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Doctoral Thesis

Integrable Models and Geometry of Target Spaces from the Partition Function of $\mathcal{N} = (2,2)$ theories on S^2

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Foreword

This thesis is based on my PhD. research published in the following papers, which are partially included in the body of the thesis:

- G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, "The Stringy Instanton Partition Function," JHEP **1401** (2014) 038, [arXiv:1306.0432 [hep-th]].
- G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, "Vortex partition functions, wall crossing and equivariant Gromov-Witten invariants," Commun. Math. Phys. **333** (2015) 2, 717, [arXiv:1307.5997 [hep-th]].
- G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, "Six-dimensional supersymmetric gauge theories, quantum cohomology of instanton moduli spaces and gl(N) Quantum Intermediate Long Wave Hydrodynamics," JHEP **1407** (2014) 141, [arXiv:1403.6454 [hep-th]].
- G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, "Quantum Cohomology and Quantum Hydrodynamics from Supersymmetric Quiver Gauge Theories," arXiv:1505.07116 [hep-th].

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

Abstract

Department of Theoretical Particle Physics

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Integrable Models and Geometry of Target Spaces from the Partition Function of $\mathcal{N} = (2,2)$ theories on S^2

by Petr VASKO

In this thesis we analyze the exact partition function for $\mathcal{N} = (2,2)$ supersymmetric theories on the sphere S^2. Especially, its connection to geometry of target spaces of a gauged linear sigma model under consideration is investigated. First of all, such a model has different phases corresponding to different target manifolds as one varies the Fayet-Iliopoulos parameters. It is demonstrated how a single partition function includes information about geometries of all these target manifolds and which operation corresponds to crossing a wall between phases. For a fixed phase we show how one can extract from the partition function the $\mathcal{I}$ function, a central object of Givental's formalism developed to study mirror symmetry. It is in some sense a more fundamental object than the exact Kähler potential, since it is holomorphic in the coordinates of the moduli space (in a very vague sense it is a square root of it), and the main advantage is that one can derive it from the partition function in a more effective way. Both these quantities contain genus zero Gromov–Witten invariants of the target manifold. For manifolds where mirror construction is not known (this happens typically for targets of non-abelian gauged linear sigma models), this method turns out to be the only available one for obtaining these invariants. All discussed features are illustrated on numerous examples throughout the text.

Further, we establish a way for obtaining the effective twisted superpotential based on studying the asymptotic behavior of the partition function for large radius of the sphere. Consequently, it allows for connecting the gauged linear sigma model with a quantum integrable system by applying the Gauge/Bethe correspondence of Nekrasov and Shatashvili. The dominant class of examples we study are "ADHM models“, i.e. gauged linear sigma models with target manifold the moduli space of instantons (on $\mathbb{C}^2$ or $\mathbb{C}^2/\Gamma$). For the case of a unitary gauge group we were able to identify the related integrable system, which turned out to be the Intermediate Long Wave system describing hydrodynamics of two layers of liquids in a channel. It has two interesting limits, the Korteweg–deVries integrable system (limit of shallow water with respect to the wavelength) and Benjamin–Ono integrable system (deep water limit). Another integrable model that naturally enters the scene is the (spin) Calogero–Sutherland model. We examine relations among energy eigenvalues of the latter, the spectrum of integrals of motion for Benjamin–Ono and expectation values of chiral correlators in the ADHM model.

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Contents

Introduction

Exact results in Quantum Field Theory (QFT) are quite rare and thus very valuable, since they can explore strongly coupled regions in the parameter space, which are inaccessible by perturbation theory. Completing such a calculation amounts to summing all perturbative corrections as well as non-perturbative ones induced by instantons. A rather hopeless task for a general QFT and still a big challenge even for special (but large enough) families of theories. Nevertheless, examples of theories exist, where exact results for particular classes of observables are achieved. There are different non-perturbative methods, but especially in more than two dimensions they require some amount of supersymmetry. We will concentrate on a specific technique going under the name of *supersymmetric localization*. As the name suggests it applies only to theories which posses some supersymmetry. Indeed, our viewpoint about simplicity of a QFT has evolved in the last decades. Nowadays, most of the researchers in this field would point to the $\mathcal{N} = 4$ super-Yang–Mills as being the simplest (interacting) QFT in four dimensions, instead of the scalar ϕ^4 theory that used to play this role in the past. What makes supersymmetric theories special in the space of all QFTs and why are they computationally more accessible? The general principle says that more symmetry always implies simplicity and gives hope for solving the theory completely (even at the expense of introducing more fields leading to, at first sight, more complicated Lagrangians). The other, more concrete, reasons are:

(i) supersymmetry equips the target manifold with canonical geometrical structures which are rigid and thus can be often uniquely fixed

