SISSA–INTERNATIONAL SCHOOL OF ADVANCED STUDIES

DOCTORAL THESIS

Exploring Spacetime Phenomenology:

From Lorentz Violations to Experimental Tests of Non-locality

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Declaration of Authorship

The research presented in this thesis was mainly conducted in SISSA – International School for Advanced Studies between September 2012 and June 2016. This thesis is the result of the author own work, as well as the outcome of scientific collaborations stated below, except where explicit reference is made to the results of others. The content of this thesis is based on the following research papers published in refereed Journals, conference proceedings or preprints available on arxiv.org:

- A. Belenchia, S. Liberati, and A. Mohd
  Emergent gravitational dynamics in a relativistic Bose-Einstein condensate
  In: **Phys. Rev. D**90.10, p. 104015, arXiv: 1407.7896 [gr-qc].

- A. Belenchia, D.M.T. Benincasa, and S. Liberati
  Nonlocal Scalar Quantum Field Theory from Causal Sets
  In: **JHEP** 03, p. 036. arXiv: 1411.6513 [gr-qc].

- A. Belenchia, D.M.T. Benincasa, A. Marcianó, and L. Modesto
  Spectral Dimension from Causal Set Nonlocal Dynamics
  In: **Phys.Rev. D93** (2016) 044017, arXiv: 1507.00330 [gr-qc]

- A. Belenchia, A. Gambassi and S. Liberati
  Lorentz violation naturalness revisited
  In: **JHEP** (2016), arXiv: 1601.06700 [hep-th]

- A. Belenchia, D.M.T. Benincasa, Stefano Liberati, Francesco Marin, Francesco Marino, and Antonello Ortolan
  Tests of Quantum Gravity induced non-locality via opto-mechanical quantum oscillators
  In: **Phys.Rev.Lett.**116 (2016), arXiv: 1512.02083 [gr-qc]

- A. Belenchia,
  Universal behaviour of generalized Causal-set d'Alembertian in curved spacetime
  In: **Class.Quant.Grav.** (2016), arXiv: 1510.04665 [gr-qc]

- A. Belenchia, D.M.T. Benincasa, and F. Dowker
  The continuum limit of a 4-dimensional causal set scalar d'Alembertian
  In: arXiv: 1510.04656 [gr-qc]

- A. Belenchia
  Causal set, non-locality and phenomenology
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- A. Belenchia, D.M.T. Benincasa, E. Martin-Martinez and M. Saravani
  Low-Energy Signatures of Non-Local Field Theories
  Accepted in: *PRD rapid communication*; In: arXiv:1605.0397

"A mathematical friend of mine said to me the other day half in jest: "The mathematician can do a lot of things, but never what you happen to want him to do just at the moment." Much the same often applies to the theoretical physicist when the experimental physicist calls him in. What is the reason for this peculiar lack of adaptability?"

Albert Einstein

SISSA–INTERNATIONAL SCHOOL OF ADVANCED STUDIES

Abstract

Doctor of Philosophy

Exploring Spacetime Phenomenology:
From Lorentz Violations to Experimental Tests of Non-locality

by Alessio BELENCHIA

This thesis deals primarily with the phenomenology associated to quantum aspects of spacetime. In particular, it aims at exploring the phenomenological consequences of a fundamental discreteness of the spacetime fabric, as predicted by several quantum gravity models and strongly hinted by many theoretical insights.

The first part of this work considers a toy-model of emergent spacetime in the context of analogue gravity. The way in which a relativistic Bose–Einstein condensate can mimic, under specific configurations, the dynamics of a scalar theory of gravity will be investigated. This constitutes proof-of-concept that a legitimate dynamical Lorentzian spacetime may emerge from non-gravitational (discrete) degrees of freedom. Remarkably, this model will emphasize the fact that in general, even when arising from a relativistic system, any emergent spacetime is prone to show deviations from exact Lorentz invariance. This will lead us to consider Lorentz Invariance Violations as first candidate for a discrete spacetime phenomenology.

Having reviewed the current constraints on Lorentz Violations and studied in depth viable resolutions of their apparent naturalness problem, the second part of this thesis focusses on models based on Lorentz invariance. In the context of Casual Set theory, the coexistence of Lorentz invariance and discreteness leads to an inherently nonlocal scalar field theory over causal sets well approximating a continuum spacetime. The quantum aspects of the theory in flat spacetime will be studied and the consequences of its non-locality will be spelled out. Noticeably, these studies will lend support to a possible dimensional reduction at small scales and, in a classical setting, show that the scalar field is characterized by a universal non-minimal coupling when considered in curved spacetimes.

Finally, the phenomenological possibilities for detecting this non-locality will be investigated. First, by considering the related spontaneous emission of particle detectors, then by developing a phenomenological model to test nonlocal effects using opto-mechanical, non-relativistic systems. In both cases, one could be able to cast in the near future stringent bounds on the non-locality scale.

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List of Abbreviations

LHS	**L**eft **H**and **S**ide
RHS	**R**ight **H**and **S**ide
GR	**G**eneral **R**elativity
QG	**Q**uantum **G**ravity
QFT	**Q**uantum **F**ield **T**heory
LIV	**L**orentz **I**nvariance **V**iolations
AG	**A**nalogue **G**ravity
SSB	**S**pontaneous **S**ymmetry **B**reaking
CS	**C**ausal **S**et
UV	**U**ltra **V**iolet
IR	**I**nfra **R**ed
GFT	**G**roup **F**ield **T**heory
LQG	**L**oop **Q**uantum **G**ravity
MDR	**M**odified **D**ispersion **R**elation
ST	**S**tring **T**heory
LQG	**S**tring **F**ield **T**heory
SM	**S**tandard **M**odel
GCD	**G**eneralized **C**ausal set **D**'Alembertians
HP	**H**uygens **P**rinciple
FRW	**F**riedmann **R**obertson **W**alker
EEP	**E**instein **E**quivalence **P**rinciple
QND	**Q**uantum **N**on **D**emolition
GUP	**G**eneralized **U**certainty **P**rinciple

To my family

Preface

Theory is when we know everything but nothing works. Praxis is when everything works but we do not know why. We always end up by combining theory with praxis: nothing works and we do not know why.

Albert Einstein

Understanding the small-scale structure of spacetime is one of the main goals of research in Quantum Gravity (QG). In particular, the question whether it is fundamentally a continuum — as in General Relativity (GR) and Quantum Field Theory (QFT) — or discrete, dates back to Riemann [189]. Over the last 30 years, QG research has undergone a rapid development and many theoretical advancements have been made with the aim to address at least some of the fundamental aspects of the problem. However, the two pillars of modern physics, i.e., GR and Quantum Theory, have been shown to be difficult to be reconciled to the point that a fully consistent theory of QG has not yet been reached.

One of the major obstacles in QG lies in the fact that the deviations from standard physics — Standard Model of particles physics and cosmology, GR and QFT in flat space — due to quantum gravitational effects are expected to occur at very high-energy/short-lengths, i.e., at the Planck scale, far outside our observational capabilities. This expectation led the community to focus on mathematical consistency of the theories as a quality measure. This landscape started to change at the end of the 90s when different proposals for testing QG effects began to emerge and the new field of Quantum Gravity Phenomenology (QGP) blossomed. At present, there exists a plethora of QG models and theories, more or less developed, and various ideas on how to test them through observations and experiments. In most cases, constraints have been placed on the free parameters of the models. It should be noted that, although different from a direct detection of QG effects, casting bounds on free parameters is undoubtedly a valuable goal, since it brings QG back to the realm of falsifiable theories and, as the history of physics teaches, it can set theoretical research on the right path.

This thesis investigates whether spacetime may be fundamentally discrete, addressing the phenomenological signatures of such possibility. In particular, various aspects related to discreteness of spacetime and phenomenology will be taken into consideration, ranging from toy models of emergent spacetime and Lorentz Invariance Violations (LIV) to discrete, Lorentz Invariant spacetime models and their phenomenology. In this way, a panoramic view of the phenomenological avenues that lead towards a better understanding of spacetime structure is provided, with particular emphasis placed on models which preserve Lorentz Invariance (LI).

Arrangement of the thesis

This thesis is arranged into seven main chapters. Chapter 1 is an introduction to analogue models, LIV and Causal Set (CS) theory that is preparatory for the rest of the work.

The other chapters are mainly based on original research and form part of the work done during the Ph.D.

Chapter 2 considers an analogue model consisting of a relativistic Bose–Einstein condensate. This chapter is mainly based on [35]. It is argued that, in general, LIV cannot be avoided in the emergent spacetime even when the underline system is LI. It is shown that, by fine-tuning the model is possible to hide such LIV and at the same time to also emerge a gravitational dynamics for the effective spacetime due to the back-reation of perturbations. This is a proof-of-concept that analogue models can mimic not only the kinematical features of QFT in curved spacetime but also gravitational dynamics — although simpler than GR dynamics. Since the model requires a sort of fine-tuning, it appears that LIV can be hidden only in very peculiar cases, and to leading order in the approximations used, whereas they are bound to be present in general.

In Chapter 3, LIV are further investigated in the effective field theory (EFT) framework. In particular, the Chapter focusses on the naturalness problem of LIV. Such a problem, although presents in general, can be tamed by considering a separation of scales between the QG scale — assumed to be the Planck scale — and an EFT LI scale. Furthermore, LIV in the case in which dissipation is present — an unavoidable fact if the LIV arise from a dynamical process — are considered. This chapter is mainly based on [37].

From Chapter 4 onwards we will be concerned with models that do not violate LI. In particular, Causal Set theory and the construction of an effective non-local scalar field theory in flat spacetime, derived from causal sets, are deal with in Chapter 4. This is mainly based on [35].

Chapter 5 explores two different aspects of the non-local field theory derived from causal sets. In particular, both a classical and a quantum aspect are considered. Namely, the local limit of the discrete CS d'Alembertians in curved spacetime is obtained and connected with possible violations of the Einstein Equivalence Principle (EEP). Furthermore, the spectral dimension of the quantum theory is computed which shows evidence of spontaneous dimensional reduction, typical of several QG models. This Chapter is based on [34, 40].

In Chapter 6, two different ways to phenomenologically test non-locality with low-energy systems are investigated. The first exploits Unruh–DeWitt particle detectors in inertial motion. A brief discussion on test of Hyugens' principle violations will be presented whereas a discussion of the uniformly accelerating case is presented in the conclusions. The second way consist in employing non-relativistic, macroscopic quantum systems. In particular, a proposal to experimentally test non-local modification of dynamics with opto-mechanical systems is advanced. It will be shown which effects are bound to emerge due to the non-locality and the forecast for bounds that could be achieved in the near future. Chapter 6 is mainly based on [38, 41] and unpublished material.

The final chapter draws some conclusion and highlight possible future directions.

A note on notation

Throughout this work the metric signature used is $(- + ++)$ apart from Chapter 3 in which the other convention is used. This is so since the latter convention is the usual one employed in the particle physics community to which Chapter 3 — and the work [37] on which the chapter is based — is aimed. Concerning Chapter 2 and 5, in which curved spacetime are considered, the differential geometry conventions are those of Wald in ref. [224].

Chapter 1

Introduction

Per me si va ne la città dolente,
per me si va ne l'etterno dolore,
per me si va tra la perduta gente...
Lasciate ogni speranza, voi ch'intrate

Dante Alighieri
Divina Commedia

1.1 Quantum Gravity and Phenomenology

Our understanding of Nature is based on the two pillars of modern physics, General Relativity and Quantum Theory. The former is a classical theory describing the interplay between matter and spacetime as summarized by the words of Wheeler: *Spacetime tells matter how to move; matter tells spacetime how to curve*; the latter is an intrinsically probabilistic theory that describes the microscopic world and that *I think I can safely say that nobody understands* (cit. R. Feymann).

These two theories are far apart and trying to bring them together has proven to be a tall order. Nevertheless, we know that they are not the final word on Nature for fairly good reasons. On one hand, General Relativity marks its own limit with the prediction of singularities both in gravitational collapses and at the origin of the Universe. On the other hand, Quantum Theory, when formulated in a relativistic framework — i.e., QFT — and applied to the gravitational field, is plague by divergences. For these reasons the search for a consistent theory of quantum gravity is almost as old as GR and Quantum Mechanics themselves, the first attempts dating back to the early 30s (see [191] and references therein).

The characteristic scale at which quantum gravitational effects are expected to be relevant is the Planck scale $E_P \approx 10^{19}$ GeV ($\ell_P \approx 10^{-35}$m). This can be understood by various means, e.g., by considering when the perturbation of the metric δg, induced by localizing the maximum amount of energy E allowed by quantum mechanics — i.e., up to its Compton wavelength — is non-negligible [1, 115]. Using Einstein's equations, which relate curvature and energy density, and the fact that the Compton length is given by $c\Delta x = \hbar/E$ we see that

$$\delta g \approx \frac{GE^2}{c^3\hbar},$$

and $\delta g \approx 1$ occurs at the Planck energy given by $E_P = \sqrt{\hbar c^3/G} \approx 10^{19}$ GeV— the associated Planck length $\ell_P \approx 10^{-35}$m can also be defined. The magnitude of such a Planck scale seems so far out of experimental range that quantum gravity tests were considered completely unfeasible until recently. Historically, the research in QG had a resurgence of interest in the 80s but until the end of the 90s it was mainly based on mathematical consistence of the various models and the possibility of anchoring the research field to observations was hardly considered. As stated by Isham in 1995 [122]:

> *The feature of quantum gravity that challenges its very right to be considered as a genuine branch of theoretical physics is the singular absence of any observed property of the world that can be identified unequivocally as the result of some interplay between general relativity and quantum theory. This problem stems from the fact that the natural Planck length has the extremely small value of approximately* 10^{-35} *m which is well beyond the range of any foreseeable laboratory-based experiments. Indeed, this simple dimensional argument suggests strongly that the only physical regime where effects of quantum gravity might be studied directly is in the immediate post big-bang era of the universe—which is not the easiest thing to probe experimentally.*

In the past 20 years, though, the field of Quantum Gravity Phenomenology [11] has blossomed. Its main aim is to connect quantum gravity ideas with experiments and observations in the hope to shed some light on the fundamental structure of spacetime, or at least give theoreticians some pointers. Presently, the two major proposal for a quantum theory of gravity are String Theory (ST) [183] and Loop Quantum Gravity (LQG) [190], to which a variety of other approaches — Causal Set Theory [58], Causal Dynamical Triangulation [9], Group Field Theory (GFT) [169], to name few — have to be added. All these approaches are far from being complete and various questions remain to be solved. Nevertheless, they have inspired well-defined phenomenological models aimed to probe specific features the fundamental theory might have [11, 115].

Among the most notable examples of phenomenological avenues being explored we can list

- Violations of Lorentz Invariance (and of other exact symmetries)
- Deformation of Special Relativity
- Space-time foam
- Deformed field dynamics
- Generalized Uncertainty principle
- Quantum Cosmology

It is beyond the scope of this thesis to review all the various possibilities to test quantum gravitational effects, both directly and indirectly. In what follows we will be mostly interested on the effects which may arise from assuming the fundamental structure of spacetime to be discrete.

A final remark is in order. A common trend in QG phenomenology is that the attention has been focused mainly on astronomical, high-energy and cosmological experiments/observations. This is justified by the very nature of the effects that one often tries to test — e.g., departures from relativistic symmetries — and the observation that by integrating over very large distances and times we could be able to observe the pile up of otherwise tiny effects. However, recent proposals, such as test of the generalized uncertainty principle [32] or of the fate of the Equivalence Principle in quantum physics [8], which use low-energy, macroscopic quantum systems have been advanced. In connection to this point, in the last chapter of this work we propose a phenomenological model to test non-locality which employs macroscopic quantum oscillators and with the potential to achieve near Planck-scale sensitivity.

1.1.1 Discreteness of spacetime

The fact that our concept of spacetime as we know it — i.e., a continuum differentiable manifold endowed with a metric and affine structure — ceases to be valid around the Planck scale, or at least that some new physics has to manifest at such scale, seems to be a common feature of approaches to QG. In particular, the existence of a minimal length scale, usually identified with the Planck scale, emerges from both *gedankenexperiment* and various QG theories. The generality of the arguments from which this result is derived make it an almost model-independent feature of QG.

The presence of a minimal length scale, below which spacetime points cannot be resolved, can be derived by combining arguments of special relativity, general relativity and quantum mechanics. As an example, consider the Heisenberg microscope[1]. In its standard version [111] this gedankenexperiment shows that the uncertainty in the position of a particle observed through a photon is $\Delta x \gtrsim 1/\Delta p_x$. When including Newtonian gravity an additional source of uncertainty comes into existence due to the gravitational attraction between the particle and the photon. The particle accelerate towards the photon — which has energy ω — with acceleration $\approx l_P^2\omega/r^2$ and acquire a velocity $\approx l_P^2\omega/r$ during the time r (note that $c = 1$) of strong interaction. Thus, the particle travels a distance $l_P^2\omega$ in the unknown direction in which the photon moves during the interaction time. Projecting along the x-axis $\Delta x \gtrsim l_P^2 \Delta p_x$ and combining with the quantum mechanical uncertainty

$$\Delta x \gtrsim l_P. \tag{1.1}$$

This last equation shows that the Planck scale sets a limit to the resolution of the particle position. The same result can be obtained with more refined examples and we refer the reader to [96, 116] for a detailed analysis.

The existence of such a minimal scale has led many physicist to consider the possibility that spacetime maybe fundamentally discrete[2] at the Planck scale. One of the first to seriously consider such a possibility was Heisenberg [116] in the 30s, but remarkably already Riemann [189] exhibited doubts about the reality of the continuum and later on Einstein pointed out how the continuum could be the source of problems in reconciling GR and quantum theory.

[1] We follow here [96, 116].

[2] Note that the existence of a minimal length scale does not imply, *a priori*, that spacetime has to be discrete. As an example, in Asymptotic Safe Gravity the Planck scale plays still the role of minimal scale even if no discrete structure for the spacetime is invoked, see [175].

There are also various contradictions in existing theories that speak for abandoning the notion of a continuous spacetime. Quoting Sorkin [202, 205] they can be dubbed as the "three infinities"

- **Singularity of classical GR**: singularities in GR are inevitable under fairly reasonable physical assumptions on the collapsing matter. They do not belong to the spatiotemporal manifold and signal a breakdown of the theory, i.e., the fact that GR has to be replaced by a more fundamental theory. A discrete spacetime is expected to not present or, at least, to allow for a fully predictive description of singularities.

- **Infinities of Quantum Field Theories**: UV divergences that manifest themselves both in loop integrals due to arbitrary high-momenta and in the correlations of quantum fields which blow up in the coincidence limit. Some of these divergences become even more troublesome when QFTs are formulated in curved spacetime [48]. It is clear that they are related to the small-scale behaviour of the theory and, in turn, to the continuum nature of spacetime. In a discrete setting we expect a natural cut-off to tame the divergences and render the theories finite.

- **Black hole (entanglement) entropy**: whereas the Bekenstein–Hawking entropy of black holes, $S = k_B A/4\ell_P^2$ — where A is the area of the horizon — is finite, it indicates a limit to how much information can be stored in spacetime and connects this limit to the Planck scale. When instead the entanglement entropy of fields living in a black hole background is calculated the result, again proportional to the horizon area, is divergent unless a small-distance cut-off is introduced. If the cut-off is chosen at the Planck scale, then this entropy has the same magnitude of the Bekenstein-Hawking entropy. Whether this new entropy can be interpreted as the whole black hole entropy or just as a part of it is unsettled, but surely this contribution is present. The divergence of entanglement entropy is a general result of QFT both in flat and curved spacetime and, again, can be traced back to the continuum nature of spacetime. A discrete structure of spacetime should introduce a natural cut-off and ensure the finiteness of the entanglement entropy. Furthermore, note that a mere minimal length scale present in a continuum theory, whereas able to regularised the correlations of the field theory, is not guaranteed to allow for a finite entanglement entropy [162].

Motivated by these theoretical problems, various approaches to quantum gravity exploit the idea of discreteness. However, they differ in the kind of discrete structures they invoke and the meaning itself of *discrete spacetime* is not unambiguous. First of all, we can distinguish between models that use discrete structures as mathematical tools to be removed at the end of the calculations, e.g., Causal Dynamical Triangulation and Regge Calculus; and models in which the discreteness is physical, e.g., Causal Set theory, Spin-foam models and GFT. Secondly, a distinction ought to be made between approaches that assume spacetime to be discrete from the outset, e.g., GFT and CS theory; and others that, starting from the continuum, find discrete structures upon quantization, e.g., the original version of

LQG. All these discrete models have to face at least two general problems. The first one concerns the *emergence* of a continuum spacetime. The word emergence has been abused in the recent QG literature and different meanings are attached to it by different communities. However, at the more basic level, what we mean is that any discrete model has to answer the question of how to recover the continuum physics starting from a discrete structure that, in principle, should make no reference to the continuum spacetime to start with. The second problem concerns phenomenology. How can we test the hypothesis that spacetime is fundamentally discrete? What are the new phenomenological effects that we can hope to verify in order to make these theories *scientific*?

1.2 Analogue Gravity

How classical spacetime, such as the one we experience, emerges from models of discrete/quantum spacetime is a pivotal question for QG theories. Given the complexity of the problem, a starting point for this investigation could be the development of toy-models which capture some aspects of the emergence of spacetime and gravity. In this context, analogue models of gravity play a relevant role. These models are provided by several condensed-matter/optical systems in which the excitations propagate in a relativistic fashion on an emergent pseudo-Riemannian geometry induced by the medium [21] — see below. Thus, they serve as an example of emergence of spacetime from discrete systems, and constitute a natural arena where one can develop useful tools to tackle these problems. Indeed, since the seminal work of Unruh [218] analogue models of gravity have set a fruitful stage in which issues related to emergent gravity can be studied and have opened up the possibility of experimentally simulating phenomena expected within quantum field theory on curved spacetime, e.g., analogue Hawking radiation [22, 124, 218] and cosmological particle production [210, 228, 230][3].

1.2.1 The ancestor of all analogue models

To illustrate the basic ideas of the analogue gravity program consider the simplest analogue model: sound waves in a moving fluid [218]. At the level of geometrical acoustic, sound rays which move with velocity $c_s\mathbf{n}$ — where $\mathbf{n}$ is a unit-norm vector — with respect to the fluid, which in turn moves with velocity $\mathbf{v}$ with respect to the laboratory, define a sound cone in spacetime given by

$$-c_s^2 dt^2 + (d\mathbf{x} - \mathbf{v}dt)^2 = 0,$$

associated with a conformal class of (in general curved spacetime) metrics

$$g_{\mu\nu}(t, \mathbf{x}) \equiv \Omega^2 \begin{bmatrix} -(c_s^2 - v^2) & \vdots & -\mathbf{v}^T \\ \cdots\cdots & . & \cdots\cdots \\ -\mathbf{v} & \vdots & \mathcal{I} \end{bmatrix}, \tag{1.2}$$

[3]For a comprehensive review of the subject we refer the reader to [21, 141]

where Ω^2 is an undetermined conformal factor and $\mathcal{I}$ is the 3×3 identity matrix. Sound waves are dragged along by moving fluids, thus they can never escape/enter supersonic flow regions. In conjunction with the emergence of an acoustic (class of) metric(s), supersonic flow regions are seen to be the analogue of black/white hole regions. These configurations are dubbed *dumb holes*.

In order to further appreciate the mathematical analogy between semiclassical and analogue gravity, it is necessary to consider physical acoustics. A central result in analogue gravity is the fact that, under rather generic assumptions on the fluid and its motion — in particular, a barotropic and inviscid fluid with an irrotational flow — the propagation of sound waves is described by

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0, \tag{1.3}$$

where ϕ is the velocity potential describing acoustic perturbations in the fluid. This equation is the d'Alembert equation of a scalar field in a curved spacetime with metric eq. (1.2) and $\Omega = \rho/c_s$ (where ρ is the density of the fluid).

Acoustic perturbations, which propagate on top of a moving background fluid, feel a curved spacetime metric or, in other words, propagate as massless (classical) scalar fields on a curved, Lorentzian spacetime. The metric of this *emergent* spacetime, g, takes the name of *acoustic metric* and the same name is widely used in the analogue gravity literature, even when the system considered has nothing to do with acoustic waves in fluid. In fact, it can be shown that the emergence of a Lorentzian signature metric is a characteristic of a large class of systems [23] of which moving fluid are a rather significant example for what concern experimental viability [187, 227, 229].

1.2.2 Bose–Einstein condensate analogue

Among the various analogue systems, a preeminent role has been played by Bose–Einstein condensates (BECs) in recent years. This is because BECs are macroscopic quantum systems whose phonons/quasi-particle excitations can be meaningfully treated quantum mechanically and hence used to fully simulate QFT on curved spacetime phenomena [22, 94, 95].

A BEC is a particular phase of a system of identical bosons in which a single energy level has a macroscopic occupation number and it is typically realized by using ultra-cold atoms [177]. Being a non-relativistic, many-particle, quantum system it can be properly described in the second quantization formalism. The evolution of a BEC, in the dilute gas approximation, and neglecting the external potential, is generated by the Hamiltonian

$$H = \int d\mathbf{x}\hat{\Psi}^\dagger(t,\mathbf{x})\left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + \frac{\kappa}{2}|\hat{\Psi}|^2\right]\hat{\Psi}(t,\mathbf{x})$$

where μ is the chemical potential and κ is the strength of the two-body interaction proportional to the scattering length. The condensation process is often treated in the mean field approximation, i.e., it is assumed that the field operator acquires a non-vanishing expectation value on the ground state $|\Omega\rangle$ of the system

$$\langle\Omega|\hat{\Psi}(x)|\Omega\rangle = \psi(x).$$

This motivate the (Bogolyubov) decomposition of the field operator as

$$\hat{\Psi} \approx \psi \hat{\mathbb{I}} + \hat{\chi},$$

where the field operator $\hat{\chi}$ describes the quantum fluctuations around the condensate, i.e., the non-condensate fraction of atoms (assumed to be small). The ground state of the system is the vacuum state for the quasi-particles (phonons) on top of the condensate. Thus, the condensation can be seen as the field operator acquiring a non-vanishing vev, i.e., a spontaneous symmetry breaking (SSB) of the global $U(1)$ symmetry of the system. This SSB gives rise to a massless Goldston boson that is identified with the phonons which propagate on top of the condensate.

At lowest order in the fluctuations, the dynamics of the condensate is described by the Gross-Pitaevskii equation

$$i\hbar \frac{\partial}{\partial t} \psi(t, \mathbf{x}) = \left(-\frac{\hbar^2}{2m} \nabla^2 - \mu + \kappa |\psi(t, \mathbf{x})|^2 \right) \psi(t, \mathbf{x}), \tag{1.4}$$

in which the back-reaction of the fluctuations as been completely neglected. Using the Madelung representation for the condensate field

$$\psi(t, \mathbf{x}) \equiv \rho(t, \mathbf{x}) e^{-i\theta(t,\mathbf{x})/\hbar}$$

it is possible to rewrite eq. (1.4) in a fluid form,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1.5}$$

$$m \partial_t \mathbf{v} + \nabla \left(\frac{m v^2}{2} - \mu + \kappa \rho - \underbrace{\nabla \cdot \frac{\hbar^2}{2m} \frac{\nabla \sqrt{\rho}}{\sqrt{\rho}}}_{\text{Quantum potential}} \right) = 0, \tag{1.6}$$

where we have defined $\mathbf{v} \equiv \frac{\nabla \theta}{m}$, the (locally) irrotational velocity field. These equations are the continuity and Euler equation, respectively, for an ideal, inviscid fluid with irrotational flow apart from the quantum potential term in eq. (1.6). When this term can be neglected — the so called hydrodynamical regime — the assumptions considered in the case of physical acoustic are met. Thus, the excitations over the condensate propagate as a massless scalar field on a curved spacetime described by an acoustic metric of the form eq. (1.2) with

$$\Omega^2 = \frac{\rho}{m c_s},$$

where $c_s^2 \equiv \kappa \rho / m$ is the sound velocity, i.e., the speed of phonons in the condensate . This shows that BECs are good candidates for analogue models of gravity where, most importantly, the excitations which feel the acoustic metric are genuinely of a quantum nature.

The spectrum of excitations in a BEC, without neglecting the quantum potential, can be found by diagonalizing the equation for the $\hat{\chi}$ field using Bogolyubov mode transformations. The final result is that phonons have a dispersion relations given by[4]

$$\omega^2 = c_s^2 k^2 + \frac{\hbar^2 k^4}{(2m)^2}. \tag{1.7}$$

This expression shows that

- phonons have a gapless spectrum, as expected since they are the Goldston bosons of the $U(1)$ SSB
- at low energies[5], the dispersion relation is linear $\omega^2 \approx c_s^2 k^2$. In this limit phonons behave as relativistic particles, with c_s playing the role of the speed of light. This is consistent with the emergence of an acoustic spacetime as seen above
- at high energies, the phonons' dispersion relation become the one of the non-relativistic atoms which constitute the system. There is an interpolation between the regime $\lambda \gg \lambda_c$, where we have relativistic massless quasi-particles, and $\lambda \ll \lambda_c$, where we recover the non-relativistic constituent of the system. The transition scale is determined by the healing length $\lambda_c \equiv \hbar/(mc_s)$.

The dispersion relation clearly shows that, even if the low-energy dynamics of phonons possesses Lorentz symmetry — with c_s as the invariant speed — this is only an approximate symmetry bound to be violated at high energies, where the Galilean symmetry is present. It is interesting to ask whether starting from a relativistic BEC one can, in general, solve the violations of Lorentz invariance just described.

The relativistic BEC as an analogue model was first analyzed in [88]. There it was shown that a relativistic BEC can indeed be seen as an analogue model, even if out of reach of near future experiments. The way in which the system is treated is analogous to the non-relativistic case. Using the mean-field approximation to split the field operator — denoted now by $\hat{\phi}$ — in a condensed (classical) part ϕ and a quantum (fractional) fluctuations $\hat{\psi}$ as $\hat{\phi} \approx \phi(1+\hat{\psi})$ the relativistic version of the Gross-Pitaevskii equation is given by

$$\left(\Box - m^2\right)\varphi - 2\lambda|\varphi|^2\varphi = 0, \tag{1.8}$$

where m is the mass of the interacting Bose field considered, a $\lambda\phi^4$ interaction is assumed and no external potential is considered. The dynamics of the fluctuations is determined by

$$\left[i\hbar u^\mu \partial_\mu - T_\rho - mc_0^2]\hat{\psi}\right] = mc_0 \hat{\psi}^\dagger, \tag{1.9}$$

where the condensate field has been expressed in Madelung form $\phi = \sqrt{\rho}e^{i\theta}$, $u^\mu \equiv \hbar\partial^\mu\theta/m$ is proportional to the $U(1)$-symmetry current, $c_0^2 \equiv \lambda\rho\hbar^2/m^2$ incorporates the strength of the interaction, and

$$T_\rho \equiv -\frac{\hbar^2}{2m\rho}\partial^\mu(\rho\partial_\mu)$$

---

[4] Assuming for simplicity the condensate at rest.

[5] More precisely, for wavelengths greater than the healing length, i.e., the characteristic length-scale of the BEC's dynamics given by $\hbar/(mc_s)$.

is the relativistic equivalent of the quantum potential. After some manipulations, a single equation for $\hat{\psi}$ can be obtained and is given by

$$\left\{[i\hbar u^{\mu}\partial_{\mu}+T_{\rho}]\frac{1}{c_0^2}[-i\hbar u^{\mu}\partial_{\mu}+T_{\rho}]-\frac{\hbar^2}{\rho}\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\right\}\hat{\psi}=0, \tag{1.10}$$

Finally, when T_{ρ} can be neglected, phonons — the gapless excitations, see below — propagate as massless scalar fields on a curved spacetime characterized by the acoustic metric given by

$$g_{\mu\nu}=\rho\frac{c}{c_s}\left[\eta_{\mu\nu}+(1-\frac{c_s^2}{c^2})\frac{v_{\mu}v_{\nu}}{c^2}\right], \tag{1.11}$$

where c_s is the speed of sound and $v^{\mu}=cu^{\mu}/\,\|u\|$ is the velocity of the fluid flow.

In this system, the dispersion relation for the perturbations around the condensate can be obtained from eq. (1.10). Assuming for simplicity the condensate to be at rest, and ρ, c_0 and u^0 to be constant both in space and time the dispersion relation is

$$\begin{aligned}\omega_{\pm}^2 =c^2 &\left\{k^2+2\left(\frac{mu^0}{\hbar}\right)^2\left[1+\left(\frac{c^0}{u^0}\right)^2\right]\right. \\ &\left.\pm 2\left(\frac{mu^0}{\hbar}\right)\sqrt{k^2+\left(\frac{mu^0}{\hbar}\right)\left[1+\left(\frac{c^0}{u^0}\right)^2\right]^2}\right\}.\end{aligned} \tag{1.12}$$

This is a generalization of the non-relativistic Bogolyubov dispersion relation eq. (1.7) and shows that in a relativistic BEC there are both gapless (ω_-) and gapped (ω_+) excitations. The gapped excitations are typical of the relativistic systems since they arise from the presence of anti-particles — thus they disappear in the non-relativistic limit — and do not propagate in general on the acoustic geometry, when it can be defined. For what concern the gapless excitations, which feel the acoustic geometry, their dispersion relation shows that[6]

- in the low-momentum limit and when T_{ρ} can be neglected: $\omega_-\approx c_s k$, i.e., the standard phononic dispersion relation is recovered
- in the high-momentum limit: $\omega_{\pm}\approx ck$, i.e., the dispersion relation of the relativistic bosons which constitute the system is found

In the case of a relativistic BEC there is again an interpolation between a low-energy relativity group, with invariant velocity given by the speed of sound, and the high-energy one characterized by the speed of light. Thus, even if the system from which the acoustic spacetime emerges is relativistic, the emergent physics is still only approximately Lorentz invariant due to the interpolating behaviour between two relativity groups with different limit velocities.

[6]See [88] for a more detailed analysis.

1.2.3 Emergent dynamics?

Most of the research on analogue gravity so far has dealt with the questions related to the emergence of a background spacetime and quantum field theory on it. The analogue of gravitational dynamics is generally missing, i.e., the spacetime that emerges has a dynamics which cannot be cast in the form of background independent geometric equations. It goes without saying that QG theories have to be able to not only emerge a classical manifold as a, well-defined, low-energy limit of whatever structure they deem as fundamental but also the dynamics of general relativity has to emerge.

In ref. [100] a first step in this direction was undertaken by showing how Newtonian-like dynamics can emerge from a non-relativistic BEC analogue model. In order to ask dynamical questions the back-reaction of quasi-particles, the would-be gravitating matter, on the condensate has to be considered. This can be accomplished by using an improvement of the Gross-Pitaevskii equation eq. (1.4), the so called Bogolyubov-de Gennes equation which reads

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu + \kappa|\psi|^2\right) + 2\kappa\left(\tilde{n}\psi + \tilde{m}\psi^*\right), \tag{1.13}$$

where $\tilde{n} = \langle\hat{\chi}^\dagger\hat{\chi}\rangle$ and $\tilde{m} = \langle\hat{\chi}\hat{\chi}\rangle$ are the anomalous density and mass, respectively. Given the non-relativistic nature of the underlying system, the hope is to, at most, emerge Newtonian-like dynamics for the emergent spacetime. In turn, this implies that the gravitating matter should be massive. Thus, it is necessary to introduce a soft $U(1)$-breaking term in order to promote the phonons to Pseudo-Goldston bosons and make them acquire a mass. As discussed in ref. [100], there are rather physical situations in which such $U(1)$-breaking happens[7].

Once the $U(1)$ soft breaking is introduced, the analysis of both the emergent acoustic metric and the hamiltonian of the quasi-particles — in the low-momentum limit, and for nearly homogeneous condensate wave functions — leads to the identification of an analogue gravitational potential defined by

$$\Phi_{grav}(x) = \frac{(\mu+4\lambda)(\mu+2\lambda)}{2\lambda m}u(x),$$

where λ is the strength of the soft breaking term and $u(x)$ is the non homogeneous part of the condensate's wave function. Finally, the dynamics of this potential can be deduced from the Bogoliubov-de Gennes equation and is given in terms of a generalized Poisson equation with a cosmological constant term (see ref. [100] for further discussions),

$$\left(\nabla^2 - \frac{1}{L^2}\right)\Phi_{grav} = 4\pi G_N\rho_{matter} + \Lambda. \tag{1.14}$$

In this equation ρ_{matter} and Λ are determined by the anomalous mass and density, $L^2 = \hbar^2/(4m(\mu+\lambda))$ and the emergent Newton constant G_N is given in terms of the system parameters.

This model proves that it is possible to emerge a gravity-like dynamics in analogue models based on BEC by considering the back-reaction of the excitations on the very same emergent spacetime in which they propagate,

[7] The $U(1)$ symmetry is associated with particles conservation.

in the spirit of general relativity. Chapter 2 of this thesis offers the first example of an analogue model able to simulate not only the kinematic — i.e., QFT on curved spacetime — but also the dynamics of a relativistic gravity theory, even if not as rich as general relativity. The system exploited will be a relativistic BEC, like the one introduced in the previous section.

1.3 Lorentz Invariance Violations

At the end of Sec. 1.2.2, we found that even when considering a relativistic analogue system the emergent physics is only approximately Lorentz Invariant. This tells us that discreteness and Lorentz Invariance (LI) are not easy to reconcile and that Lorentz Invariance Violations (LIV) might be, if present, an hint toward (certain kinds of) spacetime discreteness. More in general, symmetries play a fundamental role in theoretical physics. In particular, spacetime symmetries are at the basis of quantum field theory and general relativity. It is then natural to investigate and question the very nature of these symmetries, in particular whether they are exact or accidental, i.e., emerging in the low-energy world that we can access with present experiments.

(Local) Lorentz invariance is one of the best tested symmetries of Nature even if the Lorentz group, being non-compact, makes its probing process an endless task. As far as we can say, Lorentz symmetry presently appears to be an exact symmetry since tests of Lorentz invariance violations have provided stringent bounds on possible violations [11, 140, 150]. So, in principle, one could wonder about the point in questioning this symmetry. The answer, as is well known, comes from the fact that various quantum gravity proposals seem to entail violations (or modifications) of Lorentz symmetry at the fundamental level. Among the many examples are models inspired by string theory [137], spacetime-foam models [15, 86], semi-classical spin-networks calculations in LQG [92][8], non-commutative geometry [66] and emergent gauge bosons models [50]. Furthermore, there are also alternative models of gravity which incorporate Lorentz breaking as a choice of preferred frame, like Einstein-Aether theory [125] and Horava gravity [113][9]; and, as previously discussed, condensed matter analogues of emergent spacetime [21] which present LIV as a characteristic feature.

It is then imperative to test if such violations can be detected or, alternatively, used to rule out some QG scenario. In fact, any viable theory of quantum gravity/spacetime needs to carefully treat LI in order to recover it in the low-energy limit. This is far from trivial. Among the various QG theories there are some that are not affected by this problem since they assume LI from the outset, as in the case of CS theory [57, 58], while other proposals modify the action of the Lorentz group by making it non-linear [11]. It is worth mentioning that in the latter approach, known as doublyspecial relativity (DSR), modified dispersions relations such as the ones considered in the following, are expression of an extended symmetry group. Accordingly, naturalness arguments — topic of Chapter 3 — typical of LIV effective field theory do not straightforwardly apply.

[8] See however ref. [192] for what concerns loop quantum gravity.

[9] The latter is claim to give a renormalizable theory of gravity UV complete. As such, it is directly relevant for the quantum gravity community.

For what concerns other approaches to QG, the quest for how to recover LI at low energies should be of primary importance and it is, in many cases, an open issue. Last but not least, it should be noted that discreteness of spacetime — present in many QG models as previously argued — is something that *a priori* is hard to reconcile with Lorentz invariance, thus making LIV phenomenology central for discrete spacetime models.

1.3.1 An experimental window on Quantum Gravity

Searches for Lorentz Invariance Violations motivated by quantum gravity are a prominent example of, successful, quantum gravity phenomenology. Despite the fact that ideas concerning LIV were already present in the early 50s [75], only in recent times we have managed to have a fully fledged phenomenological setting to study LIV in a systematic way. In most QG models which involve LIV, these enter through modified dispersion relations (MDRs) like

$$E^2 = p^2 + m_A^2 + \sum_{n=1}^{\infty} \xi_A^{(n)} \frac{p^n}{M^{n-2}}, \tag{1.15}$$

where it is assumed that rotational invariance is preserved for simplicity[10], the subscript A labels different particles — leaving open the possibility that the LIV coefficients are different for different particles — and dissipative effects are not taken into account[11]. The scale M appearing in these MDRs is usually identify with the Planck mass, inspired by the idea that violations of LI are due to QG effects. This being the case, in order to observe significant deviations from exact LI we need either high energies (still well below the Planck scale), propagation of signals for long time/distances, or to look at particular reactions which can be modified — or even allowed only — in presence of LIV. Here we report a (incomplete) list of possibilities that could permit us to take a peek into the QG realm [140, 150]:

- in vacuum dispersion and vacuum birefringence
- normally forbidden reactions allowed by LV terms, e.g., photon decay, vacuum Čerenkov effect and gravitational Čerenkov
- shifting of existing threshold reactions, e.g., GZK reaction
- synchroton radiation
- neutrino oscillations

Whereas MDRs constitute a kinematical framework for studies of LIV, with the merit of allowing for constraints which are applicable to quite a large class of scenarios, most of the constraints on LIV depend on the dynamics underlying the choice of MDR. A natural framework that allows one to consider dynamical questions, without knowing the detail of the underlying QG theory, is the Effective Field Theory (EFT) one. Indeed, EFTs provide a way to extend the Standard Model (SM) of particle physics including all LV operators of mass dimension 3 and 4, i.e., (power counting)

[10] We refer the reader to [140, 150] for further reviews on LIV, both theoretical and experimental.

[11] We will come back to this point later in this section. See also Chapter 3.

renormalizable, and higher than 4. This extension goes under the name of Standard Model Extension (SME) [67]. Just to give an idea of the accuracy to which LIV have been tested in this framework we report in Table 1.1 constraints available on the SME [140].

Order	photon	e^-/e^+	Protrons	Neutrinos[a]
n=2	N.A.	$O(10^{-16})$	$O(10^{-20})$	$O(10^{-8} \div 10^{-10})$
n=3	$O(10^{-16})$ (GRB)	$O(10^{-16})$	$O(10^{-14})$	$O(40)$
n=4	$O(10^{-8})$	$O(10^{-8})$	$O(10^{-6})$	$O(10^{-7})^*$

TABLE 1.1: Typical strengths of the available constrains on the SME at different n orders for rotational invariant, neutrino flavor independent LIV operators, from [140]. Updated to 2013.

It has to be mentioned that, some QG models which entail LIV cannot a priori be described by a low-energy EFT and, as such, can avoid most of the stringent bounds on LIV. For a discussion of this point we refer the interested reader to [11, 140]. In the following we will stick to the more conservative EFT framework.

As we noted before, in the example of MDR in eq. (1.15) dissipative terms are neglected and only dispersive ones are considered. This has also been the case in the majority of the literature on LIV, overlooking a worthy window on quantum gravity. As shown in [173], dissipative terms have to be present for consistency with the Kramer's relations, at least as far as LIV are suppose to have a dynamical origin. Dissipative effects can also be expected as a consequence of emergent-spacetime models [142]. Note that, considering these effects in the MDRs leads to stringent constraints on the coefficients of the dissipative terms from astronomical observations. In the seminal work [142], the authors show that the bound on a possible "viscosity of spacetime" term is able to push the scale of such a dissipative effect way beyond the Planck scale. Thus, they conclude, *any viable emergent spacetime scenario should provide a hydrodynamical description of the spacetime close to that of a superfluid* (i.e., with zero viscosity).

We will come back to this point in Chapter 3, where we consider dissipative effects and their naturalness.

1.3.2 The naturalness problem of LIV

The EFT approach, in the form of the Standard Model Extension, has been an invaluable tool in testing Lorentz Invariance to an extreme accuracy. However, the same approach raises also a new theoretical problem: why is LI such a good symmetry at low energies? In fact, in EFT radiative corrections will generically allow the unsuppressed percolation of higher dimensional Lorentz violating terms into the lower dimensional ones, giving rise to mass dimension 3 and 4 operators, which are already in contrast with observational bounds [69]. As such, we are left with three possibilities: i) accept that LI is an exact symmetry of Nature; ii) existence of an extreme fine-tuning in the LI violating terms at high energies; iii) existence of protection mechanisms, e.g., custodial symmetries, which prohibit the unsuppressed percolation. The first possibility will be discuss at length in the second part of this thesis and by definition falls outside of LIV phenomenology. Among the second and third possibility, the former is much less appealing than the latter, even if it would not be the first fine-tuning problem in theoretical physics[12].

We briefly discuss now which kind of protection mechanisms have been envisaged for resolving this fine-tuning problem (see ref. [140] for an extended discussion and ref. [2] for recent developments).

SUSY as a custodial symmetry: In this case supersymmetry is considered as a custodial symmetry which forbids lower dimensional LI violating operators. Although SUSY is strictly related to Lorentz symmetry, it is still possible to violate the latter while preserving the former. In this sense, SUSY is allowed to play the role of custodial symmetry for LIV. It has been shown that, when imposing unbroken SUSY on LIV EFTs, the lower dimensional LI violating operators which are admitted have mass dimension 5 and they do not produce lower dimensional ones via radiative corrections [56, 105]. However, SUSY is certainly broken in Nature and SUSY breaking allows for the percolation of LIV to renormalizable operators, though in this case the percolation is suppressed by ratio of the SUSY-breaking scale with the Planck one. Mass dimension three operators, despite the suppressing factor, are problematic and lead to fine-tuning in order to agree with observations, but they can be eliminated by imposing CPT symmetry in QED. For what concerns mass dimension 4 operators instead, they are sufficiently suppressed provided SUSY breaking happens below 100 TeV. The case of SUSY, and other models involving a spontaneous symmetry breaking [198], can serve as a paradigmatic example for the possibility of new (a priori Lorentz invariant) physics between the electroweak scale and an eventual Lorentz breaking at the Planck scale.

Gravitational confinement: Another possible solution to the LIV fine-tuning problem is to restrict the violation to the gravity sector of the theory, leaving the matter sector LI, as in the case proposed by Pospelov and Shang [185]. In particular, a separation between the Planck and the LIV scale in the gravity sector — with the latter taken to be smaller than the former — was used to show how percolation can be tamed. This proposal relies on the Planck mass suppressed vertices that appear in the matter sector due to the coupling with gravity and on the fact that LIV in the gravitational sector is not so stringently tested [185].

RG flow and strong coupling: Finally, a third possible way of achieving an infrared protection from high-energy LIV is the one envisaged by Nielsen and collaborators in the seventies, which makes use of renormalization-group techniques [163, 164]. While the standard logarithmic running towards a LI theory in the infrared is generically not fast enough to be compatible with current observations, it was nonetheless noticed that a strong coupling close to the Planck scale can sufficiently enhance the running such that almost exact LI is rapidly achieved. This is the basic idea behind the proposal of ref. [33].

[12] The cosmological constant, the mass of the Higgs and various hierarchy problems in particle physics are concrete examples of other fine-tuning problems.

Let us stress that, the presence of the naturalness problem does not detract anything from phenomenological studies. In fact, test of Lorentz symmetry — which basically ignore this problem — are valuable in view of the fact that a definitive QG theory is missing and over restrict our models would not be a long-sighted choice[13].

In Chapter 3 the naturalness problem will be revisited by showing, in a neat way, that a separation between the EFT scale and the Lorentz violation one can tame the percolations — as often argued in the literature. Furthermore, the case in which dissipative terms are involved, largely overlooked in the literature, will be considered.

1.4 Causal Set Theory and Non-Locality

Causal Set (CS) theory is an approach to quantum gravity based on discreteness and causal order, which preserves Lorentz Invariance. The original idea can be traced back to the late 80s [58][14] and it is based on results which show the fundamental nature of causal order in Lorentzian geometry.

Given a spacetime, i.e., a couple $(\mathcal{M}, g)$ composed of a differentiable manifold $\mathcal{M}$ and a metric g, the causal order is defined by the set of spacetime points $\mathcal{M}$ — now seen as merely a set of events, without its standard manifold-like topological character — and a partial order relation (in spatiotemporal terms, given two points $x, y \in \mathcal{M}$, $x \prec y$ means that x is in the causal past of y). It is known that, starting from $(M, \prec)$, it is possible to recover all the mathematical structures of spacetime geometry [108, 144, 174, 202]: its topology, differential structure and its metric up to a conformal factor.

The failure of the reconstruction theorems to account for the conformal factor, i.e., the necessity to define a measure on spacetime in order to recover the volume information and fix it, was one of the motivations for the assumption of discreteness which is at the basis of the theory. In fact, assuming a discrete structure permits to gain the volume information in a very intuitive way: by *counting elements* [202]; thus, a discrete structure might be able to incorporate all the information needed to recover geometry. As already discussed, there are also various physical reasons to assume the small-scale structure of spacetime to be discrete and they come from quantum mechanical arguments. The natural discretization scale emerging from these arguments — the Planck scale — is such that $l_P \to 0$ if $\hbar \to 0$, i.e., spacetime discreteness is inherently quantum [202].

Causal Set theory combines discreteness and causal order to produce a discrete structure on which a quantum theory of spacetime can be based. In particular, the Planck scale cut-off is implemented through the covariantly well-defined spacetime volume l_P^4. This leads to the definition of causal set as a locally finite partial order. In other words, a causal set is a set $\mathcal{C}$ endowed with a partial order relation $\preceq$ satisfying

1. reflexivity: $x \preceq x$

[13] As already commented, some approaches to QG cannot be formulated in the EFT framework. This allows them to elude, *a priori*, the naturalness problem.

[14] Similar constructions were proposed independently in the same period [161, 213].

2. acyclicity: if $x \preceq y \preceq x \quad \Rightarrow \quad x = y$
3. transitivity: $\forall x, y, z \in \mathcal{C} \quad x \preceq y \preceq z \ \Rightarrow \ x \preceq z$
4. **local finiteness**: $\forall x, y \in \mathcal{C}$ with $x \preceq y$ the cardinality of $I(x, y) \equiv \{z \in \mathcal{C} | x \preceq z \preceq y\}$ is finite

The last axiom, local finiteness, characterizes causal sets. In fact, the first three axioms are valid for every spacetime — without close causal curves — and it is the last one which encodes the fact that causal sets are *discrete*.

The basic hypothesis of the causal set program is that, at small scales the continuum spacetime which we experience is superseded by a discrete structure and it is recovered only as a macroscopic approximation. Thus, continuum spacetime is an emergent concept in CS theory.

1.4.1 Kinematics

Kinematical results — i.e., results which make no reference to any particular dynamical law — in CS theory are numerous, we focus here on the specific question: when is a causal set well approximated by a Lorentzian manifold? The answer to this question is clearly central to the development of a phenomenology of causal sets.

Given a spacetime, the discretization procedure chosen has to respect the *Volume-Number* correspondence, i.e., the possibility to recover volume information by counting elements, which is at the basis of the CS program. It is important to note that, a naïve regular discretization is never compatible with this correspondence, as it can be shown by boosting a regular lattice [212]. In order to implement the volume-number correspondence a random lattice discretization is needed.

A causal set $\mathcal{C}$ can be generated from a Lorentzian spacetime $(\mathcal{M}, g)$ via the *sprinkling* process. This consists of a random Poisson process of selecting points in $\mathcal{M}$, with density ρ, in such a way as to respect the volume-number correspondence on average, i.e., the expected number of points in a spacetime region of volume V is ρV. The sampled points are then endowed with the casual order of $(\mathcal{M}, g)$ restricted to the points, see figure 1.1. A causal set $\mathcal{C}$ is said to be well-approximated by a spacetime $(\mathcal{M}, g)$ — $\mathcal{M} \approx \mathcal{C}$ — if it can arise with *high probability* by sprinkling into $\mathcal{M}$.

The sprinkling process is not a dynamical process from which causal sets arise, i.e., at this level no dynamical law that describes the causal set has been introduced. It is purely kinamatical. Nonetheless, this process allows one to describe, e.g., the dynamics of fields propagating on causal sets which are well-approximated by spacetimes of interest. In particular, for the rest of this work we will be mainly concerned with causal sets that are well-approximated by d-dimensional Minkowski spacetime.

1.4.2 Lorentz Invariance and Non-locality

Among the various attractive features of causal sets as a discrete foundation for quantum gravity, the one which truly distinguishes the theory is undoubtedly (local) Lorentz Invariance.

Since the theory is discrete, it is important to define what Lorentz Invariant actually means. LI is refereed to the continuum approximation to the

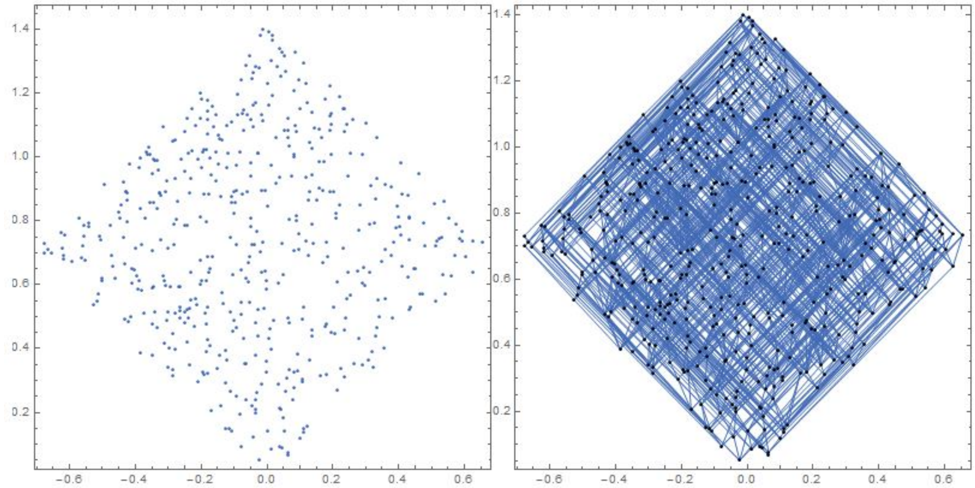

FIGURE 1.1: Sprinkling of 500 points in a 2d diamond of Minkowski spacetime. In the right panel are shown also the links between points, i.e., the irreducible relations not implied by the others via transitivity.

discrete structure rather than to the causal set itself. Whenever a continuum is a good approximation, discreteness must not, in and of itself, serve to distinguish a local Lorentz frame at any point [79, 112]. Given its importance, the quest for LI has been formulated in mathematically rigorous terms and takes the form of a theorem in [57]. The proof is based on two observations: causal information is Lorentz invariant and so is the Poisson distribution since probabilities depend only on the covariantly defined spacetime volume. These observations already show that the whole sprinkling process is Lorentz Invariant. However, the fact that the sprinkling process is LI does not, a priori, ensure that the single realization does not pick a preferred frame. In [57] this gap was filled by proving that each individual realization of the sprinkling process is LI[15]. The proof works by showing that there does not exist, in Minkowski spacetime, a measurable equivariant map — i.e., that commutes with Lorentz transformations — which can associate a preferred direction to sprinklings, and it is based on the non-compact nature of the Lorentz group. In spacetime other than Minkowski, the existence of local Lorentz Invariance can be claimed on similar grounds.

On top of the theoretical arguments, Causal sets' phenomenological models — some of which will be briefly illustrated below — exhibit Lorentz invariance, thus proving that the symmetry is preserved in physically relevant circumstances.

Non-locality in Causal Sets

Preserving Lorentz Invariance while assuming a fundamental discreteness of spacetime comes at a price: A fundamental non-locality of causal sets. To see this, consider the nearest neighbours to a given point in a causal set well-approximated by Minkowski spacetime. These will lie roughly on the

[15]In fact, the proof is even more general and shows that the sprinklings cannot be used to pick even only a time direction. Furthermore, as a corollary to the theorem, it is shown that a sprinkling cannot determine a preferred location in Minkowski spacetime, thus extending the results to the whole Poincaré group.

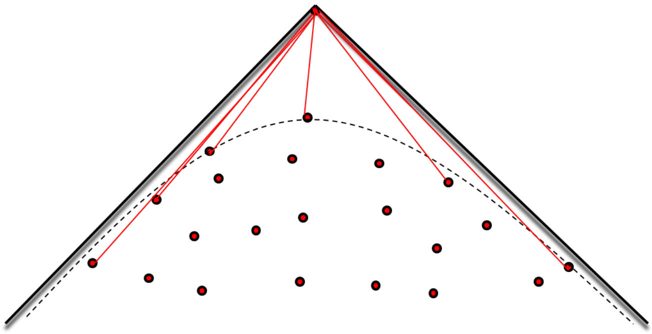

FIGURE 1.2: Considering the causal past of a given point, the nearest neighbours to it, i.e., the points connected with the chosen one by links, lie roughly on the hyperboloid lying one Planck unit of proper time away from that point.

hyperboloid lying one Planck unit of proper time away from that point and therefore will be infinite in number, see figure 1.2. While this is a dramatic departure from local physics, due to the causality properties of the underlying discrete structure no violations of causality whatsoever occur.

This non-locality manifests itself also in the definition of a non-local d'Alembertian for a scalar field on the causal set. The non-local d'Alembertian is a discrete operator that reduces in the continuum limit to the standard (local) wave operator. The precise form of this correspondence is given by performing an average of the causal set d'Alembertian over all sprinklings of Minkowski, giving rise to a non-local, retarded, Lorentz invariant linear operator in the continuum, $\Box_{nl}$, whose non-locality is parametrised by a scale l_n. In particular, this *non-locality* scale is conjectured to be much larger than the discreteness one [203] — in order to tame the fluctuations of the discrete operator — and this fact opens up interesting phenomenological possibilities. Locality is restored in the limit $l_n \to 0$ in which $\Box_{nl} \to \Box$. The non-locality scale l_n is the free parameter of the theory on which phenomenological bounds should be cast. The general expression for the non-local d'Alembertians in flat spacetime of dimension d was introduced in ref. [17] and is given by

$$\Box_\rho^{(d)}\phi(x) = \rho^{\frac{2}{d}}\left(\alpha^{(d)}\phi(x) + \rho\,\beta^{(d)}\sum_{n=0}^{N_d} C_n^{(d)}\int_{J^-(x)} d^d y\,\frac{(\rho V(x,y))^n}{n!}e^{-\rho V(x,y)}\phi(y)\right), \tag{1.16}$$

where N_d is a dimension dependent positive integer, $\rho = 1/l_n^d$, $V(x,y)$ is the volume of the causal interval between x and y, $J^-(x)$ indicates the causal past of x and the coefficients $\alpha^{(d)}$, $\beta^{(d)}$ and $C_n^{(d)}$ can be found in equations (12)-(15) of ref. [80]. In ref. [35] the canonical quantization of the non-local, free scalar field theory was performed. A different quantization scheme was investigated, considering non-local field theories that share some common features with the ones derived from causal sets, with similar results in [195].

Non-locality in this thesis

Most dynamical system in Nature are described by differential equations involving at most two derivatives in time. However, several models of QG predict a non-local dynamics governed by equations of motion with infinitely many higher order derivatives — e.g., String Field Theory (SFT) [158, 214], field theories on non-commutative spacetimes [78], QFTs with a minimal length scale [114] and CS theory as exemplified by eq. (1.16). The focus of the present work is on dynamical equation of motions with infinitely many powers of $\Box$, i.e., $\Box \to f(\Box)$. On one hand, the presence of an infinite number of derivatives, in contrast to higher than second (in time) but finite derivatives, is justified by the fact that in this way it is possible to evade the assumptions of the Ostrogradsky theorem [172] and thus avoid the related dynamical instabilities. On the other, considering infinite powers of $\Box$ is consistent with (local) Lorentz Invariance. As it will result clear in Chapter 4, CS theory belongs to a class of theories for which $f(\Box)$ is non-analytic, whereas in the example of SFT the function is analytic. In the second part of this thesis, the non-local scalar field theory derived from CS theory and the phenomenology associated to the non-locality of both analytic and non-analytic functions will be investigated in detail.

Finally, even though causal set theory motivates our study of (a specific realization of) non-locality, it should be noted that this feature can be seen to emerge in various QG scenarios, albeit in different forms. For example non-locality has been considered: to cure the divergences of QFTs [87]; in AdS-CFT [147], string theory [61] and spin-foam models/LQG [146, 186]; to find a way out of the information-loss problem [98]; as a characteristic feature of non-commutative geometry [78]; for applications in both the early Universe [29] and late times cosmology [143]; and it appears also as a feature of quantum field theories with a minimal-length scale [114]. Given that QG models which aim at preserving LI while postulating some form of discreteness suggest to consider dynamical random lattices[16] and that the argument exemplified in figure 1.2 rests on the random nature of points' locations as well as the non-compacteness of Lorentz group, it is tempting to conjecture that non-locality is a direct consequence of Lorentz Invariant discreteness [20, 91, 112, 206].

1.4.3 CS Phenomenology (so far)

Despite a quantum dynamics for the theory is presently missing, CS theory offers a broad range of phenomenological models and predictions based on its kinematical playground. This may seem restrictive, however it should be noted that most of the approaches to QG have difficulties to answer dynamical questions and most of the QG phenomenology is based on kinematical properties expected from QG models.

Here we report on a (non exhaustive) list of phenomenological works and characteristic effects of CS quantum gravity:

- (Heuristic) prediction of the **cosmological constant magnitude** [204]. This prediction dates back to 1990 when the acceleration of the expansion of the Universe was far from being discovered.

- **Ever present** Λ: a proposal for a model of fluctuating cosmological constant that could possibly be in agreement with recent data of BOSS, the Baryon Oscillation Spectroscopic Survey, part of the Sloan Digital Sky Survey [3, 4, 31, 72, 90]

[16] See discussion in Sec. 7 of [197]

- Particles **swerves**: massive point particles which propagate on a causal set will swerve due to the discreteness of the background. This process leads to the *unique* LI diffusion equation in momentum space [81, 178]. This equation presents only one free parameter that has been constrained using astrophysical data [149]
- **Polarization diffusion**: extension of the previous model to massless particles which has been confronted with CMB polarization data [70]

The swerves process, although valuable for pointing out the unique LI diffusion equation in momentum space, considers the highly idealized case of point particles propagating on causal sets. A more realistic model should take into account the propagation of fields on causal sets. In this respect, it is clear that the operators in eq. (1.16), and in particular their non-locality, offers the possibility for new phenomenological studies. We will come back to this topic in Chapter 4 of this thesis.

1.4.4 Dynamics

Even though this thesis will not touch upon the dynamics of CS theory, if not transversely, it is worth spending some words on it given its central role for any QG theory. In fact, at present among the various proposals for a QG theory, none can claim to have the dynamics under control.

Causal sets are suited for being histories in a sum-over-histories — i.e. a path-integral — approach to QG dynamics. Indeed, a discrete theory of spacetime could succeed in the attempt to give a (rigorous) meaning to the QG partition function

$$\mathcal{Z} = \int_{\text{Geometries,Topologies?...}} D[g] e^{iS[g]} \longrightarrow \mathcal{Z} = \sum_{\mathcal{C}} e^{iS_{BDG}},$$

where the last sum is over CSs and S_{BDG} stands for Benincasa–Dowker–Glaser action [44, 80].

