start with the gauge case. The action (8.40) splits into two parts, $CS = CS^{(2)} + CS^{(3)}$, of order two and three, respectively, in the gauge field A. The second term is expected to come from the 3-point function of the gauge current. Gauge invariance splits as follows

$$\delta^{(0)}CS^{(2)} = 0, \qquad \delta^{(1)}CS^{(2)} + \delta^{(0)}CS^{(3)} = 0. \tag{8.94}$$

These equations reflect themselves in the conservation laws, which also split into two equations. The conservation law for the 2-point function is simply the vanishing of the divergence (on any index) of the latter, while for the 3-point function it does not consist in the vanishing of the divergence of the latter, but involves also contributions from 2-point functions. More precisely

$$\begin{aligned}&\partial_x^\mu \langle 0|\mathcal{T} J_\mu^a(x) J_\nu^b(y) J_\lambda^c(z)|0\rangle \\ &= i f^{abc'} \delta(x-y) \langle 0|\mathcal{T} J_\nu^{c'}(x) J_\lambda^c(z)|0\rangle + f^{acc'} \delta(x-z) \langle 0|\mathcal{T} J_\lambda^{c'}(x) J_\nu^b(y)|0\rangle,\end{aligned} \tag{8.95}$$

which in momentum space becomes

$$-iq^\mu \tilde{J}^{abc}_{\mu\nu\lambda}(k_1,k_2) + f^{abc'} \tilde{J}^{c'c}_{\nu\lambda}(k_2) + f^{acc'} \tilde{J}^{c'b}_{\lambda\nu}(k_1) = 0, \tag{8.96}$$

where $q = k_1 + k_2$ and $\tilde{J}^{ab}_{\mu\nu}(k)$ and $\tilde{J}^{abc}_{\mu\nu\lambda}(k_1,k_2)$ are Fourier transform of the 2- and 3-point functions, respectively.

8.3.2 Gravitational CS term

Let us consider next the gravitational CS case. Much as in the previous case we split the action (8.57) in pieces according to the number of $h_{\mu\nu}$ contained in them. This time however the number of pieces is infinite:

$$CS_g = \kappa \int d^3x\, \epsilon^{\mu\nu\lambda} \left(\partial_\mu \omega_\nu^{ab} \omega_{\lambda ba} + \frac{2}{3} \omega_{\mu a}{}^b \omega_{\nu b}{}^c \omega_{\lambda c}{}^a \right) = CS_g^{(2)} + CS_g^{(3)} + \dots, \tag{8.97}$$

where

$$CS_g^{(2)} = \frac{\kappa}{2} \int d^3x\, \epsilon_{\sigma\nu\rho}\, h^{\lambda\rho} \left(\partial^\sigma \partial_\lambda \partial_b h^{b\nu} - \partial^\sigma \Box h_\lambda^\nu \right) \tag{8.98}$$

and

$$\begin{aligned} CS_g^{(3)} \;=\; & \frac{\kappa}{4} \int d^3x\, \epsilon^{\mu\nu\lambda} \Big(2\partial_a h_{\nu b} \partial_\lambda h_\sigma^b \partial_\mu h^{\sigma a} - 2\partial_a h_\mu^b \partial^c h_{b\nu} \partial^a h_{c\lambda} - \frac{2}{3} \partial_a h_\mu^b \partial_b h_\nu^c \partial_c h_\lambda^a \\ & -2\partial_\mu \partial^b h_\nu^a (h_a^c \partial_c h_{b\lambda} - h_b^c \partial_c h_{a\lambda}) + \partial_\mu \partial^b h_\nu^a (h_\lambda^c \partial_a h_{bc} - \partial_a h_\lambda^c h_{bc}) \\ & +\partial_\mu \partial^b h_\nu^a \left(\partial_b h_\lambda^c h_{ac} - h_\lambda^c \partial_b h_{ac} \right) - h_\lambda^\rho h_\rho^a \partial_\mu \left(\Box h_{a\nu} - \partial_a \partial_b h_\nu^b \right) \Big). \end{aligned} \tag{8.99}$$

Invariance of CS_g under diffeomorphisms also splits into infinite many relations. The first two, which are relevant to us here, are

$$\delta_\xi^{(1)} CS_g^{(0)} = 0, \qquad \delta_\xi^{(1)} CS_g^{(0)} + \delta_\xi^{(0)} CS_g^{(1)} = 0, \tag{8.100}$$

where ξ is the parameter of diffeomorphisms. Similar relations hold also for Weyl transformations.

Such splittings correspond to the splittings of the Ward identities for diffeomorphisms and Weyl transformations derived from the generating function (8.22). The lowest order WI is just the vanishing of the divergence of the 2-point e.m. tensor correlators. The next to lowest order involves 2-point as well as 3-point functions of the e.m. tensor:

$$\begin{aligned} & \langle 0|\mathcal{T}\{\partial^\mu T_{\mu\nu}(x) T_{\lambda\rho}(y) T_{\alpha\beta}(z)\}|0\rangle \\ & = i \Big\{ 2 \frac{\partial}{\partial x^{(\alpha}} \left[\delta(x-z) \langle 0|\mathcal{T}\{T_{\beta)\nu}(x) T_{\lambda\rho}(y)\}|0\rangle \right] + 2 \frac{\partial}{\partial x^{(\lambda}} \left[\delta(x-y) \langle 0|\mathcal{T}\{T_{\rho)\nu}(x) T_{\alpha\beta}(z)\}|0\rangle \right] \\ & - \frac{\partial}{\partial x_\tau} \delta(x-z) \eta_{\alpha\beta} \langle 0|\mathcal{T}\{T_{\tau\nu}(x) T_{\lambda\rho}(y)\}|0\rangle - \frac{\partial}{\partial x_\tau} \delta(x-y) \eta_{\lambda\rho} \langle 0|\mathcal{T}\{T_{\tau\nu}(x) T_{\alpha\beta}(z)\}|0\rangle \\ & + \frac{\partial}{\partial x^\nu} \delta(x-z) \langle 0|\mathcal{T}\{T_{\lambda\rho}(y) T_{\alpha\beta}(x)\}|0\rangle + \frac{\partial}{\partial x^\nu} \delta(x-y) \langle 0|\mathcal{T}\{T_{\lambda\rho}(x) T_{\alpha\beta}(z)\}|0\rangle \Big\}. \end{aligned} \tag{8.101}$$

In momentum space, denoting by $\tilde{T}_{\mu\nu\lambda\rho}(k)$ and by $\tilde{T}_{\mu\nu\lambda\rho\alpha\beta}(k_1, k_2)$ the 2-point and 3-point function, respectively, this formula becomes

$$\begin{aligned} iq^\mu \tilde{T}_{\mu\nu\lambda\rho\alpha\beta}(k_1, k_2) \;=\; & 2q_{(\alpha} \tilde{T}_{\beta)\nu\lambda\rho}(k_1) + 2q_{(\lambda} \tilde{T}_{\rho)\nu\alpha\beta}(k_2) - \eta_{\alpha\beta} k_2^\tau \tilde{T}_{\tau\nu\lambda\rho}(k_1) \\ & -\eta_{\lambda\rho} k_1^\tau \tilde{T}_{\tau\nu\alpha\beta}(k_2) + k_{2\nu} \tilde{T}_{\alpha\beta\lambda\rho}(k_1) + k_{1\nu} \tilde{T}_{\lambda\rho\alpha\beta}(k_2), \end{aligned} \tag{8.102}$$

where round brackets denote symmetrization normalized to 1.

From the action term (8.99), by differentiating three times with respect to $h_{\mu\nu}(x)$,$h_{\lambda\rho}(y)$ and $h_{\alpha\beta}(z)$ and Fourier-transforming the result one gets a sum of local terms in momentum space (see Appendix H.6), to be compared with the IR and UV limit of the 3-point e.m. tensor correlator.

8.3.3 CS term for the B field

Here we would like to understand the nature of the "CS-like" terms obtained in the IR and UV limits, and especially to understand how it is possible that they are different in the spin-3 case, unlike what we saw in spin-1 and spin-2 [4]. For this purpose, we use a higher-spin "geometric" construction originally developed in [81]. In the spin-3 case the linearised "Christoffel connection" is given by the so-called second affinity defined by

$$\begin{aligned}\Gamma_{\alpha_1\alpha_2;\beta_1\beta_2\beta_3} \;=\; & \frac{1}{3}\Big\{\partial_{\alpha_1}\partial_{\alpha_2}B_{\beta_1\beta_2\beta_3} - \frac{1}{2}\left(\partial_{\alpha_1}\partial_{\beta_1}B_{\alpha_2\beta_2\beta_3} + \partial_{\alpha_1}\partial_{\beta_2}B_{\alpha_2\beta_1\beta_3}\right. \\ & \left. +\partial_{\alpha_1}\partial_{\beta_3}B_{\alpha_2\beta_1\beta_2} + \partial_{\alpha_2}\partial_{\beta_1}B_{\alpha_1\beta_2\beta_3} + \partial_{\alpha_2}\partial_{\beta_2}B_{\alpha_1\beta_1\beta_3} + \partial_{\alpha_2}\partial_{\beta_3}B_{\alpha_1\beta_1\beta_2}\right) \\ & +\partial_{\beta_1}\partial_{\beta_2}B_{\alpha_1\alpha_2\beta_3} + \partial_{\beta_1}\partial_{\beta_3}B_{\alpha_1\alpha_2\beta_2} + \partial_{\beta_2}\partial_{\beta_3}B_{\alpha_1\alpha_2\beta_1}\Big\}. \end{aligned} \tag{8.103}$$

Under the gauge transformation (8.12) this "connection" transforms as

$$\delta_\Lambda \Gamma_{\alpha\beta;\mu\nu\rho} = \partial_\mu\partial_\nu\partial_\rho\Lambda_{\alpha\beta}. \tag{8.104}$$

The natural generalisation of (the quadratic part of) the spin-1 and the spin-2 CS action term to the spin-3 case is given by

$$\begin{aligned} I_{\rm CS}[B] \;\equiv\; & \int d^3x\, \epsilon^{\mu\sigma\nu}\, \Gamma^{\alpha\beta}{}_{;\mu\rho\lambda}\, \partial_\sigma \Gamma^{\rho\lambda}{}_{;\nu\alpha\beta} \\ =\; & \frac{1}{3}\int d^3x\, \epsilon_{\mu\sigma\nu}\big(\partial_\alpha\partial_\beta B^{\mu\alpha\beta}\,\partial^\sigma\partial_\rho\partial_\lambda B^{\nu\rho\lambda} + 2\,\partial_\alpha\Box B^{\mu\alpha\beta}\,\partial^\sigma\partial_\rho B^{\nu\rho}{}_\beta \\ & + \Box B^{\mu\alpha\beta}\,\partial^\sigma\Box B^\nu{}_{\alpha\beta}\big) + \text{(boundary terms)}. \end{aligned} \tag{8.105}$$

From (8.104) directly follows that this CS term is gauge invariant (up to boundary terms). In the spin-3 case one can construct another 5-derivative CS term by using Fronsdal tensor (or spin-3 "Ricci tensor") defined by

$$\begin{aligned} \mathcal{R}_{\mu\nu\rho} \;\equiv\; & \Gamma^\alpha{}_{\alpha;\mu\nu\rho} \\ =\; & \frac{1}{3}\big\{\Box B_{\mu\nu\rho} - \partial^\alpha(\partial_\mu B_{\alpha\nu\rho} + \partial_\nu B_{\alpha\rho\mu} + \partial_\rho B_{\alpha\mu\nu}) \\ & + \partial^\mu\partial^\nu B_{\rho\alpha}{}^\alpha + \partial^\rho\partial^\mu B_{\nu\alpha}{}^\alpha + \partial^\nu\partial^\rho B_{\mu\alpha}{}^\alpha\big\}. \end{aligned} \tag{8.106}$$

Using this tensor one can defined another CS action term

$$\begin{aligned} I'_{\rm CS}[B] \;\equiv\; & \int d^3x\, \epsilon^{\mu\sigma\nu}\, \mathcal{R}_{\mu\rho\lambda}\,\partial_\sigma \mathcal{R}_\nu{}^{\rho\lambda} \\ =\; & \frac{1}{9}\int d^3x\, \epsilon_{\mu\sigma\nu}\big(2\,\partial_\alpha\partial_\beta B^{\mu\alpha\beta}\,\partial^\sigma\partial_\rho\partial_\lambda B^{\nu\rho\lambda} + 2\,\partial_\alpha\Box B^{\mu\alpha\beta}\,\partial^\sigma\partial_\rho B^{\nu\rho}{}_\beta \\ & - 2\,\partial_\alpha\partial_\beta B^{\mu\alpha\beta}\,\partial^\sigma\Box B^{\nu\rho}{}_\rho + \Box B^{\mu\alpha\beta}\,\partial^\sigma\Box B^\nu{}_{\alpha\beta} \\ & + \Box B^{\mu\alpha}{}_\alpha\,\partial^\sigma\Box B^{\nu\rho}{}_\rho\big) + \text{(boundary terms)}. \end{aligned} \tag{8.107}$$

The presence of two CS terms in the spin-3 case explains why there is a priori no reason to expect from UV and IR limits to lead to the same CS-like term.

Now it is easy to see that the following combination

$$5\, I_{\rm CS}[B] - 3\, I'_{\rm CS}[B] \tag{8.108}$$

[4] In the literature one can find two kinds of generalizations of the CS action in 3d to higher spins (for a general review on higher spin theories, see [70–77]). One leads to quadratic equations of motion, the other to higher derivative equations of motion. The first kind of theories are nicely summarized in [78]. The second kind of theories, to our best knowledge, was introduced in [39] (following [79]). This splitting was already shadowed in [80].

exactly gives the effective action term (8.91) which we obtained from the one-loop calculation.

To understand why the combination (8.108) represents a generalization of the spin-2 CS term (gravitational CS term), one has to take into account the symmetry under generalized Weyl (g-Weyl) transformations, which for spin-3 is given by (8.13). It can be shown that the CS terms (8.105) and (8.107) are not g-Weyl-invariant, but that (8.108) is the *unique g-Weyl-invariant* linear combination thereof.

It is then not surprising that the effective current $J_{\mu\nu\rho}$ obtained from (8.108) is proportional to the spin-3 "Cotton tensor" studied in [82]. It is the gauge- and g-Weyl- invariant conserved traceless totally symmetric tensor with the property that if it vanishes then the gauge field is g-Weyl-trivial. With this we have completed the demonstration that on the linear level the spin-3 CS term is a natural generalisation of the spin-2 CS term.

For completeness we add that the combination

$$\frac{1}{192\pi}\left(-\frac{41}{3}I_{\rm CS}[B]+9I'_{\rm CS}[B]\right) \tag{8.109}$$

reproduces (8.92), which is not g-Weyl invariant.

8.4 Three-point gauge current correlator: odd parity part

In this section we explicitly compute the 3-point current correlator of the current $J^a_\mu(x)$. The 3-point correlator for currents is given by the triangle diagram. The three external momenta are q, k_1, k_2. q is incoming, while k_1, k_2 are outgoing and of course momentum conservation implies $q = k_1 + k_2$. The relevant Feynman diagram is

$$\tilde{J}^{1,abc}_{\mu\nu\lambda}(k_1,k_2) = i\int\frac{d^3p}{(2\pi)^3}{\rm Tr}\left(\gamma_\mu T^a\frac{1}{\not p - m}\gamma_\nu T^b\frac{1}{\not p-\not k_1 - m}\gamma_\lambda T^c\frac{1}{\not p-\not q-m}\right) \tag{8.110}$$

to which we have to add the cross graph contribution $\tilde{J}^{2,abc}_{\mu\nu\lambda}(k_1,k_2) = \tilde{J}^{1,acb}_{\mu\lambda\nu}(k_2,k_1)$. From this we extract the odd parity part and perform the integrals. The general method is discussed in subsection 8.5.2, here we limit ourselves to the results. Such results have already been presented in [35], but since they are important for the forthcoming discussion we summarize them below. For simplicity we set $k_1^2 = k_2^2 = 0$, so the total energy of the process is $E = \sqrt{q^2} = \sqrt{2k_1\cdot k_2}$.

Near the IR fixed point we obtain a series expansion of the type

$$\tilde{J}^{abc(odd)}_{\mu\nu\lambda}(k_1,k_2) \approx i\frac{1}{32\pi}\sum_{n=0}^{\infty}\left(\frac{E}{m}\right)^{2n} f^{abc}\tilde{I}^{(2n)}_{\mu\nu\lambda}(k_1,k_2) \tag{8.111}$$

and, in particular, the relevant term in the IR is

$$I^{(0)}_{\mu\nu\lambda}(k_1,k_2) = -12\epsilon_{\mu\nu\lambda}. \tag{8.112}$$

The first thing to check is conservation. The current (8.2) should be conserved also at the quantum level, because no anomaly is expected in this case. It is evident that the contraction with q^μ does not give a vanishing result, as we expect because we must include also the contribution from the 2-point functions, (8.95). But even including such contributions we get

$$-\frac{3}{8\pi}f^{abc}q^{\mu}\epsilon_{\mu\nu\lambda}+\frac{1}{4\pi}f^{abc}\epsilon_{\nu\lambda\sigma}k_2^{\sigma}+\frac{1}{4\pi}f^{abc}\epsilon_{\nu\lambda\sigma}k_1^{\sigma}\neq 0. \tag{8.113}$$

Conservation is violated by a local term. Thus we can recover it by adding to $I^{(0)}_{\mu\nu\lambda}(k_1,k_2)$ a term $4\epsilon_{\mu\nu\lambda}$. This corresponds to correcting the effective action by adding a counterterm

$$-2\int dx\epsilon^{\mu\nu\lambda}f^{abc}A_{\mu}^{a}A_{\nu}^{b}A_{\lambda}^{c}. \tag{8.114}$$

Adding this to the result from the 2-point correlator we reconstruct the full CS action (8.40).