(ii) states that saturate the BPS bound (BPS protected operators) form representations of the supersymmetry algebra with smaller dimension than states that do not saturate this bound, so these different classes of representations can not mix under renormalization; typically BPS protected operators have highly constrained quantum corrections, which are feasible to compute

(iii) symmetry between bosons and fermions implies cancellations occurring in perturbation theory, requiring thus to compute just a very small part of the spectrum of bosonic and fermionic differential (kinetic) operators

(iv) path integrals for supersymmetric theories are independent of a certain class of deformations, which allows us to use this invariance to simplify the computation drastically and eventually to evaluate the path integral exactly

It is the last point that is crucial in derivation of the supersymmetric localization formula. However, from a historical point of view, localization techniques have been known before to mathematicians in the realm of finite dimensional integrals. It is the statement that for a certain class of integrals the saddle point approximation leads to a precise answer. The main results in this field have been collected in a couple of localization theorems by Duistermaat–Heckman [1], Berline–Vergne [2] and Atiyah–Bott [3], which are dated in the early 80's. Soon after, these ideas have been generalized to the infinite dimensional setting by Witten and applied to path integrals of supersymmetric field theories, though at first exclusively to topological ones [4, 5]. This is so because interesting quantities like partition functions, Wilson loops and other observables suffer from infra-red divergences that need to be regularized. A convenient way to do it is by considering the theory on a compact manifold instead of flat space. However, at that time topological twisting was the only way how to define a supersymmetric theory on a rather general curved manifold.

A substantial renaissance of supersymmetric localization techniques appeared after the work of Pestun [6] in 2007. The method of localization actually stayed unchanged, what got significantly enriched was the class of theories to which it could have been applied. People learned how to define supersymmetric theories on curved manifolds without performing topological twisting. It consists of deforming the Lagrangian as well as supersymmetry variations by terms proportional to the curvature of the manifold, while keeping the action invariant under some amount of supersymmetry. At first, it was done by hand, on a case by case basis, just later got partially systematized in a series of papers initiated by [7] based on supergravity considerations. The new invention opened a bright window for obtaining new results and indeed their number grew fast. Many different theories were considered on various curved manifolds (mostly spheres or orbifolds of them) and diverse observables were computed for them. After eight years the literature is already vast, so we do not attempt to provide a list of references here. Special volumes of review papers have been dedicated to this topic, the reader is suggested to see [8] for a rich list of references.

Let us explore the contents of point (iv) by deriving the supersymmetric localization formula and later we explain what are the results good for. Suppose that we want to compute a vacuum expectation value of some operator $\mathcal{O}[\Phi]$, which is a functional of the set of fields Φ in the theory

$$\langle\mathcal{O}\rangle = \int \{D\Phi\} e^{-S[\Phi]}\mathcal{O}[\Phi]. \tag{1}$$

One selects a localizing supercharge $\mathcal{Q}$, i.e. a (suitably chosen) linear combination of conserved supercharges. The supersymmetry algebra implies $\{\mathcal{Q},\mathcal{Q}\} = \mathcal{B}$, where $\mathcal{B}$ is some bosonic symmetry (even generator in the superalgebra). A necessary condition for the localization theorem to hold is invariance of the operator $\mathcal{O}[\Phi]$ under the localizing supercharge $\mathcal{Q}$, $\mathcal{Q}\mathcal{O}[\Phi] = 0$, as well as of the action, which holds by assumption of supersymmetry. Then one can deform the action $S[\Phi] \to S[\Phi] + tS_{def}[\Phi]$, where $S_{def}[\Phi] = \mathcal{Q}V[\Phi]$ with $V[\Phi]$ a fermionic functional and t a real parameter. The deformation action $S_{def}[\Phi]$ should be positive-semidefinite and should not change the asymptotic behavior of the original action $S[\Phi]$ at infinity of field space. Then one can show, under the condition $\mathcal{Q}S_{def}[\Phi] = \frac{1}{2}\mathcal{B}V[\Phi] = 0$, the independence of the path integral of such a deformation

$$\langle\mathcal{O}\rangle = \int \{D\Phi\} e^{-S[\Phi]}\mathcal{O}[\Phi] = \int \{D\Phi\} e^{-\left(S[\Phi]+tS_{def}[\Phi]\right)}\mathcal{O}[\Phi]. \tag{2}$$