Whereas a line of research tries to define the dynamics of causal sets from first principles [188], another one tries to obtain an action for causal sets which correspond to the Hilber–Einstein action in the continuum. This second way was paved in [44], and since then several other works in that direction have appeared [45, 60, 80, 102]. The causal set action can be derived starting from the causal set non-local d'Alembertian in curved spacetime. In this thesis we will be concerned with the flat spacetime case but we will briefly discuss the generalization to curved spacetime.

Chapter 2

Analogue Gravity as a toy model of Emergent spacetime

And I cherish more than anything else the Analogies, my most trustworthy masters. They know all the secrets of Nature, and they ought to be least neglected in Geometry

Johannes Kepler

The reasons why spacetime — and gravity — should be an emergent phenomenon go from the non-renormalizability of GR to the impossibility, as for now, to consistently quantize gravity. Even more striking are the links between thermodynamics and gravitation — black-hole thermodynamics [28, 109, 110]; membrane description of event horizon [71, 216]; the AdS/CFT correspondence [145]; the gravity/fluid duality. In particular, in a seminal work [127] Jacobson showed that Einstein equations can be derived from the thermodynamics of local causal horizons paving the way to the idea that spacetime/gravity could be a coarse-grained (a.k.a. thermodynamical) description of a more fundamental microscopic theory. In a recent work [126] by the same author, a new derivation of the Einstein equation has been obtained from the point of view of statistical physics, i.e., exploiting a principle of maximum entropy, lending further support to the emergence paradigm.

While the emergent gravity paradigm might seem at odd with some approaches to quantum gravity it is not necessarily incompatible with them. In particular, emergence akin to that in condensed-matter systems has been increasingly studied within the quantum gravity community in order to understand the emergence of spacetime and gravity from different fundamental ontologies[1]. In this sense, emergent gravity settings might end up being more a completion of quantum gravity scenarios rather than a drastic alternative.

In this context Analogue models, and in particular analogue models based on fluid mechanics or the fluid dynamic approximation to BECs, are specific examples of *emergent physics* in which the microscopic level is well understood. As such, they are useful for providing hints as to how such a procedure might work in a more fundamental theory of quantum

[1] See also [24] for a recent extension of these ideas to electromagnetism

gravity. In fact, an active stream of works takes literally the idea of Bose-Einstein condensation at the basis for the emergence of spacetime from pre-geometric structures [82, 83, 99, 117, 168, 170, 171, 197] and largely exploits analogue gravity techniques.

As discussed in Sec. 1.2.2, most of the research on analogue gravity so far has dealt with questions related to the emergence of a background spacetime and quantum field theory on it, i.e., QFT on a fixed background. This has its intrinsic value since it permits to investigate important effects like Hawking radiation [22, 124, 218] and cosmological particle production [228, 230] — otherwise out of our reach — in laboratory systems fully under control. However, in general the emergent spacetime has a dynamics which cannot be cast in the form of background-independent geometric equations, precluding the possibility of going beyond fixed background scenarios. Nonetheless, there are have been in recent times attempts of reproducing the emergence of some gravitational dynamics within analogue gravity systems (see e.g. [100, 129, 222, 223]).

The BEC model with $U(1)$ soft-breaking terms, described in the Sec. 1.2.3, is an example of analogue model with emergent dynamics encoded in a modified Poisson equation. Noticeably, this equation is sourced, as in Newtonian gravity, by the density of the quasi-particles (the analogue of the matter in this system) while a cosmological constant is also present due to the back-reaction of the atoms which are not part of the condensate (the so called depletion factor) [89, 199]. While the appearance of an analogue gravitational dynamics in a BEC system is remarkable, it is not a surprise that this analogue system is able to produce only Newtonian-like gravity since it is based on the non-relativistic BEC. Nonetheless, a derivation of relativistic gravitational dynamics in analogue models has been missing so far.

Remarkably, BEC can also be described within a completely relativistic framework [88], as noted in Sec. 1.2.2. It is then natural to expect that a relativistic BEC (rBEC) might provide a suitable model for the relativistic dynamics of an emergent spacetime. This is the subject of this Chapter.

2.1 Complex scalar field theory: relativistic BEC

Let us start by considering the general theory for a relativistic Bose–Einstein condensate. This is generically described by a complex scalar field endowed with an internal $U(1)$ symmetry which ends up to be spontaneously broken below some critical temperature [46, 106, 107, 134]. Note that in the present case it is not necessary to introduce $U(1)$-breaking terms in order to recover some gravitational dynamics, in contrast to the non-relativistic case [199]. Indeed, the $U(1)$ breaking was necessary in the non-relativistic BEC case since in Newtonian-like gravity massless particles cannot be treated and the back-reaction of massive particles is needed. Since we expect to emerge a relativistic gravity dynamics, even only massless gravitating quasi-particles would be enough.

For the general treatment of the rBEC we shall closely follow [134], to which we refer the reader for a detailed analysis. The Lagrangian of the system is given by

$$\mathcal{L} = -\eta^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{2.1}$$

The theory has a $U(1)$-invariance under phase rotation of the fields. The corresponding conserved current is given by

$$j_\mu = i(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger). \tag{2.2}$$

Space integral of the zeroth component of current gives the conserved charge,

$$Q = i\int \mathrm{d}^3x(\phi^\dagger\partial_t\phi - \phi\partial_t\phi^\dagger). \tag{2.3}$$

To describe the theory at a finite temperature $T = 1/\beta$ we Wick-rotate the time $\tau = -it$ and periodically identify the fields with a period $\tau = \beta$. Instead of using a complex field, it is convenient to use the real and imaginary parts of ϕ as dynamical variables: $(\phi_1 + i\phi_2)/\sqrt{2}$. Defining the momentum conjugate to the fields as $\pi_i = \partial\phi_i/\partial t$ for $i = 1, 2$, the partition function at a finite value of charge is then given by

$$\mathcal{Z} = \mathcal{N}\int \mathrm{D}\pi_1\mathrm{D}\pi_2\mathrm{D}\phi_1\mathrm{D}\phi_2 \exp\left[\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3x\left(i\pi_1\dot{\phi}_1 + i\pi_2\dot{\phi}_2 - [\mathcal{H} - \mu\mathcal{Q}]\right)\right], \tag{2.4}$$

where μ is the chemical potential sourcing the charge density $\mathcal{Q} = \phi_2\pi_1 - \phi_1\pi_2$ in the system and $\mathcal{H}$ is the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left(\pi_1^2 + \pi_2^2 + (\vec{\nabla}\phi_1)^2 + (\vec{\nabla}\phi_2)^2 + m^2\left(\phi_1^2 + \phi_2^2\right)\right) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2. \tag{2.5}$$

The total amount of charge at equilibrium can be obtained from the partition function as

$$Q = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\mathcal{Z}. \tag{2.6}$$

In the laboratory, one prepares the system with some net amount of charge Q and the value of μ is obtained by inverting eq. (2.6).

The integral over momenta in eq. (2.4) is a Gaussian integral. Hence, the momenta can be integrated away. This gives

$$\mathcal{Z} = \mathcal{N}_\beta\int \mathrm{D}\phi_1\mathrm{D}\phi_2 \exp\left[-\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3x\ \mathcal{L}_{\text{eff}}\right], \tag{2.7}$$

where $\mathcal{N}_\beta$ is a β dependent constant, and $\mathcal{L}_{\text{eff}}$ is the effective Lagrangian of the theory given by

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + (\vec{\nabla}\phi_1)^2 + (\vec{\nabla}\phi_2)^2\right) + i\mu(\phi_2\dot{\phi}_1 - \phi_1\dot{\phi}_2) + V(\phi) \tag{2.8}$$

where $V(\phi)$ is the effective potential given by

$$V(\phi) = \frac{1}{2}(m^2 - \mu^2)(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 \tag{2.9}$$

From the form of the effective potential it is clear that at a given β if $\mu > m$ then the system is in the broken $U(1)$ phase and the condensate has formed. It can be shown [134] that this phase transition is second order and the critical temperature is given by

$$T_c = \frac{3}{\lambda}\left(\mu^2 - m^2\right). \tag{2.10}$$

Later we shall be interested in the massless limit for which the critical temperature is given by $T_c = 3\mu^2/\lambda$. Thus, in the massless case, a non-zero chemical potential is necessary in order for the $U(1)$ symmetry to be broken and the condensate to be formed at a finite non-zero critical temperature. In the following we shall always consider the system to be at temperatures $T \ll T_c$, so that thermal effects can be safely neglected.

2.2 Relativistic BEC as an analogue gravity model

The effective Lagrangian of eq. (2.8) can be rewritten in terms of the complex valued fields as

$$\mathcal{L}_{\text{eff}} = -\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 + \mu^2\phi^*\phi + i\mu(\phi^*\partial_t\phi - \phi\partial_t\phi^*) \tag{2.11}$$

The equation of motion for ϕ is obtained by variation with respect to ϕ^* and we get,

$$\left(-\Box + m^2 - \mu^2 - 2i\mu\frac{\partial}{\partial t}\right)\phi + 2\lambda(\phi^*\phi)\phi = 0. \tag{2.12}$$

We can factor out explicitly the dependence on the chemical potential and write the field as

$$\phi = \varphi e^{i\mu t}. \tag{2.13}$$

This gets rid of the μ dependent terms and we get

$$\left(\Box - m^2\right)\varphi - 2\lambda|\varphi|^2\varphi = 0. \tag{2.14}$$

This was the starting equation in ref. [88] where the acoustic metric was first derived.

Following our discussion in Sec. 1.2.2, let us decompose the field φ as $\varphi = \varphi_0(1+\psi)$, where φ_0 is the condensed part of the field ($\langle\varphi\rangle = \varphi_0$), which we take to be real, and ψ is the fractional fluctuation. The reality of the condensate order parameter is the crucial assumption here. We will comment on this in the discussion section. Note that ψ is instead complex and $\langle\psi\rangle = 0$. It can be written in terms of its real and imaginary parts $\psi = \psi_1 + i\psi_2$. Substituting this decomposition in eq. (2.14) and taking the expectation value we get the equation of motion for the condensate

$$(\Box - m^2)\varphi_0 - 2\lambda\varphi_0^3 - 2\lambda\varphi_0^3\left[3\langle\psi_1^2\rangle + \langle\psi_2^2\rangle\right] = 0, \tag{2.15}$$

where we have assumed that the cross-correlation of the fluctuations vanish, i.e., $\langle\psi_1\psi_2\rangle = 0$. This is justified a posteriori by equations (2.19), which show that ψ_1 and ψ_2 do not interact with each other at the order of approximation we are working. eq. (2.15) determines the dynamics of the condensate taking into account the backreaction of the fluctuations. It is the relativistic generalization of the Gross–Pitævskii equation [181].

2.2.1 Dynamics of perturbations: acoustic metric

Having determined the dynamics of the condensate we now want to calculate the equations of motion for the perturbations themselves. To this end, we insert $\varphi = \varphi_0(1 + \psi_1 + i\psi_2)$ in eq. (2.14) and expand it to linear order in ψ's. Using the Gross–Pitævskii equation to that order and separating the real and imaginary parts we get the equation of motion for ψ_1 and ψ_2,

$$\Box\psi_1 + 2\eta^{\mu\nu}\partial_\mu(\ln\varphi_0)\partial_\nu\psi_1 - 4\lambda\varphi_0^2\psi_1 = 0, \tag{2.16a}$$

$$\Box\psi_2 + 2\eta^{\mu\nu}\partial_\mu(\ln\varphi_0)\partial_\nu\psi_2 = 0. \tag{2.16b}$$

We therefore see that ψ_2 is the massless mode, which is the Goldstone boson of the broken $U(1)$ symmetry, while ψ_1 is the massive mode with mass $2\varphi_0\sqrt{\lambda}$. We now define a "acoustic" metric, which is conformal to the background Minkowski,

$$g_{\mu\nu} = \varphi_0^2\,\eta_{\mu\nu}. \tag{2.17}$$

The relation between the d'Alembertian operators for $g_{\mu\nu}$ and $\eta_{\mu\nu}$ is given by,

$$\Box_g = \frac{1}{\varphi_0^2}\Box + \frac{2}{\varphi_0^2}\,\eta^{\mu\nu}\,\partial_\mu(\ln\varphi_0)\,\partial_\nu. \tag{2.18}$$

Equations (2.16) can be written in terms of the d'Alembertian of $g_{\mu\nu}$ as

$$\Box_g\psi_1 - 4\lambda\psi_1 = 0, \tag{2.19a}$$

$$\Box_g\psi_2 = 0. \tag{2.19b}$$

We see from eqs. (2.19) that the fluctuations propagate on a curved metric, called the acoustic metric, which in this case is conformal to the background Minkowski space eq. (2.17). Note that in this derivation there was no low-momentum approximation needed in order to derive the acoustic metric.

2.3 Relation to previous results

The following section is devoted to the connection with the previous results on relativistic BEC as analogue models presented in [88] (see also Sec. 1.2.2).

As noted earlier, eq. (2.14) correspond to eq. (1.8), i.e., the starting equation of ref. [88] in which the acoustic metric felt by the perturbations of condensate was derived for the first time. Furthermore, that work showed that the acoustic metric coincides with the one derived in ref. [221][2] for the relativistic flow of an inviscid, irrotational fluid with a barotropic equation of state. The perturbations of such a fluid propagate on an acoustic geometry

[2] See also [47, 159] for an earlier derivation

which is disformally related to the background Minkowski space,

$$g_{\mu\nu} = \rho \frac{c}{c_s} \left[\eta_{\mu\nu} + \left(1 - \frac{c_s^2}{c^2} \right) \frac{v_\mu v_\nu}{c^2} \right] , \tag{2.20}$$

where c_s is the speed of sound and $v^\mu = cu^\mu / \|u\|$ is the velocity of the fluid flow. Here $u^\mu \equiv \frac{\hbar}{m} \eta^{\mu\nu} \partial_\nu \theta$ is the usual four vector directly associated to the spacetime dependence of the phase of the background field written in the Madelung form $\varphi = \sqrt{\rho} e^{i\theta}$ (see [88] for a detailed discussion). Given the disformal form of the acoustic metric found in all these studies, it might seem quite surprising that the acoustic metric in the present case is conformal to the flat space. More importantly, there is no Lorentz violation in the dynamics of the perturbations in our system: perturbations experience the same acoustic metric both at low and high momenta. Since the acoustic metric is conformally flat they propagate with the "speed of sound" equal to c (the speed of light).

The point is that we have assumed φ_0 to be real which is tantamount to have a constant phase θ. It is indeed possible to start from the general equations of Sec. 1.2.2 and ref. [88] and try to see what happens in the limit in which the phase of the order parameter becomes a spacetime constant (in particular zero for simplicity). The results of this kind of limiting procedure are the following: first of all the dispersion relations in eq. (1.12) becomes the dispersion relations for a massless and a massive mode that one can derive from eq. (2.16); secondly, the parameter b, used in ref. [88] to define the low momentum limit — i.e., the approximation in which the acoustic metric can be derived —, approaches infinity in the present case so that the low momentum limit is always satisfied. With the same limiting procedure it is also possible to show that the speed of sound becomes equal to the speed of light as it is in our current treatment and as should be expected by the dispersion relations which do not show anymore the Lorentz violating terms. Finally, another quantity that remain well define despite of the limit is the fluid four velocity, in fact one can easily see that $v^\mu v_\mu = -c^2$ and v^μ is finite. This explanation has the weakness to not be straightforwardly applicable to massless particles, that we shall assume later, as the definition of u^μ becomes singular in this limit. But it seems to be possible to take the massless limit at the end of the calculation when no quantity directly depends on the mass. The discussion of the massless boson gas condensation would need a separate treatment in case one wants to purse the fluid analogy.

The previous discussion shows that the limiting procedure is well defined. The final step then is to see how the acoustic metric can be read off from the perturbation equations in such a limit of constant phase and if this metric is really conformally flat. For doing this is sufficient to start from eq. (1.9) and take the limit of constant phase, then one obtain

$$(\Box + \eta^{\mu\nu} \partial_\mu \ln \rho \partial_\nu)\psi - 2\lambda\rho(\psi + \psi^\dagger) = 0$$

which is equivalent to eq. (2.16), where ρ corresponds to φ_0^2. From this equation we already know that it is possible to read out the conformally flat acoustic metric felt by the perturbations. Note also that the same conclusion can be obtained starting from eq. (2.20) and looking at the case in which $c_s = c$. It should be noted that, the equality between the speed of sound and the speed of light gives rise to the fact that in this system there has no interpolation phase between two relativity groups with two limit speeds (c_s in the IR limit, c in the UV one), thus the relativity group remains always the same at any energy. This is hence an example of a model of emergent space-time where the low and high energy regime share the same Lorentz invariance. This point is not trivial since, as far as we know, there is no toy model of emergent spacetime in which Lorentz violation is screened in this way at the lowest order of perturbation theory. The case at hand shows that it can be possible at the price of some non-trivial conditions on the background system, i.e., fine-tuning in a certain sense.

While the above discussion shows how the current result is related to that of Sec. 1.2.2, we should also stress that the starting formalisms are indeed different. In Sec. 1.2.2 and ref. [88] the condensation was assumed *a priori* and the rest followed, here we have used the Grand Canonical formalism that is more suited to show that a condensation actually happens and also permits to derive the critical temperature for the interacting case. The crucial feature is that this formalism allowed us to single out explicitly the chemical potential that gives to us a mass scale that we will use in the next sections in order to rescale the fields.

2.4 Emergent Nordström gravity

In Sec. 2.2.1 we saw that the fluctuations of the condensate, also called the quasi-particle excitations, are oblivious of the flat background metric. They instead experience a curved geometry dictated by the condensate and the background. On the other hand, they back-react on the condensate through the relativistic generalization of the Gross–Pitaevskii equation (2.15). It is natural to ask if it is possible to have a geometric description of the dynamics of the condensate too.

The Ricci tensor of the acoustic metric (2.17) can be calculated to be

$$R_g = -6\,\frac{\Box\,\varphi_0}{\varphi_0^3}. \tag{2.21}$$

Dividing the relativistic Gross–Pitaevskii equation by φ_0^3, eq. (2.15) can be written as

$$R_g + 6\frac{m^2}{\varphi_0^2} + 12\lambda = \langle T_{\rm qp}\rangle, \tag{2.22}$$

where we have defined $\langle T_{\rm qp}\rangle := -12\lambda\left[\,3\,\langle\psi_1^2\rangle + \langle\psi_2^2\rangle\,\right]$ and the subscript “qp" reminds us that this quantity is determined by the quasi-particle excitations of the condensate.

Eq. (2.22) is evidently reminiscent of the Einstein–Fokker equation describing Nordström gravity [73, 101],

$$R + \Lambda = 24\pi\frac{G_{\rm N}}{c^4}\,T, \tag{2.23}$$

where R and T are, respectively, the Ricci scalar and the trace of the stress-energy tensor of matter. Unfortunately, the gravitational analogy of our equation is spoiled by the mass term. Therefore we will consider our system in the zero mass limit. Notice that, as discussed earlier, this limit does not spoil the presence of a condensate (see eq. (2.10)) or the uniqueness of the Lorentz group for constituents and excitations found in Sec. 2.3. We shall come back to the physical reasons for this limit in the discussion section.

The striking resemblance of equations (2.22) with zero mass term and (2.23) should not distract us from the need of one more step before comparing them. Indeed, the dimensions of the various quantities appearing in eq. (2.22) are not canonical and need to be fixed for such comparison to be meaningful. This is due to the fact that, as is usual in the analogue gravity literature, our acoustic metric is a dimensional quantity because φ_0 is dimensional. The fractional perturbations ψ_1 and ψ_2, on the other hand, are dimensionless. We therefore need rescaling of the fields in order to have a dimensionless metric and (mass) dimension one scalar fields propagating on it.

Field redefinition

We are going to redefine the fields in such a way to have a dimensionless acoustic metric and mass dimension one scalar fields propagating on it. The dimension of the field is given by $[\phi] = \sqrt{ML/T^2}$, the chemical potential has the dimension of an energy and $[\lambda] = T^2/ML^3$. First of all we redefine the background field (the condensate part) φ_0 as

$$\tilde{\varphi}_0 = \frac{\sqrt{\hbar c}}{\mu}\varphi_0, \tag{2.24}$$

in such a way to render it dimensionless. Note that, this redefinition is unique since only one mass scale given by the chemical potential is present. Analogously, we redefine the perturbation field as

$$\tilde{\psi} = \frac{\mu}{\sqrt{\hbar c}}\psi. \tag{2.25}$$

The new acoustic metric is given by $\tilde{g}_{\mu\nu} = \tilde{\varphi}_0^2\eta_{\mu\nu}$ and the background equation becomes

$$\tilde{R} + 12\lambda\frac{\mu^2}{c\hbar} = 0.$$

We can call cosmological constant the factor $\Lambda_{\text{eff}} \equiv 12\lambda\mu^2/c\hbar$ which has in fact the right dimension, i.e., $1/L^2$. Finally, from dimensional arguments, the only combination of constants of the model with the right dimension for giving rise to the emergent gravitational constant is $\hbar c^5/\mu^2 \propto G_{\text{eff}}$, and so the would be Planck mass is dimensionally set by μ/c^2, see Sec. 2.4.1.

The upshot of the dimensional analysis is that we need to scale the fields φ_0 and ψ as in eqs. (2.24) and (2.25). Using these rescaled quantities, and omitting from now on the tildes, we can rewrite eq. (2.22) (with $m = 0$) in the form of eq. (2.23), i.e.,

$$R + \Lambda_{\text{eff}} = \langle T_{\text{qp}} \rangle, \tag{2.26}$$

where $\Lambda_{\text{eff}} \equiv 12\lambda\frac{\mu^2}{c\hbar}$ and T_{qp} here, and in the following, is the same expression as in (2.22) but with the mass dimension one fields in eqs. (2.24) and (2.25). Equations of motion of the quasi-particles (2.16) can also be rewritten in terms of the rescaled fields as

$$\Box_g \psi_1 - \frac{4\lambda\mu^2}{\hbar c}\psi_1 = 0, \tag{2.27a}$$

$$\Box_g \psi_2 = 0, \tag{2.27b}$$

where all quantities, including the $\Box_g$ operator, now pertain to those of the rescaled fields.

2.4.1 Stress Energy Tensor and Newton constant

As a final step in order to verify the emergence of a true Nordström gravity theory from our system we still need to prove that the expression indicated via $T_{\rm qp}$ is indeed related to the trace of the stress energy tensor for the analogue matter fields — i.e., the quasiparticles — which we indicate with T. This turns out to be indeed the case and the proportionality factor relating these quantities allows us to identify the effective gravitational constant $G_{\rm eff}$ for this analogue system. In the following we refer the reader mainly to appendix A for technicalities and just state the main results.

It should be noted that, it is possible to rewrite the effective action (2.11) in a geometric form in terms of the acoustic metric (2.17) (see box below) and use this action for computing the stress energy tensor for the perturbations. In particular, we want to compute it by varying this action with respect to the acoustic metric, i.e.

$$T_{\mu\nu} \equiv -\frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{S}_2\right)}{\delta g^{\mu\nu}}, \tag{2.28}$$

where $\mathcal{S}_2$ is the quadratic (in perturbations) part of the action (see eq. (2.35)) as the linear part $\mathcal{S}_1$ can be shown to give no contribution to the trace of the stress energy tensor (see appendix A).

Action in geometrical form

The action for background field and perturbations can be put in a geometrical form, suitable for the computation of the stress energy tensor in eq. (2.28), by making explicit use of the acoustic metric. Here we use the non-redefined fields and natural units. Using $\phi = \varphi_0(1+\psi)$ the effective Lagrangian (2.11) — after getting rid of the μ dependent term — can be written as

$$\mathcal{L}_{\rm eff} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{3,4}, \tag{2.29}$$

where the number in the suffixes represents the number of the perturbation fields in the Lagrangians and

$$\mathcal{L}_0 = -\eta^{\mu\nu}\partial_\mu\varphi_0\partial_\nu\varphi_0 - m^2\varphi_0^2 - \lambda\varphi_0^4 \tag{2.30}$$

$$\mathcal{L}_1 = \left(-\eta^{\mu\nu}\partial_\mu\varphi_0\partial_\nu\varphi_0 - m^2\varphi_0^2 - 2\lambda\varphi_0^4\right)(\psi^* + \psi) - \eta^{\mu\nu}\partial_\mu\varphi_0\varphi_0\partial_\nu\psi \tag{2.31}$$

$$- \eta^{\mu\nu}\partial_\mu\varphi_0\varphi_0\partial_\nu\psi^* \tag{2.32}$$

$$\mathcal{L}_2 = \left(-\eta^{\mu\nu}\partial_\mu\varphi_0\partial_\nu\varphi_0 - m^2\varphi_0^2\right)(\psi^*\psi) - \lambda\varphi_0^4\left(\psi\psi + \psi^*\psi^* + 4\psi^*\psi\right)$$
$$- \eta^{\mu\nu}\varphi_0^2\partial_\mu\psi^*\partial_\nu\psi - \eta^{\mu\nu}\varphi_0\partial_\mu\varphi_0\psi^*\partial_\nu\psi - \eta^{\mu\nu}\partial_\mu\psi^*\varphi_0\partial_\nu\varphi_0\psi$$

$$\mathcal{L}_{3,4} = -\lambda\varphi_0^4\left(2\psi^*\psi\psi + 2\psi^*\psi^*\psi + \psi^*\psi^*\psi\psi\right) \tag{2.33}$$

Employing the fact that $g_{\mu\nu} = \varphi_0^2 \eta_{\mu\nu}$, $\sqrt{-g} = \varphi_0^4$ and throughout integration by parts of the terms in the $\mathcal{L}_i$ with $i = 1, 2$ it is possible to rewrite the action of the theory, up to quadratic terms in the perturbations, in a geometrical form (we refer for further details to Appendix B of ref. [35]). The action of the theory, when $m = 0$, is given in geometrical form by the following expression

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{6}R - \frac{1}{6}R(\psi + \psi^*) - \frac{1}{6}R(\psi^*\psi)\right. \tag{2.34}$$
$$\left. -\lambda\left[1 + 2(\psi^* + \psi) + \psi\psi + \psi^*\psi^* + 4\psi^*\psi\right] - g^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi \right\}.$$

Finally, in terms of the redefined fields the quadratic (in perturbations) part of the action, entering eq. (2.28), is given by

$$\mathcal{S}_2 = -\int d^4x\sqrt{-g}\left\{\frac{1}{6}R(\psi^*\psi) + \frac{1}{12}\Lambda\left[\psi\psi + \psi^*\psi^* + 4\psi^*\psi\right] + g^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi\right\}. \tag{2.35}$$

The final result for the expectation value of the trace of the stress-energy tensor in the background of $g_{\mu\nu} = \left(\mu^2/\hbar c\right)\varphi_0^2\eta_{\mu\nu}$ is given by

$$\langle T\rangle = -2\lambda\frac{\mu^2}{c\hbar}\left[3\langle\psi_1^2\rangle + \langle\psi_2^2\rangle\right] = \frac{1}{6}\frac{\mu^2}{c\hbar}\langle T_{qp}\rangle. \tag{2.36}$$

From this last expression it is evident that the RHS of eq. (2.26) is given by $6c\hbar\langle T\rangle/\mu^2$ and, hence, the emergent Nordström gravity equation is exactly of the form (2.23) with the identification $G_{\rm eff} = \hbar c^5/(4\pi\mu^2)$. This value corresponds to an emergent analogue Planck scale $M_{\rm Pl} = \mu\sqrt{4\pi}/c^2$.

We have thus succeeded in expressing the dynamics of the background for the rBEC analogue model in a geometric language

$$R + \Lambda_{\rm eff} = 24\pi\frac{G_{\rm eff}}{c^4}\langle T\rangle. \tag{2.37}$$

The dynamics of the acoustic metric appears to be sourced by the expectation value of the trace of the stress-energy tensor of the perturbations on top of the condensate, which play the role of the matter. These matter fields in turn propagate relativistically on the conformally flat acoustic metric (2.17) with equations (2.27). Note that, this is analogous to the non-relativistic case of Sec. 1.2.3 in which the (massive) quasi-particles were the source in the modified Poisson equation (1.14).

A final comment is deserved by the emergent, positive, cosmological constant term $\Lambda_{\rm eff}$. The quantity of interest for what concern the usual cosmological constant problem is the ratio between the energy density associated to the (emergent) cosmological constant $\epsilon_{\Lambda_{\rm eff}} \sim \left(\Lambda_{\rm eff}c^4\right)/G_{\rm eff}$ and the emergent Planck energy density $\epsilon_{\rm pl} \sim c^7/\left(\hbar G_{\rm eff}^2\right)$. In the present case this ratio is given by

$$\frac{\epsilon_{\Lambda_{\rm eff}}}{\epsilon_{\rm pl}} \simeq \frac{3\lambda\hbar c}{\pi}. \tag{2.38}$$

This ratio is proportional to $\lambda\hbar$, thus it is clearly pretty small due to the presence of Planck constant and of the assumption of weak interactions. Of course, in principle this term can be "renormalised" by the vacuum contribution of the matter fields, basically the vacuum expectation value $\langle T \rangle$, and in principle even change sign. It is however non-trivial, and beyond the scope of the present Chapter, to split the ground state in a matter and vacuum part as it is not an eigenstate of the number operator (which in the present relativistic system is not conserved).

2.5 Summary

In this Chapter a relativistic Bose–Einstein condensation, in a theory of massless complex scalar field with a quartic coupling, has been considered. Below the critical temperature the $U(1)$ symmetry is broken resulting in the non-zero value of the expectation value of the field — the condensate. We showed that the dynamics of the condensate is described by the relativistic generalisation of the Gross–Pitaevskii equation given by eq. (2.15). The fluctuations of the condensate experience the presence of the condensate through the acoustic metric (2.17) which, with the particular background state chosen here, turns out to be conformal to the flat Minkowski metric. The propagation of the two components of the perturbation is described by eqs. (2.27), which are just the Klein-Gordon equations for massive and massless scalar fields on the curved background provided by the acoustic metric. Perturbations in turn gravitate through the trace of their stress-energy tensor which is calculated in detail in appendix A. The dynamics of the acoustic metric is governed by the analogue Einstein–Fokker equation (2.26), which is the equation of motion for the Nordström gravity with cosmological constant. To the best of our knowledge, this is the first study of the emergence of Lorentz invariant dynamics for the emergent spacetime in an analogue model[3]. As a side remark, note that the emergence of only conformally flat analogue spacetimes is in no way a trivial result since cosmological solutions in GR are conformally flat as well and nevertheless they incorporate characteristic features like expansion of the Universe and cosmological particle creation.

2.6 Discussion

The central assumption that has permitted us to carry out the geometrical interpretation of the model is the reality of the order parameter. Thanks to this it was possible to have a conformally flat acoustic metric and to rewrite the background equation in a geometrical form. In the general case in which the order parameter is complex — see Sec. 1.2.2 — it does not seem to be much hope to cast the non-linear Klein-Gordon equation for the background in a geometrical form, although an acoustic metric can still be derived (2.20). This is due to the fact that the general disformal acoustic metric depends both on the (derivative of the) phase and the modulus of the order parameter, whereas the background equation is too simple to describe the dynamics of both of them. Thus, this equation cannot be recast in a background independent form. The reality of the condensate, on the other hand, leaves only one degree of freedom to play with. Hence, at the

[3]See also ref. [99].

very best, one can hope to recover a scalar theory of gravity, such as Nordström theory, in this limit. It would be interesting to further characterise the particular background state that has to be chosen in order to recover a gravitational dynamics.

Another approximation which we have taken in order to emerge Nordström gravity is the zero mass limit of the underlying atoms. Although we have seen that the massless case is not pathological from the point of view of Bose–Einstein condensation, one should be aware that, strictly speaking, such a limit is not necessary. Indeed, it is sufficient to require for the mass term in eq. (2.22) to be negligible with respect to the others, though this would call for a careful analysis beyond the scope of this work. But why the mass term ruins the geometrical interpretation of the equation? Let us just notice that this term breaks the conformal invariance of the background equation (2.15). Similarly, the addition of higher order interactions (see also discussion below) would break the conformal invariance of (2.15) and spoil the possibility to recast the equation in a geometric form. It would be interesting to further investigate this apparent link and pinpoint the exact connection (if any) between conformal invariance of the background equation and its viability for a geometric interpretation.

From a pure EFT point of view it is clear that other interaction terms, apart from the $\lambda\phi^4$ considered, are admissible. As just mentioned, higher mass dimension interaction terms ϕ^n as well as a cubic term (which could be discarded anyway by parity arguments), would end up spoiling the geometrical interpretation of the theory. However, while in principle the aforementioned higher order interactions are allowed, there are good physical reasons for the $\lambda\phi^4$ interaction to be the most relevant one. In fact, such term models two body interactions which are generically dominant in dilute systems as the condensate which we have considered here. Thus, higher order interaction terms will not only be irrelevant from an EFT point of view, but will be associated to many-body interactions which will be generically subdominant.

It is also interesting that we obtain quite naturally a cosmological constant term whose size is set by the coupling constant λ and the chemical potential μ. Remarkably, the emergent cosmological constant is such that the ratio between its energy density and the energy density associated to the emergent Planck length eq. (2.38) is small. Thus, there is no "cosmological constant problem" — in the sense of unnatural smallness — in this emergent gravity model. This result is in close analogy with the non-relativistic case discussed in Sec. 1.2.3[4]. It is however important to stress that in the present, relativistic, case the recovery of a cosmological constant term is strongly dependent on the choice of the particular interaction characterising the initial Lagrangian (2.1), i.e., the $\lambda\phi^4$ one. Instead, as discussed in [89], the small, negative, cosmological constant term found in the non-relativistic BEC is basically due to the depletion factor, i.e., to the ever present atoms which are not in the condensate phase. This is a pure quantum effect due to the quantum inequivalence of the phonon and atomic vacua.

The relativistic case shows a "bare" gravitational constant term, simply

[4] See also ref. [89] for further discussion.

stemming from the ϕ^4 term, which is there independently from the vacuum expectation value $\langle T \rangle$ contribution (the relativistic generalisation of the term associated to depletion in the non-relativistic BEC). Of course it is possible to recover the non-relativistic BEC case from the relativistic BEC (see [88]): the dimensional bare coupling constant $\Lambda_{\text{eff}} = 12\lambda\mu^2/c\hbar$ will go to zero as $c \to \infty$ and only the "depletion" contribution will remain.

Finally, Nordström gravity is only a scalar theory of gravity and has been falsified by experiments, for example, it does not predict the bending of light. However, it is the only other known theory in 4 dimensions that satisfies the strong equivalence principle [74]. With the aim of getting closer to emerge General Relativity, one necessarily needs to look for richer Lagrangians than that in eq. (2.1). Of course, emergence of a theory characterised by spin-2 graviton would open the door to a possible conflict with the Weinberg–Witten theorem [225]. However, one may guess that analogue models (or analogue model inspired systems) will generically lead to Lagrangians which show Lorentz invariance and background-independence only as approximate symmetries for the lowest order in the perturbative expansion. The relativistic model proposed here shows that, at least at the level of linear perturbations, such symmetries are realised both in the equations of the linear perturbations as well as in those describing the dynamics of the background. As such it might serve as toy model for the use of emergent gravity scenarios in investigating, e.g., geometrogenesis — here corresponding to the condensation process [167] — or the nature of spacetime singularities in this framework.

We have shown how Lorentz Invariance is preserved in this rBEC model to first order in perturbation theory. It is nevertheless true that this result holds only perturbatively and, moreover, only in the particular case of a real order parameter. This can be seen as a sort of fine-tuning of the model which ensure LI. However, as discussed in Sec. 1.2.2, in general a rBEC will show violations of LI in the dispersion relation of the quasi-particles which feel the acoustic metric. Thus, it is evident that LIV are naturally expected in these kind of emergent gravity models and any theory which tries to implement these ideas has to carefully dealt with the strong observational constraints on LIV and their naturalness problem. Indeed, LIV once present at high-energy can percolate — via radiative corrections — into the low-energy physics realm giving rise to a fine-tuning problem which require some suppression mechanism to be dealt with. In the next Chapter we shall indeed analyse this issue in depth.

Chapter 3

Lorentz Invariance Violations naturalness

A single part of physics occupies the lives of many men, and often leaves them dying in uncertainty.

Voltaire

In the previous Chapter we have seen that LIV are an expected feature of analogue gravity models which can be avoided only by, in a certain sense, fine-tuning the system to do so. As discussed in Sec. 1.3 these violations, while extremely interesting as a window on QG, are currently strongly constrained by observations. Thus, Lorentz Symmetry appears to be an exact symmetry of Nature, as far as we can say. Nevertheless, being the Lorentz group non-compact and Lorentz symmetry so fragile to (naïve) discretization of spacetime, neglecting the possibility of (tiny) LIV to be present around the Planck scale seems an overstatement.

However, even if LIV are conjectured only in the far ultra-violet (UV), in the EFT framework, still they would lead generically to large low-energy effects. In a seminal work [69], Collins *et al.* showed that in a generic QFT, seen as an effective field theory, Lorentz violations in the UV can percolate in the infra-red (IR) without being suppressed and leading to unacceptably large effects. Here the term "percolation" refers to the fact that, in an EFT setting, even if one starts by adding only LIV operators of mass dimension larger than 4, radiative corrections will generate mass dimension four (and, generically, mass dimension three) operators. Then the percolation is said to be unsuppressed if there is no small amplitude suppressing the effects of these operators (in addition to usual coupling constants). This is a peculiarity of LIV. Indeed, in the case of LI theories, the physics at high energies affects the IR physics only via renormalization of the bare couplings of the theory. The existence of this percolation can be easily understood from the EFT point of view. Indeed, in EFT every operator respecting the fundamental symmetries of the theory can be present. Accordingly, even if a certain operator is not present in the UV, radiative corrections can give rise to mass dimension four (and in general three) Lorentz invariance violating operators (see also refs. [93, 119, 182]) once higher-order LIV operators are allowed in the theory.

The conclusion of ref. [69] was that if LIV is admitted, then it is necessary to accept an unnaturally and extreme fine-tuning of the theory, in order to respect the stringent experimental constraints on LIV at low energies. From this point of view, LIV can be considered a new fine-tuning problem to be added to the various ones existing in particle physics and cosmology, i.e., we have a LIV naturalness problem (see Sec. 1.3.2).

Clearly, various solutions to this fine-tuning problem have been proposed in the literature as we discussed in Sec. 1.3.2. Furthermore, before proceeding, let us mention as a cautionary note, that the low-energy effective field theory paradigm, as well as the related naturalness arguments, are not always capable of capturing the correct physics. As an example, consider the effective field theory prediction for the magnitude of the cosmological constant, which is out by more than 120 orders of magnitude compared with observations. Without a direct measurement, one would have expected a naturalness problem for the cosmological constant. The observational evidence that the latter has such a small value seems to suggest, instead, a breakdown of EFT or the presence of yet to be understood symmetries at intermediate energies between the TeV and the Planck scale. In the following we will focus only on the EFT description but keeping in mind that a breakdown of an EFT-based intuition might apply also in the case of LIV naturalness.

In this Chapter we first review the argument of refs. [68, 69], presenting detailed calculations and showing how, in some special cases, a cancellation of the percolation can be achieved. Then, we attack the problem from a different perspective, illustrating in a simple way how in the case in which there is a large separation between the EFT validity scale Λ and the LIV scale M, the percolation can be suppressed and discussing how this suppression generically scales with energy. In particular, we find that for dispersion relations with leading LIV terms of order $(E/M)^{2n}$ the percolation scales as $\Delta c \propto (\Lambda/M)^{2n}$. This implies that if any such scenario could be successfully applied to the Standard Model (SM), values of Λ below 10^{10} GeV would be sufficient to reconcile the most interesting (CPT invariant) case $n = 1$ with current observational constraints (see further discussion in sections 3.3 and 3.5). Finally, we also consider a dissipative case, in the spirit of ref. [173], and show that while the dissipative behaviour — i.e., the presence of imaginary contributions— does not percolate, a dispersive one does. We also demonstrate that a scale separation can hinder such percolation with basically the same behaviour as the one we find in the dispersive case. This Chapter is based on [37].

3.1 LIV percolation: previous results

In this section we briefly review, for the readers' convenience, the results of ref. [69]. This work considers a model of a scalar ϕ and a fermion ψ coupled via a Yukawa interaction

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m_\phi^2}{2}\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m_\psi)\psi + g\phi\bar{\psi}\psi, \tag{3.1}$$

where g is the dimensionless coupling constant[1]. Beyond tree-level, the theory is made finite by a cutoff on spatial momenta (in a given preferred frame), implemented as a modification of the free propagators. As a first important point, we emphasise here that the scale entering as a (LIV) cutoff

[1]Hereafter we consider natural units, with $c = 1$.

for the UV behaviour of the theory is the LIV scale itself, i.e., the same scale as the one appearing in the modified dispersion relations (MDR). Moreover, this scale can be identified with the Planck scale. Once such a cutoff is introduced, one assumes for the scalar and fermion propagators in momentum space (see refs. [68, 69] for details)

$$\frac{i}{\not{p} - m_\psi + i\epsilon} \to \frac{i f\left(|\mathbf{p}|/\Lambda\right)}{\not{p} - m_\psi + \Delta\left(|\mathbf{p}|, \Lambda\right) + i\epsilon}, \tag{3.2}$$

$$\frac{i}{p^2 - m_\phi^2 + i\epsilon} \to \frac{i \tilde{f}\left(|\mathbf{p}|/\Lambda\right)}{p^2 - m_\phi^2 + \tilde{\Delta}\left(|\mathbf{p}|, \Lambda\right) + i\epsilon}, \tag{3.3}$$

where $|\mathbf{p}|$ is the modulus of the 3-momentum and the cutoff functions $f\left(|\mathbf{p}|/\Lambda\right)$ and $\tilde{f}\left(|\mathbf{p}|/\Lambda\right)$ are such that they approach 1 as $|\mathbf{p}|/\Lambda \ll 1$, in order to reproduce low-energy physics, while they vanish sufficiently rapidly as $|\mathbf{p}|/\Lambda \gg 1$, in order to render the theory UV finite. The functions $\tilde{\Delta}$ and Δ appearing in the denominators, instead, come from actual proposals for MDR that usually appear in the quantum gravity (phenomenology) literature [11, 140, 150]. These functions are such that they vanish for $|\mathbf{p}|/\Lambda \ll 1$, again to recover the low-energy physics. Actually, in ref. [69], Δ and $\tilde{\Delta}$ are not introduced since it is argued that they will not affect the argument. This is due to the fact that the LIV effects are seen as producing a natural cutoff at large (spatial) momenta which is already produced by the cutoff functions f and $\tilde{f}$. However, note that neglecting Δ and $\tilde{\Delta}$, i.e., the MDR, is possible only at the price of identifying the EFT and the MDR scales. This is tantamount to assuming no new physics between the SM scale and the Planck one.

It has been objected that introducing a new scale between the low-energy and the Planck one would be intrinsically against the philosophy of quantum gravity phenomenology models entailing Lorentz breaking in the UV, which hinge on the persistence at low energies of these Planck scale effects [176]. However, we do not see a problem in conjecturing such a hierarchy of scales (we rather found difficult to conceive that no new physics will be present from the TeV to the Planck scale): the effective field theory framework is perfectly capable to account for these scenarios. For these reasons, later on in this Chapter we will drop such an identification by requiring the LIV scale which enters the MDR to be different from the cutoff scale of the theory that we will introduce in a Lorentz-invariant way. Let us stress that this does not imply any notion of intermediate Lorentz invariance between the Planck scale and currently tested energies. Indeed, in the scenario considered here all the physics below the Planck scale is Lorentz breaking. What we are envisaging here is that there could be some exact symmetry of nature, such as SUSY, which could be broken below some energy scale $\Lambda \ll M$ and that the new physics associated with this symmetry is not *per se* inducing Lorentz breaking operators suppressed by a scale other than M (so for $M \to \infty$ all the physics should be LI).

The unsuppressed percolation of LIV from the UV to the IR was considered in ref. [69] on the scalar field by computing its one-loop self-energy. As anticipated, we briefly review below the full computation. In the presence of LIV, the inverse propagator Π of the scalar field including one-loop corrections can be written as

$$\Pi(p) = A + Bp^2 + \xi p^\mu p^\nu W_\mu W_\nu + \Pi^{(LI)}(p^2) + \mathcal{O}(p^4/\Lambda^2), \tag{3.4}$$

where W^μ is a unit timelike background vector field permitting to write a LIV expression in a covariant form[2]. The third term on the R.H.S. of eq. (3.4) contains the unsuppressed LIV, i.e., it results in a different limit velocity of the scalar field[3] (with respect to c) while $\Pi^{(LI)}$ is a Lorentz invariant term (see ref. [68]). Our aim here is to compute ξ.

In what follows we will use the propagators in eqs. (3.2) and (3.3), this means that we set the calculation in the preferred reference frame characterised by the 4-velocity W^μ. In this case, the coefficient ξ of interest is obtained as

$$\xi = \left[\frac{\partial^2 \Pi}{\partial (p^0)^2} + \frac{\partial^2 \Pi}{\partial (p^1)^2}\right]_{p=0}. \tag{3.5}$$

In section 3.1.1 we briefly consider for pedagogical purpose the LI case, while in section 3.1.2 we consider the LIV case originally studied in ref. [69].

3.1.1 LI case

When there is no LIV, ξ is expected to vanish. In fact, this can be seen directly from eq. (3.5). Computing the diagram in figure 3.1, we find

$$\begin{aligned}\Pi(p^2) &= -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{tr[(\not{k} - \not{p} + m_\psi)(\not{k} + m_\psi)]}{[(k-p)^2 - m_\psi^2][k^2 - m_\psi^2]} \\ &= -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{4(k^2 - p\cdot k) + 4m_\psi^2}{[(k-p)^2 - m_\psi^2][k^2 - m_\psi^2]},\end{aligned} \tag{3.6}$$

and so

$$\begin{aligned}\xi &= \left[\frac{\partial^2 \Pi}{\partial (p^0)^2} + \frac{\partial^2 \Pi}{\partial (p^1)^2}\right]_{p=0} \\ &= \frac{-ig^2}{\pi^4} \int d^4k \frac{[(k^0)^2 + (k^1)^2](k^2 + 3m^2)}{(k^2 - m^2)^4},\end{aligned} \tag{3.7}$$

where hereafter we set $m \equiv m_\psi$ to simplify the notation. Although the integral in eq. (3.7) is formally logarithmically divergent (by power counting), the fact that it actually vanishes can be understood from a symmetry argument [69]. Indeed rotating in the Euclidean space and using four-dimensional spherical symmetry it is straightforward to see that the angular part of the integral implies $\xi = 0$.

[2] Recall that when speaking of Lorentz invariance in field theory we always refer to active Lorentz invariance, see the discussion in ref. [150].

[3] The explicit value of the parameter ξ resulting from LIV will be dependent on the coupling constant of the theory, as it is due to radiative corrections. Then, considering more rich theories one can see that the different particles will have different limit speeds in such a way that a simple rescaling of the limit velocity cannot eliminate this effect.

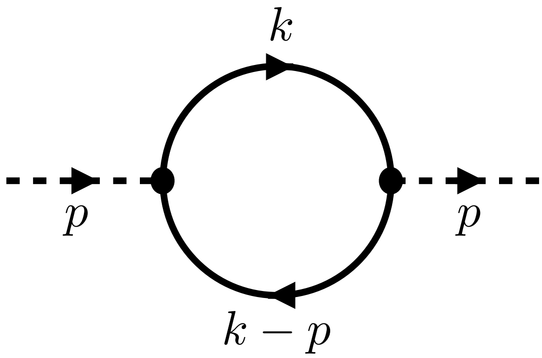

FIGURE 3.1: One-loop self-energy for the scalar field ϕ which is represented by dashed lines, whereas solid lines represent the fermion field ψ.

3.1.2 LIV case

We present now some details of the computation in the presence of LIV. We will relegate some technicalities to appendix B.1 while showing here the main steps in a self-contained way.

The diagram of interest is given in figure 3.1. The LIV is introduced via a cutoff function f and, according to the discussion above, we can use

$$\frac{i}{\not{p} - m + i\epsilon} \to \frac{i f(\mathbf{p}^2/\Lambda^2)}{\not{p} - m + i\epsilon} \tag{3.8}$$

for the fermionic propagator[4] with $f(0) = 1$ and $f(\infty) = 0$ (i.e., UV finiteness). In this case we will write for the propagator $G(p) = f(\mathbf{p}^2/\Lambda^2)S_0(p)$ where S_0 is the standard fermion propagator. Moreover, note that we are using $\mathbf{p}^2/\Lambda^2$ as argument of the LIV cutoff function f instead of $|\mathbf{p}|/\Lambda$ used in ref. [69]. This choice is motivated by computational simplicity and clearly it does not affect the general behaviour of the final result.

In order to determine ξ it is first convenient to calculate

$$\begin{aligned} c_{ab} &\equiv \frac{\partial}{\partial p^a}\frac{\partial}{\partial p^b}\Pi(p)\bigg|_{p=0} \\ &= -ig^2 \int \frac{d^4k}{(2\pi)^4}\, tr\left[\left(\frac{\partial}{\partial p^a}\frac{\partial}{\partial p^b}G(k-p)\right)G(k)\right]\bigg|_{p=0} \\ &= ig^2 \int \frac{d^4k}{(2\pi)^4} tr\left[\frac{\partial G(k)}{\partial k^a}\frac{\partial G(k)}{\partial k^b}\right], \end{aligned} \tag{3.9}$$

where[5] in the last line we have used that $\partial/\partial p^a = -\partial/\partial k^a$, performed an integration by parts and discarded a boundary term in view of the asymptotic properties of f. Using the above expression we can now express ξ as[6]

[4] Here we neglect the MDR Δ and $\tilde{\Delta}$ and introduce only the cutoff function, without loss of generality, due to the fact that the scale which appears in the MDR is assumed to be the same as the cutoff scale.

[5] The trace in eq. (3.9) is due to the fermionic loop of Fig. 3.1, i.e., the one which appear in eq. (3.6).

[6] Actually, in this expression, we could have used c_{22} or c_{33} instead of c_{11}, as they are all equal because we are assuming Lorentz breaking only in the boost, i.e., that the rotation symmetry in space is unbroken.

$$\xi \equiv c_{00} + c_{11}. \tag{3.10}$$

Details for the calculation of the trace

Here we calculate the trace appearing on the last line of eq. (3.9) which determines the values of c_{aa}. In order to do so we use the following relations involving the LIV fermion propagator G, the standard fermion propagator S_0, the cutoff function $f(\mathbf{k}^2/\Lambda^2)$ and the LIV cutoff Λ (these quantities are introduced in the main text after eq. (3.8)):

$$\frac{\partial G(k)}{\partial k^a} = \frac{\partial S_0(k)}{\partial k^a} f + S_0(k) \frac{\partial f(\mathbf{k}^2/\Lambda^2)}{\partial k^a}, \tag{3.11}$$

$$\frac{\partial S_0(k)}{\partial k^a} = i\eta_{aa} \left(\frac{\gamma^a}{k^2 - m^2} - 2k^a \frac{\not{k} + m}{(k^2 - m^2)^2} \right), \tag{3.12}$$

$$\frac{\partial f}{\partial k^a} = \frac{2k^a}{\Lambda^2} f'(\mathbf{k}^2/\Lambda^2)(1 - \delta_{a0}), \tag{3.13}$$

where η_{ab} is the Minkowski metric and δ_{ab} is the usual Kronecker delta. For simplicity, in these expressions we do not write explicitly the $+i\epsilon$ factors which characterises the denominators of propagators, as they are inconsequential for the present discussion. Using the above relations it is possible to express explicitly $\partial G(k)/\partial k^a$. Inserting this expression in the trace we conclude that the structure of the trace in eq. (3.9) is given by a sum of two terms, one proportional to $k_a k_b$ and the other to the flat spacetime metric η_{ab} as reported in the following in eq. (3.14).

Note that the trace in eq. (3.9) is such that (see box above for details)

$$tr \left[\frac{\partial G(k)}{\partial k^a} \frac{\partial G(k)}{\partial k^b} \right] = k_a k_b F + \eta_{ab} G, \tag{3.14}$$

where F and G are scalar functions and η_{ab} is the flat spacetime metric. From here it is already clear that c_{ab} does not vanish only if $a = b$. Indeed, while the second term on the R.H.S. is zero when $a \neq b$, the first one vanishes, in view of its symmetry, after the integration in eq. (3.9). Using the expressions in the previous box and the properties of gamma matrices, we can compute the trace in eq. (3.14) when $a = b$ as

$$\begin{aligned} tr \left[\frac{\partial G(k)}{\partial k^a} \frac{\partial G(k)}{\partial k^a} \right] &= 16(k^a)^2(k^2 + m^2) \\ &\cdot \left[\frac{f'\,(1 - \delta_{a0})}{\Lambda^2(k^2 - m^2)^2} + \frac{f^2}{(k^2 - m^2)^2} - \frac{2f'\,f}{\Lambda^2(k^2 - m^2)^3} \eta_{aa}(1 - \delta_{a0}) \right] \\ &+ 4\eta_{aa} \frac{f^2}{(k^2 - m^2)^2} \\ &+ 16(k^a)^2 \frac{f\,\eta_{aa}}{(k^2 - m^2)} \left[\frac{f'(1 - \delta_{a0})}{(k^2 - m^2)} - \eta_{aa} \frac{f}{(k^2 - m^2)^2} \right]. \end{aligned} \tag{3.15}$$

Given eq. (3.15), we can now compute the coefficients of interest. For doing this it is convenient to Wick rotate $k^0 \to ik^0$ to work in Euclidean space[7] and to compute first the integral over k^0, since the unknown cutoff function f does not depend on it. Defining, for convenience,

$$A = \mathbf{k}^2 + m^2$$

(where $\mathbf{k}$ indicates the 3-momentum) and using eqs. (B.2) and (B.3) we finally arrive at

$$c_{00} = g^2 \int \frac{d^3k}{(2\pi)^3} f^2 \left(-\frac{1}{A^{3/2}} + \frac{m^2}{A^{5/2}} \right), \tag{3.16}$$

and, after some algebra at

$$\begin{aligned} c_{ii} = -\, g^2 \int \frac{d^3k}{(2\pi)^3} \Bigg[\frac{(k^i)^2}{\Lambda^2} \left(-\frac{8(f')^2}{\Lambda^2 A^{3/2}} k^2 - 4\frac{f\,f'}{A^{5/2}}(-k^2 + 2m^2) \right) \\ + \frac{5m^2 f^2 (k^i)^2}{A^{7/2}} - \frac{f^2}{A^{3/2}} \Bigg], \end{aligned} \tag{3.17}$$

where, hereafter, $k = |\mathbf{k}|$ and the argument of f and its derivative f' is understood to be k^2/Λ^2. In order to compute ξ according to eq. (3.10) we exploit the spherical symmetry of the problem and integrate on the angular variables, making explicit the fact that c_{ii} is actually independent of i. In this way we conclude that

$$\begin{aligned} \xi = -\, \frac{g^2}{2\pi^2} \int_0^\infty dk\, k^2 \Bigg[f^2 \frac{m^2}{A^{7/2}} \left(\frac{5}{3} k^2 - A \right) - \frac{8k^4}{3\Lambda^4 A^{3/2}} (f')^2 \\ + \frac{4k^2(k^2 - 2m^2)}{3\Lambda^2 A^{5/2}} f\, f' \Bigg]. \end{aligned} \tag{3.18}$$

This expression can be cast in a simpler form by introducing the dimensionless variable $y = k^2/\Lambda^2$ and the ratio $\rho = m^2/\Lambda^2$. Accordingly,

$$\begin{aligned} \xi = -\, \frac{g^2}{2\pi^2} \Bigg[-\frac{4}{3} \int_0^\infty dy\, y^{5/2} (f')^2 \frac{1}{(y+\rho)^{3/2}} \\ + \frac{2}{3} \int_0^\infty dy\, y^{3/2}\, f\, f' \frac{y - \rho}{(y+\rho)^{5/2}} \Bigg], \end{aligned} \tag{3.19}$$

where now f is a function of y, $f' \equiv df(y)/dy$ and we have used that

$$\int_0^\infty dy \frac{\sqrt{y}}{2} f^2 \left(\frac{2}{3} y - \rho \right) \frac{\rho}{(y+\rho)^{7/2}} = \frac{2}{3} \int_0^\infty dy\, \frac{y^{3/2}}{(y+\rho)^{5/2}} f\, f'\, \rho,$$

which holds for the first integral on the first line of eq. (3.19) up to vanishing boundary terms. We are interested here in the IR percolation of LIV, i.e. in the value of ξ in the formal limit $\Lambda \to \infty$, corresponding to having Λ much larger than any mass scale m, which also implies $\rho \to 0$. In this limit,

[7] Wick rotating here comes without problems. Indeed, the location of the poles of the integrand in the complex k^0-plane is exactly the same as in standard QFT and therefore there are no obstructions for rotating the integration contour on the imaginary axis. Alternatively, one can avoid Wick rotation and instead use the residue theorem taking care of the $i\epsilon$ terms.

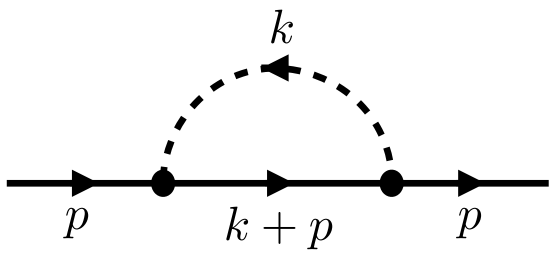

FIGURE 3.2: One-loop self-energy for the fermion. The dashed line represents the scalar field ϕ while solid ones represent the fermion ψ.

eq. (3.19) gives

$$\xi = \frac{g^2}{6\pi^2}\left[1 + 4\int_0^\infty dy\, y\, (f'(y))^2\right]. \tag{3.20}$$

Taking into account that ref. [69] considers a cutoff function $\hat{f}$ (denoted therein simply by f) which depends on $x \equiv |\mathbf{k}|/\Lambda$ and not on $y = x^2$ as we do here, with $\hat{f}(x) = f(x^2)$ the previous equation becomes

$$\xi = \frac{g^2}{6\pi^2}\left[1 + 2\int_0^\infty dx\, x\, (\hat{f}'(x))^2\right], \tag{3.21}$$

which indeed coincides with the result reported in eq. (A.2) of ref. [69], after the change of notation $\hat{f} \to f$. Let us note that, given the absence of MDR, this result can be also seen as a recovery of the well-known fact that a Lorentz invariance violating cutoff leaves a “LIV memory” even when it is formally removed.

We emphasise here that ξ can be interpreted as the fractional deviation with respect to the speed of light $c = 1+\mathcal{O}(g^2)$ (assumed to equal one at tree level for convenience) in a perturbative expansion in the coupling constant (see refs. [5, 219]), i.e.,

$$\frac{\Delta c}{c} = \xi + \mathcal{O}(g^4). \tag{3.22}$$

This relationship is valid also in the case of the fermion discussed in the next section. For this reason, hereafter we will indicate by Δc (instead of ξ) the quantity representing the LIV percolation.

3.2 Fermion self-energy

In this section we focus on the LIV percolation on the fermion and therefore we consider the fermion self-energy at one loop. We will use the previous setting concerning the propagators. In this case we obtain some interesting results beyond corroborating the generality of the argument in ref. [69] (see also ref. [219] for a similar analysis).

The relevant diagram contributing to the self-energy Σ of the fermion is represented in figure 3.2. Note that, since now one fermion and one scalar are involved in the loop, we can choose which field carries the LIV, i.e., which cutoff function f or $\tilde{f}$ to introduce. These options give rise to different cases that we analyze separately further below. In particular, we assume in full generality that the scalar and fermion fields have not only unequal masses m_ϕ and m_ψ but also different LIV cutoff functions f and $\tilde{f}$, respectively, still with no MDR as assumed in ref. [69]. Then, we specialise the corresponding general expressions to particular cases.

The self-energy Σ is given by

$$\Sigma(p) = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{\not{k} + \not{p} + m_\psi}{(k+p)^2 - m_\psi^2} \tilde{f}\left(|\mathbf{k}+\mathbf{p}|^2/\Lambda^2\right) \frac{f\left(|\mathbf{k}|^2/\Lambda^2\right)}{k^2 - m_\phi^2}. \tag{3.23}$$

This expression can be decomposed in three parts, i.e.,

$$\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{3.24}$$

where

$$\Sigma_1 = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{m_\psi}{(k+p)^2 - m_\psi^2} \tilde{f}\left(|\mathbf{k}+\mathbf{p}|^2/\Lambda^2\right) \frac{f\left(|\mathbf{k}|^2/\Lambda^2\right)}{k^2 - m_\phi^2} \equiv m_\psi \chi_1(p), \tag{3.25}$$

$$\Sigma_2 = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}}{(k+p)^2 - m_\psi^2} \tilde{f}\left(|\mathbf{k}+\mathbf{p}|^2/\Lambda^2\right) \frac{f\left(|\mathbf{k}|^2/\Lambda^2\right)}{k^2 - m_\phi^2} \equiv \chi_2(p), \tag{3.26}$$

$$\Sigma_3 = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{\not{p}}{(k+p)^2 - m_\psi^2} \tilde{f}\left(|\mathbf{k}+\mathbf{p}|^2/\Lambda^2\right) \frac{f\left(|\mathbf{k}|^2/\Lambda^2\right)}{k^2 - m_\phi^2} \equiv \not{p}\chi_3(p). \tag{3.27}$$

The one-loop form of the fermion inverse propagator in momentum space is

$$\not{p} - m_\psi + \Sigma(p) = \not{p} - (1 - \chi_1(0))m_\psi + c_0\gamma^0 p^0 - c_i \gamma^i p^i, \tag{3.28}$$

where γ^a are gamma matrices and we have expanded the self-energy around $p = 0$, neglecting higher-order terms. In order to extract the LIV we will compute the coefficients

$$c_a = \frac{1}{4}\, tr \left[\gamma^a \frac{\partial \Sigma}{\partial p^a}\right]\Bigg|_{p=0}. \tag{3.29}$$

They are analogous to the c_{aa} coefficients we determined in the case of the scalar; a LIV amounts to a non vanishing

$$\Delta c \equiv c_0 - c_i, \tag{3.30}$$

where $i = \{1, 2, 3\}$ stands for a spatial index. As before, the result will be independent of the particular i chosen since rotational invariance is unbroken. The fact that in standard LI QFT this quantity vanishes can be checked directly by using, e.g., dimensional regularization. Moreover, note that the coefficients c_a are alternatively given by

$$c_a = \chi_3(0) + \frac{1}{4} tr \left[\gamma^a \frac{\partial \chi_2(p)}{\partial p^a}\right]\Bigg|_{p=0} = \chi_3(0) + \tilde{c}_a, \tag{3.31}$$

with $a = 0, 1, 2, 3$ and therefore the violation in eq. (3.30) can also be written as $\Delta c = \tilde{c}_0 - \tilde{c}_i$.