This breakdown of conservation is surprising, therefore it is important to understand where it comes from. To this end we consider the full theory for finite m. The contraction of the 3-point correlator with q^{μ} is given by

$$q^{\mu}\tilde{J}^{abc}_{\mu\nu\lambda}(k_1,k_2)=-i\int\frac{d^3p}{(2\pi)^3}\mathrm{Tr}\left(\not{q}T^{a}\frac{1}{\not{p}-m}\gamma_{\nu}T^{b}\frac{1}{\not{p}-\not{k}_1-m}\gamma_{\lambda}T^{c}\frac{1}{\not{p}-\not{q}-m}\right). \tag{8.115}$$

Replacing $\not{q}=(\not{p}-m)-(\not{p}-\not{q}-m)$ considerably simplifies the calculation. The final result for the odd parity part (after adding the cross diagram contribution, $1\leftrightarrow 2, b\rightarrow c, \nu\leftrightarrow\lambda$) is

$$\begin{aligned}q^{\mu}\tilde{J}^{abc}_{\mu\nu\lambda}(k_1,k_2)= \quad & -\tfrac{i}{4\pi}f^{abc}\epsilon_{\lambda\nu\sigma}k_1^{\sigma}\,\tfrac{2m}{k_1}\mathrm{arccoth}\left(\tfrac{2m}{k_1}\right)\\ & -\tfrac{i}{4\pi}f^{abc}\epsilon_{\lambda\nu\sigma}k_2^{\sigma}\,\tfrac{2m}{k_2}\mathrm{arccoth}\left(\tfrac{2m}{k_2}\right).\end{aligned} \tag{8.116}$$

Therefore, as far as the odd part is concerned, the 3-point conservation (8.96) reads

$$\begin{aligned}&-iq^{\mu}\tilde{J}^{(odd)abc}_{\mu\nu\lambda}(k_1,k_2)+f^{abd}\tilde{J}^{(odd)dc}_{\nu\lambda}(k_2)+f^{acd}\tilde{J}^{(odd)db}_{\lambda\nu}(k_1)\\ &\quad=-\frac{1}{4\pi}f^{abc}\epsilon_{\lambda\nu\sigma}\left(k_1^{\sigma}\frac{2m}{k_1}\mathrm{arccoth}\left(\frac{2m}{k_1}\right)+k_2^{\sigma}\frac{2m}{k_2}\mathrm{arccoth}\left(\frac{2m}{k_2}\right)\right)\\ &\quad\quad+\frac{1}{4\pi}f^{abc}\epsilon_{\lambda\nu\sigma}\left(k_1^{\sigma}\frac{2m}{k_1}\mathrm{arccoth}\left(\frac{2m}{k_1}\right)+k_2^{\sigma}\frac{2m}{k_2}\mathrm{arccoth}\left(\frac{2m}{k_2}\right)\right)=0.\end{aligned} \tag{8.117}$$

Thus conservation is secured for any value of the parameter m. The fact that in the IR limit we find a violation of the conservation is an artifact of the procedure we have used (in particular of the limiting procedure) and we have to remedy by subtracting suitable counterterms from the effective action. These subtractions are to be understood as (part of) the definition of our regularization procedure.

Something similar can be done also for the UV limit. However in the UV limit the resulting effective action has a vanishing coupling $\sim\frac{m}{E}$, unless we consider an $N\rightarrow\infty$ limit theory, as outlined above. In order to guarantee invariance under large gauge transformations we have also to fine tune the limit in such a way that the κ coupling be an integer. But even in the UV we meet the problem of invariance breaking.

We will meet the same problem below for the 3-point function of the e.m. tensor.

8.5 Three-point e.m. correlator: odd parity part

We go now to the explicit calculation of the 3-point e.m. tensor correlator. The three-point function is given by three contributions, the bubble diagram, the triangle diagram and the cross triangle diagram. We will focus in the sequel only on the odd parity part.

8.5.1 The bubble diagram: odd parity

The bubble diagram is constructed with one V_{gff} vertex and one V_{ggff} vertex. It has an incoming line with momentum $q = k_1 + k_2$ with Lorentz indices μ, ν, and two outgoing lines have momenta k_1, k_2 with Lorentz labels λ, ρ and α, β, respectively. The internal running momentum is denoted by p. The corresponding contribution is

$$D_{\lambda\rho\alpha\beta\mu\nu}(k_1,k_2) = \frac{i}{128}\int \frac{d^3p}{(2\pi)^3}\mathrm{Tr}\left[\frac{1}{\not p - m}t_{\lambda\rho\alpha\beta\sigma}(k_2-k_1)^\sigma \frac{1}{\not p - \not q - m}((2p_\mu - q_\mu)\gamma_\nu + \mu \leftrightarrow \nu)\right] \quad (8.118)$$

where

$$t_{\lambda\rho\alpha\beta\sigma} = \eta_{\lambda\alpha}\epsilon_{\rho\beta\sigma} + \eta_{\lambda\beta}\epsilon_{\rho\alpha\sigma} + \eta_{\rho\alpha}\epsilon_{\lambda\beta\sigma} + \eta_{\rho\beta}\epsilon_{\lambda\alpha\sigma}. \quad (8.119)$$

The odd part gives (the metric is Lorentzian)

$$\begin{aligned}\tilde{D}_{\lambda\rho\alpha\beta\mu\nu}(k_1,k_2) &= \frac{m}{256\pi}t_{\lambda\rho\alpha\beta\sigma}(k_2-k_1)^\sigma\left(\eta_{\mu\nu}\left(2m - \frac{q^2-4m^2}{|q|}\mathrm{arctanh}\frac{|q|}{2m}\right)\right.\\ &\left.+q_\mu q_\nu\left(\frac{2m}{q^2} + \frac{q^2-4m^2}{|q|^3}\mathrm{arctanh}\frac{|q|}{2m}\right)\right). \quad (8.120)\end{aligned}$$

Saturating with q^μ we get

$$q^\mu \tilde{D}_{\lambda\rho\alpha\beta\mu\nu}(k_1,k_2) = \frac{m^2}{256\pi}t_{\lambda\rho\alpha\beta\sigma}(k_2-k_1)^\sigma 2q_\nu. \quad (8.121)$$

This corresponds to an anomaly

$$\mathcal{A}_\xi \sim \int d^3x\, \partial_\nu\xi^\nu \epsilon_{\rho\beta\sigma}h^{\lambda\rho}\partial^\sigma h^\beta_\lambda \quad (8.122)$$

which we have to subtract. This gives

$$\tilde{D}_{\lambda\rho\alpha\beta\mu\nu}(k_1,k_2) = \frac{1}{256\pi}t_{\lambda\rho\alpha\beta\sigma}(k_2-k_1)^\sigma\left(q_\mu q_\nu - \eta_{\mu\nu}q^2\right)\left(\frac{2m^2}{q^2} + m\frac{q^2-4m^2}{|q|^3}\mathrm{arctanh}\frac{|q|}{2m}\right). \quad (8.123)$$

Taking the limit of the form factor (last round brackets) for $\frac{m}{|q|} \to 0$ (UV), we find 0 (the linear term in $\frac{m}{|q|}$ vanishes). Taking the limit $\frac{m}{|q|} \to \infty$ (IR) we find

$$\tilde{D}^{(IR)}_{\lambda\rho\alpha\beta\mu\nu}(k_1,k_2) = \frac{2}{3}\frac{1}{256\pi}t_{\lambda\rho\alpha\beta\sigma}(k_2-k_1)^\sigma\left(q_\mu q_\nu - \eta_{\mu\nu}q^2\right). \quad (8.124)$$

This corresponds to the action term

$$\sim \int d^3x\,(\Box h - \partial_\mu\partial_\nu h^{\mu\nu})\, t_{\lambda\rho\alpha\beta\sigma}\left(h^{\lambda\rho}\partial^\sigma h^{\alpha\beta} - \partial^\sigma h^{\lambda\rho}\, h^{\alpha\beta}\right). \quad (8.125)$$

8.5.2 Triangle diagram: odd parity

It is constructed with three V_{gff} vertices. It has again an incoming line with momentum $q = k_1 + k_2$ with Lorentz indices μ, ν. The two outgoing lines have momenta k_1, k_2 with

Lorentz labels λ, ρ and α, β, respectively. The contribution is formally written as

$$\tilde{T}^{(1)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2) = -\frac{1}{512}\int \frac{d^3p}{(2\pi)^3}\,\mathrm{tr}\left[\left(\frac{1}{\not p - m}\left((2p-k_1)_\lambda\gamma_\rho + (\lambda\leftrightarrow\rho)\right)\,\frac{1}{\not p - \not k_1 - m}\right.\right.$$
$$\left.\left.\times\left((2p-2k_1-k_2)_\alpha\gamma_\beta + (\alpha\leftrightarrow\beta)\right)\,\frac{1}{\not p - \not q - m}\left((2p-q)_\mu\gamma_\nu + (\mu\leftrightarrow\nu)\right)\right)\right], \quad (8.126)$$

to which the cross graph contribution $\tilde{T}^{(2)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2) = \tilde{T}^{(1)}_{\mu\nu\lambda\rho\alpha\beta}(k_2,k_1)$ has to be added.

The odd part of (8.126) is

$$\tilde{T}^{(1,odd)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2) = -\frac{m}{512}\int \frac{d^3p}{(2\pi)^3}\,\mathrm{tr}\left[\not p\gamma_\rho(\not p - \not k_1)\gamma_\beta\gamma_\nu + \gamma_\rho(\not p - \not k_1)\gamma_\beta(\not p - \not q)\gamma_\nu\right.$$
$$\left. + \not p\gamma_\rho\gamma_\beta(\not p - \not q)\gamma_\nu + m^2\gamma_\rho\gamma_\beta\gamma_\nu\right]\frac{(2p-k_1)_\lambda(2p-2k_1-k_2)_\alpha(2p-q)_\mu}{(p^2-m^2)((p-k_1)^2-m^2)((p-q)^2-m^2)}, \quad (8.127)$$

where the symmetrization $\lambda\leftrightarrow\rho, \alpha\leftrightarrow\beta, \mu\leftrightarrow\nu$ is understood. In order to work out (8.127) we introduce two Feynman parameters: u integrated between 0 and 1, and v integrated between 0 and $1-u$. The denominator in (8.127) becomes

$$\left[(p-(1-u)k_1 - vk_2)^2 + u(1-u)k_1^2 + v(1-v)k_2^2 + 2uv\,k_1\!\cdot\! k_2 - m^2\right]^3.$$

After taking the traces, we get

$$\begin{aligned}\tilde{T}^{(1,odd)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2) \;=\; & \frac{im}{128}\int_0^1 du\int_0^{1-u} dv\int\frac{d^3p}{(2\pi)^3}\left[\epsilon_{\rho\sigma\nu}\left(-2p_\beta k_1^\sigma + k_{1\beta}q^\sigma + q_\beta k_1^\sigma\right)\right.\\ & \left. + 2\epsilon_{\sigma\beta\nu}p_\rho k_2^\sigma + \epsilon_{\rho\beta\nu}\left(-5p^2 + (2p-q)\!\cdot\! k_1 + m^2\right) + \eta_{\rho\nu}\epsilon_{\sigma\beta\tau}k_1^\sigma k_2^\tau\right]\\ & \cdot\frac{(2p-k_1)_\lambda(2p-2k_1-k_2)_\alpha(2p-q)_\mu}{[(p-(1-u)k_1-vk_2)^2+\Delta)^3}, \end{aligned} \quad (8.128)$$

where $\Delta = u(1-u)\,k_1^2 + v(1-v)\,k_2^2 + 2uv\,k_1\!\cdot\! k_2 - m^2$.

So we can shift $p \to p' = p - (1-u)k_1 - vk_2$ and integrate over p'. The p-integrals can be easily carried out, see Appendix H.4. Unfortunately we are not able to integrate over u and v in an elementary way. So, one way to proceed is to use Mathematica, which however is not able to integrate over both u and v unless some simplification is assumed. Therefore we choose the condition $k_1^2 = 0 = k_2^2$. In this case the total energy of the process is $E = \sqrt{q^2} = \sqrt{2k_1\!\cdot\! k_2}$.

An alternative way is to use Mellin-Barnes representation for the propagators in (8.127) and proceed in an analytic as suggested by Boos and Davydychev, [83–85]. This second approach is discussed in Appendix H.5. In all the cases we were able to compare the two methods they give the same results (up to trivial terms).

8.5.3 The IR limit

The IR limit corresponds to $m \to \infty$ or, better, $\frac{m}{E} \to \infty$ where $E = \sqrt{2k_1\!\cdot\! k_2}$. In this limit we find one divergent term $\mathcal{O}(m^2)$ and a series in the parameter $\frac{m}{E}$ starting from the 0-th order term. The $\mathcal{O}(m^2)$ term is (after adding the cross contribution)

$$\sim \frac{m^2}{16\pi}\Big[16\epsilon_{\sigma\beta\nu}k_2^\sigma\left(\eta_{\rho\lambda}\eta_{\alpha\mu}+\eta_{\rho\alpha}\eta_{\lambda\mu}+\eta_{\rho\mu}\eta_{\lambda\alpha}\right)+16\epsilon_{\sigma\rho\nu}k_1^\sigma\left(\eta_{\beta\lambda}\eta_{\alpha\mu}+\eta_{\beta\alpha}\eta_{\lambda\mu}+\eta_{\beta\mu}\eta_{\lambda\alpha}\right)$$
$$-\epsilon_{\rho\beta\nu}\left(-\frac{112}{3}(k_1-k_2)_\mu\eta_{\lambda\alpha}+\frac{16}{3}(11k_1+7k_2)_\alpha\eta_{\lambda\mu}-\frac{16}{3}(7k_1+11k_2)_\lambda\eta_{\alpha\mu}\right)\Big]. \quad (8.129)$$

This term has to be symmetrized under $\mu\leftrightarrow\nu, \lambda\leftrightarrow\rho, \alpha\leftrightarrow\beta$. It is a (non-conserved) local term. It must be subtracted from the action. Once this is done the relevant term for us is the 0-th order one. Let us call it $\tilde{T}^{(odd,IR)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2)$. Its lengthy explicit form is written down in Appendix H.6.2.

If we compare this expression plus the contribution from the bubble diagram with the one obtained from $CS_g^{(3)}$ in H.6.1, which it is expected to reproduce, we see that they are different. This is not surprising in view of the discussion of the gauge case: the next to leading order of the relevant CS action is not straightaway reproduced by the relevant 3-point correlators, but need corrections. This can be seen also by contracting $\tilde{T}^{(odd,IR)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2)$ with q^μ and inserting it in the WI (8.102): the latter is violated.

Now let us Fourier antitransform $\tilde{T}^{(odd,IR)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2)$ and insert the result in the $W[g]$ generating function. We obtain a local action term of 3rd order in $h_{\mu\nu}$, which we may call $\widetilde{CS}_g^{(3)}$. Having in mind (8.94), we find instead

$$\delta_\xi^{(1)}CS_g^{(2)}+\delta_\xi^{(0)}\widetilde{CS}_g^{(3)}=\mathcal{Y}^{(2)}(\xi)\neq 0, \quad (8.130)$$

where $\mathcal{Y}^{(2)}(\xi)$ is an integrated local expression quadratic in $h_{\mu\nu}$ and linear in the diffeomorphism parameter ξ. It is clear that in order to reproduce (8.94) we must add counterterms to the action, as we have done in the analogous case in section 8.4. The question is whether this is possible. We can proceed as follows, we subtract from (8.130) the second equation in (8.94) and obtain

$$\delta_\xi^{(0)}(\widetilde{CS}_g^{(3)}-CS_g^{(3)})=\mathcal{Y}^{(2)}(\xi). \quad (8.131)$$

Therefore $\widetilde{CS}_g^{(3)}-CS_g^{(3)}$ is the counterterm we have to subtract from the action in order to satisfy (8.94) and simultaneously reconstruct the gravitational Chern-Simons action up to the third order. The procedure seems to be tautological, but this is simply due to the fact that we already know the covariant answer, i.e. the gravitational CS action, otherwise we would have to work our way through a painful analysis of all the terms in $\mathcal{Y}^{(2)}(\xi)$ and find the corresponding counterterms[5].

The just outlined procedure is successful but somewhat disappointing. For the purpose of reproducing $CS_g^{(3)}$ the overall three-point calculation seems to be rather ineffective. One can say that the final result is completely determined by the two-point function analysis. Needless to say it would be preferable to find a regularization as well as a way to take the IR and UV limits that do not break covariance. We do not know it this is possible.

On the other hand the three-point function analysis is important for other reasons. For brevity we do not report other explicit formulas about the coefficients of the expansion in $\frac{E}{m}$ and $\frac{m}{E}$. They all look like correlators, which may be local and non-local. The analysis of these expressions opens a new subject of investigation.

[5]Of course in the process of defining the regularization and the IR limiting procedure, we are allowed to subtract all the necessary counterterms (with the right properties) except fully covariant action terms (like the CS action itself, for one).

8.6 Conclusion

In this paper we have calculated two- and three-point functions of currents in the free massive fermion model in 3d. We have mostly done our calculations with two different methods, as explained in Appendix H.4 and H.5, and obtained the same results. When the model is coupled to an external gauge potential and metric, respectively, we have extracted from them, in the UV and the IR limit, CS action terms for gauge and gravity in 3d. We have also coupled the massless fermion model to higher spin potentials and explicitly worked out the spin 3 case, by obtaining a very significant new result in the UV limit: the action reconstructed from the two-point current correlator is a particular case of higher spin action introduced a long time ago by [39]; this is one of the possible generalizations of the CS action to higher spin. It is of course expected that carrying out analogous calculations for higher spin currents we will obtain the analogous generalizations of Chern-Simons to higher spin. Our result for the spin 3 case in the IR is an action with a higher spin gauge symmetry, different from the UV one; we could not recognize it as a well-known higher spin action.

