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# Some results on quasistatic evolution problems for unidirectional processes

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*Non affannatevi dunque dicendo: Che cosa mangeremo? Che cosa berremo? Che cosa indosseremo? Di tutte queste cose si preoccupano i pagani; il Padre vostro celeste infatti sa che ne avete bisogno. Cercate prima il regno di Dio e la sua giustizia, e tutte queste cose vi saranno date in aggiunta.*

Mt 6, 31–33

# Abstract

The present thesis is devoted to the study of some models of quasistatic evolutions for materials, in the presence of unidirectional phenomena, such as damage and fracture. In particular, these models concern the coupling between damage and plasticity, and the growth of brittle and cohesive fractures in antiplane linearized elasticity.

As for the coupling between damage and plasticity, we consider:

1. an evolution based on *global* minimization, that combines a gradient damage model similar to the one by Mielke and Roubíček (Math. Models Methods Appl. Sci., 2006) and the model of perfect plasticity studied by Dal Maso, DeSimone and Mora (Arch. Ration. Mech. Anal., 2006);
2. an evolution based on *local* minimization, that employs the technique of vanishing viscosity, as done by Knees, Rossi, and Zanini (Math. Models Methods Appl. Sci., 2013) in a damage model, in a modeling framework similar to that of model 1;
3. an evolution based on *global* minimization for *gradient* plasticity coupled with damage, where gradient plasticity follows the model by Gurtin and Anand (J. Mech. Phys. Solids, 2005), see also Giacomini and Lussardi (SIAM J. Math. Anal., 2008).

In these three models, the internal variable which describes the damage affects both the elastic tensor and the plastic yield surface (namely the region where the internal stresses are constrained to lie). The coupling between damage and plasticity allows for describing a *fatigue phenomenon*. Indeed, it is possible to require that the history of plastic strain up to the current state influences the future evolution of damage, in such a way that it is easier to damage zones interested by plastic cycles. This issue is discussed in models 1 and 2. The model 3 accounts in particular for the interplay between dislocation density and damage growth; the behavior considered complies with the mechanism of microcrack formation and coalescence by dislocation pile-up, investigated for long time by the mechanical community.

The work on brittle fracture concerns the viscous approximation of quasistatic crack growth, where the crack is not prescribed a priori, but chosen in a class of admissible curves. In this framework, the results by Lazzaroni and Toader (Math. Models Methods Appl. Sci., 2011), in which the admissible cracks are suitable $C^{1,1}$ curves, are extended to a larger class $\mathcal{S}$ of cracks, introduced by Racca (Asymptot. Anal., 2014). The cracks in $\mathcal{S}$ may have many connected components, each of them being the union of a certain number of branches that are regular curves of the type considered by Lazzaroni and Toader. Moreover, some geometric restrictions are imposed in order to guarantee that $\mathcal{S}$ is closed with respect to the Hausdorff convergence, that the number of connected components and of branches is uniformly bounded in $\mathcal{S}$, and that

the uncracked part of the body is always a connected set. These conditions allow for cracks displaying branching and kinking.

In the work on cohesive fracture, the crack is contained in a prescribed surface. In our model, both the density of the energy dissipated in the fracture process and the maximal surface tension between the two sides of the crack depend on the total variation of the amplitude of the jump. Thus, any change in the crack opening entails a loss of energy and a decrease in the maximal tensile stress, until the crack is complete: then the energy is no longer dissipated and there is no surface stress on the fracture set. In particular this implies a fatigue phenomenon, i.e., a complete fracture may be produced by oscillation of small jumps. The decrease in time of the maximal tensile stress corresponds to unidirectionality. The main result in this context is the existence of a globally stable quasistatic evolution. The original feature of our approach is that the limit evolution is formulated only in terms of functions, even if it is obtained passing through a very weak notion of solution, that involves *Young measures*. Indeed, since the energy dissipated in the fracture process is bounded, there is a lack of controls on the variations of the jump of the approximate evolutions, so that one cannot employ the Helly Theorem for functions.

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# Contents

# Introduction

The present thesis is devoted to the study of quasistatic evolutions for systems characterized by the presence of unidirectional phenomena.

In solid mechanics, the term *quasistatic* refers to evolutions that happen slowly enough for the system to remain in internal equilibrium. They are idealizations and are admissible only if the relevant processes of the system take place on a much slower time scale than the internal relaxation. In particular, in these evolutions the inertial effects are negligible. Every quasistatic evolution is *rate independent*, namely it is invariant under time rescaling. More precisely, consider a process that associates to a given time dependent input function $t \ni [0,T] \mapsto \ell(t)$ a time dependent output $t \ni [0,T] \mapsto q(t)$. This process is said rate independent if, given a sufficiently smooth time reparametrization $s\colon [0,T] \mapsto [0,T]$, the reparametrized input $\ell \circ s$ corresponds to the reparametrized output $q \circ s$. Then the time dependent mapping $\ell \mapsto q$ may be illustrated by curves lying on the $(\ell, q)$-plane, which are followed during the evolution, independently of the rate at which the input changes.

Friction, damage, crack propagation, plasticity, delamination, solid-solid phase change, ferromagnetism, ferroelectricity, are just a few examples of rate independent phenomena (under the assumption of slow external loading). Because of their relevance in applications, the analysis of these systems has attracted much attention by the mathematical community, since the 1970s. In these years, J.J. Moreau [**83, 84**] studied the so-called *sweeping processes*, with particular focus on elastoplasticity, B. Halphen and Q.C. Nguyen [**53**] introduced the notion of *generalized standard materials* to consider also phase transformations, magnetization, piezoelectric effects, damage, and fracture, while M.A. Krasnosel'skiĭ and A.V. Pokrovskiĭ [**65**] developed a mathematical theory for the *hysteresis* operators.

In recent years, the mathematical theory of rate independent systems has been enriched by two different abstract approaches, introduced by Mielke, Theil, and Levitas in [**81, 80**] (see also [**74**] and references therein), and by Efendiev and Mielke in [**40**] (refined by Mielke, Rossi, and Savaré e.g. in [**75, 76, 77**]), respectively. Either approach gives rise to a different notion of solution.

In order to describe them one may consider a very simple example of rate independent system: a heavy block sliding on a rough surface, subject to friction. Assume that the block is pulled by means of a linearly elastic rope, in such a way that, for a fixed one-dimensional frame, the input of the system is the position $\ell(t)$ of the free end of the rope, while $q(t)$ is the position of the block, or more precisely the insertion point of the rope. The roughness of the surface is such that the block may begin to move only when the force exerted by the rope reaches (in modulus) some critical activation threshold $\tau > 0$. Above this threshold, the block slides rigidly, in the sense that $\ell - q$ is constant. In particular, no inertial effects are to be considered. By assuming that the tension of the rope equals $k(\ell - q)$ (with $k > 0$), the evolution of this system can be described by the system of relations

$$|k(\ell - q)| \leq \tau, \quad \dot{q}(f - k(\ell - q)) \leq 0 \quad \text{for every } f \in [-\tau, \tau]. \tag{0.0.1}$$

The first condition states that the tension of the rope is always below the threshold $\tau$. The second one asserts that if the tension is (in modulus) strictly less than $\tau$ then the block does not move. If the tension is exactly at the threshold, then $\ell - q$ is constant. It is immediate to check that the latter system is rate independent: by doubling the speed at which $\ell$ moves, the effect on $q$ is doubled in speed. Indeed, as soon as the tension of the rope goes below the threshold, the block stops by the dry friction between itself and the rough surface, as far as all motions are so slow that inertia can be neglected. Notice the occurence of a so-called *hysteresis* phenomenon, namely the output depends both on present and on past inputs: $q(t)$ follows $\ell(t)$ with some delay, due to the fact that when the direction of $\ell$ changes some time is needed to reach again the threshold (e.g. when the tension passes from $\tau$ to $-\tau$ the block is steady).

It is easily seen that, defined $\mathcal{Q} := \mathbb{R}$, $\mathcal{E}(t,q) := k\frac{q^2}{2} - k\ell(t)q$, and $\mathcal{R}(q) := \tau|q|$ for every $q \in \mathcal{Q}$, (0.0.1) is equivalent to the conditions

$$\partial_q \mathcal{E}(t, q(t))\, w + \mathcal{R}(w) \geq 0 \quad \text{for every } w \in \mathcal{Q},\ t > 0, \tag{0.0.2a}$$

$$\partial_q \mathcal{E}(t, q(t))\, \dot{q}(t) + \mathcal{R}(\dot{q}(t)) = 0 \quad \text{for every } t > 0, \tag{0.0.2b}$$

which can be rewritten in a compact form as

$$\partial \mathcal{R}(\dot{q}(t)) + \partial_q \mathcal{E}(t, q(t)) \ni 0 \quad \text{for every } t > 0. \tag{0.0.3}$$

In the above equation $\partial\mathcal{R}$ is the (convex analysis) subdifferential of the convex function $\mathcal{R}$, a set defined by

$$\zeta \in \partial\mathcal{R}(\eta) \Longleftrightarrow \mathcal{R}(w) - \mathcal{R}(\eta) \geq \zeta(w - \eta) \quad \text{for every } w \in \mathcal{Q}.$$

The abstract formulation (0.0.3) is very general and it appears naturally in a wide range of applications, both in the finite-dimensional and in the infinite-dimensional setting. Notice that the rate independence corresponds to the *positive one-homogeneity* of $\mathcal{R}$. Indeed, the differential problem (0.0.3) is the reference for rate independent systems. However, given a certain vector space $\mathcal{Q}$, an energy $\mathcal{E}$, and a dissipation $\mathcal{R}$, the problem is in general non-smooth and very often fails to admit strong solutions.

This calls for a weak notion of solution. The approach by Mielke, Theil, and Levitas is based on the notion of the so-called *energetic solutions*, which are characterized by the two principles of global energy minimization and conservation of energy. The main feature of such a formulation is that it is *derivative-free*; moreover, it coincides with (0.0.3) when $q \mapsto \mathcal{E}(t,q)$ is convex and the evolution is sufficiently regular in time. More precisely, assuming $\mathcal{E}$ convex in the second variable, from (0.0.2a) one deduces the *global stability condition*

$$\mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \widetilde{q}) + \mathcal{R}(\widetilde{q} - q(t)) \quad \text{for every } \widetilde{q} \in \mathcal{Q}, \tag{0.0.4a}$$

while integrating (0.0.2b) and by the chain rule one obtain the *energy-dissipation balance*

$$\mathcal{E}(t,q(t)) + \int_0^t \mathcal{R}(\dot q(s))\,\mathrm{d}s = \mathcal{E}(0,q(0)) + \int_0^t \partial_t\mathcal{E}(s,q(s))\,\mathrm{d}s\,,$$

which may be rewritten as

$$\mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{R}}(q,[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_t\mathcal{E}(s,q(s))\,\mathrm{d}s\,, \tag{0.0.4b}$$

where

$$\mathrm{Diss}_{\mathcal{R}}(q,[0,t]) := \sup\left\{\sum_{j=1}^N \mathcal{R}(q(t_j)-q(t_{j-1}))\colon N\in\mathbb{N},\ 0\le t_0<t_1<\cdots<t_N\le t\right\}$$

is the energy dissipated in $[0,t]$. The global stability condition states that no competitor state $\widetilde q$ can be preferred to the current state $q(t)$ in terms of energy gain versus dissipation. The energy balance reflects the fact that the energy at time $t$ plus the energy dissipated in the time interval $[0,t]$ (the sum in the left-hand side of (0.0.4b)) equals the initial energy plus the work supplied by external actions (the right-hand side of (0.0.4b)). Notice that the dissipated energy is the variation of $q$ in $[0,t]$ with respect to the seminorm $\mathcal{R}$.

The differential inclusion (0.0.3) may be generalized to the case when $\mathcal{R}$ depends also on $q(t)$, by

$$\partial_{\dot q}\mathcal{R}(q(t),\dot q(t)) + \partial_q\mathcal{E}(t,q(t)) \ni 0 \quad \text{for } t>0\,, \tag{0.0.5}$$

where $\partial_{\dot q}$ is the subdifferential with respect to the second variable of $\mathcal{R}$. (Notice that the notation is consistent with $\partial_q\mathcal{E}$, since for a differentiable function the subdifferential contains only the differential. Moreover, rate independence corresponds to $\mathcal{R}$ positively one-homogeneous in $\dot q$.)

When $\mathcal{R}$ depends on two variables, in order to simplify the setting one may assume that there exists a *dissipation potential*, namely a function $\mathcal{D}$ such that

$$\mathcal{R}(v,z) = \partial_v\mathcal{D}(v)[z]\,. \tag{0.0.6}$$

In this situation, provided that

$$\mathcal{F}(t,q) := \mathcal{E}(t,q) + \mathcal{D}(q) \tag{0.0.7}$$

is convex, (0.0.5) is equivalent to

$$\mathcal{F}(t,q(t)) \le \mathcal{F}(t,\widetilde q) \quad \text{for every } \widetilde q\in\mathcal{Q}\,, \tag{0.0.8a}$$

$$\mathcal{F}(t,q(t)) = \mathcal{F}(0,q(0)) + \int_0^t \partial_t\mathcal{E}(s,q(s))\,\mathrm{d}s\,. \tag{0.0.8b}$$

Both (0.0.4) and (0.0.8) may be rewritten in terms of a *dissipation distance* $\Delta(q_1,q_2)$, for which

$$\mathcal{E}(t,q(t)) \le \mathcal{E}(t,\widetilde q) + \Delta(q(t),\widetilde q) \quad \text{for every } \widetilde q\in\mathcal{Q}\,, \tag{ST}$$

and

$$\mathcal{E}(t,q(t)) + \mathrm{Diss}_{\Delta}(q,[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_t\mathcal{E}(s,q(s))\,\mathrm{d}s\,, \tag{EB}$$

where

$$\mathrm{Diss}_\Delta(q,[0,t]) := \sup\left\{\sum_{j=1}^N \Delta(q(t_{j-1}),q(t_j))\colon N\in\mathbb{N},\ 0\le t_0<t_1<\dots<t_N\le t\right\}.$$

Since it does not contain derivatives, the energetic formulation is more general (in the case of a nonconvex energy) than the differential inclusion (0.0.3). Indeed, it is directly adapted to the case of non-smooth potentials, and could be modified in order to encompass evolutions $t\mapsto q(t)$ discontinuous in time. Hence, the formulation (ST) + (EB) is considered the reference for the energetic solutions. The main advantage of such an abstract formulation is the possibility to provide general mathematical methods for a large class of applications. In **[79]** the existence and approximation theory for energetic solutions is carefully described: this is based on time discretization and incremental minimum problems, and it is effective from the viewpoint of Numerical Analysis.

Despite its interesting features, the concept of energetic solution seems nowadays not completely satisfactory out of the realm of convex energies. The crucial point is that global stability turns out to be a too strong constraint for evolution. In particular, it is not difficult to show examples of double well energies for which the request of global minimization causes unphysical jumps in time of the evolution, which may overtake energy barriers (see, for instance, [**73**, Ex. 6.1] and [**79**, Ex. 1.8.1]). In such cases the jumps of the system happen also earlier than the jumps really observed.

The reasons above motivate the research of notions of solutions based on *local minimization*. However, in recent years various types of local stability concepts have been proposed, and it is still unclear which of them should be preferred. A fruitful idea in order to go beyond energetic solutions has been to see rate independent evolutions as limits of solutions to some rate dependent systems containing a viscous dissipation that tends to zero, following the approach by Efendiev and Mielke **[40]**. Namely, the *$\varepsilon$-viscous solutions* $q_\varepsilon$ satisfy the differential inclusion (compare with (0.0.5))

$$\partial_{\dot q}\left[\mathcal{R}(q_\varepsilon(t),\dot q_\varepsilon(t))+\frac{\varepsilon}{2}\mathcal{V}(\dot q_\varepsilon(t))\right]+\partial_q\mathcal{E}(t,q_\varepsilon(t))\ni 0\quad\text{for } t>0\,, \tag{0.0.9}$$

where $\mathcal{V}$ is a quadratic, coercive, and continuous viscosity potential. This is called *vanishing-viscosity approach.*

Solutions to (0.0.9) usually behave better than those of (0.0.5); in **[75, 76, 77]** the authors investigated the hypoteses that permit to determine and approximate limit evolutions as $\varepsilon\to 0$. The general strategy relies on the fact that (under suitable regularity assumptions) the solutions to (0.0.9) are regular in time and satisfy an energy-dissipation balance. The limit evolutions strongly depend on the choice of the quadratic viscous dissipation $\mathcal{V}$, as observed in **[68]** and **[100]**. Assuming for simplicity that $\mathcal{Q}$ is a reflexive Banach space, and that $\mathcal{V}(q)=\|q\|^2_{\mathcal{Q}}$, the balance reads

$$\mathcal{E}(t,q_\varepsilon(t))+\int_0^t\left(\mathcal{R}(q_\varepsilon(s),\dot q_\varepsilon(s))+\varepsilon\|\dot q_\varepsilon(s)\|_{\mathcal{Q}}\right)\mathrm{d}s=\mathcal{E}(0,q_0)+\int_0^t\partial_t\mathcal{E}(s,q_\varepsilon(s))\,\mathrm{d}s\,, \tag{0.0.10}$$

where $q_\varepsilon(0)$ equals the initial datum $q_0$. Hence $q_\varepsilon\in H^1(0,T;\mathcal{Q})\subset C([0,T];\mathcal{Q})$. However, the $H^1$ estimate is not uniform in $\varepsilon$. Then the approach is to consider the graphs $\{(t,q_\varepsilon(t))\colon t\in$

$[0,T]\}$ and possibly find a uniform bound for their arc-lengths, with respect to the viscous norm, namely the arc-lengths $\widehat{s}_\varepsilon(t) := t + \int_0^t \|\dot{q}_\varepsilon(\tau)\|_{\mathcal{Q}}\,\mathrm{d}\tau$. In the positive case, one reparametrizes using the inverse functions $\widehat{t}_\varepsilon := \widehat{s}_\varepsilon^{-1}$, obtaining a family of contractions. The limit of the $\widehat{s}_\varepsilon$ gives a time scale, which is slower than $t$, in which is described the limit evolution. The limit evolution may still exhibit jumps in time, but due to the viscous approximation jumps happen later and energy barriers are not overtaken.

The limit solution is called *parametrized BV solution* (see [**76**]) or also *rescaled quasistatic viscosity evolution*. The main characteristics of this evolution may be pointed out by assuming that there exists a dissipation potential $\mathcal{D}$ for $\mathcal{R}$, namely (0.0.6) holds. In this simplified setting, in [**86**] it is proved that the parametrized $BV$ solution is a Lipschitz function $[0,S] \ni s \to (\widehat{t}(s), \widehat{q}(s))$, with $\widehat{t}'(s) \geq 0$, $\widehat{t}'(s) + \|\widehat{q}'(s)\| \leq 1$, satisfying the following:

$$\|\partial_q \mathcal{F}(\widehat{t}(s), \widehat{q}(s))\|_{\mathcal{Q}'} = 0 \quad \text{for } s \text{ s.t. } \widehat{t}'(s) > 0\,, \tag{STv}$$

$$\mathcal{F}(\widehat{t}(S), \widehat{q}(S)) = \mathcal{F}(0, q_0) - \int_0^S \|\partial_q \mathcal{F}(\widehat{t}(s), \widehat{q}(s))\|_{\mathcal{Q}'}\,\mathrm{d}s + \int_0^S \partial_t \mathcal{F}(\widehat{t}(s), \widehat{q}(s))\,\widehat{t}'(s)\,\mathrm{d}s\,, \tag{EBv}$$

where $\mathcal{F}$ is as in (0.0.7). Notice that the evolution is defined by means of a parametrization of the "extended graph"; with this choice, a jump in the original time scale is represented by "vertical parts" of the extendend graph, where $\widehat{t}(s) = t$ for $s \in [\tau^-, \tau^+]$ and $\widehat{q}(\tau^-) \neq \widehat{q}(\tau^+)$. The path between the two states $\widehat{q}(\tau^-)$ and $\widehat{q}(\tau^+)$ satisfies a suitable equation obtained by (EBv), which keeps track of the viscous approximation. Outside the jumps, where $\widehat{t}'(s) > 0$, the process is rate independent since, by (STv), $\partial_q \mathcal{F}(\widehat{t}(s), \widehat{q}(s)) = 0$ (then up to a reparametrization one obtains (EB), plus a first order stability condition).

This is a sketch of the general approach to vanishing viscosity solutions. A comparative analysis for different notions of local solutions may be found in [**79**, Section 1.8].

The abstract approaches described above are a powerful tool to understand the features of rate independence, and provide a reference framework for several different behaviors. Nevertheless, it is non always trivial to adapt the general schemes to particular rate independent evolutions. Some models have required the development of new mathematical tools. In this context, a very large class of processes that has been investigated in recent years is that of *unidirectional phenomena*, such as damage and fracture (see e.g. [**78, 11, 106, 41, 105, 61, 60**] and [**36, 18, 44, 31, 33, 59, 62, 67**], respectively). In such phenomena there is an order parameter (e.g. the damage internal variable, or the fracture length) that may only increase (or may only decrease) in time. Notice that in mechanics their are often referred to also as irreversible phenomena. (For this reason we adopt this terminology in the following chapters of the thesis.) However, the adjective "irreversible" is ambiguous. Indeed, in thermodynamics, it has a different, wider meaning indicating dissipation of energy implying irreversibility of time. Thus, nonunidirectional processes are still irreversible in this sense. Perfect plasticity (without damage) is an example of this ambiguity.

This thesis presents some results on unidirectional evolutions, in particular on the coupling between plasticity and damage, and on fractures, both of brittle and of cohesive type. The remaining part of this introduction is devoted to a brief description of the problems studied and of the results obtained.

## Elastoplasticity and damage

Plasticity and damage describe the inelastic behavior of materials in response to applied forces, respectively accounting for permanent deformations and for discontinuities on microscales, both of surface type (microcracks) and of volume type (microvoids). Damage affects the elastic response of the material with respect to loading and unloading, whilst plasticity produces residual deformations that persist after complete unloading. It is natural to consider the coupling between this two phenomena since, in spite of their different macroscopical implications, their initial causes are identical, in particular in metals they are originated by movement and accumulation of dislocations (cf. [**69**, Chapter 7]). Moreover, the combination of damage and plasticity provides a better description of e.g. cyclic loading, see for instance [**58**, Section 3.6] or [**69**, Section 7.5]. For these reasons such interaction is the subject of many engineering and numerical papers. Nevertheless, there are few analytical papers on the evolution of damage and plasticity together (see e.g. [**97**, Chapter 9], [**2**], [**93**], and the references in [**79**, Section 4.3.2]), even if in literature there are several works on evolution of plasticity without damage (cf. e.g. [**28**], [**98**], [**43**], [**52**], [**48**], [**71**]), and damage models that refer to purely elastic materials (cf. e.g. [**78**], [**106**], [**61**]).

The result presented here are obtained in the context of small strain elastoplasticity, namely assuming that the displacement is a small perturbation of the equilibrium configuration. The models considered here are:

1. an evolution based on *global* minimization, that combines a gradient damage model as in [**78**] and the model of perfect plasticity in [**28**];
2. an evolution based on *local* minimization, that employs the technique of vanishing viscosity (as done by Knees, Rossi, and Zanini in the damage model [**61**]) in a modeling framework similar to that of model 1;
3. an evolution based on *global* minimization for *gradient* plasticity coupled with damage (where gradient plasticity is formulated as in [**52**], see also [**48**]).

In the formulations of plasticity (without damage), the unknowns are the vector field $u$ of the *displacement* with respect to the equilibrium configuration and the matrix fields of the *elastic and plastic strains*, denoted respectively by $e$ and $p$. The *total strain* $\mathrm{E}u = \frac{\nabla u + \nabla u^T}{2}$ is decomposed as

$$\mathrm{E}u = e + p\,.$$

Such functions are defined on $[0,T] \times \Omega$, with $[0,T]$ the time interval where the process is observed, and $\Omega \subset \mathbb{R}^n$ is the reference configuration of the material. The additive decomposition of the total strain is a linear approximation of the multiplicative decomposition in finite strain plasticity; accordingly, the usual condition that plasticity is volume preserving consists, in small strain plasticity, in requiring that $p(x,t) \in \mathbb{M}^{n\times n}_D$, where $\mathbb{M}^{n\times n}_D$ is the space of deviatoric (i.e., trace free) matrices. The data are a prescribed displacement $w(t)$ on the Dirichlet part of the boundary $\partial_D\Omega \subset \partial\Omega$, surface forces $g(t)$ (on $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$), and volume forces $f(t)$. Moreover an initial condition is assigned. The common fundamental ideas are the following:

- elasticity corresponds to the restorative (internal) forces, expressed by the Cauchy stress matrix $\sigma$, that depends linearly on the elastic strain according to Hooke's Law, i.e.,

$$\sigma := \mathbb{C}e\,;$$

- the evolution of the plastic strain (and of the plastic strain gradient, in model 3) is governed by a threshold criterion; specifically some part of the Cauchy stress tensor (and some higher-order stress tensors, in model 3) lies into a *constraint* set $K$, and, when it reaches the boundary of this set, plasticity may evolve.

The fourth-order *linearized elasticity tensor* $\mathbb{C}$ and the set $K$ are constitutive parameters for the material considered.

Due to wear, any body undergoes a degradation: microscopically, this may be interpreted in terms of broken interatomic bonds. It is natural that both elastic and plastic stiffness are smaller when the number of broken bonds is higher (*softening* behavior). Thus a further internal variable $\alpha$ is introduced, accounting for the percentage of unbroken bonds in a neighborhood of any point (thus $\alpha(t,x) \in [0,1]$, where 1 corresponds to a perfectly sound material). Both $\mathbb{C}$ and $K$ depend on the *damage variable* $\alpha$, increasingly. Accounting for this dependence, the strong formulation of perfect plasticity reads

$$\begin{cases} Eu(t,x) = e(t,x) + p(t,x) & \\ u(t,x) = w(t,x) & \text{for } x \in \partial_D\Omega\,, \end{cases} \tag{sf1}$$

$$\sigma(t,x) := \mathbb{C}(\alpha(t,x))e(t,x)\,, \tag{sf2}$$

$$\begin{cases} -\operatorname{div}\, \sigma(t,x) = f(t,x) & \\ \sigma(t)\nu = g(t) & \text{for } x \in \partial_N\Omega\,, \end{cases} \tag{sf3}$$

$$\sigma_D(t,x) \in K(\alpha(t,x))\,, \tag{sf4}$$

$$\dot p(t,x) \in N_{K(\alpha(t,x))}(\sigma_D(t,x))\,. \tag{sf5}$$

Above, $\sigma_D$ is the deviatoric part of $\sigma$ (the projection onto the space of null trace matrices) and $N_K(\sigma)$ is the (convex analysis) normal cone to the set $K$ at $\sigma$. The tensor $\mathbb{C}$ is nondecreasing in $\alpha$ with respect to the Löwner order of tensors, whilst the constraint set $K$ is nondecreasing with respect to the inclusion of sets. Both $\mathbb{C}$ and $K$ are bounded from below with respect to the damage variable, so that some stiffness remains even when $\alpha$ is null: this corresponds to consider incomplete damage. For the complete set of hypotheses on $\mathbb{C}$ and $K$ we refer the reader to Section 2.1 (and to the first part of Chapter 3).

In gradient damage models some energetic terms, depending only on $\alpha$ and $\nabla\alpha$, are introduced. In particular, the term in $\nabla\alpha$ has not only a mechanical interpretation (see for instance [**106**, Section 1]), but also regularizing effects for the damage variable. The damage regularization considered here (and in the approach by vanishing viscosity) is strongly enough to guarantee that the damage variable is continuous in $\Omega$. Due to this choice, the strong formulation for the damage evolution has a nontrivial expression. For this reason, in the exposition a weaker criterion is employed. This is the following:

$$\begin{cases} \dot\alpha(t) \le 0 \text{ in } \Omega\,, \\ \langle \partial_\alpha \mathcal{E}(\alpha(t), e(t)), \beta\rangle \ge 0 \qquad \text{for every } \beta \le 0 \text{ s.t. } \mathcal{E}(\beta,\eta) < \infty \text{ for any } \eta \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \\ \langle \partial_\alpha \mathcal{E}(\alpha(t), e(t)), \dot\alpha(t)\rangle \ge 0\,, \end{cases} \tag{sf6}$$

where the total energy $\mathcal{E}$ is introduced below. Notice that the expression above has a form similar to (0.0.3). Indeed, as it will be seen later, it can be recast in terms of a dissipation $\mathcal{R}$, which is unbounded: this entails the expected entropy condition, i.e., that $\alpha$ in nonincreasing in time. Moreover, $\mathcal{R}$ admits an associated dissipation potential. Following [**2**], we choose to include the potential for $\mathcal{R}$ in the energy (similarly to (0.0.7)), and to write separately the unidirectionality condition.

For technical reasons, in the approaches by global and local minimization two different damage regularization are employed. Thus, for the reader's convenience, two notations $\mathcal{E}^G$ and $\mathcal{E}^L$ are used for global and local minimization models, respectively. We will see below that these energies could be generalized to account for a *fatigue phenomenon*.

The *total internal energy* for our global minimization model

$$\mathcal{E}^G(\alpha, e) := \mathcal{Q}(\alpha, e) + D(\alpha) + \|\nabla\alpha\|^\gamma_{L^\gamma}\,, \quad \gamma > n\,, \tag{0.0.12}$$

is the sum of the *stored elastic energy* $\mathcal{Q}(\alpha, e) = \frac{1}{2}\int_\Omega \mathbb{C}(\alpha)e\cdot e\,\mathrm{d}x$, now depending also on damage, of a continuous functional $D(\alpha) = \int_\Omega d(\alpha)\,\mathrm{d}x$, and of a regularizing gradient term. When $\mathcal{E}^G$ is finite, it turns out that $\alpha \in W^{1,\gamma}(\Omega)$ and in particular $\alpha \in C(\overline{\Omega})$. Then the duality in (sf6) is rigorously written as a duality between $W^{1,\gamma}(\Omega)$ and its dual space. The $W^{1,\gamma}$ damage regularization (with $\gamma > n$) is employed also in [**78**] and more recently in [**60**], for example.

The work of the external loading is

$$\langle \mathcal{L}(t), u\rangle := \int_\Omega f(t)\cdot u\,\mathrm{d}x + \int_{\partial_N\Omega} g(t)\cdot u\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

We now introduce in more detail the terms related to the plastic dissipation. The starting point is considering for any $\alpha$ (and $\xi \in \mathbb{M}^{n\times n}_D$) the support function of $K(\alpha)$

$$H(\alpha,\xi) := \sup_{\sigma\in K(\alpha)} \sigma\cdot\xi\,.$$

This function is convex and positively one-homogeneous in $\xi$. If $p \in L^1(\Omega;\mathbb{M}^{n\times n}_D)$, the plastic potential is $\mathcal{H}(\alpha,p) = \int_\Omega H(\alpha(x), p(x))\,\mathrm{d}x$. Due to the one-homogeneity of $H(\alpha,\xi)$ and to the lack of compactness of $L^1(\Omega;\mathbb{M}^{n\times n}_D)$, the plastic strain $p$ is set in the space of $\mathbb{M}^{n\times n}_D$-valued Borel measures $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$. Employing this space for $p$ the incremental minimization procedure admits solutions. Therefore, since $H$ is also convex in $\xi$, the *plastic potential* is defined as a convex function of a measure, by

$$\mathcal{H}(\alpha,p) := \int_{\Omega\cup\partial_D\Omega} H\left(\alpha(x), \frac{\mathrm{d}p}{\mathrm{d}|p|}(x)\right)\mathrm{d}|p|(x)\,, \qquad \text{for every } p \in M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)\,, \tag{0.0.13}$$

where $p/|p|$ is the Radon-Nikodym derivative of the measure $p$ with respect to its variation $|p|$. Given $\alpha\colon [0,T]\to C(\overline{\Omega})$ and $p\colon [0,T]\to M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$, the *plastic dissipation* in a time interval $[s,t]$ is defined as

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;s,t) := \sup\Big\{\sum_{j=1}^{N}\mathcal{H}(\alpha(t_j),p(t_j)-p(t_{j-1})):\ \ s=t_0<t_1<\dots<t_N=t\,,\ N\in\mathbb{N}\Big\}. \tag{0.0.14}$$

It is proven in Section 1.2 that

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;s,t) = \int_s^t \mathcal{H}(\alpha(\tau),\dot p(\tau))\,\mathrm{d}\tau \tag{0.0.15}$$

whenever $\alpha$ is nondecreasing in time, $\alpha(t)$ is uniformly bounded in $W^{1,\gamma}(\Omega)$, and $p$ is absolutely continuous from $[0,T]$ into $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$.

To ease the reading, when $\alpha\in C(\overline{\Omega};[0,1])$ does not depend on time, the symbol $\widehat{\mathcal{V}}_H$ is used instead of $\mathcal{V}_H$, hence

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha,p;s,t) = \int_s^t \mathcal{H}(\alpha,\dot p(\tau))\,\mathrm{d}\tau\,. \tag{0.0.16}$$

Finally, the "generalized energy" is defined by

$$\mathcal{E}^G_\lambda(\alpha,e;p,t) := \mathcal{E}^G(\alpha,e)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha,p;0,t)\,, \tag{0.0.17}$$

where $\lambda\in[0,1]$ is a parameter of the model. Then (sf6) is replaced by

$$\begin{cases}\dot\alpha(t)\le 0 \text{ in } \Omega\,,\\ \langle\partial_\alpha\mathcal{E}^G_\lambda(\alpha(t),e(t);p,t),\beta\rangle\ge 0 \qquad \text{for every } \beta\in W^{1,\gamma}(\Omega)\,,\ \beta\le 0\,,\\ \langle\partial_\alpha\mathcal{E}^G_\lambda(\alpha(t),e(t);p,t),\dot\alpha(t)\rangle\ge 0\,.\end{cases} \tag{sf6'}$$

The mechanical meaning of $\mathcal{E}^G_\lambda$ is discussed below. Notice that when $\lambda=0$ the generalized energy reduces to $\mathcal{E}^G$.

Given $\lambda\in[0,1]$, external forces $f(t)$, $g(t)$, a prescribed boundary displacement $w(t)$, and an initial condition $(\alpha_0,u_0,e_0,p_0)$, with suitable regularity assumptions, a *quasistatic evolution* for the energetic formulation is a function $t\mapsto(\alpha(t),u(t),e(t),p(t))$ fulfilling the following conditions:

$(\mathrm{qs0})^G$ *unidirectionality* : for every $x\in\Omega$

$$t\in[0,T]\mapsto\alpha(t,x)\quad\text{is nonincreasing}\,;$$

$(\mathrm{qs1})^G$ *global stability*: the function $t\mapsto p(t)$ from $[0,T]$ into $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$ has bounded variation, $(u(t),e(t),p(t))\in A(w(t))$ for every $t\in[0,T]$, and

$$\mathcal{E}^G_\lambda(\alpha(t),e(t);p,t)-\langle\mathcal{L}(t),u(t)\rangle\le\mathcal{E}^G_\lambda(\beta,\eta;p,t)-\langle\mathcal{L}(t),v\rangle+\mathcal{H}(\beta,q-p(t))$$

for every $\beta\le\alpha(t)$ and $(v,\eta,q)\in A(w(t))$, where

$$A(w) := \{(u,e,p)\colon Eu=e+p \text{ in } \Omega,\ p=(w-u)\odot\nu\,\mathcal{H}^{n-1}\text{ on }\partial\Omega\}$$

is the set of admissible displacements with respect to a boundary datum $w$;

(qs2)$^G$ *energy balance*: for every $t \in [0,T]$

$$\mathcal{E}^G_\lambda(\alpha(t), e(t); p, t) - \langle \mathcal{L}(t), u(t)\rangle + (1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) =$$
$$\mathcal{E}^G(\alpha(0), e(0)) - \langle \mathcal{L}(0), u(0)\rangle + \int_0^t \langle \sigma(s), E\dot{w}(s)\rangle_{L^2}\,\mathrm{d}s$$
$$- \int_0^t \langle \dot{\mathcal{L}}(s), u(s)\rangle_{L^2}\,\mathrm{d}s - \int_0^t \langle \mathcal{L}(s), \dot{w}(s)\rangle_{L^2}\,\mathrm{d}s\,,$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$.

The functions $f$, $g$, and $w$ are assumed absolutely continuous in time, with respect to their target spaces (chosen such that the duality products make sense). The initial condition satisfies the global stability. The set of admissible displacements is introduced as a relaxation for the set of triples $(u,e,p)$ satisfying (sf1), in order to perform the incremental minimization scheme. (The same motivation leads to set $p$ into the space of Radon measures.) The existence of quasistatic evolutions for the energetic formulation is proven in Theorem 2.3.3.

We now discuss the mechanical meaning of the generalized energy. As mentioned above, the case of positive $\lambda$ accounts for a fatigue phenomenon. Since the present approach is based on energy minimization, it is important to detect under which condition the term with $\lambda$ in $\mathcal{E}^G_\lambda$ is minimized.

To fix the ideas, consider the simplified case when $p$ is absolutely continuous from $[0,T]$ into $L^1(\Omega;\mathbb{M}^{n\times n}_D)$ and the constraint sets are balls with radius $V(\alpha)$, so that $H(\alpha,\xi) = V(\alpha)|\xi|$. By (0.0.15) and (0.0.16), it follows that (recall that $\alpha(t) \equiv \overline{\alpha}$)

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha}, p; 0, t) = \int_\Omega V(\overline{\alpha}(x))\Big(\int_0^t |\dot{p}(s,x)|\,\mathrm{d}s\Big)\,\mathrm{d}x\,,$$

that is a space integral of the product between a nondecreasing function of $\overline{\alpha}$ and the *cumulated plastic strain* $x \mapsto \int_0^t |\dot{p}(s,x)|\,\mathrm{d}s$. This shows that it is easier to damage portions of the material where the cumulated plastic strain is larger, i.e., parts more affected by plastic evolution until $t$. The typical example of a fatigue phenomenon is *cycling loading*: even if the plastic strain remains small, the material may break by effect of plastic cycles. Tuning $\lambda$ between zero and one, different effects of the plasticity on the damage process are described; setting $\lambda = 0$ leads to an energy balance analogous to the one of [**29**], while the choice $\lambda = 1$ was instead prescribed in [**2, 3**].

Notice that the coupling between linearized perfect plasticity and damage fits into the abstract scheme of rate independent systems described above. Indeed, consider as $\mathcal{Q}$ the ambient space for the triples of damage, displacement, and plastic strain $(\beta, v, q)$, and

$$\mathcal{E}_\lambda(t, v, q, \beta; p|_{(0,t)}) = \frac{1}{2}\int_\Omega \mathbb{C}(\beta)(\mathrm{E}v - q)\cdot(\mathrm{E}v - q)\,\mathrm{d}x + \int_\Omega \beta(x)\,\mathrm{d}x + \|\nabla\beta\|^\gamma_{L^\gamma}$$
$$+ \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\beta, p; 0, t) - \langle \mathcal{L}(t), v\rangle\,,$$

$$\mathcal{R}(\beta, \dot{\beta}, \dot{q}) = \mathcal{H}(\beta, \dot{q}) + \mathcal{R}^1(\beta, \dot{\beta})\,, \quad \mathcal{R}^1(\widehat{v}, z) = \begin{cases} \int_\Omega -d'(\widehat{v}(x))z(x)\,\mathrm{d}x\,, & \text{if } z \le 0 \text{ a.e. in } \Omega \\ +\infty\,, & \text{otherwise}\,. \end{cases}$$

Recalling that, if $\dot q$ is a function, $\mathcal{H}(\beta,\dot q)=\int_\Omega H(\beta,\dot q)\,\mathrm{d}x$, the strong formulation reads

$$\begin{cases}\partial_u\mathcal{E}_\lambda=0\,,\\ \partial_{\dot p}\mathcal{R}+\partial_p\mathcal{E}_\lambda\ni 0\,,\\ \partial_{\dot\alpha}\mathcal{R}+\partial_\alpha\mathcal{E}_\lambda\ni 0\,.\end{cases}$$

In particular, the differential inclusions above correspond to (sf3), (sf4)-(sf5), and (sf6'), respectively. The fact that $\mathcal{R}=\infty$ when $\dot\beta$ is somewhere positive enforces the unidirectionality for the damage variable. Moreover, in the finiteness domain of $\mathcal{R}^1$, $D$ is the corresponding dissipation potential (see (0.0.6)). This justifies the choice of including it in the energy (see also (0.0.7)), that is also done in [**2**].

As a technical note, observe that the monotonicity in time of $\alpha$ and the softening property of $\mathcal{H}$ are essential to prove that $\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;0,t)$ is indeed nondecreasing with respect to refinements of the partition $\mathcal{P}$ of $[0,t]$. In other plasticity models with an internal variable, e.g. Cam-Clay plasticity [**29**], this property (which is fundamental in our approach to energetic solutions) does not hold since $\alpha$ may increase in time.

Since the functional appearing in the global stability property $(\mathrm{qs1})^G$ is not globally convex, the energetic solution is in general not smooth in time but one can only say that it has bounded variation. In the case where it is sufficiently regular, it is proven that the weak formulation complies with the strong formulation. However, due to lack of convexity, the approach by vanishing viscosity is useful in this case. This is the subject of a work in collaboration with Giuliano Lazzaroni, that is presented below.

The total internal (generalized) energy for our local minimization model is

$$\mathcal{E}^L_\lambda(\alpha,e;p,t):=\mathcal{Q}(\alpha,e)+D(\alpha)+\frac{\kappa}{2}\,|\alpha|^2_{m,2}+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha,p;0,t)\,,$$

with

$$|\alpha|^2_{m,2}=\sum_{|\beta|=m}\|D^\beta\alpha\|_2^2\,,\quad m:=\left[\frac{n}{2}\right]+1\,.$$

The only difference with respect to the energy for the globally stable evolution (see (0.0.12) and (0.0.17)) is the damage regularization, which is now of Hilbert type. Thus the target space of $\alpha$ is now $H^m(\Omega)$. The choice of $m$ still gives the embedding into the space of continuous functions. The plastic potential is the same as before and all the remaining notations are the same. Under regularity assumptions, the quasistatic evolution has to satisfy the conditions (sf1)–(sf5), plus the Kuhn-Tucker conditions

$$\begin{cases}\dot\alpha(t)\le 0 \text{ in } \Omega\,,\\ \langle\partial_\alpha\mathcal{E}^L_\lambda(\alpha(t),e(t);p,t),\beta\rangle\ge 0 \qquad \text{for every } \beta\in W^{1,\gamma}(\Omega)\,,\ \beta\le 0\,,\\ \langle\partial_\alpha\mathcal{E}^L_\lambda(\alpha(t),e(t);p,t),\dot\alpha(t)\rangle\ge 0\,.\end{cases}\tag{sf6''}$$

As mentioned above, the functional in the global stability property $(\mathrm{qs1})^G$ is not convex. This is due to the presence of damage. In contrast, in perfect plasticity the evolution is regular. For these reasons, the viscosity involves only the damage variable. The choice of the viscous dissipation is a delicate point: too strong regularizations might lead to trivial viscous approximate evolutions. For this reason it has not been considered a quadratic dissipation for $\alpha$ in its reference space $H^m(\Omega)$, but in $L^2(\Omega)$. The same viscous penalization is adopted also in vanishing viscosity for damage without plasticity **[61]**. We refer also below to this paper.

As in the approach by global minimization, the starting point is discretizing the time interval into $k+1$ subintervals and solving $k$ incremental minimum problems (with a fixed viscous penalization). In this way, for every vanishing viscosity parameter $\varepsilon > 0$, one obtains a family of piecewise affine approximate evolutions. The major difficulty is to find suitable a priori estimates for these approximate evolutions, in order to pass to the limit first as $k$ tends to $\infty$ and then as $\varepsilon$ tends to $0$. In **[61]** the following estimates are proven for the piecewise affine interpolations $\alpha_{\varepsilon,k}$:

$$\int_0^T \|\dot{\alpha}_{\varepsilon,k}(s)\|^2_{H^m(\Omega)} \leq C_\varepsilon \quad \text{and} \quad \int_0^T \|\dot{\alpha}_{\varepsilon,k}(s)\|_{H^m(\Omega)} \leq C\,. \tag{0.0.18}$$

As seen in page 4, the estimates above would be trivial with an $H^m$ regularization. In contrast, with the $L^2$-viscosity one obtains easily only the estimates with $L^2$ norm. The proof of (0.0.18) is based on a discrete Gronwall lemma.

In order to follow an approach similar to the one in **[61]** also in the presence of plasticity, the idea is to control the time discrete derivatives of the strain by the time discrete derivatives of the damage and of the boundary datum. This is motivated by the fact that, in the framework of perfect plasticity, one can prove an analogous control. By the improved "perfect plasticity estimate", obtained in Lemma 3.1.6, not only (0.0.18) follows, but also that all the approximate viscous evolutions are uniformly $H^1$ in time, independently of the time discretization, and uniformly $W^{1,1}$ in time with respect to $\varepsilon > 0$.

Basing on the estimates in (0.0.18), one can argue as in the abstract scheme. In particular, the existence of *rescaled quasistatic viscosity evolutions* is proven by means of an arc-length reparametrization (see Theorem 3.3.6).

A *rescaled quasistatic viscosity evolution in the time interval* $[0,S]$ is a 5-tuple of Lipschitz functions $(\alpha^\circ, u^\circ, e^\circ, p^\circ, t^\circ)$ from $[0,S]$ into $H^m(\Omega;[0,1]) \times BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D) \times [0,T]$ such that, setting for every $s \in [0,S]$

$$\sigma^\circ(s) := \mathbb{C}(\alpha^\circ(s))e^\circ(s)\,, \quad w^\circ(s) := w(t^\circ(s))\,, \quad \mathcal{L}^\circ(s) := \mathcal{L}(t^\circ(s))\,, \quad \text{and}$$

$$U^\circ := \{s \in [0,S] : t^\circ \text{ is constant in a neighbourhood of } s\}\,,$$

the following conditions are satisfied:

$(\mathrm{qs0})^L$ *unidirectionality:* $t^\circ$ is nondecreasing and surjective, and for every $x \in \Omega$

$$[0,S] \ni s \mapsto \alpha^\circ(s,x) \quad \text{is nonincreasing;}$$

$(\mathrm{qs1a})^L$ *global minimality for fixed damage:* for every $s \in [0,S]$, $(u^\circ(s), e^\circ(s), p^\circ(s))$ solves

$$\min_{(v,\eta,q)\in A(w^\circ(s))} \left\{\mathcal{Q}(\alpha^\circ(s),\eta) - \langle \mathcal{L}^\circ(s), v\rangle + \mathcal{H}(\alpha^\circ(s), q - p^\circ(s))\right\}\,;$$

$(\mathrm{qs1b})^L$ *Kuhn-Tucker inequality in* $[0,S]\setminus U^\circ$*:* for every $s \in [0,S]\setminus U^\circ$

$$\langle \partial_\alpha \mathcal{E}^L_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \beta\rangle \geq 0 \quad \text{for every } \beta \in H^m(\Omega)\,,\ \beta \leq 0\,;$$

$(\mathrm{qs2})^L$ *energy balance*: for every $s \in [0,S]$

$$\mathcal{E}^L_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s) - \langle \mathcal{L}^\circ(s), u^\circ(s)\rangle + (1-\lambda)\int_0^s \mathcal{H}(\alpha^\circ(\tau), \dot p^\circ(\tau))\,\mathrm{d}\tau$$
$$+ \int_{(0,s)\cap U^\circ} \|\dot\alpha^\circ(\tau)\|_2 \Psi(\alpha^\circ(\tau), e^\circ(\tau); p^\circ, \tau)\,\mathrm{d}\tau$$
$$= \mathcal{E}^L(\alpha_0, e_0) - \langle \mathcal{L}(0), u_0\rangle + \int_0^s \Big[\langle \sigma^\circ(\tau), E\dot w^\circ(\tau)\rangle_{L^2} - \langle \dot{\mathcal{L}}^\circ(\tau), u^\circ(\tau)\rangle_{L^2} - \langle \mathcal{L}^\circ(\tau), \dot w^\circ(\tau)\rangle_{L^2}\Big]\,\mathrm{d}\tau\,,$$

where the convention $0\cdot\infty = 0$ is adopted and

$$\Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s) = \sup_{\beta\in H^m_-(\Omega),\, \|\beta\|_2\le 1} \langle -\partial_\alpha \mathcal{E}^L_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \beta\rangle\,.$$

The jumps are described in the slow time scale by the set $U^\circ$. The global minimality $(\mathrm{qs1})^G$ is replaced by a global minimality with respect to plasticity plus a first order stability condition with respect to the damage that holds outside the jumps, where the evolution is rate independent. The term $\Psi$, corresponding to jumps in time, involves indeed the $L^2$ norm of the damage variable derivative, so that it keeps track of the $L^2$-viscous approximation. In fact, $\Psi$ is a sort of *unilateral slope* in $L^2$ of the energy with respect to admissible variations of damage. It is the analogue of the term in (STv), with the restrictions that viscosity norm $L^2$ is considered instead of the $H^m$ norm, and that there is a condition on the sign of $\beta$, that comes from the unidirectionality of damage. In Lemma 3.2.5 it is proven that $\Psi$ indeed represents how far the current configuration is from satisfying the first order stability condition. Precisely, it holds that

$$\Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s) = \min\{\|g\|_{L^2} \colon \langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s) + g, \beta\rangle \ge 0\ \forall \beta \in H^m_-(\Omega)\}\,.$$

Thus $\Psi$ is the $L^2$ distance of the partial derivative with respect to $\alpha$ evaluated in the current configuration (this belongs to the dual of $H^m(\Omega)$) from the set of the elements in $(H^m(\Omega))'$ for which $(\mathrm{qs1b})^L$ holds. See also Remark 3.2.6 for further comments about $\Psi$.

The main motivation for the study of *strain gradient plasticity* is that of some features of plastic materials that are not well described by the theory of perfect plasticity.

For instance, perfect plasticity allows for displacement fields that jump on $(n-1)$-dimensional surfaces of the domain $\Omega$. These are not observed in reality, where the thickness of the sharp transitions between two zones with different displacements cannot be smaller than a certain quantity, usually in the range 500 nm–50 $\mu$m. Such transition regions are called *plastic shear bands*. In strain gradient plasticity, some controls for the gradient of plastic strain are assumed: this gives more regularity for the total strain, and prevents displacement from displaying jump surfaces.

There are several papers devoted to strain gradient plasticity, and many different ways to deal with the gradient of the plastic strain. (See for instance [**1, 17, 46, 56, 51, 52**].)

Another advantage of strain gradient plasticity theories is the possibility to account more directly for *dislocations*. Dislocations are line defects within a crystal structure that are characterized by two vectors: the Burgers vector, $b$, that measures the slip displacement associated with the line defect, and a unit vector $t$, that points in the direction of the dislocation line.

There are two main types of dislocations: edge dislocations, where $b$ and $t$ are perpendicular, and screw dislocations, where the two vectors are parallel. In the most general case the dislocation line lies at an arbitrary angle to its Burgers vector and the dislocation has a mixed edge and screw character.

As mentioned in page 6, dislocations are the microscopical responsible for plasticity. From the mathematical point of view, these objects can be characterized by means of a first order differential operator applied to the plastic strain. Since strain gradient plasticity deals with the first order derivatives of the plastic strain, it is useful in order to describe the interaction among dislocations, and to capture size effects, such as strengthening and strain hardening, caused by these defects.

In particular, the strict relation among dislocations and damage motivates the study of the coupling between gradient plasticity and damage. Since the early works on damage and dislocations in the 1950s (see e.g. [**110, 101, 21, 96**]), it has been pointed out as the pile-up of these defects is responsible for microcrack formation and coalescence, that is for creation and evolution of damage.

The gradient plasticity model employed here for the coupling between plasticity and damage is the one proposed by Gurtin and Anand [**52**]. This model shares some common characteristics with perfect plasticity, and it has been studied in the framework of energetic solutions by Giacomini and Lussardi in [**48**] and [**47**].

The strong formulation is now introduced. For clarity of notation, the label "$GA$" indicates that the corresponding objects or properties refer to the gradient plasticity model coupled with damage. Moreover, to shorten the notation, the dependence on $x$ in the strong formulation is no longer written explicitely.

The additive decomposition of the total strain and the condition on the prescribed boundary displacement do not change with respect to the cases above. Then we require:

$$\begin{cases} Eu(t) = e(t) + p(t) & \\ u(t) = w(t) & \text{on } \partial_D\Omega\,. \end{cases} \tag{sfGA1}$$

The starting point, as in the approach of Gurtin and Anand [**52**], is to consider $\dot{e}(t)$, $\dot{p}(t)$, and $\nabla\dot{p}(t)$ as *independent rate-like kinematical descriptors* with conjugated internal forces given by some tensors $\sigma(t)$, $\sigma^p(t)$, and $\mathbb{K}^p(t)$ such that the (internal) power expenditure within a subdomain $\mathcal{B} \subset \Omega$ at a time $t$ is expressed by

$$\mathcal{W}_{\text{int}}(\mathcal{B}, t) = \int_{\mathcal{B}} \sigma(t)\cdot\dot{e}(t) + \sigma^p(t)\cdot\dot{p}(t) + \mathbb{K}^p(t)\cdot\nabla\dot{p}(t)\,\mathrm{d}x\,. \tag{0.0.19}$$

Then the stress configuration of the system is described by $\sigma(t)$, which is the usual Cauchy stress, by a second order tensor $\sigma^p(t)$, and by a third order tensor $\mathbb{K}^p(t)$. (By " $\cdot$ " we denote the scalar product between tensors of the same order, independently of the order.)

The elastic tensor $\mathbb{C}$ depends on $\alpha$ and it is expressed in terms of the shear and the bulk moduli $\mu$ and $k$:

$$\mathbb{C}(\alpha)e := 2\mu(\alpha)e_D + k(\alpha)(\text{tr } e)I\,.$$

By the softening character of damage, $\mu$ and $k$ are assumed nondecreasing with respect to $\alpha$.

The total energy density for our model is

$$\psi = \mu(\alpha)|e_D|^2 + \frac{1}{2}k(\alpha)|\,\mathrm{tr}\; e|^2 + \frac{L^2}{2}\mu(\alpha)|\,\mathrm{curl}\; p|^2 + \frac{\ell^2}{2}|\nabla\alpha|^2 + d(\alpha)\,,$$

where $d$ is a continuous nonnegative function, and $L$, $\ell$ are length scales. The first three terms of $\psi$ correspond to the free energy density assumed by Gurtin and Anand, where the elastic moduli depends on the damage. Notice that the dependence on the damage gradient is quadratic. The part of energy density depending only on $\alpha$ is denoted by $d$, as in the cases above.

The term $\frac{L^2}{2}\mu(\alpha)|\,\mathrm{curl}\; p|^2$ is the density of energy stored by the *geometrically necessary dislocations*. Indeed, the energy stored per unit length by a dislocation is proportional to $\mu|b|^2$, see e.g. in [**57**, Section 4.4] and [**69**, Section 1]. The *macroscopic Burgers tensor* $\mathrm{curl}\, p$ measures the incompatibility of the field $p$ and, for every unit vector $m$, $(\mathrm{curl}\, p)\, m$ is the Burgers vector, measured per unit area, associated with small loops orthogonal to $m$, namely with those dislocation whose lines pierce the plane with normal $m$ (see [**52**, Section 3]); then $\mathrm{curl}\; p$ provides a measure of the dislocation density.

As in [**52**], we define an energetic higher-order stress $\mathbb{K}^p_{\mathrm{en}}$, associated to dislocations, as the symmetric-deviatoric part in the first two components (cf. (1.1.1)) of the partial derivative of $\psi$ with respect to $\mathrm{curl}\, p$, and the dissipative higher order stress $\mathbb{K}^p_{\mathrm{diss}}$ is the remaining part of $\mathbb{K}^p$. Considering also that $\sigma$ depends on $e$ as in perfect plasticity, the consitutive relations become:

$$\begin{cases}\sigma(t) := \mathbb{C}(\alpha(t))e(t)\,,\\ \mathbb{K}^p_{\mathrm{en}}(t)\cdot\nabla A := \mu(\alpha(t))L^2\mathrm{curl}\, p(t)\cdot\mathrm{curl}\, A \text{ for every } \mathbb{M}^{n\times n}_{sym}\text{-valued function } A\,,\\ \mathbb{K}^p_{\mathrm{diss}} := \mathbb{K}^p - \mathbb{K}^p_{\mathrm{en}}\,.\end{cases} \tag{sfGA2}$$

The equilibrium conditions are derived by imposing a balance between the power of the internal forces (0.0.19) and the one of the external forces usually considered in gradient plasticity. They are the following:

$$\begin{cases}-\,\mathrm{div}\;\sigma(t) = f(t) \quad \text{in}\;\Omega\,, \quad \sigma(t)\nu = g(t) \quad \text{on}\;\partial_N\Omega\,,\\ \sigma^p(t) = \sigma_D(t) + \mathrm{div}\;\mathbb{K}^p(t) \quad \text{in}\;\Omega\,,\\ \mathbb{K}^p(t)\nu = 0 \quad \text{on}\;\partial\Omega\,.\end{cases} \tag{sfGA3}$$

Above, $\nu$ is the outward normal to $\Omega$. For every subbody $\mathcal{B}$ with outward normal $\nu$, the deviatoric matrix $\mathbb{K}^p\nu$ represents the surface density of microtractions associated to the plastic strain. We refer to [**51**, Sections 9 and 11] for the connection between microtractions and thermodynamic force between dislocations. As in [**52**, Section 8] we assume that the microscopic power expenditure at the boundary is null.

As for the flow rule for plasticity, also the microscopic dissipative force $\mathbb{K}^p_{\mathrm{diss}}$ associated to $\nabla p$ has to be considered. Then the constraint set $K(\alpha(t,x))$ is an ellipsoid in the product space between the deviatoric matrices and the third order tensors that are symmetric-deviatoric in the first two components. It holds:

$$(\sigma^p(t,x), \mathbb{K}^p_{\rm diss}(t,x)) \in K(\alpha(t,x)) := \left\{ (A, \mathbb{B}) : \frac{|A|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{B}|^2}{l^2 S_2(\alpha(t,x))^2} \le 1 \right\}, \quad \text{(sfGA4)}$$

$$(\dot p(t,x), \nabla \dot p(t,x)) \in N_{K(\alpha(t,x))}\big((\sigma^p(t,x), \mathbb{K}^p_{\rm diss}(t,x))\big)\,. \quad \text{(sfGA5)}$$

Notice that one can deal with two different softening-type behaviors corresponding to different directions of the generalized constraint sets.

The energy is the space integral of $\psi$, so

$$\mathcal{E}^{GA}(\alpha, e, \mathrm{curl}\, p) := \mathcal{Q}_1(\alpha, e) + \mathcal{Q}_2(\alpha, \mathrm{curl}\, p) + \frac{\ell^2}{2}\|\nabla\alpha\|^2_{L^2} + D(\alpha)\,,$$

with

$$\mathcal{Q}_1(\alpha, e) := \frac{1}{2}\int_\Omega \mathbb{C}(\alpha) e\cdot e \,\mathrm{d}x\,,\ \ \mathcal{Q}_2(\alpha, \mathrm{curl}\,p) := \frac{L^2}{2}\int_\Omega \mu(\alpha)|\mathrm{curl}\,p|^2\,\mathrm{d}x\,,\ \ D(\alpha) := \int_\Omega d(\alpha)\,\mathrm{d}x\,.$$

As in perfect plasticity with damage, some Kuhn-Tucker conditions are required for the evolution of damage:

$$\begin{cases} \dot\alpha(t) \le 0 \text{ in } \Omega\,, \\ \langle \partial_\alpha \mathcal{E}^{GA}(\alpha(t), e(t), \mathrm{curl}\,p(t)), \beta\rangle \ge 0 \quad \text{for every } \beta\in H^1(\Omega)\,,\ \beta\le 0\,, \\ \langle \partial_\alpha \mathcal{E}^{GA}(\alpha(t), e(t), \mathrm{curl}\,p(t)), \dot\alpha(t)\rangle \ge 0\,. \end{cases} \quad \text{(sfGA6)}$$

Notice that in order to minimize $\mu(\alpha)|\,\mathrm{curl}\,p|^2$ (namely to minimize $\mathcal{Q}_2$) it is convenient to damage portions of the material with high dislocation density. (Recall that $\mu$ is nondecreasing.) This type of interplay between damage and dislocations complies with the models of microcrack formation and coalescence by dislocation pile-up. Moreover, the $H^1$ damage regularization employed here is natural from a mechanical point of view (see for instance [**2**] and [**70**]). This is an improvement with respect to the elastoplastic-damage models presented above.

In order to write the energetic formulation for the Gurtin-Anand model coupled with damage, the expression of the plastic dissipation is needed.

The plastic potential is defined similarly to the one in perfect plasticity with damage, starting from the support function of the constraint set. Now, $\nabla p$ is a measure, so $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$. Therefore, for any $\alpha\in H^1(\Omega)$ and $p\in BV(\Omega;\mathbb{M}^{n\times n}_D)$, we define

$$\mathcal{H}^{GA}(\alpha, p) := \int_\Omega \sqrt{S_1(\alpha)^2|p|^2 + l^2 S_2(\alpha)^2|\nabla p|^2}\,\mathrm{d}x + l\int_\Omega S_2(\widetilde\alpha)\,\mathrm{d}|\mathrm{D}^s p|\,.$$

Here, $\nabla p$ and $\mathrm{D}^s p$ are the absolutely continuous and the singular part of $\mathrm{D}p$ with respect to the Lebesgue measure $\mathcal{L}^n$, and $\widetilde\alpha$ is the precise representative of $\alpha$, which is well defined at $\mathcal{H}^{n-1}$–a.e. $x\in\Omega$. Observe that when $l=0$ $\mathcal{H}^{GA}$ reduces to a particular case of the potential in (0.0.13). The variation $\mathcal{V}^{GA}_{\mathcal{H}}(\alpha, p; s, t)$ is defined as in (0.0.14), starting from $\mathcal{H}^{GA}(\alpha, p)$.

The main result in this framework (see Theorem 4.1.5) is the existence of a *quasistatic evolution for the Gurtin-Anand model coupled with damage*, namely of a function

$$[0,T]\ni t\mapsto (\alpha(t), u(t), e(t), p(t)) \in H^1(\Omega;[0,1])\times W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times BV(\Omega;\mathbb{M}^{n\times n}_D)$$

that satisfies the following conditions:

$(\mathrm{qs0})^{GA}$ *unidirectionality* : for every $x\in\Omega$ the function $[0,T]\ni t\mapsto \alpha(t,x)$ is nonincreasing;

$(\mathrm{qs1})^{GA}$ *global stability*: $(u(t), e(t), p(t))$ is admissible for the boundary condition $w(t)$ (i.e., its energy is finite and (sfGA1) hold) for every $t \in [0,T]$ and

$$\mathcal{E}^{GA}(\alpha(t), e(t), \mathrm{curl}\, p(t)) - \langle \mathcal{L}(t), u(t)\rangle \leq \mathcal{E}^{GA}(\beta, \eta, \mathrm{curl}\, q) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}^{GA}(\beta, q - p(t))$$

for every $\beta \leq \alpha(t)$ and every triple $(v, \eta, q)$ admissible for $w(t)$.

$(\mathrm{qs2})^{GA}$ *energy balance*: the function $t \mapsto p(t)$ from $[0,T]$ into $BV(\Omega; \mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t \in [0,T]$

$$\begin{aligned}
&\mathcal{E}^{GA}(\alpha(t), e(t), \mathrm{curl}\, p(t)) - \langle \mathcal{L}(t), u(t)\rangle + \mathcal{V}^{GA}_{\mathcal{H}}(\alpha, p; 0, t) \\
&= \mathcal{E}^{GA}(\alpha(0), e(0), \mathrm{curl}\, p(0)) - \langle \mathcal{L}(0), u(0)\rangle + \int_0^t \langle \sigma(s), \mathrm{E}\dot{w}(s)\rangle \,\mathrm{d}s \\
&- \int_0^t \langle \dot{\mathcal{L}}(s), u(s)\rangle \,\mathrm{d}s - \int_0^t \langle \mathcal{L}(s), \dot{w}(s)\rangle \,\mathrm{d}s\,.
\end{aligned}$$

An important issue in the existence proof is to obtain the lower semicontinuity of the plastic potential with respect to both variables. Indeed, now $\alpha$ is no longer continuous and then Reshetnyak's Theorem does not apply.

Moreover, in Chapter 4 the limit evolution as the length scales $l$ and $L$ tend to zero is studied, following the anlaysis in **[48]** for the case without damage. Here, we require a stronger regularization, specifically a $W^{1,\gamma}$ regularization ($\gamma > n$). In the limit we obtain quasistatic evolution for perfect plasticity coupled with damage (see Theorem 4.5.1). In fact, the gradient plasticity formally reduces to perfect plasticity when $L = l = 0$.

The last part of Chapter 4, which is unpublished, contains the proofs of a new Reshetnak-type lower semicontinuity theorem (Theorem 4.6.1) and of a result (Theorem 4.6.6), that in our opinion is an important step toward the existence of globally stable quasistatic evolutions for elastoplasticity coupled with damage, where an $H^1$-regularization for damage replaces the $W^{1,\gamma}$-regularization, $\gamma > n$, in the energy (0.0.12).

## Brittle and cohesive fractures

Fracture is a typical example of unidirectional phenomenon. The fracture process is the result of the competition between the energy of the unfractured body and the work needed to create a new crack or to extend an existing one. (This work corresponds to a dissipated energy.)

The choice of the dissipation due to the crack growth determines whether the fracture model is *brittle* or *cohesive*. In brittle fracture, the energy spent to produce a crack only depends on the geometry of the crack itself, the simplest case being a surface energy proportional to the measure of the crack set (Mumford-Shah energy). In contrast, cohesive dissipations, introduced in **[9]**, also depend on the crack opening, i.e., on the difference between the traces of the displacement on the two sides of the crack. From this point of view, fracture is regarded as a gradual process, where the material is considered completely cracked at a point only when the amplitude of the jump of the displacement is sufficiently large.

Fracture models are examples of quasistatic *free discontinuity problems*, where the unknown is a pair $(u, \Gamma)$, with $\Gamma$ a closed set and $u$ a (sufficiently) smooth function on $\Omega \setminus \Gamma$. Fracture models are closely related to damage models. Indeed, many materials undergo a damage process before a crack is formed or grows. Moreover, Ambrosio and Tortorelli [**7**] proved that the Mumford-Shah energy can be obtained as the limit of functionals describing the energy of a damage model; the fracture set is then replaced by a damage variable, which continuously interpolates between sound and cracked material. In the passage to the limit from damage to fracture, the damaged regions concentrate in narrow strips along manifolds of codimension one.

The first to develop a theory for fractures based on an energetic approach was Griffith, in [**50**]. Griffith criterion is based on the notion of *energy release rate*, that is the opposite of the derivative of the energy associated with the solution when the crack length varies. It states that:

- the crack growth is unidirectional, i.e., the crack is nondecreasing in time;
- the energy release rate never exceeds the toughness;
- the fracture can grow only if the energy release rate equals the toughness.

This criterion gives the Kuhn-Tucker condition for brittle fracture, analogous to the ones for damage.

In the following, two models on brittle and cohesive fracture are presented. The two results are obtained in the framework of antiplane linearized elasticity. In the antiplane setting the configuration of the body is an infinite cylinder $\Omega\times\mathbb{R}$, with $\Omega \subset \mathbb{R}^n$ ($n = 2$ being the physically relevant case), and the deformation $v\colon \Omega\times\mathbb{R} \to \Omega\times\mathbb{R}$ takes the form $v(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n, x_{n+1} + u(x_1, \dots, x_n))$, where $u\colon \Omega \to \mathbb{R}$ is the vertical displacement. Outside the cracked region, the material is assumed to be linearly elastic. Then the energy of the uncracked body is

$$\frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x\,.$$

In particular, we are considering *quasistatic fractures*, thus neglecting the dynamical effects in the uncracked region. The material may present cracks of the form $\widehat{\Gamma}\times\mathbb{R}$, where $\widehat{\Gamma}$ is contained in $\Omega$.

The first model, studied in Chapter 5, is based on local minimization. The cohesive model, analyzed in Chapter 6, is based on global minimization.

The viscous approximation of quasistatic crack growth has been studied in the literature only for $n = 2$, thus the deformations are functions of two variables and the cracks are represented as one-dimensional sets. Moreover, the crack must be sufficiently regular. More precisely, in [**59, 62**] the crack evolves on a given, smooth, prescribed curve, while in [**66, 67**] the crack is not prescribed a priori, but chosen in a class of admissible curves of class $C^{1,1}$.

In the present work, obtained in collaboration with Giuliano Lazzaroni, the previous results by Lazzaroni and Toader [**67**] are extended to a larger class $\mathcal{S}$ of cracks, introduced by Racca [**90**]. The cracks in $\mathcal{S}$ may have many connected components, each of them being the union of a certain number of branches that are regular curves of the type considered in [**67**]. Moreover, some geometric restrictions are imposed in order to guarantee that $\mathcal{S}$ is closed with respect to the Hausdorff convergence, that the number of connected components and of branches is uniformly bounded in $\mathcal{S}$, and that the uncracked part of the body is always a connected set. These conditions allow for cracks displaying branching and kinking.

The viscous regularization employed here is a quadratic penalization of the elongation of any connected component. The passage to the limit as the time discretization step tends to zero was already studied in [**90**] in order to prove the existence of viscously regularized evolutions satisfying an energy inequality. Here the work initiated there is completed by showing the energy-dissipation balance at viscous level and by passing to the limit as the viscous parameter tends to zero.

We now briefly describe the main properties of the limit evolution, referring the reader to Theorem 5.4.4 for the existence result and for more details. As one expects from the general properties of evolutions obtained by viscous approximation (see page 5), in the reparametrized time scale there is an at most countable number of intervals each corresponding to a jump. In the continuity set, where the process is quasistatic, the Griffith principle is satisfied for any branch with the following limitation: the second law of Griffith holds only when the branch tip does not meet a certain set of exceptional points. Such exceptional points are of two types: either they are points of branching or kinking, or they are points where the evolution stops because of the geometric restrictions on the cracks; moreover, also the limits of exceptional points of the approximating viscous evolutions have to be included among the exceptional points of the quasistatic evolution. Because of the restrictions on the class of admissible cracks, it turns out that the exceptional points are in a finite number.

A full understanding of the Griffith principle at singular points would require to characterize the limit of the energy release rates of a sequence of irregular cracks converging in the sense of Hausdorff. However, the characterizations of the energy release rate of a crack at an irregular point given in [**85, 19, 20, 8**] do not provide the desired continuity properties.

The remaining part of the introduction is devoted to the description of the results about cohesive fracture, obtained in collaboration with Giuliano Lazzaroni and Gianluca Orlando. We point out this interesting and original feature: the limit evolution is formulated only in terms of functions, even if it is obtained passing through a very weak notion of solution, that involves *Young measures*.

As mentioned before, in cohesive fracture the energy dissipated during the fracture process depends on the evolution of the jump, denoted by $[u(t)]\colon \Gamma \to \mathbb{R}$, Moreover, there are very different models prescribing various behaviors of the body when the size of the jump decreases. In the model presented here, some energy is dissipated also when the size of the jump decreases (until a maximal dissipation is reached), because of the contact between the two sides of the crack. In this respect, the behavior of our system differs from those considered in the mathematical literature on quasistatic cohesive fracture. For instance, when the crack opening decreases one may assume that no energy is dissipated [**37**] or that some dissipated energy is recovered [**15, 4**]. To describe the response of the system to loading, first consider the situation where $[u(0)] = 0$ on $\Gamma$ and $t \mapsto [u(t)]$ is nondecreasing on $\Gamma$ in a time interval $[0, t_1]$. In this case, the energy dissipated in $[0, t_1]$ is

$$\int_\Gamma g\big(\big|[u(t_1)]\big|\big)\, \mathrm{d}\mathcal{H}^{n-1}\,,$$

where $g\colon [0,+\infty) \to [0,+\infty)$ is a concave (thus nondecreasing) function satisfying: $g(0)=0$; $g'(0)$ exists, finite; and $g(\xi) \to \kappa \in (0,\infty)$ as $\xi \to \infty$. If, afterwards, $t \to [u(t)]$ is nonincreasing in the interval $[t_1,t_2]$, there is still some dissipated energy in $[t_1,t_2]$, which amounts to

$$\int_\Gamma g\big(\big|[u(t_1)]\big| + \big|[u(t_2)]-[u(t_1)]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1} - \int_\Gamma g\big(\big|[u(t_1)]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

As a consequence, a complete fracture (corresponding to $g=\kappa$) may occur not only after a large crack opening, but even after oscillations of small jumps (e.g. by a cyclic loading).

In fact, on the contact area between the two parts of the material, the repeated relative surface motion can induce damage by a fatigue process. In applications, this wear phenomenon is known as *fretting* [**22**] and occurs as a result of relative sliding motion of the order from nanometres to millimetres.

The motivation for the study of this type of cohesive fracture comes indeed from the fatigue phenomenon observed in the coupling between perfect plasticity and damage (see above in this introduction, and Chapters 2 and 3). The limit of the energy of such a model, when damage is forced to concentrate on hypersurfaces, has been studied by Dal Maso, Orlando, and Toader [**34**] and gives rise to cohesive surface energies, which inspired the incremental minimum scheme below.

Given an initial condition $u(0)=u_0$ and a time-dependent Dirichlet datum $w(t)$ on $\partial_D\Omega$, for every $k$, $u_k^i$ and $V_k^i$ are defined recursively by

$$\begin{aligned}
&u_k^i \in \operatorname{argmin}\left\{\frac12\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Gamma g\Big(V_k^{i-1} + \big|[u]-[u_k^{i-1}]\big|\Big)\,\mathrm{d}\mathcal{H}^{n-1} \ :\ u = w(i\tfrac{T}{k}) \text{ on } \partial_D\Omega\right\},\\
&V_k^i := V_k^{i-1} + \big|[u_k^i]-[u_k^{i-1}]\big|\,,
\end{aligned}$$

where $u_k^0 := u_0$ and $\mathcal{V}_k^0 = \big|[u_0]\big|$. The function $V_k^i$ describes the cumulated jump of the approximate evolutions at each point of $\Gamma$. The piecewise constant interpolations of $u_k^i$ and $V_k^i$ in time are denoted by $u_k(t)$ and $V_k(t)$, respectively.

The main difficulty in the passage to limit as $k\to\infty$ is the lack of controls on $V_k(t)$. In fact, by the usual a priori estimates, it can be only inferred that $\int_\Gamma g(V_k(t))\,\mathrm{d}\mathcal{H}^{n-1}$ is uniformly bounded, but this gives no information on the equi-integrability of $V_k(t)$, since $g$ is bounded. (This would not be the case if $g$ had e.g. linear growth as in a model for perfect plasticity constrained on $\Gamma$.) In the first instance, in order to pass to the limit as $k\to\infty$, the only chance is to employ compactness properties of the wider class of Young measures (as already done in [**15**]). Indeed, because of the monotonicity of $V_k(t)$, a Helly-type selection principle [**15**, Theorem 3.20] guarantees that $V_k(t)$ converges to a Young measure $\nu(t) = (\nu^x(t))_{x\in\Gamma}$ for every $t$, up to a subsequence independent of $t$.

As for the displacements, from the uniform a priori bounds it follows that there is a subsequence $u_{k_j}(t)$ weakly converging to a function $u(t)$. Yet the subsequence $k_j = k_j(t)$ may depend on $t$. Despite this technical inconvenience, one might employ the methods of [**32**] to derive the energy balance. However, a different approach is followed here; this allows for obtain the convergence on a subsequence independent of the time instant. In order to keep track of the relation between $V_k(t)$ and $[u_k(t)]$, the idea is to pass to the limit in the unidirectionality relation

$$V_k(t) \geq V_k(s) + \big|[u_k(t)] - [u_k(s)]\big| \quad \text{for any } s \leq t\,. \tag{0.0.20}$$

However $u_k(t)$ and $u_k(s)$ may converge along different subsequences! This difficulty is solved by rewriting the previous inequality as a system of two inequalities

$$V_k(t) + [u_k(t)] \geq V_k(s) + [u_k(s)] \quad \text{for any } s \leq t\,, \tag{0.0.21}$$

$$V_k(t) - [u_k(t)] \geq V_k(s) - [u_k(s)] \quad \text{for any } s \leq t\,. \tag{0.0.22}$$

In fact, it is now possible to pass to the limit in these relations by means of the Helly-type theorem in [**15**, Theorem 3.20], extracting a further subsequence (not relabeled) independent of $t$ and exploiting the monotonicity of $V_k(t) \pm [u_k(t)]$. Thus (0.0.20) holds for $\nu$ and $u$ instead of $V_k$ and $u_k$, namely

$$\nu(t) \succeq \nu(s) \oplus \big|[u(t)] - [u(s)]\big| \quad \text{for every } s \leq t\,. \tag{0.0.23}$$

(In the equation above $\oplus$ is a suitable notion of sum between Young measure that generalizes the sum between functions.) Moreover, thanks to this trick it turns out that the limit jump $[u(t)]$ is identified without extracting further subsequences. Ultimately, also the displacement $u(t)$ is the limit of the whole sequence $u_k(t)$, since $u(t)$ is the solution of a minimum problem among functions with prescribed jump $[u(t)]$. This property is relevant for the approximation of the solutions.

At this point of the analysis, one can pass to the limit in the global stability and in the energy balance, obtaining that $(u(t), \nu(t))$ complies with a weak notion of quasistatic evolution. Specifically, the variation of the jump on $\Gamma$ is replaced by the Young measure $\nu(t)$.

The last step of the existence proof is to improve the properties of the limit quasistatic evolution: we obtain that it is charachterized by the two conditions of global stability and energy balance, that do not involve Young measures, but only $u$ and the function $V_u(t)$, that is the (pointwise) variation of the jump of $u(t)$ (see Theorem 6.1.9). By (0.0.23), it follows immediately that $\nu(t)$ is "greater" than the Young measure concentrated on $V_u(t)$. On the other hand, the concavity of the dissipation energy density $g$ allows for proving the global stability also for $V_u(t)$. From these two facts the energy balance with $V_u(t)$ can be derived, and then (by comparing the two energy balances) it turns out that the limit measure $\nu(t)$ is concentrated on the limit function $V_u(t)$ in the interval where $g$ is strictly increasing (namely until $V_u(t)$ reaches the complete fracture threshold). More precisely, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ either the limit measure $\nu^x(t)$ is concentrated on $V_u(t;x)$, or it is supported where $g$ is constant, i.e., where the energy is no longer dissipated. So also the limit of the discrete variations $V_k(t)$ is characterized.

This introduction is concluded by a remark on the unidirectionality of this cohesive model.

Differently from the other problems presented above, in the discussion about this cohesive fracture model we did not start with the strong formulation, but with energetic considerations.

Nevertheless, under regularity assumptions, the strong formulation is derived (see Propositions 6.1.7 and 6.1.8). This requires that for any $t \in [0,T]$ the following hold:

$$\begin{cases} \Delta u(t) = 0 & \text{in } \Omega \setminus \Gamma\,, \\ u(t) = w(t) & \text{on } \partial_D \Omega\,, \\ \partial_\nu u(t) = 0 & \text{on } \partial_N \Omega\,, \end{cases} \qquad \text{for any } t \in [0,T]\,, \tag{sfC1}$$

$$|\partial_\nu u(t)| \leq g'(V_u(t)) \quad \text{on } \Gamma\,, \tag{sfC2}$$

$$\partial_\nu u(t) \in g'(V_u(t))\,\mathrm{Sign}\big([\dot u(t)]\big) \quad \text{on } \Gamma\,, \tag{sfC3}$$

where we denote by Sign the multifunction given by

$$\mathrm{Sign}(\xi) := \begin{cases} 1 & \text{if } \xi > 0\,, \\ [-1,1] & \text{if } \xi = 0\,, \\ -1 & \text{if } \xi < 0\,. \end{cases}$$

In particular, for any $x \in \Gamma$ the surface tension at $x$ lies in the set $[-g'(V_u(t;x)), g'(V_u(t;x))]$, and the jump may vary only when the surface tension reaches the boundary of this set. Therefore (sfC2) and (sfC3) are the Kuhn-Tucker conditions for the growth of cohesive fracture.

The variation of the jumps $V_u(t)$ is thus the relevant memory variable describing the fracture process. Since $V_u(t)$ is nondecreasing, this type of cohesive fracture is unidirectional (recall that $g$ is concave). Indeed, consider a subinterval $[t_1, t_2] \subset [0,T]$ where the jump $t \mapsto [u(t)]$ is not constant in a part of $\Gamma$: even if $u(t_1) = u(t_2)$, the variation $V_u$ increases, so that the constraint set shrinks, namely maximal tensile stress decreases. In this respect our model differs from those studied in [**37, 15**], where, in general, cyclic loadings result into reversible evolutions. For these reasons, also the present model is unidirectional, like the ones discussed in this introduction.

The structure of the thesis is the following: in Chapter 1 the notation is fixed and some preliminary results are stated; Chapter 2 and Chapter 3 concern the study of the coupling between linearized perfect plasticity and damage, by a global and a local minimization approach, respectively; in Chapter 4 the Gurtin-Anand plasticity with damage is considered; Chapter 5 and Chapter 6 are devoted to the study of fracture models, of brittle and cohesive type, respectively.

Every chapter corresponds to a bibliographical reference (except for Chapter 1): the works related to Chapters 2, 3, 4, 5, and 6 are [**23**], [**26**], [**24**], [**25**], and [**27**], respectively.

CHAPTER 1

# Preliminary results

In this chapter we fix some notation and we collect some abstract results which will be useful in the sequel.

## 1.1. Notation and general preliminaries

**Matrices.** We denote by $\mathbb{M}^{n\times n}$ (respectively by $\mathbb{M}^{n\times n\times n}$) the space of $n\times n$ real matrices (resp. third order tensors) endowed with the Euclidean scalar product $\xi\cdot\eta := \sum_{i,j}\xi_{ij}\eta_{ij}$ (resp. $\mathbb{A}\cdot\mathbb{B} := \sum_{i,j,k}\mathbb{A}_{ijk}\mathbb{B}_{ijk}$) and with the corresponding Euclidean norm $|\xi| := (\xi\cdot\xi)^{1/2}$. Moreover $\mathbb{M}^{n\times n}_{sym}$ denotes the subspace of symmetrix matrices and $\mathbb{M}^{n\times n}_{D}$ the subspace of trace free matrices in $\mathbb{M}^{n\times n}_{sym}$. Given $\xi\in\mathbb{M}^{n\times n}_{sym}$, its orthogonal projection on $\mathbb{M}^{n\times n}_{D}$ is the deviator $\xi_D := \xi - \frac{1}{n}(\mathrm{tr}\,\xi)I$.

The symmetrized gradient of an $\mathbb{R}^n$-valued function $u(x)$ is the $\mathbb{M}^{n\times n}_{sym}$-valued function $\mathrm{E}u(x)$ with components $E_{ij}u := \frac{1}{2}(\mathrm{D}_j u_i + \mathrm{D}_i u_j)$, where $\mathrm{D}_i$ denotes the derivative $\frac{\partial}{\partial x_i}$ for $1\le i\le n$.

The gradient, the divergence, and the curl of a $\mathbb{M}^{n\times n}$-valued function $\xi(x) = (\xi_{ij}(x))$ are defined as

$$(\nabla\xi)_{ijk} := \mathrm{D}_k\,\xi_{ij}\,,\quad (\mathrm{div}\;\xi)_i := \sum_j \mathrm{D}_j\,\xi_{ij}\,,\quad (\mathrm{curl}\;\xi)_{ij} := \sum_{p,q}\varepsilon_{ipq}\,\mathrm{D}_p\,\xi_{jq}\,,$$

where $\varepsilon_{ipq}$ is the standard permutation symbol.

We say that a third order tensor $\mathbb{A} = (a_{ijk})$ is *symmetric deviatoric in its first two components*, and we write $\mathbb{A}\in\mathbb{M}^{n\times n\times n}_{D}$, if

$$a_{ijk} = a_{jik} \quad\text{and}\quad \sum_p a_{ppk} = 0\,. \tag{1.1.1}$$

The divergence of a $\mathbb{M}^{n\times n\times n}$-valued function $\mathbb{A}(x) = (a_{ijk}(x))$ is the $\mathbb{M}^{n\times n}$-valued function given by

$$(\mathrm{div}\;\mathbb{A})_{ij} := \sum_k \mathrm{D}_k\,a_{ijk}\,.$$

By " $\cdot$ " we denote the scalar product between tensors of the same order, independently of the order.

**Measures and function spaces.** We denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$ and by $\mathcal{H}^s$ the $s$-dimensional Hausdorff measure, for every $s>0$. Given a locally compact subset $B$ of $\mathbb{R}^n$ and a finite dimensional Hilbert space $X$, we use the symbol $M_b(B;X)$ for the space of bounded $X$-valued Radon measures on $B$, the indication of $X$ being omitted when $X=\mathbb{R}$. This space is endowed with the norm $\|\mu\|_1 := |\mu|(B)$, where $|\mu|\in M_b(B)$ is the total variation of the measure $\mu$. For every $\mu \in M_b(B;X)$ we denote by $\mu^a$ and $\mu^s$ the absolutely continuous and the singular part of $\mu$ with respect to $\mathcal{L}^n$. By the Riesz Representation Theorem, $M_b(B;X)$ can be regarded as the dual of $C_0(B;X)$, the space of continuous functions $\varphi\colon B \to X$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$ (see, e.g., [**95**, Theorem 6.19]). The weak$^*$ topology of $M_b(B;X)$ is defined using this duality. Moreover we say that a sequence $(\mu_k)_k \subset M_b(B;X)$ converges strictly to a bounded Radon measure $\mu$ if and only if it converges in the weak$^*$ topology and $|\mu_k|(B) \to |\mu|(B)$. We use the symbol $\|\cdot\|_p$ for the $L^p$ norm and $\|\cdot\|_{1,q}$ for the norm of the Sobolev spaces $W^{1,q}$. Notice that if $L^1(B;X)$ is identified with the space of bounded measures $\mu$ with $\mu^s = 0$ (considering the density of $\mu^a$ with respect to $\mathcal{L}^n$), then $\|\cdot\|_1$ coincides with the induced norm, so that the notation is consistent. Throughout the thesis we adopt the brackets $\langle\cdot,\cdot\rangle$ to denote the product between dual spaces, the arrows $\to$, $\rightharpoonup$, and $\overset{*}{\rightharpoonup}$ for the strong, weak, and weak$^*$ convergences, respectively, and $\overset{s}{\to}$ for the strict convergence of measures.

Given an open subset $U$ of $\mathbb{R}^n$ the space $BV(U;X)$ is the set of the functions $u \in L^1(U;X)$ whose distributional derivative $\mathrm{D}u$ is a vector-valued bounded Radon measure. This is a Banach space with respect to the norm

$$\|u\|_{BV} := \|u\|_1 + \|\mathrm{D}u\|_1 .$$

A sequence $(u_k)_k$ converges to $u$ weakly$^*$ in $BV$ if and only if $u_k \to u$ in $L^1$ and $\mathrm{D}u_k \overset{*}{\rightharpoonup} \mathrm{D}u$ in $M_b$. We recall that if $U$ is bounded and has Lipschitz boundary then every bounded sequence in $BV(U;X)$ has a weakly$^*$ convergent subsequence and $BV(U;X)$ is continuously embedded into $L^q(U;X)$ for every $1 \leq q \leq \frac{n}{n-1}$, the embedding being compact for $1 \leq q < \frac{n}{n-1}$. For the general theory of $BV$ functions we refer to [**6**].

For every $u \in L^1(U;\mathbb{R}^n)$, with $U$ open in $\mathbb{R}^n$, let $Eu$ be the $\mathbb{M}^{n\times n}_{sym}$-valued distribution on $U$ whose components are defined by $E_{ij}u = \frac{1}{2}(D_j u_i + D_i u_j)$. The space $BD(U)$ of functions with *bounded deformation* is the space of all $u \in L^1(U;\mathbb{R}^n)$ such that $Eu \in M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with respect to the norm

$$\|u\|_1 + \|Eu\|_1 .$$

It is possible to prove that $BD(U)$ is the dual of a normed space (see [**104**] and [**72**]), and this defines the weak$^*$ topology of $BD(U)$. A sequence $u_k$ converges to $u$ weakly$^*$ in $BD(U)$ if and only if $u_k \to u$ strongly in $L^1(U;\mathbb{R}^n)$ and $Eu_k \rightharpoonup Eu$ weakly$^*$ in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. If $U$ is a bounded open set with Lipschitz boundary, for every function $u \in BD(U)$ the trace of $u$ on $\partial U$ belongs to $L^1(\partial U;\mathbb{R}^n)$. It will always be denoted by the same symbol $u$. If $u_k$, $u \in BD(U)$, $u_k \to u$ strongly in $L^1(U;\mathbb{R}^n)$, and $\|Eu_k\|_1 \to \|Eu\|_1$, then $u_k \to u$ strongly in $L^1(\partial U;\mathbb{R}^n)$ (see [**103**, Chapter II, Theorem 3.1]). Moreover (see [**103**, Proposition 2.4 and Remark 2.5]), there exists a constant $C > 0$, depending on $U$, such that

$$\|u\|_{1,U} \leq C\,\|u\|_{1,\partial U} + C\,\|Eu\|_{1,U} , \tag{1.1.2}$$

$\|\cdot\|_{p,B}$ being the $L^p$ norm of a function with domain a Borel set $B$. For the general properties of $BD(U)$ we refer to [**103**].

**Hausdorff distance.** The Hausdorff distance $\mathrm{d}_{\mathscr{H}}$ is defined for two compact sets $K_1, K_2$ by

$$\mathrm{d}_{\mathscr{H}}(K_1, K_2) := \max\left\{\sup_{x\in K_1} \mathrm{dist}\,(x, K_2)\,, \sup_{x\in K_2} \mathrm{dist}\,(x, K_1)\right\}\,,$$

with the conventions $\mathrm{d}_{\mathscr{H}}(x, \emptyset) = \mathrm{diam}\,(\Omega)$ and $\sup\emptyset = 0$. A sequence $\Gamma_k$ of compact subsets of $\overline{\Omega}$ *converges to* $\Gamma$ *in the Hausdorff metric* if $\mathrm{d}_H(\Gamma_k;\Gamma) \to 0$. In this case we write $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$.

**Capacity.** We recall some facts about the theory of capacity, referring to [**54**] for a complete treatment of the subject. Given an open subset $U$ of $\mathbb{R}^n$ and $1 \le q < +\infty$, for every $E \subset U$ the $q$-capacity of $E$ in $U$ is defined by

$$C_q(E, U) := \inf\left\{\int_U |\nabla u|^q \,\mathrm{d}x\colon u \in W^{1,q}_0(U),\ u \ge 1 \text{ a.e. in a neighbourhood of } E\right\}\,.$$

We shall use the shorter notation $C_q(E)$ when there is no ambiguity on the domain. The $q$-capacity is indeed a Carathéodory outer measure such that if $1 < q < n$ and $C_q(E) = 0$, then the Hausdorff dimension of $E$ is at most $n - q$. We say that a real valued function $u$ is $C_q$-quasicontinuous in $U$ if for every $\varepsilon > 0$ there is an open set $G$ such that $C_q(G) < \varepsilon$ and the restriction of $u$ to $U \setminus G$ is continuous. A sequence of real valued functions $u_k$ converges $C_q$-quasiuniformly in $U$ to $u$ if for every $\varepsilon > 0$ there is an open set $G$ such that $C_q(G) < \varepsilon$ and $u_k \to u$ uniformly in $U \setminus G$. For every $(u_k)_k \subset C(U) \cap W^{1,q}(U)$ that is a Cauchy sequence in $W^{1,q}(U)$, there exist a function $u \in W^{1,q}(U)$ and a subsequence converging locally $C_q$-quasiuniformly (namely, quasiuniformly in the compact subsets of $U$) to $u$. It follows that such a limit $u$ is $C_q$-quasicontinuous, that $u_k \to u$ pointwise $C_q$-quasieverywhere in $U$ (that is, pointwise except on a set of $C_q$-capacity zero), and that every $W^{1,q}$ function admits a quasicontinuous representative uniquely defined up to a $C_q$-negligible set. For every $u \in W^{1,q}(U)$ its precise representative $\widetilde{u}$, that is defined as the approximate limit of $u$ in the Lebesgue points and takes value zero elsewhere, is a $C_q$-quasicontinuous representative of $u$. When $u_k \rightharpoonup u$ in $W^{1,q}(U)$ there exists a subsequence $(u_j)_j$ such that $\widetilde{u_j} \to \widetilde{u}$ in $\mu$-measure, for every $\mu$ nonnegative bounded Radon measure that vanishes on all $C_q$-negligible Borel sets (cf. [**16**, Proposition 3.5 and Remark 3.4]). These results hold also for vector-valued functions, as one can see considering each component.

## 1.2. Some auxiliary results

In this section we provide some abstract lemmas, which are used in Chapters 2, 3, and 4. In particular, we analyse the particular variation used to define the plastic dissipation both for perfect plasticity and gradient plasticity, when coupled with damage, and we show a property of monotone functions with values in $L^p$ spaces. Moreover, we prove a compactness result for functions from a time interval into a space with separable predual, such that their first time derivative is not strongly measurable. The section is concluded with a generalization of the Riesz Representation Theorem for bounded linear functionals acting on the space of continuous functions.

**A "weighted" variation.** Let $X$ be a Banach space, $F$ a set, and $\mathcal{G}\colon F\times X\to\mathbb{R}^+\cup\{0\}$. Given $\alpha\colon[0,T]\to F$, $p\colon[0,T]\to X$, $a,b\in[0,T]$ with $a<b$, and $\mathcal{P}:=\{t_i\}_{0\le i\le N}$ with $a=t_0<t_1<\cdots<t_N=b$, we define

$$\mathcal{V}_{\mathcal{G}}^{\mathcal{P}}(\alpha,p;a,b):=\sum_{i=1}^{N}\mathcal{G}(\alpha(t_i),p(t_i)-p(t_{i-1}))\,.$$

and the $\mathcal{G}$*-variation of* $p$ *with respect to* $\alpha$ *on* $[a,b]$ as

$$\begin{aligned}\mathcal{V}_{\mathcal{G}}(\alpha,p;a,b):=&\sup\Big\{\sum_{i=1}^{N}\mathcal{G}(\alpha(t_i),p(t_i)-p(t_{i-1}))\colon a=t_0<t_1<\cdots<t_N=b,\ N\in\mathbb{N}\Big\}\\=&\sup\Big\{\mathcal{V}_{\mathcal{G}}^{\mathcal{P}}(\alpha,p;a,b)\colon\mathcal{P}\text{ partition of }[a,b]\Big\}\,.\end{aligned}\tag{1.2.1}$$

When $\alpha(t)=\overline{\alpha}\in F$ for every $t$ we use the symbols $\widehat{\mathcal{V}}_{\mathcal{G}}$ and $\widehat{\mathcal{V}}_{\mathcal{G}}^{\mathcal{P}}$ instead of $\mathcal{V}_{\mathcal{G}}$ and $\mathcal{V}_{\mathcal{G}}^{\mathcal{P}}$, so that

$$\widehat{\mathcal{V}}_{\mathcal{G}}(\overline{\alpha},p;a,b):=\mathcal{V}_{\mathcal{G}}(\alpha,p;a,b)|_{\alpha(t)=\overline{\alpha}}\,,\quad\widehat{\mathcal{V}}_{\mathcal{G}}^{\mathcal{P}}(\overline{\alpha},p;a,b):=\mathcal{V}_{\mathcal{G}}^{\mathcal{P}}(\alpha,p;a,b)|_{\alpha(t)=\overline{\alpha}}\,.$$

Let us assume that

$$\mathcal{G}(\alpha(t_2),f)\le\mathcal{G}(\alpha(t_1),f),\ \text{for every}\quad 0\le t_1\le t_2\le T,\ f\in X\,,\tag{1.2.2a}$$

$$\mathcal{G}(\beta,0)=0,\ \text{for every}\quad\beta\in F\,,\tag{1.2.2b}$$

$$\mathcal{G}(\beta,f_1+f_2)\le\mathcal{G}(\beta,f_1)+\mathcal{G}(\beta,f_2),\quad\text{for every }\beta\in F,\ f_1,f_2\in X\,.\tag{1.2.2c}$$

LEMMA 1.2.1. *With the notations and assumptions above, it follows that:*

(1) *If* $\mathcal{P}_1,\mathcal{P}_2$ *are partitions of* $[a,b]$, *with* $\mathcal{P}_1\subset\mathcal{P}_2$, *then*

$$\mathcal{V}_{\mathcal{G}}^{\mathcal{P}_1}(\alpha,p;a,b)\le\mathcal{V}_{\mathcal{G}}^{\mathcal{P}_2}(\alpha,p;a,b)\,.$$

(2) *For every* $p\colon[a,b]\to X$ *piecewise constant and continuous from the right, with discontinuities at the points* $s_1,\dots,s_N$ *with* $a<s_1<s_2<\cdots<s_N\le b$,

$$\mathcal{V}_{\mathcal{G}}(\alpha,p;a,b)=\sum_{i=1}^{N}\mathcal{G}(\alpha(s_i),p(s_i)-p(s_{i-1}))\,,$$

*where* $s_0:=a$.

(3) *For every* $a\le t_1<t_2<t_3\le b$,

$$\mathcal{V}_{\mathcal{G}}(\alpha,p;t_1,t_3)=\mathcal{V}_{\mathcal{G}}(\alpha,p;t_1,t_2)+\mathcal{V}_{\mathcal{G}}(\alpha,p;t_2,t_3)\,.$$

(4) *Assume in addition that* $F$ *is a measurable topological space,* $X$ *is the dual of a separable Banach space* $Y$, $p\in AC([a,b];X)$, $\alpha\colon[a,b]\to F$ *is continuous for a.e.* $t\in[a,b]$, *and*

$$\mathcal{G}(\beta,tf)=t\mathcal{G}(\beta,f)\ \textit{for every}\ \beta\in F,\ f\in X,\ \textit{and}\ t>0\,,\tag{1.2.3a}$$

$$f\mapsto\mathcal{G}(\beta,f)\ \textit{is weakly}^*\ \textit{lower semicontinuous in}\ X\ \textit{for every}\ \beta\in F\,,\tag{1.2.3b}$$

$$\mathcal{G}(\beta_k,f)\to\mathcal{G}(\beta,f)\quad\textit{for every}\ \beta_k\to\beta\ \textit{in}\ F\ \textit{and}\ f\in X\,.\tag{1.2.3c}$$

*Then* $t \mapsto \mathcal{G}(\alpha(t), \dot{p}(t))$ *is measurable and*

$$\mathcal{V}_{\mathcal{G}}(\alpha, p; a, b) = \int_a^b \mathcal{G}(\alpha(t), \dot{p}(t))\,\mathrm{d}t\,. \tag{1.2.4}$$

PROOF. **Proof of (1)** It is enough to see that for every $a \le t_1 \le t_2 \le t_3 \le b$

$$\mathcal{G}(\alpha(t_3), p(t_3) - p(t_1)) \le \mathcal{G}(\alpha(t_3), p(t_3) - p(t_2)) + \mathcal{G}(\alpha(t_2), p(t_2) - p(t_1))\,.$$

This is true because, by (1.2.2c), $\mathcal{G}(\alpha(t_3), p(t_3)-p(t_1)) \le \mathcal{G}(\alpha(t_3), p(t_3)-p(t_2))+\mathcal{G}(\alpha(t_3), p(t_2)-p(t_1))$; apply then (1.2.2a) to the second term in the right-hand side.

**Proof of (2)** Observe firstly that given a partition $\mathcal{P} := \{t_i\}_{0\le i\le M}$ of $[a,b]$ it is possible to choose a set of indices $1 \le i_1 < i_2 < \cdots < i_k \le N$ such that

$$\mathcal{V}_{\mathcal{G}}^{\mathcal{P}}(\alpha, p; a, b) \le \sum_{j=1}^{k} \mathcal{G}(\alpha(s_{i_j}), p(s_{i_j}) - p(s_{i_j-1}))\,. \tag{1.2.5}$$

In fact, if $s_i \le t_j < t_{j+1} < s_{i+1}$, then

$$\mathcal{G}(\alpha(t_{j+1}), p(t_{j+1}) - p(t_j)) = \mathcal{G}(\alpha(t_{j+1}), p(s_i) - p(s_i)) = 0\,,$$

while if $s_i \le t_j < s_{i+1} < \cdots < s_{i+l} \le t_{j+1} < s_{i+l+1}$ it follows that

$$\mathcal{G}(\alpha(t_{j+1}), p(t_{j+1}) - p(t_j)) = \mathcal{G}(\alpha(t_{j+1}), p(s_{i+l}) - p(s_i)) \le \mathcal{G}(\alpha(s_{i+l}), p(s_{i+l}) - p(s_i))\,,$$

by (1.2.2b) and (1.2.2a). From (1) and (1.2.5), for every $\mathcal{P}$ partition of $[a,b]$ the inequalities

$$\mathcal{V}_{\mathcal{G}}^{\mathcal{P}}(\alpha, p; a, b) \le \sum_{i=1}^{N} \mathcal{G}(\alpha(s_i), p(s_i) - p(s_{i-1})) \le \mathcal{V}_{\mathcal{G}}(\alpha, p; a, b)$$

hold. The conclusion follows by taking the supremum over the partitions of $[a,b]$.

**Proof of (3)** It is always true that $\mathcal{V}_{\mathcal{G}}(\alpha, p; t_1, t_3) \ge \mathcal{V}_{\mathcal{G}}(\alpha, p; t_1, t_2) + \mathcal{V}_{\mathcal{G}}(\alpha, p; t_2, t_3)$ because for every partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[t_1, t_2]$ and $[t_2, t_3]$, $\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2$ is a partition of $[t_1, t_3]$. On the other hand, for every $\mathcal{P}$ partition of $[t_1, t_3]$, $\widetilde{\mathcal{P}} := \mathcal{P} \cup \{t_2\}$ is the union of two partitions of $[t_1, t_2]$ and $[t_2, t_3]$ respectively; since, by (1),

$$\mathcal{V}_{\mathcal{G}}^{\mathcal{P}}(\alpha, p; a, b) \le \mathcal{V}_{\mathcal{G}}^{\widetilde{\mathcal{P}}}(\alpha, p; a, b)\,,$$

the latter inequality holds.

**Proof of (4)** From (1.2.2c), (1.2.3a), and (1.2.3b), we have that for every $\beta \in F$ the function $f \mapsto \mathcal{G}(\beta, f)$ is weakly$^*$ lower semicontinuous, convex and positively one-homogeneous. Then, by [**55**, Theorem 5], for every $\beta \in F$ there exists a bounded closed convex set $\mathcal{K}_\beta \subset Y$ such that

$$\mathcal{G}(\beta, f) = \sup_{y\in\mathcal{K}_\beta} \langle y, f\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $X$ and $Y$. Being $Y$ separable, we get $\mathcal{G}(\beta, f) = \sup_{y\in\mathcal{K}_\beta^0}\langle y, f\rangle$, where $\mathcal{K}_\beta^0$ is a countable dense subset of $\mathcal{K}_\beta$.

Since $p \in AC([a,b];X)$, the weak$^*$-limit

$$\dot{p}(t) := w^*\text{-}\lim_{s\to t} \frac{p(s)-p(t)}{s-t}$$

exists for a.e. $t \in [a,b]$, and then the function $t \mapsto \langle y, \dot{p}(t)\rangle$ is measurable for every $y \in Y$. Therefore $t \mapsto \mathcal{G}(\beta, \dot{p}(t))$ is measurable for every $\beta \in F$. Moreover, from [**28**, Theorem 7.1],

$$\widehat{\mathcal{V}}_{\mathcal{G}}(\beta, p; t_1, t_2) = \int_{t_1}^{t_2} \mathcal{G}(\beta, \dot{p}(t))\,\mathrm{d}t\,, \tag{1.2.6}$$

for every $a \le t_1 < t_2 \le b$ and every $\beta \in F$.