The quantities of interest are given by

$$c_a = \frac{\partial}{\partial p^a}(-ig^2)\int \frac{d^4k}{(2\pi)^4}\frac{\eta^{ab}(k_b+p_b)}{(k+p)^2-m_\psi^2}\tilde{f}\left(|\mathbf{k}+\mathbf{p}|^2/\Lambda^2\right)\left.\frac{f\left(|\mathbf{k}|^2/\Lambda^2\right)}{k^2-m_\phi^2}\right|_{p=0} \tag{3.32}$$

$$= (-ig^2)\int \frac{d^4k}{(2\pi)^4}\left.\left\{\frac{\tilde{f}f}{(k^2-m_\psi^2)(k^2-m_\phi^2)}+k^a\left(\frac{\left.\frac{\partial\tilde{f}}{\partial p^a}\right|_{p=0}}{k^2-m_\psi^2}-\frac{2k^a\eta_{aa}}{(k^2-m_\psi^2)}\tilde{f}\right)\frac{f}{k^2-m_\phi^2}\right\}\right|_{p=0},$$

where in the last line the dependence of f and $\tilde{f}$ on $y \equiv |\mathbf{k}|^2/\Lambda^2$ is understood. By using the chain rule one has

$$\left.\frac{\partial\tilde{f}}{\partial p^a}\right|_{p=0} = \begin{cases} 0 & \text{for} \quad a=0, \\ \frac{2k^a}{\Lambda^2}\tilde{f}' & \text{for} \quad a\neq 0 \end{cases} \tag{3.33}$$

where $\tilde{f}' = d\tilde{f}(y)/dy$, i.e., this term is present only for $a \neq 0$.

Now both c_0 and c_i can be read from eq. (3.32):

$$c_0 = (-ig^2)\int \frac{d^4k}{(2\pi)^4}\left\{\tilde{f}f\left[\frac{1}{(k^2-m_\psi^2)(k^2-m_\phi^2)}-\frac{2(k^0)^2}{(k^2-m_\psi^2)^2(k^2-m_\phi^2)}\right]\right\}, \tag{3.34}$$

$$c_i = (-ig^2)\int \frac{d^4k}{(2\pi)^4}\left\{\tilde{f}f\left[\frac{1}{(k^2-m_\psi^2)(k^2-m_\phi^2)}+\frac{2(k^i)^2}{(k^2-m_\psi^2)^2(k^2-m_\phi^2)}\right]\right.$$
$$\left.+\frac{2(k^i)^2}{(k^2-m_\psi^2)(k^2-m_\phi^2)}f\tilde{f}'\frac{1}{\Lambda^2}\right\}, \tag{3.35}$$

and therefore Δc from eq. (3.30) is given by

$$\begin{aligned}\Delta c =& 2ig^2\int\frac{d^4k}{(2\pi)^4}\tilde{f}f\frac{(k^0)^2+(k^i)^2}{(k^2-m_\psi^2)^2(k^2-m_\phi^2)} \\ &+ig^2\int\frac{d^4k}{(2\pi)^4}\frac{2(k^i)^2/\Lambda^2}{(k^2-m_\psi^2)(k^2-m_\phi^2)}f\tilde{f}' \equiv \mathcal{P}+\mathcal{Q},\end{aligned} \tag{3.36}$$

in which we emphasise the presence of two contributions $\mathcal{P}$ and $\mathcal{Q}$. In order to proceed further we can Wick rotate the integration domain in the complex k^0-plane and compute the integral over k^0 since, as in the previous section, the cutoff functions f and $\tilde{f}$ are independent of it. The integration can be done using the formulas reported in appendix B.1. Consider first term denoted by $\mathcal{P}$ in eq. (3.36). We can simplify it by a change of variables analogous to the one in the previous section. In particular, we define $A \equiv k^2+m_\psi^2$, $B \equiv k^2+m_\phi^2$, $z = k^2/m_\psi^2$, $R \equiv m_\phi^2/m_\psi^2$, $\rho = m_\psi/\Lambda$, where

$k = |\mathbf{k}|$, and we get

$$\mathcal{P} = -\frac{g^2}{\pi^2}\int_0^\infty dk\; k^2 \tilde{f}\left(k^2/\Lambda^2\right) f\left(k^2/\Lambda^2\right)\left\{\frac{1}{4\sqrt{A}(\sqrt{A}+\sqrt{B})^2} - \frac{(2\sqrt{A}+\sqrt{B})k^2}{12A^{3/2}(\sqrt{A}+\sqrt{B})^2\sqrt{B}}\right\}$$

$$= -\frac{g^2}{\pi^2}\int_0^\infty dz \frac{\sqrt{z}}{2}\tilde{f}\left(\rho^2 z\right) f\left(\rho^2 z\right)\left\{\frac{1}{4\sqrt{z+1}(\sqrt{z+1}+\sqrt{z+R})^2} + \right. \left. - \frac{(2\sqrt{z+1}+\sqrt{z+R})z}{12(z+1)^{3/2}(\sqrt{z+1}+\sqrt{z+R})^2\sqrt{z+R}}\right\}. \tag{3.37}$$

In order to extract the possible IR percolation we consider the limit $\rho \to 0$, with generic mass ratio R, exactly as we did in section 3.1.2. Taking into account the properties of the cutoff functions f and $\tilde{f}$ for vanishing arguments, i.e., $f, \tilde{f} \to 1$, the remaining integral gives

$$\mathcal{P}(\rho \ll 1) = -\frac{g^2}{48\pi^2}, \tag{3.38}$$

which turns out to be independent of the mass ratio R.

The second term in eq. (3.36), i.e. $\mathcal{Q}$, can now be calculated after performing a Wick rotation followed by the integration over k^0 and by the same change of variables as above

$$\begin{aligned}\mathcal{Q} &= -g^2\int\frac{d^3k}{(2\pi)^3} f\tilde{f}'\frac{2(k^i)^2}{\Lambda^2}\int\frac{dk^0}{(2\pi)}\frac{1}{((k^0)^2+A)((k^0)^2+B)} \\ &= -g^2\int\frac{d^3k}{(2\pi)^3} f\tilde{f}'\frac{(k^i)^2}{\Lambda^2}\frac{1}{2}\frac{1}{A\sqrt{B}+B\sqrt{A}} \\ &= -\frac{g^2}{12\pi^2}\int_0^\infty dz\sqrt{z}\rho^2 z f(\rho^2 z)\tilde{f}'(\rho^2 z)\frac{1}{(z+1)\sqrt{z+R}+(z+R)\sqrt{z+1}},\end{aligned} \tag{3.39}$$

where ρ, R and z are given right before eq. (3.37). This expression can be simplified as

$$\mathcal{Q} = -\frac{g^2}{12\pi^2}\int_0^\infty dz\sqrt{z} z f(\rho^2 z)\frac{d}{dz}\left[\tilde{f}(\rho^2 z)\right]\frac{1}{(z+1)\sqrt{z+R}+(z+R)\sqrt{z+1}}. \tag{3.40}$$

In order to proceed further with the calculation of $\mathcal{Q}$ we need to consider below specific choices for the functions f and $\tilde{f}$. Note, however, that having generically $\mathcal{P}, \mathcal{Q} \neq 0$ suggests that the percolation will be unsuppressed, i.e., that $\Delta c \neq 0$ unless a cancellation occurs.

3.2.1 Particles with equal masses ($R = 1$) and same violation ($f = \tilde{f}$)

As a first simplification, we assume that both the masses m_ϕ and m_ψ and the LIV cutoff functions f and $\tilde{f}$ of the fields are equal. Though a priori there is no reason for the latter assumption, one could argue that QG affects both fermionic and bosonic fields in exactly the same way, hence suggesting $f = \tilde{f}$. In this case we can use that $f \cdot f'(z) = 1/2\; d(f^2)/dz$, and an integration by parts of eq. (3.40) yields

$$\mathcal{Q} = \frac{g^2}{12\pi^2}\int_0^\infty dz \frac{1}{2} f^2(\rho^2 z) \frac{d}{dz}\left(\frac{\sqrt{z}z}{2(z+1)^{3/2}}\right), \tag{3.41}$$

where the contribution of the boundary terms stemming from the integration vanishes due to the behaviour of the function f, while the remaining part can be integrated and, in the limit $\rho \to 0$ (which implies $f \to 1$), it gives $g^2/(48\pi^2)$ which is equal and opposite to $\mathcal{P}$ in eq. (3.37) and therefore (see eq. (3.36))

$$\Delta c(\rho \ll 1) = 0. \tag{3.42}$$

Remarkably, in this case, the percolation on the fermion is absent albeit the calculation of section 3.1 shows that this is not the case for the scalar. In order to understand how general this fact is, we consider below the case in which the two particles still carry the same LIV ($f = \tilde{f}$) but have different masses.

3.2.2 Particles with different masses ($R \neq 1$) and same violation ($f = \tilde{f}$)

Under the assumption $f = \tilde{f}$, eq. (3.39) can be integrated by parts, as done above. The associated boundary terms vanish and one is left with

$$\mathcal{Q} = +\frac{g^2}{12\pi^2}\int_0^\infty dz \frac{1}{2} f^2(\rho^2 z) \frac{d}{dz}\left(\frac{\sqrt{z}z}{(z+1)\sqrt{z+R} + (z+R)\sqrt{z+1}}\right). \tag{3.43}$$

In spite of having $R \neq 1$, this integral still gives $g^2/(48\pi^2)$ for $\rho \to 0$ (i.e. $m_\psi \ll \Lambda$) and therefore Δc vanishes as in eq. (3.42), independently of the values of the masses $m_{\phi,\psi}$. This case is more interesting than the previous one $m_\phi = m_\psi$ as it suggests that, with an heavy scalar field $m_\phi \gg m_\psi$, the low-energy physics of the fermion field will not be affected by the LIV unsuppressed percolation on the scalar computed in section 3.1, because no percolation is present on the fermion and indeed the scalar can be integrated out. As such, it would be interesting to check whether this scenario could be extended to the SM and Higgs field.

3.2.3 Violation only on the scalar field ($\tilde{f} = 1$)

Finally, we want to specialise eqs. (3.2) and (3.3) to the case $\tilde{f} = 1$, in which only the scalar propagator carries the LIV. Note that, the corresponding expression for Δc cannot be derived directly from the ones discussed above, as they assume that $\tilde{f}$ vanishes for large values of its argument, which is not the case here. Starting from eq. (3.36) we see that $\mathcal{Q} = 0$ and therefore the only contribution to Δc is due to $\mathcal{P}$ in eq. (3.37) with $\tilde{f} = 1$. Accordingly, the result of the integration in the IR limit is given by

$$\Delta c(\rho \ll 1) = -\frac{g^2}{48\pi^2}. \tag{3.44}$$

This shows that again there is an unsuppressed percolation as in ref. [69] (and in accordance with ref. [219]).

3.3 Separation of scales

Now that we have reviewed and extended the results presented in the literature we are going to show how the introduction of a LI cutoff, in addition to a LIV modified dispersion relation, can hinder the percolation. As anticipated in Sec. 1.3.2, the quest for a mechanism able to prevent the IR percolation of LIV is not new (see, e.g., refs. [140, 160]). In particular, there were proposals based on having supersymmetry as a custodial symmetry. What we are going to show in the following is somehow related to this custodial symmetries protection mechanism. The idea is that, if there is a separation between the EFT validity scale Λ (i.e., the scale of possible new physics beyond the SM) and the LIV scale[8] M, with $\Lambda < M$, then the IR percolation is suppressed by a power of the ratio Λ/M which eventually controls its magnitude. This result, from the EFT perspective, is rather natural because the introduction of a new mass scale Λ gives the possibility to have a (small) dimensionless ratio.

Note that, we are not arguing here that the one discussed below is a protection mechanism which works for the entire Standard Model (SM) nor that there will be room in the SM for such a large scale separation to suppress low-energy LIV in a way which complies with the strong bounds coming from observational data. What we want to emphasise, via a toy-model computation, is that the separation of scales could be one, or part of a, solution to the naturalness problem of LIV and in this way show the validity of some heuristic ideas presented in the literature.

In order to investigate how this separation of scales hinders the percolation, we consider again the model defined by eq. (3.1). In particular we introduce a LI cutoff (as explained below) called Λ that represents the scale of validity of the EFT description and we consider the case in which the LIV is carried only by the scalar field and is encoded in a MDR through the scale M possibly associated with some QG scenario. We will then be interested in the case in which both scales are larger than the masses of the particles in the problem and analyze the effect of the separation of scales on the LIV percolation. To be concrete, we choose some particular forms of the MDR in order to be able to carry out numerically the calculation of the resulting percolation. However, we will argue that the result is largely independent of these choices. Note that, the case under study is physically interesting since the one-loop scalar self-energy does not receive a contribution from the fermion loop as we assume that the fermion field is LI at the tree-level. Accordingly, we have to consider only the fermion self-energy depicted in figure 3.2 .

The calculation of Δc is similar to the one presented in section 3.2, with the difference that we want to introduce here a new LI cutoff Λ that represents the scale of validity of the EFT, i.e., the scale of new, Lorentz-invariant physics. This is done in two different ways: The first is a LI sharp cutoff on the 4-momentum, which does not break LI although it clearly breaks Poincaré invariance. The second is a LI cutoff introduced via a smooth non-local function which can be thought of as deriving from a fundamentally non-local theory in which the non-locality improves the UV behaviour

[8] M can be assumed to coincide with the Planck scale due to the fact that we can expect LIV coming from the scale at which our concept of spacetime as a pseudo-Riemannian manifold is questionable together with the associated symmetries.

of the theory (see, however, section 3.3.2 for additional comments). Noticeably, these non-local features have been suggested in the quantum gravity literature as a possible low-energy signature of the microscopic nature of spacetime, see Chapter 6. In both cases we will conveniently implement the cutoff only after rotation in Euclidean space (see also the discussion in section 3.3.2).

Before considering the presence of two different scales Λ and M let us make a remark on the case with a single scale investigated in the previous sections. Note that the MDR itself can serve as a LIV cutoff at the scale set by M. Indeed, it can be shown (see appendix B.2) that upon increasing M we recover a finite percolation as predicted in ref. [69] but with a value of Δc which depends on the specific form of the MDR used. In particular, considering a MDR for the scalar field of the form given by eq. (3.3) with $\tilde{f} = 1$ and

$$\Delta(|\mathbf{k}|, M) \equiv -|\mathbf{k}|^2 \left(\frac{|\mathbf{k}|^2}{M^2}\right)^n , \tag{3.45}$$

with $n > 0$, which is typically encountered in QG phenomenology, Δc approaches

$$\Delta c(m_{\phi,\psi}/M \ll 1) = -\frac{g^2}{48\pi^2}\frac{n+1}{n}, \tag{3.46}$$

for large values of M compared to the mass scales in the problem, see appendix B.2 and compare with eq. (3.44). From this expression it is clear that the percolation is always larger in modulus (by a numerical factor) than the one found in the absence of MDR but using a LIV cutoff on the spatial momenta as in ref. [69]. Indeed, the present case in which the MDR effectively introduces a LIV regularization in the UV of the spatial part of the integral, is intrinsically different from the one considered in ref. [69] given that there (and in sections 3.1 and 3.2 here) the cutoff function is such that it renders the theory UV finite independently of the order of the radiative corrections considered while the MDR cannot achieve this in general.

3.3.1 Sharp LI cutoff

First, we consider the case of a sharp cutoff Λ on the four momentum. We work in the Euclidean space where this cutoff has the effect of restricting the integration inside a sphere of radius Λ. The computation of Δc follows the same lines as in section 3.2 and, in fact, we arrive at an equation similar to eq. (3.36):

$$\Delta c = 2ig^2 \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{(k^0)^2 + (k^i)^2}{(k^2 - m_\psi^2)^2 \left(k^2 - m_\phi^2 + \Delta(|\mathbf{k}|, M)\right)}, \tag{3.47}$$

where $\mathbf{k} = \left\{k^1, k^2, k^3\right\}$ is the 3-momentum and Δ is given by eq. (3.45). This expression is still in Minkowski space and the subscript Λ in the integral indicates that a sharp cutoff is implemented, i.e., that the domain of integration is restricted as specified further below. Here the LIV is entirely introduced by the MDR of the scalar field and we use M as the LIV scale while Λ as the LI cutoff scale. Now we perform a Wick rotation[9] and use

[9]It is easy to check that the locations of the poles of eq. (3.47) in the complex k^0-plane are such that a rotation of the integration contour is possible.

$4d$ spherical coordinates in order to evaluate the integrals. Accordingly, eq. (3.47) becomes

$$\begin{aligned}\Delta c &= -2g^2 \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{(k^0)^2 - (k^1)^2}{\left(k^2 + m_\psi^2\right)^2 \left(k^2 + \mathbf{k}^2 \left(\frac{\mathbf{k}^2}{M^2}\right)^n + m_\phi^2\right)} \\ &= -\frac{g^2}{2\pi^3} \int_0^\Lambda dk \, \frac{k^5}{\left(m_\psi^2 + k^2\right)^2} \int_0^\pi d\phi \frac{\cos^2\phi - (\sin^2\phi)/3}{\left(k^2 + m_\phi^2 + \dfrac{k^{2+2n}\sin^{2+2n}\phi}{M^{2n}}\right)} \sin^2\phi,\end{aligned} \tag{3.48}$$

where k stand for the modulus of the 4-momentum in the Euclidean space[10]. Note that the sharp cutoff is implemented in the Euclidean space by restricting the integration to Euclidean momenta with modulus $k \leq \Lambda$. In particular after the rescaling $k = \Lambda z$ we find

$$\Delta c = -\frac{g^2}{2\pi^3} \int_0^1 dz \, \frac{z^5}{(z^2 + R_\psi)^2} \int_0^\pi d\phi \, \sin^2\phi \frac{\cos^2\phi - \left(\sin^2\phi\right)/3}{z^2 + R_\phi + \lambda^n z^{2+2n} \sin^{2+2n}\phi}, \tag{3.49}$$

where $\lambda \equiv \Lambda^2/M^2$ and $R_{\phi,\psi} = m_{\phi,\psi}^2/\Lambda^2$. These variables are particularly convenient for studying the large-scale separation regime $m_{\phi,\psi} \ll \Lambda \ll M$ which simply amounts at imposing $R_{\phi,\psi} \ll 1$ and $\lambda \ll 1$. In this regime the denominator in eq. (3.49) can be expanded around $\lambda = 0$

$$\frac{1}{z^2 + R_\phi + \lambda^n z^{2+2n} \sin^{2+2n}\phi} = \frac{1}{z^2 + R_\phi}\left[1 - \frac{\lambda^n z^{2+2n}\sin^{2+2n}\phi}{z^2 + R_\phi} + \mathcal{O}(\lambda^{2n})\right]. \tag{3.50}$$

Plugging this expansion in eq. (3.49), performing the angular integration and noting that the latter vanishes at the zeroth order we have

$$\Delta c = -\frac{g^2}{2\pi^3}\lambda^n \frac{(1+n)\sqrt{\pi}\Gamma\left(n + 5/2\right)}{3\Gamma(4+n)} \int_0^1 \frac{dy}{2} \frac{y^{3+n}}{(y + R_\psi)^2 (y + R_\phi)^2} + \mathcal{O}(\lambda^{2n}), \tag{3.51}$$

where we introduced $y = z^2$, see box below for details on the y-integration.

This equation clearly shows that the violation $\Delta c \propto (\Lambda/M)^{2n}$ is suppressed whenever $M \gg \Lambda$. The actual degree of suppression depends on the separation between the LIV scale M (which can be identified with the Planck scale) and the scale Λ of the LI new physics as well as on the specific form of the MDR, i.e., on n. In particular, the suppression increases upon increasing the mass dimension of the LIV operators responsible for the MDR as the algebraic dependence on the small ratio Λ/M has the same power as the one with which M appears in the MDR. This means that, as expected, the percolation is weaker for LIV coming from mass dimension 8 operators compared to the one due to mass dimension 5 operators. Moreover, eq. (3.51) clearly shows that Δc (up to first order in λ^n) is symmetric with respect to the exchange of R_ψ and R_ϕ; the limit in which both fields are massless is finite and the dependence of Δc on $R_{\psi,\phi}$ is rather weak at least as long as $m_{\psi,\phi} \ll \Lambda$, M.

[10] In the numerator of the integrand of this equation, $(k^0)^2 - (k^1)^2$ corresponds to considering $i = 1$ in eq. (3.30); however, the result is clearly independent of the choice of $i \in \{1, 2, 3\}$.

y-integration in eq. (3.51)

We need to compute

$$Q_n(R_\phi, R_\psi) \equiv \int_0^1 \frac{dy}{2} \frac{y^{3+n}}{(y+R_\psi)^2(y+R_\phi)^2}, \tag{3.52}$$

where $R_{\psi,\phi} \geq 0$. Note that Q_n can be calculated as

$$Q_n(R_\psi, R_\phi) = \frac{\partial^2}{\partial R_\psi \partial R_\phi} \frac{S_n(R_\psi) - S_n(R_\phi)}{R_\phi - R_\psi} \tag{3.53}$$

where

$$S_n(R_{\phi,\psi}) \equiv \int_0^1 \frac{dy}{2} \frac{y^{3+n}}{y + R_{\phi,\psi}}; \tag{3.54}$$

the remaining integral S_n is simpler and can be expressed in terms of hypergeometric ${}_2F_1$ functions. However, for convenience, we omit the lengthy explicit expression of the resulting $Q_n(R_\psi, R_\phi)$, which can straightforwardly determined as explained above. Here we only note that when both fields are massless, i.e. $R_{\psi,\phi} = 0$ its expression is particularly simple: $Q_n(0,0) = 1/(2n)$.

Finally, we analyzed numerically also the regime $M \ll \Lambda$, i.e. $\lambda \gg 1$. Note that in order to do so the change of variable $k = Mz$ is more convenient instead of the one done right before eq. (3.49). The finite unsuppressed percolation in this case approaches the value we already found in the case without the LI cutoff, i.e., eq. (3.46) (see the discussion in appendix B.2). The numerical results for $n = 1/2,\ 1$ and 2 are shown in figure 3.3, in which we assume $R_\psi = R_\phi = 10^{-12}$ (i.e., equal masses).

3.3.2 Smooth LI cutoff

We close this section by considering an alternative way to introduce a LI cutoff based on non-local theories. By non-local we mean here that the kinetic term of one of the fields contains also a pseudo-differential operator of infinite order, i.e., an infinite number of spacetime derivatives acting on the field. This kind of theories are potentially unstable [85, 233], but the Ostrogradski theorem does not apply straightforwardly. Hence they might be stable and therefore we assume this to be the case here; moreover various works argue that they can have a better behaviour in the UV with respect to standard QFTs, while preserving the low-energy limit (see refs. [130, 135] and also refs. [157, 217] and references therein). Moreover, as discussed in Sec. 1.4.2, some form of non-locality seems to be a common feature to different approaches to quantum gravity, although each theory has its own peculiarity. More importantly, non-local theories can be Poincaré invariant while introducing a UV cutoff — which can account for spacetime discreteness — which is what we are seeking here.

The possibility of such a LI regulator for QFT is also considered in ref. [68] as a way out from the apparent unavoidable link between granularity of spacetime and LIV. However, in that work it was claimed that these theories suffer from causality violations and, as such, they are not viable. Actually, the very same refs. [128, 130] discussed in ref. [68] not only show that there

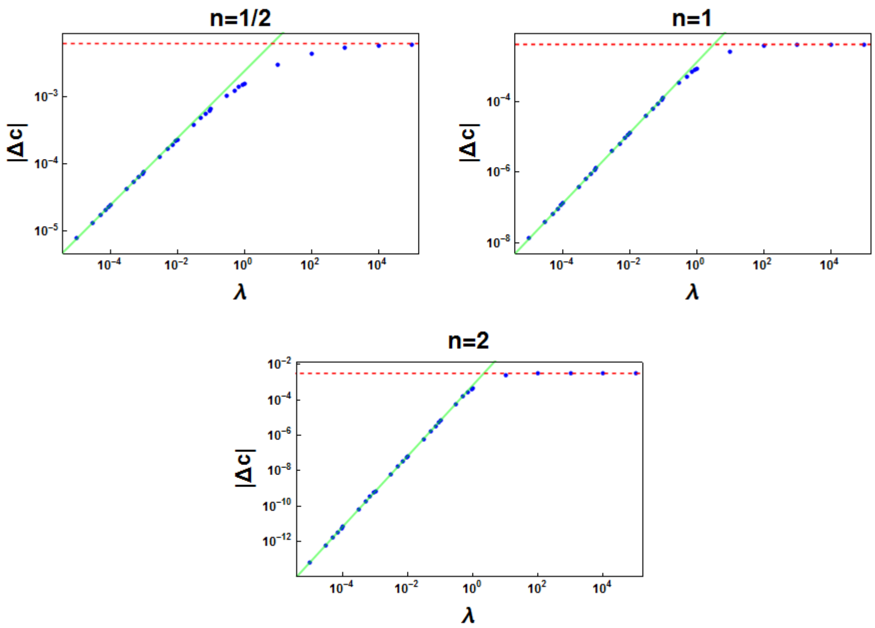

FIGURE 3.3: Dependence of $|\Delta c|$ on $\lambda \equiv (\Lambda/M)^2$ in the case of a sharp cutoff (see eq. (3.49)) with $n = 1/2, 1, 2$, $R_\phi = R_\psi = 10^{-12}$ and $g = 1$. Symbols correspond to numerical data while the solid lines correspond to eq. (3.51) which well describes the violation at small values of λ. The dashed lines correspond to eq. (3.46), i.e., to the values approached by $|\Delta c|$ for $\lambda \gg 1$. The violation is suppressed whenever $M \gg \Lambda$, i.e., $\lambda \ll 1$, which is exactly the separation of scales invoked various times in the literature. This behaviour agrees with what is heuristically expected on the basis of the physical intuition discussed in the main text.

are causality violations but also that they arise only at high energies (comparable with the non-locality scale), where the theory, if interpreted as an effective one, should anyhow cease to be valid. In this sense we shall take this smooth cutoff as not being fundamental but simply assume it to be a consequence of an EFT type description of a more fundamental theory. It is easy to see that the introduction of a smooth cutoff function f in Euclidean space[11] is tantamount to replacing eq. (3.48) by

$$\Delta c = -\frac{g^2}{2\pi^3}\int_0^\infty dk\,\frac{k^5 f\left(-k^2/\Lambda^2\right)}{\left(m_\psi^2+k^2\right)^2}\int_0^\pi d\phi\frac{\cos^2\phi-\left(\sin^2\phi\right)/3}{\left(k^2+m_\phi^2+\frac{k^{2+2n}\sin^{2+2n}\phi}{M^{2n}}\right)}\sin^2\phi, \tag{3.55}$$

Note that contrary to sections 3.1 and 3.2 here f depends on both the timelike and spacelike part of the 4-momentum in a LI way[12]

The behaviour of Δc for large separation of scale $M \gg \Lambda$ can be obtained again using the expansion in eq. (3.50), the only difference with respect to eq. (3.51) being in the integration on y which now runs up to infinity and is regulated by the cutoff function, i.e.

$$\int_0^1 \frac{dy}{2}\frac{y^{3+n}}{(y+R_\psi)^2(y+R_\phi)^2} \longrightarrow \int_0^\infty \frac{dy}{2}\frac{y^{3+n}f(-y)}{(y+R_\psi)^2(y+R_\phi)^2}. \tag{3.56}$$

We have calculated numerically eq. (3.55) after the suitable rescaling of the coordinates already introduced after eq. (3.49). The results for $n = 1/2, 1, 2$ and the cutoff function $f(-x) = e^{-x^2}$ confirm those of the previous paragraph and are reported in figure 3.4. The violation is again suppressed as $\Delta c \propto (\Lambda/M)^{2n}$, while the asymptotic values for $\Lambda \gg M$ agree with eq. (3.46) as demonstrated in appendix B.2.

3.4 Dissipation and LIV naturalness

In the previous section we focussed on the standard picture in which the dispersion relation (in our case of the scalar field) is modified by some effects of LIV physics motivated by QG scenarios. However, general arguments [173] show that if LIV emerges dynamically from UV kinetic terms or interactions with heavy fields which are traced out in the low-energy description, then dissipative effects will also unavoidably arise. In this section we therefore focus on their influence on the percolation of LIV. Note that dissipation does not necessarily spoil unitarity, since it can emerge from bona-fide Hamiltonian models after tracing out degrees of freedom. Following these lines, ref. [142] investigated the phenomenology of dissipative effects in fields propagation, showing that strong constraints can be cast on the lower-order transport coefficients while higher-order ones are basically unconstrained. A possible percolation of higher-order dissipative terms can hence be considered an opportunity for strengthening current constraints and a motivation for our investigation. A viable model will

[11] Actually, the smooth cutoff has to be enforced after the rotation in the Euclidean space since the loop integrals make sense only there, see the discussion in refs. [87, 193]

[12] See ref. [211] for a case in which a cutoff function dependent on both energy and momenta is introduced, even if in a way which still violates LI.

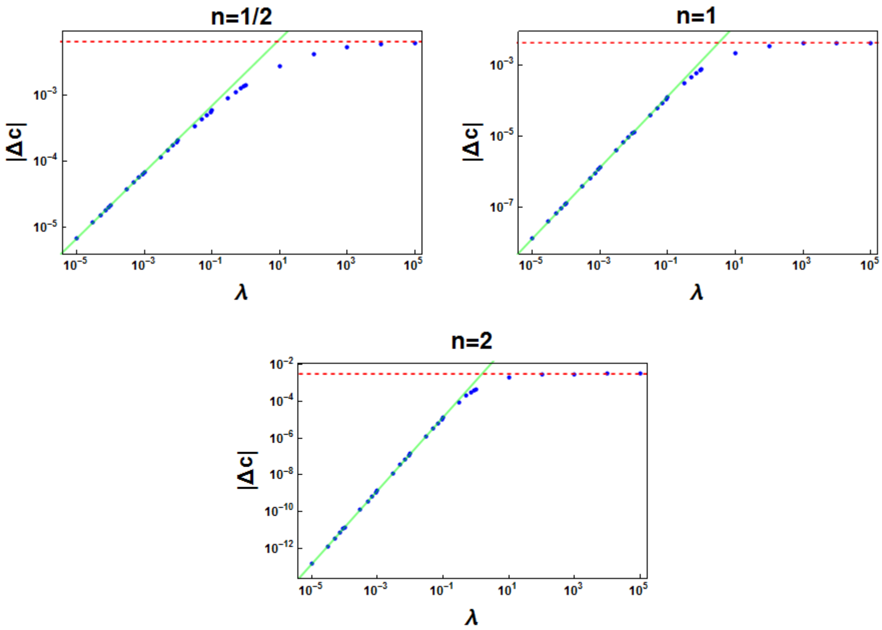

FIGURE 3.4: Dependence of $|\Delta c|$ on $\lambda \equiv (\Lambda/M)^2$ (see eq. (3.55)) for the same values of parameters as in figure 3.3, i.e. $n = 1/2, 1, 2$, $R_\phi = R_\psi = 10^{-12}$ and $g = 1$ but with a LI Gaussian cutoff function. Symbols correspond to numerical data while the solid lines correspond to the theoretical prediction $\Delta c \propto (\Lambda/M)^{2n}$ coming from eq. (3.51) after the replacement in eq. (3.56). The dashed lines correspond to eq. (3.46), i.e. to the values approached by $|\Delta c|$ for $\lambda \gg 1$. The qualitative features of these curves are the same as in figure 3.3.

anyhow require dissipation not to percolate strongly given the currently available constraints. In this sense we shall explore also in this case the effectiveness of a protection mechanism based on a separation of scales.

In order to study the percolation of LIV in the presence of dissipation in the scalar field, we first address the validity of the argument of ref. [69] when there is only one relevant cutoff scale in the problem. Then we show how the picture changes when we introduce one additional scale, having the same physical interpretation as the one discussed in section 3.3. We discuss this issue both with a sharp and a smooth LI cutoff (in the same fashion as in section 3.3).

3.4.1 The general setting

For concreteness, we consider again the model in eq. (3.1) in which dissipation affects only the scalar field. Following ref. [173], its propagator is given by

$$G_F(\omega, k) = \frac{i}{\omega^2 - k^2 - m_\phi^2 + i|\omega|k^{2+2n}/M^{1+2n}}, \tag{3.57}$$

where ω indicates the time-like component of the momentum and k the modulus of the space-like components. The parameter $n \geq 0$ is introduced for later convenience and M is a mass scale. This expression can be derived from dynamical models in which heavy degrees of freedom have been traced out (see, e.g., sections 2.3 and 2.4 of ref. [173]). Note, in addition, that the dispersion relation associated with this propagator is reminiscent of the one of dissipative fluids as reported, e.g., in ref. [142]. In particular, by varying the value of n we can change the first dissipative (and dispersive) term which appears in the dispersion relation (in the same way as in ref. [142]). In analogy with the dispersive case discussed in section 3.3, we expect Δc to depend algebraically on Λ/M to some power which increases upon increasing n.

The calculation of Δc follows the same lines as in section 3.2, leading to an expression similar to eq. (3.36):

$$\begin{aligned}\Delta c =& ig^2 8\pi \int \frac{dk}{(2\pi)^3} k^2 \\ & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2 + k^2/3}{(\omega^2 - k^2 - m_\psi^2 + i\epsilon)^2 \left(\omega^2 - k^2 - m_\phi^2 + i\epsilon + i|\omega|k^{2+2n}/M^{1+2n}\right)},\end{aligned} \tag{3.58}$$

where, taking into account the rotational symmetry, we have done the angular integration in the three spatial directions. Note that this expression refers still to Minkowski space while we do not specify, for the moment, the cutoff function that we are using. Due to the form of the dissipative term in the denominator it is not straightforward to perform a Wick rotation towards the Euclidean space. However, it is possible to show (both numerically and analytically) that Δc given above actually takes real values and the integral can be performed in Euclidean space. In order to do this, we can split the ω integration into two parts and use the symmetry of the argument for $\omega \to -\omega$ to write

$$\Delta c = 2ig^2 8\pi \int \frac{dk}{(2\pi)^3} k^2 \tag{3.59}$$
$$\int_0^\infty \frac{d\omega}{2\pi} \frac{\omega^2 + k^2/3}{(\omega^2 - k^2 - m_\psi^2 + i\epsilon)^2 \left(\omega^2 - k^2 - m_\phi^2 + i\epsilon + i\omega k^{2+2n}/M^{1+2n}\right)}.$$

Now it is clear that there are no poles in the first quadrant of the complex ω-plane. Accordingly, the integral done along the path which includes the positive real ω-axis up to a certain arbitrarily large value Ω, the quarter of circumference of radius Ω centered in the origin O of the plane and contained in its first quadrant and finally the path from $i\Omega$ to O along the imaginary axis, gives zero. Noting that the integral along the quarter of circle vanishes as $\Omega \to \infty$, we conclude that the integral along the real axis equals the one along the imaginary axis, i.e., its Wick rotation. As a result (see appendix B.3 for more details).

$$\Delta c = -g^2 16\pi \int \frac{dk}{(2\pi)^3} k^2 \tag{3.60}$$
$$\int_0^\infty \frac{d\omega}{2\pi} \frac{\omega^2 - k^2/3}{(+\omega^2 + k^2 + m_\psi^2)^2 \left(\omega^2 + k^2 + m_\phi^2 + \omega k^{2+2n}/M^{1+2n}\right)}$$
$$= -\frac{g^2}{\pi^3} \int d\rho \frac{\rho^5}{\left(\rho^2 + m_\psi^2\right)^2} \int_0^{\frac{\pi}{2}} d\phi \frac{\sin^2\phi \left[\cos^2\phi - \left(\sin^2\phi\right)/3\right]}{\rho^2 + m_\phi^2 + \frac{\rho^{3+2n}\sin^{2+2n}\phi\cos\phi}{M^{1+2n}}},$$

where in the last line we have used polar coordinates $\omega = \rho\cos\phi$, $k = \rho\sin\phi$ where ρ is the modulus of the (Euclidean) 4-momentum. Note that the integration in ϕ is now from 0 to $\pi/2$ ensuring that we are integrating only over the half line with $\omega > 0$. In the following sections we specify the form of the LI cutoff which is implicit in eq. (3.59) and which introduces a second scale into the problem.

Before proceeding, we note that the dissipative dispersion relation considered above can actually be seen as an effective LIV regulator for the integral analogous to the ones we considered in section 3.3; accordingly we could investigate the limit $M \to \infty$ in the previous equations and study the resulting percolation of LIV without introducing the new LI scale Λ. In accordance with ref. [69] we expect a non-vanishing value of Δc. In this case, it can be shown that the actual value of Δc, in this limit, depends on the value of n in the MDR (see appendix B.2) similarly to what was observed in section 3.3 for the dispersive case, and is given by

$$\Delta c = -\frac{g^2}{48\pi^2} \frac{n - 1/2}{n + 1/2}. \tag{3.61}$$

Note, in addition, that Δc vanishes (up to this order) for $n = 1/2$.

3.4.2 Sharp LI cutoff

After Wick rotating into Euclidean space, we consider a LI cutoff such as the one discussed in section 3.3.1. Accordingly, eq. (3.60) becomes

$$\Delta c = -\frac{g^2}{\pi^3}\int_0^{\Lambda} d\rho \frac{\rho^5}{\left(\rho^2+m_\psi^2\right)^2}\int_0^{\frac{\pi}{2}} d\phi \frac{\sin^2\phi\left[\cos^2\phi-\left(\sin^2\phi\right)/3\right]}{\rho^2+m_\phi^2+\frac{\rho^{3+2n}\sin^{2+2n}\phi\cos\phi}{M^{1+2n}}} \tag{3.62}$$
$$= -\frac{g^2}{\pi^3}\int_0^{1} dz \frac{z^5}{\left(z^2+R_\psi\right)^2}\int_0^{\frac{\pi}{2}} d\phi \frac{\sin^2\phi\left[\cos^2\phi-\left(\sin^2\phi\right)/3\right]}{z^2+R_\phi+z^{3+2n}\lambda^{n+1/2}\sin^{2+2n}\phi\cos\phi},$$

where in the second line we have done the change of variables introduced in eq. (3.49), i.e., $\rho = \Lambda z$ in order to have dimensionless variables, while we introduced $\lambda = \Lambda^2/M^2$ and $R_{\phi,\psi} = m^2_{\phi,\psi}/\Lambda^2$ as we did in section 3.3.

As in section 3.3.1 the behaviour of Δc for $\lambda \ll 1$ can be obtained by expanding the integrand in eq. (3.62) around $\lambda = 0$. In this case, after plugging this expansion back in eq. (3.62) and using that the angular integration of the zeroth-order term vanishes we have

$$\Delta c = -\frac{g^2}{\pi^3}\frac{2n-1}{3(2n+7)(2n+5)}\lambda^{n+1/2}\int_0^1 \frac{dy}{2}\frac{y^{n+7/2}}{(y+R_\psi)^2(y+R_\phi)^2} + \mathcal{O}(\lambda^{1+2n}), \tag{3.63}$$

where $y = z^2$. Analogously to what we observed after eq. (3.61), Δc vanishes (actually at all orders in λ) for $n = 1/2$.

y-integration in eq. (3.63)

We would like to calculate the following integral

$$\int_0^1 \frac{dy}{2}\frac{y^{n+7/2}}{(y+R_\psi)^2(y+R_\phi)^2}, \tag{3.64}$$

which actually corresponds to $Q_{n+1/2}(R_\phi, R_\psi)$ defined in eq. (3.52). Accordingly, it can also be expressed in terms of $S_{n+1/2}(R_{\phi,\psi})$ as in eq. (3.53) and, in turn, in terms of hypergeometric ${}_2F_1$ functions. For convenience, we do not report its lengthy expression here but only quote the value eq. (3.64) takes in the massless case $R_{\phi,\psi} = 0$, i.e., $Q_{n+1/2}(0,0) = 1/(2n+1)$.

The percolation $\Delta c \propto (\Lambda/M)^{n+1/2}$ in eq. (3.63) decreases upon decreasing Λ/M analogously to what happens in the pure dispersion case discussed in section 3.3; the small ratio Λ/M controls the behaviour of Δc for $\Lambda \ll M$ with the same algebraic power as the one of the LIV scale M in the MDR considered, see eq. (3.57). This demonstrates that, also in the presence of dissipation, the percolation can be tamed by a large separation of scales protecting in this way low-energy physics. Moreover, as in section 3.3, eq. (3.63) clearly shows that Δc (up to first order in λ^n) is symmetric with respect to the exchange of R_ψ and R_ϕ; the limit in which both fields are massless is finite while the dependence of Δc on R_ψ, R_ϕ is rather weak as long as $m_\psi, m_\phi \ll \Lambda, M$. Finally, we note that the percolation of dissipation is purely real and as such is qualitatively similar to that induced by dispersion.

3.4.3 Smooth LI cutoff

In this section we consider the case of a smooth LI cutoff, instead of the sharp one investigated in the previous subsection, adopting the same approach as in the dispersive case discussed in section 3.3. The expression of interest in this case is

$$\Delta c = -\frac{g^2}{\pi^3}\int_0^\infty d\rho\, e^{-\rho^2/\Lambda^2}\frac{\rho^5}{\left(\rho^2+m_\psi^2\right)^2}\int_0^{\frac{\pi}{2}} d\phi \frac{\sin^2\phi\left(\cos^2\phi-\sin^2\phi/3\right)}{\rho^2+m_\phi^2+\frac{\rho^{3+2n}\sin^{2+2n}\phi\cos\phi}{M^{1+2n}}}$$

$$= -\frac{g^2}{\pi^3}\int_0^\infty dz\, e^{-z^2}\frac{z^5}{\left(z^2+R_\psi\right)^2}\int_0^{\frac{\pi}{2}} d\phi \frac{\sin^2\phi\left(\cos^2\phi-\sin^2\phi/3\right)}{z^2+R_\phi+z^{3+2n}\lambda^{\frac{1+2n}{2}}\sin^{2+2n}\phi\cos\phi}, \quad (3.65)$$

where the smooth cutoff is introduced in the Euclidean space via the Gaussian function with parameter Λ, see also eq. (3.55). On the second line we introduced the same parameters as in eq. (3.55). Repeating the analysis done before we conclude that also in this case $\Delta c \propto (\Lambda/M)^{n+1/2}$ and therefore Δc is suppressed with a wide separation of scales $\Lambda \ll M$. The only difference compared to eq. (3.63) is in the radial integration in y which runs up to ∞ after the introduction of the cutoff function, as in eq. (3.56). We have computed numerically the integral eq. (3.65) as a function of $\lambda = \Lambda/M$ for various values of n. The result for Δc with $R_{\phi,\psi} = 10^{-12}$ is shown in figure 3.5 together with its asymptotic behaviours. As in the dispersive case, we have confirmed that the value approached by Δc for large λ, i.e. for $\Lambda \gg M$ with fixed $R_{\phi,\psi}$ is given by eq. (3.61) both for sharp and smooth cutoffs, in agreement with the values approached when only the MDR is present as a regulator and with the analytical argument presented in appendix B.2. This agrees with the result of ref. [69] of unsuppressed LIV percolation.

3.5 Summary and Discussion

In this Chapter we have revisited the problem of the IR percolation of Lorentz invariance violations due to radiative corrections. As previously discussed, this problem is crucial for phenomenological models that want to deal with LIV, even if it does not detract anything from experimental efforts to constraint LIV.

We have considered a model consisting of a scalar and a fermionic field coupled by a Yukawa interaction, which was already used in the literature for studying the naturalness problem of LIV and which provides an essential version of the structure of the scalar sector of the standard model of particle physics.

In sections 3.1 and 3.2, we discussed the instance in which only the UV Lorentz breaking scale is present in the problem in addition to the mass scales of the particles. In this case one can model the LIV by introducing a Lorentz-breaking cutoff which eliminates large momenta in a certain preferred reference frame and this is the way the problem was tackled in previous studies [219]. We revisited the original calculation of ref. [69], making explicit various steps as well as considering the percolation of LIV on

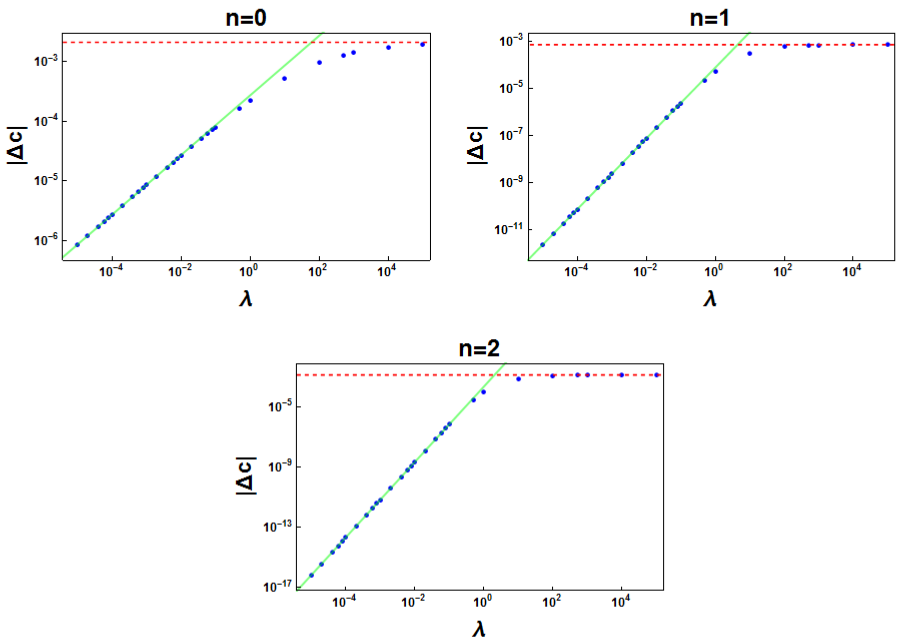

FIGURE 3.5: Dependence of $|\Delta c|$ on $\lambda \equiv (\Lambda/M)^2$ in the presence of dissipation with a smooth Gaussian cutoff (see eq. (3.65)) with $n = 0, 1, 2$ and the same values of the parameters as in figures 3.3 and 3.4, i.e. $R_\phi = R_\psi = 10^{-12}$ and $g = 1$. Symbols correspond to numerical data while the solid lines correspond to the theoretical prediction $\Delta c \propto (\Lambda/M)^{n+1/2}$ coming from expanding in λ for $\lambda \ll 1$. The dashed lines correspond to eq. (3.61), i.e. to the values approached by Δc for $\lambda \gg 1$. Note that, these values depend on n while the violation is suppressed for $M \gg \Lambda$..

the fermion field. In this case, our results agree with ref. [219], though the present analysis is more general. Indeed, we make no assumption concerning the masses of the fields and we consider also a cutoff for both the fermionic and scalar fields. The results reported in section 3.2.2 show a possible cancellation of the LIV percolation even in rather generic situations. Indeed, if the cutoff function is the same for both these fields — conceivable if (quantum) gravity affects all species in the same way —, the percolation on the fermion is completely suppressed, at least up to one loop in perturbation theory. Clearly, unsuppressed percolation are anyhow present on the scalar due to fermionic loops. In this respect, the case with unequal scalar and fermion masses m_ϕ and m_ψ, respectively, and in particular with $m_\phi \gg m_\psi$ is the most physically interesting. Indeed, in an EFT approach, the heavy scalar has to be traced out for describing low-energy physics which, in this way, becomes unaffected by the unsuppressed percolation on the scalar field, at least at this order in perturbation theory.

In sections 3.3 and 3.4 we have considered the case in which a scale M of LIV and a scale Λ are present, where Λ is a LI cutoff representing the scale of validity of the EFT. In particular, we assume that between such a Planck scale M (setting the Lorentz breaking) and the low-energy physics there exist some extra (per se) Lorentz invariant physics. The LI cutoff Λ has been introduced via both a sharp and a smooth cutoff function in Euclidean momentum space. The second choice being motivated by non-local theories which aim to improve the UV behavior of QFTs. We have shown that if these two scales are well separated, i.e., $\Lambda \ll M$, the percolation is suppressed. While this result could have been expected on mathematical and physical grounds, we determine here the scaling behaviour of the percolation for various MDR: In particular in section 3.3 we consider the case in which $\Lambda/M \gg 1$ with various modified dispersion relations Δ given by eq. (3.45). Heuristically this case is expected to be equivalent to the one investigated in refs. [69, 219] due to the fact that the MDRs themselves act as cutoffs. Indeed, it can be seen from eq. (3.46) that we find an unsuppressed percolation but the value of the corresponding Δc (up to one loop) depends on the detail of the MDR, a fact that was not noticed in previous discussions. Secondly, we consider the case of $\Lambda/M \ll 1$ and show that the percolation depends linearly on Λ/M to the power with which M appears in the MDR, see figures 3.3 and 3.4, a fact which is quite interesting from the phenomenological point of view. Indeed, most available models of quantum gravity seem to preserve CPT symmetry: this means that CPT odd operators appearing in the MDR are disfavored whereas CPT even mass dimension 5 or 6 operators can be shown to give negligible contributions or terms of the type p^4/M^2 [140]. In this case, we have shown that one could expect a possibly strong suppression of the form Λ^2/M^2. If finally M is identified with the Planck mass and we consider a constraint on the LIV dimensionless parameter of the order of 10^{-18} (from constraints on the neutrino-electron sector of the SM, see ref. [59]) we see that the EFT scale Λ has to be simply less than 10^7 TeV for evading current constraints.

Last but not least, we have considered also the instance in which the scalar field is affected by dissipation. This case has not been treated extensively in the literature in spite of the fact that interesting phenomenology can be extracted by allowing dissipative MDR (see refs. [142, 173]) and that from the theoretical point of view dissipation must be present if LIV arises dynamically [173]. In section 3.4 (see eq. (3.60) and the discussion thereof) we found the unexpected result that no percolation of these dissipative effects occurs, i.e., that no imaginary contribution emerges as a consequence of the LIV percolation, neither in Δc of the fermion nor in its mass, as the one-loop correction to it turns out to be real (see appendix B.3). Accordingly, the mass correction does not imply a stringent constraint on the LIV percolation. Moreover, we also show that a separation of scales suppresses again the LIV percolation in the same fashion as it does in the purely dispersive case of section 3.3, see figure 3.5.

While the idea that separation of scales can prevent unsuppressed LIV is not entirely new, the present results highlights how such a mechanism works while extending it to the interesting case of field theories with effective dissipation, largely overlooked in literature.

Chapter 4

Causal Set theory and Non-locality

Well, I never heard it before, but it sounds uncommon nonsense.

Lewis Carroll
Alice in Wonderland

Recent years have witnessed growing activity in the field of quantum gravity phenomenology [11, 116, 140, 150], with unforeseeable success having been achieved in testing possible breakdowns of spacetime symmetries close to the Planck scale. In particular, as we discussed both in the introduction and in the previous Chapter, tests of EFT with ultraviolet departures from Local Lorentz Invariance have been proven invaluable in providing severe constraints on such scenarios [140]. Nonetheless, the very same successes of these studies — with the tight constraints imposed on LIV — together with the LIV naturalness problem, demand to take in serious consideration the possibility that Lorentz symmetry might be an exact symmetry of Nature.

Causal Set theory, introduced in Sec. 1.4, is the perfect candidate for a QG phenomenology investigation both for its simplicity as well as for its, almost unique, characteristic of being a discrete but Lorentz invariant theory of emergent spacetime. In particular, in order to marry fundamental discreteness with (local) Lorentz invariance the causal set is subject to a kinematic randomness — made concrete by the concept of *sprinkling*, see Sec. 1.4 — which forces us to give up locality at the fundamental level, thus opening a mostly unexplored window for quantum gravity phenomenology. This inherent non-locality is evident in the definition of d'Alembert's operator on a causal set, defined by constructing a finite difference equation in which linear combinations of the value of the field at neighbouring points are taken. Since the number of nearest neighbours, next nearest neighbours, etc., is very large (infinite), the corresponding expression looks highly non-local. Nonetheless, the resulting operator can be shown to be approximately local, with the non-locality confined to scales of order the discreteness scale.

As briefly discussed in Sec. 1.4.2, the precise form of this correspondence is given by performing an average of the causal set d'Alembertian over all sprinklings of a given spacetime, giving rise to a non-local, retarded, Lorentz invariant linear operator in the continuum, $\Box_{nl}$, whose non-locality is parametrised by the discreteness scale l, and is such that it reduces to the local continuum d'Alembertian, $\Box$, in the limit $l \to 0$. Hence,

$\Box_{nl}$ encodes an averaged effect of the underlying spacetime discreteness on the propagation of scalar fields.

It was soon realised however [203], that although the mean of this operator has the correct continuum limit, its discrete counterpart suffers from large/unacceptable fluctuations — growing with $1/l^d$, i.e., with the inverse discreteness scale to the power of spacetime dimensions. This was solved by introducing a new length scale[1], $l_n \gg l$, over which the original discrete expressions were "smeared out". When averaged over sprinklings these operators lead to non-local d'Alembertians where the non-locality is now confined to scales of order l_n rather than l. As such, nonlocal corrections to the exact continuum d'Alembertian are expected to still be relevant at scales where the continuum description of the spacetime is already valid to a good approximation. Again, locality is restored in the limit $l_n \to 0$. Note that at this stage the length scale l_n enters as a purely phenomenological parameter, introduced in order to make the discrete propagation physically meaningful, amenable to phenomenological constraints.

The focus of this Chapter is to study the non-local field theory for a scalar field whose dynamics are given by the non-local d'Alembertian for which we take $l_n \neq 0$. In particular, we firstly review properties of the continuum non-local d'Alembertian derived from its CS, discrete counterpart. Then, in Section 4.2 we extend the definition of the non-local d'Alembertian to include massive fields and in Section 4.3 we study the retarded Green functions found in [17] in two and four dimensions. Interestingly enough, we show that the theory presents violations of the Huygens' principle, which are relevant for future phenomenological studies[2]. In Section 4.4 we construct the quantum field theory for a scalar field based on the aforementioned non-local d'Alembertian in two and four dimensions. Finally, in Section 4.5 we discuss the physical implications of our study and future perspectives about the phenomenology apt to constraint them. The mesoscopic regime where non-local effects start to play an important role are those of major interest and require a proper phenomenological study which we will address in the last Chapter of this thesis.

4.1 Nonlocal d'Alembertians

The first construction of a d'Alembertian operator on a causal set appeared in a seminal paper by Sorkin [203]. This was later extended to 4 dimensions and curved spacetimes, [44], and subsequently to all other dimensions [80, 102] and with an arbitrary number of layers[3] [17]. All such operators have continuum counterparts, $\Box_{nl}$, obtained by averaging the discrete operators over all sprinklings of a given spacetime. As anticipated above, the non-locality of the continuum operators is parametrised by a length scale l_n, taken to be much larger than the discreteness scale l in order to damp fluctuations (see box below). This new mesoscopic scale could provide interesting phenomenology, since the non-locality of the d'Alembertian would survive at scales in which the continuum description of spacetime is approximately

[1] In eq. (1.16) of Sec. 1.4.2 we considered directly this new length scale.

[2] See [52, 133] for recent results concerning these violations and their relevance in cosmology in order to discriminate between different QG models.

[3] See discussion after eq. (4.1).

valid, hence the continuum non-local d'Alembertian would describe the dynamics of a scalar field more accurately. Only at much smaller scales, of order l, would the discrete description then be necessary.

Given a causal set $\mathcal{C}$ and a scalar field $\phi : \mathcal{C} \to \mathbb{R}$ on it, let us consider the operators defined by[4]

$$(B^{(d)}\phi)(x) = l^{-2} \left(a\,\phi(x) + \sum_{n=0}^{L_{max}} b_n \sum_{y \in I_n(x)} \phi(y) \right), \tag{4.1}$$

where d is the spacetime dimension, a, b_n are dimension dependent coefficients and l is the discreteness scale. $I_n(x)$ represents the set of past n-th neighbours of x, where a point y is said to be an n-th past neighbour of x if the cardinality of the set $Int(x, y) = \{z \in \mathcal{C} : x \prec z \prec y\}$ is equal to n. In the literature the first sum in eq. (4.1) is referred to as sum over *layers*, where each I_n is a layer. The operators in eq. (4.1) are derived under the following physical assumptions [17]

1. **Linearity**: the result of the action of the operator on a scalar field at an element of the causal set should be a linear combination of the values of the fields at other elements
2. **Retardedness**: the result of the action of the operator on a scalar field at an element of the causal set should depend on the values of the field in the past of that element
3. **Label invariance**: the operator should be invariant under the relabellings of the causal set elements
4. **Neighbourly democracy**: all n-th neighbours of x should contribute to $(B_\rho^{(d)}\phi)(x)$ with the same coupling

These operators are well defined for a general causal set $\mathcal{C}$ but are particularly relevant for causal sets that well approximate continuum spacetimes. Indeed, given a generic spacetime $(\mathcal{M}, g)$, the average of the discrete operators over all Poisson sprinklings of $\mathcal{M}$ leads to a continuum operator given by[5]

$$\begin{aligned}\Box_{nl}^{(d)}\phi(x) \equiv \mathbb{E}(B_\rho^{(d)}\phi)(x) &= \rho^{2/d} a\,\phi(x) \\ &+ \rho^{(2+d)/d} \sum_{n=0}^{L_{max}^{(d)}} \frac{b_n}{n!} \int_{J^-(x)} \sqrt{-g} e^{-\rho V(x,y)} [\rho V(x,y)]^n \phi(y) d^d y,\end{aligned} \tag{4.2}$$

where $\mathbb{E}$ stands for average over sprinklings, $L_{max}^{(d)}$ is a dimension dependent positive integer, $J^-(x)$ is the causal past of x and $V(x, y)$ is the spacetime volume of the causal interval between x and y. Here ρ should corresponds to $1/l^d$, however we will consider it related to the non-locality scale, i.e., $\rho \equiv 1/l_n^d$, see box below.

[4] Here we follow the notation of [17].
[5] For further details see [212] and references therein.

Fluctuations and the non-locality scale

Note that, as discussed in the introduction to this Chapter, even if the averaged operators converge to the local ones when l goes to zero, the fluctuations actually tend to increase. To tame this problem, the new scale l_n was introduced in [203]. In particular, new discrete operators have been found which supersede the ones in eq. (4.1). These new operators involve a smearing out of the sum over layers and are given by [44, 80, 203]

$$(B_{nl}^{(d)}\phi)(x) = \frac{\epsilon^{2/d}}{l^2}\left(a\,\phi(x) + \epsilon\sum_{n=0}^{\infty} f_d(n,\epsilon)\sum_{y\in I_n(x)}\phi(y)\right), \tag{4.3}$$

where $\epsilon = (l/l_n)^d$, and

$$f_d(n,\epsilon) = (1-\epsilon)^n \sum_{m=0}^{L_{max}} b_m \binom{n}{m}\left(\frac{\epsilon}{1-\epsilon}\right)^m$$

is the smearing function. Once averaged over sprinklings of a given spacetime, the operators in eq. (4.3) give the same continuum operators as in eq. (4.2) with $\rho = 1/l_n$. For the rest of this Chapter we will be interested in the continuum version of these new operators, i.e., eq. (4.2).

It was shown in [80, 203] that, for particular choices of coefficients $\{a, b_n\}$, the operators in eq. (4.2) reduce to the standard wave operator in flat spacetime in the local limit, i.e., for $\rho \to \infty$. The operators which satisfy this condition are called Generalized Causal Set D'Alembertians (GCD). Once $\{a, b_n\}$ are identified, these operators can also be studied for a generic curved spacetime [34, 80, 102]. In this Chapter we will consider specifically the minimal operators in two and four dimensions. These are the original operators which were introduced in [44, 203] and they form a well-defined subset of all the GCD. In particular, minimal here refers to the fact that $L_{max}^{(d)}$ is the smallest integer for which the continuum limit can be recovered, i.e.,

$$\lim_{\rho\to\infty} \Box_{nl}\phi(x) = \Box\phi(x),$$

e.g., $L_{max}^{2d} = 2$ and $L_{max}^{4d} = 3$. It should be noted that, for the minimal case the set of coefficients $\{a, b_n\}$ is unique whereas for every non-minimal values of $L_{max}^{(d)}$ there are infinitely many choices of $\{a, b_n\}$[6]. In this sense minimal operators comprise an interesting subset of all GCD. Nonetheless, the quantization procedure which we outline in the following can be trivially generalized to non-minimal operators.

The retarded, minimal, non-local d'Alembertian in d-dimensional Minkowski spacetime $\mathbb{M}^d$ is given by eq. (4.2) where $\sqrt{g} = 1$ — the coefficients $\{a, b_n\}$ in two and four dimensions can be found in [17]. This can be rewritten as

$$\Box_{nl}^{(d)}\phi(x) = \int_{J^-(x)} d^d y\, K^{(d)}(x,y)\phi(y), \tag{4.4}$$

[6] For details see [17].

where

$$K^{(d)}(x,y) := \rho^{\frac{2}{d}} \left(a\delta(x,y) + \rho \sum_{n=0}^{L^d_{max}} b_n \frac{(\rho V(x,y))^n}{n!} e^{-\rho V(x,y)} \right). \tag{4.5}$$

Translational symmetry of $\mathbb{M}^d$ implies that $K^{(d)}(x,y) = K^{(d)}(x-y)$; therefore, letting $w = y - x$,

$$\Box^{(d)}_{nl}\phi(x) = \int_{J^-(0)} d^d w\, K^{(d)}(-w) e^{w\cdot\partial_x}\phi(x) = f^{(d)}(-\Box)\phi(x), \tag{4.6}$$

where we used the fact that since $K^{(d)}$ is Lorentz invariant, it is function of $w^2 := w \cdot w$, and we defined

$$f^{(d)}(-\Box) := \int_{J^-(0)} d^d w\, K^{(d)}(-w) e^{w\cdot\partial_x}. \tag{4.7}$$

The non-local equations of motion of a massless field, $\phi(x)$, living in d-dimensional Minkowski spacetime can therefore be written as

$$f^{(d)}(-\Box)\phi(x) = 0. \tag{4.8}$$

The action of $f(-\Box)$ on ϕ can be defined in different ways, depending on the analytic properties of f: If $f(z)$ is everywhere analytic then we can represent it as the convergent power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{4.9}$$

Otherwise we may define it through its action on Laplace transforms, which is how we will proceed.

The Laplace transform of $f(-\Box)$ in d-dimensions is given by [17]

$$f^{(d)}(k^2) = \rho^{2/d} \left(a + 2(2\pi)^{d/2-1} Z^{\frac{2-d}{4}} \sum_{n=0}^{L^(_{max}d)} \frac{b_n}{n!} \gamma_d^n \int_0^\infty ds\, s^{d(n+1/2)} e^{-\gamma_d s^d} K_{\frac{d}{2}-1}(Z^{1/2}s) \right), \tag{4.10}$$

where

$$Z = \frac{k\cdot k}{\rho^{2/d}}, \qquad \gamma_d = \frac{\left(\frac{\pi}{4}\right)^{\frac{d-1}{2}}}{d\,\Gamma\left(\frac{d+1}{2}\right)}, \tag{4.11}$$

and K_ν is the modified Bessel function of the second kind containing a cut along the negative real axis and for which we assume the principal value. Plane wave solutions $e^{ik\cdot x}$ to $f(-\Box)\phi = 0$ lie in the kernel of $f(k^2)$. Aslanbeigi *et al.* [17] showed that in two and four dimensions $f^{(d)}(k^2) = 0$ iff $k^2 = 0$ and $k^2 = 0, \zeta_4, \zeta_4^*$ respectively, where $\Re(\zeta_4) < 0$ and $\Im(\zeta_4) > 0$ and the existence of the complex mass poles was determined numerically. Applying their analysis to other dimensions we obtain Table 4.1 for $2 \leq d \leq 7$

where $\Re(\eta_d) < 0$ and $\Im(\eta_d) > 0$, $d = 4, \ldots, 7$. [7] A simple dimensional

[7] It should be noted that for $d \geq 4$ the roots of $f^{(d)}$ away from the origin were found numerically.