Beside the results concerning effective actions terms in the UV and IR limit, there are other interesting aspects of the correlators we obtain as intermediate steps. For instance, the odd parity current correlators at fixed points are conformal invariant and are limits of a free theory, but they cannot be obtained from any free theory using the Wick theorem. There are also other interesting and not understood aspects. For instance, the two-point e.m. tensor correlators of the massive model can be expanded in series of E/m or m/E, where E is the relevant energy, the coefficients in the expansion being proportional always to the same conformal correlator. For the three-point functions the situation is more complicated, there is the possibility of different limits and the expansion coefficients are also nonlocal. Still, however, we have a stratification similar to the one in the two-point functions with coefficient that look like conserved three-point correlators (but have to be more carefully evaluated). One would like to know what theories these correlators refer to.

Finally it would be interesting to embed the massive fermion model in an AdS_4 geometry. One can naively imagine the AdS_4 space foliated by 3d submanifolds with constant geodesic distance from the boundary and a copy of the theory defined on each submanifold with a mass depending of the distance. This mass could be generated, for instance, by the vev of a pseudoscalar field. This and the previous question certainly deserve further investigation.

Appendix H

Details of computations and useful facts

H.1 Gamma matrices in 3d

In $2+1$ dimensions we may take the gamma matrices, [42], as

$$\gamma_0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_1 = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \gamma_2 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{H.1}$$

They satisfy the Clifford algebra relation for the anticommutator of gamma matrices, namely

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu},$$

For the trace of three gamma matrices we have

$$\mathrm{tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\right) = -2i\epsilon_{\mu\nu\rho},$$

Properties of gamma matrices in 3d

$$\begin{gathered}
\mathrm{tr}\left(\gamma_\mu\gamma_\nu\right) = 2\eta_{\mu\nu}, \\
\mathrm{tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\right) = -2i\epsilon_{\mu\nu\rho}, \\
\mathrm{tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) = 2\left(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right), \\
\{\gamma^\mu, \Sigma^{\rho\sigma}\} = -i\epsilon^{\mu\rho\sigma}.
\end{gathered}$$

$$\gamma_\sigma\gamma_\mu\gamma_\nu = -i\epsilon_{\sigma\mu\nu} + \eta_{\mu\sigma}\gamma_\nu - \eta_{\sigma\nu}\gamma_\mu + \eta_{\mu\nu}\gamma_\sigma \tag{H.2}$$

$$\mathrm{tr}\left(\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\right) = -2i\left(\epsilon_{\mu\nu\lambda}\eta_{\sigma\rho} + \eta_{\mu\nu}\epsilon_{\sigma\lambda\rho} - \eta_{\mu\lambda}\epsilon_{\sigma\nu\rho} + \eta_{\nu\lambda}\epsilon_{\sigma\mu\rho}\right) \tag{H.3}$$

Identity for ϵ and η tensors :

$$\eta_{\mu\nu}\epsilon_{\lambda\rho\sigma} - \eta_{\mu\lambda}\epsilon_{\nu\rho\sigma} + \eta_{\mu\rho}\epsilon_{\nu\lambda\sigma} - \eta_{\mu\sigma}\epsilon_{\nu\lambda\rho} = 0$$

Finally, to make contact with the spinorial label notation of ref.[39] one may use the symmetric matrices

$$(\tilde\gamma_0)_{\alpha\beta} = i(\gamma_0)_\alpha{}^\gamma\epsilon_{\gamma\beta}, \qquad (\tilde\gamma_1)_{\alpha\beta} = (\gamma_1)_\alpha{}^\gamma\epsilon_{\gamma\beta}, \qquad (\tilde\gamma_2)_{\alpha\beta} = -(\gamma_2)_\alpha{}^\gamma\epsilon_{\gamma\beta}, \tag{H.4}$$

where ϵ is the antisymmetric matrix with $\epsilon_{12} = -1$, and write

$$h_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} = h_{abc}(\tilde\gamma^a)_{\alpha_1\alpha_2}(\tilde\gamma^b)_{\alpha_3\alpha_4}(\tilde\gamma^c)_{\alpha_5\alpha_6}, \qquad \partial_a(\tilde\gamma^a\epsilon)_\alpha{}^\beta = \partial_\alpha{}^\beta, \qquad \text{etc.} \tag{H.5}$$

Wick rotation Among the various conventions for the Wick rotation to compute Feynman diagram, we think the simplest one is given by the following formal rule on the metric: $\eta_{\mu\nu} \to -\eta^{(E)}_{\mu\nu}$. This implies

$$k^2 \to -(k^{(E)})^2, \qquad p_\mu p_\nu \to \frac{1}{3}(p^{(E)})^2\, \eta^{(E)}_{\mu\nu}, \dots$$

We have also to multiply any momentum integral by i. For the sake of simplicity we always understand the superscript $^{(E)}$.

H.2 Invariances of the 3d free massive fermion

In the theory defined by 8.4 there is a problem connected with the presence of $\sqrt{g} = e$ in the action. When defining the Feynman rules we face two possibilities: 1) either we incorporate $\sqrt{e}$ in the spinor field ψ, so that the factor $\sqrt{g}$ in fact disappears from the action, or, 2), we keep the action as it is.

In the first case we define a new field $\Psi = \sqrt{e}\psi$. The new action becomes

$$S = \int d^3x \left[i\bar{\Psi} E^\mu_a \gamma^a \nabla_\mu \Psi - m\bar{\Psi}\psi \right]. \tag{H.6}$$

due essentially to the fact that $\nabla_\lambda g_{\mu\nu} = 0$. The action is still diff-invariant provided Ψ transforms as

$$\delta_\xi \Psi = \xi^\mu \partial_\mu \Psi + \frac{1}{2} \nabla_\mu \xi^\mu \Psi \tag{H.7}$$

In the case $m = 0$ we also have Weyl invariance with

$$\delta_\omega \Psi = \frac{1}{2}\omega\Psi, \qquad \text{instead of} \qquad \delta_\omega \psi = \omega\psi, \tag{H.8}$$

So the simmetries are classically preserved while passing from ψ to Ψ. From a quantum point of view this might seem a Weyl transformation of Ψ, but it is not accompanied by a corresponding Weyl transformation of the metric. So it is simply a field redefinition, not a symmetry operation.

Alternative 1) is the procedure of Delbourgo-Salam. The action can be rewritten

$$S = \int d^3x \left[\frac{i}{2} \bar{\Psi} E^\mu_a \gamma^a \overleftrightarrow{\partial} \Psi - m\bar{\Psi}\Psi + \frac{1}{2} E^\mu_a \omega_{\mu bc} \epsilon^{abc} \bar{\Psi}\Psi \right]. \tag{H.9}$$

In this case we have one single graviton-fermion-fermion vertex V_{gff} represented by

$$\frac{i}{8}\left[(p+p')_\mu \gamma_\nu + (p+p')_\nu \gamma_\mu \right] \tag{H.10}$$

and one single 2-gravitons–2-fermions vertex V_{ggff} given by

$$\frac{1}{16} t_{\mu\nu\mu'\nu'\lambda} (k-k')^\lambda \tag{H.11}$$

where

$$t_{\mu\nu\mu'\nu'\lambda} = \eta_{\mu\mu'}\epsilon_{\nu\nu'\lambda} + \eta_{\nu\mu'}\epsilon_{\mu\nu'\lambda} + \eta_{\mu\nu'}\epsilon_{\nu\mu'\lambda} + \eta_{\nu\nu'}\epsilon_{\mu\mu'\lambda}, \tag{H.12}$$

the fermion propagator being

$$\frac{i}{\not p - m + i\epsilon}$$

The convention for momenta are the same as in [19, 37].

Alternative 2) introduces new vertices. In this case the Lagrangian can be written

$$\begin{aligned} L &= \frac{i}{2}\bar{\psi}\,\gamma^a \overleftrightarrow{\partial}_a \psi - i\,m\,\bar{\psi}\psi \\ &+\frac{i}{4}\bar{\psi}\,\gamma^a h_a^\mu \overleftrightarrow{\partial}_\mu \psi + \frac{i}{4}h_\lambda^\lambda\,\bar{\psi}\,\gamma^a \overleftrightarrow{\partial}_a \psi - \frac{i}{2}h_\lambda^\lambda\,m\,\bar{\psi}\psi \\ &+\frac{i}{8}h_\lambda^\lambda\,\bar{\psi}\,\gamma^a\,h_a^\mu \overleftrightarrow{\partial}_\mu \psi - \frac{1}{16}\bar{\psi}\,h_c^\lambda \partial_a h_{\lambda b}\psi\,\epsilon^{abc} \end{aligned} \tag{H.13}$$

As a consequence we have three new vertices. A vertex V'_{gff} coming from the mass term

$$-\frac{i}{2}\,m\,\eta_{\mu\nu}\mathbf{1}, \tag{H.14}$$

another V''_{gff} coming from the kinetic term

$$\frac{i}{4}\eta_{\mu\nu}(\not p + \not p') \tag{H.15}$$

and a new V'_{ggff}

$$\frac{i}{8}\eta_{\mu'\nu'}\left[(p+p')_\mu\gamma_\nu + (p+p')_\nu\gamma_\mu\right] \tag{H.16}$$

An obvious conjecture is that the two procedures lead to the same results, up to trivial terms. But this has still to be proved.

In this paper we follow alternative 1 only.

H.3 Perturbative cohomology

In this Appendix we define the form of local cohomology which is needed in perturbative field theory. Let us start from the gauge transformations.

$$\delta A = d\lambda + [A,\lambda], \qquad \delta\lambda = -\frac{1}{2}[\lambda,\lambda]_+, \qquad \delta^2 = 0, \qquad \lambda = \lambda^a(x)T^a \tag{H.17}$$

To dovetail the perturbative expansion it is useful to split it. Take A and λ infinitesimal and define the perturbative cohomology

$$\begin{aligned} &\delta^{(0)}A = d\lambda, \qquad \delta^{(0)}\lambda = 0, \qquad (\delta^{(0)})^2 = 0 \\ &\delta^{(1)}A = [A,\lambda], \qquad \delta^{(1)}\lambda = -\frac{1}{2}[\lambda,\lambda]_+ \\ &\delta^{(0)}\delta^{(1)} + \delta^{(1)}\delta^{(0)} = 0, \qquad (\delta^{(1)})^2 = 0 \end{aligned} \tag{H.18}$$

The full coboundary operator for diffeomorphisms is given by the transformations

$$\delta_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\nu, \qquad \delta_\xi\xi^\mu = \xi^\lambda\partial_\lambda\xi^\mu \tag{H.19}$$

with $\xi_\mu = g_{\mu\nu}\xi^\nu$. We can introduce a perturbative cohomology, or graded cohomology, using as grading the order of infinitesimal, as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^\mu_\lambda h^{\lambda\nu} + \ldots \tag{H.20}$$

The analogous expansions for the vielbein is

$$e_\mu^a = \delta_\mu^a + \chi_\mu^a + \frac{1}{2}\psi_\mu^a + \dots,$$

Since $e_\mu^a \eta_{ab} e_\nu^b = h_{\mu\nu}$, we have

$$\chi_{\mu\nu} = \frac{1}{2}h_{\mu\nu}, \qquad \psi_{\mu\nu} = -\chi_\mu^a \chi_{a\nu} = -\frac{1}{4}h_\mu^\lambda h_{\lambda\nu}, \qquad \dots \tag{H.21}$$

This leads to the following expansion for the spin connection ω_μ^{ab}

$$\begin{aligned}\omega_\mu^{ab} &= \frac{1}{2}e^{\nu a}\left(\partial_\mu e_\nu^b - \partial_\nu e_\mu^b\right) - \frac{1}{2}e^{\nu b}\left(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a\right) - \frac{1}{2}e^{\rho a}e^{\sigma b}\left(\partial_\rho e_{\sigma c} - \partial_\sigma e_{\rho c}\right)e_\mu^c \qquad \text{(H.22)}\\ &= -\frac{1}{2}\left(\partial^a h_\mu^b - \partial^b h_\mu^a\right) - \frac{1}{8}\left(h^{\sigma a}\partial_\mu h_\sigma^b - h^{\sigma b}\partial_\mu h_\sigma^a\right) + \frac{1}{4}\left(h^{\sigma a}\partial_\sigma h_\mu^b - h^{\sigma b}\partial_\sigma h_\mu^a\right)\\ &\quad -\frac{1}{8}\left(h^{\sigma a}\partial_\sigma h_\mu^b - h^{\sigma b}\partial_\sigma h_\sigma^a\right) - \frac{1}{8}h_\mu^c\left(\partial^a h_c^b - \partial^b h_c^a\right)\\ &\quad -\frac{1}{8}\left(\partial^b(h_\mu^\lambda h_\lambda^a) - \partial^a(h_\mu^\lambda h_\lambda^b)\right) + \dots\end{aligned}$$

Inserting the above expansions in (H.19) we see that we have a grading in the transformations, given by the order of infinitesimals. So we can define a sequence of transformations

$$\delta_\xi = \delta_\xi^{(0)} + \delta_\xi^{(1)} + \delta_\xi^{(2)} + \dots$$

At the lowest level we find immediately

$$\delta_\xi^{(0)} h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \delta_\xi^{(0)} \xi_\mu = 0 \tag{H.23}$$

and $\xi_\mu = \xi^\mu$. Since $(\delta_\xi^{(0)})^2 = 0$ this defines a cohomology problem.

At the next level we get

$$\delta_\xi^{(1)} h_{\mu\nu} = \xi^\lambda \partial_\lambda h_{\mu\nu} + \partial_\mu \xi^\lambda h_{\lambda\nu} + \partial_\nu \xi^\lambda h_{\mu\lambda}, \qquad \delta_\xi^{(1)} \xi^\mu = \xi^\lambda \partial_\lambda \xi^\mu \tag{H.24}$$

One can verify that

$$(\delta_\xi^{(0)})^2 = 0 \qquad \delta_\xi^{(0)}\delta_\xi^{(1)} + \delta_\xi^{(1)}\delta_\xi^{(0)} = 0, \qquad (\delta_\xi^{(1)})^2 = 0 \tag{H.25}$$

Proceeding in the same way we can define an analogous sequence of transformations for the Weyl transformations. From $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $\delta_\omega h_{\mu\nu} = 2\omega g_{\mu\nu}$ we find

$$\delta_\omega^{(0)} h_{\mu\nu} = 2\omega \eta_{\mu\nu}, \qquad \delta_\omega^{(1)} k_{\mu\nu} = 2\omega h_{\mu\nu}, \qquad \delta_\omega^{(2)} h_{\mu\nu} = 0, \dots \tag{H.26}$$

as well as $\delta_\omega^{(0)}\omega = \delta_\omega^{(1)}\omega = 0, \dots$.

Notice that we have $\delta_\xi^{(0)}\omega = 0, \delta_\xi^{(1)}\omega = \xi^\lambda \partial_\lambda \omega$. As a consequence we can extend (H.25) to

$$(\delta_\xi^{(0)} + \delta_\omega^{(0)})(\delta_\xi^{(1)} + \delta_\omega^{(1)}) + (\delta_\xi^{(1)} + \delta_\omega^{(1)})(\delta_\xi^{(0)} + \delta_\omega^{(0)}) = 0 \tag{H.27}$$

and $\delta_\xi^{(1)}\delta_\omega^{(1)} + \delta_\omega^{(1)}\delta_\xi^{(1)} = 0$, which together with the previous relations make

$$(\delta_\xi^{(0)} + \delta_\omega^{(0)} + \delta_\xi^{(1)} + \delta_\omega^{(1)})^2 = 0 \tag{H.28}$$

For what concerns the higher tensor field $B_{\mu\nu\lambda}$ in this paper we use only the lowest order transformations given by (8.12) and (8.13).

H.4 Useful integrals

The Euclidean integrals over the momentum p we use for the 2-point function are:

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2+\Delta)^2} = \frac{1}{8\pi}\frac{1}{\sqrt{\Delta}}, \tag{H.29}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{p^2}{(p^2+\Delta)^2} = -\frac{3}{8\pi}\sqrt{\Delta}, \tag{H.30}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{p^4}{(p^2+\Delta)^2} = \frac{5}{8\pi}\Delta^{3/2}, \tag{H.31}$$

where $\Delta = m^2 + x\,(1-x)\,k^2$ and for the 3-point functions

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2+\Delta)^3} = \frac{1}{32\pi}\frac{1}{\Delta^{\frac{3}{2}}} \tag{H.32}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{p^2}{(p^2+\Delta)^3} = \frac{3}{32\pi}\frac{1}{\sqrt{\Delta}} \tag{H.33}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{p^4}{(p^2+\Delta)^3} = \frac{15}{32\pi}\sqrt{\Delta} \tag{H.34}$$

where $\Delta = m^2 + u(1-u)k_1^2 + v(1-v)k_2^2 + 2uvk_1\cdot k_2$. In these formulae x, u, v are Feynman parameters.