Let us fix $\varepsilon > 0$. There exist points $t_0, \dots, t_N$, with $a = t_0 < t_1 < t_2 < \dots < t_N \le b$, such that

$$\mathcal{V}_{\mathcal{G}}(\alpha, p; 0, t) - \varepsilon \le \sum_{i=1}^{N} \mathcal{G}(\alpha(t_i), p(t_i) - p(t_{i-1}))\,. \tag{1.2.7}$$

For every $k \in \mathbb{N}$ we consider the set $\big(a + i\frac{b-a}{k}\big)_{i=0}^{k} \cup (t_j)_{j=1}^{N} =: s_0^k < s_1^k < \dots < s_{M(k)}^K$, with $s_0^k = a$, and we define $\alpha_k$ as

$$\alpha_k(t) := \alpha(s_{j+1}) \quad \text{when } t \in (s_j, s_{j+1}]$$

and $\alpha_k(a) = \alpha(a)$. In other words $\alpha_k$ is the left-continuous piecewise constant interpolation of $\alpha$ with nodes $(s_j)_j$. By construction

$$\alpha_k(t_j) = \alpha(t_j) \quad \text{for every } j \in \{1, \dots, N\} \tag{1.2.8}$$

and by (1.2.2a) and (1.2.3c) we get that for every $f \in X$

$$\mathcal{G}(\alpha_k(s), f) \le \mathcal{G}(\alpha_{k+1}(s), f) \le \mathcal{G}(\alpha(s), f) \tag{1.2.9}$$

for every $s \in [a,b]$, and

$$\mathcal{G}(\alpha_k(s), f) \to \mathcal{G}(\alpha(s), f) \tag{1.2.10}$$

for every $s$ continuity point of $\alpha$.

Since the functions $\alpha_k$ are piecewise constant, from the point (3) and (1.2.6) we have that

$$\begin{aligned}\mathcal{V}_{\mathcal{G}}(\alpha_k, p; a, b) &= \sum_{j=1}^{M(k)} \mathcal{V}_{\mathcal{G}}(\alpha_k, p; s_{j-1}^k, s_j^k) = \sum_{j=1}^{M(k)} \widehat{\mathcal{V}}_{\mathcal{G}}(\alpha_k(s_j^k), p; s_{j-1}^k, s_j^k)\\ &= \sum_{j=1}^{M(k)} \int_{s_{j-1}}^{s_j} \mathcal{G}(\alpha_k(s_j^k), \dot{p}(t))\,\mathrm{d}t = \int_a^b \mathcal{G}(\alpha_k(t), \dot{p}(t))\,\mathrm{d}t\,.\end{aligned} \tag{1.2.11}$$

Moreover the fact that $\alpha$ is continuous for a.e. $t \in [a,b]$ and (1.2.10) imply that

$$t \mapsto \mathcal{G}(\alpha(t), \dot{p}(t)) \text{ is measurable}\,,$$

as well as

$$\int_a^b \mathcal{G}(\alpha(t), \dot{p}(t))\,\mathrm{d}t = \lim_{k\to\infty} \int_a^b \mathcal{G}(\alpha_k(t), \dot{p}(t))\,\mathrm{d}t\,, \tag{1.2.12}$$

using the Monotone Convergence Theorem.

By (1.2.7), (1.2.8), and (1.2.9) we obtain

$$\mathcal{V}_{\mathcal{G}}(\alpha, p; a, b) - \varepsilon \leq \sum_{i=1}^{N} \mathcal{G}(\alpha_k(t_i), p(t_i) - p(t_{i-1})) \leq \mathcal{V}_{\mathcal{G}}(\alpha_k, p; a, b) \leq \mathcal{V}_{\mathcal{G}}(\alpha, p; a, b)\,,$$

and using (1.2.11) and (1.2.12) we can pass to the limit as $k \to \infty$ and get

$$\mathcal{V}_{\mathcal{G}}(\alpha, p; a, b) - \varepsilon \leq \int_a^b \mathcal{G}(\alpha(t), \dot{p}(t))\,\mathrm{d}t \leq \mathcal{V}_{\mathcal{G}}(\alpha, p; a, b)\,.$$

We therefore conclude since $\varepsilon$ is arbitrary. □

**A remark about monotone functions from time into $L^p$ spaces.**

LEMMA 1.2.2. *Let $(X, \mu)$ a measure space with $\mu(X) < \infty$, and $\alpha\colon [0,T] \to L^\infty(X, \mu)$ such that $\|\alpha(t)\|_\infty \leq M$ for every $t \in [0,T]$ and*

$$\alpha(t_2) \leq \alpha(t_1)\; \mu\text{-a.e. in } X \text{ for every } t_1 \leq t_2\,. \tag{1.2.13}$$

*Then there exists a countable set $E \subset [0,T]$ such that for every $1 \leq p < \infty$ the function $\alpha$ is continuous in every $t \in [0,T] \setminus E$ with respect to the $L^p(X, \mu)$ norm.*

PROOF. For every $s \in (0,T]$ and $t \in [0,T)$ we define

$$\alpha^-(s) := \inf_{n\in\mathbb{N}} \alpha(t_n^-), \quad \alpha^+(t) := \sup_{n\in\mathbb{N}} \alpha(t_n^+)\,,$$

where $t_n^- < s$ and $t < t_n^+$ are sequences in $[0,T]$ convergent to $s$ and $t$, and

$$\alpha^-(0) := \alpha(0), \quad \alpha^+(T) := \alpha(T)\,.$$

By (1.2.13) these definitions are well posed. Indeed, let for instance $t < s_n^+$ be a sequence that converges to $t$, and $\widetilde{\alpha}(t^+) := \sup_{n\in\mathbb{N}} \alpha(s_n^+)$. For every $m \in \mathbb{N}$, there exists $n_m$ such that $t < s_n^+ \leq t_m^+$ for every $n \geq n_m$: therefore $\widetilde{\alpha}(t^+) \geq \alpha(s_n^+) \geq \alpha(t_m^+)$ for every $m$, and $\widetilde{\alpha}(t^+) \geq \alpha(t^+)$, taking the supremum over $m$. The opposite inequality follows by interchanging the two sequences. Moreover for every $t \in [0,T]$

$$\alpha(t_n^+) \to \alpha^+(t), \quad \alpha(t_n^-) \to \alpha^-(t) \quad \text{strongly in } L^p(X, \mu)\,, \tag{1.2.14}$$

by Monotone Convergence Theorem and (1.2.13) again, and

$$\alpha^-(t) \geq \alpha(t) \geq \alpha^+(t)\,,$$

for every $t \in [0,T]$. Let us consider now the function

$$g(t) := \int_X \big(\alpha^-(t) - \alpha^+(t)\big)\,\mathrm{d}\mu\,.$$

It takes values in $\mathbb{R}^+ \cup \{0\}$ and for every $t_1 < \cdots < t_k \in E := \{t \in [0,T] |\, g(t) > 0\}$ we get, using in particular (1.2.13), that

$$\sum_{i=1}^{k} g(t_i) \leq \int_X \big(\alpha^-(t_1) - \alpha^+(t_k)\big)\,\mathrm{d}\mu \leq 2M\mu(X)\,.$$

By a standard argument, we deduce that $E$ is a countable set. By definition of $E$, $\alpha^+(t) = \alpha^-(t) = \alpha(t)\,\mu$-a.e. for every $t \in [0,T] \setminus E$ and we conclude by (1.2.14). □

**A compactness result.** If $X$ is a reflexive space it is well known that $L^2(0,T;X)$ is isomorphic to the dual space of $L^2(0,T;X')$, where $X'$ is the dual space of $X$. We now consider the case when $X$ is only the dual of a separable Banach space $Y$: every function in $L^2(0,T;X)$ is in the dual of $L^2(0,T;Y)$ but the limit (in the sense of the dual of $L^2(0,T;Y)$) of a converging sequence in $L^2(0,T;X)$ could be *weakly$^*$ measurable* but not strongly measurable.

A function $f\colon (0,T)\to X$ is said weakly$^*$ measurable if $(0,T)\ni t\mapsto \langle f(t),g\rangle$ is measurable for every $g\in Y$. Let us denote

$$L^2_w(0,T;X) := \{p\colon [0,T]\to X \text{ weakly}^* \text{ measurable} : t\mapsto \|p(t)\|\in L^2(0,T)\}.$$

Adapting the proof of [**109**, Theorem IV.1.8] we can see that there is an algebraic isomorphism $\mathcal{I}$ between the dual space of $L^2(0,T;Y)$ and $L^2_w(0,T;X)$ given, for every $p\in L^2_w(0,T;X)$ and $\varphi\in L^2(0,T;Y)$, by

$$\mathcal{I}(p)(\varphi) := \int_0^T \langle p(t),\, \varphi(t)\rangle\,\mathrm{d}t, \text{ with } \|\mathcal{I}(p)\|^2 = \int_0^T \|p(t)\|^2\,\mathrm{d}t\,.$$

This defines the weak$^*$ convergence in $L^2_w(0,T;X)$. In the following we study the space of functions with distributional time derivative in $L^2_w(0,T;X)$. In Section 3.2 the lemma below is applied to the case of $X=M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$ and $Y=C_0(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$. Notice that $Y$ can be identified with the space of functions in $C(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ vanishing on $\overline{\partial_N\Omega}$.

LEMMA 1.2.3. *Let $X$ be the dual space of a separable Banach space $Y$ and let*

$$H^1_w(0,T;X) := \Big\{p\in L^2_w(0,T;X) : \exists\,\widehat{p}\in L^2_w(0,T;X) \text{ s.t. for every } \varphi\in C^1_c((0,T);Y)$$
$$\int_0^T \langle p(t),\, \partial_t\varphi(t)\rangle\,\mathrm{d}t = -\int_0^T \langle \widehat{p}(t),\, \varphi(t)\rangle\,\mathrm{d}t\Big\}. \tag{1.2.15}$$

*Then every $p\in H^1_w(0,T;X)$ admits a unique representative absolutely continuous into $X$, its distributional derivative $\widehat{p}$ is characterized by*

$$\widehat{p}(t) = w^*\text{-}\lim_{s\to t}\frac{p(s)-p(t)}{s-t} =: \dot p(t) \text{ for a.e. } t\in(0,T)\,, \tag{1.2.16}$$

*and*

$$\|p\|_{C^{0,1/2}([0,T];X)} \le C\big(\|p(\cdot)\|_2 + \|\dot p(\cdot)\|_2\big)\,, \tag{1.2.17}$$

*with $C$ independent of $p\in H^1_w(0,T;X)$.*

*Moreover, for every sequence $\{p_k\}_k\subset H^1_w(0,T;X)$ with $\|p_k(\cdot)\|_2+\|\dot p_k(\cdot)\|_2\le C$ for every $k$, there exists a function $p\in H^1_w(0,T;X)$ such that, up to a subsequence,*

$$p_k(t)\overset{*}{\rightharpoonup} p(t) \text{ weakly}^* \text{ in } X \text{ for every } t\in[0,T]\,,\quad \dot p_k\overset{*}{\rightharpoonup}\dot p \text{ weakly}^* \text{ in } L^2_w(0,T;X)\,.$$

PROOF. Let $\rho$ be the standard mollifier in $\mathbb{R}$ and $\rho_k(t) := k\,\rho(\frac{t}{k})$. For every $t_1\le t_2\in[0,T]$, $\psi\in Y$, we take in (1.2.15) $\varphi_k(t)=\psi\,\omega_k(t)$, where $\omega_k$ is the convolution product between $\rho_k$ and the indicator function of $[t_1,t_2]$, and let $k$ tend to $+\infty$. Then we get that for every $p\in H^1_w(0,T;X)$

$$\langle p(t_2)-p(t_1),\psi\rangle = \int_{t_1}^{t_2}\langle\widehat{p}(s),\psi\rangle\,\mathrm{d}s\,. \tag{1.2.18}$$

Since $\int_{t_1}^{t_2} \langle \widehat{p}(s), \psi \rangle \, \mathrm{d}s \leq \int_{t_1}^{t_2} \|\widehat{p}(s)\| \, \mathrm{d}s$ for every $\|\psi\| \leq 1$, it follows that

$$\|p(t_2) - p(t_1)\| \leq \int_{t_1}^{t_2} \|\widehat{p}(s)\| \, \mathrm{d}s \,, \tag{1.2.19}$$

and then $p$ is absolutely continuous, $s \mapsto \|\widehat{p}(s)\|$ being in $L^2$. Then [**28**, Lemma 7.1] implies that for a.e. $t \in (0,T)$ the weak$^*$ limit $\dot{p}(t)$ defined in (1.2.16) exists. Let us now consider the function $h(t) := \|p(t)\|$: we have

$$|h(t) - h(s)| \leq \|p(t) - p(s)\| \,,$$

and therefore, by (1.2.19) and the Hölder inequality, $h \in H^1(0,T)$ and $|\dot{h}(t)| \leq \|\dot{p}(t)\|$ for a.e. $t \in (0,T)$. From the Sobolev embedding theorem for real valued functions (1.2.17) follows.

By (1.2.18) and a standard argument that uses the separability of $Y$, we obtain that for a.e. $t \in (0,T)$ it holds

$$\lim_{s \to t} \Big\langle \frac{p(s) - p(t)}{s - t}, \psi \Big\rangle = \langle \widehat{p}(t), \psi \rangle \text{ for every } \psi \in Y \,,$$

and then (1.2.16) follows.

By (1.2.17), every sequence $\{p_k\}_k$ as in the statement is equibounded in $C^{0,1/2}([0,T];X)$, and in particular $\|p_k(t)\| \leq M$ for every $k$ and $t$. It is now well known that, since $Y$ is separable, there exists a distance $\mathrm{d}_M$ on $B_M$, the ball of $X$ with radius $M$ centered in the origin, inducing the weak$^*$ convergence, and the metric space $(B_M, \mathrm{d}_M)$ is complete. Then the Arzelà-Ascoli Theorem implies that there exists $p \in C^{0,1/2}([0,T];X)$ such that, up to a subsequence,

$$p_k(t) \overset{*}{\rightharpoonup} p(t) \text{ in } X, \text{ for every } t \in [0,T] \,.$$

Since $\|\dot{p}_k(\cdot)\|_2 \leq C$, there exists $q \in L^2_w(0,T;X)$ such that, up to a subsequence,

$$\dot{p}_k \rightharpoonup q \text{ weakly}^* \text{ in } L^2_w(0,T;X) \,.$$

This implies that for every $\varphi \in C_c^\infty((0,T);Y)$

$$\int_0^T \langle q(t), \, \varphi(t) \rangle \, \mathrm{d}t = -\int_0^T \langle p(t), \, \partial_t \varphi(t) \rangle \, \mathrm{d}t \,,$$

and therefore $q = \dot{p}$. This concludes the proof. □

**A generalization of the Riesz Representation Theorem.** The following lemma is a generalization of the Riesz Representation Theorem for bounded linear functionals acting on the space of continuous functions. It is employed in Lemma (3.2.5).

LEMMA 1.2.4. *Let $B$ be an open bounded subset of $\mathbb{R}^n$, and let $S$ be a distribution on $B$ such that*

$$\langle S, \beta \rangle \leq C \|\beta\|_p \quad \textit{for every } \beta \in C_c^\infty(B) \,, \tag{1.2.20}$$

*with $C > 0$ and $p \in [1, \infty)$. Then there exists a unique pair $(g, \mu)$ such that $g \in L^{p'}(B)$, with $\frac{1}{p'} + \frac{1}{p} = 1$, $g \geq 0$, $\mu \in M^+(B)$ (namely $\mu$ is a nonnegative measure on $B$), $g \, \mathrm{d}x$ and $\mu$ are mutually singular, and*

$$\langle S, \beta \rangle = \int_B g \, \beta \, \mathrm{d}x - \int_B \beta \, \mathrm{d}\mu \quad \textit{for every } \beta \in C_c^\infty(B) \,. \tag{1.2.21}$$

Proof. In the following we will use the notation $C_0^+(B) := \{\beta \in C_0(B) :\ \beta \geq 0\}$, and the analogous for $C_0^-(B)$.

Recall that every $\beta \in C_0^+(B)$ can be approximated uniformly (and thus in $L^p$-norm) from below in $C_c^\infty(B) \cap C_0^+(B)$. We define

$$\langle S^+, \beta\rangle := \sup_{\substack{\varphi \in C_c^\infty(B) \\ 0 \leq \varphi \leq \beta}} \langle S, \varphi\rangle \quad \text{for every } \beta \in C_0^+(B)\,, \tag{1.2.22}$$

which satisfies

$$0 \leq \langle S^+, \beta\rangle \leq C\|\beta\|_p$$

for every $\beta \in C_0^+(B)$ by (1.2.20). Following [**94**, Proposition 24], we extend $S^+$ by setting

$$\langle S^+, \beta\rangle := -\langle S^+, -\beta\rangle \quad \text{for every } \beta \in C_0^-(B)$$

and we see that the functional $S^+$ is linear and positive on $C_0(B)$. Moreover

$$|\langle S^+, \beta\rangle| = |\langle S^+, \beta^+\rangle - \langle S^+, \beta^-\rangle| \leq 2C\|\beta\|_p \quad \text{for every } \beta \in C_0(B)\,,$$

and thus there exists $g \in L^{p'}(B)$ such that

$$\langle S^+, \beta\rangle = \int_B g\,\beta\,\mathrm{d}x \quad \text{for every } \beta \in C_0(B)\,. \tag{1.2.23}$$

Since $\langle S^+, \beta\rangle \in \mathbb{R}$ for every $\beta$, the distribution

$$\langle S^-, \beta\rangle := \langle S^+, \beta\rangle - \langle S, \beta\rangle \quad \text{for every } \beta \in C_c^\infty(B) \tag{1.2.24}$$

is well defined and by (1.2.22) we obtain

$$\langle S^-, \beta\rangle \geq 0 \quad \text{for every } C_c^\infty(B) \cap C_0^+(B)\,.$$

It is well known from the theory of distributions that there exists a nonnegative measure $\mu \in M^+(B)$ such that

$$\langle S^-, \beta\rangle = \int_B \beta\,\mathrm{d}\mu \quad \text{for every } C_c^\infty(B)\,. \tag{1.2.25}$$

Collecting (1.2.23), (1.2.24), and (1.2.25) we find that $g$ and $\mu$ satisfy the properties as in the statement. Since every measure is uniquely decomposed into a nonnegative and a nonpositive part, the uniqueness of $g$ and $\mu$ follows. Thus the proof is concluded. $\square$

CHAPTER 2

# Globally stable evolution for perfect plasticity coupled with damage

### Overview of the chapter

In this chapter we study a model for the interplay between linearized perfect plasticity [**28**] and damage [**78**], based on global stability. As already discussed in the Introduction, to which we refer in this overwiew, the main result is the existence of a quasistatic evolution for the model, in the framework of energetic solutions. The evolution is characterized by the conditions $(\mathrm{qs0})^G$, $(\mathrm{qs1})^G$, $(\mathrm{qs2})^G$ (that throughout the chapter are denoted without "$G$"), and under regularity assumptions it satisfies (sf1),... ,(sf5), (sf6'). Notice that in this chapter we assume for simplicity that the external forces are null, namely that the evolution is driven only by the prescribed displacement on the Dirichlet boundary. The issue of dealing with a general external loading is usually addressed by imposing some conditions on the forces, called *safe load conditions*; the techniques are well-established and are employed in Chapter 4. The results of this chapter are published in [**23**].

The chapter contains four sections: firstly, the main objects employed in the coupling between perfect plasticity and damage are introduced; the second section includes the results needed to solve the incremental minimum problems and to assure convergence of the stability properties in the continuous time limit; the third one is devoted to prove the existence result; in the last section we show qualitative properties of the evolution.

## 2.1. Mechanical assumptions for perfect plasticity coupled with damage

The object introduced below and the corresponding properties are useful not only in this chapter, but also in Chapter 3. In this introductive section, only the common features will be described. Actually, the hypoteses in the two models are slightly different, so that in Section 2.2 the additional hypoteses for the approach based on global minimization are considered, whilst in Chapter 3 the additional assumptions (with respect to those one introduced here) for the vanishing viscosity approach will be specified in the introductory section.

**The body and its displacement.** We consider an elastoplastic body whose *reference configuration* is a bounded, connected, open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N$. We assume that $\partial_D\Omega$ and $\partial_N\Omega$ are relatively open, $\partial_D\Omega \cap \partial_N\Omega = \emptyset$, $\mathcal{H}^{n-1}(N) = 0$, and

$$\partial_D\Omega \neq \emptyset\,. \qquad (2.1.1)$$

We assume that the Kohn-Temam condition is satisfied (topological notions refer here to the relative topology of $\partial\Omega$):

$$\begin{aligned} \partial(\partial_D\Omega) = \partial(\partial_N\Omega) \quad & \text{is a } (n-2)\text{-dimensional } C^2 \text{ manifold,} \\ \partial\Omega \quad & \text{is } C^2 \text{ in a neighborhood of } \partial(\partial_D\Omega) = \partial(\partial_N\Omega)\,. \end{aligned} \tag{2.1.2}$$

The only role of this condition is to assure (2.1.21) below; another sufficient condition for (2.1.21) is for instance the one considered in [**43**, Theorem 6.6].

The *displacement* of the body is represented by a function $u \in BD(\Omega)$, so $Eu$ is the corresponding linearized *strain*.

We study the evolution of the body under time-dependent external loading. Here we consider only Dirichlet boundary conditions on $\partial_D\Omega$: such a choice notably simplifies the exposition. For including forces in a related model we refer to e.g. [**29**] and [**99**]. The prescribed boundary displacement is extended into the domain $\Omega$; at every time it is thus a function in $H^1(\mathbb{R}^n;\mathbb{R}^n)$, whose trace on $\partial_D\Omega$ is the prescribed boundary value. For the time regularity of the boundary condition, see (2.1.27).

**The elastic and the plastic strain.** Given a displacement $u \in BD(\Omega)$ and a boundary datum $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, the *elastic* and the *plastic strain*, denoted by $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p \in M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)$, respectively, are assumed to satisfy the following *weak kinematic compatibility conditions*

$$\mathrm{E}u = e + p \text{ in } \Omega\,, \tag{2.1.3a}$$

$$p = (w-u)\odot\nu\,\mathcal{H}^{n-1} \text{ on } \partial_D\Omega\,. \tag{2.1.3b}$$

The *set of admissible displacements and strains* for a given boundary datum $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$ on $\partial_D\Omega$ is

$$A(w) := \{(u,e,p) \in BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D) \colon (2.1.3) \text{ holds}\}\,.$$

The *space of admissible plastic strains* is defined by

$$\begin{aligned} \Pi(\Omega) := \{p \in M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D) \colon \exists\,(u,w,e) \in BD(\Omega)\times H^1(\mathbb{R}^n;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \\ \text{s.t. } (u,e,p) \in A(w)\}\,. \end{aligned} \tag{2.1.4}$$

**The damage variable and the associated dissipation.** Following Frémond's concept [**45**], the damage state of the body is represented by an internal variable $\alpha\colon \Omega \to [0,1]$, where $\alpha = 1$ marks the sound material and $\alpha = 0$ the most damaged state. Due to the regularizing term in Chapters 2 and 3, the damage will be uniformly continuous on $\Omega$ at each time instant. Therefore, from now on we define all energy terms for $\alpha \in C(\overline{\Omega})$.

Given $\alpha_0 \in C(\overline{\Omega})$, we denote the *admissible damage states* by

$$\mathcal{D}(\alpha_0) := \{\alpha \in C(\overline{\Omega}) \colon 0 \le \alpha \le \alpha_0 \text{ in } \overline{\Omega}\}\,,$$

so that

$$\mathcal{D}(\alpha_2) \subset \mathcal{D}(\alpha_1) \text{ for every } \alpha_2 \in \mathcal{D}(\alpha_1)\,.$$

Irreversibility is formulated in the following way: if $\alpha_0$ is the current damage state, then all future damage states are in $\mathcal{D}(\alpha_0)$. The total energy includes the energy dissipated by the body during the damage process. As we have seen in the Introduction, this can be related to a positive one-homogeneous dissipation $\mathcal{R}$. Then, for $\alpha \in C(\overline{\Omega})$, we define

$$D(\alpha) := \int_\Omega d(\alpha(x))\,\mathrm{d}x\,, \tag{2.1.5}$$

The hypotheses on $d$ will be such that the damage will be constrained to assume positive values (cf. (2.2.3) and (3.0.19)).

**The stored elastic energy.** For $(\alpha, e) \in C(\overline{\Omega};[0,1]) \times L^2(\Omega, \mathbb{M}^{n\times n}_{sym})$, the *stored elastic energy* is

$$\mathcal{Q}(\alpha, e) := \frac{1}{2}\int_\Omega \mathbb{C}(\alpha(x))e(x)\cdot e(x)dx = \tfrac{1}{2}\langle \mathbb{C}(\alpha)e, e\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{sym})}\,. \tag{2.1.6}$$

Following [**28**], [**43**], and [**99**], we assume that

$$\mathbb{C}(\alpha)\xi := \mathbb{C}_D(\alpha)\xi_D + \varsigma(\alpha)(\mathrm{tr}\,\xi)I\,, \tag{2.1.7}$$

where $\mathbb{C}_D \in L^\infty([0,1]; Sym(\mathbb{M}^{n\times n}_D;\mathbb{M}^{n\times n}_D))$, $\varsigma \in L^\infty([0,1])$, and

$$\mathbb{C}\colon [0,1] \to Lin(\mathbb{M}^{n\times n}_{sym};\mathbb{M}^{n\times n}_{sym}) \text{ is Lipschitz}\,, \tag{2.1.8a}$$

$$\alpha \mapsto \mathbb{C}(\alpha)\xi\cdot\xi \text{ is nondecreasing for every } \xi \in \mathbb{M}^{n\times n}_{sym}\,, \tag{2.1.8b}$$

$$\gamma_1|\xi|^2 \le \mathbb{C}(\alpha)\xi\cdot\xi \le \gamma_2|\xi|^2 \quad \text{for every } \alpha\in[0,1], \xi \in \mathbb{M}^{n\times n}_{sym}\,, \tag{2.1.8c}$$

where $\gamma_1$, $\gamma_2$ are positive constants independent of $\alpha$ and $Sym(\mathbb{M}^{n\times n}_D;\mathbb{M}^{n\times n}_D)$ is the set of symmetric endomorphisms on $\mathbb{M}^{n\times n}_D$. In particular, this implies

$$|\mathbb{C}(\alpha)\xi| \le 2\gamma_2|\xi|\,. \tag{2.1.9}$$

Assumption (2.1.8b) reflects the fact that the stiffness decreases as the material passes from the sound to the fully damaged state; at this last stage there is still elastic elastic response, by (2.1.8c), and thus the material is not completely damaged. Given $\alpha \in C(\overline{\Omega};[0,1])$, it is well known that the function $e \mapsto \mathcal{Q}(\alpha, e)$ is weakly lower semicontinuous on $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$.

In fact, (2.1.7) is not needed to prove existence results in plasticity, see e.g. [**102**]. Nevertheless, (2.1.7) is assumed for mechanical reasons, since purely volumetric deformations do not affect plastic behavior.

**The constraint sets and their support functions.** The dissipation related to plasticity is defined through the so-called constraint sets, which are subsets of $\mathbb{M}^{n\times n}_D$ containing the admissible stresses. The coupling between damage and plasticity is reflected in the dependence of such sets on the damage variable. In a softening framework, we require the constraint sets $(K(\alpha))_{\alpha\in[0,1]}$ to fulfill the following conditions:

$$K(\alpha) \subset \mathbb{M}^{n\times n}_D \text{ is closed and convex for every } \alpha\in[0,1]\,, \tag{2.1.10a}$$

$$B_{\widehat{r}}(0) \subset K(\alpha_1) \subset K(\alpha_2) \subset B_{\widehat{R}}(0) \text{ for every } 0 \le \alpha_1 \le \alpha_2 \le 1\,, \tag{2.1.10b}$$

$$U \subset \mathbb{M}^{n\times n}_D \text{ open } \Longrightarrow \{\alpha\in[0,1] \,|\, K(\alpha)\cap U \neq \emptyset\} \text{ and } \{\alpha\in[0,1]\,|\,K(\alpha)\subset U\} \text{ relatively open} \tag{2.1.10c}$$

with $0 < r < R$.

Let us consider the function $H\colon [0,1] \times \mathbb{M}^{n\times n}_D \to \mathbb{R}^+ \cup \{0\}$ defined by

$$H(\alpha, \xi) := \sup_{\sigma \in K(\alpha)} \sigma \cdot \xi \quad \text{for every } \alpha \in [0,1]\,,$$

namely $\xi \mapsto H(\alpha, \xi)$ is the support function of $K(\alpha)$. Arguing as in [**98**, Proposition 2.4], we can show that (2.1.10c) implies that

$$\alpha \mapsto H(\alpha, \xi) \text{ is continuous for every } \xi \in \mathbb{M}^{n\times n}_D\,. \tag{2.1.11}$$

Then we get, from (2.1.10), that the four conditions below are simultaneously satisfied:

$$H \text{ is continuous}\,, \tag{2.1.12a}$$

$$\alpha \mapsto H(\alpha, \xi) \text{ is nondecreasing for every } \xi \in \mathbb{M}^{n\times n}_D\,, \tag{2.1.12b}$$

$$\xi \mapsto H(\alpha, \xi) \text{ is convex and positively one-homogeneous for every } \alpha \in [0,1]\,, \tag{2.1.12c}$$

$$r|\xi| \le H(\alpha, \xi) \le R|\xi| \text{ for every } \alpha \in [0,1] \text{ and every } \xi \in \mathbb{M}^{n\times n}_D\,. \tag{2.1.12d}$$

Indeed, by [**55**, Theorem 5], we have that (2.1.10a) and (2.1.10b) are equivalent to (2.1.12b), (2.1.12c), and (2.1.12d). Since the functions $\xi \mapsto H(\alpha, \xi)$ are convex with respect to $\xi$ for every $\alpha$ and locally equi-bounded with respect to $\alpha$ by (2.1.12d), condition (2.1.10) is equivalent to (2.1.12a).

**The plastic potential.** The *plastic potential* $\mathcal{H}\colon C(\overline{\Omega}; [0,1]) \times M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D) \to \mathbb{R}$ is defined by

$$\mathcal{H}(\alpha, p) := \int_{\Omega \cup \partial_D \Omega} H\left(\alpha(x), \frac{\mathrm{d}p}{\mathrm{d}\mu}(x)\right) \mathrm{d}\mu(x)\,, \tag{2.1.13}$$

where $\mu \in M_b(\Omega \cup \partial_D\Omega)^+$ is a measure such that $p \ll \mu$ and $\frac{\mathrm{d}p}{\mathrm{d}\mu}$ is the Radon-Nikodym derivative of $p$ with respect to $\mu$; since $H(\alpha(x), \cdot)$ is one-homogeneous, the definition is actually independent of $\mu$. We refer to [**49**] for the theory of convex functions of measures. By [**6**, Proposition 2.37]

$$p \mapsto \mathcal{H}(\alpha, p) \text{ is convex and positively one-homogeneous for every } \alpha \in C(\overline{\Omega}; [0,1])\,.$$

In particular,

$$\mathcal{H}(\alpha, p_1 + p_2) \le \mathcal{H}(\alpha, p_1) + \mathcal{H}(\alpha, p_2) \tag{2.1.14}$$

for every $\alpha \in C(\overline{\Omega}; [0,1])$ and $p_1, p_2 \in M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$. Since

$$\left|\frac{\mathrm{d}p}{\mathrm{d}|p|}(x)\right| = 1 \quad \text{for } |p|\text{-a.e. } x \in \Omega \cup \partial_D\Omega\,, \tag{2.1.15}$$

by (2.1.12d) we have

$$r\|p\|_1 \le \mathcal{H}(\alpha, p) \le R\|p\|_1\,. \tag{2.1.16}$$

Moreover, by continuity of $H$, there exists a modulus of continuity $\omega$, namely an increasing function defined on $\mathbb{R}^+ \cup \{0\}$ which vanishes at $0$, such that

$$|H(\alpha_1(x), \xi) - H(\alpha_2(x), \xi)| \le \omega(|\alpha_1(x) - \alpha_2(x)|)\,, \tag{2.1.17}$$

for every $\alpha_1$, $\alpha_2 \in C(\overline{\Omega})$, $x \in \Omega$, and $\xi \in \mathbb{M}^{n\times n}_D$ with $|\xi| = 1$. Then, from (2.1.15) we obtain

$$|\mathcal{H}(\alpha_2, p) - \mathcal{H}(\alpha_1, p)| \leq \omega(\|\alpha_1 - \alpha_2\|_\infty)\|p\|_1 \tag{2.1.18}$$

for every $\alpha_1$, $\alpha_2 \in C(\overline{\Omega})$.

LEMMA 2.1.1. *Let $\alpha_k$ and $p_k$ be sequences in $C(\overline{\Omega})$ and $M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ such that $\alpha_k \to \alpha$ uniformly and $p_k \rightharpoonup p$ weakly* in $M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$. Then*

$$\mathcal{H}(\alpha, p) \leq \liminf_{k\to\infty} \mathcal{H}(\alpha_k, p_k)\,. \tag{2.1.19}$$

PROOF. From (2.1.18) we obtain

$$\mathcal{H}(\alpha_k, p_k) \geq \mathcal{H}(\alpha, p_k) - \omega(\|\alpha_k - \alpha\|_\infty)\|p_k\|_1\,.$$

The lower semicontinuity result follows now from the weak* convergence of $p_k$ and Reshetnyak's Lower Semicontinuity Theorem (see [**91**, Theorem 2]). □

**Stress-strain duality.** We now recall the notion of stress-strain duality, basing on [**64**], [**28**], and the more recent extension to Lipschitz boundaries [**43**], to which we refer for the properties mentioned below. We define

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})\colon \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)\,,\ \sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)\}$$

and, for $\sigma \in \Sigma(\Omega)$ and $p \in \Pi(\Omega)$,

$$\langle[\sigma_D : p], \varphi\rangle := -\int_\Omega \varphi\sigma\cdot(e - Ew)\,\mathrm{d}x - \int_\Omega \sigma\cdot[(u-w)\odot\nabla\varphi]\,\mathrm{d}x - \int_\Omega \varphi(\operatorname{div}\sigma)\cdot(u-w)\,\mathrm{d}x \tag{2.1.20}$$

for every $\varphi \in C^\infty_c(\mathbb{R}^n)$, where $u$ and $e$ are such that $(u, e, p) \in A(w)$. (The definition is indeed independent of $u$ and $e$.) Under the previous assumptions $\sigma \in L^r(\Omega; \mathbb{M}^{n\times n}_{sym})$ for every $r < \infty$, $u \in L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, and $[\sigma_D : p]$ is a bounded Radon measure such that $\|[\sigma_D : p]\|_1 \leq \|\sigma_D\|_\infty\|p\|_1$ in $\mathbb{R}^n$. Using the restriction to $\Omega \cup \partial_D\Omega$, we also define

$$\langle\sigma_D \,|\, p\rangle := [\sigma_D : p](\Omega \cup \partial_D\Omega)\,.$$

For $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^n)$, we denote by $[\sigma\nu]$ the normal trace on $\partial\Omega$, in general defined as a distribution. When $\sigma \in C(\Omega; \mathbb{M}^{n\times n}_{sym})$ we have $[\sigma\nu] = \sigma\nu$ where the right-hand side is the pointwise product between the matrix $\sigma(x)$ and the normal vector $\nu(x)$ at each $x \in \partial\Omega$. By (2.1.20), if $[\sigma\nu] \in L^\infty(\partial_N\Omega;\mathbb{R}^n)$ and (2.1.2) holds, we obtain the integration-by-parts formula

$$\langle\sigma_D \,|\, p\rangle = -\langle\sigma, e - Ew\rangle - \langle\operatorname{div}\sigma, u-w\rangle + \langle[\sigma\nu], u-w\rangle_{\partial_N\Omega} \quad \text{for every } \sigma \in \Sigma(\Omega) \text{ and } (u,e,p) \in A(w)\,. \tag{2.1.21}$$

For $\alpha \in C(\overline{\Omega})$ let

$$\mathcal{K}_\alpha(\Omega) := \{\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})\colon \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)\,,\ \sigma_D(x) \in K(\alpha(x)) \text{ for a.e. } x \in \Omega\}\,.$$

Since the multifunction $\alpha \in [0,1] \mapsto K(\alpha)$ is continuous, from [**43**, Proposition 3.9] (which holds also if $\operatorname{div}\sigma$ is not identically 0) it follows that for every $\sigma \in \mathcal{K}_\alpha(\Omega)$

$$H\Big(\alpha, \frac{\mathrm{d}p}{\mathrm{d}|p|}\Big)|p| \geq [\sigma_D : p] \quad \text{as measures on } \Omega \cup \partial_D\Omega\,, \tag{2.1.22}$$

and, arguing as in [**98**, Theorem 3.6 and Corollary 3.8], we deduce that, for every $p \in \Pi(\Omega)$

$$\mathcal{H}(\alpha, p) = \sup_{\sigma \in \mathcal{K}_\alpha(\Omega)} \langle \sigma_D \,|\, p \rangle \,. \tag{2.1.23}$$

**The plastic dissipation.** We are now in a position to define the dissipation related to plasticity. A function $p\colon [0,T] \to M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ will be regarded as a function defined on the time interval $[0,T]$ with values in the dual of the space $C_0(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$. This space can be identified with the space of functions in $C(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ vanishing on $\overline{\partial_N\Omega}$. For every $s,t \in [0,T]$ with $s \leq t$ the total variation of $p$ on $[s,t]$ is

$$\mathcal{V}(p; s, t) := \sup\left\{ \sum_{j=1}^{N} \|p(t_j) - p(t_{j-1})\|_1 : \ s = t_0 < t_1 < \cdots < t_N = t, \ N \in \mathbb{N} \right\}. \tag{2.1.24}$$

Let $\alpha\colon [0,T] \to C(\overline{\Omega}; [0,1])$. The *plastic dissipation* in the time interval $[s,t]$ is defined by

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; s, t) := \sup\left\{ \sum_{j=1}^{N} \mathcal{H}(\alpha(t_j), p(t_j) - p(t_{j-1})) : \ s = t_0 < t_1 < \cdots < t_N = t\,, \ N \in \mathbb{N} \right\}. \tag{2.1.25}$$

To ease the reading, when $\alpha \in C(\overline{\Omega}; [0,1])$ does not depend on time we use the following notation:

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha, p; s, t) := \sup\left\{ \sum_{j=1}^{N} \mathcal{H}(\alpha, p(t_j) - p(t_{j-1})) : \ s = t_0 < t_1 < \cdots < t_N = t, \ N \in \mathbb{N} \right\}. \tag{2.1.26}$$

**The prescribed boundary displacement.** The external loading will consist only in Dirichlet boundary conditions on $\partial_D\Omega$. However, similar results to those showed here hold also in the presence of external forces, under suitable regularity assumptions on $\partial\Omega$ and uniform safe load conditions, like the ones in [**29**, Section 2]. This task is addressed in Chapter 4 (see Section 4.1).

We assume that the prescribed boundary displacement $w$ depends on time and satisfies the regularity assumption

$$w \in AC([0,T]; H^1(\mathbb{R}^n; \mathbb{R}^n))\,, \tag{2.1.27}$$

so that the time derivative $t \mapsto \dot{w}(t)$ belongs to $L^1(0,T; H^1(\mathbb{R}^n; \mathbb{R}^n))$ and its strain $t \mapsto E\dot{w}(t)$ belongs to $L^1(0,T; L^2(\mathbb{R}^n; \mathbb{M}^{n\times n}_{sym}))$. For the main properties of absolutely continuous functions with values in reflexive Banach spaces we refer to [**13**, Appendix].

## 2.2. The minimization problem

The mechanical assumptions employed in this chapter have been introduced in Section 2.1. The total energy and the generalized energy account for the damage regularization and for the fatigue term. They have been already mentioned in the introduction, and called by $\mathcal{E}^G$ and $\mathcal{E}^G_\lambda$; since in this chapter these are the unique energy considered, in the following they are referred to as $\mathcal{E}$ and $\mathcal{E}_\lambda$.

In this section we study the minimization problem employed in the incremental formulation of the quasistatic evolution corresponding to a given parameter $\lambda \in [0,1]$. Therefore we deal with a problem of the type

$$\operatorname{argmin}\{\mathcal{E}_\lambda(\alpha, e; \overline{q}, t) + \mathcal{H}(\alpha, p - \overline{p})\colon (\alpha, (u,e,p)) \in \mathcal{D}(\overline{\alpha}) \times A(w)\}\,, \tag{2.2.1}$$

where

$$\mathcal{E}_\lambda(\alpha, e; \overline{q}, t) := \mathbb{Q}(\alpha, e) + D(\alpha) + \|\nabla\alpha\|_\gamma^\gamma + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha, \overline{q}; 0, t)\,. \tag{2.2.2}$$

The data are the current values $\overline{\alpha} \in W^{1,\gamma}(\Omega)$ and $\overline{p} \in \Pi(\Omega)$ of the damage variable and the plastic strain, and the updated value $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$ of the boundary displacement; if $\lambda > 0$ we consider as an additional datum a function $\overline{q}\colon [0,t] \to M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ with bounded variation, which represents the evolution of the plastic strain up to the current time $t$. Solving this problem, we get the updated values $\alpha$, $u$, $e$, and $p$ of damage, displacement, elastic and plastic strain.

First we show the existence and the main properties of the solutions to (2.2.1). The second part of the section is devoted to prove a stability property with respect to variations of the data.

Throughout this section, we suppose that (2.1.8), (2.1.10), and (2.1.27) hold when $\lambda = 0$. We assume that $\mathbb{C}(\alpha)$ and $K(\alpha)$ are defined also in $(-\infty, 0)$ and that there they take constant values $\mathbb{C}(0)$ and $K(0)$, respectively. Furthermore, we consider $d$ in (2.1.5) such that

$$d \in C(\mathbb{R}; \mathbb{R}^+ \cup \{0\}),\ d(s) > d(0) \text{ for } s < 0\,. \tag{2.2.3}$$

When $\lambda > 0$ we make the following additional assumption on $H$:

$$\xi \mapsto H(\alpha_2, \xi) - H(\alpha_1, \xi) \text{ is convex, for every } \alpha_1 \le \alpha_2\,. \tag{2.2.4}$$

Notice that, if we consider a multiplicative setting for the constraint sets, then (2.2.4) holds. In other words, (2.2.4) holds if we set

$$K(\alpha) := V(\alpha)K(1)\,, \tag{2.2.5}$$

where $B_r(0) \subset K(1) \subset B_R(0)$, $K(1)$ is closed and convex, and

$$V\colon \mathbb{R} \to [m, M] \text{ is Lipschitz, nondecreasing, and constant in } (-\infty, 0] \text{ and } [1, +\infty)$$

with $r, R, m, M$ positive constants.

Let us prove the existence of a solution to (2.2.1).

THEOREM 2.2.1 (Existence of solutions to the incremental problem). *Let $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\overline{\alpha} \in W^{1,\gamma}(\Omega)$, $\overline{p} \in \Pi(\Omega)$, and $\overline{q}\colon [0,t] \to M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ with bounded variation. Then (2.2.1) has a solution. Moreover, if $\overline{\alpha} \in W^{1,\gamma}(\Omega;[0,1])$, then for every $(\alpha,(u,e,p))$ solution of (2.2.1) we have that $\alpha \in W^{1,\gamma}(\Omega;[0,1])$.*

PROOF. Let $(\alpha_k, (u_k, e_k, p_k)) \in \mathcal{D}(\overline{\alpha}) \times A(w)$ be a minimizing sequence for problem (2.2.1).

By (2.1.8c) and (2.1.16) the sequences $\alpha_k$, $e_k$, and $p_k$ are bounded in $W^{1,\gamma}(\Omega)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$, respectively. Since $Eu_k = e_k + p_k$ in $\Omega$, it follows that $Eu_k$ is bounded in $M_b(\Omega; \mathbb{M}^{n\times n}_{sym})$. Since $(w - u_k) \odot \nu\, \mathcal{H}^{n-1} = p_k$ is bounded in $M_b(\partial\Omega; \mathbb{M}^{n\times n}_D)$, the traces of $u_k$ are bounded in $L^1(\partial\Omega;\mathbb{R}^n)$. Therefore $u_k$ is bounded in $BD(\Omega)$ by (1.1.2).

Up to extracting a subsequence, we may assume that $u_k \rightharpoonup u$ weakly* in $BD(\Omega)$, $e_k \rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $p_k \rightharpoonup p$ weakly* in $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$. By [**28**, Lemma 2.1], we have $(u,e,p)\in A(w)$.

The existence of solutions to (2.2.1) now follows from the lower semicontinuity of $\mathcal{H}$ (see (2.1.19)), which in turns imply the lower semicontinuity of $\mathcal{E}_\lambda$. Notice that if $\alpha\neq\alpha^+ := \alpha\vee 0$ then

$$\mathcal{E}_\lambda(\alpha^+,e;\overline{q},t)=\mathcal{E}(\alpha^+,e)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha^+,\overline{q};0,t)<\mathcal{E}_\lambda(\alpha,e;\overline{q},t)=\mathcal{E}(\alpha,e)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha,\overline{q};0,t)\,,$$

and this is enough to conclude that $\alpha$ takes values in $[0,1]$ if $\overline{\alpha}\in W^{1,\gamma}(\Omega;[0,1])$. □

The following lemma is not only useful in Lemma 2.2.3 below, but also in the proof of the stability for the approximate solutions in Theorem 2.3.3, when $\lambda=0$.

LEMMA 2.2.2. *If* $(\alpha,(u,e,p))$ *solves* (2.2.1) *then*

$$\mathcal{E}_\lambda(\alpha,e;\overline{q},t)\le\mathcal{E}_\lambda(\widehat{\alpha},\widehat{e};\overline{q},t)+\mathcal{H}(\widehat{\alpha},\widehat{p}-p)\,, \tag{2.2.6}$$

*for every* $(\widehat{\alpha},(\widehat{u},\widehat{e},\widehat{p}))\in\mathcal{D}(\alpha)\times A(w)$.

PROOF. Let $(\widehat{\alpha},(\widehat{u},\widehat{e},\widehat{p}))\in\mathcal{D}(\alpha)\times A(w)$. Then, since $\alpha\le\overline{\alpha}$, this quadruple belongs to $\mathcal{D}(\overline{\alpha})\times A(w)$ too. From our hypotesis, $\mathcal{E}_\lambda(\alpha,e;\overline{q},t)\le\mathcal{E}_\lambda(\widehat{\alpha},\widehat{e};\overline{q},t)+\mathcal{H}(\widehat{\alpha},\widehat{p}-\overline{p})-\mathcal{H}(\alpha,p-\overline{p})$, and by (2.1.14) and (2.1.12b), $\mathcal{H}(\widehat{\alpha},\widehat{p}-\overline{p})\le\mathcal{H}(\widehat{\alpha},\widehat{p}-p)+\mathcal{H}(\widehat{\alpha},p-\overline{p})\le\mathcal{H}(\widehat{\alpha},\widehat{p}-p)+\mathcal{H}(\alpha,p-\overline{p})$. Thus we conclude. □

We now derive some differential conditions for a triple $(u,e,p)$ such that $(\alpha,(u,e,p))$ solves (2.2.1), from a characterization of the solutions to (2.2.6).

LEMMA 2.2.3. *Let* $w\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\overline{q}\colon[0,t]\to M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$, $\alpha\in W^{1,\gamma}(\Omega)$, $(u,e,p)\in A(w)$ *satisfy* (2.2.6), $\overline{q}$ *having bounded variation. Then*

$$-\mathcal{H}(\alpha,q)\le\langle\mathbb{C}(\alpha)e,\eta\rangle\le\mathcal{H}(\alpha,-q)$$

*for every* $(v,\eta,q)\in A(0)$, *and*

$$\mathbb{C}(\alpha)e\in\mathcal{K}_\alpha(\Omega),\ \operatorname{div}(\mathbb{C}(\alpha)e)=0\quad in\ \Omega\,.$$

PROOF. Let us assume fix $(v,\eta,q)\in A(0)$. Since for every $\varepsilon\in\mathbb{R}$

$$(\alpha,(u+\varepsilon v,e+\varepsilon\eta,p+\varepsilon q))\in\mathcal{D}(\alpha)\times A(w)\,,$$

we have

$$\mathcal{Q}(\alpha,e+\varepsilon\eta)+\mathcal{H}(\alpha,\varepsilon q)\ge\mathcal{Q}(\alpha,e)\ \text{ for every }\varepsilon\in\mathbb{R}\,.$$

The positive homogeneity of $\mathcal{H}$ implies

$$\mathcal{Q}(\alpha,e\pm\varepsilon\eta)+\varepsilon\mathcal{H}(\alpha,\pm q)\ge\mathcal{Q}(\alpha,e)\ \text{ for every }\varepsilon>0\,.$$

Dividing by $\varepsilon$ and passing to the limit as $\varepsilon\to0$, we recover the former condition.

In order to get the latter one we can argue as in the first part of [**28**, Proposition 3.5], using the integration by parts formula (2.1.21). □

The following lemma shows, for pairs $(\alpha, (u, e, p))$ that satisfy (2.2.6), the Hölder dependence of $u$ and $e$ on $\alpha$, $p$, and $w$.

LEMMA 2.2.4. *For $i = 1, 2$ let $w_i \in H^1(\mathbb{R}^n, \mathbb{R}^n)$. Suppose that $(\alpha_i, (u_i, e_i, p_i))$ satisfies (2.2.6) with boundary datum $w = w_i$, and let*

$$\omega_{12} := \|\alpha_2 - \alpha_1\|_\infty + \|p_2 - p_1\|_1^{1/2} + \|Ew_2 - Ew_1\|_2 .$$

*Then*

$$\|e_2 - e_1\|_2 \leq C\, \omega_{12} , \tag{2.2.7}$$

*where $C$ is a positive constant depending on $\|e_1\|_2$, $R$, $\gamma_1$, $\gamma_2$, and $\Omega$.*

PROOF. We modify the proof of [**28**, Theorem 3.8], considering that here $\mathbb{C}$ depends on $\alpha$. Let

$$\begin{aligned} v &:= (u_2 - w_2) - (u_1 - w_1), \\ \eta &:= (e_2 - Ew_2) - (e_1 - Ew_1), \\ q &:= p_2 - p_1 . \end{aligned}$$

Since $(v, \eta, q) \in A(0)$, by Lemma 2.2.3 it follows that

$$\begin{aligned} -\mathcal{H}(\alpha_1, p_2 - p_1) \leq & \langle \mathbb{C}(\alpha_1) e_1, \eta \rangle, \\ & \langle \mathbb{C}(\alpha_2) e_2, \eta \rangle \leq \mathcal{H}(\alpha_2, p_1 - p_2) . \end{aligned}$$

Adding term by term and using (2.1.12d), we obtain

$$\langle \mathbb{C}(\alpha_2)(e_2 - e_1), \eta) \rangle \leq \langle [\mathbb{C}(\alpha_1) - \mathbb{C}(\alpha_2)] e_1, \eta \rangle + 2R \|p_2 - p_1\|_1 .$$

Observe that above we have put an extra term $-\langle \mathbb{C}(\alpha_2) e_1, \eta \rangle$ on both sides. From the definition of $\eta$,

$$\begin{aligned} \langle \mathbb{C}(\alpha_2)(e_2 - e_1), e_2 - e_1) \rangle \leq & \langle \mathbb{C}(\alpha_2)(e_2 - e_1), Ew_2 - Ew_1 \rangle + \langle [\mathbb{C}(\alpha_1) - \mathbb{C}(\alpha_2)] e_1, e_2 - e_1 \rangle \\ & + \langle [\mathbb{C}(\alpha_1) - \mathbb{C}(\alpha_2)] e_1, Ew_1 - Ew_2 \rangle + 2R \|p_2 - p_1\|_1 . \end{aligned}$$

By (2.1.8), this implies

$$\begin{aligned} 2\gamma_1 \|e_2 - e_1\|_2^2 \leq & 2\gamma_2 \|e_2 - e_1\|_2 \|Ew_2 - Ew_1\|_2 \\ & + \|e_1\|_2 \|\alpha_k^i - \alpha_k^{i-1}\|_\infty (\|e_2 - e_1\|_2 + \|\mathrm{E}w_k^{i+1} - \mathrm{E}w_1\|_2) + 2R \|p_2 - p_1\|_1 , \end{aligned}$$

which yields (2.2.7) by the Cauchy inequality. ☐

REMARK 2.2.5. We can also deduce the continuous dependence on $\alpha$, $p$, and $w$ of $u$, expressed (with the same notation as above) by

$$\begin{aligned} \|Eu_2 - Eu_1\|_1 &\leq C\, (\omega_{12} + \|p_2 - p_1\|_1) , \\ \|u_2 - u_1\|_1 &\leq C\, (\omega_{12} + \|p_2 - p_1\|_1 + \|w_2 - w_1\|_2) , \end{aligned}$$

arguing as in the final part of [**28**, Theorem 3.8].

We now show some stability results for the solutions of problems of the type (2.2.1) with respect to the weak convergence of the data. To ease the reading we first consider, in Theorem 2.2.6, the case $\lambda = 0$, and then we study, in Lemma 2.2.7, the additional term that appears when $\lambda > 0$. The result for the case $\lambda > 0$ (Theorem 2.2.8) follows from this lemma, arguing as in Theorem 2.2.6.

THEOREM 2.2.6 (Stability, case $\lambda = 0$). *Let* $w_k \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\alpha_k \in W^{1,\gamma}(\Omega)$, *and* $(u_k, e_k, p_k) \in A(w_k)$ *for every* $k$. *Assume that* $\alpha_k \rightharpoonup \alpha_\infty$ *weakly in* $W^{1,\gamma}(\Omega)$, $u_k \rightharpoonup u_\infty$ *weakly* in* $BD(\Omega)$, $e_k \rightharpoonup e_\infty$ *weakly in* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_k \rightharpoonup p_\infty$ *weakly* in* $M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)$, $w_k \rightharpoonup w_\infty$ *weakly in* $H^1(\mathbb{R}^n;\mathbb{R}^n)$. *Then* $(u_\infty, e_\infty, p_\infty) \in A(w_\infty)$. *If, in addition,*

$$\mathcal{E}(\alpha_k, e_k) \le \mathcal{E}(\widehat{\alpha}_k, \widehat{e}_k) + \mathcal{H}(\widehat{\alpha}_k, \widehat{p}_k - p_k) \tag{2.2.9}$$

*for every* $k$ *and every* $(\widehat{\alpha}_k, (\widehat{u}_k, \widehat{e}_k, \widehat{p}_k)) \in \mathcal{D}(\alpha_k) \times A(w_k)$, *then*

$$\mathcal{E}(\alpha_\infty, e_\infty) \le \mathcal{E}(\alpha, e) + \mathcal{H}(\alpha, p - p_\infty) \tag{2.2.10}$$

*for every* $(\alpha, (u, e, p)) \in \mathcal{D}(\alpha_\infty) \times A(w_\infty)$.

PROOF. The fact that $(u_\infty, e_\infty, p_\infty) \in A(w_\infty)$ follows by [**28**, Lemma 2.1].

We fix $\alpha \in \mathcal{D}(\alpha_\infty)$ and $(u, e, p) \in A(w_\infty)$, and test (2.2.9) by

$$\begin{aligned}
\widehat{\alpha}_k &:= \alpha \wedge \alpha_k\,,\\
\widehat{u}_k &:= u - u_\infty + u_k\,,\\
\widehat{e}_k &:= e - e_\infty + e_k\,,\\
\widehat{p}_k &:= p - p_\infty + p_k\,.
\end{aligned}$$

Then $\widehat{\alpha}_k \rightharpoonup \alpha$ and $\alpha \vee \alpha_k \rightharpoonup \alpha_\infty$ weakly in $W^{1,\gamma}(\Omega)$, $\widehat{u}_k \rightharpoonup u$ weakly* in $BD(\Omega)$, $\widehat{e}_k \rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\widehat{p}_k \rightharpoonup p$ weakly* in $M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)$.

Since for every $\alpha \in W^{1,\gamma}(\Omega)$ and every $e_1, e_2 \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ we have

$$\mathcal{Q}(\alpha, e_1) - \mathcal{Q}(\alpha, e_2) = \frac{1}{2}\langle \mathbb{C}(\alpha)(e_1 + e_2), e_1 - e_2\rangle \tag{2.2.11}$$

and for every $\alpha, \beta \in W^{1,\gamma}(\Omega)$

$$\|\nabla(\alpha \vee \beta)\|_\gamma^\gamma + \|\nabla(\alpha \wedge \beta)\|_\gamma^\gamma = \|\nabla\alpha\|_\gamma^\gamma + \|\nabla\beta\|_\gamma^\gamma\,,$$

(2.2.9) can be rewritten, adding to both sides the term $-\mathcal{Q}(\widehat{\alpha}_k, e_k)$, as

$$\begin{aligned}
\gamma_k :=& \mathcal{Q}(\alpha_k, e_k) - \mathcal{Q}(\widehat{\alpha}_k, e_k) + D(\alpha_k) + \|\nabla(\alpha \vee \alpha_k)\|_\gamma^\gamma - \|\nabla\alpha\|_\gamma^\gamma\\
&\le \frac{1}{2}\langle \mathbb{C}(\widehat{\alpha}_k)(e - e_\infty + 2e_k), e - e_\infty\rangle + D(\widehat{\alpha}_k) + \mathcal{H}(\widehat{\alpha}_k, p - p_\infty) =: \eta_k\,.
\end{aligned}$$

From (2.1.8a), for every $\alpha_1, \alpha_2 \in C(\Omega)$ and $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$

$$|\mathcal{Q}(\alpha_1, e) - \mathcal{Q}(\alpha_2, e)| \le \mathrm{Lip}(\mathbb{C})\|\alpha_1 - \alpha_2\|_\infty \|e\|_2^2\,.$$

Therefore,

$$\liminf_{k\to\infty} \mathcal{Q}(\alpha_k, e_k) - \mathcal{Q}(\widehat{\alpha}_k, e_k) = \liminf_{k\to\infty} \mathcal{Q}(\alpha_\infty, e_k) - \mathcal{Q}(\alpha, e_k)$$
$$= \liminf_{k\to\infty} \frac{1}{2}\langle[\mathbb{C}(\alpha_\infty) - \mathbb{C}(\alpha)]e_k, e_k\rangle\,.$$

Since $\alpha \in \mathcal{D}(\alpha_\infty)$, by (2.1.8b) we have that $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \mapsto [\mathbb{C}(\alpha_\infty) - \mathbb{C}(\alpha)]e{\cdot}e$ is a positive semidefinite quadratic form. Hence, by lower semicontinuity,

$$\liminf_{k\rightharpoonup\infty} \gamma_k \geq \mathcal{Q}(\alpha_\infty, e_\infty) - \mathcal{Q}(\alpha, e_\infty) + D(\alpha_\infty) + \|\nabla(\alpha_\infty)\|^\gamma_\gamma - \|\nabla\alpha\|^\gamma_\gamma\,.$$

On the other hand,

$$\lim_{k\rightharpoonup\infty} \eta_k = \frac{1}{2}\langle\mathbb{C}(\alpha)(e + e_\infty), e - e_\infty\rangle + D(\alpha) + \mathcal{H}(\alpha, p - p_\infty)$$
$$= \mathcal{Q}(\alpha, e) - \mathcal{Q}(\alpha, e_\infty) + D(\alpha) + \mathcal{H}(\alpha, p - p_\infty)\,.$$

This concludes the proof. □

From now on we treat the case $\lambda > 0$.

Lemma 2.2.7. *In addition to* (2.1.5)*,* (2.2.3)*,* (2.1.8)*,* (2.1.10)*, and* (2.1.27)*, let us assume also* (2.2.4)*. Let $\beta_k$ and $\widehat{\beta}_k$ be two sequences in $C(\overline{\Omega})$ such that $\beta_k \to \beta_\infty$ and $\widehat{\beta}_k \to \beta$ uniformly in $\overline{\Omega}$, and $\widehat{\beta}_k \in \mathcal{D}(\beta_k)$ for every $k$. Moreover let $q_k$, $q$ be functions from $[0,t]$ into $M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ such that $q_k(s) \rightharpoonup q(s)$ weakly* in $M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ for every $s \in [0,t]$. Then*

$$\widehat{\mathcal{V}}_\mathcal{H}(\beta_\infty, q; 0, t) - \widehat{\mathcal{V}}_\mathcal{H}(\beta, q; 0, t) \leq \liminf_{k\to\infty}[\widehat{\mathcal{V}}_\mathcal{H}(\beta_k, q_k; 0, t) - \widehat{\mathcal{V}}_\mathcal{H}(\widehat{\beta}_k, q_k; 0, t)]\,. \tag{2.2.12}$$

Proof. Let us consider the functionals $\widetilde{\mathcal{H}}_k$ and $\widetilde{\mathcal{H}}$ from $M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ into $\mathbb{R}^+ \cup \{0\}$ defined, for every $p \in M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$, by

$$\widetilde{\mathcal{H}}(p) := \mathcal{H}(\beta_\infty, p) - \mathcal{H}(\beta, p)\,,$$
$$\widetilde{\mathcal{H}}_k(p) := \mathcal{H}(\beta_k, p) - \mathcal{H}(\widehat{\beta}_k, p)\,.$$

By (2.2.4), $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}_k$ are convex, positively one-homogeneous (and consequently subadditive), and weakly* lower semicontinuous, thanks to Reshetnyak's Lower Semicontinuity Theorem. We now show that

$$\widehat{\mathcal{V}}_\mathcal{H}(\beta_\infty, q; 0, t) - \widehat{\mathcal{V}}_\mathcal{H}(\beta, q; 0, t) = \mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t)\,, \tag{2.2.13}$$
$$\widehat{\mathcal{V}}_\mathcal{H}(\beta_k, q; 0, t) - \widehat{\mathcal{V}}_\mathcal{H}(\widehat{\beta}_k, q; 0, t) = \mathcal{V}_{\widetilde{\mathcal{H}}_k}(q; 0, t)\,, \tag{2.2.14}$$

for every $k$. Indeed, let us fix $\varepsilon > 0$ and let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be three partitions of $[0,t]$ such that

$$\widehat{\mathcal{V}}^{\mathcal{P}_1}_\mathcal{H}(\beta_\infty, q; 0, t) > \widehat{\mathcal{V}}_\mathcal{H}(\beta_\infty, q; 0, t) - \varepsilon\,,$$
$$\widehat{\mathcal{V}}^{\mathcal{P}_2}_\mathcal{H}(\beta, q; 0, t) > \widehat{\mathcal{V}}_\mathcal{H}(\beta, q; 0, t) - \frac{\varepsilon}{2}\,,$$
$$\mathcal{V}^{\mathcal{P}_3}_{\widetilde{\mathcal{H}}}(q; 0, t) > \mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t) - \frac{\varepsilon}{2}\,.$$

It follows that

$$\mathcal{V}_{\widetilde{\mathcal{H}}}(q;0,t) \geq \mathcal{V}^{\mathcal{P}_1}_{\widetilde{\mathcal{H}}}(q;0,t) = \widehat{\mathcal{V}}^{\mathcal{P}_1}_{\mathcal{H}}(\beta_\infty,q;0,t) - \widehat{\mathcal{V}}^{\mathcal{P}_1}_{\mathcal{H}}(\beta,q;0,t) > \widehat{\mathcal{V}}_{\mathcal{H}}(\beta_\infty,q;0,t) - \varepsilon - \widehat{\mathcal{V}}_{\mathcal{H}}(\beta,q;0,t)\,.$$

On the other hand, we get

$$\begin{aligned}\widehat{\mathcal{V}}_{\mathcal{H}}(\beta_\infty,q;0,t) - \widehat{\mathcal{V}}_{\mathcal{H}}(\beta,q;0,t) &> \widehat{\mathcal{V}}^{\mathcal{P}_2\cup\mathcal{P}_3}_{\mathcal{H}}(\beta_\infty,q;0,t) - \widehat{\mathcal{V}}^{\mathcal{P}_2}_{\mathcal{H}}(\beta,q;0,t) - \frac{\varepsilon}{2}\\ &\geq \widehat{\mathcal{V}}^{\mathcal{P}_2\cup\mathcal{P}_3}_{\mathcal{H}}(\beta_\infty,q;0,t) - \widehat{\mathcal{V}}^{\mathcal{P}_2\cup\mathcal{P}_3}_{\mathcal{H}}(\beta,q;0,t) - \frac{\varepsilon}{2}\\ &= \mathcal{V}^{\mathcal{P}_2\cup\mathcal{P}_3}_{\widetilde{\mathcal{H}}}(q;0,t) - \frac{\varepsilon}{2} > \mathcal{V}_{\widetilde{\mathcal{H}}}(q;0,t) - \varepsilon\,,\end{aligned}$$

where the second inequality follows from Lemma 1.2.1(1) and the last one comes from the subadditivity of $\widetilde{\mathcal{H}}$. This concludes the verification of (2.2.13). The proof of (2.2.14) is analogous.

Arguing as in Lemma 2.1.1 we have that

$$\widetilde{\mathcal{H}}(p) \leq \liminf_{k\to\infty} \widetilde{\mathcal{H}}_k(p_k)\,,$$

for every $p_k \rightharpoonup p$ weakly* in $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$. Hence

$$\mathcal{V}_{\widetilde{\mathcal{H}}}(q;0,t) \leq \liminf_{k\to\infty} \mathcal{V}_{\widetilde{\mathcal{H}}_k}(q_k;0,t)\,,$$

and we conclude by (2.2.13) and (2.2.14). □

THEOREM 2.2.8 (Stability, case $\lambda > 0$). *Besides* (2.1.5), (2.2.3), (2.1.8), (2.1.10), *and* (2.1.27), *assume also* (2.2.4). *Let* $w_k \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\alpha_k \in W^{1,\gamma}(\Omega)$, $(u_k,e_k,p_k) \in A(w_k)$, *and* $q_k$ *be functions from* $[0,t]$ *into* $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$ *of bounded variation, for every* $k$. *Suppose that* $\alpha_k \rightharpoonup \alpha_\infty$ *weakly in* $W^{1,\gamma}(\Omega)$, $u_k \rightharpoonup u_\infty$ *weakly* in* $BD(\Omega)$, $e_k \rightharpoonup e_\infty$ *weakly in* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_k \rightharpoonup p_\infty$ *weakly* in* $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$, $w_k \rightharpoonup w_\infty$ *weakly in* $H^1(\mathbb{R}^n;\mathbb{R}^n)$, *and* $q_k(s) \rightharpoonup q(s)$ *weakly* in* $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$ *for every* $s\in[0,t]$. *Then* $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$. *If, in addition,*

$$\mathcal{E}(\alpha_k,e_k) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k,q_k;0,t) \leq \mathcal{E}(\widehat{\alpha}_k,\widehat{e}_k) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\widehat{\alpha}_k,q_k;0,t) + \mathcal{H}(\widehat{\alpha}_k,\widehat{p}_k - p_k)$$

*for every* $k$ *and every* $(\widehat{\alpha}_k,(\widehat{u}_k,\widehat{e}_k,\widehat{p}_k)) \in \mathcal{D}(\alpha_k)\times A(w_k)$, *then*

$$\mathcal{E}(\alpha_\infty,e_\infty) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\infty,q;0,t) \leq \mathcal{E}(\alpha,e) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha,q;0,t) + \mathcal{H}(\alpha,p-p_\infty)\,,$$

*for every* $(\alpha,(u,e,p))\in\mathcal{D}(\alpha_\infty)\times A(w_\infty)$.

PROOF. We can argue as in the proof of Theorem 2.2.6, choosing the same test functions, and adding to $\gamma_k$ the term $\lambda(\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k,q_k;0,t) - \widehat{\mathcal{V}}_{\mathcal{H}}(\widehat{\alpha}_k,q_k;0,t))$. The sequence of these terms is lower semicontinuous by Lemma 2.2.7 and this is enough to conclude. □

## 2.3. Quasistatic evolution

Fixed $\lambda\in[0,1]$, we now consider the problem of existence of globally stable quasistatic evolutions, where the time-dependent data are (only) Dirichlet boundary conditions $w \in AC([0,T];H^1(\mathbb{R}^n;\mathbb{R}^n))$. The functions $\alpha$, $u$, $e$, $p$ will be then functions from $[0,T]$ into the functional spaces $W^{1,\gamma}(\Omega;[0,1])$, $BD(\Omega)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$, respectively.

The parameter $\lambda$ accounts for the interplay between damage growth and cumulation of plastic strain. When $\lambda = 1$ it is more convenient to damage material parts more affected by plastic evolution up to the current time. The physical meaning of $\lambda$ will be explained in detail in Section 5, where we will study the properties of the evolutions. The case $\lambda = 1$ corresponds to the model of [**2**] and [**3**], with a different gradient damage regularization.

The existence of quasistatic evolutions is shown in Theorem 2.3.3, the main result of the chapter.

DEFINITION 2.3.1. Let $\lambda \in [0,1]$. A *quasistatic evolution (corresponding to $\lambda$)* is a function $t \mapsto (\alpha(t), u(t), e(t), p(t))$ from $[0,T]$ into $W^{1,\gamma}(\Omega;[0,1]) \times BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)$ that satisfies the following conditions:

(qs0) *irreversibility* : for every $x \in \Omega$

$$t \in [0,T] \mapsto \alpha(t,x) \quad \text{is nonincreasing;} \tag{2.3.1}$$

(qs1) *global stability*: the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)$ has bounded variation, $(u(t), e(t), p(t)) \in A(w(t))$ for every $t \in [0,T]$, and

$$\mathcal{E}(\alpha(t), e(t)) + \lambda \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t), p; 0, t) \leq \mathcal{E}(\beta, \eta) + \lambda \widehat{\mathcal{V}}_{\mathcal{H}}(\beta, p; 0, t) + \mathcal{H}(\beta, q - p(t)) \tag{2.3.2}$$

for every $(\beta, (v, \eta, q)) \in \mathcal{D}(\alpha(t)) \times A(w(t))$;

(qs2) *energy balance*: for every $t \in [0,T]$

$$\begin{aligned} &\mathcal{E}(\alpha(t), e(t)) + \lambda \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t), p; 0, t) + (1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) \\ &\qquad = \mathcal{E}(\alpha(0), e(0)) + \int_0^t \langle \sigma(s), E\dot{w}(s) \rangle \, \mathrm{d}s \,, \end{aligned} \tag{2.3.3}$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$ and $\mathcal{V}_{\mathcal{H}}$, $\widehat{\mathcal{V}}_{\mathcal{H}}$ are defined in (2.1.25) and (2.1.26), respectively.

REMARK 2.3.2. The integral in (2.3.3) is well defined.
Indeed, from (2.3.2) it follows that $t \mapsto (\alpha(t), u(t), e(t), p(t))$ is a solution to the problem

$$\operatorname{argmin} \{ \mathcal{E}_\lambda(\beta, \eta; p, t) + \mathcal{H}(\beta, q - p(t)) \colon (\beta, (v, \eta, q)) \in \mathcal{D}(\alpha(t)) \times A(w(t)) \} \,,$$

for every $t \in [0,T]$, where $\mathcal{E}_\lambda$ is defined in (2.2.2). In view of Lemma 2.2.4, choosing $e_2 = e(t)$ for every $t \in [0,T]$ and $e_1 = e(0)$, we can observe that

$$\sup_{t \in [0,T]} \|e(t)\|_2 \leq C \,, \tag{2.3.4}$$

where $C$ is independent of time.

Let us now verify the measurability of $t \mapsto e(t)$. This follows from Lemma 2.2.4 if we show that $t \mapsto \alpha(t)$ is continuous for a.e. $t$ with respect to the uniform convergence in $\overline{\Omega}$, since $t \mapsto p(t)$ is strongly continuous into $M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)$ for a.e. $t$, having bounded variation. Now, by the irreversibility condition and the fact that for every $t \in [0,T]$ the function $\alpha(t)$ takes values in $[0,1]$ we have, using Lemma 1.2.2, that there exists a countable set $E \subset [0,T]$ such that $\alpha$ is continuous in every $t \in [0,T] \setminus E$ with respect to the $L^p$ norm, with $1 \leq p < \infty$. In other words, for every $t \in [0,T] \setminus E$

$$\alpha(s) \to \alpha(t) \text{ in } L^p(\Omega) \text{ as } s \to t \,.$$

From the stability condition, choosing $\beta \equiv 0$ and $(v, \eta, q) = (u(t), e(t), p(t))$, and using (2.3.4), it follows that

$$\sup_{t\in[0,T]} \|\nabla\alpha(t)\|_\gamma^\gamma < C$$

with $C$ independent of $t \in [0,T]$. Then, by the Urysohn Property, $\alpha$ is continuous in every $t \in [0,T] \setminus E$ with respect to the weak convergence in $W^{1,\gamma}$, i.e., for every $t \in [0,T] \setminus E$

$$\alpha(s) \rightharpoonup \alpha(t) \text{ weakly in } W^{1,\gamma}(\Omega) \text{ as } s \to t\,.$$

The above convergence is uniform in $\overline{\Omega}$ by the compact Sobolev embedding.

Then $e$ and $\sigma$ belong to $L^\infty(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$. Finally, by (2.1.27), it follows that $\dot{w} \in L^1(0,T; H^1(\mathbb{R}^n; \mathbb{R}^n))$, and we conclude.

Theorem 2.3.3 (Existence of quasistatic evolutions). *Let* $\lambda \in [0,1]$ *and assume* (2.1.5), (2.2.3), (2.1.8), (2.1.10), *and* (2.1.27). *If* $\lambda > 0$ *assume also* (2.2.4). *Let* $(\alpha_0, (u_0, e_0, p_0)) \in W^{1,\gamma}(\Omega; [0,1]) \times A(w(0))$ *satisfy the stability condition*

$$\mathcal{E}(\alpha_0, e_0) \le \mathcal{E}(\beta, \eta) + \mathcal{H}(\beta, q - p_0) \tag{2.3.5}$$

*for every* $(\beta, (v, \eta, q)) \in \mathcal{D}(\alpha_0) \times A(w(0))$. *Then there exists a quasistatic evolution* $t \mapsto (\alpha(t), u(t), e(t), p(t))$ *corresponding to* $\lambda$ *such that* $\alpha(0) = \alpha_0$, $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$.

Proof. The proof is based on discrete time approximation and is split into several steps.

**Approximate solutions.** Let us fix a sequence of subdivisions $(t_k^i)_{0\le i\le k}$ of the interval $[0,T]$, with

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T\,,$$
$$\lim_{k\to\infty} \max_{1\le i\le k} (t_k^i - t_k^{i-1}) = 0\,.$$

For every $k$ we define the approximate solutions $\alpha_k$, $u_k$, $e_k$, and $p_k$ by induction as follows. We set $(\alpha_k^0, (u_k^0, e_k^0, p_k^0)) := (\alpha_0, (u_0, e_0, p_0)) \in W^{1,\gamma}(\Omega; [0,1]) \times A(w(0))$ and for $i = 1, \dots, k$ we define $(\alpha_k^i, (u_k^i, e_k^i, p_k^i))$ as a solution to the incremental problem

$$\operatorname{argmin}\{\mathcal{E}_\lambda(\beta, \eta; p_k, t_k^{i-1}) + \mathcal{H}(\beta, q - p_k^{i-1})\colon (\beta, (v, \eta, q)) \in \mathcal{D}(\alpha_k^{i-1}) \times A(w_k^i)\}\,, \tag{2.3.6}$$

where $w_k^i := w(t_k^i)$ and, according to (2.2.2) and using Lemma 1.2.1(2),

$$\mathcal{E}_\lambda(\beta, \eta; p_k, t_k^{i-1}) = \mathcal{E}(\beta, \eta) + \lambda \sum_{j=1}^{i-1} \mathcal{H}(\beta, p_k^j - p_k^{j-1})\,,$$

with $p_k(t) := p_k^h$, $h$ being the largest integer such that $t_k^h \le t$. The existence of a solution to this problem and the fact that $\alpha_k^i \in W^{1,\gamma}(\Omega; [0,1])$ for every $k \in \mathbb{N}$ and $0 \le i \le k$ follow from Theorem 2.2.1.

For every $t \in [0,T]$ we define the piecewise constant interpolations

$$\alpha_k(t) := \alpha_k^i,\ \ u_k(t) := u_k^i,\ \ e_k(t) := e_k^i,\ \ \sigma_k(t) := \mathbb{C}(\alpha_k^i) e_k^i,\ \ w_k(t) := w_k^i\,, \tag{2.3.7}$$

where $i$ is the largest integer such that $t_k^i \le t$. By definition $t \mapsto \alpha_k(t)$ is nonincreasing, $\alpha_k(t) \in W^{1,\gamma}(\Omega;[0,1])$ and $(u_k(t), e_k(t), p_k(t)) \in A(w_k(t))$ for every $t \in [0,T]$. By Lemma 1.2.1(2) it follows that

$$\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^i) = \mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^{i-1}) + \lambda\mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) . \tag{2.3.8}$$

Then (2.3.6) implies that

$$\begin{aligned}\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^i) + (1-\lambda)\mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) &= \mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^{i-1}) + \mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) \\ &\le \mathcal{E}_\lambda(\beta, \eta; p_k, t_k^{i-1}) + \mathcal{H}(\beta, q - p_k^{i-1})\end{aligned} \tag{2.3.9}$$

for every $k$, $1 \le i \le k$, and $(\beta, (v, \eta, q)) \in \mathcal{D}(\alpha_k^i) \times A(w_k^i)$. Since

$$\begin{aligned}\mathcal{H}(\beta, q - p_k^{i-1}) &\le \mathcal{H}(\beta, p_k^i - p_k^{i-1}) + \mathcal{H}(\beta, q - p_k^i) \\ &\le \lambda\mathcal{H}(\beta, p_k^i - p_k^{i-1}) + (1-\lambda)\mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) + \mathcal{H}(\beta, q - p_k^i) ,\end{aligned}$$

from (2.3.9) we get the discrete formulation of global stability

$$\mathcal{E}_\lambda(\alpha_k(t), e_k(t); p_k, t) \le \mathcal{E}_\lambda(\beta, \eta; p_k, t) + \mathcal{H}(\beta, q - p_k(t)) \tag{2.3.10}$$

for every $k$, $t \in [0,T]$, and $(\beta, (v, \eta, q)) \in \mathcal{D}(\alpha_k(t)) \times A(w_k(t))$. Notice that if $\lambda = 0$ the equation (2.3.10) follows directly from Lemma 2.2.2.

**The discrete energy inequality.** We now derive an energy estimate for the solutions of the incremental problems. Let us fix $i \in \{1, \dots, k\}$ and for a given integer $h$ with $1 \le h \le i$ let $v := u_k^{h-1} - w_k^{h-1} + w_k^h$ and $\eta := e_k^{h-1} - Ew_k^{h-1} + Ew_k^h$.

Since $(\alpha_k^{h-1}, (v, \eta, p_k^{h-1})) \in \mathcal{D}(\alpha_k^{h-1}) \times A(w_k^h)$, by the minimality condition (2.3.6) we obtain

$$\begin{aligned}&\mathcal{E}_\lambda(\alpha_k^h, e_k^h; p_k, t_k^{h-1}) + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \le \mathcal{E}_\lambda(\alpha_k^{h-1}, e_k^{h-1}; p_k, t_k^{h-1}) \\ &+ \mathcal{Q}(\alpha_k^{h-1}, Ew_k^h - Ew_k^{h-1}) + \langle \mathbb{C}(\alpha_k^{h-1}) e_k^{h-1}, Ew_k^h - Ew_k^{h-1} \rangle ,\end{aligned} \tag{2.3.11}$$

where we have used the identity

$$\mathcal{Q}(\alpha, e_1 + e_2) = \mathcal{Q}(\alpha, e_1) + \mathcal{Q}(\alpha, e_2) + \langle \mathbb{C}(\alpha) e_1, e_2 \rangle$$

for every $\alpha \in W^{1,\gamma}(\Omega)$ and $e_1$, $e_2 \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. From the absolute continuity of $w$ with respect to $t$ we obtain

$$w_k^h - w_k^{h-1} = \int_{t_k^{h-1}}^{t_k^h} \dot{w}(t)\, \mathrm{d}t ,$$

where we use a Bochner integral of a function with values in $H^1(\mathbb{R}^n; \mathbb{R}^n)$. This implies that

$$Ew_k^h - Ew_k^{h-1} = \int_{t_k^{h-1}}^{t_k^h} E\dot{w}(t)\, \mathrm{d}t , \tag{2.3.12}$$

where the integral is again in the sense of Bochner and the target space is $L^2(\mathbb{R}^n; \mathbb{M}^{n\times n}_{sym})$. By (2.1.8c) and (2.3.12) we get

$$\mathcal{Q}(\alpha_k^{h-1}, Ew_k^h - Ew_k^{h-1}) \le \gamma_2 \Big( \int_{t_k^{h-1}}^{t_k^h} \|E\dot{w}(t)\|_2\, \mathrm{d}t \Big)^2 .$$

From (2.3.8), (2.3.11), and (2.3.12) it follows that

$$\begin{aligned}\mathcal{E}_\lambda(\alpha_k^h, e_k^h; p_k, t_k^h) + (1-\lambda)\mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) &\le \mathcal{E}_\lambda(\alpha_k^{h-1}, e_k^{h-1}; p_k, t_k^{h-1}) \\ &+ \int_{t_k^{h-1}}^{t_k^h} \langle \mathbb{C}(\alpha_k^{h-1}) e_k^{h-1}, E\dot{w}(t)\rangle \,\mathrm{d}t + \omega_k \int_{t_k^{h-1}}^{t_k^h} \|E\dot{w}(t)\|_2 \,\mathrm{d}t\,,\end{aligned} \tag{2.3.13}$$

where

$$\omega_k := \gamma_2 \max_{1\le h\le k} \int_{t_k^{h-1}}^{t_k^h} \|E\dot{w}(t)\|_2 \,\mathrm{d}t \;\to 0$$

by the absolute continuity of the integral. Iterating now inequality (2.3.13) for $1 \le h \le i$, we have

$$\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^i) + (1-\lambda)\sum_{h=1}^{i} \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \le \mathcal{E}(\alpha_0, e_0) + \int_0^{t_k^i} \langle \sigma_k(s), E\dot{w}(s)\rangle \,\mathrm{d}s + \delta_k\,, \tag{2.3.14}$$

with $\delta_k := \omega_k \int_0^T \|E\dot{w}(t)\|_2 \,\mathrm{d}t \to 0$.

**A priori estimates.** Using the hypoteses (2.1.8c) and (2.1.12d) in the left-hand side, as well as (2.1.9) and the fact that the function $t \mapsto \|E\dot{w}(t)\|_2$ is integrable on $[0,T]$ in the right-hand side, we find

$$\begin{aligned}&\gamma_1 \|e_k(t)\|_2^2 + D(\alpha_k(t)) + \|\nabla\alpha_k(t)\|_\gamma^\gamma + r(1-\lambda)\sum_{h=1}^{i} \|p_k^h - p_k^{h-1}\|_1 \\ &\le \mathcal{E}(\alpha_0, e_0) + 2\gamma_2 \sup_{t\in[0,T]} \|e_k(t)\|_2 \int_0^T \|E\dot{w}(s)\|_2 \,\mathrm{d}s + \delta_k\end{aligned}$$

for every $k$ and $t \in [t_k^1, T]$, where $i$ is the largest integer such that $t_k^i \le t$.

Thus, by the Cauchy inequality,

$$\sup_{t\in[0,T]} \|e_k(t)\|_2 \le C\,. \tag{2.3.15}$$

Henceforth, $C$ denotes a suitable constant depending only on $\gamma_1$, $\gamma_2$, $r$, and on the functions $\alpha_0$, $e_0$, and $w$. We immediately deduce that

$$\sup_{t\in[0,T]} \|\nabla\alpha_k(t)\|_\gamma^\gamma \le C\,, \tag{2.3.16}$$

and, from the fact that $t \mapsto p_k(t)$ is constant on the intervals $[t_k^{h-1}, t_k^h[$, that

$$\mathcal{V}(p_k; 0, T) = \sum_{i=1}^{k} \|p_k^i - p_k^{i-1}\|_1 \le C\,. \tag{2.3.17}$$

**Passage to the limit.** Since the functions $\alpha_k$ are nonincreasing in time and take values in $[0,1]$, by virtue of (2.3.16) we can apply the generalized version of the classical Helly Theorem given in [**39**, Helly Theorem] to conclude that there exist a subsequence, still denoted $\alpha_k$, and a function $\alpha\colon [0,T] \to W^{1,\gamma}(\Omega;[0,1])$ nonincreasing in time such that $\alpha_k(t) \to \alpha(t)$ strongly in $L^1(\Omega)$ for every $t \in [0,T]$. By (2.3.16) and the Urysohn Property we have weak convergence in $W^{1,\gamma}(\Omega)$ and thus uniform convergence in $\overline{\Omega}$. In particular (qs0) holds.

In the same way, using now (2.3.17) and [**28**, Lemma 7.2], we can assume that there exists $p\colon [0,T] \to M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ with bounded variation on $[0,T]$ such that $p_k(t) \rightharpoonup p(t)$ weakly* in $M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$, for every $t \in [0,T]$.

Following the same argument used in the proof of Theorem 2.2.1, by (2.3.15) and (2.3.17) we can deduce that

$$\sup_{t\in[0,T]} \|u_k(t)\|_{BD(\Omega)} \le C\,. \tag{2.3.18}$$

Let us fix $t \in [0,T]$. From (2.3.15) and (2.3.18) it follows that there exist an increasing sequence $k_j$ (possibly depending on $t$) and two functions $\widehat{u} \in BD(\Omega)$ and $\widehat{e} \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ such that $u_{k_j}(t) \rightharpoonup \widehat{u}$ weakly* in $BD(\Omega)$ and $e_{k_j}(t) \rightharpoonup \widehat{e}$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. By (2.3.10) we can apply Theorem 2.2.8 (or Theorem 2.2.6 if $\lambda = 0$) and find that $(\alpha(t), (\widehat{u}, \widehat{e}, p(t)))$ is a solution to the problem

$$\operatorname{argmin}\{\mathcal{E}_\lambda(\beta, \eta; p, t) + \mathcal{H}(\beta, q - p(t))\colon (\beta, (v, \eta, q)) \in \mathcal{D}(\alpha(t)) \times A(w(t))\}\,.$$

In particular $(\widehat{u}, \widehat{e})$ minimizes the functional $(v, \eta) \mapsto \mathcal{Q}(\alpha(t), \eta)$, which is strictly convex in $\eta$, on the convex set $K := \{(v, \eta)\colon (v, \eta, p(t)) \in A(w(t))\}$. Then $(\widehat{u}, \widehat{e})$ is uniquely determined, using also Korn's inequality; defining $(u(t), e(t)) := (\widehat{u}, \widehat{e})$, we have that $u_k(t) \rightharpoonup u(t)$ in $BD(\Omega)$ and $e_k(t) \rightharpoonup e(t)$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Therefore (qs1) holds.

To prove that $t \mapsto (\alpha(t), u(t), e(t), p(t))$ is a quasistatic evolution it remains to show the energy balance (qs2).

**Energy balance.** We consider now the asymptotics of the discrete energy inequality (2.3.14). Later we will show that also the equality holds in the limit.

Since $p_k$ is piecewise constant and continuous from the right, $\alpha_k$ is nonincreasing, and (2.1.12b) holds, by Lemma 1.2.1(2) we have

$$\mathcal{V}_\mathcal{H}(\alpha_k, p_k; 0, t) = \sum_{h=1}^{i} \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1})\,, \tag{2.3.19}$$

where $i$ is the largest integer such that $t_k^i \le t$. From the lower semicontinuity of $\mathcal{H}$ (Lemma 2.1.1) and the definition of plastic dissipation (2.1.25) it follows that

$$\mathcal{V}_\mathcal{H}(\alpha, p; 0, t) \le \liminf_{k\to\infty} \mathcal{V}_\mathcal{H}(\alpha_k, p_k; 0, t), \text{ and } \widehat{\mathcal{V}}_\mathcal{H}(\alpha(t), p; 0, t) \le \liminf_{k\to\infty} \widehat{\mathcal{V}}_\mathcal{H}(\alpha_k(t), p_k; 0, t)\,. \tag{2.3.20}$$

Moreover, since $\alpha_k(t) \to \alpha(t)$ uniformly in $\overline{\Omega}$ and $e_k(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ for every $t \in [0,T]$, by (2.1.8), (2.1.27), (2.3.15), and the Dominated Convergence Theorem we get

$$\int_0^t \langle \sigma(s), E\dot{w}(s)\rangle\,\mathrm{d}s = \lim_{k\to\infty} \int_0^{t_k^i} \langle \sigma_k(s), E\dot{w}(s)\rangle\,\mathrm{d}s\,, \tag{2.3.21}$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$ for every $s \in [0,T]$.

Collecting (2.3.19)–(2.3.21), from (2.3.14) and the lower semicontinuity of the remaining terms the inequality

$$\mathcal{E}_\lambda(\alpha(t), e(t); p, t) + (1-\lambda)\mathcal{V}_\mathcal{H}(\alpha, p; 0, t) \le \mathcal{E}(\alpha(0), e(0)) + \int_0^t \langle \sigma(s), E\dot{w}(s)\rangle\,\mathrm{d}s$$

follows, for every $t \in [0,T]$.

Conversely, let us fix $t \in [0,T]$ and let $(s_k^i)_{0\le i\le k}$ be a sequence of subdivisions of the interval $[0,t]$ satisfying

$$0 = s_k^0 < s_k^1 < \cdots < s_k^{k-1} < s_k^k = t\,,$$
$$\lim_{k\to\infty} \max_{1\le i\le k} (s_k^i - s_k^{i-1}) = 0\,.$$

For every $i = 1,\dots,k$ let $v := u(s_k^i) - w(s_k^i) + w(s_k^{i-1})$ and $\eta := e(s_k^i) - Ew(s_k^i) + Ew(s_k^{i-1})$. Since $(\alpha(s_k^i), (v,\eta,p(s_k^i))) \in \mathcal{D}(\alpha(s_k^{i-1})) \times A(w(s_k^{i-1}))$, by the global stability (2.3.2) we have

$$\begin{aligned}\mathcal{E}_\lambda(\alpha(s_k^{i-1}), e(s_k^{i-1}); p, s_k^{i-1}) \le\ & \mathcal{E}_\lambda(\alpha(s_k^i), e(s_k^i); p, s_k^{i-1}) + \mathcal{Q}(\alpha(s_k^i), Ew(s_k^{i-1}) - Ew(s_k^i))\\ & - \langle \sigma(s_k^i), Ew(s_k^i) - Ew(s_k^{i-1})\rangle + \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1}))\,.\end{aligned} \tag{2.3.22}$$

By definition of $\widehat{\mathcal{V}}_{\mathcal{H}}$ it follows that

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(s_k^i), p; 0, s_k^{i-1}) + \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1})) \le \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(s_k^i), p; 0, s_k^i)\,,$$

and then, recalling the definition of $\mathcal{E}_\lambda$, (2.3.22) implies that

$$\begin{aligned}\mathcal{E}_\lambda(\alpha(s_k^{i-1}), e(s_k^{i-1}); p, s_k^{i-1}) \le\ & \mathcal{E}_\lambda(\alpha(s_k^i), e(s_k^i); p, s_k^i) + \mathcal{Q}(\alpha(s_k^i), Ew(s_k^{i-1}) - Ew(s_k^i))\\ & - \langle \sigma(s_k^i), Ew(s_k^i) - Ew(s_k^{i-1})\rangle + (1-\lambda)\mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1}))\,.\end{aligned}$$

Now, following the same argument used in (2.3.13), we find that there exists a sequence $\omega_k \to 0^+$ such that

$$\begin{aligned}\mathcal{E}_\lambda(\alpha(s_k^{i-1}),\ e(s_k^{i-1}); p, s_k^{i-1}) \le\ & \mathcal{E}_\lambda(\alpha(s_k^i), e(s_k^i); p, s_k^i) + (1-\lambda)\mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1}))\\ & - \int_{s_k^{i-1}}^{s_k^i} \langle \sigma(s_k^i), E\dot w(t)\rangle dt + \omega_k \int_{s_k^{i-1}}^{s_k^i} \|E\dot w(t)\|_2\, \mathrm{d}t\,.\end{aligned}$$

On $[0,t]$ we define the piecewise constant function $\overline{\sigma}_k(s) := \sigma(s_k^i)$, where $i$ is the smallest index such that $s \le s_k^i$.

Since $\sum_i \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1})) \le \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t)$, iterating the last inequality for $1 \le i \le k$ we obtain

$$\mathcal{E}(\alpha(0), e(0)) \le \mathcal{E}_\lambda(\alpha(t), e(t); p, t) + (1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) - \int_0^t \langle \overline{\sigma}_k(s), E\dot w(s)\rangle ds + \delta_k\,,$$

where $\delta_k := \omega_k \int_0^T \|E\dot w(s)\|_2\, \mathrm{d}s$. By Remark 2.3.2 the set of discontinuity points of $s \mapsto \alpha(s)$ and $s \mapsto e(s)$ is at most countable, and $\|\alpha(s)\|_\infty$, $\|e(s)\|_2$ are uniformly bounded in $s$. Therefore $\overline{\sigma}_k(s) \to \sigma(s)$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ for a.e. $s$, so that

$$\int_0^t \langle \sigma(s), E\dot w(s)\rangle\, \mathrm{d}s = \lim_{k\to\infty} \int_0^t \langle \overline{\sigma}_k(s), E\dot w(s)\rangle\, \mathrm{d}s\,,$$

by Dominated Convergence Theorem. This concludes the proof. □

## 2.4. Qualitative properties of quasistatic evolutions

In this section we show some qualitative properties of quasistatic evolutions, whose existence is proved in Theorem 2.3.3.

First, in Proposition 2.4.1, we deduce that $t \mapsto u(t)$, $t \mapsto e(t)$, and $t \mapsto p(t)$ are continuous, with respect to the norms of their spaces, at the continuity points for $t \mapsto \alpha(t)$ with respect to the uniform convergence in $\overline{\Omega}$. Then the time discontinuities of the quasistatic evolutions are at most countable, by Remark 2.3.2. This regularity in time of $\alpha$ also permits to say that $\mathcal{H}(\alpha(\bar{t}), \dot{p}(\bar{t}))$ represents the rate of plastic dissipation at $\bar{t}$, and then to understand the physical meaning of the term in $\lambda$ in (qs1) (cf. Remark 2.4.2).

In Corollary 2.4.3 we derive from (qs1) Euler conditions with respect to the variation of $u$, $e$, and $p$, corresponding to equilibrium and stress constraint properties. In the last part of the section we assume suitable regularity properties on $\mathbb{C}$, $D$ and $\mathcal{H}$, and absolute continuity of the evolutions. In Proposition 2.4.4 is shown an Euler condition for $\alpha$ and the differential counterpart of the energy balance (qs2): together with the irreversibility, these are Kuhn-Tucker conditions (see e.g. **[89]** for this terminology) governing the evolution of the damage variable $\alpha$. Moreover, it is deduced the Hill's maximum plastic work principle that, if $p$ is regular enough, implies the Prandtl-Reuss flow rule with damage.

Throughout this section, we suppose that (2.1.5), (2.2.3), (2.1.8), (2.1.10), and (2.1.27) hold when $\lambda = 0$; when $\lambda > 0$ we will assume also (2.2.4).

Except for countable many instants, every quasistatic evolution is continuous in time, as shown in the following result.

Proposition 2.4.1. *Every quasistatic evolution $t \mapsto (\alpha(t), u(t), e(t), p(t))$ is strongly continuous from $[0,T]$ into $C(\overline{\Omega};[0,1]) \times BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)$ except for a countable subset of $[0,T]$, which is the set of discontinuity points of $\alpha$ with respect to the uniform convergence in $\overline{\Omega}$.*

Proof. From the energy balance condition (qs2), written for a time interval $[s,t]$, we deduce

$$\begin{aligned}&Q(\alpha(t),e(t)) - Q(\alpha(s),e(s)) + \mathcal{H}(\alpha(t), p(t)-p(s))\\&\le \int_s^t \langle \sigma(\tau), E\dot{w}(\tau)\rangle \,\mathrm{d}\tau + D(\alpha(s)) - D(\alpha(t)) + \|\nabla(\alpha(s))\|_\gamma^\gamma - \|\nabla(\alpha(t))\|_\gamma^\gamma\,,\end{aligned}$$

using also (1) of Lemma 1.2.1 both for $(1-\lambda)\mathcal{V}_\mathcal{H}(\alpha, p; s,t)$ and for $\lambda\widehat{\mathcal{V}}_\mathcal{H}(\alpha(t), p; s,t)$.

Notice now that

$$D(\alpha(s)) - D(\alpha(t)) + \|\nabla(\alpha(s))\|_\gamma^\gamma - \|\nabla(\alpha(t))\|_\gamma^\gamma \le 0\,.$$

Indeed, if the term above were strictly positive, from (2.1.12b) and (2.1.8b) we would have

$$\mathcal{E}(\alpha(t),e(s)) + \lambda\widehat{\mathcal{V}}_\mathcal{H}(\alpha(t),p;0,t) < \mathcal{E}(\alpha(s),e(s)) + \lambda\widehat{\mathcal{V}}_\mathcal{H}(\alpha(s),p;0,t)\,,$$

which contradicts (qs1) since $(\alpha(t),(u(s),e(s),p(s))) \in \mathcal{D}(\alpha(s)) \times A(w(s))$.

Then

$$Q(\alpha(t),e(t))-Q(\alpha(s),e(s))+\mathcal{H}(\alpha(t),p(t)-p(s))\le\int_s^t\langle\sigma(\tau),E\dot w(\tau)\rangle\,\mathrm{d}\tau \tag{2.4.1}$$

Now, by Lemma 2.2.3 it follows that

$$-\langle\sigma(s),e(t)-e(s)-(Ew(t)-Ew(s))\rangle\le\mathcal{H}(\alpha(s),p(t)-p(s))\,, \tag{2.4.2}$$

because $(u(t)-u(s)-(w(t)-w(s)),e(t)-e(s)-(Ew(t)-Ew(s)),p(t)-p(s))\in A(0)$. Summing (2.4.1) and (2.4.2) we get

$$\begin{aligned}Q(\alpha(s),e(t)-e(s))\le\frac{1}{2}\langle\big[\mathbb{C}(\alpha(s))-\mathbb{C}(\alpha(t))\big]e(t),e(t)\rangle-\langle\sigma(s),Ew(t)-Ew(s)\rangle\\+\int_s^t\langle\sigma(\tau),E\dot w(\tau)\rangle\,\mathrm{d}\tau+\mathcal{H}(\alpha(s),p(t)-p(s))-\mathcal{H}(\alpha(t),p(t)-p(s))\end{aligned}$$

which implies

$$\|e(t)-e(s)\|_2^2\le C\Big(\|\alpha(t)-\alpha(s)\|_\infty+\omega(\|\alpha(t)-\alpha(s)\|_\infty)+\|Ew(t)-Ew(s)\|_2\Big)\,, \tag{2.4.3}$$

where $\omega$ was introduced in (2.1.17) and $C$ depends on $\mathrm{Lip}(\mathbb{C})$, $\gamma_1$, $\gamma_2$, and $\sup_t\|e(t)\|_2$ (recall that, from (qs2), the variation of $p$ is bounded by such a $C$).

By (2.4.1), (2.1.16), and (2.4.3), we obtain

$$\|p(t)-p(s)\|_1\le\widetilde{C}\Big(\|\alpha(t)-\alpha(s)\|_\infty+\omega(\|\alpha(t)-\alpha(s)\|_\infty)+\|Ew(t)-Ew(s)\|_2\Big)\,,$$

$\widetilde{C}$ depending on $C$, $r$, and $\sup_t\|Ew(t)\|_2$. An analogous estimate holds for $u$, arguing as in [**28**, Theorem 3.8]. Then we conclude by Remark 2.3.2, where it is stated that the discontinuity points of $t\mapsto\alpha(t)$ with respect to the uniform convergence in $\overline{\Omega}$ are countable many. ☐

In order to establish the differential formulation of the energy balance the following remark turns to be useful. Moreover it allows us to explain the role of $\lambda$ in the model.

REMARK 2.4.2. If in addition $p\in AC([0,T];M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D))$ then

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;0,t)=\int_0^t\mathcal{H}(\alpha(s),\dot p(s))\,\mathrm{d}s \tag{2.4.4}$$

for every $t\in[0,T]$.

Indeed this follows from Lemma 1.2.1(4), since $\alpha\colon[0,T]\to\mathrm{C}(\overline{\Omega};[0,1])$ has at most countable many discontinuity points.

In the light of (2.4.4), we point out that the term in $\lambda\widehat{\mathcal{V}}_{\mathcal{H}}$ in (qs1) makes it easier to damage, at a given instant $t$, a part of the material more affected by plastic evolution until $t$: indeed, if $p\in AC([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_D))$ and $\beta\in C(\overline{\Omega};[0,1])$, we get that

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\beta,p;0,t)=\int_\Omega\int_0^t H(\beta(x),\dot p(s,x))\,\mathrm{d}s\,\mathrm{d}x\,.$$

To fix the ideas, let us consider the simplest case of a multiplicative setting (see (2.2.5)) where $K(1)=B(1)$, the unit ball of $\mathbb{M}^{n\times n}_D$. Here the above formula reads as

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\beta,p;0,t)=\int_\Omega V(\beta(x))\Big(\int_0^t|\dot p(s,x)|\,\mathrm{d}s\Big)\,\mathrm{d}x\,.$$

By the monotonicity property of $V$, in order to minimize $\widehat{\mathcal{V}}_{\mathcal{H}}(\beta,p;0,t)$ in (qs1) it is convenient to take $\beta$ smaller when the *cumulated plastic strain* $\int_0^t |\dot{p}(s,\cdot)|\,\mathrm{d}s$ is greater. Therefore the parameter $\lambda$ is related to a fatigue phenomenon; when $\lambda$ increases the cumulated plastic strain affects more seriously the damage growth.

The stability condition (qs1) and Lemma 2.2.3 imply the following result, which states Euler conditions with respect to variations of $u$, $e$, and $p$: (2.4.5a) is the equilibrium condition, while (2.4.5b) gives a constraint for the elastic stress.

COROLLARY 2.4.3. *Let $t\in[0,T]\mapsto(\alpha(t),u(t),e(t),p(t))$ be a quasistatic evolution corresponding to $\lambda\in[0,1]$. Then we have that for every $t\in[0,T]$:*

$$\mathrm{div}\,(\sigma(t)) = 0 \ \textit{in}\ \Omega\,, \tag{2.4.5a}$$

$$\sigma(t)\in\mathcal{K}_{\alpha(t)}(\Omega)\,. \tag{2.4.5b}$$

Let us now assume the multiplicative setting, namely (2.2.5) holds, $C^1$ regularity for $\mathbb{C}$, $D$, $V$, and absolute continuity for the quasistatic evolution. Then we can obtain a differential condition also for the damage variable $\alpha$ and a differential formulation of the energy balance.

PROPOSITION 2.4.4. *Besides the assumptions* (2.1.5), (2.2.3), (2.1.8), *and* (2.1.10), *let us assume that*

$$d\in C^1(\mathbb{R})\,, \tag{2.4.6a}$$

$$\mathbb{C}\in C^1(\mathbb{R};Lin(\mathbb{M}^{n\times n}_{sym};\mathbb{M}^{n\times n}_{sym}))\,, \tag{2.4.6b}$$

$$K(\alpha)=V(\alpha)K(1),\ \textit{with}\ K(1)\ \textit{closed and convex,}\ B_r(0)\subset K(1)\subset B_R(0),\ V\in C^1(\mathbb{R})\,. \tag{2.4.6c}$$

*Let $t\in[0,T]\mapsto(\alpha(t),u(t),e(t),p(t))$ be a quasistatic evolution corresponding to $\lambda\in[0,1]$ absolutely continuous into $W^{1,\gamma}(\Omega;[0,1])\times BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$. Then for every $t$ the functional $\beta\mapsto\widehat{\mathcal{V}}_{\mathcal{H}}(\beta,p;0,t)$ belongs to $C^1(C(\overline{\Omega}))$ and $W^{1,\gamma}(\Omega)\ni\beta\mapsto\mathcal{E}_\lambda(\beta,e(t);p,t)$ is differentiable at $\alpha(t)$ with Gâteaux derivative in the direction $\beta\in W^{1,\gamma}(\Omega)$*

$$\begin{aligned}\langle\partial_\alpha\mathcal{E}_\lambda(\alpha(t),e(t);p,t),\beta\rangle &= \frac{1}{2}\langle\mathbb{C}'(\alpha(t))\beta e(t),e(t)\rangle+\langle D'(\alpha(t)),\beta\rangle\\ &\quad+\gamma\int_\Omega|\nabla\alpha(t)|^{\gamma-2}\nabla\alpha(t)\cdot\nabla\beta\,\mathrm{d}x+\lambda\left\langle\partial_\alpha\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t),p;0,t),\beta\right\rangle\,.\end{aligned} \tag{2.4.7}$$

*Moreover the following hold:*

$$\langle\partial_\alpha\mathcal{E}_\lambda(\alpha(t),e(t);p,t),\beta\rangle\geq 0 \tag{2.4.8}$$

*for every $t\in[0,T]$ and $\beta\in W^{1,\gamma}(\Omega)$, $\beta\leq 0$ in $\Omega$,*

$$\langle\partial_\alpha\mathcal{E}_\lambda(\alpha(t),e(t);p,t),\dot{\alpha}(t)\rangle = 0\,, \tag{2.4.9}$$

*and*

$$\mathcal{H}(\alpha(t),\dot{p}(t)) = \langle(\sigma(t))_D\colon\dot{p}(t)\rangle\,, \tag{2.4.10}$$

*for a.e. $t\in(0,T)$, with $\sigma(t):=\mathbb{C}(\alpha(t))e(t)$.*

PROOF. By Dominated Convergence Theorem and (2.4.6) it follows that $\beta \mapsto \widehat{\mathcal{V}}_{\mathcal{H}}(\beta, p; 0, t) \in C^1(C(\overline{\Omega}))$ and that $W^{1,\gamma}(\Omega) \ni \beta \mapsto \mathcal{E}_\lambda(\beta, e(t); p, t)$ is differentiable at $\alpha(t)$ with Gâteaux derivative given by (2.4.7).