TABLE 4.1: Roots of $f(z)$

$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
0	0	$0, \zeta_4, \zeta_4^*$	$0, \zeta_5, \zeta_5^*$	$0, \zeta_6, \zeta_6^*, \eta_6, \eta_6^*$	$0, \zeta_7, \zeta_7^*, \eta_7, \eta_7^*$

argument, confirmed by numerics, suggests that $\zeta_d, \eta_d \propto \rho^{2/d}$ as is expected given that ρ is the only dimensionful parameter in the theory. The table above suggests that the number of zeros of $f^{(d)}$ grows with the number of dimensions, with a new pair appearing every time the dimensionality of the spacetime goes up by two. Assuming this pattern continues for all d, then the number of zeros of the minimal d'Alembertian in d-dimensions would be $(d-1)$ for d even, and $(d-2)$ for d odd. Note that the complex mass solutions appear in complex conjugate pairs ensuring that the theory is CPT invariant – the pole structure of the propagator in the complex k^0-plane has to be symmetric about the real axis. This is expected since our action (see Equation 4.18 below) is CPT-invariant: C being trivial for a real scalar field theory, and PT because the non-local d'Alembertian $\Box_{nl}$ is a function of the spacetime volume only, which is PT-invariant in Minkowski spacetime.

We will see in Section 4.4 that in order to establish the number of initial conditions required to specify a state of the system (equivalently the number of propagating degrees of freedom), one needs to know the degree of the roots of $f(z)$. Consider therefore the case $d = 4$, which we will be concerned with in Section 4.4.2. In this case there exist three roots: $z = 0, \zeta_4, \zeta_4^*$. The root at the origin, $z = 0$, is a branch point whose degree can be easily established to be < 1, since $\lim_{z\to 0} z f^{-1}(z) = 0$. Had we known the exact location of roots $z = \zeta_4, \zeta_4^*$, then a similar analysis would have given their degree. However, since we are only able to establish their existence numerically, we have had to resort to numerics to compute their degree. The numerical analysis is given in the box below, and suggests that the roots at $\pm\Omega, \pm\Omega^*$, where $\Omega := k^0 = \sqrt{\mathbf{k}^2 - \zeta_4}$ and $\Omega^* = \sqrt{\mathbf{k}^2 - \zeta_4^*}$, are of order 1. We will therefore assume this to be the case throughout the rest of this article. [8]

Degree of Singularity of Complex Mass Poles

Here we provide evidence that the complex mass poles appearing in the $4d$ propagator are of order 1.

[8] If the poles are of higher order, then the coefficients in the field expansion will have an explicit time dependence.

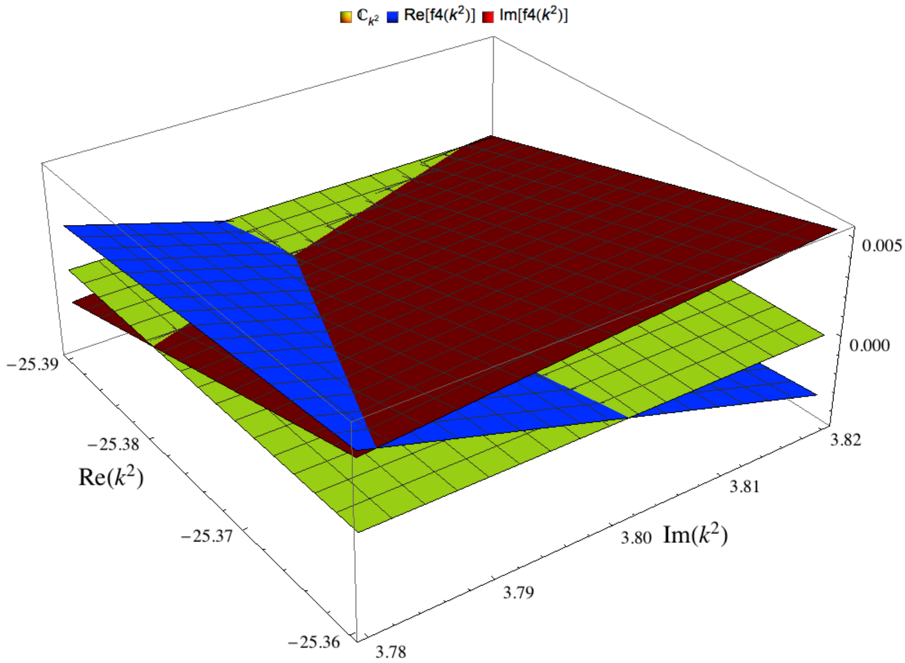

FIGURE 4.1: Real and imaginary part together. The two planes intersect with the complex-plane in what seems like a single point representing one of the complex roots of $f^{(4)}(z)$.

The existence of these poles was established numerically using Cauchy's argument principle, as such we only know their location in the complex k^2 plane approximately. In order to establish their nature we probe the behaviour of $f(z)$ in the vicinity of these zeros numerically, and provide evidence that $f(z)$ is linear in z in these neighbourhoods, see figure 4.1. A linear behaviour in z corresponds to the inverse having simple poles, which is what we assume in the following of this Chapter.

It is interesting to note that the (formal) infrared expansion of (4.7) in two and four dimensions is given by [9]

$$f^{(2)}(-\Box) = \Box - \frac{\Box^2}{2\rho}\left[\gamma + \ln\left(\frac{-\Box}{2\rho}\right)\right] + \dots, \tag{4.12}$$

and

$$f^{(4)}(-\Box) = \Box - \frac{3}{2\pi\sqrt{6}}\frac{\Box^2}{\sqrt{\rho}}\left[3\gamma - 2 + \ln\left(\frac{3\Box^2}{2\pi\rho}\right)\right] + \dots, \tag{4.13}$$

respectively (here γ is Euler–Mascheroni's constant). [10] Therefore the nonlocality is manifest even in the first order corrections to the standard continuum d'Alembertian in the IR limit.

[9] The derivation of these expansions, together with the full series, is given in Appendix C.1

[10] The same expressions appear in [131], where the author obtains the same corrections as above, and the same power series expressions found in Appendix C.1.

4.2 Massive Extension: Nonlocal Klein-Gordon Equation

In this section we generalise the nonlocal wave equation (4.8) to include massive fields.

Consider the following naive massive extension to (4.8)

$$\left[f(-\Box) - m^2\right]\phi(x) = 0, \tag{4.14}$$

where $m^2 \in \mathbb{R}^+$. This equation does not admit plane wave solutions (for finite ρ) with $k^2 = -m^2$, since $f(k^2) \in \mathbb{C}$ if $k^2 < 0$ (although it does reduce to the standard Klein-Gordon equation in the limit $\rho \to \infty$). Instead, we define the equation of motion of the massive field to be

$$f(-\Box + m^2)\phi(x) = 0. \tag{4.15}$$

We call this the non-local KG equation. Unlike Equation (4.14), this does admit plane wave solutions with massive dispersion relations $k^2 = -m^2$, as well as possessing the correct continuum limit:

$$\lim_{\rho\to\infty} f(-\Box + m^2)\phi(x) = (\Box - m^2)\phi(x). \tag{4.16}$$

Indeed, from Section 4.1 we know that (ignoring, if any, the complex mass solutions which will simply undergo a translation along the real axis) $f(z) = 0$ if $z = 0$. Hence, in momentum space, we have that $f(k^2 + m^2) = 0$ if $k^2 = -m^2$ for any ρ. Note that the functional form of f has not changed so the branch cut remains but with the branch point now shifted to $k^2 = -m^2$. Table 4.1 can therefore be trivially extended to include the massive case to give

TABLE 4.2: Roots of $f(z + m^2)$

$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$-m^2$	$-m^2$	$-m^2, \tilde\zeta_4, \tilde\zeta_4^*$	$-m^2, \tilde\zeta_5, \tilde\zeta_5^*$	$-m^2, \tilde\zeta_6, \tilde\zeta_6^*, \tilde\eta_6, \tilde\eta_6^*$	$-m^2, \tilde\zeta_7, \tilde\zeta_7^*, \tilde\eta_7, \tilde\eta_7^*$

where $\tilde\zeta_d = \zeta_d - m^2$ and $\tilde\eta_d = \eta_d - m^2$.

An interesting problem for the causal set community is to determine the inverse Laplace transform of this function, which would correspond to the massive version of Equation (1.16) in position space. This would lead to a definition of the KG equation, as well as of the mass term, on a causal set. Even if some advancement in this direction have been made (see box below[11]), a definite answer has still to be reached.

Discrete Mass Operators

Here we present candidate discrete Klein-Gordon operators. These operators are similar to the original wave operators derived from Causal Set theory. However, we now demand that, in the local limit, the averaged operators

[11] The material in the box appeared in a talk given by the author of this thesis at the workshop *Prospects for causal set quantum gravity*, http://icms.org.uk/workshops/prospects

reduce to the standard massive Klein-Gordon operator. This permits also to define a discrete mass term for a scalar field propagating on a causal set.

To obtain these discrete, massive operators the strategy is to consider continuum (averaged) versions, which reduce to the local Klein-Gordon — in the local limit — and are obtained using the results of [36]. Then, these operators are discretized giving rise to the sought after discrete operators. In the following we report the expressions for the massive, discrete, non-local Klein-Gordon operators in two and four dimensions

$$B_m^{(d=2)}\phi(x) = -2\rho\phi(x) + 4\rho\left(\sum_{y\in L_1} -(2-\frac{1}{2}\frac{m^2}{\rho})\sum_{y\in L_2} +(1-2\frac{m^2}{\rho})\sum_{y\in L_3} + \frac{3}{2}\frac{m^2}{\rho}\sum_{y\in L_4}\right)\phi(y);$$

$$B_m^{(d=4)}\phi(x) = \sqrt{\rho}\alpha\phi(x) + \sqrt{\rho}\sum_{n=0}^{L'_{max}} c_n \sum_{y\in L_n(x)}\phi(y) + m^2 \sum_{n=1}^{L_{max}+1} n b_{n-1}\sum_{y\in L_n(x)}\phi(y),$$

where L'_{max} and L_{max} are always greater or equal than the minimal number of layers. As it is evident from these expressions, a massive operator always requires a layer more than the corresponding minimal massless operator.

It should be noted that, these operators are not unique in the same way as the wave operators discussed in the main text. Whether the proposed expressions can actually present, once averaged, a physical pole for $k^2 = -m^2$ when $\rho \neq \infty$ is still an open issue and it will require a spectral analysis of these operators — like the one performed in [17] for the massless case. Furthermore, how to discretize the operators $f(-\Box + m^2)$, which will have the correct pole structure, remains an open problem since the previous expressions do not seem to give these operators once averaged.

In the main text we will be concern mostly with the massless case though our expressions can be easily generalized to the case of $f(-\Box + m^2)$ which, in light of the previous discussion, can be considered just a phenomenological model that is not derived directly from Causal Set and does not have a discrete counterpart, as for now.

4.3 Huygens' Principle and the Nonlocal d'Alembertians

Huygens' principle (HP) can be interpreted as stating that in spacetime dimensions $d = 2n + 2$, $n > 0$, the propagators of the wave equation $\Box G = \delta$ have support on the light-cone, while in dimensions $d = 2n + 1$ they also have support *inside* the light-cone. The case $d = 2$ is degenerate and somewhat counter intuitive since the Green functions are constant inside the light cone.

In a recent stream of works, violations of the HP were shown to be relevant for both QG phenomenology and quantum communication [51, 52, 133]. In the seminal work [133] it is shown that, when such violations occur, it is possible to have information transmission without energy exchange. Furthermore, two observers — Alice and Bob — which can perform local measurements on the field, by coupling or not their Unruh–DeWitt detectors, can communicate through a massless field even when timelike separated[12], giving rise to a normally forbidden quantum communication channel. Since violations of the HP happens in general when spacetime is curve, in [51, 52] the case of a flat Friedmann–Robertson–Walker (FRW) Universe has been considered. In this case the authors show that violations of HP can be used as a sensitive probe of non-standard gravitational dynamics. In particular, these violations are employed to distinguish between the standard FRW dynamics and the modified one dictated by Loop Quantum Cosmology bouncing Universe. All these results prove that HP violations are a relevant and new avenue for QG phenomenology to be explored.

It is interesting to ask therefore if such HP holds in the case where $\Box$ is replaced by $f(-\Box)$. A priori there is no reason why it should, since the analytic properties of $f^{-1}(-\Box)$ are in general very different from those of $\Box^{-1}$. Indeed, using the explicit form for the retarded Green function in d-dimensions given in [17], it is possible to show that for $d = 4$ the retarded Green function has support inside the light cone. Explicitly the d-dimensional Green function is given by

$$\rho^{2/d} G(x-y) \overset{x \succ y}{=} \frac{2(-1)^{1+\frac{d}{2}} \tau_{xy}^{1-\frac{d}{2}}}{(2\pi)^{d/2}} \rho^{2/d} \int_0^\infty d\xi \, \xi^{d/2} \frac{\operatorname{Im}\left[f(-\xi^2 + i\epsilon)\right]}{|f(-\xi^2+i\epsilon)|^2} J_{\frac{d}{2}-1}(\tau_{xy}\xi), \tag{4.17}$$

where $J_n(z)$ is the Bessel function of the first kind, and $x \succ y$ means x lies to the future of y. A numerical plot of this function against proper time τ is given in figure 4.2. [13] Note how the function is non vanishing for nonzero values of proper time but decays to zero rapidly for $\tau \gg \rho^{-1/d}$ (in the plot $\rho = 1$). This is to be expected since the Green function (4.17) reduces to the standard, local Green function which has no support inside the light cone, in the limit $\rho \to \infty$. This plot therefore demonstrates the failure of HP in $4d$. Since all other even dimensional Green functions are similar in form one should expect this to be true for all $d = 2n + 4$, $n \geq 0$. Even though HP does not strictly apply in $d = 2$, it is nonetheless interesting to see how the nonlocal Green function is modified in this case. The plot of G_R against proper time τ is shown in figure 4.3. As one can see the function is no longer constant for all $\tau > 0$ but, crucially, it tends to a constant in the limit $\tau/\rho \to \infty$.

The above observations, though purely classical at this point, are important for the quantum theory. Indeed, the Pauli-Jordan function – defined as the vacuum expectation value of the commutator of fields – is given by i times the difference between the retarded and advanced propagators. Hence for a $4d$ theory based on such non-local dynamics, it will fail to vanish inside the light-cone, unlike a QFT based on local dynamics given by just $\Box$.

[12] If the HP holds Bob, can receive Alice's message only if his wordline intersects the light cone of Alice.

[13] In the plot for $d = 2$ we have corrected the expression for the Green function by an overall factor of 2. This is because Equation (4.17) for $d = 2$ was tending to -1/4 in the limit $\tau/\rho \to \infty$, rather than -1/2, which is the value that the retarded Green function of $\Box$ takes inside the light cone in $2d$. We therefore believe that (at least in the 2 dimensional case), Equation (4.17) is off by an overall factor of 2.

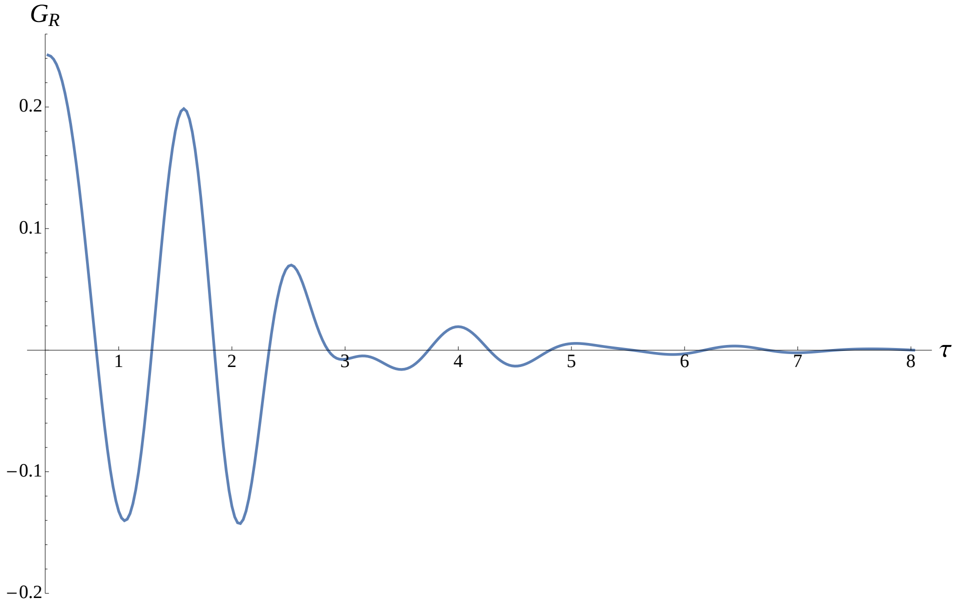

FIGURE 4.2: 4 dimensional retarded, non-local Green function as a function of proper time τ for $\rho = 1$. Note how the function has non-zero support inside the light cone ($\tau > 0$) but decreases rapidly and asymptotes 0 in the limit $\tau/\rho \to \infty$. For small $\tau \sim O(\rho)$ we therefore have large deviations from the local retarded Green function in $4d$, but recover the usual result in the limit.

For what concerns the phenomenological implications of HP violations, it should be noted that the works present in literature always consider either flat spacetime — in lower than four dimensions — or curved spacetime and cosmological observations. Indeed, these are the cases in which HP violations usually show up. However, we have shown that in presence of non-locality akin to the one of Causal Set theory, HP violations occur even in $4d$ flat spacetime. This clearly open to the possibility to study such violations in laboratories[14] looking for deviations from standard local QFTs. In Chapter 6 we will make the first step towards the study of HP violations by investigating the response of a single Unruh–DeWitt detector coupled to the non-local field.

4.4 Free Scalar Nonlocal QFT

In the following sections we construct free scalar quantum field theories based on the nonlocal dynamics defined by $f(-\Box)$. We closely follow the formalism set out in [27] for canonically quantising nonlocal field equations. Everything we say can be extended to the massive case by simply replacing $f(-\Box)$ by $f(-\Box + m^2)$, with all the caveats discussed previously in Sec. 4.2.

[14] With all the caveats of extending the non-local model to gauge theories.

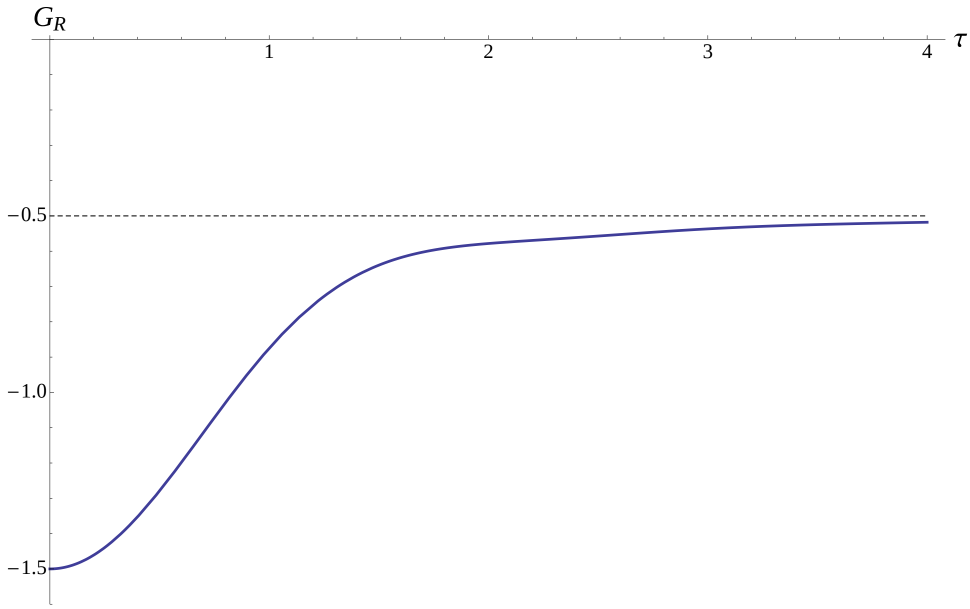

FIGURE 4.3: 2 dimensional retarded, non-local Green function as a function of proper time τ for $\rho = 1$. Note how for $\tau \sim O(\rho)$ the non-local retarded Green function deviates from the usual local one, but asymptotes the local Green function in the limit $\tau/\rho \to \infty$.

We begin with the (free) non-local Lagrangian [15]

$$\mathcal{L} = \phi(x) f(-\Box)\phi(x), \tag{4.18}$$

which leads to the following Euler-Lagrange equations (see Section 2 of [27] and references therein)

$$f(-\Box)\phi(x) = 0. \tag{4.19}$$

A general solution to the equations of motion can be written as [26]

$$\phi(x) = \int_\Gamma dk^0 \int d^{d-1}\mathbf{k}\, \frac{a(k)}{f(k^2)} e^{ik\cdot x}, \tag{4.20}$$

where Γ runs from $-\infty$ to ∞ for $\Im(k^0) > s$ and ∞ to $-\infty$ for $\Im(k) < -s$ for $f^{-1}(k^2)$ analytic in $\{k^0 \in \mathbb{C} \mid |\Im(k^0)| > s\}$, and $a(k)$ is an entire analytic function (note that for $a(k) = 1$ this contour defines the usual Pauli-Jordan function, which is in the kernel of the d'Alembertian). [16] By continuously deforming the path Γ around the singularities of f^{-1}, Equation (4.20) can be rewritten as

$$\phi(x) = \int d^d k\, \theta(k^0)\Delta(f^{-1})(a(k)e^{ik\cdot x} - a^*(k)e^{-ik\cdot x}) + \sum_{i=1}^{N} \int_{\Gamma_i} d^d k\, \frac{a(k)}{f(k^2)} e^{ik\cdot x}, \tag{4.21}$$

[15] Note that this kind of non-locality does not imply a violation of micro-causality. As such it is fundamentally different from the kind of non-locality often considered in QFT, e.g., [147].

[16] $f^{-1}(z)/z^s$ also has to be bounded continuous in $\{k \in \mathbb{C} \mid |\Im(k)| \geq s\}$, see [26].

where we used reality conditions on ϕ to determine that $a(-k) = a(k)^*$, and we have defined the discontinuity functional across the branch cut in f^{-1} to be

$$\Delta(f^{-1}) := f^{-1}(-(k^0 + i\epsilon)^2 + k^2) - f^{-1}(-(k^0 - i\epsilon)^2 + k^2), \tag{4.22}$$

and the Γ_i's are loops surrounding isolated singularities of f^{-1}. It is interesting to note that the general solution (4.21), when $f^{-1}(z)$ contains a branch cut (as is the case here), is *not* a linear superposition of plane waves with dispersion relation $k^2 = 0$ only. Rather, the functional $i\Delta$ in general gives non-zero weight to all $k^2 = -m^2$ with $m^2 \geq 0$, hence $\phi(x)$ is a superposition of all possible (massless and massive) plane waves. Nonetheless (ignoring the existence of complex mass solutions for the time being), it would not be correct to state that plane waves with $k^2 < 0$ are part of the basis of the solution space, since they themselves do not satisfy the equations of motion. It is also incorrect to view plane waves with $k^2 = 0$ as a basis since the expansion of a general solution clearly also contains plane waves with $k^2 \neq 0$.

Naively one might think that the infinite number of derivatives present in the equations of motion requires having to specify an infinite number of initial conditions to determine a solution. However, it can be shown that a large class of infinite order differential equations admit a well-defined initial value problem [30], requiring only a finite number of initial conditions; the precise number depending on the number of poles in the propagator and their degree. Equivalently, poles in the propagator can be used to determine the number of degrees of freedom; with every pole of degree ≤ 1 representing a single degree of freedom. So, for example, in 4-dimensions where the number of roots of $f(k^2)$ is 3, one needs 6 initial conditions to specify a solution: two for the pole at $k^2 = 0$ and two for each of the complex mass roots $k^2 = \zeta_4, \zeta_4^*$. As such one should expect 3 propagating degrees of freedom. In the quantum theory these degrees of freedom are quantised and appear as states in the physical Hilbert space of the theory. Thus, coming back to our $4d$ example, we expect the Hilbert space of the (free) quantum theory to contain states associated to the 3 poles in the propagator. In fact, as we will see in Section 4.4.2, it is possible to construct the quantum theory such that states associated to the complex mass modes do not appear asymptotically – in a sense, they behave as if propagating in a medium that acts as a perfect absorber – while the massless ones do, as is suggested by the above argument. In Appendix C.2 we explicitly check that the continuum of massive modes associated to the cut $k^2 < 0$ do not appear in the asymptotic states of the (free) quantum theory.

In [76], the canonical quantisation of theories containing fractional powers of the d'Alembertian was performed. Although ultimately successful, this method is not without difficulties, having an infinite set of second class constraints to solve and ill-defined Poisson brackets between the conjugate variables. We therefore quantise our system following D. Barci *et. al* [27]. Their method consists in observing that Schwinger's quantisation method implies the Hamiltonian must be the generator of time evolution at the level of quantum theory. This is equivalent to imposing Heisenberg's equations of motion on the quantum field: $\dot{\phi} = i[H, \phi]$. [17] The definition of a vacuum state proceeds as usual for the massless modes associated to the branch point at $k^2 = 0$, since their creation and annihilation operators possess canonical commutation relations. As for the complex mass modes, their creation and annihilation operators possess peculiar commutation relations (akin to those between q and p in standard quantum mechanics) and therefore require further care in defining the vacuum state and the corresponding Fock space. This will be analysed in Subsection 4.4.2 below.

In general dimension, d, the Hamiltonian can be shown to be given by (c.f. [54])

$$H = -\int d^{d-1}\mathbf{x} \int d^dk \int d^dk'\, e^{i(k+k')\cdot x}\phi(k)\phi(k')k'^0 \frac{k'^0 - k^0}{k^2 - k'^2}(f(k^2) - f(k'^2)) \tag{4.23}$$

$$+ \int d^{d-1}\mathbf{x}\, \mathcal{L},$$

where the last term vanishes when the field is on-shell.

4.4.1 2 Dimensions

In two dimensions the momentum space d'Alembertian can be expressed in the relatively simple form

$$f(k^2) = -k^2 e^{k^2/2\rho}\, E_2(k^2/2\rho), \tag{4.24}$$

where E_2 is the Exponential Integral of the second kind:

$$E_2(z) := z e^{-z} \int_0^\infty \frac{e^{-t}}{(t+z)^2} dt, \tag{4.25}$$

containing a branch cut along $z \leq 0$ [103].

To find the general solution (Equation (4.20)), we note that as a function of $k^0 \in \mathbb{C}$, f^{-1} is analytic for $\Im(k^0) > 0$, hence we can deform the contour Γ (see figure 4.4) such that

$$\phi(x) = \int_{-\infty}^{\infty} d^2k\, \theta(k^0)\Delta(f^{-1}) \left(a(k)e^{ik\cdot x} - a(k)^* e^{-ik\cdot x}\right). \tag{4.26}$$

After some manipulations the discontinuity functional for $k^0 > 0$ can be rewritten as

$$i\Delta(f^{-1}) = \lim_{\epsilon \to 0} \frac{2\, e^{-k^2/2\rho}}{k^2} \frac{\Im[E_2((k^2 + i\epsilon)/2\rho)]}{|E_2((k^2 + i\epsilon)/2\rho)|^2}. \tag{4.27}$$

Figure 4.5 shows a plot of (4.27) for $\rho = 1$. Note that all modes have positive weight, and hence positive energy with respect to the Hamiltonian defined below (see Equation (4.33)). Consistency with $\Box_{nl} \to \Box$ as $\rho \to \infty$ implies

[17] Although this quantisation procedure is not canonical, Barci and Oxman found that when applied to theories containing fractional powers of the d'Alembertian, their results were consistent with those of [76].

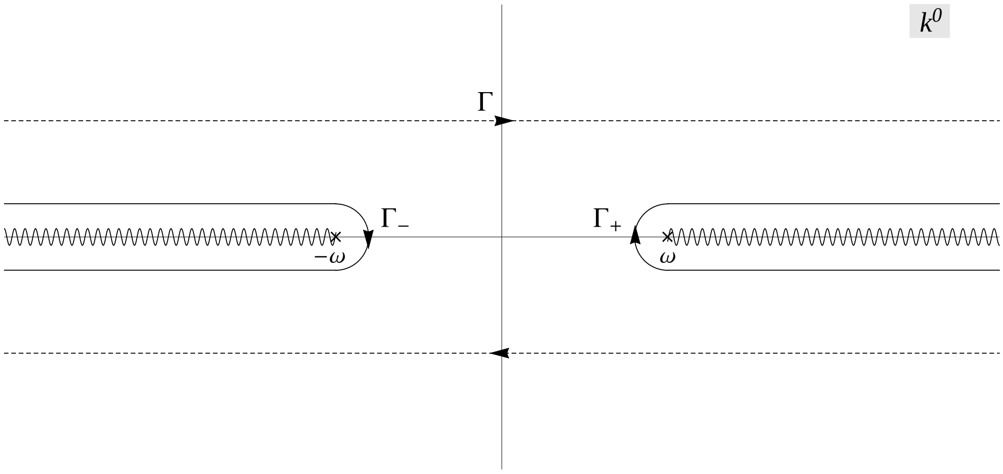

FIGURE 4.4: Pole structure of $f^{-1}(k^2)$ in $2d$ in the complex k^0 plane, together with a choice of contour, Γ, which gives a solution to the equations of motion. The continuous paths represent a deformation of Γ: $\Gamma = \Gamma_+ \cup \Gamma_-$.

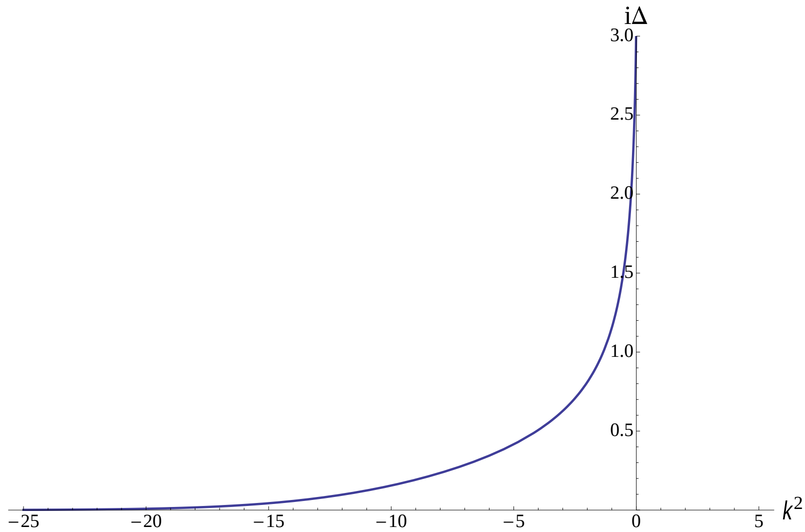

FIGURE 4.5: Plot of $i\Delta(f^{-1})$ as a function of k^2 for $k^0 > 0$ and $\rho = 1$. Note how the function blows up at the origin, ensuring that massless modes provide the greatest contribution, and is non zero for all $k^2 < 0$ but rapidly decays to zero.

that $i\Delta(k^2) \to \delta(k^2)$ in the limit $\rho \to \infty$. Indeed one can show

$$\lim_{\rho\to\infty} \Delta(z) = \lim_{\epsilon\to 0}\left\{-\frac{1}{z-i\epsilon} + \frac{1}{z+i\epsilon} + \frac{i}{\rho}\mathrm{Im}\left[\frac{z+i\epsilon}{z-i\epsilon}\left(\ln\left(\frac{z+i\epsilon}{2\rho}\right)+\gamma\right)\right] + O(\rho^{-2})\right\}. \tag{4.28}$$

The zeroth order term is just a representation of the delta function (i.e. $-2\pi i\delta(z)$), hence we recover the standard local theory in the limit $\rho \to \infty$. The second term represents the first order correction in the $1/\rho$ series expansion, and it is interesting to note that this term already contains a branch cut, and therefore implies the presence of a continuum of massive modes.

Substituting (4.26) into (4.23) and integrating over x and k'^1 we find the on-shell Hamiltonian

$$H = -\int dk \int_\Gamma dk^0 \int_{\Gamma'} dk'^0\, e^{-i(k^0+k'^0)t}\frac{k'^0}{k'^0+k^0}a(k^0,k)a(k'^0,-k) \cdot (f^{-1}(k'^0,-k) - f^{-1}(k^0,k)). \tag{4.29}$$

Without loss of generality, the contour Γ' is chosen such that it runs above Γ for $\Im(k^0) > 0$ and below for $\Im(k^0) < 0$. It is then straightforward to show using Cauchy's residue theorem that

$$H = 2\pi\int_{-\infty}^{\infty} d^2k\, k^0\delta^+G(k)(a(k)a(k)^* + a(k)^*a(k)), \tag{4.30}$$

where $\delta^+G(k) := i\theta(k^0)\Delta(f^{-1})$. Note that, unlike the local massless theory where only massless modes ($k^2 = 0$) appear in the Hamiltonian (i.e., where $\delta^+G(k) \sim \delta(k^2)$), here all modes with $k^2 \in \mathrm{supp}(\delta^+G(k))$ are present, implying the existence of a continuum of massive modes. This property is similar to that of interacting local QFTs (c.f. the Källen-Lehman spectral representation), except that our theory contains these modes already at tree level and in the absence of any interactions. We will see later however that these modes are not present in the asymptotic states of the quantum theory [18]; which is consistent with the fact that the theory possesses a single propagating degree of freedom coming from the only singularity of the propagator at $k^2 = 0$.

The quantum theory is defined by promoting the a's and a^*'s to operators: $\hat{a}$ and $\hat{a}^\dagger$ respectively, and imposing Heisenberg's equations of motion. We find

$$[H, a] = -k^0 a, \qquad [H, a^\dagger] = k^0 a^\dagger. \tag{4.31}$$

For $k^0 > 0$, $a(k)$ and $a(k)^\dagger$ correspond to raising and lowering operators respectively (we have dropped hats to denote operators). The vacuum is defined to be the state such that

$$a(k)|0\rangle = 0, \quad \forall k^2 \in \mathrm{supp}(\delta^+G),\ k^0 > 0. \tag{4.32}$$

Finally we define the normal ordered Hamiltonian

$$: H := 4\pi\int d^2k\, k^0\delta^+G(k)a(k^0,k)^\dagger a(k^0,k) \tag{4.33}$$

[18]The term asymptotic comes from the definition of asymptotic field operators in interacting QFTs as defined by Greenberg [104]

for which $|0\rangle$ is the zero energy eigenstate. Using this in (4.31) we obtain the commutation relations

$$[a(k), a(k')] = [a(k)^\dagger, a(k')^\dagger] = 0, \qquad 4\pi\delta^+ G(k)[a(k), a(k')^\dagger] = \delta^{(2)}(k - k') \tag{4.34}$$

for $k^2,\ k'^2 \in \mathrm{supp}(\delta^+ G(k))$.

Having defined the vacuum state we can now define the Wightman function,

$$\begin{aligned} W(x-y) &:= \langle 0|\phi(x)\phi(y)|0\rangle \\ &= \frac{1}{4\pi}\int d^2k\, \delta^+ G(k) e^{ik\cdot(x-y)} \qquad (4.35) \\ &= \frac{i}{4\pi}\int_{\Gamma_+} d^2k\, f^{-1}(k^2) e^{ik\cdot(x-y)}, \qquad (4.36) \end{aligned}$$

and the Feynman propagator of the theory,

$$\begin{aligned} G_F(x-y) &:= \langle 0|T\{\phi(x)\phi(y)\}|0\rangle \\ &= \theta(x^0 - y^0)W(x-y) + \theta(y^0 - x^0)W(y-x) \qquad (4.37) \\ &= \frac{i}{4\pi}\int_{\Gamma_F} d^2k\, f^{-1}(k^2) e^{ik\cdot(x-y)}, \qquad (4.38) \end{aligned}$$

where $\Gamma_F = \theta(x^0 - y^0)\Gamma_+ - \theta(y^0 - x^0)\Gamma_-$.

4.4.2 4 Dimensions

The four dimensional momentum space d'Alembertian is given by

$$\begin{aligned} f^{(4)}(k^2) = -\frac{4}{\sqrt{6}}\sqrt{\rho} + \frac{16}{\sqrt{6}}\pi\rho^{3/2}\int_0^\infty d\tau\, \frac{\tau^2}{\sqrt{k^2}} e^{-\rho V_4} K_1(\tau\sqrt{k^2}) \qquad (4.39) \\ \cdot \left(1 - 9\rho V_4 + 8\rho^2 V_4^2 - \frac{4}{3}\rho^3 V_4^3\right), \end{aligned}$$

where $V_4 = \pi\tau^4/24$ (from here on we will drop the superscript denoting the dimension we are working in). This function has zeros at $k^2 = 0, \zeta_4, \zeta_4^*$ and a branch cut along $k^2 \le 0$ (see Section 4.1), therefore a general solution to (4.19) is

$$\phi(x) = \int d^4k\, \theta(k^0)\Delta(f^{-1})(a(k)e^{ik\cdot x} - a(k)^* e^{-ik\cdot x}) + \sum_{i=1}^{4}\int_{\Gamma_i} d^4k\, \frac{a(k)}{f(k^2)} e^{ik\cdot x}, \tag{4.40}$$

where again $\Delta(f^{-1}) = f^{-1}(-(k^0 + i\epsilon)^2 + k^2) - f^{-1}(-(k^0 - i\epsilon)^2 + k^2)$ and the Γ_i are loops surrounding the isolated singularities of f^{-1} at $k^0 = \pm\Omega, \pm\Omega^*$ (recall $\Omega = \sqrt{\mathbf{k}^2 - \zeta_4}$), obtained by continuously deforming the contour Γ, see figure 4.6. The discontinuity functional $i\Delta$ is shown in figure 4.7. Note that unlike the $2d$ case, this functional is *not* positive definite, implying that some modes will have negative energy with respect to the non-local Hamiltonian defined below (Equation (4.44)).

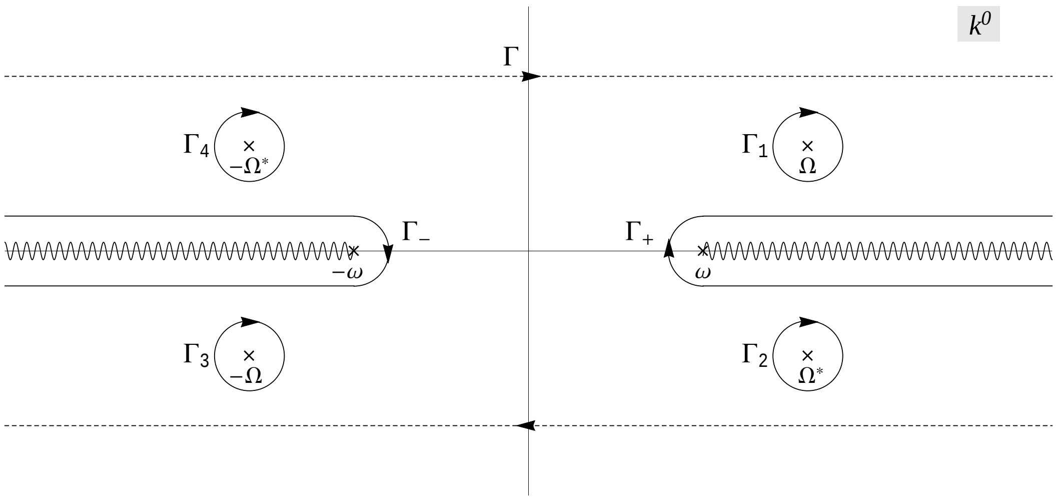

FIGURE 4.6: Pole structure of $f^{-1}(k^2)$ in $4d$ in the complex k^0 plane, together with a choice of contour, Γ, which gives a solution to the equations of motion. The continuous paths represent a deformation of Γ, i.e. $\Gamma = \Gamma_+ \cup \Gamma_- \cup_{i=1}^{4} \Gamma_i$.

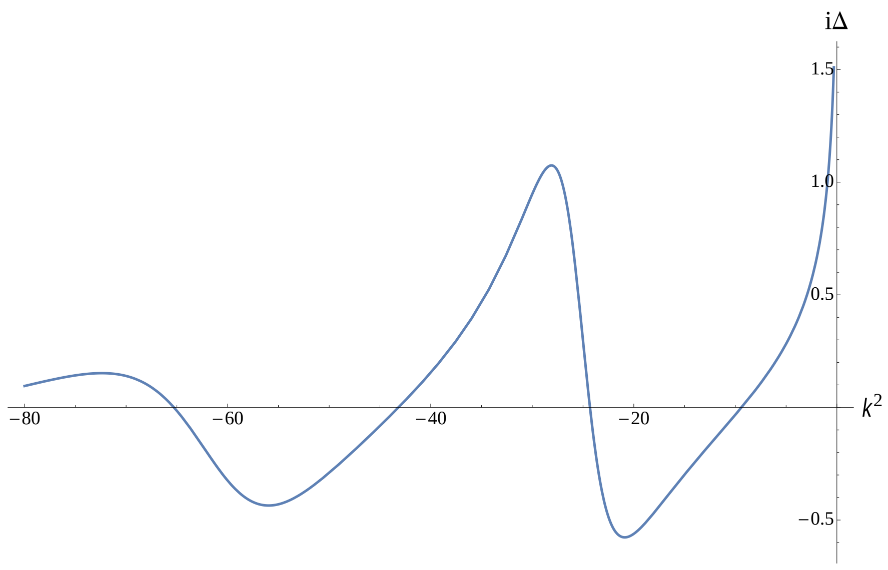

FIGURE 4.7: Numerical plot of $i\Delta(f^{-1})$ as a function of k^2 for $\rho = 1$. Note how the function blows up at the origin, ensuring that massless modes provide the dominant contribution, and rapidly decays in the limit $k^2/\sqrt{\rho} \to \infty$. Note also that, unlike the case $d = 2$, this functional fails to be positive definite for all $k^2 < 0$.

Assuming the complex poles of f^{-1} are simple (see Section 4.1), we denote the residues of f^{-1} by

$$g_1(\Omega, \mathbf{k}) = \mathrm{Res}[f^{-1}(k^0, \mathbf{k}),\ k^0 = \Omega], \tag{4.41}$$
$$g_2(\Omega^*, \mathbf{k}) = \mathrm{Res}[f^{-1}(k^0, \mathbf{k}),\ k^0 = \Omega^*], \tag{4.42}$$

the other two residues being trivially related to these. Using Cauchy's residue theorem and imposing reality conditions we can then write (4.40) as

$$\begin{aligned}\phi(x) = &\int d^4k\, \theta(k^0)\Delta(f^{-1})(a(k)e^{ik\cdot x} - a(k)^* e^{-ik\cdot x}) \\ &-2\pi i \int d^3\mathbf{k}\, \big(g_1(\Omega, \mathbf{k})a(\Omega, \mathbf{k})e^{i\kappa\cdot x} - g_1(\Omega, \mathbf{k})a(\Omega^*, \mathbf{k})^* e^{-i\kappa\cdot x} \\ &\qquad + g_1(\Omega, \mathbf{k})^* a(\Omega^*, \mathbf{k})e^{i\kappa^*\cdot x} - g_1(\Omega, \mathbf{k})^* a(\Omega, \mathbf{k})^* e^{-i\kappa^*\cdot x}\big), \end{aligned} \tag{4.43}$$

where $\kappa = (\Omega, \mathbf{k})$ and $\kappa^* = (\Omega^*, \mathbf{k})$. Substituting this into (4.23), integrating over $\mathbf{x}$ and $\mathbf{k}'$, and again choosing the contour Γ' such that it runs above Γ for $\Im(k^0) > 0$ and below for $\Im(k^0) < 0$ we find the on-shell Hamiltonian

$$\begin{aligned} H = &\,2\pi \int d^4k\, k^0 \delta^+ G(k)\, (a(k)a(k)^* + a(k)^* a(k)) \\ &+ 4\pi^2 \int d^3\mathbf{k}\, \Omega\, g_1(\Omega, \mathbf{k})\{a(\Omega, \mathbf{k}), a(\Omega^*, \mathbf{k})^*\} + \text{h.c.}, \end{aligned} \tag{4.44}$$

where $\delta^+ G(k) = i\theta(k^0)\Delta(f^{-1})$.

The quantum theory is defined by promoting the a's and a^*'s to operators and by imposing Heisenberg's equations of motion:

$$[H, a(k)] = -k^0 a(k), \qquad [H, a(k)^\dagger] = k^0 a(k)^\dagger, \tag{4.45}$$
$$[H, c_\mathbf{k}] = -\Omega\, c_\mathbf{k}, \qquad [H, b_\mathbf{k}] = \Omega\, b_\mathbf{k}, \tag{4.46}$$
$$[H, b_\mathbf{k}^\dagger] = -\Omega^*\, b_\mathbf{k}^\dagger, \qquad [H, c_\mathbf{k}^\dagger] = \Omega^*\, c_\mathbf{k}^\dagger, \tag{4.47}$$

where we defined $c_\mathbf{k} := a(\Omega, \mathbf{k})$, $b_\mathbf{k} := a(\Omega^*, \mathbf{k})^\dagger$, and we have dropped hats to denote operators. For $k^0 > 0$, $a(k)$ and $a(k)^\dagger$ correspond to raising and lowering operators respectively; while the operators $c_\mathbf{k}$, $b_\mathbf{k}$ and their hermitian conjugates cannot be interpreted as raising and lowering operators for modes of energy Ω and Ω^* respectively, as we will see shortly.

Substituting (4.44) into (4.47) and defining $\beta^{-1} := \Omega g_1(\Omega, \mathbf{k})$ we find the set of commutation relations

$$4\pi\delta^+ G(k)[a(k), a(k')^\dagger] = \delta(k - k'), \quad [a(k), a(k')] = [a(k)^\dagger, a(k')^\dagger] = 0, \tag{4.48}$$

$$[c_\mathbf{k}, b_{\mathbf{k}'}] = \frac{\Omega\beta}{8\pi^2}\delta(\mathbf{k} - \mathbf{k}'), \quad [c_\mathbf{k}, c_{\mathbf{k}'}] = [b_\mathbf{k}, b_{\mathbf{k}'}] = [c_\mathbf{k}, b_{\mathbf{k}'}^\dagger] = \cdots = 0, \tag{4.49}$$

with all commutators between the a's and b's or c's vanishing.

From the commutation relations one can see that, unlike the $2d$ theory, this theory has two distinct sectors that we name the bradyonic/luxonic sector, corresponding to the creation and annihilation operators $a(k)$ and

$a(k)^\dagger$ respectively, and the complex mass sector corresponding to the $b_{\mathbf{k}}$, $c_{\mathbf{k}}$ and their Hermitean conjugates. In the former sector, which we refer to as the BL-sector (B for bradyons and L for luxons), the vacuum state $|0\rangle$ is defined in the usual way

$$a(k)|0\rangle = 0, \quad \forall k^2 \in \text{supp}(\delta^+ G),\ k^0 > 0. \tag{4.50}$$

To define the vacuum state of the latter sector we follow the analysis of Bollini and Oxman [55] closely. A suitable representation is

$$d_{\mathbf{k}} \to z, \quad d^\dagger_{\mathbf{k}} \to z^*, \quad b_{\mathbf{k}} \to -i\frac{d}{dz}, \quad b^\dagger_{\mathbf{k}} \to -i\frac{d}{dz^*}, \tag{4.51}$$

for each $\mathbf{k}$, where $d_{\mathbf{k}} := \frac{8\pi^2 i}{\beta\Omega} c_{\mathbf{k}}$. The inner product defined to be

$$\langle f|g\rangle = \int dz \int dz^* f g^*. \tag{4.52}$$

In this representation the complex-mass sector Hamiltonian becomes

$$H = \int d^3\mathbf{k} - \Omega\left(z\frac{d}{dz} + \frac{1}{2}\right) + \Omega^*\left(z^*\frac{d}{dz^*} + \frac{1}{2}\right). \tag{4.53}$$

Similarly one can show that complex mass sector momentum-density operator is

$$p_{\mathbf{k}} = \frac{i\mathbf{k}}{2}\left\{z, -i\frac{\partial}{\partial z}\right\} + h.c. = \mathbf{k}\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right), \tag{4.54}$$

so that its zero eigenvalue eigenfunctions are given by functions $f = f(zz^*)$. The vacuum state, $|0\rangle$, is defined to be the zero momentum eigenfunction of the Hamiltonian with zero energy, and can be shown to be given by

$$f_0(zz^*) = \frac{1}{\sqrt{zz^*}}. \tag{4.55}$$

Note that in this sector the energy is *not* proportional to Ω, nor $|\Omega|$. In fact, the eigenvalue equation for zero momentum eigenfunctions: $h_{\mathbf{k}} f_E(zz^*) = E f_E(zz^*)$, can be shown to have solutions (up to a normalisation factor)

$$f_E = (zz^*)^{iE-1/2}, \tag{4.56}$$

for $E \in \mathbb{R}$. Hence the energy spectrum for a given $\mathbf{k}$, rather than being discrete as for the BL-sector, is the whole real line. In this representation we have the following two-point functions

$$\langle 0|d_{\mathbf{k}} b_{\mathbf{k}'}|0\rangle = -\langle 0|b_{\mathbf{k}'} d_{\mathbf{k}}|0\rangle = \frac{i}{2}\delta(\mathbf{k}-\mathbf{k}'), \tag{4.57}$$

$$\langle 0|d^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k}'}|0\rangle = -\langle 0|b^\dagger_{\mathbf{k}'} d^\dagger_{\mathbf{k}}|0\rangle = \frac{i}{2}\delta(\mathbf{k}-\mathbf{k}'). \tag{4.58}$$

To compute the Wightman function and the Feynman propagator we first rewrite our general solution (4.40) as

$$\begin{aligned}\phi(x) = &-i\int d^4k\delta^+G(k)\left[a(k)e^{ik\cdot x}-a(k)^\dagger e^{-ik\cdot x}\right] \\ &-2\pi i\int d^3\mathbf{k}\left[-\frac{i}{8\pi^2}d_k e^{i\kappa\cdot x}-\frac{1}{\beta\Omega}b_k e^{-i\kappa\cdot x}\right] \\ &-2\pi i\int d^3\mathbf{k}\left[\frac{1}{\beta^*\Omega^*}b_k^\dagger e^{i\kappa^*\cdot x}-\frac{i}{8\pi^2}d_k^\dagger e^{-i\kappa^*\cdot x}\right] =: \varphi_{\mathrm{BL}}(x)+\varphi(x)+\overline{\varphi}(x).\end{aligned} \tag{4.59}$$

Since the BL and complex mass sectors do not mix (and neither do φ and $\overline{\varphi}$ within the complex-mass sector), we can look at their corresponding two-point functions separately. These are

$$\begin{aligned}\langle 0|\varphi_{\mathrm{BL}}(x)\varphi_{\mathrm{BL}}(y)|0\rangle &= \frac{1}{4\pi}\int d^4k\delta^+G(k)e^{ik\cdot(x-y)} \\ &= \frac{i}{4\pi}\int_{\Gamma_+} d^4k f^{-1}(k^2)e^{ik\cdot(x-y)},\end{aligned} \tag{4.60}$$

$$\begin{aligned}\langle 0|\varphi(x)\varphi(y)|0\rangle &= \frac{1}{4}\int d^3\mathbf{k}\frac{1}{\beta\Omega}\left(e^{i\kappa\cdot(x-y)}-e^{-i\kappa\cdot(x-y)}\right) \\ &= \frac{i}{2}\int_{\Gamma_1} d^4k\frac{\sin\left(k\cdot(x-y)\right)}{f(k^2)},\end{aligned} \tag{4.61}$$

$$\begin{aligned}\langle 0|\overline{\varphi}(x)\,\overline{\varphi}(y)|0\rangle &= \frac{1}{4}\int d^3\mathbf{k}\frac{1}{\beta^*\Omega^*}\left(e^{i\kappa^*\cdot(x-y)}-e^{-i\kappa^*\cdot(x-y)}\right) \\ &= \frac{i}{2}\int_{\Gamma_2} d^4k\frac{\sin\left(k\cdot(x-y)\right)}{f(k^2)}.\end{aligned} \tag{4.62}$$

Hence the full Wightman function is given by

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle =& \frac{1}{4\pi}\int d^4k\delta^+G(k)e^{ik\cdot(x-y)} \\ &-\frac{i}{2}\int d^3\mathbf{k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left(\frac{1}{\beta\Omega}\sin(\Omega\Delta t)-\frac{1}{\beta^*\Omega^*}\sin(\Omega^*\Delta t)\right),\end{aligned} \tag{4.63}$$

and the Feynman propagator

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \frac{i}{4\pi}\int_{\Gamma_F} d^4k\frac{e^{ik\cdot(x-y)}}{f(k^2)}+G_W(x-y)+G_W^*(x-y), \tag{4.64}$$

where

$$G_W(x) = -\frac{i}{2}\mathrm{sgn}(t)\int d^3\mathbf{k}\frac{1}{\beta\Omega}e^{-i\mathbf{k}\cdot\mathbf{x}}\sin\left(\Omega t\right), \tag{4.65}$$

$$G_W^*(x) = \frac{i}{2}\mathrm{sgn}(t)\int d^3\mathbf{k}\frac{1}{\beta^*\Omega^*}e^{i\mathbf{k}\cdot\mathbf{x}}\sin\left(\Omega^* t\right), \tag{4.66}$$

are known as the Wheeler propagators for φ and $\overline{\varphi}$ respectively, and $\Gamma_F = \theta(x^0-y^0)\Gamma_+ - \theta(y^0-x^0)\Gamma_-$ is the Feynman contour defined in the previous section. The Wheeler propagator lacks the on-shell singular contribution of free field excitations (see Section 2 of [55]). [19] Physically this means that these modes do not propagate asymptotically.

Hystory of the Wheeler propagator

Wheeler and Feynman originally introduced this propagator in an attempt to provide relativistic action-at-a-distance interpretation of electrodynamics: the Wheeler-Feynman absorber theory [231]. One of the upshots of this interpretation was that it used both retarded and advanced propagators and was therefore explicitly time-reversal invariant. In their theory charged particles act as both emitters (via retarded solutions) and absorbers (via advanced solutions) of radiation, hence the use of the Wheeler propagator. Nonetheless they were able to argue that, provided there exist a sufficiently large number of charged particles in the Universe absorbing radiation emitted by any one particle, the overall field propagator is retarded, thus recovering causality. The physical picture of having a large number of (absorbing) charges can be simply captured by imposing that the field vanish at infinity. In our theory, the asymptotic absence of these modes arises by construction, i.e., by the way we decided to quantise the complex mass sector. Whether this can be given sensible physical interpretation, other than the fact that it ensures that these instabilities are not present asymptotically, remains an open issue whose solution probably lies in the study of the interacting theory.

4.4.3 Renormalisation

As was originally pointed out by Sorkin [203] the above propagator can be used to define a Lorentz invariant regularisation tool for QFTs. Indeed, as was fleshed out by Aslanbeigi *et al.* [17], the position space propagator $G(x, y)$ contains a δ-function type singularity in the coincidence limit $y \to x$, due to the constant term appearing in the UV expansion of $f(z)$, see Equation (3.19) in [17]. Subtracting this constant in momentum space leads to a regularised propagator, call it $G_{reg}(z)$, for which all loop integrals are finite. In the interacting theory one would therefore have to replace f^{-1} with G_{reg} in order to obtain a UV finite theory (see also discussion in Chapter 5).

4.5 Summary and Outlook

We have canonically quantised free massless scalar fields, satisfying nonlocal equations of motion, in two and four dimensions. In both dimensions we find a continuum of massive modes arising from the cut discontinuity along $k^2 \leq 0$ of the momentum space Green function, $f^{-1}(k^2)$, similar to what was found in previously studied free, non-local, massless QFTs [27]. This feature will be present in any dimension, since the existence of the branch cut is a generic feature of the Laplace transform of retarded Lorentz invariant functions [77]. In spite of the existence of these massive modes we

[19] Effectively the Wheeler propagator is a sum of $1/f(k^2)$ with delta functions $\delta(k^0 \pm \Omega^{(*)})$ such that the singular behaviour of $1/f$ at its complex roots is cancelled by the delta functions.

showed that, in accordance with the analysis of Barnaby and Kamran [30], only massless states appear in the asymptotic state space of the quantum theory. In four dimensions, where the solution space to the non-local equations of motion is augmented by the conjugate pair of complex mass modes, we found that by constructing the quantum theory appropriately the states associated to these modes are propagated via the Wheeler propagator, ensuring that they do not appear asymptotically and thus removing possible instabilities in the theory. [20]

Note that in [195], Aslanbeigi and Saravani consider non-local scalar field theories inspired by Causal Set theory. In particular, they investigate operators which present neither complex poles nor negative definite two-point functions (as it happens here in $2d$). This stable, by construction, theories are quatized via the Schwinger–Keldysh formalism and the final results — form of the general solution, Hamiltonian and two-point functions — agree with the ones presented in this Chapter. Finally, in [194, 195] also the interacting case is considered.

The above results lead to a wide range of interesting questions that in our mind require further exploration. First, it would be interesting to find out what role the continuum of massive modes plays in the interacting theory, and whether the improved UV behaviour of the $4d$ theory (after renormalisation, see Section 4.4.3) is due to its presence. To this end note that in two dimensions the discontinuity function $\delta^+ G(k)$ is positive definite, ensuring that all massive modes in its support have positive energy, while in four dimensions it is not. It is natural therefore to ask whether the improved UV behaviour in four dimensions is related to this fact, or whether the existence of such negative energy states end up spoiling the theory irrevocably, e.g., by breaking unitarity.

Secondly, it remains to be shown that the complex mass modes present in the $4d$ theory do not spoil unitarity themselves. Similar work in this direction was done by Bollini and Oxman [53], who showed that a higher order theory containing a conjugate pair of complex mass solutions is unitary. Whether their analysis will continue to hold in the interacting theory, and whether it is applicable to our case remains to be investigated.

Thirdly it would be interesting to compare the above results with those of [132] in which the causet Feynman propagator is constructed. This could shed light on the Sweety-Salty duality defined in Section 3.2 of [131]. Doing this will require one to (at least) evaluate the expressions for the Feynman propagator numerically.

Finally the possible phenomenological consequences of these nonlocal field theories are manifold. Recall from Section 4.3 that one of the features of the $4d$ theory is that it fails to satisfy Huygens' principle, i.e., there is a "leakage" inside the light cone of the field emitted from a delta function source. As discussed before, this could prove fruitful in testing the theory, since one can envisage performing high precision tests of radiation emitted by very localised sources to check if such afterglow is present. A key

[20] It is interesting to note that Barnaby and Kamran [30] suggest an alternative solution to the instabilities problem (i.e., the existence of complex mass solutions to (4.19)), consisting in specifying a contour in the complex k^0-plane which does not enclose the unstable modes. Their claim being that a choice of contour is an integral part in the construction of a nonlocal QFT. If our analysis of the unstable modes continues to stand strong once interactions are introduced, then it might not be necessary to exclude such modes by hand.

feature of this phenomenon, due to its intrinsic Lorentz invariance, is that it will be frequency independent. Another interesting phenomenological avenue is to study the non-relativistic limit of (4.15). It is well known that the Schrödinger equation is the non-relativistic limit of the Klein-Gordon equation. Therefore, the non-relativistic limit of (4.15) will give rise to a non-local generalisation of the Schrödinger equation whose non-locality is again parametrised by ρ and is such that the usual Schrödinger equation is recovered in the limit $\rho \to \infty$ [41]. This will be the focus of Chapter 6, albeit in the more manageable case of analytic $f(\Box - m^2)$.

Chapter 5

Curved Spacetime and Dimensional reduction in Causal Set theory

No great discovery was ever made without a bold guess.

Isaac Newton

In this Chapter, which builds upon the previous one, we are going to consider two different and separate aspects of non-local Causal set d'Alembertians. Firstly, we will consider a completely classical problem, i.e., finding the local limit of the entire family of Generalized Causal set d'Alembertians (GCD) (introduced in Sec. 4.1) in curved spacetime. In doing so, we will show that Einstein Equivalence principle (EEP) might be violated by these operators. Secondly, we will turn to a genuinely quantum problem and analyze the running of the spectral dimension associated to the non-local d'Alembertians in flat spacetime. In this context, we will show that dimensional reduction to $2d$ happens in general due to the properties of these operators.

5.1 Curved Spacetime d'Alembertians

In Chapter 4 the general expression for the retarded, non-local, continuum operator $\Box_{nl}$ was given in eq. (4.2) in terms of a generic Lorentzian metric g. However, in that occasion we focused our attention on the properties of this operators in flat spacetime. Here we show that all GCD, when averaged over all sprinklings on a given curved spacetime, reduce in the local limit to the covariant d'Alembertian plus a term proportional to the scalar curvature. In particular, they reduce to

$$\Box_g \phi(x) - \frac{1}{2} R(x)\phi(x)$$

in the local limit for all dimensions, i.e., the factor $-R/2$ is universal.

This result is a generalization of what founded in [80, 102] for the minimal operators which, as we discussed in Sec. 4.1, are only a subset of all GCD. We will discuss the implications of this result in relation to the Einstein Equivalence Principle in the conclusions of this Chapter.

Before starting, let us note that the presence of the Ricci scalar in the local limit of these non-local operators gave rise to the proposal for a curvature estimator in CS theory and, as a related result, for a Causal Set action [44, 45]. This paves the way for the study of a path-integral formulation of CS theory, as discussed in Sec. 1.4.4.

5.1.1 General set-up

In this section we introduce the family of GCD [17] and outline the general set-up used in the rest of the first part of this Chapter.

Given a causal set $\mathcal{C}$ and a scalar field $\phi : \mathcal{C} \to \mathbb{R}$ on it, let us consider the operators defined in eq. (4.1)

$$(B_\rho^{(d)}\phi)(x) = \rho^{2/d}\left(a\,\phi(x) + \sum_{n=0}^{L_{max}^{(d)}} b_n \sum_{y\in I_n(x)} \phi(y)\right).$$

As discussed in Sec. 4.1, the form of these operators is derived under rather general and reasonable physical assumptions [17] and their average version, i.e., eq. (4.2), reduces to the standard wave operator in flat spacetime, in the local limit, for particular choices of coefficients $\{a, b_n\}$. Once these coefficients are identified, the operators can also be studied for a generic curved spacetime [36, 80, 102].