The argument goes as follows

$$\begin{aligned}\frac{d}{dt}\langle\mathcal{O}\rangle &= -\int \{D\Phi\}\left(\mathcal{Q}V[\Phi]\right)\mathcal{O}[\Phi] e^{-\left(S[\Phi]+tS_{def}[\Phi]\right)} \\ &= -\int \{D\Phi\}\mathcal{Q}\left(V[\Phi]\mathcal{O}[\Phi]e^{-\left(S[\Phi]+tS_{def}\right)}\right) = 0,\end{aligned} \tag{3}$$

which vanishes by application of an analog of Stokes' theorem for the $\mathcal{Q}$-exact integrand. Having this freedom of deformations at our disposal, we can evaluate the path integral at any value of t we like. Obviously, the choice $t \to \infty$ facilitates the computation essentialy, since in this limit the exact result is obtained by the saddle point approximation around the extrema of $S_{def}[\Phi]$. By our assumption on semi-positive definiteness of $S_{def}[\Phi]$ the extrema are achieved on the localization locus $\mathcal{L} = \{\Phi_* \mid \mathcal{Q}V[\Phi_*] = 0\}$. Then the final result is obtained just by computing the quadratic fluctuations around the localization locus

$$\langle\mathcal{O}\rangle = \int_{\mathcal{L}} \{D\Phi_*\}\mathcal{O}[\Phi_*]e^{-S[\Phi_*]}\left[\operatorname{sdet}\left(\frac{\delta^2 S_{def}}{\delta\Phi^2}[\Phi_*]\right)\right]^{-1}, \tag{4}$$

where sdet$(\cdot)$ denotes the super-determinant (Berezinian). The most powerful conclusions can be drawn from this formula, when the localization locus $\mathcal{L}$ degenerates to constant field configurations. In that case we end up with a finite dimensional integral and the problem is reduced to solving a *matrix model*. Supersymmetry enters (4) twice in a crucial way. First of all, due to (iv), it allows us to write an equality sign instead of an asymptotic equality, which would happen in a general QFT! And then, by applying (iii), it significantly facilitates the computation of sdet$(\cdot)$. One can concentrate only a very small part of the spectrum of bosonic/fermionic kinetic terms that does not cancel. Such a simplification turns out to be priceless for practical purposes on a general curved manifold.

Let us comment about the dependence of the correlator $\langle\mathcal{O}\rangle$ on running coupling parameters. We can split the original action into a $\mathcal{Q}$-exact piece and the rest, $S(\sigma,\tau) = S_{\mathcal{Q}-\mathrm{ex}}(\sigma) + S_{\mathrm{not}\ \mathcal{Q}-\mathrm{ex}}(\tau)$ [1]; here σ,τ are two sets of coupling parameters. As we just showed the output of path integration does not depend on $\mathcal{Q}$-exact terms in the action, therefore the correlator will depend only on the couplings τ. In particular, it might happen that the quantity under consideration will be independent of running coupling parameters (or depend only on those which undergo just a one-loop renormalization). In that case we have found a renormalization group invariant. Such observables are very powerful, since we can evaluate them at an arbitrary energy scale and we are guaranteed that they stay constant along the renormalization group flow (or behave in a very controlled manner). Of course, the natural thing to do is to perform the calculation at a point where the theory has a weakly coupled description.

Now we arrived at the question of applications. The major use is found in the very difficult branch of investigations, namely testing of various duality conjectures. In general, a duality is a map between different QFTs. Genuinely, it maps strongly coupled regions to weakly coupled ones and vice versa, thus is really hard to prove. Exact results obtained by supersymmetric localization come to rescue, however one should keep in mind that we are testing only a narrow class of observables. To illustrate this topic better let us sketch two very famous examples.

Seiberg–Witten solution and electric-magnetic duality conjectures. The exact low energy effective action for a four dimensional gauge theory with eight supercharges was obtained by Seiberg and Witten some time ago [9]. It is described in terms of a single holomorphic function $\mathcal{F}(a)$ called the prepotential, where a are coordinates on the moduli space of vacua in the Coulomb branch. This function is fixed by a geometrical structure on the target manifold going under the name of special Kähler geometry, an instance that we saw in point (i). Their derivation was crucially based on the conjecture of electric-magnetic duality, exchanging the coupling as $\tau_{IR}^{-1} \leftrightarrow \widetilde{\tau}_{IR}$. So it was of great value when Nekrasov confirmed this result by computing the exact partition function Z for these theories using localization [10] and relating it to the prepotential of Seiberg and

[1] Actually, $S_{\mathcal{Q}-\mathrm{ex}}$ is often a convenient choice for S_{def}.

Witten as $\mathcal{F}(a) = -\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2 \log Z(a|\epsilon_1,\epsilon_2)$ [11]; ϵ_1, ϵ_2 are regulating parameters characterizing the Ω-background and the limit reduces this background to $\mathbb{R}^4$.

Precision tests of gauge/holography (AdS/CFT like) dualities. The most striking checks were performed in the original AdS/CFT setting [12], claiming equivalance between a gauge theory and a superstring theory. On the gauge theory side one considers a $U(N)$ $\mathcal{N}=4$ super-Yang–Mills theory living on the boundary of an AdS_5 space while on the gravity side one has a type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. It is in the limit of infinite rank gauge group, where checks can be done. This corresponds to free strings ($g_s \to 0$, α' stays finite) and in this regime both sides are supposed to be integrable. For a very readable introduction to these topics see [13]. Since $\mathcal{N}=4$ SYM is just a special case of $\mathcal{N}=2$ gauge theories, one can enlarge this equivalence and test different exact results for $\mathcal{N}=2$ theories against generalized holography predictions. The AdS/CFT correspondence has evolved in many different directions since its discovery. Thus a similar way of thinking applies at present time to theories with various amount of supersymmetry in diverse dimensions.