Let $t \in [0,T]$ and $\beta \in W^{1,\gamma}(\Omega)$, with $\beta \leq 0$ in $\Omega$. Using $(\alpha(t) + h\beta, (u(t), e(t), p(t)))$ as a test pair in (qs1) for every $h > 0$, we get

$$\frac{\mathcal{E}_\lambda(\alpha(t) + h\beta, e(t); p, t) - \mathcal{E}_\lambda(\alpha(t), e(t); p, t)}{h} \geq 0\,,$$

and taking the limit as $h \to 0$ we deduce (2.4.8).

By [**28**, Lemma 5.5] we have that for a.e. $t \in (0,T)$

$$(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in A(\dot{w}(t))\,. \tag{2.4.11}$$

Thus, by (2.4.5a), (2.4.11), and the integration by parts formula (2.1.21) we get

$$\langle (\sigma(t))_D \,|\, \dot{p}(t) \rangle = \langle \sigma(t), E\dot{w}(t) - \dot{e}(t) \rangle \tag{2.4.12}$$

and by (qs2), recalling (2.4.4), it follows that for a.e. $t \in (0,T)$

$$\langle \sigma(t), \dot{e}(t) \rangle + \mathcal{H}(\alpha(t), \dot{p}(t)) + \langle \partial_\alpha \mathcal{E}_\lambda(\alpha(t), e(t); p, t), \dot{\alpha}(t) \rangle = \langle \sigma(t), E\dot{w}(t) \rangle\,. \tag{2.4.13}$$

From (2.4.12) and (2.4.13) we obtain that

$$\mathcal{H}(\alpha(t), \dot{p}(t)) - \langle (\sigma(t))_D \,|\, \dot{p}(t) \rangle + \langle \partial_\alpha \mathcal{E}_\lambda(\alpha(t), e(t); p, t), \dot{\alpha}(t) \rangle = 0 \tag{2.4.14}$$

for a.e. $t \in (0,T)$. Since by (2.4.5b) and (2.1.22) it follows that

$$\mathcal{H}(\alpha(t), \dot{p}(t)) - \langle (\sigma(t))_D \,|\, \dot{p}(t) \rangle \geq 0\,,$$

we conclude (2.4.9) and (2.4.10) by (2.4.14) and (2.4.8). □

We can now use the maximal dissipation property (2.4.10) (also called Hill's maximum plastic work principle) to show the validity of the elastoplastic flow rule $\mathcal{L}^n$–a.e. on the support $\{|\dot{p}(t)| > 0\}$ of the measure $\dot{p}(t)$. The following remark is useful to prove Proposition 2.4.6.

REMARK 2.4.5. From (2.4.5b), (2.1.22), and (2.4.10) we deduce that for a.e. $t \in (0,T)$

$$H\Big(\alpha(t), \frac{\mathrm{d}\dot{p}(t)}{\mathrm{d}|\dot{p}(t)|}\Big)|\dot{p}(t)| = [\sigma_D(t) : \dot{p}(t)] \quad \text{as measures on } \Omega \cup \partial_D\Omega\,, \tag{2.4.15}$$

where the measure denoted by square brackets is defined in (2.1.20).

PROPOSITION 2.4.6 (Flow rule). *In the hypoteses of Proposition 2.4.4, for a.e.* $t \in (0,T)$

$$\frac{\mathrm{d}\dot{p}(t)}{\mathrm{d}|\dot{p}(t)|}(x) \in N_{K(\alpha(t,x))}(\sigma_D(t,x)) \quad \textit{for } \mathcal{L}^n\textit{–a.e. } x \in \{|\dot{p}(t)| > 0\}\,, \tag{2.4.16}$$

*where* $\sigma_D(t,x)$ *denotes the value of* $\sigma_D(t)$ *at the point* $x$ *and* $N_{K(\alpha(t,x))}(\sigma_D(t,x))$ *is the normal cone to the closed convex set* $K(\alpha(t,x))$ *at* $\sigma_D(t,x)$. *In particular, if* $\dot{p}(t) \in L^1(\Omega)$ *for a.e.* $t \in (0,T)$, *we have that*

$$\dot{p}(t,x) \in N_{K(\alpha(t,x))}(\sigma_D(t,x)) \quad \textit{for } \mathcal{L}^n\textit{–a.e. } x\,. \tag{2.4.17}$$

PROOF. It is enough to argue as in the proof of [**43**, Theorem 3.13]. □

CHAPTER 3

# Vanishing viscosity approach for perfect plasticity coupled with damage

## Overview of the chapter

The study of the interplay between linearized perfect plasticity [**28**] and damage [**78**], started in Chapter 2, is extended now by a vanishing viscosity approach to the existence of a quasistatic evolution satisfying, under regularity assumptions, the conditions (sf1),... ,(sf5), and (sf6") in the Introduction.

Some further regularity hypoteses on the elastoplastic parameters are assumed with respect to the presentation in Section 2.1. Beside these comments, the chapter is divided into four sections, respectively concerning the discrete time approximation, the existence of viscous approximation for the evolutions, the existence of a limit evolutions as the viscosity parameter vanishes, and the properties of this limit evolution.

The results of this chapter, obtained in collaboration with Giuliano Lazzaroni, are published in [**26**].

**Mechanical assumptions.** Throughout the chapter, we refer to Section 2.1 for the mechanical preliminaries. We now specify the additional hypoteses required, in particular the stronger regularity of the elastoplastic parameters. Notice that the energy considered here differs from the one employed in Chapter 2 only for the damage regularization.

Given a damage state $\alpha \in C(\overline{\Omega};[0,1])$ and an elastic strain $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, the *total energy* of the configuration is

$$\mathcal{E}(\alpha,e) := \begin{cases} \mathcal{Q}(\alpha,e) + D(\alpha) + \frac{\kappa}{2}|\alpha|^2_{m,2} & \text{if } \alpha \in H^m(\Omega;[0,1])\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.0.18}$$

where $D$ and $\mathcal{Q}$ are defined in (2.1.5) and (2.1.6), respectively. For the density $d$ of the integral functional $D$, we assume that

$$d \in C^2((0,+\infty);\mathbb{R}^+) \cap C([0,+\infty);\mathbb{R}^+ \cup \{+\infty\})\,, \tag{3.0.19a}$$

$$s^{2n} d(s) \to +\infty \text{ as } s \to 0^+\,. \tag{3.0.19b}$$

As for $\mathbb{C}$ in (2.1.6), beside (2.1.7) and (2.1.8), we require the additional condition that

$$\mathbb{C}\colon [0,1] \to Lin(\mathbb{M}^{n\times n}_{sym};\mathbb{M}^{n\times n}_{sym}) \text{ is of class } C^{1,1}\,. \tag{3.0.20}$$

In other words, (2.1.8a) is strenghtened by requiring that $\mathbb{C}$ is not only Lipschitz, but of class $C^{1,1}$.

The regularization for the damage variable is proportional, through the positive constant $\kappa$, to the seminorm

$$|\alpha|_{m,2}^2 := \sum_{|\beta|=m} \|D^\beta \alpha\|_2^2\,, \quad m := \left[\tfrac{n}{2}\right] + 1\,, \tag{3.0.21}$$

where $[\cdot]$ denotes the integer part. The corresponding scalar product is

$$\langle \alpha_1, \alpha_2 \rangle_{m,2} := \sum_{|\beta|=m} \langle D^\beta \alpha_1, D^\beta \alpha_2 \rangle_2\,.$$

We recall that $|\cdot|_{m,2}$ is a seminorm on the space $H^m(\Omega)$ and that the norm

$$\|\cdot\|_{m,2} := \|\cdot\|_2 + |\cdot|_{m,2}$$

is equivalent to the usual norm in $H^m(\Omega)$ defined by $\|\alpha\|_{H^m(\Omega)} := \sum_{|\beta|\leq m} \|D^\beta\alpha\|_2$. In particular, if a state has finite energy, the corresponding damage variable is in $H^m(\Omega)$, which is compactly embedded in $C(\overline{\Omega})$.

The previous assumptions imply that $\mathcal{E}$ is lower semicontinuous with respect to the uniform convergence of the damage variable and the weak$^*$-$L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ convergence of the elastic strain. Moreover, for every $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ the functional $H^m(\Omega;[0,1]) \ni \alpha \mapsto \mathcal{E}(\alpha, e)$ is differentiable and

$$\langle \partial_\alpha \mathcal{E}(\alpha, e), \beta \rangle = \tfrac{1}{2}\langle \mathbb{C}'(\alpha)\beta e, e\rangle + \langle \partial D(\alpha), \beta\rangle + \kappa \langle \alpha, \beta \rangle_{m,2} \tag{3.0.22}$$

for every $\beta \in H^m(\Omega)$, where $\partial D(\alpha) \in M_b(\overline{\Omega})$ is the differential of $D$ at $\alpha$, given by $\langle \partial D(\alpha), \beta \rangle = \int_\Omega d'(\alpha(x))\beta(x)\,\mathrm{d}x$.

Now, we state some conditions that allows us to differentiate also the plastic potential $\mathcal{H}$ with respect to $\alpha$. The constraint sets $(K(\alpha))_{\alpha\in[0,1]}$ are required to satisfy (2.1.10). Moreover, the condition (2.1.10c) is replaced by the following one, which is stronger:

$$\mathrm{d}_{\mathscr{H}}(K(\alpha_1), K(\alpha_2)) \leq C_K |\alpha_1 - \alpha_2| \text{ for every } \alpha_1,\, \alpha_2 \in [0,1]\,, \tag{3.0.23}$$

where $C_K$ is a positive constant and $\mathrm{d}_{\mathscr{H}}$ is the Hausdorff distance.

Therefore, for the support functions $H(\alpha,\xi) := \sup_{\sigma\in K(\alpha)} \sigma : \xi$, not only (2.1.12) hold, but also that

$$0 \leq H(\alpha_2,\xi) - H(\alpha_1,\xi) \leq C_K(\alpha_2 - \alpha_1) \text{ for } 0 \leq \alpha_1 \leq \alpha_2 \leq 1 \text{ and } \xi \in \mathbb{M}^{n\times n}_D\,, |\xi| = 1\,, \tag{3.0.24}$$

as proved in lemma below.

LEMMA 3.0.7. *If $K$ satisfies* (2.1.10) *and* (3.0.23)*, then* (3.0.24) *holds for $H$.*

PROOF. Let us fix $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $\xi \in \mathbb{M}^{n\times n}_D$ with $|\xi| = 1$, and let $\Big(E_1, \dots, E_N\Big)$, $N := \frac{n(n+1)}{2} - 1$, be an orthonormal basis of $\mathbb{M}^{n\times n}_D$ with $\xi = E_1$. Hence, for every $\alpha$

$$H(\alpha, E_1) = \max_{\sigma\in K(\alpha)} \sigma^1\,,$$

where $\sigma^i$ is the $i$-th component of $\sigma$ in the choosen basis. Since the constraint sets are closed we have that $K(\alpha_2)$ is contained in the tubular neighbourhood $\bigcup_{\sigma\in K(\alpha_1)} \overline{B}(\sigma, \mathrm{d}_{\mathscr{H}}(K(\alpha_1), K(\alpha_2)))$ of $K(\alpha_1)$. Then for all $\sigma \in K(\alpha_2)$ we have $\sigma^1 \leq H(\alpha_1, E_1) + \mathrm{d}_{\mathscr{H}}(K(\alpha_1), K(\alpha_2))$; assuming the opposite would imply $\sigma \notin \bigcup_{\sigma \in K(\alpha_1)} \overline{B}(\sigma, \mathrm{d}_{\mathscr{H}}(K(\alpha_1), K(\alpha_2)))$. Taking the supremum for $\sigma \in K(\alpha_2)$ we get

$$H(\alpha_2, \xi) - H(\alpha_1, \xi) \leq \mathrm{d}_{\mathscr{H}}(K(\alpha_1), K(\alpha_2)) \text{ for every } |\xi| = 1,$$

and together with (2.1.10b) and (3.0.23) we get (3.0.24). □

For some of the results (case $\lambda \in (0,1]$ in the following Sections) we will make the additional assumptions that

$$\xi \mapsto H(\alpha_2, \xi) - H(\alpha_1, \xi) \text{ is convex for every } 0 \leq \alpha_1 \leq \alpha_2 \leq 1\,, \tag{3.0.25a}$$

$$\alpha \mapsto H(\alpha, \xi) \in C^{1,1}([0,1]) \text{ and } |\partial_\alpha H(\alpha_2, \xi) - \partial_\alpha H(\alpha_1, \xi)| \leq \overline{C}_K |\alpha_1 - \alpha_2| \text{ for } |\xi| = 1\,, \tag{3.0.25b}$$

with $\overline{C}_K$ uniform with respect to $\alpha$ and $\xi$.

All the previous assumptions are satisfied in the usual multiplicative example: $K(\alpha) := V(\alpha)K(1)$, for $V \in C^{1,1}([0,1]; [\overline{m}, \overline{M}])$ nondecreasing, $0 < \overline{m} < \overline{M}$, so $H(\alpha, \xi) = V(\alpha)H(1, \xi)$.

The plastic potential $\mathcal{H}(\alpha, p)$, introduced in (2.1.13), now has also the property that

$$0 \leq \mathcal{H}(\alpha_2, p) - \mathcal{H}(\alpha_1, p) \leq C_K \|\alpha_1 - \alpha_2\|_\infty \|p\|_1 \quad \text{for } 0 \leq \alpha_1 \leq \alpha_2 \leq 1\,. \tag{3.0.26}$$

Furthermore, under the additional hypothesis (3.0.25), the functional $C(\overline{\Omega}; [0,1]) \ni \alpha \mapsto \mathcal{H}(\alpha, p)$ is differentiable, $\partial_\alpha H$ is convex in the second variable, and $\partial_\alpha \mathcal{H}(\alpha, p) \in M_b(\overline{\Omega})$ is given by

$$\langle \partial_\alpha \mathcal{H}(\alpha, p), \beta \rangle = \int_{\Omega \cup \partial_D \Omega} \partial_\alpha H\Big(\alpha(x), \frac{\mathrm{d}p}{\mathrm{d}|p|}(x)\Big) \beta(x) \,\mathrm{d}|p|(x) \quad \text{for every } \beta \in C(\overline{\Omega})\,; \tag{3.0.27}$$

thus by (3.0.25b)

$$\|\partial_\alpha \mathcal{H}(\alpha, p)\|_1 \leq \overline{R} \|p\|_1\,,$$

for a suitable $\overline{R}$ depending only on $H$, and

$$\|\partial_\alpha \mathcal{H}(\alpha_1, p) - \partial_\alpha \mathcal{H}(\alpha_2, p)\|_1 \leq \overline{C}_K \|\alpha_1 - \alpha_2\|_\infty \|p\|_1\,. \tag{3.0.28}$$

Consider now the plastic dissipation $\mathcal{V}_\mathcal{H}(\alpha, p; s, t)$ in the time interval $[s,t]$, defined in (2.1.25). By Lemma 1.2.1, if $p \in AC([s,t]; M_b(\Omega \cup \partial_D \Omega; \mathbb{M}^{n\times n}_D))$ and $\alpha$ is strongly continuous (with respect to the strong topology of $C(\overline{\Omega})$) at a.e. $\tau \in [s,t]$, then

$$\mathcal{V}_\mathcal{H}(\alpha, p; s, t) = \int_s^t \mathcal{H}(\alpha(\tau), \dot{p}(\tau)) \,\mathrm{d}\tau\,.$$

Notice that the condition on $\alpha$ is satisfied if $\alpha \in L^\infty(0,T; H^m(\Omega))$ and it is increasing in time, using Lemma 1.2.2 and the compact embedding of $H^m(\Omega)$ into $C(\overline{\Omega})$. Moreover, under the additional assumption (3.0.25), the functional $C(\overline{\Omega}; [0,1]) \ni \alpha \mapsto \widehat{\mathcal{V}}_\mathcal{H}(\alpha, p; s, t)$ is differentiable and

$$\Big\langle \partial_\alpha \widehat{\mathcal{V}}_\mathcal{H}(\alpha, p; s, t), \beta \Big\rangle = \int_s^t \langle \partial_\alpha \mathcal{H}(\alpha, \dot{p}(\tau)), \beta \rangle \,\mathrm{d}\tau \tag{3.0.29}$$

for every $\beta \in C(\overline{\Omega})$. (See also (3.0.27) for the expression of $\partial_\alpha \mathcal{H}(\alpha, \dot{p}(\tau))$.)

Consequently, we can deduce some properties for the generalized energy

$$\mathcal{E}_\lambda(\alpha, e; p, t) := \mathcal{E}(\alpha, e) + \lambda \widehat{\mathcal{V}}_\mathcal{H}(\alpha, p; 0, t)\,, \tag{3.0.30}$$

where $\lambda \in [0,1]$ (recall that, when the parameter $\lambda$ varies between zero and one, we account for different possible couplings between damage and plasticity). We notice here that, assuming $p \in AC([0,t]; M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D))$ and (3.0.25), by (3.0.22) and (3.0.29) the functional $H^m(\Omega;[0,1]) \ni \alpha \mapsto \mathcal{E}_\lambda(\alpha, e; p, t)$ is differentiable and

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha, e; p, t), \beta\rangle = \tfrac{1}{2}\langle \mathbb{C}'(\alpha)\beta e, e\rangle + \langle \partial D(\alpha), \beta\rangle + \kappa\langle \alpha, \beta\rangle_{m,2} + \lambda \int_0^t \Big\langle \partial_\alpha \mathcal{H}(\alpha, \dot{p}(s)), \beta\Big\rangle \,\mathrm{d}s \tag{3.0.31}$$

for every $\beta \in H^m(\Omega)$.

Eventually, we state some conditions for the initial values $\alpha_0$, $u_0$, $e_0$, $p_0$ for damage, displacement, elastic, and plastic strain, respectively. Precisely, we require that

$$\alpha_0 \in H^{2m}(\Omega;[c,1])\,,\ e_0 \in L^4(\Omega)\,,\ (u_0, e_0, p_0) \in A(w(0))\,,\ \sigma_0 \in \mathcal{K}_{\alpha_0}(\Omega)\,,\ \mathrm{div}\,\sigma_0 = 0 \text{ in } \Omega\,, \tag{3.0.32}$$

with $c > 0$ and $\sigma_0 := \mathbb{C}(\alpha_0)e_0$. Then the differential $\partial_\alpha \mathcal{E}(\alpha_0, e_0)$ given according to (3.0.22) by

$$\langle \partial_\alpha \mathcal{E}(\alpha_0, e_0), \beta\rangle = \tfrac{1}{2}\langle \mathbb{C}'(\alpha_0)\beta e_0, e_0\rangle_2 + \langle \partial D(\alpha_0), \beta\rangle + \kappa\langle \alpha_0, \beta\rangle_{m,2}$$

for every $\beta \in C(\overline{\Omega})$, is represented by an $L^2$ function.

## 3.1. Discrete-time viscous approximation

To show the existence of quasistatic evolutions, we adopt the well-known method of vanishing viscosity, thus we study an approximate problem containing a viscous term driven by a (small) parameter $\varepsilon > 0$. The existence of viscous approximations is proved by time-discretization through an incremental scheme. Therefore, we divide the time interval introducing $k$ equispaced nodes, solve a unilateral minimum problem (3.1.1) including the viscous dissipation, and take a piecewise affine interpolant of the resulting approximants; this is contained in the present section, together with some a-priori estimates on the approximants, which allow the passage to the limit as $k \to \infty$ and as $\varepsilon \to 0$, performed respectively in Section 3.2 and 3.3.

In particular, for the piecewise affine interpolants we show, using an argument developed in **[99]**, that the time derivatives of the strains are bounded by the time derivatives of the damage and of the external loading, up to a multiplicative constant independent of $k$ and $\varepsilon$ (see Lemma 3.1.6). Combining this estimate with arguments similar to **[61]** allows us to prove that the approximate evolutions are $H^1$ in time uniformly with respect to $k$ for $\varepsilon$ fixed and that they are absolutely continuous in time, uniformly with respect to $\varepsilon$, too.

Henceforth, we always assume that (3.0.19), (2.1.8), (2.1.10) hold and that $w$ and $(\alpha_0, u_0, e_0, p_0)$ satisfy (2.1.27) and (3.0.32), respectively. For some of the results (case $\lambda \in (0,1]$) we will require also (3.0.25).

**The incremental scheme.** We set a sequence of subdivisions of the interval $[0,T]$ by fixing equispaced nodes $(t^i_k)_{0\le i\le k}$,

$$t^i_k := \tfrac{i}{k}T\,.$$

For every $k$, we set $(\alpha_k^0,(u_k^0,e_k^0,p_k^0)) := (\alpha_0,(u_0,e_0,p_0))$ and for $i=1,\dots,k$ we define $(\alpha_k^i,(u_k^i,e_k^i,p_k^i))$ as a solution to the incremental problem

$$\min\left\{\mathcal{E}_\lambda(\beta,\eta;p_k,t_k^{i-1})+\mathcal{H}(\beta,q-p_k^{i-1})+\tfrac{\varepsilon}{2\tau}\|\beta-\alpha_k^{i-1}\|_2^2:\ (\beta,(u,\eta,q))\in\mathcal{D}(\alpha_k^{i-1})\times A(w_k^i)\right\}, \tag{3.1.1}$$

where $\tau=\tau_k:=\frac{1}{k}$ and we have used the following interpolants:

$$w_k^i := w(t_k^i) \quad \text{for every } i=0,\dots,k,$$

$$p_k(t):=p_k^j+\tfrac{t-t_k^j}{\tau}(p_k^{j+1}-p_k^j) \quad \text{for } t\in[t_k^j,t_k^{j+1}) \text{ and } j=0,\dots,k-1. \tag{3.1.2}$$

We remark that, according to (3.0.30) and using (2.1.14) to evaluate the dissipation of a piecewise affine function,

$$\mathcal{E}_\lambda(\beta,e;p_k,t_k^i)=\mathcal{E}(\beta,e)+\lambda\sum_{j=1}^{i}\mathcal{H}(\beta,p_k^j-p_k^{j-1}) \quad \text{for } i=1,\dots,k. \tag{3.1.3}$$

The existence of solutions to problem (3.1.1) can be proved as in Theorem 2.2.1 with straightforward modifications to account for the viscous term. In the following Lemma we collect some properties of discrete solutions which follow from Lemmas 2.2.2 and 2.2.3, [**28**, Theorem 3.6], and [**99**, Lemma 3.2].

LEMMA 3.1.1. *If $(\alpha_k^i,(u_k^i,e_k^i,p_k^i))$ is a solution to problem* (3.1.1)*, then the following equivalent conditions hold:*

*(a) $-\mathcal{H}(\alpha_k^i,p)\le\langle\mathbb{C}(\alpha_k^i)e_k^i,e\rangle\le\mathcal{H}(\alpha_k^i,-p)$ for every $(u,e,p)\in A(0)$,*

*(b) $\mathbb{C}(\alpha_k^i)e_k^i\in\mathcal{K}_{\alpha_k^i}(\Omega)$, $\operatorname{div}(\mathbb{C}(\alpha_k^i)e_k^i)=0$ in $\Omega$, $[(\mathbb{C}(\alpha_k^i)e_k^i)\nu]=0$ on $\partial_N\Omega$.*

*Moreover,*

$$\mathcal{E}_\lambda(\alpha_k^i,e_k^i;p_k,t_k^{i-1})+\tfrac{\varepsilon}{2\tau}\|\alpha_k^i-\alpha_k^{i-1}\|_2^2\le\mathcal{E}_\lambda(\beta,\eta;p_k,t_k^{i-1})+\mathcal{H}(\beta,q-p_k^i)+\tfrac{\varepsilon}{2\tau}\|\beta-\alpha_k^{i-1}\|_2^2$$

*for every $(\beta,(u,\eta,q))\in\mathcal{D}(\alpha_k^i)\times A(w_k^i)$, and*

$$\mathcal{Q}(\alpha_k^i,e_k^i)+\mathcal{Q}(\alpha_k^i,\eta-e_k^i)\le\mathcal{Q}(\alpha_k^i,\eta)+\mathcal{H}(\alpha_k^i,q-p_k^i) \tag{3.1.4}$$

*for every $(u,\eta,q)\in A(w_k^i)$.*

Notice that we shall employ in the sequel only the latter of the equivalent conditions (a) and (b) above. We define the following piecewise constant and piecewise affine interpolants:

$$\begin{aligned}&\underline{\alpha}_k(t):=\alpha_k^i,\ \underline{u}_k(t):=u_k^i,\ \underline{e}_k(t):=e_k^i,\\ &\underline{p}_k(t):=p_k^i,\ \underline{\sigma}_k(t):=\mathbb{C}(\alpha_k^i)e_k^i,\ \underline{w}_k(t):=w_k^i \quad \text{for } t\in[t_k^i,t_k^{i+1}),\end{aligned} \tag{3.1.5a}$$

$$\begin{aligned}&\overline{\alpha}_k(t):=\alpha_k^i,\ \overline{u}_k(t):=u_k^i,\ \overline{e}_k(t):=e_k^i,\\ &\overline{p}_k(t):=p_k^i,\ \overline{\sigma}_k(t):=\mathbb{C}(\alpha_k^i)e_k^i,\ \overline{w}_k(t):=w_k^i \quad \text{for } t\in(t_k^{i-1},t_k^i],\end{aligned} \tag{3.1.5b}$$

$$\begin{aligned}&\alpha_k(t):=\alpha_k^i+\tfrac{t-t_k^i}{\tau}(\alpha_k^{i+1}-\alpha_k^i),\ u_k(t):=u_k^i+\tfrac{t-t_k^i}{\tau}(u_k^{i+1}-u_k^i),\ e_k(t):=e_k^i+\tfrac{t-t_k^i}{\tau}(e_k^{i+1}-e_k^i),\\ &\sigma_k(t):=\mathbb{C}(\alpha_k(t))e_k(t),\ w_k(t):=w_k^i+\tfrac{t-t_k^i}{\tau}(w_k^{i+1}-w_k^i) \quad \text{for } t\in[t_k^i,t_k^{i+1}).\end{aligned} \tag{3.1.5c}$$

(Recall also the definition of $p_k$ from (3.1.2).) Definitions (3.1.5a) and (3.1.5c) are given for $i = 0 \dots k{-}1$, and (3.1.5b) for $i = 1 \dots k$ instead. We define $\underline{\alpha}_k(T) = \alpha_k(T) := \alpha_k^k$ and $\overline{\alpha}_k(0) := \alpha_0$, and the same for the other interpolants. By definition $\underline{\alpha}_k$, $\overline{\alpha}_k$, and $\alpha_k$ are non-increasing in time; moreover, $(\underline{u}_k(t), \underline{e}_k(t), \underline{p}_k(t)) \in A(\underline{w}_k(t))$, $(\overline{u}_k(t), \overline{e}_k(t), \overline{p}_k(t)) \in A(\overline{w}_k(t))$, and $(u_k(t), e_k(t), p_k(t)) \in A(w_k(t))$ for every $t \in [0, T]$. We shall also use the notation

$$\underline{\tau}_k(t) := t_k^i \text{ if } t \in [t_k^i, t_k^{i+1}), \quad \overline{\tau}_k(t) := t_k^{i+1} \text{ if } t \in (t_k^i, t_k^{i+1}].$$

**The discrete energy inequality.** We now derive an energy estimate for the solutions of the incremental problems. Let us fix $i \in \{1, \dots, k\}$ and for a given integer $h$ with $1 \le h \le i$ let $u := u_k^{h-1} - w_k^{h-1} + w_k^h$ and $\eta := e_k^{h-1} - Ew_k^{h-1} + Ew_k^h$. Since $(\alpha_k^{h-1}, (u, \eta, p_k^{h-1})) \in \mathcal{D}(\alpha_k^{h-1}) \times A(w_k^h)$, by the minimality condition (3.1.1) we obtain

$$\begin{aligned} &\mathcal{E}_\lambda(\alpha_k^h, e_k^h; p_k, t_k^{h-1}) + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) + \tfrac{\varepsilon}{2\tau}\|\alpha_k^h - \alpha_k^{h-1}\|_2^2 \\ &\quad \le \mathcal{E}_\lambda(\alpha_k^{h-1}, e_k^{h-1}; p_k, t_k^{h-1}) + \langle \sigma_k^{h-1}, Ew_k^h - Ew_k^{h-1}\rangle + \mathcal{Q}(\alpha_k^{h-1}, Ew_k^h - Ew_k^{h-1}), \end{aligned} \tag{3.1.6}$$

where we have used the identity

$$\mathcal{Q}(\alpha, e_1 + e_2) = \mathcal{Q}(\alpha, e_1) + \langle \mathbb{C}(\alpha) e_1, e_2 \rangle + \mathcal{Q}(\alpha, e_2),$$

which holds for every $\alpha \in H^m(\Omega; [0,1])$ and $e_1$, $e_2 \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. From the absolute continuity of $w$ with respect to $t$ we obtain

$$w_k^h - w_k^{h-1} = \int_{t_k^{h-1}}^{t_k^h} \dot w(t)\, \mathrm{d}t,$$

using the notion of Bochner integral for functions with values in $H^1(\mathbb{R}^n; \mathbb{R}^n)$. This implies that

$$Ew_k^h - Ew_k^{h-1} = \int_{t_k^{h-1}}^{t_k^h} E\dot w(t)\, \mathrm{d}t, \tag{3.1.7}$$

where the integral is again in the sense of Bochner and the target space is $L^2(\mathbb{R}^n; \mathbb{M}^{n\times n}_{sym})$. By the continuity of $\mathcal{Q}$ and (3.1.7) we get

$$\mathcal{Q}(\alpha_k^{h-1}, Ew_k^h - Ew_k^{h-1}) \le \gamma_2 \Big( \int_{t_k^{h-1}}^{t_k^h} \|E\dot w(t)\|_2\, \mathrm{d}t \Big)^2.$$

Since

$$\lambda \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k^h, p_k, 0, t_k^{h-1}) + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) = \lambda \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k^h, p_k, 0, t_k^h) + (1-\lambda)\mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}), \tag{3.1.8}$$

from (3.1.3), (3.1.6), and (3.1.7) it follows that

$$\begin{aligned} &\mathcal{E}_\lambda(\alpha_k^h, e_k^h; p_k, t_k^h) + (1-\lambda)\mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) + \frac{\varepsilon}{2\tau}\|\alpha_k^h - \alpha_k^{h-1}\|_2^2 \\ &\quad \le \mathcal{E}_\lambda(\alpha_k^{h-1}, e_k^{h-1}; p_k, t_k^{h-1}) + \int_{t_k^{h-1}}^{t_k^h} \langle \sigma_k^{h-1}, E\dot w(t)\rangle\, \mathrm{d}t + \omega_k \int_{t_k^{h-1}}^{t_k^h} \|E\dot w(t)\|_2\, \mathrm{d}t, \end{aligned}$$

where

$$\omega_k := \gamma_2 \max_{1\le h\le k} \int_{t_k^{h-1}}^{t_k^h} \|E\dot w(t)\|_2\, \mathrm{d}t \to 0$$

by the absolute continuity of the integral. Iterating now the latter inequality for $1 \le h \le i$ amounts to the following property.

PROPOSITION 3.1.2. *For every* $i = 1, \dots, k$

$$\begin{aligned} &\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^i) + (1-\lambda)\sum_{h=1}^{i} \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) + \sum_{h=1}^{i} \frac{\varepsilon}{2\tau}\|\alpha_k^h - \alpha_k^{h-1}\|_2^2 \\ &\le \mathcal{E}(\alpha_0, e_0) + \int_0^{t_k^i} \langle \underline{\sigma}_k(s), E\dot{w}(s)\rangle \, \mathrm{d}s + \delta_k \,, \end{aligned} \tag{3.1.9}$$

*where* $\delta_k := \omega_k \int_0^T \|E\dot{w}(t)\|_2 \,\mathrm{d}t \to 0$.

**A priori estimates.** Using (2.1.8c) and (2.1.12d) in the left-hand side of (3.1.9), as well as (2.1.9) and the fact that the function $t \mapsto \|E\dot{w}(t)\|_2$ is integrable on $[0,T]$ in the right-hand side, we find that for every $k \in \mathbb{N}$ and $t \in (t_k^{i-1}, t_k^i]$

$$\begin{aligned} &\gamma_1\|\underline{e}_k(t)\|_2^2 + D(\underline{\alpha}_k(t)) + \frac{\kappa}{2}|\underline{\alpha}_k(t)|_{m,2}^2 + r(1-\lambda)\sum_{h=1}^{i}\|p_k^h - p_k^{h-1}\|_1 + \frac{\varepsilon}{2}\int_0^{t_k^i}\|\dot{\alpha}_k(s)\|_2^2\,\mathrm{d}s \\ &\le \mathcal{E}(\alpha_0, e_0) + 2\gamma_2 \sup_{t\in[0,T]}\|\underline{e}_k(t)\|_2 \int_0^T \|E\dot{w}(s)\|_2\,\mathrm{d}s + \delta_k \,. \end{aligned}$$

Thus, by the Cauchy inequality,

$$\sup_{t\in[0,T]} \|\underline{e}_k(t)\|_2 \le C \,. \tag{3.1.10}$$

Henceforth, $C$ denotes a suitable constant depending only on $\gamma_1, \gamma_2, r$, and on the functions $\alpha_0$, $e_0$, and $w$. We immediately deduce that

$$\sup_{t\in[0,T]} D(\underline{\alpha}_k(t)) \le C \,, \tag{3.1.11a}$$

$$\sup_{t\in[0,T]} \|\underline{\alpha}_k(t)\|_{m,2} \le C \,, \tag{3.1.11b}$$

$$\varepsilon \int_0^T \|\dot{\alpha}_k(s)\|_2^2\,\mathrm{d}s \le C \,, \tag{3.1.11c}$$

and, from the definitions of the interpolants, that

$$\mathcal{V}(\underline{p}_k; 0,T) = \mathcal{V}(\overline{p}_k; 0,T) = \mathcal{V}(p_k; 0,T) = \sum_{i=1}^{k}\|p_k^i - p_k^{i-1}\|_1 \le C \,. \tag{3.1.12}$$

Notice that analogous estimates to (3.1.10), (3.1.11a), (3.1.11b) also hold for the other interpolants from (3.1.5b) and (3.1.5c).

Next we show a bound from below on the damage variable, thanks to assumption (3.0.19).

LEMMA 3.1.3. *There exists* $m_0 > 0$ *independent of* $\varepsilon$, $k$, $t$, *such that*

$$\underline{\alpha}_k(t) \ge m_0 \,, \quad \overline{\alpha}_k(t) \ge m_0 \,, \quad \alpha_k(t) \ge m_0 \quad \text{in } \overline{\Omega} \tag{3.1.13}$$

*for every* $k \in \mathbb{N}$ *and* $t \in [0,T]$.

PROOF. By (3.1.11b) and the continuous immersion $H^m(\Omega) \subset C^{0,1/2}(\Omega)$, cf. (3.0.21), there exists $\widetilde{C}$ independent of $\varepsilon$, $k$, $t$, with

$$|\underline{\alpha}_k(t,x) - \underline{\alpha}_k(t,y)| \leq \widetilde{C}|x-y|^{1/2} \quad \text{for every } x, y \in \Omega\,.$$

Let $M > 0$; by (3.0.19b), there exists $\overline{\delta} > 0$ such that $d(\delta) > M\delta^{-2n}$ for every $0 < \delta \leq \overline{\delta}$. Assume now that (3.1.13) does not hold, so we can find $k \in \mathbb{N}$, $t \in [0,T]$, and $\overline{x} \in \Omega$ such that $\underline{\alpha}_k(t,\overline{x}) < \frac{\overline{\delta}}{2}$. Then $\underline{\alpha}_k(t,x) < \overline{\delta}$ for every $x \in B\big(\overline{x}, \big(\overline{\delta}/(2\widetilde{C})\big)^2\big)$. Therefore, $D(\underline{\alpha}_k(t)) > M\omega_n/\big(2\widetilde{C}\big)^{2n}$, where $\omega_n$ is the measure of the unit ball in $\mathbb{R}^n$. Since $M$ is arbitrary, this contradicts (3.1.11a) and proves the thesis for $\underline{\alpha}_k$. The other statements are analogous. □

By minimality, we get some differential conditions on the damage variable, which correspond to a discrete approximation of the Kuhn-Tucker conditions appearing in the following sections (cf. Definitions 3.2.1 and 3.3.1, and Propositions 3.2.3 and 3.3.5). We recall that we assume (3.0.25) when $\lambda \neq 0$; in that case we obtain the Kuhn-Tucker conditions (3.1.15). If $\lambda = 0$, we would still be able to deduce (3.1.15) assuming (3.0.25); however, without that hypothesis, we can obtain the weaker version (3.1.14), which is sufficient for the subsequent applications of the lemma.

LEMMA 3.1.4. *Let* $\varepsilon > 0$, $k \in \mathbb{N}$, *and* $t \in (0,T) \setminus \{t_k^1, \dots, t_k^{k-1}\}$.
**Case** $\lambda = 0$**.** *We have*

$$\langle \partial_\alpha \mathcal{E}(\overline{\alpha}_k(t), \overline{e}_k(t)), \beta \rangle + \varepsilon \langle \dot{\alpha}_k(t), \beta \rangle_2 \geq 0 \tag{3.1.14a}$$

*for every* $\beta \in H^m_-(\Omega) := \{\beta \in H^m(\Omega) \colon \beta \leq 0 \text{ in } \Omega\}$, *and*

$$\langle \partial_\alpha \mathcal{E}(\overline{\alpha}_k(t), \overline{e}_k(t)), \dot{\alpha}_k(t) \rangle + \varepsilon \|\dot{\alpha}_k(t)\|_2^2 \leq C_K\, \tau \|\dot{\alpha}_k(t)\|_\infty \|\dot{p}_k(t)\|_1\,, \tag{3.1.14b}$$

*with* $C_K$ *introduced in* (3.0.23).

**Case** $\lambda \in (0,1]$**.** *Under the additional assumption* (3.0.25) *we have*

$$\langle \partial_\alpha \mathcal{E}_\lambda(\overline{\alpha}_k(t), \overline{e}_k(t); p_k, \underline{t}_k(t)), \beta \rangle + \tau \langle \partial_\alpha \mathcal{H}(\overline{\alpha}_k(t), \dot{p}_k(t)), \beta \rangle + \varepsilon \langle \dot{\alpha}_k(t), \beta \rangle_2 \geq 0 \tag{3.1.15a}$$

*for every* $\beta \in H^m_-(\Omega)$, *and*

$$\langle \partial_\alpha \mathcal{E}_\lambda(\overline{\alpha}_k(t), \overline{e}_k(t); p_k, \underline{t}_k(t)), \dot{\alpha}_k(t) \rangle + \varepsilon \|\dot{\alpha}_k(t)\|_2^2 = -\tau \langle \partial_\alpha \mathcal{H}(\overline{\alpha}_k(t), \dot{p}_k(t)), \dot{\alpha}_k(t) \rangle\,. \tag{3.1.15b}$$

PROOF. Let us denote $\dot{\alpha}_k^i := \frac{\alpha_k^i - \alpha_k^{i-1}}{\tau}$. By (3.1.13), for every $\beta \in H^m_-(\Omega)$ there exists $\overline{\delta} \in (0,1)$ such that $\alpha_k^i + \delta\beta > 0$ in $\overline{\Omega}$ for every $k$, $i$, and $0 < \delta \leq \overline{\delta}$, which implies $\alpha_k^i + \delta\beta \in \mathcal{D}(\alpha_k^{i-1})$. By minimality of $\alpha_k^i$

$$\begin{aligned} 0 \leq\ & \mathcal{E}_\lambda(\alpha_k^i + \delta\beta, e_k^i; p_k, t_k^{i-1}) + \mathcal{H}(\alpha_k^i + \delta\beta, p_k^i - p_k^{i-1}) + \tfrac{\varepsilon}{2\tau}\|\alpha_k^i + \delta\beta - \alpha_k^{i-1}\|_2^2 \\ & - \Big(\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^{i-1}) + \mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) + \tfrac{\varepsilon}{2\tau}\|\alpha_k^i - \alpha_k^{i-1}\|_2^2\Big)\,. \end{aligned}$$

If $\lambda = 0$, dividing by $\delta$ and letting $\delta$ tend to 0, we get (3.1.14a), since $\mathcal{H}(\alpha_k^i + \delta\beta, p_k^i - p_k^{i-1}) \leq \mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1})$ by (3.0.26) (recall also the regularity assumptions on $\mathbb{C}$ and $D$). If $\lambda > 0$, exploiting (3.0.25) and its consequences (3.0.27) and (3.0.28), we deduce (3.1.15a) using also (2.1.12c).

Moreover, $\alpha_k^i - \delta\,\dot\alpha_k^i \in \mathcal{D}(\alpha_k^{i-1})$ for $\delta < \tau$, so

$$\begin{aligned}0 \leq \mathcal{E}_\lambda(\alpha_k^i - \delta\,\dot\alpha_k^i, e_k^i; p_k, t_k^{i-1}) + \mathcal{H}(\alpha_k^i - \delta\,\dot\alpha_k^i, p_k^i - p_k^{i-1}) + \tfrac{\varepsilon}{2\tau}\|\alpha_k^i - \alpha_k^{i-1} + \delta\,\dot\alpha_k^i\|_2^2 \\ - \Big(\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^{i-1}) + \mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) + \tfrac{\varepsilon}{2\tau}\|\alpha_k^i - \alpha_k^{i-1}\|_2^2\Big)\,.\end{aligned}$$

If $\lambda = 0$ we get

$$\langle \partial_\alpha \mathcal{E}(\overline{\alpha}_k(t), \overline{e}_k(t)), \dot\alpha_k(t)\rangle + \varepsilon\|\dot\alpha_k(t)\|_2^2 - \tau\langle \underline{\partial}_\alpha^+ \mathcal{H}(\overline{\alpha}_k(t), \dot p_k(t)), -\dot\alpha_k(t)\rangle \leq 0\,,$$

where

$$\langle \underline{\partial}_\alpha^+ \mathcal{H}(\overline{\alpha}_k(t), \dot p_k(t)), -\dot\alpha_k(t)\rangle := \liminf_{\delta\to 0^+} \frac{\mathcal{H}(\overline{\alpha}_k(t) - \delta\dot\alpha_k(t), \dot p_k(t)) - \mathcal{H}(\overline{\alpha}_k(t), \dot p_k(t))}{\delta}\,,$$

and then (3.1.14b) follows by (3.0.26). If $\lambda \in (0,1]$, since we have already proved (3.1.15a) we get (3.1.15b) using again (3.0.25). □

The next remark will turn out to be useful in the sequel.

REMARK 3.1.5. Differentiating (3.1.8) with respect to the damage variable, we get that for every $\lambda \in (0,1]$ and $\beta \in H^m(\Omega)$

$$\begin{aligned}&\lambda\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha}_k(t), p_k; 0, \underline{\tau}_k(t)), \beta\rangle + \tau\langle \partial_\alpha \mathcal{H}(\overline{\alpha}_k(t), \dot p_k(t)), \beta\rangle \\ &\quad = \lambda\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha}_k(t), p_k; 0, \overline{\tau}_k(t)), \beta\rangle + (1-\lambda)\tau\langle \partial_\alpha \mathcal{H}(\overline{\alpha}_k(t), \dot p_k(t)), \beta\rangle\,,\end{aligned}$$

and then

$$\begin{aligned}&\langle \partial_\alpha \mathcal{E}_\lambda(\overline{\alpha}_k(t), \overline{e}_k(t); p_k, \underline{\tau}_k(t)), \beta\rangle + \tau\langle \partial_\alpha \mathcal{H}(\overline{\alpha}_k(t), \dot p_k(t)), \beta\rangle \\ &\quad = \langle \partial_\alpha \mathcal{E}_\lambda(\overline{\alpha}_k(t), \overline{e}_k(t); p_k, \overline{\tau}_k(t)), \beta\rangle + (1-\lambda)\tau\langle \partial_\alpha \mathcal{H}(\overline{\alpha}_k(t), \dot p_k(t)), \beta\rangle\,.\end{aligned} \tag{3.1.16}$$

The following lemma permits to bound the norm of $\dot u_k$, $\dot e_k$, and $\dot p_k$ by the norm of $\dot\alpha_k^i$ and $\dot w_k^i$ times a constant independent of $k$ and $\varepsilon$; this will be very useful to get the estimates in Propositions 3.1.7 and 3.1.8. In the proof we adapt an argument developed in [**99**, Lemma 3.3].

LEMMA 3.1.6. *For every $k \in \mathbb{N}$ and $0 \leq i \leq k-1$ let*

$$\widetilde{\omega}_k^i := \|\alpha_k^{i+1} - \alpha_k^i\|_\infty + \|Ew_k^{i+1} - Ew_k^i\|_2\,.$$

*Then there exists a positive constant $C$ independent of $\varepsilon$, $k$, and $i$ such that*

$$\|e_k^{i+1} - e_k^i\|_2 \leq C\,\widetilde{\omega}_k^i\,, \tag{3.1.17a}$$

$$\|p_k^{i+1} - p_k^i\|_1 \leq C\,\widetilde{\omega}_k^i\,, \tag{3.1.17b}$$

$$\|Eu_k^{i+1} - Eu_k^i\|_1 \leq C\,\widetilde{\omega}_k^i\,, \tag{3.1.17c}$$

$$\|u_k^{i+1} - u_k^i\|_{BD} \leq C(\widetilde{\omega}_k^i + \|w_k^{i+1} - w_k^i\|_2)\,. \tag{3.1.17d}$$

*In particular, dividing by $\tau$, we have that for every $t \in (0,T) \setminus \{t_k^1, \dots, t_k^{k-1}\}$*

$$\|\dot e_k(t)\|_2 + \|\dot p_k(t)\|_1 + \|E\dot u_k(t)\|_1 \leq 3C(\|\dot\alpha_k(t)\|_\infty + \|E\dot w_k(t)\|_2)\,.$$

*Finally, for every $t \in (0,T) \setminus \{t_k^1, \dots, t_k^{k-1}\}$*

$$\mathcal{H}(\overline{\alpha}_k(t), \dot p_k(t)) \leq \langle (\overline{\sigma}_k(t))_D | \dot p_k(t)\rangle + C\tau\big(\|\dot\alpha_k(t)\|_\infty^2 + \|E\dot w_k(t)\|_2^2\big)\,. \tag{3.1.18}$$

Proof. By Lemma 3.1.1 we obtain $\sigma_k^i := \mathbb{C}(\alpha_k^i)e_k^i \in \mathcal{K}_{\alpha_k^i}(\Omega)$, div $\sigma_k^i = 0$ in $\Omega$, and $[\sigma_k^i\nu] = 0$ on $\partial_N\Omega$ for every $k$ and $i$. One can easily see that

$$(\alpha_k^{i+1}, (u_k^i + w_k^{i+1} - w_k^i, e_k^i + E(w_k^{i+1} - w_k^i), p_k^i)) \in \mathcal{D}(\alpha_k^i) \times A(w_k^{i+1}),$$

and then by minimality

$$\mathcal{Q}(\alpha_k^{i+1}, e_k^{i+1}) + \mathcal{H}(\alpha_k^{i+1}, p_k^{i+1} - p_k^i) \leq \mathcal{Q}(\alpha_k^{i+1}, e_k^i + E(w_k^{i+1} - w_k^i)). \tag{3.1.19}$$

Since for every $\alpha \in H^m(\Omega;[0,1])$ and every $e_1, e_2 \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$

$$\mathcal{Q}(\alpha, e_1) - \mathcal{Q}(\alpha, e_2) = \tfrac{1}{2}\langle \mathbb{C}(\alpha)(e_1 + e_2), e_1 - e_2\rangle, \tag{3.1.20}$$

recalling the integration-by-parts formula (2.1.21) and condition (b) of Lemma 3.1.1, by (3.1.19) we infer that

$$\begin{aligned}\mathcal{H}(\alpha_k^{i+1}, p_k^{i+1} - p_k^i) \leq{}& \langle \sigma_k^{i+1}, E(w_k^{i+1} - w_k^i) - (e_k^{i+1} - e_k^i)\rangle + \mathcal{Q}(\alpha_k^{i+1}, E(w_k^{i+1} - w_k^i))\\ &- \langle \mathbb{C}(\alpha_k^{i+1})(e_k^{i+1} - e_k^i), E(w_k^{i+1} - w_k^i)\rangle + \mathcal{Q}(\alpha_k^{i+1}, e_k^{i+1} - e_k^i)\\ ={}&\langle (\sigma_k^{i+1})_D \,|\, p_k^{i+1} - p_k^i\rangle + \mathcal{Q}(\alpha_k^{i+1}, E(w_k^{i+1} - w_k^i))\\ &- \langle \mathbb{C}(\alpha_k^{i+1})(e_k^{i+1} - e_k^i), E(w_k^{i+1} - w_k^i)\rangle + \mathcal{Q}(\alpha_k^{i+1}, e_k^{i+1} - e_k^i).\end{aligned} \tag{3.1.21}$$

By (3.1.21), using (2.1.16) and the Cauchy inequality we have

$$r\|p_k^{i+1} - p_k^i\|_1 \leq C_1\big(\|E(w_k^{i+1} - w_k^i)\|_2 + \|e_k^{i+1} - e_k^i\|_2\big), \tag{3.1.22}$$

where $C_1$ depends on $\gamma_2$ introduced in (2.1.8c) and on the constant in (3.1.10).

Testing (3.1.4) by $(u_k^{i+1} - (w_k^{i+1} - w_k^i), e_k^{i+1} - E(w_k^{i+1} - w_k^i), p_k^{i+1}) \in A(w_k^i)$, by simple algebraic manipulations we obtain

$$\mathcal{Q}(\alpha_k^i, e_k^i) + \mathcal{Q}(\alpha_k^i, e_k^{i+1} - e_k^i) + \langle \sigma_k^i, E(w_k^{i+1} - w_k^i)\rangle \leq \mathcal{Q}(\alpha_k^i, e_k^{i+1}) + \mathcal{H}(\alpha_k^i, p_k^{i+1} - p_k^i).$$

Using (3.1.19), it follows that

$$\begin{aligned}\mathcal{Q}(\alpha_k^i, e_k^{i+1} - e_k^i) \leq{}& \mathcal{Q}(\alpha_k^i, e_k^{i+1}) - \mathcal{Q}(\alpha_k^i, e_k^i) + \mathcal{Q}(\alpha_k^{i+1}, e_k^i) - \mathcal{Q}(\alpha_k^{i+1}, e_k^{i+1})\\ &+ \langle \big[\mathbb{C}(\alpha_k^{i+1}) - \mathbb{C}(\alpha_k^i)\big]\, e_k^i, E(w_k^{i+1} - w_k^i)\rangle + \mathcal{Q}(\alpha_k^{i+1}, E(w_k^{i+1} - w_k^i))\\ &+ \mathcal{H}(\alpha_k^i, p_k^{i+1} - p_k^i) - \mathcal{H}(\alpha_k^{i+1}, p_k^{i+1} - p_k^i).\end{aligned} \tag{3.1.23}$$

Notice now that, employing again (3.1.20),

$$\mathcal{Q}(\alpha_k^i, e_k^{i+1}) - \mathcal{Q}(\alpha_k^i, e_k^i) + \mathcal{Q}(\alpha_k^{i+1}, e_k^i) - \mathcal{Q}(\alpha_k^{i+1}, e_k^{i+1}) = \tfrac{1}{2}\langle \big[\mathbb{C}(\alpha_k^i) - \mathbb{C}(\alpha_k^{i+1})\big]\,(e_k^i + e_k^{i+1}), (e_k^{i+1} - e_k^i)\rangle. \tag{3.1.24}$$

By (3.1.23), (3.1.24), (2.1.8c), (3.0.26), and the Cauchy inequality, we deduce

$$\begin{aligned}\gamma_1\|e_k^{i+1} - e_k^i\|_2^2 \leq{}& \tfrac{\gamma_1}{4}\|e_k^{i+1} - e_k^i\|_2^2 + \tfrac{\gamma_1 r^2}{4(C_1)^2}\|p_k^{i+1} - p_k^i\|_1^2\\ &+ C_2\Big(\|\alpha_k^{i+1} - \alpha_k^i\|_\infty^2 + \|E(w_k^{i+1} - w_k^i)\|_2^2\Big),\end{aligned} \tag{3.1.25}$$

with $C_2$ depending on the constant in (3.1.10), $w$, Lip($\mathbb{C}$), $C_K$, $r$, $\gamma_1$, $\gamma_2$, $\kappa$. Thus (3.1.17a) and (3.1.17b) follow from (3.1.22) and (3.1.25). Arguing as in [**28**, Theorem 3.8], we obtain also (3.1.17c) and (3.1.17d). Finally, using (3.1.17) and the Cauchy inequality, we get (3.1.18) from (3.1.21). □

Combining Lemma 3.1.6 and some arguments from [**61**, Proposition 4.1], we prove that for $\varepsilon$ fixed the functions $\alpha_k$ are bounded in $H^1(0,T;H^m(\Omega))$, uniformly in $k$.

PROPOSITION 3.1.7. *There exists a positive constant $C$ independent of $\varepsilon$, $k$, and $t$ such that for every $\varepsilon>0$, $k\in\mathbb{N}$, $t\in(0,T)\setminus\{t_k^1,\dots,t_k^{k-1}\}$*

$$\varepsilon\|\dot\alpha_k(t)\|_2 \le Ce^{\frac{C}{\varepsilon}\overline{\tau}_k(t)}\,, \tag{3.1.26a}$$

$$\varepsilon\int_0^{\overline{\tau}_k(t)}\|\dot\alpha_k(s)\|^2_{m,2}\,\mathrm{d}s \le Ce^{\frac{C}{\varepsilon}\overline{\tau}_k(t)}\,. \tag{3.1.26b}$$

PROOF. We start from (3.1.14), denoting $\dot\alpha_k^i:=\frac{\alpha_k^i-\alpha_k^{i-1}}{\tau}$, $\dot e_k^i=\frac{e_k^i-e_k^{i-1}}{\tau}$, $\dot p_k^i=\frac{p_k^i-p_k^{i-1}}{\tau}$, and $\dot w_k^i=\frac{w_k^i-w_k^{i-1}}{\tau}$ for every $k\in\mathbb{N}$ and $1\le i\le k$. Let now fix $k$ and $i$. First we consider the case $2\le i\le k$; the case $i=1$ needs a slightly different treatment and will be considered below.

We take (3.1.14b) in the case $\lambda=0$, respectively (3.1.15b) in the case $\lambda\in(0,1]$, evaluated at $t\in(t_k^{i-1},t_k^i)$, thus $\overline{\alpha}_k(t)=\alpha_k^i$ and $\dot\alpha_k(t)=\dot\alpha_k^i$. Then we subtract (3.1.14a) (resp. (3.1.15a)) evaluated at $t\in(t_k^{i-2},t_k^{i-1})$, (thus $\overline{\alpha}_k(t)=\alpha_k^{i-1}$ and $\dot\alpha_k(t)=\dot\alpha_k^{i-1}$), and tested by $\beta:=\dot\alpha_k^i$. Recall that the differentiability of $\mathcal{E}_\lambda(\cdot,e;p,t)$ follows from (3.0.25), which is assumed if $\lambda\in(0,1]$, while for $\lambda=0$ the energy reduces to $\mathcal{E}(\alpha,e)$ and some terms disappear, see also above. We obtain that for every $2\le i\le k$ and every $\lambda\in[0,1]$

$$\begin{aligned}\varepsilon\langle\dot\alpha_k^i-\dot\alpha_k^{i-1},\dot\alpha_k^i\rangle+\kappa\langle\alpha_k^i-\alpha_k^{i-1},\dot\alpha_k^i\rangle_{m,2}\le&-\Big[\tfrac12\langle\mathbb{C}'(\alpha_k^i)\dot\alpha_k^ie_k^i,e_k^i\rangle-\tfrac12\langle\mathbb{C}'(\alpha_k^{i-1})\dot\alpha_k^ie_k^{i-1},e_k^{i-1}\rangle\Big]\\&-\langle\partial D(\alpha_k^i)-\partial D(\alpha_k^{i-1}),\dot\alpha_k^i\rangle+C_K\,\tau\|\dot\alpha_k^i\|_\infty\|\dot p_k^i\|_1\\&-\lambda\sum_{h=1}^{i-1}\big\langle\partial_\alpha\mathcal{H}(\alpha_k^i,p_k^h-p_k^{h-1})-\partial_\alpha\mathcal{H}(\alpha_k^{i-1},p_k^h-p_k^{h-1}),\dot\alpha_k^i\big\rangle\,.\end{aligned}$$

When $\lambda\in(0,1]$ the inequality stated above follows from (3.0.31) and (3.1.16) by neglecting the term $(1-\lambda)\langle\partial_\alpha\mathcal{H}(\alpha_k^{i-1},p_k^{i-1}-p_k^{i-2}),\dot\alpha_k^i\rangle$, which is negative by the softening assumption (2.1.10b) and by the monotonicity in time of $\alpha_k$. Therefore

$$\begin{aligned}\varepsilon\langle\dot\alpha_k^i-\dot\alpha_k^{i-1},\dot\alpha_k^i\rangle+\kappa\langle\alpha_k^i-\alpha_k^{i-1},\dot\alpha_k^i\rangle_{m,2}\le&\ \tfrac12\Big|\langle[\mathbb{C}'(\alpha_k^i)-\mathbb{C}'(\alpha_k^{i-1})]\dot\alpha_k^ie_k^i,e_k^i\rangle\Big|\\&+\tfrac12\Big|\langle\mathbb{C}'(\alpha_k^{i-1})\dot\alpha_k^ie_k^i,e_k^i\rangle-\langle\mathbb{C}'(\alpha_k^{i-1})\dot\alpha_k^ie_k^{i-1},e_k^{i-1}\rangle\Big|\\&+C\tau\|\dot\alpha_k^i\|^2_\infty\big(1+\mathcal{V}(\overline{p}_k;0,\underline{\tau}_k(t))\big)+C_K\,\tau\|\dot\alpha_k^i\|_\infty\|\dot p_k^i\|_1\,,\end{aligned}$$

taking into account the regularity assumptions on $\mathbb{C}$, $D$, $H$ (see (2.1.8), (3.0.20), (3.0.19), (2.1.12), (3.0.24) (3.0.25)). Using the fact that $2a(a-b)\ge a^2-b^2$ for every $a,b$, we get

$$\varepsilon\langle\dot\alpha_k^i-\dot\alpha_k^{i-1},\dot\alpha_k^i\rangle\ge\frac{\varepsilon}{2}\Big(\|\dot\alpha_k^i\|_2^2-\|\dot\alpha_k^{i-1}\|_2^2\Big)\,,$$

and then, using (3.1.12), we obtain that

$$\begin{aligned}\frac{\varepsilon}{2}\Big(\|\dot\alpha_k^i\|_2^2-\|\dot\alpha_k^{i-1}\|_2^2\Big)+\tau\kappa|\dot\alpha_k^i|^2_{m,2}&\le C\tau\Big(\|\dot\alpha_k^i\|^2_\infty+\|\dot\alpha_k^i\|_\infty\|\dot e_k^i\|_2+\|\dot\alpha_k^i\|_\infty\|\dot p_k^i\|_1\Big)\\&\le C\tau\Big(\|\dot\alpha_k^i\|^2_\infty+\|E\dot w_k^i\|_2^2\Big)\end{aligned} \tag{3.1.27}$$

for every $2\le i\le k$, where $C$ depends on the $C^{1,1}$ norm of $\mathbb{C}$, $D$, $H$ (if $\lambda\in(0,1]$), and on the constants $r$, $\gamma_1$, $\gamma_2$. Notice that in the last inequality we have used Lemma 3.1.6.

Since $\partial_\alpha \mathcal{E}(\alpha_0, e_0) \in L^2(\Omega)$, using (3.1.14b) we get

$$\begin{aligned} \varepsilon \|\dot\alpha_k^1\|_2^2 &\le \langle -\partial_\alpha \mathcal{E}(\alpha_0, e_0), \dot\alpha_k^1\rangle - \Big[\tfrac{1}{2}\big(\langle \mathbb{C}'(\alpha_k^1)\dot\alpha_k^1 e_k^1, e_k^1\rangle - \langle \mathbb{C}'(\alpha_0)\dot\alpha_k^1 e_0, e_0\rangle_{L^2}\big) \\ &\qquad + \langle \partial D(\alpha_k^1) - \partial D(\alpha_0), \dot\alpha_k^1\rangle + \kappa\langle \alpha_k^1 - \alpha_0, \dot\alpha_k^1\rangle_{m,2} + C_K\,\tau \|\dot p_k^1\|_1 \|\dot\alpha_k^1\|_\infty\Big] \\ &\le \frac{1}{2\varepsilon}\|\partial_\alpha \mathcal{E}(\alpha_0, e_0)\|_2^2 + \frac{\varepsilon}{2}\|\dot\alpha_k^1\|_2^2 - \tau\kappa |\dot\alpha_k^1|_{m,2}^2 + C\tau\big(\|\dot\alpha_k^1\|_\infty^2 + \|E\dot w_k^1\|_2^2\big)\,, \end{aligned} \tag{3.1.28}$$

arguing as before, since $(\alpha_0, (u_0, e_0, p_0))$ satisfies (3.0.32). We can read (3.1.28) as

$$\frac{\varepsilon}{2}\|\dot\alpha_k^1\|_2^2 + \tau\kappa|\dot\alpha_k^1|_{m,2}^2 \le C\Big(\frac{1}{\varepsilon} + \tau\big(\|\dot\alpha_k^1\|_\infty^2 + \|E\dot w_k^1\|_2^2\big)\Big)\,. \tag{3.1.29}$$

Since $H^m(\Omega)$ is compactly embedded into $L^\infty(\Omega)$, for every $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$\|\cdot\|_\infty^2 \le \delta|\cdot|_{m,2}^2 + C_\delta\|\cdot\|_2^2\,. \tag{3.1.30}$$

For every $2 \le h \le k$, summing (3.1.27) for $2 \le i \le h$ and (3.1.29) and taking into account (3.1.30) for $\delta = \frac{1}{2C}$, we get

$$\frac{\varepsilon}{2}\|\dot\alpha_k^h\|_2^2 + \frac{\tau\kappa}{2}\sum_{i=1}^h |\dot\alpha_k^i|_{m,2}^2 \le C\left(\frac{1}{\varepsilon} + \tau\Big(\sum_{i=1}^h \|\dot\alpha_k^i\|_2^2 + \sum_{i=1}^h \|E\dot w_k^i\|_2^2\Big)\right). \tag{3.1.31}$$

Adding $\frac{\tau\kappa}{2}\sum_{i=1}^h \|\dot\alpha_k^i\|_2^2$ to both sides of (3.1.31), it follows that for every $t \in (0,T)\setminus\{t_k^1, \dots, t_k^{k-1}\}$,

$$\begin{aligned} \frac{\varepsilon}{2}\|\dot\alpha_k(t)\|_2^2 + \frac{\kappa}{2}\int_0^{\overline{\tau}_k(t)} \|\dot\alpha_k(s)\|_{m,2}^2\,\mathrm{d}s &\le C\left(\frac{1}{\varepsilon} + \int_0^{\overline{\tau}_k(t)} \|\dot\alpha_k(s)\|_2^2\,\mathrm{d}s + \int_0^{\overline{\tau}_k(t)} \|E\dot w_k(s)\|_2^2\,\mathrm{d}s\right) \\ &\le C\left(\frac{1}{\varepsilon} + 1 + \int_0^{\overline{\tau}_k(t)} \|\dot\alpha_k(s)\|_2^2\,\mathrm{d}s\right), \end{aligned} \tag{3.1.32}$$

where we have used (2.1.27) in the last inequality. Now Gronwall's Inequality implies that

$$\varepsilon\|\dot\alpha_k(t)\|_2^2 \le C\Big(\frac{1}{\varepsilon} + 1\Big)e^{\frac{C}{\varepsilon}\overline{\tau}_k(t)}\,,$$

for every $t \in (0,T) \setminus \{t_k^1, \dots, t_k^{k-1}\}$. We recover (3.1.26a) multiplying with $\varepsilon$ and taking the square root. Now (3.1.26b) follows from (3.1.26a) and (3.1.32). ☐

Arguing as in [**61**, Proposition 4.3] we improve the estimate of Proposition 3.1.7 and show that the functions $\alpha_k$ are bounded in $AC([0,T], H^m(\Omega))$ by a constant independent of $\varepsilon$ and $k$.

PROPOSITION 3.1.8. *There exists a positive constant $C$ independent of $\varepsilon$, $k$, and $t$ such that for every $0 < \varepsilon < 1$ and $k \in \mathbb{N}$, with $k \ge (4\varepsilon)^{-1}$,*

$$\int_0^t \|\dot\alpha_k(s)\|_{m,2}\,\mathrm{d}s \le C\,.$$

PROOF. Let $\dot\alpha_k^i := \frac{\alpha_k^i - \alpha_k^{i-1}}{\tau}$ and $\dot w_k^i = \frac{w_k^i - w_k^{i-1}}{\tau}$ for every $k \in \mathbb{N}$ and $1 \le i \le k$. From the inequality

$$\|\dot\alpha_k^i\|_2\Big(\|\dot\alpha_k^i\|_2 - \|\dot\alpha_k^{i-1}\|_2\Big) \le \langle \dot\alpha_k^i - \dot\alpha_k^{i-1}, \dot\alpha_k^i\rangle_2\,,$$

arguing as done for (3.1.27), we get that for every $2 \le i \le k$

$$\varepsilon\|\dot\alpha_k^i\|_2\Big(\|\dot\alpha_k^i\|_2 - \|\dot\alpha_k^{i-1}\|_2\Big) + \tau\kappa|\dot\alpha_k^i|_{m,2}^2 \le C\tau\Big(\|\dot\alpha_k^i\|_\infty^2 + \|E\dot w_k^i\|_2^2\Big)\,. \tag{3.1.33}$$

By the compact embedding of $H^m(\Omega)$ into $L^\infty(\Omega)$, for every $\delta > 0$ there exists a constant $C(\delta) > 0$ such that

$$\|\cdot\|_\infty^2 \le \delta|\cdot|_{m,2}^2 + C_\delta\|\cdot\|_1^2 \le \delta|\cdot|_{m,2}^2 + \widetilde C_\delta\|\cdot\|_1\|\cdot\|_2\,, \tag{3.1.34}$$

since $\Omega$ is bounded. Adding a term $\tau\kappa\|\dot\alpha_k^i\|_2^2$ to both sides of (3.1.33) and using (3.1.34) with $\delta = \frac{1}{2C}$, we obtain that

$$\varepsilon\|\dot\alpha_k^i\|_2\Big(\|\dot\alpha_k^i\|_2 - \|\dot\alpha_k^{i-1}\|_2\Big) + \frac{\kappa}{2}\tau\|\dot\alpha_k^i\|_{m,2}^2 \le C\tau\Big(\|\dot\alpha_k^i\|_1\|\dot\alpha_k^i\|_2 + \|E\dot w_k^i\|_2^2\Big)\,,$$

for $2 \le i \le k$. Multiplying the inequality above by $2/\varepsilon$ and taking into account that $\|\dot\alpha_k^i\|_{m,2}^2 \ge \|\dot\alpha_k^i\|_2^2$, we have that

$$2\|\dot\alpha_k^i\|_2\Big(\|\dot\alpha_k^i\|_2 - \|\dot\alpha_k^{i-1}\|_2\Big) + \frac{\tau\kappa}{2\varepsilon}\|\dot\alpha_k^i\|_2^2 + \frac{\tau\kappa}{2\varepsilon}\|\dot\alpha_k^i\|_{m,2}^2 \le \frac{2\tau C}{\varepsilon}\Big(\|\dot\alpha_k^i\|_1\|\dot\alpha_k^i\|_2 + \|E\dot w_k^i\|_2^2\Big) \tag{3.1.35}$$

for $2 \le i \le k$. We now set

$$a_i := \|\dot\alpha_k^i\|_2, \quad b_i := \Big(\frac{\tau\kappa}{2\varepsilon}\Big)^{1/2}\|\dot\alpha_k^i\|_{m,2}, \quad c_i := \Big(\frac{2\tau C}{\varepsilon}\Big)^{1/2}\|E\dot w_k^i\|_2 \quad d_i := \frac{\tau C}{\varepsilon}\|\dot\alpha_k^i\|_1 \quad \zeta := \frac{\tau\kappa}{4\varepsilon}$$

for $2 \le i \le k$. This definition allows us to recast (3.1.35) in the form

$$2a_i(a_i - a_{i-1}) + 2\zeta a_i^2 + b_i^2 \le c_i^2 + 2a_i d_i\,,$$

and so to follow the proof performed in [**61**]. Indeed, by a discrete Gronwall-type inequality with weights (see [**61**, Lemma 4.1], to which we refer for all details), we conclude that

$$\begin{aligned}\frac{\kappa}{2\varepsilon}\sum_{i=2}^h \tau(1+\zeta)^{2(i-h)-1}\|\dot\alpha_k^i\|_{m,2}^2 \le 2(1+\zeta)^{-2h}\|\dot\alpha_k^1\|_2^2 + 16C\zeta\sum_{i=2}^h(1+\zeta)^{2(i-h)-1}\|E\dot w_k^i\|_2^2\\ + 4C^2\Big(\sum_{i=2}^h\frac{\tau}{\varepsilon}(1+\zeta)^{i-h-1}\|\dot\alpha_k^i\|_1\Big)^2,\end{aligned} \tag{3.1.36}$$

for $2 \le h \le k$. We bound the right hand side of (3.1.36) with

$$\left[\sqrt2(1+\zeta)^{-h}\|\dot\alpha_k^1\|_2 + \Big(1 + 16C\zeta\sum_{i=2}^k(1+\zeta)^{2(i-h)-1}\|E\dot w_k^i\|_2^2\Big) + 2C\sum_{i=2}^h\frac{\tau}{\varepsilon}(1+\zeta)^{i-h-1}\|\dot\alpha_k^i\|_1\right]^2,$$

using the fact that for every $a$, $b$, $c > 0$

$$a^2 + b^2 + c^2 \le (a+b+c)^2 \le (a + (1+b^2) + c)^2\,.$$

In order to estimate from below the left hand side of (3.1.36), we appeal to the Hölder inequality,

$$\frac{\kappa}{2\varepsilon}\sum_{i=2}^h\tau(1+\zeta)^{2(i-h)-1}\|\dot\alpha_k^i\|_{m,2} \le \Big(\frac{\kappa}{2\varepsilon}\sum_{i=2}^h\tau(1+\zeta)^{2(i-h)-1}\Big)^{\frac12}\Big(\frac{\kappa}{2\varepsilon}\sum_{i=2}^h\tau(1+\zeta)^{2(i-h)-1}\|\dot\alpha_k^i\|_{m,2}^2\Big)^{\frac12}.$$

Evaluating the geometric sum and using the fact that $\frac{\tau\kappa}{2\varepsilon} \leq 2\zeta \leq 2\kappa$, we deduce that

$$\frac{1}{2\varepsilon}\sum_{i=2}^{h}\tau(1+\zeta)^{2(i-h)-1} \leq \kappa + 1\,.$$

Therefore from (3.1.36) we obtain

$$\frac{1}{\varepsilon}\sum_{i=2}^{h}\tau(1+\zeta)^{2(i-h)-1}\|\dot\alpha_k^i\|_{m,2} \leq C\bigg(1 + (1+\zeta)^{-h}\|\dot\alpha_k^1\|_2 + \zeta\sum_{i=2}^{h}(1+\zeta)^{2(i-h)-1}\|E\dot w_k^i\|_2^2 + \sum_{i=2}^{h}\frac{\tau}{\varepsilon}(1+\zeta)^{i-h-1}\|\dot\alpha_k^i\|_1\bigg)\,. \tag{3.1.37}$$

Now we multiply both sides of (3.1.37) by $\tau$ and sum over $h = 2, \dots, k$. Recalling the formula of the geometric sum we get for every $k \in \mathbb{N}$ and $\rho_2, \rho_3, \dots, \rho_k \geq 0$

$$\frac{1}{\varepsilon}\sum_{h=2}^{k}\tau\sum_{i=2}^{h}\tau(1+\zeta)^{2(i-h)-1}\rho_i = 4\,\frac{1+\zeta}{2+\zeta}\sum_{i=2}^{k}\tau\rho_i\big(1-(1+\zeta)^{2(i-k)-2}\big)\,, \tag{3.1.38}$$

where we have changed the order of the sums; this identity will be used to rewrite the left hand side of (3.1.37) setting $\rho_i = \|\dot\alpha_k^i\|_{m,2}$ and the third term in the right hand side setting $\rho_i = \|E\dot w_k^i\|_2^2$. Moreover, for the second and the fourth summand in the right hand side of (3.1.37) we have

$$\sum_{h=2}^{k}\tau(1+\zeta)^{-h}\|\dot\alpha_k^1\|_2 \leq \frac{8\varepsilon}{\kappa}\|\dot\alpha_k^1\|_2 \tag{3.1.39}$$

and

$$\sum_{h=2}^{k}\tau\sum_{i=2}^{h}\frac{\tau}{\varepsilon}(1+\zeta)^{i-h-1}\|\dot\alpha_k^i\|_1 \leq \frac{4}{\kappa}\sum_{i=2}^{k}\tau\|\dot\alpha_k^i\|_1\,. \tag{3.1.40}$$

Collecting (3.1.37)–(3.1.40) we obtain that

$$\sum_{i=2}^{k}\tau\|\dot\alpha_k^i\|_{m,2} \leq C\bigg(T + \varepsilon\|\dot\alpha_k^1\|_2 + \sum_{i=2}^{k}\tau\|E\dot w_k^i\|_2^2 + \sum_{i=2}^{k}\tau\|\dot\alpha_k^i\|_1\bigg) + \sum_{i=2}^{k}\tau(1+\zeta)^{2(i-k)-2}\|\dot\alpha_k^i\|_{m,2}\,.$$

The last term in the equation above is estimated with (3.1.37), so we get

$$\sum_{i=2}^{k}\tau\|\dot\alpha_k^i\|_{m,2} \leq C\bigg(T + \varepsilon\|\dot\alpha_k^1\|_2 + \sum_{i=2}^{k}\tau\|E\dot w_k^i\|_2^2 + \sum_{i=2}^{k}\tau\|\dot\alpha_k^i\|_1\bigg)\,. \tag{3.1.41}$$

We are now left to estimate the term with $i = 1$. From (3.1.29) and (3.1.34) it follows that

$$\tau|\dot\alpha_k^1|_{m,2}^2 \leq C\left(\frac{1}{\varepsilon} + \tau\|\dot\alpha_k^1\|_1^2 + \tau\|E\dot w_k^1\|_2^2\right)\,.$$

Multiplying by $\tau$, since $\frac{\tau}{2\varepsilon} \leq 2$ we get

$$\tau^2|\dot\alpha_k^1|_{m,2}^2 \leq C\left(1 + \tau^2\|\dot\alpha_k^1\|_1^2 + \tau^2\|E\dot w_k^1\|_2^2\right) \leq C\left(1 + \tau\|\dot\alpha_k^1\|_1 + \tau\|E\dot w_k^1\|_2\right)^2\,,$$

and then

$$\tau\|\dot\alpha_k^1\|_{m,2} \le C\left(1+\tau\|\dot\alpha_k^1\|_1+\tau\|E\dot w_k^1\|_2\right)+\tau\|\dot\alpha_k^1\|_2 \le C\left(1+\tau\|\dot\alpha_k^1\|_1+\tau\|E\dot w_k^1\|_2+\varepsilon\|\dot\alpha_k^1\|_2\right)\,. \tag{3.1.42}$$

Summing up (3.1.41) and (3.1.42) gives

$$\begin{aligned}\sum_{i=1}^k \tau\|\dot\alpha_k^i\|_{m,2} &\le C\Big(T+\varepsilon\|\dot\alpha_k^1\|_2+\sum_{i=1}^k\tau\|E\dot w_k^i\|_2^2+\sum_{i=1}^k\tau\|\dot\alpha_k^i\|_1\Big)\\ &\le C\Big(1+\int_0^T\|E\dot w_k(s)\|_2^2+\int_\Omega \alpha_0(x)-\alpha_k^k(x)\,\mathrm{d}x\Big)\,,\end{aligned}$$

where in the last inequality we have used the fact that $\alpha_k^i\le\alpha_k^{i-1}$ and (3.1.26a) for $t\in(0,t_k^1)=(0,\tau)$, taking into account that $\frac{\tau}{\varepsilon}\le 4$. Thus we conclude, recalling (2.1.27), (3.1.11b), and the fact that $C$ is independent of $\varepsilon$, $k$, and $t$. □

Remark 3.1.9. Using Lemma 3.1.6, by Proposition 3.1.7 we get that

$$\varepsilon\int_0^T\|\dot e_k(s)\|_2^2\,\mathrm{d}s\le C,\quad \varepsilon\int_0^T\|\dot p_k(s)\|_1^2\,\mathrm{d}s\le C,\quad \varepsilon\int_0^T\|\dot u_k(s)\|_{BD}^2\,\mathrm{d}s\le C\,, \tag{3.1.43}$$

while by Proposition 3.1.8 it follows that

$$\int_0^T\|\dot e_k(s)\|_2\,\mathrm{d}s\le C,\quad \int_0^T\|\dot p_k(s)\|_1\,\mathrm{d}s\le C,\quad \int_0^T\|\dot u_k(s)\|_{BD}\,\mathrm{d}s\le C \tag{3.1.44}$$

for $4k\varepsilon>1$, where $C$ is a constant independent of $\varepsilon$, $k$, and $t$.

Remark 3.1.10. We conclude this section with a short discussion on the choice of the regularizing term in (3.0.18). Let us consider the general case of a damage regularization in a Banach space $X$, namely, whenever the energy is finite, the damage variable belongs to $X$. In order to differentiate the energy with respect to time, a priori estimates for $\alpha_k$ in the space $W^{1,1}(0,T;X)$ should be derived, in such a way that $\dot\alpha$ is in duality with $\partial_\alpha\mathcal{E}\in X'$, cf. (3.0.22). In the present context, following **[61]**, we exploit the Hilbert structure of the space $X=H^m(\Omega)$, see Lemma 3.1.8. Instead, the choice $X=W^{1,\gamma}(\Omega)$ $(\gamma>n)$ considered in **[60]** (for damage without plasticity) provides only a uniform estimate for $\alpha_k$ in $W^{1,1}(0,T;H^1(\Omega))$. For this reason in **[60]** the evolution fulfills only an energy inequality, see also Remark 3.4 therein.

## 3.2. Viscous evolutions

In this section we pass to the limit in the discrete-time problems as the time step converges to zero. For every fixed $\varepsilon>0$ we then find a quadruple $(\alpha_\varepsilon,u_\varepsilon,e_\varepsilon,p_\varepsilon)$ satisfying for every $t\in[0,T]$:

- admissibility and equilibrium conditions, with $\alpha_\varepsilon$ nonincreasing in time;
- a first order stability condition in the damage, referred to as Kuhn-Tucker inequality;
- an energy balance including viscous dissipation.

Such quadruples are called $\varepsilon$-approximate viscous evolutions (see Definition 3.2.1 and Theorem 3.2.8). We also prove some crucial estimates for the passage to the limit as viscosity vanishes, which will be studied in Section 3.3.

We start introducing the notion of $\varepsilon$-approximate viscous evolution. Notice that when $X$ is the dual space of a Banach space $Y$ we denote

$$L^2_w(0,T;X) := \{p\colon [0,T] \to X \text{ weakly}^* \text{ measurable} : t \mapsto \|p(t)\| \in L^2(0,T)\}\,,$$

with $f\colon (0,T) \to X$ weakly* measurable if and only if $(0,T) \ni t \mapsto \langle f(t), g\rangle$ is measurable for every $g \in Y$, and

$$H^1_w(0,T;X) := \Big\{p \in L^2_w(0,T;X)\colon \exists \widehat{p} \in L^2_w(0,T;X) \text{ s.t. for every } \varphi \in C^1_c((0,T);Y)$$
$$\int_0^T \langle p(t),\, \partial_t\varphi(t)\rangle \,\mathrm{d}t = -\int_0^T \langle \widehat{p}(t),\, \varphi(t)\rangle \,\mathrm{d}t \Big\}\,.$$

DEFINITION 3.2.1. Let (3.0.19), (2.1.8), (2.1.10) hold, and let $w$ be as in (2.1.27). We say that a function $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$ from $[0,T]$ into $H^m(\Omega;[0,1]) \times BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)$ is an *$\varepsilon$-approximate viscous evolution* if

$$\begin{aligned} &\alpha_\varepsilon \in H^1(0,T;H^m(\Omega))\,, &&e_\varepsilon \in H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))\,,\\ &u_\varepsilon \in H^1_w(0,T;BD(\Omega)), &&p_\varepsilon \in H^1_w(0,T;M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D))\,, \end{aligned} \tag{3.2.1}$$

and, setting $\sigma_\varepsilon(t) := \mathbb{C}(\alpha_\varepsilon(t))e_\varepsilon(t)$ for every $t \in [0,T]$, the following conditions are satisfied:

$(\mathrm{ev0})_\varepsilon$ *irreversibility*: for every $x \in \Omega$

$$[0,T] \ni t \mapsto \alpha_\varepsilon(t,x) \quad \text{is nonincreasing}\,;$$

$(\mathrm{ev1})_\varepsilon$ *kinematic condition and equilibrium*: for every $t \in [0,T]$

$$(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in A(w(t))\,, \quad \mathrm{div}\,\sigma_\varepsilon(t) = 0 \text{ in } \Omega\,, \quad [\sigma_\varepsilon\nu] = 0 \text{ on } \partial_N\Omega\,;$$

$(\mathrm{ev2})_\varepsilon$ *stress constraint*: for every $t \in [0,T]$

$$\sigma_\varepsilon(t) \in \mathcal{K}_{\alpha_\varepsilon(t)}(\Omega)\,;$$

$(\mathrm{ev3})_\varepsilon$ *Kuhn-Tucker inequality*: for a.e. $t \in (0,T)$

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t), \beta\rangle + \varepsilon\langle\dot\alpha_\varepsilon(t), \beta\rangle \geq 0 \quad \text{for every } \beta \in H^m_-(\Omega) = \{\beta \in H^m(\Omega)\colon \beta \leq 0 \text{ in } \Omega\}\,;$$

$(\mathrm{ev4})_\varepsilon$ *energy balance*: for every $t \in [0,T]$

$$\begin{aligned} &\mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t) + (1-\lambda)\int_0^t \mathcal{H}(\alpha_\varepsilon(s), \dot p_\varepsilon(s))\,\mathrm{d}s + \varepsilon\int_0^t \|\dot\alpha_\varepsilon(s)\|_2^2\,\mathrm{d}s\\ &= \mathcal{E}(\alpha_0, e_0) + \int_0^t \langle \sigma_\varepsilon(s), E\dot w(s)\rangle\,\mathrm{d}s\,. \end{aligned}$$

REMARK 3.2.2. By [**98**, Theorem 3.10], conditions $(\mathrm{ev1})_\varepsilon$ and $(\mathrm{ev2})_\varepsilon$ are equivalent to the following global minimality condition for fixed damage variable: for every $t \in [0,T]$, $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in A(w(t))$ and

$$\mathcal{Q}(\alpha_\varepsilon(t), e_\varepsilon(t)) \leq \mathcal{Q}(\alpha_\varepsilon(t), \eta) + \mathcal{H}(\alpha_\varepsilon(t), q - p_\varepsilon(t)) \quad \text{for every } (v,\eta,q) \in A(w(t))$$

Two characterizations of the approximate viscous evolutions are given below: the first ensures in particular that the damage variable satisfies the Kuhn-Tucker conditions, while the second will be useful in the proof of Theorem 3.2.8.

PROPOSITION 3.2.3. *Let* $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$ *be a function satisfying the conditions* (3.2.1), $(\mathrm{ev0})_\varepsilon$ – $(\mathrm{ev3})_\varepsilon$, *with* $\alpha_\varepsilon(t) \in H^m(\Omega;[0,1])$. *Then* $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$ *is an* $\varepsilon$*-approximate viscous evolution, i.e. it satisfies the energy balance* $(\mathrm{ev4})_\varepsilon$, *if and only if any of the conditions below holds true:*

$(\mathrm{ev4'})_\varepsilon$ *for a.e.* $t \in (0,T)$ *the following hold:*

– Kuhn-Tucker equality:

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t), \dot\alpha_\varepsilon(t)\rangle + \varepsilon \|\dot\alpha_\varepsilon(t)\|_2^2 = 0\,; \tag{3.2.2}$$

– Hill's maximum plastic work principle:

$$\mathcal{H}(\alpha_\varepsilon(t),\, \dot p_\varepsilon(t)) = \langle \sigma_\varepsilon(t))_D |\, \dot p_\varepsilon(t)\rangle\,.$$

$(\mathrm{ev4''})_\varepsilon$ energy inequality: *for every* $t \in [0,T]$

$$\begin{aligned}\mathcal{E}_\lambda(\alpha_\varepsilon(T), e_\varepsilon(T); p_\varepsilon, T) &+ (1-\lambda)\int_0^T \mathcal{H}(\alpha_\varepsilon(t),\, \dot p_\varepsilon(t))\,\mathrm{d}t + \varepsilon\int_0^T \|\dot\alpha_\varepsilon(t)\|_2^2\,\mathrm{d}t\\ &\le \mathcal{E}(\alpha_0, e_0) + \int_0^T \langle \sigma_\varepsilon(t), E\dot w(t)\rangle\,\mathrm{d}t\,.\end{aligned}$$

PROOF. **Ad** $(\mathbf{ev4})_\varepsilon \iff (\mathbf{ev4'})_\varepsilon$**:** From the absolute continuity of $\alpha_\varepsilon$, $e_\varepsilon$, and $p_\varepsilon$, we obtain that the function $t \mapsto \mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t)$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t) = \langle \partial_\alpha \mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t), \dot\alpha_\varepsilon(t)\rangle + \lambda\mathcal{H}(\alpha_\varepsilon(t), \dot p_\varepsilon(t)) + \langle \sigma_\varepsilon(t), \dot e_\varepsilon(t)\rangle \tag{3.2.3}$$

for a.e. $t \in (0,T)$. Property $(\mathrm{ev1})_\varepsilon$ and [**28**, Lemma 5.5] imply that

$$(\dot u_\varepsilon(t), \dot e_\varepsilon(t), \dot p_\varepsilon(t)) \in A(\dot w(t)) \text{ for a.e. } t \in (0,T)\,,$$

so that, from the integration by parts formula (2.1.21),

$$\langle \sigma_\varepsilon(t), \dot e_\varepsilon(t)\rangle = \langle \sigma_\varepsilon(t), E\dot w(t)\rangle - \langle (\sigma_\varepsilon(t))_D |\, \dot p_\varepsilon(t)\rangle \tag{3.2.4}$$

for a.e. $t \in (0,T)$. Then $(\mathrm{ev4})_\varepsilon$ is equivalent to

$$-(1-\lambda)\mathcal{H}(\alpha_\varepsilon(t),\, \dot p_\varepsilon(t)) - \varepsilon\|\dot\alpha_\varepsilon(t)\|_2^2 + \langle \sigma_\varepsilon(t), E\dot w(t)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t)$$

for a.e. $t \in (0,T)$, which is also equivalent to

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t), \dot\alpha_\varepsilon(t)\rangle + \varepsilon\|\dot\alpha_\varepsilon(t)\|_2^2 + \mathcal{H}(\alpha_\varepsilon(t),\, \dot p_\varepsilon(t)) - \langle (\sigma_\varepsilon(t))_D |\, \dot p_\varepsilon(t)\rangle = 0 \tag{3.2.5}$$

for a.e. $t \in (0,T)$. Now, from $(\mathrm{ev2})_\varepsilon$ and (2.1.23) it follows that

$$\langle (\sigma_\varepsilon(t))_D |\, \dot p_\varepsilon(t)\rangle \le \mathcal{H}(\alpha_\varepsilon(t), \dot p_\varepsilon(t))\,, \tag{3.2.6}$$

since $\dot p_\varepsilon(t) \in \Pi(\Omega)$ for a.e. $t \in [0,T]$. Then, using $(\mathrm{ev3})_\varepsilon$ with $\beta = \dot\alpha_\varepsilon$, we get that (3.2.5) is equivalent to $(\mathrm{ev4'})_\varepsilon$.

**Ad** $(\mathbf{ev4})_\varepsilon \iff (\mathbf{ev4''})_\varepsilon$**:** Let us prove that $(\mathrm{ev4''})_\varepsilon$ implies $(\mathrm{ev4})_\varepsilon$, the converse being trivial. Gathering $(\mathrm{ev3})_\varepsilon$ with $\beta = \dot\alpha_\varepsilon(t)$, (3.2.3), (3.2.4), and (3.2.6), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t) \ge -(1-\lambda)\mathcal{H}(\alpha_\varepsilon(t),\, \dot p_\varepsilon(t)) - \varepsilon\|\dot\alpha_\varepsilon(t)\|_2^2 + \langle \sigma_\varepsilon(t), E\dot w(t)\rangle$$

for a.e. $t \in (0,T)$. Integrating, we get for every $0 \le t_1 \le t_2 \le T$ the inequality

$$\begin{aligned}\mathcal{E}_\lambda(\alpha_\varepsilon(t_2), e_\varepsilon(t_2); p_\varepsilon, t_2) + (1-\lambda)\int_{t_1}^{t_2} \mathcal{H}(\alpha_\varepsilon(s), \dot p_\varepsilon(s))\,\mathrm{d}s + \varepsilon\int_{t_1}^{t_2} \|\dot\alpha_\varepsilon(s)\|_2^2\,\mathrm{d}s \\ \ge \mathcal{E}_\lambda(\alpha_\varepsilon(t_1), e_\varepsilon(t_1); p_\varepsilon, t_1) + \int_{t_1}^{t_2} \langle \sigma_\varepsilon(s), E\dot w(s)\rangle\,\mathrm{d}s\,,\end{aligned}$$

which implies the energy balance $(\mathrm{ev4})_\varepsilon$ in view of $(\mathrm{ev4''})_\varepsilon$. This concludes the proof. □

Using the Kuhn-Tucker conditions, we can rewrite the energy balance as in the following Remark.

REMARK 3.2.4. Let $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$ be an $\varepsilon$-approximate viscous evolution. From $(\mathrm{ev3})_\varepsilon$ and (3.2.2) it follows that

$$\varepsilon\|\dot\alpha_\varepsilon(t)\|_2 = \sup_{\beta\in F}\langle -\partial_\alpha\mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t), \beta\rangle = -\inf_{\beta\in F}\langle \partial_\alpha\mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t), \beta\rangle \tag{3.2.7}$$

for a.e. $t \in (0,T)$, where

$$F := \{\beta \in H^m_-(\Omega) \colon \|\beta\|_2 \le 1\}\,.$$

Indeed, by $(\mathrm{ev3})_\varepsilon$

$$\varepsilon\langle\dot\alpha_\varepsilon(t), \beta\rangle \ge \langle -\partial_\alpha\mathcal{E}_\lambda(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t), \beta\rangle\,,$$

for every $\beta \in H^m_-(\Omega)$, while (3.2.2) implies that the supremum in (3.2.7) is a maximum, attained for $\beta = \frac{\dot\alpha_\varepsilon(t)}{\|\dot\alpha_\varepsilon(t)\|_2}$ if $\|\dot\alpha_\varepsilon(t)\|_2 > 0$.

Then, by (3.2.7), $(\mathrm{ev4})_\varepsilon$ reads as

$$\begin{aligned}\mathcal{E}_\lambda(\alpha_\varepsilon(T), e_\varepsilon(T); p_\varepsilon, T) + (1-\lambda)\int_0^T \mathcal{H}(\alpha_\varepsilon(t),\, \dot p_\varepsilon(t))\,\mathrm{d}t + \int_0^T \|\dot\alpha_\varepsilon(t)\|_2\Psi(\alpha_\varepsilon(t), e_\varepsilon(t); p_\varepsilon, t)\,\mathrm{d}t \\ = \mathcal{E}(\alpha_0, e_0) + \int_0^T \langle\sigma_\varepsilon(t), E\dot w(t)\rangle\,\mathrm{d}t\,,\end{aligned} \tag{3.2.8}$$

where

$$\Psi(\alpha, e; p, t) := \Phi(\partial_\alpha\mathcal{E}_\lambda(\alpha, e; p, t))\,, \tag{3.2.9}$$

for every $\alpha \in H^m(\Omega; [0,1])$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $p \in AC([0,T]; M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D))$, $t \in [0,T]$, with

$$\Phi(f) := \sup_{\beta\in F}\langle -f, \beta\rangle \quad \text{for every } f \in (H^m(\Omega))'\,. \tag{3.2.10}$$

Notice that $\Psi(\alpha, e; p, t) \in [0, +\infty]$.

In the following lemma we characterize the operator $\Phi$ introduced above.

LEMMA 3.2.5. *Let $\Phi$ be the operator defined in* (3.2.10), *and let*

$$G := \{h \in (H^m(\Omega))' \colon \langle h, \beta\rangle \ge 0 \quad \textit{for every } \beta \in H^m_-(\Omega)\}$$

*and*

$$\mathrm{d}_2(f, G) := \min\{\|g\|_2 \colon g \in L^2(\Omega),\ f + g \in G\} \quad \textit{for every } f \in (H^m(\Omega))'\,, \tag{3.2.11}$$

*which is well defined for every $f$. Then*

$$\Phi(f) = \mathrm{d}_2(f,G) \quad \textit{for every } f \in (H^m(\Omega))' . \tag{3.2.12}$$

Proof. Let us fix $f \in (H^m(\Omega))'$.

**Proof of $\Phi(f) \le \mathrm{d}_2(f,G)$.** Let $\mathrm{d}_2(f,G) < +\infty$ and $g \in L^2(\Omega)$ such that $f+g \in G$: we have that

$$\langle -f, \beta\rangle \le \int_\Omega g\,\beta\,\mathrm{d}x \le \|g\|_2 \quad \text{for every } \beta \in F$$

and we conclude by definition of $\Phi(f)$ and $\mathrm{d}_2(f,G)$.

**Proof of $\mathrm{d}_2(f,G) \le \Phi(f)$.** We can assume $\Phi(f) < +\infty$; then

$$\langle f, \beta\rangle \le \Phi(f)\|\beta\|_2 \quad \text{for every } \beta \in H^m_+(\Omega) = \{\beta \in H^m(\Omega) \colon \beta \ge 0 \text{ in } \Omega\} . \tag{3.2.13}$$

Let $B \subset \mathbb{R}^n$ be an open set such that $\overline{\Omega} \subset B$ and

$$\langle S, \beta\rangle := \langle f, \beta|_\Omega\rangle \quad \text{for every } \beta \in H^m_0(B) ; \tag{3.2.14}$$

by (3.2.13)

$$\langle S, \beta\rangle \le \Phi(f)\|\beta\|_{L^2(B)} \quad \text{for every } \beta \in H^m_0(B) .$$

By Lemma 1.2.4 we get that there exists a unique pair $(g,\mu)$ with $g \in L^2(B)$, $g \ge 0$ and $\mu \in M^+(B)$ (namely $\mu$ is a nonnegative measure on $B$) such that $g\,\mathrm{d}x$ and $\mu$ are mutually singular and

$$\langle S, \beta\rangle = \int_B g\,\beta\,\mathrm{d}x - \int_B \beta\,\mathrm{d}\mu \quad \text{for every } \beta \in H^m_0(B) ;$$

in particular the former property implies that $\int_B g\,\mathrm{d}\mu^a = 0$. Using (3.2.14) we have that $g \in L^2(\Omega)$ and $\mu \in M^+(\overline{\Omega})$. Therefore

$$\langle f, \beta\rangle = \int_\Omega g\,\beta\,\mathrm{d}x - \int_{\overline{\Omega}} \beta\,\mathrm{d}\mu \quad \text{for every } \beta \in H^m(\Omega) ;$$

then $-\langle g, \cdot\rangle_2 + f \in G$, and this gives

$$\mathrm{d}_2(f,G) \le \|g\|_2 . \tag{3.2.15}$$

Let us fix $\varepsilon > 0$. We claim that there exists $\beta \in C^\infty_c(\Omega) \cap F$ such that

$$-\int_\Omega g\,\beta\,\mathrm{d}x > \|g\|_2 - \varepsilon \quad \text{and} \quad -\int_\Omega \beta\,\mathrm{d}\mu^a < \varepsilon , \tag{3.2.16}$$

where $\mu = \mu^a + \mu^s$ is the decomposition of $\mu \in M^+(\overline{\Omega})$ into its absolutely continuous and its singular part (with respect to $\mathcal{L}^n$). Indeed, we can first consider $h \in L^\infty(\Omega)$ with compact support such that $-\frac{g}{\|g\|_2} \le h \le 0$ and

$$-\int_\Omega g\,h\,\mathrm{d}x > \|g\|_2 - \tfrac{\varepsilon}{2} , \tag{3.2.17}$$

for instance $h = \big(-1_{\Omega_k}\frac{g}{\|g\|_2}\big) \vee (-k)$ for $\Omega_k$ compact such that $\Omega \subset \Omega_k + B(0,\frac{1}{k})$ and $k \in \mathbb{N}$ large enough. Then we set $h_k := \frac{h*\varrho_k}{1\vee\|h*\varrho_k\|_2}$ for a suitable $k\in\mathbb{N}$ (here $\rho_k(t) := k\,\rho(\frac{t}{k})$, with $\rho$ the standard mollifier in $\mathbb{R}$), so that $h_k \in F$, $\|h_k\|_\infty \le \|h\|_\infty$ for every $k$, and

$$\lim_{k\to\infty}\int_\Omega h_k\,\mathrm{d}\mu^a = \int_\Omega h\,\mathrm{d}\mu^a = 0\,, \tag{3.2.18}$$

where the first equality follows by Dominated Convergence Theorem and the second from $-\frac{g}{\|g\|_2} \le h \le 0$ and $\int_\Omega g\,\mathrm{d}\mu^a = 0$. Since $h_k \to h$ in $L^2(\Omega)$, for $k$ large $\|h-h_k\|_2 < \frac{\varepsilon}{2\|g\|_2}$ and then we get (3.2.16) by (3.2.17) and (3.2.18).

Let us now consider $\mu^s \in M^+(\overline{\Omega})$; let $E$ be the set on which $\mu^s$ is concentrated, and $K$ be a compact subset of $E$ such that

$$\mu^s(E\setminus K) < \tfrac{\varepsilon}{\|\beta\|_\infty}\,. \tag{3.2.19}$$

Since $\mathcal{L}^n(E)=0$, for every $\eta>0$ we can find an open set $U$ such that $K\subset U$ and $\mathcal{L}^n(U)<\eta$. Let us take $\varphi\in C^\infty_c(\Omega;[0,1])$ such that $\varphi=0$ in $K$ and $\varphi=1$ in $\Omega\setminus U$, and let $\overline{\beta} := \beta\,\varphi$. Then $\overline{\beta}\in F$ and we can assume that $\overline{\beta}$ satisfies (3.2.16), choosing $\eta$ sufficiently small: then

$$\Phi(f) \ge -\langle f,\overline{\beta}\rangle = -\int_\Omega g\,\overline{\beta} + \int_\Omega \overline{\beta}\,\mathrm{d}\mu^a + \int_{E\setminus K}\overline{\beta}\,\mathrm{d}\mu^s > \|g\|_2 - 3\varepsilon \ge \mathrm{d}_2(f,G) - 3\varepsilon\,,$$

by (3.2.15), (3.2.16), and (3.2.19). The proof is concluded since $\varepsilon$ is arbitrary. □

Remark 3.2.6. The identity (3.2.12) can be used to connect the notions of solutions provided in [**61**] and in [**87**], in the context of damage (without plasticity). The energy balance has the same structure for the two evolutions; the term related to the energy dissipated during jumps in the energy balance of [**61**] is given in terms of $d_2(\cdot,G)$, while the one in [**87**] is given in terms of $\Phi$.

The following Lemma states some semicontinuity properties that will be useful for the proof of Theorem 3.2.8. For the reader's convenience we give the proof following the lines of [**29**, Lemmas 6.1 and 6.2], to which we refer for full details.

Lemma 3.2.7. *Let $\beta_k$, $\beta\in C([0,T];C(\overline{\Omega};[0,1]))$ such that*

$$\beta_k \to \beta \text{ in } C([0,T];C(\overline{\Omega}))\,, \tag{3.2.20}$$

*and $q_k$, $q\in H^1(0,T;M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D))$ such that*

$$q_k(t) \stackrel{*}{\rightharpoonup} q(t) \text{ in } M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D) \tag{3.2.21}$$

*for every $t\in[0,T]$ and*

$$\int_0^T \|\dot q_k(t)\|_1\,\mathrm{d}t + \int_0^T \|\dot q(t)\|_1\,\mathrm{d}t \le C \tag{3.2.22}$$

*for $C$ independent of $k$. Then for every $t\in[0,T]$*

$$\int_0^t \mathcal{H}(\beta(s),\dot q(s))\,\mathrm{d}s \le \liminf_{k\to\infty}\int_0^t \mathcal{H}(\beta_k(s),\dot q_k(s))\,\mathrm{d}s \tag{3.2.23}$$

*and*

$$\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\beta_1,q;0,t),\beta_2\rangle \le \liminf_{k\to\infty}\Big[\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\beta_1,q_k;0,t),\beta_2\rangle\Big] \tag{3.2.24}$$

*for every* $\beta_1 \in C(\overline{\Omega};[0,1])$ *and* $\beta_2 \in C(\overline{\Omega};[0,\infty))$.

PROOF. Let us fix $t \in [0,T]$ and define $\mu_k,\ \mu \in M_b((0,t)\times(\Omega\cup\partial_D\Omega);\mathbb{M}^{n\times n}_D)$ by setting

$$\langle\varphi,\mu_k\rangle := \int_0^t \langle\varphi(s,\cdot),\dot q_k(s)\rangle\,\mathrm{d}s \ \text{ and } \ \langle\varphi,\mu\rangle := \int_0^t \langle\varphi(s,\cdot),\dot q(s)\rangle\,\mathrm{d}s$$

for every $\varphi \in C_0((0,t)\times(\Omega\cup\partial_D\Omega);\mathbb{M}^{n\times n}_D)$. Using (3.2.21) and (3.2.22) it is possible to see that

$$\mu_k \rightharpoonup \mu \quad \text{weakly}^* \text{ in } M_b((0,t)\times(\Omega\cup\partial_D\Omega);\mathbb{M}^{n\times n}_D) \tag{3.2.25}$$

by uniform approximation, cf. [**29**, Lemma 6.1].