Sample calculation As an example of our calculations we explain here some details of the derivation in 8.2.2. To make sense of the integral in (8.43) we have to go Euclidean, which implies $p^2 \to -p^2, k^2 \to -k^2, \eta_{\mu\nu} \to -\eta_{\mu\nu}$ and $d^3p \to i d^3p$. Therefore

$$\tilde{T}^{(odd)}_{\mu\nu\lambda\rho}(k) = \frac{m}{32}\int_0^1 dx \int \frac{d^3p}{(2\pi)^3}\left[\epsilon_{\sigma\nu\rho}\,k^\sigma\,\frac{\frac{4}{3}p^2\eta_{\mu\lambda} + (2x-1)^2 k_\mu k_\lambda}{[p^2+m^2+x(1-x)k^2]^2} + \begin{pmatrix}\mu \leftrightarrow \nu\\ \lambda \leftrightarrow \rho\end{pmatrix}\right]. \tag{H.35}$$

Next we use the appropriate Euclidean integrals above to integrate over p and get

$$\tilde{T}^{(odd)}_{\mu\nu\lambda\rho}(k) = -\frac{m}{256\pi}\int_0^1 dx\,\epsilon_{\sigma\nu\rho}\,k^\sigma \times\left(4\eta_{\mu\lambda}(m^2+x(1-x)k^2)^{\frac{1}{2}} + k_\mu k_\lambda \frac{(2x-1)^2}{(m^2+x(1-x)k^2)^{\frac{1}{2}}}\right) + \begin{pmatrix}\mu \leftrightarrow \nu\\ \lambda \leftrightarrow \rho\end{pmatrix} \tag{H.36}$$

The x integrals are well defined:

$$\int_0^1 dx\,(m^2+x(1-x)k^2)^{\frac{1}{2}} = \frac{1}{2}m + \frac{1}{4}\frac{k^2+4m^2}{|k|}\arctan\frac{|k|}{2m} \tag{H.37}$$

$$\int_0^1 dx\,\frac{(2x-1)^2}{(m^2+x(1-x)k^2)^{\frac{1}{2}}} = -2\frac{m}{k^2} + \frac{k^2+4m^2}{|k|^3}\arctan\frac{|k|}{2m} \tag{H.38}$$

Therefore the result is

$$\begin{aligned}
\tilde{T}^{(odd)}_{\mu\nu\lambda\rho}(k) &= \frac{m}{256\pi}\,\epsilon_{\sigma\nu\rho}\,k^\sigma\left[-\eta_{\mu\lambda}\left(2m+\frac{k^2+4m^2}{|k|}\arctan\frac{|k|}{2m}\right)\right.\\
&\left.+\frac{k_\mu k\nu}{k^2}\left(-2m+\frac{k^2+4m^2}{|k|}\arctan\frac{|k|}{2m}\right)\right]+\begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix}\\
&= \frac{m}{256\pi}\,\epsilon_{\sigma\nu\rho}\,k^\sigma\left[2m\left(-\eta_{\mu\lambda}-\frac{k_\mu k_\lambda}{k^2}\right)\right.\\
&\left.+\left(-\eta_{\mu\lambda}+\frac{k_\mu k_\lambda}{k^2}\right)\frac{k^2+4m^2}{|k|}\arctan\frac{|k|}{2m}\right]+\begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix} \qquad \text{(H.39)}
\end{aligned}$$

The final step is to return to the Lorentzian metric, $k^2\to -k^2$ and $\eta_{\mu\nu}\to-\eta_{\mu\nu}$, $\arctan\frac{|k|}{2m}\to i\,\mathrm{arctanh}\frac{|k|}{2m}$.

H.5 An alternative method for Feynman integrals

An alternative method to calculate Feynman diagrams was introduced in a series of paper by A. I .Davydychev and collaborators, [83–85]. The basic integral to be computed in our case are

$$J_2(d;\alpha,\beta;m)=\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-m^2)^\alpha\left((p-k)^2-m^2\right)^\beta} \qquad \text{(H.40)}$$

and

$$J_3(d;\alpha,\beta,\gamma;m)=\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-m^2)^\alpha((p-k_1)^2-m^2)^\beta((p-q)^2-m^2)^\gamma}, \qquad \text{(H.41)}$$

with $q=k_1+k_2$. Following [83–85] these can be expressed, via the Mellin-Barnes representation of the propagator, as

$$\begin{aligned}
J_2(d;\alpha,\beta;m) &= \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{(-m^2)^{\frac{d}{2}-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\\
&\times\int\frac{du}{2\pi i}\left(-\frac{k^2}{m^2}\right)^u\Gamma(-u)\,\frac{\Gamma(\alpha+u)\Gamma(\beta+u)\Gamma\left(\alpha+\beta-\frac{d}{2}+u\right)}{\Gamma(\alpha+\beta+2u)} \qquad \text{(H.42)}
\end{aligned}$$

and

$$\begin{aligned}
J_3(d;\alpha,\beta,\gamma;m) &= \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{(-m^2)^{\frac{d}{2}-\alpha-\beta-\gamma}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\\
&\times\int\frac{ds}{2\pi i}\frac{dt}{2\pi i}\frac{du}{2\pi i}\left(-\frac{k_1^2}{m^2}\right)^s\left(-\frac{q^2}{m^2}\right)^t\left(-\frac{k_2^2}{m^2}\right)^u\Gamma(-s)\Gamma(-t)\Gamma(-u)\\
&\times\frac{\Gamma(\alpha+\beta+\gamma-\frac{d}{2}+s+t+u)\Gamma(\alpha+s+t)\Gamma(\beta+s+u)+\Gamma(\gamma+t+u)}{\Gamma(\alpha+\beta+\gamma+2s+2t+2u)}. \qquad \text{(H.43)}
\end{aligned}$$

The integrals run from $-i\infty$ to $i\infty$ along vertical contours that separate the positive poles of the Γ's from the negative ones. Positive poles are those of $\Gamma(-u)$ in the case of J_2 or those of $\Gamma(-s)\Gamma(-t)\Gamma(-u)$ in the case of J_3, negative poles are the others. It is clear that the contours of integration must cross the real axis just to the left of the origin. The contours close either to the left or to the right in such a way as to assure convergence of the series.

Let us analyse more closely the case of J_2 to better understand how this works. Using the duplication formula of the gamma function, i.e.

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{H.44}$$

we are able to recast (H.42) into the form

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}(-m^2)^{\frac{d}{2}-\alpha-\beta}\frac{\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+1}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)}\int\frac{du}{2\pi i}\left(-\frac{k^2}{4m^2}\right)^u \times\Gamma(-u)\frac{\Gamma(\alpha+u)\Gamma(\beta+u)\Gamma\left(\alpha+\beta-\frac{d}{2}+u\right)}{\Gamma\left(\frac{\alpha+\beta}{2}+u\right)\Gamma\left(\frac{\alpha+\beta+1}{2}+u\right)}. \tag{H.45}$$

Assuming $\left|\frac{k^2}{4m^2}\right| < 1$ (IR region), we must close the contour of integration on the right $(\mathrm{Re}(u) > 0)$ in order to guarantee convergence of the result and by doing so we will pick-up the poles of $\Gamma(-u)$. For $\alpha = \beta = 1$ and $d = 3$ we obtain

$$J_2^{\mathrm{IR}}(3;1,1;m) = \frac{i}{8\pi|m|}\sum_{j=0}^{\infty}\left(\frac{k^2}{4m^2}\right)^j\frac{1}{2j+1} = \frac{i}{4\pi|k|}\mathrm{arctanh}\left(\sqrt{\frac{k^2}{4m^2}}\right). \tag{H.46}$$

On the other hand, assuming $\left|\frac{k^2}{4m^2}\right| > 1$ (UV region), we need to close the integration contour on the left. For $\alpha = \beta = 1$ and $d = 3$, we will have poles at $u = -\frac{1}{2}$ and at $u = -1, -2, -3, \ldots$, hence

$$\begin{aligned} J_2^{\mathrm{UV}}(3;1,1;m) &= \frac{i}{8\pi|m|}\left(i\pi\frac{|m|}{|k|} + \sum_{j=1}^{\infty}\left(\frac{4m^2}{k^2}\right)^j\frac{1}{(2j-1)}\right) \\ &= -\frac{1}{8|k|} + \frac{i}{4\pi|k|}\mathrm{arctanh}\left(\sqrt{\frac{4m^2}{k^2}}\right). \end{aligned} \tag{H.47}$$

As far as (H.43) is concerned, in this paper we are interested in particular in the IR region, which is the one where m^2 is much larger than k_1^2, k_2^2, q^2 in the case of J_3. This requires that the relevant powers s, t, u in the integrands be positive, and, so, the contours must close around the poles of the positive real axis, that is the poles of $\Gamma(-s)\Gamma(-t)\Gamma(-u)$. An easy calculation gives

$$J_3(d;\alpha,\beta,\gamma;m) = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}(-m^2)^{\frac{d}{2}-\alpha-\beta-\gamma}\frac{\Gamma(\alpha+\beta+\gamma-\frac{n}{2})}{\Gamma(\alpha+\beta+\gamma)} \times\Phi_3\left[\begin{matrix}\alpha+\beta+\gamma-\frac{n}{2},\alpha,\beta,\gamma\\ \alpha+\beta+\gamma\end{matrix}\middle|\frac{k_1^2}{m^2},\frac{q^2}{m^2},\frac{k_2^2}{m^2}\right], \tag{H.48}$$

where Φ_3 is a generalized Lauricella function:

$$\Phi_3\left[\begin{matrix}a_1,a_2,a_3,a_4\\ c\end{matrix}\middle|z_1,z_2,z_3\right] = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{(a_1)_{j_1+j_2+j_3}(a_2)_{j_1+j_2}(a_3)_{j_1+j_3}(a_4)_{j_2+j_3}}{(c)_{2j_1+2j_2+2j_3}} \tag{H.49}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochammer symbol. The leading term in the IR is clearly the one given by $j_1 = j_2 = j_3 = 0$, i.e. by setting $\Phi_3 = 1$ in (H.48).

In general, we need to evaluate not only (H.41) but more general integrals

$$J_{3,\mu_1\dots\mu_M}(d;\alpha,\beta,\gamma;m) = \int \frac{d^dp}{(2\pi)^d} \frac{p_{\mu_1}\dots p_{\mu_M}}{(p^2-m^2)^\alpha((p-k_1)^2-m^2)^\beta((p-q)^2-m^2)^\gamma} \tag{H.50}$$

One can prove by induction that the following formula holds in general

$$\begin{aligned} J_{3,\mu_1\dots\mu_M}(d;\alpha,\beta,\gamma;m) = &\sum_{\substack{\lambda,\kappa_1,\kappa_2,\kappa_3 \\ 2\lambda+\sum\kappa_i=M}} \left(-\frac{1}{2}\right)^\lambda (4\pi)^{M-\lambda} \left\{[\eta]^\lambda [q_1]^{\kappa_1} [q_2]^{\kappa_2} [q_3]^{\kappa_3}\right\}_{\mu_1\dots\mu_M} \\ &\times (\alpha)_{\kappa_1} (\beta)_{\kappa_2} (\gamma)_{\kappa_3} J_3(d+2(M-\lambda);\alpha+\kappa_1,\beta+\kappa_2,\gamma+\kappa_3;m), \end{aligned} \tag{H.51}$$

where the symbol $\left\{[\eta]^\lambda [q_1]^{\kappa_1}\dots[q_N]^{\kappa_N}\right\}_{\mu_1\dots\mu_M}$ stands for the complete symmetrization of the objects inside the curly brackts, for example

$$\{\eta q_1\}_{\mu_1\mu_2\mu_3} = \eta_{\mu_1\mu_2}q_{1\mu_3} + \eta_{\mu_1\mu_3}q_{1\mu_2} + \eta_{\mu_2\mu_3}q_{1\mu_1}.$$

H.6 Third order gravity CS and 3-point e.m. correlator

In this appendix we collect the result concerning the odd parity 3-point function of the e.m. tensor and its relation to the third order term in gravitational CS action.

H.6.1 The third order gravitational CS

From the action term (8.99), by differentiating three times with respect to $h_{\mu\nu}(x)$,$h_{\lambda\rho}(y)$ and $h_{\alpha\beta}(z)$ and Fourier-transforming the result one gets the sum of the following local terms in momentum space (they feature in the same order they appear in (8.99),

$$\begin{aligned} \frac{\kappa}{4}\frac{i}{4}k_1^\sigma k_2^\tau \quad &\Big(\epsilon_{\mu\sigma\tau}(q_\alpha\eta_{\nu\lambda}\eta_{\rho\beta} - q_\rho\eta_{\nu\alpha}\eta_{\lambda\beta}) + \epsilon_{\lambda\sigma\tau}(k_{1\alpha}\eta_{\mu\rho}\eta_{\nu\beta} - k_{1\nu}\eta_{\mu\alpha}\eta_{\rho\beta}) \\ &+\epsilon_{\alpha\sigma\tau}(k_{2\nu}\eta_{\mu\rho}\eta_{\lambda\beta} - k_{2\lambda}\eta_{\mu\beta}\eta_{\nu\rho})\Big) \end{aligned} \tag{H.52}$$

$$\begin{aligned} \frac{\kappa}{4}\frac{i}{4}\epsilon_{\mu\lambda\alpha}\Big(\quad &-k_1\cdot k_2\,(k_{1\rho}\eta_{\beta\nu} - k_{2\beta}\eta_{\rho\nu} + (k_2-k_1)_\nu\eta_{\beta\rho}) \\ &+k_2^2\,(\eta_{\beta\rho}k_{1\nu} - \eta_{\nu\rho}k_{1\beta}) + k_1^2\,(\eta_{\beta\nu}k_{2\rho} - \eta_{\beta\rho}k_{2\nu})\Big) \end{aligned} \tag{H.53}$$

$$\frac{\kappa}{4}\frac{i}{4}\epsilon_{\mu\lambda\alpha}\,(k_{1\beta}q_\rho k_{2\nu} - k_{1\nu}q_\beta k_{2\rho}) \tag{H.54}$$

$$\begin{aligned} \frac{\kappa}{4}\frac{i}{4}\Big(\quad &\epsilon_{\mu\alpha\sigma}q^\sigma(q_\beta\eta_{\nu\lambda} - q_\lambda\eta_{\beta\nu})k_{2\rho} + \epsilon_{\mu\lambda\sigma}q^\sigma(q_\rho\eta_{\nu\beta} - q_\beta\eta_{\nu\rho})k_{1\alpha} \\ &+\epsilon_{\lambda\alpha\sigma}k_1^\sigma(k_{1\beta}\eta_{\mu\rho} - k_{1\mu}\eta_{\beta\rho})k_{2\nu} + \epsilon_{\alpha\lambda\sigma}k_2^\sigma(k_{2\rho}\eta_{\mu\beta} - k_{2\mu}\eta_{\beta\rho})k_{1\nu} \\ &+\epsilon_{\mu\lambda\sigma}k_1^\sigma(k_{1\nu}\eta_{\beta\rho} - k_{1\beta}\eta_{\rho\nu})q_\alpha + \epsilon_{\mu\alpha\sigma}k_2^\sigma(k_{2\nu}\eta_{\rho\beta} - k_{2\rho}\eta_{\beta\nu})q_\lambda\Big) \end{aligned} \tag{H.55}$$

$$-\frac{\kappa}{4}\frac{i}{8}\Big(\quad \epsilon_{\nu\rho\sigma}\left(q^\sigma q_\alpha(k_{2\mu}-k_{1\mu})\eta_{\lambda\beta}-k_1^\sigma k_{1\alpha}(k_{2\lambda}+q_\lambda)\eta_{\mu\beta}\right)$$
$$+\epsilon_{\nu\beta\sigma}\left(q^\sigma q_\lambda(k_{1\mu}-k_{2\mu})\eta_{\alpha\rho}-k_2^\sigma k_{2\lambda}(k_{1\alpha}+q_\alpha)\eta_{\mu\rho}\right)$$
$$+\epsilon_{\beta\rho\sigma}\left(k_1^\sigma k_{1\mu}(k_{2\lambda}+q_\lambda)\eta_{\alpha\nu}-k_2^\sigma k_{2\mu}(k_{1\alpha}+q_\alpha)\eta_{\lambda\nu}\right)\Big) \quad \text{(H.56)}$$

$$-\frac{\kappa}{4}\frac{i}{8}\Big(\quad \epsilon_{\sigma\lambda\alpha}(\eta_{\beta\mu}\eta_{\rho\nu}(k_1^\sigma k_1\cdot k_2-k_2^\sigma k_2\cdot q)+\eta_{\beta\nu}\eta_{\rho\mu}(k_1^\sigma k_1\cdot q-k_2^\sigma k_1\cdot k_2))$$
$$+\epsilon_{\sigma\mu\alpha}(\eta_{\beta\lambda}\eta_{\nu\rho}(q^\sigma q\cdot k_2+k_2^\sigma k_1\cdot k_2)+\eta_{\beta\rho}\eta_{\nu\lambda}(-q^\sigma k_1\cdot q+k_2^\sigma q\cdot k_2))$$
$$+\epsilon_{\sigma\mu\lambda}(\eta_{\nu\beta}\eta_{\rho\alpha}(q^\sigma q\cdot k_1+k_1^\sigma k_1\cdot k_2)+\eta_{\nu\alpha}\eta_{\rho\beta}(-q^\sigma k_2\cdot q+k_1^\sigma q\cdot k_1))\Big) \quad \text{(H.57)}$$

$$\frac{\kappa}{4}\frac{i}{8}\Big[\quad \epsilon_{\sigma\beta\nu}\eta_{\mu\rho}k_2^\sigma\left(\eta_{\alpha\lambda}k_2^2-k_{2\lambda}k_{2\alpha}\right)+\epsilon_{\sigma\beta\lambda}\eta_{\mu\rho}k_2^\sigma\left(\eta_{\alpha\nu}k_2^2-k_{2\nu}k_{2\alpha}\right)$$
$$+\epsilon_{\sigma\rho\nu}\eta_{\mu\beta}k_1^\sigma\left(\eta_{\alpha\lambda}k_1^2-k_{1\lambda}k_{1\alpha}\right)+\epsilon_{\sigma\rho\alpha}\eta_{\mu\beta}k_1^\sigma\left(\eta_{\lambda\nu}k_1^2-k_{1\nu}k_{1\lambda}\right)$$
$$-\epsilon_{\sigma\nu\beta}\eta_{\alpha\rho}\eta_{\mu\lambda}q^\sigma q^2-\epsilon_{\sigma\nu\rho}\eta_{\alpha\mu}\eta_{\beta\lambda}q^\sigma q^2+\epsilon_{\sigma\mu\rho}\eta_{\alpha\lambda}q^\sigma q_\beta q_\nu+\epsilon_{\sigma\mu\alpha}\eta_{\beta\rho}q^\sigma q_\lambda q_\nu\Big] \quad \text{(H.58)}$$

These terms must be simmetrized under $\mu \leftrightarrow \nu, \lambda \leftrightarrow \rho, \alpha \leftrightarrow \beta$. They are expected to correspond to odd-parity 3-point e.m. tensor correlator.