Relations defining the $\{a, b_n\}$ coefficients

GCD were studied in flat spacetime in [17], where relations defining the coefficients $\{a, b_n\}$ were found via a spectral analysis. We give an overview of these relations, in even dimensions, which are fundamental for our discussion.
The coefficients $\{a, b_n\}$ in even dimensions, defining $d = 2N + 2$ with $N = 0, 1, \ldots$, are determined by the following equations

$$\sum_{n=0}^{L_{max}} \frac{b_n}{n!}\Gamma(n + \frac{k+1}{N+1}) = 0, \quad k = 0, 1, \ldots, N+1 \tag{5.1a}$$

$$a + \frac{2(-1)^{N+1}\pi^N}{N! D^2 C_D} \sum_{n=0}^{L_{max}} b_n \psi(n+1) = 0 \tag{5.1b}$$

$$\sum_{n=0}^{L_{max}} \frac{b_n}{n!}\Gamma(n + \frac{N+2}{N+1})\psi(n + \frac{N+2}{N+1}) = \frac{2(-1)^N (N+1)!}{\pi^N} D^2 C_D^{\frac{N+2}{N+1}}, \tag{5.1c}$$

where $C_d = \frac{(\pi/4)^{\frac{d-1}{2}}}{d\Gamma(\frac{d+1}{2})}$, Γ is the Gamma function and ψ stems for the Digamma function.
Note that we have $N+4$ equations. This means that in the minimal cases, given by $L_{max}^{(d)} = \frac{D+2}{2}$, we have a unique solution corresponding to the minimal operators. In the non-minimal cases instead, the number of equations is less than the unknowns making the system under-determined therefore admitting an infinite number of solutions, in accordance with the discussion in Sec. 4.1.

The continuum operators given by eq. (4.2) can be rewritten in the following form[1]

$$\Box_{nl}\phi(x) = \rho^{2/d}a\,\phi(x) + \rho^{(2+d)/d}\hat{O}\underbrace{\int_{J^-(x)}\sqrt{-g}\;e^{-\rho V(x,y)}\phi(y)d^dy}_{I^{(d)}}, \quad (5.2)$$

where the operator $\hat{O}$ is defined as

$$\hat{O} = \sum_{n=0}^{L_{max}} \frac{b_n}{n!}(-1)^n H_n, \quad (5.3)$$

and $H_n \equiv \rho^n \partial^n/\partial\rho^n$. Note that $\hat{O}\rho^n \propto \rho^n$.

In the following we use the geometrical set-up of [36] (see also [80]) and we assume that the scalar field has compact support of size smaller than the curvature radius and such that it varies slowly on scales of the order of the non-locality scale l_n. In [36] it was shown that the finite contributions to the local limit come from the so called *near region* (W_1), i.e., the region of the past light cone of a chosen point x, that is a neighbourhood of the point itself, see figure 5.1. Motivated by this result (see also discussion below) we focus on the integral $I^{(d)}$, in eq. (5.2), restricted to W_1 defined by

$$W_1 := \left\{y \in supp(\phi) \cap J^-(x) : 0 \le u \le v \le \tilde{a}\right\},$$

where $\tilde{a} > 0$ is chosen small enough for the expansions (introduced below) to be valid but such that the near region is much larger than the non-locality scale, i.e., $\rho\,\tilde{a}^d \gg 1$ (for more details on the geometrical constructions we refer the reader to [36]). In this region we use (past pointing) null Riemann normal coordinates $(u, v, \varphi_1, ..., \varphi_{d-2})$ defined by $u = (-y^0 - r)/\sqrt{2}$, $v = (-y^0 + r)\sqrt{2}$, where $r = \sqrt{\sum_{i=1}^{d-1} y^{i2}}$ and $\{y^\mu\}$ are the RNC.

The integral $I^{(d)}$ restricted to the near region in these coordinates takes the form

$$I^d_{W_1} = \int_0^{\tilde{a}}\int_0^v du \int d\Omega_{d-2}\frac{(v-u)^{d-2}}{2^{(d-2)/2}}\sqrt{-g(y)}\phi(y)e^{-\rho V(y)}. \quad (5.4)$$

As already discussed in [80] (see Sec.3 therein) assuming the size of the compact support of the field to be smaller than the curvature radius implies that all the curvature corrections to the flat space integral will be small. We can assume then that eq. (5.2) will converge to a local result as $\rho \to \infty$ as in the flat case. Thus, by dimensional arguments, the limit $\rho \to \infty$ of eq. (5.2) will be a linear combination of the d'Alembertian and the Ricci scalar curvature.

What we will show in the following section is that, when the aforementioned local limit exists, the limit will be $\Box_g - R/2$ for all GCD. The existence of a local limit is already assumed in earlier work [80, 102], where higher order terms were neglected due to this assumption and the dimensional arguments. In this Chapter, even if we stick to the same assumption, we argue that the local limit always exists based on the results in [36, 44] (see

[1]In the following, we are going to suppress the superscript indicating the dimensional dependence when no confusion is possible.

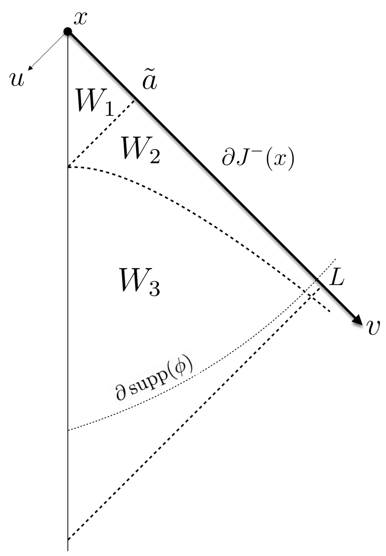

FIGURE 5.1: Graphical representation of the geometrical construction with $d-2$ dimensions suppressed. The dotted line represents the boundary of the intersection of the support of the field and the past light cone of the origin. Regions W_2 and W_3 are called *down the light cone region* and *far region* respectively and are not considered in this work (see for further details [36]) . W_1 is the near region that we consider.

appendix D.1 and the following discussion). It should be noted that our proof strictly holds true when the assumption that the compact support of the field is much smaller than the curvature scale is fulfilled. This implies that we do not have to consider regions $W_{2,3}$ at all, in the limit. However, this assumption could be relaxed by including regions $W_{2,3}$. Although such a proof goes beyond the scope of this thesis, note that in [36] the complete proof of the existence of a local limit in $4d$ curved spacetimes for the minimal operator is given considering all the terms and all the regions $W_{1,2,3}$. Since only the properties of the operator $\hat{O}$ are used to complete the proof, there is no reason why the same construction cannot be extended in all dimensions and to non-minimal operators. This lends strong support to the conjecture that the universality we are going to prove in what follows extends to configurations like the one in figure 5.1, where the support of the field is not restricted to be in the near region.

In order to proceed, we expand the volume element and the volume of causal intervals around the origin up to order R^2 terms (see [80, 97]) and the field up to second derivative terms

$$\sqrt{-g} = 1 - \frac{1}{6}R_{\mu\nu}(0)y^\mu y^\nu + \mathcal{O}(R^2) \equiv 1 + \delta\sqrt{-g} + \mathcal{O}(R^2), \tag{5.5}$$

$$V(y) = V_0^d\left(1 - \frac{d}{24(d+1)(d+2)}R(0)\tau^2 + \frac{d}{24(d+1)}R_{\mu\nu}(0)y^\mu y^\nu + \mathcal{O}(R^2)\right) \tag{5.6}$$

$$\equiv V_0 + \delta V + \mathcal{O}(R^2),$$

$$\phi(y) = \phi(0) + y^\mu \phi_{,\mu}(0) + \frac{1}{2} y^\nu y^\mu \phi_{,\nu\mu}(0) + y^\mu y^\nu y^\sigma \Phi_{\mu\nu\sigma}(y), \tag{5.7}$$

where $\tau^2 = 2uv$, $V_0^d = C_d \tau^d = 2^{d/2} C_d (uv)^{d/2} \equiv c_d (uv)^{d/2}$ is the volume of a causal interval in d-dimensional Minkowski spacetime between the origin and the point with Cartesian coordinates $\{y^\mu\}$ and $\Phi_{\mu\nu\sigma}(y)$ is a smooth function of y. Using these expansions we can then write eq. (5.4) as

$$\begin{aligned} I^{(d)} = & \int_0^{\tilde{a}} dv \int_0^v du \frac{(v-u)^{d-2}}{2^{(d-2)/2}} \int d\Omega_{d-2} \\ & \Big[(1 + \delta\sqrt{-g}) \cdot (\phi(0) + y^\mu \phi_{,\mu}(0) + \frac{1}{2} y^\nu y^\mu \phi_{,\nu\mu}(0)) \cdot (1 - \rho \delta V) \\ & + (1 + \delta\sqrt{-g}) \cdot (\phi(0) + y^\mu \phi_{,\mu}(0) + \frac{1}{2} y^\nu y^\mu \phi_{,\nu\mu}(0)) \cdot \sum_{k=2}^{\infty} \frac{(-\rho \delta V)^k}{k!} \Big] e^{-\rho V_0}. \end{aligned} \tag{5.8}$$

where we neglected terms $\mathcal{O}(R^2)$ and with more than two derivatives of the field coming from eqs. (5.5), (5.6) and (5.7). Since we are interested only in the local limit in the following we also neglect all terms $\mathcal{O}(R^2)$ and $R\,\partial\phi$, $R\,\partial^2\phi$ in eq. (5.8). Note that these terms are relevant when the nonlocality scale is not strictly vanishing (see [36] for details in $4d$ with the minimal operator). We consider these terms schematically in appendix D.1, and argue that they do not contribute in the local limit.

Performing the integration over the spherical coordinates and using that

$$\int d\Omega_{d-2} (y^j)^2 = \frac{1}{d-1} \int d\Omega_{d-2} = \omega_{d-1},$$

where ω_{d-1} is the volume of the Euclidean ball of radius one in $d-1$ dimensions, we arrive at

$$\begin{aligned} I^{(d)} = & \int_0^{\tilde{a}} dv \int_0^v du \frac{(v-u)^{d-2}}{2^{(d-2)/2}} \\ & \big[(d-1)\omega_{d-1}\phi(0) + (d-1)\omega_{d-1} y^0 \phi_{,0}(0) \\ & + \frac{(u+v)^2}{2} \omega_{d-1}(d-1) \left(\frac{1}{2}\phi_{,00}(0) - \frac{1}{6} R_{00}(0)\phi(0) - \rho\, \phi(0) c_d (uv)^{d/2} \frac{d}{24(d+1)} R_{00}(0) \right) \\ & + \frac{(v-u)^2}{2} \omega_{d-1} \left(\frac{1}{2}\phi_{,ii}(0) - \frac{1}{6} R_{ii}(0)\phi(0) - \rho\, \phi(0) c_d (uv)^{d/2} \frac{d}{24(d+1)} R_{ii}(0) \right) \\ & + \omega_{d-1}(d-1)\rho\, \phi(0) c_d (uv)^{1+d/2} \frac{2d}{24(d+2)(d+1)} R(0) \Big] e^{-\rho V_0}, \end{aligned} \tag{5.9}$$

where repeated indices are summed over and we have used

$$V_0 = C_d \tau^d = c_d (uv)^{d/2}.$$

For later convenience note that

$$\omega_{d-1} = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2} + 1\right)}. \tag{5.10}$$

5.1.2 Universality of $-R/2$ factor:

In order to prove the universality of the $-R/2$ factor we will construct operators $\hat{O}$ such that (see eq. (5.2))

$$\lim_{\rho\to+\infty} \Box_{nl}\phi = \Box_g\phi + aR\phi \tag{5.11}$$

and then prove that $a = -1/2$ for the entire family of GCD. In particular, we will prove this result for even dimensions. The proof can be easily extended in odd dimensions, see [34] for further details.

We firstly construct operators $\hat{O}$ that annihilate terms in eq. (5.9) that would give rise to divergences in the local limit. In this way, we are going to recover eq. (5.1a). Then we will choose $\hat{O}$ such that we recover the d'Alembertian in the local limit and prove that this implies $a = -1/2$ in eq. (5.11) for the entire family of GCD.

Let us start by noting that, all the integrals appearing in eq. (5.9) (for $d > 2$) are of the general form

$$I_{m,n} = \int_0^{\tilde{a}} dv \int_0^{v} du\, u^m v^n e^{-\rho\, c_d (u\, v)^{d/2}}, \tag{5.12}$$

with non-negative m, n and $m + n = d-2,\ d-1,\ d,\ 2d$. This can be recognized to be true by a binomial expansion of the term $(u-v)^{d-2}$ in eq. (5.9).

In even dimensions we show that, eq. (5.12) with $m \neq n$ gives divergent contributions in the local limit. Constructing the operators $\hat{O}$ such that they annihilate these divergences ensures that they also annihilate divergent terms coming from eq. (5.12) with $m = n$. However when $m = n$ logarithmic terms, which are not annihilated by $\hat{O}$, are also present and they lead to the finite contributions in the local limit.

We discuss first the case of even dimensions with $d > 2$ in detail and then briefly analyze the $2d$ case.

Case $m \neq n$

Consider $I_{m,n}$ for $m \neq n$,

$$\begin{aligned} I_{m\neq n} = {}& \frac{\tilde{a}^{-m}}{(m-n)d/2}\frac{1}{(c_d\rho)^{\frac{2+2m+n}{d}}}\left[\tilde{a}^n (c_d\rho)^{\frac{n}{d}}\left(-\Gamma\left(\frac{1+m}{d/2}\right)+\Gamma\left(\frac{1+m}{d/2},\tilde{a}^d c_d\rho\right)\right)\right. \\ &\left. +\tilde{a}^m (c_d\rho)^{\frac{m}{d}}\left(\Gamma\left(\frac{2+m+n}{d}\right)-\Gamma\left(\frac{2+m+n}{d},\tilde{a}^d c_d\rho\right)\right)\right], \end{aligned} \tag{5.13}$$

where $\Gamma(\cdot,\cdot)$ are incomplete Gamma functions which are exponentially vanishing in the local limit. The terms potentially divergent in the local limit are of the form

$$\rho^{-\frac{2+2m}{d}}, \quad \rho^{-\frac{2+m+n}{d}},$$

for $m + n \neq 2d$ and

$$\rho^{-\frac{2+2m}{d}+1}, \quad \rho^{-\frac{2+m+n}{d}+1},$$

for $m + n = 2d$. Note that, terms with $m + n = 2d$ are multiplied by ρ in eq. (5.9) and m is always at least $d/2$ thanks to the presence of the $(uv)^{d/2}$ factor.

Given the factor $\rho^{\frac{2+d}{d}}$ in eq. (5.2), a sensible requirement for the operator $\hat{O}$ is to annihilate terms proportional to $\rho^{-\alpha}$ with

$$\frac{2+d}{d} - \alpha \geq 0.$$

Indeed those are the terms that diverge for $\rho \to \infty$, i.e., in the local limit. The need for annihilate $\rho^{(d+2)/d}$ will be fully clarified in the next section. We consider the two relevant subcases, $m+n \neq d-1$ and $m+n = d-1$, separately since the second one shows why we require $\hat{O}$ to annihilate also $\rho^{(d+2)/d}$, which in principle should give a finite contribution in the local limit.

When $m+n \neq D-1$, the relevant terms in $I_{m,n}$ can be written as $\rho^{-p/D}$ with p a positive and even integer. We require that $\hat{O}$ annihilates terms for which

$$\frac{2+d}{d} - \frac{p}{d} \geq 0 \Rightarrow \; p \leq 2+d, \tag{5.14}$$

i.e.,

$$\hat{O}\rho^{-\frac{2(k+1)}{d}} = 0, \quad k = 0, 1, 2, \ldots, \frac{d}{2}. \tag{5.15}$$

These are $(d+2)/2$ requirements, i.e., $N+2$ requirements (defining $d = 2N+2$) exactly as many as in eq. (5.1a). Using

$$\begin{aligned} H_n\left(\frac{1}{\rho^{\frac{2(k+1)}{d}}}\right) &= \frac{1}{\rho^{\frac{2(k+1)}{d}}}(-1)^n \prod_{s=0}^{n-1}(\frac{2(k+1)}{d} + s) \\ &= (-1)^n \frac{1}{\rho^{\frac{2(k+1)}{d}}} \frac{2(k+1)}{d} \frac{\Gamma[\frac{2(k+1)}{d} + n]}{\Gamma[1 + \frac{2(k+1)}{d}]}. \end{aligned} \tag{5.16}$$

one can show that (see eq. (5.3))

$$\begin{aligned} \hat{O}\rho^{-\frac{2(k+1)}{d}} &= \sum_{n=0}^{L_{max}} \frac{b_n}{n!} \frac{2(k+1)}{d} \frac{\Gamma[\frac{2(k+1)}{d} + n]}{\Gamma[1 + \frac{2(k+1)}{d}]} \\ &= \frac{2(k+1)}{d} \frac{1}{\Gamma[1 + \frac{2(k+1)}{d}]} \sum_{n=0}^{L_{max}} \frac{b_n}{n!} \Gamma\left[\frac{2(k+1)}{d} + n\right], \end{aligned} \tag{5.17}$$

which proves the equivalence of eqs. (5.15) and (5.1a).

For $m+n = D-1$, consider the term

$$\frac{\phi_{,0}(0)}{\sqrt{2}}\left(-\int_0^{\tilde{a}} dv \int_0^v du \frac{(v-u)^{d-2}}{2^{(d-2)/2}}(u+v)e^{-\rho'(u\,v)^{d/2}}\right), \tag{5.18}$$

where $\rho' \equiv c_d \rho$ and we now define $d = 2M$. Using the change of variables

$$\begin{aligned} x &\equiv u^M, \\ y &\equiv v^M, \end{aligned} \tag{5.19}$$

and the binomial expansion, the general integrals appearing in eq. (5.18) are ($M > 1$ since $d > 2$)

$$\int_0^{\tilde{a}^M} dy \int_0^y dx\, y^{\frac{M-1-k}{M}} x^{\frac{2+k-M}{M}} = \tag{5.20}$$
$$\frac{M\rho'^{-1-\frac{1}{2M}}\Gamma\left(1+\frac{1}{2M}\right)}{3+2k-2M} - \frac{\tilde{a}^{2M-3-2k}M\rho'^{-\frac{2+k}{M}}\Gamma\left(\frac{2+k}{M}\right)}{3+2k-2M}$$
$$-\frac{M\rho'^{-1-\frac{1}{2M}}\Gamma\left(1+\frac{1}{2M},\tilde{a}^{2M}\rho'\right)}{3+2k-2M} + \frac{\tilde{a}^{2M-3-2k}M\rho'^{-\frac{2+k}{M}}\Gamma\left(\frac{2+k}{M},\tilde{a}^{2M}\rho'\right)}{3+2k-2M},$$

$$\int_0^{\tilde{a}^M} dy \int_0^y dx\, y^{\frac{M-k}{M}} x^{\frac{1+k-M}{M}} = \tag{5.21}$$
$$\frac{M\rho'^{-1-\frac{1}{2M}}\Gamma\left(1+\frac{1}{2M}\right)}{1+2k-2M} - \frac{\tilde{a}^{2M-1-2k}M\rho'^{-\frac{1+k}{M}}\Gamma\left(\frac{1+k}{M}\right)}{1+2k-2M}$$
$$-\frac{M\rho'^{-1-\frac{1}{2M}}\Gamma\left(1+\frac{1}{2M},\tilde{a}^{2M}\rho'\right)}{1+2k-2M} + \frac{\tilde{a}^{2M-1-2k}M\rho'^{-\frac{1+k}{M}}\Gamma\left(\frac{1+k}{M},\tilde{a}^{2M}\rho'\right)}{3+2k-2M}.$$

Note that since k and M are integers the denominators never vanish. The incomplete Gamma functions do not contribute in the local limit, thus the relevant terms are

$$\rho^{-\frac{d+1}{d}}, \quad \rho^{-\frac{2+k}{M}},$$

from eq. (5.20) and

$$\rho^{-\frac{d+1}{d}}, \quad \rho^{-\frac{1+k}{M}},$$

from eq. (5.21). To summarize:

- Terms proportional to $\rho^{-\frac{2+k}{M}} = \rho^{-\frac{4+2k}{d}}$ give a divergent or a constant term in the local limit. These terms are such that

  $$\frac{d+2}{d} - \frac{4+2k}{d} \geq 0 \Rightarrow k \leq M-1,$$

  i.e., they are of the form $\rho^{-p/d}$ with p even and less than or equal to $d+2$ (see eq. (5.14)), as such they are already annihilated by $\hat{O}$, see eqs. (5.14) and (5.15).

- Terms proportional to $\rho^{-\frac{d+1}{d}}$ sum to zero

$$-\rho^{-\frac{d+1}{d}}\frac{d}{2}\Gamma(1+\frac{1}{d})\sum_{k=0}^{d-2}\binom{d-2}{k}(-1)^k\frac{4+4k-2d}{(3+2k-d)(1+2k-d)} = 0. \tag{5.22}$$

We see that $\hat{O}\rho^{-(d+2)/d} = 0$ comes from requiring the correct IR behavior of the operator, in the sense of obtaining $\square$ rather than some other combination of derivatives.

Case $m = n$

Let us consider the terms with $m = n$. Note that m can assume the values $\frac{d-2}{2}, \frac{d}{2}, d$, where the $I_{d,d}$ are also multiplied by a factor of ρ. The terms of interest in eq. (5.9) are given by

$$\frac{1}{2^{\frac{d-2}{2}}}\Big\{(d-1)\omega_{d-1}\phi(0)A_0 I_{\frac{d-2}{2},\frac{d-2}{2}} \tag{5.23}$$
$$+A_1\left[\frac{1}{2}\omega_{d-1}\left(\frac{d-1}{2}\phi_{,00}(0)-\frac{1}{6}\phi(0)(d-1)R_{00}\right)\right]I_{d/2,d/2}$$
$$+A_2\left[\frac{1}{2}\omega_{d-1}\left(\frac{1}{2}\phi_{,ii}-\frac{1}{6}\phi(0)R_{ii}\right)\right]I_{d/2,d/2}$$
$$+A_1\left[\frac{\omega_{d-1}}{2}\left(-\frac{d}{24(d+1)}c_d\phi(0)(d-1)R_{00}\right)\right]\rho I_{d,d}$$
$$+A_2\left[\frac{\omega_{d-1}}{2}\left(-\frac{d}{24(d+1)}c_d\phi(0)R_{ii}\right)\right]\rho I_{d,d}$$
$$+A_5\left[\frac{2d}{24(d+1)(d+2)}\omega_{d-1}(d-1)c_d\phi(0)R\right]\rho I_{d,d}\Big\},$$

where

$$A_0 \equiv \binom{d-2}{\frac{d-2}{2}}(-1)^{\frac{d-2}{2}}, \tag{5.24}$$
$$A_1 \equiv \left[\binom{d-2}{\frac{d-4}{2}}(-1)^{\frac{d-4}{2}}+\binom{d-2}{\frac{d-2}{2}}2(-1)^{\frac{d-2}{2}}+\binom{d-2}{\frac{d}{2}}(-1)^{\frac{d}{2}}\right], \tag{5.25}$$
$$A_2 \equiv \left[\binom{d-2}{\frac{d-4}{2}}(-1)^{\frac{d-4}{2}}-\binom{d-2}{\frac{d-2}{2}}2(-1)^{\frac{d-2}{2}}+\binom{d-2}{\frac{d}{2}}(-1)^{\frac{d}{2}}\right], \tag{5.26}$$
$$A_5 \equiv \binom{d-2}{\frac{d-2}{2}}(-1)^{\frac{d-2}{2}}. \tag{5.27}$$

The general $I_{m,m}$ can be computed with the change of variables in eq. (5.19),

$$I_{d/2,d/2} = \frac{2}{d^2}\rho'^{-\frac{d+2}{d}}\left[G^{3,0}_{2,3}\left(\tilde{a}^d\rho' \middle| \begin{matrix}1,1\\ 0,0,1+\frac{2}{d}\end{matrix}\right)\right. \tag{5.28}$$
$$\left.+\Gamma\left(1+\frac{2}{d}\right)\left(\log(\tilde{a}^d\rho')-\psi\left(1+\frac{2}{d}\right)\right)\right],$$
$$I_{\frac{d-2}{2},\frac{d-2}{2}} = \frac{4}{d^2}\frac{\Gamma\left(0,\tilde{a}^d\rho'\right)+\log(\tilde{a}^d\rho')+\gamma}{2\rho'}, \tag{5.29}$$
$$I_{d,d} = \frac{2}{d^2}\rho'^{-\frac{2}{d}-2}\left[G^{3,0}_{2,3}\left(\tilde{a}^d\rho' \middle| \begin{matrix}1,1\\ 0,0,2+\frac{2}{d}\end{matrix}\right)\right. \tag{5.30}$$
$$\left.+\Gamma\left(2+\frac{2}{d}\right)\left(\log(\tilde{a}^d\rho')-\psi\left(2+\frac{2}{d}\right)\right)\right],$$

where again $\rho' = c_d\rho$. The only terms that give finite contributions in the local limit are the logarithmic ones[2]. Indeed, terms proportional to powers of ρ are annihilated[3] by $\hat{O}$ (see eq. (5.15)), whereas terms containing the $G^{3,0}_{2,3}$ Meijer's G-function do not contribute since these functions decay exponentially fast in the local limit (see [103]).

[2]Actually, the logarithmic term in eq. (5.29) serves the purpose of eliminating the constant term appearing on the RHS of eq. (5.2), see next section.

[3]Note that the terms in eq. (5.30) are multiplied by ρ.

First condition: eliminating the constant

In order to not have divergences in the local limit, we need to cancel the first term appearing on the RHS of eq. (5.2). From eq. (5.23), we have to impose

$$\rho^{\frac{2+d}{d}}\frac{1}{2^{\frac{d-2}{2}}}(d-1)\omega_{d-1}A_0\hat{O}\underbrace{\frac{Log[\tilde{a}^d c_d\rho]}{2(\frac{d}{2})^2 c_d\rho}}_{\subset I_{\frac{d-2}{2},\frac{d-2}{2}}} = -\rho^{2/d}a. \tag{5.31}$$

Using expression eq. (5.10), $d = 2N+2$, $c_d = 2^{N+1}C_{2N+2}$ and noting that

$$A_0 = (-1)^N\frac{(2N)!}{(N!)^2}$$

we can rewrite (5.31) as

$$\frac{1}{2^N}(-1)^N\frac{(2N)!}{(N!)^2}(2N+1)\frac{2(4\pi)^N N!}{(2N+1)!}\frac{1}{2(N+1)^2 2^{N+1}C_{2N+2}}\hat{O}\frac{Log[\tilde{a}^d c_d\rho]}{\rho} = -\frac{a}{\rho}. \tag{5.32}$$

It can be proven that this last equation is equivalent to eq. (5.1b), see appendix D.2.1 for details.

Second condition: finding the d'Alembertian

We now proceed by choosing $\hat{O}$ such that we recover the d'Alembertian in the local limit and prove that this implies $a = -1/2$ in eq. (5.11) for the entire family of GCD. In order to obtain the d'Alembertian from terms involving two derivatives of the field in eq. (5.9) we require that (see eq. (5.23))

$$\lim_{\rho\to\infty}\left\{\rho^{\frac{d+2}{d}}\frac{1}{2^{\frac{d}{2}}}\omega_{d-1}\left[\frac{d-1}{2}\phi_{,00}A_1+\frac{1}{2}\phi_{,ii}A_2\right]\hat{O}I_{d/2,d/2}\right\} = \Box\phi. \tag{5.33}$$

This is equivalent to considering the action of $\hat{O}$ on the logarithmic term in eq. (5.28) (see discussion thereafter). From $-(d-1)/2\ A_1 = A_2/2$ we have

$$\frac{1}{2^{\frac{d}{2}}}\omega_{d-1}\left[\frac{d-1}{2}\phi_{,00}A_1+\frac{1}{2}\phi_{,ii}A_2\right] = \frac{1}{2^{\frac{d-2}{2}}}\frac{1}{2}\omega_{d-1}\frac{A_2}{2}\Box\phi(0), \tag{5.34}$$

and using eq. (5.34) in eq. (5.33) we obtain

$$\rho^{\frac{d+2}{d}}\hat{O}\frac{\Gamma[\frac{d+2}{d}]}{2(\frac{d}{2})^2(c_d)^{\frac{d+2}{d}}\rho^{\frac{d+2}{d}}}\log(\tilde{a}^d c_d\rho) = \frac{2^{\frac{d+2}{2}}}{\omega_{d-1}A_2}. \tag{5.35}$$

It can be shown that this last equation is equivalent to eq. (5.1c), see appendix D.2.2 for details. This result should not come as a surprise since we had required to obtain the d'Alembertian in the first place. Up to now, we have bridged the gap between the formalism of [17] and the one of [36, 44, 80]. We now show that the conditions we have found imply $a = -1/2$ in eq. (5.11), i.e., that the factor $-R/2$ is universal for all GCD (in even dimensions).

Third condition: universal factor

Finally we must consider terms in eq. (5.23) that contain curvatures and are given by

$$\frac{1}{2^{\frac{d-2}{2}}}\left\{-\frac{\omega_{d-1}}{12}A_2R\phi(0)I_{d/2,d/2}\right. \tag{5.36}$$
$$-\frac{\omega_{d-1}}{2}A_2\frac{d}{24(d+1)}c_dR\phi(0)\rho I_{d,d}$$
$$\left.+\omega_{d-1}(d-1)A_5\frac{2d}{24(d+1)(d+2)}c_dR\phi(0)\rho I_{d,d}\right\},$$

where we used $A_1 = -A_2/(d-1)$. Note that the action of $\hat{O}$ on the first term in eq. (5.36) is completely determined by eq. (5.35) and gives

$$-\frac{R}{3}\phi. \tag{5.37}$$

For the other terms we need to compute $\hat{O}\rho I_{d,d}$. Since we are interested in the local limit we focus on the logarithmic term of $I_{d,d}$ (see eq. (5.30) and discussion thereafter)

$$\hat{O}\left(\frac{2}{d^2c_d^{\frac{2d+2}{d}}}\Gamma(\frac{2d+2}{d})\frac{\log(\tilde{a}^d\rho)}{\rho^{\frac{d+2}{d}}}\right). \tag{5.38}$$

It is easy to see that eq. (5.38) is determined by eq. (5.35) since

$$\rho^{\frac{d+2}{d}}\hat{O}\left(\frac{2}{d^2}\Gamma(\frac{2d+2}{d})\frac{\log(\tilde{a}^d\rho)}{\rho^{\frac{d+2}{d}}}\right) \tag{5.39}$$
$$=\rho^{\frac{d+2}{d}}\frac{\Gamma(\frac{2d+2}{d})}{\Gamma(\frac{d+2}{d})c_d}\hat{O}\left(\frac{\Gamma[\frac{d+2}{d}]}{2(\frac{d}{2})^2(c_d)^{\frac{d+2}{d}}\rho^{\frac{d+2}{d}}}\log(\tilde{a}^dc_d\rho)\right)$$
$$=\frac{\Gamma(\frac{2d+2}{d})}{\Gamma(\frac{d+2}{d})c_d}\frac{2^{\frac{d+2}{2}}}{\omega_{d-1}A_2}.$$

Using eq. (5.39) we find that the last two terms in eq. (5.36) give in the local limit

$$-\frac{1}{6}\frac{d+2}{2d+2}R\phi(0) \tag{5.40}$$
$$\frac{d-1}{d+1}\frac{1}{3}\frac{A_5}{A_2}R\phi(0), \tag{5.41}$$

where it can be shown that $A_5/A_2 = d/(4-4d)$. Summing eq. (5.37) and eq. (5.40) we finally obtain the universal factor

$$\boxed{\left(-\frac{1}{3}-\frac{1}{6}\frac{d+2}{2d+2}-\frac{d}{12(d+1)}\right)R(0)\phi(0) = -\frac{1}{2}R(0)\phi(0).} \tag{5.42}$$

Hence, we have proven that all GCD in even dimensions reduce to $(\Box - R/2)\phi$ in the local limit.

$2d$ case:

The only difference with the previous sections is that r is no more a non-negative radial coordinate and eq. (5.9) is replaced by

$$I(2) = \int_0^{\tilde{a}} dv \int_0^{\tilde{a}} du \left[\phi(0) + r\phi_{,r} + \frac{-u-v}{\sqrt{2}}\phi_{,0} \right. \\ + \frac{1}{2}\frac{(u+v)^2}{2}\phi_{,00} + \frac{1}{2}\frac{(v-u)^2}{2}\phi_{,rr} + r\frac{(v-u)}{\sqrt{2}}\phi_{,0r} \\ - \frac{1}{6}R_{rr}\phi\frac{(v-u)^2}{2} - \frac{1}{6}R_{00}\phi\frac{(v+u)^2}{2} - \frac{1}{6}R_{0r}\phi 2r\frac{-u-v}{\sqrt{2}} \\ \left. -\rho\phi uv\left(-\frac{R}{72}uv + \frac{R_{00}}{36}\frac{(v+u)^2}{2}\right)\right] e^{-\rho uv}. \tag{5.43}$$

With this clarification, it is possible to proceed in the analysis as in the previous sections. In particular, requiring $\hat{O}$ to annihilate the diverging terms (in the local limit) we arrive at eq. (5.15) for $k = 0, 1$. As in the previous section, the requirement of annihilating ρ^{-2} comes from eliminating the term with $\phi_{,0}$.

When $m = n$ the only relevant terms in the local limit are again the logarithmic ones, analogously to $d > 2$. From $I_{0,0}$ the logarithmic term is given by $\rho^{-1}\log(\tilde{a}^2\rho)$. The condition for eliminating the first term in eq. (5.2) can be obtained following the previous discussion, and proved to be equivalent to eq. (5.1b). Considering the integrals $I_{1,1}$ and $\rho I_{2,2}$, the only relevant term is $\rho^{-2}\log(\tilde{a}^2\rho)$ in both cases and the same calculations of the $d > 2$ case can be applied, which completes the proof.

5.1.3 Summary

We have studied *Generalized Causal Set d'Alembertians* in curved spacetime. In particular, we have shown that, when a local limit exists, all GCD give

$$\Box_g\phi(x) - \frac{1}{2}R(x)\phi(x),$$

in this limit.

In doing so we have bridged the gap between the formalism of [36, 44, 80] and the one of [17], showing how the equations found in [17] via a spectral analysis of the non-local operators in flat spacetime translate into properties of $\hat{\mathcal{O}}$ in the set-up of [36, 44, 80]. We have also shown that the requirements that lead to the right local limit in the flat case are sufficient to ensure the appearance of the universal $-R/2$ factor in curved spacetime for all GCD. The present result is an independent proof of the universality of the $-R/2$ factor for the entire family of non-local operators inspired by Causal set theory. Moreover, this result shows that the universal factor is not related to the minimality condition but to the physical requirements that characterize the operators.

It should be noted that the assumptions made — in particular compact support of the field — are ubiquitous in the literature [36, 44, 80, 102]. In order to weaken them, further studies are required. Extending the analysis of [36] to all GCD would allow one to fully take into account the case in which the support of the field is not restricted to the near region. Even better, a spectral analysis similar to that in [17] would remove the assumption of compact support altogether and, as such, deserves further investigation.

5.2 Dimensional Reduction

In GR, and more generically in classical physics, the dimension of spacetime is a fix and unambiguous concept. Indeed, this fixed "by hand" dimension of spacetime can be seen in GR as the last vestige of background structures (together with the non-dynamical topology). It is then reasonable to believe that in a theory of quantum gravity, where the concept itself of spacetime as a smooth manifold may be ill-defined, even dimension should be an emergent concept. This call for a precise definition of what we call dimension, i.e., we need dimensional estimators [64] (see also the box in Sec. 5.2.2).

Several dimensional estimators have been defined in the quantum gravity literature [63, 64]. The striking results is that different QG theories, also employing different estimators, all point towards a reduction of the effective dimension of spacetime at small length-scales. In fact, dimensional reduction has been found in causal dynamical triangulations [10], asymptotically safe quantum gravity [138], loop quantum gravity [152], superrenormalizable gravity [153, 154, 156], noncommutative geometry [16, 42], minimal length scenario [155] and high temperature string theory [19]; and it has been recently shown to have relevant implications in cosmology [13]. Dimensional reduction thus appears to be a relevant and ubiquitous feature of quantum gravity (see [65]).

In recent years, the study of the spectral dimension as a tool to investigate the small scale structure of spacetime has attracted much attention and analysis of this quantity in the aforementioned approaches to QG all point towards a running of the spectral dimension at short scales [62] and dimensional reduction. However, the causal set approach appears to be an outlier. Indeed, results by Eichhorn and Mizera [84] (EM from here on) show that the spectral dimension of a causal set, despite running, increases at small scales. Their analysis relies on computing the spectral dimension from the return probability, P, of a random walker on a causal set with the spectral dimension[4] defined as $d_{EM} = -2\partial \ln P(s)/\partial \ln s$. They then argue that the increasing behaviour of d_{EM} at short scales is due to the radical nonlocality present in causal sets. This is the exact same non-locality which arises because of the interplay between Lorentz invariance and discreteness and appears in the construction of d'Alembertians governing the dynamics of scalar fields on causal sets (see previous Chapter). An obvious question then is whether the spectral dimension computed from such d'Alembertians, using the usual heat kernel techniques, corroborates the EM result.

The remaining of this Chapter is devoted to compute the spectral dimension of Minkowski spacetime from non-local d'Alembertians and show that dimensional reduction happens in every dimension. In particular, we determine both the small and large scale behaviour of the spectral dimension in all dimensions analytically, and for particular choices of nonlocal

[4] They also defined a "causal" spectral dimension using the meeting probability of two random walkers, P_M, both moving forward in time, which give similar results.

GCD — i.e., the minimal ones — in 2, 3 and 4 dimensions we numerically compute the full dependence of the spectral dimension on the diffusion parameter.

5.2.1 Momentum Space Nonlocal d'Alembertians

The momentum space representation of eq. (4.2) was computed in [17] and is given in eq. (4.10) which we report here for the reader convenience

$$f^{(d)}(k^2) = \rho^{2/d}\left(a + 2(2\pi)^{d/2-1} Z^{\frac{2-d}{4}} \sum_{n=0}^{L^{(d)}_{max}} \frac{b_n}{n!}\gamma_d^n \int_0^\infty ds\, s^{d(n+1/2)} e^{-\gamma_d s^d} K_{\frac{d}{2}-1}(Z^{1/2}s)\right). \tag{5.44}$$

These functions simplify considerably in the limits where $k^2 \ll \rho^{2/d}$ and $k^2 \gg \rho^{2/d}$, i.e., the infrared and ultraviolet limit respectively. By construction the IR behaviour is the same for all operators, i.e., $f^{(d)}(k^2) \to -k^2$ as $k^2 \to 0$. The UV limit on the other hand is dimension dependent, and can be shown to be

$$f^{(d)}(k^2) \to \alpha\rho^{2/d} + \beta\rho^{\frac{2}{d}+1}(k^2)^{-d/2} + \dots, \tag{5.45}$$

as $k \to \infty$, where α and β are dimension dependent constants. Note that since $f^{(d)}$ goes to a constant in the UV, the Green function (4.17) possesses a delta function divergence in the coincidence limit. This divergence can be regularised by subtracting the constant $(\alpha\rho^{2/d})^{-1}$ from the momentum space Green function [17]. Inverting back gives a *regularised* momentum space d'Alembertian

$$f^{(d)}_{reg} := \frac{\alpha\rho^{2/d} f^{(d)}}{\alpha\rho^{2/d} - f^{(d)}}, \tag{5.46}$$

which now has the following UV behaviour

$$f^{(d)}_{reg} \to -\frac{\alpha^2}{\beta}\rho^{\frac{2}{d}-1}(k^2)^{\frac{d}{2}} + \dots. \tag{5.47}$$

Note that this regularisation procedure is manifestly Lorentz invariant and is physically motivated by the underlying theory being a theory on the *discrete* causal set, where a coincidence limit does not exist.

The operators we started with in eq. (4.2) are *retarded* Lorentz invariant operators, and their Laplace transforms are therefore defined in the limit $\Im(k^0) \to 0^+$ [17, 77, 195], i.e., in the upper half complex k^0-plane. Their inverse Fourier transforms therefore yield the (unique) retarded Green functions to (4.2) and are given by the integral

$$G^{(d)}(x,y) = \int_{\Gamma_R} \frac{dk^0}{2\pi} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \frac{e^{ik\cdot(x-y)}}{f^{(d)}(k^2)}, \tag{5.48}$$

where Γ_R is a contour running from $-\infty$ to ∞ in the upper half complex k^0 plane such that all singularities of f^{-1} lie below the contour, see figure 5.2.

In the next section, in order to compute the spectral dimension using heat-kernel techniques, we will need to Wick rotate the d'Alembertian, f_{reg}, or equivalently its retarded propagator. However, such a Wick rotation cannot be performed on the retarded propagator because the contour, Γ_R,

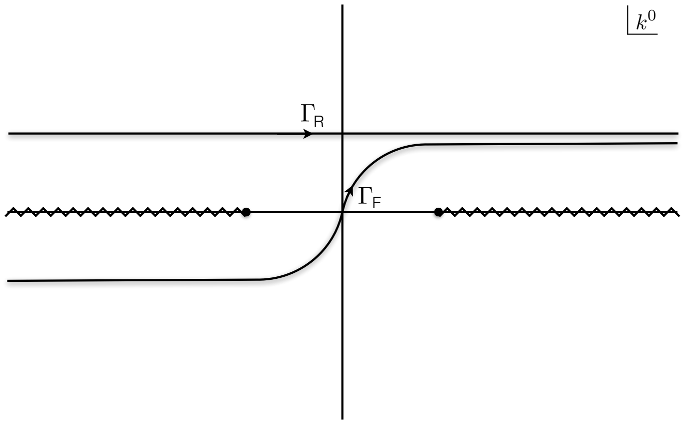

FIGURE 5.2: A schematic diagram of the singular structure of $1/f(k^2)$ in the complex k^0 plane, including two contours, Γ_R and Γ_F, which define the retarded and Feynman propagators respectively. Note that before we are able to define the latter we must first analytically continue $f(k^2)$ to the lower half the plane. The contour Γ_R will clearly encounter an obstruction under a Wick rotation, $k^0 \to -ik^0$, while Γ_F won't.

would cross singularities. To avoid this issue one must use the Feynman propagator whose contour can be Wick rotated without crossing any singularities (see figure 5.2). To define this propagator we first analytically continue f to the whole complex plane and then define the Feynman propagator to be (5.48) with f_{reg} replaced by its analytically continued version and Γ_R replaced by Γ_F. Although this propagator is not a Green function of the original retarded d'Alembertians, which indeed only admit unique retarded propagators, two independent studies of free scalar QFTs based on such nonlocal dynamics, i.e., the analysis of Chapter 4 (based on [35]) and [195], support that it is the correct Feynman propagator for such theories.

5.2.2 Spectral dimension

In this section we provide both numerical and analytical results for the spectral dimension computed from the regularised Laplace transform of the non-local d'Alembertian $f^{(d)}_{reg}$. The numerical analysis will be restricted to the minimal cases in 2, 3 and 4 spacetime dimensions d.

We first recall that the spectral dimension is defined as

$$d_s = -2\frac{\partial \ln(K(s))}{\partial \ln(s)}, \tag{5.49}$$

where $K(s) := \mathrm{Tr}(K(s;x,y))$ is the return probability for a particle, the diffusion process of which is dictated by the Laplacian of interest $\mathcal{D}$ via the equation

$$\frac{\partial}{\partial s}K(s;x,y) + \mathcal{D}K(s;x,y) = 0, \tag{5.50}$$

with initial condition $K(0;x,y) = \delta(x-y)$. Therefore, given a modified d'Alembertian, one has to first find the corresponding Laplacian in Euclidean signature, and then consider the heat equation, on the manifold of interest, determined by the Laplacian at hand [43]. Equation (5.49) can be interpreted as describing diffusion of a fictitious particle on the euclideanised spacetime, the solutions of which are given by the heat kernel. The latter, denoted as $K(s; x, y)$, physically represents the probability density of diffusion from x to y in time s. The trace of K then gives the return probability for the diffusing particle. In flat d-dimensional Minkowski spacetime with standard local d'Alembertian for example, the spectral dimension coincides with the Hausdorff dimension d of the spacetime, for all diffusion times.

More on spectral dimension and other dimensional estimators

The spectral dimension as a dimensional estimator finds its rationale in the fact that, for any space in which the diffusion process eq. (5.50) can be defined and for the standard Laplacian the trace of the heat kernel behaves like[65]

$$K(s) \sim (4\pi s)^{-d/2},$$

where d is the topological dimension. Thus, when considering a modified Laplacian, as in our case, the spectral dimension might be interpreted as the *(fictitious) topological dimension of that simple diffusion process governed by the standard Laplacian that most closely approximates $K(s)$ at the indicated value of the diffusion time s*, cit.[207]. It should be noted that, the spectral dimension encodes information on both the small-scale behaviour of QFTs and the large-scale structure of spacetime. This is so since the heat kernel encodes such information. In particular, for large diffusion times we are probing the IR regime of our theory, in this case the spectral dimension will run due to curvature effects. In an intermediate regime or, as in our case, in absence of curvature we expect the spectral dimension to agree with the topological one. Finally, for small diffusion times the running of the spectral dimension encodes information on the kinematic of the theory in the UV. Despite not being the unique generalization of dimension, d_s is particularly interesting since: it is directly related to the Green function of the theory — indeed the heat kernel can be shown to be the Laplace transform of the propagator of the theory[65, 162]

$$G(x, y) = \int_0^\infty ds K(s; x, y);$$

it can serves to find the dispersion relation characterizing a given theory [207]; and it has been proven to be the Haussdorf dimension of momentum space [12].

Other dimensional estimators which are present in the literature include: the Haussdorf dimension, the Myrheim-Mayer dimension, the "walk" dimension, various thermodynamical dimensions, "causal" spectral dimension and several others — see [64] and references therein. Recently, other dimensional estimators have been introduced which deserve further studies. In particular, in [14] the so called *thermal dimension* is introduced which is claimed to be more suitable of the spectral dimension for describing dimensional reduction, whereas still related to (deformed) d'Alembertians. Indeed, this new estimator does not require the Euclideanization of the theory which could lead, in some cases, to ambiguous results. Another dimensional estimator of interest, the *Unruh dimension*, has been introduced in [7]. This estimator is related to the spectral dimension, even though not identical to it, and has the virtue to be directly connected to a physical observables, i.e., to the rate of a unifromly accelerated Unruh detector.

Analytical results

Consider now a free massless scalar field in d-dimensional Minkowski spacetime, with dynamics defined by a member of the regularised nonlocal d'Alembertians, namely $f^{(d)}_{reg}$. The trace of the heat kernel, $K(s)$, is given by

$$K(s) = \int \frac{d^d k}{(2\pi)^d} e^{s\, f^{(d)}_{reg}} = C_d \int_0^\infty dk\, k^{d-1}\, e^{s\, f^{(d)}_{reg}}\,, \tag{5.51}$$

where C_d is a dimension dependent constant, and we have analytically continued to Euclidean signature.

The IR behaviour of $f^{(d)}_{\text{reg}} \to -k^2$, as $k^2 \to 0$, ensures that for large diffusion times the spectral dimension will flow to the value of the Hausdorff dimension d. In the UV instead we have that $f^{(d)}_{reg} \approx k^d$, so that the trace of the heat kernel can be written as

$$\int_\lambda^\infty dk\, k^{d-1}\, e^{-k^d s} = \frac{1}{s\, d}(e^{-\lambda^d s})\,, \tag{5.52}$$

up to irrelevant numerical factors. In the above integral we have introduced an IR cutoff λ, which has been chosen large enough that the UV approximation for $f^{(d)}_{reg}$ is still valid, i.e., $\lambda \gg l^{-1}$. Substituting this back into (5.49), we find

$$d_s^{UV} = 2 + 2\lambda^d s \to 2\,, \qquad \text{as } s \to 0\,. \tag{5.53}$$

This shows that the improved UV behaviour of the regularised nonlocal Green functions leads to dimensional reduction to $d_s = 2$ for all spacetime dimensions considered. Furthermore, the linear behaviour in s with large coefficient λ^d is in accordance with the numerical evidence provided in figure 5.3. It is important to note that (5.53) is only valid for small values (with respect to the non-locality scale l_n) of the diffusion parameter $s \ll l_n$.

Finally, we should pay attention to the fact that the universal dimensional reduction to $d_s = 2$, crucially relies on the regularisation of the Green functions. One might therefore ask what happens should one decide not to perform the regularisation? The first hint that something will go wrong with the computation of d_s, arises from the fact that $f^{(d)}(k^2) \to$ constant as $k^2 \to 0$, so that the integral in (5.51) clearly diverges. To analyse the small scale behaviour of d_s we must therefore introduce both an IR cutoff λ and a UV cutoff Λ. Then

$$K(s) \approx \lim_{\lambda\to 0,\, \Lambda\to\infty} \int_\lambda^\Lambda dk\, k^{d-1} e^{-c_d s} \qquad = \lim_{\lambda\to 0,\, \Lambda\to\infty} e^{-c_d s} \frac{\Lambda^d - \lambda^d}{d},$$

where we used the fact that in the UV, $f^{(d)}$ goes like a dimensionful and dimension dependent constant, c_d, which therefore dominates the integral. Substituting this into (5.49) and ignoring issues about the order of limits, we find that $d_s = 2c_d s$. This result is in accordance with the numerical simulations we have performed, and clearly leads to a spectral dimension that does not make much physical sense. Indeed, it does not asymptote the Hausdorff dimension d for $s \gg l_n$, but actually diverges.

One could have argued from the beginning that without regularisation the spectral dimension would be meaningless, and that it would possess a linear dependence in s for all s. For instance, consider the spectral dimension in a spacetime with $d = 2$. Then,

$$-2s\frac{\int_0^\infty dz\, f^{(2)}(z)e^{f^{(2)}(z)s}}{\int_0^\infty \mathrm{d}z\, e^{f^{(2)}(z)s}} = 4\rho s\,, \tag{5.54}$$

where $z = k^2$. To obtain (5.54) we split both the numerator and denominator by introducing an IR cutoff L big enough to approximate the integrand with its UV constant behaviour and use the fact that $\int_0^\infty \mathrm{d}z\; e^{f_2(z)s}$ diverges. The factor of 4 obtained in the final expression is specific to the minimal non-local d'Alembertian in $d = 2$.

Numerical results

We numerically calculate the spectral dimension in 2, 3 and 4 dimensions using the regularised minimal non-local d'Alembertians. These are given in terms of the unregularised d'Alembertians [17]:

$$f^{(2)}(k^2) = a^{(2)}\rho + 2\rho \tag{5.55}$$
$$\cdot\sum_{n=0}^{2}\frac{b_n^{(2)}}{n!}\left(\frac{\sqrt{\pi}}{4}\right)^n\int_0^\infty d\xi\; \xi^{2n+1}e^{-\frac{\sqrt{\pi}}{4}\xi^2}K_0\left(\sqrt{\frac{k^2}{\rho}}\;\xi\right),$$

$$f^{(3)}(k^2) = a^{(3)}\rho^{2/3} + 2(2\pi)^{1/2}\rho^{5/6}(k^2)^{-1/4} \tag{5.56}$$
$$\cdot\sum_{n=0}^{2}\frac{b_n^{(3)}}{n!}\left(\frac{\pi}{12}\right)^n\int_0^\infty d\xi\; \xi^{3n+3/2}e^{-\frac{\pi}{12}\xi^3}K_{1/2}\left(\sqrt{k^2}\rho^{-1/3}\xi\right),$$

$$f^{(4)}(k^2) = a^{(4)}\sqrt{\rho} + 4\pi\left(\frac{k^2}{\sqrt{\rho}}\right)^{-1/2} \tag{5.57}$$
$$\cdot\sum_{n=0}^{3}\frac{b_n^{(4)}}{n!}\left(\frac{\pi}{24}\right)^n\int_0^\infty d\xi\; \xi^{4n+2}e^{-\frac{\pi}{24}\xi^4}K_1\left(\sqrt{\frac{k^2}{\sqrt{\rho}}}\;\xi\right),$$

in $d = 2, 3$ and 4 respectively, where K_n are modified Bessel functions of second type and

$$a^{(2)} = -2,\; b_0^{(2)} = 4,\; b_1^{(2)} = -8,\; b_2^{(2)} = 4,$$

$$a^{(3)} = -\frac{1}{\Gamma(5/3)}\left(\frac{\pi}{3\sqrt{2}}\right)^{2/3},\; b_0^{(3)} = \frac{1}{\Gamma(5/3)}\left(\frac{\pi}{3\sqrt{2}}\right)^{2/3},$$

$$b_1^{(3)} = -\frac{27}{16}\left(\frac{\pi}{2\sqrt{3}}\right)^{2/3}\frac{1}{\Gamma(2/3)},\; b_2^{(3)} = \frac{9}{8}\left(\frac{\pi}{\sqrt{6}}\right)^{2/3}\frac{1}{\Gamma(2/3)}.$$

$$a^{(4)} = \frac{-4}{\sqrt{6}},\; b_0^{(4)} = \frac{4}{\sqrt{6}},\; b_1^{(4)} = \frac{-36}{\sqrt{6}},\; b_2^{(4)} = \frac{64}{\sqrt{6}},\; b_3^{(4)} = \frac{-32}{\sqrt{6}},$$

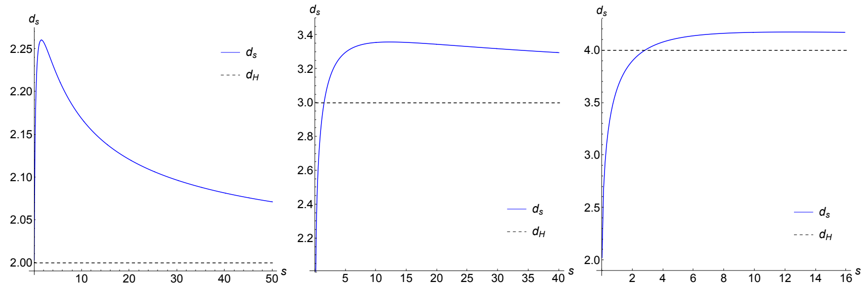

FIGURE 5.3: From top to bottom we have the spectral dimension d_s as a function of diffusion time s for $\rho = 1$ for $d = 2, 3$ and 4 respectively. The dashed line represents the value of the Hausdorff dimension. In the 2d plot the spectral dimension can clearly be seen to interpolate between $d_s = 2$ at short scales and $d_s = 2$ at large scales, with a maximum occurring for $s \approx \rho^{1/2}$. The large scale asymptotic behaviour and the maximum are not as clear in the 3 and 4 dimensional cases due to poorer numerics in $d > 2$. Nonetheless one can see the short scale limit to $d_s = 2$, a maximum for $s \approx 10\rho^{1/d}$, and a large scale limit which is very slowly asymptoting the Hausdorff dimension. These numerical results corroborate the analytical analysis of Section 5.2.2 and provide evidence for the behaviour of d_s interpolating between short and large scales

as obtained via equation (5.46). The results given in figure 5.3 show that in $d = 2, 3, 4$ the spectral dimension approaches 2 in the limit $s \to 0$, increases to a maximum greater than the large scale Hausdorff dimension when $s \sim O(l_n)$, and then decays back to the Hausdorff dimension as $s \to \infty$. [5]

5.2.3 Summary

We have studied the spectral dimension arising from non-local d'Alembertians derived from causal-set theory. We have shown that after regularising these d'Alembertians by removing an unphysical [6] divergence in the coincidence limit, dimensional reduction is present. In particular, the spectral dimension goes to 2 at short scales in every dimension. The small scale dimensional reduction can be seen to arise from the improved UV behaviour of the propagator for such non-local theories. We have also provided numerical evidence for d_s as a function of diffusion time s for the minimal regularised d'Alembertians in $d = 2, 3$ and 4, confirming our analytical estimates in the limits $s \ll l_n$ and $s \gg l_n$. These simulations show that in all three cases considered the spectral dimension possesses a maximum on scales of order l_n, and asymptotes the Hausdorff dimension from above. This is an indication of super-diffusive behaviour on intermediate scales — where effects due to the non-locality scale start to become relevant

[5]Note that in $d = 3, 4$ the maximum occurs on scales $s \sim 10l_n$ rather than l_n and in the $s \to \infty$ the decay back to the Hausdorff dimension is not as marked as in 2d. We believe that these minor differences are due to the higher degree of numerical approximation present in their analysis.

[6]From the point of view of the fundamental discrete theory, in which the coincidence limit is ill-defined.

— a feature similar to that found in [208] in the context of causal dynamical triangulations.

5.3 Conclusions and Outlook

In this Chapter we have investigated two different problems connected to nonlocal d'Alembertians in Causal Set theory.

In the first part of the Chapter, we have studied the local limit of the operators in eq. (4.2) in curved spacetime finding that they all reduce to

$$\Box_g \phi - \frac{R}{2}\phi$$

in the limit. It would be interesting to study the connection of this result with the Einstein equivalence principle. Indeed, the EEP for a scalar field coupled to gravity requires there to be a non-minimal coupling in order to hold true. This point was discussed in [201], where the authors proved, without relying on conformal invariance, that in $4d$ the required coupling has to coincide with the conformal one, i.e., $1/6$. Thus, the value of the universal factor for GCD has to be carefully considered in light of the EEP,[7] also in view of possible phenomenological consequences.

In particular, for non-conformal couplings in $4d$ (as in our case) wave tails propagating inside the light cone are present for massless fields [201]. This amount to violations of Huygens' principle which can have interesting effects (see discussion in Sec. 4.3). Notice that in the present case of curved spacetime the violations of Huygens' principle are due to the non-conformal coupling that arises in the local limit. This is different to what discussed in Sec. 4.3 where such violations are due to the non-locality scale and are thus not present in the local limit.

More problematic would be the case of massive fields without conformal coupling in which case massive particles could be allowed to propagate on the light cone (see [201]) in clear contradiction with the local special relativistic description of physics dictated by EEP. It would be interesting to find a Causal set version of the Klein-Gordon operator (see discussion in Sec. 4.2) and see the curved spacetime version of this operator in the local limit. We speculate that, a massive operator arising from causal set will have a local limit in curved spacetime with a curvature term different from the one of the massless case studied in this Chapter. Whether the new term would (or could) be compatible with EEP, therefore avoiding the problematic propagation along the null cone, is entirely an open question.

Finally, since the family of GCD is derived from a set of precise physical assumptions (see Sec. 4.1) it is tempting to understand which of these need to be relaxed/modified in order to obtain a different local limit and maybe recover the conformal coupling in $4d$.

In the second part of this Chapter we have studied the behaviour of the spectral dimension associated with nonlocal d'Alembertians and shown evidences of dimensional reduction. To conclude with, we would like to

[7] Note that the value of the universal factor cannot be interpreted *a priori* as violating the EEP since other effects may come about when quantum corrections are taken into account, e.g., the running of the coupling.

speculate on the significance and validity of these results. Recall first that the UV behaviour $f_{reg}^{(d)} \to (k^2)^{d/2}$ was key in ensuring that $d_s(s) \to 2$ as $s \to 0$ in all dimensions. Now, it was shown in [17] and [35], where the quantum theory of free scalar fields with nonlocal dynamics was studied, that these operators lead to Wightman functions of the form

$$W(x,y) = \int_0^\infty d\mu^2 \rho(\mu^2) \int \frac{d^d k 2\pi}{(2\pi)^d} \theta(k^0)\delta(k^2+\mu^2) e^{ik\cdot(x-y)},$$

where $\rho(-k^2) = \mathrm{Im}(f^{(d)}(k^2))/|f^{(d)}(k^2)|^2$, a form reminiscent of the Källen-Lehman representation of the Wightman function in interacting QFTs.[8] Unlike the Källen-Lehman representation of local interacting QFTs, however, the spectral function ρ arising from the causet-derived nonlocal d'Alembertians defined in [17] is not a positive function in general. In fact, evidence suggests that for $d > 2$ all operators of this kind will lead to a non-positive spectral function ρ. This implies the existence of negative norm states in the quantum description of these theories (see Chapter 4).

As was explicitly stated by Weinberg (see ref. [226] Section 10.7, p.460), a consequence of the Källen-Lehman spectral representation, and the positivity of the spectral function $\rho(\mu^2)$ (which in standard local QFT is given by the sum of squares of matrix elements of the field-observable), is that the propagator cannot vanish for $|k^2| \to \infty$ faster than the bare propagator $1/k^2$. Since the Wightman function of nonlocal theories of the kind studied here can be written *à la* Källen-Lehman, a natural question is whether the improved UV behaviour of the propagator, and therefore the small scale dimensional reduction in the spectral dimension, are inextricably linked to negative norm states in the quantum theory. If this were the case, it would raise doubts as to the physical relevance of our result in the regime $s \ll l_n$.

Knowing about the issue of non-positive spectral functions associated to causet derived nonlocal d'Alembertians, Aslanbeigi and Saravani studied QFTs of generalised versions of the operators in [17], where they imposed the positivity of $\rho(\mu^2)$ from the beginning [195]. Hence, from Weinberg's argument one would expect that their newly defined $\tilde{f}_{reg}^{(4)}(k^2)$ goes like k^2 as $k^2 \to \infty$, which is indeed the case. Because of this the spectral dimension does not reduce to 2 at small scales but rather starts off at 4, increases, and then asymptotes back to 4 from above at large scales (much like the $2d$ case considered in Section 5.2.2). Although interesting in its own right, the specific theory they consider does not fall within the family of d'Alembertians used in this Chapter, and as such cannot be (at least trivially) seen as continuum approximations to the more fundamental operators living on an underlying causal set, in the way Sorkin had originally constructed them.

A possible way out of these issues arising in the quantum theory is to take the non-locality scale l_n to correspond to the causet fundamental discreteness scale l. In this case one would expect that on scales $s \sim l$ the diffusing particle would cease to see a continuum spacetime, and would start feeling the underlying discreteness of the causal set. Therefore, at these scales the analysis performed would cease to be applicable. In order to study the behaviour of the spectral dimension on scales $s < l$, one would

[8] We have ignored contributions coming from complex mass poles in the propagator since they are not relevant for the current discussion and their inclusion does not change the conclusion of this argument.

instead have to resort to methods to be deployed directly to the causal set itself, e.g., Eichhorn *et al.*'s method [84].

We conclude by noting that our analysis shows an increasing $d_s(s)$ as $s \to l_n^+$, which then only starts to decrease after it has gone through $s \sim O(l_n)$. This might be a hint of the existence of a universal description of the spectral dimension which interpolates between our results and the EM spectral dimension. However, our finding of dimensional reduction is corroborated by other hints towards it in Causal Set [63] which involve different dimension estimators. Thus, the two spectral dimensions — the EM one and d_s studied here — might encode different information about the small-scale structure of spacetime in this theory[9] and as such they both deserve further investigation.

[9] See [63] for an interpretation of the result of EM consistent with dimensional reduction.

Chapter 6

Non-locality phenomenology via Quantum Systems

We are all agreed that your theory is crazy. The question which divides us is whether it is crazy enough to have a chance of being correct. My own feeling is that it is not crazy enough.

Niels Bohr

A vast uncharted territory lies between the scale presently tested in high energy accelerators, $\approx 10^{13}$ eV, and the Planck scale, $M_{\rm Pl} = 1.22 \times 10^{28}$ eV, where quantum gravitational physics is expected to become relevant. As we discussed at length in the previous chapters of this thesis, QG phenomenology attempts to bridge this gap by connecting models of quantum gravity with observations/experiments. Given the high energies at which QG effects are expected, most of the studies in QG phenomenology have concentrated on high-energy physics or cosmological/astrophysical observations involving high energies/long distances. However, with the advent of novel quantum technologies and the enhancement of control of quantum systems, new windows on the microstructure of spacetime are opening, which involve the usage of quantum features and systems [32, 38, 41, 179]. In this Chapter we will concentrate on two examples of this emerging paradigm and consider both a low-energy (non-relativistic) quantum system and a "macroscopic", non-relativistic opto-mechanical system.