Since $s\mapsto|\dot q(s)|$ is weakly$^*$ measurable from $(0,t)$ into $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$, we define $\nu_k,\ \nu \in M_b((0,t)\times(\Omega\cup\partial_D\Omega))$ by

$$\langle\varphi,\nu_k\rangle := \int_0^t \langle\varphi(s,\cdot),|\dot q_k(s)|\rangle\,\mathrm{d}s \ \text{ and } \ \langle\varphi,\nu\rangle := \int_0^t \langle\varphi(s,\cdot),|\dot q(s)|\rangle\,\mathrm{d}s$$

for every $\varphi \in C_0((0,t)\times(\Omega\cup\partial_D\Omega))$. As in [**29**, Lemma 6.1], we have that $\mu_k \ll \nu_k$, $\mu \ll \nu$ and

$$\begin{aligned}
\int_0^t \mathcal{H}(\beta(s),\dot q_k(s))\,\mathrm{d}s &= \int_{(0,t)\times(\Omega\cup\partial_D\Omega)} H\Big(\beta(s,x),\frac{\mathrm{d}\mu_k}{\mathrm{d}\nu_k}(s,x)\Big)\,\mathrm{d}\nu_k(s,x)\,,\\
\int_0^t \mathcal{H}(\beta(s),\dot q(s))\,\mathrm{d}s &= \int_{(0,t)\times(\Omega\cup\partial_D\Omega)} H\Big(\beta(s,x),\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(s,x)\Big)\,\mathrm{d}\nu(s,x)\,.
\end{aligned}$$

By Reshetnyak's Lower Semicontinuity Theorem and (3.2.25), we have

$$\int_0^t \mathcal{H}(\beta(s),\dot q(s))\,\mathrm{d}s \le \liminf_{k\to\infty}\int_0^t \mathcal{H}(\beta(s),\dot q_k(s))\,\mathrm{d}s\,. \tag{3.2.26}$$

In order to get (3.2.23), it is enough to observe that (3.0.26) gives

$$\Big|\int_0^t \mathcal{H}(\beta_k(s),\dot q_k(s))\,\mathrm{d}s - \int_0^t \mathcal{H}(\beta(s),\dot q_k(s))\,\mathrm{d}s\Big| \le C_K \sup_{s\in[0,t]}\|\beta_k(s)-\beta(s)\|_\infty \int_0^t \|\dot q_k(s)\|_1\,\mathrm{d}s\,,$$

and the same holds replacing $q_k$ with $q$. Then we get (3.2.23) by (3.2.20), (3.2.22), and (3.2.26).

We can argue similarly to prove (3.2.24), noticing that

$$\begin{aligned}
\langle\partial_\alpha\widehat{\mathcal{V}}_{\mathcal{H}}(\beta_1,q_k;0,t),\beta_2\rangle &= \int_{(0,t)\times(\Omega\cup\partial_D\Omega)} \partial_\alpha H\Big(\beta_1(x),\frac{\mathrm{d}\mu_k}{\mathrm{d}\nu_k}(s,x)\Big)\beta_2(x)\,\mathrm{d}\nu_k(s,x)\,,\\
\langle\partial_\alpha\widehat{\mathcal{V}}_{\mathcal{H}}(\beta_1,q;0,t),\beta_2\rangle &= \int_{(0,t)\times(\Omega\cup\partial_D\Omega)} \partial_\alpha H\Big(\beta_1(x),\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(s,x)\Big)\beta_2(x)\,\mathrm{d}\nu(s,x)\,,
\end{aligned}$$

and applying Reshetnyak's Lower Semicontinuity Theorem, since $(x,\xi)\mapsto\partial_\alpha H(\beta_1(x),\xi)\beta_2(x)$ is a nonnegative continuous function positively 1-homogeneous and convex in the second variable. This allows us to conclude. □

We prove now the existence of a family of absolutely continuous $\varepsilon$-approximate viscous evolutions according to Definition 3.2.1, satisfying in addition a uniform bound on the $L^1$-norm of the time derivative.

THEOREM 3.2.8. *Assume* (3.0.19), (2.1.8), (2.1.10), (2.1.27), (3.0.32) *for given* $\alpha_0, u_0, e_0, p_0$ *and, if* $\lambda \in (0,1]$, *also* (3.0.25). *There exists a family* $\{(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)\}_{\varepsilon>0}$ *of* $\varepsilon$*-approximate viscous evolutions such that* $(\alpha_\varepsilon(0), u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) = (\alpha_0, u_0, e_0, p_0)$ *and*

$$\int_0^T \|\dot\alpha_\varepsilon(t)\|_{m,2}\,\mathrm{d}t + \int_0^T \|\dot e_\varepsilon(t)\|_2\,\mathrm{d}t \le C \tag{3.2.27}$$

*with* $C$ *independent of* $\varepsilon$.

PROOF. The proof is divided in subsequent steps.

**Time-discretization and time-continuous limit.** Let us fix $\varepsilon > 0$. Starting with the given initial condition $(\alpha_0, u_0, e_0, p_0)$ we consider the incremental problems (3.1.1) in correspondence with the parameter $\varepsilon > 0$, thus obtaining a sequence of approximate solutions

$$\alpha_{k,\varepsilon} \equiv \alpha_k, \quad u_{k,\varepsilon} \equiv u_k \quad e_{k,\varepsilon} \equiv e_k \quad p_{k,\varepsilon} \equiv p_k\,.$$

We use the same notation of Section 3.1 for their piecewise constant interpolants.

From (3.1.26b) we have

$$\|\overline\alpha_k - \alpha_k\|_{L^\infty(0,T;H^m(\Omega))} \le \tau^{1/2}\|\dot\alpha_k\|_{L^2(0,T;H^m(\Omega))} \le C_\varepsilon \tau^{1/2}\,,$$

and the same holds for $u_k$, $e_k$, and $p_k$, by Remark 3.1.9. By standard compactness results and Helly's Theorem, there exist $\alpha_\varepsilon \in H^1(0,T;H^m(\Omega))$ and $e_\varepsilon \in H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ such that (up to subsequences)

$$\alpha_k \rightharpoonup \alpha_\varepsilon \text{ in } H^1(0,T;H^m(\Omega))\,, \quad e_k \rightharpoonup e_\varepsilon \text{ in } H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))\,, \tag{3.2.28}$$

$$\alpha_k(t) \rightharpoonup \alpha_\varepsilon(t) \text{ in } H^m(\Omega)\,, \quad e_k(t) \rightharpoonup e_\varepsilon(t) \text{ in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \quad \text{for every } t \in [0,T]\,, \tag{3.2.29}$$

and

$$\alpha_k \to \alpha_\varepsilon \text{ in } C([0,T];C(\overline\Omega))\,, \tag{3.2.30}$$

since $H^m(\Omega)$ is compactly embedded into $C(\overline\Omega)$.

In particular, since $\dot\alpha_k \rightharpoonup \dot\alpha_\varepsilon$ in $L^1(0,T;H^m(\Omega))$ and $\dot e_k \rightharpoonup \dot e_\varepsilon$ in $L^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, by Proposition 3.1.8 and (3.1.44) there exists a constant $C$ independent of $\varepsilon > 0$ such that

$$\int_0^T \|\dot\alpha_\varepsilon(t)\|_{m,2}\,\mathrm{d}t + \int_0^T \|\dot e_\varepsilon(t)\|_2\,\mathrm{d}t \le C$$

for every $\varepsilon > 0$.

Taking into account (3.1.43) and the fact that $p_k(0) = p_0$ and $u_k(0) = u_0$ for every $k$, from Lemma 1.2.3 it follows that there exist $p_\varepsilon \in H^1_w(0,T;M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D))$, $u_\varepsilon \in H^1(0,T;L^q(\Omega;\mathbb{R}^n))$, with $1 \le q < \frac{n}{n-1}$, and $E_\varepsilon \in H^1_w(0,T;M_b(\Omega;\mathbb{M}^{n\times n}_{sym}))$ such that, for a suitable subsequence,

$$\dot p_k \overset{*}{\rightharpoonup} \dot p \text{ in } L^2_w(0,T;M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D))\,,$$
$$\dot u_k \rightharpoonup \dot u_\varepsilon \text{ in } L^2(0,T;L^q(\Omega;\mathbb{R}^n))\,, \quad E\dot u_k \overset{*}{\rightharpoonup} \dot E_\varepsilon \text{ in } L^2_w(0,T;M_b(\Omega;\mathbb{M}^{n\times n}_{sym}))\,,$$

and

$$p_k(t) \overset{*}{\rightharpoonup} p_\varepsilon(t) \text{ in } M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)\,, \tag{3.2.31}$$
$$u_k(t) \rightharpoonup u_\varepsilon(t) \text{ in } L^1(\Omega;\mathbb{R}^n), \quad Eu_k(t) \overset{*}{\rightharpoonup} E_\varepsilon(t) \text{ in } M_b(\Omega;\mathbb{M}^{n\times n}_{sym})$$

for every $t \in [0,T]$. This implies that $Eu_\varepsilon(t) = E_\varepsilon(t)$ for every $t \in [0,T]$, hence

$$u_\varepsilon \in H^1_w(0,T;BD(\Omega)), \quad u_k(t) \overset{*}{\rightharpoonup} u_\varepsilon(t) \text{ in } BD(\Omega) \text{ for every } t \in [0,T]. \tag{3.2.32}$$

Let us now prove that $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$ is an $\varepsilon$-approximate viscous evolution. The irreversibility condition $(\mathrm{ev0})_\varepsilon$ holds by (3.2.29) and the monotonicity in time of the $\alpha_k$. We can assume that (3.2.29), (3.2.31), and (3.2.32) hold for the same subsequence and thus $(\mathrm{ev1})_\varepsilon$ follows by [**28**, Lemma 2.1] and by the fact that $w_k(t) \to w(t)$ in $H^1(\mathbb{R}^n;\mathbb{R}^n)$ for every $t$ ($w$ being continuous into $H^1(\mathbb{R}^n;\mathbb{R}^n)$).

We now prove $(\mathrm{ev2})_\varepsilon$. Let us fix $t \in [0,T]$. For

$$\widehat{\sigma}_k(t,x) := \Pi_{K(\alpha_\varepsilon(t,x))}(\sigma_k(t,x)),$$

$\Pi_{K(\alpha_\varepsilon(t,x))}$ being the projection onto $K(\alpha_\varepsilon(t,x))$, we have by (3.0.23) that

$$|\overline{\sigma}_k(t,x) - \widehat{\sigma}_k(t,x)| \le C_K |\overline{\alpha}_k(t,x) - \alpha_\varepsilon(t,x)|$$

for every $x$ such that $\overline{\sigma}_k(t,x) \in K(\overline{\alpha}_k(t,x))$, and then

$$\|\overline{\sigma}_k(t) - \widehat{\sigma}_k(t)\|_\infty \le C_K \|\overline{\alpha}_k(t) - \alpha_\varepsilon(t)\|_\infty .$$

We now recall that $\overline{\sigma}_k(t) \in \mathcal{K}_{\overline{\alpha}_k(t)}(\Omega)$, $\overline{\sigma}_k(t) \rightharpoonup \sigma_\varepsilon(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $\overline{\alpha}_k(t) \to \alpha_\varepsilon(t)$ uniformly in $\overline{\Omega}$ for every $t \in [0,T]$. Therefore $\mathcal{K}_{\alpha_\varepsilon(t)}(\Omega) \ni \widehat{\sigma}_k(t) \rightharpoonup \sigma_\varepsilon(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\sigma_\varepsilon(t) \in \mathcal{K}_{\alpha_\varepsilon(t)}(\Omega)$, by convexity of the sets $K(\alpha)$.

By Proposition 3.2.3, it is enough to prove the energy inequality $(\mathrm{ev4''})_\varepsilon$ and the Kuhn-Tucker inequality $(\mathrm{ev3})_\varepsilon$.

**Proof of the energy inequality $(\mathrm{ev4''})_\varepsilon$.** From the absolute continuity of $\alpha_k$, $e_k$, and $p_k$, we get that $t \mapsto \mathcal{E}_\lambda(\alpha_k(t), e_k(t); p_k, t)$ is absolutely continuous and for every $k \in \mathbb{N}$, $t \in (0,T) \setminus \{t^1_k, \dots, t^{k-1}_k\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\alpha_k(t), e_k(t); p_k, t) = \langle \partial_\alpha \mathcal{E}_\lambda(\alpha_k(t), e_k(t); p_k, t), \dot{\alpha}_k(t)\rangle + \lambda \mathcal{H}(\alpha_k(t), \dot{p}_k(t)) + \langle \sigma_k(t), \dot{e}_k(t)\rangle .$$

We first consider the case $\lambda \in (0,1]$. By (3.1.15b) and (3.1.16)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\alpha_k(t), e_k(t); p_k, t) = -\varepsilon \|\dot{\alpha}_k(t)\|_2^2 + \lambda \mathcal{H}(\alpha_k(t), \dot{p}_k(t)) + \langle \sigma_k(t), \dot{e}_k(t)\rangle + \delta_k(t), \tag{3.2.33}$$

where

$$\begin{aligned}
\delta_k(t) := &- (1-\lambda)\tau \langle \partial_\alpha \mathcal{H}(\overline{\alpha}_k(t), \dot{p}_k(t)), \dot{\alpha}_k(t)\rangle - \tfrac{1}{2}\langle \big[\mathbb{C}'(\overline{\alpha}_k(t)) - \mathbb{C}'(\alpha_k(t))\big]\dot{\alpha}_k(t) e_k(t), e_k(t)\rangle \\
&- \tfrac{1}{2}\Big[\langle \mathbb{C}'(\overline{\alpha}_k(t))\dot{\alpha}_k(t)\overline{e}_k(t), \overline{e}_k(t)\rangle - \langle \mathbb{C}'(\overline{\alpha}_k(t))\dot{\alpha}_k(t) e_k(t), e_k(t)\rangle\Big] \\
&- \langle \partial D(\overline{\alpha}_k(t)) - \partial D(\alpha_k(t)), \dot{\alpha}_k(t)\rangle - \kappa \langle \overline{\alpha}_k(t) - \alpha_k(t), \dot{\alpha}_k(t)\rangle_{m,2} \\
&- \lambda \Big[\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha}_k(t), p_k; 0, t), \dot{\alpha}_k(t)\rangle - \langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k(t), p_k; 0, t), \dot{\alpha}_k(t)\rangle\Big] .
\end{aligned}$$

Let us estimate $\delta_k(t)$. First remark that

$$\begin{aligned}\int_0^T & \left|\tfrac{1}{2}\langle\big[\mathbb{C}'(\overline{\alpha}_k(t))-\mathbb{C}'(\alpha_k(t))\big]\dot{\alpha}_k(t)e_k(t),e_k(t)\rangle\right| + \left|\langle\partial D(\overline{\alpha}_k(t))-\partial D(\alpha_k(t)),\dot{\alpha}_k(t)\rangle\right|\\ &+\lambda\left|\langle\partial_\alpha\widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha}_k(t),p_k;0,t),\dot{\alpha}_k(t)\rangle-\langle\partial_\alpha\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k(t),p_k;0,t),\dot{\alpha}_k(t)\rangle\right|\\ &+\kappa\left|\langle\overline{\alpha}_k(t)-\alpha_k(t),\dot{\alpha}_k(t)\rangle_{m,2}\right|\,\mathrm{d}t\\ &\le C\|\overline{\alpha}_k-\alpha_k\|_{L^\infty(0,T;H^m(\Omega))}\int_0^T\|\dot{\alpha}_k(t)\|_{H^m(\Omega)}\,,\end{aligned}$$

where $C$ depends on $D$, $\overline{C}_K$, $\kappa$, $\sup_t\|e_k(t)\|_2$, $\mathcal{V}_{\mathcal{H}}(p_k;0,T)$, and on the $C^{1,1}$ norm of $\mathbb{C}$. Moreover,

$$\langle\partial_\alpha\mathcal{H}(\overline{\alpha}_k(t),\dot{p}_k(t)),\dot{\alpha}_k(t)\rangle\le C\big(\|\dot{\alpha}_k(t)\|_\infty^2+\|\dot{p}_k(t)\|_1^2\big)$$

and

$$\begin{aligned}\int_0^T & \left|\langle\mathbb{C}'(\overline{\alpha}_k(t))\dot{\alpha}_k(t)\overline{e}_k(t),\overline{e}_k(t)\rangle-\langle\mathbb{C}'(\overline{\alpha}_k(t))\dot{\alpha}_k(t)e_k(t),e_k(t)\rangle\right|\mathrm{d}t\\ &\le C\|\overline{e}_k-e_k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\int_0^T\|\dot{\alpha}_k(t)\|_\infty\,.\end{aligned}$$

Therefore, by Lemma 3.1.6 we get

$$\int_0^T\delta_k(t)\,\mathrm{d}t\le C\Big(\sup_{t\in[0,T]}\|\overline{\alpha}_k-\alpha_k\|_{m,2}+\sup_{t\in[0,T]}\|\overline{e}_k-e_k\|_2+\tau\Big)\int_0^T\|\dot{\alpha}_k(t)\|^2_{H^m(\Omega)}+\|E\dot{w}_k(t)\|_2^2\,\mathrm{d}t\,. \tag{3.2.34}$$

In the case $\lambda=0$ we obtain, using (3.1.14b),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\alpha_k(t),e_k(t))\le-\varepsilon\|\dot{\alpha}_k(t)\|_2^2+\langle\sigma_k(t),\dot{e}_k(t)\rangle+\delta_k'(t)\,, \tag{3.2.35}$$

with

$$\begin{aligned}\delta_k'(t):=&\,C\tau\|\dot{\alpha}_k(t)\|_\infty\|\dot{p}_k(t)\|_1-\langle\partial D(\overline{\alpha}_k(t))-\partial D(\alpha_k(t)),\dot{\alpha}_k(t)\rangle-\kappa\,\langle\overline{\alpha}_k(t)-\alpha_k(t),\dot{\alpha}_k(t)\rangle_{m,2}\\ &-\frac{1}{2}\Big[\langle\mathbb{C}'(\overline{\alpha}_k(t))\dot{\alpha}_k(t)\overline{e}_k(t),\overline{e}_k(t)\rangle-\langle\mathbb{C}'(\alpha_k(t))\dot{\alpha}_k(t)e_k(t),e_k(t)\rangle\Big]\,,\end{aligned}$$

so that

$$\int_0^T\delta_k'(t)\,\mathrm{d}t\le C\Big(\sup_{t\in[0,T]}\|\overline{\alpha}_k-\alpha_k\|_{m,2}+\sup_{t\in[0,T]}\|\overline{e}_k-e_k\|_2+\tau\Big)\int_0^T\|\dot{\alpha}_k(t)\|^2_{H^m(\Omega)}+\|E\dot{w}_k(t)\|_2^2\,\mathrm{d}t\,. \tag{3.2.36}$$

The rest of the proof is common for both cases $\lambda=0$ and $\lambda\in(0,1]$.

Now, we have that

$$\begin{aligned}\langle\sigma_k(t),\dot{e}_k(t)\rangle=&\,\langle\overline{\sigma}_k(t),\dot{e}_k(t)\rangle+\langle\big[\mathbb{C}(\alpha_k(t))-\mathbb{C}(\overline{\alpha}_k(t))\big]e_k(t),\dot{e}_k(t)\rangle\\ &+\langle\mathbb{C}(\overline{\alpha}_k(t))(e_k(t)-\overline{e}_k(t)),\dot{e}_k(t)\rangle\end{aligned} \tag{3.2.37}$$

with

$$\begin{aligned}\int_0^T & \left|\langle\big[\mathbb{C}(\alpha_k(t))-\mathbb{C}(\overline{\alpha}_k(t))\big]e_k(t),\dot{e}_k(t)\rangle+\langle\mathbb{C}(\overline{\alpha}_k(t))(e_k(t)-\overline{e}_k(t)),\dot{e}_k(t)\rangle\right|\\ &\le C\sup_t\big(\|\alpha_k(t)-\overline{\alpha}_k(t)\|_\infty+\|e_k(t)-\overline{e}_k(t)\|_2\big)\int_0^T\|\dot{e}_k(t)\|_2\,\mathrm{d}t\,.\end{aligned} \tag{3.2.38}$$

Since, by definition of interpolants and Lemma 3.1.1, div $\overline{\sigma}_k(t) = 0$ and $(\dot{u}_k(t), \dot{e}_k(t), \dot{p}_k(t)) \in A(\dot{w}_k(t))$ for every $t \in [0, T]$, it follows from the integration by parts formula (2.1.21) that

$$\langle \overline{\sigma}_k(t), \dot{e}_k(t) \rangle = \langle \overline{\sigma}_k(t), E\dot{w}_k(t) \rangle - \langle (\overline{\sigma}_k(t))_D | \dot{p}_k(t) \rangle \,. \tag{3.2.39}$$

By (3.1.18) (recall also (3.0.26)), for a.e. $t \in (0, T)$

$$\begin{aligned} -\langle (\overline{\sigma}_k(t))_D | \dot{p}_k(t) \rangle \leq & -\mathcal{H}(\alpha_k(t), \dot{p}_k(t)) + C\tau\big(\|\dot{\alpha}_k(t)\|_\infty^2 + \|E\dot{w}_k(t)\|_2^2\big) \\ & + C_K \sup_t \|\alpha_k(t) - \overline{\alpha}_k(t)\|_\infty \|\dot{p}_k(t)\|_1 \,. \end{aligned} \tag{3.2.40}$$

Gathering (3.2.37), (3.2.38), (3.2.39), and (3.2.40), it follows that

$$\begin{aligned} \int_0^T \langle \sigma_k(t), \dot{e}_k(t) \rangle \, \mathrm{d}t \leq & \int_0^T \langle \overline{\sigma}_k(t), E\dot{w}_k(t) \rangle \, \mathrm{d}t - \int_0^T \mathcal{H}(\alpha_k(t), \dot{p}_k(t)) \, \mathrm{d}t \\ & + C \sup_t \big(\|\alpha_k(t) - \overline{\alpha}_k(t)\|_\infty + \|e_k(t) - \overline{e}_k(t)\|_2\big) \int_0^T \|\dot{e}_k(t)\|_2 \, \mathrm{d}t \\ & + C\tau \int_0^T \big(\|\dot{\alpha}_k(t)\|_\infty^2 + \|E\dot{w}_k(t)\|_2^2\big) \, \mathrm{d}t + C_K \sup_t \|\alpha_k(t) - \overline{\alpha}_k(t)\|_\infty \int_0^T \|\dot{p}_k(t)\|_1 \, \mathrm{d}t \,. \end{aligned} \tag{3.2.41}$$

Integrating (3.2.33) (resp. (3.2.35)) between 0 and $T$, by (3.2.34) (resp. (3.2.36)) and (3.2.41) we get that

$$\begin{aligned} & \mathcal{E}_\lambda(\alpha_k(T), e_k(T); p_k, T) + (1-\lambda) \int_0^T \mathcal{H}(\alpha_k(t), \dot{p}_k(t)) \, \mathrm{d}t + \varepsilon \int_0^T \|\dot{\alpha}_k(t)\|_2^2 \, \mathrm{d}t \\ & \leq \mathcal{E}(\alpha_0, e_0) + \int_0^T \langle \overline{\sigma}_k(t), E\dot{w}_k(t) \rangle \, \mathrm{d}t + \eta_k \,, \end{aligned} \tag{3.2.42}$$

with

$$\eta_k := C\Big( \sup_{t \in [0,T]} \|\overline{\alpha}_k - \alpha_k\|_{m,2} + \sup_{t \in [0,T]} \|\overline{e}_k - e_k\|_2 + \tau \Big) \int_0^T \|\dot{\alpha}_k(t)\|_{m,2}^2 + \|E\dot{w}_k(t)\|_2^2 \, \mathrm{d}t \,,$$

taking into account Lemma 3.1.6. By (3.2.30), (3.2.31), and (3.1.43) we can apply Lemma 3.2.7 obtaining that

$$\int_0^T \mathcal{H}(\alpha_\varepsilon(T), \dot{p}_\varepsilon(t)) \, \mathrm{d}t \leq \liminf_{k \to \infty} \int_0^T \mathcal{H}(\alpha_k(T), \dot{p}_k(t)) \, \mathrm{d}t \,, \tag{3.2.43a}$$

$$\int_0^T \mathcal{H}(\alpha_\varepsilon(t), \dot{p}_\varepsilon(t)) \, \mathrm{d}t \leq \liminf_{k \to \infty} \int_0^T \mathcal{H}(\alpha_k(t), \dot{p}_k(t)) \, \mathrm{d}t \,. \tag{3.2.43b}$$

Since $\overline{\sigma}_k(t) \rightharpoonup \sigma_\varepsilon(t)$ for every $t \in [0, T]$ and $E\dot{w}_k(t) \to E\dot{w}(t)$ in $L^2(\Omega)$ for a.e. $t \in (0, T)$, by (2.1.27), we have that

$$\int_0^T \langle \overline{\sigma}_k(t), E\dot{w}_k(t) \rangle \, \mathrm{d}t \longrightarrow \int_0^T \langle \sigma_\varepsilon(t), E\dot{w}(t) \rangle \, \mathrm{d}t \quad \text{as } k \to \infty \tag{3.2.44}$$

by the Dominated Convergence Theorem.

Convergence (3.2.28) gives

$$\int_0^T \|\dot{\alpha}_\varepsilon(t)\|_2^2 \, \mathrm{d}t \leq \liminf_{k \to \infty} \int_0^T \|\dot{\alpha}_k(t)\|_2^2 \, \mathrm{d}t \,. \tag{3.2.45}$$

By (3.2.42), (3.2.43), (3.2.44), (3.2.45), and the semicontinuity of $\mathcal{E}$, we get the inequality $(\text{ev4''})_\varepsilon$.

**Proof of the Kuhn-Tucker inequality $(\mathbf{ev3})_\varepsilon$.** Let us consider $\beta \in L^\infty(0,T;H^m(\Omega))$ such that $\beta(t) \in H^m_-(\Omega)$ for a.e. $t \in (0,T)$. We can say that for every $\lambda \in [0,1]$ and a.e. $t \in (0,T)$.

$$\begin{aligned} 0 \le\ & \tfrac{1}{2}\langle \mathbb{C}'(\alpha_\varepsilon(t))\beta(t)\overline{e}_k(t), \overline{e}_k(t)\rangle + \langle \partial D(\overline{\alpha}_k(t)), \beta(t)\rangle + \kappa\, \langle \overline{\alpha}_k(t), \beta(t)\rangle_{m,2} + \varepsilon \langle \dot{\alpha}_k(t), \beta(t)\rangle_2 \\ & + \lambda \langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t), p_k; 0, t), \beta(t)\rangle + \tfrac{1}{2}\langle \big[\mathbb{C}'(\overline{\alpha}_k(t)) - \mathbb{C}'(\alpha_\varepsilon(t))\big]\overline{e}_k(t), \overline{e}_k(t)\rangle \\ & + \lambda \langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha}_k(t), p_k; 0, t) - \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t), p_k; 0, t), \beta(t)\rangle\,, \end{aligned} \tag{3.2.46}$$

using (3.1.14a) in the case $\lambda = 0$ and (3.1.15a) when $\lambda \in (0,1]$. By (3.1.43), (3.2.31), and by choice of $\beta$, Lemma 3.2.7 gives

$$-\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t), p_\varepsilon; 0, t), \beta(t)\rangle \le \liminf_{k\to\infty} \Big[-\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t), p_k; 0, t), \beta(t)\rangle\Big]\,. \tag{3.2.47}$$

for a.e. $t \in (0,T)$.

In addition, by weak lower semicontinuity of positive semidefinite quadratic forms, we get that for a.e. $t \in (0,T)$

$$-\langle \mathbb{C}'(\alpha_\varepsilon(t))\beta(t)e_\varepsilon(t), e_\varepsilon(t)\rangle \le \liminf_{k\to\infty}\Big[-\langle \mathbb{C}'(\alpha_\varepsilon(t))\beta(t)\overline{e}_k(t), \overline{e}_k(t)\rangle\Big]\,. \tag{3.2.48}$$

By (3.2.47), (3.2.48), and Fatou's Lemma, we have that

$$\begin{aligned} &-\int_0^T \Big[\tfrac{1}{2}\langle \mathbb{C}'(\alpha_\varepsilon(t))\beta(t)e_\varepsilon(t), e_\varepsilon(t)\rangle + \lambda\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t), p_\varepsilon; 0, t), \beta(t)\rangle\Big]\,\mathrm{d}t \\ &\le \liminf_{k\to\infty}\int_0^T -\Big[\tfrac{1}{2}\mathbb{C}'(\alpha_\varepsilon(t))\beta(t)\overline{e}_k(t), \overline{e}_k(t)\rangle + \lambda\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t), p_k; 0, t), \beta(t)\rangle\Big]\,\mathrm{d}t\,. \end{aligned} \tag{3.2.49}$$

The fact that $\alpha_k \rightharpoonup \alpha_\varepsilon$ in $H^1([0,T];L^2(\Omega))$ implies that $\dot{\alpha}_k \rightharpoonup \dot{\alpha}_\varepsilon$ in $L^2([0,T];L^2(\Omega))$ and then

$$\int_0^T \langle \dot{\alpha}_k(t), \beta(t)\rangle\,\mathrm{d}t \longrightarrow \int_0^T \langle \dot{\alpha}_\varepsilon(t), \beta(t)\rangle\,\mathrm{d}t\,. \tag{3.2.50}$$

Since $\overline{\alpha}_k(t) \rightharpoonup \alpha_\varepsilon(t)$ weakly in $H^m(\Omega)$ for every $t$, it follows that

$$\langle \partial D(\overline{\alpha}_k(t)), \beta(t)\rangle \longrightarrow \langle \partial D(\alpha_\varepsilon(t)), \beta(t)\rangle \;\text{ and }\; \langle \overline{\alpha}_k(t), \beta(t)\rangle_{m,2} \longrightarrow \langle \alpha_\varepsilon(t), \beta(t)\rangle_{m,2}$$

for every $t$, thus by the Dominated Convergence Theorem

$$\int_0^T \Big[\langle \partial D(\overline{\alpha}_k(t)), \beta(t)\rangle + \kappa\,\langle \overline{\alpha}_k(t), \beta(t)\rangle_{m,2}\Big]\,\mathrm{d}t \longrightarrow \int_0^T \Big[\langle \partial D(\alpha_\varepsilon(t)), \beta(t)\rangle + \kappa\,\langle \alpha_\varepsilon(t), \beta(t)\rangle_{m,2}\Big]\,\mathrm{d}t\,. \tag{3.2.51}$$

Notice now that

$$\begin{aligned} &\Big|\langle \big[\mathbb{C}'(\overline{\alpha}_k(t)) - \mathbb{C}'(\alpha_\varepsilon(t))\big]\beta(t)\overline{e}_k(t), \overline{e}_k(t)\rangle + \lambda\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha}_k(t), p_k; 0, t) - \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t), p_k; 0, t), \beta(t)\rangle\Big| \\ &\quad \le C\|\overline{\alpha}_k(t) - \alpha_\varepsilon(t)\|_\infty \|\beta(t)\|_\infty\,, \end{aligned} \tag{3.2.52}$$

where $C$ depends on an upper bound for the $C^{1,1}$ norm of $\mathbb{C}$ and $C_K$ (if $\lambda \in (0,1]$), $\sup_t \|\bar{e}_k(t)\|_2$, and $\mathcal{V}_{\mathcal{H}}(p_k;0,t)$. Integrating (3.2.46) from 0 and $T$ and passing to the limit as $k \to \infty$, we deduce from (3.2.49), (3.2.50), (3.2.51), and (3.2.52) that

$$0 \le \int_0^T \frac{1}{2}\Big[\langle \mathbb{C}'(\alpha_\varepsilon(t))\beta(t)e_\varepsilon(t), e_\varepsilon(t)\rangle + \langle \partial D(\alpha_\varepsilon(t)), \beta(t)\rangle + \kappa\,\langle \alpha_\varepsilon(t), \beta(t)\rangle_{m,2} + \lambda\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t), p_\varepsilon; 0, t), \beta(t)\rangle\Big]\,\mathrm{d}t + \varepsilon\int_0^T \langle \dot\alpha_\varepsilon(t), \beta(t)\rangle\,\mathrm{d}t\,.$$

We now fix $\beta \in H^m_-(\Omega)$ and set $\beta(t) := 1_A(t)\beta$ where $A$ is a measurable subset of $[0,T]$. Since $A$ is arbitrary, we find

$$\tfrac{1}{2}\langle \mathbb{C}'(\alpha_\varepsilon(t))\beta e_\varepsilon(t), e_\varepsilon(t)\rangle + \langle \partial D(\alpha_\varepsilon(t)), \beta\rangle + \kappa\,\langle \alpha_\varepsilon(t), \beta\rangle_{m,2} + \lambda\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t), p_\varepsilon; 0, t), \beta\rangle + \varepsilon\langle \dot\alpha_\varepsilon(t), \beta\rangle \ge 0\,,$$

for $t \in [0,T] \setminus E_\beta$, where $E_\beta$ is a negligible set depending on $\beta$. Thanks to the separability of $H^m_-(\Omega)$, it is easily seen that the inequality holds for every $t \in [0,T] \setminus E$, where $E$ is a negligible set independent of $\beta$. Then the Kuhn-Tucker inequality (ev3)$_\varepsilon$ is proved. □

REMARK 3.2.9. By (3.1.10) and (3.2.29), there exists $C$ independent of $\varepsilon$ such that for every $\varepsilon$

$$\sup_{t\in[0,T]} \|\sigma_\varepsilon(t)\|_2 \le C\,.$$

Then, the energy balance (ev4)$_\varepsilon$ and (2.1.16) imply that

$$\int_0^T \|\dot p_\varepsilon(t)\|_1\,\mathrm{d}t \le C \tag{3.2.53}$$

for every $\varepsilon > 0$, $C$ being independent of $\varepsilon$.

## 3.3. Rescaled quasistatic viscosity evolutions

In this section we study the asymptotic behavior of $\varepsilon$-approximate viscous evolutions as $\varepsilon$ tends to 0 using the rescaling technique of [**40, 76, 29**]. Thanks to estimates (3.2.27) and (3.2.53) in Theorem 3.2.8 and Remark 3.2.9, the total arclength of the graphs of the functions $t \mapsto (\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in H^m(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ is uniformly bounded in $\varepsilon$. Then the inverse functions of the arclength reparametrizations converge uniformly to a map $t^\circ$, up to subsequences.

Using to the "slow" time scale $s = (t^\circ)^{-1}(t)$ and passing to the limit as $\varepsilon \to 0$, we obtain a *rescaled quasistatic viscosity evolution*. In the intervals where the original time $t = t^\circ(s)$ increases, such an evolution behaves as a "0-approximate viscous evolution", namely conditions (ev0)$_\varepsilon, \dots,$ (ev4)$_\varepsilon$ hold with $\varepsilon = 0$.

DEFINITION 3.3.1. Let us assume (3.0.19), (2.1.8), (2.1.10), and let $w$ be as in (2.1.27). We say that a 5-tuple of Lipschitz functions $(\alpha^\circ, u^\circ, e^\circ, p^\circ, t^\circ)$ from $[0,S]$ into $H^m(\Omega;[0,1]) \times BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D) \times [0,T]$ is a *rescaled quasistatic viscosity evolution in the time interval* $[0,S]$ with datum $w$ if, setting for every $s \in [0,S]$

$$\sigma^\circ(s) := \mathbb{C}(\alpha^\circ(s))e^\circ(s)\,, \quad w^\circ(s) := w(t^\circ(s))\,, \quad \text{and}$$

$$U^\circ := \{s \in [0,S] \colon t^\circ \text{ is constant in a neighbourhood of } s\}\,,$$

the following conditions are satisfied:

(ev0) *irreversibility:* $t^\circ$ is nondecreasing and surjective, and for every $x \in \Omega$

$$[0,S] \ni s \mapsto \alpha^\circ(s,x) \quad \text{is nonincreasing;}$$

(ev1) *kinematic condition and equilibrium:* for every $s \in [0,S]$

$$(u^\circ(s), e^\circ(s), p^\circ(s)) \in A(w^\circ(s))\,, \quad \operatorname{div} \sigma^\circ(s) = 0 \text{ in } \Omega\,, \quad [\sigma^\circ(s)\nu] = 0 \text{ on } \partial_N\Omega\,;$$

(ev2) *stress constraint:* for every $s \in [0,S]$

$$\sigma^\circ(s) \in \mathcal{K}_{\alpha^\circ(s)}(\Omega)\,;$$

(ev3) *Kuhn-Tucker inequality in* $[0,S] \setminus U^\circ$*:* for every $s \in [0,S] \setminus U^\circ$

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \beta\rangle \geq 0 \quad \text{for every } \beta \in H^m_-(\Omega)\,;$$

(ev4) *energy balance*: for every $s \in [0,S]$

$$\mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s) + (1-\lambda)\int_0^s \mathcal{H}(\alpha^\circ(\tau), \dot{p}^\circ(\tau))\,\mathrm{d}\tau + \int_0^s \|\dot\alpha^\circ(\tau)\|_2 \Psi(\alpha^\circ(\tau), e^\circ(\tau); p^\circ, \tau)\,\mathrm{d}\tau$$
$$= \mathcal{E}(\alpha_0, e_0) + \int_0^s \langle \sigma^\circ(\tau), E\dot{w}^\circ(\tau)\rangle\,\mathrm{d}\tau\,,$$

where $\Psi$ is defined in (3.2.9) and we use the convention $0 \cdot \infty = 0$.

Remark 3.3.2. By [**29**, Remark 4.2] the integrals in (ev4) make sense. Moreover, by definition of $\Psi$ (see also Remark 3.2.4 and Lemma 3.2.5) and (ev3) we have that

$$\Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s) = \mathrm{d}_2(\partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), G) = \sup_{\beta \in F}\langle -\partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \beta\rangle = 0 \tag{3.3.1}$$

for every $s \in [0,S] \setminus U^\circ$.

Remark 3.3.3. By [**98**, Theorem 3.10], conditions (ev1) and (ev2) are equivalent to the following global minimality condition for fixed damage variable: for every $s \in [0,S]$,

$$(u^\circ(s), e^\circ(s), p^\circ(s)) \in A(w^\circ(s))$$

and

$$\mathcal{Q}(\alpha^\circ(s), e^\circ(s)) \leq \mathcal{Q}(\alpha^\circ(s), \eta) + \mathcal{H}(\alpha^\circ(s), q - p^\circ(s)) \quad \text{for every } (v, \eta, q) \in A(w^\circ(s))$$

Remark 3.3.4. The energy balance (ev4) shows the role of the parameter $\lambda \in [0,1]$. In fact, notice that the damage variable acts differently in the second and in the third summand of the left-hand side, since it is computed at the final point of the interval in the former case, whilst it is variable in the latter. Therefore, if we derive in $s$ and take into account the cancellation, from the two dissipative integrals we obtain

$$\mathcal{H}(\alpha^\circ(s), \dot{p}^\circ(s)) + \lambda\,\langle \partial_\alpha \mathcal{H}(\alpha^\circ(s), \dot{p}^\circ(s)), \dot\alpha^\circ(s)\rangle\,,$$

the first term being the dissipation potential related to plasticity, the second one giving a contribution to the dissipation potential related to the damage variable: the latter is damped by the parameter $\lambda$. Tuning $\lambda$ between zero and one, we account for different effects of the plasticity on the damage process; indeed, the bigger is $\lambda$, the easier it is to damage a portion of the material affected by plastic strain's changes. Thus the parameter $\lambda$ is related to a fatigue phenomenon. Setting $\lambda = 0$ leads to an energy balance analogous to the one of [**29**]; the choice $\lambda = 1$ was instead prescribed in [**2, 3**].

Below we give two characterizations of the notion of rescaled quasistatic viscosity evolution: the first will be employed to derive a condition of Kuhn-Tucker type for the damage variable and a weak formulation of the Prandtl-Reuss flow rule; the second will be useful in the proof of Theorem 3.3.6.

Proposition 3.3.5. *Let* $(\alpha^\circ, u^\circ, e^\circ, p^\circ, t^\circ)$ *be a 5-tuple of Lipschitz functions from* $[0, S]$ *into* $H^m(\Omega; [0,1]) \times BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D) \times [0, T]$ *satisfying* (ev0)–(ev3). *Then* $(\alpha^\circ, u^\circ, e^\circ, p^\circ, t^\circ)$ *is a rescaled quasistatic viscosity evolution, i.e. it satisfies the energy balance* (ev4), *if and only if any of the two following conditions holds true:*

(ev4') *for a.e.* $s \in (0, S)$ *the following hold:*

– generalized Kuhn-Tucker equality:

$$\langle -\partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \dot\alpha^\circ(s)\rangle = \|\dot\alpha^\circ(s)\|_2 \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\,; \tag{3.3.2a}$$

– Hill's maximum plastic work principle:

$$\mathcal{H}(\alpha^\circ(s), \dot p^\circ(s)) = \langle (\sigma^\circ(s))_D, \dot p^\circ(s)\rangle\,. \tag{3.3.2b}$$

(ev4'') energy inequality:

$$\mathcal{E}_\lambda(\alpha^\circ(S), e^\circ(S); p^\circ, S) + (1-\lambda)\int_0^S \mathcal{H}(\alpha^\circ(s), \dot p^\circ(s))\,\mathrm{d}s + \int_0^S \|\dot\alpha^\circ(s)\|_2 \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\,\mathrm{d}s$$
$$\leq \mathcal{E}(\alpha_0, e_0) + \int_0^S \langle \sigma^\circ(s), E\dot w^\circ(s)\rangle\,\mathrm{d}s\,.$$

Proof. **Ad (ev4)$\iff$ (ev4'):** Since $\alpha^\circ$, $e^\circ$, $p^\circ$ are Lipschitz, $s \mapsto \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s)$ is absolutely continuous and for a.e. $s \in (0, S)$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s) = \langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \dot\alpha^\circ(s)\rangle + \langle \sigma^\circ(s), \dot e^\circ(s)\rangle + \lambda \mathcal{H}(\alpha^\circ(s), \dot p^\circ(s))\,. \tag{3.3.3}$$

Moreover, property (ev1) and [**28**, Lemma 5.5] give that

$$(\dot u^\circ(s), \dot e^\circ(s), \dot p^\circ(s)) \in A(\dot w^\circ(s)) \quad \text{for a.e. } s \in (0, S)\,,$$

and then the integration by parts formula (2.1.21) implies

$$\langle (\sigma^\circ(s))_D \,|\, \dot p^\circ(s)\rangle = \langle \sigma^\circ(s), E\dot w^\circ(s)\rangle - \langle \sigma^\circ(s), \dot e^\circ(s)\rangle \tag{3.3.4}$$

for a.e. $s \in (0, S)$. Then (ev4) holds if and only if

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s) = -\,(1-\lambda)\mathcal{H}(\alpha^\circ(s), \dot p^\circ(s)) + \langle \sigma^\circ(s), E\dot w^\circ(s)\rangle$$
$$-\,\|\dot\alpha^\circ(s)\|_2 \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\,,$$

which in turn is equivalent to

$$\begin{aligned}&\langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \dot\alpha^\circ(s)\rangle - \|\dot\alpha^\circ(s)\|_2 \inf_{\beta\in F} \langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \beta\rangle \\ &\quad + \mathcal{H}(\alpha^\circ(s), \dot p^\circ(s)) - \langle (\sigma^\circ(s))_D \,|\, \dot p^\circ(s)\rangle = 0\,,\end{aligned} \tag{3.3.5}$$

see (3.2.9) for the definition of $\Psi$. Now, by (ev2) and (2.1.23), and since $\dot p^\circ(s) \in \Pi(\Omega)$ for a.e. $s$, we can say that

$$\langle (\sigma^\circ(s))_D \,|\, \dot p^\circ(s)\rangle \leq \mathcal{H}(\alpha^\circ(s), \dot p^\circ(s)) \tag{3.3.6}$$

for a.e. $s \in (0, S)$. Then (3.3.5) is equivalent to (ev4').

**Ad (ev4)$\Longleftrightarrow$ (ev4"):** It is obvious that (ev4) implies (ev4"); let us prove the converse. By (3.3.3), (3.3.4), and (3.3.6) we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s) \geq &- (1-\lambda)\mathcal{H}(\alpha^\circ(s), \dot p^\circ(s)) + \|\dot\alpha^\circ(s)\|_2 \inf_{\beta\in F} \langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \beta\rangle \\ &+ \langle \sigma^\circ(s), E\dot w^\circ(s)\rangle\end{aligned}$$

for a.e. $s \in (0, S)$. Integrating, we get for every $0 \leq s_1 \leq s_2 \leq S$ the inequality

$$\begin{aligned}&\mathcal{E}_\lambda(\alpha^\circ(s_2), e^\circ(s_2); p^\circ, s_2) + (1-\lambda)\int_{s_1}^{s_2} \mathcal{H}(\alpha^\circ(s), \dot p^\circ(s))\,\mathrm{d}s + \int_{s_1}^{s_2} \|\dot\alpha^\circ(s)\|_2 \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\,\mathrm{d}s \\ &\quad \geq \mathcal{E}_\lambda(\alpha^\circ(s_1), e^\circ(s_1); p^\circ, s_1) + \int_{s_1}^{s_2} \langle \sigma^\circ(s), E\dot w^\circ(s)\rangle\,\mathrm{d}s\,,\end{aligned}$$

which implies the energy balance (ev4) thanks to (ev4"). This concludes the proof. ☐

The following theorem is the main result of the chapter.

THEOREM 3.3.6. *Assume* (3.0.19), (2.1.8), (3.0.20), (2.1.10), (3.0.23), *and let* $w$ *and* $\alpha_0$, $u_0$, $e_0$, $p_0$ *satisfy* (2.1.27) *and* (3.0.32) *respectively. If* $\lambda \in (0, 1]$, *assume also* (3.0.25). *Then there exist* $S > 0$ *and a rescaled quasistatic viscosity evolution in the time interval* $[0, S]$ *according to Definition 3.3.1 such that* $(\alpha_0, u_0, e_0, p_0, 0) = (\alpha^\circ(0), u^\circ(0), e^\circ(0), p^\circ(0), t^\circ(0))$.

PROOF. The proof is divided in subsequent steps.

**Viscous approximation.** Let $\{(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)\}_{\varepsilon>0}$ be a family of $\varepsilon$-approximate viscous evolutions satisfying (3.2.27), whose existence follows from Theorem 3.2.8. For every $\varepsilon > 0$ and $t \in [0, T]$ let us define the function

$$s^\circ_\varepsilon(t) := t + \int_0^t \|\dot\alpha_\varepsilon(s)\|_{m,2}\,\mathrm{d}s + \int_0^t \|\dot e_\varepsilon(s)\|_2\,\mathrm{d}s + \int_0^t \|\dot p_\varepsilon(s)\|_1\,\mathrm{d}s\,.$$

It is easy to see that $s^\circ_\varepsilon$ is absolutely continuous, increasing, bijective on its domain, and

$$s^\circ_\varepsilon(t_2) - s^\circ_\varepsilon(t_1) \geq t_2 - t_1 \quad \text{for every } 0 \leq t_1 \leq t_2 \leq S_\varepsilon := s^\circ_\varepsilon(T)\,.$$

Let $t^\circ_\varepsilon \colon [0, S_\varepsilon] \mapsto [0, T]$ be the inverse of $s^\circ_\varepsilon$. By (3.2.27) and (3.2.53), it follows that $\sup_\varepsilon S_\varepsilon < +\infty$ and then, up to a subsequence, $S_\varepsilon \to S$ as $\varepsilon \to 0$, with $S \geq T$, since $S_\varepsilon(T) \geq T$. For every $\varepsilon > 0$, define the rescaled functions on $[0, S_\varepsilon]$ by

$$\begin{aligned}\alpha^\circ_\varepsilon(s) &:= \alpha_\varepsilon(t^\circ_\varepsilon(s))\,, & u^\circ_\varepsilon(s) &:= u_\varepsilon(t^\circ_\varepsilon(s))\,, & e^\circ_\varepsilon(s) &:= e_\varepsilon(t^\circ_\varepsilon(s))\,, \\ p^\circ_\varepsilon(s) &:= p_\varepsilon(t^\circ_\varepsilon(s))\,, & \sigma^\circ_\varepsilon(s) &:= \sigma_\varepsilon(t^\circ_\varepsilon(s))\,, & w^\circ_\varepsilon(s) &:= w(t^\circ_\varepsilon(s))\,.\end{aligned} \tag{3.3.7}$$

Up to assuming that the rescaled functions and $t^\circ_\varepsilon$ take their value at $S_\varepsilon$ also in $(S_\varepsilon, \overline{S}]$, with $\overline{S} := \sup_{\varepsilon>0} S_\varepsilon$, we may consider them to be defined on the fixed time interval $[0, S]$.

By compactness we may assume that $t^\circ_\varepsilon$ converges weakly* in $W^{1,\infty}((0,S);[0,T])$ to a function $t^\circ$ such that $t^\circ(0) = 0$ and

$$0 \le t^\circ(s_2) - t^\circ(s_1) \le s_2 - s_1 \quad \text{for every } 0 \le s_1 \le s_2 \le S\,.$$

By the uniform convergence of $t^\circ_\varepsilon$ to $t^\circ$ we immediately get that for every $s \in [0, S]$

$$w^\circ_\varepsilon(s) \to w^\circ(s) \quad \text{in } H^1(\mathbb{R}^n;\mathbb{R}^n)\,,$$

where we recall that $w^\circ(s) = w(t^\circ(s))$. From the definitions of $s^\circ_\varepsilon$ and $t^\circ_\varepsilon$ we obtain that

$$\|\alpha^\circ_\varepsilon(s_2) - \alpha^\circ_\varepsilon(s_1)\|_{m,2} + \|e^\circ_\varepsilon(s_2) - e^\circ_\varepsilon(s_1)\|_2 + \|p^\circ_\varepsilon(s_2) - p^\circ_\varepsilon(s_1)\|_1 \le s_2 - s_1 \tag{3.3.8}$$

for every $0 \le s_1 < s_2 \le S$. Arguing as in [**29**, proof of (5.29)–(5.32)] and using (3.3.8) we see that there exist a quadruple of functions $(\alpha^\circ, u^\circ, e^\circ, p^\circ)$ from $[0, S]$ into $H^m(\Omega) \times BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)$, such that, up to a (not relabeled) subsequence of $\alpha^\circ_\varepsilon$, $u^\circ_\varepsilon$, $e^\circ_\varepsilon$, $p^\circ_\varepsilon$, it holds

$$\alpha^\circ_\varepsilon(s_\varepsilon) \rightharpoonup \alpha^\circ(s) \text{ weakly in } H^m(\Omega)\,, \tag{3.3.9a}$$
$$u^\circ_\varepsilon(s_\varepsilon) \rightharpoonup u^\circ(s) \text{ weakly}^* \text{ in } BD(\Omega)\,, \tag{3.3.9b}$$
$$e^\circ_\varepsilon(s_\varepsilon) \rightharpoonup e^\circ(s) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \tag{3.3.9c}$$
$$p^\circ_\varepsilon(s_\varepsilon) \rightharpoonup p^\circ(s) \text{ weakly}^* \text{ in } M_b(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_D)\,, \tag{3.3.9d}$$

for every $s \in [0, S]$ and $s_\varepsilon \to s$. Moreover $(u^\circ(s), e^\circ(s), p^\circ(s)) \in A(w^\circ(s))$, div $\sigma^\circ(s) = 0$, and

$$\alpha^\circ_\varepsilon \to \alpha^\circ \;\text{ in } C([0,S]; C(\overline{\Omega}))\,. \tag{3.3.10}$$

In particular (ev0) and (ev1) follow. By lower semicontinuity we obtain from (3.3.8) that

$$\|\alpha^\circ(s_2) - \alpha^\circ(s_1)\|_{m,2} + \|e^\circ(s_2) - e^\circ(s_1)\|_2 + \|p^\circ(s_2) - p^\circ(s_1)\|_1 \le s_2 - s_1 \tag{3.3.11}$$

for every $0 \le s_1 < s_2 \le S$, hence

$$\|\dot\alpha^\circ(s)\|_{m,2} + \|\dot e^\circ(s)\|_2 + \|\dot p^\circ(s)\|_1 \le 1 \quad \text{for a.e. } s \in [0,S]\,.$$

We now define

$$s^\circ_-(t) := \sup\{s \in [0,S] \colon t^\circ(s) < t\} \quad \text{for } t \in (0,T]\,,$$
$$s^\circ_+(t) := \inf\{s \in [0,S] \colon t^\circ(s) > t\} \quad \text{for } t \in [0,T)\,,$$

and $s^\circ_-(0) := 0$, $s^\circ_+(T) := S$. Then

$$s^\circ_-(t) \le \liminf_{\varepsilon\to 0} s^\circ_\varepsilon(t) \le \limsup_{\varepsilon\to 0} s^\circ_\varepsilon(t) \le s^\circ_+(t) \quad \text{and} \quad t^\circ(s^\circ_-(t)) = t = t^\circ(s^\circ_+(t))$$

for every $t \in [0,T]$,

$$s^\circ_-(t^\circ(s)) \le s \le s^\circ_+(t^\circ(s))$$

for every $s \in [0,S]$, the set

$$S^\circ := \{t \in [0,T] \colon s^\circ_-(t) < s^\circ_+(t)\} \tag{3.3.12}$$

is at most countable, and

$$U^\circ = \bigcup_{t\in S^\circ} (s^\circ_-(t), s^\circ_+(t))\,, \tag{3.3.13}$$

where $U^\circ$ is defined in (3.3.1). Moreover, for every $t\in[0,T]\setminus S^\circ$,

$$u_\varepsilon(t) \rightharpoonup u^\circ(s^\circ_-(t)) \text{ weakly}^* \text{ in } BD(\Omega)\,, \tag{3.3.14a}$$

$$e_\varepsilon(t) \rightharpoonup e^\circ(s^\circ_-(t)) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \tag{3.3.14b}$$

$$p_\varepsilon(t) \rightharpoonup p^\circ(s^\circ_-(t)) \text{ weakly}^* \text{ in } M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)\,, \tag{3.3.14c}$$

$$\alpha_\varepsilon(t) \to \alpha^\circ(s^\circ_-(t)) \text{ strongly in } C(\overline{\Omega})\,. \tag{3.3.14d}$$

These convergences will be used at the end of the proof.

From (ev2)$_\varepsilon$ and (3.3.7) we have

$$\sigma^\circ_\varepsilon(s)\in\mathcal{K}_{\alpha^\circ_\varepsilon(s)} \quad \text{for every } s\in[0,S]\,,$$

thus the convexity of $K(\alpha)$ for every $\alpha\in[0,1]$, (3.0.23) and (3.3.9) imply (ev2). By Proposition 3.3.5, in order to show that $(\alpha^\circ,u^\circ,e^\circ,p^\circ,t^\circ)$ is a rescaled quasistatic viscosity evolution it remains to prove only (ev3) and inequality (ev4").

**Proof of (ev3).** Setting

$$A^\circ := \Big\{s\in[0,S]\colon\, \Psi(\alpha^\circ(s),e^\circ(s);p^\circ,s)>0\Big\}\,, \tag{3.3.15}$$

in order to get (ev3) it is enough to show that $A^\circ\subset U^\circ$.

Arguing as in the proof of the energy inequality (ev4")$_\varepsilon$ in Theorem 3.2.8 and using (3.3.9c), (3.3.9d), (3.3.10), we see that for every $s\in[0,S]$ and $\beta\in H^m_-(\Omega)$

$$\langle -\partial_\alpha\mathcal{E}_\lambda(\alpha^\circ(s),e^\circ(s);p^\circ,s),\beta\rangle \le \liminf_{\varepsilon\to0}\langle -\partial_\alpha\mathcal{E}_\lambda(\alpha^\circ_\varepsilon(s),e^\circ_\varepsilon(s);p^\circ_\varepsilon,s),\beta\rangle\,,$$

thus

$$\Psi(\alpha^\circ(s),e^\circ(s);p^\circ,s)\le\liminf_{\varepsilon\to0}\Psi(\alpha^\circ_\varepsilon(s),e^\circ_\varepsilon(s);p^\circ_\varepsilon,s)\,. \tag{3.3.16}$$

Moreover, for every $\beta\in H^m_-(\Omega)$ $s\mapsto\langle\partial_\alpha\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha^\circ(s),p^\circ;0,s),\beta\rangle$ is continuous, being an integral function. Together with (3.3.11), this implies that $s\mapsto\langle -\partial_\alpha\mathcal{E}_\lambda(\alpha^\circ(s),e^\circ(s);p^\circ,s),\beta\rangle$ is continuous for every $\beta\in H^m_-(\Omega)$, and consequently that

$$s\mapsto\Psi(\alpha^\circ(s),e^\circ(s);p^\circ,s) \text{ is lower semicontinuous}\,. \tag{3.3.17}$$

Thus, $A^\circ$ is open.

We now set $D^\circ := \{s\in(0,S)\colon\, \dot t^\circ(s)=0\}$ and prove that

$$\limsup_{\varepsilon\to0}\dot t^\circ_\varepsilon(s)>0 \quad \text{for a.e. } s\in(0,S)\setminus D^\circ\,. \tag{3.3.18}$$

Indeed, assuming the opposite, we could find a measurable set $A\subset(0,S)\setminus D^\circ$ with positive measure such that

$$\lim_{\varepsilon\to0}\dot t^\circ_\varepsilon(s)=0 \quad \text{for every } s\in A\,,$$

$t^\circ_\varepsilon$ being nondecreasing. Since the functions $t^\circ_\varepsilon$ are 1-Lipschitz, the Dominated Convergence Theorem implies that

$$\lim_{\varepsilon\to0}\int_A\dot t^\circ_\varepsilon(s)\,\mathrm{d}s=0\,.$$

On the other hand,

$$\lim_{\varepsilon\to 0}\int_A \dot t^\circ_\varepsilon(s)\,\mathrm{d}s = \int_A \dot t^\circ(s)\,\mathrm{d}s\,,$$

because $t^\circ_\varepsilon \rightharpoonup t^\circ$ weakly$^*$ in $W^{1,\infty}$. But

$$\int_A \dot t^\circ(s)\,\mathrm{d}s > 0\,,$$

since $\dot t^\circ(s) > 0$ for a.e. $s\in(0,S)\setminus D^\circ$. Then (3.3.18) is proved.

Since $\mathcal{H}$ is 1-homogeneous in the second variable, the reparametrization $t = t^\circ_\varepsilon(s)$ gives

$$\int_0^{t^\circ_\varepsilon(S)} \mathcal{H}(\alpha_\varepsilon(t),\dot p_\varepsilon(t))\,\mathrm{d}t = \int_0^S \mathcal{H}(\alpha^\circ_\varepsilon(s),\dot p^\circ_\varepsilon(s))\,\mathrm{d}s\,. \tag{3.3.19}$$

By (3.0.29), for every $s\in[0,S]$ and $\beta\in C(\overline{\Omega})$

$$\langle\partial_\alpha\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\varepsilon(t^\circ_\varepsilon(s)),p_\varepsilon;0,t^\circ_\varepsilon(s)),\beta\rangle = \langle\partial_\alpha\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha^\circ_\varepsilon(s),p^\circ_\varepsilon;0,s),\beta\rangle\,, \tag{3.3.20}$$

thus

$$\langle\partial_\alpha\mathcal{E}_\lambda(\alpha_\varepsilon(t^\circ_\varepsilon(s)),e_\varepsilon(t^\circ_\varepsilon(s));p_\varepsilon,t^\circ_\varepsilon(s)),\beta\rangle = \langle\partial_\alpha\mathcal{E}_\lambda(\alpha^\circ_\varepsilon(s),e^\circ_\varepsilon(s);p^\circ_\varepsilon,s),\beta\rangle\,. \tag{3.3.21}$$

By (3.3.16)

$$\begin{aligned}0\le\Psi(\alpha^\circ(s),e^\circ(s);p^\circ,s)&\le\liminf_{\varepsilon\to0}\Psi(\alpha^\circ_\varepsilon(s),e^\circ_\varepsilon(s);p^\circ_\varepsilon,s)=\liminf_{\varepsilon\to0}\varepsilon\|\dot\alpha_\varepsilon(t^\circ_\varepsilon(s))\|_2\\&=\liminf_{\varepsilon\to0}\varepsilon\frac{\|\dot\alpha^\circ_\varepsilon(s)\|_2}{\dot t^\circ_\varepsilon(s)}=0\end{aligned}$$

for a.e. $s\in(0,S)\setminus D^\circ$, where the first equality follows from (3.2.7), (3.2.9), and (3.3.21) and the last from (3.3.8) and (3.3.18). Therefore for a.e. $s\in A^\circ$ we have $\dot t^\circ(s)=0$. Since $A^\circ$ is open by (3.3.17), every $s\in A^\circ$ has an open neighborhood where $\dot t^\circ=0$; then $A^\circ\subset U^\circ$ since $t^\circ$ is Lipschitz and hence absolutely continuous.

**Proof of the energy inequality (ev4").** Using the change of variable $t=t^\circ_\varepsilon(s)$ in the left-hand side of (3.2.8), we get by (3.3.19), (3.3.20), and (3.3.21)

$$\begin{aligned}\mathcal{E}_\lambda(\alpha^\circ_\varepsilon(S),e^\circ_\varepsilon(S);p^\circ_\varepsilon,S)+(1-\lambda)\int_0^S\mathcal{H}(\alpha^\circ_\varepsilon(s),\dot p^\circ_\varepsilon(s))\,\mathrm{d}s+\int_0^S\|\dot\alpha^\circ_\varepsilon(s)\|_2\Psi(\alpha^\circ_\varepsilon(s),e^\circ_\varepsilon(s);p^\circ_\varepsilon,s)\,\mathrm{d}s\\=\mathcal{E}(\alpha_0,e_0)+\int_0^{t^\circ_\varepsilon(S)}\langle\sigma_\varepsilon(t),E\dot w(t)\rangle\,\mathrm{d}t\,.\end{aligned} \tag{3.3.22}$$

By (3.3.9d), (3.3.10), (3.3.11), and using Lemma 3.2.7 we deduce that

$$\int_0^S\mathcal{H}(\alpha^\circ(s),\dot p^\circ(s))\,\mathrm{d}s\le\liminf_{\varepsilon\to0}\int_0^S\mathcal{H}(\alpha^\circ_\varepsilon(s),\dot p^\circ_\varepsilon(s))\,\mathrm{d}s\,, \tag{3.3.23a}$$

$$\int_0^S\mathcal{H}(\alpha^\circ(S),\dot p^\circ(s))\,\mathrm{d}s\le\liminf_{\varepsilon\to0}\int_0^S\mathcal{H}(\alpha^\circ_\varepsilon(S),\dot p^\circ_\varepsilon(s))\,\mathrm{d}s\,. \tag{3.3.23b}$$

Let us now prove that

$$\int_{A^\circ}\|\dot\alpha^\circ(s)\|_2\,\Psi(\alpha^\circ(s),e^\circ(s);p^\circ,s)\,\mathrm{d}s\le\liminf_{\varepsilon\to0}\int_{A^\circ}\|\dot\alpha^\circ_\varepsilon(s)\|_2\,\Psi(\alpha^\circ_\varepsilon(s),e^\circ_\varepsilon(s);p^\circ_\varepsilon,s)\,\mathrm{d}s\,. \tag{3.3.24}$$

For every compact set $C \subset A^\circ$ and every continuous function $\psi\colon C \to [0,+\infty)$ such that

$$\Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s) > \psi(s) \quad \text{for every } s \in C\,,$$

by the compactness of $C$ and (3.3.16), for $\varepsilon$ sufficiently small we get

$$\Psi(\alpha^\circ_\varepsilon(s), e^\circ_\varepsilon(s); p^\circ_\varepsilon, s) > \psi(s) \quad \text{for every } s \in C\,.$$

We now claim that

$$\int_C \|\dot\alpha^\circ(s)\|_2\, \psi(s)\,\mathrm{d}s \le \liminf_{\varepsilon\to 0} \int_C \|\dot\alpha^\circ_\varepsilon(s)\|_2\, \psi(s)\,\mathrm{d}s$$

for every compact $C \subset A^\circ$ and every continuous function $\psi\colon C \to [0,+\infty)$. This can be proved as in [**29**, Lemma 6.4] using (3.3.8) and (3.3.9a) and noticing that for every $\varphi \in C_c(\Omega)$ with $\|\varphi\|_2 = 1$ the functions $s \mapsto \langle\varphi, \dot\alpha^\circ_\varepsilon(s)\rangle$ are equi-Lipschitz on $[0,S]$ and converge to $s \mapsto \langle\varphi, \dot\alpha^\circ(s)\rangle$ for every $s$. By (3.3.17) and a standard approximation argument, (3.3.24) follows.

Let us now consider the left-hand side of (3.3.22): by (3.3.9), (3.3.23), and (3.3.24) we have

$$\begin{aligned}
&\mathcal{E}_\lambda(\alpha^\circ(S), e^\circ(S); p^\circ, S) + (1-\lambda)\int_0^S \mathcal{H}(\alpha^\circ(s), \dot p^\circ(s))\,\mathrm{d}s + \int_0^S \|\dot\alpha^\circ(s)\|_2 \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\,\mathrm{d}s \\
&\quad\le \liminf_{\varepsilon\to 0}\Bigg[\mathcal{E}_\lambda(\alpha^\circ_\varepsilon(S), e^\circ_\varepsilon(S); p^\circ_\varepsilon, S) + (1-\lambda)\int_0^S \mathcal{H}(\alpha^\circ_\varepsilon(s), \dot p^\circ_\varepsilon(s))\,\mathrm{d}s \\
&\qquad\qquad + \int_0^S \|\dot\alpha^\circ_\varepsilon(s)\|_2 \Psi(\alpha^\circ_\varepsilon(s), e^\circ_\varepsilon(s); p^\circ_\varepsilon, s)\,\mathrm{d}s\Bigg]\,.
\end{aligned} \tag{3.3.25}$$

As for the right-hand side, by (3.3.14) and the Dominated Convergence Theorem,

$$\int_0^T \langle\sigma^\circ(s^\circ_-(t)), E\dot w(t)\rangle\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^{t^\circ_\varepsilon(S)} \langle\sigma_\varepsilon(t), E\dot w(t)\rangle\,\mathrm{d}t\,. \tag{3.3.26}$$

Since $t^\circ$ is nondecreasing and Lipschitz, by (2.1.27) the function $w^\circ$ is absolutely continuous and

$$E\dot w^\circ(s) = E\dot w(t^\circ(s))\,\dot t^\circ(s) \quad \text{for a.e. } s \in [0,S]\,.$$

Hence

$$\int_0^T \langle\sigma^\circ(s^\circ_-(t)), E\dot w(t)\rangle\,\mathrm{d}t = \int_0^S \langle\sigma^\circ(s^\circ_-(t^\circ(s))), E\dot w(t^\circ(s))\,\dot t^\circ(s)\rangle\,\mathrm{d}s = \int_0^S \langle\sigma^\circ(s), E\dot w^\circ(s)\rangle\,\mathrm{d}s\,. \tag{3.3.27}$$

The last equality holds since $\dot t^\circ(s) = 0$ for a.e. $s \in U^\circ$ and $s^\circ_-(t^\circ(s)) = s$ for a.e. $s \in [0,S] \setminus U^\circ$. (The only exceptions are the points of the form $s = s^\circ_+(t)$ for $t \in S^\circ$.) From (3.3.22), (3.3.25), (3.3.26), and (3.3.27) we get finally the energy inequality (ev4”). Thus the proof is completed. □

Remark 3.3.7. From (3.3.2a) and (3.3.15) we immediately get the classical Kuhn-Tucker conditions in $[0,S] \setminus A^\circ$:

- For every $s \in [0,S] \setminus A^\circ$

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \beta \rangle \geq 0 \quad \text{for every } \beta \in H^m_-(\Omega)\,.$$

- For a.e. $s \in [0,S] \setminus A^\circ$

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \dot\alpha^\circ(s) \rangle = 0\,.$$

## 3.4. Properties of rescaled quasistatic viscosity evolutions

In the following we highlight some properties of rescaled viscosity evolutions, whose existence has been proved in Section 3.3 by time rescaling [**40, 76, 29**].

In the first part of this section we study what happens when the original time scale $t = t^\circ(s)$ is constant, i.e., in the jumping regime. In Lemma 3.4.1 we observe that if the damage variable is constant in a subinterval of $U^\circ$, then also the other variables are constant. On the other hand, if $\dot\alpha^\circ > 0$ in an interval then, up to a further time rescaling, the evolution is governed formally by $(\mathrm{ev0})_\varepsilon, \dots, (\mathrm{ev4})_\varepsilon$ with $\varepsilon = 1$ (see Proposition 3.4.3 and Remark 3.4.4).

Moreover, exploiting the results [**28, 43, 98**] in Proposition 3.4.5 we recover a weak formulation of the Prandtl-Reuss flow rule, in the presence of damage. Together with conditions (ev1) and (ev2), this flow rule characterizes the perfect plasticity.

Finally, following [**30**], we come back to the original time variable $t$ and correspondingly we define the notion of *quasistatic viscosity evolution*. Such an evolution satisfies an energy balance with terms depending only on $t$; the energy dissipated during the jumping regime is thus concentrated on the jump instants. The state after a jump is known through the slow time scale description, which allows then evaluating the dissipation.

Henceforth we assume that $(\alpha^\circ, u^\circ, e^\circ, p^\circ, t^\circ)$ is a rescaled viscosity evolution in the time interval $[0,S]$ with datum $w$, and we use the notation of Section 3.3.

LEMMA 3.4.1. *If $\dot\alpha^\circ(s) = 0$ in $\Omega$ for every $s$ in an interval $(s_1, s_2) \subset U^\circ$, then*

$$u^\circ(s) = u^\circ(s_1)\,, \quad e^\circ(s) = e^\circ(s_1)\,, \quad p^\circ(s) = p^\circ(s_1)\,, \quad t^\circ(s) = t^\circ(s_1) \quad \textit{for every } s \in (s_1, s_2)\,.$$

*In other words, the evolution is trivial in $(s_1, s_2)$. Moreover, it cannot happen that $(s_1, s_2)$ is a connected component of the set $A^\circ$ defined in* (3.3.15).

PROOF. Let $(s_1, s_2) \subset U^\circ$ be such that $\dot\alpha^\circ(s) = 0$ in $\Omega$ for every $s \in (s_1, s_2)$; by definition of $U^\circ$ we have that

$$t^\circ(s) = t^\circ(s_1)\,, \quad w^\circ(s) = w^\circ(s_1) \quad \text{for every } s \in (s_1, s_2)\,, \tag{3.4.1}$$

and by assumption

$$\alpha^\circ(s) = \alpha^\circ(s_1) \quad \text{for every } s \in (s_1, s_2) \tag{3.4.2}$$

in the interval $(s_1, s_2)$. By [**98**, Theorem 3.10], (ev1) and (ev2) are equivalent to the fact that the triple $(u^\circ(s), e^\circ(s), p^\circ(s))$ solves the minimum problem

$$\min_{(u,e,p) \in A(w^\circ(s_1))} \{ \mathcal{Q}(\alpha^\circ(s_1), e) + \mathcal{H}(\alpha^\circ(s_1), p - p^\circ(s)) \}$$

for every $s \in (s_1, s_2)$. Moreover, in view of (3.4.1) and (3.4.2), we can write the energy balance in the time interval $(s_1, s_2)$ as

$$\mathcal{E}(\alpha^\circ(s_1), e^\circ(s_2)) + \int_{s_1}^{s_2} \mathcal{H}(\alpha^\circ(s_1), \dot{p}^\circ(\tau))\, \mathrm{d}\tau = \mathcal{E}(\alpha_0, e_0)\,.$$

Thus $(u^\circ, e^\circ, p^\circ)$ is a quasistatic evolution in perfect plasticity (for heterogeneous materials) according to [**98**, Definition 3.13] with $\mathbb{C} = \mathbb{C}(\alpha^\circ(s_1))$, $K = K(\alpha^\circ(s_1))$ and constant external loading in $(s_1, s_2)$. Then by [**98**, Theorem 3.14] we deduce

$$u^\circ(s) = u^\circ(s_1)\,, \quad e^\circ(s) = e^\circ(s_1)\,, \quad p^\circ(s) = p^\circ(s_1) \quad \text{for every } s \in (s_1, s_2)\,.$$

In order to prove the final statement, assume that $\dot\alpha^\circ(s) = 0$ in $\Omega$ for every $s$ in a connected component $(s_1, s_2)$ of $A^\circ$. This implies $\partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s) = \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s_1), e^\circ(s_1); p^\circ, s_1)$ for every $s \in [s_1, s_2]$, which is impossible by definition of $A^\circ$: indeed, $\Psi(\alpha^\circ(s_i), e^\circ(s_i); p^\circ, s_i) = 0$ for $i = 1, 2$ and $\Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s) > 0$ for $s \in (s_1, s_2)$. □

We now show a variational inequality describing the jumping regime and further reparametrize it.

Proposition 3.4.2. *For a.e.* $s \in (0, S)$

$$\|\dot\alpha^\circ(s)\|_2 \langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \beta - \dot\alpha^\circ(s)\rangle + \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\langle \dot\alpha^\circ(s), \beta - \dot\alpha^\circ(s)\rangle_2 \geq 0 \tag{3.4.3}$$

*for every* $\beta \in H^m_-(\Omega)$.

*In particular, if* $\dot\alpha^\circ(s) \leq -C < 0$ *in* $\Omega$, *then*

$$\|\dot\alpha^\circ(s)\|_2 \langle -\partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \beta\rangle = \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\langle \dot\alpha^\circ(s), \beta\rangle_2$$

*for every* $\beta \in H^m(\Omega)$.

Proof. In this proof it is convenient to use the characterization (3.2.12) of $\Psi$ in terms of $\mathrm{d}_2$. Let us consider the nontrivial case when $\dot\alpha^\circ(s)$ is not identically zero. Assume that $g \in L^2(\Omega)$ realizes the distance $\mathrm{d}_2(\partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), G)$, i.e., $g + \partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s) \in G$ and

$$\|g\|_2 = \mathrm{d}_2(\partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), G) = \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\,.$$

By (3.3.2a) we get

$$\|g\|_2\, \|\dot\alpha^\circ(s)\|_2 = \langle -\partial_\alpha \mathcal{E}_\lambda(\alpha^\circ(s), e^\circ(s); p^\circ, s), \dot\alpha^\circ(s)\rangle \leq \int_\Omega g\, \dot\alpha^\circ(s)\, \mathrm{d}x \leq \|g\|_2\, \|\dot\alpha^\circ(s)\|_2\,,$$

where the first inequality above follows from (3.2.11) and the fact that $\dot\alpha^\circ(s) \in H^m_-(\Omega)$. Hence, by the Cauchy inequality $g$ is proportional to $\dot\alpha^\circ(s)$, and so

$$g = \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\, \frac{\dot\alpha^\circ(s)}{\|\dot\alpha^\circ(s)\|_2}\,.$$

Therefore (3.4.3) follows from (3.2.11) and (3.3.2a). The last assertion follows by substituting $\beta$ with $\delta\beta + \dot\alpha^\circ(s)$ in (3.4.3) for suitable $\delta > 0$. □

PROPOSITION 3.4.3. *Let* $(s_1, s_2)$ *be an interval in* $A^\circ$ *(defined in* (3.3.15)*) containing no subintervals where* $\|\dot\alpha^\circ(s)\|_2 = 0$ *for a.e.* $s$*. Setting*

$$\varrho(s) := \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\,, \tag{3.4.4}$$

*and*

$$r^\sharp(s) := \int_{\frac{s_1+s_2}{2}}^{s} \frac{\|\dot\alpha^\circ(\sigma)\|_2}{\varrho(\sigma)}\,\mathrm{d}\sigma \quad \textit{for } s \in (s_1, s_2)\,,$$

*it turns out that* $r^\sharp$ *is locally Lipschitz and strictly monotone, and we call* $s^\sharp$ *its inverse function. Then*

$$\alpha^\sharp(r) := \alpha^\circ(s^\sharp(r)) \quad \textit{for } r \in r^\sharp((s_1, s_2))$$

*has bounded variation and is continuous into* $H^m(\Omega)$*, and*

$$\|\dot\alpha^\circ(s^\sharp(r))\|_2^2\Big[\langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\sharp(r), e^\sharp(r); p^\sharp, r), \beta - \dot\alpha^\sharp(r)\rangle + \langle \dot\alpha^\sharp(r), \beta - \dot\alpha^\sharp(r)\rangle_2\Big] \geq 0 \tag{3.4.5}$$

*for a.e.* $r \in r^\sharp((s_1, s_2))$*.*

PROOF. By (3.3.15), (3.3.17) and (3.4.4) it follows that for every compact set $K \subset A^\circ$ there exists $\delta_K > 0$ such that $\varrho(s) \geq \delta_K$ for $s \in K$. Thus $r^\sharp$ is locally Lipschitz on $(s_1, s_2)$ and in particular $\mathcal{L}^n(r^\sharp(E)) = 0$ for every $E \subset (s_1, s_2)$ such that $\mathcal{L}^n(E) = 0$. Moreover $r^\sharp$ is strictly increasing, because by assumption every subinterval in $(s_1, s_2)$ has a subset of positive measure where $\|\dot\alpha^\circ(s)\|_2 > 0$. This implies that $s^\sharp$ is continuous and strictly increasing, and $\alpha^\sharp$ is continuous and has bounded variation, $\alpha^\circ$ being Lipschitz.

Therefore, using the change of variables $s = s^\sharp(r)$ in (3.4.3) and the analogous of (3.3.21), we obtain that for a.e. $r \in (r_1, r_2) := r^\sharp((s_1, s_2))$

$$\|\dot\alpha^\circ(s^\sharp(r))\|_2 \langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\sharp(r), e^\sharp(r); p^\sharp, r), \beta - \dot\alpha^\circ(s^\sharp(r))\rangle + \varrho(s^\sharp(r))\langle \dot\alpha^\circ(s^\sharp(r)), \beta - \dot\alpha^\circ(s^\sharp(r))\rangle_2 \geq 0 \tag{3.4.6}$$

for every $\beta \in H^m_-(\Omega)$. Since $\alpha^\sharp$ has bounded variation in $H^m(\Omega)$, it is $H^m(\Omega)$-weakly differentiable at a.e. $r \in (r_1, r_2)$, and the chain rule

$$\dot\alpha^\sharp(r) = \dot\alpha^\circ(s^\sharp(r))\dot s^\sharp(r) = \dot\alpha^\circ(s^\sharp(r))\frac{\varrho(s^\sharp(r))}{\|\dot\alpha^\circ(s^\sharp(r))\|_2} \quad \text{a.e. in } \Omega$$

holds for a.e. $r$ such that $\|\dot\alpha^\circ(s^\sharp(r))\|_2 > 0$. Thus for a.e. $r \in (r_1, r_2)$

$$\|\dot\alpha^\circ(s^\sharp(r))\|_2\, \dot\alpha^\sharp(r) = \dot\alpha^\circ(s^\sharp(r))\varrho(s^\sharp(r)) \quad \text{a.e. in } \Omega\,. \tag{3.4.7}$$

By (3.4.7), the inequality (3.4.6) reads as

$$\|\dot\alpha^\circ(s^\sharp(r))\|_2\Big[\langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\sharp(r), e^\sharp(r); p^\sharp, r), \beta - \dot\alpha^\circ(s^\sharp(r))\rangle + \langle \dot\alpha^\sharp(r), \beta - \dot\alpha^\circ(s^\sharp(r))\rangle_2\Big] \geq 0$$

for every $\beta \in H^m_-(\Omega)$; so by using again (3.4.7) we get (3.4.5), since $\varrho(s^\sharp(r)) > 0$ for a.e. $r \in (r_1, r_2)$. This concludes the proof. ☐

REMARK 3.4.4. In addition to the hypoteses above, let us assume that $\|\dot\alpha^\circ(s)\|_2 > 0$ for every $s \in (s_1, s_2)$ and that for every $K$ compact set in $(s_1, s_2)$ there exists $\delta_K > 0$ such that $\|\dot\alpha^\circ(s)\|_2 \geq \delta_K$ for $s \in K$. Then $r^\sharp$ is locally bi-Lipschitz, $\alpha^\sharp$ is locally Lipschitz, and

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\sharp(r), e^\sharp(r); p^\sharp, r), \beta - \dot\alpha^\sharp(r)\rangle + \langle \dot\alpha^\sharp(r), \beta - \dot\alpha^\sharp(r)\rangle_2 \geq 0 \quad \text{for a.e. } r \in r^\sharp((s_1, s_2))\,.$$

In particular, this variational inequality is equivalent to

$$\begin{cases} \langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\sharp(r), e^\sharp(r); p^\sharp, r), \beta\rangle + \langle \dot\alpha^\sharp(r), \beta\rangle_2 \geq 0 \quad \text{for a.e. } r \in r^\sharp((s_1, s_2)), \\ \langle \partial_\alpha \mathcal{E}_\lambda(\alpha^\sharp(r), e^\sharp(r); p^\sharp, r), \dot\alpha^\sharp(r)\rangle + \|\dot\alpha^\sharp(r)\|_2^2 = 0\,. \end{cases} \tag{3.4.8}$$

Thus, in those intervals of $A^\circ$, $(\alpha^\sharp, u^\sharp, e^\sharp, p^\sharp, t^\sharp) := (\alpha^\circ, u^\circ, e^\circ, p^\circ, t^\circ) \circ s^\sharp$ is a 1-*approximate viscous evolution*, in the sense that the evolution satisfies the same properties $(\mathrm{ev1})_\varepsilon$–$(\mathrm{ev4'})_\varepsilon$ of an $\varepsilon$-approximate viscous evolution, with $\varepsilon = 1$. In particular, (3.4.8) is the analogous of the Kuhn-Tucker conditions $(\mathrm{ev3})_\varepsilon$ and (3.2.2).

We now prove a weak formulation of the Prandtl-Reuss flow rule: together with conditions (ev1) and (ev2) in Definition 3.3.1, this corresponds to the formulation of quasistatic evolution for perfect plasticity in the presence of damage.

Proposition 3.4.5 (Maximum plastic work principle and flow rule). *From* (3.3.2b), (ev2), *and* (2.1.22) *we easily deduce the* maximum plastic work principle*:*

$$H\Big(\alpha^\circ(s), \frac{\mathrm{d}\dot p^\circ(s)}{\mathrm{d}|\dot p^\circ(s)|}\Big)|\dot p^\circ(s)| = [(\sigma^\circ(s))_D : \dot p^\circ(s)] \quad \textit{as measures on } \Omega \cup \partial_D\Omega,$$

*for a.e.* $s \in (0, S)$, *where the measure denoted by square brackets has been introduced in* (2.1.20). *Moreover, defining* $\mu(s) := \mathcal{L}^n + |\dot p^\circ(s)|$ *for every* $s \in [0, S]$, *there exists* $\widehat\sigma^\circ_D(s) \in L^\infty_{\mu(s)}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ *for a.e.* $s \in (0, S)$ *such that*

$$\widehat\sigma^\circ_D(s) = \sigma^\circ_D(s) \quad \mathcal{L}^n\text{–a.e. on } \Omega\,,$$

$$[\sigma^\circ_D(s) : \dot p^\circ(s)] = \Big(\widehat\sigma^\circ_D(s) \cdot \frac{\mathrm{d}\dot p^\circ(s)}{\mathrm{d}|\dot p^\circ(s)|}\Big)|\dot p^\circ(s)| \quad \text{on } \Omega \cup \partial_D\Omega\,,$$

$$\frac{\mathrm{d}\dot p^\circ(s)}{\mathrm{d}|\dot p^\circ(s)|}(x) \in N_{K(\alpha^\circ(s,x))}(\widehat\sigma^\circ_D(s,x)) \quad \text{for } |\dot p^\circ(s)|\text{–a.e. } x \in \Omega \cup \partial_D\Omega\,,$$

*where* $\widehat\sigma^\circ_D(s,x)$ *denotes the value of* $\widehat\sigma^\circ_D(s)$ *at the point* $x$ *and* $N_{K(\alpha^\circ(s,x))}(\sigma^\circ_D(s,x))$ *is the normal cone to the closed convex set* $K(\alpha^\circ(s,x))$ *at* $\sigma^\circ_D(s,x)$.

Proof. It is enough to repeat the same construction of the precise representative of the stress as in [**28**, Theorem 6.4], using [**98**, Lemma 3.16]. To this end, notice that in [**43**, Theorem 6.2] it is proved that the density of the $\mathcal{L}^n$–absolutely continuous part of $[\sigma_D : p]$ is $\sigma_D : p_a$, where $p_a$ is the density of the $\mathcal{L}^n$–absolutely continuous part of a plastic strain $p$ and $\sigma$ is an elastic stress, and that [**98**, Lemma 3.16] does not use the regularity of $\Omega$. □

From now on we study the evolutions in terms of the original variable $t$.

Definition 3.4.6. Let us assume (3.0.19), (2.1.8), (3.0.18), (2.1.10), and (2.1.27) for a given $w$. We say that $(\alpha, u, e, p)$ is a *quasistatic viscosity evolution* with datum $w$ if there exists a rescaled viscosity evolution $(\alpha^\circ, u^\circ, e^\circ, p^\circ, t^\circ)$ with the same datum such that $t^\circ : [0, S] \to [0, T]$ and for every $t \in [0, T]$

$$\alpha(t) = \alpha^\circ(s^\circ_-(t))\,, \quad u(t) = u^\circ(s^\circ_-(t))\,, \quad e(t) = e^\circ(s^\circ_-(t))\,, \quad p(t) = p^\circ(s^\circ_-(t))\,,$$

where we recall that $s^\circ_-(t) := \sup\{s \in [0,S] \colon t^\circ(s) < t\}$. Moreover, we denote

$$\sigma(t) := \sigma^\circ(s^\circ_-(t))\,.$$

By continuity with respect to time of rescaled viscosity evolutions and by left continuity of $s^\circ_-$, the functions introduced above are left-continuous in the norm topologies of their target spaces. Since

$$\lim_{h\to 0} s^\circ_-(t+h) = s^\circ_+(t)$$

for every $t \in [0,T]$, the right limits $\alpha(t^+)$, $u(t^+)$, $e(t^+)$, and $p(t^+)$ in their norm topologies satisfy

$$\alpha(t^+) = \alpha^\circ(s^\circ_+(t))\,, \quad u(t^+) = u^\circ(s^\circ_+(t))\,, \quad e(t^+) = e^\circ(s^\circ_+(t))\,, \quad p(t^+) = p^\circ(s^\circ_+(t))\,. \tag{3.4.10}$$

Notice that $p\colon [0,T] \to M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ has bounded variation, since $p^\circ$ is Lipschitz and $s^\circ_-$ is nondecreasing. Then we define $\mu$ as the unique Radon measure on $[0,T]$ such that

$$\mu([0,t]) = \mathcal{V}(p;0,t)\,,$$

for every continuity point $t$ of $t \mapsto \mathcal{V}(p;0,t)$, with $\mathcal{V}(p;0,t)$ the total variation of $p$ on $[0,T]$ introduced in (2.1.24). By the continuity properties of $p$, we have that $\mu(\{t\}) = 0$ for every $t \notin S^\circ$ (recall (3.3.12)), and then the diffuse part $\mu_d$ of $\mu$ satisfies

$$\mu_d = \mu - \sum_{\tau \in S^\circ} \mu(\{t\})\delta_\tau\,,$$

where $\delta_\tau$ is the unit mass at $\tau$.

By [**30**, Theorem 7.1], there is a unique (up to $\mu$-equivalence) function $\nu_p\colon [0,T] \to M_b(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ such that for every $\varphi \in C_0(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_D)$ the function $t \mapsto \langle \nu_p(t), \varphi\rangle$ is $\mu$-integrable and

$$\langle p(b) - p(a)\rangle = \int_a^b \langle \nu_p(t), \varphi\rangle \,\mathrm{d}\mu(t)$$

for every $a$, $b \in [0,T]$, with $a \le b$, such that $\mu(\{a\}) = \mu(\{b\}) = 0$. Moreover,

$$\|\nu_p(t)\|_1 \le 1$$

for $\mu$-a.e. $t \in [0,T]$.

Proposition 3.4.7. *Let $(\alpha, u, e, p)$ be a quasistatic viscosity evolution with datum $w$. Then*

$$\mathcal{E}(\alpha(\tau), e(\tau)) - \mathcal{E}(\alpha(\tau^+), e(\tau^+)) \ge 0 \tag{3.4.11}$$

*for every $\tau \in S^\circ \cap [0,T)$, and*

$$\begin{aligned} &\mathcal{E}(\alpha(T), e(T)) + \lambda \int_0^T \mathcal{H}(\alpha(T), \nu_p(t))\,\mathrm{d}\mu_d(t) + (1-\lambda)\int_0^T \mathcal{H}(\alpha(t), \nu_p(t))\,\mathrm{d}\mu_d(t) \\ &+ \sum_{\tau \in S^\circ \cap [0,T)} \Big(\mathcal{E}(\alpha(\tau), e(\tau)) - \mathcal{E}(\alpha(\tau^+), e(\tau^+))\Big) = \mathcal{E}(\alpha_0, e_0) + \int_0^T \langle \sigma(t), E\dot{w}(t)\rangle\,\mathrm{d}t\,. \end{aligned} \tag{3.4.12}$$

Proof. For every $\tau \in S^\circ \cap [0,T)$ evaluating the energy balance (ev4) in $(s^\circ_-(\tau), s^\circ_+(\tau)) \subset U^\circ$ gives, since $\dot{t}^\circ = 0$ in $U^\circ$,

$$\begin{aligned}&\int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)} \Big(\lambda \mathcal{H}(\alpha^\circ(s^\circ_-(T)), \dot{p}^\circ(s)) + (1-\lambda)\mathcal{H}(\alpha^\circ(s), \dot{p}^\circ(s)) + \|\dot{\alpha}^\circ(s)\|_2 \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\Big)\,\mathrm{d}s \\ &\quad = \mathcal{E}(\alpha^\circ(s^\circ_-(\tau)), e^\circ(s^\circ_-(\tau))) - \mathcal{E}(\alpha^\circ(s^\circ_+(\tau)), e^\circ(s^\circ_+(\tau)))\,.\end{aligned} \tag{3.4.13}$$

By definition of quasistatic viscosity evolutions and (3.4.10), we get immediately (3.4.11). Moreover, arguing as in [**30**, Lemma 5.5] we deduce

$$\begin{aligned}&\int_{(0,s^\circ_-(T))\setminus U^\circ} \Big(\lambda \mathcal{H}(\alpha^\circ(s^\circ_-(T)), \dot{p}^\circ(s)) + (1-\lambda)\mathcal{H}(\alpha^\circ(s), \dot{p}^\circ(s))\Big)\,\mathrm{d}s \\ &\quad = \int_0^T \Big(\lambda \mathcal{H}(\alpha(T), \nu_p(t)) + (1-\lambda)\mathcal{H}(\alpha(t), \nu_p(t))\Big)\,\mathrm{d}\mu_d(t)\,.\end{aligned} \tag{3.4.14}$$

The energy balance (ev4) in $(0, s^\circ_-(T))$ reads

$$\begin{aligned}&\mathcal{E}(\alpha^\circ(s^\circ_-(T)), e^\circ(s^\circ_-(T))) + \lambda \int_0^{s^\circ_-(T)} \mathcal{H}(\alpha^\circ(s^\circ_-(T)), \dot{p}^\circ(s))\,\mathrm{d}s + (1-\lambda)\int_0^{s^\circ_-(T)} \mathcal{H}(\alpha^\circ(s), \dot{p}^\circ(s))\,\mathrm{d}s \\ &+ \int_{(0,s^\circ_-(T))\cap U^\circ} \|\dot{\alpha}^\circ(s)\|_2 \Psi(\alpha^\circ(s), e^\circ(s); p^\circ, s)\,\mathrm{d}s = \mathcal{E}(\alpha_0, e_0) + \int_0^{s^\circ_-(T)} \langle \sigma^\circ(s), E\dot{w}^\circ(s)\rangle\,\mathrm{d}s\,,\end{aligned}$$

hence we deduce (3.4.12) from (3.4.13) and (3.4.14), recalling (3.3.13) and the definition of quasistatic viscosity evolution. □

Remark 3.4.8. Neglecting the positive viscous terms in (3.4.12) an energy inequality can be written in every subinterval $[t_1, t_2]$ of $[0,T]$. This inequality holds as an equality, also with $\mu_d = \mu$, in every subinterval $[t_1, t_2]$ such that $[t_1, t_2] \cap S^\circ = \emptyset$, with $S^\circ$ introduced in (3.3.12).

CHAPTER 4

# Globally stable evolution for strain gradient plasticity coupled with damage

### Overview of the chapter

In this chapter we consider the coupling between the strain gradient plasticity model proposed by Gurtin and Anand [**52**] and a damage model as in [**78**].

We prove the existence of a quasistatic evolution satisfying the conditions $(\mathrm{qs0})^{GA}$, $(\mathrm{qs1})^{GA}$, $(\mathrm{qs2})^{GA}$ (see the Introduction). Moreover, we show the connection between the present evolution and the one whose existence is proven in Chapter 2, and we proof of a new Reshetnak-type lower semicontinuity theorem that may be useful for perfect plasticity with damage.

The notation of this chapter is independent of that of Chapters 2 and 3. In particular, in the last section we refer by the symbol "pp" to the energy, the plastic potential, and the admissible configurations for perfect plasticity with damage. However, since the strong formulation is presented in the Introduction, the label for its conditions are used also throghout the chapter.

The structure of the chapter is the following: in Section 4.1 we introduce the model starting from the mathematical formulation of the classical Gurtin-Anand model provided in [**47**], we give the definition of quasistatic evolutions, and state the existence result, which is proved in Sections 4.2 and 4.3. The relation between strong and energetic formulation of the evolution is studied in Section 4.4, while Section 4.5 is devoted to the asymptotic analysis for vanishing strain gradient terms. The last part (Section 4.6) contains the proof of a new Reshetnak-type lower semicontinuity theorem (Theorem 4.6.1) and of a result (Theorem 4.6.6), that in our opinion is an important step toward the existence of globally stable quasistatic evolutions for elastoplasticity coupled with damage, where the damage regularization is $H^1$.

## 4.1. Quasistatic evolutions for the Gurtin-Anand model coupled with damage

In this section we introduce the weak formulation of our model, corresponding to the strong formulation described in the Introduction, and we specify the mathematical framework adopted.

**The reference configuration.** The reference configuration of the elasto-plastic body considered is a bounded, open, and Lipschitz set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, whose boundary is decomposed as

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega, \quad \partial_D\Omega \cap \partial_N\Omega = \emptyset\,, \tag{H1.1}$$

$\partial_D\Omega$ being the part of $\partial\Omega$ where the displacement is prescribed, while traction forces are applied on $\partial_N\Omega$. Here $\partial_D\Omega$ and $\partial_N\Omega$ are open (in the relative topology), with the same boundary $\Gamma$

such that

$$\mathcal{H}^{n-2}(\Gamma) < +\infty\,. \tag{H1.2}$$

**The external loading.** We consider an evolution up to a time $T > 0$, driven by an absolutely continuous loading: this is given by an imposed boundary displacement (in the sense of trace on $\partial_D\Omega$)

$$w \in AC(0,T; H^1(\Omega;\mathbb{R}^n))\,, \tag{H2.1}$$

and by volume and surface forces (on $\partial_N\Omega$) with densites

$$f \in AC(0,T; L^n(\Omega;\mathbb{R}^n))\,, \quad g \in AC(0,T; L^n(\partial_N\Omega;\mathbb{R}^n))\,. \tag{H2.2}$$

For every $t \in [0,T]$ we define $\mathcal{L}(t)\colon W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n) \to \mathbb{R}$ as

$$\langle \mathcal{L}(t), u\rangle := \int_\Omega f(t)\cdot u \,\mathrm{d}x + \int_{\partial_N\Omega} g(t)\cdot u\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

It is easily seen that $\mathcal{L}(t)$ is linear and continuous on $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$.

**Admissible configurations.** As usual in linarized plasticity, the variables

$$u\colon [0,T]\times\overline{\Omega} \to \mathbb{R}^n\,, \quad e\colon [0,T]\times\Omega \to \mathbb{M}^{n\times n}_{sym}\,, \quad p\colon [0,T]\times\Omega \to \mathbb{M}^{n\times n}_D\,,$$

denoting the displacement and the elastic and plastic strains respectively, satisfy for every $t \in [0,T]$ the additive strain decomposition

$$\mathrm{E}u(t) = e(t) + p(t) \quad \text{ in } \Omega\,,$$

that corresponds to small strain assumptions ($\mathrm{E}u = \frac{\nabla u + \nabla u^T}{2}$ is the linearized strain). Given $\overline{w} \in H^1(\Omega;\mathbb{R}^n)$, an admissible configuration relative to $\overline{w}$ is a triple $(u,e,p)$ such that

$$u \in W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\,, \quad e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \quad p \in BV(\Omega;\mathbb{M}^{n\times n}_D)\,, \tag{4.1.1a}$$

$$\mathrm{E}u = e + p \quad \text{in } \Omega\,, \quad u = \overline{w} \quad \text{on } \partial_D\Omega\,, \quad \operatorname{curl} p \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \tag{4.1.1b}$$

the second equality in (4.1.1b) being in the sense of traces. The set of admissible configurations is then

$$A(\overline{w}) := \{(u,e,p) :\ (4.1.1)\,\text{hold}\}\,.$$

Notice that if $u\colon \Omega \to \mathbb{R}^n$ measurable, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$ satisfy (4.1.1b), then $u \in W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ by properties of $BV$ functions and Korn's inequality.

**The damage variable.** The damage state of the body is described by a scalar internal variable

$$\alpha\colon [0,T]\times\Omega \to \mathbb{R}\,.$$

We shall see that during the evolution $\alpha(t) \in H^1(\Omega;[0,1])$ for every $t \in [0,T]$, by the expression of our total energy. At a given $x \in \Omega$, as $\alpha(\cdot,x)$ decreases from 1 to 0, the material point $x$ passes from a sound state to a fully damaged one.

**The elastic energy.** In our formulation the elastic shear and bulk moduli of the body, denoted respectively by $\mu$ and $k$, depend on the damage state $\alpha$. We assume that they are Lipschitz and nondecreasing functions defined on $\mathbb{R}$ and constant in $\mathbb{R}^-$ with

$$\mu(\alpha) > c > 0, \quad 2\mu(\alpha) + k(\alpha) > c \quad \text{for every } \alpha \in [0,1]. \tag{H3}$$

This corresponds to say that the stiffness decreases as the damage grows and that an elastic response is present even in the most damaged state. Defining for every $\alpha \in \mathbb{R}$ the elastic tensor $\mathbb{C}(\alpha)$ by

$$\mathbb{C}(\alpha)e := 2\mu(\alpha)e_D + k(\alpha)(\text{tr } e)I\,, \tag{H4}$$

the assumptions above imply that

$$\mathbb{C}\colon \mathbb{R} \to Lin(\mathbb{M}^{n\times n}_{sym}; \mathbb{M}^{n\times n}_{sym}) \text{ is Lipschitz and } \mathbb{C}(\mathbb{R}^-) = \{\mathbb{C}(0)\}\,, \tag{H5.1}$$

$$\alpha \mapsto \mathbb{C}(\alpha)\,\xi\cdot\xi \quad \text{is nondecreasing for every}\, \xi \in \mathbb{M}^{n\times n}_{sym}\,, \tag{H5.2}$$

$$\gamma_1|\xi|^2 \leq \mathbb{C}(\alpha)\,\xi\cdot\xi \leq \gamma_2|\xi|^2 \quad \text{for every}\, \alpha \in \mathbb{R}\,,\ \xi \in \mathbb{M}^{n\times n}_{sym} \tag{H5.3}$$

for suitable positive constants $\gamma_1$ and $\gamma_2$. The elastic energy is

$$\mathcal{Q}_1(\alpha, e) := \frac{1}{2}\int_\Omega \mathbb{C}(\alpha)\,e\cdot e\,\mathrm{d}x\,. \tag{H6}$$

**The energy stored by the dislocations.** As explained in [**52**, Section 3], the macroscopic Burgers tensor curl $p$ measures the incompatibility of the field $p$ and it provides a measure of the dislocation density. Following the approach of Gurtin-Anand, the energy stored by the dislocations is given by

$$\mathcal{Q}_2(\alpha, \operatorname{curl} p) := \frac{L^2}{2}\int_\Omega \mu(\alpha)|\operatorname{curl} p|^2\,\mathrm{d}x\,, \tag{H7}$$

with $L > 0$ a length scale and $\mu$ the shear modulus. Notice that, since $\mu$ is nondecreasing, in order to minimize $\mu(\alpha)|\operatorname{curl}\, p|^2$ it is convenient to damage portions of the material with high dislocation density.

REMARK 4.1.1. Let us consider the functionals $\mathcal{Q}_1$ and $\mathcal{Q}_2$: their densities are the functions $(\alpha,\xi) \mapsto \frac{1}{2}\mathbb{C}(\alpha)\,\xi\cdot\xi$ and $(\alpha,\xi) \mapsto \frac{L^2}{2}\mu(\alpha)|\xi|^2$, convex in $\xi$ and continuous. Then the Ioffe-Olach Semicontinuity Theorem (cf. [**14**, Theorem 2.3.1]) gives that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are lower semicontinuous with respect to the strong convergence of the first variable in $L^1(\Omega)$ and the weak convergence of the second variable in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, namely for $i \in \{1,2\}$

$$\mathcal{Q}_i(\alpha,\eta) \leq \liminf_{k\to\infty} \mathcal{Q}_i(\alpha_k,\eta_k) \quad \text{for every } \alpha_k \to \alpha \text{ in } L^1(\Omega),\ \eta_k \rightharpoonup \eta \text{ in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,. \tag{4.1.2}$$

**The total energy.** The total energy of a quadruple $(\alpha,u,e,p)$ such that $\alpha \in H^1(\Omega)$ and $(u,e,p) \in A(\overline{w})$ for some $\overline{w}$ is given by:

$$\mathcal{E}(\alpha, e, \operatorname{curl}\, p) := \mathcal{Q}_1(\alpha,e) + \mathcal{Q}_2(\alpha, \operatorname{curl}\, p) + \frac{\ell^2}{2}\|\nabla\alpha\|_2^2 + D(\alpha)\,,$$

where $\ell > 0$ is an internal length and

$$D(\alpha) := \int_\Omega d(\alpha)\,\mathrm{d}x\,, \tag{H8.1}$$

with

$$d\colon \mathbb{R} \to \mathbb{R}^+ \cup \{0\} \text{ continuous and } d(x) > d(0) \text{ for } x < 0\,. \tag{H8.2}$$

We include in the total energy the function $D$ and a quadratic gradient damage term. This choice is motivated by [**88**], where an analogous expression of (elastic) strain work is derived for an isotropic material in absence of prestress, under the assumption that the strain work depends also on $\nabla\alpha$, by an expansion up to the second order in the strain and in $\nabla\alpha$. The term $D(\alpha)$ is related to the energy dissipated during the damage growth up to $\alpha$.

**The plastic dissipation.** We now introduce a term which accounts for the energy dissipated in the evolution of plasticity. Let us first define the *plastic potential* $\mathcal{H}$ for every $(\alpha, p) \in H^1(\Omega) \times BV(\Omega; \mathbb{M}^{n\times n}_D)$ as

$$\mathcal{H}(\alpha,p) := \int_\Omega \sqrt{S_1(\alpha)^2|p|^2 + l^2 S_2(\alpha)^2 |\nabla p|^2}\,\mathrm{d}x + l \int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mathrm{D}^s p|\,, \tag{H9}$$

with $\widetilde{\alpha}$ the precise representative of $\alpha$, which is well defined at $\mathcal{H}^{n-1}$–a.e. $x \in \Omega$ (indeed it is the $C_2$-quasicontinuous representative of $\alpha$), and $\nabla p$ and $\mathrm{D}^s p$ the absolutely continuous and the singular part of $\mathrm{D}p$ with respect to the Lebesgue measure $\mathcal{L}^n$. We recall that

$$\int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mathrm{D}^s p| = \int_{J_p} S_2(\widetilde{\alpha})\,|p^+ - p^-|\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mathrm{D}^c p|\,,$$

where $J_p$ is the jump set of $p$, the functions $p^+$ and $p^-$ are the approximate upper and lower limit of $p$, respectively, and $\mathrm{D}^c p$ is the Cantor part of $\mathrm{D}p$ (see [**6**, Section 3.9]). We assume for $i \in \{1,2\}$

$$S_i\colon \mathbb{R} \to \mathbb{R} \quad \text{bounded, Lipschitz and nondecreasing,} \quad S_i(\alpha) = S_i(0) > 0 \quad \text{for } \alpha < 0\,. \tag{H10}$$

This definition of $\mathcal{H}$ is a generalization of the one in [**48**], where $S_1(\alpha) = S_2(\alpha) = S^0_Y > 0$. Notice that for every $\alpha$ in $H^1(\Omega)$ and $p_1$, $p_2$ in $BV(\Omega;\mathbb{M}^{n\times n}_D)$

$$\mathcal{H}(\alpha, p_1 + p_2) \leq \mathcal{H}(\alpha, p_1) + \mathcal{H}(\alpha, p_2)$$

and $\mathcal{H}$ is positively 1-homogeneous in $p$. Moreover, for every $\alpha$ in $H^1(\Omega)$ and $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$

$$r\|p\|_{BV} \leq \mathcal{H}(\alpha,p) \leq R\|p\|_{BV}\,, \tag{4.1.3}$$

where $r := S_1(0) \wedge (l S_2(0))$ and $R := \sup_\mathbb{R} S_1 \vee (l \sup_\mathbb{R} S_2)$.