H.6.2 The IR limit of the 3-point e.m. correlator

The 0-th order term, after adding the cross contribution, is given (up to an overall multiplicative factor of $\frac{1}{128\cdot 32\pi}$) by

$$\tilde{T}^{(odd,IR)}_{\mu\nu\alpha\beta\lambda\rho}(k_1,k_2)=\frac{1}{256\pi}\sum_{i=1}^{4}\mathcal{T}^{(i)}_{\mu\nu\lambda\rho\alpha\beta}(k_1,k_2) \quad \text{(H.59)}$$

where

$$\begin{aligned}\mathcal{T}^{(1)}_{\mu\nu\lambda\rho\alpha\beta}(k_1,k_2) &= -\epsilon_{\sigma\beta\nu}k_2^\sigma\Big[\frac{4}{3}k_1\cdot k_2\left(\eta_{\rho\lambda}\eta_{\alpha\mu}+\eta_{\rho\alpha}\eta_{\lambda\mu}+\eta_{\rho\mu}\eta_{\lambda\alpha}\right)+\frac{4}{3}q_\alpha k_{2\mu}\eta_{\rho\lambda}-\frac{4}{3}k_{1\alpha}k_{2\lambda}\eta_{\rho\mu}\\ &\quad -\frac{2}{3}\eta_{\lambda\mu}(q_\alpha k_{1\rho}+k_{1\alpha}q_\rho+k_{1\alpha}k_{2\rho})+\frac{2}{3}\eta_{\lambda\alpha}(2q_\rho k_{1\mu}+k_{1\rho}(k_1-k_2)_\mu)\\ &\quad +\frac{4}{3}k_{1\mu}q_\lambda\eta_{\alpha\rho}+\frac{2}{3}\eta_{\mu\alpha}(2q_\rho q_\lambda+k_{1\rho}q_\lambda+q_\rho k_{2\lambda}+k_{2\rho}k_{2\lambda})\Big]\\ &+ \frac{2}{3}\epsilon_{\sigma\beta\nu}k_1^\sigma k_{2\rho}\Big[(k_1-k_2)_\mu\eta_{\lambda\alpha}+(q+k_2)_\lambda\eta_{\mu\alpha}-(q+k_1)_\alpha\eta_{\lambda\mu}\Big]\end{aligned} \quad \text{(H.60)}$$

$$\begin{aligned}\mathcal{T}^{(2)}_{\mu\nu\lambda\rho\alpha\beta}(k_1,k_2) &= -\epsilon_{\sigma\rho\nu}k_1^\sigma\Big[\frac{4}{3}k_1\cdot k_2\left(\eta_{\beta\lambda}\eta_{\alpha\mu}+\eta_{\alpha\beta}\eta_{\lambda\mu}+\eta_{\beta\mu}\eta_{\lambda\alpha}\right)+\frac{4}{3}q_\lambda k_{1\mu}\eta_{\beta\alpha}-\frac{4}{3}k_{1\alpha}k_{2\lambda}\eta_{\beta\mu}\\ &\quad +\frac{2}{3}\eta_{\lambda\mu}(2q_\alpha q_\beta+k_{1\alpha}q_\beta+q_\alpha k_{2\beta}+k_{1\alpha}k_{1\beta})+\frac{2}{3}\eta_{\lambda\alpha}(2q_\beta k_{2\mu}+k_{2\beta}(k_2-k_1)_\mu)\\ &\quad +\frac{4}{3}k_{2\mu}q_\alpha\eta_{\lambda\beta}-\frac{2}{3}\eta_{\mu\alpha}(q_\beta k_{2\lambda}+k_{2\beta}q_\lambda+k_{2\lambda}k_{1\beta})\Big]\\ &- \frac{2}{3}\epsilon_{\sigma\rho\nu}k_2^\sigma k_{1\beta}\Big[(k_2-k_1)_\mu\eta_{\lambda\alpha}-(q+k_2)_\lambda\eta_{\mu\alpha}+(q+k_1)_\alpha\eta_{\lambda\mu}\Big]\end{aligned} \quad \text{(H.61)}$$

$$\begin{aligned}\mathcal{T}^{(3)}_{\mu\nu\lambda\rho\alpha\beta}(k_1,k_2) &= \epsilon_{\rho\beta\nu}\Big[\frac{74}{15}k_1\cdot k_2(k_1-k_2)_\mu\eta_{\alpha\lambda}-\frac{1}{3}k_1\cdot k_2(15k_2+44k_1)_\alpha\eta_{\lambda\mu}\\ &+\frac{1}{3}k_1\cdot k_2(44k_2+15k_1)_\lambda\eta_{\alpha\mu}-\frac{1}{15}k_{1\alpha}k_{1\lambda}(11k_1+47k_2)_\mu\\ &+\frac{1}{15}k_{2\alpha}k_{2\lambda}(4k_2+7k_1)_\mu+\frac{1}{5}k_{1\alpha}k_{2\lambda}(k_2-k_1)_\mu+\frac{1}{15}k_{2\alpha}k_{1\lambda}(37k_1+3k_2)_\mu\Big]\end{aligned} \tag{H.62}$$

$$\begin{aligned}\mathcal{T}^{(4)}_{\mu\nu\lambda\rho\alpha\beta} &= -\eta_{\rho\nu}\epsilon_{\sigma\beta\tau}k_1^\sigma k_2^\tau\left(\frac{2}{3}\eta_{\mu\alpha}(k_1+2k_2)_\lambda+\frac{2}{3}\eta_{\lambda\alpha}(k_1-k_2)_\mu-\frac{2}{3}\eta_{\mu\lambda}(2k_1+k_2)_\alpha\right)\\ &-\eta_{\beta\nu}\epsilon_{\sigma\rho\tau}k_2^\sigma k_1^\tau\left(-\frac{2}{3}\eta_{\mu\alpha}(k_1+2k_2)_\lambda+\frac{2}{3}\eta_{\lambda\alpha}(k_2-k_1)_\mu+\frac{2}{3}\eta_{\mu\lambda}(2k_1+k_2)_\alpha\right).\end{aligned} \tag{H.63}$$

This must be simmetrized under $\mu\leftrightarrow\nu, \lambda\leftrightarrow\rho, \alpha\leftrightarrow\beta$. The IR limit is entirely local.

Appendix I

Feynman integrals

In this appendix we are going to review the results presented in [83–87]. The two fundamental facts that are going to be used repeatedly in these notes are the Mellin-Barnes representation of $1/\left(k^2-m^2\right)^\alpha$

$$\frac{1}{(k^2-m^2)^\alpha}=\frac{1}{(k^2)^\alpha}\frac{1}{\Gamma(\alpha)}\int\limits_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\left(-\frac{m^2}{k^2}\right)^s\Gamma(-s)\,\Gamma(\alpha+s)\,, \tag{I.1}$$

and the Barnes's Lemma

$$\int\limits_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(c-s)\,\Gamma(d-s)=\frac{\Gamma(a+c)\,\Gamma(a+d)\,\Gamma(b+c)\,\Gamma(b+d)}{\Gamma(a+b+c+d)}. \tag{I.2}$$

I.1 Derivation of $I_2(d,\nu_1,\nu_2)$ with Feynman parametrization

Here we are going to compute the integral

$$I_2(d,\nu_1,\nu_2)\equiv\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2)^{\nu_1}\left[(p-k)^2\right]^{\nu_2}}. \tag{I.3}$$

We are going to use the Feynman parametrization to perform the computation. The useful identity is

$$\frac{1}{A^{\nu_1}B^{\nu_2}}=\frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\,\Gamma(\nu_2)}\int_0^1 dx\frac{x^{\nu_1-1}(1-x)^{\nu_2-1}}{[xA+(1-x)\,B]^{\nu_1+\nu_2}}. \tag{I.4}$$

Using (I.4) in (I.3) we find

$$I_2(d,\nu_1,\nu_2)=\frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\,\Gamma(\nu_2)}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\frac{x^{\nu_2-1}(1-x)^{\nu_1-1}}{\left[x(p-k)^2+(1-x)\,p^2\right]^{\nu_1+\nu_2}}. \tag{I.5}$$

The denominator of the last expression can be written as

$$x(p-k)^2+(1-x)\,p^2=(p-xk)^2-x(x-1)\,k^2$$

and by doing the change of variable $\ell = p - xk$ we can compute the integral in the momentum

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{\left(\ell^2 - x\left(x-1\right)k^2\right)^{\nu_1+\nu_2}} = \frac{(-1)^{\nu_1+\nu_2}\, i}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma\left(\nu_1+\beta\right)} \left(-\frac{1}{x\left(1-x\right)k^2}\right)^{\nu_1+\nu_2-\frac{d}{2}}$$
$$= \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \left(k^2\right)^{\frac{d}{2}-\nu_1-\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma\left(\nu_1+\beta\right)} x^{\frac{d}{2}-\nu_1-\nu_2} \left(1-x\right)^{\frac{d}{2}-\nu_1-\nu_2}. \tag{I.6}$$

Plugging back (I.6) in (I.5) we obtain

$$I_2\left(d,\nu_1,\nu_2\right) = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \left(k^2\right)^{\frac{d}{2}-\nu_1-\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \int_0^1 dx x^{\frac{d}{2}-\nu_1-1} \left(1-x\right)^{\frac{d}{2}-\nu_2-1}.$$

In the last expression we can identify the Euler beta function

$$B\left(\nu_1,\nu_2\right) = \int_0^1 dx\, x^{\nu_1-1}\left(1-x\right)^{\nu_2-1} = \frac{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)}{\Gamma\left(\nu_1+\nu_2\right)},$$

and the final result is

$$I_2\left(d,\nu_1,\nu_2\right) = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \left(k^2\right)^{\frac{d}{2}-\nu_1-\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-\nu_1\right)\Gamma\left(\frac{d}{2}-\nu_2\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)\Gamma\left(d-\nu_1-\nu_2\right)}. \tag{I.7}$$

I.2 Derivation of $I_2\left(d,\nu_1,\nu_2;m\right)$

Here we are going to compute the integral

$$I_2\left(d,\nu_1,\nu_2;m\right) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\left(p^2-m^2\right)^{\nu_1}\left[\left(p-k\right)^2-m^2\right]^{\nu_2}}. \tag{I.8}$$

The main idea will be to rewrite the integrand as

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{\left(p^2\right)^{\nu_1}\left[\left(p-k\right)^2\right]^{\nu_2}} \frac{1}{\left(1-\frac{m^2}{p^2}\right)^{\nu_1}\left(1-\frac{m^2}{(p-k)^2}\right)^{\nu_2}}$$

and use the Mellin-Barnes representation of the factors $1/\left(1+A\right)^{\alpha}$, namely

$$\frac{1}{\left(1-\frac{m^2}{p^2}\right)^{\nu_1}\left(1-\frac{m^2}{(p-k)^2}\right)^{\nu_2}}$$
$$= \frac{1}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \int\limits_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\frac{dt}{2\pi i} \left(-\frac{m^2}{p^2}\right)^s \left(-\frac{m^2}{\left(p-k\right)^2}\right)^t \Gamma\left(-s\right)\Gamma\left(-t\right)\Gamma\left(s+\nu_1\right)\Gamma\left(t+\nu_2\right).$$

Plugging this expression back in our integral we have

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{dt}{2\pi i}\left(-m^2\right)^{s+t}\Gamma(-s)\Gamma(-t)\Gamma(s+\nu_1)\Gamma(t+\nu_2)\times$$
$$\times\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2)^{\nu_1+s}\left[(p-k)^2\right]^{\nu_2+t}},$$

where we identify the integral $I_2(d,\nu_1+s,\nu_2+t;m)$ which we computed in the section I.1:

$$\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2)^{\nu_1+s}\left[(p-k)^2\right]^{\nu_2+t}}$$
$$=\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\left(k^2\right)^{\frac{d}{2}-\nu_1-\nu_2-s-t}\frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}+s+t\right)\Gamma\left(\frac{d}{2}-\nu_1-s\right)\Gamma\left(\frac{d}{2}-\nu_2-t\right)}{\Gamma(\nu_1+s)\Gamma(\nu_2+t)\Gamma(d-\nu_1-\nu_2-s-t)}.$$

Performing the change of variables $-u=\nu_1+\nu_2-\frac{d}{2}+s+t$ we get

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{\left(-m^2\right)^{\frac{d}{2}-\nu_1-\nu_2}}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{du}{2\pi i}\left(-\frac{k^2}{m^2}\right)^u\Gamma(-s)\Gamma(-u)$$
$$\times\frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}+u+s\right)\Gamma\left(\frac{d}{2}-\nu_1-s\right)\Gamma(\nu_1+u+s)}{\Gamma\left(\frac{d}{2}+u\right)}.$$

We may use Barnes's lemma to compute the integration over s

$$\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\Gamma(-s)\Gamma\left(\frac{d}{2}-\nu_1-s\right)\Gamma(\nu_1+u+s)\Gamma\left(\nu_1+\nu_2-\frac{d}{2}+u+s\right)$$
$$=\frac{\Gamma(\nu_1+u)\Gamma(\nu_2+u)\Gamma\left(\frac{d}{2}+u\right)\Gamma\left(\nu_1+\nu_2-\frac{d}{2}+u\right)}{\Gamma(\nu_1+\nu_2+2u)}$$

and we are left with

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{\left(-m^2\right)^{\frac{d}{2}-\nu_1-\nu_2}}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{-i\infty}^{i\infty}\frac{du}{2\pi i}\left(-\frac{k^2}{m^2}\right)^u\Gamma(-u)\frac{\Gamma(\nu_1+u)\Gamma(\nu_2+u)\Gamma\left(\nu_1+\nu_2-\frac{d}{2}+u\right)}{\Gamma(\nu_1+\nu_2+2u)}.$$

After rewriting the gamma function in the denominator of the integrand using the doubling formula

$$\Gamma(2z)=2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

we find

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \frac{\left(-m^2\right)^{\frac{d}{2}-\nu_1-\nu_2}}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{-i\infty}^{i\infty} \frac{du}{2\pi i} \left(-\frac{k^2}{m^2}\right)^u$$
$$\times \Gamma(-u) \frac{\Gamma(\nu_1+u)\Gamma(\nu_2+u)\Gamma\left(\nu_1+\nu_2-\frac{d}{2}+u\right)}{2^{\nu_1+\nu_2+2u-1}\pi^{-\frac{1}{2}}\Gamma\left(\frac{\nu_1+\nu_2}{2}+u\right)\Gamma\left(\frac{\nu_1+\nu_2+1}{2}+u\right)}. \quad \text{(I.9)}$$

It is convenient to rewrite our last expression using Pochammer symbols

$$(a)_\nu \equiv \frac{\Gamma(a+\nu)}{\Gamma(a)},$$

which gives us

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \left(-m^2\right)^{\frac{d}{2}-\nu_1-\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1+\nu_2)} \int_{-i\infty}^{i\infty} \frac{du}{2\pi i} \left(-\frac{k^2}{4m^2}\right)^u \Gamma(-u) \frac{(\nu_1)_u (\nu_2)_u \left(\nu_1+\nu_2-\frac{d}{2}\right)_u}{\left(\frac{\nu_1+\nu_2}{2}\right)_u \left(\frac{\nu_1+\nu_2+1}{2}\right)_u}.$$

Assuming $\left|\frac{k^2}{4m^2}\right| < 1$, we close the contour on the right and we pick-up the poles of $\Gamma(-u)$. The integral becomes the following sum

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \left(-m^2\right)^{\frac{d}{2}-\nu_1-\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1+\nu_2)} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{k^2}{4m^2}\right)^j \frac{(\nu_1)_j (\nu_2)_j \left(\nu_1+\nu_2-\frac{d}{2}\right)_j}{\left(\frac{\nu_1+\nu_2}{2}\right)_j \left(\frac{\nu_1+\nu_2+1}{2}\right)_j}$$

which is the definition of a generalized hypergeometric function. Our final result is

$$I_2(d,\nu_1,\nu_2;m) = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \left(-m^2\right)^{\frac{d}{2}-\nu_1-\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1+\nu_2)} {}_3F_2\left(\begin{matrix} \nu_1,\, \nu_2,\, \nu_1+\nu_2-\frac{d}{2} \\ \frac{\nu_1+\nu_2}{2},\, \frac{\nu_1+\nu_2+1}{2} \end{matrix} \,\middle|\, \frac{k^2}{4m^2} \right).$$

I.3 Mellin-Barnes representation of $I_3(d;\nu_1,\nu_2,\nu_3)$

Here we are going to find the Mellin-Barnes representation of the integral

$$I_3(d;\nu_1,\nu_2,\nu_3) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{\left[(q_1+p)^2\right]^{\nu_1} \left[(q_2+p)^2\right]^{\nu_2} \left[(q_3+p)^2\right]^{\nu_3}}, \quad \text{(I.10)}$$

which comes from a triangle diagram with momenta $k_1 = q_3 - q_2$, $k_2 = q_1 - q_3$ and $k_3 = q_2 - q_1$ all ingoing. This parametrization of the external momenta is nice because it makes conservation manifest.