It would feel incomplete not to mention a relation between the partition function of Nekrasov for a four dimensional gauge theory with eight supercharges and a non-supersymmetric conformal field theory in two dimensions, the Liouville theory. This is known as the AGT correspondence. Pestun computed an exact partition function for $\mathcal{N}=2$ theories on S^4 [6], which is tightly connected to Nekrasov's function as

$$\mathbf{Z}_{S^4} = \int da |Z(a)|^2. \tag{5}$$

Then we can state the AGT correspondence in a way relating the full 4-point correlator in Liouville theory with the partition function of $\mathcal{N}=2$ $SU(2)$ gauge theory with four fundamental hypermultiplets on S^4

$$\langle e^{2\alpha_4\phi(\infty)} e^{2\alpha_3\phi(1)} e^{2\alpha_2\phi(q)} e^{2\alpha_1\phi(0)} \rangle \propto \mathbf{Z}_{S^4}(m_1,\ldots,m_4|\tau), \tag{6}$$

where the proportionality constant is known and there is a dictionary among parameters on both sides. Since we are going to meet Nekrasov partition functions and their deformations in following chapters, we felt obliged to briefly introduce this important part of the story.

Maybe somewhat surprisingly, the exact results stemming from supersymmetric localization proved to be equally useful in more mathematical subjects like geometry of moduli spaces, knot theory, topological invariants and further topics. In this thesis we concentrate on one concrete piece of the bigger mosaic. The exact partition function of $\mathcal{N} = (2,2)$ theories on S^2 obtained by supersymmetric localization in [14, 15] is thoroughly analyzed.

Outline of the thesis

In Chapter 1 we lay down the basics of $\mathcal{N} = (2,2)$ supersymmetric theories in flat space. Chapter 2 shows how to deform them in order to keep invariance under four supercharges on S^2. Then we briefly review the localization computation to arrive at the result for the partition function on the sphere[2]. At the end of this chapter we comment about the possibility of reducing the number of supercharges, defining thus the analog of $\mathcal{N} = (0,2)$ theories from flat space on the sphere. This is interesting because the partition function would allow us to study their properties which are still unknown to a great extent. However, it seems rather challenging to put these theories on S^2.

Shortly before we started this project, it was suggested that the partition function computes exactly the Kähler potential of the target space and hence contains the genus zero Gromov–Witten invariants of the target manifold [17]. In Chapter 3 we pursue this direction further and connect the partition function with Givental's formalism developed to study mirror symmetry. We also show how a single partition function encompasses different phases of the gauged linear sigma model (GLSM) and their target space geometries as one varies the Fayet-Iliopoulos parameters. In Chapter 4 we observe that the effective twisted superpotential for a GLSM under consideration can be extracted from the partition function as well, studying its asymptotics for large radius of S^2. Then by the Nekrasov–Shatashvili correspondence one can associate a quantum integrable system to it.

One model, the ADHM GLSM, which recurs throughout the whole body of this thesis and by means of which we illustrate the main features of techniques we developed is a GLSM with target space the moduli space of instantons. Chapter 5 defines these models for all classical gauge groups, the corresponding partition functions are computed and Bethe equations of associated quantum integrable systems are listed. Only for the case of a unitary gauge group we were able to identify the related integrable model. In that event it is the Intermediate Long Wave (ILW) integrable system. The unitary ADHM is analyzed in detail in Chapter 6, both from the point of view of the geometry of its target space and the integrability point of view. Chapter 7 studies a further generalization of the unitary ADHM, allowing for instantons on asymptotically locally Euclidean spaces. Luckily, also an appropriate generalization of ILW was available in the literature. It turns out that a spin Calogero–Sutherland model is linked to these topics as well, so we

[2] The final result for the partition function holds as well for an ellipsoid as was shown in [16], even if the action and supersymmetry variations get changed.

comment on connections among these three models. Appendices are reserved for more technical issues and proofs of some statements.

Chapter 1

$\mathcal{N} = (2, 2)$ Supersymmetry basics

In the rest of this thesis we will focus exclusively on supersymmetric quantum field theories in two dimensions with four real supercharges, two of positive chirality while the remaining two of negative chirality. Such theories will be said to posses $\mathcal{N} = (2, 2)$ supersymmetry. This section is intended to summarize the elementary facts to make the text self contained. Many more details can be found in various beautiful review papers or books, in particular we are following the exposition given in [18]. Readers who are familiar with these topics can skip this section if desired. In the following, we introduce the concept of superspace, whose symmetries define the (graded) algebra of symmmetry generators. Studying its representation theory provides us with basic building blocks, the field multiplets. These can be conveniently packaged within the superfield formalism; the components of a superfield furnish a representation of the $\mathcal{N} = (2, 2)$ supersymmetry algebra. Next, we move to construct supersymmetric actions and provide a list of basic models.