In particular, this Chapter is devoted to investigate the phenomenology of non-locality, intended as discussed in Sec. 1.4.2, in models in which LI is held as a guiding principle, by way of low-energy laboratory-based quantum systems. It is generally believed that a successful marriage between fundamental discreteness and local Lorentz invariance requires some form of non-locality (c.f., Sec. 1.4.2). Indeed, disparate approaches to quantum gravity where this marriage occurs explicitly lead to nonlocal modifications of standard local dynamics [49, 136, 203]. Thus the search for nonlocal LI effects is linked to the search for the fundamental microstructure of spacetime.

Explicit realisations arising from quantum gravity models aside, the appearance of nonlocal dynamics, of the type discussed in this thesis, can be expected on very general grounds: LI together with the requirement that the dynamics not suffer from classical instabilities, effectively singles out two types of dynamics, standard local dynamics (1st or 2nd order in space and time) and nonlocal dynamics (∞-order in space and time) — the infinite number of time derivatives being needed in order to avoid Ostrogardsky's theorem [172] (see also Sec. 1.4.2).

In the previous two Chapters we have already encountered nonlocal wave operators arising from Causal set quantum gravity. These $f(\Box)$ operators are characterized by the fact that the function f — in momentum space — presents a branch-cut for timelike momenta. In contrast, other examples of LI nonlocal operators considered in the extant literature [30] are characterized by analytic functions in the complex momentum space. Thus, we can think of these nonlocal operators as belonging to two different families. In the first part of this Chapter we will stick to the *á la Causet* non-locality case, i.e., f with a branch-cut on timelike momenta, and study a model of Unruh–DeWitt particle detector coupled to such a nonlocal scalar field. In particular, we will analyse the spontaneous emission regime of the detector and illustrate how a low-energy system could potentially be used to cast bounds on the non-locality scale beyond current high-energy physics constraints. In the second part of the Chapter, we will move to the other family of nonlocal operators we are considering, i.e., analytic nonlocal operators. In this case, the analyticity of the operators will permit us to investigate the non-relativistic limit of a "general" nonlocal Klein-Gordon operator. In this way, a nonlocal Schödinger operator can be obtained and the corresponding equation solved. We will solve the nonlocal Schödinger equation perturbatively in presence of an harmonic potential and sketch the constraints which could be cast on the non-locality scale by way of opto-mechanical "macroscopic" quantum systems.

6.1 Non-Locality Phenomenology via Unruh–DeWitt detectors

In this first part we consider a pointlike Unruh–DeWitt detector coupled to an *á la Causal set* nonlocal scalar field. The non-analytic functions considered here contain a branch-cut, i.e., a 1-dimensional subspace of the complex plane where the function has a discontinuity. In the Green function this branch-cut corresponds to a continuum of massive modes, even though the original field itself is massless (see Chapter 4). Note that this is similar to what happens to the Green function of local interacting QFTs [226][1]. As we will discuss below, the presence of this continuum of massive modes modifies all n-point functions of the theory, thus giving rise to non-trivial modifications to many physical observables.

The fact that the low-energy behaviour of particle detectors is sensitive to high-energy effects was recently pointed out by Louko and Husain [118]. They showed that some features of low-energy particle detectors can be sensitive to violations of Lorenz invariance at high energies. For example, in [118] it is shown that polymer quantization (motivated by loop quantum gravity) may induce a Lorentz violation at high energies that is perceived by low-energy detectors (below current ion-collider energy scales). More concretely, they found low-energy Lorentz violations in the response of atoms modelled as Unruh–DeWitt detectors (which capture the features of the atom-light interactions [6, 148]) for a general family of quantum fields with modified dispersion relations at high (Planckian) energies.

[1] We will comment on this more in the following.

In contrast to [118], we will consider nonlocal theories with non-analytic fs that, crucially, preserve Lorentz Invariance (LI). It should be noted that LI violations are strictly constrained by various experimental observations, as we discussed in Sec. 1.3.1 and Chapter 3, making theories that preserve LI particularly appealing [140, 150].

Unless otherwise stated, we use natural units $c = \hbar = 1$.

6.1.1 Nonlocal Dynamics: Wightman functions

We study a real scalar field obeying a special class of nonlocal dynamics given by real, retarded, Poincaré invariant wave operators, $\widetilde{\Box} \equiv f(\Box)$ similar to the ones encountered in Chapter 4. The retarded nature of these operators implies that f is non-analytic [77]. Whereas interest in this particular kind of operators can be traced back to the original construction of R. Sorkin of a d'Alembertian operator on a causal set [203], in this Chapter we will not consider specifically Causal set's nonlocal d'Alembertians. Indeed, as already discussed in Chapter 4, these operators present in four dimensions both complex mass poles and a non-positive definite two-point function. Despite we have argued that the consequences of these *a priori* potentially unstable features are far from being clear, to be on the safe side we will consider only nonlocal operators *inspired* by CS theory, i.e., with the same kind of branch-cut non-locality but without the unwanted features. These kind of operators were considered in [194, 195] for both free and interacting scalar fields and quantum field theories based on this family of dynamics have been constructed using different quantization schemes. All these quantization schemes lead to the same quantum theory, which coincides with the one presented in Chapter 4, i.e., the two point functions, Hamiltonian, etc., coincide a part from the complex mass poles contributions[2].

We have already studied the Green functions of nonlocal theories in Chapter 4 and 5, where we focused on the Feynman and retarded Green functions. For the sake of the present Chapter we need to consider the two-point Wighmann function for the nonlocal theory (c.f., Eq. (4.35) and (4.60)). This function $D^{(+)}(x, y) := \langle 0|\phi(x)\phi(y)|0\rangle$ for the free theory is given by [195]

$$D^{(+)}(x-y) = \int \frac{d^4k}{(2\pi)^4} \widetilde{W}(k^2) e^{ik\cdot(x-y)}, \tag{6.1}$$

where

$$\widetilde{W}(k^2) = \frac{2\text{Im}(f)\theta(k^0)}{|f|^2}, \tag{6.2}$$

[2] It should be noted that, in the Causal set case, for dimensions grater than three the Wightman function is not positive definite. This does not happen in the operators considered in this Chapter. However, the formal structure of the two-point functions as well as the Hamiltonian are the same in both cases.

correspond to what we named $\delta^+ G$ in Chapter 4. The Wightman function can be re-written as

$$D^{(+)}(x-y) = \int \frac{d^4k}{(2\pi)^4} 2\pi\theta(k^0)\delta(k^2)e^{ik\cdot(x-y)} \tag{6.3}$$
$$+ \int_0^\infty d\mu^2 \rho(\mu^2) \int \frac{d^4k}{(2\pi)^4} 2\pi\theta(k^0)\delta(k^2+\mu^2)e^{ik\cdot(x-y)},$$

where $2\pi\widetilde{\rho}(-k^2) = \widetilde{W}(k^2)$ and $\widetilde{\rho}(\mu^2) = \delta(\mu^2) + \rho(\mu^2)$ is a sort of *spectral density*[3]. It can be seen that $D^{(+)}$ is a sum of two parts, one is the standard Wightman function for a local massless scalar field, $D_0^{(+)}$, and the other is an integral over the Wightman function of a local massive field, $G_\mu^{(+)}$, weighted by the finite part of the discontinuity function, $\rho(\mu^2)$.

For every choice of $f(\Box)$ there corresponds a specific spectral density ρ. We focus here in two different kinds of d'Alembertians whose discontinuity functions are given by

$$\rho(\mu^2) = \lim_{\epsilon\to 0^+} \frac{-2\, e^{l_n^2\mu^2/2}}{\mu^2} \frac{\Im[E_2(l_n^2(-\mu^2+i\epsilon)/2))]}{|E_2(-l_n^2\mu^2/2)|^2}. \tag{6.4}$$

and

$$\rho(\mu^2) = l_n^2 e^{-\alpha l_n^2 \mu^2}. \tag{6.5}$$

where α is an order one numerical coefficient [195]. The former choice of ρ can be shown to give rise to a stable interacting QFT [195], while the latter is a much simpler function which captures all the fundamental features of (6.4) (see [17, 194]) and allows us to check that our results are largely independent of the specific form of the discontinuity function. Note that the asymptotic limit of the discontinuity function for small masses is given by $\rho(\mu^2) = l_n^2$ [194], while for large masses it is exponentially suppressed (see Appendix B of [17]).

6.1.2 Unruh–DeWitt detector and its Coupling to the field

Extracting spatiotemporal information from quantum fields is notoriously a complicate task. The Unuh–Dewitt model of particle detector allows to extract localized spatiotemporal information from quantum fields by locally coupling a two-level system to the field itself. Furthermore, the model permits to define the concept of particles even in the ambiguous case of curved spacetime, as summarized by the words of Unruh *a particle is what a particles detector detects*. This model has been successful for the study of effects like Unruh effect and Hawking radiation.

Here we will be interested in the idealized model of a pointlike two-level (non-relativistic) system which couples to the scalar nonlocal field through its monopole momentum. This model, despite being idealize, has

[3] The name refer to the spectral representation of local interacting field theories. We will come back to the similarity with this case in the following.

been a valuable guide in many studies ranging from QFT in cruved spacetime to Relativistic Quantum Information, and the investigation of correlations characteristic of the quantum vacuum of relativistic fields. The interaction of our nonlocal field with a two-level $\{|g\rangle, |e\rangle\}$ Unruh-DeWitt detector is described by the interaction Hamiltonian

$$H = g\,\chi(\tau/T)m(\tau)\phi[x(\tau)], \tag{6.6}$$

where g is a small coupling constant and $x^\mu(\tau)$ are the detector's worldline coordinates parametrised by proper time τ. We will consider in the following only inertial detectors, as such the proper time is given by the Minkowski coordinate time t. In this Hamiltonian, m is the detector's monopole moment given by $|e\rangle\langle g|e^{i\Omega t} + |g\rangle\langle e|e^{-i\Omega t} = \sigma_+ e^{i\Omega t} + \sigma_- e^{-i\Omega t}$, where $\sigma_\pm$ are the ladder operators and Ω is the energy difference between the two detector's states. Furthermore, we have included a switching function χ that controls the time dependence of the detector's coupling strength, and is strongly supported for a timescale T. This detector model captures the fundamental features of the light-matter interaction in the absence of angular momentum exchange [6, 148], as it can be seen expanding the interaction Hamiltonian in $\sigma_\pm$ and $a^{(\dagger)}$ operators — relative to the detector and field respectively — and confronting it with the matter-light interaction Hamiltonian in the electric dipole approximation.

Having specified the interaction Hamiltonian we can study the response of the detector when coupled to the field, i.e., the excitation (or decay) probability of the detector due to the interaction with the field. The response function of the Unruh-Dewitt detector is [48, 209]

$$\mathcal{F}(\Omega, T) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\Omega\Delta\tau} D^{(+)}(\Delta\tau)\chi\Big(\frac{\tau}{T}\Big)\chi\Big(\frac{\tau'}{T}\Big). \tag{6.7}$$

Using (6.3) one can see that the response function (6.7) splits into the response function of a detector coupled to a local massless scalar field, $\mathcal{F}_0$, plus the response function of a local massive scalar field, $\mathcal{F}_\mu$, integrated over μ weighted by $\rho(\mu^2)$. We can therefore write (6.7) as

$$\mathcal{F}(\Omega, T) = \mathcal{F}_0(\Omega, T) + \int_0^\infty d\mu^2 \rho(\mu^2)\mathcal{F}_\mu(\Omega, T). \tag{6.8}$$

Since the first term is common to both local and nonlocal theories, in what follows we will study the relative difference in the detector's response, i.e.,

$$\Delta(l_n, \Omega, T) := \frac{\mathcal{F}(\Omega, T) - \mathcal{F}_0(\Omega, T)}{\mathcal{F}_0(\Omega, T)}. \tag{6.9}$$

It is a well known fact that, in a local QFT, an inertial detector in the ground state, switched on for an infinite time, $T \to \infty$, will not click because of Poincaré invariance. A straightforward calculation, along the lines of [48], shows that this is also true in the nonlocal theories studied in this Chapter. Indeed, consider the rate of the detector which is given by the Fourier transform of the correlation function (up to a detector dependent multiplicative factor),

$$\dot{\mathcal{F}}(\Omega) = \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Omega\Delta\tau} G^{(+)}(\Delta\tau), \tag{6.10}$$

where $\Omega > 0$ since we are looking at the excitation probability of the detector, i.e., at its clicks. Choosing a frame in which the detector is at rest we have

$$\dot{\mathcal{F}}(\Omega) = \int_0^\infty \rho(\mu^2) \int \frac{d^4k}{(2\pi)^4} 2\pi\Theta(k^0)\delta(k^2+\mu^2) \underbrace{\int_{-\infty}^{\infty} d\Delta\tau e^{-i(k^0+\Omega)\Delta\tau}}_{\delta(k^0+\Omega)} . \quad (6.11)$$

It is then clear, since $k^0 + \Omega > 0$, that the rate is vanishing. This should not come as a surprise given that such theories are also Poincaré invariant (and stable) by construction.

We now ask what happens when the inertial detector is switched on for a finite time, T, which we implement by inserting non-trivial switching functions $\chi(\tau/T)$ in the Unruh–DeWitt interaction. Within this context, the most interesting case is that of spontaneous emission, i.e., when the detector starts out in an excited state, since in this case there can be differences between the behaviour of the detector coupled to local and nonlocal field theories even in the limit $T \to \infty$. Furthermore, spontaneous emission is a well-understood, experimentally accessible phenomenon [196].

We will assume that the non-locality (length-)scale is much smaller than any other length scale in the problem. In particular we assume that $|\Omega|l_n \ll 1,\ T/l_n \gg 1$, where the first condition defines the “low-energy" condition, and the second ensures that the detector is switched on for a reasonable amount of time.

We first consider the behaviour of the detector's vacuum response and the spontaneous emission in short detector timescales, where we consider the dependence of the results on the shape of the switching function. Secondly, we will analyze the more relevant and experimentally accessible case of spontaneous emission when the detector interacts with the field for long times compared to the detector's Heisenberg time Ω^{-1}. In this experimentally accessible regime the detector's response is independent of the details of the switching function. We will show how a low-energy detector can resolve non-locality scales with a precision comparable to a high-energy particle collider experiment.

6.1.3 Vacuum response and short time ($|\Omega|T \ll 1$) spontaneous emission

The behaviour of the relative response (6.9) can be readily analyzed for $|\Omega|T \ll 1$ regardless of the sign of Ω. This regime corresponds to a rapid switching of the detector. Using the following dimensionless variables $t = \tau/T$, $k = Tp$, $m = T\mu$, and defining the Fourier transform of the switching function as $\widetilde{\chi}(\omega) = \int dt\, e^{-i\omega t}\chi(t)$ the detector response is given by

$$\mathcal{F} - \mathcal{F}_0 = \frac{1}{T^2}\int dm^2 \rho(m^2/T^2) \int d^4k dt dt'\ \delta(k^2+m^2) e^{-i(\Omega T + k^0)(t-t')}\chi(t)\chi(t'), \quad (6.12)$$

$$= \frac{1}{T^2}\int dm^2 \rho(m^2/T^2) \int d^4k\ \delta(k^2+m^2)|\tilde{\chi}(k^0+\Omega T)|^2. \quad (6.13)$$

where in the second line we have used the definition of Fourier transform. We then have

$$\begin{aligned}\mathcal{F}-\mathcal{F}_0 &= \frac{1}{T^2}\int_0^\infty dk^0 \int_0^{k^0} dk 4\pi k^2 \rho(\frac{(k^0)^2-k^2}{T^2})|\tilde{\chi}(k^0+\Omega T)|^2 \\ &= \frac{1}{T^2}\int_0^\infty dk^0 \int_0^{k^0} dk 4\pi k^2 l_n^2 e^{-l_n\frac{(k^0)^2-k^2}{T^2}}|\tilde{\chi}(k^0+\Omega T)|^2 \\ &= \frac{2\pi}{l_n}\int_0^\infty dk^0 |\tilde{\chi}(k^0+\Omega T)|^2 (k^0 l_n - T D_+(k^0 l_n/T)) \\ &= \frac{2\pi T}{l_n}\int_{\Omega T}^\infty dx |\tilde{\chi}(x)|^2 \left[\frac{x l_n}{T} - \Omega l_n - D_+\left(\frac{x l_n}{T}-\Omega l_n\right)\right] \\ .\end{aligned} \tag{6.14}$$

where D_+ is the so called Dawson function[4], $x = k^0 + \Omega T$, and in the second line we have specialized to the exponential spectral function while in the third we have used that

$$\int_0^{k^0} dk k^2 e^{k^2 l_n^2/T^2} = \frac{T^2}{2 l_n^3}\left[k^0 l_n - T D_+(k^0 l_n/T)\right]. \tag{6.15}$$

Now expanding the Dawson function around zero we get

$$\mathcal{F}-\mathcal{F}_0 = \frac{2\pi T}{l_n}\int_{\Omega T}^\infty dx |\tilde{\chi}(x)|^2 \frac{2}{3}\left(\frac{x l_n}{T} - \Omega l_n + \mathcal{O}\left((x l_n/T - \Omega l_n)^5\right)\right). \tag{6.16}$$

It is easy to see that the leading term is

$$\frac{4\pi}{3}\frac{l_n^2}{T^2}\int_0^\infty dk^0 (k^0)^3 \, |\tilde{\chi}(k^0)|^2, \tag{6.17}$$

if we take $\Omega = 0$. Note that the approximations done are valid as far as the Fourier transform of the switching function decay for $x \gg 1$ faster than polynomially (see also discussion at the end of next section). Moreover, although rigorously speaking (6.16) was obtained for the exponential spectral function (6.5), the asymptotic result (6.17) is also confirmed by a numerical analysis with the spectral function in (6.4) (see Tabable 6.1).

Finally, in order to get the response for the massless field, we can replace $\rho(\mu^2) = \delta(\mu^2)$ and follow the same steps to get

$$\mathcal{F}_0 = 4\pi \int_0^\infty dx \, x |\tilde{\chi}(x)|^2. \tag{6.18}$$

Thus, the relative response in this limit scales like l_n^2/T^2 and its fine-details clearly depend on the specific switching function. It should be noted that, whereas the (Fourier transform of the) exponential switching function — considered in Table 6.1 — does not satisfy the condition of fall-off previously discussed, it is still possible to estimate the behaviour of the detector's response in such a case and it turns out to be, at leading order, the same.

[4] It is defined as $D_+(x) = e^{-x^2}\int_0^x e^{y^2} dy$, see [103].

$\chi(t)$	$e^{-\|t\|}$	$\frac{\sin(t)}{t}$	$\frac{1}{t^2+1}$	e^{-t^2}
$\Omega > 0,\ \Omega T \gg 1$	$\approx l_n^2/T^2$	0	$\frac{l_n^2}{T^2}e^{-2\Omega T}$	$\frac{e^{-\frac{\Omega^2 T^2}{2}} l_n^2}{\Omega^4 T^6}$
$\|\Omega\| T \ll 1$	$\approx l_n^2/T^2$	l_n^2/T^2	l_n^2/T^2	l_n^2/T^2
$\Omega < 0,\ \|\Omega\| T \gg 1$	$T l_n^2 \|\Omega\|^3$	$T l_n^2 \|\Omega\|^3$	$T l_n^2 \|\Omega\|^3$	$T l_n^2 \|\Omega\|^3$

TABLE 6.1: Detector's response $\mathcal{F} - F_0$ for various switching functions (taking a dimensionless argument $t = \tau/T$) and for both the exponential spectral function eq. (6.5) and the causal sets inspired spectral function eq. (6.4).

For what concern the case $\Omega > 0$ (corresponding to studying the detector's spontaneous excitation probability due to the 'vacuum noise' of the field) and $\Omega T \gg 1$, a full analytical treatment has proven elusive. We summarize our findings in Table 6.1.

6.1.4 Spontaneous emission

Consider now the case in which $\Omega < 0$, corresponding to the process of spontaneous emission. We are interested here in the regime characterized by $|\Omega| T \gg 1$, which corresponds to assuming that the detector is turned on for times much larger than the Heisenberg time of the atomic system. In this regime, we expect the detector's response to be largely independent of the specific form of the switching function.

Using the dimensionless variables previously defined one can show that

$$\mathcal{F} - \mathcal{F}_0 = \frac{1}{T^2} \int dm^2 \rho(m^2/T^2) \times \int d^4k\, \delta(k^2 + m^2) |\tilde{\chi}(k^0 + \Omega T)|^2. \tag{6.19}$$

For switching functions whose Fourier transform decays asymptotically faster than polynomially (e.g., Gaussian, Lorentzian or sinc), and assuming that $|\Omega| T \gg 1$ and $|\Omega| l_n \ll 1$, we get the asymptotic result

$$\mathcal{F} - \mathcal{F}_0 \approx \frac{4\pi}{3} T l_n^2 |\Omega|^3 \left(\left[1 + \mathcal{O}(l^2\Omega^2)\right] \int_{-\infty}^{\infty} \mathrm{d}x |\tilde{\chi}(x)|^2 \right). \tag{6.20}$$

Performing a similar calculation in the local, massless case yields

$$\mathcal{F}_0 = 2\pi |\Omega| T \int_{-\infty}^{\infty} dx\, |\tilde{\chi}(x)|^2. \tag{6.21}$$

Finally we find that the relative response of eq. (6.9) goes like

$$\Delta(l_n, \Omega, T) \approx \frac{2}{3} c^{-2} |\Omega|^2 l_n^2, \tag{6.22}$$

where we have reintroduced the speed of light for dimensional reasons. Although, rigorously speaking, (6.20) was obtained for the exponential spectral function (6.5), the asymptotic result (6.22) is also confirmed by a numerical analysis with the spectral function in (6.4) (see Table 6.1). It should be noted that, the use of switching functions whose Fourier transform decays faster than polynomially is just a sufficient condition for (6.22) to hold: We can see in Table 6.1 that (6.22) also applies to all the switching modalities considered, including the exponential switching function, whose Fourier transform decays polynomially.

An exact result

Finally, we can recover the previous result in the limit of infinite time coupling by simply removing the switching functions from the picture. In this case we are forced to consider, instead of the response function of the detector, the detector's rate — as it is usual in these kind of calculations [48].

In order to compute the contribution to the detector rate coming from the second term on the RHS of eq. (6.3), we need to evaluate the rate for a detector coupled to a massive local field[5]. This is given by

$$\frac{-i\mu}{4\pi^2}\int_{-\infty}^{\infty} d\Delta\tau \frac{K_1(i\mu(\Delta\tau - i\epsilon))}{\Delta\tau - i\epsilon} e^{-i\Omega\Delta\tau}, \tag{6.23}$$

where we have used the explicit form of the Wightmann function of a massive field in position space. Note that, $K_1(z)$ has two branch points at $z = 0, \infty$, and a branch-cut between them. In particular, the branch-cut can be placed on the negative real axis $Re(z) < 0$, corresponding to $Im(\Delta\tau) > \epsilon$.

Performing the following change of variables $z = \mu\sqrt{-(\tau - i\epsilon)^2}$ we arrive at

$$\frac{-i\mu}{4\pi^2}\int_{\mathcal{C}} dz \frac{K_1(z)}{z} e^{-i\Omega\tau/\mu}. \tag{6.24}$$

The branch-cut is now located on the negative real axis. Since for large z we have $K_1(z) \approx e^{-z}/\sqrt{z}$, when $\Omega > -\mu$ we can close the contour in the right half-plane and the integration gives zero. However, when $\Omega < -\mu$, the contour can be deformed as in figure 6.1 and, using that

$$\frac{1}{z}K_1(z) = \frac{1}{z^2} + \ln(z/2)\frac{I_1(z)}{z} + \mathcal{P}(z), \tag{6.25}$$

where $\mathcal{P}(z)$ is a series of terms z^n, we are left with the integral

$$\frac{-i\mu}{4\pi^2}\int_{\mathcal{C}'} dz \left(\frac{1}{z^2} + \ln(z)\frac{I_1(z)}{z}\right) e^{-i\Omega\tau/\mu}. \tag{6.26}$$

Finally, considering the branch-cut of the logarithm we arrive at

$$\frac{-i\mu}{4\pi^2}\left[2\pi i\left(\frac{-\Omega}{\mu}\right) + 2\pi i\int_0^{\infty} dr \frac{I_1(-r)}{r} e^{-|\Omega| r/\mu}\right], \tag{6.27}$$

where we have used the residue theorem for the first term and changed variable $z = -r$ in the second one. This expression can be computed and, adding the fact that it is valid only for $\Omega < -\mu$, we conclude that the rate of the detector is given by

$$\dot{\mathcal{F}}_\mu = \frac{1}{2\pi}\Theta(-\Omega - \mu)\sqrt{\Omega^2 - \mu^2}. \tag{6.28}$$

[5] From private communication with Jorma Louko.

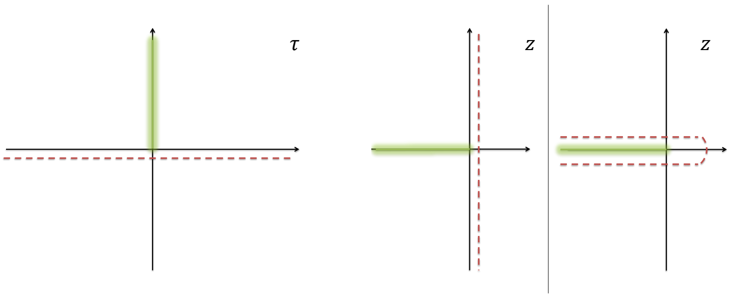

FIGURE 6.1: Contour in the τ complex plane (left) and contour in the complex z plane and its deformation when $\Omega < -\mu$ (right). The green line represent the branch cut.

Finally, we can compute the relative rate,

$$\frac{\dot{\mathcal{F}}_{nonlocal} - \dot{\mathcal{F}}_{local}}{\dot{\mathcal{F}}_{local}} = \frac{\int_0^{|\Omega|^2} d\mu^2 \rho(\mu^2) \frac{1}{2\pi} \sqrt{\Omega^2 - \mu^2}}{\frac{1}{2\pi} |\Omega| \Theta(-\Omega)}. \tag{6.29}$$

Since in the nonlocal case the integration is cut-offed by $|\Omega|$, the assumption $|\Omega| \ll l_n^{-1}$ implies that we are considering $\mu^2 l_n^2 \ll 1$ in the integration. With this in mind, and remembering the behaviour of the spectral function in this limit, we arrive at

$$\Delta \approx \frac{2}{3} |\Omega|^2 l_n^2, \tag{6.30}$$

consistently with our previous result eq. (6.22).

6.1.5 Discussion

In all the physically reasonable regimes studied for an inertial detector coupled for a finite time to a nonlocal field, we find that the nonlocal contribution to the detector's response is polynomial in the non-locality scale, i.e., $\propto l_n^2$. The behaviour of the relative response (eq. (6.9)) in different regimes is reported in figure 6.2. This result is independent of the specific form of the switching function $\chi(t)$. The fact that nonlocal effects are not exponentially suppressed opens up interesting phenomenological windows.

In the case $|\Omega| T \ll 1$, we see from Table 6.1 that the detector's response (and, indeed, also the relative response) is independent of the detector's gap, at leading order. As for the case of vacuum excitation with large ΩT we find that while the polynomial scaling with the non-locality scale persists, the response is in general dependent on the details of the switching function. This is not surprising given that a non-trivial dependence also occurs in the standard local, massless case.

In the case of spontaneous emission with $|\Omega| T \gg 1$, we see from eq. (6.20) that the nonlocal contribution to the detector's response grows like $T\Omega^3$, a fact which can be used to amplify the signature of nonlocality in an experimental setting. Note that this regime is particularly interesting because spontaneous emission for times greater than the detector's Heisenberg time is an experimentally very well understood process [196] (indeed spontaneous emission is far easier to observe than vacuum noise).

Substituting in some realistic numbers we can estimate the expected magnitude of the nonlocal signal. Consider an experimental tolerance for

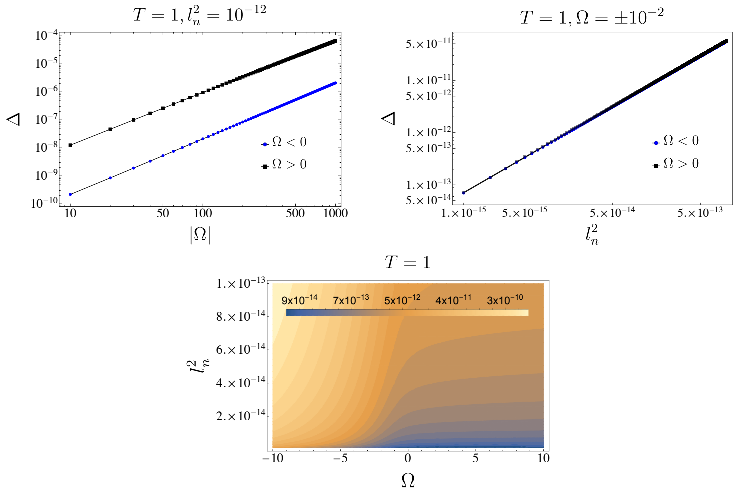

FIGURE 6.2: Detector's relative response, Δ (eq. (6.9)) for the exponential switching function and spectral function eq. (6.4). From left to right we have: a) $|\Omega|T \gg 1$ for both positive (blue circles) and negative (black squares) Ω, i.e. vacuum noise and spontaneous emission respectively; b) $|\Omega|T \ll 1$ for both positive (blue circles) and negative (black squares) Ω. The two data sets overlap which is consistent with the behaviour reported in table 6.1 for $|\Omega|T \ll 1$; c) Logarithmic-scaled contour plot of Δ. Note from plot (a) that although the vacuum response ($\Omega > 0$) has a larger relative difference compared to spontaneous emission ($\Omega < 0$), measuring the latter is experimentally easier.

the relative response of $\Delta \sim 10^{-10}$. Such a tolerance implies that the experimenter has the ability to repeat the experiment of the order of billions of times and accumulate statistics in order to distinguish the two probability distributions. Then using a frequency gap of the order of 10^{22}Hz, corresponding to γ-ray transitions, we can cast a bound on $l_n \lesssim 10^{-19}$m. Note that this constraint is of the same order as present constraints on non-locality coming from LHC data [49].

At first sight these numbers may seem experimentally far fetched, but recall that we are analyzing the process of spontaneous emission and we could have a large number of events. Let us analyze some realistic experimental testbeds in nuclear physics. Consider for example $^{20}_{11}$Na. This nuclear species has a half-life of $T_{1/2}$ ~500 ms and decays to electromagnetically excited, highly unstable, $^{20}_{10}$Ne, which then spontaneously decays to its ground state emitting $\sim$ 11 MeV gamma radiation [120, 121]. Suppose now that one has $\sim$ 20 grams of $^{20}_{11}$Na ($\sim N_{\rm A} \approx 6\times10^{23}$ atoms), then according to the radioactive decay laws the number of gamma emission events in time τ is given by

$$N_\gamma(\tau) = N_{\rm A}(1 - e^{-\frac{\tau}{T_{1/2}}\frac{1}{\ln 2}}). \tag{6.31}$$

But in a time of $\tau \sim 10$s, $N_\gamma \simeq N_{\rm A} \sim 10^{23} \gg \Delta^{-1}$. Assuming that gamma ray detection is not 100% efficient (which it is not), and in particular assuming a very conservative 0.1 % experimental detection efficiency (at least one order of magnitude more conservative than realistic estimates [200]), there are still orders of magnitude more detection events than Δ^{-1}. In other words a low energy nuclear physics experiment ($\sim$10 MeV scale) would already yield a higher resolution than the LHC experiments. In theory, following the reasoning above, if we assume that we have 200 grams $^{20}_{11}$Na (i.e., we have $\sim 10N_A$ of the nuclear species) a very conservative estimate for this number of emission events yields that the detectable relative response would be of order $\Delta \sim 10^{-23}$, which in turn implies that the experiment could detect non-locality scales of $l_n \lesssim 10^{-25}$ m, six orders of magnitude better than the resolution of the LHC. Furthermore, there are more than a dozen different nuclear species that provide a reliable source of spontaneous emission of gamma rays [121], so the use of $^{20}_{11}$Na provides just one possible example.

Due to the similarity of eq. (6.3) with the standard Källén–Lehmann representation for interacting theories [123], one may wonder whether it is possible to discern the nonlocal contribution to spontaneous emission from the similar effect that would arise through interaction with a secondary massive field. In fact one can show that such a contribution, in the case of long time spontaneous emission, vanishes unless the massive field's mass $2m < |\Omega|$. In fact, the rate due to interaction with a massive field can be computed using eq. (6.28) and as it is clear from this equation, only fields with mass μ less than the energy gap $|\Omega|$ contribute to the transition rate of the detector in the long time spontaneous emission. Therefore, for EM nuclear decay, considering $|\Omega| < 2m_e \sim 1$ MeV (where m_e is the electron mass) would suffice to guarantee that the only non-trivial contribution to (6.7) comes from l_n. Doing so would worsen the bound on l_n discussed above by one order of magnitude – which is still better than the LHC bound – but it would also greatly increase the number of experimentally viable nuclear species. Furthermore, contributions from local massive fields can in principle always be accounted for *a priori* and subtracted when defining Δ (see eq. (6.9)).

6.2 Non-Locality Phenomenology via Opto-Mechanical Oscillators

As anticipated in the introduction of this Chapter, in this second part we are going to consider nonlocal kinetic operators characterized by *analytic* functions $f(\Box)$ and their phenomenology. In particular, let us consider a free, massive scalar field in flat spacetime. Its standard (local) dynamics are given by the Klein-Gordon (KG) equation, $(\Box + m^2)\phi(x) = 0$, and they are *stable*. Now, a modification of this dynamical law, which respects LI and avoids generic Ostrogardsky instabilities, must necessarily take the form $(\Box + m^2) \to f(\Box + m^2)$, where f is some non-polynomial function, i.e., the dynamics become *nonlocal*, with stability depending on the specific choice of f[6]. It follows therefore that if specific models of QG lead to modifications of standard local dynamics while simultaneously claiming to preserve LI, then they must do so through nonlocal dynamics of the kind just described, in accordance with the previous discussions.

The definition of f must contain a characteristic, covariantly defined scale, which allows for a suitable power law expansion characterising the

[6]As stressed e.g. in [30], requiring the function f^{-1} to have only a single pole at the physical mass value ensures the absense of ghosts.

deviation from the standard local field equations. It is important to note that this scale, i.e., the non-locality length scale l_n, need not be related to the quantum/discreteness scale normally associated to QG, i.e., the Planck scale. In fact, in Chapter 4 we have already seen that in CS theory this is not the case, l_n being some mesoscopic length scale lying somewhere between the TeV scale and the Planck scale; a fact which is of particular relevance within the context of casting phenomenological constraints.

In what follows we shall describe a novel, promising way to test non-local dynamics of this type by way of opto-mechanical oscillators, i.e., employing one of the simpler, and yet fruitful, physical system: the harmonic oscillator. In particular, we will perform a perturbative study of the effects of modified equations of motion on the evolution of opto-mechanical quantum oscillators based on the following methodological steps. In order to compare our analysis to actual experiments on the aforementioned systems, we must first derive the non-relativistic limit of an effective nonlocal massive scalar field theory with dynamics given by an analytic f, showing that the evolution of a quantum system is governed by a modified (nonlocal) Schrödinger equation. This approach allows us to overcome long standing issues about how to relate the evolution of macroscopic objects to the effects of non-locality arising from a potential discreteness of spacetime. We proceed by performing a perturbative expansion around the local regime adapted to the specific experimental setting we are interested in. We then solve for the evolution of the wave function and compute the behaviour of the relevant physical observables. We show that a characteristic signature due to a periodic squeezing is introduced by the first nonlocal correction to coherent states of the mechanical oscillator. Using this feature we discuss the constraints already available and provide forecasts for those deducible from future experiments. In particular, we establish that sensitivity close to the Planck scale can be achieved, thus severely constraining, or even ruling out, models of QG in which the non-locality scale is larger than the Planck scale.

6.2.1 Framework

Consider a free complex, massive, scalar nonlocal QFT defined by

$$\mathcal{L} = \phi(x)^* f(\Box + \mu^2)\phi(x) + c.c., \tag{6.32}$$

where $\Box = c^{-2}\partial_t^2 - \nabla^2$ and $\mu = mc/\hbar$. In order for the theory to be physically sensible we assume that the following conditions hold:

a. $f(k^2) = 0$ iff $k^2 = 0$: this property ensures that there exist no classical runaway solutions and, when f is entire, no ghosts.

b. the nonlocal QFT must be unitary: conservation of probability.

c. the nonlocal QFT must possess a global $U(1)$ symmetry: this condition ensures that (some form of) a probabilistic interpretation can be given to the wave function.

The function f is entire analytic so that it can be expanded as

$$f(z) = \sum_{j=1}^{\infty} b_j z^j. \tag{6.33}$$

Implicit in the definition of f is the non-locality scale l_n which, in the local limit $l_n \to 0$, sends[7] $f(\Box + \mu^2) \to \Box + \mu^2$. In particular, we have that $b_j \propto l_n^{2j-2}$.

Upon making the ansatz $\phi(x) = e^{-i\frac{mc^2}{\hbar}t}\psi(t, \boldsymbol{x})$ and taking the limit $c \to \infty$ we find

$$\mathcal{L}_{\text{NR}} = \psi^*(t, \boldsymbol{x}) f(\mathcal{S}')\psi(t, \boldsymbol{x}) + c.c., \tag{6.34}$$

where NR stands for non-relativistic, $\mathcal{S}' = -\frac{2m}{\hbar^2}\mathcal{S}$, and

$$\mathcal{S} = i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 \tag{6.35}$$

is the usual Schrödinger operator.

A generalisation of the Euler–Lagrange equations for nonlocal Lagrangians of this kind gives

$$f(\mathcal{S})\psi(t, \boldsymbol{x}) = 0, \tag{6.36}$$

as the equations of motion, where the nonlocal Schrödinger operator is defined as[8]

$$f(\mathcal{S}) \equiv \frac{-\hbar^2}{2m} f(\mathcal{S}') = \mathcal{S} + \sum_{j=2}^{\infty} b_j \left(\frac{-2m}{\hbar^2}\right)^{j-1} \mathcal{S}^j. \tag{6.37}$$

A potential can be trivially introduced by adding a potential $V(\psi^*\psi)$ to the Lagrangian. We simplify notation by setting $b_j = l_n^{2j-2} a_j$ so that eq. (6.37) becomes

$$f(\mathcal{S}) = \sum_{n=1}^{\infty} (-2m/\hbar^2)^{j-1} a_j l_n^{2j-2} \mathcal{S}^j, \tag{6.38}$$

where $a_1 = b_1 = 1$.

Depending on the precise form of the function f, the conserved charge associated to the symmetry $\psi \to e^{i\alpha}\psi$ may or may not be positive semidefinite. As stated in (c), in what follows we will assume that such a condition is satisfied. In any case, one can show that the conserved current is given by a perturbative expansion in the non-locality scale[9]

$$j^0_{NL} = a_1\psi^*\psi - ia_2 l_n^2 \frac{2m}{\hbar}\psi^*\overleftrightarrow{\partial_t}\psi - a_2 l_n^2\psi^*\nabla^2\psi - a_2\psi\nabla^2\psi^* + O(l_n^4) \tag{6.39}$$

$$j^i_{NL} = -ia_1\frac{\hbar}{2m}\psi^*\overleftrightarrow{\nabla}\psi + ia_2 l_n^2\frac{\hbar}{2m}\psi^*\overleftrightarrow{\nabla}^3\psi + 2a_2 l_n^2\dot{\psi}^*\overleftrightarrow{\nabla}\psi + O(l_n^4), \tag{6.40}$$

where $f\overleftrightarrow{\nabla}^j g = \sum_{i=0}^{j}(-)^i \nabla^i f \nabla^{j-i} g$, whose zeroth order term is the usual $j^\mu = (\psi^*\psi, -ia_1\frac{\hbar}{2m}\psi^*\overleftrightarrow{\nabla}\psi)$.

[7] This in turn implies $b_1 = 1$.

[8] This is in analogy with the local case in which the Klein-Gordon operator reduces, in the non-relativistic limit, to $-2m/\hbar^2\mathcal{S}$.

[9] This can be better formulated in terms of the dimensionless parameter ϵ which we introduce in the following.

6.2.2 Perturbative analysis

Having laid down the foundations for a nonlocal Schrödinger evolution of a single particle quantum system, we now turn to the problem of solving the nonlocal differential equation in the presence of a potential $V(x)$.

We wish to solve the nonlocal equation

$$f(\mathcal{S})\psi(t,x) = V(x)\psi(t,x), \tag{6.41}$$

where $f(\mathcal{S})$ is some analytic function given by (6.38), and $V(x)$ some physically reasonable potential. Given a specific choice of f and, crucially, a non-trivial potential V, this (pseudo)-differential equation is incredibly hard to solve.[10] To circumvent this problem we choose to solve the nonlocal differential equation *perturbatively* as follows.

In order to cast (6.41) in a form amenable to a perturbative analysis we first note that the introduction of a non-trivial potential $V(x)$ will also introduce a new scale which, for reasons that will become clear shortly, we denote by Ω and whose dimensions are $1/\text{Time}$. With this new scale one can construct a dimensionless parameter $\epsilon := m\Omega l_k^2/\hbar$ which, for physically reasonable m, Ω and l_n, is much smaller than unity. We can now use ϵ to expand $f(\mathcal{S})$ as:

$$f(\mathcal{S}) = \mathcal{S} - \frac{2a_2}{\hbar\Omega}\epsilon\,\mathcal{S}^2 + \sum_{n=3}^{\infty} a_n \left(\frac{-2}{\hbar\Omega}\right)^{n-1} \epsilon^{n-1}\mathcal{S}^n. \tag{6.42}$$

Next we will assume that (6.42) admits solutions of the form

$$\psi = \sum_{n=0}^{\infty} \epsilon^n \psi_n. \tag{6.43}$$

Substituting (6.43) into (6.42) we find the following differential equations order by order

$$O(1): \quad (\mathcal{S} - V)\psi_0 = 0 \tag{6.44}$$
$$O(\epsilon): \quad (\mathcal{S} - V)\psi_1 = J_1 \tag{6.45}$$
$$O(\epsilon^2): \quad (\mathcal{S} - V)\psi_2 = J_2 \tag{6.46}$$
$$\text{etc.,} \tag{6.47}$$

where J_i, $i = 1, 2, \ldots$ are source terms. Note the the i-th source term depends on the solution of the $(i-1)$th order problem, for example

$$J_1 = \frac{2a_2}{\hbar\Omega}\mathcal{S}^2\psi_0, \tag{6.48}$$
$$J_2 = \frac{-4a_3}{\hbar^2\Omega^2}\mathcal{S}^3\psi_1. \tag{6.49}$$

Implicit in the above analysis is the assumption that ψ_0 – a solution to the standard Schrödinger equation – is also an approximate solution to the

[10] Indeed the *local* Schrödinger equation itself is hard enough to solve for generic potentials.

nonlocal equation, i.e.

$$|(f(\mathcal{S}) - V)\psi_0| = O(\epsilon) \ll 1, \quad \forall t, \ x. \tag{6.50}$$

In words this assumption forces f to be such that its solutions, for reasonable choices of the potential V, are perturbations around solutions to the local Schrödinger equation. This ensures that tensions with well-tested quantum physics are avoided.

In summary we do the following:

- We take a generic $f(\mathcal{S})$ (which we assume satisfies (6.50)) and potential $V(x)$.
- Using the scale introduced by the potential we construct a small dimensionless parameter ϵ.
- We expand $f(\mathcal{S})$ in ϵ and assume the solution can be written as (6.43).
- We solve the problem order by order in ϵ as in (6.47).

Finally, one should check for consistency that each higher order term $\epsilon^n\psi^n$ is indeed smaller than the previous one $\epsilon^{n-1}\psi^{n-1}$ for each n (up to the relevant order of interest).

6.2.3 Nonlocal Schrödinger Equation in (1+1)D With Harmonic Oscillator Potential

We shall now consider the case of a harmonic oscillator in 1-dimension whose evolution is assumed to be described by the above nonlocal Schrödinger equation with a harmonic potential. This study is motivated both by its simplicity and ubiquity in physics, and in view of its application to the actual experiments involving systems that are effectively 1-dimensional.

Hence, we wish to solve the nonlocal equation

$$f(\mathcal{S})\psi(t,x) = V(x)\psi(t,x), \tag{6.51}$$

where V is a harmonic potential $V(x) = \frac{1}{2}m\Omega^2x^2$, m is the mass of the system and Ω its natural angular frequency. Following the steps laid out in the previous section we construct the dimensionless parameter $\epsilon \equiv m\Omega l_k^2/\hbar$ and $f(\mathcal{S})$ as:

$$\left(\mathcal{S} - \frac{2a_2}{\hbar\Omega}\epsilon\,\mathcal{S}^2 + \sum_{n=3}^{\infty} a_n \left(\frac{-2}{\hbar\Omega}\right)^{n-1} \epsilon^{n-1}\mathcal{S}^n\right)\psi(t,x) = \frac{1}{2}m\Omega^2x^2\psi(t,x), \tag{6.52}$$

and assume that ψ can be written as (6.43). In order to keep the notation as clear as possible we define the following dimensionless variables $\hat{t} = \Omega t$ and $\hat{x} = \sqrt{\Omega m/\hbar}\,x$, so that (6.52) becomes

$$\left(\hat{\mathcal{S}} + \sum_{n=2}^{\infty} a_n\epsilon^{n-1}(-2)^{n-1}\hat{\mathcal{S}}^n\right)\psi = \frac{1}{2}\hat{x}^2\psi. \tag{6.53}$$

Throughout the rest of this section we will use these dimensionless variables but will drop the hat for notational simplicity. It is worth noticing that the (square-root of the) dimensionless parameter ϵ, which we use to define the perturbative expansion, represents the ratio between l_n and the width of the oscillator's ground-state wavefunction $x_{\text{zpm}} = \sqrt{\hbar/m\Omega}$. Furthermore, through ϵ, the mass parameter enters the nonlocal dynamics of the harmonic oscillator breaking the usual Ω scaling. This dependence on m suggests that massive quantum systems could be the ideal setting for detecting such non-locality, which is supported by our findings.

We will be interested in solutions that are perturbations around coherent states since these are easiest to realise within the experimental settings we have in mind, and furthermore include the harmonic oscillator's ground state as a specific case. We therefore choose

$$\psi_0 := \psi_\alpha(t,x) = \frac{1}{\pi^{1/4}} \exp\left[\sqrt{2}\alpha e^{-it}x - \frac{1}{2}\alpha^2 e^{-2it} - \frac{\alpha^2}{2} - \frac{it}{2} - \frac{x^2}{2}\right], \quad (6.54)$$

where, without loss of generality, α can be taken real, and $(\mathcal{S} - x^2/2)\psi_0 = 0$.

Next we want to solve the differential equation at order ϵ. To this end we substitute ψ_0 into (6.45) to find

$$(\mathcal{S} - \frac{1}{2}x^2)\psi_1(t,x) = 2a_2\mathcal{S}^2\psi_0(t,x). \quad (6.55)$$

In order to solve this differential equation we use the following ansatz

$$\psi_1(t,x) = \psi_0(t,x)\left[c_0(t) + c_1(t)x + c_2(t)x^2 + c_3(t)x^3 + c_4(t)x^4\right]. \quad (6.56)$$

Plugging this into eq. (6.55), we obtain a system of ordinary differential equations for the time dependent coefficients $c_i(t)$. We then solve these equations imposing the initial condition $\psi_1(0,x) = 0$. It should be noted that, this initial condition corresponds to assuming the initial state (at time $t = 0$) to be the unperturbed coherent state ψ_0, solution of the local Schödinger equation. The coefficients $c_i(t)$ that we found are given by[11]

$$c_0(t) = -\frac{1}{32}a_2 e^{-8it}\left(-\alpha^4 + 8\alpha^4 e^{2it} - 8\alpha^4 e^{6it} + \alpha^4 e^{8it} + 6\alpha^2 e^{2it} - 20\alpha^2 e^{4it}\right. \quad (6.57)$$
$$\left. -14\alpha^2 e^{6it} + 28\alpha^2 e^{8it} - 3e^{4it} - 4e^{6it} + 7e^{8it}\right),$$

$$c_1(t) = -\frac{\alpha a_2 e^{-7it}\left(\alpha^2 - 6\alpha^2 e^{2it} + 3\alpha^2 e^{4it} + 2\alpha^2 e^{6it} - 3e^{2it} + 4e^{4it} - e^{6it}\right)}{4\sqrt{2}}, \quad (6.58)$$

$$c_2(t) = \frac{1}{8}a_2 e^{-6it}\left(3\alpha^2 - 12\alpha^2 e^{2it} + 9\alpha^2 e^{4it} - 3e^{2it} - 2e^{4it} + 5e^{6it}\right), \quad (6.59)$$

$$c_3(t) = -\frac{\alpha a_2 e^{-5it}\left(-1 + e^{2it}\right)^2}{2\sqrt{2}}, \quad (6.60)$$

$$c_4(t) = -\frac{1}{8}a_2 e^{-4it}\left(-1 + e^{4it}\right). \quad (6.61)$$

[11] In solving the systems of ordinary differential equations for the coefficients $c_i(t)$ a naïve solution will show linear in time terms which are unbounded. This is clearly an artifact of the truncation as can be seen by solving eq. (6.53), at first order in ϵ, numerically. We refer the reader to Appendix E and [39] for further details.

The same procedure used for finding these coefficients can be iterated in order to solve the order ϵ^2 eq. (6.46) with an ansatz similar to the one in eq. (6.56)[12].

As it is clear from the expressions of the c_i coefficients, and eqs. (6.63-6.66) in the following section, our results depend on α whenever it is different from zero, i.e., apart from the case of the ground state of the system. Albeit this implies that there could be an amplification effect, while using coherent states, with respect to considering the ground state; it also poses some limit to the validity of the perturbative approach. This can be understood by looking at the distance between the unperturbed state and the perturbed one at order ϵ which, for $|\alpha| > 1$, goes like

$$\|\psi_0 - (\psi_0 + \epsilon\psi_1)\| \propto \epsilon^2|\alpha|^8.$$

For the perturbative expansion to be valid, i.e., for the solutions at order ϵ to be close to the standard quantum mechanical evolution, $\epsilon\alpha^4 \ll 1$ is required. This is corroborated by solving the order ϵ^2 equation, as described above. In fact, at order ϵ^2 the same condition on α appears to be sufficient to ensure the validity of the perturbative approach. For what concerns $|\alpha| < 1$ the perturbative method has to be safe. Finally, we have checked that the order ϵ solution remains smaller, in modulus, than the order zero solution both in space and in time in the region relevant for the system[13].

Now that we have a first order perturbative solution to eq. (6.52), we can go on with computing expectation values and variances of physical observables in this state. In particular, in the next section we will compute the expectation value and variance of both position and momentum. Before doing this, recall that since $\langle\psi|\psi\rangle$ is not conserved by the first perturbative term and in order to have a well-defined probability distribution, we normalise the wavefunction $\psi_0 + \epsilon\psi_1$ using its own norm, i.e., we shall define the probability density as

$$\rho(t,x) = \frac{\psi^*(t,x)\psi(t,x)}{\int_{-\infty}^{\infty} |\psi|^2 \mathrm{d}x}, \tag{6.62}$$

such that $\int_{-\infty}^{\infty} dx\, \rho(x) = 1$. It should be noted that, for the ground state this normalization factor is one at order ϵ, i.e., $\langle\psi_0|\psi_1\rangle = 0$, while in the case of a generic coherent state an order ϵ time dependent correction will be present. The above normalisation factor ensures that even in this case we a have a meaningful probability distribution.

[12] In particular in this case a polynomial of order 8 ought to be considered.

[13] The same has been checked also for the order ϵ^2 solution. Clearly, due to the form of the ansatz, at spatial infinity the wavefunction ψ_1 will grow bigger than ψ_0. However, this region is irrelevant for the system under study.

6.2.4 Spontaneous Squeezing of States

Given our probability distribution (6.62) we can now compute the mean and variance of the position and momentum of the particle. We find

$$\langle x\rangle = \sqrt{2}\alpha\cos(t)\left(1+\frac{1}{4}\epsilon\alpha^2 a_2\left[\cos(2t)-1\right]\right)+\mathcal{O}(\epsilon^2), \tag{6.63}$$

$$\langle p\rangle = \sqrt{2}\alpha\sin(t)\left(1+\frac{1}{4}\epsilon\, a_2\ \left[\alpha^2(7+3\cos(2t))-2\right]\right)+\mathcal{O}(\epsilon^2), \tag{6.64}$$

$$\mathrm{Var}(x) = \frac{1}{2}\left(1-\epsilon a_2\left[\left(6\alpha^2-1\right)\sin^2(t)\right]\right)+\mathcal{O}(\epsilon^2), \tag{6.65}$$

$$\mathrm{Var}(p) = \frac{1}{2}\left(1+\epsilon a_2\left[\left(6\alpha^2-1\right)\sin^2(t))\right]\right)+\mathcal{O}(\epsilon^2), \tag{6.66}$$

which are plotted in figures 6.3, 6.4. It is interesting to note that the expectation values of x and p in the ground state, i.e., when $\alpha = 0$, are left unchanged from the standard local case to first order[14] in ϵ. However, the variance of x and p is modified to order ϵ always, except for the peculiar case where $\alpha = \pm 1/\sqrt{6}$. Given that also in this case the mean values of x and p still show corrections of order ϵ we are led to consider this just an accident. Indeed, this is confirmed by going to order ϵ^2, where the variances acquire again corrections with respect to the local result — this time of order ϵ^2, consistently.

Significantly, we observe that $\mathrm{Var}(x)\mathrm{Var}(p) = 1/4 + O(\epsilon^2)$, thus the perturbed state is still a state of minimum uncertainty. It undergoes a spontaneous, cyclic, time dependent squeezing in position and momenta, where the name squeezing is justified in view of the previous observation on its minimal uncertainty. This is shown in figure 6.4 which also sketches the experimental procedure to measure the variance.

6.2.5 A bird eye view on quantum optomechanics

Before passing to illustrate which are the forecasts and present constraints on the non-locality scale offered by optomechanical oscillators, a brief introduction on at least the main ideas of quantum optomechanics deserves some attention. In this section we go through the basics of optomechanics and we follow ref. [18] to which we refer the reader for an exhaustive review on the argument.

Optomechanics, as the name suggests, deals with the interaction between radiation and mechanical systems. The easiest configuration which can be considered is an optical cavity with a movable end-mirror[15]. The latter behaves as a mechanical oscillator under the action of the radiation-pressure. The general setting is shown in figure 6.5. Here are considered only one mode of the radiation field, with frequency ω_{cav} chosen between the set of allowed resonances of the cavity $\omega_{m,cav} = m\pi c/L$, where m is an integer and L the cavity lenght; and one mechanical mode with frequency Ω. The radiation mode is usually taken to be the closest to resonance with the driving laser (which frequency is ω_L) whereas the choice of the mechanical mode is more arbitrary and can be experimentally performed. The

[14] The same happens also at the next order.

[15] A great variety of configurations of experimental interest exists, some of which involve superconducting circuits, cold atoms, optical microspheres, ect.

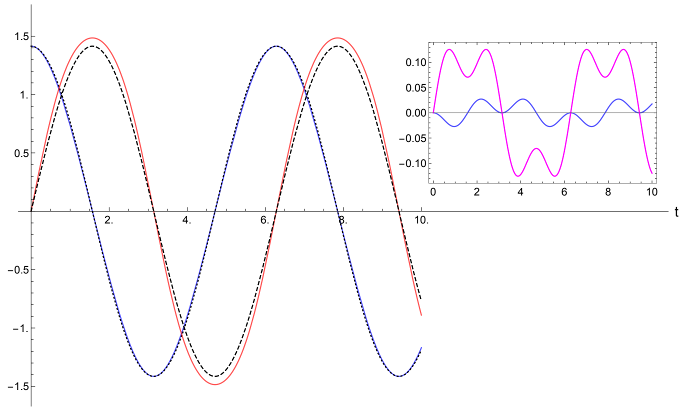

FIGURE 6.3: Time dependence of the mean position and momentum for a coherent state. Here we fix $\alpha = 1$, $a_2 = 1$ and $\epsilon = 10^{-1}$, the last choice is in order to exaggerate the effect of nonlocality. The continuous blue and red lines represent the mean position and momentum in eqs. (6.63), (6.64) respectively. The black dotted and black dashed lines represent the mean position and momentum for the standard coherent state respectively. The insert on the top right shows the order ϵ corrections to the mean position (light-blue) and momentum (magenta) of the standard coherent state.

optical cavity is characterized by the photon density decay rate κ, while the mechanical mode by the energy damping rate Γ_m and the mechanical quality factor $Q_m = \Omega/\Gamma_m$. To see the relevance of the latter, consider the effect of dissipation on the average phonon number $\bar{n}$ for the mechanical mode coupled to a high-temperature bath

$$\frac{d\bar{n}}{dt} = -\Gamma_m(\bar{n} - \bar{n}_{th}), \tag{6.67}$$

where $\bar{n}_{th}$ is the average phonon number of the environment. If the mechanical oscillator starts out in the ground state then $\bar{n}(t) = \bar{n}_{th}(1 - e^{-\Gamma_m t})$ and the rate at which the oscillator heats out of the ground state is given by the thermal decoherence rate

$$\frac{k_B T_{bath}}{\hbar Q_m}. \tag{6.68}$$

Higher is the mechanical quality factor lower is the decoherence rate, which is an important condition for neglecting the effects of dechoerence during the observation of the system. In particular, $Q_m/\bar{n}_{th}$ represents the number of coherent oscillations in the presence of thermal decoherence, i.e., the time for which decoherence effects due to the thermal environment can be safely neglected.

The system illustrated above can be described quantum mechanically as a system of two harmonic oscillators, the radiation mode with annihilation operator $\hat{a}$ and the mechanical one with annihilation operator $\hat{b}$, which interact parametrically through radiation-pressure force. In particular, the cavity frequency is controlled by the displacement $\hat{x}$ of the mechanical system, i.e., $\omega_{cav} = \omega_{cav}(x) \approx \omega_{cav} + x\partial\omega_{cav}/\partial x + ...$ and the linear term only is

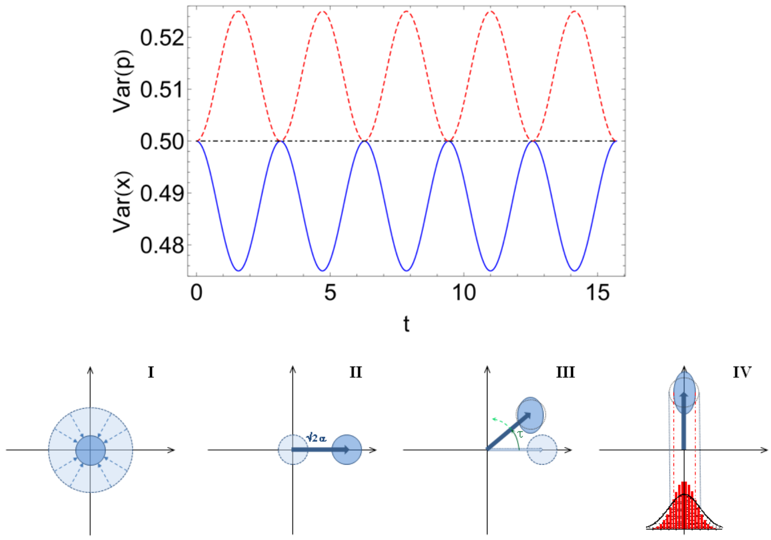

FIGURE 6.4: Time dependence of the variances of a coherent state for $\alpha = 1$ and $a_2\epsilon = 10^{-2}$. The continuous (blue) and dashed (red) lines represent the position and momentum variances respectively. The black dot-dashed line is the standard value, 1/2, of both variances in the local theory. Below the plot we sketch in $x - p$ phase diagrams the proposed experimental procedure for measuring the variance of x, involving (I) cooling the oscillator down to $\langle n \rangle \ll 1$, (II) a pulsed excitation in a well-defined coherent state, (III) free evolution for a time τ, (IV) the measurement of x in a time interval shorter than the oscillation period. Steps (II) and (III) should last much less than the thermal decoherence time. The cycle is iterated several times, the variance of the measurements of x is calculated, then τ is changed and the whole measurement procedure repeated.

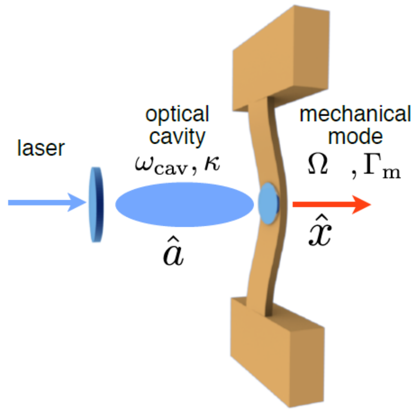

FIGURE 6.5: General setting of an optomechanical system. From [18].

usually retained [18]. The interaction Hamiltonian is then given by

$$\hat{H}_{int} = -\hbar g_0 \hat{a}^\dagger \hat{a}(\hat{b} + \hat{b}^\dagger), \tag{6.69}$$

where we have used that $\hat{x} = x_{zpm}(\hat{b} + \hat{b}^\dagger)$ with $x_{zpm} = (\hbar/2m\Omega)^{1/2}$ the mechanical zero-point fluctuations and $g_0 = -x_{zpm}\partial\omega_{cav}/\partial x$ the vacuum optomechanical coupling strength. As it is clear, the interaction Hamiltonian is of the form $-\hat{F}\hat{x}$ with $\hat{F} \approx \omega_{cav}\hat{a}^\dagger\hat{a}/L$ the radiation-pressure force.

Moving to a frame rotating with the laser frequency we can finally write down the total Hamiltonian of the system as

$$\hat{H} = -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\Omega\hat{b}^\dagger\hat{b} - \hbar g_0\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger), \tag{6.70}$$

where $\Delta = \omega_L - \omega_{cav}$ is the laser detuning.

Most of the experiments to date can be described by the linearized approximate description of optomechanics, which basically casts apart the strong laser drive, splitting the radiation operator as $\hat{a} = \alpha + \delta\hat{a}$, where $\alpha = \sqrt{\bar{n}_{cav}}$ ($\bar{n}_{cav}$ is the photon average number of the coherent state representing the laser field) and $\delta\hat{a}$ is a fluctuating term representing a shifted oscillator in its ground state. With this splitting — and if necessary by shifting the origin of the mechanical displacement — the interaction Hamiltonian becomes

$$\hat{H}_{int} \approx -\hbar g(\delta\hat{a}^\dagger + \delta\hat{a})(\hat{b}^\dagger + \hat{b}), \tag{6.71}$$

where now $g = g_0\sqrt{\bar{n}_{cav}}$.

Finally, let us discuss the different regimes in which the system can work.

- **Red Detuning $\Delta \approx -\Omega$** : this regime is relevant for cooling and quantum state transfer between the radiation and mechanical mode. This is so since, in the rotating wave approximation (RWA), the interaction becomes $-\hbar g(\delta\hat{a}^\dagger\hat{b} + \delta\hat{a}\hat{b}^\dagger)$ also known as beam-splitter Hamiltonian in quantum optics.