Our first step will be to use the Feynman parametrization

$$\frac{1}{A^{\nu_1}B^{\nu_2}C^{\nu_3}} = \frac{\Gamma(\nu_t)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_0^1 \frac{\prod_{i=1}^3 dx_i x_i^{\nu_i-1}\delta\left(1-\sum_i x_i\right)}{\left[x_1 A + x_2 B + x_3 C\right]^{\nu_t}}, \quad \text{(I.11)}$$

where $\nu_t = \nu_1+\nu_2+\nu_3$, to recast (I.10) in the form

$$\frac{\Gamma(\nu_t)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int \frac{d^d p}{(2\pi)^d} \int_0^1 \frac{\prod_{i=1}^3 dx_i x_i^{\nu_i-1}\delta\left(1-\sum_i x_i\right)}{\left[x_1(q_1+p)^2 + x_2(q_2+p)^2 + x_3(q_3+p)^2\right]^{\nu_t}}.$$

The denominator can be written as

$$\left(p+\sum_i x_iq_i\right)^2 + x_1x_2\left(q_1-q_2\right)^2 + x_1x_3\left(q_1-q_3\right)^2 + x_2x_3\left(q_2-q_3\right)^2,$$

which can be written in terms of the external momenta k_i as

$$\left(p+\sum_i x_iq_i\right)^2 + x_1x_2k_3^2 + x_1x_3k_2^2 + x_2x_3k_1^2 \equiv \ell^2-\Delta.$$

Now we may perform the momentum integration

$$\int \frac{d^dp}{(2\pi)^d}\frac{1}{\left(\ell^2-\Delta\right)^{\nu_t}} = \frac{(-1)^{\nu_t}\,i}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(\nu_t-\frac{d}{2}\right)}{\Gamma\left(\nu_t\right)}\left(\frac{1}{\Delta}\right)^{\nu_t-\frac{d}{2}},$$

hence we have up to now

$$I_3\left(d;\nu_1,\nu_2,\nu_3\right) = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(\nu_t-\frac{d}{2}\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)\Gamma\left(\nu_3\right)}\int_0^1\frac{\prod_{i=1}^3 dx_ix_i^{\nu_i-1}\delta\left(1-\sum_i x_i\right)}{\left(x_1x_2k_3^2+x_1x_3k_2^2+x_2x_3k_1^2\right)^{\nu_t-\frac{d}{2}}}. \tag{I.12}$$

As we said in the introduction, the Mellin-Barnes representation of $1/\left(A+B\right)^\lambda$ is

$$\frac{1}{(A+B)^\lambda} = \frac{1}{A^\lambda}\frac{1}{\Gamma(\lambda)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\left(\frac{B}{A}\right)^s\Gamma(-s)\,\Gamma(s+\lambda)\,.$$

Of course that it generalizes to expressions of the type $1/\left(A+B+C\right)^\lambda$ for which we find

$$\frac{1}{(A+B+C)^\lambda} = \frac{1}{A^\lambda}\frac{1}{\Gamma(\lambda)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\left(\frac{B+C}{A}\right)^s\Gamma(-s)\,\Gamma(s+\lambda)$$

and we use again the Mellin-Barnes representation to rewrite $1/\left(B+C\right)^{-s}$. We obtain

$$\frac{1}{A^\lambda}\frac{1}{\Gamma(\lambda)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{1}{A^s}\left(\frac{1}{B^{-s}}\frac{1}{\Gamma(-s)}\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\left(\frac{C}{B}\right)^t\Gamma(-t)\,\Gamma(t-s)\right)\Gamma(-s)\,\Gamma(s+\lambda)\,.$$

In our last expression we shift $s\to s+t$ to find

$$\frac{1}{(A+B+C)^\lambda} = \frac{1}{A^\lambda}\frac{1}{\Gamma(\lambda)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{dt}{2\pi i}\left(\frac{B}{A}\right)^s\left(\frac{C}{A}\right)^t\Gamma(-t)\,\Gamma(-s)\,\Gamma(\lambda+s+t)\,. \tag{I.13}$$

Using (I.13) to rewrite the denominator inside the integration over the x_i in (I.12) we find

$$\frac{1}{\left(x_1x_2k_3^2+x_1x_3k_2^2+x_2x_3k_1^2\right)^{\nu_t-\frac{d}{2}}}$$
$$=\frac{1}{\left(x_1x_2k_3^2\right)^{\nu_t-\frac{d}{2}}}\frac{1}{\Gamma\left(\nu_t-\frac{d}{2}\right)}\int\limits_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{dt}{2\pi i}\left(\frac{x_3k_2^2}{x_2k_3^2}\right)^s\left(\frac{x_3k_1^2}{x_1k_3^2}\right)^t\Gamma\left(-t\right)\Gamma\left(-s\right)\Gamma\left(\nu_t-\frac{d}{2}+s+t\right).$$

Now we are left with the integration on the x_i

$$\int_0^1\prod_{i=1}^3 dx_i\delta\left(1-\sum_i x_i\right)x_1^a x_2^b x_3^c,$$

with

$$a=\frac{d}{2}-\nu_2-\nu_3-1-t,$$
$$b=\frac{d}{2}-\nu_1-\nu_3-1-s,$$
$$c=\nu_3-1+s+t.$$

We first do the integral with respect to x_3 to deal with the delta function

$$\int_0^1 dx_1dx_2x_1^ax_2^b\left(1-x_1-x_2\right)^c$$

and now we do the change of variables

$$x_1=\eta\xi,\quad x_2=\eta\left(1-\xi\right),$$

which has Jacobian η, and we find

$$\left(\int_0^1 d\eta\eta^{a+b+1}\left(1-\eta\right)^c\right)\left(\int_0^1 d\xi\xi^a\left(1-\xi\right)^b\right)=\frac{\Gamma\left(a+b+2\right)\Gamma\left(c+1\right)}{\Gamma\left(a+b+c+3\right)}\frac{\Gamma\left(a+1\right)\Gamma\left(b+1\right)}{\Gamma\left(a+b+2\right)}.$$

Hence

$$\int_0^1\prod_{i=1}^3 dx_i\delta\left(1-\sum_i x_i\right)x_1^ax_2^bx_3^c=\frac{\Gamma\left(a+1\right)\Gamma\left(b+1\right)\Gamma\left(c+1\right)}{\Gamma\left(a+b+c+3\right)}.$$

Putting everything together we have

$$I_3\left(d;\nu_1,\nu_2,\nu_3\right)=\frac{i^{1-d}}{\left(4\pi\right)^{\frac{d}{2}}}\frac{\left(k_3^2\right)^{\frac{d}{2}-\nu_t}}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)\Gamma\left(\nu_3\right)\Gamma\left(d-\nu_t\right)}\int\limits_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{dt}{2\pi i}\left(\frac{k_1^2}{k_3^2}\right)^t\left(\frac{k_2^2}{k_3^2}\right)^s\Gamma\left(-t\right)\Gamma\left(-s\right)$$
$$\times\Gamma\left(\frac{d}{2}-\nu_2-\nu_3-t\right)\Gamma\left(\frac{d}{2}-\nu_1-\nu_3-s\right)\Gamma\left(\nu_3+s+t\right)\Gamma\left(\nu_t-\frac{d}{2}+s+t\right).\quad\text{(I.14)}$$

I.3.1 Some exact results

I.3.1.1 $d = \sum \nu_i$

When $\sum \nu_i = d$, $I_3(d; \nu_1, \nu_2, \nu_3)$ becomes particularly simple (see [87]), namely

$$I_3(d; \nu_1, \nu_2, \nu_3)|_{\sum \nu_i = d} = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \prod_{i=1}^{3} \frac{\Gamma\left(\frac{d}{2} - \nu_i\right)}{\Gamma(\nu_i)} \left(k_i^2\right)^{\nu_i - \frac{d}{2}}. \tag{I.15}$$

For example, for $d = 3$ and $\nu_1 = \nu_2 = \nu_3 = 1$ we have

$$I_3(3; 1, 1, 1) = -\frac{1}{8} \frac{1}{k_1 k_2 k_3}.$$

I.3.1.2 $d = 4$ and $\nu_i = 1$

For $d = 4$ and $\nu_i = 1$, the Mellin-Barnes representation of I_3 (I.14) becomes

$$I_3(4; 1, 1, 1) = \frac{i}{16\pi^2} \frac{1}{k_3^2} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{dt}{2\pi i} \left(\frac{k_1^2}{k_3^2}\right)^t \left(\frac{k_2^2}{k_3^2}\right)^s \Gamma(-t)^2 \Gamma(-s)^2 \Gamma(1 + s + t)^2. \tag{I.16}$$

Performing the integrals over s and t we arrive at (vide [87])

$$I_3(4; 1, 1, 1) = \frac{i}{16\pi^2} \frac{1}{k_3^2} \Phi\left(\frac{k_1^2}{k_3^2}, \frac{k_2^2}{k_3^2}\right), \tag{I.17}$$

where

$$\Phi(x, y) = \frac{1}{\lambda} \left[2\left(\mathrm{Li}_2(-\rho x) + \mathrm{Li}_2(-\rho y)\right) + \ln(\rho x) \ln(\rho y) + \ln\frac{y}{x} \ln\frac{1 + \rho y}{1 + \rho x} + \frac{\pi^2}{3}\right], \tag{I.18}$$

where $\mathrm{Li}_2(x)$ is Euler's dilogarithm and

$$\lambda \equiv \sqrt{(1 - x - y)^2 - 4xy}, \quad \rho = \frac{1}{1 - x - y + \lambda}.$$

I.4 The Triple-K representation

I.4.1 The Triple-K representation of $I_3(d, \nu_1, \nu_2, \nu_3)$

Here we are going to reproduce a derivation of the triple-K representation of $I_3(d, \nu_1, \nu_2, \nu_3)$ presented in the appendix A.3 of [14]. Our starting point is

$$I_3(d; \nu_1, \nu_2, \nu_3) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{(q_1 + p)^{2\nu_1} (q_2 + p)^{2\nu_2} (q_3 + p)^{2\nu_3}}, \tag{I.19}$$

where we choose the auxiliary external momenta q_i such that the physical external momenta are given by $k_1 = q_1 - q_3$, $k_2 = q_2 - q_1$ ans $k_3 = q_3 - q_2$. Using Schwinger parameters

$$\frac{1}{A^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty ds\, s^{\alpha - 1} e^{-sA}, \quad \alpha > 0, \tag{I.20}$$

we are able to write

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{\prod_{i=1}^{3}(q_i+p)^{2\nu_i}} = \frac{1}{\prod_{i=1}^{3}\Gamma(\nu_i)} \int \frac{d^d p}{(2\pi)^d} \int_{\mathbb{R}^3_+} d\vec{s}\, s_1^{\nu_1-1} s_2^{\nu_2-1} s_3^{\nu_3-1} \times \exp\left[-\left(s_1(q_1+p)^2 + s_2(q_2+p)^2 + s_3(q_3+p)^2\right)\right]. \tag{I.21}$$

Following [14], we will denote $s_t = s_1 + s_2 + s_3$ and rewrite the expression in the exponent as

$$s_1(q_1+p)^2 + s_2(q_2+p)^2 + s_3(q_3+p)^2 = s_t \ell^2 + \Delta, \tag{I.22}$$

where

$$\ell = p + \frac{q_1 s_1 + q_2 s_2 + q_3 s_3}{s_t}, \quad \Delta = \frac{(q_1-q_2)^2 s_1 s_2 + (q_1-q_3)^2 s_1 s_3 + (q_2-q_3)^2 s_2 s_3}{s_t}. \tag{I.23}$$

Now the integral (I.21) becomes

$$\frac{1}{\prod_{i=1}^{3}\Gamma(\nu_i)} \int_{\mathbb{R}^3_+} d\vec{s}\, s_1^{\nu_1-1} s_2^{\nu_2-1} s_3^{\nu_3-1} e^{-\Delta} \int \frac{d^d \ell}{(2\pi)^d} e^{-s_t \ell^2}. \tag{I.24}$$

For any s_t such that $\mathrm{Re}(s_t) > 0$ we have

$$\int \frac{d^d \ell}{(2\pi)^d} e^{-s_t \ell^2} = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \frac{1}{s_t^{\frac{d}{2}}}, \tag{I.25}$$

hence

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{\prod_{i=1}^{3}(q_i+p)^{2\nu_i}} = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \frac{1}{\prod_{i=1}^{3}\Gamma(\nu_i)} \int_{\mathbb{R}^3_+} d\vec{s}\, s_t^{-\frac{d}{2}} s_1^{\nu_1-1} s_2^{\nu_2-1} s_3^{\nu_3-1} e^{-\Delta}. \tag{I.26}$$

The integral representation (I.26) is akin to what we would get by using Feynman parametrisation instead of Schwinger. Now we are going to transform this integral representation which contains three integrals into one that only contains one, namely, the triple-K representation. We start by defining $\nu_t = \nu_1 + \nu_2 + \nu_3$ and performing the following change of variables:

$$s_j = \frac{v_1 v_2 + v_1 v_3 + v_2 v_3}{2 v_j} = \frac{V}{2 v_j}, \quad j = 1, 2, 3. \tag{I.27}$$

One should notice that these transformations imply

$$d\vec{s} = \frac{V^3}{8 v_1^2 v_2^2 v_3^2} d\vec{v}, \quad s_t = \frac{V^2}{2 v_1 v_2 v_3}, \quad \Delta = \frac{1}{2}\left[(q_2-q_3)^2 v_1 + (q_1-q_3)^2 v_2 + (q_1-q_2)^2 v_3\right]. \tag{I.28}$$

Using these facts we find

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}} \frac{2^{\frac{d}{2}-\nu_t}}{\prod_{i=1}^{3}\Gamma(\nu_i)} \int_{\mathbb{R}^3_+} d\vec{v}\, V^{\nu_t-d} \prod_{j=1}^{3} v_j^{\frac{d}{2}-\nu_j-1} e^{-\frac{1}{2}Q_j^2 v_j}, \tag{I.29}$$

where we introduced the notation $Q_1 \equiv q_3 - q_2 = k_3$, $Q_2 \equiv q_1 - q_3 = k_1$ and $Q_3 \equiv q_2 - q_1 = k_2$ in order to have a more compact expression. The next step is to notice that we can write

$$V = v_1 v_2 v_3 \left(\frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3}\right), \tag{I.30}$$

hence

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{2^{\frac{d}{2}-\nu_t}}{\prod_{i=1}^{3}\Gamma(\nu_i)}\int_{\mathbb{R}^3_+}d\vec{v}\left(\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}\right)^{\nu_t-d}\prod_{j=1}^{3}v_j^{-\frac{d}{2}+\nu_t-\nu_j-1}e^{-\frac{1}{2}Q_j^2v_j}. \tag{I.31}$$

Now we may use the Schwinger parametrisation to rewrite the term $\left(\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}\right)^{\nu_t-d}$, i.e.

$$\left(\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}\right)^{\nu_t-d}=\frac{1}{\Gamma(d-\nu_t)}\int_0^\infty dt\,t^{d-\nu_t-1}\prod_{j=1}^{3}e^{-\frac{t}{v_j}}. \tag{I.32}$$

Inserting this expression into (I.31) we have

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{2^{\frac{d}{2}-\nu_t}}{\Gamma(d-\nu_t)\prod_{i=1}^{3}\Gamma(\nu_i)}\int_0^\infty dt\,t^{d-\nu_t-1}\int_{\mathbb{R}^3_+}d\vec{v}\prod_{j=1}^{3}v_j^{-\frac{d}{2}+\nu_t-\nu_j-1}e^{-\frac{1}{2}Q_j^2v_j-\frac{t}{v_j}}. \tag{I.33}$$

Now we perform a new change of variables and define

$$u_j=\frac{1}{2}Q_j^2v_j, \tag{I.34}$$

thus

$$\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{2^{-d+\nu_t}}{\Gamma(d-\nu_t)\prod_{i=1}^{3}\Gamma(\nu_i)}\int_0^\infty dt\,t^{d-\nu_t-1}\int_{\mathbb{R}^3_+}d\vec{u}\prod_{j=1}^{3}Q_j^{d-2\nu_t+2\nu_j}u_j^{-\frac{d}{2}+\nu_t-\nu_j-1}e^{-u_j-\frac{tQ_j^2}{2u_j}}. \tag{I.35}$$

We may recognize the integral form of the modified Bessel function of second type $K_\nu(z)$, namely

$$K_\nu(z)=\frac{1}{2}\left(\frac{z}{2}\right)^\nu\int_0^\infty e^{-u-\frac{z^2}{4u}}u^{-\nu-1}du,\qquad |\arg z|<\frac{\pi}{4}. \tag{I.36}$$

In fact,

$$Q_j^{d-2\nu_t+2\nu_j}\int_0^\infty du_j u_j^{-\frac{d}{2}+\nu_t-\nu_j-1}e^{-u_j-\frac{tQ_j^2}{2u_j}}=$$

$$=2^{\frac{d}{2}-\nu_t+\nu_j+1}\left(\sqrt{2t}\right)^{-\frac{d}{2}+\nu_t-\nu_j}Q_j^{\frac{d}{2}-\nu_t+\nu_j}K_{\frac{d}{2}-\nu_t+\nu_j}\left(\sqrt{2t}Q_j\right)$$

Therefore

$$I_3(d;\nu_1,\nu_2,\nu_3)=\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{2^{-\frac{d}{2}+4}}{\Gamma(d-\nu_t)\prod_{i=1}^{3}\Gamma(\nu_i)}\int_0^\infty dx\,x^{\frac{d}{2}-1}\prod_{j=1}^{3}Q_j^{\frac{d}{2}-\nu_t+\nu_j}K_{\frac{d}{2}-\nu_t+\nu_j}(xQ_j), \tag{I.37}$$

where we have defined $x=\sqrt{2t}$. The integral representation (I.37) is the triple-K representation of $I_3(d;\nu_1,\nu_2,\nu_3)$. For brevity, let us introduce a notation to talk about the triple-K integral. Let

$$I_{\alpha\{\beta_1,\beta_2,\beta_3\}}(k_1,k_2,k_3)=\int_0^\infty dx\,x^\alpha\prod_{j=1}^{3}k_j^{\beta_j}K_{\beta_j}(x\,k_j). \tag{I.38}$$

Recall that $Q_1 = k_3$, $Q_2 = k_1$ and $Q_3 = k_2$, hence

$$I_3\left(d;\nu_1,\nu_2,\nu_3\right) = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{2^{-\frac{d}{2}+4}}{\Gamma\left(d-\nu_t\right)\prod_{i=1}^{3}\Gamma\left(\nu_i\right)} I_{\frac{d}{2}-1\left\{\frac{d}{2}-\nu_t+\nu_j\right\}}\left(k_3,k_1,k_2\right). \tag{I.39}$$

I.4.2 Analytical properties and regularization of triple-K integrals

In this section we are going to study the analytical properties of the triple-K integral in the Euclidean. Most of the material presented here is based on the references [14, 49, 54].