1.1 Superspace and superfields

Let us consider a field theory on $\mathbb{R}^2$ with coordinates $\{x^0, x^1\}$. Besides these usual bosonic coordinates we introduce additional complex Graßmann (fermionic) coordinates

$$\theta^+,\ \theta^-,\ \overline{\theta}^+,\ \overline{\theta}^-,$$

which are related to each other by complex conjugation, $(\theta^\pm)^* = \overline{\theta}^\pm$. The $\pm$ index refers to the chirality (spin) under a Lorentz transformation in case of a Lorentz signature while to holomorphic (anti-holomorphic) supercoordinates in case of Euclidean signature. The fermionic nature of these coordinates means that they anticommute. Then the $(2, 2)$ *superspace* is the space equipped with a coordinate system $\{x^0, x^1, \theta^\pm, \overline{\theta}^\pm\}$.

Superfields are functions on superspace. Any such function can be expanded in monomials of the Graßmann coordinates and such expansion terminates as a consequence of the anticommutativity of these; indeed a general function contains $2^4 = 16$ terms. We will see that individual supersymmetry multiplets will be obtained by imposing certain shortening conditions on this general function.

In order to express supersymmetry variations and define different kinds of superfields, it is useful to introduce two sets of differential operators on superspace. The first set being

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\partial_\pm \tag{1.1}$$

$$\overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\partial_\pm, \tag{1.2}$$

where $\partial_\pm$ is a differentiation with respect to $x^\pm := x^0 \pm x^1$. They satisfy the anti-commutation relations $\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} = -2i\partial_\pm$; all other anti-commutators vanish. The second set is

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\partial_\pm \tag{1.3}$$

$$\overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\partial_\pm. \tag{1.4}$$

These operators anti-commute with the $\mathcal{Q}, \overline{\mathcal{Q}}$ system and satisfy their own relations $\{D_\pm, \overline{D}_\pm\} = 2i\partial_\pm$, where other combinations again vanish. This algebra admits an automorphism group $U(1)_L \times U(1)_R$, or by regrouping the generators, $U(1)_V \times U(1)_A$. These are the vector and axial R-rotations and act on a superfield like

$$e^{i\alpha F_V}: \quad \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \longmapsto e^{i\alpha q_V}\mathcal{F}(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\overline{\theta}^\pm) \tag{1.5}$$

$$e^{i\beta F_A}: \quad \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \longmapsto e^{i\beta q_A}\mathcal{F}(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\overline{\theta}^\pm) \tag{1.6}$$

with q_V and q_A the vector and axial R-charge of the superfield $\mathcal{F}$, respectively. Then the supersymmetric variation is defined as

$$\delta := \epsilon_+\mathcal{Q}_- - \epsilon_-\mathcal{Q}_+ - \overline{\epsilon}_+\overline{\mathcal{Q}}_- + \overline{\epsilon}_-\overline{\mathcal{Q}}_+. \tag{1.7}$$

By assumption, a supersymmetric action is invariant under this transformation. Applying the Noether procedure, we find four conserved supercurrents $G^\mu_\pm, \overline{G}^\mu_\pm$ given by

$$\delta\int d^2x\mathcal{L} = \int d^2x\left(\partial_\mu\epsilon_+G^\mu_- - \partial_\mu\epsilon_-G^\mu_+ - \partial_\mu\overline{\epsilon}_+\overline{G}^\mu_- + \partial_\mu\overline{\epsilon}_-\overline{G}^\mu_+\right). \tag{1.8}$$

Integrating their time component on a fixed time slice yields the corresponding conserved supercharges

$$Q_\pm = \int dx^1 G^0_\pm \tag{1.9}$$

$$\overline{Q}_\pm = \int dx^1 \overline{G}^0_\pm. \tag{1.10}$$

In addition, as in any Poincare invariant quantum field theory, there are also conserved charges

$$H,\ P,\ M$$

corresponding to time translations, spatial translations and rotations. Moreover if the action is invariant under vector and axial R-symmetries, there are Noether charges

$$F_V,\ F_A$$

as well.