- **Blue Detuning** $\Delta \approx \Omega$: this regime is relevant for creating correlations between the two modes. The interaction Hamiltonian in the RWA takes the form of a two-mode squeezing interaction.

- **On Resonance Regime** $\Delta = 0$: the interaction is represented by

$$(\delta\hat{a}^\dagger + \delta\hat{a})\hat{x}. \tag{6.72}$$

  The mechanical position leads to a phase shift of the light field. Basically, in this regime the cavity is operated as an interferometer and can be used to read out the mechanical motion in which we are interested.

It should be noted that, in the on-resonance regime, when the system is used to perform weak continuous measurements of the mechanical motion, the measurement is limited by basic principles of quantum mechanics. Indeed, following the trajectory $x(t)$ of the oscillator corresponds to simultaneously measure two non-commuting quadratures, which is limited in precision by Heisenberg's principle. This fundamental limitation takes the name of Standard Quantum Limit. Ways to go beyond this limit include measuring a single observable, i.e., a single quadrature of the motion, with arbitrary precision. This can be achieved for observables which commute with the Hamiltonian of the system and, in turn, it can be implemented by a displacement measurement (at $\Delta = 0$) with a laser beam which intensity is modulated at Ω or through stroboscopic periodic observations. These kind of measurements are called Quantum Non Demolition (QND) measurements and, together with quantum tomography techniques, can be used to reconstruct the state of the mechanical oscillator by obtaining its Wigner function in phase-space[16].

6.2.6 Present constraints and forecasts

We finally consider the constraints imposed by both existing opto-mechanical experiments and experiments that will be performed in the near future. Let us begin by noting that bounds on the non-locality scale have already been obtained by comparing nonlocal relativistic EFTs to the 8 TeV LHC data [49], in which the authors find $l_n \leq 10^{-19}$m.

Returning to the non-relativistic setting, experiments have recently achieved the goal of cooling a mechanical oscillator down to thermal occupation numbers below unity, i.e., to conditions close to the ground state. In such systems, a wave function can be associated to an effective coordinate describing the elastic vibration mode, or, to reasonable approximation, the center-of-mass of an oscillating portion of a body. The coordinate is then measured by the mode of the radiation field (the *meter*), whose eigenfrequency $\omega_{cav}(x)$ depends on such coordinate, as illustrated in the previous discussion.

The expressions that we derived for the mean values and variances contain dimensionless coefficients, a_n, that depend on the specific model. For the sake of clarity, in the following we set $a_2 = 1$. We start our analysis by comparing the measured variance of x with our corresponding prediction for the ground state. Taking the time average of (6.65), with $\alpha = 0$, we infer that it should differ from its standard value by $\epsilon/2$. For a meaningful

[16] See Section VI.B.1 of [18] for further details.

comparison, the experimental system needs to be well within the perturbative regime, i.e., $\epsilon \ll 1$. This implies that the experiment should achieve an oscillator energy close to its standard ground value — namely with an average occupation number $\langle n \rangle \ll 1$. In these conditions, eq. (6.65) allows one to derive an upper limit to ϵ of the form $\epsilon < 2(2\langle x^2 \rangle_{\text{measured}} - 1)$. The experiments with macroscopic oscillators that enter the quantum regime more deeply (ref. [165] achieved $\langle n \rangle < 0.07$ and ref. [151] achieved $\langle n \rangle = 0.02$), do not provide a measurement of the variance of x. Other experiments [139, 180, 215, 232], which use the drum mode of aluminum membranes, measure instead $\langle x^2 \rangle$ of the oscillator coupled to microwave radiation that is used to cool and monitor its motion. In the best case scenario, with $\langle n \rangle < 0.1$ [139], eq. (6.65) yields the upper limit $\epsilon < 0.2$. Using $x_{\text{zpm}} \simeq 5$ fm (the mass is ~ 50 pg [215] and the frequency $\Omega/2\pi = 15$ MHz), we derive $l_n < 2 \times 10^{-15}$ m. This is reasonably close to the constraint obtained at LHC and has the extra virtue of being independent of it.

A further comparison between our model and experiments can be based on the evolution of coherent states. As shown in Eqs. (6.63)-(6.64), our model predicts some third-harmonic distortion proportional to ϵ and α^3: the effect of non-locality is thus amplified, with respect to the ground state, by the coherent amplitude α. Unfortunately, to our knowledge no experiment could so far realise a quantum coherent state, i.e., a coherent state produced from the ground state, where thermal fluctuations are negligible. Treating such cases rigorously would require a generalisation of eq. (6.63)-(6.64) to thermal states, something far from being straightforward.

However, the situation is not as hopeless as it might seem. A recent set of experiments developed highly isolated quantum oscillators, i.e., with a high mechanical quality factor, to constrain potential Planck scale deviations from the standard uncertainty relations [32] — the so called Generalized Uncertainty Principle (GUP). These systems also suffer from the issue of only being able to generate thermal coherent states, but are presently undergoing improvements which should allow them to enter a truly quantum regime. Interestingly, we can already provide first estimates of the strength of the constraints achievable via these experiments. In ref. [32], the oscillator is excited in order to create a coherent state with large amplitude α, then the time evolving position x is monitored through weak coupling to a *meter* field. The third-harmonic component of $x(t)$ is accurately evaluated as a function of α. Such third-harmonic should be null in a perfectly harmonic oscillator, with standard dynamics. On the contrary, its presence is predicted by Eqs. (6.63), namely with a ratio between third- and first-harmonic amplitudes (third-harmonic distortion) equal to $\epsilon\alpha^2/8$. A third harmonic can also be interpreted as a consequence of the deformed commutator analysed in ref. [32], with a third-harmonic distortion equal to $\alpha^2\beta/4$ where β is a deformation parameter. A non-null third-harmonic distortion is indeed found in ref. [32], and it is explained as the effect of structural non-linearities. Assuming no accidental cancellation between non-linearities and deformed dynamics, and attributing, in the worst case, the overall effect to the deformed dynamics, ref. [32] derives a series of upper limits for β for different oscillator parameters (frequency and mass). Comparing the above expressions for the third-harmonic distortion, we can directly translate the upper limits on β to upper limits on ϵ ($\beta \to \epsilon/2$) and

l_n ($\epsilon \to m\Omega l_n^2/\hbar$). Such limits range from $l_n \lesssim 2 \times 10^{-22}$ m (for an oscillator with $m = 2 \times 10^{-11}$ kg and $\Omega = 7.5 \times 10^5$ Hz), to $l_n \lesssim 1 \times 10^{-29}$ m ($m = 3 \times 10^{-5}$ kg and $\Omega = 5.6 \times 10^3$ Hz).

Such figures suggest that similar experiments, once performed on pre-cooled, quantum oscillators — all improvements within technical reach and currently under realisation — could explore the interesting region between the electroweak and Planck scales and cast constraints stronger than those achieved at LHC by several orders of magnitudes. Let us remark that while our best forecast falls short by roughly six orders of magnitude from $l_p = 10^{-35}$m, figures of this order could already be effective for constraining scenarios where the non-locality scale is larger than the Planck one[17].

The experiments just described are useful to set constraints on nonlocal effects but, should they be observed, can hardly be used to claim the discovery of such phenomena, since a third-harmonic distortion can be attributed to several effects. On the other hand, the spontaneous, oscillating squeezing is a much more meaningful phenomenon that cannot be generated by environmental effects. Of crucial importance for a potential experimental test is that the oscillation has a precise phase relation with the evolution of the average position in a coherent state, a property that can be exploited in synchronous detections.

A realistic experiment could be based on repeated measurement cycles on a mechanical oscillator, including: a) the cooling down to $\langle n \rangle \ll 1$, b) the pulsed excitation in a well-defined coherent state and c) after a delay τ, the measurement of x (see the sketch in figure 6.4). Note that the use of pulsed techniques [220], and the estimate of time-evolving indicators on iterated measurements [184], are indeed at the forefront of experimental research in optomechanics. In this sense, the class of experiments under preparation are already in the right regime to severely constrain (or in the best scenario confirm) nonlocal physics at sub-Planckian scales.

6.3 Conclusions

In this Chapter we have considered two kind of phenomenological studies of non-locality which promise to offer clear signatures and/or tight constraints. As we empathized in the introduction, LI non-locality seems to be intimately related to spacetime discreteness as it is suggested by general arguments as well as explicit examples. As such, the non-locality phenomenology illustrated here can serve as a direct probe also of the discreteness of spacetime, even though of more general applicability.

Both the systems considered can be addressed as non-relativistic and of low-energy (with respect to the expected scales of non-locality and spacetime discreteness). In the first part of the Chapter, we have considered a model of Unruh–DeWitt inertial detector coupled to an *á la Causal set* nonlocal scalar field for a finite time. Already in the local case, an inertial detector coupled for a finite time has a non-vanishing excitation rate. However, in the nonlocal case this rate is changed by the non-locality — in particular due to the branch-cut characterizing the particular non-locality considered — that gives rise to an additional response which, in turn, is only polynomially suppressed in the non-locality scale. Indeed, for the cases considered

[17] See e.g. the relative discussion in [35].

(eqns. (6.4) and (6.5)), we gave both numerical and analytical evidence that the detector's relative response depends quadratically on the non-locality scale, and argued that this result should hold for any exponentially suppressed spectral function ρ. Interestingly, from the point of view of a practical implementation, it is the case of spontaneous emission which is far easier to experimentally realize than the vacuum response. We exploited this fact to show that experimentally feasible setups – involving detectors with energy gaps of the order of MeVs (e.g., gamma emission following the β decay of $^{20}_{11}$Na) – can potentially probe non-locality scales of the order of $l_n \lesssim 10^{-25}$ m, six orders of magnitude better than a TeV-scale experiment at the LHC [49]. This paves the way for low-energy experimental tests of high-energy theories and models of quantum gravity.

In the second part of the Chapter, we have shown that experiments employing opto-mechanical quantum harmonic oscillators can cast very severe constraints on the scale of non-locality associated to several QG scenarios which respect local Lorentz invariance. This was achieved by considering the non-relativistic limit of a nonlocal EFT characterised by an analytic function of the Klein-Gordon operator. The so derived nonlocal Schrödinger equation was solved perturbatively in the experimentally relevant case of a harmonic oscillator potential. The perturbed solutions showed a characteristic periodic squeezing which cannot be introduced by environmental effects or by dissipative behaviour due to the finiteness of the quality factor. This provides a clear signature of what one should seek for in this class of experiments, and opens up a new channel for quantum gravity phenomenology. Improvements of the oscillators used in [32] are under development with the precise aim to test the features presented here. These new experiments could soon reach sensitivities close to the Planck scale, and will therefore offer a concrete possibility of observing quantum gravity induced effects, or at least rule out a whole class of candidate theories.

From the previous discussion it appears that, either employing field correlations through low-energy detectors or using macroscopic quantum systems (c.f., the dependence of ϵ on the mass of the oscillator) we could soon reach sensitivities far beyond the ones of high-energy experiments. This is, in our view, extremely interesting since it paves the way to novel, laboratory-based, QG phenomenology, aimed at unveiling the microscopic structure of spacetime while using features of quantum systems over which one can have an accurate control.

Chapter 7

Conclusions

'The Answer to the Great Question... Of Life, the Universe and Everything... Is... Forty-two,' said Deep Thought, with infinite majesty and calm.

Douglas Adams
The Hitchhiker's Guide to the Galaxy

In concluding this work it seems useful to summarize the key points analyzed so far. The fact that General Relativity is not the final description of Nature was extensively argued in the introductory Chapter. Making GR compatible with quantum theory, in the attempt to solve the problems and controversies of both theories, is the aim of quantum gravity. However, the lack of observational constraints and contact with experiments has slowed down the advancement in QG and highly speculative routes have been followed.

The central topic of this thesis has been QG phenomenology, i.e., the recently blossomed research field that tries to put general features of QG models and their expected effects to the test. The search for effects which originates from quantum features of spacetime and can, *in principle*, leave a trace in controllable quantum systems has been a significant part of our research.

At the heart of several approaches to QG there is the idea that the description of spacetime as a smooth, 4-dimensional manifold has to be given up at the microscopic level and that spacetime itself could be an emergent concept. In Chapter 2, an analogue model based on a relativistic BEC was considered, providing a toy-model of such an emergence. Analogue Gravity models have been extensively studied at the kinematical level for what concerns mimicking QFT on curved spacetime effects, such as Hawking radiation and cosmological particle production. However, in this work the analogy was extended to the dynamical level showing that, for a particular choice of the model's parameters, a relativistic BEC is able to mimic Nördostrom scalar gravity with a cosmological constant term. This is in accordance with previous results which show that a non-relativistic BEC (with a soft $U(1)$ breaking term) can mimic Newtonian-like gravity. Nevertheless, the fact that no LIV for the quasi-particles seem to be present in the particular case considered is the result of a sort of fine-tuning (related to the choice of a real VEV). Indeed, for a relativistic BEC in a general configuration LIV will be present in the phononic dispersion relation. In conclusion, it appears that analogue models based on BEC are capable of mimicking the dynamics of gravitational theories, presenting an example of emergent gravity from a condensed matter system. However, they are way too simple to be able to capture the complexity of GR and the presence of LIV can be avoided only in special cases.

The fact that LIV require fine-tuning in order to make theories viable has been addressed in detail in Chapter 3. In particular, even if LIV are assumed to occur at very high-energies (e.g., at the Planck scale) they do percolate, unsuppressed, onto operators of lower mass-dimension through radiative corrections in EFT. This give rise to the naturalness problem of LIV, i.e., it seems that an unwanted and unreasonable fine-tuning of the parameters of the theory is required in order to render the theory compatible with the stringent constraints on LIV currently existing. In Chapter 3, after a review of the naturalness problem, we saw that when a second, LI scale is introduced in the problem the percolation can be suppressed. While this way out from the naturalness problem is illustrated by way of a Yukawa theory of a scalar and a fermion, one can argue that the basic protection mechanism is quite general. In addition, we also presented a novel analysis of the less studied case of LIV induced by UV dissipative terms. Also in this case we saw that an intermediate EFT scale could play a crucial role in taming dangerous percolation to low-energies.

Even if possible to tame, the problem of LIV naturalness and the existing experimental constraints motivate the study of discrete theories of spacetime which, crucially, preserve LI. Indeed, discreteness of spacetime is usually hard to reconcile with LI. However, CS theory is an exception and the coexistence of LI and spacetime discreteness gives rise to a characteristic non-locality. This non-locality is parametrized, in an effective scalar field model, by a non-locality scale which is assumed to be a mesoscopic scale, i.e., greater than the Planck scale. Thus it plays a central role in this thesis. In fact, the non-locality scale is the free parameter on which phenomenological bounds have to be cast. In Chapter 4, the scalar field theory in flat spacetime characterized by CS nonlocal wave operators is analyzed and a quantization procedure is laid down. It is also shown that in four dimensions the Huygens' principle is violated which, in turn, could entail interesting phenomenological consequences. In Chapter 5, the nonlocal operators are studied in curved spacetime and at the classical level. It is shown that in the local limit, i.e., when the non-locality scale vanishes, all CS nonlocal d'Alembertians — which constitute an infinite family of operators in every dimension — reduce to $\Box_g - R/2$, where $\Box_g$ is the covariant d'Alembertian and R the Ricci scalar. This shows, on the one hand that the local limit is universal for all the operators derived under specific physical assumptions and, on the other hand, that a tension with the EEP — which requires a conformal coupling in four dimensions to hold true — is likely to be present. The second part of the Chapter considers the same operators in flat spacetime and analyses the behaviour of the spectral dimension associated with them. The spectral dimension is a dimensional estimator largely used in the QG literature to show dimensional reduction in several approaches to QG. Indeed, the flow of the spectral dimension in the UV (i.e., when considering small diffusion times) to values smaller than the topological dimension is a characteristic shared by so many approaches that it is considered a reliable prediction of QG. For the CS operators, it has been shown that the spectral dimension converges in the UV to 2 in every dimension.

Since the non-locality of kinetic operators is also present in other approaches to QG, the last Chapter of this thesis considers two different low-energy systems which can manifest distinctive signatures of it. Firstly, the non-locality *á la Causal set*, i.e., characterized by non-analytic nonlocal operators, is shown to modify the response of an Unruh–DeWitt particle detector coupled to the nonlocal field. The case of spontaneous emission is particularly interesting compared to the harder to observe vacuum noise excitation of the detector. Plugging in realistic numbers, it appears that the model for spontaneous emission could allow one to bound the non-locality scale up to 10^{-25}m, which is six order of magnitude better than the constraint from LHC data. Secondly, non-locality carried by analytic operators has been considered. In this case, a nonlocal Schödinger equation has been derived as the non-relativistic limit of a nonlocal Klein-Gordon. Then, this equation has been solved perturbatively in a harmonic potential considering the deformations of standard coherent states. It is shown that, the corrected states manifest a spontaneous time-periodic squeezing which represents a characteristic feature due to the non-locality. This model is appealing since it can be realized in opto-mechanical experiments and current data, and extrapolations, allows one to forecast bounds for the non-locality scale only six order of magnitude away from the Planck scale.

Outlook

The field of QG phenomenology is, by its own nature, vast and mostly uncharted. Thus, this thesis represents a small (but hopefully significant) part of a bigger effort to connect QG with observations and experiments. The final goal is indeed very ambitious: the understanding of the microscopic (quantum) structure of space and time.

The results presented in this thesis raise various questions. In particular

- Is it possible to cook-up an analogue model complex enough to reproduce the dynamics of Einstein gravity and circumvent the possible problems with the Weinberg-Witten theorem[1]?
- It would be interesting to further consider the phenomenology of dissipative LIV by including them in the SME in a consistent way.
- At the same time, understanding to what extent the non-locality scale can play a role in the resolution of the LIV naturalness problem deserves further studies.
- What is the role of the complex mass poles in the CS inspired nonlocal QFT? Do they ruin the unitarity of the theory once interactions are introduced? And what about the branch-cut states in an interacting theory?
- As discussed in the main text, the construction of a discrete Klein-Gordon operator in CS theory could shed light on the meaning of mass in a discrete spacetime setting. Furthermore, it would permit to study the local limit of a massive operator in the hope of singling out the operators which satisfy the EEP.

[1] See also the caveat imposed by [147]

- Dimensional reduction is a common denominator of several QG models. The nonlocal wave operators studied in this work suggest that CS theory also shares this feature. However, previous results employing random walks on the lattice predict a divergence of the spectral dimension for small diffusion times. How these two spectral dimensions — the one calculated from the nonlocal wave operator and the one coming from the random walk — are related is still an open question. There are already other hints towards the fact that CS theory exhibits dimensional reduction [7, 63] and it would be interesting to study the UV behaviour of other Green functions, e.g., the Hadamard one, in order to acquire further evidence.

Last but not least, the final Chapter of this thesis has pointed out two interesting low-energy systems which can offer the ideal test-field for QG phenomenology studies. In particular, projects in this direction which we intend to explore in the future include the following.

- Since a single inertial detector has already proven to be sensitive to the non-locality scale, the study of the signaling probability between two detectors seems promising. In particular, in [133] it was shown that the signaling probability between two detectors which are purely timelike related, and communicate through a massless field, is different from zero only when the Huygens' principle is violated. Since in a local 4-dimensional theory the HP holds, whereas in the 4-dimensional nonlocal theories *á la Causal set* it is violated, a study of the signaling probability in this spatiotemporal configuration of the detectors would constitute an acid test for non-locality.

- The uniformly accelerated detector constitute the testbed for studies of the Unruh effect and deserves some attention. Some preliminary results for what concerns the fate of this effect in the nonlocal case have already appeared in the literature [7]. Given the form of the Wightman function (see eq. (6.3)) it is easy to see that the detector rate will be given by

$$\mathcal{R}(\Omega) = \int_0^\infty d\mu^2 \tilde{\rho}(\mu^2) \mathcal{R}_\mu(\Omega), \tag{7.1}$$

  where $\mathcal{R}_\mu(\Omega)$ is the rate for a uniformly accelerated detector coupled to a free scalar field of mass μ. It is clear from this expression that there will be a deviation from the standard detector rate but with no modifications of the Unruh temperature. Thus, it would be interesting to determine the dependence of the deformed rate on the non-locality scale.

- The uniformly accelerated detector is a very idealized model. A more realistic case would be to consider a detector that is switched on and off, something which can be modeled through a switching function like the ones used in Chapter 6. As we have seen in that Chapter, allowing for a switching of the detector introduces transient effects which could be sensitive to the non-locality scale present in the problem.

- Finally, the opto-mechanical system which will enable us to test the spontaneous squeezing of states is under development. In order to have a more detailed description of what could be measured and a better account of various effects neglected in the current work, it would be useful to have a Hamiltonian description of the nonlocal and non-relativistic model. This would allow one to study the master equation describing the system while in interaction with the environment, i.e., as an open quantum system.

In conclusion, the work presented here encompasses different topics related to QG phenomenology, from LIV to analogue models as toy-models of emergent spacetime and CS theory to phenomenology by way of low-energy quantum systems. Many avenues remain to be explored in a field which aims to pin-point subtle effects using clever and sophisticated methods. The current development of new quantum control techniques and systems, such as opto-mechanical and quantum optics ones, opens new interesting possibilities for QG phenomenology to be explored in the future.

Appendix A

Stress Energy Tensor for the relativistic BEC

In this appendix we report the detailed calculation for the stress energy tensor of quasi-particles, and its trace, employed in Chapter 2. The quantity of interest is given by

$$T_{\mu\nu} \equiv -\frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{S}_2\right)}{\delta g^{\mu\nu}}. \tag{A.1}$$

where $\mathcal{S}_2$ is the quadratic (in perturbations) part of the action in eq.(2.34) expressed in terms of the redifined fields, i.e.,

$$\begin{aligned}\mathcal{S}_2 &\equiv \frac{1}{c}\int d^4x\sqrt{-g}\mathcal{L}_2^{geom} \\ &= -\int d^4x\sqrt{-g}\left\{\frac{1}{6}R(\psi^*\psi) + \frac{1}{12}\Lambda\left[\psi\psi + \psi^*\psi^* + 4\psi^*\psi\right] + g^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi\right\}.\end{aligned} \tag{A.2}$$

Furthermore, we will show that the linear part of the action does not give any contribution to the stress energy tensor. We need the following relations

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{A.3a}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box_g\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}, \tag{A.3b}$$

$$\int d^4x\sqrt{-g}[f\delta R] = \int d^4x\sqrt{-g}[fR_{\mu\nu} + g_{\mu\nu}\Box_g f - \nabla_\mu\nabla_\nu f]\delta g^{\mu\nu}, \tag{A.3c}$$

where the third one follow from the second integrating by parts and neglecting boundary terms. Then we have

$$\begin{aligned}\delta\mathcal{S}_2 = -\frac{1}{c}\int d^4x\sqrt{-g}\Big\{&\frac{1}{6}R_{\mu\nu}\psi^*\psi + \frac{1}{6}g_{\mu\nu}\Box_g\psi^*\psi + \frac{2}{6}g_{\mu\nu}\nabla^a\psi^*\nabla_a\psi \\ &+\frac{1}{6}g_{\mu\nu}\psi^*\Box_g\psi - \frac{1}{6}\nabla_\mu\nabla_\nu\psi^*\psi - \frac{1}{6}\nabla_\nu\psi^*\nabla_\mu\psi - \frac{2}{6}\nabla_\mu\psi^*\nabla_\nu\psi \\ &-\frac{\Lambda}{12}\frac{1}{2}g_{\mu\nu}\left[\psi\psi + \psi^*\psi^* + 4\psi^*\psi\right] - \frac{1}{2}g_{\mu\nu}\partial_\alpha\psi^*\partial^\alpha\psi + \partial_\mu\psi^*\partial^\nu\psi \\ &-\frac{1}{6}R\frac{1}{2}g_{\mu\nu}\psi^*\psi\Big\}\,\delta g^{\mu\nu}.\end{aligned} \tag{A.4}$$

The stress energy tensor is simply given by

$$\begin{aligned} T_{\mu\nu} =& \frac{1}{6}G_{\mu\nu}\psi^*\psi + \frac{1}{6}g_{\mu\nu}\Box_g\psi^*\psi + \frac{2}{6}g_{\mu\nu}\nabla^a\psi^*\nabla_a\psi + \frac{1}{6}g_{\mu\nu}\psi^*\Box_g\psi \\ &- \frac{1}{6}\nabla_\mu\nabla_\nu\psi^*\psi - \frac{1}{6}\nabla_\nu\psi^*\nabla_\mu\psi - \frac{2}{6}\nabla_\mu\psi^*\nabla_\nu\psi \\ &- \frac{\Lambda}{12}\frac{1}{2}g_{\mu\nu}\left[\psi\psi + \psi^*\psi^* + 4\psi^*\psi\right] - \frac{1}{2}g_{\mu\nu}\partial_\alpha\psi^*\partial^\alpha\psi + \partial_\mu\psi^*\partial^\nu\psi, \end{aligned} \tag{A.5}$$

and its trace by

$$\begin{aligned} T =& -\left(\frac{R+\Lambda}{6}\right)\psi^*\psi - \frac{\Lambda}{6}\left[\psi\psi + \psi^*\psi^* + 3\psi^*\psi\right] \\ &+ \Box_g\psi^*\psi\left(\frac{2}{3} - \frac{1}{6}\right) + \psi^*\Box_g\psi\left(\frac{2}{3} - \frac{1}{6}\right) + \partial_\alpha\psi^*\partial^\alpha\psi\left(-\frac{4}{3} + \frac{4}{3}\right). \end{aligned} \tag{A.6}$$

Finally, using the background and the perturbations equations

$$R + \Lambda = 0, \tag{A.7a}$$

$$\Box_g\psi = \frac{\Lambda}{6}\left(\psi + \psi^*\right), \tag{A.7b}$$

and splitting the field ψ in real and imaginary part, we end up with

$$T = -2\lambda\frac{\mu^2}{c\hbar}\left[3\psi_1^2 + \psi_2^2\right]. \tag{A.8}$$

To conclude this appendix we show, as anticipated, that the linear (in the perturbations) part of the action gives no contribution to the stress tensor. The linear part is given by

$$\mathcal{S}_1 \propto -\int d^4x\sqrt{-g}\left\{\frac{1}{6}R(\psi^* + \psi) + \frac{1}{6}\Lambda(\psi + \psi^*)\right\}. \tag{A.9}$$

Following the same steps as before we have

$$\begin{aligned} \delta\mathcal{S}_1 \propto& -\int d^4x\sqrt{-g}\left\{\frac{1}{6}R_{\mu\nu}(\psi^* + \psi) + \frac{1}{6}g_{\mu\nu}(\Box_g\psi^* + \Box_g\psi)\right. \\ &\left.-\frac{1}{6}\left(\nabla_\mu\nabla_\nu\psi^* + \nabla_\mu\nabla_\nu\psi\right) - \frac{1}{6}R\frac{1}{2}(\psi^* + \psi) - \frac{\Lambda}{6}(\psi^* + \psi)\frac{1}{2}g_{\mu\nu}\right\}\delta g^{\mu\nu}. \end{aligned} \tag{A.10}$$

It is easy to see which is the contribution to the trace of the stress energy tensor given by the linear term

$$T^{(1)} = -\left(\frac{R+\Lambda}{6}\right)(\psi + \psi^*) + \frac{1}{2}\left(\Box_g\psi^* + \Box_g\psi - \frac{2\Lambda}{6}(\psi + \psi^*)\right), \tag{A.11}$$

and that, using the background and perturbations equations, $T^{(1)}$ vanishes.

Appendix B

Addendum to Chapter 3

B.1 Useful integrals for sections 3.1 and 3.2

Here we collect some formulas which are used throughout Chapter3, in particular in eqs. (3.9) and (3.36) in order to derive eqs. (3.16), (3.17), (3.37) and (3.39).

In particular, concerning the integrals in section 3.1, let us indicate

$$I_n \equiv \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{1}{((k^0)^2 + \mathbf{k}^2 + m^2)^n} = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{1}{((k^0)^2 + A)^n}, \tag{B.1}$$

where we are working in the Euclidean space, where $A \equiv \mathbf{k}^2 + m^2$ and $\mathbf{k}$ is the spatial three momentum. These kind of integrals are needed for calculating c_{11} according to eq. (3.9), taking into account also eq. (3.15). They can be explicitly calculated and in particular one finds

$$I_1 = \frac{1}{2\sqrt{A}}, \quad I_2 = \frac{1}{4A^{3/2}}, \quad I_3 = \frac{3}{16A^{5/2}}, \quad I_4 = \frac{5}{32A^{7/2}}. \tag{B.2}$$

For calculating c_{00} from the same expressions, instead, we need also the following integrals which, in turn, can be expressed in terms of the previous ones:

$$\int \frac{dk^0}{2\pi} \frac{(k^0)^2}{((k^0)^2 + A)^3} = I_2 - A\, I_3 = \frac{1}{16A^{3/2}}, \tag{B.3}$$

$$\int \frac{dk^0}{2\pi} (k^0)^2 \frac{k^2 - m^2}{((k^0)^2 + A)^3} = (I_2 - A\, I_3) - 2m^2(I_3 - A\, I_4) = \frac{1}{16A^{3/2}}(1 - m^2 A^{-1}),$$

which are used in deriving eq. (3.16).

Concerning section 3.2, instead, in order to calculate the k^0 integration which leads to eqs. (3.37) and (3.39) from eq. (3.36) we have used the following identities,

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{(k^0)^2}{((k^0)^2 + A)^2((k^0)^2 + B)} = \frac{1}{4\sqrt{A}(\sqrt{A} + \sqrt{B})^2}, \tag{B.4}$$

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{1}{((k^0)^2 + A)^2((k^0)^2 + B)} = \frac{(2\sqrt{A} + \sqrt{B})}{4A^{3/2}(\sqrt{A} + \sqrt{B})^2\sqrt{B}}, \tag{B.5}$$

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{1}{((k^0)^2 + A)^2((k^0)^2 + B)} = \frac{1}{2\left(A\sqrt{B} + \sqrt{A}B\right)}, \tag{B.6}$$

where $A = \mathbf{k}^2 + m_\psi^2$ and $B = \mathbf{k}^2 + m_\phi^2$. Note that these relations hold also for the case with modified dispersion relations (MDR) discussed in section 3.3 with suitable corresponding definition of A and B.

B.2 Asymptotic behaviours

In this appendix we demonstrate eqs. (3.46) and (3.61) concerning the asymptotic behaviour of Δc for $m_{\psi,\phi} \ll M$ in the case in which the scale M of the MDR (see eq. (3.45)) is the only cutoff scale present, with the additional assumption that $\Lambda \gg M$ when the LI cutoff scale Λ is also present.

B.2.1 Dispersive case

Let us consider eq. (3.48), rescale the momenta by M and consider the case in which $\Lambda \gg M$ and $m_{\psi,\phi} \ll M$; for convenience hereafter we assume $m_\psi = m_\phi = 0$, then we have

$$\Delta c = -\frac{g^2}{4\pi^3} \int_0^\infty dy \int_0^\pi d\phi \, \sin^2\phi \, \frac{\cos^2\phi - (\sin^2\phi)/3}{y + y^{1+n}\sin^{2+2n}\phi} \equiv -\frac{g^2}{4\pi^3} I, \tag{B.7}$$

where we defined $y = k^2/M^2$. The integral on y runs up to ∞ both in the case with a sharp and smooth cutoff as we are anyhow interested in the limit $\Lambda/M \gg 1$. Moreover we will see that the result is actually determined by the behaviour of the integrand around $y = 0$ which is not affected by the way the cutoff is imposed. The integral I defined in eq. (B.7) can be regularised by introducing a $\epsilon > 0$

$$I = \lim_{\epsilon \to 0} \int_\epsilon^\infty dy \int_0^\pi d\phi \frac{\mathcal{P}(\phi)}{y + y^{1+n}C}, \tag{B.8}$$

where we defined $C = \sin^{2+2n}\phi$ and

$$\mathcal{P}(\phi) = \sin^2\phi \left[\cos^2\phi - (\sin^2\phi)/3\right]. \tag{B.9}$$

Note that $C > 0$ and

$$\int_0^\pi d\phi \, \mathcal{P}(\phi) = 2\int_0^{\pi/2} d\phi \, \mathcal{P}(\phi) = 0. \tag{B.10}$$

Let us first consider the integral over y, which is logarithmically divergent as $\epsilon \to 0$. In fact

$$\begin{aligned} \int_\epsilon^\infty dy \frac{1}{y + y^{1+n}C} &= \int_{\epsilon C^{1/n}}^\infty d\chi \frac{1}{\chi + \chi^{1+n}} \\ &= -\log\epsilon + \int_0^1 d\chi \left(\frac{1}{\chi+\chi^{1+n}} - \frac{1}{\chi}\right) + \int_1^\infty d\chi \frac{1}{\chi + \chi^{1+n}} \\ &\quad - \frac{1}{n}\log C + \mathcal{O}(\epsilon), \end{aligned} \tag{B.11}$$

where on the first line $\chi = C^{1/n} y$, while on the second one we have taken the limit $\epsilon \to 0$ of the first integral since it is finite for $n > 0$. The first three terms on the last line do not depend on ϕ and due to eq. (B.10), they do not contribute to I in eq. (B.8), while the last term yields

$$I = -2\frac{n+1}{n}\int_0^{\pi} d\phi\, \mathcal{P}(\phi)\log(\sin\phi). \tag{B.12}$$

This integral can be calculated by using its symmetry around $\phi = \pi/2$, the change of variable $x = \sin\phi$, and the fact that $\log x = \lim_{\epsilon\to 0}(x^{\epsilon}-1)/\epsilon$, which allows us to express I in terms of

$$i_\alpha \equiv \int_0^1 dx \frac{x^\alpha}{\sqrt{1-x^2}} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)} \qquad (\alpha > -1), \tag{B.13}$$

and conclude that

$$I = \frac{n+1}{n}\frac{\pi}{12}. \tag{B.14}$$

Accordingly,

$$\Delta c = -\frac{g^2}{48\pi^2}\frac{n+1}{n}, \tag{B.15}$$

in accordance with eq. (3.46). Note that, as anticipated, the integral I is essentially determined by the algebraic singularity of the integrand as a function of $y \to 0$ and this shows that the eventual result is independent of how the LI cutoff Λ is actually imposed, as this affects the integrand only at $y \simeq 1$.

B.2.2 Dissipative case

Equation (3.61) can be derived from eq. (3.60) by following the same steps as above. Indeed, eq. (3.60) can be written as (as we are interested in the IR limit we set $m_{\psi,\phi} = 0$)

$$\Delta c = -\frac{g^2}{2\pi^3}\int_0^\infty dy \int_0^{\pi/2} d\phi \frac{\mathcal{P}(\phi)}{y + y^{3/2+n}\sin^{2+2n}\phi\cos\phi} \equiv -\frac{g^2}{2\pi^3}\tilde{I}, \tag{B.16}$$

where we defined $y = k^2/M^2$ and $\mathcal{P}(\phi)$ is given in eq. (B.9). Defining now $\sin^{2+2n}\phi\cos\phi \equiv C$ we see that the y-integration in $\tilde{I}$ is formally analogous to the one in I defined in eq. (B.8) and has a decomposition similar to eq. (B.11) with $n \to n+1/2$. Then, using eq. (B.10) we remain with only the angular part to be computed. This is given by

$$\int_0^{\pi/2} d\phi \mathcal{P}(\phi)\left[(2n+2)\log(\sin\phi) + \log(\cos\phi)\right]. \tag{B.17}$$

The first term is the one already computed in eq. (B.12), giving $-(2n+2)\pi/48$ while the second term is analogously evaluated to be $\pi/16$. The final result is then

$$\Delta c = -\frac{g^2}{2\pi^2(n+1/2)}\left(\frac{2n-1}{48}\right) = -\frac{g^2}{48\pi^2}\frac{n-1/2}{n+1/2},$$

in accordance with eq. (3.61). Again, as before, the entire contribution to the integral $\tilde{I}$ comes from the region of integration around $y = 0$ and this shows that the result is actually independent of how the LI cutoff Λ is imposed.

B.3 Reality and pole structure in the dissipative case

In this appendix we analyze the location of the poles and the possibility of performing the Wick rotation in section 3.4. Moreover, we discuss also the correction to the mass of the fermion arising in the dissipative case due to its self-energy.

First, we study the location of the poles of the integrand and the reality of the following integral

$$i\int_0^\infty d\omega \frac{\omega^2+\frac{k^2}{3}}{(\omega^2-k^2-m_\psi^2+i\epsilon)^2\left(\omega^2-k^2-m_\psi^2+i\epsilon+i\omega\frac{k^{2+2n}}{M^{1+2n}}\right)}, \tag{B.18}$$

which is relevant for the discussion in section 3.4 (see eq. (3.59)). Note that the primitive of this integral (which does not show any singularity within the domain of integration, as long as $\epsilon \neq 0$) can be calculated explicitly (however, we do not report here its lengthy expression) and one can show that in the limit $\epsilon \to 0^+$ the corresponding integral is indeed finite and real. In addition its subsequent integration in k (see eq. (3.59)) is finite and therefore this limit can be taken from the outset. The location of the poles of the integrand in the complex ω-plane, is easy to determine by studying the zeros of the denominator. Doing so it turns out that the poles are always located outside the region with both $\mathrm{Re}[\omega]$ and $\mathrm{Im}[\omega]$ positive in such a way that the integral on a close contour entirely within this region gives zero due to Cauchy's theorem. Now given the structure of the integrand it is easy to see that on the arch of circumference of large radius Ω which lies within that first quadrant, the integral is bounded by $\approx 1/\Omega^3$, so that it vanishes in the limit $\Omega \to \infty$. Then we can Wick rotate the integral in eq. (B.18), obtaining finally

$$-\int_0^\infty d\omega \frac{\omega^2-\frac{k^2}{3}}{(+\omega^2+k^2+m_\psi^2)^2\left(\omega^2+k^2+m_\psi^2+\omega\frac{k^{2+2n}}{M^{1+2n}}\right)}, \tag{B.19}$$

which is used for deriving eq. (3.60).

Mass correction

The same reasoning as above can be applied for studying the correction to the fermion mass. The latter is encoded in $\chi_1(0)$ of eq. (3.28) (considering in this case a dissipative term in the scalar propagator, see eq. (3.57), with $f = \tilde{f} = 1$), i.e.

$$-ig^2\int_0^\infty \frac{dk}{(2\pi)^3}4\pi k^2 \int_{-\infty}^\infty \frac{d\omega}{2\pi}\frac{1}{(\omega^2-k^2-m_\psi^2+i\epsilon)(\omega^2-k^2-m_\phi^2+i\epsilon+i|\omega|k^{2+2n}/M^{1+2n})}. \tag{B.20}$$

The symmetry under $\omega \to -\omega$ of the integrand and the same splitting of the integral as the one invoked in deriving eq. (3.59) can be used in this expression in order to arrive at

$$-8\pi i g^2 \int_0^\infty \frac{dk}{(2\pi)^3} k^2 \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{(\omega^2 - k^2 - m_\psi^2 + i\epsilon)(\omega^2 - k^2 - m_\phi^2 + i\epsilon + i\omega k^{2+2n}/M^{1+2n})}. \tag{B.21}$$

At this point it is again possible to calculate the primitive of the ω-integral and show that in the limit $\epsilon \to 0$ the whole expression is real. The poles of the integrand are located at the same positions as above and this permit again to perform the Wick rotation. The successive integration in k is finite (apart from the case in which both fields are massless for which there is an IR divergence, as expected). This shows that the mass does not acquire an imaginary part (at least at one loop).

Appendix C

Addendum to Chapter 4

C.1 Series Form of the Non-Local Operator

In this appendix we show how to rewrite the Laplace transform of the $2d$ and $4d$ non-local d'Alembert operator in series form. This allows one to easily determine the IR corrections to the exact continuum d'Alembertian.

C.1.1 4 Dimensions

Following the notation in [17] we begin by rewriting Equation (4.39)

$$\tilde{f}^{(4)}(Z) := -\frac{4}{\sqrt{6}}+\frac{16}{\sqrt{6}}\pi Z^{-1/2}\int_0^\infty ds\, s^2 e^{-C_4 s^4} K_1\left(Z^{1/2}s\right)\left(1-9C_4s^4+8C_4^2s^8-\frac{4}{3}C_4^3s^{12}\right), \tag{C.1}$$

where $f^4(k^2) = \sqrt{\rho}\tilde{f}^{(4)}(Z)$, $Z = k^2/\sqrt{\rho}$ and $C_4 = \pi/24$. The IR condition then takes the form $Z \ll 1$. From the power series expansion of K_1 (see Equation 10.31.1 of [166]) one has

$$(Z^{1/2}s)^{-1}K_1(Z^{1/2}s) = (Zs^2)^{-1} + \frac{1}{2}\ln\left(\frac{1}{2}Z^{1/2}s\right)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}Zs^2\right)^k}{k!\Gamma(k+2)} - \frac{1}{4}\sum_{k=0}^{\infty}\left(\psi(k+1)+\psi(k+2)\right)\frac{\left(\frac{1}{4}Zs^2\right)^k}{k!(k+1)!}. \tag{C.2}$$

As shown in Appendix A of [17], the first two terms of this series ($k = 0, 1$), when substituted back into Equation (C.1), give $-Z$. The procedure can be continued for generic values of k in order to obtain the series form of the entire Laplace transform of the non-local d'Alembertian. The series can be compactly written as

$$\tilde{f}^{(4)}(Z) = -Z + \sum_{k=2}^{\infty} c_k(Z) Z^k, \tag{C.3}$$

where the kth term of the series is given by,

$$c_k(Z) = -\frac{1}{\Gamma(k+2)^2} 2^{-\frac{k+1}{2}} 3^{-\frac{k+1}{2}} \pi^{-\frac{k}{2}} \Big(16(k+1)\Gamma(\frac{k}{2}+3) \\ +k^3\Gamma(\frac{k}{2}) \left(\ln(2\pi/3) - k^2\ln(2\pi/3) - 2k - k\ln(2\pi/3) + 2(k^2+k-1)\ln(Z)\right) \\ +\Gamma(1+\frac{k}{2})(-36 - 56k - 16k^2 + 8k^3 + k\ln(\frac{4\pi^2}{9}) - 4k\ln(Z) \\ +2k(k-1)(k+1)^2\psi(1+\frac{k}{2}) - 8k(k-1)(k+1)^2\psi(1+k)) \Big). \quad \text{(C.4)}$$

Note that c_k can be written as $c_k = a_k + b_k \ln(Z)$ for constants a_k and b_k.

C.1.2 2 Dimensions

In $2d$ we use the power series expansion of K_0 given by

$$K_0\left(z^{1/2}\right) = -\ln\left(\frac{1}{2}z^{1/2}s\right)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}zs^2\right)^k}{k!\Gamma(k+1)} + \frac{1}{2}\sum_{k=0}^{\infty}2\psi(k+1)\frac{\left(\frac{1}{4}zs^2\right)^k}{(k!)^2}, \quad \text{(C.5)}$$

to solve for

$$\tilde{f}^{(2)}(Z) = \frac{f^2(k^2)}{\rho} = -2 + 2\int_0^{\infty} ds\, s\, e^{-s^2/2}\left(4 - 4s^2 + \frac{1}{2}s^4\right) K_0\left(Z^{1/2}s\right), \quad \text{(C.6)}$$

order by order. The terms corresponding to $k = 0, 1$ in the expansion of the modified Bessel function combine together to give the IR behaviour of (C.6):

$$\tilde{f}^{(2)}(Z) = -Z + \ldots. \quad \text{(C.7)}$$

The higher order terms give rise to a power series in which the kth term has the form:

$$a_k Z^k + b_k \ln(Z) Z^k := \frac{2^{1-k}Z^k(1 - 2k + k(k-1)\ln(2) - k(k-1)\ln(Z) + k(k-1)\psi(k+1))}{\Gamma(k+1)}. \quad \text{(C.8)}$$

Hence

$$\tilde{f}^{(2)}(Z) = -Z + \sum_{k=2}^{\infty}\left(a_k + b_k\ln(Z)\right) Z^k. \quad \text{(C.9)}$$

Equations (4.12) and (4.13) can then be easily obtained by truncating these series at $k = 2$.

C.2 Asymptotic States

Here we show that asymptotic states of the BL sector only contain massless states. We follow the analysis of Barci and Oxman [25].

The fields $\phi(x)$, solutions to the nonlocal equations of motion (4.19), are examples of *generalised free fields*, introduced in [104]. [1] These are said to be associated with asymptotic modes of mass m if the asymptotic in/out fields

[1] In 4 dimensions the definitions don't fully match since the generalised free fields of Greenberg have to have positive weight $\delta^+ G$. This condition is not essential for this part

$$\phi^{\pm}_{m^2}(x) := \lim_{\tau\to\pm\infty} \int_{y^0=\tau} d^3\mathbf{y} \left(\phi(y) \frac{\overleftrightarrow{\partial}}{\partial y^0} \Delta_{m^2}(x-y) \right) \tag{C.10}$$

satisfy the free field commutation relations

$$\left[\phi^{\pm}_{m^2}(x), \phi^{\pm}_{m^2}(y)\right] = i\Delta_{m^2}(x-y). \tag{C.11}$$

Recall that the general solution to the nonlocal equations of motion (4.19) in the BL sector is given by

$$\phi(x) = \int d^d k\, \delta^+ G(k) \left(a(k) e^{ik\cdot x} - a(k)^\dagger e^{-ik\cdot x} \right). \tag{C.12}$$

Then the Pauli-Jordan function is

$$i\Delta_{NL}(x-y) \equiv [\phi(x), \phi(y)] = \frac{i}{4\pi} \int d\mu^2 \delta^+ G(\mu^2) \Delta_{\mu^2}(x-y), \tag{C.13}$$

where $\Delta_{\mu^2}(x-y)$ is the Pauli-Jordan function for a free field of mass μ satisfying the KG equation. Now,

$$\begin{aligned}\left[\phi^{\pm}_{m^2}(x), \phi^{\pm}_{m^2}(y)\right] = &\lim_{\tau,\tau'\to\pm\infty} \int_{z^0=\tau} d^3\mathbf{z} \int_{z'^0=\tau} d^3\mathbf{z}' \Big\{ \left[\phi(z), \phi(z')\right] \frac{\partial}{\partial z^0} \Delta_{m^2}(x-z) \frac{\partial}{\partial z'^0} \Delta_{m^2}(y-z') \\ &- \left[\phi(z), \pi(z')\right] \frac{\partial}{\partial z^0} \Delta_{m^2}(x-z) \Delta_{m^2}(y-z') - \left[\pi(z), \phi(z')\right] \Delta_{m^2}(x-z) \frac{\partial}{\partial z'^0} \Delta_{m^2}(y-z') \\ &+ \left[\pi(z), \pi(z')\right] \Delta_{m^2}(x-z) \Delta_{m^2}(y-z') \Big\}.\end{aligned} \tag{C.14}$$

The four terms in the integrand are

$$(1) = \frac{i}{4\pi} \int d\mu^2 \delta^+ G(\mu^2) \Delta_{\mu^2}(z-z') \frac{\partial}{\partial z^0} \Delta_{m^2}(x-z) \frac{\partial}{\partial z'^0} \Delta_{m^2}(y-z'), \tag{C.15}$$

$$(2) = -\frac{i}{4\pi} \int d\mu^2 \delta^+ G(\mu^2) \frac{\partial}{\partial z'^0} \Delta_{\mu^2}(z-z') \frac{\partial}{\partial z^0} \Delta_{m^2}(x-z) \Delta_{m^2}(y-z'), \tag{C.16}$$

$$(3) = -\frac{i}{4\pi} \int d\mu^2 \delta^+ G(\mu^2) \frac{\partial}{\partial z^0} \Delta_{\mu^2}(z-z') \Delta_{m^2}(x-z) \frac{\partial}{\partial z'^0} \Delta_{m^2}(y-z'), \tag{C.17}$$

$$(4) = \frac{i}{4\pi} \int d\mu^2 \delta^+ G(\mu^2) \frac{\partial}{\partial z^0} \frac{\partial}{\partial z'^0} \Delta_{\mu^2}(z-z') \Delta_{m^2}(x-z) \Delta_{m^2}(y-z'). \tag{C.18}$$

Integrating the sum of the first two terms and taking the limit gives

$$\frac{i}{4\pi} \lim_{\tau\to\pm\infty} \int_{z^0=\tau} d^3\mathbf{z} \frac{\partial}{\partial z^0} \Delta_{m^2}(x-z) \int d\mu^2 \delta^+ G(\mu^2) \begin{cases} \Delta_{\mu^2}(z-y), & \text{if } m^2 = \mu^2 \\ 0, & \text{otherwise} \end{cases} \tag{C.19}$$

---

of the analysis, but its relaxation will need to be dealt with carefully once interactions are introduced.

Doing the same for the sum of the last two terms

$$\frac{i}{4\pi}\lim_{\tau\to\pm\infty}\int_{z^0=\tau} d^3\mathbf{z}\Delta_{m^2}(x-z)\frac{\partial}{\partial z^0}\int d\mu^2\delta^+G(\mu^2)\begin{cases}-\Delta_{\mu^2}(z-y), & \text{if } m^2=\mu^2\\ 0, & \text{otherwise}\end{cases}, \tag{C.20}$$

where we used the fact $\Delta_{\mu^2}(y-z) = -\Delta_{\mu^2}(z-y)$. Putting these together we find

$$-\frac{i}{4\pi}\lim_{\tau\to\pm\infty}\int_{z^0=\tau} d^3\mathbf{z}\int d\mu^2\delta^+G(\mu^2)\Delta_{m^2}(x-z)\frac{\overleftrightarrow{\partial}}{\partial z^0}\begin{cases}\Delta_{\mu^2}(z-y), & \text{if } \mu^2=m^2\\ 0, & \text{otherwise}\end{cases}. \tag{C.21}$$

Since $\mu^2 = m^2$ is a set of measure zero, Equation (C.21) is non zero only at delta-function type singularities of $\delta^+G(k)$, i.e. for $\mu^2 = 0$. Using the following identity

$$\lim_{\tau\to\pm\infty}\int_{y^0=\tau} d^3\mathbf{y}\left(\Delta_{m_0^2}(x-y)\frac{\overleftrightarrow{\partial}}{\partial y^0}\Delta_{m^2}(z-y)\right) = \begin{cases}\Delta_{m_0^2}(x-z), & \text{if } m^2=m_0^2\\ 0, & \text{otherwise}\end{cases}, \tag{C.22}$$

we find that the asymptotic in/out fields satisfy massless free field commutation relations

$$[\phi_0^\pm(x), \phi_0^\pm(y)] = i\Delta_0(x-y). \tag{C.23}$$

Appendix D

Addendum to Chapter 5

D.1 Previously Neglected Terms

In this appendix we show for completeness that the terms in eq. (5.8) which were neglected in the main text due to dimensional arguments, are indeed irrelevant in the local limit. Terms containing unknown functions (like the ones that appear in the expansion of field, metric and volume) and also the infinite series are not fully taken into account in this way, but this goes beyond the scope of the present thesis. However, we see no reasons why the results in [36] should not extend to all dimensions and for non-minimal operators, given that they rely only on properties of $\hat{O}$.

Let us start by considering the terms in eq. (5.8) given by

$$\int_0^{\tilde{a}} dv \int_0^v du \frac{(v-u)^{D-2}}{2^{(D-2)/2}} \int d\Omega_{D-2} \tag{D.1}$$
$$\left[-\frac{1}{6}R_{\mu\nu}(0)y^\mu y^\nu \cdot (y^\mu \phi_{,\mu}(0) + \frac{1}{2}y^\nu y^\mu \phi_{,\nu\mu}(0))\right.$$
$$\left. + \left(y^\mu \phi_{,\mu}(0) + \frac{1}{2}y^\nu y^\mu \phi_{,\nu\mu}(0)\right)(-\rho\delta V)\right] e^{-\rho V_0},$$

Using spherical symmetry and with a schematic way of writing we can classify these terms based on powers of u and v (up to the common exponential factor) as

$$(v-u)^{D-2} \cdot \begin{cases} tr^2, t^3 \\ \rho\tau^{D+2}r^2 \\ \rho\tau^{D+2}r^2 t, \rho\tau^{D+2}t^3 \\ \rho\tau^{D+2}r^4, \rho\tau^{D+2}r^2t^2, \rho\tau^{D+2}t^4 \\ \rho\tau^D r^6, \rho\tau^D r^4 t^2, \rho\tau^D t^6 \end{cases} \tag{D.2}$$

The general term is of the usual form $I_{m,n}$ with $m+n = (D+1, 2D+2, 2D+3, 2D+4)$. The remaining terms in eq. (5.8) are given by

$$\int_0^{\tilde{a}} dv \int_0^v du \frac{(v-u)^{D-2}}{2^{(D-2)/2}} \int d\Omega_{D-2} \tag{D.3}$$
$$\left[\left(\phi(0) + y^\mu \phi_{,\mu}(0) + \frac{1}{2}y^\nu y^\mu \phi_{,\nu\mu}(0)\right)\delta\sqrt{-g} \cdot (-\rho\delta V)\right. \tag{D.4}$$
$$\left. + (1+\delta\sqrt{-g}) \cdot (\phi(y)) \cdot \sum_{k=2}^{\infty} \frac{(-\rho\delta V)^k}{k!}\right] e^{-\rho V_0}.$$

In schematic form

$$(v-u)^{D-2}\cdot\begin{cases}\rho\tau^{D+2}r^2,\,\rho\tau^{D+2}t^2\\ \rho\tau^{D+2}r^2t,\,\rho\tau^{D+2}t^3\\ \rho\tau^{D+2}r^4,\,\rho\tau^{D+2}r^2t^2,\,\rho\tau^{D+2}t^4\\ \rho\tau^{D}r^4,\,\rho\tau^{D}r^2t^2,\,\rho\tau^{D}t^4\\ \rho\tau^{D}r^4t,\,\rho\tau^{D}r^2t^3,\,\rho\tau^{D}t^5\\ \rho\tau^{D}r^6,\,\rho\tau^{D}r^2t^4,\,\rho\tau^{D}r^4t^2,\,\rho\tau^{D}t^6,\end{cases} \tag{D.5}$$

for terms in the first line of eq. (D.3) and

$$(v-u)^{D-2}\cdot\begin{cases}\rho^k\tau^{Dk+2k},\,\rho^k\tau^{Dk}y^2k\\ \rho^k\tau^{Dk+2k}y,\,\rho^k\tau^{Dk}y^{2k+1}\\ \rho^k\tau^{Dk+2k}y^2,\,\rho^k\tau^{Dk}y^{2k+2}\\ \rho^k\tau^{Dk+2k}y^3,\,\rho^k\tau^{Dk}y^{2k+3}\\ \rho^k\tau^{Dk+2k}y^4,\,\rho^k\tau^{Dk}y^{2k+4},\end{cases} \tag{D.6}$$

for terms in the second line, where $k \geq 2$ and y can be both r and t (with r always appearing in even powers due to spherical symmetry). Terms in eqs. (D.5) and (D.6) contain $I_{m,n}$ with

$$m+n=\begin{cases}2D+2,2D+3,2D+4,(k+1)D+2(k-1)\\ (k+1)D+2k-1,(k+1)D+2k,(k+1)D+2k+1,(k+1)D+2(k+1)).\end{cases}$$

Now that we have collected all the terms of interest we can study their contributions in the local limit. The general terms that appear in the previous section are of the form $I_{m,n}$ (multiplied by some power of ρ). We need to differentiate two cases.

D.1.1 Case $n \neq m$

The relevant terms in the local limit are proportional to

$$\rho^{-\frac{2+2m}{D}},\rho^{-\frac{2+m+n}{D}}$$

if $m+n=D+1$,

$$\rho^{-\frac{2+2m-D}{D}},\rho^{-\frac{2+m+n-D}{D}}$$

for other values of $m+n$ not involving $k \geq 2$ and

$$\rho^{-\frac{2+2m-Dk}{D}},\rho^{-\frac{2+m+n-Dk}{D}}$$

for the terms in eq. (D.6). Given the definition of $\hat{O}$ (see eq. (5.3)) only terms proportional to $\rho^{-\alpha}$ with $\alpha \leq \frac{D+2}{D}$ can give divergent or finite contributions in the local limit.

- For $m+n=D+1$ the only terms that could give problems are the ones proportional to $\rho^{-\frac{2+2m}{D}}$, but these are annihilated by $\hat{O}$, see eq. (5.15).
- For all the other terms not involving k, there is always a factor of $\tau^D = (uv)^{D/2}$, i.e. m (or n) is always of the form $D/2+x$ with x

an integer. Possible divergent (or finite) terms are proportional to $\rho^{-\frac{2+2m-D}{D}} = \rho^{-\frac{2+2x}{D}}$ and are annihilated by $\hat{O}$.

- Finally, for the terms in eq. (D.6): the ones proportional to $\rho^{-\frac{2+m+n-Dk}{D}}$ do not give any contribution in the local limit; the ones proportional to $\rho^{-\frac{2+2m-Dk}{D}}$ are annihilated by $\hat{O}$ since there is always a factor τ^{Dk}, i.e. $m = kD/2 + x$ with x integer.

D.1.2 Case $m = n$

In this case all the terms are multiplied by ρ or ρ^k and it is easy to see (by directly computing the integrals using eq. (5.19)) that the relevant terms are

$$\rho^{\frac{D-2-2m}{D}}, \rho^{\frac{Dk-2-2m}{D}}$$

and

$$\rho^{\frac{D-2-2m}{D}} \log(\tilde{a}^D \rho), \rho^{\frac{Dk-2-2m}{D}} \log(\tilde{a}^D \rho).$$

It can be shown that

$$\hat{O}\rho^{-\alpha} \log(c\rho) \propto \rho^{-\alpha} \log(c\rho),$$

thus these terms do not give any finite contribution in the local limit.

D.2 Inductive Proofs

In this appendix we collect some details of the proof that were omitted in sec. 5.1.2.

D.2.1 Equivalence of eqs. (5.32) and (5.1b)

To prove the equivalence we need first of all to compute $\hat{O}\left(Log[\tilde{a}^D c_D \rho]/\rho\right)$. From eq. (5.3) we see that it is sufficient to compute the general

$$H_n \frac{Log[c\rho]}{\rho}, \tag{D.7}$$

where c is a constant:

$$H_n \frac{Log[c\rho]}{\rho} = \frac{1}{\rho}\left(A_n^1 + A_n^2 \log[c\rho]\right), \tag{D.8}$$
$$A_n^1 = (-1)^{n+1} n!(\psi(n+1) + \gamma), \tag{D.9}$$
$$A_n^2 = (-1)^n n!.$$

We prove eq. (D.8) by induction. Assuming that the n-th term is of the form in eq. (D.8), we want to prove that the $n+1$-th term is of the same form. From the definition of H_n

$$H_{n+1}(\cdot) \equiv \rho^{n+1} \frac{\partial}{\partial \rho^{n+1}}(\cdot) = -n\, H_n(\cdot) + H(H_n(\cdot)), \tag{D.10}$$

so that

$$H_{n+1}\frac{Log[c\rho]}{\rho} = -\frac{n}{\rho}\left(A_n^1 + A_n^2\log[c\rho]\right) \tag{D.11}$$
$$+ A_1^n\left(-\frac{1}{\rho}\right) + A_2^n\frac{1-\log[c\,\rho]}{\rho}$$
$$= \frac{1}{\rho}\left[(-nA_1^n - A_1^n + A_2^n) + (-nA_2^n - A_2^n)\log[c\rho]\right].$$

Consider the two new coefficients in the above expression, the first one is

$$(-nA_1^n - A_1^n + A_2^n) = \tag{D.12}$$
$$= (-1)^{n+1+1}(n+1)!(\psi(n+1)+\gamma+\frac{n!}{(n+1)!})$$
$$= (-1)^{n+1+1}(n+1)!(\psi(n+1+1)+\gamma) \equiv A_1^{n+1},$$

where in the last line we used $\psi(x+1) = \psi(x) + \frac{1}{x}$. The second coefficient is

$$(-nA_2^n - A_2^n) = \tag{D.13}$$
$$= -(-1)^n(n+1)n! = (-1)^{n+1}(n+1)! \equiv A_2^n.$$

This conclude the inductive proof of eq. (D.8).

Inserting eq. (D.8) in eq. (5.32) we have

$$\frac{1}{2^N}(-1)^N\frac{(2N)!}{(N!)^2}(2N+1)\frac{2(4\pi)^N N!}{(2N+1)!}\frac{1}{2(N+1)^2 2^{2N+1}C_{2N+2}}. \tag{D.14}$$
$$\sum_{n=0}^{L_{max}}\frac{b_n}{n!}(-1)^n\left[(-1)^{n+1}n!(\psi(n+1)+\gamma)+(-1)^n n!\log[c_D\tilde{a}^D\rho]\right] = -a.$$

The sum $\sum_{n=0}^{L_{max}} b_n$ appearing in the above expression vanishes, see eq. (5.1a) with $k = \frac{D-2}{2} = N$, therefor eq. (D.14) reduces to

$$a + \frac{2(-1)^{N+1}\pi^N}{C_D N! D^2}\sum_{n=0}^{L_{max}} b_n\psi(n+1) = 0, \tag{D.15}$$

i.e. eq. (5.1b).

D.2.2 Equivalence of eqs. (5.35) and (5.1c)

To prove the equivalence we need first of all to compute $\hat{O}\left[\log(\tilde{a}^D c_D\rho)/\rho^{(D+2)/D}\right]$. We prove by induction that

$$H_n\frac{Log[c\rho]}{\rho^{\frac{D+2}{D}}} = \frac{1}{\rho^{\frac{D+2}{D}}}\left(B_n^1 + B_n^2\log[c\rho]\right), \tag{D.16}$$

$$B_n^1 = (-1)^{n+1}\frac{1}{\Gamma(\frac{D+2}{D})}\left[\Gamma(n+\frac{D+2}{D})\psi(n+\frac{D+2}{D}) - \psi(\frac{D+2}{D})\Gamma(n+\frac{D+2}{D})\right], \tag{D.17}$$

$$B_n^2 = (-1)^n\frac{\Gamma(n+\frac{D+2}{D})}{\Gamma(\frac{D+2}{D})}.$$

Assume that the n-th term is of the above form, then the $n+1$-th term is given by

$$H_{n+1}\frac{Log[c\rho]}{\rho^{\frac{D+2}{D}}} = \frac{1}{\rho^{\frac{D+2}{D}}}\left[\left(-nB_n^1 + B_n^1\frac{-2-D}{D} + B_2^n\right) + \left(-nB_n^2 - \frac{D+2}{D}B_n^2\right)\log(c\rho)\right]. \quad \text{(D.18)}$$

The coefficients in the above expression are such that

$$\begin{aligned}&\left(-nB_n^1 + B_n^1\frac{-2-D}{D} + B_n^2\right) \\ &= (-1)^{n+1+1}\frac{1}{\Gamma(\frac{D+2}{D})}\left[\Gamma(n+1+\frac{D+2}{D})\psi(n+1+\frac{D+2}{D})\right. \\ &\left.-\psi(\frac{D+2}{D})\Gamma(n+1+\frac{D+2}{D})\right] \\ &\equiv B_{n+1}^1,\end{aligned} \quad \text{(D.19)}$$

$$\begin{aligned}&\left(-nB_n^2 - \frac{D+2}{D}B_n^2\right) = \\ &= (-1)^{n+1}\frac{\Gamma(n+1+\frac{D+2}{D})}{\Gamma(\frac{D+2}{D})} \equiv B_{n+1}^2,\end{aligned} \quad \text{(D.20)}$$

where we used $\psi(x+1) = \psi(x) + 1/x$ and $\Gamma(1+x) = x\Gamma(x)$. This concludes the proof by induction of eq. (D.16).