The triple-K integral is given by

$$I_{\alpha\{\beta_1,\beta_2,\beta_3\}}\left(k_1,k_2,k_3\right) = k_1^{\beta_1}k_2^{\beta_2}k_3^{\beta_3}\int_0^{\infty} dx\, x^{\alpha} K_{\beta_1}\left(x\, k_1\right) K_{\beta_2}\left(x\, k_2\right) K_{\beta_3}\left(x\, k_3\right). \tag{I.40}$$

To study the convergence properties of (I.40) we need to study the bahaviour of the integrand i) for large x and ii) for small x.

For large x: The asymptotic behaviour of $K_\beta\left(x\right)$ for large x is (vide [**milne1972handbook**], p. 378):

$$K_\beta\left(x\right) = \sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}} + \cdots,$$

hence the asymptotic form of the integrand in (I.40) for large x is

$$\sim x^{\alpha-\frac{3}{2}}e^{-x\left(k_1+k_2+k_3\right)}$$

which requires the sum of the norm of the external momenta to be positive for convergence of the integral, i.e. $k_1 + k_2 + k_3 > 0$.

For small x: The modified Bessel function of second type K is defined in terms of the modified Bessel function of first type I as

$$K_\beta\left(x\right) = \frac{1}{2}\frac{\pi}{\sin\left(\beta\pi\right)}\left[I_{-\beta}\left(x\right) - I_\beta\left(x\right)\right] \tag{I.41}$$

and I is given by the series

$$I_\beta\left(x\right) = \left(\frac{x}{2}\right)^\beta \sum_{n=0}^{\infty}\frac{x^{2n}}{2^{2j} j!\Gamma\left(\beta+n+1\right)}, \quad \forall\beta. \tag{I.42}$$

The series expansion of $K_\beta\left(x\right)$ in x comes directly from (I.41) and (I.42), and is decomposed into two series:

$$K_\beta\left(x\right) = \left(\frac{x}{2}\right)^\beta\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}n!}\Gamma\left(-n-\beta\right)x^{2n}\right) + \left(\frac{x}{2}\right)^{-\beta}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}n!}\Gamma\left(-n+\beta\right)x^{2n}\right). \tag{I.43}$$

If β is an integer, the two gamma functions will contribute with divergent terms and in that case we define

$$K_\beta\left(x\right) = \lim_{\epsilon\to 0} K_{\beta+\epsilon}\left(x\right), \quad \beta\in\mathbb{Z}.$$

We will write the expression (I.43) more compactly as

$$K_{\beta}\left(x\right)=\sum_{\sigma=\pm 1}\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{2^{\sigma\beta+2n+1}n!}\Gamma\left(-n-\sigma\beta\right)x^{\sigma\beta+2n}. \tag{I.44}$$

Using (I.44), the integrand of (I.40) becomes

$$x^{\alpha}\prod_{j=1}^{3}k_j^{\beta_j}K_{\beta_j}\left(x\,k_j\right)$$
$$=\sum_{\sigma_j=\pm 1}\sum_{n_j=0}^{\infty}\left(\prod_{i=1}^{3}\frac{\left(-1\right)^{n_i}}{2^{\sigma_i\beta_i+2n_i+1}n_i!}\Gamma\left(-n_i-\sigma_i\beta_i\right)k_i^{(1+\sigma_i)\beta_i+2n_i}\right)x^{\alpha+\sum_{j=1}^{3}(\sigma_j\beta_j+2n_j)}. \tag{I.45}$$

From the expression above we see that in order for the integral to converge in the vicinity of $x=0$, we need

$$\alpha+1+|\beta_1|+|\beta_2|+|\beta_2|>0. \tag{I.46}$$

When the condition (I.46) is violated we may try and define the integral through an analytical continuation. The regularization scheme that we are going to consider is

$$\alpha\to\tilde{\alpha}=\alpha+u\epsilon,\quad \beta_j\to\tilde{\beta}_j=\beta_j+v\epsilon,\quad j=1,2,3, \tag{I.47}$$

where u and v are arbitrary parameters and ϵ is our regulator. Different choices of u and v corresponds to different schemes. From the expression (I.45) we see that the integrand has a Frobenius expansion of the form

$$\sum_{\eta}c_{\eta}x^{\eta},$$

with

$$\eta=\tilde{\alpha}+\sum_{j=1}^{3}\left(\sigma_j\tilde{\beta}_j+2n_j\right)=-1+\left(\alpha+1+\sum_{j=1}^{3}\sigma_j\beta_j+2\sum_{j=1}^{3}n_j\right)+\epsilon\left(u+v\sum_{j=1}^{3}\sigma_j\right). \tag{I.48}$$

Let us introduce an arbitrary energy scale μ and split the integral (I.40) into an upper part and a lower part, i.e.

$$I_{\tilde{\alpha}\{\tilde{\beta}_1,\tilde{\beta}_2,\tilde{\beta}_3\}}\left(k_1,k_2,k_3\right)=\int_0^{\mu^{-1}}dx\sum_{\eta}c_{\eta}x^{\eta}+\int_{\mu^{-1}}^{\infty}dx\,x^{\tilde{\alpha}}\prod_{j=1}^{3}k_j^{\tilde{\beta}_j}K_{\tilde{\beta}_j}\left(x\,k_j\right). \tag{I.49}$$

The full integral must be independent of μ, which must hold true order by order in the ϵ-expansion. As we already established, the second integral will not contribute with any divergencies, hence it can only contribute with terms of order ϵ^0 and higher. Since the regulators guarantee that $\eta>-1$, the evaluation of the first integral in (I.49) gives

$$I_{\tilde{\alpha}\{\tilde{\beta}_1,\tilde{\beta}_2,\tilde{\beta}_3\}}\left(k_1,k_2,k_3\right)=\sum_{\eta}c_{\eta}\frac{\mu^{-(\eta+1)}}{\eta+1}+\int_{\mu^{-1}}^{\infty}dx\,x^{\tilde{\alpha}}\prod_{j=1}^{3}k_j^{\tilde{\beta}_j}K_{\tilde{\beta}_j}\left(x\,k_j\right).$$

Now we are going to show that divergencies are present only if $\eta=-1+w\epsilon$, for some finite w. Indeed, let us suppose that $\eta=m+w\epsilon$, for $m\neq-1$. In this case the term $1/\left(\eta+1\right)$ is regular as $\epsilon\to 0$, thus any divergency must come from c_{η} but any such a singularity would be μ-dependent since it would be multiplied by $\mu^{-(\eta+1)}=\mu^{-(m+1)}\left(1+\mathcal{O}\left(\epsilon\right)\right)$. The only way to cancel this μ-dependence is to have $m = -1$. From (I.48) we now establish that we only have singularities when

$$\alpha + 1 + \sigma_1\beta_1 + \sigma_2\beta_2 + \sigma_3\beta_3 = -2n_{\sigma_1,\sigma_2,\sigma_3}, \quad n_{\sigma_1,\sigma_2,\sigma_3} = n_1 + n_2 + n_3. \tag{I.50}$$

From (I.48) we also read the possible values of w in our regularization scheme, namely

$$w = \{u - 3v, u - v, u + v, u + 3v\}. \tag{I.51}$$

The different possibilities for w come from the different possibilities of combinations of the signs σ_i in order to satisfy (I.50).

Types of singularities: One very interesting implication of the singularity condition (I.50) is that different sign combinations imply different powers of the momenta in the coefficient of the singularities, which can be read from (I.45), namely

$$k_1^{(1+\sigma_1)\tilde{\beta}_1+2n_1} k_2^{(1+\sigma_2)\tilde{\beta}_2+2n_2} k_3^{(1+\sigma_3)\tilde{\beta}_3+2n_3}, \quad n_1, n_2, n_3 \in \{0, 1, 2, \dots\}. \tag{I.52}$$

Recall that *local* terms in a 3-point function are those that are analytical on all three momenta. If an expression is analytical on only two of the three momenta we call it *semi-local*, otherwise we call it *non-local*. In configuration space, local contributions correspond to the product of delta functions or derivatives of delta functions, semi-local contributions present a delta function or derivative of a delta function times some contribution which has support at non-coincident points and non-local contributions present no delta function and is purely composed by functions with support at non-coincident points.

From (I.52) we see that given a solution of (I.50) with signs $\{\sigma_1, \sigma_2, \sigma_3\}$ we have a different type of singularity. As a matter of fact, the classification is as follows:

$$\begin{array}{ccc} \{- - -\} & \longrightarrow & \text{local singularity} \\ \{+ - -\}, \{- + -\}, \{- - +\} & \longrightarrow & \text{semi-local singularity} \\ \{- + +\}, \{+ - +\}, \{+ + -\}, \{+ + +\} & \longrightarrow & \text{non-local singularity} \end{array}$$

For a full analysis of the meaning of these singularities in correlation functions in momentum space, see [49]. Summarizing the discussion, in the context of computation of correlation functions of local operators in a QFT, the presence of local singularities indicate the need of renormalization and call for the introduction of local counterterms. The presence of semi-local singularities indicate the need of renormalization of the sources of the local operators. Non-local singularities are not true singularities of the theory, are only spurious singularities of the triple-K representation and are not present in the final result. For instance, if a Feynman integral is proportional to a triple-K integral that has a singularity of type $\{+ + +\}$, there will be also a gamma function in the proportionality constant that diverges, in this case $\Gamma(d - \nu_t)$, vide (I.39).

I.5 Relating I_4 to I_3

Let us consider the box diagram

$$I_4(d; \nu_i) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{\prod_{i=1}^4 \left[(q_i + p)^2\right]^{\nu_i}},$$

where the physical momenta k_i are given by $k_i = q_i - q_{i+1}$ and $q_5 \equiv q_1$.

Repeting the steps used for the computation of I_3 we find

$$I_4(d;\nu_i)=\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(\nu_t-\frac{d}{2}\right)}{\prod_{i=1}^4\Gamma(\nu_i)}\int_0^1\frac{\prod_{i=1}^4 dx_i x_i^{\nu_i-1}\delta\left(1-\sum_i x_i\right)}{\left[\sum_{i<j}x_ix_j\left(q_i-q_j\right)^2\right]^{\nu_t-\frac{d}{2}}}. \tag{I.53}$$

Explictly, the denominator of (I.53) is given by

$$x_1x_2k_1^2+x_1x_3s+x_1x_4k_4^2+x_2x_3k_2^2+x_2x_4t+x_3x_4k_3^2,$$

where we have used the Mandelstam variables

$$s=(k_1+k_2)^2\,,\quad t=(k_1+k_4)^2=(k_2+k_3)^2\,.$$

Performing the change of variables

$$x_4=\Lambda,\quad x_i=(1-\Lambda)\,y_i,\quad i=1,2,3,$$

we find

$$I_4(d;\nu_i)=\frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(\nu_t-\frac{d}{2}\right)}{\prod_{i=1}^4\Gamma(\nu_i)}\int_0^1\prod_{i=1}^3 dy_i y_i^{\nu_i-1}\delta\left(1-\sum_i y_i\right)$$
$$\int_0^1 d\Lambda\frac{(1-\Lambda)^{2-\nu_t+\frac{d}{2}}}{\left[\Lambda\left(y_1k_4^2+y_2t+y_3k_3^2\right)+(1-\Lambda)\left(y_1y_2k_1^2+y_1y_3s+y_2y_3k_2^2\right)\right]^{\nu_t-\frac{d}{2}}}. \tag{I.54}$$

The integration over Λ is of the form

$$\int_0^1 d\Lambda\frac{(1-\Lambda)^{2-\nu_t+\frac{d}{2}}}{\left[\Lambda A+(1-\Lambda)B\right]^{\nu_t-\frac{d}{2}}}=\left(\frac{d}{2}+3-\nu_t\right)^{-1}B^{\frac{d}{2}-\nu_t}{}_2F_1\left(1,\nu_t-\frac{d}{2},\frac{d}{2}+4-\nu_t;1-\frac{A}{B}\right). \tag{I.55}$$

I.5.1 $d=4$, $\nu_i=1$

In the particular case of $d=4$ and $\nu_i=1$, the integration over Λ (I.55) gives

$$\int_0^1 d\Lambda\frac{1}{\left[\Lambda A+(1-\Lambda)B\right]^2}=\frac{1}{AB}.$$

Hence

$$I_4(4;1,1,1,1)=\frac{i}{16\pi^2}\int_0^1\frac{\prod_{i=1}^3 dy_i\delta\left(1-\sum_i y_i\right)}{\left(y_1k_4^2+y_2t+y_3k_3^2\right)\left(y_1y_2k_1^2+y_1y_3s+y_2y_3k_2^2\right)}. \tag{I.56}$$

Now we will use the double Mellin-Barnes representation of $(A+B+C)^{-\lambda}$

$$\frac{1}{(A+B+C)^\lambda}=\frac{1}{A^\lambda}\frac{1}{\Gamma(\lambda)}\int_{-i\infty}^{i\infty}\frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\left(\frac{B}{A}\right)^{z_1}\left(\frac{C}{A}\right)^{z_2}\Gamma(-z_1)\Gamma(-z_2)\Gamma(\lambda+z_1+z_2)$$

to rewrite the term $\left(y_1k_4^2+y_2t+y_3k_3^2\right)^{-1}$. We find

$$\frac{1}{\left(y_1k_4^2+y_2t+y_3k_3^2\right)}=\frac{1}{y_2t}\int\limits_{-i\infty}^{i\infty}\frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\left(\frac{y_1k_4^2}{y_2t}\right)^{z_1}\left(\frac{y_3k_3^2}{y_2t}\right)^{z_2}\Gamma\left(-z_1\right)\Gamma\left(-z_2\right)\Gamma\left(1+z_1+z_2\right). \tag{I.57}$$

Inserting (I.57) into (I.56) we find

$$I_4\left(4;1,1,1,1\right)=\frac{i}{16\pi^2}\frac{1}{t}\int\limits_{-i\infty}^{i\infty}\frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\left(\frac{k_4^2}{t}\right)^{z_1}\left(\frac{k_3^2}{t}\right)^{z_2}\Gamma\left(-z_1\right)\Gamma\left(-z_2\right)\Gamma\left(1+z_1+z_2\right)$$
$$\int_0^1\prod_{i=1}^{3}dy_i\delta\left(1-\sum_i y_i\right)\frac{y_1^{z_1}y_2^{-z_1-z_2-1}y_3^{z_2}}{\left(y_1y_2k_1^2+y_1y_3s+y_2y_3k_2^2\right)}.$$

Comparing the integral over y_i with (I.12) we notice that

$$\int_0^1\prod_{i=1}^{3}dy_i\delta\left(1-\sum_i y_i\right)\frac{y_1^{z_1}y_2^{-z_1-z_2-1}y_3^{z_2}}{\left(y_1y_2k_1^2+y_1y_3s+y_2y_3k_2^2\right)}=$$
$$=4\pi i\Gamma\left(z_1+1\right)\Gamma\left(z_2+1\right)\Gamma\left(-z_1-z_2\right)I_3\left(2;z_1+1,-z_1-z_2,z_2+1\right).$$

Now we use the expression (I.15) to find

$$\int_0^1\prod_{i=1}^{3}dy_i\delta\left(1-\sum_i y_i\right)\frac{y_1^{z_1}y_2^{-z_1-z_2-1}y_3^{z_2}}{\left(y_1y_2k_1^2+y_1y_3s+y_2y_3k_2^2\right)}=\frac{1}{s}\left(\frac{k_1^2}{s}\right)^{z_2}\left(\frac{k_2^2}{s}\right)^{z_1}\Gamma\left(-z_1\right)\Gamma\left(-z_2\right)\Gamma\left(1+z_1+z_2\right).$$

Thus

$$I_4\left(4;1,1,1,1\right)=\frac{i}{16\pi^2}\frac{1}{st}\int\limits_{-i\infty}^{i\infty}\frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\left(\frac{k_2^2k_4^2}{st}\right)^{z_1}\left(\frac{k_1^2k_3^2}{st}\right)^{z_2}\Gamma\left(-z_1\right)^2\Gamma\left(-z_2\right)^2\Gamma\left(1+z_1+z_2\right)^2.$$

From (I.16) we know that

$$\int\limits_{-i\infty}^{i\infty}\frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\left(\frac{k_2^2k_4^2}{st}\right)^{z_1}\left(\frac{k_1^2k_3^2}{st}\right)^{z_2}\Gamma\left(-z_1\right)^2\Gamma\left(-z_2\right)^2\Gamma\left(1+z_1+z_2\right)^2=\Phi\left(\frac{k_1^2k_3^2}{st},\frac{k_2^2k_4^2}{st}\right).$$

Finally, we have

$$I_4\left(4;1,1,1,1\right)=\frac{i}{16\pi^2}\frac{1}{st}\Phi\left(\frac{k_1^2k_3^2}{st},\frac{k_2^2k_4^2}{st}\right),$$

where

$$\Phi\left(x,y\right)=\frac{1}{\lambda}\left[2\left(\mathrm{Li}_2\left(-\rho x\right)+\mathrm{Li}_2\left(-\rho y\right)\right)+\ln\left(\rho x\right)\ln\left(\rho y\right)+\ln\frac{y}{x}\ln\frac{1+\rho y}{1+\rho x}+\frac{\pi^2}{3}\right], \tag{I.58}$$

where $\mathrm{Li}_2\left(x\right)$ is Euler's dilogarithm and

$$\lambda\equiv\sqrt{\left(1-x-y\right)^2-4xy},\quad \rho=\frac{1}{1-x-y+\lambda}.$$

I.5.2 $d = 3$ and $\nu_i = 1$

In the particular case of $d = 3$ and $\nu_i = 1$, the integration over Λ (I.55) gives

$$\int_0^1 d\Lambda \frac{(1-\Lambda)^{-\frac{1}{2}}}{\left[\Lambda A + (1-\Lambda) B\right]^{\frac{5}{2}}} = \frac{2}{3}\frac{1}{AB^{\frac{3}{2}}} + \frac{4}{3}\frac{1}{A^2 B^{\frac{1}{2}}}.$$

Now we will use the double Mellin-Barnes representation of $(A + B + C)^{-\lambda}$

$$\frac{1}{(A+B+C)^{\lambda}} = \frac{1}{A^{\lambda}}\frac{1}{\Gamma(\lambda)} \int_{-i\infty}^{i\infty} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i} \left(\frac{B}{A}\right)^{z_1} \left(\frac{C}{A}\right)^{z_2} \Gamma(-z_1)\Gamma(-z_2)\Gamma(\lambda + z_1 + z_2)$$

to rewrite the terms $\left(y_1 k_4^2 + y_2 t + y_3 k_3^2\right)^{-1}$ and $\left(y_1 k_4^2 + y_2 t + y_3 k_3^2\right)^{-2}$. We find

$$\frac{1}{\left(y_1 k_4^2 + y_2 t + y_3 k_3^2\right)} = \frac{1}{y_2 t} \int_{-i\infty}^{i\infty} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i} \left(\frac{y_1 k_4^2}{y_2 t}\right)^{z_1} \left(\frac{y_3 k_3^2}{y_2 t}\right)^{z_2} \Gamma(-z_1)\Gamma(-z_2)\Gamma(1 + z_1 + z_2),$$

$$\frac{1}{\left(y_1 k_4^2 + y_2 t + y_3 k_3^2\right)^2} = \frac{1}{y_2^2 t^2} \int_{-i\infty}^{i\infty} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i} \left(\frac{y_1 k_4^2}{y_2 t}\right)^{z_1} \left(\frac{y_3 k_3^2}{y_2 t}\right)^{z_2} \Gamma(-z_1)\Gamma(-z_2)\Gamma(2 + z_1 + z_2).$$

I.6 Reducing tensor integrals to scalar ones

In this appendix we are going to review the derivation of a formula to relate tensor one-loop integrals with scalar ones presented in [84].