Finally, the time has come to spell out the $\mathcal{N} = (2,2)$ superalgebra in full detail. The conserved charges of the theory satisfy the relations

$$\begin{gathered}
Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0 \\
\{Q_\pm, \overline{Q}_\pm\} = H \pm P \\
\{\overline{Q}_+, \overline{Q}_-\} = Z,\ \{Q_+, Q_-\} = Z^* \\
\{Q_-, \overline{Q}_+\} = \widetilde{Z},\ \{Q_+, \overline{Q}_-\} = \widetilde{Z}^* \\
[iM, Q_\pm] = \mp Q_\pm,\ \left[iM, \overline{Q}_\pm\right] = \mp \overline{Q}_\pm \\
[iF_V, Q_\pm] = -iQ_\pm,\ \left[iF_V, \overline{Q}_\pm\right] = i\overline{Q}_\pm \\
[iF_A, Q_\pm] = \mp iQ_\pm,\ \left[iF_A, \overline{Q}_\pm\right] = \pm i\overline{Q}_\pm
\end{gathered}$$

as long as $Z, \widetilde{Z}$ are central, i.e. they commute with all other generators in the theory. Hence they are called central charges. An immediate consequence of the algebra is that Z must vanish if F_V is conserved and on the other hand $\widetilde{Z}$ is forced to vanish whenever F_A is conserved. The above algebra is invariant under a $\mathbb{Z}_2$ outer automorphism acting on the generators as

$$Q_- \longleftrightarrow \overline{Q}_-,\ F_V \longleftrightarrow F_A,\ Z \longleftrightarrow \widetilde{Z}$$

and keeping all other fixed. A pair of $\mathcal{N} = (2,2)$ quantum field theories is defined to be *mirror* if the isomorphism of Hilbert spaces exchanges the generators by the above automorphism.

Now, we move to study representations of the $\mathcal{N} = (2,2)$ superalgebra. For instance take an operator ϕ satisfying $\left[\overline{Q}_\pm, \phi\right] = 0$. Then, acting by other generators

$$\psi_\pm := [iQ_\pm, \phi], \quad F := \{Q_+, [Q_-, \phi]\}$$

we can construct a representation $(\phi, \psi_+, \psi_-, F)$ called a *chiral multiplet.* All the components can be merged into a single object called a chiral superfield. So let us provide a list of superfields that will be used for building actions invariant under $\mathcal{N} = (2,2)$ supersymmetry.

Chiral superfield Φ is a superfield that satisfies the constraint

$$\overline{D}_\pm \Phi = 0 \tag{1.11}$$

and can be expanded into components

$$\Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) = \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm), \tag{1.12}$$

where $y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm$. To be explicit we can expand further around the point $x^\pm$ with the result

$$\begin{aligned}\Phi = \phi &- i\theta^+\overline{\theta}^+\partial_+\phi - i\theta^-\overline{\theta}^-\partial_-\phi - \theta^+\theta^-\overline{\theta}^-\overline{\theta}^+\partial_+\partial_-\phi \\ &+ \theta^+\psi_+ - i\theta^+\theta^-\overline{\theta}^-\partial_-\psi_+ + \theta^-\partial_-\psi_- - i\theta^-\theta^+\overline{\theta}^+\partial_+\psi_- + \theta^+\theta^- F.\end{aligned} \tag{1.13}$$

A product of two chiral superfields is again a chiral superfield and a supersymmetric variation of a chiral superfield is still chiral, which consistently implies transformation rules for individual component fields

$$\begin{aligned}\delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+ \\ \delta\psi_\pm &= \pm 2i\overline{\epsilon}_\mp\partial_\pm\phi + \epsilon_\pm F \\ \delta F &= -2i\overline{\epsilon}_+\partial_-\psi_+ - 2i\overline{\epsilon}_-\partial_+\psi_-.\end{aligned} \tag{1.14}$$

The complex conjugate $\overline{\Phi}$ of a chiral superfield obeys the equation

$$D_\pm\overline{\Phi} = 0 \tag{1.15}$$

and is called an *anti-chiral superfield.*

Twisted chiral superfield Y is defined by the condition

$$\overline{D}_+ Y = D_- Y = 0 \tag{1.16}$$

and has the form

$$\begin{aligned} Y(x^\mu,\theta^\pm,\overline{\theta}^\pm) &= v(\widetilde{y}^\pm)+\theta^+\overline{\chi}_+(\widetilde{y}^\pm)+\overline{\theta}_-\chi_-(\widetilde{y}^\pm)+\theta^+\overline{\theta}^-E(\widetilde{y}^\pm) \\ &= v-i\theta^+\overline{\theta}^+\partial_+v+i\theta^-\overline{\theta}^-\partial_-v+\theta^+\overline{\theta}^+\theta^-\overline{\theta}^-\partial_+\partial_-v \\ &+\theta^+\overline{\chi}_++i\theta^+\theta^-\overline{\theta}^-\partial_-\overline{\chi}_++\overline{\theta}^-\chi_--i\overline{\theta}^-\theta^+\overline{\theta}^+\partial_+\chi_-+\theta^+\overline{\theta}^-E, \end{aligned} \tag{1.17}$$

where $\widetilde{y}=x^\pm\mp-\theta^\pm\overline{\theta}^\pm$. In a similar fashion, twisted chiral superfields are closed under multiplication and supersymmetric variation. The transformation rules for components read

$$\begin{aligned} \delta v &= \overline{\epsilon}_+\chi_- - \epsilon_-\overline{\chi}_+ \\ \delta\overline{\chi}_+ &= 2i\overline{\epsilon}_-\partial_+v+\overline{\epsilon}_+E \\ \delta\chi_- &= -2i\epsilon_+\partial_-v+\epsilon_-E \\ \delta E &= -2i\epsilon_+\partial_-\overline{\chi}_+-2i\overline{\epsilon}_-\partial_+\chi_-. \end{aligned} \tag{1.18}$$