We now have

$$\hat{O}\left[\log(\tilde{a}^D c_D\rho)/\rho^{(D+2)/D}\right] = \sum_{n=0}^{L_{max}}\frac{b_n}{n!}(-1)^n(B_1^n + B_2^n\log(\tilde{a}^D c_D\rho)), \quad \text{(D.21)}$$

where $\sum_{n=0}^{L_{max}}\frac{b_n}{n!}(-1)^n B_2^n = 0$ (see eq. (5.1a) with $k = D/2$). Using this eq. (5.35) became

$$\sum_{n=0}^{L_{max}}\frac{b_n}{n!}(-1)^n B_1^n = \frac{2^{N+2}}{\frac{2N!(4\pi)^N}{(2N+1)!}}\frac{2(N+1)^2 2^{N+2} C_D^{\frac{N+2}{N+1}}}{\Gamma(\frac{D+2}{D})}\frac{1}{A_2}, \quad \text{(D.22)}$$

where we used the expressions for ω_{D-1} and c_D. The RHS of the above expression can be further simplified observing that

$$A_2 = (-1)^{N+1}2\cot(2N)!\frac{\frac{2N+1}{N+1}}{(N!)^2}.$$

In this way we obtain for eq. (5.35)

$$\sum_{n=0}^{L_{max}}\frac{b_n}{n!}(-1)^n B_1^n = \frac{2}{\pi^N}C_D^{\frac{N+2}{N+1}}D^2(N+1)!\frac{(-1)^{N+1}}{\Gamma(\frac{D+2}{D})}. \quad \text{(D.23)}$$

Finally, using that $\sum_{n=0}^{L_{max}}\frac{b_n}{n!}(-1)^n\Gamma(n+\frac{D+2}{D}) = 0$ (see eq. (5.1a) with $k = D/2$) and the expression of B_1^n from eq. (D.16) we conclude that eq. (5.35) is equivalent to eq. (5.1c).

Appendix E

Addendum to Chapter 6

In order to take into account the secular terms appearing in the coefficients c_i the method of multiple scales should be used. We will not go into the details here (we refer the reader to [39]). However, in order to confirm the reliability of our solution in Eq.(6.56) with eq.(6.57), we have checked that, solving numerically the non-local Schrödinger equation at order ϵ, there are no solutions growing in time as fast as ϵt. To this end, we solved numerically the equation $\left(\mathcal{S} - 2a_2\epsilon\mathcal{S}^2 - x^2/2\right)\psi(t,x) = 0$ in the rectangular domain $[0 \le t \le 25] \times [-6 \le x \le 6]$ of the spacetime plane, with $2a_2\epsilon = 10^{-3}$ and periodic boundary conditions in space. In addition, we set the initial conditions

$$\psi(0,x) = \frac{1}{\pi^{1/4}} \exp\left[\sqrt{2}x - \frac{x^2}{2} - 1\right]$$
$$\left.\frac{d}{dt}\psi(t,x)\right|_{t=0} = -\frac{ie^{-\frac{x^2}{2}+\sqrt{2}x-1}\left(2\sqrt{2}x-1\right)}{2\pi^{1/4}},$$

which represent the $\alpha = 1$ coherent state.

The numerical solution was calculated using the implicit Euler method of the partial differential equation solver provided by *Mathematica*. To quantify numerical errors in the discrete space domain, we introduce the Chebyshev distance between solutions ψ_1 and ψ_2 as

$$D(t_k) \equiv \max_j \left\{ |\psi_1(t_k, x_j) - \psi_2(t_k, x_j)| \right\}. \tag{E.1}$$

To calculate $D(t_k)$, we set the space mesh size to 10^{-2} and the time mesh size to 10^{-1}.

Fig. E.1 reports the plots of the relative maximum distances among the numerical, analytical solutions and analytical solution with secular terms, and clearly shows that we have properly discarded the secular terms in the polynomial coefficients $c_i(t)$ of ψ_1. We stress that the small mismatch in Fig. E.1 between numerical and analytical solutions is due to the accumulation of numerical errors at large time, and not to secular terms. In fact, the distance is clearly not growing as fast as ϵt. As a final remark, we also point out that there is a good agreement between mean and variance of position and momentum in eqs.(6.63)-(6.66) evaluated with $\alpha = 1$, $2a_2\epsilon = 10^{-3}$, and the same quantities estimated by means of the numerical solution.

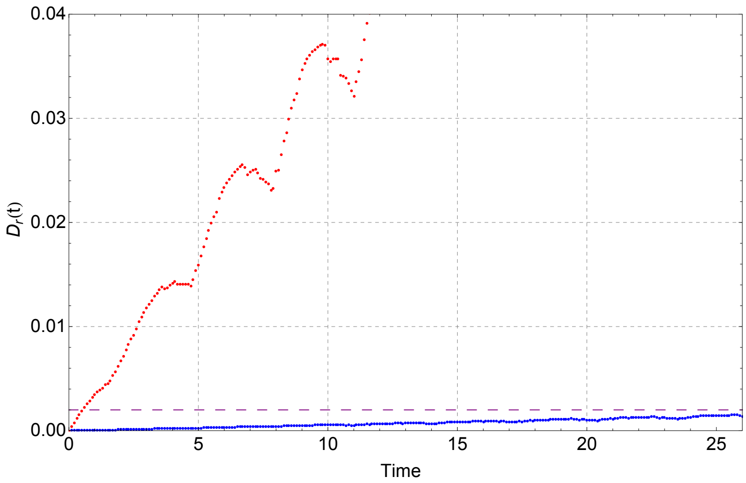

FIGURE E.1: Time dependence of the relative distance $D_r(t_k) \equiv D(t_k)/\max_j\{|\psi(t_k,x_j)|\}$ among perturbed solutions of non-local Schrödinger. For the numerical solution, we fix $\alpha = 1$ and $a_2\epsilon = 2\times10^{-3}$. The horizontal purple line is the relative distance threshold 2×10^{-3}. Blue dots represent the relative distance between ψ_1 used in Chapter6 and numerical solutions. Red dots represent the relative distance between numerical and solution with secular terms in the $c_i(t)$.

Bibliography

[1] Ronald J. Adler. "Six easy roads to the Planck scale". In: *Am. J. Phys.* 78 (2010), pp. 925–932. DOI: 10.1119/1.3439650. arXiv: 1001.1205 [gr-qc].

[2] Niayesh Afshordi. "Why is High Energy Physics Lorentz Invariant?" In: (2015). arXiv: hep-th/1511.07879 [hep-th].

[3] Maqbool Ahmed and Rafael Sorkin. "Everpresent Λ. II. Structural stability". In: *Phys.Rev.* D87.6 (2013), p. 063515. DOI: 10.1103/PhysRevD.87.063515. arXiv: 1210.2589 [gr-qc].

[4] Maqbool Ahmed et al. "Everpresent Lambda". In: *Phys. Rev.* D69 (2004), p. 103523. eprint: astro-ph/0209274.

[5] Jorge Alfaro. "Quantum gravity and Lorentz invariance deformation in the standard model". In: *Phys. Rev. Lett.* 94 (2005), p. 221302. DOI: 10.1103/PhysRevLett.94.221302. arXiv: hep-th/0412295 [hep-th].

[6] Álvaro M. Alhambra, Achim Kempf, and Eduardo Martín-Martínez. "Casimir forces on atoms in optical cavities". In: *Phys. Rev. A* 89 (3 Mar. 2014), p. 033835. DOI: 10.1103/PhysRevA.89.033835. URL: http://link.aps.org/doi/10.1103/PhysRevA.89.033835.

[7] Natalia Alkofer et al. "Quantum Gravity signatures in the Unruh effect". In: (2016). arXiv: 1605.08015 [gr-qc].

[8] Brett Altschul et al. "Quantum tests of the Einstein Equivalence Principle with the STE–QUEST space mission". In: *Advances in Space Research* 55.1 (2015), pp. 501–524. ISSN: 0273-1177. DOI: http://dx.doi.org/10.1016/j.asr.2014.07.014. URL: http://www.sciencedirect.com/science/article/pii/S0273117714004384.

[9] J. Ambjorn, J. Jurkiewicz, and R. Loll. "Emergence of a 4D world from causal quantum gravity". In: *Phys. Rev. Lett.* 93 (2004), p. 131301. DOI: 10.1103/PhysRevLett.93.131301. arXiv: hep-th/0404156.

[10] J. Ambjorn, J. Jurkiewicz, and R. Loll. "Spectral dimension of the universe". In: *Phys. Rev. Lett.* 95 (2005), p. 171301. DOI: 10.1103/PhysRevLett.95.171301. arXiv: hep-th/0505113.

[11] Giovanni Amelino-Camelia. "Quantum-Spacetime Phenomenology". In: *Living Rev. Rel.* 16 (2013), p. 5. DOI: 10.12942/lrr-2013-5. arXiv: gr-qc/0806.0339 [gr-qc].

[12] Giovanni Amelino-Camelia et al. "Dimensional reduction in momentum space and scale-invariant cosmological fluctuations". In: *Phys. Rev.* D88.10 (2013), p. 103524. DOI: 10.1103/PhysRevD.88.103524. arXiv: 1309.3999 [gr-qc].

[13] Giovanni Amelino-Camelia et al. “Dimensional reduction in the sky”. In: *Phys. Rev.* D87.12 (2013), p. 123532. DOI: 10.1103/PhysRevD.87.123532. arXiv: 1305.3153 [gr-qc].

[14] Giovanni Amelino-Camelia et al. “Thermal dimension of quantum spacetime”. In: (2016). arXiv: 1602.08020 [hep-th].

[15] G. Amelino-Camelia et al. “Tests of quantum gravity from observations of gamma-ray bursts”. In: *Nature* 393 (1998), pp. 763–765. DOI: 10.1038/31647. arXiv: astro-ph/9712103 [astro-ph].

[16] Michele Arzano and Tomasz Trzesniewski. “Diffusion on κ-Minkowski space”. In: *Phys.Rev.* D89.12 (2014), p. 124024. DOI: 10.1103/PhysRevD.89.124024. arXiv: 1404.4762 [hep-th].

[17] Siavash Aslanbeigi, Mehdi Saravani, and Rafael D. Sorkin. “Generalized causal set d’Alembertians”. In: *JHEP* 1406 (2014), p. 024. DOI: 10.1007/JHEP06(2014)024. arXiv: 1403.1622 [hep-th].

[18] Markus Aspelmeyer, Tobias J. Kippenberg, and Florian Marquardt. “Cavity optomechanics”. In: *Rev. Mod. Phys.* 86 (4 Dec. 2014), pp. 1391–1452. DOI: 10.1103/RevModPhys.86.1391. URL: http://link.aps.org/doi/10.1103/RevModPhys.86.1391.

[19] Joseph J Atick and Edward Witten. “The Hagedorn transition and the number of degrees of freedom of string theory”. In: *Nuclear Physics B* 310.2 (1988), pp. 291–334.

[20] Benjamin Bahr, Bianca Dittrich, and Song He. “Coarse graining free theories with gauge symmetries: the linearized case”. In: *New J. Phys.* 13 (2011), p. 045009. DOI: 10.1088/1367-2630/13/4/045009. arXiv: 1011.3667 [gr-qc].

[21] Carlos Barcelo, Stefano Liberati, and Matt Visser. “Analogue gravity”. In: *Living Rev.Rel.* 8 (2005), p. 12. arXiv: gr-qc/0505065 [gr-qc].

[22] Carlos Barcelo, Stefano Liberati, and Matt Visser. “Probing semiclassical analog gravity in Bose-Einstein condensates with widely tunable interactions”. In: *Phys.Rev.* A68 (2003), p. 053613. DOI: 10.1103/PhysRevA.68.053613. arXiv: cond-mat/0307491 [cond-mat].

[23] Carlos Barcelo, Matt Visser, and Stefano Liberati. “Einstein gravity as an emergent phenomenon?” In: *Int.J.Mod.Phys.* D10 (2001), pp. 799–806. DOI: 10.1142/S0218271801001591. arXiv: gr-qc/0106002 [gr-qc].

[24] Carlos Barceló et al. “Electromagnetism as an emergent phenomenon: a step-by-step guide”. In: (2014). arXiv: 1407.6532 [gr-qc].

[25] D. G. Barci and L. E. Oxman. “Asymptotic States in Non-Local Field Theories”. In: *Mod.Phys.Lett.* A12 (1997), pp. 493–500. eprint: hep-th/9611147. URL: http://arxiv.org/abs/hep-th/9611147.

[26] D. G. Barci et al. “Non-Local Pseudo-Differential Operators”. In: (1996). eprint: hep-th/9606183. URL: http://arxiv.org/abs/hep-th/9606183.

[27] D.G. Barci, L.E. Oxman, and M. Rocca. "Canonical quantization of non-local field equations". In: *Int.J.Mod.Phys.* A11 (1996), pp. 2111–2126. eprint: hep-th/9503101. URL: http://arxiv.org/abs/hep-th/9503101.

[28] James M. Bardeen, B. Carter, and S.W. Hawking. "The Four laws of black hole mechanics". In: *Commun.Math.Phys.* 31 (1973), pp. 161–170. DOI: 10.1007/BF01645742.

[29] Neil Barnaby. "Nonlocal Inflation". In: *Can. J. Phys.* 87 (2009), pp. 189–194. DOI: 10.1139/P08-089. arXiv: 0811.0814 [hep-th].

[30] Neil Barnaby and Niky Kamran. "Dynamics with Infinitely Many Derivatives: The Initial Value Problem". In: *JHEP* 0802 (2008), p. 008. eprint: 0709.3968. URL: http://arxiv.org/abs/0709.3968.

[31] John D. Barrow. "A Strong Constraint on Ever-Present Lambda". In: *Phys. Rev.* D75 (2007), p. 067301. DOI: 10.1103/PhysRevD.75.067301. arXiv: gr-qc/0612128 [gr-qc].

[32] Mateusz Bawaj et al. "Probing deformed commutators with macroscopic harmonic oscillators". In: (2014). [Nature Commun.6,7503(2015)]. DOI: 10.1038/ncomms8503. arXiv: 1411.6410 [gr-qc].

[33] Grigory Bednik, Oriol Pujolás, and Sergey Sibiryakov. "Emergent Lorentz invariance from Strong Dynamics: Holographic examples". In: *JHEP* 11 (2013), p. 064. DOI: 10.1007/JHEP11(2013)064. arXiv: hep-th/1305.0011 [hep-th].

[34] Alessio Belenchia. "Universal behavior of generalized causal set d'Alembertians in curved spacetime". In: *Classical and Quantum Gravity* 33.13 (2016), p. 135011. URL: http://stacks.iop.org/0264-9381/33/i=13/a=135011.

[35] Alessio Belenchia, Dionigi M. T. Benincasa, and Stefano Liberati. "Nonlocal Scalar Quantum Field Theory from Causal Sets". In: *JHEP* 03 (2015), p. 036. DOI: 10.1007/JHEP03(2015)036. arXiv: 1411.6513 [gr-qc].

[36] Alessio Belenchia, Dionigi M.T. Benincasa, and Fay Dowker. "The continuum limit of a 4-dimensional casual set scalar d'Alembertian". In: (2015).

[37] Alessio Belenchia, Andrea Gambassi, and Stefano Liberati. "Lorentz violation naturalness revisited". In: *Journal of High Energy Physics* 2016.6 (2016), pp. 1–33. ISSN: 1029-8479. DOI: 10.1007/JHEP06(2016)049. URL: http://dx.doi.org/10.1007/JHEP06(2016)049.

[38] Alessio Belenchia et al. "Low-Energy Signatures of Nonlocal Field Theories". In: (2016). arXiv: 1605.03973 [quant-ph].

[39] Alessio Belenchia et al. "Quantum gravity phenomenology via Non-local Schrödinger Equations". In: *In preparation* (2016).

[40] Alessio Belenchia et al. "Spectral Dimension from Causal Set Non-local Dynamics". In: (2015). arXiv: gr-qc/1507.00330 [gr-qc].

[41] Alessio Belenchia et al. "Testing Quantum Gravity Induced Non-locality via Optomechanical Quantum Oscillators". In: *Phys. Rev. Lett.* 116 (16 Apr. 2016), p. 161303. DOI: 10.1103/PhysRevLett.116.161303. URL: http://link.aps.org/doi/10.1103/PhysRevLett.116.161303.

[42] Dario Benedetti. "Fractal properties of quantum spacetime". In: *Phys.Rev.Lett.* 102 (2009), p. 111303. DOI: 10.1103/PhysRevLett.102.111303. arXiv: 0811.1396 [hep-th].

[43] Dario Benedetti and Joe Henson. "Spectral geometry as a probe of quantum spacetime". In: *Phys. Rev.* D80 (2009), p. 124036. DOI: 10.1103/PhysRevD.80.124036. arXiv: 0911.0401 [Unknown].

[44] Dionigi M. T. Benincasa and Fay Dowker. "The Scalar Curvature of a Causal Set". In: *Phys. Rev. Lett.* 104 (2010), p. 181301. DOI: 10.1103/PhysRevLett.104.181301. arXiv: 1001.2725 [Unknown].

[45] Dionigi M. T. Benincasa, Fay Dowker, and Bernhard Schmitzer. "The Random Discrete Action for 2-Dimensional Spacetime". In: *Class. Quant. Grav.* 28 (2011), p. 105018. DOI: 10.1088/0264-9381/28/10/105018. arXiv: 1011.5191 [gr-qc].

[46] Jeremy Bernstein and Scott Dodelson. "Relativistic Bose gas". In: *Phys.Rev.Lett.* 66 (1991), pp. 683–687. DOI: 10.1103/PhysRevLett.66.683.

[47] Neven Bilic. "Relativistic acoustic geometry". In: *Class.Quant.Grav.* 16 (1999), pp. 3953–3964. DOI: 10.1088/0264-9381/16/12/312. arXiv: gr-qc/9908002 [gr-qc].

[48] Nicholas David Birrell and Paul Charles William Davies. *Quantum fields in curved space*. 7. Cambridge university press, 1984.

[49] Tirthabir Biswas and Nobuchika Okada. "Towards {LHC} physics with nonlocal Standard Model". In: *Nuclear Physics B* 898 (2015), pp. 113–131. ISSN: 0550-3213. DOI: http://dx.doi.org/10.1016/j.nuclphysb.2015.06.023. URL: http://www.sciencedirect.com/science/article/pii/S0550321315002382.

[50] James Bjorken. "Emergent gauge bosons". In: *Proceedings to the workshops: What comes beyond the standard model 2000, 2001. Volume 1*. 2001. arXiv: hep-th/0111196 [hep-th]. URL: http://www-public.slac.stanford.edu/sciDoc/docMeta.aspx?slacPubNumber=SLAC-PUB-9063.

[51] Ana Blasco et al. "Timelike information broadcasting in cosmology". In: *Phys. Rev. D* 93 (2 Jan. 2016), p. 024055. DOI: 10.1103/PhysRevD.93.024055. URL: http://link.aps.org/doi/10.1103/PhysRevD.93.024055.

[52] Ana Blasco et al. "Violation of the Strong Huygens' Principle and Timelike Signals from the Early Universe". In: *Phys. Rev. Lett.* 114.14 (2015), p. 141103. DOI: 10.1103/PhysRevLett.114.141103. arXiv: 1501.01650 [quant-ph].

[53] C. G. Bollini and L. E. Oxman. "Unitarity and Complex Mass Fields". In: *International Journal of Modern Physics A* 08.18 (1993), pp. 3185–3198. DOI: 10.1142/S0217751X93001272. URL: http://www.worldscientific.com/doi/abs/10.1142/S0217751X93001272.

[54] CG Bollini and JJ Giambiagi. "Lagrangian procedures for higher order field equations". In: *Revista Brasileira de Fisica* 17 (1987), p. 14.

[55] C.G. Bollini and L.E. Oxman. "Propagator for Complex-Mass Fields". In: *International Journal of Modern Physics A* 07.27 (1992), pp. 6845–6855. DOI: 10.1142/S0217751X92003148. URL: http://www.worldscientific.com/doi/abs/10.1142/S0217751X92003148.

[56] Pavel A. Bolokhov, Stefan Groot Nibbelink, and Maxim Pospelov. "Lorentz violating supersymmetric quantum electrodynamics". In: *Phys. Rev.* D72 (2005), p. 015013. DOI: 10.1103/PhysRevD.72.015013. arXiv: hep-ph/0505029 [hep-ph].

[57] Luca Bombelli, Joe Henson, and Rafael D. Sorkin. "Discreteness without symmetry breaking: A theorem". In: *Mod. Phys. Lett.* A24 (2009), pp. 2579–2587. DOI: 10.1142/S0217732309031958. arXiv: gr-qc/0605006.

[58] Luca Bombelli et al. "Space-time as a causal set". In: *Phys. Rev. Lett* 59 (1987), p. 521.

[59] Enrico Borriello et al. "Stringent constraint on neutrino Lorentz-invariance violation from the two IceCube PeV neutrinos". In: *Phys. Rev.* D87.11 (2013), p. 116009. DOI: 10.1103/PhysRevD.87.116009. arXiv: astro-ph/1303.5843 [astro-ph.HE].

[60] Michel Buck et al. "Boundary Terms for Causal Sets". In: (2015). arXiv: 1502.05388 [gr-qc].

[61] Gianluca Calcagni and Leonardo Modesto. "Nonlocality in string theory". In: *J. Phys.* A47.35 (2014), p. 355402. DOI: 10.1088/1751-8113/47/35/355402. arXiv: hep-th/1310.4957 [hep-th].

[62] Gianluca Calcagni, Leonardo Modesto, and Giuseppe Nardelli. "Quantum spectral dimension in quantum field theory". In: (2014). arXiv: 1408.0199 [hep-th].

[63] S. Carlip. "Dimensional reduction in causal set gravity". In: (2015). arXiv: 1506.08775 [gr-qc].

[64] S. Carlip. "Spontaneous Dimensional Reduction in Quantum Gravity". In: (2016). arXiv: 1605.05694 [gr-qc].

[65] Steven Carlip. "The Small Scale Structure of Spacetime". In: (2009), pp. 69–84. arXiv: 1009.1136 [gr-qc].

[66] Sean M. Carroll et al. "Noncommutative Field Theory and Lorentz Violation". In: *Phys. Rev. Lett.* 87 (14 Sept. 2001), p. 141601. DOI: 10.1103/PhysRevLett.87.141601. URL: http://link.aps.org/doi/10.1103/PhysRevLett.87.141601.

[67] D. Colladay and V. Alan Kostelecký. "Lorentz-violating extension of the standard model". In: *Phys. Rev. D* 58 (11 Oct. 1998), p. 116002. DOI: 10.1103/PhysRevD.58.116002. URL: http://link.aps.org/doi/10.1103/PhysRevD.58.116002.

[68] John Collins, Alejandro Perez, and Daniel Sudarsky. "Lorentz invariance violation and its role in quantum gravity phenomenology". In: *Approaches to Quantum Gravity: Towards a New Understanding of Space and Time*. Ed. by D. Oriti. Cambridge University Press, 2006. arXiv: hep-th/0603002 [hep-th].

[69] John Collins et al. "Lorentz invariance and quantum gravity: an additional fine-tuning problem?" In: *Phys. Rev. Lett.* 93 (2004), p. 191301. DOI: 10.1103/PhysRevLett.93.191301. arXiv: gr-qc/0403053 [gr-qc].

[70] Carlo Contaldi, Fay Dowker, and Lydia Philpott. "Polarization Diffusion from Spacetime Uncertainty". In: (2010). arXiv: 1001.4545 [Unknown].

[71] T. Damour. "Black Hole Eddy Currents". In: *Phys.Rev.* D18 (1978), pp. 3598–3604. DOI: 10.1103/PhysRevD.18.3598.

[72] Timothée Delubac et al. "Baryon acoustic oscillations in the Lyα forest of BOSS DR11 quasars". In: *Astron. Astrophys.* 574 (2015), A59. DOI: 10.1051/0004-6361/201423969. arXiv: 1404.1801 [astro-ph.CO].

[73] Nathalie Deruelle. "Nordstrom's scalar theory of gravity and the equivalence principle". In: *Gen.Rel.Grav.* 43 (2011), pp. 3337–3354. DOI: 10.1007/s10714-011-1247-x. arXiv: 1104.4608 [gr-qc].

[74] Eolo Di Casola, Stefano Liberati, and Sebastiano Sonego. "Weak equivalence principle for self-gravitating bodies: A sieve for purely metric theories of gravity". In: *Phys.Rev.* D89 (2014), p. 084053. DOI: 10.1103/PhysRevD.89.084053. arXiv: 1401.0030 [gr-qc].

[75] Paul Adrien Maurice Dirac. "Is there an aether?" In: (1951).

[76] RLPG Do Amaral and EC Marino. "Canonical quantization of theories containing fractional powers of the d'Alembertian operator". In: *Journal of Physics A: Mathematical and General* 25.19 (1992), p. 5183.

[77] Alberto Gonzalez Dominguez and Susana Elena Trione. "On the Laplace transforms of retarded, Lorentz-invariant functions". In: *Advances in Mathematics* 31.1 (1979), pp. 51–62.

[78] Michael R. Douglas and Nikita A. Nekrasov. "Noncommutative field theory". In: *Rev. Mod. Phys.* 73 (4 Nov. 2001), pp. 977–1029. DOI: 10.1103/RevModPhys.73.977. URL: http://link.aps.org/doi/10.1103/RevModPhys.73.977.

[79] Fay Dowker, Raquel Garcia, and Rafael D. Sorkin. *Hilbert space from quantum measure theory*. in preparation. 2005.

[80] Fay Dowker and Lisa Glaser. "Causal set d'Alembertians for various dimensions". In: *Class. Quant. Grav.* 30 (2013), p. 195016. DOI: 10.1088/0264-9381/30/19/195016. arXiv: 1305.2588 [gr-qc].

[81] Fay Dowker, Joe Henson, and Rafael D. Sorkin. "Quantum gravity phenomenology, Lorentz invariance and discreteness". In: *Mod. Phys. Lett.* A19 (2004), pp. 1829–1840. DOI: 10.1142/S0217732304015026. arXiv: gr-qc/0311055.

[82] Olaf Dreyer. "Emergent general relativity". In: (2006). arXiv: gr-qc/0604075 [gr-qc].

[83] Olaf Dreyer. "Why things fall". In: *PoS* QG-PH (2007), p. 016. arXiv: 0710.4350 [gr-qc].

[84] Astrid Eichhorn and Sebastian Mizera. "Spectral dimension in causal set quantum gravity". In: *Class.Quant.Grav.* 31 (2014), p. 125007. DOI: 10.1088/0264-9381/31/12/125007. arXiv: 1311.2530 [gr-qc].

[85] D. A. Eliezer and R. P. Woodard. "The Problem of Nonlocality in String Theory". In: *Nucl. Phys.* B325 (1989), p. 389. DOI: 10.1016/0550-3213(89)90461-6.

[86] J.R Ellis, N.E Mavromatos, and A.S Sakharov. "Synchrotron radiation from the Crab Nebula discriminates between models of space–time foam". In: *Astroparticle Physics* 20.6 (2004), pp. 669–682. ISSN: 0927-6505. DOI: http://dx.doi.org/10.1016/j.astropartphys.2003.12.001. URL: http://www.sciencedirect.com/science/article/pii/S0927650503002627.

[87] D. Evens et al. "Nonlocal regularizations of gauge theories". In: *Phys. Rev.* D43.2 (1991), pp. 499–519. DOI: 10.1103/PhysRevD.43.499.

[88] Serena Fagnocchi et al. "Relativistic Bose-Einstein Condensates: a New System for Analogue Models of Gravity". In: *New J.Phys.* 12 (2010), p. 095012. DOI: 10.1088/1367-2630/12/9/095012. arXiv: 1001.1044 [gr-qc].

[89] Stefano Finazzi, Stefano Liberati, and Lorenzo Sindoni. "Cosmological Constant: A Lesson from Bose-Einstein Condensates". In: *Phys.Rev.Lett.* 108 (2012), p. 071101. DOI: 10.1103/PhysRevLett.108.071101. arXiv: 1103.4841 [gr-qc].

[90] Andreu Font-Ribera et al. "Quasar-Lyman α Forest Cross-Correlation from BOSS DR11 : Baryon Acoustic Oscillations". In: *JCAP* 1405 (2014), p. 027. DOI: 10.1088/1475-7516/2014/05/027. arXiv: 1311.1767 [astro-ph.CO].

[91] Rodolfo Gambini and Jorge Pullin. "Emergence of stringlike physics from Lorentz invariance in loop quantum gravity". In: *Int. J. Mod. Phys.* D23.12 (2014), p. 1442023. DOI: 10.1142/S0218271814420231. arXiv: 1406.2610 [gr-qc].

[92] Rodolfo Gambini and Jorge Pullin. "Nonstandard optics from quantum space-time". In: *Phys. Rev. D* 59 (12 May 1999), p. 124021. DOI: 10.1103/PhysRevD.59.124021. URL: http://link.aps.org/doi/10.1103/PhysRevD.59.124021.

[93] Rodolfo Gambini, Saeed Rastgoo, and Jorge Pullin. "Small Lorentz violations in quantum gravity: do they lead to unacceptably large effects?" In: *Class. Quant. Grav.* 28 (2011), p. 155005. DOI: 10.1088/0264-9381/28/15/155005. arXiv: gr-qc/1106.1417 [gr-qc].

[94] L.J. Garay et al. "Black holes in Bose-Einstein condensates". In: *Phys.Rev.Lett.* 85 (2000), pp. 4643–4647. DOI: 10.1103/PhysRevLett.85.4643. arXiv: gr-qc/0002015 [gr-qc].

[95] L.J. Garay et al. "Sonic black holes in dilute Bose-Einstein condensates". In: *Phys.Rev.* A63 (2001), p. 023611. DOI: 10.1103/PhysRevA.63.023611. arXiv: gr-qc/0005131 [gr-qc].

[96] Luis J. Garay. "Quantum gravity and minimum length". In: *Int. J. Mod. Phys.* A10 (1995), pp. 145–166. DOI: 10.1142/S0217751X95000085. arXiv: gr-qc/9403008 [gr-qc].

[97] G. W. Gibbons and S. N. Solodukhin. "The Geometry of small causal diamonds". In: *Phys. Lett.* B649 (2007), pp. 317–324. DOI: 10.1016/j.physletb.2007.03.068. arXiv: hep-th/0703098 [HEP-TH].

[98] Steven B. Giddings. "Black hole information, unitarity, and nonlocality". In: *Phys. Rev. D* 74 (10 Nov. 2006), p. 106005. DOI: 10.1103/PhysRevD.74.106005. URL: http://link.aps.org/doi/10.1103/PhysRevD.74.106005.

[99] Steffen Gielen, Daniele Oriti, and Lorenzo Sindoni. "Homogeneous cosmologies as group field theory condensates". In: (2013). arXiv: 1311.1238 [gr-qc].

[100] Florian Girelli, Stefano Liberati, and Lorenzo Sindoni. "Gravitational dynamics in Bose Einstein condensates". In: *Phys.Rev.* D78 (2008), p. 084013. DOI: 10.1103/PhysRevD.78.084013. arXiv: 0807.4910 [gr-qc].

[101] Domenico Giulini. "What is (not) wrong with scalar gravity?" In: (2006). arXiv: gr-qc/0611100 [gr-qc].

[102] Lisa Glaser. "A closed form expression for the causal set d'Alembertian". In: (Nov. 2013). eprint: 1311.1701. URL: http://arxiv.org/abs/1311.1701.

[103] IS Gradshteyn and IM Ryzhik. "Tables of Integrals, Series, and Products, Academic, San Diego, 2007". In: *Received July* 2 (2009).

[104] O. W Greenberg. "Generalized free fields and models of local field theory". In: *Annals of Physics* 16.2 (Nov. 1961), pp. 158–176. DOI: http://dx.doi.org/10.1016/0003-4916(61)90032-X. URL: http://www.sciencedirect.com/science/article/pii/000349166190032X.

[105] Stefan Groot Nibbelink and Maxim Pospelov. "Lorentz violation in supersymmetric field theories". In: *Phys. Rev. Lett.* 94 (2005), p. 081601. DOI: 10.1103/PhysRevLett.94.081601. arXiv: hep-ph/0404271 [hep-ph].

[106] Howard E. Haber and H. Arthur Weldon. "Finite Temperature Symmetry Breaking as Bose-Einstein Condensation". In: *Phys.Rev.* D25 (1982), p. 502. DOI: 10.1103/PhysRevD.25.502.

[107] Howard E. Haber and H. Arthur Weldon. "Thermodynamics of an Ultrarelativistic Bose Gas". In: *Phys.Rev.Lett.* 46 (1981), p. 1497. DOI: 10.1103/PhysRevLett.46.1497.

[108] S. W. Hawking, A. R. King, and P. J. McCarthy. "A new topology for curved space-time which incorporates the causal, differential, and conformal structures". In: *J. Math. Phys.* 17 (1976), pp. 174–181.

[109] S.W. Hawking. "Black hole explosions". In: *Nature* 248 (1974), pp. 30–31. DOI: 10.1038/248030a0.

[110] S.W. Hawking. "Particle Creation by Black Holes". In: *Commun.Math.Phys.* 43 (1975), pp. 199–220. DOI: 10.1007/BF02345020.

[111] W. Heisenberg. *The Physical Principles of the Quantum Theory*. Dover Books on Physics and Chemistry. Dover Publications, 1949. ISBN: 9780486601137. URL: https://books.google.it/books?id=NzMBh4ZxKJsC.

[112] Joe Henson. "Discovering the Discrete Universe". In: (2010). arXiv: 1003.5890 [Unknown].

[113] Petr Ho řava. "Quantum gravity at a Lifshitz point". In: *Phys. Rev. D* 79 (8 Apr. 2009), p. 084008. DOI: 10.1103/PhysRevD.79.084008. URL: http://link.aps.org/doi/10.1103/PhysRevD.79.084008.

[114] S. Hossenfelder. "A Note on Quantum Field Theories with a Minimal Length Scale". In: *Class. Quant. Grav.* 25 (2008), p. 038003. DOI: 10.1088/0264-9381/25/3/038003. arXiv: 0712.2811 [hep-th].

[115] Sabine Hossenfelder. "Experimental Search for Quantum Gravity". In: *Workshop on Experimental Search for Quantum Gravity NORDITA, Stockholm, Sweden, July 12-16, 2010*. 2010. arXiv: 1010.3420 [gr-qc]. URL: https://inspirehep.net/record/873239/files/arXiv:1010.3420.pdf.

[116] Sabine Hossenfelder. "Minimal Length Scale Scenarios for Quantum Gravity". In: *Living Rev. Rel.* 16 (2013), p. 2. DOI: 10.12942/lrr-2013-2. arXiv: 1203.6191 [gr-qc].

[117] B.L. Hu. "Can spacetime be a condensate?" In: *Int.J.Theor.Phys.* 44 (2005), pp. 1785–1806. DOI: 10.1007/s10773-005-8895-0. arXiv: gr-qc/0503067 [gr-qc].

[118] Viqar Husain and Jorma Louko. "Low Energy Lorentz Violation from Modified Dispersion at High Energies". In: *Phys. Rev. Lett.* 116 (6 Feb. 2016), p. 061301. DOI: 10.1103/PhysRevLett.116.061301. URL: http://link.aps.org/doi/10.1103/PhysRevLett.116.061301.

[119] Roberto Iengo, Jorge G. Russo, and Marco Serone. "Renormalization group in Lifshitz-type theories". In: *JHEP* 11 (2009), p. 020. DOI: 10.1088/1126-6708/2009/11/020. arXiv: hep-th/0906.3477 [hep-th].

[120] P. D. Ingalls. "Decay of Na-20 to gamma-ray emitting states of Ne-20". In: *Phys. Rev. C* 14 (1976), pp. 254–260. DOI: 10.1103/PhysRevC.14.254.

[121] International Atomic Energy Agency, International Nuclear Structure and Decay Data Network. https://www-nds.iaea.org/. In: ().

[122] C. J. Isham. "Structural issues in quantum gravity". In: *General relativity and gravitation. Proceedings, 14th International Conference, Florence, Italy, August 6-12, 1995*. 1995. arXiv: gr-qc/9510063 [gr-qc]. URL: http://alice.cern.ch/format/showfull?sysnb=0211403.

[123] Claude Itzykson and Jean-Bernard Zuber. *Quantum field theory*. Courier Corporation, 2006.

[124] Ted Jacobson. "Black hole radiation in the presence of a short distance cutoff". In: *Phys.Rev.* D48 (1993), pp. 728–741. DOI: 10.1103/PhysRevD.48.728. arXiv: hep-th/9303103 [hep-th].

[125] Ted Jacobson. "Einstein-aether gravity: A Status report". In: *PoS* QG-PH (2007), p. 020. arXiv: 0801.1547 [gr-qc].

[126] Ted Jacobson. "Entanglement Equilibrium and the Einstein Equation". In: *Phys. Rev. Lett.* 116 (20 May 2016), p. 201101. DOI: 10.1103/PhysRevLett.116.201101. URL: http://link.aps.org/doi/10.1103/PhysRevLett.116.201101.

[127] Ted Jacobson. "Thermodynamics of space-time: The Einstein equation of state". In: *Phys.Rev.Lett.* 75 (1995), pp. 1260–1263. DOI: 10.1103/PhysRevLett.75.1260. arXiv: gr-qc/9504004 [gr-qc].

[128] Ambar Jain and Satish D. Joglekar. "Causality violation in nonlocal quantum field theory". In: *Int. J. Mod. Phys.* A19 (2004), pp. 3409–3426. DOI: 10.1142/S0217751X0401941X. arXiv: hep-th/0307208 [hep-th].

[129] G. Jannes and G.E. Volovik. "The cosmological constant: A lesson from the effective gravity of topological Weyl media". In: *JETP Lett.* 96 (2012), pp. 215–221. DOI: 10.1134/S0021364012160035. arXiv: 1108.5086 [gr-qc].

[130] Satish D. Joglekar. "Causality violation in non-local QFT". In: *Workshop Series on Theoretical High Energy Physics Roorkee, Uttaranchal, India, March 16-20, 2005*. 2006. arXiv: hep-th/0601006 [hep-th].

[131] Steven Johnston. "Correction terms for propagators and d'Alembertians due to spacetime discreteness". In: (Nov. 2014). eprint: 1411.2614. URL: http://arxiv.org/abs/1411.2614.

[132] Steven Johnston. "Feynman Propagator for a Free Scalar Field on a Causal Set". In: *Phys. Rev. Lett.* 103 (2009), p. 180401. DOI: 10.1103/PhysRevLett.103.180401. arXiv: 0909.0944 [hep-th].

[133] Robert H. Jonsson, Eduardo Martin-Martinez, and Achim Kempf. "Information transmission without energy exchange". In: *Phys. Rev. Lett.* 114.11 (2015), p. 110505. DOI: 10.1103/PhysRevLett.114.110505. arXiv: 1405.3988 [quant-ph].

[134] Joseph I. Kapusta. "Bose-Einstein Condensation, Spontaneous Symmetry Breaking, and Gauge Theories". In: *Phys.Rev.* D24 (1981), pp. 426–439. DOI: 10.1103/PhysRevD.24.426.

[135] Gary Kleppe and R. P. Woodard. "Nonlocal Yang-Mills". In: *Nucl. Phys.* B388 (1992), pp. 81–112. DOI: 10.1016/0550-3213(92)90546-N. arXiv: hep-th/9203016 [hep-th].

[136] Alexey S. Koshelev. "Modified non-local gravity". In: *Rom. J. Phys.* 57 (2012), pp. 894–900. arXiv: 1112.6410 [hep-th].

[137] V. Alan Kostelecký and Stuart Samuel. "Spontaneous breaking of Lorentz symmetry in string theory". In: *Phys. Rev. D* 39 (2 Jan. 1989), pp. 683–685. DOI: 10.1103/PhysRevD.39.683. URL: http://link.aps.org/doi/10.1103/PhysRevD.39.683.

[138] O. Lauscher and M. Reuter. "Fractal spacetime structure in asymptotically safe gravity". In: *JHEP* 0510 (2005), p. 050. DOI: 10.1088/1126-6708/2005/10/050. arXiv: hep-th/0508202 [hep-th].

[139] F. Lecocq et al. "Quantum Nondemolition Measurement of a Nonclassical State of a Massive Object". In: *Phys. Rev. X* 5 (4 Dec. 2015), p. 041037. DOI: 10.1103/PhysRevX.5.041037. URL: http://link.aps.org/doi/10.1103/PhysRevX.5.041037.

[140] Stefano Liberati. "Tests of Lorentz invariance: a 2013 update". In: *Class. Quant. Grav.* 30 (2013), p. 133001. DOI: 10.1088/0264-9381/30/13/133001. arXiv: gr-qc/1304.5795 [gr-qc].

[141] Stefano Liberati, Florian Girelli, and Lorenzo Sindoni. "Analogue Models for Emergent Gravity". In: (2009). arXiv: 0909.3834 [gr-qc].

[142] Stefano Liberati and Luca Maccione. "Astrophysical constraints on Planck scale dissipative phenomena". In: *Phys. Rev. Lett.* 112 (2014), p. 151301. DOI: 10.1103/PhysRevLett.112.151301. arXiv: 1309.7296 [gr-qc].

[143] Michele Maggiore. "Nonlocal Infrared Modifications of Gravity. A Review". In: (2016). arXiv: 1606.08784 [hep-th].

[144] David B. Malament. "The class of continuous timelike curves determines the topology of spacetime". In: *J. Math. Phys.* 18 (1977), pp. 1399–1404.

[145] Juan Maldacena. "The Large-N Limit of Superconformal Field Theories and Supergravity". In: *International Journal of Theoretical Physics* 38.4 (1999), pp. 1113–1133. ISSN: 1572-9575. DOI: 10.1023/A:1026654312961. URL: http://dx.doi.org/10.1023/A:1026654312961.

[146] Fotini Markopoulou and Lee Smolin. "Disordered locality in loop quantum gravity states". In: *Class. Quant. Grav.* 24 (2007), pp. 3813–3824. DOI: 10.1088/0264-9381/24/15/003. arXiv: gr-qc/0702044 [gr-qc].

[147] Donald Marolf. "Emergent Gravity Requires Kinematic Nonlocality". In: *Phys. Rev. Lett.* 114 (3 Jan. 2015), p. 031104. DOI: 10.1103/PhysRevLett.114.031104. URL: http://link.aps.org/doi/10.1103/PhysRevLett.114.031104.

[148] Eduardo Martín-Martínez, Miguel Montero, and Marco del Rey. "Wavepacket detection with the Unruh-DeWitt model". In: *Phys. Rev. D* 87 (6 Mar. 2013), p. 064038. DOI: 10.1103/PhysRevD.87.064038. URL: http://link.aps.org/doi/10.1103/PhysRevD.87.064038.

[149] David Mattingly. "Causal sets and conservation laws in tests of Lorentz symmetry". In: *Phys. Rev.* D77 (2008), p. 125021. DOI: 10.1103/PhysRevD.77.125021. arXiv: 0709.0539 [gr-qc].

[150] David Mattingly. "Modern tests of Lorentz invariance". In: *Living Rev. Rel.* 8 (2005), p. 5. DOI: 10.12942/lrr-2005-5. arXiv: gr-qc/0502097 [gr-qc].

[151] Seán M. Meenehan et al. "Pulsed Excitation Dynamics of an Optomechanical Crystal Resonator near Its Quantum Ground State of Motion". In: *Phys. Rev. X* 5 (4 Oct. 2015), p. 041002. DOI: 10.1103/PhysRevX.5.041002. URL: http://link.aps.org/doi/10.1103/PhysRevX.5.041002.

[152] Leonardo Modesto. "Fractal Structure of Loop Quantum Gravity". In: *Class.Quant.Grav.* 26 (2009), p. 242002. DOI: 10.1088/0264-9381/26/24/242002. arXiv: 0812.2214 [gr-qc].

[153] Leonardo Modesto. "Super-renormalizable Multidimensional Quantum Gravity". In: (2012). arXiv: 1202.3151 [hep-th].

[154] Leonardo Modesto. "Super-renormalizable Quantum Gravity". In: *Phys. Rev.* D86 (2012), p. 044005. DOI: 10.1103/PhysRevD.86.044005. arXiv: 1107.2403 [hep-th].

[155] Leonardo Modesto and Piero Nicolini. "Spectral dimension of a quantum universe". In: *Phys. Rev. D* 81 (10 May 2010), p. 104040. DOI: 10.1103/PhysRevD.81.104040. URL: http://link.aps.org/doi/10.1103/PhysRevD.81.104040.

[156] Leonardo Modesto and Leslaw Rachwal. "Super-renormalizable and finite gravitational theories". In: *Nucl. Phys.* B889 (2014), pp. 228–248. DOI: 10.1016/j.nuclphysb.2014.10.015. arXiv: 1407.8036 [hep-th].

[157] Leonardo Modesto and Leslaw Rachwal. "Universally finite gravitational and gauge theories". In: *Nucl. Phys.* B900 (2015), pp. 147–169. DOI: 10.1016/j.nuclphysb.2015.09.006. arXiv: hep-th/1503.00261 [hep-th].

[158] Nicolas Moeller and Barton Zwiebach. "Dynamics with Infinitely Many Time Derivatives and Rolling Tachyons". In: *Journal of High Energy Physics* 2002.10 (2002), p. 034. URL: http://stacks.iop.org/1126-6708/2002/i=10/a=034.

[159] V. Moncrief. "Stability of stationary, spherical accretion onto a Schwarzschild black hole". In: *Astrophys. J.* 235 (1980), pp. 1038–1046.

[160] Robert C. Myers and Maxim Pospelov. "Experimental challenges for quantum gravity". In: *Proceedings, 3rd International Symposium on Quantum theory and symmetries (QTS3)*. 2004, pp. 732–744. arXiv: gr-qc/0402028 [gr-qc].

[161] J. Myrheim. "STATISTICAL GEOMETRY". In: (1978).

[162] Dmitry Nesterov and Sergey N. Solodukhin. "Short-distance regularity of Green's function and UV divergences in entanglement entropy". In: *JHEP* 09 (2010), p. 041. DOI: 10.1007/JHEP09(2010)041. arXiv: 1008.0777 [hep-th].

[163] Holger Bech Nielsen and Masao Ninomiya. "β-Function in a non-covariant Yang-Mills theory". In: *Nucl. Phys. B* 141.1 (1978), pp. 153–177.

[164] Holger Bech Nielsen and I. Picek. "Redei Like Model and Testing Lorentz Invariance". In: *Phys. Lett.* B114 (1982), p. 141. DOI: 10.1016/0370-2693(82)90133-2.

[165] Aaron D O'Connell et al. "Quantum ground state and single-phonon control of a mechanical resonator". In: *Nature* 464.7289 (2010), pp. 697–703.

[166] F. W. J. Olver et al., eds. *NIST Handbook of Mathematical Functions*. New York, NY: Cambridge University Press, 2010.

[167] Daniele Oriti. "Disappearance and emergence of space and time in quantum gravity". In: *Stud.Hist.Philos.Mod.Phys.* 46 (2014), pp. 186–199. DOI: 10.1016/j.shpsb.2013.10.006. arXiv: 1302.2849 [physics.hist-ph].

[168] Daniele Oriti. "Group field theory as the microscopic description of the quantum spacetime fluid: A New perspective on the continuum in quantum gravity". In: *PoS* QG-PH (2007), p. 030. arXiv: 0710.3276 [gr-qc].

[169] Daniele Oriti. "The Group field theory approach to quantum gravity: Some recent results". In: *AIP Conf. Proc.* 1196 (2009), pp. 209–218. DOI: 10.1063/1.3284386. arXiv: 0912.2441 [hep-th].

[170] Daniele Oriti, Daniele Pranzetti, and Lorenzo Sindoni. "Horizon entropy from quantum gravity condensates". In: (2015). arXiv: 1510.06991 [gr-qc].

[171] Daniele Oriti, Lorenzo Sindoni, and Edward Wilson-Ewing. "Emergent Friedmann dynamics with a quantum bounce from quantum gravity condensates". In: (2016). arXiv: 1602.05881 [gr-qc].

[172] M Ostrogradski. "Member academic science st". In: *Petersbourg* 1.385 (1850), p. 18502.

[173] Renaud Parentani. "Constructing QFT's wherein Lorentz invariance is broken by dissipative effects in the UV". In: *PoS* QG-PH (2007), p. 031. arXiv: hep-th/0709.3943 [hep-th].

[174] Onkar Parrikar and Sumati Surya. "Causal Topology in Future and Past Distinguishing Spacetimes". In: *Class. Quant. Grav.* 28 (2011), p. 155020. DOI: 10.1088/0264-9381/28/15/155020. arXiv: 1102.0936 [gr-qc].

[175] Roberto Percacci and Gian Paolo Vacca. "Asymptotic Safety, Emergence and Minimal Length". In: *Class. Quant. Grav.* 27 (2010), p. 245026. DOI: 10.1088/0264-9381/27/24/245026. arXiv: 1008.3621 [hep-th].

[176] Alejandro Perez and Daniel Sudarsky. "Comments on challenges for quantum gravity". In: *Phys. Rev. Lett.* 91 (2003), p. 179101. DOI: 10.1103/PhysRevLett.91.179101. arXiv: gr-qc/0306113 [gr-qc].

[177] Christopher J Pethick and Henrik Smith. *Bose-Einstein condensation in dilute gases*. Cambridge university press, 2002.

[178] Lydia Philpott, Fay Dowker, and Rafael D. Sorkin. "Energy-momentum diffusion from spacetime discreteness". In: *Phys. Rev.* D79 (2009), p. 124047. DOI: 10.1103/PhysRevD.79.124047. arXiv: 0810.5591 [gr-qc].

[179] Igor Pikovski et al. "Probing Planck-scale physics with quantum optics". In: *Nature Physics* 8.5 (2012), pp. 393–397.

[180] J.-M. Pirkkalainen et al. "Squeezing of Quantum Noise of Motion in a Micromechanical Resonator". In: *Phys. Rev. Lett.* 115 (24 Dec. 2015), p. 243601. DOI: 10.1103/PhysRevLett.115.243601. URL: http://link.aps.org/doi/10.1103/PhysRevLett.115.243601.

[181] L.P. Pitaevskii and S. Stringari. *Bose-Einstein Condensation*. International Series of Monographs on Physics. Clarendon Press, 2003. ISBN: 9780198507192. URL: http://books.google.it/books?id=7J-ZngEACAAJ.

[182] Joseph Polchinski. "Comment on [arXiv:1106.1417] 'Small Lorentz violations in quantum gravity: do they lead to unacceptably large effects?'" In: *Class. Quant. Grav.* 29 (2012), p. 088001. DOI: 10.1088/0264-9381/29/8/088001. arXiv: gr-qc/1106.6346 [gr-qc].

[183] Joseph Polchinski. *String theory: Volume 2, superstring theory and beyond*. Cambridge university press, 1998.

[184] A. Pontin et al. "Dynamical Two-Mode Squeezing of Thermal Fluctuations in a Cavity Optomechanical System". In: *Phys. Rev. Lett.* 116 (10 Mar. 2016), p. 103601. DOI: 10.1103/PhysRevLett.116.103601. URL: http://link.aps.org/doi/10.1103/PhysRevLett.116.103601.

[185] Maxim Pospelov and Yanwen Shang. "On Lorentz violation in Horava-Lifshitz type theories". In: *Phys. Rev.* D85 (2012), p. 105001. DOI: 10.1103/PhysRevD.85.105001. arXiv: hep-th/1010.5249 [hep-th].

[186] Chanda Prescod-Weinstein and Lee Smolin. "Disordered Locality as an Explanation for the Dark Energy". In: *Phys. Rev.* D80 (2009), p. 063505. DOI: 10.1103/PhysRevD.80.063505. arXiv: 0903.5303 [hep-th].

[187] Maurício Richartz et al. "Rotating black holes in a draining bathtub: superradiant scattering of gravity waves". In: *Phys. Rev.* D91.12 (2015), p. 124018. DOI: 10.1103/PhysRevD.91.124018. arXiv: 1411.1662 [gr-qc].

[188] D. P. Rideout and R. D. Sorkin. "Classical sequential growth dynamics for causal sets". In: *Phys. Rev. D* 61 (2 Dec. 1999), p. 024002. DOI: 10.1103/PhysRevD.61.024002. URL: http://link.aps.org/doi/10.1103/PhysRevD.61.024002.

[189] Bernhard Riemann. *Über die Hypothesen, welche der Geometrie zu Grunde liegen*. Riemann's Habiliationsschrift, Göttingen, 1954. 1868.

[190] Carlo Rovelli. "Loop quantum gravity". In: *Living Rev. Rel* 1.1 (1998), pp. 41–135.

[191] Carlo Rovelli. "Notes for a brief history of quantum gravity". In: *Recent developments in theoretical and experimental general relativity, gravitation and relativistic field theories. Proceedings, 9th Marcel Grossmann Meeting, MG'9, Rome, Italy, July 2-8, 2000. Pts. A-C*. 2000, pp. 742–768. arXiv: gr-qc/0006061 [gr-qc]. URL: http://alice.cern.ch/format/showfull?sysnb=2193246.

[192] Carlo Rovelli and Simone Speziale. "Lorentz covariance of loop quantum gravity". In: *Phys. Rev.* D83 (2011), p. 104029. DOI: 10.1103/PhysRevD.83.104029. arXiv: gr-qc/1012.1739 [gr-qc].

[193] Gaitri Saini and Satish D. Joglekar. "Bound on nonlocal scale from g-2 of muon in a nonlocal WS model". In: *Z. Phys.* C76 (1997), pp. 343–353. DOI: 10.1007/s002880050559. arXiv: hep-ph/9701405 [hep-ph].

[194] Mehdi Saravani and Niayesh Afshordi. "Off-shell Dark Matter: A Cosmological relic of Quantum Gravity". In: (2016). arXiv: 1604.02448 [gr-qc].

[195] Mehdi Saravani and Siavash Aslanbeigi. “Dark matter from spacetime nonlocality”. In: *Phys. Rev. D* 92 (10 Nov. 2015), p. 103504. DOI: 10.1103/PhysRevD.92.103504. URL: http://link.aps.org/doi/10.1103/PhysRevD.92.103504.

[196] M.M.O. Scully and M.S. Zubairy. *Quantum Optics*. Cambridge University Press, 1997. ISBN: 9780521434584. URL: http://books.google.ca/books?id=UtdfQgAACAAJ.

[197] Lorenzo Sindoni. “Emergent Models for Gravity: an Overview of Microscopic Models”. In: *SIGMA* 8 (2012), p. 027. DOI: 10.3842/SIGMA.2012.027. arXiv: 1110.0686 [gr-qc].

[198] Lorenzo Sindoni. “The Higgs mechanism in Finsler spacetimes”. In: *Phys. Rev.* D77 (2008), p. 124009. DOI: 10.1103/PhysRevD.77.124009. arXiv: gr-qc/0712.3518 [gr-qc].

[199] Lorenzo Sindoni, Florian Girelli, and Stefano Liberati. “Emergent gravitational dynamics in Bose-Einstein condensates”. In: (2009). arXiv: 0909.5391 [gr-qc].

[200] B.P. Singh and H.C. Evans. “Relative efficiency of Ge(Li) gamma ray detectors from 0.5 to 12 MeV”. In: *Nuclear Instruments and Methods* 97.3 (1971), pp. 475–482. ISSN: 0029-554X. DOI: http://dx.doi.org/10.1016/0029-554X(71)90249-7. URL: http://www.sciencedirect.com/science/article/pii/0029554X71902497.

[201] S. Sonego and V. Faraoni. “Coupling to the curvature for a scalar field from the equivalence principle”. In: *Class. Quant. Grav.* 10 (1993), pp. 1185–1187. DOI: 10.1088/0264-9381/10/6/015.

[202] Rafael D. Sorkin. “Causal sets: Discrete gravity (Notes for the Valdivia Summer School)”. In: *Lectures on Quantum Gravity, Proceedings of the Valdivia Summer School, Valdivia, Chile, January 2002*. Ed. by Andrés Gomberoff and Don Marolf. Plenum, 2005. eprint: gr-qc/0309009.

[203] Rafael D. Sorkin. “Does locality fail at intermediate length-scales”. In: (2007). arXiv: gr-qc/0703099 [GR-QC].

[204] Rafael D. Sorkin. “First steps with causal sets”. In: *Proceedings of the ninth Italian Conference on General Relativity and Gravitational Physics, Capri, Italy, September 1990*. Ed. by R. Cianci et al. World Scientific, Singapore, 1991, pp. 68–90.

[205] Rafael D. Sorkin. “Forks in the road, on the way to quantum gravity”. In: *Int. J. Theor. Phys.* 36 (1997), pp. 2759–2781. eprint: gr-qc/9706002.

[206] Rafael D. Sorkin. “Is the cosmological 'constant' a nonlocal quantum residue of discreteness of the causal set type?” In: *AIP Conf.Proc.* 957 (2007), pp. 142–153. DOI: 10.1063/1.2823750. arXiv: 0710.1675 [gr-qc].

[207] Thomas P. Sotiriou, Matt Visser, and Silke Weinfurtner. “From dispersion relations to spectral dimension - and back again”. In: *Phys.Rev.* D84 (2011), p. 104018. DOI: 10.1103/PhysRevD.84.104018. arXiv: 1105.6098 [hep-th].

[208] Thomas P. Sotiriou, Matt Visser, and Silke Weinfurtner. "Spectral dimension as a probe of the ultraviolet continuum regime of causal dynamical triangulations". In: *Phys.Rev.Lett.* 107 (2011), p. 131303. DOI: 10.1103/PhysRevLett.107.131303. arXiv: 1105.5646 [gr-qc].

[209] L. Sriramkumar and T. Padmanabhan. "Response of finite time particle detectors in noninertial frames and curved space-time". In: *Class. Quant. Grav.* 13 (1996), pp. 2061–2079. DOI: 10.1088/0264-9381/13/8/005.

[210] Jeff Steinhauer. "Observation of self-amplifying Hawking radiation in an analog black hole laser". In: *Nature Phys.* 10 (2014), p. 864. DOI: 10.1038/NPHYS3104. arXiv: 1409.6550 [cond-mat.quant-gas].

[211] D. Sudarsky and J.A. Caicedo. "On the proposals of Lorentz invariance violation resulting from a quantum-gravitational granularity of space-time". In: *J. Phys. Conf. Ser.* 24 (2005), pp. 69–76. DOI: 10.1088/1742-6596/24/1/009.

[212] Sumati Surya. "Directions in Causal Set Quantum Gravity". In: (Mar. 2011). eprint: http://arxiv.org/abs/1103.6272.

[213] Gerard 't Hooft. "QUANTUM GRAVITY: A FUNDAMENTAL PROBLEM AND SOME RADICAL IDEAS". In: *NATO Sci. Ser. B* 44 (1979), pp. 323–345.

[214] Washington Taylor and Barton Zwiebach. "D-branes, tachyons, and string field theory". In: *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings*. 2003, pp. 641–759. arXiv: hep-th/0311017 [hep-th].

[215] JD Teufel et al. "Sideband cooling of micromechanical motion to the quantum ground state". In: *Nature* 475.7356 (2011), pp. 359–363.

[216] Kip S. Thorne, R.H. Price, and D.A. Macdonald. *Black holes: the Membrane paradigm*. Ed. by Kip S. Thorne. 1986.

[217] E. T. Tomboulis. "Nonlocal and quasi-local field theories". In: (2015). arXiv: 1507.00981 [hep-th].

[218] W.G. Unruh. "Experimental black hole evaporation". In: *Phys.Rev.Lett.* 46 (1981), pp. 1351–1353. DOI: 10.1103/PhysRevLett.46.1351.

[219] Luis F. Urrutia. "Corrections to flat-space particle dynamics arising from space granularity". In: *Lect. Notes Phys.* 702 (2006). [,299(2005)], pp. 299–345. DOI: 10.1007/3-540-34523-X_11. arXiv: hep-ph/0506260 [hep-ph].

[220] MR Vanner et al. "Cooling-by-measurement and mechanical state tomography via pulsed optomechanics". In: *Nature communications* 4 (2013).

[221] Matt Visser and Carmen Molina-Paris. "Acoustic geometry for general relativistic barotropic irrotational fluid flow". In: *New J.Phys.* 12 (2010), p. 095014. DOI: 10.1088/1367-2630/12/9/095014. arXiv: 1001.1310 [gr-qc].

[222] G.E. Volovik. "z=3 Lifshitz-Horava model and Fermi-point scenario of emergent gravity". In: *JETP Lett.* 89 (2009), pp. 525–528. DOI: 10.1134/S0021364009110010. arXiv: 0904.4113 [gr-qc].

[223] G.E. Volovik and M.A. Zubkov. "Emergent Horava gravity in graphene". In: *Annals Phys.* 340 (2014), pp. 352–368. DOI: 10.1016/j.aop.2013.11.003. arXiv: 1305.4665 [cond-mat.mes-hall].

[224] Robert M. Wald. *General Relativity*. 1984.

[225] S. Weinberg and E. Witten. "Limits on massless particles". In: *Phys. Lett. B* 96 (1980), pp. 59–62. DOI: 10.1016/0370-2693(80)90212-9.

[226] Steven Weinberg. *The Quantum Theory of Fields, Volume 1: Foundations*. Cambridge University Press, 2005.

[227] Silke Weinfurtner et al. "Classical aspects of Hawking radiation verified in analogue gravity experiment". In: *Lect. Notes Phys.* 870 (2013), pp. 167–180. DOI: 10.1007/978-3-319-00266-8_8. arXiv: 1302.0375 [gr-qc].

[228] Silke Weinfurtner et al. "Cosmological particle production in emergent rainbow spacetimes". In: *Class.Quant.Grav.* 26 (2009), p. 065012. DOI: 10.1088/0264-9381/26/6/065012. arXiv: 0801.2673 [gr-qc].

[229] Silke Weinfurtner et al. "Measurement of stimulated Hawking emission in an analogue system". In: *Phys. Rev. Lett.* 106 (2011), p. 021302. DOI: 10.1103/PhysRevLett.106.021302. arXiv: 1008.1911 [gr-qc].

[230] Silke Weinfurtner et al. "On the phenomenon of emergent spacetimes: An instruction guide for experimental cosmology". In: *PoS* QG-PH (2007), p. 044. arXiv: 0804.1346 [gr-qc].

[231] John Archibald Wheeler and Richard Phillips Feynman. "Interaction with the Absorber as the Mechanism of Radiation". In: *Rev. Mod. Phys.* 17 (2-3 Apr. 1945), pp. 157–181. DOI: 10.1103/RevModPhys.17.157. URL: http://link.aps.org/doi/10.1103/RevModPhys.17.157.

[232] Emma Edwina Wollman et al. "Quantum squeezing of motion in a mechanical resonator". In: *Science* 349.6251 (2015), pp. 952–955.

[233] R. P. Woodard. "The Theorem of Ostrogradsky". In: (2015). arXiv: hep-th/1506.02210 [hep-th].