Given $\alpha\colon [s,t] \to H^1(\Omega)$ and $p\colon [s,t] \to BV(\Omega;\mathbb{M}^{n\times n}_D)$ evolutions of damage and plastic strain in a time interval $[s,t]$, the *plastic dissipation* corresponding is defined as the $\mathcal{H}$*-variation of* $p$ *with respect to* $\alpha$ on $[s,t]$, namely

$$\mathcal{V}_\mathcal{H}(\alpha,p;s,t) := \sup\left\{\sum_{j=1}^N \mathcal{H}(\alpha(t_j), p(t_j) - p(t_{j-1}))\colon s = t_0 < t_1 < \dots < t_N = t\,,\ N \in \mathbb{N}\right\}. \tag{4.1.4}$$

We denote the variation of $p$ on $[s,t]$ by

$$\mathcal{V}(p;s,t) := \sup\left\{\sum_{j=1}^N \|p(t_j) - p(t_{j-1})\|_{BV}\colon s = t_0 < t_1 < \dots < t_N = t\,,\ N \in \mathbb{N}\right\}.$$

**The safe load conditions.** Besides the assumptions (H2), we require that the forces satisfy the following *strong safe load condition*: for every $t \in [0,T]$ there exists $\varrho(t) \in L^n(\Omega; \mathbb{M}^{n\times n}_{sym})$ such that

$$\begin{cases} -\operatorname{div} \varrho(t) = f(t) & \text{in } \Omega \\ \varrho(t)\nu = g(t) & \text{on } \partial_N \Omega \end{cases} \tag{H11.1}$$

and there exists $c_0 > 0$ such that for every $A \in \mathbb{M}^{n\times n}_D$ with $|A| \le c_0$ we have

$$|A + \varrho_D(t)| \le S_1(0) \wedge S_2(0) \quad \text{a.e. in } \Omega\,. \tag{H11.2}$$

We also assume that the functions $t \mapsto \varrho(t)$ and $t \mapsto \varrho_D(t)$ are absolutely continuous from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$ respectively. Notice that the second equality in (H11.1) is well defined in the dual of the space of traces on $\partial_N\Omega$ of $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ since $\varrho(t)$ and div $\varrho(t)$ are in $L^n$ for every $t$, and that for every $(u,e,p) \in A(w)$ the representation formula

$$\langle \mathcal{L}(t), u\rangle = -\langle \varrho(t)\nu, w\rangle_{\partial_D\Omega} + \int_\Omega \varrho(t)\cdot e \,\mathrm{d}x + \int_\Omega \varrho_D(t)\cdot p\,\mathrm{d}x \tag{4.1.5}$$

holds, where $\langle\cdot\,,\cdot\rangle$ denotes the pairing between $H^{-1/2}(\partial_D\Omega;\mathbb{R}^n)$ and $H^{1/2}(\partial_D\Omega;\mathbb{R}^n)$ (here we use $\mathcal{H}^{n-2}(\Gamma) < \infty$).

Remark 4.1.2. The conditions (H11) are standard assumptions in the study of evolutions in perfect plasticity and strain gradient plasticity, when there are not null external forces (see e.g. [**28**, Equations (2.17) and (2.18)] and [**48**, Equations (4.13) and (4.14)]). However, as observed in [**43**, Remark 2.9], it is not a trivial issue the feasibility, for a given pair $(f(t), g(t))$ of loads, of finding a stress tensor $\varrho(t)$ satisfying (H11). The safe load conditions are important in order to provide the following coercivity estimate for the plastic dissipation:

$$\mathcal{H}(\alpha,p) - \int_\Omega \varrho_D(t)\cdot p\,\mathrm{d}x \ge \tfrac{c_0}{2}\|p\|_1 + \min\{l\tfrac{c_0}{2}, lS_2(0)\}\|\mathrm{D}p\|_1 \tag{4.1.6}$$

for every $t\in[0,T]$, $\alpha \in H^1(\Omega)$, and $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$. This can be obtained adapting the proof of [**48**, Lemma 4.3], and it is based on the fact that $\varrho_D(t)$ belongs to the ball centered in the origin of $\mathbb{M}^{n\times n}_D$ with radius $(S_1(0)\wedge S_2(0)) - c_0$. From (4.1.6), it is immediate to deduce that

$$\mathcal{H}(\alpha,p) - \int_\Omega \varrho_D(t)\cdot p\,\mathrm{d}x \ge C(c_0, l, S_2(0))\|p\|_{BV}\,. \tag{4.1.7}$$

**Quasistatic evolutions.** We are now ready to give the definition of quasistatic evolution for the present model. We define, for given $\alpha \in H^1(\Omega)$ and $\overline{w} \in H^1(\Omega;\mathbb{R}^n)$,

$$\mathcal{A}(\alpha, w) := \{(\beta,u,e,p)\colon \beta\in H^1(\Omega),\ \beta\le\alpha, \text{ and } (u,e,p)\in A(\overline{w})\}\,. \tag{4.1.8}$$

Definition 4.1.3. A *quasistatic evolution for the Gurtin-Anand model coupled with damage* is a function

$$[0,T]\ni t\mapsto (\alpha(t),u(t),e(t),p(t))\in H^1(\Omega;[0,1])\times W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times BV(\Omega;\mathbb{M}^{n\times n}_D)$$

that satisfies the following conditions:

(qs0) *irreversibility* : for every $x\in\Omega$ the function $[0,T]\ni t\mapsto \alpha(t,x)$ is nonincreasing;

(qs1) *global stability*: for every $t \in [0,T]$ we have $(u(t),e(t),p(t)) \in A(w(t))$ and

$$\mathcal{E}(\alpha(t),e(t),\operatorname{curl} p(t)) - \langle \mathcal{L}(t),u(t)\rangle \le \mathcal{E}(\beta,\eta,\operatorname{curl} q) - \langle \mathcal{L}(t),v\rangle + \mathcal{H}(\beta,q-p(t))$$

for every $(\beta,v,\eta,q) \in \mathcal{A}(\alpha(t),w(t))$;

(qs2) *energy balance*: the function $t \mapsto p(t)$ from $[0,T]$ into $BV(\Omega;\mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t\in[0,T]$

$$\begin{aligned}\mathcal{E}(\alpha(t),e(t),&\operatorname{curl} p(t)) - \langle \mathcal{L}(t),u(t)\rangle + \mathcal{V}_{\mathcal{H}}(\alpha,p;0,t)\\ &= \mathcal{E}(\alpha(0),e(0),\operatorname{curl} p(0)) - \langle \mathcal{L}(0),u(0)\rangle + \int_0^t \langle \sigma(s),\mathrm{E}\dot{w}(s)\rangle\,\mathrm{d}s\\ &\quad - \int_0^t \langle \dot{\mathcal{L}}(s),u(s)\rangle\,\mathrm{d}s - \int_0^t \langle \mathcal{L}(s),\dot{w}(s)\rangle\,\mathrm{d}s\,,\end{aligned}$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$.

Remark 4.1.4. We shall prove in Lemma 4.3.1 that such an evolution is measurable and the integrals in (qs2) are well defined.

We now state the main result of the chapter, that will be proved in Sections 4.2 and 4.3.

Theorem 4.1.5 (Existence of quasistatic evolutions). *Assume* (H1)*,* (H2)*,* (H3)–(H6)*,* (H8)–(H10) *and* (H11)*, and let* $(\alpha_0,(u_0,e_0,p_0)) \in H^1(\Omega;[0,1]) \times A(w(0))$ *satisfy the stability condition*

$$\mathcal{E}(\alpha_0,e_0,\operatorname{curl} p_0) - \langle \mathcal{L}(0),u_0\rangle \le \mathcal{E}(\beta,\eta,\operatorname{curl} q) - \langle \mathcal{L}(0),v\rangle + \mathcal{H}(\beta,q-p_0)$$

*for every* $(\beta,v,\eta,q)\in\mathcal{A}(\alpha_0,w(0))$. *Then there exists a quasistatic evolution for the Gurtin-Anand model coupled with damage* $t\mapsto(\alpha(t),u(t),e(t),p(t))$ *such that* $\alpha(0)=\alpha_0$, $u(0)=u_0$, $e(0)=e_0$, $p(0)=p_0$.

## 4.2. The minimization problem

This section is focused on the minimization problem employed in the construction of time discrete approximations for a quasistatic evolution. If $\overline{\alpha}\in H^1(\Omega;[0,1])$ and $\overline{p}\in BV(\Omega;\mathbb{M}^{n\times n}_D)$ are the current values of the damage variable and of the plastic strain, and $w\in H^1(\Omega;\mathbb{R}^n)$, $f\in L^n(\Omega;\mathbb{R}^n)$, $g\in L^n(\partial_N\Omega;\mathbb{R}^n)$ are the updated values of the boundary displacement and of the body and surface loads, the updated values of the internal variables $\alpha,u,e,p$ are obtained by solving the problem

$$\operatorname{argmin}\{\mathcal{E}(\alpha,e,\operatorname{curl} p) - \langle\mathcal{L},u\rangle + \mathcal{H}(\alpha,p-\overline{p}) :\ (\alpha,u,e,p)\in\mathcal{A}(\overline{\alpha},w)\}\,, \tag{4.2.1}$$

where

$$\langle\mathcal{L},u\rangle := \int_\Omega f\cdot u\,\mathrm{d}x + \int_{\partial_N\Omega} g\cdot u\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.2.2}$$

First we show the existence of solutions to this problem and their main properties, and afterwards a stability property of the solutions with respect to variations of the data.

The following semicontinuity theorem will be used several times in the following, for instance to prove the existence of solutions to (4.2.1). Notice that in the case when the energy includes a gradient damage term $\|\nabla\alpha\|_\gamma^\gamma$, with $\gamma>n$ the result follows easily from Reshetnyak's Lower

Semicontinuity Theorem (see Lemma 2.1.1). Instead, for the current regularization $\|\nabla\alpha\|_2^2$, the proof relies on the specific form of $\mathcal{H}$; in particular we use the fact that $\mathrm{D}p$ is the gradient of a $BV$ function and then it vanishes on sets with dimension lower than $n-1$.

THEOREM 4.2.1. *The plastic potential $\mathcal{H}$ defined in* (H9) *is lower semicontinuous with respect to the weak–$H^1(\Omega)$ convergence of $\alpha_k$ and the weak$^*$–$BV(\Omega;\mathbb{M}^{n\times n}_D)$ convergence of $p_k$, namely*

$$\mathcal{H}(\alpha,p) \leq \liminf_{k\to\infty} \mathcal{H}(\alpha_k,p_k) \tag{4.2.3}$$

*for every $\alpha_k \rightharpoonup \alpha$ in $H^1(\Omega)$ and $p_k \overset{*}{\rightharpoonup} p$ in $BV(\Omega;\mathbb{M}^{n\times n}_D)$.*

PROOF. Let $(\alpha_k)_k$ and $(p_k)_k$ be two sequences in $H^1(\Omega)$ and $BV(\Omega;\mathbb{M}^{n\times n}_D)$ such that $\alpha_k \rightharpoonup \alpha$ in $H^1(\Omega)$ and $p_k \overset{*}{\rightharpoonup} p$ in $BV(\Omega;\mathbb{M}^{n\times n}_D)$. We divide the proof into two steps, starting from the case when the functions $p_k$ are uniformly bounded, that is $\|p_k\|_\infty < M$, for a suitable $M>0$.

**Step 1 ($p_k$ uniformly bounded).** Notice that for $\beta\in H^1(\Omega)\cap L^\infty(\Omega)$ and $q\in BV(\Omega;\mathbb{M}^{n\times n}_D)\cap L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$ we have that $\beta\,q\in BV(\Omega;\mathbb{M}^{n\times n}_D)$ and

$$\mathrm{D}(\beta\,q) = \widetilde{\beta}\,\mathrm{D}q + q\otimes\nabla\beta \qquad \text{in } M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\,, \tag{4.2.4}$$

where $\widetilde{\beta}$ is the precise representative of $\beta$. Indeed it is well-known that this formula holds for $\beta\in C^1(\Omega)$; thus we can argue by approximation, considering a sequence $(\beta_k)_k\subset C^1(\Omega)$ uniformly bounded in $L^\infty(\Omega)$ such that $\beta_k\to\beta$ in $H^1(\Omega)$. Therefore the total variations $\|\mathrm{D}(\beta_k\,q)\|_1$ are uniformly bounded and then up to a subsequence

$$\mathrm{D}(\beta_k\,q) \overset{*}{\rightharpoonup} \mathrm{D}(\beta\,q) \quad \text{in } M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\,.$$

Moreover, up to a further subsequence, $\beta_k\to\beta$ pointwise $C_2$-quasieverywhere (see Section 2.1), which implies $\beta_k(x)\to\widetilde{\beta}(x)$ for $|\mathrm{D}q|$–a.e. $x\in\Omega$; then we recover (4.2.4) by using the fact that $q\in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$ and the Dominated Convergence Theorem for the convergence of the right-hand side.

We now take $q=p_k$, $\beta=S_i(\alpha_k)$, and recall that $S_i$ are bounded and Lipschitz maps (cf. (H10)). Since $S_i(\alpha_k)\to S_i(\alpha)$ in $L^2(\Omega)$ and the sequences $(S_i(\alpha_k))_k$ are equibounded in $L^\infty(\Omega)$ and in $H^1(\Omega)$, we get that $S_i(\alpha_k)\to S_i(\alpha)$ in $L^r(\Omega)$ for every $r\in[1,+\infty)$ and $S_i(\alpha_k)\rightharpoonup S_i(\alpha)$ in $H^1(\Omega)$, for $i=1,2$. In particular

$$S_i(\alpha_k)p_k \to S_i(\alpha)p \quad \text{in } L^1(\Omega;\mathbb{M}^{n\times n}_D)\,. \tag{4.2.5}$$

Evaluating (4.2.4) with $q=p_k$ and $\beta=S_2(\alpha_k)$ we get

$$\mathrm{D}(S_2(\alpha_k)p_k) = S_2(\widetilde{\alpha}_k)\mathrm{D}p_k + p_k\otimes\nabla(S_2(\alpha_k)) \qquad \text{in } M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\,.$$

Hence the measures $\mathrm{D}(S_2(\alpha_k)p_k)$ have uniformly bounded total variations, and (4.2.5) implies that

$$\mathrm{D}(S_2(\alpha_k)p_k) \overset{*}{\rightharpoonup} \mathrm{D}(S_2(\alpha)p) \qquad \text{in } M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\,.$$

On the other hand, since $p_k \to p$ in $L^1(\Omega;\mathbb{M}^{n\times n}_D)$ and we are assuming the $p_k$ uniformly bounded, then $p_k \to p$ in $L^r(\Omega;\mathbb{M}^{n\times n}_D)$ for every $r\in[1,+\infty)$ and

$$p_k\otimes\nabla S_2(\alpha_k) \rightharpoonup p\otimes\nabla S_2(\alpha) \qquad \text{in } L^1(\Omega;\mathbb{M}^{n\times n\times n}_D)\,.$$

By difference (and (4.2.4) with $q=p$ and $\beta=\alpha$) we obtain that

$$S_2(\widetilde{\alpha}_k)\mathrm{D}p_k \overset{*}{\rightharpoonup} S_2(\widetilde{\alpha})\mathrm{D}p \quad \text{in } M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\,. \tag{4.2.6}$$

In order to prove (4.2.3), we observe that by definition $\mathcal{H}$ is the total variation of a convex function of a measure, defined in the sense of [**49**]; precisely for every $\beta\in H^1(\Omega)$ and $q\in BV(\Omega;\mathbb{M}^{n\times n}_D)$

$$\mathcal{H}(\beta,q)=\|\overline{f}\big((S_1(\beta)q,S_2(\widetilde{\beta})\mathrm{D}q)\big)\|_1\,, \tag{4.2.7}$$

where

$$\overline{f}(\xi,\mathbb{A}):=\sqrt{|\xi|^2+l^2|\mathbb{A}|^2} \quad \text{for every } (\xi,\mathbb{A})\in\mathbb{M}^{n\times n}_D\times\mathbb{M}^{n\times n\times n}_D$$

and $(S_1(\beta)q,S_2(\widetilde{\beta})\mathrm{D}q)\in M_b(\Omega;\mathbb{M}^{n\times n}_D\times\mathbb{M}^{n\times n\times n}_D)$ is the product measure of $S_1(\beta)q$ and $S_2(\widetilde{\beta})\mathrm{D}q$. From (4.2.5) and (4.2.6) it follows that

$$(S_1(\alpha_k)p_k,S_2(\widetilde{\alpha_k})\mathrm{D}p_k) \overset{*}{\rightharpoonup} (S_1(\alpha)p,S_2(\widetilde{\alpha})\mathrm{D}p) \quad \text{in } M_b(\Omega;\mathbb{M}^{n\times n}_D\times\mathbb{M}^{n\times n\times n}_D)\,.$$

In view of (4.2.7), by Reshetnyak's Lower Semicontinuity Theorem (cf. [**6**, Theorem 2.38]) applied to the convex function $\overline{f}$ and to the measures above, we get (4.2.3).

**Step 2 (General case).** We now approximate the functions $p_k$ with bounded functions, without increasing the total variation of the gradient. For every $x\in\Omega$, $q\in BV(\Omega;\mathbb{M}^{n\times n}_D)$, and $R>1$ we define

$$\varphi_R(q)(x):=\omega_R(|q(x)|)q(x)$$

where $\omega_R\in C^1(\mathbb{R}^+\cup\{0\};[0,1])$ is a nonincreasing map such that

$$\begin{aligned}\omega_R(\varrho)&=1 \quad \text{for every } \varrho\le R\,,\\ \omega_R(\varrho)&=0 \quad \text{for every } \varrho\ge \widehat{R}\,,\\ \omega_R(\varrho)+\varrho^2(\omega_R'(\varrho))^2&\le 1 \quad \text{for every } \varrho\ge 0\,.\end{aligned}$$

and $\widehat{R}(R)$ is some radius bigger than $R$. We can take for instance

$$\omega_R(\varrho)=\begin{cases}1-\frac{(\varrho-R)^2}{4(R+1)^2} & \text{for } \varrho\in[R,R+1]\,,\\ 1-\frac{1}{4(R+1)^2}-\frac{1}{2(R+1)}\ln\frac{\varrho}{R+1}=:g_R(\varrho) & \text{for } \varrho\in[R+1,(R+1)e^{\frac{4(R+1)^2-1}{2(R+1)}}]\,,\\ 0 & \text{for } \varrho\in[(R+1)e^{\frac{4(R+1)^2-1}{2(R+1)}},+\infty)\,.\end{cases}$$

The resulting function $\omega_R$ has a $C^1$ discontinuity at $(R+1)e^{\frac{4(R+1)^2-1}{2(R+1)}}$, where $g_R$ vanishes; however we can modify it near the corner to obtain a $C^1$ function by using a smooth cut-off $h_R$ such that $|h_R'(\varrho)|\le|g_R'(\varrho)|$ and $h_R(\varrho)+\varrho^2(h_R'(\varrho))^2\le 1$.

By construction $|\varphi_R(q)|\le\widehat{R}$ a.e. in $\Omega$, and we can see that $\varphi_R(q)\in BV(\Omega;\mathbb{M}^{n\times n}_D)$ with

$$|\mathrm{D}\varphi_R(q)|\le|\mathrm{D}q| \quad \text{in } M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\,. \tag{4.2.8}$$

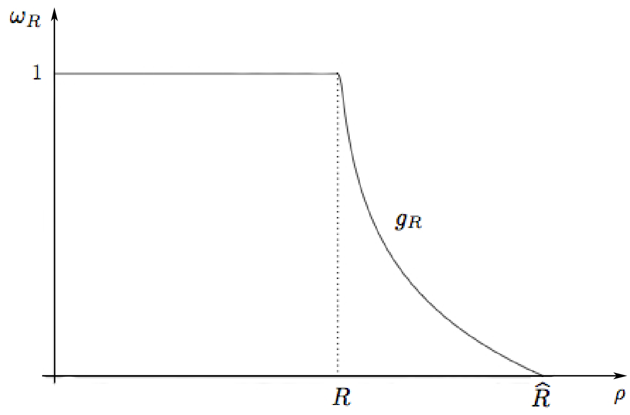


**FIGURE 1.** The cut-off function $\omega_R$.

Let us prove (4.2.8) first in the case $q \in C^1(\Omega; \mathbb{M}_D^{n\times n})$. Here we see every matrix $\xi$ as a vector in $\mathbb{R}^{n^2}$; then

$$\mathrm{D}_i(\varphi_R(q)_j) = \omega_R(|q|)\mathrm{D}_i q_j + \omega_R'(|q|)\frac{q\cdot \mathrm{D}_i q}{|q|}q_j \quad \text{in } \Omega, \text{ for every } i \in [1,n],\ j \in [1,n^2]$$

which gives

$$\begin{aligned}|\mathrm{D}(\varphi_R(q))|^2 &= (\omega_R(|q|))^2|\mathrm{D}q|^2 + (\omega_R'(|q|))^2\sum_{i=1}^n (q\cdot \mathrm{D}_i q)^2 + 2\frac{\omega_R(|q|)}{|q|}\omega_R'(|q|)\sum_{i=1}^n (q\cdot \mathrm{D}_i q)^2\\ &\le (\omega_R(|q|))^2|\mathrm{D}q|^2 + (\omega_R'(|q|))^2|q|^2|\mathrm{D}q|^2 \le |\mathrm{D}q|^2 \quad \text{in } \Omega\,,\end{aligned}$$

by the Cauchy inequality and the fact that $\omega_R$ is nonnegative and nondecreasing. Therefore the inequality (4.2.8) is proved when $q \in C^1(\Omega; \mathbb{M}_D^{n\times n})$. We now show the general case: since these measures are regular, it is sufficient to prove (4.2.8) on open sets. Given $q \in BV(\Omega; \mathbb{M}_D^{n\times n})$ and $U$ an open subset of $\Omega$, by the Anzellotti-Giaquinta Approximation Theorem (cf. [**6**, Theorem 3.9]) there exists $(q_k)_k \subset C^1(U; \mathbb{M}_D^{n\times n})$ such that $q_k \overset{*}{\rightharpoonup} q{\restriction}_U$ in $BV(U; \mathbb{M}_D^{n\times n})$ and

$$\|\mathrm{D}q\|_{1,U} = \lim_{k\to\infty}\|\nabla q_k\|_{1,U} = \lim_{k\to\infty}\int_U |\nabla q_k|\,\mathrm{d}x\,;$$

by regularity of $\omega_R$ we get that

$$\mathrm{D}(\varphi_R(q_k)) \overset{*}{\rightharpoonup} \mathrm{D}(\varphi_R(q)) \quad \text{in } M_b(U; \mathbb{M}_D^{n\times n\times n}) \tag{4.2.9}$$

as $k \to \infty$. By semicontinuity of the total variation with respect to weak$^*$ convergence the inequality (4.2.8) is proved for open sets, and this concludes the proof of (4.2.8).

By (4.2.9) we have that $\varphi_R(p_k) \overset{*}{\rightharpoonup} \varphi_R(p)$ in $BV(\Omega; \mathbb{M}_D^{n\times n})$ as $k \to \infty$; then from the Step 1 (recall that $|\varphi_R(p_k)| \le \widehat{R}$ a.e. in $\Omega$) it follows that

$$\mathcal{H}(\alpha, \varphi_R(p)) \le \liminf_{k\to\infty}\mathcal{H}(\alpha_k, \varphi_R(p_k)) \quad \text{for every } R > 1\,,$$

and we want to pass to the limit as $R \to \infty$. First we prove that for every $k$

$$\mathcal{H}(\alpha_k, \varphi_R(p_k)) \le \mathcal{H}(\alpha_k, p_k)\,. \tag{4.2.10}$$

To this end it is useful to rewrite $\mathcal{H}$ as

$$\mathcal{H}(\beta, q) = \int_\Omega S_2(\widetilde{\beta})\, \mathrm{d}\left|\big(S_1(\beta)S_2(\beta)^{-1}q, l\,\mathrm{D}q\big)\right| ,$$

where $\left|\big(S_1(\beta)S_2(\beta)^{-1}q, l\,\mathrm{D}q\big)\right|$ is the variation of the product measure

$$(S_1(\beta)S_2(\beta)^{-1}q, l\,\mathrm{D}q) = (S_1(\beta)S_2(\beta)^{-1}q, l\,\nabla q)\mathcal{L}^n + (0, l\,\mathrm{D}^c q) + (0, l\,(q^+ - q^-) \otimes \nu_q \mathcal{H}^{n-1}\lfloor_{J_q}) .$$

Since by construction $|\varphi_R(q)| \leq |q|$ a.e. in $\Omega$ for every $q \in BV(\Omega; \mathbb{M}^{n\times n}_D)$ we get by (4.2.8) that

$$|(S_1(\beta)S_2(\beta)^{-1}\varphi_R(q), l\,\mathrm{D}(\varphi_R(q)))| \leq |(S_1(\beta)S_2(\beta)^{-1}q, l\,\mathrm{D}q)| \quad \text{in } M_b(\Omega)$$

for every $\beta \in H^1(\Omega)$, $q \in BV(\Omega; \mathbb{M}^{n\times n}_D)$, and $R > 1$. Taking $\beta = \alpha_k$, $q = p_k$, and integrating the positive function $S_2(\widetilde{\alpha}_k)$, we obtain (4.2.10).

Therefore the proof is completed if we show that

$$\mathcal{H}(\alpha, p) = \lim_{R\to\infty} \mathcal{H}(\alpha, \varphi_R(p)) . \tag{4.2.11}$$

The chain rule for $BV$ functions proved in **[108]** gives in our case

$$\mathrm{D}\varphi_R(p) = \nabla\varphi_R(p)\nabla p\, \mathcal{L}^n + \nabla\varphi_R(\widetilde{p})\, \mathrm{D}^c p + (\varphi_R(p^+) - \varphi_R(p^-)) \otimes \nu_p \mathcal{H}^{n-1}\lfloor_{J_p} ,$$

where $\widetilde{p}(x)$ is the approximate limit of $p$ at any Lebesgue point $x$, and then

$$\begin{aligned}\mathcal{H}(\alpha, \varphi_R(p)) = &\int_\Omega \sqrt{S_1(\alpha)^2|\varphi_R(p)|^2 + l^2 S_2(\alpha)^2|\nabla(\varphi_R(p))|^2}\, \mathrm{d}x \\ &+ l\int_{\Omega\setminus J_p} S_2(\widetilde{\alpha})\left|\nabla\varphi_R(\widetilde{p})\frac{\mathrm{d}\mathrm{D}^c p}{\mathrm{d}|\mathrm{D}^c p|}\right| \mathrm{d}|\mathrm{D}^c p| + l\int_{J_p} S_2(\widetilde{\alpha})|\varphi_R(p^+) - \varphi_R(p^-)|\, \mathrm{d}\mathcal{H}^{n-1} .\end{aligned} \tag{4.2.12}$$

It is known from the theory of $BV$ functions that $p^+(x), p^-(x) \in \mathbb{R}$ for $\mathcal{H}^{n-1}$–a.e. $x \in \Omega$ and hence $\mathcal{H}^{n-1}$–a.e. $x \in \Omega \setminus J_p$ is a Lebesgue point for $p$. Since $\omega_R(|x|) = 1$ if $|x| \leq R$, it follows that

$$\lim_{R\to\infty} \varphi_R(p^\pm(x)) = p^\pm(x) \quad \text{for } \mathcal{H}^{n-1}\text{–a.e. } x \in J_p ,$$

and

$$\lim_{R\to\infty} \left[\nabla\varphi_R(\widetilde{p})\frac{\mathrm{d}\mathrm{D}^c p}{\mathrm{d}|\mathrm{D}^c p|}\right] = \frac{\mathrm{d}\mathrm{D}^c p}{\mathrm{d}|\mathrm{D}^c p|} \quad \text{for } |\mathrm{D}^c p|\text{–a.e. } x \in \Omega \setminus J_p .$$

By (4.2.8) we have that

$$|\nabla(\varphi_R(p))| \leq |\nabla p| , \quad \left|\nabla\varphi_R(\widetilde{p})\frac{\mathrm{d}\mathrm{D}^c p}{\mathrm{d}|\mathrm{D}^c p|}\right| \leq 1 , \quad |\varphi_R(p^+) - \varphi_R(p^-)| \leq |p^+ - p^-| .$$

Then we can pass to the limit in (4.2.12) using the Dominated Convergence Theorem and obtain (4.2.11). Therefore the proof is concluded. □

Using Theorem 4.2.1, we can prove the existence of solutions to the minimization problem (4.2.1) by applying the direct method of the Calculus of Variations.

LEMMA 4.2.2. *Problem* (4.2.1) *admits a solution, and for every* $(\alpha, u, e, p)$ *solution of* (4.2.1) *it holds that* $\alpha \in H^1(\Omega; [0, 1])$.

PROOF. Let

$$(\alpha_k, u_k, e_k, p_k) \in \mathcal{A}(\overline{\alpha}, w)$$

be a minimizing sequence for (4.2.1); by (H8.2), (H5.1), and (H10) we can assume $\alpha_k \in H^1(\Omega;[0,1])$ for every $k$. Since $(0, w, \mathrm{E}w, 0) \in \mathcal{A}(\overline{\alpha}, w)$ and

$$\mathcal{E}(0, \mathrm{E}w, 0) - \langle \mathcal{L}, w \rangle + \mathcal{H}(0, \overline{p}) =: C \in \mathbb{R}$$

we get that $\mathcal{E}(\alpha_k, e_k, \mathrm{curl}\, p_k) - \langle \mathcal{L}, u_k \rangle + \mathcal{H}(\alpha_k, p_k - \overline{p})$ is uniformly bounded in $k$ and

$$\begin{aligned} &\mathcal{E}(\alpha_k, e_k, \mathrm{curl}\, p_k) - \int_\Omega \varrho(t) \cdot e_k \,\mathrm{d}x + \mathcal{H}(\alpha_k, p_k - \overline{p}) - \int_\Omega \varrho_D(t) \cdot (p_k - \overline{p}) \,\mathrm{d}x \\ &\leq C + \int_\Omega \varrho_D(t) \cdot \overline{p} \,\mathrm{d}x - \langle \varrho(t)\nu, w \rangle_{\partial_D \Omega} \end{aligned}$$

by the representation formula (4.1.5). By definition of $\mathcal{E}$ and (4.1.7) we obtain that

$$\|\nabla \alpha_k\|_2^2 + \|e_k\|_2^2 + \|\mathrm{curl}\, p_k\|_2^2 + \|p_k - \overline{p}\|_{BV} \leq C_1 ,$$

and hence there exist $\alpha \in H^1(\Omega;[0,1])$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$ such that up to a subsequence

$$\alpha_k \rightharpoonup \alpha \quad \text{in } H^1(\Omega) , \quad e_k \rightharpoonup e \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) , \quad p_k \overset{*}{\rightharpoonup} p \quad \text{in } BV(\Omega;\mathbb{M}^{n\times n}_D) .$$

Moreover $\mathrm{curl}\, p \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and

$$\mathrm{curl}\, p_k \rightharpoonup \mathrm{curl}\, p \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) .$$

Using the embedding $BV(\Omega;\mathbb{M}^{n\times n}_D) \hookrightarrow L^{\frac{n}{n-1}}(\Omega;\mathbb{M}^{n\times n}_D)$ and Korn's inequality it follows easily from $(u_k, e_k, p_k) \in A(w)$ that $u_k$ are uniformly bounded in $W^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$: then up to a further subsequence

$$u_k \rightharpoonup u \quad \text{in } W^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$$

for a suitable $u$ such that $(u, e, p) \in A(w)$. Collecting the semicontinuity properties (4.1.2) and (4.2.3) we get that $(\alpha, u, e, p)$ is a minimizer, and the proof is completed. □

In the very same way of Lemma 2.2.2, we deduce the remark below from the properties of $\mathcal{H}$.

REMARK 4.2.3. If $(\alpha, u, e, p)$ solves (4.2.1) then

$$\mathcal{E}(\alpha, e, \mathrm{curl}\, p) - \langle \mathcal{L}, u \rangle \leq \mathcal{E}(\widetilde{\alpha}, \widetilde{e}, \mathrm{curl}\, \widetilde{p}) - \langle \mathcal{L}, \widetilde{u} \rangle + \mathcal{H}(\widetilde{\alpha}, \widetilde{p} - p) , \tag{4.2.13}$$

for every $(\widetilde{\alpha}, \widetilde{u}, \widetilde{e}, \widetilde{p}) \in \mathcal{A}(\alpha, w)$.

The following lemma states some differential conditions for a triple $(u, e, p)$ such that $(\alpha, u, e, p)$ satisfies (4.2.13). We shall make use of these conditions to recover the classical formulation of the model.

LEMMA 4.2.4. *Let* $(\alpha, u, e, p)$ *satisfy* (4.2.13). *Then*

$$\Big| \langle \sigma, \eta \rangle + \langle L^2 \mu(\alpha)\, \mathrm{curl}\, p, \mathrm{curl}\, q \rangle - \langle \mathcal{L}, v \rangle \Big| \leq \mathcal{H}(\alpha, q) \tag{4.2.14}$$

*for every* $(v,\eta,q)\in A(0)$*, where* $\sigma:=\mathbb{C}(\alpha)e$*. Moreover*

$$\begin{cases} -\operatorname{div}\sigma = f & \text{in } \Omega\,,\\ \sigma\nu = g & \text{on } \partial_N\Omega\,. \end{cases} \tag{4.2.15}$$

PROOF. Let us fix $(v,\eta,q)\in A(0)$. Since for every $\varepsilon\in\mathbb{R}$

$$(\alpha, u+\varepsilon v, e+\varepsilon\eta, p+\varepsilon q)\in\mathcal{A}(\alpha,w)\,,$$

from the remark above we have

$$\mathcal{Q}_1(\alpha,e+\varepsilon\eta)+\mathcal{Q}_2(\alpha,\operatorname{curl}(p+\varepsilon q))+\mathcal{H}(\alpha,\varepsilon q)\ge\mathcal{Q}_1(\alpha,e)+\mathcal{Q}_2(\alpha,\operatorname{curl}p)\quad\text{for every }\varepsilon\in\mathbb{R}\,.$$

Then the positive homogeneity of $\mathcal{H}$ gives that

$$\mathcal{Q}_1(\alpha,e\pm\varepsilon\eta)+\mathcal{Q}_2(\alpha,\operatorname{curl}(p\pm\varepsilon q))+\varepsilon\mathcal{H}(\alpha,\pm q)\ge\mathcal{Q}_1(\alpha,e)+\mathcal{Q}_2(\alpha,\operatorname{curl}p)\quad\text{for every }\varepsilon\in\mathbb{R}\,.$$

Dividing by $\varepsilon$ and passing to the limit as $\varepsilon\to 0$, we recover (4.2.14).

Choosing in (4.2.14) $(v,Ev,0)$ for every $v\in C^\infty(\overline{\Omega};\mathbb{R}^n)$ with $v=0$ on $\partial_D\Omega$, we get (4.2.15). Notice that the normal trace of $\sigma$ on $\partial\Omega$ is well defined in $H^{-1/2}(\partial\Omega;\mathbb{R}^n)$ since $\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ with divergence in $L^2(\Omega;\mathbb{R}^n)$. □

The lemma below will permit us to say that when both $\alpha$ and $p$ are continuous at a given time then all the evolution is continuous there. In contrast with Lemma 2.2.4, here it is not useful to write $\omega_{12}$ in terms of $\|\alpha_1-\alpha_2\|_\infty$; indeed we will consider the case when a sequence of functions $\alpha_1$ tends to a function $\alpha_2$ weakly in $H^1(\Omega)$, and this does not provide uniform convergence in $\Omega$.

LEMMA 4.2.5. *For* $i=1,2$ *let* $w_i\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $f_i\in L^n(\Omega;\mathbb{R}^n)$, $g_i\in L^\infty(\partial_N\Omega;\mathbb{R}^n)$, *and let* $\mathcal{L}_i$ *be defined by* (4.2.2) *with* $f=f_i$ *and* $g=g_i$*. Suppose that* $(\alpha_i,u_i,e_i,p_i)$ *satisfies* (4.2.13) *with data* $w=w_i$, $\mathcal{L}=\mathcal{L}_i$*, and let*

$$\begin{aligned}\omega_{12}:=\big\|[\mathbb{C}(\alpha_2)-\mathbb{C}(\alpha_1)]e_1\big\|_2+\big\|(\mu(\alpha_2)-\mu(\alpha_1))\operatorname{curl}p_1\big\|_2+\|p_2-p_1\|_{BV}^{1/2}\\ +\|p_2-p_1\|_1+\|f_2-f_1\|_n+\|g_2-g_1\|_{\infty,\partial_N\Omega}+\|\mathrm{E}w_2-\mathrm{E}w_1\|_2\,.\end{aligned}$$

*Then there exists a positive constant* $C$ *depending on* $L$, $\mu(0)$, $\gamma_1$, $\gamma_2$, $R$, $\Omega$, $\partial_N\Omega$ *such that*

$$\begin{aligned}\|e_2-e_1\|_2+\|\operatorname{curl}p_2-\operatorname{curl}p_1\|_2\le C\,\omega_{12}\,,\\ \|u_2-u_1\|_{1,\frac{n}{n-1}}\le C(\omega_{12}+\|w_2-w_1\|_2)\,.\end{aligned}\tag{4.2.16}$$

PROOF. Let

$$v:=(u_2-w_2)-(u_1-w_1)\,,\quad \eta:=(e_2-\mathrm{E}w_2)-(e_1-\mathrm{E}w_1)\,,\quad q:=p_2-p_1\,.$$

Since $(v,\eta,q)\in A(0)$, by (4.2.14) we have that

$$\begin{aligned}-\mathcal{H}(\alpha_1,p_2-p_1)\le&\langle\mathbb{C}(\alpha_1)e_1,\eta\rangle+L^2\langle\mu(\alpha_1)\operatorname{curl}p_1,\operatorname{curl}(p_2-p_1)\rangle-\langle\mathcal{L}_1,v\rangle\,,\\ &\langle\mathbb{C}(\alpha_2)e_2,\eta\rangle+L^2\langle\mu(\alpha_2)\operatorname{curl}p_2,\operatorname{curl}(p_2-p_1)\rangle-\langle\mathcal{L}_2,v\rangle\le\mathcal{H}(\alpha_2,p_2-p_1)\,.\end{aligned}$$

Gathering the inequalities above and using (4.1.3) we obtain that

$$\begin{aligned}&\langle \mathbb{C}(\alpha_2)(e_2-e_1),\eta)\rangle + L^2\int_\Omega \mu(\alpha_2)|\mathrm{curl}\,(p_2-p_1)|^2\,\mathrm{d}x\\ &\quad\le \langle [\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1,\eta\rangle + L^2\,\langle [\mu(\alpha_1)-\mu(\alpha_2)]\mathrm{curl}\,p_1,\mathrm{curl}\,(p_2-p_1)\rangle + \langle \mathcal{L}_2-\mathcal{L}_1,v\rangle\\ &\qquad + 2R\|p_2-p_1\|_{BV}\,,\end{aligned}$$

and then, by the definition of $\eta$,

$$\begin{aligned}&\langle \mathbb{C}(\alpha_2)(e_2-e_1),e_2-e_1\rangle + L^2\int_\Omega \mu(\alpha_2)|\mathrm{curl}\,(p_2-p_1)|^2\,\mathrm{d}x\\ &\quad\le \langle \mathbb{C}(\alpha_2)(e_2-e_1),\mathrm{E}w_2-\mathrm{E}w_1\rangle + \langle [\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1,e_2-e_1+(\mathrm{E}w_1-\mathrm{E}w_2)\rangle\\ &\qquad + L^2\,\langle [\mu(\alpha_1)-\mu(\alpha_2)]\mathrm{curl}\,p_1,\mathrm{curl}\,(p_2-p_1)\rangle + \langle \mathcal{L}_2-\mathcal{L}_1,v\rangle + 2R\|p_2-p_1\|_{BV}\,,\end{aligned} \tag{4.2.17}$$

Arguing as in the proof of [**28**, Theorem 3.8] we see that there exists a constant $\widehat{C}$ depending only on $\Omega$ and $\partial_N\Omega$ such that

$$|\langle \mathcal{L}_2-\mathcal{L}_1,v\rangle| \le \widehat{C}\big(\|f_2-f_1\|_n+\|g_2-g_1\|_{\infty,\partial_N\Omega}\big)\big(\|e_2-e_1\|_2+\|\mathrm{E}w_2-\mathrm{E}w_1\|_2+\|p_2-p_1\|_1\big)\,.$$

Since

$$\gamma_1\|e_2-e_1\|_2^2+L^2\,\mu(0)\|\mathrm{curl}\,(p_2-p_1)\|_2^2 \le \langle \mathbb{C}(\alpha_2)(e_2-e_1),e_2-e_1\rangle + L^2\int_\Omega \mu(\alpha_2)|\mathrm{curl}\,(p_2-p_1)|^2\,\mathrm{d}x\,,$$

we conclude the former of (4.2.16) from (4.2.17) using the Cauchy inequality. The latter estimate is easily shown using the compatibility conditions (4.1.1b) and Korn's Inequality. □

We now prove a stability result for the solutions of (4.2.13) with respect to the weak convergence of the data.

THEOREM 4.2.6 (Stability of solutions to (4.2.13)). *Let $w_k\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\mathcal{L}_k\in\big(W^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\big)'$, $\alpha_k\in H^1(\Omega;[0,1])$, and $(u_k,e_k,p_k)\in A(w_k)$ for every $k$. Assume that these sequences of functions converge weakly* (weakly for reflexive spaces) in their target spaces to functions $w_\infty$, $\mathcal{L}$, $\alpha_\infty$, $u_\infty$, $e_\infty$, and $p_\infty$, respectively. Then $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$. If, in addition,*

$$\mathcal{E}(\alpha_k,e_k,\mathrm{curl}\,p_k)-\langle \mathcal{L}_k,u_k\rangle \le \mathcal{E}(\widehat{\alpha}_k,\widehat{e}_k,\mathrm{curl}\,\widehat{p}_k)-\langle \mathcal{L}_k,\widehat{u}_k\rangle + \mathcal{H}(\widehat{\alpha}_k,\widehat{p}_k-p_k) \tag{4.2.18}$$

*for every $k$ and every $(\widehat{\alpha}_k,\widehat{u}_k,\widehat{e}_k,\widehat{p}_k)\in\mathcal{A}(\alpha_k,w_k)$, then*

$$\mathcal{E}(\alpha_\infty,e_\infty,\mathrm{curl}\,p_\infty)-\langle \mathcal{L},u_\infty\rangle \le \mathcal{E}(\alpha,e,\mathrm{curl}\,p)-\langle \mathcal{L},u\rangle + \mathcal{H}(\alpha,p-p_\infty) \tag{4.2.19}$$

*for every $(\alpha,u,e,p)\in\mathcal{A}(\alpha_\infty,w_\infty)$.*

PROOF. The fact that $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$ is immediate by the definition of admissible triple and the weak convergences assumed.

Let us now fix $(\alpha,u,e,p)\in\mathcal{A}(\alpha_\infty,w_\infty)$ and test (4.2.18) by

$$\widehat{\alpha}_k := \alpha\wedge\alpha_k\,,\quad \widehat{u}_k := u-u_\infty+u_k\,,\quad \widehat{e}_k := e-e_\infty+e_k\,,\quad \widehat{p}_k := p-p_\infty+p_k\,.$$

Indeed by assumption $(\widehat{\alpha}_k,\widehat{u}_k,\widehat{e}_k,\widehat{p}_k)\in\mathcal{A}(\alpha_k,w_k)$, and moreover $\widehat{\alpha}_k\rightharpoonup\alpha$ and $\alpha\vee\alpha_k\rightharpoonup\alpha_\infty$ in $H^1(\Omega)$, $\widehat{u}_k\rightharpoonup u$ in $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, $\widehat{e}_k\rightharpoonup e$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\widehat{p}_k\overset{*}{\rightharpoonup} p$ in $BV(\Omega;\mathbb{M}^{n\times n}_D)$.

Since for every $\alpha \in H^1(\Omega)$ and every $\eta_1, \eta_2 \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ we have that

$$\mathcal{Q}_1(\alpha, \eta_1) - \mathcal{Q}_1(\alpha, \eta_2) = \tfrac{1}{2}\langle \mathbb{C}(\alpha)(\eta_1 + \eta_2), \eta_1 - \eta_2 \rangle \,, \tag{4.2.20}$$

$$\mathcal{Q}_2(\alpha, \eta_1) - \mathcal{Q}_2(\alpha, \eta_2) = \tfrac{L^2}{2}\langle \mu(\alpha)(\eta_1 + \eta_2), \eta_1 - \eta_2 \rangle \,, \tag{4.2.21}$$

and for every $\alpha, \beta \in H^1(\Omega)$

$$\|\nabla(\alpha \vee \beta)\|_2^2 + \|\nabla(\alpha \wedge \beta)\|_2^2 = \|\nabla \alpha\|_2^2 + \|\nabla \beta\|_2^2 \,,$$

then the inequality (4.2.18) can be rewritten, adding to both sides $-\mathcal{Q}_1(\widehat{\alpha}_k, e_k) - \mathcal{Q}_2(\widehat{\alpha}_k, \operatorname{curl} p_k)$, thus obtaining

$$\begin{aligned}
\gamma_k := {} & \mathcal{Q}_1(\alpha_k, e_k) - \mathcal{Q}_1(\widehat{\alpha}_k, e_k) + \mathcal{Q}_2(\alpha_k, \operatorname{curl} p_k) - \mathcal{Q}_2(\widehat{\alpha}_k, \operatorname{curl} p_k) + \mathcal{D}(\alpha_k) \\
& + \frac{\ell^2}{2}\|\nabla(\alpha \vee \alpha_k)\|_2^2 - \frac{\ell^2}{2}\|\nabla \alpha\|_2^2 \\
\le {} & \tfrac{1}{2}\langle \mathbb{C}(\widehat{\alpha}_k)(e - e_\infty + 2e_k), e - e_\infty \rangle + \tfrac{L^2}{2}\langle \mu(\widehat{\alpha}_k) \operatorname{curl}(p - p_\infty + 2p_k), \operatorname{curl}(p - p_\infty) \rangle \\
& + \mathcal{D}(\widehat{\alpha}_k) + \mathcal{H}(\widehat{\alpha}_k, p - p_\infty) - \langle \mathcal{L}_k, u - u_\infty \rangle =: \delta_k \,.
\end{aligned}$$

Notice that for every $\eta \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$

$$\mathcal{Q}_1(\alpha_k, \eta) - \mathcal{Q}_1(\widehat{\alpha}_k, \eta) = \tfrac{1}{2}\langle [\mathbb{C}(\alpha_k) - \mathbb{C}(\widehat{\alpha}_k)]\eta, \eta \rangle \,,$$
$$\mathcal{Q}_2(\alpha_k, \eta) - \mathcal{Q}_2(\widehat{\alpha}_k, \eta) = \tfrac{L^2}{2}\langle (\mu(\alpha_k) - \mu(\widehat{\alpha}_k))\eta, \eta \rangle \,.$$

Moreover $(x, \beta, \xi) \mapsto [\mathbb{C}(\beta) - \mathbb{C}(\beta \wedge \alpha(x))]\xi \cdot \xi$ and $(x, \beta, \xi) \mapsto (\mu(\beta) - \mu(\beta \wedge \alpha(x)))|\xi|^2$ are measurable functions from $\Omega \times \mathbb{R} \times \mathbb{M}^{n\times n}_{sym}$ into $\mathbb{R}^+ \cup \{0\}$, continuous in the variable $\beta$ and convex in $\xi$. Therefore the Ioffe-Olach Semicontinuity Theorem (cf. [**14**, Theorem 2.3.1]) implies that

$$\mathcal{Q}_i(\alpha_\infty, \eta_\infty) - \mathcal{Q}_i(\alpha, \eta_\infty) \le \liminf_{k\to\infty} \left[\mathcal{Q}_i(\alpha_k, \eta_k) - \mathcal{Q}_i(\widehat{\alpha}_k, \eta_k)\right]$$

for every $i \in \{1, 2\}$ and $\eta_k \rightharpoonup \eta_\infty$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Then it follows that

$$\mathcal{E}(\alpha_\infty, e_\infty, \operatorname{curl} p_\infty) - \mathcal{Q}_1(\alpha, e_\infty) - \mathcal{Q}_2(\alpha, \operatorname{curl} p_\infty) - \frac{\ell^2}{2}\|\nabla \alpha\|_2^2 \le \liminf_{k\to\infty} \gamma_k \,. \tag{4.2.22}$$

On the other hand

$$\lim_{k\to\infty} \delta_k = \mathcal{E}(\alpha, e, \operatorname{curl} p) - \mathcal{Q}_1(\alpha, e_\infty) - \mathcal{Q}_2(\alpha, \operatorname{curl} p_\infty) - \frac{\ell^2}{2}\|\nabla\alpha\|_2^2 + \mathcal{H}(\alpha, p - p_\infty) - \langle \mathcal{L}, u - u_\infty \rangle \,. \tag{4.2.23}$$

Indeed, since $\widehat{\alpha}_k \rightharpoonup \alpha$ in $H^1(\Omega)$, up to a subsequence $k_j$ we have that $\widetilde{\widehat{\alpha}}_{k_j}(x) \to \widetilde{\alpha}(x)$ for $|\mathrm{D}(p - p_\infty)|$–a.e. $x \in \Omega$; therefore, by the Dominated Convergence Theorem

$$\lim_{j\to\infty} \mathcal{H}(\widehat{\alpha}_{k_j}, p - p_\infty) = \mathcal{H}(\alpha, p - p_\infty) \,,$$

and, since the limit is independent of the subsequence, the convergence above holds for the whole sequence. The convergence of $\mathcal{D}(\widehat{\alpha}_k)$ to $\mathcal{D}(\alpha)$ follows easily from (H8). Let us consider the first term in $\delta_k$: the simmetry of $\mathbb{C}(\beta)$ for every $\beta \in \mathbb{R}$ gives that

$$\tfrac{1}{2}\langle \mathbb{C}(\widehat{\alpha}_k)(e - e_\infty + 2e_k), e - e_\infty \rangle = \tfrac{1}{2}\langle e - e_\infty + 2e_k, \mathbb{C}(\widehat{\alpha}_k)(e - e_\infty) \rangle \,.$$

Since $\mathbb{C}(\beta)$ is bounded uniformly with respect to $\beta \in \mathbb{R}$ and $\widehat{\alpha}_k \rightharpoonup \alpha$ in $H^1(\Omega)$, by the Dominated Convergence Theorem we get that

$$\mathbb{C}(\widehat{\alpha}_k)(e - e_\infty) \to \mathbb{C}(\alpha)(e - e_\infty) \quad \text{in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym})\,.$$

From the fact that $e_k \rightharpoonup e_\infty$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ we conclude that

$$\lim_{k\to\infty} \tfrac{1}{2}\langle \mathbb{C}(\widehat{\alpha}_k)(e - e_\infty + 2e_k), e - e_\infty\rangle = \mathcal{Q}_1(\alpha, e) - \mathcal{Q}_1(\alpha, e_\infty)\,,$$

recalling (4.2.20). In the same way we get that

$$\lim_{k\to\infty} \tfrac{L^2}{2}\langle \mu(\widehat{\alpha}_k)\,\mathrm{curl}\,(p - p_\infty + 2p_k), \mathrm{curl}\,(p - p_\infty)\rangle = \mathcal{Q}_2(\alpha, \mathrm{curl}\,p) - \mathcal{Q}_2(\alpha, \mathrm{curl}\,p_\infty)$$

and then we conclude (4.2.23). Gathering (4.2.22) and (4.2.23) we get (4.2.19) and the proof is completed. □

## 4.3. Existence of quasistatic evolutions

This section is devoted to the proof of Theorem 4.1.5, basing on discrete time approximation. First we construct a sequence of approximate evolutions by solving, for the $k$-th approximant, $k$ incremental problems of the type (4.2.1) which we have studied in Section 4.2; then we show that this sequence converges in a suitable sense to a quasistatic evolution for the Gurtin-Anand model coupled with damage. Henceforth we assume the hypotheses of Theorem 4.1.5, in particular the stability condition on the initial datum $(\alpha_0, u_0, e_0, p_0)$.

Before starting the proof of the existence result, we prove that the integrals in the energy balance (qs2) of Definition 4.1.3 are well defined. This follows immediately by the following lemma.

LEMMA 4.3.1. *Let $(\alpha, u, e, p)$ be a quasistatic evolution and $\sigma(t) := \mathbb{C}(\alpha(t))e(t)$, according to Definition 4.1.3. Let $r \in [1, \infty)$. Then the functions $t \mapsto \alpha(t) \in L^r(\Omega)$, $t \mapsto u(t) \in W^{1,\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$, $t \mapsto e(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and $t \mapsto \sigma(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ are strongly continuous except at most for a countable subset of $[0, T]$, and*

$$(\alpha, u, e, p) \in L^\infty(0, T; H^1(\Omega) \times W^{1,\frac{n}{n-1}}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times BV(\Omega; \mathbb{M}^{n\times n}_D))\,.$$

PROOF. By the irreversibility condition and Lemma 1.2.2 it follows that there exists a countable set $E_1 \subset [0, T]$ such that $\alpha$ is continuous at every $t \in [0, T] \setminus E_1$ with respect to the $L^r$ norm, for every $r \in [1, \infty)$. The condition (qs2) gives that $p \in L^\infty(0, T; BV(\Omega; \mathbb{M}^{n\times n}_D))$; then by (qs1), taking $(\beta, v, \eta, q) = (0, w(t), \mathrm{E}w(t), 0)$ for every $t$, we deduce that $\alpha(t)$, $u(t)$, $e(t)$ are uniformly bounded in $H^1(\Omega)$, $W^{1,\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$, $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, respectively. Thus for every $t \in [0, T] \setminus E_1$

$$\alpha(s) \rightharpoonup \alpha(t) \quad \text{in } H^1(\Omega), \quad \alpha(s) \to \alpha(t) \quad \text{in } L^r(\Omega) \quad \text{as } s \to t\,. \tag{4.3.1}$$

Since $p$ has bounded variation into the space $BV(\Omega; \mathbb{M}^{n\times n}_D)$, the set $E_2$ of its discontinuity points is at most countable. Moreover, by the uniform bound for $\mu(\alpha)$ and $\mathbb{C}(\alpha)$, (4.3.1), and the Dominated Convergence Theorem it follows that for every $t \in [0, T] \setminus E_1$

$$\mathbb{C}(\alpha(s))e(t) \to \mathbb{C}(\alpha(t))e(t)\,, \quad \mu(\alpha(s))\mathrm{curl}\,p(t) \to \mu(\alpha(t))\mathrm{curl}\,p(t) \quad \text{in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \text{ as } s \to t\,.$$

Then, using Lemma 4.2.5 (recall that the loading is continuous in time) we obtain that $e$ and $u$ are strongly continuous in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ at every $t\in[0,T]\setminus E$, with $E=E_1\cup E_2$.

Hence, by (4.3.1), $\sigma(s)\to\sigma(t)$ in $L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$ as $s\to t$ for every $t\in[0,T]\setminus E$. Since $\mathbb{C}(\alpha)$ is uniformly bounded, and then $|\sigma(s)|\le C|e(s)|$ in $\Omega$, we deduce that this convergence is indeed strong in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, applying the Dominated Convergence Theorem. Finally, $\alpha$ is measurable into $H^1(\Omega)$ by the former of (4.3.1) and the fact that $H^1(\Omega)$ is separable. This concludes the proof. □

For every $k\in\mathbb{N}$ we define approximate evolutions $(\alpha_k,u_k,e_k,p_k)$ by induction. Let us set $t^i_k:=T\frac{i}{k}$ for $i=0,\dots,k$ and

$$(\alpha^0_k,u^0_k,e^0_k,p^0_k):=(\alpha_0,u_0,e_0,p_0)\in\mathcal{A}(1,w(0))\,.$$

For $i=1,\dots,k$ let $(\alpha^i_k,u^i_k,e^i_k,p^i_k)$ be a solution to the incremental problem

$$\operatorname{argmin}\{\mathcal{E}(\alpha,e,\operatorname{curl}p)-\langle\mathcal{L}^i_k,u\rangle+\mathcal{H}(\alpha,p-p^{i-1}_k):\,(\alpha,u,e,p)\in\mathcal{A}(\alpha^{i-1}_k,w^i_k)\}\,,\tag{4.3.2}$$

where $w^i_k:=w(t^i_k)$ and $\mathcal{L}^i_k:=\mathcal{L}(t^i_k)$. Notice that Lemma 4.2.2 ensures the existence of solutions. Then we define for $i=0,\dots,k-1$ and $t\in[t^i_k,t^{i+1}_k)$

$$\begin{gathered}\alpha_k(t):=\alpha^i_k\,,\;\; u_k(t):=u^i_k\,,\;\; e_k(t):=e^i_k\,,\;\; p_k(t):=p^i_k\,,\\ \sigma_k(t):=\mathbb{C}(\alpha^i_k)e^i_k\,,\;\; w_k(t):=w^i_k\,,\;\; \mathcal{L}_k(t):=\mathcal{L}^i_k\,,\end{gathered}\tag{4.3.3}$$

while $(\alpha_k(T),e_k(T),u_k(T),p_k(T)):=(\alpha^k_k,u^k_k,e^k_k,p^k_k)$.

The proposition below gives that these piecewise constant approximants satisfy a discretized version of the stability condition (qs1), a discretized energy inequality, and some a-priori estimates. The proof follows the line of [**48**, Proposition 6.2], with some modifications due to the presence of the damage variable.

PROPOSITION 4.3.2. *For every $k\in\mathbb{N}$ the evolution $(\alpha_k,u_k,e_k,p_k)$ defined in* (4.3.3) *satisfies the following conditions:*

(qs0)$_k$ *for every $x\in\Omega$ the function $t\in[0,T]\mapsto\alpha_k(t,x)$ is nonincreasing;*

(qs1)$_k$ *for every $t\in[0,T]$ we have $(u_k(t),e_k(t),p_k(t))\in A(w_k(t))$ and*

$$\mathcal{E}(\alpha_k(t),e_k(t),\operatorname{curl}p_k(t))-\langle\mathcal{L}_k(t),u_k(t)\rangle\le\mathcal{E}(\beta,\eta,\operatorname{curl}q)-\langle\mathcal{L}_k(t),v\rangle+\mathcal{H}(\beta,q-p_k(t))$$

*for every $(\beta,v,\eta,q)\in\mathcal{A}(\alpha_k(t),w_k(t))$;*

(qs2)$_k$ *for every $t\in[t^i_k,t^{i+1}_k)$*

$$\begin{aligned}&\mathcal{E}(\alpha_k(t),e_k(t),\operatorname{curl}p_k(t))-\langle\mathcal{L}_k(t),u_k(t)\rangle+\mathcal{V}_{\mathcal{H}}(\alpha_k,p_k;0,t)\le\mathcal{E}(\alpha_0,e_0,\operatorname{curl}p_0)-\langle\mathcal{L}(0),u_0\rangle\\&\quad+\int_0^{t^i_k}\langle\sigma_k(s),E\dot w(s)\rangle\,\mathrm{d}s-\int_0^{t^i_k}\langle\dot{\mathcal{L}}(s),u_k(s)\rangle\,\mathrm{d}s-\int_0^{t^i_k}\langle\mathcal{L}_k(s),\dot w(s)\rangle\,\mathrm{d}s+\delta_k\,,\end{aligned}$$

*where $\delta_k\to0$ as $k\to\infty$.*

*Moreover there exists a positive constant $C$ independent of $k$ and $t\in[0,T]$ such that*

$$\|\alpha_k(t)\|_{1,2}+\|u_k(t)\|_{1,\frac{n}{n-1}}+\|e_k(t)\|_2+\|\operatorname{curl}p_k(t)\|_2+\mathcal{V}(p_k;0,t)\le C\,.\tag{4.3.4}$$

PROOF. The condition $(\mathrm{qs0})_k$ holds since $\alpha_k^i \leq \alpha_k^{i-1}$. Moreover $(u_k(t), e_k(t), p_k(t)) \in A(w_k(t))$ for every $t \in [0,T]$, by definition of the approximate evolutions. By (4.3.2) and Remark 4.2.3 we get

$$\mathcal{E}(\alpha_k^i, e_k^i, \mathrm{curl}\ p_k^i) - \langle \mathcal{L}_k^i, u_k^i \rangle \leq \mathcal{E}(\beta, e, \mathrm{curl}\ p) - \langle \mathcal{L}_k^i, u \rangle + \mathcal{H}(\beta, p - p_k^i)$$

for every $k$, $i = 1, \dots, k$, and $(\beta, u, e, p) \in \mathcal{A}(\alpha_k^i, w_k^i)$, which gives $(\mathrm{qs1})_k$.

In order to prove $(\mathrm{qs2})_k$ let us fix $i \in \{1, \dots, k\}$, $t \in [t_k^{i-1}, t_k^i)$, $u := u_k^{h-1} - w_k^{h-1} + w_k^h$, and $e := e_k^{h-1} - \mathrm{E}w_k^{h-1} + \mathrm{E}w_k^h$ for a given integer $h$ with $1 \leq h \leq i$. Testing (4.3.2) for $i = h$ by $(\alpha_k^{h-1}, (u, e, p_k^{h-1})) \in \mathcal{A}(\alpha_k^{h-1}, w_k^h)$ we get

$$\begin{aligned} &\mathcal{E}(\alpha_k^h, e_k^h, \mathrm{curl}\ p_k^h) - \langle \mathcal{L}_k^h, u_k^h \rangle + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \\ &\quad\leq \mathcal{E}(\alpha_k^{h-1}, e_k^{h-1}, \mathrm{curl}\ p_k^{h-1}) + \mathcal{Q}_1(\alpha_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) + \langle \mathbb{C}(\alpha_k^{h-1}) e_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1} \rangle \\ &\qquad - \langle \mathcal{L}_k^h, u_k^{h-1} + w_k^h - w_k^{h-1} \rangle \\ &\quad= \mathcal{E}(\alpha_k^{h-1}, e_k^{h-1}, \mathrm{curl}\ p_k^{h-1}) + \int_{t_k^{h-1}}^{t_k^h} \langle \sigma_k^{h-1}, E\dot{w}(s) \rangle \,\mathrm{d}s - \langle \mathcal{L}_k^{h-1}, u_k^{h-1} \rangle \\ &\qquad - \int_{t_k^{h-1}}^{t_k^h} \langle \dot{\mathcal{L}}(s), u_k(s) \rangle \,\mathrm{d}s - \int_{t_k^{h-1}}^{t_k^h} \langle \mathcal{L}_k(s), \dot{w}(s) \rangle \,\mathrm{d}s + \delta_{k,h}\,, \end{aligned} \tag{4.3.5}$$

where

$$\delta_{k,h} := \mathcal{Q}_1(\alpha_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) - \langle \mathcal{L}_k^h - \mathcal{L}_k^{h-1}, w_k^h - w_k^{h-1} \rangle\,.$$

Iterating for $1 \leq h \leq i$ we deduce $(\mathrm{qs2})_k$, with $\delta_k = \sum_{h=1}^i \delta_{k,h}$. Indeed, since $p_k$ is piecewise constant and continuous from the right, and $\alpha_k$ is nonincreasing, the supremum in the definition of $\mathcal{V}_\mathcal{H}$ is attained by the subdivision $(t_k^h)_h$, namely (cf. Lemma 1.2.1)

$$\mathcal{V}_\mathcal{H}(\alpha_k, p_k; 0, t) = \sum_{h=1}^i \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1})\,.$$

Moreover, the absolute continuity of the loading (H2) implies that $\delta_k \to 0$ as $k \to \infty$.

Let us now prove (4.3.4). By (4.1.5) we can rewrite the inequality in (4.3.5) as

$$\begin{aligned} &\mathcal{E}(\alpha_k^h, e_k^h, \mathrm{curl}\ p_k^h) - \int_\Omega \varrho(t_k^h) \cdot e_k^h \,\mathrm{d}x - \int_\Omega \varrho_D(t_k^h) \cdot p_k^h \,\mathrm{d}x + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \\ &\quad\leq \mathcal{E}(\alpha_k^{h-1}, e_k^{h-1}, \mathrm{curl}\ p_k^{h-1}) + \mathcal{Q}_1(\alpha_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) + \langle \mathbb{C}(\alpha_k^{h-1}) e_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1} \rangle \\ &\qquad - \int_\Omega \varrho(t_k^h) \cdot (e_k^{h-1} + \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) \,\mathrm{d}x - \int_\Omega \varrho_D(t_k^h) \cdot p_k^{h-1} \,\mathrm{d}x\,. \end{aligned}$$

By the absolute continuity of $w$ and $\varrho$

$$\begin{aligned} &\mathcal{E}(\alpha_k^h, e_k^h, \mathrm{curl}\ p_k^h) - \int_\Omega \varrho(t_k^h) \cdot e_k^h \mathrm{d}x + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) - \int_\Omega \varrho_D(t_k^h) \cdot (p_k^h - p_k^{h-1}) \,\mathrm{d}x \\ &\quad\leq \mathcal{E}(\alpha_k^{h-1}, e_k^{h-1}, \mathrm{curl}\ p_k^{h-1}) - \int_\Omega \varrho(t_k^{h-1}) \cdot e_k^{h-1} \,\mathrm{d}x - \int_{t_k^{h-1}}^{t_k^h} \int_\Omega \dot{\varrho}(s) \cdot e_k(s) \,\mathrm{d}x \,\mathrm{d}s \\ &\qquad - \int_{t_k^{h-1}}^{t_k^h} \langle \varrho(t_k^h), \mathrm{E}\dot{w}(s) \rangle \,\mathrm{d}s + \int_{t_k^{h-1}}^{t_k^h} \langle \sigma_k(s), \mathrm{E}\dot{w}(s) \rangle \,\mathrm{d}s + \omega_{k,h} \end{aligned}$$

with $\omega_{k,h} := \mathcal{Q}_1(\alpha_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) \to 0$ as $k \to \infty$. Let $t \in [t_k^i, t_k^{i+1})$; summing up for $h = 1, \dots, i$ we get

$$\mathcal{E}(\alpha_k(t), e_k(t), \operatorname{curl} p_k(t)) - \int_\Omega \varrho(t_k^i) \cdot e_k(t) \mathrm{d}x + \sum_{h=1}^{i} \left[ \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) - \int_\Omega \varrho_D(t_k^h) \cdot (p_k^h - p_k^{h-1}) \,\mathrm{d}x \right]$$
$$\leq \mathcal{E}(\alpha_0, e_0, \operatorname{curl} p_0) - \int_\Omega \varrho(0) \cdot e_0 \,\mathrm{d}x - \int_0^{t_k^h} \int_\Omega \dot{\varrho}(s) \cdot e_k(s) \,\mathrm{d}x \,\mathrm{d}s - \int_0^{t_k^h} \langle \bar{\varrho}_k(s), \mathrm{E}\dot{w}(s) \rangle \,\mathrm{d}s$$
$$+ \int_0^{t_k^h} \langle \sigma_k(s), E\dot{w}(s) \rangle \,\mathrm{d}s + \omega_k$$

with $\bar{\varrho}_k(s) = \varrho(t_k^j)$ if $s \in (t_k^{j-1}, t_k^j]$ and $\omega_k = \sum_{h=1}^i \omega_{k,h} \to 0$ as $k \to \infty$. By (4.1.7) we obtain the estimate

$$\sum_{h=1}^{i} \left[ \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) - \int_\Omega \varrho_D(t_k^h) \cdot (p_k^h - p_k^{h-1}) \,\mathrm{d}x \right] \geq C(c_0, l, S_2(0)) \mathcal{V}(p_k; 0, t) \,.$$

Therefore $\|e_k(t)\|_2$ is uniformly bounded in $k$ and $t$ by the hypoteses on $\mathcal{Q}_1$ and the regularity assumptions on the external loading; hence $\alpha_k(t)$, $\mathcal{V}(p_k; 0, t)$, and $\operatorname{curl} p_k(t)$ are bounded as well. Finally, also $u_k(t)$ is bounded by Korn's inequality. This concludes the proof. ☐

The following lemma shows (in the spirit of [**28**, Theorem 4.7]) that in order to prove that an evolution satisfies Definition 4.1.3, it is sufficient to verify the irreversibility and the global stability condition (qs0), (qs1), and (qs2) as an inequality.

LEMMA 4.3.3. *Let* $(\alpha, u, e, p) \colon [0, T] \to H^1(\Omega; [0, 1]) \times W^{1, \frac{n}{n-1}}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n \times n}_{sym}) \times BV(\Omega; \mathbb{M}^{n \times n}_D)$ *be such that the conditions* (*qs*0) *and* (*qs*1) *of Definition 4.1.3 hold. Moreover assume that* $p$ *is a function with bounded variation from* $[0, T]$ *into* $BV(\Omega; \mathbb{M}^{n \times n}_D)$ *and that for every* $t \in [0, T]$

$$\begin{aligned} &\mathcal{E}(\alpha(t), e(t), \operatorname{curl} p(t)) - \langle \mathcal{L}(t), u(t) \rangle + \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) \\ &\quad \leq \mathcal{E}(\alpha(0), e(0), \operatorname{curl} p(0)) - \langle \mathcal{L}(0), u(0) \rangle + \int_0^t \langle \sigma(s), E\dot{w}(s) \rangle \,\mathrm{d}s \\ &\qquad - \int_0^t \langle \dot{\mathcal{L}}(s), u(s) \rangle \,\mathrm{d}s - \int_0^t \langle \mathcal{L}(s), \dot{w}(s) \rangle \,\mathrm{d}s \,, \end{aligned} \tag{4.3.6}$$

*where* $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$. *Then* $(\alpha, u, e, p)$ *is a quasistatic evolution for the Gurtin-Anand model coupled with damage.*

PROOF. Let us fix $t \in [0, T]$ and let us define $s_k^i := \frac{i}{k} t$ for every $k \in \mathbb{N}$ and $i = 0, 1, \dots, k$. For given $k$ and $i$ we set $u := u(s_k^i) - w(s_k^i) + w(s_k^{i-1})$ and $e := e(s_k^i) - \mathrm{E}w(s_k^i) + \mathrm{E}w(s_k^{i-1})$; from the fact that $(\alpha(s_k^i), u, e, p(s_k^i)) \in \mathcal{A}(\alpha(s_k^{i-1}), w(s_k^{i-1}))$, the global stability condition (qs1) implies

$$\mathcal{E}(\alpha(s_k^{i-1}), e(s_k^{i-1}), \operatorname{curl} p(s_k^{i-1})) - \langle \mathcal{L}(s_k^{i-1}), u(s_k^{i-1}) \rangle \leq \mathcal{E}(\alpha(s_k^i), e(s_k^i), \operatorname{curl} p(s_k^i)) - \langle \mathcal{L}(s_k^{i-1}), u \rangle$$
$$+ \mathcal{Q}_1(\alpha(s_k^i), \mathrm{E}w(s_k^{i-1}) - \mathrm{E}w(s_k^i)) - \langle \sigma(s_k^i), \mathrm{E}w(s_k^i) - \mathrm{E}w(s_k^{i-1}) \rangle + \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1})) \,.$$

This inequality can be rewritten as

$$\mathcal{E}(\alpha(s_k^{i-1}),\, e(s_k^{i-1}), \operatorname{curl} p(s_k^{i-1})) - \langle \mathcal{L}(s_k^{i-1}), u(s_k^{i-1})\rangle + \int_{s_k^{i-1}}^{s_k^i} \langle \overline{\sigma}_k(s), E\dot{w}(s)\rangle \mathrm{d}s$$
$$- \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\mathcal{L}}(s), \overline{u}_k(s)\rangle \mathrm{d}s - \int_{s_k^{i-1}}^{s_k^i} \langle \overline{\mathcal{L}}_k(s), \dot{w}(s)\rangle \mathrm{d}s + \overline{\delta}_{k,i}$$
$$\leq \mathcal{E}(\alpha(s_k^i), e(s_k^i), \operatorname{curl} p(s_k^i)) - \langle \mathcal{L}(s_k^i), u(s_k^i)\rangle + \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1}))\,,$$

where for $s \in (s_k^{i-1}, s_k^i]$ we define

$$\overline{u}_k(s) := u(s_k^i), \quad \overline{\sigma}_k(s) := \sigma(s_k^i), \quad \overline{\mathcal{L}}_k(s) := \mathcal{L}(s_k^i)$$

and

$$\overline{\delta}_{k,i} := -\mathcal{Q}_1(\alpha(s_k^i), \mathrm{E}w(s_k^{i-1}) - \mathrm{E}w(s_k^i)) - \langle \mathcal{L}(s_k^i) - \mathcal{L}(s_k^{i-1}), w(s_k^i) - w(s_k^{i-1})\rangle\,.$$

Since $\sum_i \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1})) \leq \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t)$, iterating the last inequality for $1 \leq i \leq k$ we obtain

$$\begin{aligned} &\mathcal{E}(\alpha(0), e(0), \operatorname{curl}\, p(0)) - \langle \mathcal{L}(0), u(0)\rangle + \int_0^t \langle \overline{\sigma}_k(s), E\dot{w}(s)\rangle \mathrm{d}s - \int_0^t \langle \dot{\mathcal{L}}(s), \overline{u}_k(s)\rangle \mathrm{d}s \\ &- \int_0^t \langle \overline{\mathcal{L}}_k(s), \dot{w}(s)\rangle \mathrm{d}s + \overline{\delta}_k \leq \mathcal{E}(\alpha(t), e(t), \operatorname{curl}\, p(t)) - \langle \mathcal{L}(t), u(t)\rangle + \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t)\,, \end{aligned} \tag{4.3.7}$$

where $\overline{\delta}_k := \sum_{i=1}^k \overline{\delta}_{k,i} \to 0$ as $k \to \infty$. Lemma 4.3.1 implies that $\overline{\sigma}_k(s) \to \sigma(s)$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\overline{u}_k(s) \to u(s)$ in $W^{1,\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$ for a.e. $s \in (0,t)$. Taking into account the continuity in time of the external loading and using the Dominated Convergence Theorem, the inequality (4.3.7) passes to the limit as $k \to \infty$ and we deduce that

$$\mathcal{E}(\alpha(0), e(0), \operatorname{curl}\, p(0)) - \langle \mathcal{L}(0), u(0)\rangle + \int_0^t \langle \sigma(s), E\dot{w}(s)\rangle\, \mathrm{d}s - \int_0^t \langle \dot{\mathcal{L}}(s), u(s)\rangle\, \mathrm{d}s$$
$$- \int_0^t \langle \mathcal{L}(s), \dot{w}(s)\rangle\, \mathrm{d}s \leq \mathcal{E}(\alpha(t), e(t), \operatorname{curl}\, p(t)) - \langle \mathcal{L}(t), u(t)\rangle + \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t)\,.$$

Then the energy balance (qs2) is proved. □

In the following theorem we prove that the piecewise constant interpolants defined in (4.3.3) converge in a suitable sense, up to subsequences, to a quasistatic evolution for the Gurtin-Anand model coupled with damage.

THEOREM 4.3.4. *In the hypoteses of Theorem 4.1.5, for every $k \in \mathbb{N}$ let $(\alpha_k, u_k, e_k, p_k)$ be the evolution defined in* (4.3.3)*. Then there exist a subsequence (not relabeled) and a quasistatic evolution $(\alpha, u, e, p)$ for the Gurtin-Anand model coupled with damage such that* $(\alpha(0), u(0), e(0), p(0)) =$

$(\alpha_0, u_0, e_0, p_0)$ *and for every* $t \in [0,T]$

$$\alpha_k(t) \to \alpha(t) \qquad \text{in } H^1(\Omega)\,, \tag{4.3.8a}$$

$$u_k(t) \rightharpoonup u(t) \qquad \text{in } W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\,, \tag{4.3.8b}$$

$$e_k(t) \to e(t) \qquad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \tag{4.3.8c}$$

$$p_k(t) \rightharpoonup p(t) \qquad \text{in } BV(\Omega;\mathbb{M}^{n\times n}_D)\,, \tag{4.3.8d}$$

$$\operatorname{curl} p_k(t) \to \operatorname{curl} p(t) \qquad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,. \tag{4.3.8e}$$

Proof. Since the functions $\alpha_k$ are nonincreasing in time and $\alpha_k(t,x) \in [0,1]$, we get that the $\alpha_k$ are uniformly bounded in $BV(0,T;L^1(\Omega))$. Moreover, by the a priori estimates (4.3.4), $\|\alpha_k(t)\|_{1,2} \le C$ for every $k$ and $t$. Therefore we can apply the generalized version of the classical Helly Theorem given in [**39**, Helly Theorem] to conclude that there exist a subsequence (not relabeled) and a function $\alpha\colon [0,T] \to H^1(\Omega;[0,1])$ nonincreasing in time such that $\alpha_k(t) \rightharpoonup \alpha(t)$ in $H^1(\Omega)$ for every $t\in[0,T]$. By (4.3.4) it also follows that $\mathcal{V}(p_k;0,T) \le C$ for every $k$; then [**28**, Lemma 7.2] implies the existence of $p \in BV(0,T;BV(\Omega;\mathbb{M}^{n\times n}_D))$ such that the convergence (4.3.8d) holds up to a subsequence. The uniform bound in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ for the $\operatorname{curl} p_k$ gives also that curl $p_k(t) \rightharpoonup$ curl $p(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$.

Let us fix $t\in[0,T]$. The a priori estimates on $u_k$ and $e_k$ imply that there exist two functions $\widehat{u} \in W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ and $\widehat{e} \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and an increasing sequence $(k_j)_j$ (possibly depending on $t$) such that $u_{k_j}(t) \rightharpoonup \widehat{u}$ in $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ and $e_{k_j}(t) \rightharpoonup \widehat{e}$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. By Theorem 4.2.6, the global stability condition $(\text{qs1})_k$ proved in Proposition 4.3.2 for the approximate evolutions passes to the limit, so the quadruple $(\alpha(t),\widehat{u},\widehat{e},p(t))$ is a solution to the minimization problem

$$\operatorname{argmin}\,\{\mathcal{E}(\beta,\eta,\text{curl } q) - \langle\mathcal{L}(t),v\rangle + \mathcal{H}(\beta, q-p(t)) : \ (\beta,v,\eta,q)\in\mathcal{A}(\alpha(t),w(t))\}\,.$$

In particular $(\widehat{u},\widehat{e})$ minimizes the functional $(u,e)\mapsto \mathcal{Q}_1(\alpha(t),e) - \langle\mathcal{L}(t),u\rangle$, which is strictly convex in $e$, on the convex set $K := \{(u,e) : \ (u,e,p(t)) \in A(w(t))\}$. Then $(\widehat{u},\widehat{e})$ is uniquely determined, using also Korn's inequality; if we define $(u(t),e(t)) := (\widehat{u},\widehat{e})$, we obtain that (4.3.8b) holds and that $e_k(t) \rightharpoonup e(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, without passing to further subsequences.

By construction, the quadruple $(\alpha,u,e,p)$ satisfies the conditions (qs0), (qs1) in Definition 4.1.3, and $p\in BV(0,T;BV(\Omega;\mathbb{M}^{n\times n}_D))$. By Lemma 4.3.3, it is enough to show the inequality (4.3.6) for every $t\in[0,T]$ in order to conclude that $(\alpha,u,e,p)$ is a quasistatic evolution for the Gurtin-Anand model coupled with damage.

Let us then fix $t\in[0,T]$ and consider the discrete inequality $(\text{qs2})_k$ in Proposition 4.3.2 given by

$$\mathcal{E}(\alpha_k(t),e_k(t),\text{curl } p_k(t)) - \langle\mathcal{L}_k(t),u_k(t)\rangle + \mathcal{V}_{\mathcal{H}}(\alpha_k,p_k;0,t) \le \mathcal{E}(\alpha_0,e_0,\text{curl } p_0) - \langle\mathcal{L}(0),u_0\rangle$$
$$+\int_0^{t_k^i} \langle\sigma_k(s),E\dot{w}(s)\rangle\,\mathrm{d}s - \int_0^{t_k^i}\langle\dot{\mathcal{L}}(s),u_k(s)\rangle\,\mathrm{d}s - \int_0^{t_k^i}\langle\mathcal{L}_k(s),\dot{w}(s)\rangle\,\mathrm{d}s + \delta_k\,.$$

By the approximation properties already shown, the fact that $\mathcal{L}_k(t) \to \mathcal{L}(t)$ strongly in $(W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n))'$, and the Dominated Convergence Theorem, the right-hand side converges to the right-hand side of (qs2) and

$$\langle \mathcal{L}_k(t), u_k(t)\rangle \to \langle \mathcal{L}(t), u(t)\rangle \tag{4.3.9}$$

as $k \to \infty$. On the other hand, from the lower semicontinuity of $\mathcal{H}$ proved in Lemma 4.2.3 and the definition of plastic dissipation (4.1.4) it follows that

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) \le \liminf_{k\to\infty} \mathcal{V}_{\mathcal{H}}(\alpha_k, p_k; 0, t)\,. \tag{4.3.10}$$

Moreover the weak lower semicontinuity of the energetic terms implies that

$$\mathcal{E}(\alpha(t), e(t), \mathrm{curl}\ p(t)) \le \liminf_{k\to\infty} \mathcal{E}(\alpha_k(t), e_k(t), \mathrm{curl}\ p_k(t))\,. \tag{4.3.11}$$

By (4.3.9), (4.3.10), and (4.3.11), we can pass to the limit in $(\mathrm{qs2})_k$ and deduce (4.3.6) and the existence result. Furthermore, we obtain the convergence of the total energy and thus, again by lower semicontinuity,

$$\begin{aligned}
\|\alpha(t)\|_{1,2} &= \lim_{k\to\infty} \|\alpha_k(t)\|_{1,2}\,,\\
\mathcal{Q}_1(\alpha(t), e(t)) &= \lim_{k\to\infty} \mathcal{Q}_1(\alpha_k(t), e_k(t))\,,\\
\mathcal{Q}_2(\alpha(t), \mathrm{curl}\, p(t)) &= \lim_{k\to\infty} \mathcal{Q}_2(\alpha_k(t), \mathrm{curl}\, p_k(t))\,,
\end{aligned}$$

and then (4.3.8a), (4.3.8c), (4.3.8e), by strict convexity. This concludes the proof. □

The main existence result, Theorem 4.1.5, is now a consequence of the previous theorem.

## 4.4. Properties of quasistatic evolutions and classical formulation

In this section we study the connection between the energetic formulation for the Gurtin-Anand model coupled with damage, given in Definition 4.1.3, and the strong formulation of the model, described in the Introduction (to which we refer for the notation for the strong formulation). We shall prove that, without any further regularity assumption with respect to the hypotheses of Theorem 4.1.5, the classical balance equations (sfGA3) and the constraint condition (sfGA4) are satisfied during every evolution. Moreover, under additional regularity assumptions, also the flow rule (sfGA5) holds almost everywhere in space and time, and the evolution of damage is governed by the Kuhn-Tucker type conditions (sfGA6). Notice the improved regularity is required in order to differentiate the energy balance condition, while the classical balance equations (sfGA3) and the constraint condition are obtained without any differentiation.

In the following we assume that $(\alpha, u, e, p)$ is a quasistatic evolution for the Gurtin-Anand model coupled with damage, according to Definition 4.1.3. For every $t \in [0, T]$ let $\mathbb{K}^p_{\mathrm{en}}(t) \in \mathbb{M}^{n\times n\times n}_D$ be given by

$$\mathbb{K}^p_{\mathrm{en}}(t) \cdot \nabla A = \mu(\alpha(t)) L^2 \mathrm{curl}\, p(t) \cdot \mathrm{curl}\, A \quad \text{for every } \mathbb{M}^{n\times n}_{sym}\text{-valued function } A\,, \tag{4.4.1}$$

and let $\sigma(t) := \mathbb{C}(\alpha(t)) e(t)$.

As in perfect plasticity [28], the balance equations for the Cauchy stress $\sigma$ easily follow from the global stability condition (qs1), computing the corresponding Euler equation. By Lemma 4.2.4 we get that for every $t \in [0,T]$ and every $(v,\eta,q) \in A(0)$

$$\Big|\langle\sigma(t),\eta\rangle + \langle L^2\mu(\alpha(t))\,\mathrm{curl}\,p(t),\mathrm{curl}\,q\rangle - \langle\mathcal{L}(t),v\rangle\Big| \le \mathcal{H}(\alpha(t),q)\,, \tag{4.4.2}$$

and then

$$\begin{cases} -\operatorname{div}\sigma(t) = f(t) & \text{in } \Omega\,,\\ \sigma(t)\nu = g(t) & \text{on } \partial_N\Omega\,. \end{cases}$$

Following [48], we now characterize the plastic potential $\mathcal{H}$ as the supremum of certain duality products. A similar type of characterization for the plastic potential is given also in perfect plasticity (cf. [98, Corollary 3.8] and (2.1.23)). In view of the dependence of $\mathcal{H}$ on the damage $\alpha$, we have to introduce the closed space of measures that vanishes on sets with 2-capacity zero, which was not useful in [48].

Lemma 4.4.1. *Let us define the closed linear subspace of* $M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$

$$M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D) := \{\mu \in M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\colon \mu(E)=0 \ \text{ if } C_2(E)=0\}\,,$$

*where we recall that* $C_2(E)$ *is the 2-capacity of the set* $E$*, and let us set for every* $\alpha \in H^1(\Omega)$

$$\begin{aligned}\mathcal{K}_\alpha(\Omega) := \Big\{&(A,\mathbb{B},\mathbb{L}) \in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)\times L^\infty(\Omega;\mathbb{M}^{n\times n\times n}_D)\times (M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D))'\colon\\ &\frac{|A(x)|^2}{S_1(\alpha(x))^2}+\frac{|\mathbb{B}(x)|^2}{l^2S_2(\alpha(x))^2}\le 1 \ \text{a.e. in } \Omega,\ |\langle\mathbb{L},\mu\rangle| \le l\int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mu| \ \forall\mu\Big\}\,.\end{aligned} \tag{4.4.3}$$

*Then for every* $\alpha\in H^1(\Omega)$ *and* $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$

$$\mathcal{H}(\alpha,p) = \sup_{(A,\mathbb{B},\mathbb{L})\in\mathcal{K}_\alpha(\Omega)} \langle (A,\mathbb{B},\mathbb{L}),(p,\nabla p,\mathrm{D}^s p)\rangle\,, \tag{4.4.4}$$

*where* $\langle (A,\mathbb{B},\mathbb{L}),(p,\nabla p,\mathrm{D}^s p)\rangle := \langle A,p\rangle + \langle\mathbb{B},\nabla p\rangle + \langle\mathbb{L},\mathrm{D}^s p\rangle$ *is the duality pairing between* $L^1(\Omega;\mathbb{M}^{n\times n}_D)\times L^1(\Omega;\mathbb{M}^{n\times n\times n}_D)\times M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$ *and its dual space.*

Proof. Let us fix $\alpha\in H^1(\Omega)$ and consider the function

$$\mathcal{F}(\alpha;\cdot,\cdot,\cdot)\colon L^1(\Omega;\mathbb{M}^{n\times n}_D)\times L^1(\Omega;\mathbb{M}^{n\times n\times n}_D)\times M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\to[0,+\infty[$$

defined by

$$\mathcal{F}(\alpha;A',\mathbb{B}',\mathbb{L}') = \int_\Omega \sqrt{S_1(\alpha)^2|A'|^2 + l^2S_2(\alpha)^2|\mathbb{B}'|^2}\,\mathrm{d}x + l\int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mathbb{L}'|\,.$$

This definition is well posed because $\widetilde{\alpha}\in L^\infty(\Omega;\mathbb{L})$ for every $\mathbb{L}\in M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$, and

$$\mathcal{H}(\alpha,p) = \mathcal{F}(\alpha;p,\nabla p,\mathrm{D}^s p)$$

for every $p\in BV(\Omega;\mathbb{M}^{n\times n}_D)$.

Since $\mathcal{F}(\alpha;\cdot,\cdot,\cdot)$ is strongly continuous and convex we have $\mathcal{F}(\alpha;\cdot,\cdot,\cdot) = \mathcal{F}(\alpha;\cdot,\cdot,\cdot)^{**}$, where $^*$ is the symbol for the Fenchel transformation. Moreover, using the fact that

$$\xi_1\cdot\xi_2+\zeta_1\cdot\zeta_2 =: (\xi_1,\zeta_1)\cdot(\xi_2,\zeta_2) \le \sqrt{\varepsilon^2|\xi_1|^2+\delta^2|\zeta_1|^2}\sqrt{\varepsilon^{-2}|\xi_2|^2+\delta^{-2}|\zeta_2|^2} \tag{4.4.5}$$

for every $\varepsilon, \delta > 0$, $\xi_1, \xi_2 \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}^m$, with the equality if and only if $\xi_1 = C\delta^2\xi_2$ and $\zeta_1 = C\varepsilon^2\zeta_2$ for any $C > 0$, it is not difficult to show that $\mathcal{F}^*(\alpha; \cdot, \cdot, \cdot)$ is the indicator function of the set $\mathcal{K}_\alpha(\Omega)$. Therefore we deduce that $\mathcal{F}(\alpha; \cdot, \cdot, \cdot)$ is the Fenchel transform of the indicator of $\mathcal{K}_\alpha(\Omega)$, that gives (4.4.4). □

We now derive the existence of three higher order stresses conjugated to $p(t)$, $\nabla p(t)$, $\mathrm{D}^s p(t)$ for every $t$, and prove that they satisfy the constitutive relations and the constraint condition (sfGA4) in the classical formulation.

PROPOSITION 4.4.2. *For every $t \in [0,T]$ there exists a triple $(\sigma^p(t), \mathbb{K}^p_{\text{diss}}(t), \mathbb{S}^p(t)) \in \mathcal{K}_{\alpha(t)}(\Omega)$ such that, setting $\mathbb{K}^p(t) := \mathbb{K}^p_{\text{en}}(t) + \mathbb{K}^p_{\text{diss}}(t)$, it holds the following*

$$\langle \sigma(t), \eta\rangle + \langle \sigma^p(t), q\rangle + \langle \mathbb{K}^p(t), \nabla q\rangle + \langle \mathbb{S}^p(t), \mathrm{D}^s q\rangle = \langle \mathcal{L}(t), v\rangle \quad \textit{for every } (v, \eta, q) \in A(0)\,, \tag{4.4.6}$$

*which implies the balance equations*

$$\begin{cases} \sigma^p(t) = \sigma_D(t) + \operatorname{div} \mathbb{K}^p(t) & \textit{in } \Omega\,, \\ \mathbb{K}^p(t)\nu = 0 & \textit{on } \partial\Omega\,. \end{cases} \tag{4.4.7}$$

PROOF. Let us fix $t \in [0,T]$. From the inequality (4.4.2) we can deduce that the linear functional

$$A(0) \ni (v, \eta, q) \mapsto \langle \sigma(t), \eta\rangle + \langle L^2\mu(\alpha(t)) \operatorname{curl} p(t), \operatorname{curl} q\rangle - \langle \mathcal{L}(t), v\rangle\,,$$

depends only on $q$. Indeed, since $A(0)$ is a linear space, if both $(v_1, \eta_1, q)$ and $(v_2, \eta_2, q)$ belong to $A(0)$ we have $(v_1 - v_2, \eta_1 - \eta_2, 0) \in A(0)$ and then $\langle \sigma(t), \eta_1 - \eta_2\rangle - \langle \mathcal{L}(t), v_1 - v_2\rangle = 0$. We can thus consider the linear functional

$$\varphi(q, \nabla q, \mathrm{D}^s q) := \langle \sigma(t), \eta\rangle + \langle L^2\mu(\alpha(t)) \operatorname{curl} p(t), \operatorname{curl} q\rangle - \langle \mathcal{L}(t), v\rangle \tag{4.4.8}$$

defined on the linear subspace of $L^1(\Omega; \mathbb{M}^{n\times n}_D) \times L^1(\Omega; \mathbb{M}^{n\times n\times n}_D) \times M^2_b(\Omega; \mathbb{M}^{n\times n\times n}_D)$

$$X = \{(q, \nabla q, \mathrm{D}^s q) \colon (v, \eta, q) \in A(0) \text{ for some } v \in W^{1,\frac{n}{n-1}}(\Omega; \mathbb{R}^n), \eta \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})\}\,.$$

By the Hahn-Banach Theorem for seminorms (see [**42**, Theorem 5.7]), we can extend in a continuous way $\varphi$ to the whole $L^1(\Omega; \mathbb{M}^{n\times n}_D) \times L^1(\Omega; \mathbb{M}^{n\times n\times n}_D) \times M^2_b(\Omega; \mathbb{M}^{n\times n\times n}_D)$ keeping the constraint condition in (4.4.2):

$$|\varphi(A, \mathbb{B}, \mathbb{L})| \le \int_\Omega \sqrt{S_1(\alpha(t))^2|A|^2 + l^2 S_2(\alpha(t))^2|\mathbb{B}|^2}\, \mathrm{d}x + l\int_\Omega S_2(\widetilde{\alpha}(t))\, \mathrm{d}|\mathbb{L}| \tag{4.4.9}$$

for every $(A, \mathbb{B}, \mathbb{L}) \in L^1(\Omega; \mathbb{M}^{n\times n}_D) \times L^1(\Omega; \mathbb{M}^{n\times n\times n}_D) \times M^2_b(\Omega; \mathbb{M}^{n\times n\times n}_D)$. Since $\varphi$ is linear and bounded there exist $\sigma^p(t) \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$, $\mathbb{K}^p_{\text{diss}}(t) \in L^\infty(\Omega; \mathbb{M}^{n\times n\times n}_D)$, and $\mathbb{S}^p(t) \in (M^2_b(\Omega; \mathbb{M}^{n\times n\times n}_D))'$ such that

$$\varphi(A, \mathbb{B}, \mathbb{L}) = -\langle \sigma^p(t), A\rangle - \langle \mathbb{K}^p_{\text{diss}}(t), \mathbb{B}\rangle - \langle \mathbb{S}^p(t), \mathbb{L}\rangle\,.$$

Therefore, choosing $(A, \mathbb{B}, 0)$ and $(0, 0, \mathbb{L})$ in (4.4.9) we get that $(\sigma^p(t), \mathbb{K}^p_{\text{diss}}(t), \mathbb{S}^p(t)) \in \mathcal{K}_{\alpha(t)}(\Omega)$ (recall (4.4.5) and the definition (4.4.3)). Moreover, by (4.4.8) it follows that

$$\langle \sigma(t), \eta\rangle + \langle L^2\mu(\alpha(t)) \operatorname{curl} p(t), \operatorname{curl} q\rangle - \langle \mathcal{L}(t), v\rangle = -\langle \sigma^p(t), q\rangle - \langle \mathbb{K}^p_{\text{diss}}(t), \nabla q\rangle - \langle \mathbb{S}^p(t), \mathrm{D}^s q\rangle \tag{4.4.10}$$

for every $(v,\eta,q)\in A(0)$. Hence (4.4.6) follows recalling the definition of $\mathbb{K}^p_{\rm en}(t)$.

In order to show (4.4.7) let us consider $q\in C^\infty(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ and choose $(0,-q,q)\in A(0)$ in (4.4.6). We obtain that

$$-\langle\sigma(t),q\rangle+\langle\sigma^p(t),q\rangle+\langle\mathbb{K}^p(t),\nabla q\rangle=0\,.$$

Since $q(x)\in\mathbb{M}^{n\times n}_D$ for every $x$, we can replace $\sigma(t)$ by $\sigma_D(t)$ and rewrite the inequality above as

$$\langle\sigma^p(t)-\sigma_D(t),q\rangle+\langle\mathbb{K}^p(t),\nabla q\rangle=0\,. \tag{4.4.11}$$

The former equation in (4.4.7) follows immediately; as for the latter, it is enough to integrate by parts, taking into account that the normal trace of $\mathbb{K}^p(t)$ on $\partial\Omega$ is in $H^{-1/2}(\partial\Omega;\mathbb{R}^{n\times n})$ since $\mathbb{K}^p(t)\in L^2(\Omega;\mathbb{M}^{n\times n\times n}_D)$ with divergence in $L^2(\Omega;\mathbb{M}^{n\times n}_D)$ by (4.4.1), (4.4.11), and the fact that $(\sigma^p(t),\mathbb{K}^p_{\rm diss}(t),\mathbb{S}^p(t))\in\mathcal{K}_{\alpha(t)}(\Omega)$. Hence (4.4.7) is proved and the proof is concluded. □

The classical flow rule (sfGA5) and the Kuhn-Tucker condition for the evolution of the damage can be derived by differentiating the energy balance equation (qs2); therefore some regularity assumptions are needed both on the constitutive coefficients and on the evolution. For instance, if $\alpha$ and $p$ are absolutely continuous from $[0,T]$ respectively into $C(\overline{\Omega})$ and $BV(\Omega;\mathbb{M}^{n\times n}_D)$, then, adapting the argument of Lemma 1.2.1, we have that for every $t\in[0,T]$

$$\mathcal{V}_{\mathcal{H}}(\alpha,p,0,t)=\int_0^t\mathcal{H}(\alpha(s),\dot p(s))\,\mathrm{d}s\,. \tag{4.4.12}$$

Proposition 4.4.3 (Kuhn-Tucker conditions and maximum plastic work principle). *Assume that the elastic moduli $\mu$, $k$ in* (H4)*, and the constitutive functions $d$, $S_1$, $S_2$ are of class $C^1$. Moreover let $\alpha$, $u$, $e$, $p$ be absolutely continuous from $[0,T]$ into $C(\overline{\Omega})\cap H^1(\Omega)$, $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $BV(\Omega;\mathbb{M}^{n\times n}_D)$, respectively. Then for every $t\in[0,T]$ the functional $C(\overline{\Omega})\cap H^1(\Omega)\ni\beta\mapsto\mathcal{E}(\beta,e(t),\operatorname{curl}p(t))$ is differentiable at $\alpha(t)$ with Gâteaux derivative in the direction $\beta\in C(\overline{\Omega})\cap H^1(\Omega)$ given by*

$$\begin{aligned}\langle\partial_\alpha\mathcal{E}(\alpha(t),e(t),\operatorname{curl}p(t)),\beta\rangle&=\tfrac12\langle\mathbb{C}'(\alpha(t))\,\beta\,e(t),e(t)\rangle+\tfrac{L^2}{2}\langle\mu'(\alpha(t))\,\beta\operatorname{curl}p(t),\operatorname{curl}p(t)\rangle\\&\quad+\int_\Omega d'(\alpha(t))\,\beta\,\mathrm{d}x+\ell^2\int_\Omega\nabla\alpha(t)\cdot\nabla\beta\,\mathrm{d}x\,.\end{aligned}$$

*Moreover*

$$\langle\partial_\alpha\mathcal{E}(\alpha(t),e(t),\operatorname{curl}p(t)),\beta\rangle\ge0 \tag{4.4.13}$$

*for every $t\in[0,T]$ and every $\beta\in C(\overline{\Omega})\cap H^1(\Omega)$, $\beta\le0$ in $\Omega$,*

$$\langle\partial_\alpha\mathcal{E}(\alpha(t),e(t),\operatorname{curl}p(t)),\dot\alpha(t)\rangle=0 \tag{4.4.14}$$

*for a.e. $t\in(0,T)$. Finally, for a.e. $t\in(0,T)$*

$$\begin{aligned}\mathcal{H}(\alpha(t),\dot p(t))&=\langle(\sigma^p(t),\mathbb{K}^p_{\rm diss}(t),\mathbb{S}^p(t)),(\dot p(t),\nabla\dot p(t),\mathrm{D}^s\dot p(t))\rangle\\&=\langle\sigma^p(t),\dot p(t)\rangle+\langle\mathbb{K}^p_{\rm diss}(t),\nabla\dot p(t)\rangle+\langle\mathbb{S}^p(t),\mathrm{D}^s\dot p(t)\rangle\,,\end{aligned} \tag{4.4.15}$$

*where $(\sigma^p(t),\mathbb{K}^p_{\rm diss}(t),\mathbb{S}^p(t))\in\mathcal{K}_{\alpha(t)}(\Omega)$ is given by Proposition 4.4.2.*

PROOF. The differentiability of $\beta \mapsto \mathcal{E}(\beta, e(t), \mathrm{curl}\; p(t))$ and the expression of its derivative follow from the regularity assumptions on the constitutive functions and on the evolution. Let $t \in [0,T]$ and $\beta \in C(\overline{\Omega}) \cap H^1(\Omega)$, $\beta \le 0$ in $\Omega$. For every $h > 0$, considering $(\alpha(t) + h\beta, u(t), e(t), p(t)) \in \mathcal{A}(\alpha(t), w(t))$ as a test pair in (qs1), we get

$$\frac{\mathcal{E}(\alpha(t) + h\beta, e(t), \mathrm{curl}\, p(t)) - \mathcal{E}(\alpha(t), e(t), \mathrm{curl}\, p(t))}{h} \ge 0\,.$$

Letting $h \to 0$ we obtain (4.4.13).

Since the evolution is assumed to be absolutely continuous, we can differentiate with respect to $t$ the energy balance (qs2). Recalling (4.4.12) we get that for a.e. $t$

$$\begin{aligned} &\langle \partial_\alpha \mathcal{E}(\alpha(t), e(t), \mathrm{curl}\; p(t)), \dot\alpha(t)\rangle + \langle \sigma(t), \dot e(t)\rangle + L^2 \langle \mu(\alpha(t))\,\mathrm{curl}\, p(t), \mathrm{curl}\, \dot p(t)\rangle \\ &- \langle \mathcal{L}(t), \dot u(t)\rangle + \mathcal{H}(\alpha(t), \dot p(t)) = \langle \sigma(t), E\dot w(t)\rangle - \langle \mathcal{L}(t), \dot w(t)\rangle\,. \end{aligned}$$

It is easy to see that $(\dot u(t) - \dot w(t), \dot e(t) - E\dot w(t), \dot p(t)) \in A(0)$, when it exists; thus, using (4.4.6) (cf. also (4.4.10)), the previous inequality gives that for a.e. $t$

$$\begin{aligned} 0 = \mathcal{H}(\alpha(t), \dot p(t)) &- \big(\langle \sigma^p(t), \dot p(t)\rangle + \langle \mathbb{K}^p_{\mathrm{diss}}(t), \nabla \dot p(t)\rangle + \langle \mathbb{S}^p(t), \mathrm{D}^s \dot p(t)\rangle\big) \\ &+ \langle \partial_\alpha \mathcal{E}(\alpha(t), e(t), \mathrm{curl}\; p(t)), \dot\alpha(t)\rangle\,. \end{aligned} \tag{4.4.16}$$

Since $(\sigma^p(t), \mathbb{K}^p_{\mathrm{diss}}(t), \mathbb{S}^p(t)) \in \mathcal{K}_{\alpha(t)}(\Omega)$, by (4.4.4) and (4.4.13) (recall that $\dot\alpha(t) \le 0$ in $\Omega$) the equality (4.4.16) implies (4.4.14) and (4.4.15) for a.e. $t$. ☐

REMARK 4.4.4. Notice that we can interpret the equalities (4.4.14) and (4.4.15) as two threshold criteria. Indeed, by (4.4.13) and (4.4.4), we have that $\dot\alpha(t)$ minimizes the duality product $\langle \partial_\alpha \mathcal{E}(\alpha(t), e(t), \mathrm{curl}\; p(t)), \beta\rangle$ among every $\beta \in C(\overline{\Omega}) \cap H^1(\Omega)$, $\beta \le 0$ in $\Omega$, while the supremum in (4.4.4) is attained on $(\sigma^p(t), \mathbb{K}^p_{\mathrm{diss}}(t), \mathbb{S}^p(t))$. In other words, (4.4.14) and (4.4.15) may be regarded as a principle of minimal loss of elastic stiffness and a maximum plastic work principle, respectively. The two conditions (4.4.13) and (4.4.14) are called Kuhn-Tucker conditions.

By (4.4.15) we deduce a weak form of the flow rule, expressed by the following conditions.

COROLLARY 4.4.5. *Gathering* (4.4.4) *and* (4.4.15) *we get that*

$$\langle \sigma^p(t) - A, \dot p(t)\rangle + \langle \mathbb{K}^p_{\mathrm{diss}}(t) - \mathbb{B}, \nabla \dot p(t)\rangle \ge 0 \tag{4.4.17a}$$

*for every* $(A, \mathbb{B}) \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D) \times L^\infty(\Omega; \mathbb{M}^{n\times n\times n}_D)$ *with* $\frac{|A(x)|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{B}(x)|^2}{l^2 S_2(\alpha(t,x))^2} \le 1$ *a.e. in* $\Omega$, *and*

$$\langle \mathbb{S}^p(t) - \mathbb{L}, \mathrm{D}^s \dot p(t)\rangle \ge 0 \tag{4.4.17b}$$

*for every* $\mathbb{L} \in (M^2_b(\Omega; \mathbb{M}^{n\times n\times n}_D))'$ *such that* $|\langle \mathbb{L}, \mu\rangle| \le l \int_\Omega S_2(\widetilde\alpha(t))\,\mathrm{d}|\mu|$ *for* $\mu \in M^2_b(\Omega; \mathbb{M}^{n\times n\times n}_D)$.