The complex conjugate $\overline{Y}$ of a twisted chiral superfield satisfies the constraints

$$D_+\overline{Y}=\overline{D}_-\overline{Y}=0 \tag{1.19}$$

and is called a *twisted anti-chiral superfield.*

With these two kinds of superfields introduced, we are already at a stage where it is appropriate to construct supersymmetric actions. First, consider the expression

$$S=\int d^2xd^4\theta K(\mathcal{F}_i), \tag{1.20}$$

where K is an arbitrary smooth function of the general superfields $\mathcal{F}_i$, called the *Kähler potential.* This action is invariant under the supersymmetry variation δ and is denoted as a *D-term.* Next, we can write down an *F-term*

$$\int d^2xd\theta^-d\theta^+W(\Phi_i)\Big|_{\overline{\theta}^\pm=0}. \tag{1.21}$$

Here Φ_i are chiral superfields and moreover dW must be a closed holomorphic one form, i.e. W is locally a holomorphic function called the *superpotential.* If these conditions are met, the supersymmetry variation vanishes as well. There is an analogous term for twisted chiral superfields called the *twisted F-term* given by

$$\int d^2xd\overline{\theta}^-d\theta^+\widetilde{W}(Y_i)\Big|_{\overline{\theta}^+=\theta^-=0}, \tag{1.22}$$

where $\widetilde{W}(Y_i)$ is a locally holomorphic function of the twisted chiral superfields going under the name *twisted superpotential*.

We can classify the theories according to the terms that appear in the action. If only the Kähler potential is present, thus the (twisted) superpotential is vanishing, we call such a model a *sigma model*. When the metric derived from this Kähler potential describes a flat space it is referred to as a *linear sigma model* while if the target space metric is non-trivial it is a *non-linear sigma model*. Once we also turn on a (twisted) superpotential then we will be speaking about a *Landau–Ginzburg (LG) model*.

The next natural step is the procedure of gauging, which leads us to introduce vector superfields. We focus here only on the abelian case. Indeed, consider a canonical D-term for a single chiral superfield

$$\int d^2x d^4\theta \overline{\Phi}\Phi. \tag{1.23}$$

This action has a symmetry $\Phi \to e^{i\alpha}\Phi$ with α constant. Now we promote α to a chiral superfield $A(x^\mu, \theta^\pm, \overline{\theta}^\pm)$. Then the term $\overline{\Phi}\Phi$ transforms to $\overline{\Phi}e^{-i\overline{A}+iA}\Phi$, which breaks invariance of the action. It is restored by introducing a vector superfield V transforming in such a way

$$V \to V + i(\overline{A} - A) \tag{1.24}$$

to cancel the unwanted term. Finally, the modified action

$$\int d^2x d^4\theta \overline{\Phi} e^V \Phi \tag{1.25}$$

turns out to be invariant again. We also see that V has to be real by consistency with the transformation rule.

Vector superfield is a real superfield that transforms according to (1.24). The gauge transformation can be used to eliminate some components, so that we can write for V in the Wess-Zumino gauge

$$\begin{aligned} V = {} & \theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} \\ & + i\theta^-\theta^+(\overline{\theta}^-\overline{\lambda}_- + \overline{\theta}^+\overline{\lambda}_+) + i\overline{\theta}^+\overline{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) + \theta^-\theta^+\overline{\theta}^+\overline{\theta}^- D, \end{aligned} \tag{1.26}$$

where v_0, v_1 are one-form fields, σ is a complex scalar, $\lambda_\pm$ and $\overline{\lambda}_\pm$ define a Dirac fermion field and D is a real scalar field. There is still a residual gauge symmetry with $A = \alpha(x^\mu)$ transforming

$$v_\mu(x) \to v_\mu(x) - \partial_\mu \alpha(x), \tag{1.27}$$

while keeping all other component fields fixed. A supersymmetry variation does not preserve the Wess–Zumino gauge, thus we have to perform a further gauge transformation to bring δV back to this gauge. In this way, the variations of component fields are fixed. However, we will not list them here. Just note that the chiral superfield is charged under the gauge transformation and so the variations of its components get modified as well.