I.6.1 One-loop Scalar N-point function integral with Schwinger parametrization

Here we are going to consider the scalar N-point function integral

$$I^{(N)}(d; \{\nu_j\}) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\prod_{i=1}^N D_i^{\nu_i}}, \quad D_i = (q_i + p)^2 - m^2 \tag{I.59}$$

and we are going to use the Schwinger parametrization

$$\frac{1}{D^{\lambda}} = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} d\alpha \alpha^{\lambda - 1} e^{-\alpha D} \tag{I.60}$$

to simplify it. Applying (I.60) to all the propagators of (I.59) we find

$$I^{(N)}(d; \{\nu_j\}) = \left(\prod_{i=1}^N \Gamma(\nu_i)^{-1}\right) \int \frac{d^d p}{(2\pi)^d} \int_0^{\infty} \prod_{i=1}^N d\alpha_i \alpha_i^{\nu_i - 1} e^{-\sum_{i=1}^N \alpha_i D_i}.$$

By performing the change of variables

$$\sum_{i=1}^N \alpha_i = \Lambda, \quad \alpha_i = \Lambda \beta_i, \quad \sum_{i=1}^N \beta_i = 1$$

we get

$$I^{(N)}(d;\{\nu_j\}) = \left(\prod_{i=1}^{N}\Gamma(\nu_i)^{-1}\right)\int_0^1 \prod_{i=1}^{N} d\beta_i \beta_i^{\nu_i-1}\delta\left(1-\sum_{i=1}^{N}\beta_i\right)\int\frac{d^dp}{(2\pi)^d}\int_0^\infty d\Lambda\Lambda^{\nu_t-1}e^{-\Lambda\sum_{i=1}^{N}\beta_i D_i}.$$

The integral over Λ can be computed and the result is

$$\int_0^\infty d\Lambda\Lambda^{\nu_t-1}e^{-\Lambda\sum_{i=1}^{N}\beta_i D_i} = \Gamma(\nu_t)\left(\sum_{i=1}^{N}\beta_i D_i\right)^{-\nu_t}.$$

Now we have

$$I^{(N)}(d;\{\nu_j\}) = \left(\prod_{i=1}^{N}\Gamma(\nu_i)^{-1}\right)\Gamma(\nu_t)\int_0^1 \prod_{i=1}^{N} d\beta_i \beta_i^{\nu_i-1}\delta\left(1-\sum_{i=1}^{N}\beta_i\right)\int\frac{d^dp}{(2\pi)^d}\left(\sum_{i=1}^{N}\beta_i D_i\right)^{-\nu_t},$$

where

$$\sum_{i=1}^{N}\beta_i D_i = \left(p+\sum_{i=1}^{N}\beta_i q_i\right)^2 + \sum_{j>i}\beta_i\beta_j(q_i-q_j)^2 - m^2$$

and the integral over the momentum gives us

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{\left[\ell^2+\sum_{j>i}\beta_i\beta_j(q_i-q_j)^2-m^2\right]^{\nu_t}} = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(\nu_t-\frac{d}{2}\right)}{\Gamma(\nu_t)}\left(\sum_{j>i}\beta_i\beta_j(q_i-q_j)^2-m^2\right)^{\frac{d}{2}-\nu_t},$$

which yields the following integral representation for the integral (I.59)

$$I^{(N)}(d;\{\nu_j\}) = \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(\nu_t-\frac{d}{2}\right)}{\prod_{i=1}^{N}\Gamma(\nu_i)}\int_0^1\prod_{i=1}^{N}d\beta_i\beta_i^{\nu_i-1}\delta\left(1-\sum_{i=1}^{N}\beta_i\right)\left(\sum_{j>i}\beta_i\beta_j(q_i-q_j)^2-m^2\right)^{\frac{d}{2}-\nu_t}. \tag{I.61}$$

I.6.2 Relations among N-point function integrals and tensor integrals

Notice that if we derive (I.59) with respect to any one of the parameters q_i we get

$$\frac{\partial}{\partial q_i^\mu}I^{(N)}(d;\{\nu_j\}) = -\int\frac{d^dp}{(2\pi)^d}\frac{2\nu_i(q_i+p)_\mu}{\prod_{j=1}^{N}D_j^{\nu_j+\delta_{ij}}}, \tag{I.62}$$

i.e.

$$\frac{\partial}{\partial q_i^\mu}I^{(N)}(d;\{\nu_j\}) = -2\nu_i q_{i\mu}I^{(N)}(d;\{\nu_j+\delta_{ij}\}) - 2\nu_i I_\mu^{(N)}(d;\{\nu_j+\delta_{ij}\}), \tag{I.63}$$

where

$$I_\mu^{(N)}(d;\{\nu_j\}) \equiv \int\frac{d^dp}{(2\pi)^d}\frac{p_\mu}{\prod_{j=1}^{N}D_j^{\nu_j}}.$$

From (I.63) we can read

$$I_\mu^{(N)}(d;\{\nu_j\}) = -q_{i\mu}I^{(N)}(d;\{\nu_j\}) - \frac{1}{2(\nu_i-1)}\frac{\partial}{\partial q_i^\mu}I^{(N)}(d;\{\nu_j-\delta_{ij}\}). \tag{I.64}$$

At this point we may notice that deriving with respect to any of the q_i would be equally effective to derive a formula for tensor integrals, hence from here on we will always derive with respect to q_1, without loss of generality. Another thing that we might note is that the expression (I.64) is useless if we want to compute $I_\mu^{(N)}(d;\{\nu_j\})$ and all the ν_j are one since clearly we would need to start with an expression contaning log's of the propagator. It turns out that we can improve the expression (I.64) using a few identities involving the scalar N-point integrals.

I.6.2.1 Vector integral

Using (I.61) we may verify the two identities

$$I^{(N)}(d;\{\nu_j\}) = -4\pi \sum_{k=1}^{N} \nu_k I^{(N)}(d+2;\{\nu_j+\delta_{jk}\}), \tag{I.65}$$

$$\frac{\partial}{\partial q_1^\mu} I^{(N)}(d;\{\nu_j\}) = 2\pi\nu_1 \sum_{k=2}^{N} (q_1-q_k)_\mu \nu_k I^{(N)}(d+2;\{\nu_j+\delta_{1j}+\delta_{jk}\}). \tag{I.66}$$

If we add and subtract the term $2\pi\nu_1 q_1 \nu_1 I^{(N)}(d+2;\{\nu_j+\delta_{1j}+\delta_{1j}\})$ to (I.66) and use the identity (I.65) we derive

$$\frac{\partial}{\partial q_1^\mu} I^{(N)}(d;\{\nu_j-\delta_{1j}\}) = -2(\nu_1-1)\left[q_{1\mu} I^{(N)}(d;\{\nu_j\}) + 4\pi \sum_{k=1}^{N} q_{k\mu}\nu_k I^{(N)}(d+2;\{\nu_j+\delta_{jk}\})\right]. \tag{I.67}$$

Using the identity (I.67) in (I.64) we have

$$I_\mu^{(N)}(d;\{\nu_j\}) = 4\pi \sum_{k=1}^{N} q_{k\mu}\nu_k I^{(N)}(d+2;\{\nu_j+\delta_{jk}\}). \tag{I.68}$$

I.6.2.2 Rank 2 tensor integral

Of course that we can express $I_{\mu_1\mu_2}^{(N)}(d;\{\nu_j\})$ in terms of a derivative of $I_{\mu_2}^{(N)}(d;\{\nu_j\})$, namely

$$I_{\mu_1\mu_2}^{(N)}(d;\{\nu_j\}) = -q_{1\mu_1} I_{\mu_2}^{(N)}(d;\{\nu_j\}) - \frac{1}{2(\nu_1-1)} \frac{\partial}{\partial q_1^{\mu_1}} I_{\mu_2}^{(N)}(d;\{\nu_j-\delta_{1j}\}). \tag{I.69}$$

Using the expression (I.68) in the second term of (I.69) we have

$$\begin{aligned}
\frac{\partial}{\partial q_1^{\mu_1}} I_{\mu_2}^{(N)}(d;\{\nu_j-\delta_{1j}\}) &= 4\pi \frac{\partial}{\partial q_1^{\mu_1}} \left(\sum_{k=1}^{N} q_{k\mu_2}(\nu_k-\delta_{1k}) I^{(N)}(d+2;\{\nu_j-\delta_{1j}+\delta_{jk}\})\right) \\
&= 4\pi\eta_{\mu_1\mu_2}(\nu_1-1) I^{(N)}(d+2;\{\nu_j\}) \\
&+ 4\pi \sum_{k=1}^{N} q_{k\mu_2}(\nu_k-\delta_{1k}) \frac{\partial}{\partial q_1^{\mu_1}} I^{(N)}(d+2;\{\nu_j-\delta_{1j}+\delta_{jk}\}).
\end{aligned}$$

Using (I.67) we have

$$\frac{\partial}{\partial q_1^{\mu_1}} I^{(N)}\left(d+2;\{\nu_j-\delta_{1j}+\delta_{jk}\}\right)$$
$$=-2\left(\nu_1-1+\delta_{1k}\right)\left[q_{1\mu_1} I^{(N)}\left(d+2;\{\nu_j+\delta_{jk}\}\right)+4\pi\sum_{l=1}^{N} q_{l\mu_1}\left(\nu_l+\delta_{lk}\right) I^{(N)}\left(d+4;\{\nu_j+\delta_{jk}+\delta_{jl}\}\right)\right],$$

thus

$$\begin{aligned}\frac{\partial}{\partial q_1^{\mu_1}} I_{\mu_2}^{(N)}\left(d;\{\nu_j-\delta_{1j}\}\right)=&\,4\pi\eta_{\mu_1\mu_2}\left(\nu_1-1\right) I^{(N)}\left(d+2;\{\nu_j\}\right)\\&-2\pi\sum_{k=1}^{N} q_{k\mu_2}\left(\nu_k-\delta_{1k}\right)\left(\nu_1-1+\delta_{1k}\right) q_{1\mu_1} I^{(N)}\left(d+2;\{\nu_j+\delta_{jk}\}\right)\\&-8\pi^2\sum_{k,l=1}^{N} q_{l\mu_1} q_{k\mu_2}\left(\nu_k-\delta_{1k}\right)\left(\nu_1-1+\delta_{1k}\right)\left(\nu_l+\delta_{lk}\right) I^{(N)}\left(d+4;\{\nu_j+\delta_{jk}+\delta_{jl}\}\right)\end{aligned}$$

Notice that $\left(\nu_k-\delta_{1k}\right)\left(\nu_1-1+\delta_{1k}\right)=\nu_k\left(\nu_1-1\right)$ for any $k\in\{1,\ldots,N\}$. The second sum we will split in two pieces, the one where $k=l$ and the one where $k\neq l$. We have

$$\begin{aligned}\frac{\partial}{\partial q_1^{\mu_1}} I_{\mu_2}^{(N)}\left(d;\{\nu_j-\delta_{1j}\}\right)=&\,\frac{1}{4\pi}\eta_{\mu_1\mu_2}\left(\nu_1-1\right) I^{(N)}\left(d+2;\{\nu_j\}\right)\\&-\frac{2\left(\nu_1-1\right)}{4\pi} q_{1\mu_1}\sum_{k=1}^{N} q_{k\mu_2}\nu_k I^{(N)}\left(d+2;\{\nu_j+\delta_{jk}\}\right)\\&-\frac{2\left(\nu_1-1\right)}{16\pi^2}\sum_{k=1}^{N} q_{k\mu_1} q_{k\mu_2}\nu_k\left(\nu_k+1\right) I^{(N)}\left(d+4;\{\nu_j+2\delta_{jk}\}\right)\\&-\frac{2\left(\nu_1-1\right)}{16\pi^2}\sum_{k<l}^{N}\left(q_{k\mu_1} q_{l\mu_2}+q_{l\mu_1} q_{k\mu_2}\right)\nu_k\nu_l I^{(N)}\left(d+4;\{\nu_j+\delta_{jk}+\delta_{jl}\}\right).\end{aligned}$$

Notice that we can use (I.68) to simplify the second line, namely

$$\begin{aligned}-\frac{1}{2\left(\nu_1-1\right)}\frac{\partial}{\partial q_1^{\mu_1}} I_{\mu_2}^{(N)}\left(d;\{\nu_j-\delta_{1j}\}\right)=&-\frac{1}{8\pi}\eta_{\mu_1\mu_2} I^{(N)}\left(d+2;\{\nu_j\}\right)\\&+q_{1\mu_1} I_{\mu_2}^{(N)}\left(d;\{\nu_j\}\right)\\&+\frac{1}{16\pi^2}\sum_{k=1}^{N} q_{k\mu_1} q_{k\mu_2}\nu_k\left(\nu_k+1\right) I^{(N)}\left(d+4;\{\nu_j+2\delta_{jk}\}\right)\\&+\frac{1}{16\pi^2}\sum_{k<l}^{N}\left(q_{k\mu_1} q_{l\mu_2}+q_{l\mu_1} q_{k\mu_2}\right)\nu_k\nu_l I^{(N)}\left(d+4;\{\nu_j+\delta_{jk}+\delta_{jl}\}\right).\end{aligned}$$

Plugging this result back in the (I.69) we find

$$\begin{aligned} I^{(N)}_{\mu_1\mu_2}\left(d;\{\nu_j\}\right) = &-\frac{1}{8\pi}\eta_{\mu_1\mu_2}I^{(N)}\left(d+2;\{\nu_j\}\right) \\ &+\frac{1}{16\pi^2}\sum_{k=1}^{N} q_{k\mu_1}q_{k\mu_2}\nu_k\left(\nu_k+1\right)I^{(N)}\left(d+4;\{\nu_j+2\delta_{jk}\}\right) \\ &+\frac{1}{16\pi^2}\sum_{k<l}^{N}\left(q_{k\mu_1}q_{l\mu_2}+q_{l\mu_1}q_{k\mu_2}\right)\nu_k\nu_l I^{(N)}\left(d+4;\{\nu_j+\delta_{jk}+\delta_{jl}\}\right). \end{aligned} \tag{I.70}$$

I.6.2.3 General tensor integral

It can be proven by induction that the following formula holds in general

$$\begin{aligned} I^{(N)}_{\mu_1\ldots\mu_M}\left(d;\{\nu_j\}\right) = &\sum_{\substack{\lambda,\kappa_1,\ldots,\kappa_N \\ 2\lambda+\sum\kappa_i=M}}\left(-\frac{1}{2}\right)^{\lambda}(4\pi)^{\lambda-M}\left\{[\eta]^{\lambda}[q_1]^{\kappa_1}\ldots[q_N]^{\kappa_N}\right\}_{\mu_1\ldots\mu_M} \\ &\left(\nu_1\right)_{\kappa_1}\ldots\left(\nu_N\right)_{\kappa_N}I^{(N)}\left(d+2\left(M-\lambda\right);\nu_1+\kappa_1,\ldots,\nu_N+\kappa_N\right), \end{aligned} \tag{I.71}$$

where the symbol $\left\{[\eta]^{\lambda}[q_1]^{\kappa_1}\ldots[q_N]^{\kappa_N}\right\}_{\mu_1\ldots\mu_M}$ stands for the complete symmetrization of the objects inside the curly brackts, for example

$$\{\eta q_1\}_{\mu_1\mu_2\mu_3} = \eta_{\mu_1\mu_2}q_{1\mu_3}+\eta_{\mu_1\mu_3}q_{1\mu_2}+\eta_{\mu_2\mu_3}q_{1\mu_1}.$$

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