*Indeed both* (4.4.4) *and* (4.4.15) *hold if and only if*

$$\langle \sigma^p(t) - A, \dot p(t)\rangle + \langle \mathbb{K}^p_{\mathrm{diss}}(t) - \mathbb{B}, \nabla \dot p(t)\rangle + \langle \mathbb{S}^p(t) - \mathbb{L}, \mathrm{D}^s \dot p(t)\rangle \ge 0$$

*for every* $(A, \mathbb{B}, \mathbb{L}) \in \mathcal{K}_{\alpha(t)}(\Omega)$.

We are now ready to prove that the classical flow rule (sfGA5) holds for a.e. $(t,x)$.

PROPOSITION 4.4.6 (Flow rule). *In the hypoteses of Proposition 4.4.3, let* $t \in [0,T]$ *such that* $\dot p(t)$ *and* $\nabla \dot p(t)$ *exist and let* $x \in \Omega$ *be a Lebesgue point for* $\sigma^p(t)$, $\mathbb{K}^p_{\rm diss}(t)$, $\dot p(t)$ *and* $\nabla \dot p(t)$. *Then the condition*

$$\frac{|\sigma^p(t,x)|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{K}^p_{\rm diss}(t,x)|^2}{l^2 S_2(\alpha(t,x))^2} < 1$$

*implies that*

$$(\dot p(t,x), \nabla \dot p(t,x)) = (0,0)\,,$$

*while if*

$$\frac{|\sigma^p(t,x)|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{K}^p_{\rm diss}(t,x)|^2}{l^2 S_2(\alpha(t,x))^2} = 1$$

*we have*

$$\dot p(t,x) = \lambda(t,x)\frac{\sigma^p(t,x)}{S_1(\alpha(t,x))^2}\,, \quad \nabla \dot p(t,x) = \lambda(t,x)\frac{|\mathbb{K}^p_{\rm diss}(t,x)|}{l^2 S_2(\alpha(t,x))^2}$$

*with*

$$\lambda(t,x) = \sqrt{S_1(\alpha(t,x))^2|\dot p(t,x)| + l^2 S_2(\alpha(t,x))^2|\nabla \dot p(t,x)|^2}\,.$$

PROOF. Let us fix $t$ and $x$ satisfying the assumption in the statement, and let us define the convex set

$$C_{t,x} := \left\{(F_0,\mathbb{G}_0) \in \mathbb{M}^{n\times n}_D \times \mathbb{M}^{n\times n\times n}_D \colon \frac{|F_0|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{G}_0|^2}{l^2 S_2(\alpha(t,x))^2} \le 1\right\}\,.$$

By assumption $(\sigma^p(t,x), \mathbb{K}^p_{\rm diss}(t,x)) \in C_{t,x}$. Given $(F_0,\mathbb{G}_0) \in C_{t,x}$ we set

$$(F(z),\mathbb{G}(z)) := \left(F_0\frac{S_1(\alpha(t,z))}{S_1(\alpha(t,x))}, \mathbb{G}_0\frac{S_2(\alpha(t,z))}{S_2(\alpha(t,x))}\right) \quad \text{for every } z \in \Omega\,.$$

Since $\alpha(t) \in C(\overline{\Omega})$ we get $(F,\mathbb{G}) \in C(\overline{\Omega};\mathbb{M}^{n\times n}_D)\times C(\overline{\Omega};\mathbb{M}^{n\times n\times n}_D)$; by construction $(F(x),\mathbb{G}(x)) = (F_0,\mathbb{G}_0)$ and $\frac{|F(z)|^2}{S_1(\alpha(t,z))^2} + \frac{|\mathbb{G}(z)|^2}{l^2 S_2(\alpha(t,z))^2} \le 1$ in $\Omega$. We now fix $r>0$ and test (4.4.17a) by

$$(A_r,\mathbb{G}_r) := \begin{cases} \frac{1}{2}(\sigma^p(t)+F, \mathbb{K}^p_{\rm diss}(t)+\mathbb{G}) & \text{in } B_r(x) \\ (\sigma^p(t), \mathbb{K}^p_{\rm diss}(t)) & \text{outside } B_r(x) \end{cases}$$

which is an admissible test function by convexity of the constraint set. Hence we obtain that for every $r>0$

$$\frac{1}{r^n}\left[\int_{B_r(x)} (\sigma^p(t)-F)\cdot \dot p(t)\,\mathrm{d}x + \int_{B_r(x)} (\mathbb{K}^p_{\rm diss}(t)-\mathbb{G})\cdot \nabla \dot p(t)\,\mathrm{d}x\right] \ge 0\,.$$

As $r \to 0$ we get (recall that $x$ is a Lebesgue point for the functions involved)

$$(F_0 - \sigma^p(t,x))\cdot \dot p(t,x) + (\mathbb{G}_0 - \mathbb{K}^p_{\rm diss}(t,x))\cdot \nabla \dot p(t,x) \le 0\,.$$

Since $(F_0,\mathbb{G}_0)$ is arbitrary in $C_{t,x}$, it follows that $(\dot p(t,x), \nabla \dot p(t,x))$ is in the normal cone to $C_{t,x}$ at $(\sigma^p(t,x), \mathbb{K}^p_{\rm diss}(t,x))$ and this proves the result. ☐

### 4.5. Asymptotic analysis for vanishing strain gradient effects

In this section we study the relation between the Gurtin-Anand model coupled with damage and the coupled elastoplastic damage model proposed in [**23**] and presented in Chapter 2.

In [**48**] it is proven that quasistatic evolutions for the Gurtin-Anand model converge in a suitable sense, as the strain gradient terms vanish, to evolutions for perfectly plastic bodies in the formulation of [**28**]. Then we expect, when $l, L$ tend to zero, the convergence of quasistatic evolutions in Definition 4.1.3 to evolutions for perfectly plastic bodies with damage studied in Chapter 2. Indeed the latter model corresponds, when the damage is constant in time, to the perfect plasticity model for heterogeneous materials in [**98**]. However, while the classical Gurtin-Anand formulation reduces to von Mises perfect plasticity model by setting $l$ and $L$ equal to zero (recall that $l$ is related to the thickness of the plastic shear bands and $L$ to the energy stored by the geometrically necessary dislocations), in the presence of damage the models have two different gradient damage regularizations, because in Chapter 2 and Chapter 3 the space continuity of $\alpha$ is needed. Thus we start from a coupled gradient plasticity-damage model with a regularizing term $\|\nabla\alpha\|_\gamma^\gamma$, $\gamma > n$, instead of $\|\nabla\alpha\|_2^2$. Moreover, in the model in Chapter 2 there is a term related to a fatigue phenomenon, which depends on a parameter $\lambda$. For simplicity, we do not consider here the fatigue and thus we take $\lambda = 0$.

For technical reasons (see Remark 4.5.2) we also require that the only loading is the displacement field $w$ applied to the whole of $\partial\Omega$.

Under this assumptions, Theorem 4.5.1 shows that evolutions for the Gurtin-Anand model coupled with damage converge weakly for every time to evolutions in Chapter 2.

For $l_k \to 0$ and $L_k \to 0$, let

$$\mathcal{E}_k(\beta, \eta, \operatorname{curl} q) := \mathcal{Q}_1(\beta, \eta) + \frac{L_k^2}{2}\int_\Omega \mu(\beta)|\operatorname{curl} q|^2 \,\mathrm{d}x + \|\nabla\beta\|_\gamma^\gamma + D(\beta)\,,$$
$$\mathcal{H}_k(\beta, q) := \int_\Omega \sqrt{S_1(\beta)^2|q|^2 + l_k^2 S_2(\beta)^2|\nabla q|^2}\,\mathrm{d}x + l_k\int_\Omega S_2(\widetilde{\beta})\,\mathrm{d}|\mathrm{D}^s q|$$

be the total energy and the plastic dissipation of the Gurtin-Anand model coupled with damage for the length scales $l = l_k$, $L = L_k$, $\ell = \sqrt{2}$. Moreover let

$$t \mapsto (\alpha_k(t), u_k(t), e_k(t), p_k(t)) \in W^{1,\gamma}(\Omega; [0,1]) \times W^{1,\frac{n}{n-1}}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times BV(\Omega; \mathbb{M}^{n\times n}_D)$$

be a corresponding quasistatic evolution with the prescribed displacement $w$. Namely the following conditions hold:

(qs0) *irreversibility* : for every $x \in \Omega$ the function $[0,T] \ni t \mapsto \alpha_k(t,x)$ is nonincreasing;
(qs1) *global stability*: for every $t \in [0,T]$ we have $(u_k(t), e_k(t), p_k(t)) \in A(w(t))$ and

$$\mathcal{E}_k(\alpha_k(t), e_k(t), \operatorname{curl} p_k(t)) \le \mathcal{E}_k(\beta, \eta, \operatorname{curl} q) + \mathcal{H}_k(\beta, q - p_k(t))$$

for every $(\beta, v, \eta, q) \in \mathcal{A}(\alpha(t), w(t))$;

(qs2) *energy balance*: the function $t \mapsto p_k(t)$ from $[0,T]$ into $BV(\Omega;\mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{E}_k(\alpha_k(t), e_k(t), \operatorname{curl} p_k(t)) + \mathcal{V}_{\mathcal{H}_k}(\alpha_k, p_k; 0, t) = \mathcal{E}_k(\alpha_k(0), e_k(0), \operatorname{curl} p_k(0)) + \int_0^t \langle \sigma_k(s), E\dot{w}(s)\rangle \,\mathrm{d}s\,,$$

where $\sigma_k(s) := \mathbb{C}(\alpha_k(s))e_k(s)$.

We now recall the notion of globally stable evolution for the coupled elastoplastic-damage model considered in Chapter 2, when the parameter $\lambda$ therein is zero.

The class of admissible configurations for a given boundary datum $w \in H^1(\Omega;\mathbb{R}^n)$ in perfect plasticity is the set

$$A_{\mathrm{pp}}(w) := \{(u,e,p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\colon \mathrm{E}u = e + p \text{ in } \Omega,\ p = (w-u)\odot\nu\,\mathcal{H}^{n-1} \text{ on } \partial\Omega\}\,, \tag{4.5.1}$$

and we define in analogy to (4.1.8)

$$\mathcal{A}_{\mathrm{pp}}(\alpha, w) := \{(\beta,u,e,p)\colon \beta \in W^{1,\gamma}(\Omega),\ \beta \le \alpha, \text{ and } (u,e,p) \in A_{\mathrm{pp}}(w)\}\,.$$

Here

$$BD(\Omega) := \{u \in L^1(\Omega;\mathbb{R}^n)\colon \mathrm{E}u \in M_b(\Omega;\mathbb{M}^{n\times n}_{sym})\}\,,$$

endowed with the norm

$$\|u\|_{BD} := \|u\|_1 + \|\mathrm{E}u\|_1\,,$$

is the Banach space of functions with bounded deformation on $\Omega$; for its general properties we refer to [**103**]. Notice that we use the subscripts "pp" (perfect plasticity with damage) to distinguish objects with analogous meaning in the two models, and that the term $w-u$ appearing in the definition of $A_{\mathrm{pp}}$ is intended in the sense of traces on $\partial\Omega$.

For every $\beta \in C(\overline{\Omega})$ and $q \in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ we set

$$\mathcal{H}_{\mathrm{pp}}(\beta, q) := \int_{\overline{\Omega}} S_1(\beta)\,\mathrm{d}|q|\,, \tag{4.5.2}$$

in analogy to $\mathcal{H}$. Here we adopt a multiplicative formulation for the constraint sets (indeed we are in von Mises setting). The plastic dissipation $\mathcal{V}_{\mathcal{H}_{\mathrm{pp}}}(\beta,q)$ is defined in the same way of $\mathcal{V}_{\mathcal{H}}$, starting from $\mathcal{H}_{\mathrm{pp}}$, and the total energy is

$$\mathcal{E}_{\mathrm{pp}}(\beta,\eta) := \mathcal{Q}_1(\beta,\eta) + D(\beta) + \|\nabla\beta\|_\gamma^\gamma\,,$$

with $\mathcal{Q}_1$ and $D$ as in (H6) and (H8.1).

A quasistatic evolution for the coupled perfect plasticity-damage model is a function

$$[0,T] \ni t \mapsto (\alpha(t),u(t),e(t),p(t)) \in W^{1,\gamma}(\Omega;[0,1])\times BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$$

satisfying the following conditions:

$(\mathrm{qs0})_{\mathrm{pp}}$ *irreversibility* : for every $x \in \Omega$ the function $[0,T]\ni t \mapsto \alpha(t,x)$ is nonincreasing;

$(\mathrm{qs1})_{\mathrm{pp}}$ *global stability*: for every $t \in [0,T]$ we have $(u(t),e(t),p(t)) \in A_{\mathrm{pp}}(w(t))$ and

$$\mathcal{E}_{\mathrm{pp}}(\alpha(t),e(t)) \le \mathcal{E}_{\mathrm{pp}}(\beta,\eta) + \mathcal{H}_{\mathrm{pp}}(\beta, q - p(t))$$

for every $(\beta,v,\eta,q) \in \mathcal{A}_{\mathrm{pp}}(\alpha(t),w(t))$;

$(\mathrm{qs2})_{\mathrm{pp}}$ *energy balance*: the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{E}_{\mathrm{pp}}(\alpha(t),e(t)) + \mathcal{V}_{\mathcal{H}_{\mathrm{pp}}}(\alpha,p;0,t) = \mathcal{E}_{\mathrm{pp}}(\alpha(0),e(0)) + \int_0^t \langle\sigma(s),\mathrm{E}\dot{w}(s)\rangle\,\mathrm{d}s\,,$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$.

Assuming $\Omega$ Lipschitz and (H2.1), (H3), (H5), (H8.2), and (H10), it is proven in Theorem 2.3.3 that for every initial data $(\alpha_0,u_0,e_0,p_0) \in \mathcal{A}_{\mathrm{pp}}(1,w(0))$ such that

$$\mathcal{E}_{\mathrm{pp}}(\alpha_0,e_0) \le \mathcal{E}_{\mathrm{pp}}(\beta,\eta) + \mathcal{H}_{\mathrm{pp}}(\beta,q-p_0)$$

for every $(\beta,v,\eta,q) \in \mathcal{A}_{\mathrm{pp}}(\alpha_0,w(0))$, there exists a quasistatic evolution for the coupled perfect plasticity-damage model $(\alpha,u,e,p)$ such that $(\alpha(0),u(0),e(0),p(0)) = (\alpha_0,u_0,e_0,p_0)$.

Now we consider the limit as $k\to\infty$, assuming for the initial conditions that

$$\begin{aligned} &\alpha_k(0) \rightharpoonup \alpha_0 \quad \text{in } W^{1,\gamma}(\Omega), && u_k(0) \overset{*}{\rightharpoonup} u_0 \quad \text{in } BD(\Omega),\\ &e_k(0) \rightharpoonup e_0 \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), && p_k(0) \overset{*}{\rightharpoonup} p_0 \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\end{aligned} \tag{4.5.3a}$$

for suitable $\alpha_0$, $e_0$, $u_0$, $p_0$, and

$$\mathcal{E}_k(\alpha_k(0),e_k(0),\operatorname{curl}p_k(0)) \to \mathcal{E}_{\mathrm{pp}}(\alpha_0,e_0)\,. \tag{4.5.3b}$$

Under this assumption, we can prove the convergence result below.

THEOREM 4.5.1. *Let $\Omega\subset\mathbb{R}^n$ be open, bounded, and Lipschitz; if $n\ge 3$, let $\Omega$ be also star-shaped. Assume $\partial_D\Omega=\partial\Omega$,* (H2.1), (H3), (H5), (H8.2), *and* (H10). *Moreover, for $l_k\to 0$ and $L_k\to 0$, let $(\alpha_k,u_k,e_k,p_k)$ be a quasistatic evolution for the Gurtin-Anand model coupled with damage associated with $l_k$ and $L_k$ such that the conditions* (4.5.3) *hold. Then there exists a quasistatic evolution for the perfect plasticity model coupled with damage $(\alpha,u,e,p)$ with $\alpha(0)=\alpha_0$, $u(0)=u_0$, $e(0)=e_0$, $p(0)=p_0$ such that, up to a subsequence,*

$$\alpha_k(t)\to\alpha(t) \quad \textit{in } W^{1,\gamma}(\Omega)\,, \tag{4.5.4a}$$

$$u_k(t) \overset{*}{\rightharpoonup} u(t) \quad \textit{in } BD(\Omega)\,, \tag{4.5.4b}$$

$$e_k(t)\to e(t) \quad \textit{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \tag{4.5.4c}$$

$$p_k(t) \overset{*}{\rightharpoonup} p(t) \quad \textit{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D) \tag{4.5.4d}$$

*for every $t\in[0,T]$.*

REMARK 4.5.2. An important difference with respect to the analysis in [**48**] relies on the fact that we cannot still characterize the global stability in the limit evolution by the equilibrium conditions for the Cauchy stress and the plastic constraint (see [**28**, Theorem 3.6]). This calls for the approximation in a strong sense of admissible triples for perfect plasticity with ones that are admissible for the Gurtin-Anand model. We show this relaxation result in the lemmas below both in the case of dimension two, and in dimension three under the additional assumption that the domain is star shaped. Actually, in the paper [**82**], M.G. Mora proves the approximation property for every Lipschitz domain; then Theorem 4.5.1 can be proved for this domains.

LEMMA 4.5.3 (Approximation, $n \geq 3$). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be open, bounded, star-shaped and Lipschitz. Then for every $(u, e, p) \in A_{\rm pp}(0)$ there exists a sequence of triples $(u_k, e_k, p_k) \in A(0)$ such that*

$$u_k \to u \quad \text{in } L^1(\Omega;\mathbb{R}^n), \qquad e_k \to e \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \qquad p_k \xrightarrow{s} p \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\,.$$

PROOF. Without loss of generality we can assume that $\Omega$ is star-shaped with respect to 0. For an open set $\widetilde{\Omega}$ such that $\overline{\Omega} \subset \widetilde{\Omega}$ let us define

$$\widehat{u} := \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \widetilde{\Omega}\setminus\Omega \end{cases}, \qquad \widehat{e} := \begin{cases} e & \text{in } \Omega \\ 0 & \text{in } \widetilde{\Omega}\setminus\Omega \end{cases}, \qquad \widehat{p} := \begin{cases} p & \text{in } \overline{\Omega} \\ 0 & \text{in } \widetilde{\Omega}\setminus\overline{\Omega} \end{cases},$$

For $k$ large enough we set

$$\widehat{u}_k(x) := (1+\tfrac{1}{k})^{-1} u\big((1+\tfrac{1}{k})x\big)\,, \quad \widehat{e}_k(x) := e\big((1+\tfrac{1}{k})x\big) \quad \text{for every } x \in \Omega_k := \Omega + B_{\frac{1}{k}}\,,$$

and

$$\widehat{p}_k := E\widehat{u}_k - \widehat{e}_k \quad \text{in } \Omega_k\,.$$

Then it is not difficult to see that

$$\widehat{u}_k(x) = 0 \quad \text{for every } x \in \Omega_k \setminus \big[(1+\tfrac{1}{k})^{-1}\Omega\big]\,,$$

$$|\widehat{p}_k|(\partial\Omega) = 0\,, \tag{4.5.5}$$

and that, taking the restriction of $\widehat{u}_k$, $\widehat{e}_k$, $\widehat{p}_k$ to $\overline{\Omega}$, we have

$$\widehat{u}_k \to u \quad \text{in } L^1(\Omega;\mathbb{R}^n), \qquad \widehat{e}_k \to e \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \qquad \widehat{p}_k \xrightarrow{s} p \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\,.$$

Moreover, if we regularize by convolution for every $k$ with the sequence of mollifiers $(\varrho_{\frac{1}{h}})_{\mathbb{N}\ni h>k}$, we get (taking the restrictions to $\overline{\Omega}$) a sequence of functions

$$(\widehat{u}^h_k, \widehat{e}^h_k, \widehat{p}^h_k) \in A(0) \cap C^\infty(\overline{\Omega};\mathbb{R}^n \times \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_D)$$

such that

$$\widehat{u}^h_k \to \widehat{u}_k \quad \text{in } L^1(\Omega;\mathbb{R}^n), \quad \widehat{e}^h_k \to \widehat{e}_k \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \quad \widehat{p}^h_k \xrightarrow{s} \widehat{p}_k \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$$

as $h \to \infty$. Indeed, by (4.5.5) it is enough to show that $\widehat{p}^h_k \xrightarrow{s} \widehat{p}_k$ in $M_b(\Omega;\mathbb{M}^{n\times n}_D)$, and this holds again by (4.5.5) since the regularization by convolution of a measure entails strict convergence on open subsets whose boundaries are not charged by the measure itself (see [**6**, Theorem 2.2]).

By a diagonal argument we obtain $(u_k, e_k, p_k)$ as $(\widehat{u}^{h_k}_k, \widehat{e}^{h_k}_k, \widehat{p}^{h_k}_k)$ with $h = h_k$ sufficiently large. □

We now show the relaxation property for perfect plasticity triples in a bidimensional domain. The construction of the approximants is similar to the one made in [**43**, Theorem 6.2, Step 1].

LEMMA 4.5.4 (Approximation, $n = 2$). *Let $\Omega \subset \mathbb{R}^2$ be open, bounded, and Lipschitz. Then for every $(u, e, p) \in A_{\rm pp}(0)$ there exists a sequence of triples $(u_k, e_k, p_k) \in A(0)$ such that*

$$u_k \to u \quad \text{in } L^1(\Omega;\mathbb{R}^n), \qquad e_k \to e \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \qquad p_k \xrightarrow{s} p \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\,.$$

PROOF. Let us define

$$\widehat{u} := \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega \end{cases}, \qquad \widehat{e} := \begin{cases} e & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega \end{cases}, \qquad \widehat{p} := \begin{cases} p & \text{in } \overline{\Omega} \\ 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \end{cases}.$$

Since $(u,e,p) \in A_{\text{pp}}(0)$, we get that

$$E\widehat{u} = \widehat{e} + \widehat{p} \quad \text{in } \mathbb{R}^2 .$$

Let $\{Q_{\nu_k}(x_k, r_k)\}_{k\in I}$ be a finite covering of $\partial\Omega$ made of open cubes with centers $x_k \in \partial\Omega$, side $2r_k$, with $r_k > 0$, and a face orthogonal to $\nu_k \in \mathbb{R}^2$ such that $\Omega \cap Q_{\nu_k}(x_k, r_k)$ is a Lipschitz subgraph in the direction $\nu_k$. Let $\{\phi_k\}_{k\in I}$ be an associated partition of unity of $\partial\Omega$. Then

$$\widehat{u} = \sum_{k\in I} \phi_k \widehat{u} + \left(1 - \sum_{k\in I} \phi_k\right) \widehat{u} ,$$

and the last term has a support compactly contained in $\Omega$. Set

$$\widehat{e}_k := \phi_k \widehat{e} + \nabla\phi_k \odot \widehat{u} \quad \text{and} \quad \widehat{p}_k := \phi_k \widehat{p} , \tag{4.5.6}$$

so that $\widehat{e}_k \in L^2(\mathbb{R}^2; \mathbb{M}^{2\times2}_{sym})$ (indeed $\widehat{u} \in BD(\mathbb{R}^2) \subset L^2(\mathbb{R}^2;\mathbb{R}^2)$) and $\widehat{p}_k \in M_b(\mathbb{R}^2; \mathbb{M}^{2\times2}_D)$ with

$$E(\phi_k \widehat{u}) = \widehat{e}_k + \widehat{p}_k \quad \text{in } \mathbb{R}^2 .$$

For $h \in \mathbb{N}$ so large that the support of the functions $\widehat{\phi}_k(x) := \phi_k\left(x + \frac{\nu_k}{h}\right)$ is compactly contained in $Q_{\nu_k}(x_k, r_k)$ for every $k \in I$, let us define

$$u_{k,h}(x) := \phi_k \left(x + \frac{\nu_k}{h}\right) \widehat{u} \left(x + \frac{\nu_k}{h}\right) ;$$

we also define $e_{k,h}$, $p_{h,k}$ following (4.5.6). Set

$$u_h := \sum_{k\in I} u_{k,h} + \left(1 - \sum_{k\in I} \phi_k\right)\widehat{u}, \quad e_h := \sum_{k\in I} e_{k,h} + \left(1 - \sum_{k\in I} \phi_k\right) \widehat{e} - \sum_{k\in I} \nabla\phi_k \odot \widehat{u}$$

$$p_h := \sum_{k\in I} p_{k,h} + \left(1 - \sum_{k\in I} \phi_k\right) \widehat{p}$$

Notice that

$$(u_h, e_h, p_h) \in BD(\mathbb{R}^2) \times L^2(\mathbb{R}^2; \mathbb{M}^{2\times2}_{sym}) \times M_b(\mathbb{R}^2; \mathbb{M}^{2\times2}_D)$$

with

$$Eu_h = e_h + p_h \quad \text{in } \mathbb{R}^2 ,$$

and that $u_h$, $e_h$, $p_h$ vanish outside a compact subset of $\Omega$. This last condition and fact that we have only used local translations imply that restricting to $\overline{\Omega}$

$$u_h \to u \quad \text{in } L^2(\Omega;\mathbb{R}^2) , \quad e_h \to e \quad \text{in } L^2(\Omega; \mathbb{M}^{2\times2}_{sym}) , \quad p_h \xrightarrow{s} p \quad \text{in } M_b(\overline{\Omega}; \mathbb{M}^{2\times2}_D) .$$

Moreover, if we regularize $(u_h, e_h, p_h)$ by convolution with a sequence of mollifiers $(\varrho_{\frac{1}{m}})_m$, we get for $m$ sufficiently large that

$$(u^m_h, e^m_h, p^m_h) \in C^\infty_c(\Omega; \mathbb{R}^2 \times \mathbb{M}^{2\times2}_{sym} \times \mathbb{M}^{2\times2}_D) \cap A(0) ,$$

using again that $u_h$, $e_h$, $p_h$ have compact support in $\Omega$. Recalling that the regularization by convolution of a measure entails strict convergence on open subsets whose boundaries are not charged by the measure itself, and that $p_h = 0$ on $\partial\Omega$, we have

$$u_h^m \to u_h \quad \text{in } L^2(\Omega;\mathbb{R}^2)\,, \quad e_h^m \to e_h \quad \text{in } L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})\,, \quad p_h^m \xrightarrow{s} p_h \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_D)\,,$$

and then we conclude by a diagonal argument. ☐

We are now ready to prove Theorem 4.5.1.

Proof of Theorem 4.5.1. The proof is divided into two steps.

**Step 1: Compactness and global stability.** By definition of $\mathcal{H}_k$ we have that for every $\beta \in W^{1,\gamma}(\Omega)$, $q \in BV(\Omega;\mathbb{M}^{n\times n}_D)$, and $k \in \mathbb{N}$

$$\mathcal{H}_k(\beta, q) \geq S_1(0)\|q\|_1\,,$$

and then

$$\mathcal{V}_{\mathcal{H}_k}(\alpha_k, p_k; 0, t) \geq S_1(0)\mathcal{V}_1(p_k; 0, t)\,,$$

with $\mathcal{V}_1(p_k; 0, t)$ the variation of $p_k$ with respect to $L^1(\Omega;\mathbb{M}^{n\times n}_D)$ in $(0,t)$. Then, by (4.5.3), the fact that $\mathcal{Q}_1$ is quadratic, and Korn's inequality, we get that there exists a constant $C$ independent of $k$ and $t$ such that

$$\|\alpha_k(t)\|_{1,\gamma} + \|u_k(t)\|_{BD} + \|e_k(t)\|_2 + \mathcal{V}_1(p_k; 0, t) \leq C\,. \tag{4.5.7}$$

Let $\widetilde{\Omega}$ be a smooth open set such that $\overline{\Omega} \subset \widetilde{\Omega}$, and let us define for every $k$ and $t$ the functions $\widehat{u}_k(t) \in W^{1,\frac{n}{n-1}}(\widetilde{\Omega};\mathbb{R}^n)$, $\widehat{e}_k(t) \in L^2(\widetilde{\Omega};\mathbb{M}^{n\times n}_{sym})$, and $\widehat{p}_k(t) \in BV(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$ as

$$\widehat{u}_k(t) := \begin{cases} u_k(t) & \text{in } \Omega \\ w(t) & \text{in } \widetilde{\Omega}\setminus\Omega \end{cases}, \quad \widehat{e}_k(t) := \begin{cases} e_k(t) & \text{in } \Omega \\ \mathrm{E}w(t) & \text{in } \widetilde{\Omega}\setminus\Omega \end{cases}, \quad \widehat{p}_k(t) := \begin{cases} p_k(t) & \text{in } \Omega \\ 0 & \text{in } \widetilde{\Omega}\setminus\Omega \end{cases}.$$

The $\alpha_k$ are nonincreasing in time and $\alpha_k(t,x) \in [0,1]$ with $\|\alpha_k(t)\|_{1,\gamma} \leq C$ and the functions $p_k$ from $[0,T]$ to $L^1(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$ have uniformly bounded variations; therefore, taking into account (4.5.3) we get the existence of two functions $\alpha\colon [0,T] \to W^{1,\gamma}(\Omega;[0,1])$ nonincreasing in time and $\widehat{p}\colon [0,T] \to M_b(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$ with bounded variation such that up to a subsequence (not relabeled)

$$\alpha_k(t) \rightharpoonup \alpha(t) \quad \text{in } W^{1,\gamma}(\Omega)\,, \qquad \widehat{p}_k(t) \xrightarrow{*} \widehat{p}(t) \quad \text{in } M_b(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$$

for every $t \in [0,T]$. Notice that we have applied [**28**, Theorem 7.2] considering $M_b(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$ as a subspace of $L^1(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$.

Let us fix $t \in [0,T]$. By the a priori estimate (4.5.7) we deduce that there exist an increasing sequence $(k_j)_j$ (that could depend on $t$) and two functions $\widehat{u} \in BD(\widetilde{\Omega})$ and $\widehat{e} \in L^2(\widetilde{\Omega};\mathbb{M}^{n\times n}_{sym})$ such that

$$\widehat{u}_{k_j} \xrightarrow{*} \widehat{u} \quad \text{in } BD(\widetilde{\Omega})\,, \qquad \widehat{e}_{k_j} \rightharpoonup \widehat{e} \quad \text{in } L^2(\widetilde{\Omega};\mathbb{M}^{n\times n}_{sym})\,.$$

As in [**48**, Lemma 9.1] (that holds in our assumptions on $\Omega$), we obtain that

$$u_{k_j}(t) \xrightarrow{*} \widehat{u} \quad \text{in } BD(\Omega), \quad e_{k_j}(t) \rightharpoonup \widehat{e} \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \quad p_k(t) \xrightarrow{*} p(t) \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\,, \tag{4.5.8}$$

and

$$(\widehat{u}, \widehat{e}, p(t)) \in A_{\rm pp}(w(t))\,,$$

where $p(t)$ denotes the restriction of $\widehat{p}(t)$ to $\overline{\Omega}$ and we have not relabeled the restrictions of $\widehat{u}$, $\widehat{e}$ to $\Omega$. We claim that the quadruple $(\alpha(t), \widehat{u}, \widehat{e}, p(t))$ satisfies the stability condition $(\mathrm{qs1})_{\rm pp}$, namely

$$\mathcal{E}_{\rm pp}(\alpha(t), \widehat{e}) \leq \mathcal{E}_{\rm pp}(\beta, \eta) + \mathcal{H}_{\rm pp}(\beta, q - p(t)) \tag{4.5.9}$$

for every $(\beta, (v, \eta, q)) \in \mathcal{A}_{\rm pp}(\alpha(t), w(t))$. Then, since $(\widehat{u}, \widehat{e})$ minimizes the functional $(v, \eta) \mapsto \mathcal{E}_{\rm pp}(\beta, \eta)$ on the convex set $\{(v, e) \colon (v, e, p(t)) \in A_{\rm pp}(w(t))\}$, we have that $(\widehat{u}, \widehat{e}) = (u(t), e(t))$ and

$$u_k(t) \overset{*}{\rightharpoonup} u(t) \quad \text{in } BD(\Omega)\,, \qquad e_k(t) \rightharpoonup e(t) \quad \text{in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym})\,, \tag{4.5.10a}$$

for the whole subsequence. We have already shown that

$$\alpha_k(t) \rightharpoonup \alpha(t) \quad \text{in } W^{1,\gamma}(\Omega)\,, \qquad p_k(t) \overset{*}{\rightharpoonup} p(t) \quad \text{in } M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)\,. \tag{4.5.10b}$$

Let us now prove the claim (4.5.9); since we work with a given $t$, we can neglect the dependence on $j$ in (4.5.8). By assumption, for every $k$ we have the stability condition:

$$\mathcal{E}_k(\alpha_k(t), e_k(t), \operatorname{curl} p_k(t)) \leq \mathcal{E}_k(\beta, \eta, \operatorname{curl} q) + \mathcal{H}_k(\beta, q - p_k(t)) \tag{4.5.11}$$

for every $(\beta, v, \eta, q) \in \mathcal{A}(\alpha_k(t), w_k(t))$.

Let us fix $(\beta, v_0, \eta_0, q_0) \in \mathcal{A}(\alpha(t), 0)$, and test (4.5.11) by

$$(\widehat{\alpha}_k, \widehat{v}_k, \widehat{\eta}_k, \widehat{q}_k) := (\beta \wedge \alpha_k(t), u_k(t) + v_0, e_k(t) + \eta_0, p_k(t) + q_0) \in \mathcal{A}(\alpha_k(t), w_k(t))\,.$$

Arguing as in Theorem 4.2.6 we deduce that

$$\begin{aligned}
\gamma_k :=\, & \mathcal{Q}_1(\alpha_k(t), e_k(t)) - \mathcal{Q}_1(\widehat{\alpha}_k, e_k(t)) + \mathcal{D}(\alpha_k(t)) + \|\nabla(\beta \vee \alpha_k(t))\|_\gamma^\gamma - \|\nabla \beta\|_\gamma^\gamma \\
\leq\, & \tfrac{1}{2}\langle \mathbb{C}(\widehat{\alpha}_k)(\eta_0 + 2e_k(t)), \eta_0\rangle + \tfrac{L_k^2}{2}\langle \mu(\widehat{\alpha}_k) \operatorname{curl}(q_0 + 2p_k(t)), \operatorname{curl} q_0\rangle + \mathcal{D}(\widehat{\alpha}_k) \\
& + \mathcal{H}_{\rm pp}(\widehat{\alpha}_k, q_0) + l_k \int_\Omega S_2(\widehat{\alpha}_k)\, \mathrm{d}|\mathrm{D}q_0| =: \delta_k\,.
\end{aligned} \tag{4.5.12}$$

To get the above inequality we have also used that

$$\frac{L_k^2}{2}\int_\Omega (\mu(\alpha_k(t)) - \mu(\widehat{\alpha}_k))|\operatorname{curl} p_k(t)|^2 \,\mathrm{d}x \geq 0$$

and that for every $\alpha \in W^{1,\gamma}(\Omega)$ and $p \in BV(\Omega; \mathbb{M}^{n\times n}_D)$

$$\mathcal{H}_k(\alpha, p) \leq \mathcal{H}_{\rm pp}(\alpha, p) + l_k \int_\Omega S_2(\alpha)\, \mathrm{d}|\mathrm{D}p|\,.$$

By (4.5.3) and the energy balance for $(\alpha_k, u_k, e_k, p_k)$ we get

$$\frac{L_k^2}{2}\int_\Omega \mu(\alpha_k(t))|\operatorname{curl} p_k(t)|^2 \,\mathrm{d}x \leq C\,,$$

for $C$ independent of $k$; by the Hölder inequality and the monotonicity of $\mu$ it follows that

$$L_k^2\langle \mu(\widehat{\alpha}_k) \operatorname{curl} p_k(t), \operatorname{curl} q_0\rangle \leq L_k \left(\int_\Omega L_k^2\, \mu(\alpha_k(t))|\operatorname{curl} p_k(t)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \mu(\widehat{\alpha}_k)|\operatorname{curl} q_0|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Thus, letting $k \to 0$ in (4.5.12) we obtain as in Theorem 4.2.6 the inequality

$$\mathcal{E}_{\rm pp}(\alpha(t), \widehat{e}) - \mathcal{Q}_1(\beta, \widehat{e}) - \|\nabla \beta\|_\gamma^\gamma \leq \tfrac{1}{2}\langle \mathbb{C}(\beta)(\eta_0 + 2\widehat{e}), \eta_0\rangle + \mathcal{D}(\beta) + \mathcal{H}_{\rm pp}(\beta, q_0)\,. \tag{4.5.13}$$

Let us consider a triple $(v, \eta, q) \in A_{\rm pp}(w(t))$; then $(v - \widehat{u}, \eta - \widehat{e}, q - p(t)) \in A_{\rm pp}(0)$. By Lemmas 4.5.3 and 4.5.4 there exist triples $(v_k, \eta_k, q_k) \in A(0)$ such that

$$v_k \to v - \widehat{u} \quad \text{in } L^1(\Omega; \mathbb{R}^n), \qquad \eta_k \to \eta - \widehat{e} \quad \text{in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}),$$
$$q_k \xrightarrow{s} q - p(t) \quad \text{in } M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)\,.$$

In particular Reshetnyak's Continuity Theorem (cf. [**6**, Theorem 2.39]) implies that

$$\mathcal{H}_{\rm pp}(\beta, q_k) \to \mathcal{H}_{\rm pp}(\beta, q - p(t))\,.$$

Therefore, considering $(v_k, \eta_k, q_k)$ in place of $(v_0, \eta_0, q_0)$ in (4.5.13) and taking the limit of the right-hand side as $k \to \infty$ we deduce (4.5.9).

**Step 2: Energy balance.** From (4.5.10b) it follows that

$$\mathcal{V}_{\mathcal{H}_{\rm pp}}(\alpha, p; 0, T) \leq \liminf_{k\to\infty} \mathcal{V}_{\mathcal{H}_k}(\alpha_k, p_k; 0, T)\,. \tag{4.5.14}$$

Indeed, for every $\beta_k \rightharpoonup \beta$ in $W^{1,\gamma}(\Omega)$ and $(q_k)_k \subset BV(\Omega; \mathbb{M}^{n\times n}_D)$ with $q_k \xrightharpoonup{*} q$ in $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$, it holds

$$\mathcal{H}_{\rm pp}(\beta, q) \leq \liminf_{k\to\infty} \int_{\overline{\Omega}} S_1(\beta_k)\,\mathrm{d}|q_k| = \liminf_{k\to\infty} \int_{\Omega} S_1(\beta_k(x))|q_k(x)|\,\mathrm{d}x \leq \liminf_{k\to\infty} \mathcal{H}_k(\beta_k, q_k)\,,$$

and then we get (4.5.14) by the definition of $\mathcal{V}_{\mathcal{H}_{\rm pp}}$ and $\mathcal{V}_{\mathcal{H}_k}$. By lower semicontinuity and the fact that $\mathcal{Q}_2(\alpha_k(t), \operatorname{curl} p_k(t))$ is nonnegative it follows that

$$\mathcal{E}_{\rm pp}(\alpha(t), e(t)) \leq \liminf_{k\to\infty} \mathcal{E}_k(\alpha_k(t), e_k(t), \operatorname{curl} p_k(t))\,. \tag{4.5.15}$$

Collecting (4.5.3), (4.5.14), and (4.5.15) we deduce that

$$\mathcal{E}_{\rm pp}(\alpha(T), e(T)) + \mathcal{V}_{\mathcal{H}_{\rm pp}}(\alpha, p; 0, T) \leq \mathcal{E}_{\rm pp}(\alpha(0), e(0)) + \int_0^T \langle \sigma(s), E\dot{w}(s)\rangle\,\mathrm{d}s\,.$$

From the stability condition $(\text{qs1})_{\rm pp}$, with arguments similar to those in Lemma 4.3.3 (cf. [**28**, Theorem 4.7]), we can prove that the opposite energy imbalance holds and then $(\alpha, u, e, p)$ is a quasistatic evolution for the coupled perfect plasticity-damage model. By (4.5.3), (4.5.14), (4.5.15), and the energy balance (evaluated in $[0, t]$) it follows that for every $t \in [0, T]$

$$\mathcal{E}_{\rm pp}(\alpha(t), e(t)) = \lim_{k\to\infty} \mathcal{E}_k(\alpha_k(t), e_k(t), \operatorname{curl} p_k(t))\,,$$

which implies

$$\mathcal{Q}_1(\alpha_k(t), e_k(t)) \to \mathcal{Q}_1(\alpha(t), e(t)), \quad \|\nabla \alpha_k(t)\|_\gamma \to \|\nabla \alpha(t)\|_\gamma, \quad \mathcal{Q}_2^k(\alpha_k(t), \operatorname{curl} p_k(t)) \to 0\,,$$

and then (4.5.4a) and (4.5.4c). This concludes the proof. □

## 4.6. A further lower semicontinuity theorem of Reshetnyak-type

In the asymptotic analysis for vanishing strain gradient effects, performed in the previous section, we started from a coupled gradient plasticity-damage model with a regularizing term $\|\nabla\alpha\|_\gamma^\gamma$, $\gamma > n$, instead of $\|\nabla\alpha\|_2^2$. This choice is motivated since in the model in Chapter 2 the space continuity of the damage variable $\alpha$ is needed. In particular, in that model of perfect plasticity with damage, the difficulty is to generalize Lemma 2.1.1 in order to get the lower semicontinuity (2.1.19) even when the sequences of damage variables and platic strains $\alpha_k$ and $p_k$ are, respectively, in $H^1(\Omega;[0,1])$ and in $\Pi(\Omega)$ (introduced in (2.1.4)), and such that $\alpha_k \rightharpoonup \alpha$ in $H^1(\Omega)$ and $p_k \overset{*}{\rightharpoonup} p$ in $M_b(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_D)$.

Employing such a generalization it would be possible to prove the existence of globally stable quasistatic evolutions for a model similar to the one in Chapter 2, where the only difference is the use of the weaker $H^1$-regularization for the damage variable (instead of the $W^{1,\gamma}$ regularization, $\gamma > n$), as done for the model of Gurtin-Anand plasticity coupled with damage. Indeed, also in this chapter, the crucial point in order to consider the $H^1$-regularization for the damage variable is Theorem 4.2.1.

The present section is devoted to the proofs of a new Reshetnak-type lower semicontinuity theorem (Theorem 4.6.1) and of a result (Theorem 4.6.6), that in our opinion is an important step toward the existence of globally stable quasistatic evolutions for elastoplasticity coupled with damage, where the damage regularization is $H^1$. Indeed, by Theorem 4.6.6, we get the semicontinuity of the plastic potential of Chapter 2 (see (2.1.13) and (4.5.2)), in the case when the damage variables $\alpha_k$ converge weakly in $H^1(\Omega)$, the elastic strains $\eta_k$ converge strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and the displacements $v_k$ (such that $(v_k,\eta_k,q_k)\in A_{\mathrm{pp}}(w)$ for a certain $w$, see (4.5.1)) converge weakly* in $BD(\Omega)$.

In the proof of existence of globally stable evolutions for perfect plasticity with damage, the lower semicontinuity of the plastic potential applies to sequences $(\alpha_k,(v_k,\eta_k,q_k))$ such that $\eta_k$ is bounded in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. Then, being $\mathbb{C}(\alpha)$ equicoercive with respect to $\alpha\in[0,1]$, the strong convergence for $\eta_k$ would follow for instance by an uniform bound for the stresses $\sigma_k=\mathbb{C}(\alpha_k)\eta_k$ in $H^1_{\mathrm{loc}}(\Omega;\mathbb{M}^{n\times n}_{sym})$. Actually, this stress regularity is reasonable, since it holds in the framework of perfect plasticity, without damage. (See the paper by Demyanov [**38**], and also e.g. [**10**].) With such a priori bound on the stresses, we would able to employ the lower semicontinuity theorem and to show the existence of quasistatic evolutions.

We first prove our Reshetnak-type lower semicontinuity theorem, that is in some sense a $BD$ version of Theorem 4.2.1, obtained in a $BV$ setting (see Remark 4.6.2).

Let $\Omega$ be an open, bounded, connected, and Lipschitz subset of $\mathbb{R}^n$, $n\geq 2$. We define the functional $\widehat{\mathcal{H}}\colon H^1(\Omega;[0,1])\times BD(\Omega)\to\mathbb{R}^+\cup\{0\}$ as

$$\widehat{\mathcal{H}}(\alpha,v) := \int_\Omega V(\widetilde{\alpha})\,\mathrm{d}|\mathrm{E}v|\,, \tag{4.6.1}$$

where (as usual in this thesis) $|\cdot|$ denotes the Euclidean norm (or Frobenius norm) of a matrix, $V\colon[0,1]\to[m,M]$ is continuous and nondecreasing with $m$ positive, and $\widetilde{\alpha}$ is the $C_2$-quasicontinuous representative of $\alpha$. Notice that the definition of $\widehat{\mathcal{H}}$ is well posed, since $\widetilde{\alpha}$ is defined at $\mathcal{H}^{n-1}$–a.e. $x\in\Omega$ and the measure $\mathrm{E}v$ does not charge sets of dimension less than

$n-1$. (See also the comments to the definition (H9) at page 98.) The main result of this section is the following.

THEOREM 4.6.1. *The functional* $\widehat{\mathcal{H}}$ *defined in* (4.6.1) *is lower semicontinuous with respect to the weak–*$H^1(\Omega)$ *convergence of* $\alpha_k$ *and the weak*$^*$*–*$BD(\Omega)$ *convergence of* $v_k$*, namely*

$$\widehat{\mathcal{H}}(\alpha, v) \leq \liminf_{k\to\infty} \widehat{\mathcal{H}}(\alpha_k, v_k) \tag{4.6.2}$$

*for every* $\alpha_k \rightharpoonup \alpha$ *in* $H^1(\Omega)$ *and* $v_k \overset{*}{\rightharpoonup} v$ *in* $BD(\Omega)$.

The remark below concerns the relation between Theorem 4.6.1 and Theorem 4.2.1.

REMARK 4.6.2. If in definition (H9) at page 98 the function $S_1$ is null, then

$$\mathcal{H}(\alpha, p) = \ell \int_\Omega S_2(\widetilde{\alpha})\, \mathrm{d}|\mathrm{D}p|\,, \quad \text{for } \alpha \in H^1(\Omega) \text{ and } p \in BV(\Omega; \mathbb{M}^{n\times n}_D)\,.$$

Arguing as in Theorem 4.2.1, one proves easily the lower semicontinuity of the (analogous) functional

$$H^1(\Omega;[0,1]) \times BV(\Omega;\mathbb{R}^n) \ni (\alpha, p) \mapsto \int_\Omega V(\widetilde{\alpha})\, \mathrm{d}|\mathrm{D}p|\,, \tag{4.6.3}$$

with respect to the weak-$H^1(\Omega)$ convergence of $\alpha_k$ and the weak$^*$–$BV(\Omega;\mathbb{R}^n)$ convergence of $p_k$. The functional in (4.6.3) is reminiscent of $\widehat{\mathcal{H}}$, since one considers the Frobenius norm either of the total gradient (in (4.6.3)) or of the symmetric gradient (in (4.6.1)). However, the semicontinuity property is obtained with very different techniques. Indeed, in a $BD$ framework it is not possible to use the truncation argument of Theorem 4.2.1. The proof of Theorem 4.6.1 is based instead on a slicing argument.

We give now some notation and recall some preliminary results about slicing. For more details, we refer the reader to **[5]**. For every $\xi \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ and for every set $B \subset \mathbb{R}^n$, we define

$$\Pi^\xi := \{z \in \mathbb{R}^n : z \cdot \xi = 0\} \quad \text{and} \quad B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\} \quad \text{for every } y \in \Pi^\xi\,.$$

For any scalar function $\alpha\colon \Omega \to \mathbb{R}$ and any vector function $v\colon \Omega \to \mathbb{R}^n$, their slices $\alpha^\xi_y\colon \Omega^\xi_y \to \mathbb{R}$ and $\widehat{v}^\xi_y\colon \Omega^\xi_y \to \mathbb{R}$ are defined by

$$\alpha^\xi_y(t) := \alpha(y + t\xi) \quad \text{and} \quad \widehat{v}^\xi_y := v(y + t\xi) \cdot \xi\,,$$

respectively. If $v_k$ is a sequence in $L^1(\Omega;\mathbb{R}^n)$ and $v \in L^1(\Omega;\mathbb{R}^n)$ such that $v_k \to v$ in $L^1(\Omega;\mathbb{R}^n)$, then for every $\xi \in \mathbb{S}^{n-1}$ there exists a subsequence $v_{k_j}$ such that

$$(\widehat{v_{k_j}})^\xi_y \to \widehat{v}^\xi_y \quad \text{in } L^1(\Omega^\xi_y) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi\,, \tag{4.6.4}$$

by Fubini Theorem. Let $\mu_y$ be a bounded measure in $\Omega^\xi_y$ for every $y \in \Pi^\xi$, such that for every Borel set $B \subset \Omega$ the map $y \mapsto \mu_y(B^\xi_y)$ is Borel measurable and $\mathcal{H}^{n-1}$-integrable on $\Pi^\xi$. Then the set function

$$\lambda(B) := \int_{\Pi^\xi} \mu_y(B^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y) \quad \text{for all } B \subset \Omega \text{ Borel} \tag{4.6.5}$$

is a measure, and we write

$$\lambda = \int_{\Pi^\xi} \mu_y \, \mathrm{d}\mathcal{H}^{n-1}(y) \quad \text{in } M_b(\Omega)\,.$$

It can be seen that its total variation $|\lambda|$ is given by

$$|\lambda| = \int_{\Pi^\xi} |\mu_y| \,\mathrm{d}\mathcal{H}^{n-1}(y) \quad \text{in } M_b(\Omega)\,. \tag{4.6.6}$$

A function $v \in L^1(\Omega;\mathbb{R}^n)$ belongs to $BD(\Omega)$ if and only if for every direction $\xi \in \mathbb{S}^{n-1}$

$$\widehat{v}^\xi_y \in BV(\Omega^\xi_y) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \quad \text{and} \quad \int_{\Pi^\xi} |\mathrm{D}\widehat{v}^\xi_y|(\Omega^\xi_y) \,\mathrm{d}\mathcal{H}^{n-1}(y) < +\infty\,.$$

Moreover, if $v \in BD(\Omega)$ then for every $\xi \in \mathrm{S}^{n-1}$ it holds that

$$\mathrm{E}v\,\xi\cdot\xi = \int_{\Pi^\xi} \mathrm{D}\widehat{v}^\xi_y \,\mathrm{d}\mathcal{H}^{n-1}(y) \quad \text{in } M_b(\Omega)\,.$$

In particular, by (4.6.6), we have that

$$|\mathrm{E}v\,\xi\cdot\xi| = \int_{\Pi^\xi} |\mathrm{D}\widehat{v}^\xi_y| \,\mathrm{d}\mathcal{H}^{n-1}(y) \quad \text{in } M_b(\Omega)\,. \tag{4.6.7}$$

Let $\beta \in L^1(\Omega)$, and $q \in [1,\infty)$. Then $\beta \in W^{1,q}(\Omega)$ if and only if for every $\xi \in \mathbb{S}^{n-1}$

$$\beta^\xi_y \in W^{1,q}(\Omega^\xi_y) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \quad \text{and} \quad \int_{\Pi^\xi} \Big( \int_{\Omega^\xi_y} |\nabla\beta^\xi_y(t)|^q \,\mathrm{d}t \Big) \,\mathrm{d}\mathcal{H}^{n-1}(y) < +\infty\,,$$

and if $\beta \in W^{1,q}(\Omega)$ then for every $\xi \in \mathrm{S}^{n-1}$ it holds that

$$\int_\Omega |\nabla\beta\cdot\xi|^q = \int_{\Pi^\xi} \Big( \int_{\Omega^\xi_y} |\nabla\beta^\xi_y(t)|^q \,\mathrm{d}t \Big) \,\mathrm{d}\mathcal{H}^{n-1}(y)\,. \tag{4.6.8}$$

Moreover, $\widehat{\nabla\beta}^\xi_y = \nabla\beta^\xi_y$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, and $\widetilde{\beta}^\xi_y$ (the slice of the $C_q$-quasicontinuous representative of $\beta$) is the continuous representative in the equivalence class of $\beta^\xi_y$ for any $y \in \Pi^\xi$ such that $\beta^\xi_y \in W^{1,q}(\Omega^\xi_y)$.

The proof of Theorem 4.6.1 employs some techniques developed for the proof of [**35**, Theorem 4.1]. In particular, we will use the following facts, that correspond to Proposition 2.1, Remark 2.2, and Lemma 2.3 in [**35**] (see therein for the proof of Proposition 4.6.3 and [**12**, Lemma 15.2] for the proof of Lemma 4.6.4).

Proposition 4.6.3. *For every $A \in \mathbb{M}^{n\times n}_{sym}$ we have*

$$|A| = \sup_{\xi^1,\dots,\xi^n} \left( \sum_{i=1}^n |A\xi^i\cdot\xi^i|^2 \right)^{1/2},$$

*where the supremum is taken over all orthonormal bases $\xi^1,\dots,\xi^n$ of $\mathbb{R}^n$, or equivalently over the columns of all rotations $R \in O(n)$.*

We recall also the following localization lemma.

Lemma 4.6.4. *Let $\Lambda$ be a function defined on the family of open subsets of $\Omega$, which is superadditive on open sets with disjoint compact closure. Let $\lambda$ be a positive measure on $\Omega$, and let $\varphi_j$, $j \in \mathbb{N}$, be nonnegative Borel functions such that*

$$\int_K \varphi_j \,\mathrm{d}\lambda \le \Lambda(A)$$

*for every open set* $A\subset\Omega$*, for every compact set* $K\subset A$*, and for every* $j\in\mathbb{N}$*. Then*

$$\int_K \sup_j \varphi_j \,\mathrm{d}\lambda = \sup\Big\{\sum_{j=1}^r \int_{K_j} \varphi_j\,\mathrm{d}\lambda\colon (K_j)_{j=1}^r \textit{ disjoint compact subsets of } K, r\in\mathbb{N}\Big\}\le\Lambda(A)$$

*for every open set* $A\subset\Omega$ *and for every compact set* $K\subset A$*.*

Let us consider the following functionals, defined for every direction $\xi\in\mathbb{S}^{n-1}$: for every $\alpha\in H^1(\Omega;[0,1])$, $v\in BD(\Omega)$, and $A\subset\Omega$ open,

$$\mathcal{F}_\xi(\alpha,v,A):=\int_A V(\widetilde{\alpha})\,\mathrm{d}|\mathrm{E}v\,\xi\cdot\xi|\,\mathrm{d}x=\int_{\Pi^\xi}\Big(\int_{A^\xi_y}V(\widetilde{\alpha})^\xi_y\,\mathrm{d}|\mathrm{D}\widehat{v}^\xi_y|\Big)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,. \tag{4.6.9}$$

Notice that the second equality in the formula above follows from (4.6.7).

We first prove the lower semicontinuity of these functionals, and then we deduce Theorem 4.6.1 using Proposition 4.6.3 and Lemma 4.6.4.

Proposition 4.6.5. *Let* $\xi\in\mathbb{S}^{n-1}$ *and let* $\alpha_k$*,* $\alpha\in H^1(\Omega;[0,1])$*,* $v_k$*,* $v\in BD(\Omega)$ *such that* $\alpha_k\rightharpoonup\alpha$ *in* $H^1(\Omega)$ *and* $v_k\overset{*}{\rightharpoonup}v$ *in* $BD(\Omega)$*. Then*

$$\mathcal{F}_\xi(\alpha,v,A)\le\liminf_{k\to\infty}\mathcal{F}_\xi(\alpha_k,v_k,A) \tag{4.6.10}$$

*for every open set* $A\subset\Omega$*.*

Proof. Let $\xi\in\mathbb{S}^{n-1}$, $A\subset\Omega$ open, $\alpha_k\rightharpoonup\alpha$ in $H^1(\Omega)$ and $v_k\overset{*}{\rightharpoonup}v$ in $BD(\Omega)$, and let us fix $\varepsilon>0$. By (4.6.4), up to extract a subsequence (not relabeled), we can say that, for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$,

$$(\widetilde{\alpha_k})^\xi_y\to\widetilde{\alpha}^\xi_y\,,\quad(\widehat{v_k})^\xi_y\to\widehat{v}^\xi_y\quad\text{in } L^1(\Omega^\xi_y)\,, \tag{4.6.11}$$

and that the liminf in (4.6.10) (that we may assume finite) is actually a limit.

We claim that for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$

$$\int_{A^\xi_y}V(\widetilde{\alpha}^\xi_y)\,\mathrm{d}|\mathrm{D}\widehat{v}^\xi_y|\le\liminf_{k\to\infty}\Big[\int_{A^\xi_y}V((\widetilde{\alpha_k})^\xi_y)\,\mathrm{d}|\mathrm{D}(\widehat{v_k})^\xi_y|+\varepsilon\int_{A^\xi_y}|\nabla(\widetilde{\alpha_k})^\xi_y(t)|^2\,\mathrm{d}t\Big]\,. \tag{4.6.12}$$

Indeed, for any $y\in\Pi^\xi$ there exists a subsequence $k_j$, depending on $y$, such that the liminf in the above formula is a limit, which is not restrictive to assume to be finite. Since the function $V$ is bounded and $\varepsilon$ is fixed, for a fixed $y$ the sequences $(\widetilde{\alpha_{k_j}})^\xi_y$ and $(\widehat{v_{k_j}})^\xi_y$ are bounded in $H^1(\Omega^\xi_y)$ and $BV(\Omega^\xi_y)$, respectively. This implies that, for those $y\in\Pi^\xi$ such that (4.6.11) holds, we have

$$(\widetilde{\alpha_{k_j}})^\xi_y\overset{*}{\rightharpoonup}\widetilde{\alpha}^\xi_y\quad\text{in } H^1(\Omega^\xi_y)\,,\quad(\widehat{v_{k_j}})^\xi_y\rightharpoonup\widehat{v}^\xi_y\quad\text{in } BV(\Omega^\xi_y)\,.$$

In particular, by the properties of slices of $H^1$ functions, for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ we have that $(\widetilde{\alpha_{k_j}})^\xi_y\overset{*}{\rightharpoonup}\widetilde{\alpha}^\xi_y$ uniformly and then, by Lemma 2.1.1, it follows that

$$\begin{aligned}\int_{A^\xi_y}V(\widetilde{\alpha}^\xi_y)\,\mathrm{d}|\mathrm{D}\widehat{v}^\xi_y|&\le\liminf_{j\to\infty}\int_{A^\xi_y}V((\widetilde{\alpha_{k_j}})^\xi_y)\,\mathrm{d}|\mathrm{D}(\widehat{v_{k_j}})^\xi_y|\\&\le\liminf_{k\to\infty}\Big[\int_{A^\xi_y}V((\widetilde{\alpha_k})^\xi_y)\,\mathrm{d}|\mathrm{D}(\widehat{v_k})^\xi_y|+\varepsilon\int_{A^\xi_y}|\nabla(\widetilde{\alpha_k})^\xi_y(t)|^2\,\mathrm{d}t\Big]\,.\end{aligned}$$

Thus (4.6.12) is proven. Integrating in $\Pi^\xi$ and recalling (4.6.9) and (4.6.8), we deduce by Fatou Lemma that

$$\mathcal{F}_\xi(\alpha, v, A) \leq \liminf_{k\to\infty} \mathcal{F}_\xi(\alpha_k, v_k, A) + \varepsilon \limsup_{k\to\infty} \int_A |\nabla\alpha_k \cdot \xi|^2 \, \mathrm{d}x\,.$$

Since the sequence $\alpha_k$ is bounded in $H^1(\Omega)$ and $\varepsilon$ is arbitrary, the proof is concluded. □

We are now ready to prove the main result.

Proof of Theorem 4.6.1. Let $\xi^1, \dots, \xi^n$ be an orthonormal basis of $\mathbb{R}^n$, and let us prove first that, for every $\alpha \in H^1(\Omega; [0,1])$, $v \in BD(\Omega)$, and $A \subset \Omega$ open, it holds

$$\left(\sum_{i=1}^n \mathcal{F}_{\xi^i}(\alpha, v, A)^2\right)^{1/2} \leq \int_A V(\widetilde{\alpha})\, \mathrm{d}|\mathrm{E}v|\,. \tag{4.6.13}$$

Indeed

$$\begin{aligned}\mathcal{F}_{\xi^i}(\alpha, v, A)^2 &= \left(\int_A V(\widetilde{\alpha}) \left|\frac{\mathrm{dE}v}{\mathrm{d}|\mathrm{E}v|}\xi^i \cdot \xi^i\right| \mathrm{d}|\mathrm{E}v|\right)^2 \\ &\leq \left(\int_A V(\widetilde{\alpha}) \left|\frac{\mathrm{dE}v}{\mathrm{d}|\mathrm{E}v|}\xi^i \cdot \xi^i\right|^2 \mathrm{d}|\mathrm{E}v|\right) \int_A V(\widetilde{\alpha})\, \mathrm{d}|\mathrm{E}v|\,,\end{aligned}$$

by the Hölder inequality with respect to the measure $|\mathrm{E}u|$. Summing for $i = 1, \dots, n$, we obtain that

$$\left(\sum_{i=1}^n \mathcal{F}_{\xi^i}(\alpha, v, A)^2\right)^{1/2} \leq \left(\int_A V(\widetilde{\alpha}) \sum_{i=1}^n \left|\frac{\mathrm{dE}v}{\mathrm{d}|\mathrm{E}v|}\xi^i \cdot \xi^i\right|^2 \mathrm{d}|\mathrm{E}v|\right)^{1/2} \left(\int_A V(\widetilde{\alpha})\, \mathrm{d}|\mathrm{E}v|\right)^{1/2} \leq \int_A V(\widetilde{\alpha})\, \mathrm{d}|\mathrm{E}v|\,,$$

and thus (4.6.13) is proven. Notice that in the last inequality above we have used (4.6.3) and the fact that

$$\left|\frac{\mathrm{dE}v}{\mathrm{d}|\mathrm{E}v|}(x)\right| = 1 \quad \text{for } |\mathrm{E}v|\text{-a.e. } x \in \Omega\,. \tag{4.6.14}$$

Let $\alpha_k$, $\alpha \in H^1(\Omega; [0,1])$, $v_k$, $v \in BD(\Omega)$ such that $\alpha_k \rightharpoonup \alpha$ in $H^1(\Omega)$ and $v_k \overset{*}{\rightharpoonup} v$ in $BD(\Omega)$. Thus we have to prove (4.6.2) for these sequences. Let $\Lambda$ be the function defined on every open set $A \subset \Omega$ by

$$\Lambda(A) := \liminf_{k\to\infty} \int_A V(\widetilde{\alpha_k})\, \mathrm{d}|\mathrm{E}v_k|\,.$$

Moreover, let $R_j$ be a sequence dense in $O(n)$ and let $\xi^1_j, \dots, \xi^n_j$ be the column vectors of $R_j$. Let us define the vector functions $\varphi^j = (\varphi^j_1, \dots, \varphi^j_n)$ by putting

$$\varphi^j_i(x) := V(\widetilde{\alpha}(x)) \left|\frac{\mathrm{dE}v}{\mathrm{d}|\mathrm{E}v|}(x)\, \xi^i_j \cdot \xi^i_j\right| \quad \text{for every } j \in \mathbb{N},\, i = 1, \dots, n, \text{ and } x \in \Omega\,. \tag{4.6.15}$$

Recalling (4.6.9), it holds that for every $j \in \mathbb{N}$ and $A \subset \Omega$ open

$$\left|\int_A \varphi^j \, \mathrm{d}|\mathrm{E}v|\right| = \left(\sum_{i=1}^n \mathcal{F}_{\xi^i_j}(\alpha, v, A)^2\right)^{1/2}\,. \tag{4.6.16}$$

By Proposition 4.6.5, for every $j \in \mathbb{N}$, $i = 1, \dots, n$, and $A \subset \Omega$ open, we have that

$$\mathcal{F}_{\xi_j^i}(\alpha, v, A) \leq \liminf_{k\to\infty} \mathcal{F}_{\xi_j^i}(\alpha_k, v_k, A)\,,$$

and then, by the superadditivity of the liminf, it follows that

$$\left(\sum_{i=1}^n \mathcal{F}_{\xi_j^i}(\alpha, v, A)^2\right)^{1/2} \leq \liminf_{k\to\infty} \left(\sum_{i=1}^n \mathcal{F}_{\xi_j^i}(\alpha_k, v_k, A)^2\right)^{1/2},$$

By the previous inequality, (4.6.13), and (4.6.16) we obtain that (recall also the definition of $\Lambda$)

$$\left|\int_A \varphi^j \,\mathrm{d}|\mathrm{E}v|\right| \leq \Lambda(A)\,. \tag{4.6.17}$$

Using the superadditivity of $\Lambda$, we have that

$$\begin{aligned}\int_K |\varphi^j|\,\mathrm{d}|\mathrm{E}v| &= \sup\left\{\sum_{h=1}^r \left|\int_{K^h} \varphi^j \,\mathrm{d}|\mathrm{E}v|\right| \colon (K^h)_{h=1}^r \text{ disjoint compact subsets of } K,\ r\in\mathbb{N}\right\}\\ &\leq \sup\left\{\sum_{h=1}^r \Lambda(A^h) \colon (A^h)_{h=1}^r \text{ disjoint compact subsets of } A,\ r\in\mathbb{N}\right\} \leq \Lambda(A)\end{aligned}$$

for every compact set $K$ and for every open set $A$ such that $K \subset A \subset \Omega$. Lemma 4.6.4 gives that

$$\int_K \sup_j |\varphi^j|\,\mathrm{d}|\mathrm{E}v| \leq \Lambda(A)\,, \tag{4.6.18}$$

and by Proposition 4.6.3 (recall (4.6.14) and (4.6.15)) we deduce that

$$\int_K V(\widetilde{\alpha})\,\mathrm{d}|\mathrm{E}v| \leq \Lambda(A)\,,$$

for every compact set $K$ such that $K \subset A$. Therefore we conclude the proof by the arbitrariness of $K$ and by recalling the definition of $\Lambda$. ☐

The remaining part of the section concerns the relation between the semicontinuity result proved above and the lower semicontinuity of the plastic potential in the framework of perfect plasticity with damage, denoted in this chapter by $\mathcal{H}_{\mathrm{pp}}$. In particular, arguing as in the proof of Theorem 4.6.1, we can deduce the following result.

THEOREM 4.6.6. *Let* $\alpha_k$, $\alpha \in H^1(\Omega;[0,1])$ *and* $(v_k,\eta_k,q_k)$, $(v,\eta,q) \in A_{\mathrm{pp}}(w)$ *such that* $\alpha_k \rightharpoonup \alpha$ *in* $H^1(\Omega)$, $v_k \overset{*}{\rightharpoonup} v$ *in* $BD(\Omega)$ *and* $\eta_k \to \eta$ *strongly in* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. *Then*

$$\mathcal{H}_{\mathrm{pp}}(\alpha, q) \leq \liminf_{k\to\infty} \mathcal{H}_{\mathrm{pp}}(\alpha_k, q_k)\,,$$

*where* $A_{\mathrm{pp}}(w)$ *and* $\mathcal{H}_{\mathrm{pp}}$ *are defined in* (4.5.1) *and* (4.5.2), *respectively.*

PROOF. Let us fix $\alpha_k$, $\alpha$ and $(v_k,\eta_k,q_k)$, $(v,\eta,q)$ satisfying the assumptions of the theorem. Let us see first that it is not restrictive to consider the case when $q_k$ and $q$ does not charge $\partial\Omega$, namely to prove the lower semicontinuity property of the theorem for the functional

$$\int_\Omega S_1(\beta)\,\mathrm{d}|q|\,.$$

Indeed, let $\widetilde{\Omega}$ be a smooth open set such that $\overline{\Omega} \subset \widetilde{\Omega}$, and let us define for every $k$ (and for $(v, \eta, q)$)

$$\overline{v}_k := \begin{cases} v_k & \text{in } \Omega \\ w & \text{in } \widetilde{\Omega} \setminus \Omega \end{cases}, \quad \overline{\eta}_k := \begin{cases} \eta_k & \text{in } \Omega \\ \mathrm{E}w & \text{in } \widetilde{\Omega} \setminus \Omega \end{cases}, \quad \overline{q}_k := \begin{cases} q_k & \text{in } \overline{\Omega} \\ 0 & \text{in } \widetilde{\Omega} \setminus \overline{\Omega} \end{cases}. \tag{4.6.19}$$

Then $\mathrm{E}\overline{v}_k = \overline{\eta}_k + \overline{q}_k$ and $\mathrm{E}\overline{v} = \overline{\eta} + \overline{q}$ as measures in $M_b(\widetilde{\Omega}; \mathbb{M}^{n\times n}_D)$, $\overline{v}_k \overset{*}{\rightharpoonup} \overline{v}$ in $BD(\widetilde{\Omega})$ and $\overline{\eta}_k \to \overline{\eta}$ strongly in $L^2(\widetilde{\Omega}; \mathbb{M}^{n\times n}_{sym})$, and

$$\int_{\widetilde{\Omega}} S_1(\overline{\alpha}_k)\, \mathrm{d}|\overline{q}_k| = \mathcal{H}_{\mathrm{pp}}(\alpha_k, q_k)\,,$$

where $\overline{\alpha}$ is the $H^1$ extension of $\alpha$ to $\widetilde{\Omega}$. (Notice that the same holds for $\alpha$ and $q$ and that the formula above makes sense for the precise representative of $\overline{\alpha}$, but we did not write it explicitely.)

Since $\eta_k$ converge strongly to $\eta$, for every $\xi \in \mathbb{S}^{n-1}$ there exists a subsequence $\eta_{k_j}$ such that

$$(\eta_{k_j}\xi \cdot \xi)^\xi_y \to (\eta\,\xi \cdot \xi)^\xi_y \quad \text{in } L^1(\Omega^\xi_y) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi\,, \tag{4.6.20}$$

by Fubini Theorem. Indeed for every $\eta$, every $\xi \in \mathbb{S}^{n-1}$ and every Borel set $B \subset \Omega$

$$\int_B \eta\,\xi \cdot \xi\, \mathrm{d}x = \int_{\Pi^\xi} \Big( \int_{B^\xi_y} (\eta\,\xi \cdot \xi)^\xi_y(t)\, \mathrm{d}t \Big)\, \mathrm{d}\mathcal{H}^{n-1}(y)\,.$$

Let us define, for every direction $\xi \in \mathbb{S}^{n-1}$, every $\beta \in H^1(\Omega; [0,1])$, every $\widehat{q}$ such that $(\widehat{v}, \widehat{\eta}, \widehat{q}) \in A_{\mathrm{pp}}(w)$, and every $A \subset \Omega$ open,

$$\mathcal{G}_\xi(\beta, \widehat{q}, A) := \int_A V(\widetilde{\beta})\, \mathrm{d}|\widehat{q}\,\xi \cdot \xi|\, \mathrm{d}x = \int_A V(\widetilde{\beta})\, \mathrm{d}|(\mathrm{E}\widehat{v} - \widehat{\eta})\,\xi \cdot \xi|\,. \tag{4.6.21}$$

In view of (4.6.20), we can use a slicing argument as in Proposition 4.6.5 and deduce that for every $\xi$ and $A \subset \Omega$ open,

$$\mathcal{G}_\xi(\alpha, q, A) \le \liminf_{k\to\infty} \mathcal{G}_\xi(\alpha_k, q_k, A)\,. \tag{4.6.22}$$

Now the result follows from (4.6.22), Proposition 4.6.3 and Lemma 4.6.4, arguing as in the proof of Theorem 4.6.1 with small changes. (Now $\varphi^j_i := V(\widetilde{\alpha}) \Big| \frac{\mathrm{d}q}{\mathrm{d}|q|}\, \xi^i_j \cdot \xi^i_j \Big|$ and we use the fact that $\Big|\frac{\mathrm{d}q}{\mathrm{d}|q|}(x)\Big| = 1$ for $|q|$-a.e. $x \in \Omega$, instead of (4.6.14).) ☐

REMARK 4.6.7. It is easy to see that the lower semicontinuity result of Theorem 4.6.6 also holds when the Dirichlet boundary $\partial_D\Omega$ does not coincide with the whole $\partial\Omega$, and thus the plastic potential has the form in (2.1.13).

REMARK 4.6.8. In order to prove the existence of a globally stable quasistatic evolution for a model of perfect plasticity and damage similar to that in Chapter 2 but having an $H^1$-damage regularization, it would be enough to prove the lower semicontinuity of $\mathcal{H}_{\mathrm{pp}}$ when $v_k \overset{*}{\rightharpoonup} v$ in $BD(\Omega)$ and $\eta_k \rightharpoonup \eta$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ (only weakly). The main difficulty in this case is that it is not true that for every $\xi \in \mathbb{S}^{n-1}$ there exists a subsequence $\eta_{k_j}$ such that (4.6.20) holds. To have an idea of a counterexample, consider the functions $\psi_k \colon \mathbb{R}^2 \to \mathbb{R}$ given by $\psi_k(x,y) := \sin(ky)$, and their slices with respect to the direction $\xi = (1,0)$: the sequence $k_j$ such that $(\psi_{k_j})^\xi_y \to 0$ depends on $y \in \mathbb{R}$.

Therefore a possible strategy for the existence proof is to find an a priori bound on $\eta_k$ that guarantees the strong convergence in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Since the elasticity tensor $\mathbb{C}(\alpha)$ is equicoercive with respect to $\alpha \in [0,1]$, the strong convergence for $\eta_k$ would follow for instance by an uniform bound for the stresses $\sigma_k = \mathbb{C}(\alpha_k)\eta_k$ in $H^1_{\mathrm{loc}}(\Omega; \mathbb{M}^{n\times n}_{sym})$. In the framework of perfect plasticity, without damage, an a priori bound of this type for the stresses is proven in [**10**] and [**38**].

CHAPTER 5

# Quasistatic crack growth based on viscous approximation: a model with branching and kinking

### Overview of the chapter

This chapter is devoted to the viscous approximation of quasistatic crack growth, where the crack path is not prescribed a priori. In this framework, the previous results of Lazzaroni and Toader [**67**] are extended to a larger class $\mathcal{S}$ of cracks, introduced by Racca [**90**]. The cracks in $\mathcal{S}$ may have many connected components, each of them being the union of a certain number of branches that are regular curves of the type considered in [**67**].

The structure of the chapter is the following. In Section 5.1 we give the definition of the class of admissible cracks basing on the one introduced in [**90**]; we prove some properties that come useful in the rest of the chapter, in particular an estimate on the energy release rate. Section 5.2 contains the definition of the time-incremental problems and the statements of some results borrowed from [**90**]. In Section 5.3 we pass to the time-continuous limit as $k \to \infty$, obtaining a family of viscous evolutions; in particular we prove the viscous energy balance and further properties of the viscous solutions that are needed to pass to the limit as the viscous parameter $\varepsilon$ tends to zero. The latter passage to the limit is the subject of Section 5.4, where we study rescaled evolutions.

The results of this chapter, obtained in collaboration with Giuliano Lazzaroni, are contained in [**25**].

## 5.1. The admissible cracks

In the setting of antiplane elasticity, we consider a brittle body whose reference configuration is a cylinder $\Omega \times \mathbb{R} \subset \mathbb{R}^3$, with $\Omega \subset \mathbb{R}^2$ an open, bounded, connected, Lipschitz set. The deformations of the body are of the type

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + u(x_1, x_2))\,,$$

where $u\colon \Omega \to \mathbb{R}$ is the corresponding displacement. We assume that the body can be fractured, that it has a perfectly elastic behaviour outside the cracked region, and that no force is transmitted across the crack. We now define the class of admissible cracks, denoted by $\mathcal{S}$, basing on the one introduced in [**90**]: this allows us to consider cracks with branching and kinking.

Starting from an initial fracture $\Gamma_0 \in \mathcal{S}$, we study the evolution of cracks under the requirement that the displacement $u(t)$ is equal to a prescribed function $w(t)$ on the Dirichlet part of the boundary $\partial_D\Omega$, where $w \in AC([0,T]; H^1(\Omega))$. Given $t \in [0,T]$ and $\Gamma \in \mathcal{S}$, $u(t)\colon \Omega \setminus \Gamma \to \mathbb{R}$ is the unique minimum point $u(w(t);\Gamma)$ of the elastic energy $\frac{1}{2}\|\nabla u\|_2^2$ under the condition $u = w(t)$ on $\partial_D\Omega$. The corresponding elastic energy associated to the crack $\Gamma$ and to the boundary displacement $w(t)$ is

$$\mathcal{E}(w(t);\Gamma) := \min\left\{\frac{1}{2}\|\nabla u\|_2^2 \colon u \in H^1(\Omega\setminus\Gamma),\ u = w(t) \text{ on } \partial_D\Omega\right\} = \frac{1}{2}\|\nabla u(w(t);\Gamma)\|_2^2 .$$

In the framework of Griffith's theory [**50**], the energy dissipated to open a crack is proportional to the crack length. Normalizing the proportionality constant to 1, we define the total energy corresponding to $\Gamma$ and $w(t)$,

$$\mathcal{F}(w(t);\Gamma) := \mathcal{E}(w(t);\Gamma) + \mathcal{H}^1(\Gamma) . \tag{5.1.1}$$

We now describe the class of admissible cracks $\mathcal{S}$ and its main properties, basing on [**90**]. Every admissible crack is the union of curves in the class $\mathcal{R}_\eta$, introduced in [**66, 67**], and here slightly modified.

DEFINITION 5.1.1. Let $\eta > 0$. Let $\Gamma \subset \mathbb{R}^2$ be a simple curve of class $C^{1,1}$ such that $\Omega\setminus\Gamma$ is open and connected. Given an arc-length parametrization of $\Gamma$, $\gamma\colon [0,L] \to R^2$, we call $p_1 := \gamma(0)$ and $p_2 := \gamma(L)$ the *endpoints* of $\Gamma$. We say that $\Gamma \in \mathcal{R}_\eta$ if and only if

(a) $\mathcal{H}^1(\Gamma) > 0$ and $\Gamma \subset\subset \Omega$;

(b) for every $x \in \Gamma$ there exist two open balls $B_1$, $B_2$ of radius $\eta$, such that

$$(B_1 \cup B_2) \cap (\Gamma \cup \partial\Omega) = \emptyset \quad \text{and} \quad \overline{B}_1 \cap \overline{B}_2 = \{x\} ;$$

(c) we have that $\Gamma \cap (B_\eta(q_1) \cup B_\eta(q_2)) = \emptyset$, where

$$q_i = p_i + \eta\frac{\dot\gamma(p_i)}{|\dot\gamma(p_i)|} \quad \text{for } i = 1, 2 .$$

In order to account for branching and kinking, it is convenient to introduce two types of neighborhoods of a curve $\Gamma \in \mathcal{R}_\eta$. They depend on two parameters

$$\beta \in (0, \eta/3) \quad \text{and} \quad \theta \in (0, \pi/4)$$

fixed throughout the chapter.

Let $\Gamma \in \mathcal{R}_\eta$, $\gamma\colon [0,L] \to \mathbb{R}^2$ be its arc-length parametrization, and $\dot\gamma(s)^\perp$ be normal to $\dot\gamma(s)$ with $|\dot\gamma(s)^\perp| = 1$. We define

$$\begin{aligned}\mathcal{P}_1(\Gamma, p) := &\left\{\gamma(s) + z\dot\gamma(s)^\perp \colon 0 < s \le L,\ |z| < \min\{s\tan\theta, \beta\}\right\}\\ &\cup \left\{\gamma(L) + (s-L)\dot\gamma(L) + z\dot\gamma(L)^\perp \colon L \le s < L+\beta,\ |z| < \min\{s\tan\theta, \sqrt{\beta^2 - (s-L)^2}\}\right\},\end{aligned}$$

where $p = \gamma(0)$, and

$$\mathcal{P}_2(\Gamma) := \left\{\gamma(s) + z\dot\gamma(s)^\perp \colon 0 < s < L,\ |z| < \min\{s\tan\theta, \beta, (L-s)\tan\theta\}\right\} .$$

Notice that $\mathcal{P}_1(\Gamma, p)$ and $\mathcal{P}_2(\Gamma)$ are neighborhoods of $\gamma((0, L])$ and $\gamma((0, L))$, respectively. We refer to them as the 1-sided and the 2-sided *pencil-like neighborhoods* of $\Gamma$, respectively. Moreover, two curves $\Gamma_1$, $\Gamma_2 \in \mathcal{R}_\eta$ may intersect at most in the endpoints of $\Gamma_1$ if $\mathcal{P}_2(\Gamma_1)\cap\Gamma_2 = \emptyset$, and at most in $p$ if $\mathcal{P}_1(\Gamma_1, p) \cap \Gamma_2 = \emptyset$.

We introduce a class $\widehat{\mathcal{S}}$ of connected sets, that are union of elements of $\mathcal{R}_\eta$.

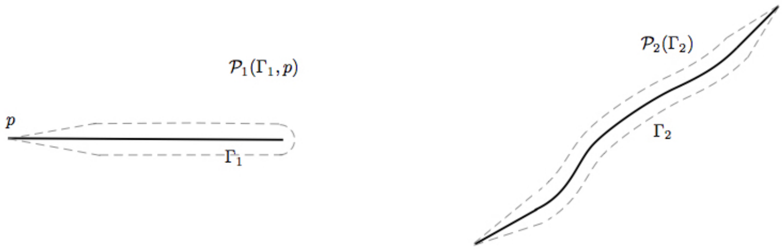


**Figure 1.** The pencil-like neighborhoods

Definition 5.1.2. The class $\widehat{\mathcal{S}}$ is given by the connected sets $K \subset \mathbb{R}^2$ such that $\Omega \setminus K$ is open and connected,

$$\mathcal{H}^1(K) \geq \frac{\beta}{\tan\theta}, \tag{5.1.2}$$

with $m \in \mathbb{N}$, and $K$ has the form

$$K = \bigcup_{j=1}^{m} \widetilde{K}_j$$

where the following hold:

(1) $\widetilde{K}_j \in \mathcal{R}_\eta$ for every $j$;
(2) if $\widetilde{K}_i \cap \widetilde{K}_j \neq \emptyset$ for $i \neq j$, then they intersect in one of their endpoints;
(3) if $\widetilde{K}_i \cup \widetilde{K}_j \in \mathcal{R}_\eta$, then there exists $\widetilde{K}_l$, $l \neq i, j$, such that $\widetilde{K}_i \cap \widetilde{K}_j \cap \widetilde{K}_l \neq \emptyset$;
(4) let $p_0$, $p_1$ be the endpoints of $\widetilde{K}_j$; if $p_0 \in \widetilde{K}_j \cap \widetilde{K}_{l_0}$ for some $l_0 \neq j$ and $p_1 \notin \widetilde{K}_l$ for any $l \neq j$, then
$$\mathcal{P}_1(\widetilde{K}_j, p_0) \cap \widetilde{K}_l = \emptyset \qquad \text{for every } l \neq j;$$
(5) let $p_0$, $p_1$ be the endpoints of $\widetilde{K}_j$; if $p_0 \in \widetilde{K}_j \cap \widetilde{K}_{l_0}$ and $p_1 \in \widetilde{K}_j \cap \widetilde{K}_{l_1}$ for some $l_0, l_1 \neq j$, then
$$\mathcal{P}_2(\widetilde{K}_j) \cap \widetilde{K}_l = \emptyset \qquad \text{for every } l \neq j;$$

We call any $\widetilde{K}_j$ a *branch* of $K$, and we define $I_1(K)$ and $I_2(K)$ as the sets of branches of $K$ satisfying the assumptions in (4) and (5), respectively.

Remark 5.1.3. It is possible to see that there exists a modulus of continuity $\omega$ (i.e., a continuous nondecreasing function $\omega\colon [0,+\infty) \to [0,+\infty)$ with $\omega(0) = 0$) such that the following holds: given $\Gamma \in \mathcal{R}_\eta$ and its arc-length parametrization $\gamma\colon [0,L] \to \mathbb{R}^2$

$$\overline{B_{\omega(s)}(\gamma(s))} \subset \mathcal{P}_1(\Gamma, \gamma(0)) \quad \text{and} \quad \overline{B_{\omega(s)\wedge\omega(L-s)}(\gamma(s))} \subset \mathcal{P}_2(\Gamma) \quad \text{for every } s \in (0,L).$$

For future convenience, without loss of generality we assume that $\omega(s) < s$ for $s > 0$.

Every admissible crack is the union of sets $K$ as in Definition 5.1.2, with some geometric restrictions.

Definition 5.1.4. Let $\Gamma$ be a set of the form

$$\Gamma = \bigcup_{j=1}^{N} K_j \tag{5.1.3}$$

with $K_j \in \widehat{\mathcal{S}}$ and $N \in \mathbb{N}$, and let us define

- the set of the *special points of* $\Gamma$

$$\mathrm{S}_\Gamma := \{x \in \Gamma \colon \exists\, v_1,\, v_2 \in \mathbb{R}^2 \text{ unit vectors tangent to } \Gamma \text{ at } x \text{ s.t. } v_1 \cdot v_2 \neq \pm 1\}\,;$$

- the set of the *crack tip points of* $\Gamma$

$$\mathrm{T}_\Gamma := \{x \in \Gamma \colon \exists\, r > 0 \text{ s.t. } \Gamma \cap \overline{B_r(x)} \in \mathcal{R}_\eta \text{ and } x \text{ is an endpoint of } \Gamma \cap \overline{B_r(x)}\}\,;$$

- the set of the *regular points of* $\Gamma$

$$\mathrm{R}_\Gamma := \Gamma \setminus (\mathrm{T}_\Gamma \cup \mathrm{S}_\Gamma) = \{x \in \Gamma \colon \exists\, r > 0 \text{ s.t. } \Gamma \cap \overline{B_r(x)} \in \mathcal{R}_\eta \text{ with } x \text{ in the relative interior of } \Gamma\}\,.$$

We say that $\Gamma \in \mathcal{S}$ if

(1) for every $j \in \{1, \dots, N\}$,

$$\begin{aligned}
&\text{if } K_j \in \mathcal{R}_\eta\,, \text{ then } \mathrm{d}(K_j, K_m) \geq \beta \text{ for } m \neq j\,;\\
&\text{if } \widetilde{K} \in I_1(K_j) \text{ and } p_0 \text{ is its endpoint s.t. } p_0 \in \mathrm{S}_\Gamma\,, \text{ then } \mathcal{P}_1(\widetilde{K}, p_0) \cap K_m = \emptyset \text{ for } m \neq j\,,\\
&\text{if } \widetilde{K} \in I_2(K_j)\,, \text{ then } \mathcal{P}_2(\widetilde{K}) \cap K_m = \emptyset \text{ for } m \neq j\,;
\end{aligned} \tag{5.1.4}$$

(2) for every $x_1 \neq x_2$ in $\mathrm{S}_\Gamma$,

$$|x_1 - x_2| \geq \beta \left( \frac{2}{\tan\theta} + 1 \right)\,. \tag{5.1.5}$$

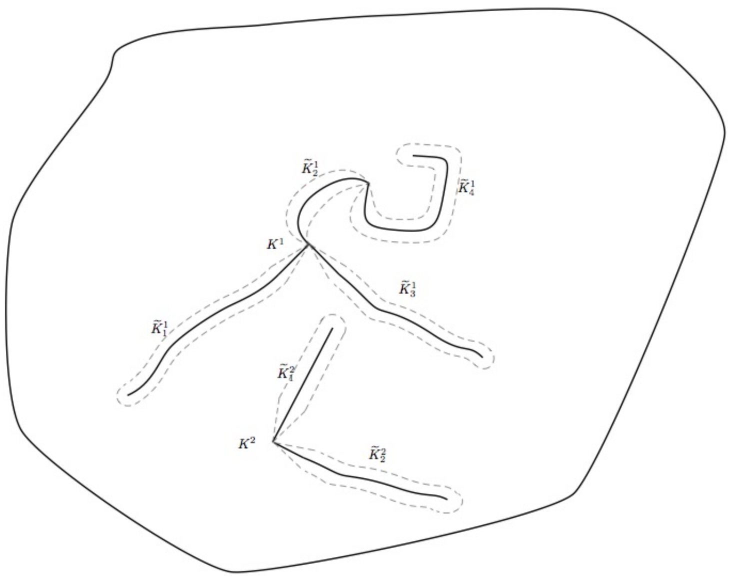


**Figure 2.** A crack $\Gamma \in \mathcal{S}$ with two connected components $K_1$ and $K_2$, with $K_i = \bigcup_j \widetilde{K}_i^j$. The pencil-like neighborhoods are delimited by dashed lines. Due to the kinked shape of the 2-sided (resp., 1-sided) pencil-like neighborhoods around both (resp., one of) the endpoints, the branching phenomenon is allowed, but there is a restriction on the number of branches. Moreover, the conditions (2) and (3) in Definition 5.1.2 describe a sort of "maximality" of each branch in the class $\mathcal{R}_\eta$ with respect to inclusion. Indeed, $\widetilde{K}_1^2 \cup \widetilde{K}_1^3 \in \mathcal{R}_\eta$ but we have two different branches $\widetilde{K}_1^2$ and $\widetilde{K}_1^3$ due to the presence of $\widetilde{K}_1^1$.

It turns out that the sets $K_j$ as in (5.1.3) are the connected components of $\Gamma$. We further underline that, if $\widetilde{K} \in I_1(K_j)$, then one of its endpoints belongs to $\mathrm{S}_\Gamma$ and the other one to $\mathrm{T}_\Gamma$. Indeed, $\mathrm{T}_\Gamma$ consists of the endpoints of the type just described and of all the endpoints of the connected components of $\Gamma$ that belong to $\mathcal{R}_\eta$.

Notice that, for every $\Gamma \in \mathcal{S}$, $\Omega \setminus \Gamma$ is connected. Indeed, $\Omega \setminus K_j$ is connected for every connected component $K_j$ of $\Gamma$, by Definition 5.1.2, and the sets $K_j$ are pairwise disjoint, by conditions (5.1.4) and (5.1.5). On the other hand, if $\Omega\setminus\Gamma$ is connected, then $\Omega\setminus K$ is connected for every connected component $K$ of $\Gamma$.

Our definition of $\mathcal{S}$ is slightly different with respect to the one in [**90**]: indeed, we have prescribed that $\Omega \setminus \Gamma$ is connected, for every $\Gamma \in \mathcal{S}$. We then have to check that this further condition is preserved under Hausdorff convergence of curves in $\mathcal{S}$. See [**90**, Lemma 4.1] for similar properties.

DEFINITION 5.1.5. Given two compact subsets $\Gamma, \Gamma' \subset \overline{\Omega}$, their *Hausdorff distance* is given by

$$\mathrm{d}_H(\Gamma';\Gamma) := \max\left\{\sup_{x\in\Gamma'} \mathrm{dist}(x,\Gamma), \sup_{x\in\Gamma} \mathrm{dist}(x,\Gamma')\right\},$$

with the conventions $\mathrm{dist}(x,\emptyset) = \mathrm{diam}\,\Omega$ and $\sup\emptyset = 0$. A sequence $\Gamma_k$ of compact subsets of $\overline{\Omega}$ *converges to* $\Gamma$ *in the Hausdorff metric* if $\mathrm{d}_H(\Gamma_k;\Gamma) \to 0$. In this case we write $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$.

PROPOSITION 5.1.6. *Let* $\Gamma_k \in \mathcal{S}$ *be such that* $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$. *Then* $\Omega \setminus \Gamma$ *is connected.*

PROOF. We first observe that, by Definition 5.1.1 and [**66**, Proposition 2.9], the class $\mathcal{R}_\eta$ is closed. Therefore if $\Gamma_k \in \mathcal{R}_\eta$ for every $k$, then $\Omega \setminus \Gamma$ is connected. Moreover, by [**90**, Lemma 3.9], the total number of branches of $\Gamma_k$ is equibounded in $k$ (see Definition 5.1.2 for the definition of branches).

By contradiction, assume that there exists an open connected set $\Omega' \subset\subset \Omega$ such that $\partial\Omega' \subset \Gamma$. Then there exist $x \in \partial\Omega'$, $\widetilde{K}^1_k$, $\widetilde{K}^2_k$ different branches of $\Gamma_k$, and $x^1_k \in \widetilde{K}^1_k$, $x^2_k \in \widetilde{K}^2_k$, such that

$$x^1_k, x^2_k \to x \tag{5.1.6}$$

and $\mathcal{H}^1(\widetilde{K}^1_k) > C_0$, $\mathcal{H}^1(\widetilde{K}^2_k) > C_0$, for a positive constant $C_0$ independent of $k$. Since $\Omega'$ is open and connected, we may assume that there exists $C_1 > 0$, independent of $k$, such that

$$\mathrm{d}(x, \widetilde{K}^1_k \cap \widetilde{K}^2_k) \geq C_1\,. \tag{5.1.7}$$

Notice that either $\widetilde{K}^1_k \cap \widetilde{K}^2_k$ is empty, or it contains only one point, which belongs to $\mathrm{S}_{\Gamma_k}$. In particular $x^1_k \neq x^2_k$.

We claim that, up to subsequences, there exists a positive constant $C_2$ such that

$$\mathrm{d}(x^1_k, \mathrm{S}_{\Gamma_k} \cap \widetilde{K}^1_k) \geq C_2 \quad \text{or} \quad \mathrm{d}(x^2_k, \mathrm{S}_{\Gamma_k} \cap \widetilde{K}^2_k) \geq C_2\,. \tag{5.1.8}$$

Indeed, by contradiction, let $\mathrm{d}(x^i_k, \mathrm{S}_{\Gamma_k} \cap \widetilde{K}^i_k) \to 0$ for $i = 1,2$, and let $y^i_k \in \mathrm{S}_{\Gamma_k} \cap \widetilde{K}^i_k$ with $|x^i_k - y^i_k| = \mathrm{d}(x^i_k, \mathrm{S}_{\Gamma_k} \cap \widetilde{K}^i_k)$. Notice that $y^1_k, y^2_k \to x$. If $y^1_k = y^2_k =: y_k$, then $y_k \in \widetilde{K}^1_k \cap \widetilde{K}^2_k$ and $y_k \to x$, in contradiction with (5.1.7). On the other hand, if $y^1_k \neq y^2_k$, we have that (5.1.5) is contradicted, since $|y^1_k - y^2_k| \to 0$. Then (5.1.8) is proved.

Assume that $x_k^1$ and $\widetilde{K}_k^1$ satisfy (5.1.8), and let $\gamma_k^1$ be an arc-length parametrization of $\widetilde{K}_k^1$. By Remark 5.1.3, we have that $\mathrm{d}(x_k^1, \widetilde{K}_k') \geq \omega(C_2)$ for any branch $\widetilde{K}_k'$ of $\Gamma_k$ different from $\widetilde{K}_k^1$. In particular,

$$|x_k^1 - x_k^2| \geq \omega(C_2)$$

for every $k$, in contradiction with (5.1.6). Therefore the result is proved. □

In the following proposition we collect the most important properties of the class of admissible cracks. These can be proved following the same arguments as in **[90]**. The property of compactness of $\mathcal{S}$ with respect to the Hausdorff convergence employs Proposition 5.1.6.

PROPOSITION 5.1.7. *The class $\mathcal{S}$ introduced in Definition 5.1.4 is compact with respect to the Hausdorff convergence, and the length of the admissible cracks is uniformly bounded, as well as the number of the branches, of the singular points and of the tip points. In particular, for every $\Gamma \in \mathcal{S}$, $\Omega \setminus \Gamma$ is the union of a uniformly bounded number of Lipschitz sets that intersect $\partial_D \Omega$.*

*Moreover, if $\Gamma_k \in \mathcal{S}$ are such that $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$, then*

*(i) $\mathcal{H}^1(\Gamma_k) \to \mathcal{H}^1(\Gamma)$;*

*(ii) for every $p \in \mathrm{T}_\Gamma$ there exists a sequence $(p_k)_k$ with $p_k \in \mathrm{T}_{\Gamma_k}$ such that $p_k \to p$;*

*(iii) if $p_k^1, p_k^2 \in \mathrm{T}_{\Gamma_k}$, $p_k^1 \neq p_k^2$ and $(p_k^1)_k$, $(p_k^2)_k$ are converging to $p \in \mathrm{T}_\Gamma$, then there exists a sequence $(y_k)_k$, with $y_k \in \mathrm{S}_{\Gamma_k}$, converging to $p$.*

We can follow the arguments of **[66]** in order to define the notion of energy release rate relative to a crack tip. First, let us introduce the extensions of a crack near a tip. In the following discussion, we fix $\Gamma \in \mathcal{S}$, $p \in \mathrm{T}_\Gamma$, and $r > 0$ as in the definition of crack tip.

DEFINITION 5.1.8. We call *extension of* $\Gamma$ *at* $p$ any $\widetilde{\Gamma} \in \mathcal{S}$ such that $\Gamma \subsetneq \widetilde{\Gamma}$,

$$\widetilde{\Gamma} \setminus \Gamma \subset\subset B_r(p) \quad \text{and} \quad \widetilde{\Gamma} \cap \overline{B_r(p)} \in \mathcal{R}_\eta \quad \text{for some } r\,.$$

REMARK 5.1.9. In the general case, there could exist points $p$ such that there are not extensions of $\Gamma$ at $p$. We denote

$$\mathrm{G}_\Gamma := \{p \in \mathrm{T}_\Gamma \colon \text{there are extensions of } \Gamma \text{ at } p\}\,. \tag{5.1.9}$$

If $\widetilde{\Gamma}$ is an extension of $\Gamma$ at $p$, let $L := \mathcal{H}^1(\widetilde{\Gamma} \setminus \Gamma)$ and let $\widetilde{\gamma}_p \colon [0, L] \to \Omega$ be the arc-length parametrization of $(\widetilde{\Gamma} \setminus \Gamma) \cup \{p\}$ such that $\widetilde{\gamma}_p(0) = p$. Then

$$(0, L] \ni s \mapsto \widetilde{\Gamma}_s^p := \Gamma \cup \widetilde{\gamma}_p((0, s])$$

is a family of extensions of $\Gamma$ in $p$ such that $\mathcal{H}^1(\widetilde{\Gamma}_s^p \setminus \Gamma) = s$.

We also use the following notation:

$$\Gamma_{p,r} := \Gamma \cap \overline{B_r(p)} \in \mathcal{R}_\eta \quad \text{with} \quad \Gamma_{p,r} \cap \partial B_r(p) =: \{q\}\,. \tag{5.1.10}$$

Let $p \in \mathrm{G}_\Gamma$ and let $\widetilde{\Gamma}$ be an extension of $\Gamma$ in $p$. It holds that

$$\widetilde{\Gamma}_{p,r} := \widetilde{\Gamma} \cap \overline{B_r(p)} \in \mathcal{R}_\eta \quad \text{with} \quad \widetilde{\Gamma}_{p,r} \cap \partial B_r(p) = \{q\}\,.$$

Let $\widetilde{\gamma}_{p,r}\colon [0, l(\widetilde{\Gamma}_{p,r})] \to \Omega$ be the arc-length parametrization of $\widetilde{\Gamma}_{p,r}$ such that $\gamma'_{p,r}(0) = q$. As showed in [**66**], the function

$$l \mapsto \mathcal{E}(w(t); \Gamma \cup \widetilde{\gamma}_{p,r}([0,l])$$

is differentiable at $l = \mathcal{H}^1(\Gamma_{p,r})$ and the value of the derivative is independent of the choice of the extension $\widetilde{\Gamma}$. In order to see these properties, one employs the Poincaré inequality in $\Omega \setminus (\Gamma \cup \widetilde{\Gamma}_{p,r})$, which holds since for every $\Gamma \subset \mathcal{S}$, $\Omega \setminus \Gamma$ is the union of a fixed number of Lipschitz sets that intersect $\partial_D \Omega$. Then the following definition is well posed.

Definition 5.1.10. Let $p \in \mathrm{G}_\Gamma$. The *energy release rate relative to* $w(t)$, $p$, and $\Gamma$ is

$$\mathcal{G}(w(t); \Gamma, p) := -\partial_l \mathcal{E}(w(t); \Gamma \cup \widetilde{\gamma}_{p,r}([0,l]))|_{l=\mathcal{H}^1(\Gamma_{p,r})}\,.$$

**Notation.** In the Sections 3, 4, 5 we will use for every $t \in [0,T]$ the notation $\mathcal{E}(t;\Gamma)$, $\mathcal{F}(t;\Gamma)$, and $\mathcal{G}(t;\Gamma,p)$ respectively for $\mathcal{E}(w(t);\Gamma)$, $\mathcal{F}(w(t);\Gamma)$, and $\mathcal{G}(w(t);\Gamma,p)$.

The following integral representation was proven in [**66**, Propositions 2.2 and 2.4].

Proposition 5.1.11. *Let* $\Gamma \in \mathcal{S}$, $p \in \mathrm{G}_\Gamma$ *and* $r > 0$ *such that*

$$\Gamma \cap \overline{B_r(p)} \in \mathcal{R}_\eta\,.$$

*Let* $\gamma$ *be the arc-length parametrization of* $\Gamma \cap \overline{B_r(p)} \in \mathcal{R}_\eta$ *with* $p = \gamma(L)$, $L = \mathcal{H}^1(\Gamma \cap \overline{B_r(p)})$. *Then*

$$\mathcal{G}(g;\Gamma,p) = \int_{\Omega\setminus\Gamma} \left[ \frac{(D_1 u)^2 - (D_2 u)^2}{2}(D_1 V^1 - D_2 V^2) + D_1 u\, D_2 u\, (D_2 V^1 + D_1 V^2) \right] \mathrm{d}x\,, \quad (5.1.11)$$

*where* $V$ *is any vector field of class* $C^{0,1}$ *with compact support in* $\Omega$ *such that* $V(\gamma(s)) = \dot{\gamma}(s)$ *for* $s$ *in a neighborhood of* $L$, *and* $u = u(g;\Gamma)$ *is the unique minimum point of the elastic energy with boundary condition* $g$ *on* $\partial_D\Omega$.

The integral representation allows us to deduce the fundamental continuity properties of the energy release rate with respect to the convergence of the curves, of the tips, and of the boundary displacements, provided that condition (5.1.10) holds uniformly.

Proposition 5.1.12. *Let* $\Gamma_0, \Gamma_k, \Gamma \in \mathcal{S}$ *with* $\Gamma_0 \subset \Gamma_k$, $\Gamma_0 \subset \Gamma$. *Moreover, let* $g_k \to g$ *in* $H^1(\Omega \setminus \Gamma_0)$ *and* $p_k \in \mathrm{G}_{\Gamma_k}$, $p \in \mathrm{G}_\Gamma$.

*Assume that* $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$, $p_k \to p$, *and that there exists* $r > 0$ *such that*

$$\Gamma_k \cap \overline{B_r(p)} \in \mathcal{R}_\eta\,. \quad (5.1.12)$$

*Then*

$$\mathcal{G}(g_k;\Gamma_k,p_k) \to \mathcal{G}(g;\Gamma,p)$$

*and there exists a positive constant* $C(\eta,r)$, *where* $\eta$ *and* $r$ *are as in* (5.1.12), *such that*

$$\mathcal{G}(g_k;\Gamma_k,p_k) \le C(\eta,r) \sup_k \|\nabla u_k\|_2^2\,. \quad (5.1.13)$$

Proof. Since $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$ and the class $\mathcal{R}_\eta$ is closed with respect to Hausdorff convergence, we get that $\Gamma_k \cap \overline{B_r(p)} \xrightarrow{\mathcal{H}} \Gamma \cap \overline{B_r(p)}$ and $\Gamma \cap \overline{B_r(p)} \in \mathcal{R}_\eta$.

Following the lines of [**66**, Theorem 2.12] and [**90**, Lemma 8.2], we extend $\Gamma_k \cap \overline{B_r(p)}$, for every $k$, and $\Gamma \cap \overline{B_r(p)}$ with a segment following the tangent direction to the curve at the tips $p_k$ and $p$. By the Implicit Function Theorem, there exist a neighborhood $U \subset B_r(p)$ of $p$ and two suitable coordinate axes such that the extended curves are parametrized in $U$ by $(x_1, \varphi_k(x_1))$ and $(x_1, \varphi(x_1))$, with $\varphi_k$, $\varphi$ of class $C^{1,1}$. Notice that, by definition of $\mathcal{R}_\eta$, we can take $U = B_{\eta\wedge r}(p)$. Indeed, if $K \in \mathcal{R}_\eta$ and $B$ is a ball of radius $\eta$, there are at most two points of $K$ such that the tangent vectors to $K$ at these points are orthogonal.

We now set

$$V_k(x) := \zeta(x)(1, \dot{\varphi}_k(x_1)), \quad V(x) := \zeta(x)(1, \dot{\varphi}(x_1)),$$

with $\zeta$ a cutoff function supported in $U$. Thus, by (5.1.11)

$$\mathcal{G}(g_k; \Gamma_k, p_k) = \int_{\Omega\setminus\Gamma_k} \left[\frac{(D_1 u_k)^2 - (D_2 u_k)^2}{2}(D_1 V_k^1 - D_2 V_k^2) + D_1 u_k\, D_2 u_k\, (D_2 V_k^1 + D_1 V_k^2)\right] \mathrm{d}x,$$

with $u_k := u(g_k; \Gamma_k)$, and an analogous identity holds for $\mathcal{G}(g; \Gamma, p)$.

Since $\Gamma_k \cap \overline{B_r(p)} \xrightarrow{\mathcal{H}} \Gamma \cap \overline{B_r(p)}$ and these are elements of $\mathcal{R}_\eta$, we get that

$$\nabla V_k \xrightharpoonup{*} \nabla V \quad \text{in } L^\infty(\Omega; \mathbb{R}^{2\times 2}).$$

Notice that there exists a positive constant $C$, depending only on $\eta$ and $r$, such that

$$|\nabla V_k| \le C, \tag{5.1.14}$$

because $\ddot{\gamma}(s)$ is bounded by $\frac{1}{\eta}$ and $\nabla\zeta$ is controlled in terms of $r$ and $\eta$, since $U = B_{\eta\wedge r}(p)$.

By [**36**, Theorem 5.1] and the Poincaré inequality, $\nabla u_k \to \nabla u$ in $L^2(\Omega; \mathbb{R}^2)$. Therefore we can pass to the limit in the identity above as $k \to \infty$. The inequality (5.1.13) follows from (5.1.14). □

## 5.2. The time-incremental problems

In this section we recall the construction of discrete-time approximated evolutions of viscous type, already presented in [**90**]. We fix a subdivision of the time interval in $k+1$ equispaced nodes and a viscosity parameter $\varepsilon > 0$, and we solve incremental minimum problems on the class $\mathcal{S}$, thus allowing for new branches and kinks. The results in [**90**, Section 4] provide some a priori estimates, useful in order to pass to the limit as $k \to \infty$ to continuous-time viscous evolutions, and a discrete Griffith principle. In Section 5.3 we show new results on the viscous solutions, which permit to pass to the limit as $\varepsilon \to 0$ in Section 5.4.

We fix a sequence of subdivisions of the interval $[0, T]$ consisting of equispaced nodes $(t_k^i)_{0\le i\le k}$,

$$t_k^i := \tfrac{i}{k} T. \tag{5.2.1}$$

We put $\tau := \frac{1}{k}$ and we define

$$\Gamma^0_{\varepsilon,k} := \Gamma_0 \in \mathcal{S}, \qquad u^0_{\varepsilon,k} := u(w(0); \Gamma_0) \equiv u_0,$$

and $(\Gamma^i_{\varepsilon,k}, u^i_{\varepsilon,k})$ as a solution to the minimum problem

$$\min\Big\{\frac{1}{2}\|\nabla u\|_2^2+\mathcal{H}^1(\Gamma)+\frac{\varepsilon}{2\tau}\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma^{i-1}_{\varepsilon,k},\Gamma)}\mathcal{H}^1(\mathfrak{c})^2\ :\ \Gamma\in\mathcal{S},\,\Gamma\supset\Gamma^{i-1}_{\varepsilon,k},\,u\in H^1(\Omega\setminus\Gamma),\,u=w(t)\ \text{ on }\partial_D\Omega\Big\}\,, \tag{5.2.2}$$

where $\mathcal{C}(\Gamma_1,\Gamma_2)$ is the set of the connected components of $\Gamma_2\setminus\Gamma_1$ for $\Gamma_1\subset\Gamma_2\in\mathcal{S}$. Equivalently one can define $\Gamma^i_{\varepsilon,k}$ as a solution to

$$\min_{\substack{\Gamma\supset\Gamma^{i-1}_{\varepsilon,k}\\ \Gamma\in\mathcal{S}}}\Big\{\mathcal{E}(t^i_k;\Gamma)+\mathcal{H}^1(\Gamma)+\frac{\varepsilon}{2\tau}\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma^{i-1}_{\varepsilon,k},\Gamma)}\mathcal{H}^1(\mathfrak{c})^2\Big\}\,, \tag{5.2.3a}$$

and

$$u^i_{\varepsilon,k}:=u(w(t^i_k);\Gamma^i_{\varepsilon,k})\,.$$

By [**90**, Lemma 4.1 and Proposition 4.2] (recall also Proposition 5.1.7), problem (5.2.3) has a solution. Let us define the piecewise constant interpolations

$$u_{\varepsilon,k}(t):=u^i_{\varepsilon,k}\,,\quad \Gamma_{\varepsilon,k}(t):=\Gamma^i_{\varepsilon,k}\,,\quad \underline{l}_{\varepsilon,k}(t):=\mathcal{H}^1(\Gamma^i_{\varepsilon,k})\qquad\text{for } t\in[t^i_k,t^{i+1}_k)\,,$$

and the piecewise affine interpolation

$$l_{\varepsilon,k}(t):=\mathcal{H}^1(\Gamma^i_{\varepsilon,k})+\frac{t-t^i_k}{\tau}\mathcal{H}^1(\Gamma^{i+1}_{\varepsilon,k}\setminus\Gamma^i_{\varepsilon,k})\qquad\text{for } t\in[t^i_k,t^{i+1}_k)\,,$$

with $u_{\varepsilon,k}(T):=u^k_{\varepsilon,k}$, $\Gamma_{\varepsilon,k}(T):=\Gamma^k_{\varepsilon,k}$, and $l_{\varepsilon,k}(T)=\underline{l}_{\varepsilon,k}(T):=\mathcal{H}^1(\Gamma_{\varepsilon,k}(T))$. Let us set also

$$\mathrm{T}_{\varepsilon,k}(t):=\mathrm{T}_{\Gamma_{\varepsilon,k}(t)}\,,\quad \mathrm{S}_{\varepsilon,k}(t):=\mathrm{S}_{\Gamma_{\varepsilon,k}(t)}\,,\quad \mathrm{R}_{\varepsilon,k}(t):=\mathrm{R}_{\Gamma_{\varepsilon,k}(t)}\,,\quad \mathrm{G}_{\varepsilon,k}(t):=\mathrm{G}_{\Gamma_{\varepsilon,k}(t)}\,.$$

As usual, a priori bounds are derived by comparing the minimum value of the functional in (5.2.2) with the one assumed for the admissible pair $(\Gamma^{i-1}_{\varepsilon,k}, u^{i-1}_{\varepsilon,k}+w(t^i_k)-w(t^{i-1}_k))$. By standard computations, and recalling that the number of connected components of curves in $\mathcal{S}$ is uniformly bounded, one gets the following estimates.

Proposition 5.2.1. *For every $\varepsilon$, $k$, and $t\in[t^i_k,t^{i+1}_k)$,*

$$\begin{aligned}\mathcal{E}(t^i_k;\Gamma_{\varepsilon,k}(t))+\mathcal{H}^1(\Gamma_{\varepsilon,k}(t))+\frac{\varepsilon}{2\tau}\sum_{j=1}^{i}\Bigg[\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma^{j-1}_{\varepsilon,k},\Gamma^j_{\varepsilon,k})}\mathcal{H}^1(\mathfrak{c})^2\Bigg]&\\ \le\mathcal{E}(0;\Gamma_0)+l_0+\int_0^{t^i_k}\langle\nabla u_{\varepsilon,k}(s),\nabla\dot w(s)\rangle\,\mathrm{d}s+\delta(k)\,,&\end{aligned}$$

*where*

$$\delta(k)=\sup_{1\le i\le k}\Big(\int_{t^i_k-1}^{t^i_k}\|\nabla\dot w(s)\|_2\,\mathrm{d}s\Big)\int_0^T\|\nabla\dot w(s)\|_2\,\mathrm{d}s\to 0\quad\text{as } k\to\infty\,.$$

*Moreover, there exists a constant $\overline{C}>0$, independent of $\varepsilon$, $k$, and $t$, such that*

$$\mathcal{E}(t;\Gamma_{\varepsilon,k}(t))+\mathcal{H}^1(\Gamma_{\varepsilon,k}(t))\le\overline{C}\,,\quad \frac{\varepsilon}{\tau}\sum_{j=1}^{i}\Bigg[\sum_{\mathfrak{c}\in\mathcal{C}(j-1,j)}\mathcal{H}^1(\mathfrak{c})^2\Bigg]\le\overline{C}\,,\quad \varepsilon\,\|l_{\varepsilon,k}\|_{H^1(0,T)}\le\overline{C}\,, \tag{5.2.4}$$

*where $\mathcal{C}(j-1,j)=\mathcal{C}(\Gamma^{j-1}_{\varepsilon,k},\Gamma^j_{\varepsilon,k})$.*

We now introduce the notion of discrete velocity, for those tips in $\mathrm{T}_{\varepsilon,k}(t_k^i)$ such that the corresponding connected component of $\Gamma^i_{\varepsilon,k} \setminus \Gamma^{i-1}_{\varepsilon,k}$ does not contain elements of $\mathrm{S}_{\varepsilon,k}(t_k^i)$.

DEFINITION 5.2.2. Let $t \in [t_k^i, t_k^{i+1})$, $p \in \mathrm{T}_{\varepsilon,k}(t) = \mathrm{T}_{\varepsilon,k}(t_k^i)$. If $p \in \mathrm{T}_{\varepsilon,k}(t_k^i) \cap \mathrm{T}_{\varepsilon,k}(t_k^{i-1})$, we set

$$v_{\varepsilon,k}(t,p) := 0\,.$$

Otherwise let $\mathfrak{c}^p_{\varepsilon,k} \in \mathcal{C}(\Gamma^{i-1}_{\varepsilon,k}, \Gamma^i_{\varepsilon,k})$ be such that $p \in \mathfrak{c}^p_{\varepsilon,k}$. If $\left[\mathfrak{c}^p_{\varepsilon,k} \setminus \Gamma^{i-1}_{\varepsilon,k}\right] \cap \mathrm{S}_{\varepsilon,k}(t_k^i) = \emptyset$, we set

$$v_{\varepsilon,k}(t,p) := \frac{\mathcal{H}^1(\mathfrak{c}^p_{\varepsilon,k})}{\tau}\,.$$

The following result is the discrete version of the Griffith principle.

PROPOSITION 5.2.3. *Let $t$ and $p \in \mathrm{G}_{\varepsilon,k}(t)$ be such that $v_{\varepsilon,k}(t,p)$ is defined as in Definition 5.2.2. Then*

$$v_{\varepsilon,k}(t,p) \geq 0 \tag{5.2.5a}$$

$$\mathcal{G}(t_k^i; \Gamma^i_{\varepsilon,k}, p) \leq 1 + \varepsilon\, v_{\varepsilon,k}(t,p) \tag{5.2.5b}$$

$$\left[-\mathcal{G}(t_k^i; \Gamma^i_{\varepsilon,k}, p) + 1 + \varepsilon\, v_{\varepsilon,k}(t,p)\right] v_{\varepsilon,k}(t,p) = 0\,. \tag{5.2.5c}$$

PROOF. Property (5.2.5a) is trivial. By (5.2.3)

$$\begin{aligned} &\mathcal{E}(t_k^i; \Gamma^i_{\varepsilon,k}) + \mathcal{H}^1(\Gamma^i_{\varepsilon,k}) + \frac{\varepsilon}{2\tau} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma^{i-1}_{\varepsilon,k}, \Gamma^i_{\varepsilon,k})} \mathcal{H}^1(\mathfrak{c})^2 \\ &\qquad \leq \mathcal{E}(t_k^i; \widetilde{\Gamma}^p_s) + \mathcal{H}^1(\widetilde{\Gamma}^p_s) + \frac{\varepsilon}{2\tau} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma^{i-1}_{\varepsilon,k}, \widetilde{\Gamma}^p_s)} \mathcal{H}^1(\mathfrak{c})^2\,, \end{aligned} \tag{5.2.6}$$

where $\widetilde{\Gamma}^p_s$ is an extension of $\Gamma^i_{\varepsilon,k}$ in $p$ (recall Definition 5.1.8) such that

$$\mathcal{H}^1(\widetilde{\Gamma}^p_s \setminus \Gamma^i_{\varepsilon,k}) = s\,. \tag{5.2.7}$$

Dividing (5.2.6) by $s$ and letting $s \to 0$, by definition of energy release rate at $p$ and by (5.2.7) we obtain (5.2.5b).

As for (5.2.5c), we can assume that $v_{\varepsilon,k}(t,p) > 0$, and then that $\mathcal{H}^1(\mathfrak{c}^p_{\varepsilon,k}) > 0$, for $\mathfrak{c}^p_{\varepsilon,k}$ as in Definition 5.2.2. Let us consider $0 < s < \mathcal{H}^1(\mathfrak{c}^p_{\varepsilon,k})$ and let $\widehat{\Gamma}^p_s$ be such that

$$\Gamma^i_{\varepsilon,k} \setminus \mathfrak{c}^p_{\varepsilon,k} \subset \widehat{\Gamma}^p_s \subset \Gamma^i_{\varepsilon,k} \quad \text{and} \quad \mathcal{H}^1(\Gamma^i_{\varepsilon,k} \setminus \widehat{\Gamma}^p_s) = s$$

This is a competitor for the minimum problem (5.2.3a), and then

$$\begin{aligned} &\mathcal{E}(t_k^i; \Gamma^i_{\varepsilon,k}) + \mathcal{H}^1(\Gamma^i_{\varepsilon,k}) + \frac{\varepsilon}{2\tau} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma^{i-1}_{\varepsilon,k}, \Gamma^i_{\varepsilon,k})} \mathcal{H}^1(\mathfrak{c})^2 \\ &\qquad \leq \mathcal{E}(t_k^i; \widehat{\Gamma}^p_s) + \mathcal{H}^1(\widehat{\Gamma}^p_s) + \frac{\varepsilon}{2\tau} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma^{i-1}_{\varepsilon,k}, \widehat{\Gamma}^p_s)} \mathcal{H}^1(\mathfrak{c})^2\,. \end{aligned}$$

Dividing by $s$ and letting $s \to 0$ we get (5.2.5c). Notice that $\mathcal{G}(t_k^i; \Gamma^i_{\varepsilon,k}, p)$ is well defined since $\left[\mathfrak{c}^p_{\varepsilon,k} \setminus \Gamma^{i-1}_{\varepsilon,k}\right] \cap \mathrm{S}_{\varepsilon,k}(t_k^i) = \emptyset$. □

## 5.3. Viscous evolutions

In this section we pass to the limit in the discrete-time problems as the time step converges to zero, for fixed $\varepsilon > 0$. We prove that there exists a continuous (with respect to the Hausdorff convergence) curve $\Gamma_\varepsilon$ such that the corresponding total energy is absolutely continuous and satisfies a suitable energy balance (which was not observed in [**90**]). Moreover, we prove a Griffith criterion for almost every time when the tips of the crack are not contained in a certain set constituted by a finite number of points, which are either limits of the singular points of the interpolations, or limit of points of the interpolations where the energy release rate is not defined.

Definition 5.3.1. Fixed $\varepsilon > 0$, a set function $[0,T] \ni t \mapsto \Gamma_\varepsilon(t) \in \mathcal{S}$ is a *viscous solution* in $\mathcal{S}$ if there exist time discretizations $\{t_k^i\}_{i=0}^k$ as in (5.2.1) and a sequence of set functions $[0,T] \ni t \mapsto \Gamma_{\varepsilon,k}(t) \in \mathcal{S}$ such that $\Gamma_{\varepsilon,k}(0) = \Gamma_0$, $\Gamma_{\varepsilon,k}$ is constant in every time interval $[t_k^i, t_k^{i+1})$, $\Gamma_{\varepsilon,k}(t_k^i) = \Gamma_{\varepsilon,k}^i$ solves (5.2.3) for $i \geq 1$, and

$$\Gamma_{\varepsilon,k}(t) \xrightarrow{\mathcal{H}} \Gamma_\varepsilon(t)$$

for every $t \in [0,T]$.

Remark 5.3.2. Let $[0,T] \ni t \mapsto \Gamma_\varepsilon(t) \in \mathcal{S}$ be a *viscous solution* in $\mathcal{S}$. Then, by [**36**, Theorem 5.1], for every $t \in [0,T]$

$$\nabla u_{\varepsilon_k}(t) \to \nabla u_\varepsilon(t) \qquad \text{in } L^2(\Omega;\mathbb{R}^2)\,,$$

where $u_\varepsilon(t) := u(t;\Gamma_\varepsilon(t))$. Moreover, since, for every $\Gamma \in \mathcal{S}$, $\Omega \setminus \Gamma$ is the union of a uniformly bounded number of Lipschitz sets that intersect $\partial_D\Omega$, we have that for every $t \in [0,T]$

$$u_{\varepsilon_k}(t) \to u_\varepsilon(t) \qquad \text{in } H^1(\Omega)\,.$$

We recall from [**90**, Proposition 5.1 and Corollary 5.2] the existence of viscous solutions and their continuity in time. We give a sketch of the proof, for the reader's convenience.

Theorem 5.3.3. *Fixed $\varepsilon > 0$, there exists a viscous solution $[0,T] \ni t \mapsto \Gamma_\varepsilon(t)$. Moreover, such a solution is continuous with respect to the Hausdorff convergence, its length $l_\varepsilon(t) := \mathcal{H}^1(\Gamma_\varepsilon(t))$ belongs to $H^1(0,T)$, and there exists a positive constant $C$, independent of $t$, such that*

$$\|u_\varepsilon(t)\|_{H^1(\Omega\setminus\Gamma_\varepsilon(t))} \leq C\,. \tag{5.3.1}$$

Proof. The existence follows from the Helly Theorem, applied to the sequence of nondecreasing set functions $\Gamma_{\varepsilon,k}$. Since, for $\varepsilon$ fixed, $\|l_{\varepsilon,k}\|_{H^1}$ are uniformly bounded by (5.2.4), we have that

$$l_{\varepsilon,k} \rightharpoonup l_\varepsilon \quad \text{in } H^1(0,T)\,.$$

On the other hand, by (i) in Proposition 5.1.7 we get $\underline{l}_{\varepsilon,k} \to \underline{l}_\varepsilon$ pointwise, and for $t \in [t_k^i, t_k^{i+1})$

$$0 \leq l_{\varepsilon,k}(t) - \underline{l}_{\varepsilon,k}(t) = \int_{t_k^i}^t \dot{l}_{\varepsilon,k}(s)\,\mathrm{d}s \leq \tau^{\frac12}\,\|l_{\varepsilon,k}\|_{H^1}^{\frac12} \leq \frac{\overline{C}}{\varepsilon}\,\tau^{\frac12}\,,$$

where $\overline{C}$ is the constant in (5.2.4). Then $\underline{l}_\varepsilon = l_\varepsilon$.

The continuity of $t \mapsto \Gamma_\varepsilon(t)$ follows from the continuity of $l_\varepsilon$ (see [**90**, Corollary 5.2]). The functions $u_{\varepsilon_k}(t)$ are bounded in $H^1(\Omega \setminus \Gamma_{\varepsilon_k}(t))$, uniformly in $k$ and $t$, by (5.2.4) and the regularity of the boundary datum $w$. Thus, in the limit we get (5.3.1). □

In the following we prove some properties of the viscous solutions. Let us set

$$\mathrm{T}_\varepsilon(t) := \mathrm{T}_{\Gamma_\varepsilon(t)}\,, \quad \mathrm{S}_\varepsilon(t) := \mathrm{S}_{\Gamma_\varepsilon(t)}\,, \quad \mathrm{R}_\varepsilon(t) := \mathrm{R}_{\Gamma_\varepsilon(t)}\,, \quad \mathrm{G}_\varepsilon(t) := \mathrm{G}_{\Gamma_\varepsilon(t)}\,.$$

Up to considering a subsequence (depending on $\varepsilon$), we may assume that the number of singular points of $\Gamma_{\varepsilon,k}(T)$ is constant, so that

$$\mathrm{S}_{\varepsilon,k}(T) = \left\{x^1_{\varepsilon,k}, \dots, x^M_{\varepsilon,k}\right\}\,,$$

and $x^j_{\varepsilon,k} \to x^j_\varepsilon$, as $k \to \infty$. Let us define the set of limit of singular points

$$\mathrm{F}_\varepsilon := \left\{ x^j_\varepsilon \colon x^j_\varepsilon = \lim_k x^j_{\varepsilon,k} \right\}\,. \tag{5.3.2}$$

Notice that $\mathrm{S}_\varepsilon(T) \subset \mathrm{F}_\varepsilon$, since the curvature of every branch of any curve in $\mathcal{S}$ is less than $\frac{1}{\eta}$, and

$$\mathrm{card}\,(\mathrm{F}_\varepsilon) \leq M = \mathrm{card}\,(\mathrm{S}_{\varepsilon,k}(T))\,.$$

(In fact, it might happen that $x^l_\varepsilon = x^j_\varepsilon$ for some $j$, so the inequality may be strict.) Fix now $j, l \in \{1, \dots, M\}$, $j \neq l$: since by (5.1.5) $|x^j_{\varepsilon,k} - x^l_{\varepsilon,k}| \geq \beta\left(\frac{2}{\tan\theta} + 1\right)$ for every $k$, we have that $|x^j_\varepsilon - x^l_\varepsilon| \geq \beta\left(\frac{2}{\tan\theta} + 1\right)$ for every $\varepsilon > 0$.

Arguing as in [**90**, Lemma 6.1], we can find a partition of $[0, T]$

$$0 = t^0_\varepsilon < t^1_\varepsilon < \cdots < t^{N_\varepsilon + 1}_\varepsilon = T$$

such that for every $t \leq t' \in [0, T]$

$$\begin{cases} \mathrm{S}_\varepsilon(t) = \mathrm{S}_\varepsilon(t') \quad \text{and} \quad \mathrm{card}\,(\mathrm{T}_\varepsilon(t)) = \mathrm{card}\,(\mathrm{T}_\varepsilon(t')) & \text{if } t, t' \in (t^n_\varepsilon, t^{n+1}_\varepsilon]\,, \\ \mathrm{S}_\varepsilon(t) \neq \mathrm{S}_\varepsilon(t') \quad \text{or} \quad \mathrm{card}\,(\mathrm{T}_\varepsilon(t)) < \mathrm{card}\,(\mathrm{T}_\varepsilon(t')) & \text{if } t \leq t^n_\varepsilon < t'\,. \end{cases}$$

We define the time intervals

$$I^n_\varepsilon := (t^n_\varepsilon, t^{n+1}_\varepsilon]\,.$$

In $I^n_\varepsilon$ we can find exactly $k_n = k_n(\varepsilon) := \mathrm{card}\,(\mathrm{T}_\varepsilon(t^{n+1}_\varepsilon))$ branches parametrized by $\gamma^{n,j}_\varepsilon \colon I^n_\varepsilon \to \Omega$ with $\gamma^{n,j}_\varepsilon(t) \in \mathrm{T}_\varepsilon(t)$, for $j = 1, \dots, k_n$. Notice that, if a connected component $\Gamma_\varepsilon(t)$ belongs to $\mathcal{R}_\eta$, it has two tips. To simplify the notation, we see such a curve as the union of two branches, so the number of branches in $\Gamma_\varepsilon(t)$ equals the total number of tips. Recall that the length of any connected component is bounded from below by (5.1.2).

Extending by continuity $\gamma^{n,j}_\varepsilon$ to $\overline{I^n_\varepsilon}$ we get

$$\gamma^{n,j}_\varepsilon(\overline{I^n_\varepsilon}) \in \mathcal{R}_\eta\,.$$

Let us define

$$\overline{I^n_\varepsilon} \ni t \mapsto l^{n,j}_\varepsilon(t) := \mathcal{H}^1(\gamma^{n,j}_\varepsilon([t^n_\varepsilon, t]))\,. \tag{5.3.3}$$

REMARK 5.3.4. From now on we will not specify the dependence on $n$ of $\gamma_\varepsilon^{n,j}$ and $l_\varepsilon^{n,j}$. More precisely, we set

$$\gamma_\varepsilon^j(t) := \gamma_\varepsilon^{n,j}(t) \quad \text{where } t \in I_\varepsilon^n\,.$$

Notice that for every $t \in [0,T]$ there exists only one index $n$ such that $t \in I_\varepsilon^n$.

Since in $I_\varepsilon^n$ there are exactly $k_n$ branches, for every $(s_1, s_2) \subset I_\varepsilon^n$

$$l_\varepsilon(s_2) - l_\varepsilon(s_1) = \mathcal{H}^1(\Gamma_\varepsilon(s_2) \setminus \Gamma_\varepsilon(s_1)) = \sum_{j=1}^{k_n} \left[l_\varepsilon^j(s_2) - l_\varepsilon^j(s_1)\right]\,,$$

which gives $l_\varepsilon^j \in H^1(I_\varepsilon^i)$ for every $j \in \{1, \dots, k_n\}$ and

$$\dot{l}_\varepsilon = \sum_{j=1}^{k_n} \dot{l}_\varepsilon^j \quad \text{a.e. in } I_\varepsilon^n\,. \tag{5.3.4}$$

Let us define for every $t \in [0,T]$ the set

$$\mathrm{B}_\varepsilon(t) := [\mathrm{T}_\varepsilon(t) \setminus \mathrm{G}_\varepsilon(t)] \cup \big\{p \in \mathrm{T}_\varepsilon(t)\colon \text{there exist } \overline{k} \in \mathbb{N},\, p_k \to p\,,\ p_k \in \mathrm{T}_{\varepsilon,k}(t) \setminus \mathrm{G}_{\varepsilon,k}(t) \text{ for } k \geq \overline{k}\big\}\,. \tag{5.3.5}$$

REMARK 5.3.5. If the approximating sequence $(p_k)_k$ in the definition above is not unique, the limit point belongs to $\mathrm{F}_\varepsilon$. Specifically, if there are $p_k, q_k \in \mathrm{T}_{\varepsilon,k}(t)$ with $p_k \neq q_k$ and $p_k \to p$, $q_k \to p$, then by Proposition 5.1.7 $p$ is limit of elements in $\mathrm{S}_{\varepsilon,k}(t)$, so $p \in \mathrm{F}_\varepsilon$.

REMARK 5.3.6. If $x \in \mathrm{B}_\varepsilon(t) \setminus \mathrm{F}_\varepsilon$, then $x \in \mathrm{T}_\varepsilon(s)$ for every $s \in [t,T]$, in particular $x \in \mathrm{B}_\varepsilon(T)$. Indeed, assume $x \in \mathrm{B}_\varepsilon(t)$ and $x \notin \mathrm{T}_\varepsilon(s)$ for some $s > t$. If $x \in \mathrm{T}_\varepsilon(t) \setminus \mathrm{G}_\varepsilon(t)$ then $x \in \mathrm{S}_\varepsilon(T) \subset \mathrm{F}_\varepsilon$, since the tip in $x$ cannot be extended smoothly (see the definition of $\mathrm{G}_\Gamma$ (5.1.8)). If $x \in \mathrm{G}_\varepsilon(t)$, by Remark 5.3.5 we can assume that there exists only one approximating sequence $(p_k)_k$ as in (5.3.5); then for the same arguments as in the case $x \in \mathrm{T}_\varepsilon(t) \setminus \mathrm{G}_\varepsilon(t)$ we have $p_k \in \mathrm{S}_{\varepsilon,k}(T)$, thus $x \in \mathrm{F}_\varepsilon$.

Let us define the set of exceptional points

$$\mathrm{E}_\varepsilon := \mathrm{F}_\varepsilon \cup \mathrm{B}_\varepsilon(T)\,. \tag{5.3.6}$$

Notice that $\mathrm{E}_\varepsilon$ is a finite set. Moreover, by Remark 5.3.6, we have

$$\mathrm{E}_\varepsilon = \mathrm{F}_\varepsilon \cup \bigcup_{t\in[0,T]} \mathrm{B}_\varepsilon(t)\,.$$

We now present the main theorem of this section, providing an energy-dissipation balance for viscous solutions. The proof will be given in the final part of the section, after some preliminary results.

THEOREM 5.3.7. *Let $t \mapsto \Gamma_\varepsilon(t) \in \mathcal{S}$ be a viscous solution as in Definition 5.3.1. Then the total energy*

$$[0,T] \ni t \mapsto \mathcal{F}(t;\Gamma_\varepsilon(t)) := \mathcal{E}(t;\Gamma_\varepsilon(t)) + \mathcal{H}^1(\Gamma_\varepsilon(t))$$

*is absolutely continuous and for every* $t \in [0,T]$

$$\begin{aligned} &\mathcal{E}(t;\Gamma_\varepsilon(t)) + \mathcal{H}^1(\Gamma_\varepsilon(t)) + \varepsilon \sum_{h=0}^{n} \left[ \int_{I_\varepsilon^h} \sum_{j=1}^{k_h} \left( \dot{l}_\varepsilon^j(s) \right)^2 \mathrm{d}s \right] + \varepsilon \int_{t_\varepsilon^n}^{t} \sum_{j=1}^{k_n} \left( \dot{l}_\varepsilon^j(s) \right)^2 \mathrm{d}s \\ &\quad = \mathcal{E}(0;\Gamma_0) + l_0 + \int_0^t \langle \nabla u_\varepsilon(s), \nabla \dot{w}(s) \rangle \,\mathrm{d}s\,, \end{aligned} \tag{5.3.7}$$

*where* $n$ *is such that* $t \in I_\varepsilon^n$. *Moreover, the following hold:*

*(i) for every* $x \in \mathrm{E}_\varepsilon$ *( see* (5.3.6) *for the definition of* $\mathrm{E}_\varepsilon$*), there exists* $J \subset [0,T]$ *closed interval (which can also reduce to a singleton and contains* $T$ *if* $x \in \mathrm{B}_\varepsilon(T)$*) such that*

$$x \in \mathrm{T}_\varepsilon(t) \quad \textit{if and only if} \quad t \in J\,;$$

*(ii) for a.e.* $t \in I_\varepsilon^n$ *such that* $\gamma_\varepsilon^j(t) \notin \mathrm{E}_\varepsilon$

$$\dot{l}_\varepsilon^j(t) \geq 0\,, \tag{5.3.8a}$$

$$\mathcal{G}(t;\Gamma(t),\gamma_\varepsilon^j(t)) \leq 1 + \varepsilon\, \dot{l}_\varepsilon^j(t)\,, \tag{5.3.8b}$$

$$\big[-\mathcal{G}(t;\Gamma(t),\gamma_\varepsilon^j(t)) + 1 + \varepsilon\, \dot{l}_\varepsilon^j(t)\big]\dot{l}_\varepsilon^j(t) = 0\,. \tag{5.3.8c}$$

REMARK 5.3.8. The theorem above implies that, if $\gamma_\varepsilon^j$ does not satisfy (5.3.8) in an interval $J \subset I_\varepsilon^n$, then $\gamma_\varepsilon^j$ is constant in $J$ and it lies on a point of the finite set $\mathrm{E}_\varepsilon$.

The first step in order to establish a viscous energy balance is the following chain rule (which was not proved in [**90**]).

PROPOSITION 5.3.9. *For every* $n \in \{0,\dots,N_\varepsilon\}$ *the elastic energy* $t \mapsto \mathcal{E}(t;\Gamma_\varepsilon(t))$ *belongs to* $\mathrm{AC}_{\mathrm{loc}}(I_\varepsilon^n)$ *and for a.e.* $t \in I_\varepsilon^n$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t;\Gamma_\varepsilon(t)) = -\sum_{j=1}^{k_n} \mathcal{G}(t;\Gamma_\varepsilon(t),\gamma_\varepsilon^j(t))\,\dot{l}_\varepsilon^j(t) + \langle \nabla u_\varepsilon(t), \nabla \dot{w}(t)\rangle\,, \tag{5.3.9}$$

*with the convention* $\mathcal{G}(t;\Gamma_\varepsilon(t),\gamma_\varepsilon^j(t))\,\dot{l}_\varepsilon^j(t) = 0$ *if* $\gamma_\varepsilon^j(t) \notin \mathrm{G}_\varepsilon(t)$. *Moreover, if* $w \in H^1([0,T];H^1(\Omega\setminus\Gamma_0))$, $\mathcal{E}(\cdot;\Gamma_\varepsilon(\cdot))$ *is* $H^1_{\mathrm{loc}}(I_\varepsilon^n)$.

PROOF. Let us fix the interval $I_\varepsilon^n$ and let $\gamma_\varepsilon^1(\overline{I_\varepsilon^i}),\dots,\gamma_\varepsilon^{k_n}(\overline{I_\varepsilon^n})$ be the branches that end with a tip. In $I_\varepsilon^n$ we can rewrite the elastic energy as

$$\mathcal{E}(t;\Gamma_\varepsilon(t)) = \widehat{\mathcal{E}}(t;l_\varepsilon^1(t),\dots,l_\varepsilon^{k_n}(t))\,, \tag{5.3.10}$$

being $\widehat{\mathcal{E}}(t;\lambda_1,\dots,\lambda_{k_n})$ the elastic energy corresponding to a boundary datum $w(t)$ and to a curve $\Gamma(\lambda_1,\dots,\lambda_{k_n}) = \Gamma_\varepsilon(t_\varepsilon^n) \cup \bigcup_{j=1}^{k_n} C_j$, where $C_j$ is the unique curve contained in $\gamma_\varepsilon^j(\overline{I_\varepsilon^n})$ with $\gamma_\varepsilon^j(t_\varepsilon^n) \ni C_j$ and length $\lambda_j$. In fact, notice that $\Gamma(l_\varepsilon^1(t),\dots,l_\varepsilon^{k_n}(t)) = \Gamma_\varepsilon(t)$.

By the properties of $\mathcal{S}$ (see Remark 5.1.3), for every $s \in I_\varepsilon^n$ there exists an open neighborhood $U$ of $\gamma_j(s)$, depending on $s$ and $j$, such that

$$\Gamma_\varepsilon(s) \cap \overline{U} \in \mathcal{R}_\eta\,.$$

Therefore, for every tip in $\mathrm{G}_\varepsilon(t)$, where the energy release rate is well defined, $\widehat{\mathcal{E}}$ has partial derivatives

$$\frac{\partial}{\partial \lambda_j}\widehat{\mathcal{E}}(t; l^1_\varepsilon(t), \dots, l^{k_n}_\varepsilon(t)) = -\mathcal{G}(t; \Gamma(l^1_\varepsilon(t), \dots, l^{k_n}_\varepsilon(t)), \gamma^j_\varepsilon(t)) = -\mathcal{G}(t; \Gamma_\varepsilon(t), \gamma^j_\varepsilon(t))\,.$$

On the other hand, if $\gamma^j_\varepsilon(t) \notin \mathrm{G}_\varepsilon(t)$, then the tip does not elongate, namely $\gamma^j_\varepsilon(s) = \gamma^j_\varepsilon(t)$ for $s \in [t, t^{n+1}_\varepsilon]$, and $\dot{l}^j_\varepsilon(s) = 0$ for a.e. $s \in (t, t^{n+1}_\varepsilon)$. Indeed, by definition of $\mathrm{G}_\varepsilon(t)$, there is not an extension (see Definition 5.1.8) of $\Gamma_\varepsilon(t)$ at $\gamma^j_\varepsilon(t)$; since we are in the interval $I^n_\varepsilon$ a kinking is not created at $\gamma^j_\varepsilon(t)$.

By (5.3.10) and Proposition 5.1.12, the functions $t \mapsto \mathcal{G}(t; \Gamma_\varepsilon(t), \gamma^j_\varepsilon(t))$ are bounded in $I^n_\varepsilon$. Recalling that

$$\frac{\partial}{\partial t}\mathcal{E}(t; \Gamma_\varepsilon(t)) = \langle \nabla u_\varepsilon(t), \nabla \dot{w}(t)\rangle\,,$$

the result follows by the chain rule. □

The following proposition refines the results of [**90**, Lemmas 5.3, 5.5, and 5.6]. We give an independent and simplified proof for the reader's convenience. In order to simplify the notation, we omit the dependence on $\varepsilon$ for the objects that depend also on $k$.

Proposition 5.3.10. *Let $t \in I^n_\varepsilon$ and let $j \in \{1, \dots, k_n\}$ be such that $\gamma^j_\varepsilon(t) \notin \mathrm{F}_\varepsilon$. Define*

$$r_j(t) := \omega(\mathrm{d}(\gamma^j_\varepsilon(t), \mathrm{F}_\varepsilon)) \wedge \eta \quad \text{and} \quad \overline{s}_j(t) := \left[t - \left(\frac{r_j(t)\varepsilon}{4\overline{C}}\right)^2\right] \vee t^n_\varepsilon\,,$$

*where $\omega$ is the modulus of continuity introduced in Remark 5.1.3, and $\overline{C}$ is the constant in (5.2.4). Then there exists $\overline{k} \in \mathbb{N}$ such that for every $k > \overline{k}$ and $s \in (\overline{s}_j(t), t]$ the following holds:*

$$\mathrm{T}_k(s) \cap \overline{B_{r_j(t)}(\gamma^j_\varepsilon(t))} \text{ contains one and only one element, called } p^j_k(s)\,, \tag{5.3.11a}$$

$$\mathrm{S}_k(s) \cap \overline{B_{r_j(t)}(\gamma^j_\varepsilon(t))} = \emptyset\,, \tag{5.3.11b}$$

$$\Gamma_k(s) \cap \overline{B_{r_j(t)}(\gamma^j_\varepsilon(t))} \in \mathcal{R}_\eta\,. \tag{5.3.11c}$$

Proof. For simplicity, in the proof $r$ and $\overline{s}$ stand for $r_j(t)$ and $\overline{s}_j(t)$, respectively. First, let us prove (5.3.11) for $s = t$. By contradiction, assume that there exist $k_h \to \infty$ such that at least one condition in (5.3.11) does not hold, for $s = t$ and $k = k_h$.

Consider first the case where (5.3.11b) does not hold, namely for every $h$ there exists $q_h \in \mathrm{S}_{k_h}(t) \cap \overline{B_r(\gamma^j_\varepsilon(t))}$. Then there esists $q$ such that, up to a subsequence, $q_h \to q$, so that $q \in \mathrm{F}_\varepsilon \cap \overline{B_r(\gamma^j_\varepsilon(t))}$, in contradiction with the definition of $r$.

If (5.3.11a) does not hold (for $s = t$ and $k = k_h$), we may assume that there exist two sequences $(p_h)_h$ and $(q_h)_h$ such that

$$p_h, q_h \in \mathrm{T}_{k_h}(t) \cap \overline{B_r(\gamma^j_\varepsilon(t))}\,, \quad p_h \to \gamma^j_\varepsilon(t)\,, \quad q_h \to q \neq \gamma^j_\varepsilon(t)\,. \tag{5.3.12}$$

Indeed, by (ii) of Proposition 5.1.7, $\gamma^j_\varepsilon(t) \in \mathrm{T}_\varepsilon(t)$ is approximated by elements $p_h \in \mathrm{T}_{k_h}(t)$. Since (5.3.11a) does not hold, for every $h$ there exists $q_h \in \mathrm{T}_{k_h}(t) \cap \overline{B_r(\gamma^j_\varepsilon(t))}$, $q_h \neq p_h$. Up to a subsequence, $q_h \to q$. If $q = \gamma^j_\varepsilon(t)$, then $\gamma^j_\varepsilon(t) \in \mathrm{F}_\varepsilon$, by (iii) of Proposition 5.1.7. This proves (5.3.12) in the case (5.3.11a) is not satisfied.

Notice that $p_h$ and $q_h$ belong to different branches $K_h^1$ and $K_h^2$ of $\Gamma_{k_h}(t)$, respectively. We have that $\mathcal{H}^1(K_h^1) \geq C$, for a positive $C$, since otherwise $\gamma_\varepsilon^j(t)$ is approximated by singular points and then $\gamma_\varepsilon^j(t) \in \mathrm{F}_\varepsilon$. Thus, let us distinguish the two cases:

$$\mathcal{H}^1(K_h^2) \to 0 \quad \text{or} \quad \mathcal{H}^1(K_h^2) \geq C\,. \tag{5.3.13}$$

In the first case, $q \in \mathrm{F}_\varepsilon \cap \overline{B_r(\gamma_\varepsilon^j(t))}$, in contradiction with the definition of $r$. In the second case, passing to the limit, it is easy to see that two different branches of $\Gamma_\varepsilon(t)$ have nonempty intersection with $\overline{B_r(\gamma_\varepsilon^j(t))}$, in contradiction with the fact that

$$r < \omega(\mathrm{d}(\gamma_\varepsilon^j(t), \mathrm{F}_\varepsilon)) \leq \omega(\mathrm{d}(\gamma_\varepsilon^j(t), \mathrm{S}_{\Gamma_\varepsilon(t)})\,.$$

(Recall Remark 5.1.3.) Then (5.3.11a) holds. Finally assume that (5.3.11c) does not hold, namely $\Gamma_{k_h}(t) \cap \overline{B_r(\gamma_\varepsilon^j(t))} \notin \mathcal{R}_\eta$ for every $h$. Then $\overline{B_r(\gamma_\varepsilon^j(t))}$ intersects at least two different branches of $\Gamma_{k_h}(t)$. (Notice that we have used the hypotesis $r < \eta$, which implies that for every branch $K$ of $\Gamma_{k_h}(t)$, $K \cap \overline{B_r(\gamma_\varepsilon^j(t))}$ is a connected component of $\Gamma_{k_h}(t) \cap \overline{B_r(\gamma_\varepsilon^j(t))}$.) Therefore we can argue as in the previous case: on the one hand, there exists a branch converging to the branch of $\gamma_\varepsilon^j(t)$; on the other hand there exists a different branch, either converging to a point $q \in \mathrm{F}_\varepsilon \cap \overline{B_r(\gamma_\varepsilon^j(t))}$, or with length bounded from below, cf. (5.3.13). This concludes the proof of (5.3.11) for $s = t$. Notice that we have proved also that $p_k^j(t) \to \gamma_\varepsilon^j(t)$.

We are now ready to prove (5.3.11) for $s \in (\overline{s}, t)$. For $k$ large

$$\mathrm{d}(p_k^j(t), \gamma_\varepsilon^j(t)) < \frac{r}{2}\,,$$

and then

$$\mathcal{H}^1(\Gamma_k(t) \cap \overline{B_r(\gamma_\varepsilon^j(t))}) \geq \frac{r}{2}\,.$$

Let us introduce

$$s_k := \min\{s \in [t_\varepsilon^n, t) \colon \Gamma_k(s) \cap \overline{B_r(\gamma_\varepsilon^j(t))} \neq \emptyset\}\,. \tag{5.3.14}$$

Notice that the set in the last definition is not empty for $k$ sufficiently large. Indeed, let $h \in \mathbb{N}$ such that $t \in [t_k^h, t_k^{h+1})$. If $t \in (t_k^h, t_k^{h+1})$, then $t_k^h$ is a competitor for $s_k$, since $\Gamma_k$ is piecewise constant. On the other hand, if $t = t_k^h$, then

$$\mathcal{H}^1(\Gamma_k(t_k^h) \setminus \Gamma_k(t_k^{h-1})) \leq \int_{t_k^{h-1}}^{t_k^h} \dot{l}_k(s)\,\mathrm{d}s \leq \frac{1}{\sqrt{k}} \left( \int_{t_k^{h-1}}^{t_k^h} \left| \dot{l}_k(s) \right|^2 \mathrm{d}s \right)^{\frac{1}{2}} \leq \frac{\overline{C}}{\varepsilon\sqrt{k}}\,, \tag{5.3.15}$$

with $\overline{C}$ the constant in (5.2.4); for $k$ sufficiently large this implies that $t_k^{h-1}$ is a competitor for $s_k$. Moreover, the minimum in (5.3.14) is attained at a node $t_k^h$, since $\Gamma_k$ is piecewise constant and continuous from the right, and

$$\frac{r}{4} \leq \mathcal{H}^1(\Gamma_k(t) \setminus \Gamma_k(s_k))\,,$$

for $k$ large, by (5.3.15).

By the monotonicity of $\Gamma_k$ and (5.3.11) for $s = t$, we obtain that for every $s \in (s_k, t]$

$$\mathrm{T}_k(s) \cap \overline{B_r(\gamma^j_\varepsilon(t))} = \{p^j_k(s)\}\,,$$
$$\mathrm{S}_k(s) \cap \overline{B_r(\gamma^j_\varepsilon(t))} = \emptyset\,,$$
$$\Gamma_k(s) \cap \overline{B_r(\gamma^j_\varepsilon(t))} \in \mathcal{R}_\eta\,.$$

Therefore, the proof is completed if we show that $s_k \le \overline{s}$.

Let $t \in (t^{h_2}_k, t^{h_2+1}_k)$ and $s_k = t^{h_1}_k$. Necessarily $h_1 < h_2$, because otherwise $\Gamma_k(t) = \Gamma_k(s_k)$. By (5.2.4), we have that

$$\frac{r}{4} \le \mathcal{H}^1(\Gamma_k(t) \setminus \Gamma_k(s_k)) = \int_{h_1\tau}^{h_2\tau} \dot{l}_k(s)\,\mathrm{d}s \le \sqrt{\frac{(h_2-h_1)}{k}} \left(\int_{h_1\tau}^{h_2\tau} \left|\dot{l}_k(s)\right|^2 \mathrm{d}s\right)^{\frac{1}{2}} \le \frac{\overline{C}}{\varepsilon}\sqrt{t - s_k}\,.$$

Then

$$s_k \le \bar{t} - \left(\frac{r\varepsilon}{4\overline{C}}\right)^2,$$

and this concludes the proof, since $s_k \ge t^n_\varepsilon$. □

Remark 5.3.11. In Proposition 5.3.10 we chose the notation $r_j(t)$ and $\overline{s}_j(t)$ since these quantities depend on $t$ and on the branch that we consider, which corresponds to a certain $j \in \{1, \dots, k_n\}$. Moreover

$$p^j_k(t) \to \gamma^j_\varepsilon(t) \tag{5.3.16}$$

for every $t \in I^n_\varepsilon$ such that $\gamma^j_\varepsilon(t) \notin \mathrm{F}_\varepsilon$.

Let us fix $n \in \{0, \dots, N_\varepsilon\}$, $j \in \{1, \dots, k_n\}$, and $t \in I^n_\varepsilon$ such that $\gamma^j_\varepsilon(t) \notin \mathrm{F}_\varepsilon$. With the notation of Proposition 5.3.10, for $k$ sufficiently large and $s \in (\overline{s}_j(t), t]$, we have that

$$\Gamma_k(s) \cap \overline{B_{r_j(t)}(\gamma^j_\varepsilon(t))} \in \mathcal{R}_\eta\,.$$

Let us consider the functions

$$s \in (\overline{s}_j(t), t] \mapsto \underline{\ell}^j_k(s) := \mathcal{H}^1(\Gamma_k(s) \cap B_{r_j(t)}(\gamma^j_\varepsilon(t)))\,,$$
$$s \in (\overline{s}_j(t), t] \mapsto \ell^j_k(s) := \mathcal{H}^1(\Gamma_k(s) \cap B_{r_j(t)}(\gamma^j_\varepsilon(t))) + \frac{s - \underline{\tau}_k(s)}{\tau}\mathcal{H}^1\left([\Gamma_k(s+\tau) \setminus \Gamma_k(s)] \cap B_{r_j(t)}(\gamma^j_\varepsilon(t))\right),$$
$$s \in (\overline{s}_j(t), t] \mapsto \ell^j_\varepsilon(s) := \mathcal{H}^1(\Gamma(s) \cap B_{r_j(t)}(\gamma^j_\varepsilon(t))) = l^j_\varepsilon(s) - \mathcal{H}^1(\gamma^j_\varepsilon([t^i_\varepsilon, t]) \setminus B_{r_j(t)}(\gamma^j_\varepsilon(t)))\,,$$

where $\underline{\tau}_k(s) := t^h_k$ if $s \in [t^h_k, t^{h+1}_k)$. Since $\Gamma_k(s) \xrightarrow{\mathcal{H}} \Gamma(s)$ for every $s \in [0, T]$, we get that

$$\underline{\ell}^j_k(s) \to \ell^j_\varepsilon(s) \quad \text{for every } s \in (\overline{s}_j(t), t]\,.$$

Arguing as in the proof of Theorem 5.3.3 we have that

$$\|\ell^j_k\|_{H^1(\overline{s}_j(t), t)} \le C\,,$$

with $C$ depending only on the data of the problem and on $\varepsilon$, and that

$$\ell^j_k(s) \to \ell^j_\varepsilon(s) \quad \text{in } (\overline{s}_j(t), t]\,, \qquad \ell^j_k \rightharpoonup \ell^j_\varepsilon \quad \text{in } H^1(\overline{s}_j(t), t)\,. \tag{5.3.17}$$

Notice that

$$\dot\ell_k^j(s) = v_k(s, p_k^j(s)), \quad \dot\ell_\varepsilon^j(s) = \dot l_\varepsilon^j(s)\,.$$

We employ the following result, proved in [**90**, Lemma 7.3], which holds since $\overline{s}_j(t) < t$.

LEMMA 5.3.12. *Let us consider* $(s_1, s_2) \subset I_\varepsilon^n$ *such that* $\gamma_\varepsilon^j(\widetilde{t}) \notin \mathrm{F}_\varepsilon$ *for every* $\widetilde{t} \in (s_1, s_2)$*. Then, for every* $\widetilde{t} \in (s_1, s_2)$ *there exists a set* $A_j(\widetilde{t}) \subset (s_1, s_2)$*, at most countable, such that* $(\overline{s}_j(t_1), t_1]$ *and* $(\overline{s}_j(t_2), t_2]$ *are disjoint for* $t_1 \neq t_2 \in A_j(\widetilde{t})$ *and*

$$(s_1, \widetilde{t}] = \bigcup_{t \in A_j(\widetilde{t})} (\overline{s}_j(t), t]\,.$$

Employing the above lemma, we deduce the following convergence result.

LEMMA 5.3.13. *For every* $(\bar t_1, \bar t_2) \subset\subset (s_1, s_2)$

$$v_k(\cdot, p_k^j(\cdot)) \rightharpoonup \dot l_\varepsilon^j \quad \textit{in } L^2(\bar t_1, \bar t_2)\,. \tag{5.3.18}$$

PROOF. Let $f \in L^2(\bar t_1, \bar t_2)$. We have that

$$\int_{\bar t_1}^{\bar t_2} v_k(s, p_k^j(s))\, f(s)\,\mathrm{d}s = \sum_{t \in A_j(\bar t_1)} \int_{(\overline{s}_j(t), t] \cap (\bar t_1, \bar t_2)} v_k(s, p_k^j(s))\, f(s)\,\mathrm{d}s\,,$$

and, by (5.3.17),

$$v_k(\cdot, p_k^j(\cdot)) \rightharpoonup \dot l_\varepsilon^j \quad \text{in } L^2(\overline{s}_j(t), t)$$

for every $t \in A_j(\bar t_1)$. Lemma 5.3.12 ensures that $A_j(\bar t_1)$ is at most countable, so the countable additivity of the integral allows us to obtain (5.3.18). □

We are now in the position to prove a Griffith criterion for viscous solutions.

PROPOSITION 5.3.14. *Let* $n \in \{0, \dots, N_\varepsilon\}$*,* $j \in \{1, \dots, k_n\}$*, and* $(s_1, s_2) \subset I_\varepsilon^n$ *such that* $\gamma_\varepsilon^j(\widetilde{t}) \notin \mathrm{E}_\varepsilon$ *for every* $\widetilde{t} \in (s_1, s_2)$*. Then*

$$(s_1, s_2) \ni t \mapsto \mathcal{G}(t; \Gamma(t), \gamma_\varepsilon^j(t)) \quad \textit{is continuous} \tag{5.3.19}$$

*and for a.e.* $t \in (s_1, s_2)$ *the following conditions hold:*

$$\dot l_\varepsilon^j(t) \geq 0\,, \tag{5.3.20a}$$

$$\mathcal{G}(t; \Gamma(t), \gamma_\varepsilon^j(t)) \leq 1 + \varepsilon\, \dot l_\varepsilon^j(t)\,, \tag{5.3.20b}$$

$$\big[-\mathcal{G}(t; \Gamma(t), \gamma_\varepsilon^j(t)) + 1 + \varepsilon\, \dot l_\varepsilon^j(t)\big]\, \dot l_\varepsilon^j(t) = 0\,. \tag{5.3.20c}$$

PROOF. For every $\Gamma \in \mathcal{S}$ and $p \in \mathrm{G}_\Gamma$, we denote

$$\mathcal{G}_k(t; \Gamma, p) := \mathcal{G}(w_k(t); \Gamma, p)\,.$$

Recalling the definition of $\mathrm{E}_\varepsilon$ (5.3.6), we employ Proposition 5.1.12 and (5.3.16) to deduce (5.3.19) and the convergence

$$\mathcal{G}_k(t; \Gamma_k(t), p_k^j(t)) \to \mathcal{G}(t; \Gamma(t), \gamma_\varepsilon^j(t)) \quad \text{for every } t \in (s_1, s_2)\,. \tag{5.3.21}$$

By (5.2.5), we have that for every $t \in (s_1, s_2)$

$$v_k(t, p_k^j(t)) \geq 0\,, \tag{5.3.22a}$$

$$\mathcal{G}_k(t; \Gamma_k(t), p_k^j(t)) \leq 1 + \varepsilon\, v_k(t, p_k^j(t))\,, \tag{5.3.22b}$$

$$\big[-\mathcal{G}_k(t; \Gamma_k(t), p_k^j(t)) + 1 + \varepsilon\, v_k(t, p_k^j(t))\big] v_k(t, p_k^j(t)) = 0\,, \tag{5.3.22c}$$

Since $\mathcal{G}_k(t; \Gamma_k(t), p_k^j(t)) \geq 0$, by (5.3.18) and (5.3.22b) the functions

$$(\bar t_1, \bar t_2) \ni s \mapsto \mathcal{G}_k(s; \Gamma_k(s), p_k^j(s))$$

are equibounded in $L^2(\bar t_1, \bar t_2)$, for every $(\bar t_1, \bar t_2) \subset\subset (s_1, s_2)$. By the pointwise convergence (5.3.21) we get that

$$\mathcal{G}_k(\cdot; \Gamma_k(\cdot), p_k^j(\cdot)) \rightharpoonup \mathcal{G}(\cdot; \Gamma(\cdot), \gamma_\varepsilon^j(\cdot)) \quad \text{in } L^2(\bar t_1, \bar t_2)\,. \tag{5.3.23}$$

Integrating (5.3.22b) in every $(\bar t_1, \bar t_2)$, and passing to the limit using (5.3.18) and (5.3.26), we obtain that

$$\int_{\bar t_1}^{\bar t_2} \mathcal{G}(s; \Gamma(s), \gamma_\varepsilon^j(s))\, \mathrm{d}s \leq \int_{\bar t_1}^{\bar t_2} 1 + \varepsilon\, \dot l_\varepsilon^j(s)\, \mathrm{d}s\,.$$

Therefore we deduce inequality (5.3.20b) in the Lebesgue points of $\dot l_\varepsilon^j$ in $(s_1, s_2)$.

Again by (5.3.23),

$$\int_{\bar t_1}^{\bar t_2} \mathcal{G}_k(s; \Gamma_k(s), p_k^j(s))\, \mathrm{d}s \to \int_{\bar t_1}^{\bar t_2} \mathcal{G}(s; \Gamma(s), \gamma_\varepsilon^j(s))\, \mathrm{d}s\,,$$

and, since $\mathcal{G}_k(t; \Gamma_k(t), p_k^j(t)) \geq 0$, we get

$$\mathcal{G}_k(\cdot; \Gamma_k(\cdot), p_k^j(\cdot)) \to \mathcal{G}(\cdot; \Gamma(\cdot), \gamma_\varepsilon^j(\cdot)) \quad \text{in } L^1(\bar t_1, \bar t_2)\,. \tag{5.3.24}$$

Moreover, the continous function $s \mapsto \mathrm{d}(\gamma_\varepsilon^j(s), \mathrm{F}_\varepsilon)$ has positive minimum in $[\bar t_1, \bar t_2]$. Then there exists a positive constant $C_0$ such that, using the notation of Proposition 5.3.10,

$$r_j(s) \geq C_0 \quad \text{for every } s \in [\bar t_1, \bar t_2]\,. \tag{5.3.25}$$

Let us now fix a subinterval $(\overline{s}_j(t), t] \subset (s_1, s_2)$. By Proposition 5.3.10, there exists $\overline{k}$ such that (5.3.11) holds for $k \geq \overline{k}$ and $s \in (\overline{s}_j(t), t]$. Thanks to (5.3.11) and to the fact that (5.1.12) holds with a radius $r = C_0$ independent of $s$ by (5.3.25), we are allowed to use Proposition 5.1.12. Therefore, by (5.1.13) and (5.3.1), we get that there exists a positive constant $C$ independent of $k \geq \overline{k}$ and $s \in (\overline{s}_j(t), t]$ such that

$$\mathcal{G}_k(s; \Gamma_k(s), p_k^j(s)) \leq C \quad \text{for every } k \geq \overline{k} \text{ and } s \in (\overline{s}_j(t), t]\,.$$

Using also (5.3.24), it follows that

$$\mathcal{G}_k(\cdot; \Gamma_k(\cdot), p_k^j(\cdot)) \to \mathcal{G}(\cdot; \Gamma(\cdot), \gamma_\varepsilon^j(\cdot)) \quad \text{in } L^q(\overline{s}_j(t), t)\,, \quad \text{for every } q \in [1, +\infty)\,. \tag{5.3.26}$$

Let us now prove (5.3.20c). It follows immediately from (5.3.20a) and (5.3.20b) that

$$\Big[-\mathcal{G}(t; \Gamma(t), \gamma_\varepsilon^j(t)) + 1 + \varepsilon\, \dot l_\varepsilon^j(t)\Big]\, \dot l_\varepsilon^j(t) \geq 0\,.$$

By (5.3.18), (5.3.22c), and (5.3.24) we deduce that

$$\begin{aligned}0 &\leq \int_{\overline{s}_j(t)}^{t} \Big[-\mathcal{G}(s;\Gamma(s),\gamma^j_\varepsilon(s))+1+\varepsilon\,\dot{l}^j_\varepsilon(s)\Big]\,\dot{l}^j_\varepsilon(s)\,\mathrm{d}s\\
&\leq \lim_{k\to\infty}\int_{\overline{s}_j(t)}^{t}\Big[-\mathcal{G}_k(s;\Gamma_k(s),p^j_k(s))+1\Big]\,v_k(s,p^j_k(s))\,\mathrm{d}s+\varepsilon\,\liminf_{k\to\infty}\int_{\overline{s}_j(t)}^{t} v_k(s,p^j_k(s))^2\,\mathrm{d}s\\
&\leq \liminf_{k\to\infty}\int_{\overline{s}_j(t)}^{t}\Big[-\mathcal{G}_k(s;\Gamma_k(s),p^j_k(s))+1+\varepsilon\,v_k(s,p^j_k(s))\Big]\,v_k(s,p^j_k(s))\,\mathrm{d}s=0\,.\end{aligned}$$

Then (5.3.20c) holds in the Lebesgue points of $\dot{l}^j_\varepsilon$ in $(s_1,s_2)$, and the proof is completed. □

We are now ready to prove Theorem 5.3.7.

Proof of Theorem 5.3.7. Let us fix $n\in\{0,\dots,N_\varepsilon\}$ and $j\in\{1,\dots,k_n\}$, and let us consider the intersections of $\gamma^j_\varepsilon$ with the set $\mathrm{F}_\varepsilon$ defined in (5.3.2): since $\Gamma_\varepsilon$ is nondecreasing and the curves of $\mathcal{S}$ have no self-intersections, if $\overline{x}\in\mathrm{F}_\varepsilon\cap\gamma^j_\varepsilon(I^n_\varepsilon)$, then there are $t^n_\varepsilon\leq\overline{t}_1\leq\overline{t}_2\leq t^{n+1}_\varepsilon$ such that

$$\gamma^j_\varepsilon(s)=\overline{x}\quad\text{ if and only if } s\in[\overline{t}_1,\overline{t}_2]\,. \tag{5.3.27}$$

By Remark 5.3.6, if $x\in\mathrm{B}_\varepsilon(T)$, we have that the tip stops in $x$ until the final time $T$, and we deduce in particular (5.3.27) for $\overline{t}_2=t^{n+1}_\varepsilon$.

Therefore (i) holds and $(t^n_\varepsilon,t^{n+1}_\varepsilon)$ is the union of a finite number of open intervals where $\gamma^j_\varepsilon(t)\notin\mathrm{E}_\varepsilon$, and of a finite number of closed intervals in each of which $\gamma^j_\varepsilon(t)$ is constant and belongs to $\mathrm{E}_\varepsilon$. Combining this observation with Proposition 5.3.14 gives the Griffith conditions (5.3.8).

When $\overline{t}_1<\overline{t}_2$ in (5.3.27), we can say that

$$\dot{l}^j_\varepsilon(s)=0\quad\text{ for } s\in(\overline{t}_1,\overline{t}_2)\,.$$

By (5.3.20c), we have that for every $n$, $j$, and for a.e. $t\in I^n_\varepsilon$

$$\Big[-\mathcal{G}(t;\Gamma(t),\gamma^j_\varepsilon(t))+1+\varepsilon\,\dot{l}^j_\varepsilon(t)\Big]\,\dot{l}^j_\varepsilon(t)=0\,.$$

Therefore, recalling (5.3.4) and (5.3.9), $\mathcal{F}(\cdot;\Gamma_\varepsilon(\cdot))$ defined in (5.1.1) is absolutely continuous in every $(s_1,s_2)\subset I^n_\varepsilon$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t;\Gamma_\varepsilon(t))&=\sum_{j=1}^{k_n}\left[-\mathcal{G}(t;\Gamma_\varepsilon(t),\gamma^j_\varepsilon(t))+1\right]\dot{l}^j_\varepsilon(t)+\langle\nabla u_\varepsilon(t),\nabla\dot{w}(t)\rangle\\
&=-\varepsilon\sum_{j=1}^{k_n}\Big(\dot{l}^j_\varepsilon(t)\Big)^2+\langle\nabla u_\varepsilon(t),\nabla\dot{w}(t)\rangle\,.\end{aligned}$$

with the convention $\mathcal{G}(t;\Gamma_\varepsilon(t),\gamma^j_\varepsilon(t))\,\dot{l}^j_\varepsilon(t)=0$ if $\gamma^j_\varepsilon(t)\notin\mathrm{G}_\varepsilon(t)$. Integrating,

$$\mathcal{F}(s_2;\Gamma_\varepsilon(s_2))-\mathcal{F}(s_1;\Gamma_\varepsilon(s_1))=\int_{s_1}^{s_2}\Big[-\varepsilon\sum_{j=1}^{k_n}\Big(\dot{l}^j_\varepsilon(s)\Big)^2+\langle\nabla u_\varepsilon(s),\nabla\dot{w}(s)\rangle\Big]\,\mathrm{d}s\,.$$

We can now pass to the limit as $s_1\to t^n_\varepsilon$ and $s_2\to t^{n+1}_\varepsilon$, in view of the continuity of $\mathcal{F}(\cdot;\Gamma_\varepsilon(\cdot))$ and of the fact that $\Big[-\varepsilon\sum_{j=1}^{k_n}\Big(\dot{l}^j_\varepsilon(t)\Big)^2+\langle\nabla u_\varepsilon(t),\nabla\dot{w}(t)\rangle\Big]\in L^1(I^n_\varepsilon)$, obtaining the energy balance in every $I_\varepsilon^n$. Since the number of such intervals is finite and $\mathcal{F}(\cdot;\Gamma_\varepsilon(\cdot))$ is continuous in $[0,T]$, summing up we obtain (5.3.7). This concludes the proof. □

## 5.4. The vanishing viscosity limit

In this section we pass to the limit in the viscous solutions as the viscosity parameter $\varepsilon$ tends to zero. The limit evolution may display jumps in time. In order to provide a better description of the system during jumps we reparametrize by arc-length the viscous solutions, in such a way that we get a family of Lipschitz evolutions. In the limit, we obtain an evolution parametrized by arc-length, where jumps are described by means of a slow time scale. This technique was already employed in [**40, 59, 29, 67, 76**].

Given a family of viscous solutions $(\Gamma_\varepsilon)_{\varepsilon>0}$ as in Definition 5.3.1, for $t\in[0,T]$ we set

$$s_\varepsilon^\circ(t) := t + \mathcal{H}^1(\Gamma_\varepsilon(t)\setminus\Gamma_0) = t + (l_\varepsilon(t) - l_0)\,. \tag{5.4.1}$$

Being $l_\varepsilon$ increasing, we get that $s_\varepsilon^\circ$ is strictly increasing and absolutely continuous and that

$$s_\varepsilon^\circ(t_2) - s_\varepsilon^\circ(t_1) \geq t_2 - t_1 \quad \text{for every } 0\leq t_1\leq t_2\leq S_\varepsilon := s_\varepsilon^\circ(T)\,.$$

Let $t_\varepsilon^\circ\colon [0,S_\varepsilon]\mapsto[0,T]$ be the inverse of $s_\varepsilon^\circ$; then $t_\varepsilon^\circ$ is strictly increasing. By the uniform bound on the length of the elements of $\mathcal{S}$, it follows that $\overline{S} := \sup_\varepsilon S_\varepsilon < +\infty$ and then, for a sequence $\varepsilon_k$, $S_{\varepsilon_k}\to S$, with $S\geq T$. By setting $t_\varepsilon^\circ(t) = t_\varepsilon^\circ(S_\varepsilon)$ on $(S_\varepsilon,\overline{S}]$, we may assume that $t_\varepsilon^\circ$ is defined on the fixed time interval $[0,S]$. For $s\in[0,S]$ we set

$$l_\varepsilon^\circ(s) := l_\varepsilon(t_\varepsilon^\circ(s))\,,\quad \Gamma_\varepsilon^\circ(s) := \Gamma_\varepsilon(t_\varepsilon^\circ(s))\,,\quad u_\varepsilon^\circ(s) := u_\varepsilon(t_\varepsilon^\circ(s))\,. \tag{5.4.2}$$

DEFINITION 5.4.1. A *rescaled approximable quasistatic evolution* is a function $s\mapsto(\Gamma^\circ(s),t^\circ(s))$, defined in $[0,S]$, with values in $\mathcal{S}\times[0,T]$, such that there is a sequence $\Gamma_{\varepsilon_k}$ of viscous solutions in $\mathcal{S}$, with $\varepsilon_k\to0$, for which the following hold:

$$\Gamma_{\varepsilon_k}^\circ(s) \xrightarrow{\mathcal{H}} \Gamma^\circ(s) \quad \text{for every } s\in[0,S]\,, \tag{5.4.3a}$$

$$t_{\varepsilon_k}^\circ \overset{*}{\rightharpoonup} t^\circ \qquad \text{in } W^{1,\infty}([0,S])\,, \tag{5.4.3b}$$

where $\Gamma_\varepsilon^\circ$ and $t_\varepsilon^\circ$ are as above, see (5.4.1)–(5.4.2).

Employing the Helly Theorem for families of nondecreasing set functions, in the following proposition we prove the existence of rescaled approximable quasistatic evolutions.

PROPOSITION 5.4.2. *There exists a rescaled approximable quasistatic evolution. Moreover, for every rescaled approximable quasistatic evolution $s\mapsto(\Gamma^\circ(s),t^\circ(s))$ the following hold (with the notation as above): the set function $s\mapsto\Gamma^\circ(s)$ is nondecreasing,*

$$l_{\varepsilon_k}^\circ \overset{*}{\rightharpoonup} l^\circ \quad \text{in } W^{1,\infty}([0,S])\,,$$

*and*

$$(t^\circ)'(s) + (l^\circ)'(s) = 1 \quad \text{for a.e. } s\in(0,S)\,, \tag{5.4.4}$$

*where $l^\circ(s) := \mathcal{H}^1(\Gamma^\circ(s))$ for every $s\in[0,S]$ and the symbol $'$ denotes the derivative with respect to $s$. Furthermore, setting $u^\circ(s) := u(t^\circ(s),\Gamma^\circ(s))$, we have that for every $s$*

$$\nabla u_{\varepsilon_k}^\circ(s) \to \nabla u^\circ(s) \quad \text{in } L^2(\Omega;\mathbb{R}^2)\,. \tag{5.4.5}$$

PROOF. By (5.4.1) we get $s = t_\varepsilon^\circ(s) + (l_\varepsilon^\circ(s) - l_0)$, and taking the derivative we obtain the identity

$$(t_\varepsilon^\circ)'(s) + (l_\varepsilon^\circ)'(s) = 1 \quad \text{for every } \varepsilon \text{ and } s\,. \tag{5.4.6}$$

Therefore $t_\varepsilon^\circ$ and $l_\varepsilon^\circ$ are families of contractions on $[0,S]$. There are a subsequence $\varepsilon_k$ and functions $t^\circ, l \in W^{1,\infty}([0,S])$ such that

$$t_{\varepsilon_k}^\circ \overset{*}{\rightharpoonup} t^\circ\,, \quad l_{\varepsilon_k}^\circ \overset{*}{\rightharpoonup} l \quad \text{in } W^{1,\infty}([0,S])\,. \tag{5.4.7}$$

Moreover, the Helly Theorem applies to the family of nondecreasing set functions $s \mapsto \Gamma_\varepsilon^\circ(s)$, so there exists $s \mapsto \Gamma^\circ(s) \in \mathcal{S}$ nondecreasing and a further subsequence of $\varepsilon_k$ (not relabeled) such that

$$\Gamma_{\varepsilon_k}^\circ(s) \xrightarrow{\mathcal{H}} \Gamma^\circ(s) \quad \text{for every } s \in [0,S]\,,$$

namely (5.4.3a) holds. By the properties of $\mathcal{S}$, this implies that $\mathcal{H}^1(\Gamma_{\varepsilon_k}^\circ(s)) \to \mathcal{H}^1(\Gamma^\circ(s))$ for every $s$. Recalling (5.4.7), we get $l = l^\circ$ and (5.4.4). Finally, (5.4.5) follows by (5.4.3a) and [**36**, Theorem 5.1]. This concludes the proof. □

In the following part of this section, we derive important properties of rescaled approximable quasistatic evolutions. We define

$$s_-^\circ(t) := \sup\{s \in [0,S] \colon t^\circ(s) < t\} \quad \text{for } t \in (0,T]\,,$$
$$s_+^\circ(t) := \inf\{s \in [0,S] \colon t^\circ(s) > t\} \quad \text{for } t \in [0,T)\,,$$

and $s_-^\circ(0) := 0$, $s_+^\circ(T) := S$. By standard arguments, we have that

$$s_-^\circ(t) \le \liminf_{\varepsilon\to 0} s_\varepsilon^\circ(t) \le \limsup_{\varepsilon\to 0} s_\varepsilon^\circ(t) \le s_+^\circ(t) \quad \text{and} \quad t^\circ(s_-^\circ(t)) = t = t^\circ(s_+^\circ(t))\,, \quad \text{for } t \in [0,T]\,,$$
$$s_-^\circ(t^\circ(s)) \le s \le s_+^\circ(t^\circ(s)) \quad \text{for } s \in [0,S]\,,$$
$$S^\circ := \{t \in [0,T] \colon s_-^\circ(t) < s_+^\circ(t)\} \quad \text{is at most countable,}$$
$$U^\circ := \{s \in [0,S] \colon t^\circ \text{ is constant in a neighborhood of } s\} = \bigcup_{t\in S^\circ} (s_-^\circ(t), s_+^\circ(t))\,. \tag{5.4.8}$$

As in the previous section, we now divide $[0,T]$ in subintervals where the number of branches of $\Gamma^\circ$ is constant. Such branches are in turn limits of branches of viscous solutions. Once these approximation properties are ready, we will adapt the arguments of [**59**] and [**67**]. Let us set

$$\mathrm{T}^\circ(s) := \mathrm{T}_{\Gamma^\circ(s)}\,, \quad \mathrm{S}^\circ(s) := \mathrm{S}_{\Gamma^\circ(s)}\,, \quad \mathrm{R}^\circ(s) := \mathrm{R}_{\Gamma^\circ(s)}\,, \quad \mathrm{G}^\circ(s) := \mathrm{B}_{\Gamma^\circ(s)}\,.$$

Up to extracting a further subsequence, we may assume that the sets $\mathrm{F}_\varepsilon$ introduced in (5.3.2) are such that

$$\mathrm{F}_\varepsilon = \{x_\varepsilon^1, \dots, x_\varepsilon^M\}\,,$$

with $M$ independent of $\varepsilon$, and $x_\varepsilon^j \to x_j$ as $\varepsilon \to 0$. Recall that $\mathrm{S}_\varepsilon(T) \subset \mathrm{F}_\varepsilon$. We define the set of limit points

$$\mathrm{F} := \left\{x^j \colon x^j = \lim_\varepsilon x_\varepsilon^j\right\}\,. \tag{5.4.9}$$

We have that $\mathrm{card}\,(\mathrm{F}) \le M$ and $|x^j - x^l| \ge \beta\left(\frac{2}{\tan\theta} + 1\right)$ for every $x^j \neq x^l \in \mathrm{F}$. Moreover, we can find a partition of $[0,S]$

$$0 = s^0 < s^1 < \dots < s^{N+1} = S$$

such that for every $s \le s' \in [0,S]$

$$\begin{cases} \mathrm{S}^\circ(s) = \mathrm{S}^\circ(s') \text{ and } \operatorname{card}(\mathrm{T}^\circ(s)) = \operatorname{card}(\mathrm{T}^\circ(s')) & \text{if } s, s' \in (s^n, s^{n+1}]\,, \\ \mathrm{S}^\circ(s) \neq \mathrm{S}^\circ(s') \text{ or } \operatorname{card}(\mathrm{T}^\circ(s)) < \operatorname{card}(\mathrm{T}^\circ(s')) & \text{if } s \le s^n < s'\,. \end{cases}$$

As in the previous section, in the time intervals

$$I_n^\circ := (s^n, s^{n+1}]$$

we can find exactly $h_n$ branches parametrized by $\gamma_{n,j}^\circ \colon I_n^\circ \to \Omega$, with $\gamma_{n,j}^\circ(s) \in \mathrm{T}^\circ(s)$, and

$$\gamma_{n,j}^\circ(\overline{I_n^\circ}) \in \mathcal{R}_\eta\,.$$

If we introduce the functions $\overline{I_n^\circ} \ni s \mapsto l_{n,j}^\circ(s) := \mathcal{H}^1(\gamma_{n,j}^\circ([s^n, s]))$, we have that for every $(s_1, s_2) \subset I_n^\circ$

$$(l^\circ)'(s) = \sum_{j=1}^{h_n} \left(l_{n,j}^\circ\right)'(s) \quad \text{in } I_n^\circ\,.$$

Thus $l_{n,j}^\circ \in W^{1,\infty}(I_n^\circ)$ for every $j \in \{1, \dots, h_n\}$, with $\left(l_{n,j}^\circ\right)'(s) \le 1$. In order to simplify the notation, in the following we omit the dependence on $n$ of $\gamma_{n,j}^\circ$ and $l_{n,j}^\circ$ (see Remark 5.3.4).

As in Section 5.3 we define for every $s \in [0,S]$ the set

$$\mathrm{B}^\circ(s) := [\mathrm{T}^\circ(s) \backslash \mathrm{G}^\circ(s)] \cup \{p \in \mathrm{T}^\circ(s) \colon \text{there exist } \varepsilon_0 > 0,\, p_\varepsilon \to p\,,\ p_\varepsilon \in \mathrm{T}_\varepsilon^\circ(s) \setminus \mathrm{G}_\varepsilon^\circ(s) \text{ for } \varepsilon < \varepsilon_0\}$$

and the set of exceptional points

$$\mathrm{E}^\circ := \mathrm{F} \cup \mathrm{B}^\circ(T)\,. \tag{5.4.10}$$

REMARK 5.4.3. As in Remark 5.3.6, we can see that if $x \in \mathrm{B}^\circ(s) \setminus \mathrm{F}$, then $x \in \mathrm{T}^\circ(\tau)$ for every $\tau \in [s, S]$, namely $x \in \mathrm{B}^\circ(S)$. In particular,

$$\mathrm{E}^\circ = \mathrm{F} \cup \bigcup_{s \in [0,S]} \mathrm{B}^\circ(s)\,.$$

The main result of this section states the properties of rescaled approximable quasistatic evolutions and will be proved at the end of this section, after a few technical steps.

THEOREM 5.4.4. *Let $(\Gamma^\circ, t^\circ)$ be a rescaled approximable quasistatic evolution as in Definition 5.4.1. Then, with the notation as above, the following hold:*

*(i) for every $x \in \mathrm{E}^\circ$ (see (5.4.10) for the definition of $\mathrm{E}^\circ$), there exists $J \subset [0,S]$ closed interval (which can also reduce to a singleton and contains $S$ if $x \in \mathrm{B}^\circ(S)$) such that*

$$x \in \mathrm{T}^\circ(s) \quad \textit{if and only if} \quad s \in J\,;$$

*(ii) if $n \in \{0, \dots, N\}$, $j \in \{1, \dots, h_n\}$, and $(s_1, s_2) \subset I_n^\circ$ are such that $\gamma_j^\circ(s) \notin \mathrm{E}^\circ$ for every $s \in (s_1, s_2)$, then*

$$(s_1, s_2) \ni s \mapsto \mathcal{G}(t^\circ(s); \Gamma^\circ(s), \gamma_j^\circ(s)) \quad \textit{is continuous} \tag{5.4.11}$$

*and for a.e.* $s \in (s_1, s_2)$

$$\big(l_j^\circ\big)'(s) \geq 0\,; \tag{5.4.12a}$$

$$\text{If } (t^\circ)'(s) > 0\,, \text{ then } \mathcal{G}(t^\circ(s); \Gamma^\circ(s), \gamma_j^\circ(s)) \leq 1\,; \tag{5.4.12b}$$

$$\text{If } \mathcal{G}(t^\circ(s); \Gamma^\circ(s), \gamma_j^\circ(s)) < 1\,, \text{ then } l_j^\circ \text{ is constant in a neighborhood of } s\,. \tag{5.4.12c}$$

*(iii) for every* $s \in [0, T]$ *it holds the energy-dissipation balance*

$$\begin{aligned}\mathcal{E}(t^\circ(s); \Gamma^\circ(s)) + l^\circ(s) = \mathcal{E}(0; \Gamma_0) + l_0 + \int_0^s \langle \nabla u^\circ(\tau), \nabla \dot{w}^\circ(\tau)\rangle \, \mathrm{d}\tau \\ + \sum_{n=0}^{\overline{n}} \int_{I_n^\circ \cap U^\circ} \sum_{j=1}^{h_n} \big(l_j^\circ\big)'(\tau) \left[\mathcal{G}(w^\circ(\tau); \Gamma^\circ(\tau), \gamma_j^\circ(\tau)) - 1\right] \mathrm{d}\tau \\ + \int_{(s^{\overline{n}}, s) \cap U^\circ} \sum_{j=1}^{h_{\overline{n}}} \big(l_j^\circ\big)'(\tau) \left[\mathcal{G}(w^\circ(\tau); \Gamma^\circ(\tau), \gamma_j^\circ(\tau)) - 1\right] \mathrm{d}\tau\,,\end{aligned} \tag{5.4.13}$$

*where* $\overline{n}$ *is such that* $s \in I_{\overline{n}}^\circ$*, and we adopt the convention* $\Big(l_j^\circ\Big)'(\tau)\, \mathcal{G}(w^\circ(\tau); \Gamma^\circ(\tau), \gamma_j^\circ(\tau)) = 0$ *if* $\gamma_j^\circ(\tau) \notin \mathrm{G}_{\Gamma^\circ(\tau)}$*; see* (5.1.9) *for the definition of* $\mathrm{G}_{\Gamma^\circ(\tau)}$*.*

REMARK 5.4.5. Assume that $\tau$ belongs to the interior part of $I_n^\circ$, so that $\gamma_j^\circ(\tau) \notin \mathrm{F}$. If $\gamma_j^\circ(\tau) \notin \mathrm{G}_{\Gamma^\circ(\tau)}$, then, by Remark 5.4.3, we have $\gamma_j^\circ(\tau) \in \mathrm{B}^\circ(S)$. So the energy release rate is not defined at $\gamma_j^\circ(\tau)$, since there are no extensions of $\Gamma^\circ(\tau)$ at $\gamma_j^\circ(\tau)$ (see Definition 5.1.8), but the tip stops at $\gamma_j^\circ(\tau)$, and then the velocity is null. This justifies the convention $\Big(l_j^\circ\Big)'(\tau)\, \mathcal{G}(w^\circ(\tau); \Gamma^\circ(\tau), \gamma_j^\circ(\tau)) = 0$ for $\gamma_j^\circ(\tau) \notin \mathrm{G}_{\Gamma^\circ(\tau)}$.

REMARK 5.4.6. Let us fix $(s_1, s_2) \subset I_n^\circ$ such that $\gamma_j^\circ(s) \notin \mathrm{E}^\circ$ for every $s \in (s_1, s_2)$. Assuming (5.4.12a) and (5.4.12b), the condition (5.4.12c) implies that for a.e. $s \in (s_1, s_2)$ the following hold:

$$\begin{aligned}&\text{If } (t^\circ)'(s) > 0 \text{ and } \big(l_j^\circ\big)' > 0\,, \text{ then } \mathcal{G}(t^\circ(s); \Gamma^\circ(s), \gamma_j^\circ(s)) = 1\\ &\text{If } (t^\circ)'(s) = 0 \text{ and } \big(l_j^\circ\big)' > 0\,, \text{ then } \mathcal{G}(t^\circ(s); \Gamma^\circ(s), \gamma_j^\circ(s)) \geq 1\,.\end{aligned}$$

In particular, in view of (5.4.11),

$$\text{if } \mathcal{G}(t^\circ(s), \Gamma^\circ(s), \gamma_j^\circ(s)) > 1\,, \text{ then } s \in U^\circ\,. \tag{5.4.14}$$

(See (5.4.8) for the definition of $U^\circ$.)

In order to prove Theorem 5.4.4, we employ the result below, which follows the lines of Proposition 5.3.10. There a crucial point was to use the fact that the discrete lengths $\underline{l}_{\varepsilon,k}$ were equi-$H^1$ for $\varepsilon$ fixed. In the current setting, the lengths $l_\varepsilon^\circ$ are equi-Lipschitz with respect to $\varepsilon$.

PROPOSITION 5.4.7. *Let* $\widetilde{s} \in I_n^\circ$ *such that* $\gamma_j^\circ(\widetilde{s}) \notin \mathrm{F}$*, and let*

$$r_j^\circ(\widetilde{s}) := \omega(\mathrm{d}(\gamma_j^\circ(\widetilde{s}), \mathrm{F})) \wedge \eta \quad \textit{and} \quad \overline{s}_j^\circ(\widetilde{s}) := \left[\widetilde{s} - \frac{r_j^\circ(\widetilde{s})}{2}\right] \vee s^n\,,$$

*where $\omega$ is the modulus of continuity introduced in Remark 5.1.3. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $s \in (\overline{s}^\circ_j(\widetilde{s}), \widetilde{s}]$ the following hold:*

$$\mathrm{T}^\circ_\varepsilon(s) \cap \overline{B_{r^\circ_j(\widetilde{s})}(\gamma^\circ_j(\widetilde{s}))} \text{ contains one and only one element, called } p^j_\varepsilon(s)\,, \tag{5.4.15a}$$

$$\mathrm{S}^\circ_\varepsilon(s) \cap \overline{B_{r^\circ_j(\widetilde{s})}(\gamma^\circ_j(\widetilde{s}))} = \emptyset\,, \tag{5.4.15b}$$

$$\Gamma^\circ_\varepsilon(s) \cap \overline{B_{r^\circ_j(\widetilde{s})}(\gamma^\circ_j(\widetilde{s}))} \in \mathcal{R}_\eta\,. \tag{5.4.15c}$$

Proof. Arguing as in the proof of Proposition 5.3.10 it is possible to prove that (5.4.15) holds for $s = \widetilde{s}$ and that there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$

$$\mathcal{H}^1(\Gamma^\circ_\varepsilon(\widetilde{s}) \cap \overline{B_{r^\circ_j(\widetilde{s})}(\gamma^\circ_j(\widetilde{s}))}) \geq \frac{r^\circ_j(\widetilde{s})}{2}\,.$$

In order to see the corresponding properties for general $s$, it is enough to show that, for

$$s_\varepsilon := \min\{s \in [s^n, \widetilde{s})\colon \Gamma^\circ_\varepsilon(s) \cap \overline{B_{r^\circ_j(\widetilde{s})}(\gamma^\circ_j(\widetilde{s}))} \neq \emptyset\}\,,$$

it holds $s_\varepsilon \leq \overline{s}^\circ_j(\widetilde{s})$ for $\varepsilon \in (0, \varepsilon_0)$. This is implied by (5.4.6), which gives

$$\frac{r}{2} \leq \mathcal{H}^1(\Gamma^\circ_\varepsilon(\widetilde{s}) \setminus \Gamma^\circ_\varepsilon(s_\varepsilon)) = \int_{s_\varepsilon}^{\widetilde{s}} (l^\circ_\varepsilon)'(s)\,\mathrm{d}s \leq \widetilde{s} - s_\varepsilon\,.$$

and concludes the proof. □

Proof of Theorem 5.4.4. Arguing as done in Theorem 5.3.7, we can prove the statement (i) and the fact that $(s^n, s^{n+1})$ is the union of a finite number of open intervals where $\gamma^\circ_j(s) \notin \mathrm{E}^\circ$, and of a finite number of closed intervals in each of which $\gamma^\circ_j(s)$ is constant and belongs to $\mathrm{E}^\circ$.

In order to show (ii), let us fix $n \in \{0, \dots, N\}$, $j \in \{1, \dots, h_n\}$, and $(s_1, s_2) \subset I^\circ_n$ such that $\gamma^\circ_j(s) \notin \mathrm{E}^\circ$ for $s \in (s_1, s_2)$. As in Proposition 5.3.14, by Propositions 5.1.12 and 5.4.7 we deduce (5.4.11) and the convergence

$$\mathcal{G}(t^\circ_\varepsilon(s); \Gamma^\circ_\varepsilon(s), p^j_\varepsilon(s)) \to \mathcal{G}(t^\circ(s); \Gamma^\circ(s), \gamma^\circ_j(s)) \quad \text{for every } s \in (s_1, s_2)\,. \tag{5.4.16}$$

By (5.4.6), the functions

$$s \in (\overline{s}^\circ_j(\widetilde{s}), \widetilde{s}] \mapsto (g^\circ_\varepsilon)^j\,(s) := \mathcal{H}^1(\Gamma^\circ_\varepsilon(s) \cap B_{r^\circ_j(\widetilde{s})}(\gamma^\circ_j(\overline{s})))$$

are 1-Lipschitz and, by (5.4.3a),

$$(g^\circ_\varepsilon)_j \overset{*}{\rightharpoonup} (g^\circ)_j \quad \text{in } W^{1,\infty}((\overline{s}^\circ_j(\widetilde{s}), \widetilde{s}])\,,$$

where

$$s \in (\overline{s}^\circ_j(\widetilde{s}), \widetilde{s}] \mapsto (g^\circ)_j\,(s) := \mathcal{H}^1(\Gamma^\circ(s) \cap B_{r^\circ_j(\widetilde{s})}(\gamma^\circ_j(\overline{s}))) = l^\circ_j(s) - \mathcal{H}^1(\gamma^\circ_j([s^n, s]) \setminus B_{r^\circ_j(\widetilde{s})}(\gamma^\circ_j(\overline{s})))\,.$$

Notice that the time derivative of $(g^\circ_\varepsilon)_j$ depends only on $\Gamma^\circ_\varepsilon$, $s$, and on $p^j_\varepsilon(s)$. Thus we define

$$v^\circ_\varepsilon(s, p^j_\varepsilon(s)) := (g^\circ_\varepsilon)'_j\,(s)\,.$$

We also observe that the time interval $I^\circ_n$ may be approximated e.g. by two different intervals $I^{n_1}_\varepsilon$, $I^{n_2}_\varepsilon$: this is due to the fact that a branch of $\Gamma_\varepsilon(T)$ may disappear in the limit as $\varepsilon \to 0$.

For this reason we will have $p^j_\varepsilon(s) = \gamma^i_\varepsilon(l^i_\varepsilon(t^\circ_\varepsilon(s)))$ for some $i$ possibly depending on $\varepsilon$ and $s$. (See (5.3.3) for the definition of $l^i_\varepsilon$). In particular,

$$v^\circ_\varepsilon(s, p^j_\varepsilon(s)) = \dot{l}^i_\varepsilon(t^\circ_\varepsilon(s))\,(t^\circ_\varepsilon)'(s)\,. \tag{5.4.17}$$

As in the previous section (see (5.3.18)), for $(\overline{s}_1, \overline{s}_2) \subset\subset I^\circ_n$ such that $\gamma^\circ_j(s) \notin \mathrm{F}$ for every $s \in (\overline{s}_1, \overline{s}_2)$, we have that

$$v^\circ_\varepsilon(\cdot, p^j_\varepsilon(\cdot)) \overset{*}{\rightharpoonup} \big(l^\circ_j\big)' \quad \text{in } L^\infty(\overline{s}_1, \overline{s}_2)\,. \tag{5.4.18}$$

By (5.4.17) and the fact that $0 < (t^\circ_\varepsilon)'(s)$, we can rewrite (5.3.20) in the new variables as

$$v^\circ_\varepsilon(s, p^j_\varepsilon(s)) \geq 0\,, \tag{5.4.19a}$$

$$(t^\circ_\varepsilon)'(s) - \mathcal{G}(t^\circ_\varepsilon(s); \Gamma^\circ_\varepsilon(s), p^j_\varepsilon(s))\,(t^\circ_\varepsilon)'(s) + \varepsilon\, v^\circ_\varepsilon(s, p^j_\varepsilon(s)) \geq 0\,, \tag{5.4.19b}$$

$$\big[(t^\circ_\varepsilon)'(s) - \mathcal{G}(t^\circ_\varepsilon(s); \Gamma^\circ_\varepsilon(s), p^j_\varepsilon(s))\,(t^\circ_\varepsilon)'(s) + \varepsilon\, v^\circ_\varepsilon(s, p^j_\varepsilon(s))\big]\, v^\circ_\varepsilon(s, p^j_\varepsilon(s)) = 0\,, \tag{5.4.19c}$$

for a.e. $s \in (s_1, s_2)$.

As in Lemma 5.3.12, for every $s \in (s_1, s_2)$ there exists a set $A^\circ_j(s) \subset (s_1, s_2)$, at most countable, such that $(\overline{s}^\circ_j(t_1), t_1]$ and $(\overline{s}^\circ_j(t_2), t_2]$ are disjoint for $t_1 \neq t_2 \in A^\circ_j(s)$ and

$$(s_1, s] = \bigcup_{\widetilde{s} \in A^\circ_j(s)} (\overline{s}^\circ_j(\widetilde{s}), \widetilde{s}]\,.$$

Let us fix a subinterval $(\overline{s}^\circ_j(\widetilde{s}), \widetilde{s}] \subset (s_1, s_2)$. By Proposition 5.4.7, there exists $\varepsilon_0$ such that (5.4.15) holds for $\varepsilon \geq \varepsilon_0$ and $s \in (\overline{s}^\circ_j(\widetilde{s}), \widetilde{s}]$. Arguing as in the proof of Proposition 5.3.14, we get that there exists a positive constant $C$ independent of $\varepsilon \geq \varepsilon_0$ and $s \in (\overline{s}^\circ_j(\widetilde{s}), \widetilde{s}]$ such that

$$\mathcal{G}(t^\circ_\varepsilon(s); \Gamma^\circ_\varepsilon(s), p^j_\varepsilon(s)) \leq C \quad \text{for every } \varepsilon \geq \varepsilon_0 \text{ and } s \in (\overline{s}^\circ_j(\widetilde{s}), \widetilde{s}]\,.$$

Employing the fact that $\mathcal{G}(t^\circ_\varepsilon(s); \Gamma^\circ_\varepsilon(s), p^j_\varepsilon(s)) \geq 0$, and (5.4.16), we have that

$$\mathcal{G}(t^\circ_\varepsilon(\cdot); \Gamma^\circ_\varepsilon(\cdot), p^j_\varepsilon(\cdot)) \to \mathcal{G}(t^\circ(\cdot); \Gamma^\circ(\cdot), \gamma^\circ_j(\cdot)) \quad \text{in } L^q(\overline{s}^\circ_j(\widetilde{s}), \widetilde{s})\,, \quad \text{for every } q \in [1, +\infty)\,. \tag{5.4.20}$$

Let $\varphi \in L^2(\overline{s}^\circ_j(\widetilde{s}), \widetilde{s})$ such that $\varphi \geq 0$. By (5.4.19b)

$$\int_{\overline{s}^\circ_j(\widetilde{s})}^{\widetilde{s}} \varphi(s)\, \big[(t^\circ_\varepsilon)'(s) - \mathcal{G}(t^\circ_\varepsilon(s); \Gamma^\circ_\varepsilon(s), p^j_\varepsilon(s))\,(t^\circ_\varepsilon)'(s) + \varepsilon\, v^\circ_\varepsilon(s, p^j_\varepsilon(s))\big]\, \mathrm{d}s \geq 0\,.$$

By (5.4.3b), (5.4.18), and (5.4.20) we can pass to the limit obtaining that

$$\int_{\overline{s}^\circ_j(\widetilde{s})}^{\widetilde{s}} \varphi(s)\, \big[1 - \mathcal{G}(t^\circ(s); \Gamma^\circ(s), \gamma^\circ_j(s))\big]\,(t^\circ)'(s)\, \mathrm{d}s \geq 0\,,$$

and then (5.4.12b) follows by the arbitrariness of $\varphi$ and $\widetilde{s}$.

Let us prove (5.4.12c). First we show that, if $\mathcal{G}(t^\circ(s); \Gamma^\circ(s), \gamma^\circ_j(s)) < 1$, then there exists $\delta > 0$, depending only on $s$, such that

$$\mathcal{G}(t^\circ_{\varepsilon_k}(\cdot); \Gamma^\circ_{\varepsilon_k}(\cdot), p^j_{\varepsilon_k}(\cdot)) < 1 \quad \text{in } (s - \delta, s + \delta) \tag{5.4.21}$$

for $k$ sufficiently large. Otherwise, assume that there exist a sequence $t_k \to s$ such that

$$\mathcal{G}(t^\circ_{\varepsilon_k}(t_k); \Gamma^\circ_{\varepsilon_k}(t_k), p^j_{\varepsilon_k}(t_k)) \geq 1\,. \tag{5.4.22}$$

Since $|t^\circ_\varepsilon(t_k) - t^\circ_\varepsilon(s)| \le |t_k - s|$ and $|l^\circ_\varepsilon(t_k) - l^\circ_\varepsilon(s)| \le |t_k - s|$ for every $\varepsilon > 0$, we are allowed to apply Proposition 5.1.12. It follows that $\mathcal{G}(t^\circ(s);\Gamma^\circ(s),\gamma^\circ_j(s)) \ge 1$, which contradicts (5.4.22). Now, by (5.4.19c) and (5.4.21), we get that $p^j_{\varepsilon_k}$ is constant in $(s-\delta, s+\delta)$ and so is $l^\circ_j$. Then (ii) is proved.

Let us now show (iii). Arguing as in Proposition 5.3.9 we have that the total energy $\mathcal{F}(t^\circ(\cdot);\Gamma^\circ(\cdot))$ is $AC_{\rm loc}(I^\circ_n)$ for every $n \in \{0,\dots,N\}$ and that for a.e. $s \in I^\circ_n = (s^n, s^{n+1}]$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}(t^\circ(s);\Gamma^\circ(s)) = \sum_{j=1}^{h_n}\left[1 - \mathcal{G}(t^\circ(s);\Gamma^\circ(s),\gamma^\circ_j(s))\right]\left(l^\circ_j\right)'(s) + \langle \nabla u^\circ(s), \nabla \dot{w}^\circ(s)\rangle\,,$$

with the convention $\mathcal{G}(t^\circ(s);\Gamma^\circ(s),\gamma^\circ_j(s))\left(l^\circ_j\right)'(s) = 0$ if $\gamma^\circ_j(s) \notin \mathrm{G}_{\Gamma^\circ(\tau)}$.

Integrating in $(s_1,s_2) \subset\subset I^\circ_n$,

$$\begin{aligned}\mathcal{F}(t^\circ(s_2);\Gamma^\circ(s_2)) - \mathcal{F}(t^\circ(s_1);\Gamma^\circ(s_1)) = &\int_{s_1}^{s_2}\sum_{j=1}^{h_n}\left[1 - \mathcal{G}(t^\circ(\tau);\Gamma^\circ(\tau),\gamma^\circ_j(\tau))\right]\left(l^\circ_j\right)'(\tau)\,\mathrm{d}\tau \\ &+ \int_{s_1}^{s_2}\langle \nabla u^\circ(\tau), \nabla \dot{w}^\circ(\tau)\rangle\,\mathrm{d}\tau\,.\end{aligned}$$

Notice that we can pass to the limit as $s_1 \to s^n$ and $s_2 \to s^{n+1}$ since the positive part of $\sum_{j=1}^{h_n}\left[1 - \mathcal{G}(t^\circ(\tau);\Gamma^\circ(\tau),\gamma^\circ_j(\tau))\right]\left(l^\circ_j\right)'(\tau)$ is less than one and we can use Monotone Convergence Theorem for the negative part. Since $\mathcal{F}(t^\circ(\cdot);\Gamma^\circ(\cdot))$ is continuous, we can then sum up over the intervals $I^\circ_n$, whose number is finite.

We are left to prove that in the last two lines of (5.4.13) there is no contribution for $\tau \notin U^\circ$. As observed before, $(s^n, s^{n+1})$ is the union of a finite number of open intervals such that $\gamma^\circ_j(s) \notin \mathrm{E}^\circ$ for every $s$ in these subintervals, and of a finite number of closed intervals in each of which $\gamma^\circ_j$ is constant and belongs to $\mathrm{E}^\circ$. If we are in an interval of the first type, by (5.4.12c) and (5.4.14),

$$\sum_{j=1}^{h_n}\left[1 - \mathcal{G}(t^\circ(s);\Gamma^\circ(s),\gamma^\circ_j(s))\right]\left(l^\circ_j\right)'(s) = 0 \quad \text{for } s \notin U^\circ\,.$$

On the other hand, if we are in an interval $[\overline{s}_1, \overline{s}_2]$ such that $\gamma^\circ_j(s) = \overline{x} \in \mathrm{E}^\circ$, then $\left(l^\circ_j\right)'(s) = 0$. (Recall also the convention adopted for the points $\overline{x} \notin \mathrm{G}_{\Gamma^\circ(s)}$.) Threfore we conclude (5.4.13) and the proof is completed. □

REMARK 5.4.8. Arguing as in [**67**, Theorem 8.7] we have that for every $n \in \{0,\dots,N\}$, $j \in \{1,\dots,h_n\}$, and $(s_1,s_2) \subset I^\circ_n$ such that $\gamma^\circ_j(s) \notin \mathrm{E}^\circ$ for every $s \in (s_1,s_2)$, there exists a continuous function $\lambda\colon (s_1,s_2) \to [0,+\infty)$, independent of $j$, such that for a.e. $s$ and every $j$

$$\lambda(s)\left(l^\circ_j\right)'(s) = \left(\mathcal{G}(t^\circ(s);\Gamma^\circ(s),\gamma^\circ_j(s)) - 1\right)^+ \quad \text{and} \quad \lambda(s)(t^\circ)'(s) = 0\,.$$

Therefore, the rescaled evolution is governed by a viscous law in $U^\circ$. This gives insight on the unstable propagations, which correspond to jumps regime in the original time scale.

CHAPTER 6

# Cohesive fracture with irreversibility and fatigue

### Overview of the charapter

The present chapter considers the problem of quasistatic evolutions for a cohesive fracture on a prescribed crack surface, in small-strain antiplane elasticity. Precisely, we study a cohesive model where the density of the energy dissipated in the fracture process depends on the total variation of the amplitude of the jump. Thus, any change in the crack opening entails a loss of energy, until the crack is complete. For this reason it may happen that oscillations of small jumps produce a complete fracture, displaying a fatigue phenomenon. As pointed out in the Introduction, the main mathematical difficulty is related to the lack of good controls on the approximate evolutions obtained by incremental minimization: this leads to pass through a weak notion of quasistatic evolution, that involves Young measures. The results of this chapter, proven in collaboration with Giuliano Lazzaroni and Gianluca Orlando, are contained in [**27**].

The notion of quasistatic evolution and the main existence result are presented in Section 6.1, which contains also some results on a strong formulation that is satisfied by the weak solutions under suitable regularity assumptions. The final part of Section 6.1 contains a short presentation of the existence proof, which is given in more detail in the remaining part of the chapter. After recalling some preliminary results on Young measures (Section 6.2), we introduce the discrete-time problems in Section 6.3 and we pass to the continuous-time limit in Section 6.4, obtaining the formulation based on Young measures. Finally, in Section 6.5 we prove the existence of quasistatic evolutions according to the notion based on functions.

In the sequel, we will often consider time-dependent functions $t \mapsto v(t)$, where $v(t)$ is a function depending on a space variable $x$. We will write $v(t;x)$ to refer to the value of $v(t)$ in $x$.

## 6.1. Assumptions on the model and statement of the main result

**Reference configuration and boundary conditions.** Throughout the chapter, $\Omega$ is a bounded, Lipschitz, open set in $\mathbb{R}^n$ representing the cross-section of a cylindrical body in the reference configuration (in the setting of antiplane shear). The cracks of the body will be contained in a prescribed crack surface $\Gamma$, where $\Gamma$ is a $(n-1)$-dimensional Lipschitz manifold in $\mathbb{R}^n$ with $0 < \mathcal{H}^{n-1}(\Gamma \cap \overline{\Omega}) < \infty$. Moreover, we assume that $\Omega \setminus \Gamma = \Omega^+ \cup \Omega^-$, where $\Omega^+$ and $\Omega^-$ are disjoint open connected sets with Lipschitz boundary. The normal $\nu(x) = \nu_\Gamma(x)$ to the surface $\Gamma$ is chosen in such a way that it coincides with the outer normal to $\partial\Omega^-$.

We consider evolutions driven by a time-dependent boundary condition assigned on the Dirichlet part of the boundary $\partial_D\Omega$. We assume that $\partial_D\Omega$ is a relatively open set of $\partial\Omega$ and that $\mathcal{H}^{n-1}(\partial_D\Omega \cap \partial\Omega^{\pm}) > 0$, in order to apply the Poincaré Inequality separately in $\Omega^+$ and $\Omega^-$. We denote by $\partial_N\Omega$ the remaining part of the boundary, i.e., $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$.

For every $w \in H^1(\Omega)$, we define the set of *admissible displacements* corresponding to $w$ by

$$A(w) := \{u \in H^1(\Omega \setminus \Gamma)\colon u = w \text{ on } \partial_D\Omega\}\,. \tag{6.1.1}$$

We assign a function $t \mapsto w(t)$ defined on $[0,T]$ with values in $H^1(\Omega)$ and we assume that

$$t \mapsto w(t) \quad \text{belongs to} \quad AC([0,T]; H^1(\Omega))\,. \tag{6.1.2}$$

**Variation of jumps and initial data.** In order to give the notion of quasistatic evolution, we introduce a function $V_u(t)$ describing the variation of the jumps on $\Gamma$ of an evolution $t \mapsto u(t)$ in a time interval $[0,t]$.

Before defining $V_u(t)$, we recall the definition of the essential supremum of a family of measurable functions, that is the least upper bound in the sense of a.e. inequality. We give this definition in the case of functions defined on the measure space $(\Gamma; \mathcal{H}^{n-1})$. Indeed, this will be the relevant setting for our model.

Definition 6.1.1. Let $(v_i)_{i\in I}$ be a family of measurable functions from $\Gamma$ to $[-\infty,\infty]$. Let $\overline{v}\colon \Gamma \to [-\infty,\infty]$ be a measurable function such that

(i) $\overline{v} \geq v_i$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma$, for every $i \in I$;
(ii) if $v\colon \Gamma \to [-\infty,\infty]$ is a measurable function such that $v \geq v_i$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma$, for every $i \in I$, then $v \geq \overline{v}$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma$.

We say that $\overline{v}$ an *essential supremum* of the family $(v_i)_{i\in I}$.

Remark 6.1.2. Given a family of measurable functions $(v_i)_{i\in I}$, there exists a unique (up to $\mathcal{H}^{n-1}$-a.e. equivalence) essential supremum $\overline{v}$ of the family $(v_i)_{i\in I}$. We denote it by $\operatorname*{ess\,sup}_{i\in I} v_i := \overline{v}$.

We now define the essential variation, namely the variation for a time-dependent family of measurable functions, in the sense of a.e. inequality. As done for the essential supremum, we give this definition in the case of functions defined on the measure space $(\Gamma; \mathcal{H}^{n-1})$.

Definition 6.1.3. Let us consider a function $t \mapsto \gamma(t)$, with $\gamma(t)\colon \Gamma \to \mathbb{R}$ measurable for every $t \in [0,T]$. For every $0 \leq t_1 \leq t_2 \leq T$, the *essential variation* of $\gamma$ in $[t_1,t_2]$ is the function $\operatorname{ess\,Var}(\gamma; t_1, t_2)\colon \Gamma \to [0,\infty]$ defined by

$$\operatorname{ess\,Var}(\gamma; t_1, t_2) := \operatorname{ess\,sup}\Big\{ \sum_{i=1}^{j} |\gamma(s_i) - \gamma(s_{i-1})| \colon j \in \mathbb{N}\,,\ t_1 = s_0 < s_1 < \dots < s_{j-1} < s_j = t_2 \Big\}\,.$$

Remark 6.1.4. The essential variation satisfies the usual property that

$$\operatorname{ess\,Var}(\gamma; t_1, t_3) = \operatorname{ess\,Var}(\gamma; t_1, t_2) + \operatorname{ess\,Var}(\gamma; t_2, t_3) \quad \mathcal{H}^{n-1}\text{-a.e. on } \Gamma\,,$$

for any $0 \leq t_1 < t_2 < t_3 \leq t$.

Given a function $t \mapsto u(t)$ defined on $[0,T]$ with values in $H^1(\Omega\backslash\Gamma)$, we define the variation $V_u(t)\colon \Gamma \to [0,\infty]$ of its jumps on $\Gamma$ with initial condition $V_0$ by

$$V_u(t) := \operatorname{ess\,Var}([u];0,t) + V_0\,, \tag{6.1.3}$$

for every $t \in [0,T]$, where $V_0\colon \Gamma \to [0,\infty]$ is an assigned measurable function.

**Initial data.** We fix an initial displacement

$$u_0 \in A(w(0)) \tag{6.1.4}$$

and a function $V_0\colon \Gamma \to [0,\infty]$ accounting for the variation of previous jumps until the initial time $t=0$. Indeed we assume that

$$V_0(x) \geq \big|[u_0(x)]\big| \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\,. \tag{6.1.5}$$

If $V_0 = \big|[u_0]\big|$, a monotone crack opening has occurred before the initial time $t=0$. In general, the crack opening may have oscillated before the initial time in such a way that its variation in time equals $V_0$. The set $\Gamma_N(0) := \{V_0 \geq \theta(x)\}$ represents the part of $\Gamma$ which is already completely broken at the beginning of the process.

**The surface energy density.** We assume that the surface energy density $g$ depends on the point on $\Gamma$ and on the history of the jump. More precisely, $g\colon \Gamma\times[0,\infty) \to [0,\infty)$ satisfies the following assumptions:

- ($g1$) $g$ is a Carathéodory integrand, i.e., $g(x,\cdot)$ is continuous for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ and $g(\cdot,\xi)$ is $\mathcal{H}^{n-1}$-measurable for every $\xi \in [0,\infty)$;
- ($g2$) $g(x,0)=0$ and $g(x,\cdot)$ is concave for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$;
- ($g3$) $\lim\limits_{\xi\to\infty} g(x,\xi) = \kappa(x) \in [\kappa_1,\kappa_2]$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Gamma$, where $\kappa_1,\kappa_2 \in (0,\infty)$;
- ($g4$) the limit
$$\lim_{\xi\to 0^+} \frac{g(x,\xi)}{\xi} =: g'(x,0)$$
exists for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ and $g'(\cdot,0) \in L^\infty(\Gamma)$.

In particular, for $\mathcal{H}^{n-1}$-a.e. $x\in\Gamma$ it turns out that $g(x,\cdot)$ is nondecreasing and can be extended to a function in $C_b([0,\infty])$ by setting $g(x,\infty) := \kappa(x)$.

It will be convenient to introduce a measurable function $\theta\colon \Gamma \to [0,\infty]$ that represents the threshold after which the function $g(x,\cdot)$ becomes constant, i.e.,

$$\theta(x) := \inf\{\xi > 0\colon g(x,\xi) = \kappa(x)\} \in (0,\infty]\,. \tag{6.1.6}$$

The function $g(x,\cdot)$ is strictly increasing if and only if $\theta(x)=\infty$.

As already discussed in the Introduction, it is convenient to write the energy dissipated by a crack opening (cf. Figure 1) as a function of the variation of the jump $V_u(t)$ defined in (6.1.3) (cf. Figure 2):

$$\int_\Gamma g(x,V_u(t))\,\mathrm{d}\mathcal{H}^{n-1}.$$

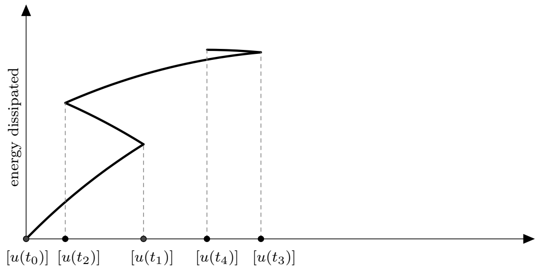


**Figure 1.** Energy dissipated by a jump $t \mapsto [u(t)]$ with a non-monotone history in a time interval $[t_0, t_4]$: $t \mapsto [u(t)]$ increases in $[t_0, t_1]$ and in $[t_2, t_3]$, whereas it decreases in $[t_1, t_2]$ and in $[t_3, t_4]$.

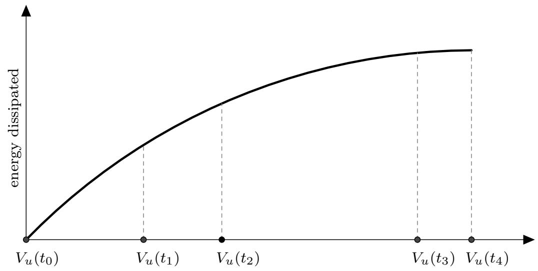


**Figure 2.** Energy dissipated as a function of the variation of the jumps $V_u(t)$ corresponding to a jump history as in Figure 1. Notice that the variation $V_u(t)$ is nondecreasing in time.

Remark 6.1.5. In the cohesive models studied in [**37**] and [**15**], the variable used to describe the energy dissipated is the supremum of the jumps reached during the evolution. This is the main point where our cohesive model differs from those considered in [**37**] and [**15**].

Cohesive models shares also some similarities with problems of delamination and adhesive contact, see e.g. [**63, 92**] for the energetic formulation of quasistatic evolutions. However, in such models the surface energy density depends on an internal variable and not on the previous history of the jump of the displacement.

**Definition of quasistatic evolution and strong formulation.** We are now in a position to give the definition of quasistatic evolution.

Definition 6.1.6. Let $w$, $u_0$, and $V_0$ be as in (6.1.2)–(6.1.5). Let $t \mapsto u(t)$ be a function defined on $[0, T]$ with values in $H^1(\Omega \setminus \Gamma)$ and let $V_u(t)$ be the variation of its jumps on $\Gamma$, defined in (6.1.3). We say that $t \mapsto u(t)$ is a *quasistatic evolution* with initial conditions $(u_0, V_0)$ and boundary datum $w$ if $u$ satisfies $u(0) = u_0$ and the following conditions:

(GS) *Global stability*: For every $t \in [0, T]$ we have $u(t) \in A(w(t))$ and

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_{\Gamma} g\big(x, V_u(t)\big) \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \widehat{u}|^2 \,\mathrm{d}x + \int_{\Gamma} g\big(x,\, V_u(t) + \big|[\widehat{u}] - [u(t)]\big|\big) \,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\widehat{u} \in A(w(t))$.

(EB) *Energy-dissipation balance*: For every $t \in [0,T]$

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} = \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_\Gamma g\big(x, V_0\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot w(s)\rangle_{L^2}\,\mathrm{d}s\,.$$

In order to give an insight into the strong formulation of the model studied in the chapter, we state two results regarding necessary conditions satisfied by a quasistatic evolution. For simplicity, we derive these differential conditions under the assumption that $g(x,\cdot)$ is of class $C^1$. We denote by $g'(x,\xi)$ the derivative of $g(x,\xi)$ with respect to $\xi$.

Proposition 6.1.7. *Assume that $g(x,\cdot)$ is of class $C^1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Let $t \mapsto u(t)$ be a function defined on $[0,T]$ with values in $H^1(\Omega\setminus\Gamma)$ and satisfying* (GS)*. Then for every $t\in[0,T]$ the following hold:*

(i) *The function $u(t)$ is a weak solution to the problem*

$$\begin{cases} \Delta u(t) = 0 & \text{in } \Omega\setminus\Gamma\,,\\ u(t) = w(t) & \text{on } \partial_D\Omega\,,\\ \partial_\nu u(t) = 0 & \text{in } H^{-\frac{1}{2}}(\partial_N\Omega)\,. \end{cases}$$

(ii) *Let $u(t)^+ := u(t)|_{\Omega^+}$ and $u(t)^- := u(t)|_{\Omega^-}$. Then $\partial_\nu u(t)^+ = \partial_\nu u(t)^-$ in $H^{-\frac{1}{2}}(\Gamma)$.*

(iii) *Let $\partial_\nu u(t) := \partial_\nu u(t)^+ = \partial_\nu u(t)^-$. Then $\partial_\nu u(t) \in L^\infty(\Gamma)$ and*

$$|\partial_\nu u(t;x)| \le g'(x, V_u(t;x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\,. \tag{6.1.7}$$

To keep the presentation clear, the proof of Proposition 6.1.7 is given in Section 6.5.

Condition (iii) in Proposition 6.1.7 expresses the fact that the surface tension on $\Gamma$ due to the displacement is constrained to stay below a suitable threshold. This threshold decreases in time, since $g'(x,\cdot)$ is nonincreasing and $V_u(\cdot\,;x)$ is nondecreasing in time. However, this condition is static and is not enough to characterise an evolution.

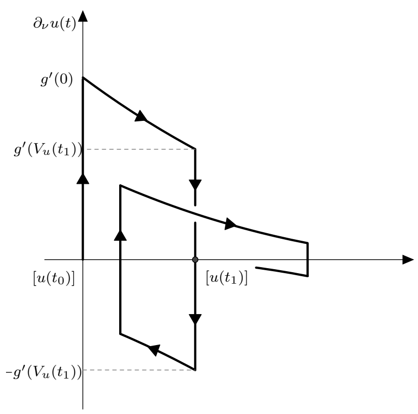


**Figure 3.** Crack opening versus surface tension corresponding to a jump history as in Figure 1.

Nonetheless, in the following proposition we employ the energy-dissipation balance to show that the evolution satisfies a flow rule: in the points where a crack opening grows, the surface tension actually must reach the maximal threshold. (See Figure 3 for a possible evolution of the surface tension.) The result is proved under regularity assumptions on the evolution $t \mapsto u(t)$. To make the statement concise, we denote by Sign the multifunction given by

$$\operatorname{Sign}(\xi) := \begin{cases} 1 & \text{if } \xi > 0\,, \\ [-1,1] & \text{if } \xi = 0\,, \\ -1 & \text{if } \xi < 0\,, \end{cases}$$

PROPOSITION 6.1.8. *Assume that $g(x,\cdot)$ is of class $C^1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Let $t \mapsto u(t)$ be a quasistatic evolution in the sense of Definition 6.1.6 and assume that $u \in AC([0,T]; H^1(\Omega \setminus \Gamma))$. Then*

$$\partial_\nu u(t;x) \in g'(x, V_u(t;x)) \operatorname{Sign}\big([\dot u(t;x)]\big) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma \text{ and a.e. } t \in [0,T]\,,$$

*where $[\dot u(t)]$ is the derivative in time of $[u(t)]$ with respect to the strong topology in $L^2(\Gamma)$.*

Proposition 6.1.8 is proved in Section 6.5.

**Statement of the main result.** We now introduce the tools needed to state our main result, which concern the existence of a quasistatic evolution and the approximation by means of discrete-time evolutions.

As usual in the proof of existence of quasistatic evolutions for rate-independent systems, we construct discrete-time evolutions by solving incremental minimum problems. For every $k \in \mathbb{N}$, let us consider a subdivision of the time interval $[0,T]$ given by $k{+}1$ nodes

$$0 = t_k^0 < t_k^1 < \dots < t_k^{k-1} < t_k^k = T, \quad \lim_{k\to\infty} \max_{1\le i\le k} |t_k^i - t_k^{i-1}| = 0\,,$$

and let us define $w_k^i := w(t_k^i)$.

We assume that the initial condition $(u_0, V_0)$ is globally stable, namely

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \, dx + \int_\Gamma g\big(x, V_0\big)\, d\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \widehat u|^2 \, dx + \int_\Gamma g\big(x,\, V_0 + \big|[\widehat u]{-}[u_0]\big|\big)\, d\mathcal{H}^{n-1}\,, \tag{6.1.8}$$

for every $\widehat u \in A(w(0))$.

As the first step of the incremental process, we set $u_k^0 := u_0$ and $V_k^0 := V_0$. Let $i \in \{1,\dots,k\}$ and assume that we know $u_k^h$ and $V_k^h$ for $h = 0,\dots,i-1$. Then we define $u_k^i$ as a solution to the problem

$$\min_u \left\{ \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2 \, dx + \int_\Gamma g\big(x,\, V_k^{i-1} + \big|[u]{-}[u_k^{i-1}]\big|\big)\, d\mathcal{H}^{n-1} \colon u \in A(w_k^i) \right\}, \tag{6.1.9}$$

and we set

$$V_k^i := V_k^{i-1} + \big|[u_k^i]{-}[u_k^{i-1}]\big| = V_0 + \sum_{j=1}^{i} \big|[u_k^j]{-}[u_k^{j-1}]\big|\,. \tag{6.1.10}$$

The existence of a solution to (6.1.9) is obtained by employing the direct method of the Calculus of Variations.

The discrete-time evolutions are then defined as piecewise constant interpolations of the solutions to the incremental problems. Namely, we set

$$u_k(t) := u_k^i\,, \quad V_k(t) := V_k^i\,, \quad w_k(t) := w_k^i \quad \text{for } t_k^i \le t < t_k^{i+1} \tag{6.1.11}$$

and $u_k(T) := u_k^k$, $V_k(T) := V_k^k$, $w_k(T) := w(T)$.

Passing to the limit as $k \to \infty$, we prove that $u_k$ converges to a quasistatic evolution $u$. A major point of our result is that the convergence holds for a subsequence independent of $t$. We also provide a convergence result for the variations of the jumps. Specifically, the truncated functions $V_k(t) \wedge \theta$ converge to $V_u(t) \wedge \theta$, where $\theta$ is as in (6.1.6), and $\wedge$ denotes the minimum between two functions. We remark that when $V_u(t;x)$ overcomes the threshold $\theta(x)$, we have no control on $V_u(t;x)$, which may increase without further dissipation of energy. Moreover, we obtain that $t \mapsto u(t)$ and $t \mapsto V_u(t)$ are continuous (in a suitable sense), except for countably many times.

These results are stated in the following theorem, whose proof is given in Section 6.5.

THEOREM 6.1.9 (Existence and approximation of quasistatic evolutions). *Assume that $g$ satisfies $(g1)$–$(g4)$. Let $w$, $u_0$, and $V_0$ be as in* (6.1.2)–(6.1.5) *and assume that $(u_0, V_0)$ is globally stable in the sense of* (6.1.8). *Consider the piecewise constant evolutions $t \mapsto u_k(t)$ and the piecewise constant variations $t \mapsto V_k(t)$ defined in* (6.1.11). *Then there exist a subsequence (independent of $t$ and not relabelled) and a quasistatic evolution $t \mapsto u(t)$ with initial conditions $(u_0, V_0)$ and boundary datum $w$ such that, for every $t \in [0,T]$,*

$$u_k(t) \to u(t) \qquad \textit{strongly in } H^1(\Omega \setminus \Gamma)\,, \tag{6.1.12}$$

$$V_k(t) \wedge \theta \to V_u(t) \wedge \theta \quad \textit{in measure}\,, \tag{6.1.13}$$

*where $V_u(t)$ is the function defined in* (6.1.3) *and $\theta$ is given in* (6.1.6).

*Moreover, there exists a set $E \subset [0,T]$, at most countable, such that, for every $t \in [0,T] \setminus E$ and every $s \to t$,*

$$u(s) \to u(t) \qquad \textit{strongly in } H^1(\Omega \setminus \Gamma)\,. \tag{6.1.14}$$

$$V_u(s) \wedge \theta \to V_u(t) \wedge \theta \quad \textit{in measure}\,. \tag{6.1.15}$$

We underline that, if $\theta(x)$ is finite and $V_u(t;x) \ge \theta(x)$, the material is completely broken at $x$. Therefore $V_u(t) \wedge \theta$, appearing in the theorem above, is the relevant state variable for the system.

REMARK 6.1.10. If $\theta \in L^\infty(\Gamma)$, then the convergence in (6.1.13) and (6.1.15) is also strong in $L^p(\Gamma)$ for every $p \in [1, \infty)$. In contrast, if $\theta \equiv \infty$ (that is $g(x, \cdot)$ is strictly increasing for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$), then $V_k(t) \to V_u(t)$ in measure as $k \to \infty$ and $V_u(s) \to V_u(t)$ in measure as $s \to t$.

**Guidelines for the proof of the main result.** The main difficulty in the passage to the continuous-time limit as $k \to \infty$ is that we lack of controls on $V_k(t)$. In fact, by (6.1.9), we can only infer that $\int_\Gamma g(x, V_k(t))\, \mathrm{d}\mathcal{H}^{n-1}$ is uniformly bounded, but this gives no information on $V_k(t)$, since $g$ is bounded. For this reason we resort to a weaker notion of quasistatic evolution, where the variation of jumps on $\Gamma$ is replaced by a Young measure. Notwithstanding, after establishing the properties of such an evolution, we are able to show that the Young measure found in the limit is concentrated on a function. Eventually, we obtain a quasistatic evolution in the sense of Definition 6.1.6. We describe here the strategy followed to prove Theorem 6.1.9.

Following the scheme of the proof of existence of energetic solutions to rate-independent systems [**79**], the starting point of our analysis is to obtain a global stability and an energy-dissipation inequality for the discrete-time evolutions $t \mapsto u_k(t)$ (Proposition 6.3.1). As usual, the energy-dissipation inequality provides a priori bounds in $H^1(\Omega \setminus \Gamma)$ for the functions $u_k(t)$, independently of $k$ and $t$. In order to study the limit of the functions $V_k(t)$, it is convenient to introduce the Young measures concentrated on the graph of $V_k(t)$, namely

$$\nu_k(t) := \delta_{V_k(t)} \in \mathcal{Y}(\Gamma; [0, \infty]) \quad \text{for every } t \in [0, T]\,. \tag{6.1.16}$$

We refer to Section 6.2 for the notation and the basic properties of Young measures. Since the functions $V_k(t)$ are nondecreasing with respect to $t$, we can apply a Helly-type selection principle (proved in [**15**]) to infer that the Young measures $\nu_k(t)$ converge narrowly to a Young measure $\nu(t) \in \mathcal{Y}(\Gamma; [0, \infty])$ on a subsequence independent of $t$. Thanks to the a priori bounds on $u_k(t)$, it is possible to extract a subsequence $k_j(t)$ (depending on $t$) such that $u_{k_j(t)}(t)$ converges to $u(t)$ weakly in $H^1(\Omega \setminus \Gamma)$. These convergences allow us to pass to the limit in the global stability of the discrete-time evolutions (Proposition 6.3.4), and thus to deduce that $t \mapsto (u(t), \nu(t))$ satisfies a suitable notion of global stability (condition (GSY) in Definition 6.4.1).

Afterwards, we show that the evolution $t \mapsto (u(t), \nu(t))$ satisfies an energy-dissipation balance (condition (EBY) in Definition 6.4.1). One inequality in this balance is a consequence of the energy-dissipation inequality of the discrete-time evolutions $t \mapsto u_k(t)$. On the contrary, the proof of the opposite inequality requires a thorough analysis. The main reason is that the Helly Selection Principle adopted before does not give any information about the relation between the Young measure $\nu(t)$ and $V_u(t)$. This relation is though encoded in a property satisfied by $t \mapsto \nu(t)$ (the *irreversibility* condition (IRY) in Definition 6.4.1), that we derive from the analogous condition $(\mathrm{IRY})_k$ for the approximating Young measures $t \mapsto \nu_k(t)$. This property relates $\nu(t)$ to $[u(t)]$ and allows us to conclude the proof of the other inequality in the energy-dissipation balance by employing the global stability.

In addition, we prove that $u_k(t)$ actually converges to $u(t)$ strongly in $H^1(\Omega \setminus \Gamma)$ on a subsequence independent of $t$. This convergence result is proved in Section 6.4 by showing that the jump $\gamma(t) := [u(t)]$ is determined *de facto* independently of $t$ (cf. equation (6.4.9)). Indeed this implies that the function $u(t)$ is the unique solution of a minimum problem among functions with a prescribed jump $\gamma(t)$ (Proposition 6.4.6). With similar arguments, we prove that $t \mapsto u(t)$ is continuous in $t$ except for a countable set $E \subset [0, T]$.

Finally, in Section 6.5 we prove that $u$ is actually a quasistatic evolution in the sense of Definition 6.1.6. Notice that for this step we need the assumption on the concavity of $g(x, \cdot)$. Moreover, this allows us to prove that the Young measure $\nu(t)$ (suitably truncated with $\theta$) is concentrated on the function $V_u(t)$. As a consequence of this fact, we are able to deduce also the convergences in (6.1.13) and (6.1.15) in Theorem 6.1.9.

## 6.2. Preliminary results about Young measures

**Probability measures.** Let $\Xi$ be a metric space. We denote by $M_b^+(\Xi)$ the set of positive bounded measures, and by $\mathcal{P}(\Xi)$ the set of probability measures on $\Xi$. The space $M_b^+(\Xi)$ can be put in duality with the space of bounded continuous functions $C_b(\Xi)$ by defining

$$\langle f, \mu \rangle := \int_{\Xi} f(\xi)\mu(\mathrm{d}\xi) = \int_{\Xi} f(\xi)\, \mathrm{d}\mu(\xi)\,, \tag{6.2.1}$$

for every $\mu \in M_b^+(\Xi)$ and $f \in C_b(\Xi)$.

If $\Xi$ is a separable metric space and $\mu \in M_b^+(\Xi)$, the support of $\mu$ is the smallest closed subset of $\Xi$ where the measure $\mu$ is concentrated, i.e.,

$$\operatorname{supp}(\mu) := \bigcap_{\substack{C \text{ closed} \\ \mu(\Xi \setminus C)=0}} C\,.$$

Let $\Xi_1$ and $\Xi_2$ be two metric spaces, let $\varphi : \Xi_1 \to \Xi_2$ be a Borel map, and let $\mu \in M_b^+(\Xi_1)$. The *push-forward* of $\mu$ through the map $\varphi$ is the measure $\varphi_\# \mu \in M_b^+(\Xi_2)$ defined by $\varphi_\#\mu(A) := \mu(\varphi^{-1}(A))$ for every $A \in \mathscr{B}(\Xi_2)$.

We will later deal with measures in the space $M_b^+([-\infty,\infty])$, where $[-\infty,\infty]$ is endowed with the metric induced by an increasing homeomorphism

$$\phi\colon [-\infty,\infty] \to [-1,1]\,, \tag{6.2.2}$$

e.g. $\phi(\xi) := \frac{2}{\pi}\arctan(\xi)$. Measures in $M_b^+([-\infty,\infty])$ are in duality with bounded continuous functions $f \in C_b([-\infty,\infty])$, i.e., continuous functions with a finite limit at $\pm\infty$.

We also recall that for every probability measure $\mu \in \mathcal{P}([-\infty,\infty])$ we can define the *cumulative distribution function* $F_\mu \colon [-\infty,\infty] \to [0,1]$ by

$$F_\mu(\xi) := \mu([-\infty,\xi]) \quad \text{for every } \xi \in [-\infty,\infty]\,. \tag{6.2.3}$$

By the right continuity of $F_\mu$, it is possible to define its *pseudo-inverse* $F_\mu^{[-1]} \colon [0,1] \to [-\infty,\infty]$ by

$$F_\mu^{[-1]}(m) := \min\{\xi \in \mathbb{R} : F_\mu(\xi) \geq m\}\,. \tag{6.2.4}$$

**Young measures.** For an introduction to the general theory of Young measures we refer, e.g., to [**107**]. Here we recall some basic notions and properties. Let us fix a metric space $\Xi$.

Definition 6.2.1. The collection of *Young measures* on $\Gamma{\times}\Xi$ with respect to the measure $\mathcal{H}^{n-1}$ is the set

$$\mathcal{Y}(\Gamma;\Xi) := \{\nu \in M_b^+(\Gamma{\times}\Xi) : \pi^\Gamma_\# \nu = \mathcal{H}^{n-1} \llcorner \Gamma\}\,,$$

where $\pi^\Gamma \colon \Gamma{\times}\Xi \to \Gamma$ is the projection on $\Gamma$.

Remark 6.2.2. We recall that a family $(\nu^x)_{x\in\Gamma}$ of probability measures $\nu^x \in \mathcal{P}(\Xi)$ parametrised on $\Gamma$ is said to be *measurable* if the function $x \mapsto \nu^x(A)$ is $\mathcal{H}^{n-1}$-measurable for every

$A \in \mathscr{B}(\Xi)$. By the Disintegration Theorem (see [**6**, Theorem 2.28]), it is always possible to associate a measurable family of probability measures $(\nu^x)_{x\in\Gamma}$ with a Young measure $\nu \in \mathcal{Y}(\Gamma; X)$ in such a way that

$$\int_{\Gamma\times\Xi} f(x,\xi)\,\mathrm{d}\nu = \int_{\Gamma}\int_{\Xi} f(x,\xi)\nu^x(\mathrm{d}\xi)\,\mathrm{d}\mathcal{H}^{n-1} \quad \text{for every } f \in L^1_\nu(\Gamma\times X)\,. \tag{6.2.5}$$

Moreover, the family $(\nu^x)_{x\in\Gamma}$ is unique up to $\mathcal{H}^{n-1}$-negligible sets, i.e., if $(\widehat{\nu}^x)_{x\in\Gamma}$ is any other measurable family of probability functions satisfying (6.2.5), then $\widehat{\nu}^x = \nu^x$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.

If $\nu = (\nu^x)_{x\in\Gamma} \in \mathcal{Y}(\Gamma;\Xi)$, for every $f \in C_b(\Gamma\times\Xi)$ the duality between $\nu$ and $f$ reads

$$\int_{\Gamma\times\Xi} f(x,\xi)\,\mathrm{d}\nu(x,\xi) = \int_{\Gamma}\int_{\Xi} f(x,\xi)\nu^x(\mathrm{d}\xi)\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma}\langle f(x,\cdot),\nu^x\rangle\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

EXAMPLE 6.2.3. The simplest example of a Young measure is obtained by fixing a measurable function $v\colon \Gamma \to \Xi$ and by considering the Young measure *concentrated* on the graph of the function $v$, identified by the measurable family of probability measures $\delta_v := (\delta_{v(x)})_{x\in\Gamma}$.

We will consider the space $\mathcal{Y}(\Gamma;\Xi)$ endowed with the *narrow* topology.

DEFINITION 6.2.4. We say that $\nu_j$ converges *narrowly* to $\nu$ (and denote $\nu_j \rightharpoonup \nu$) if and only if

$$\int_{\Gamma}\langle f(x,\cdot),\nu_j^x\rangle\,\mathrm{d}\mathcal{H}^{n-1} \to \int_{\Gamma}\langle f(x,\cdot),\nu^x\rangle\,\mathrm{d}\mathcal{H}^{n-1}, \tag{6.2.6}$$

for every $f \in C_b(\Gamma\times\Xi)$.

REMARK 6.2.5. If $\Xi$ is a compact metric space, by [**107**, Theorem 2] the convergence in (6.2.6) also holds for every *Carathéodory integrand* $f$, i.e., a measurable function such that $f(x,\cdot) \in C_b(\Xi)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ and such that $x \mapsto \|f(x,\cdot)\|_\infty$ belongs to $L^1(\Gamma)$.

The narrow convergence for concentrated Young measures is characterised in the following proposition. For the proof, we refer to [**107**, Proposition 6].

PROPOSITION 6.2.6. *Assume that $\Xi$ is a compact metric space. Let $v_j, v\colon \Gamma \to \Xi$ be measurable functions. Then $\delta_{v_j} \rightharpoonup \delta_v$ if and only if $v_j \to v$ in measure.*

REMARK 6.2.7. In the case where $\Xi$ is $[-\infty,\infty]$ endowed with the metric induced by $\phi$ in (6.2.2), then $v_j \to v$ in measure if and only if $\mathcal{H}^{n-1}(\{|\phi(v_j)-\phi(v)| \geq \varepsilon\}) \to 0$ for every $\varepsilon > 0$.

The following compactness result holds (cf. [**107**, Theorem 2]).

THEOREM 6.2.8. *Assume that $\Xi$ is a compact metric space. Then $\mathcal{Y}(\Gamma;\Xi)$, endowed with the narrow topology, is sequentially compact.*

REMARK 6.2.9. The assumption on the compactness of the space $\Xi$ is crucial to guarantee the compactness of $\mathcal{Y}(\Gamma;\Xi)$ with respect to the narrow convergence. For instance, if $\Xi = \mathbb{R}$, it may happen that a sequence $\nu_j \in \mathcal{Y}(\Gamma;\mathbb{R})$ has some mass escaping to infinity.

We will later need to infer the compactness of sequences $\nu_j \in \mathcal{Y}(\Gamma;\mathbb{R})$ with no tightness assumptions. Thus, we will consider a compactification of $\mathbb{R}$, i.e., we will regard $\nu_j$ as Young measures in $\mathcal{Y}(\Gamma;[-\infty,\infty])$. In this way, we can conclude that a subsequence of $\nu_j \in \mathcal{Y}(\Gamma;\mathbb{R})$ (not relabelled) converges narrowly to a Young measure $\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])$.

To deal with these Young measures, it is convenient to introduce the map

$$\Phi\colon \Gamma\times[-\infty,\infty] \to \Gamma\times[-1,1]\,, \quad \Phi(x,\xi) := (x,\phi(x))\,, \tag{6.2.7}$$

where $\phi$ is the homeomorphism defined in (6.2.2). In this way, for every $\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])$ we have $\Phi_\#\nu \in \mathcal{Y}(\Gamma;[-1,1])$. The elements of $\mathcal{Y}(\Gamma;[-\infty,\infty])$ are in duality with functions $f \in C_b(\Gamma\times[-\infty,\infty])$, i.e., such that $f\circ\Phi^{-1} \in C_b(\Gamma\times[-1,1])$.

**Translation.** We now recall how to shift real-valued Young measures. For every measurable function $\gamma\colon \Gamma \to \mathbb{R}$ we define the translation map $\mathcal{S}^\gamma\colon \Gamma\times[-\infty,\infty] \to \Gamma\times[-\infty,\infty]$ by $\mathcal{S}^\gamma(x,\xi) := (x,\xi+\gamma(x))$, with the usual convention that $a \pm \infty = \pm\infty$ for every $a \in \mathbb{R}$. For every $\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])$ we set

$$\nu \oplus \gamma := \mathcal{S}^\gamma_\#\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])\,, \tag{6.2.8}$$

$$\nu \ominus \gamma := \mathcal{S}^{(-\gamma)}_\#\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])\,. \tag{6.2.9}$$

REMARK 6.2.10. Let $\nu_j,\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])$ be such that $\nu_j \rightharpoonup \nu$ and let $\gamma\colon \Gamma \to \mathbb{R}$ be a measurable function. By Remark 6.2.5 we have $\nu_j \oplus \gamma \rightharpoonup \nu \oplus \gamma$.

Moreover, if $\gamma,\gamma_j\colon \Gamma \to \mathbb{R}$ are such that $\gamma_j \to \gamma$ in measure, then it is easy to see that $\nu_j \oplus \gamma_j \rightharpoonup \nu \oplus \gamma$.

**Truncation.** We now introduce the notion of truncation of Young measures. This will be employed in Section 6.5. Given a Young measure $\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])$ and a measurable function $\theta\colon \Gamma \to [-\infty,\infty]$, we consider the map $\mathcal{T}^\theta\colon \Gamma\times[-\infty,\infty] \to \Gamma\times[-\infty,\infty]$ given by

$$\mathcal{T}^\theta(x,\xi) := (x,\xi\wedge\theta(x)) \tag{6.2.10}$$

and we say that $\mathcal{T}^\theta_\#\nu$ is the *truncation* of $\nu$ by $\theta$.

REMARK 6.2.11. In this case, the cumulative distribution function of the measure $(\mathcal{T}^\theta_\#\nu)^x$ is given by

$$F_{(\mathcal{T}^\theta_\#\nu)^x}(\xi) = \begin{cases} F_{\nu^x}(\xi) & \text{if } \xi < \theta(x)\,,\\ 1 & \text{if } \xi \geq \theta(x)\,, \end{cases}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Moreover, if $\nu_j \rightharpoonup \nu$ in $\mathcal{Y}(\Gamma;[-\infty,\infty])$, then by Remark 6.2.5 we have $\mathcal{T}^\theta_\#\nu_j \rightharpoonup \mathcal{T}^\theta_\#\nu$ in $\mathcal{Y}(\Gamma;[-\infty,\infty])$.

**Partial order.** Following [**15**, Definition 3.10], we introduce a partial order in the space of Young measures on $\Gamma\times\mathbb{R}$. We recall here the definition of this order and its main properties.

DEFINITION 6.2.12. Let $\nu_1 = (\nu_1^x)_{x\in\Gamma}$, $\nu_2 = (\nu_2^x)_{x\in\Gamma} \in \mathcal{Y}(\Gamma;\mathbb{R})$. We say that $\nu_1 \preceq \nu_2$ if one of the following equivalent conditions is satisfied:

(i) for every Carathéodory integrand $f\colon \Gamma\times\mathbb{R} \to \mathbb{R}$ nondecreasing with respect to the second variable we have

$$\int_\Gamma \langle f(x,\cdot),\nu_1^x\rangle\,\mathrm{d}\mathcal{H}^{n-1} \leq \int_\Gamma \langle f(x,\cdot),\nu_2^x\rangle\,\mathrm{d}\mathcal{H}^{n-1}\,;$$

(ii) $F_{\nu_1^x}(\xi) \geq F_{\nu_2^x}(\xi)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ and for every $\xi \in \mathbb{R}$.

REMARK 6.2.13. If $\nu_1$ and $\nu_2$ are concentrated on some measurable functions $\gamma_1$ and $\gamma_2$, respectively, then

$$\nu_1 \preceq \nu_2 \qquad \text{if and only if} \qquad \gamma_1(x) \leq \gamma_2(x) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\,.$$

The partial order $\preceq$ is naturally extended to Young measures $\mathcal{Y}(\Gamma;[-\infty,\infty])$ by employing the homeomorphism $\Phi$ defined in (6.2.7). Namely, for every $\nu_1$, $\nu_2 \in \mathcal{Y}(\Gamma;[-\infty,\infty])$ we have $\nu_1 \preceq \nu_2$ if and only if $\Phi_\#\nu_1 \preceq \Phi_\#\nu_2$.

In the following we recall the definition of supremum of a family of Young measures. (See [**15**, Proposition 3.16] for the existence of such a Young measure.)

DEFINITION 6.2.14. Let $(\nu_i)_{i\in I}$ be a family of Young measures in $\mathcal{Y}(\Gamma;[-\infty,\infty])$. We say that $\overline{\nu} \in \mathcal{Y}(\Gamma;[-\infty,\infty])$ is the *supremum* over $i \in I$ of the family $(\nu_i)_{i\in I}$, and we write

$$\overline{\nu} = \sup_{i\in I} \nu_i\,,$$

if the following two conditions hold:

(i) $\overline{\nu} \succeq \nu_i$ for every $i \in I$;
(ii) if $\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])$ such that $\nu \succeq \nu_i$ for every $i \in I$, then $\nu \succeq \overline{\nu}$.

REMARK 6.2.15. In the case where $\nu_i$ are concentrated on measurable functions $v_i\colon \Gamma \to [-\infty,\infty]$, $i \in I$, we have

$$\sup_{i\in I} \delta_{v_i} = \delta_{\overline{v}}\,,$$

where $\overline{v} = \operatorname*{ess\,sup}_{i\in I} v_i$ (cf. [**15**, Remark 3.17]).

REMARK 6.2.16. If a map $t \mapsto \nu(t)$ from $[0,T]$ to $\mathcal{Y}(\Gamma;[-\infty,\infty])$ is nondecreasing with respect to $\preceq$, then there exists a countable set $E \subset [0,T]$ such that $t \mapsto \nu(t)$ is continuous in $[0,T] \setminus E$. The proof of this fact is an easy consequence of [**15**, Lemma 3.19].

We conclude this section by recalling the Helly Selection Principle for Young measures [**15**, Theorem 3.20], a key tool for the proof of our result. Notice that [**15**, Theorem 3.20] is stated for Young measures with values in $\mathbb{R}$ instead of $[-\infty,\infty]$.

THEOREM 6.2.17. *Let $t \mapsto \nu_k(t)$ be a sequence of maps from $[0,T]$ to $\mathcal{Y}(\Gamma;[-\infty,\infty])$ that are nondecreasing with respect to $\preceq$. Then there exists a subsequence $\nu_{k_j}$, independent of $t$, and a nondecreasing map $t \mapsto \nu(t)$ from $[0,T]$ to $\mathcal{Y}(\Gamma;[-\infty,\infty])$ such that $\nu_{k_j}(t) \rightharpoonup \nu(t)$, as $j \to \infty$, for every $t \in [0,T]$.*

PROOF. The result follows from a straightforward application of [**15**, Theorem 3.20] to the sequence of nondecreasing maps $\Phi_\#\nu_k(t) \in \mathcal{Y}(\Gamma;[-1,1])$, where $\Phi$ is the homeomorphism $\Phi$ defined in (6.2.7). □

## 6.3. Discrete-time evolutions

We study here the discrete-time evolutions already introduced in Section 6.1.

Let $u_k(t)$, $V_k(t)$, and $w_k(t)$ be the piecewise constant interpolations given in (6.1.11). Let $\nu_k(t) \in \mathcal{Y}(\Gamma;[0,\infty])$ be the Young measures concentrated on $V_k(t)$ defined in (6.1.16). In the following proposition we state the main properties satisfied by such approximate evolutions and we provide a priori bounds for $u_k(t)$.

PROPOSITION 6.3.1. *The discrete evolutions $t \mapsto u_k(t)$ defined in* (6.1.11) *satisfy the following conditions:*

$(\mathrm{GS})_k$ *Global stability: For every $t \in [0,T]$ we have $u_k(t) \in A(w_k(t))$ and*

$$\frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla u_k(t)|^2\,\mathrm{d}x + \int\limits_{\Gamma} g\big(x, V_k(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla \widehat{u}|^2\,\mathrm{d}x + \int\limits_{\Gamma} g\big(x,\, V_k(t) + \big|[\widehat{u}]-[u_k(t)]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1},$$

*for every $\widehat{u} \in A(w_k(t))$.*

$(\mathrm{EI})_k$ *Energy-dissipation inequality: There exists a sequence $\eta_k$ with $\eta_k \to 0$ as $k \to \infty$ such that for every $t \in [0,T]$ we have*

$$\begin{aligned}
&\frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla u_k(t)|^2\,\mathrm{d}x + \int\limits_{\Gamma} g\big(x, V_k(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad \le \frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla u_0|^2\,\mathrm{d}x + \int\limits_{\Gamma} g\big(x, V_0\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^{t_k^i} \langle \nabla u_k(s), \nabla \dot{w}(s)\rangle_{L^2}\,\mathrm{d}s + \eta_k\,,
\end{aligned}$$

*where $i \in \{0,\dots,k\}$ is the largest integer such that $t_k^i \le t$.*

*Moreover, there exists a constant $C > 0$ independent of $k$ and $t$ such that*

$$\|u_k(t)\|_{H^1(\Omega\setminus\Gamma)} \le C \quad \text{for every } k \in \mathbb{N} \text{ and } t \in [0,T]\,. \tag{6.3.1}$$

PROOF. In order to prove the global stability $(\mathrm{GS})_k$, we notice that if $i$ is the largest integer such that $t_k^i \le t$, then by (6.1.10) we get that

$$\begin{aligned}
V_k(t) + \big|[\widehat{u}] - [u_k(t)]\big| &= V_k^i + \big|[\widehat{u}] - [u_k^i]\big| = V_k^{i-1} + \big|[u_k^i] - [u_k^{i-1}]\big| + \big|[\widehat{u}] - [u_k^i]\big| \\
&\ge V_k^{i-1} + \big|[\widehat{u}] - [u_k^{i-1}]\big|\,.
\end{aligned}$$

Then we infer $(\mathrm{GS})_k$ by the fact that $u_k(t) = u_k^i$ is a solution to (6.1.9) and by the monotonicity of $g(x,\cdot)$.

Let us prove the energy-dissipation inequality $(\mathrm{EI})_k$. Let us fix $t \in [0,T]$, $k \in \mathbb{N}$, and $i \in \{1,\dots,k\}$ as in the statement (the case $i = 0$ being trivial). For $1 \le h \le i$, the function $u_k^{h-1} - w_k^{h-1} + w_k^h$ is an admissible competitor for the minimum problem (6.1.9) solved by $u_k^h$.

Hence

$$\begin{aligned}\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u_k^h|^2\,\mathrm{d}x+\int_\Gamma g\big(x,V_k^h\big)\,\mathrm{d}\mathcal{H}^{n-1}&\le\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u_k^{h-1}|^2\,\mathrm{d}x+\int_\Gamma g\big(x,V_k^{h-1}\big)\,\mathrm{d}\mathcal{H}^{n-1}\\&\quad+\int_{\Omega\setminus\Gamma}\nabla u_k^{h-1}\cdot(\nabla w_k^h-\nabla w_k^{h-1})\,\mathrm{d}x+\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla w_k^h-\nabla w_k^{h-1}|^2\,\mathrm{d}x\\&\le\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u_k^{h-1}|^2\,\mathrm{d}x+\int_\Gamma g\big(x,V_k^{h-1}\big)\,\mathrm{d}\mathcal{H}^{n-1}\\&\quad+\int_{t_k^{h-1}}^{t_k^h}\langle\nabla u_k(s),\nabla\dot w(s)\rangle_{L^2}\,\mathrm{d}s+\frac{1}{2}\Big(\int_{t_k^{h-1}}^{t_k^h}\|\nabla\dot w(s)\|_{L^2}\,\mathrm{d}s\Big)^2,\end{aligned}\tag{6.3.2}$$

where we used our assumption (6.1.2) on $w$ to deduce that

$$\nabla w_k^h-\nabla w_k^{h-1}=\int_{t_k^{h-1}}^{t_k^h}\nabla\dot w(s)\,\mathrm{d}s\,,$$

as a Bochner integral in $L^2$. Summing up the inequalities given by (6.3.2) for $h=1,\dots,i$, we get $(\mathrm{EI})_k$ with

$$\eta_k:=\frac{1}{2}\Big(\max_{1\le h\le k}\int_{t_k^{h-1}}^{t_k^h}\|\nabla\dot w(s)\|_{L^2}\,\mathrm{d}s\Big)\Big(\int_0^T\|\nabla\dot w(s)\|_{L^2}\,\mathrm{d}s\Big).$$

In particular, from $(\mathrm{EI})_k$ we readily deduce that there exists a constant $C>0$ independent of $k$ and $t$ such that $\|\nabla u_k(t)\|_{L^2}\le C$. Then, by the Poincaré inequality, we get (6.3.1) (up to changing the name of the constant). ☐

REMARK 6.3.2. It is convenient to express the properties satisfied by $u_k(t)$ also in terms of the Young measures $\nu_k(t)\in\mathcal{Y}(\Gamma;[0,\infty])$ defined in (6.1.16). In Section 6.4, we will pass to the limit in these conditions.

$(\mathrm{IRY})_k$ *Irreversibility*: $\nu_k(t)\succeq\nu_k(s)\oplus\big|[u_k(t)]-[u_k(s)]\big|$ for every $s,t\in[0,T]$ with $s\le t$.

$(\mathrm{GSY})_k$ *Global stability*: For every $t\in[0,T]$ we have $u_k(t)\in A(w_k(t))$ and

$$\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u_k(t)|^2\,\mathrm{d}x+\int_\Gamma\langle g(x,\cdot),\nu_k^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1}\le\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla\widehat u|^2\,\mathrm{d}x+\int_\Gamma\langle g(x,\cdot),\widehat\nu_k^x\rangle\,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\widehat u\in A(w_k(t))$, where $\widehat\nu_k:=\nu_k(t)\oplus\big|[\widehat u]-[u_k(t)]\big|\in\mathcal{Y}(\Gamma;[0,\infty])$.

$(\mathrm{EIY})_k$ *Energy-dissipation inequality*: For every $t\in[0,T]$

$$\begin{aligned}&\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u_k(t)|^2\,\mathrm{d}x+\int_\Gamma\langle g(x,\cdot),\nu_k^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1}\\&\quad\le\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u_0|^2\,\mathrm{d}x+\int_\Gamma g\big(x,V_0\big)\,\mathrm{d}\mathcal{H}^{n-1}+\int_0^{t_k^i}\langle\nabla u_k(s),\nabla\dot w(s)\rangle_{L^2}\,\mathrm{d}s+\eta_k\,,\end{aligned}$$

where $i \in \{0, \dots, k\}$ is the largest integer such that $t_k^i \le t$.

Notice that $(\mathrm{GSY})_k$ trivially implies that

$$\frac{1}{2} \int\limits_{\Omega \setminus \Gamma} |\nabla u_k(t)|^2 \,\mathrm{d}x + \int\limits_{\Gamma} \langle g(x,\cdot), \nu_k^x(t) \rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2} \int\limits_{\Omega \setminus \Gamma} |\nabla \widehat{u}|^2 \,\mathrm{d}x + \int\limits_{\Gamma} \langle g(x,\cdot), \widehat{\nu}^x \rangle \,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\widehat{u} \in A(w_k(t))$ and for every $\widehat{\nu} \in \mathcal{Y}(\Gamma;[0,\infty])$ with $\widehat{\nu} \succeq \nu_k(t) \oplus \big|[\widehat{u}] - [u_k(t)]\big|$.

Remark 6.3.3. By passing to the limit as $k \to \infty$ in $(\mathrm{IRY})_k$, we may formally obtain the irreversibility condition for the continuous-time quasistatic evolution. (See Definition 6.4.1 in Section 6.4 below.) Unfortunately, it is not immediate to rigorously pass to the limit in $(\mathrm{IRY})_k$: as we shall see below, in the construction of the continuous-time evolution the jumps $[u_k(t)]$ converge to $[u(t)]$ on subsequences possibly depending on $t$, thus precluding the possibility to have convergence on the same subsequence for both $[u_k(t)]$ and $[u_k(s)]$ in $(\mathrm{IRY})_k$. For this reason, we reformulate $(\mathrm{IRY})_k$ in a more convenient way. We start by noticing that the condition

$$V_k(t) \ge V_k(s) + \big|[u_k(t)] - [u_k(s)]\big| \quad \text{for every } s,t \in [0,T] \text{ with } s \le t\,,$$

is equivalent to the system of inequalities

$$V_k(t) + [u_k(t)] \ge V_k(s) + [u_k(s)] \quad \text{for every } s,t \in [0,T] \text{ with } s \le t\,, \tag{6.3.3}$$

$$V_k(t) - [u_k(t)] \ge V_k(s) - [u_k(s)] \quad \text{for every } s,t \in [0,T] \text{ with } s \le t\,. \tag{6.3.4}$$

Let us notice that since $V_0 \ge \big|[u_0]\big|$ by (6.1.5), we have $V_k(t) + [u_k(t)] \ge 0$ and $V_k(t) - [u_k(t)] \ge 0$ for every $t \in [0,T]$. In terms of the Young measures $\nu_k$, the inequalities (6.3.3) and (6.3.4) are equivalent to stating that the functions

$$t \mapsto \nu_k(t) \oplus [u_k(t)] =: \lambda_k^{\oplus}(t) \in \mathcal{Y}(\Gamma;[0,\infty])\,, \tag{6.3.5}$$

$$t \mapsto \nu_k(t) \ominus [u_k(t)] =: \lambda_k^{\ominus}(t) \in \mathcal{Y}(\Gamma;[0,\infty]) \tag{6.3.6}$$

are nondecreasing with respect to $t$. Thanks to the Helly Selection Principle for Young measures (Theorem 6.2.17), (6.3.5) and (6.3.6) are easier to handle than $(\mathrm{IRY})_k$, as we shall see later in Section 6.4.

We conclude this section with the following proposition, which shall be used to pass to the limit in $(\mathrm{GSY})_k$ as $k \to \infty$.

Proposition 6.3.4. *Let $w_k \rightharpoonup w$ weakly in $H^1(\Omega)$. Let $v_k \in A(w_k)$ and $v \in H^1(\Omega \setminus \Gamma)$ be such that $v_k \rightharpoonup v$ weakly in $H^1(\Omega \setminus \Gamma)$ and let $\mu_k, \mu \in \mathcal{Y}(\Gamma;[0,\infty])$ be such that $\mu_k \rightharpoonup \mu$. Let us assume that that for every $k \in \mathbb{N}$*

$$\frac{1}{2} \int\limits_{\Omega \setminus \Gamma} |\nabla v_k|^2 \,\mathrm{d}x + \int\limits_{\Gamma} \langle g(x,\cdot), \mu_k^x \rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2} \int\limits_{\Omega \setminus \Gamma} |\nabla \widehat{v}|^2 \,\mathrm{d}x + \int\limits_{\Gamma} \langle g(x,\cdot), \widehat{\mu}_k^x \rangle \,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{6.3.7}$$

*for every $\widehat{v} \in A(w_k)$, where $\widehat{\mu}_k := \mu_k \oplus \big|[\widehat{v}] - [v_k]\big| \in \mathcal{Y}(\Gamma;[0,\infty])$. Then $v \in A(w)$ and*

$$\frac{1}{2} \int\limits_{\Omega \setminus \Gamma} |\nabla v|^2 \,\mathrm{d}x + \int\limits_{\Gamma} \langle g(x,\cdot), \mu^x \rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2} \int\limits_{\Omega \setminus \Gamma} |\nabla \widehat{v}|^2 \,\mathrm{d}x + \int\limits_{\Gamma} \langle g(x,\cdot), \widehat{\mu}^x \rangle \,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{6.3.8}$$

*for every* $\widehat{v} \in A(w)$, *where* $\widehat{\mu} := \mu \oplus \big|[\widehat{v}] - [v]\big| \in \mathcal{Y}(\Gamma; [0,\infty])$.

Proof. By the continuity of the trace operator on $\partial_D\Omega$ with respect to the weak convergence in $H^1(\Omega \setminus \Gamma)$ we have $v \in A(w)$. To prove (6.3.8), fix $\widehat{v} \in A(w)$. Define $\widehat{\mu} := \mu \oplus \big|[\widehat{v}] - [v]\big| \in \mathcal{Y}(\Gamma; [0,\infty])$ and

$$\begin{aligned} \widehat{v}_k &:= v_k + \widehat{v} - v \in A(w_k), \\ \widehat{\mu}_k &:= \mu_k \oplus \big|[\widehat{v}] - [v]\big| = \mu_k \oplus \big|[\widehat{v}_k] - [v_k]\big|. \end{aligned} \tag{6.3.9}$$

Since $v_k \rightharpoonup v$ and $\mu_k \rightharpoonup \mu$, by Remark 6.2.10 we have

$$\widehat{v}_k \rightharpoonup \widehat{v} \quad \text{weakly in } H^1(\Omega \setminus \Gamma), \tag{6.3.10}$$

$$\widehat{\mu}_k \rightharpoonup \widehat{\mu} \quad \text{narrowly.} \tag{6.3.11}$$

From (6.3.7) we get that

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla v_k|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \mu_k^x\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \widehat{v}_k|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \widehat{\mu}_k^x\rangle \,\mathrm{d}\mathcal{H}^{n-1}. \tag{6.3.12}$$

We now use a classical quadratic trick. By (6.3.9), we infer that

$$\begin{aligned} \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla v_k|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \widehat{v}_k|^2 \,\mathrm{d}x &= \frac{1}{2}\int_{\Omega\setminus\Gamma} (\nabla v_k - \nabla \widehat{v}_k)\cdot(\nabla v_k + \nabla \widehat{v}_k) \,\mathrm{d}x \\ &= \frac{1}{2}\int_{\Omega\setminus\Gamma} (\nabla v - \nabla \widehat{v})\cdot(2\nabla v_k + \nabla \widehat{v} - \nabla v) \,\mathrm{d}x. \end{aligned} \tag{6.3.13}$$

Thanks to (6.3.11) we deduce that

$$\int_\Gamma \langle g(x,\cdot), \widehat{\mu}_k^x\rangle \,\mathrm{d}\mathcal{H}^{n-1} \to \int_\Gamma \langle g(x,\cdot), \widehat{\mu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1}. \tag{6.3.14}$$

Since $v_k \rightharpoonup v$ and $\mu_k \rightharpoonup \mu$, by (6.3.12)–(6.3.14) we have

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} (\nabla v - \nabla \widehat{v})\cdot(\nabla v + \nabla \widehat{v}) \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \mu^x\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \int_\Gamma \langle g(x,\cdot), \widehat{\mu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1},$$

from which we easily conclude that (6.3.8) holds. □

## 6.4. Quasistatic evolution in the setting of Young measures

In this section we study the continuous-time limit of the discrete evolutions $u_k(t)$ constructed in Section 6.3. The limit of the sequence of (Young measures concentrated on) functions $\nu_k(t)$ defined in (6.1.16) can only be found in the space of Young measures $\mathcal{Y}(\Gamma; [0,\infty])$. For this reason we require a definition of quasistatic evolution in a generalised sense.

Definition 6.4.1. Let $w$, $u_0$, and $V_0$ be as in (6.1.2)–(6.1.5). A *quasistatic evolution in the sense of Young measures* with initial conditions $(u_0, V_0)$ and boundary datum $w$ is a function $t \mapsto (u(t), \nu(t))$ defined in $[0,T]$ with values in $H^1(\Omega\setminus\Gamma) \times \mathcal{Y}(\Gamma; [0,\infty])$ that satisfies $u(0) = u_0$, $\nu(0) = \delta_{V_0}$, and the following conditions:

(IRY) *Irreversibility*: $\nu(t) \succeq \nu(s) \oplus \big|[u(t)] - [u(s)]\big|$ for every $s, t \in [0,T]$ with $s \le t$.

(GSY) *Global stability*: For every $t \in [0,T]$, $u(t) \in A(w(t))$ and

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \widehat{u}|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \widehat{\nu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\widehat{u} \in A(w(t))$, where $\widehat{\nu} := \nu(t) \oplus \big|[\widehat{u}] - [u(t)]\big| \in \mathcal{Y}(\Gamma;[0,\infty])$.

(EBY) *Energy-dissipation balance*: For every $t \in [0,T]$

$$\begin{aligned}\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x &+ \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ &= \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_\Gamma g(x,V_0) \,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2} \,\mathrm{d}s.\end{aligned}$$

Remark 6.4.2. In order to recognise the connection with the classical notion of quasistatic evolution, we notice that $t \mapsto u(t)$ is a quasistatic evolution (Definition 6.1.6) if and only if $t \mapsto (u(t), \delta_{V_u(t)})$ is a quasistatic evolution in the sense of Young measures (Definition 6.4.1), where $V_u(t)$ is the function defined in (6.1.3). Indeed, the irreversibility condition (IRY) of Definition 6.4.1 automatically holds for $t \mapsto \delta_{V_u(t)}$ by definition of essential variation. Moreover, (GS) and (EB) correspond to (GSY) and (EBY), since the Young measure considered in this case is concentrated on $V_u(t)$.

Remark 6.4.3. Notice that (GSY) trivially implies that

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \widehat{u}|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \widehat{\nu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\widehat{u} \in A(w(t))$ and for every $\widehat{\nu} \in \mathcal{Y}(\Gamma;[0,\infty])$ with $\widehat{\nu} \succeq \nu(t) \oplus \big|[\widehat{u}] - [u(t)]\big|$.

Moreover we underline that (IRY) is a stronger condition than the monotonicity of $t \mapsto \nu(t)$ and dictates a relationship between $\nu$ and $[u]$.

In the following theorem we prove the existence of a quasistatic evolution in the sense of of Young measures. As explained in Section 6.1, this result will be then improved in Section 6.5 by showing that the truncated Young measures $\mathcal{T}^\theta_\# \nu(t)$ are concentrated on the function $V_u(t) \wedge \theta$ which represents the cumulation of the jumps on $\Gamma$.

Theorem 6.4.4 (Existence of quasistatic evolutions in the sense of Young measures). *Assume that $g$ satisfies (g1)–(g4) and let $w$, $u_0$, and $V_0$ be as in* (6.1.2), (6.1.4), *and* (6.1.5). *Assume that the pair $(u_0, \delta_{V_0})$ is globally stable, i.e.,* (6.1.8) *holds. Then there exists a quasistatic evolution in the sense of Young measures $t \mapsto (u(t), \nu(t))$ with initial conditions $(u_0, V_0)$ and boundary datum $w$.*

In the rest of this section, we give a proof of Theorem 6.4.4.

**Construction of the evolution.** Let us consider the Young measures $\nu_k(t)$ defined in (6.1.16). The starting point of the proof is the construction of a limit of $\nu_k(t)$ as $k \to \infty$. Since the functions $t \mapsto \nu_k(t) \in \mathcal{Y}(\Gamma;[0,\infty])$ are increasing with respect to the order $\preceq$, we can apply Theorem 6.2.17 to deduce that there exists a subsequence (independent of $t$ and still denoted by $\nu_k$) and an increasing function $t \mapsto \nu(t)$ from $[0,T]$ to $\mathcal{Y}(\Gamma;[0,\infty])$ such that

$$\nu_k(t) \rightharpoonup \nu(t) \quad \text{narrowly for every } t \in [0,T]\,. \tag{6.4.1}$$

Unfortunately, the convergence in (6.4.1) is not enough to guarantee that the irreversibility condition (IRY) holds for $\nu(t)$. In other words, it is nontrivial to pass to the limit in the discrete version of the irreversibility condition $(\text{IRY})_k$. Nonetheless, by Remark 6.3.3, we know that the functions $t \mapsto \lambda_k^{\oplus}(t)$ and $t \mapsto \lambda_k^{\ominus}(t)$ are increasing. Hence we can apply again Theorem 6.2.17 and deduce that there exists a subsequence independent of $t$ (not relabelled) and two increasing functions $t \mapsto \lambda_{\oplus}(t) \in \mathcal{Y}(\Gamma;[0,\infty])$ and $t \mapsto \lambda_{\ominus}(t) \in \mathcal{Y}(\Gamma;[0,\infty])$ such that

$$\lambda_k^{\oplus}(t) \rightharpoonup \lambda_{\oplus}(t) \quad \text{narrowly for every } t \in [0,T]\,, \tag{6.4.2}$$

$$\lambda_k^{\ominus}(t) \rightharpoonup \lambda_{\ominus}(t) \quad \text{narrowly for every } t \in [0,T]\,. \tag{6.4.3}$$

The monotonicity of both the functions $\lambda_{\oplus}$ and $\lambda_{\ominus}$ encodes the irreversibility of the process in the continuous-time evolution.

We are now in a position to construct a limit of the sequence $u_k(t)$. Thanks to (6.3.1), we have $\|u_k(t)\|_{H^1(\Omega\setminus\Gamma)} \leq C$, where the constant $C$ is independent of $k$ and $t$. Let $t \in [0,T]$ and let $k_j(t)$ be a subsequence of $k$ such that

$$u_{k_j(t)}(t) \rightharpoonup u(t) \quad \text{weakly in } H^1(\Omega \setminus \Gamma)\,, \tag{6.4.4}$$

for some function $u(t) \in H^1(\Omega \setminus \Gamma)$.

A priori, the function $u(t)$ depends on the subsequence $k_j(t)$ such that (6.4.4) holds. Nevertheless, we will prove below the following result.

Remark 6.4.5. Actually, we shall prove that

$$u_k(t) \to u(t) \quad \text{strongly in } H^1(\Omega \setminus \Gamma) \tag{6.4.5}$$

on the whole sequence (independent of $t$) found by the Helly Selection Principle (cf. (6.4.1)–(6.4.3)).

We remark that also the topology of the convergence is improved. The convergence in (6.4.5) will be proved later in this section by showing that the function $u(t)$ is characterised as the unique solution to a minimum problem (Proposition 6.4.6). The convergence with respect to the strong topology of $H^1(\Omega \setminus \Gamma)$ will be a consequence of the energy-dissipation balance (EBY).

**Proof of irreversibility.** We can now infer (IRY) from the monotonicity of the functions $\lambda_{\oplus}$ and $\lambda_{\ominus}$ obtained in (6.4.2) and (6.4.3). Indeed, from (6.4.4) we deduce that $[u_{k_j(t)}] \to [u(t)]$ strongly in $L^2(\Gamma)$. By (6.4.1) and by Remark 6.2.10 this implies that $\lambda^{\oplus}_{k_j(t)}(t) = \nu_{k_j(t)}(t) \oplus [u_{k_j(t)}(t)] \rightharpoonup \nu(t) \oplus [u(t)]$. Thus, from (6.4.2) we deduce that

$$\lambda_{\oplus}(t) = \nu(t) \oplus [u(t)]\,, \tag{6.4.6}$$

and therefore that the function $t \mapsto \nu(t) \oplus [u(t)]$ is increasing. Similarly one can prove that $\lambda_{\ominus}(t) = \nu(t) \ominus [u(t)]$ and that $t \mapsto \nu(t) \ominus [u(t)]$ is increasing. Therefore, for every $s, t \in [0,T]$

with $s \leq t$ we have

$$\nu(t) \oplus [u(t)] \succeq \nu(s) \oplus [u(s)] \,,$$
$$\nu(t) \ominus [u(t)] \succeq \nu(s) \ominus [u(s)] \,.$$

It is immediate to see that the previous inequalities imply (IRY).

In order to prove (6.4.5), it is convenient to make the following key observations:

- the Young measures $\lambda_\oplus(t)$ and $\nu(t)$ are obtained as limits of a sequence independent of $t$;
- the jump $[u(t)]$ can be recovered just from $\lambda_\oplus(t)$ and $\nu(t)$ thanks to (6.4.6).

We now make precise the previous statements. We start by observing that if $x \in \Gamma$ is such that $\lambda^x_\oplus(t) = \nu^x(t) = \delta_\infty$, then $[u(t;x)]$ is not uniquely determined by (6.4.6). For this reason we introduce the set

$$\Gamma_N(t) := \{x \in \Gamma \colon \nu^x(t) \succeq \delta_{\theta(x)}\} \,, \tag{6.4.7}$$

which corresponds to the subset of $\Gamma$ where the material is completely fractured. For $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus \Gamma_N(t)$ there exists a mass $m_x \in (0,1]$ such that $F^{[-1]}_{\nu^x(t)}(m_x) \in [0, \theta(x))$, where $F^{[-1]}_{\nu^x(t)}$ is the pseudo-inverse of the cumulative distribution function $F_{\nu^x(t)}$ of $\nu^x(t)$ (cf. (6.2.3) and (6.2.4)). In particular, we have that $F^{[-1]}_{\nu^x(t)}(m_x)$ is finite. By (6.4.6) and by the definition of pseudo-inverse, it is easy to see that

$$F^{[-1]}_{\lambda^x_\oplus(t)}(m_x) - F^{[-1]}_{\nu^x(t)}(m_x) = [u(t;x)] \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma \setminus \Gamma_N(t) \,. \tag{6.4.8}$$

(We remark that, if instead $x \in \Gamma_N(t)$, it may happen that $\nu^x(t) = \delta_\infty$, and thus $F^{[-1]}_{\nu^x(t)}(m) = \infty$ for every $m \in (0,1]$. This does not allow us to infer (6.4.8).) Therefore, we can define a measurable function $\gamma(t) \colon \Gamma \setminus \Gamma_N(t) \to \mathbb{R}$ by

$$\gamma(t;x) := F^{[-1]}_{\lambda^x_\oplus(t)}(m_x) - F^{[-1]}_{\nu^x(t)}(m_x) \,, \tag{6.4.9}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus \Gamma_N(t)$. We stress that the function $\gamma(t)$ is obtained independently of the subsequence $k_j(t)$. The proof of (6.4.5) will be continued after the proof of (GSY) and (EBY).

**Proof of global stability.** The global stability (GSY) directly follows from Proposition 6.3.4, since $u_{k_j(t)}(t)$ and $\nu_{k_j(t)}(t)$ satisfy condition $(\mathrm{GSY})_k$ and by (6.4.4) and (6.4.1).

In general, the function $u(t)$ is not uniquely determined by (GSY), because $u(t)$ appears both in the left-hand side and in the right-hand side of (GSY); specifically, $\widehat{\nu}$ depends on $u(t)$. However, we have shown that the jump of $u(t)$ is given by the function $\gamma(t)$ defined in (6.4.9) independently of the subsequence $k_j(t)$. This allows us to prove the following result.

PROPOSITION 6.4.6. *The function $u(t)$ obtained in* (6.4.4) *is the unique solution to the minimum problem*

$$\min_{\widehat{u}} \Big\{ \frac{1}{2} \int\limits_{\Omega\setminus\Gamma} |\nabla \widehat{u}|^2 \,\mathrm{d}x \colon \widehat{u} \in A(w(t)) \text{ such that } [\widehat{u}(x)] = \gamma(t;x) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma \setminus \Gamma_N(t) \Big\} \,, \tag{6.4.10}$$

*where* $\Gamma_N(t)$ *is the set defined in* (6.4.7) *and* $\gamma(t)$ *is the function defined in* (6.4.9).

REMARK 6.4.7. Notice that Proposition 6.4.6 holds true also when $\mathcal{H}^{n-1}(\Gamma\backslash\Gamma_N(t)) = 0$, i.e., when the material is completely fractured on the whole surface $\Gamma$. In this case, the competitors in (6.4.10) are all functions $\widehat{u} \in A(w(t))$ (without any constraint on the jump).

PROOF OF PROPOSITION 6.4.6. We have already observed (see (6.4.8)) that $[u(t)] = \gamma(t)$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma \setminus \Gamma_N(t)$. Let us fix $\widehat{u} \in A(w(t))$ such that $[\widehat{u}] = \gamma(t) = [u(t)]$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma \setminus \Gamma_N(t)$. Setting $\widehat{\nu} := \nu(t) \oplus \big|[\widehat{u}] - [u(t)]\big|$, by (GSY) we have

$$\frac{1}{2}\int_{\Omega\backslash\Gamma} |\nabla u(t)|^2 \, \mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(t)\rangle \, \mathrm{d}\mathcal{H}^{n-1} \leq \frac{1}{2}\int_{\Omega\backslash\Gamma} |\nabla \widehat{u}|^2 \, \mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \widehat{\nu}^x\rangle \, \mathrm{d}\mathcal{H}^{n-1}. \quad (6.4.11)$$

Since $\widehat{\nu}^x \succeq \nu^x(t) \succeq \delta_{\theta(x)}$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_N(t)$ and since $g(x,\xi) = \kappa(x)$ for every $\xi \in [\theta(x), \infty]$, we deduce that

$$\int_{\Gamma_N(t)} \langle g(x,\cdot), \nu^x(t)\rangle \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma_N(t)} \langle g(x,\cdot), \widehat{\nu}^x\rangle \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma_N(t)} \kappa(x) \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

Therefore (6.4.11) is equivalent to

$$\frac{1}{2}\int_{\Omega\backslash\Gamma} |\nabla u(t)|^2 \, \mathrm{d}x + \int_{\Gamma\backslash\Gamma_N(t)} \langle g(x,\cdot), \nu^x(t)\rangle \, \mathrm{d}\mathcal{H}^{n-1} \leq \frac{1}{2}\int_{\Omega\backslash\Gamma} |\nabla \widehat{u}|^2 \, \mathrm{d}x + \int_{\Gamma\backslash\Gamma_N(t)} \langle g(x,\cdot), \widehat{\nu}^x\rangle \, \mathrm{d}\mathcal{H}^{n-1}.$$

Since $[\widehat{u}] = [u(t)]$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma \setminus \Gamma_N(t)$, we have $\widehat{\nu}^x = \nu^x(t)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus \Gamma_N(t)$, hence the previous inequality reads

$$\frac{1}{2}\int_{\Omega\backslash\Gamma} |\nabla u(t)|^2 \, \mathrm{d}x \leq \frac{1}{2}\int_{\Omega\backslash\Gamma} |\nabla \widehat{u}|^2 \, \mathrm{d}x \,.$$

This proves that $u(t)$ is a solution to (6.4.10).

The argument to prove uniqueness is standard: if $u_1$ and $u_2$ were two different solutions to (6.4.10), then $\widehat{u} := \frac{1}{2}(u_1 + u_2)$ would be an admissible competitor; by strict convexity,

$$\frac{1}{2}\int_{\Omega\backslash\Gamma} |\nabla \widehat{u}|^2 \, \mathrm{d}x = \frac{1}{2}\int_{\Omega\backslash\Gamma} \Big|\frac{\nabla u_1 + \nabla u_2}{2}\Big|^2 \, \mathrm{d}x < \frac{1}{4}\int_{\Omega\backslash\Gamma} |\nabla u_1|^2 \, \mathrm{d}x + \frac{1}{4}\int_{\Omega\backslash\Gamma} |\nabla u_2|^2 \, \mathrm{d}x = \frac{1}{2}\int_{\Omega\backslash\Gamma} |\nabla u_1|^2 \, \mathrm{d}x \,.$$

This contradicts the minimality. □

REMARK 6.4.8. The minimum problem (6.4.10) is independent of the subsequence $k_j(t)$. As a consequence, we have shown that if $k_j(t)$ is such that $u_{k_j(t)} \rightharpoonup u(t)$, then $u(t)$ is the unique solution to (6.4.10). Thus $u(t)$ does not depend on $k_j(t)$, and this implies that

$$u_k(t) \rightharpoonup u(t) \quad \text{weakly in } H^1(\Omega \setminus \Gamma) \quad \text{for every } t \in [0,T] \quad (6.4.12)$$

on the whole sequence (independent of $t$) found by the Helly Selection Principle (cf. (6.4.1)–(6.4.3)). In particular, by (6.3.1) we have

$$\|u(t)\|_{H^1(\Omega\backslash\Gamma)} \leq C \,. \quad (6.4.13)$$

After proving the energy-dissipation balance, it will turn out that the convergence is strong in $H^1(\Omega \setminus \Gamma)$.

**Proof of energy-dissipation balance.** Before proving (EBY), we show that the function $t \mapsto u(t)$ is continuous with respect to the weak topology for almost every time. This result allows for a simple proof of the energy-dissipation balance.

LEMMA 6.4.9. *There exists a countable set $E \subset [0,T]$ such that for every $t \in [0,T] \setminus E$*

$$u(s) \rightharpoonup u(t) \quad \text{weakly in } H^1(\Omega \setminus \Gamma)\,, \tag{6.4.14}$$

$$\nu(s) \rightharpoonup \nu(t) \quad \text{narrowly in } \mathcal{Y}(\Gamma;[0,\infty])\,. \tag{6.4.15}$$

*as $s \to t$.*

PROOF. Since the functions $t \mapsto \lambda_\oplus(t)$ and $t \mapsto \nu(t)$ are nondecreasing, we can find a countable set $E \subset [0,T]$ such that both $\lambda_\oplus$ and $\nu$ are continuous (with respect to the narrow topology) in $t$ for every $t \in [0,T] \setminus E$. (See Remark 6.2.16.) Thus, given $t \in [0,T] \setminus E$ and a sequence $s_k \to t$, we have

$$\lambda_\oplus(s_k) \rightharpoonup \lambda_\oplus(t)\,, \quad \nu(s_k) \rightharpoonup \nu(t)\,. \tag{6.4.16}$$

Thanks to (6.4.13), we can extract a subsequence (not relabelled) such that

$$u(s_k) \rightharpoonup u^* \quad \text{weakly in } H^1(\Omega \setminus \Gamma) \tag{6.4.17}$$

for some $u^* \in H^1(\Omega \setminus \Gamma)$. By Proposition 6.3.4, we infer that $u^* \in A(w(t))$ and

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u^*|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \widehat{u}|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \widehat{\nu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1}\,,$$

for every $\widehat{u} \in A(w(t))$, where $\widehat{\nu} = \nu(t) \oplus \big|[\widehat{u}] - [u^*]\big|$.

On the other hand, by (6.4.6), we have $\lambda_\oplus(s_k) = \nu(s_k) \oplus [u(s_k)]$. By (6.4.16), (6.4.17), and Remark 6.2.10 we deduce that $\lambda_\oplus(t) = \nu(t) \oplus [u^*]$. Hence, by (6.4.9), we obtain that $[u^*(x)] = \gamma(t;x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus \Gamma_N(t)$. Therefore, arguing as in the proof of Proposition 6.4.6, we infer that $u^*$ is a solution to the minimum problem (6.4.10). By uniqueness of the solution we get $u^* = u(t)$, which concludes the proof. ☐

REMARK 6.4.10. Lemma 6.4.9 will be improved in Proposition 6.4.12 below by showing that the continuity actually holds with respect to the strong topology.

Let us now prove (EBY). We start with proving the inequality

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_\Gamma g(x, V_0) \,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2} \,\mathrm{d}s\,.
\end{aligned} \tag{6.4.18}$$

By (6.4.1), (6.4.12), and by $(\mathrm{EIY})_k$, for every $t \in [0,T]$ we have

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad\le \liminf_{k\to\infty} \Big[\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu_k^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1}\Big] \\ &\quad\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_\Gamma g(x,V_0) \,\mathrm{d}\mathcal{H}^{n-1} + \limsup_{k\to\infty} \int_0^{t_k^i} \langle \nabla u_k(s), \nabla \dot w(s)\rangle_{L^2} \,\mathrm{d}s\,, \end{aligned} \tag{6.4.19}$$

where $i \in \{0,\dots,k\}$ is the largest integer such that $t_k^i \le t$. Thanks to (6.4.12) we know that

$$\langle \nabla u_k(s), \nabla \dot w(s)\rangle_{L^2} \to \langle \nabla u(s), \nabla \dot w(s)\rangle_{L^2} \quad \text{for every } s \in [0,t]\,.$$

Moreover, from (6.3.1) we deduce that

$$\langle \nabla u_k(s), \nabla \dot w(s)\rangle_{L^2} \le \|\nabla u_k(s)\|_{L^2} \|\nabla \dot w(s)\|_{L^2} \le C\|\nabla \dot w(s)\|_{L^2}\,,$$

for every $s \in [0,T]$. By our assumption (6.1.2) on $w$, the function $t \mapsto \nabla \dot w(t)$ is $L^1([0,T];L^2(\Omega\setminus\Gamma))$, so we can apply the Dominated Convergence Theorem to infer that

$$\limsup_{k\to\infty} \int_0^{t_k^i} \langle \nabla u_k(s), \nabla \dot w(s)\rangle_{L^2} \,\mathrm{d}s = \lim_{k\to\infty} \int_0^t \langle \nabla u_k(s), \nabla \dot w(s)\rangle_{L^2} \,\mathrm{d}s = \int_0^t \langle \nabla u(s), \nabla \dot w(s)\rangle_{L^2} \,\mathrm{d}s\,. \tag{6.4.20}$$

Together with (6.4.19), the previous inequality yields (6.4.18).

We now exploit the global stability to prove, for a fixed $t \in [0,T]$, the opposite inequality

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad\ge \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_\Gamma g(x,V_0) \,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot w(s)\rangle_{L^2} \,\mathrm{d}s\,. \end{aligned} \tag{6.4.21}$$

For every $k \in \mathbb{N}$, let us consider the subdivision of the time interval $[0,t]$ given by the $k+1$ equispaced nodes

$$s_k^h := \tfrac{h}{k} t \quad \text{for } h = 0,\dots,k\,.$$

Let $h \in \{1, \dots, k\}$. By the irreversibility condition (IRY), we have $\nu(s_k^h) \succeq \nu(s_k^{h-1}) \oplus \big|[u(s_k^h)] - [u(s_k^{h-1})]\big| =: \widehat{\nu}_h$. Since $u(s_k^h) - w(s_k^h) + w(s_k^{h-1}) \in A(w(s_k^{h-1}))$, by (GSY) we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(s_k^{h-1})|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(s_k^{h-1})\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(s_k^h)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \widehat{\nu}_h^x\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad - \int_{\Omega\setminus\Gamma} \nabla u(s_k^h)\cdot(\nabla w(s_k^h) - \nabla w(s_k^{h-1}))\,\mathrm{d}x + \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla w(s_k^h) - \nabla w(s_k^{h-1})|^2 \,\mathrm{d}x \\
&\quad \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(s_k^h)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(s_k^h)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad - \int_{s_k^{h-1}}^{s_k^h} \langle \nabla \overline{u}^k(s), \nabla \dot{w}(s)\rangle_{L^2} \,\mathrm{d}s + \frac{1}{2}\Big(\int_{s_k^{h-1}}^{s_k^h} \|\nabla \dot{w}(s)\|_{L^2}\,\mathrm{d}s\Big)^2,
\end{aligned} \tag{6.4.22}$$

where

$$\overline{u}^k(s) := u(s_k^h) \quad \text{for every } s \in (s_k^{h-1}, s_k^h]\,.$$

Summing up the inequalities given by (6.4.22) for $h = 1, \dots, k$, we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad \ge \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_\Gamma g(x, V_0)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla \overline{u}_k(s), \nabla \dot{w}(s)\rangle_{L^2}\,\mathrm{d}s - \overline{\eta}_k\,,
\end{aligned}$$

where

$$\overline{\eta}_k := \frac{1}{2}\Big(\max_{1\le h\le k} \int_{s_k^{h-1}}^{s_k^h} \|\nabla \dot{w}(s)\|_{L^2}\,\mathrm{d}s\Big)\Big(\int_0^T \|\nabla \dot{w}(s)\|_{L^2}\,\mathrm{d}s\Big)\,.$$

In order to infer (6.4.21), we notice that by Lemma 6.4.9 we have $\overline{u}^k(s) \rightharpoonup u(s)$ for almost every $s \in [0, t]$, and therefore

$$\lim_{k\to\infty} \int_0^t \langle \nabla \overline{u}^k(s), \nabla \dot{w}(s)\rangle_{L^2}\,\mathrm{d}s = \int_0^t \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2}\,\mathrm{d}s\,,$$

by the Dominated Convergence Theorem. This concludes the proof of (EBY) and of Theorem 6.4.4.

**Approximation of the evolution and continuity for almost every time.** Thanks to (EBY), we prove the convergence of the approximating evolutions (6.4.5) and we improve Lemma 6.4.9.

PROPOSITION 6.4.11. *We have*

$$u_k(t) \to u(t) \quad \textit{strongly in } H^1(\Omega \setminus \Gamma)$$

*on the whole sequence (independent of t) such that* (6.4.1)–(6.4.3) *hold.*

PROOF. By (6.4.1) and (6.4.12), for every $t \in [0,T]$ we have

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{k\to\infty}\left[\frac{1}{2}\int_\Omega |\nabla u_k(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu_k^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1}\right]. \tag{6.4.23}$$

On the other hand, by (6.4.20), (EBY), and $(\mathrm{EIY})_k$ we get

$$\begin{aligned}
\limsup_{k\to\infty} &\left[\frac{1}{2}\int_\Omega |\nabla u_k(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu_k^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1}\right] \\
&\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_\Gamma g(x,V_0) \,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot w(s)\rangle_{L^2} \,\mathrm{d}s \\
&= \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned} \tag{6.4.24}$$

Thus all inequalities in (6.4.23) and (6.4.24) are equalities. Since

$$\int_\Gamma \langle g(x,\cdot), \nu_k^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \to \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1},$$

we have $\|\nabla u_k(t)\|_{L^2} \to \|\nabla u(t)\|_{L^2}$. Thanks to (6.4.12), this concludes the proof. □

PROPOSITION 6.4.12. *There exists a countable set* $E \subset [0,T]$ *such that for every* $t \in [0,T]\setminus E$

$$u(s) \to u(t) \quad \textit{strongly in } H^1(\Omega\setminus\Gamma)\,, \tag{6.4.25}$$

$$\nu(s) \rightharpoonup \nu(t) \quad \textit{narrowly in } \mathcal{Y}(\Gamma;[0,\infty])\,. \tag{6.4.26}$$

*as* $s \to t$.

PROOF. By (EBY) we have for every $s,t \in [0,T]$

$$\begin{aligned}
\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x &+ \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\
&= \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(s)|^2 \,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(s)\rangle \,\mathrm{d}\mathcal{H}^{n-1} + \int_s^t \langle \nabla u(r), \nabla \dot w(r)\rangle_{L^2} \,\mathrm{d}r\,.
\end{aligned}$$

Thus, if $t$ is a continuity point for the nondecreasing function $s \mapsto \nu(s)$, we have $\|\nabla u(s)\|_{L^2} \to \|\nabla u(t)\|_{L^2}$ as $s \to t$, since $r \mapsto \langle \nabla u(r), \nabla w(r)\rangle_{L^2}$ is in $L^1([0,T])$ by (6.1.2) and (6.4.13). By Lemma 6.4.9, this gives the desired convergence. □

## 6.5. Proof of the main result

This section is devoted to the proof of Theorem 6.1.9. Besides, we also give a proof of Proposition 6.1.7 and of Proposition 6.1.8 regarding the strong formulation of the quasistatic evolution.

In Section 6.4 we have shown the existence of a quasistatic evolution $(u(t), \nu(t))$ in the sense of Young measures. We will now exploit the concavity of $g(x,\cdot)$ to prove that the very same displacement $t \mapsto u(t)$ found in Section 6.4 is also a quasistatic evolution in the sense of Definition 6.1.6. We recall that $g(x,\cdot)$ is strictly increasing in the interval $[0,\theta(x)]$, $\theta(x)$ is the threshold defined in (6.1.6). This allows us to prove that the Young measure $\nu(t)$ truncated by $\theta$ (see (6.2.10) for the definition) is actually concentrated on $V_u(t)\wedge\theta$, i.e., $V_u(t)\wedge\theta$ is the limit of $V_k(t)\wedge\theta$.

Proof of Theorem 6.1.9. By Theorem 6.4.4 and Proposition 6.4.11, we know that there exists a quasistatic evolution in the sense of Young measures $t \mapsto (u(t),\nu(t))$ such that, for every $t\in[0,T]$, we have (6.1.12) and

$$\delta_{V_k(t)} = \nu_k(t) \rightharpoonup \nu(t) \quad \text{in } \mathcal{Y}(\Gamma;[0,\infty])\,, \tag{6.5.1}$$

up to a subsequence independent of $t$ (not relabelled).

In order to prove (GS), we first prove that

$$\nu(t) \succeq \delta_{V_u(t)} \quad \text{for every } t\in[0,T]\,. \tag{6.5.2}$$

By definition of $V_u(t)$ and Remark 6.2.15, it is enough to show that for any partition $P$ of $[0,t]$, $P=\{0=s_0<s_1<\cdots<s_{j-1}<s_j=t\}$, we have

$$\nu(t) \succeq \delta_{V^P(t)}\,, \tag{6.5.3}$$

where

$$V^P(t) := V_0 + \sum_{i=1}^{j} \big|[u(s_i)]-[u(s_{i-1})]\big|\,.$$

The irreversibility condition (IRY) satisfied by $s\mapsto\nu(s)$ yields

$$\nu(s_i) \succeq \nu(s_{i-1}) \oplus \big|[u(s_i)]-[u(s_{i-1})]\big| \quad \text{for } i=1,\dots,j\,. \tag{6.5.4}$$

Employing (6.5.4) inductively, we obtain the chain of inequalities

$$\begin{aligned}
\nu(t) = \nu(s_j) &\succeq \nu(s_{j-1}) \oplus \big|[u(s_j)]-[u(s_{j-1})]\big| \\
&\succeq \nu(s_{j-2}) \oplus \Big(\big|[u(s_{j-1})]-[u(s_{j-2})]\big| + \big|[u(s_j)]-[u(s_{j-1})]\big|\Big) \succeq \dots \\
&\succeq \nu(s_1) \oplus \sum_{i=2}^{j} \big|[u(s_i)]-[u(s_{i-1})]\big| \\
&\succeq \nu(0) \oplus \sum_{i=1}^{j} \big|[u(s_i)]-[u(s_{i-1})]\big| = \delta_{V^P(t)}\,,
\end{aligned}$$

and thus (6.5.2) holds true.

Recalling the definition of cumulative distribution function (6.2.3), we have $F_{\delta_{V_u(t;x)}}(\xi) = 0$ for $\xi < V_u(t;x)$. Thus, by (ii) in Definition 6.2.12, we deduce that

$$\operatorname{supp}\nu^x(t) \subset [V_u(t;x), \infty] \tag{6.5.5}$$

for every $t \in [0,T]$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.

We are now in a position to prove that $t \mapsto u(t)$ satisfies the global stability condition (GS). We start by fixing $t \in [0,T]$ and $\widehat{u} \in A(w(t))$, and by setting

$$\widehat{\nu} := \nu(t) \oplus \big|[\widehat{u}] - [u(t)]\big|\,. \tag{6.5.6}$$

Condition (GSY) for $t \mapsto (u(t), \nu(t))$ gives

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2\,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \widehat{u}|^2\,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \widehat{\nu}^x\rangle\,\mathrm{d}\mathcal{H}^{n-1},$$

and thus (GS) follows if we show that

$$\int_\Gamma \Big(\langle g(x,\cdot), \widehat{\nu}^x\rangle - \langle g(x,\cdot), \nu^x(t)\rangle\Big)\,\mathrm{d}\mathcal{H}^{n-1} \le \int_\Gamma \Big(g\big(x,\, V_u(t) + \big|[\widehat{u}]-[u(t)]\big|\big) - g\big(x, V_u(t)\big)\Big)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{6.5.7}$$

In order to prove (6.5.7), notice that by (6.5.5) and (6.5.6) we have

$$\begin{aligned} \langle g(x,\cdot), \widehat{\nu}^x\rangle - \langle g(x,\cdot), \nu^x(t)\rangle &= \int_{[0,\infty]} \Big(g\big(x,\, \xi + \big|[\widehat{u}(x)]-[u(t;x)]\big|\big) - g(x,\xi)\Big)\nu^x(t)(\mathrm{d}\xi) \\ &= \int_{[V_u(t;x),\infty]} \Big(g\big(x,\, \xi + \big|[\widehat{u}(x)]-[u(t;x)]\big|\big) - g(x,\xi)\Big)\nu^x(t)(\mathrm{d}\xi)\,, \end{aligned} \tag{6.5.8}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Since $g(x,\cdot)$ is a concave function, for every $\xi \ge V_u(t;x)$ it holds

$$g\big(x,\, \xi+\big|[\widehat{u}(x)]-[u(t;x)]\big|\big) - g(x,\xi) \le g\big(x,\, V_u(t;x)+\big|[\widehat{u}(x)]-[u(t;x)]\big|\big) - g\big(x, V_u(t;x)\big)\,. \tag{6.5.9}$$

Let us observe that the right hand side in the inequality above does not depend on $\xi$. Therefore, by (6.5.8), (6.5.9), and recalling that $\nu^x(t)$ is a probability measure for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, we deduce (6.5.7). This completes the proof of (GS).

Let us now prove that $t \mapsto u(t)$ satisfies (EB). Arguing as in the proof of (6.4.21), using (GS) it is possible to see that

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2\,\mathrm{d}x + \int_\Gamma g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\qquad \ge \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2\,\mathrm{d}x + \int_\Gamma g\big(x, V_0\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2}\,\mathrm{d}s\,. \end{aligned}$$

On the other hand, the opposite inequality follows immediately from (EBY) since by (6.5.2) we have

$$\int_\Gamma g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \le \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1}.$$

Therefore, $t \mapsto u(t)$ is a quasistatic evolution in the sense of Definition 6.1.6.

We now claim that the truncation $\mathcal{T}^\theta_\# \nu(t)$ (see (6.2.10) for the definition) is concentrated on $V_u(t) \wedge \theta$. To this end, we compare (EB) and (EBY), and deduce that for every $t \in [0,T]$

$$\int_\Gamma g\big(x, V_u(t)\big)\, \mathrm{d}\mathcal{H}^{n-1} = \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle\, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.5.10}$$

Since by (6.5.2) and Definition 6.2.12 we have $g\big(x, V_u(t;x)\big) \leq \langle g(x,\cdot), \nu^x(t)\rangle$, equality (6.5.10) implies that

$$g\big(x, V_u(t;x)\big) = \langle g(x,\cdot), \nu^x(t)\rangle \tag{6.5.11}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Let us now fix $t$ and let $x$ be such that (6.5.5) holds. To prove the claim, we need to show that if $V_u(t;x) < \theta(x)$, then $\nu^x(t)\big((V_u(t;x), \infty]\big) = 0$. Let us assume, on the contrary, that $\nu^x(t)\big((V_u(t;x), \infty]\big) = c \in (0,1]$. By (6.5.5) we know that

$$\langle g(x,\cdot), \nu^x(t)\rangle = g(x, V_u(t;x))(1-c) + \int_{(V_u(t;x),\infty]} g(x,\xi)\, \nu^x(t)(\mathrm{d}\xi)\,,$$

and thus

$$\langle g(x,\cdot), \nu^x(t)\rangle - g\big(x, V_u(t;x)\big) = \int_{(V_u(t;x),\infty]} \Big(g(x,\xi) - g(x, V_u(t;x))\Big)\, \nu^x(t)(\mathrm{d}\xi)\,. \tag{6.5.12}$$

Since $g(x,\cdot)$ is strictly increasing in $[0, \theta(x)]$ and $\nu^x(t)\left((V_u(t;x), \infty]\right) > 0$, we get that the right-hand side in (6.5.12) is strictly positive. This contradicts (6.5.11), and therefore we have proved that $\mathcal{T}^\theta_\# \nu(t)$ is concentrated on $V_u(t) \wedge \theta$.

Eventually, using also (6.5.1) and Remark 6.2.11, we deduce that

$$\delta_{V_k(t)\wedge\theta} = \mathcal{T}^\theta_\# \nu_k(t) \rightharpoonup \mathcal{T}^\theta_\# \nu(t) = \delta_{V_u(t)\wedge\theta} \quad \text{in } \mathcal{Y}(\Gamma; [0,\infty])\,. \tag{6.5.13}$$

By Proposition 6.2.6, (6.5.13) is equivalent to (6.1.13).

As for the proof of (6.1.14) and (6.1.15), we notice that by Proposition 6.4.12 there exists a set $E$, at most countable, such that we have (6.1.14) and $\nu(s) \rightharpoonup \nu(t)$ in $\mathcal{Y}(\Gamma; [0,\infty])$, for $t \in [0,T] \setminus E$ and $s \to t$. The convergence in (6.1.15) then follows with an argument analogous to the one used to show (6.1.13).

This concludes the proof. □

Remark 6.5.1. In the proof of Theorem 6.1.9, we have shown that $\mathcal{T}^\theta_\# \nu(t) = \delta_{V_u(t)\wedge\theta}$. In particular, this allows us to rewrite the set $\Gamma_N(t)$ introduced in (6.4.7) (corresponding to the part of $\Gamma$ where the material is completely fractured) in terms of the variation of the jumps $V_u(t)$ and the threshold $\theta$. Namely, we have

$$\Gamma_N(t) = \{x \in \Gamma \colon V_u(t;x) \geq \theta(x)\}\,.$$

We now give the proof of the results concerning the strong formulation of the quasistatic evolution discussed in Section 6.1. The derivation of the Euler-Lagrange conditions follows by standard arguments illustrated below.

Proof of Proposition 6.1.7. Let consider the set $\Gamma_N(t) = \{x \in \Gamma \colon V_u(t;x) \geq \theta(x)\}$. Let $\psi \in H^1(\Omega \setminus \Gamma)$ with $\psi = 0$ on $\partial_D\Omega$ and let $\varepsilon \in \mathbb{R}$. Since

$$\int_{\Gamma_N(t)} g\big(x, V_u(t)\big)\, \mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma_N(t)} \kappa(x)\, \mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma_N(t)} g\big(x,\, V_u(t) + \big|\varepsilon[\psi]\big|\big)\, \mathrm{d}\mathcal{H}^{n-1}$$

and $u(t) + \varepsilon\psi \in A(w(t))$, by (GS) we have

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2\, \mathrm{d}x + \int_{\Gamma\setminus\Gamma_N(t)} g\big(x, V_u(t)\big)\, \mathrm{d}\mathcal{H}^{n-1} \\ &\qquad \leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t) + \varepsilon\nabla\psi|^2\, \mathrm{d}x + \int_{\Gamma\setminus\Gamma_N(t)} g\big(x,\, V_u(t) + \big|\varepsilon[\psi]\big|\big)\, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

Since $g$ is of class $C^1$, deriving the previous inequality with respect to $\varepsilon$ for $\varepsilon > 0$ and $\varepsilon < 0$, we get

$$-\int_{\Gamma\setminus\Gamma_N(t)} g'\big(x, V_u(t)\big)\big|[\psi]\big|\, \mathrm{d}\mathcal{H}^{n-1} \leq \int_{\Omega\setminus\Gamma} \nabla u(t)\cdot\nabla\psi\, \mathrm{d}x \leq \int_{\Gamma\setminus\Gamma_N(t)} g'\big(x, V_u(t)\big)\big|[\psi]\big|\, \mathrm{d}\mathcal{H}^{n-1}.$$

Using the fact that $g'(x,\xi) = 0$ for $\xi \geq \theta(x)$, we also get

$$-\int_{\Gamma} g'\big(x, V_u(t)\big)\big|[\psi]\big|\, \mathrm{d}\mathcal{H}^{n-1} \leq \int_{\Omega\setminus\Gamma} \nabla u(t)\cdot\nabla\psi\, \mathrm{d}x \leq \int_{\Gamma} g'\big(x, V_u(t)\big)\big|[\psi]\big|\, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.5.14}$$

By (6.5.14) for arbitrary $\psi \in H^1(\Omega)$ with $\psi = 0$ on $\partial_D\Omega$ and $\psi = 0$ in $\Omega^-$, we infer that $\Delta u(t) = 0$ in $\Omega^+$ and $\partial_\nu u(t) = 0$ in $H^{-\frac{1}{2}}(\partial_N\Omega \cap \partial\Omega^+)$. With similar arguments, we obtain analogous properties in $\Omega^-$ and we eventually deduce (i).

Let us prove (ii). Since $\nu_\Gamma$ is chosen in such a way that it coincides with the outer normal to $\partial\Omega^-$, by definition of normal derivative of the function $u(t)^+ = u(t)|_{\Omega^+}$ on $\Gamma$ we have that $\partial_\nu u(t)^+ \in H^{-\frac{1}{2}}(\Gamma)$ is given by

$$\langle \partial_\nu u(t)^+, \psi^+\rangle = -\int_{\Omega^+} \nabla u(t)\cdot\nabla\psi^+\, \mathrm{d}x\,,$$

for every $\psi^+ \in H^1(\Omega^+)$ with $\psi^+ = 0$ on $\partial_D\Omega \cap \partial\Omega^+$. Similarly, the normal derivative $\partial_\nu u(t)^- \in H^{-\frac{1}{2}}(\Gamma)$ is given by

$$\langle \partial_\nu u(t)^-, \psi^-\rangle = \int_{\Omega^-} \nabla u(t)\cdot\nabla\psi^-\, \mathrm{d}x\,,$$

for every $\psi^- \in H^1(\Omega^-)$ with $\psi^- = 0$ on $\partial_D\Omega \cap \partial\Omega^-$. Hence, by testing (6.5.14) with functions $\psi \in H^1(\Omega \setminus \Gamma)$ with $\psi = 0$ on $\partial_D\Omega$ and $[\psi] = 0$ on $\Gamma$, we infer

$$-\langle \partial_\nu u(t)^+, \psi\rangle + \langle \partial_\nu u(t)^-, \psi\rangle = 0\,,$$

which implies (ii) by the arbitrariness of $\psi$.

In order to prove (iii), we note that since $g'(x,\xi) \leq g'(x,0)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ and for every $\xi \in [0,\infty]$, by inequality (6.5.14) we get

$$\big|\langle \partial_\nu u(t), [\psi]\rangle\big| \leq \|g'(\cdot,0)\|_{L^\infty}\|[\psi]\|_{L^1},$$

for every $\psi \in H^1(\Omega \setminus \Gamma)$ with $\psi = 0$ on $\partial_D\Omega$. Thus $\partial_\nu u(t)$ is a linear and continuous operator on the space $X := \{[\psi] \colon \psi \in H^1(\Omega \setminus \Gamma)$ such that $\psi = 0$ on $\partial_D\Omega\}$. By density of $X$ in $L^1(\Gamma)$, this implies that $\partial_\nu u(t)$ can be extended to a linear and continuous operator on $L^1(\Gamma)$, and hence $\partial_\nu u(t) \in L^\infty(\Gamma)$. From (6.5.14) we deduce that

$$-\int_\Gamma g'\big(x, V_u(t)\big)|z| \,\mathrm{d}\mathcal{H}^{n-1} \leq -\int_\Gamma \partial_\nu u(t) z \,\mathrm{d}\mathcal{H}^{n-1} \leq \int_\Gamma g'\big(x, V_u(t)\big)|z| \,\mathrm{d}\mathcal{H}^{n-1},$$

for every $z \in L^1(\Gamma)$. This concludes the proof of (iii). □

In order to give a proof of Proposition 6.1.8, we need to prove the following lemma regarding the differentiability in time of the essential variation of a function that is absolutely continuous in time with values in $L^2(\Gamma)$.

LEMMA 6.5.2. *Let* $\gamma \in AC([0,T]; L^2(\Gamma))$. *Then* $\operatorname{ess\,Var}(\gamma; 0, \cdot) \in AC([0,T]; L^2(\Gamma))$ *and*

$$\lim_{s\to t} \frac{\operatorname{ess\,Var}(\gamma; s, t)}{t-s}(x) = |\dot\gamma(t; x)| \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma \text{ and for a.e. } t \in [0,T]\,, \tag{6.5.15}$$

*where the limit and the derivative* $\dot\gamma$ *are defined with respect to the strong topology in* $L^2(\Gamma)$.

PROOF. We fix $s, t \in [0,T]$ with $s < t$ and we consider a partition of the interval $[s,t]$, namely $s = s_0 < \cdots < s_j = t$. By the absolute continuity of $\gamma$, for every $i = 1, \ldots, j$ we have

$$|\gamma(s_i; x) - \gamma(s_{i-1}; x)| = \left| \int_{s_{i-1}}^{s_i} \dot\gamma(\tau; x) \,\mathrm{d}\tau \right| \leq \int_{s_{i-1}}^{s_i} \big|\dot\gamma(\tau; x)\big| \,\mathrm{d}\tau \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\,,$$

where the integrals are Bochner integrals and $\dot\gamma(\tau)$ is the derivative in $L^2(\Gamma)$ of $\gamma(\tau)$. Summing up the previous inequalities for $i = 1, \ldots, j$, we obtain

$$\sum_{i=1}^{j} |\gamma(s_i; x) - \gamma(s_{i-1}; x)| \leq \int_s^t |\dot\gamma(\tau; x)| \,\mathrm{d}\tau \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\,. \tag{6.5.16}$$

By Definition 6.1.3, (6.5.16) implies that

$$\operatorname{ess\,Var}(\gamma; s, t)(x) \leq \int_s^t |\dot\gamma(\tau; x)| \,\mathrm{d}\tau \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\,. \tag{6.5.17}$$

In particular, choosing $s = 0$ in (6.5.17) we deduce that $\operatorname{ess\,Var}(\gamma; 0, t)$ belongs to $L^2(\Gamma)$, for every $t \in [0,T]$. By taking the $L^2$ norm in (6.5.17) we infer

$$\|\operatorname{ess\,Var}(\gamma; s, t)\|_{L^2} \leq \int_s^t \|\dot\gamma(\tau)\|_{L^2} \,\mathrm{d}\tau\,.$$

Since the function $\tau \mapsto \|\dot\gamma(\tau)\|_{L^2}$ belongs to $L^1([0,T]; \mathbb{R})$, we conclude that $\operatorname{ess\,Var}(\gamma; 0, \cdot) \in AC([0,T]; L^2(\Gamma))$.

We now compute the derivative of $\operatorname{ess\,Var}(\gamma; 0, \cdot)$. Since $\frac{1}{t-s}\int_s^t |\dot\gamma(\tau)| \,\mathrm{d}\tau \to |\dot\gamma(t)|$ strongly in $L^2(\Gamma)$ as $s \to t$, dividing all terms in (6.5.17) by $t - s$ and letting $s \to t$ we deduce that

$$\lim_{s\to t} \frac{\operatorname{ess\,Var}(\gamma; s, t)}{t-s}(x) \leq |\dot\gamma(t; x)|$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. On the other hand, since $\{s,t\}$ is a particular partition of the interval $[s,t]$, by definition of essential variation we have

$$|\gamma(t;x)-\gamma(s;x)| \le \operatorname{ess\,Var}(\gamma;s,t)(x)\,,$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Dividing by $t-s$ and letting $s \to t$ in the inequality above, we obtain (6.5.15). □

We are now in a position to prove Proposition 6.1.8.

Proof of Proposition 6.1.8. Since by assumption $u \in AC([0,T];H^1(\Omega\setminus\Gamma))$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\setminus\Gamma}|\nabla u(t)|^2\,\mathrm{d}x = \int_{\Omega\setminus\Gamma}\nabla u(t)\cdot\nabla\dot u(t)\,\mathrm{d}x\,. \tag{6.5.18}$$

Moreover we claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma} g(x,V_u(t))\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma} g'\big(x,V_u(t)\big)\big|[\dot u(t)]\big|\,\mathrm{d}\mathcal{H}^{n-1}. \tag{6.5.19}$$

Let us prove (6.5.19). The absolute continuity of $u$ implies that $[u] \in AC([0,T];L^2(\Gamma))$. Let us consider the set $\Gamma_N(0) = \{x\in\Gamma\colon V_0(x)\ge\theta(x)\}$. Thanks to Lemma 6.5.2 and by the definition (6.1.3) of $V_u(t)$, for every $t\in[0,T]$ we have $V_u(t;x)<\infty$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Gamma\setminus\Gamma_N(0)$. Then, since $g(x,\xi)=\kappa(x)$ for $\xi\in[\theta(x),\infty]$, since $g(x,\cdot)$ is monotone, and since $V_u(t)$ is monotone in $t$,

$$\int_{\Gamma}\frac{g\big(x,V_u(t+h)\big)-g\big(x,V_u(t)\big)}{h}\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma\setminus\Gamma_N(0)}\frac{g\big(x,V_u(t+h)\big)-g\big(x,V_u(t)\big)}{h}\,\mathrm{d}\mathcal{H}^{n-1}.$$

Since $V_u(t+h;x)-V_u(t;x) = \operatorname{ess\,Var}([u];t,t+h)(x)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Gamma\setminus\Gamma_N(0)$ and $g'(x,V_u(t;x))=0$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Gamma_N(0)$, by taking the limit as $h\to0^+$ in the previous equality, by Lemma 6.5.2, and since $g$ is of class $C^1$, we eventually deduce (6.5.19).

The equalities (6.5.18) and (6.5.19) combined with (EB) imply that

$$\int_{\Omega\setminus\Gamma}\nabla u(t)\cdot\nabla(\dot u(t)-\dot w(t))\,\mathrm{d}x + \int_{\Gamma} g'(x,V_u(t))\big|[\dot u(t)]\big|\,\mathrm{d}\mathcal{H}^{n-1} = 0\,.$$

Since $\dot u(t)-\dot w(t)=0$ on $\partial_D\Omega$, by definition of $\partial_\nu u(t)$ we obtain

$$\int_{\Gamma}\partial_\nu u(t)[\dot u(t)]\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma} g'(x,V_u(t))\big|[\dot u(t)]\big|\,\mathrm{d}\mathcal{H}^{n-1},$$

and thus

$$\int_{\{[\dot u(t)]\neq0\}}\big(g'(x,V_u(t))\operatorname{Sign}([\dot u(t)])-\partial_\nu u(t)\big)[\dot u(t)]\,\mathrm{d}\mathcal{H}^{n-1} = 0\,.$$

By (iii) in Proposition 6.1.7, this proves the claim. □

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