The superfield

$$\Sigma := \overline{D}_+ D_- V \tag{1.28}$$

is invariant under the transformation (1.24) and is denoted as the *superfield strength* of V. One can easily check that it satisfies

$$\overline{D}_+ \Sigma = D_- \Sigma = 0 \tag{1.29}$$

and hence is a twisted chiral superfield. In this case it has the expansion

$$\Sigma = \sigma(\widetilde{y}) + i\theta^+ \overline{\lambda}_+(\widetilde{y}) - i\overline{\theta}^- \lambda_-(\widetilde{y}) + \theta^+ \overline{\theta}^- \left[D(\widetilde{y}) - i v_{01}(\widetilde{y}) \right], \tag{1.30}$$

where v_{01} is the field strength of v_μ, $v_{01} := \partial_0 v_1 - \partial_1 v_0$. Naturally, we can construct twisted F-terms out of Σ. Provided the gauge group contains a $U(1)$ factor (and since our discussion here is for abelian gauge groups, this is just the case), there is one distinguished twisted F-term, where the superfield strength enters linearly

$$\widetilde{W}_{FI,\theta} = -t\Sigma \tag{1.31}$$

with $t = \xi - i\theta$ a complex number; ξ is a Fayet-Iliopoulos term and θ is called a theta angle.

Having introduced this last building block, we can write down a supersymmetric Lagrangian for a vector multiplet minimally coupled to a charged chiral multiplet

$$\mathcal{L} = \int d^4\theta \left(\overline{\Phi} e^V \Phi - \frac{1}{2e^2} \overline{\Sigma}\Sigma \right) + \frac{1}{2} \left(-t \int d\overline{\theta}^- d\theta^+ \Sigma + c.c \right). \tag{1.32}$$

Such a model will be refered to as a *gauged linear sigma model* (GLSM). Here it was shown for an abelian gauge group only.

With this Lagrangian we finish our very basic introduction to $\mathcal{N} = (2,2)$ supersymmetry. Of course, much more could have been said, interested readers can consult appropriate references, for instance [19] or [18], where a detailed treatment is presented. To conclude, it is worth mentioning that $\mathcal{N} = (2,2)$ supersymmetry in two dimensions can be obtained by dimensionally reducing $\mathcal{N} = 1$ theories in four dimensions.

Chapter 2

$\mathcal{N} = (2,2)$ supersymmetry on S^2 and the exact partition function

The aim of this chapter is to review the results that were derived in [14, 15]. The authors have shown that an $\mathcal{N} = (2,2)$ supersymmetric theory can be placed on a two sphere S^2 while still preserving four real supercharges. After defining the theory on S^2, they computed the exact partition function using localization technique. Below, we want to summarize the main steps and formulae that will be important later. We wish to stress that the exact partition function on S^2 will be the main character for further developments.

2.1 $\mathcal{N} = (2,2)$ supersymmetry on S^2

The two sphere S^2 of radius r (as any two dimensional pseudo-Riemannian manifold) is conformally flat. The global superconformal algebra in two dimensions $\mathfrak{osp}(2|2,\mathbb{C})$ is parametrized by four conformal Killing spinors[1]. They are complex Dirac spinors while a minimal spinor in Euclidean signature is a complex Weyl spinor, which results in eight conserved superconformal charges. Two out of the four conformal Killing spinors are positive

$$D_\mu \epsilon = \frac{i}{2r}\gamma_\mu \epsilon, \tag{2.1}$$

while the remaining two are negative $D_\mu \epsilon = -\frac{i}{2r}\gamma_\mu \epsilon$. We make a choice and focus on the positive ones. The other alternative is also allowed and would define a consistent theory.

[1] Let us remind that S^2 does not admit any Killing spinors, however admits conformal Killing spinors.

The subalgebra of the superconformal algebra generated by the positive conformal Killing spinors is $\mathfrak{osp}^*(2|2) \simeq \mathfrak{su}(2|1)$, i.e. a compact real form of $\mathfrak{osp}(2|2,\mathbb{C})$. We define this simple $\mathfrak{su}(2|1)$ superalgebra as $\mathcal{N}=(2,2)$ *Euclidean supersymmetry algebra on* S^2. Its bosonic subalgebra is $\mathfrak{su}(2)\oplus\mathfrak{u}(1)_R$, where the $\mathfrak{su}(2)$ factor generates isometries of S^2 and $\mathfrak{u}(1)_R$ is the vector R-symmetry. As opposed to flat space it is part of the algebra, not an automorphism of it. All supersymmetry variations can be obtained from the known superconformal algebra by restricting to the positive conformal Killing spinors only.

2.1.1 Details about the $\mathcal{N}=(2,2)$ superalgebra on S^2

This subsection is intended for readers who wish to see a detailed construction of this superalgebra. It is also a prerequisite to Section 2.4, where we comment about the possibility to reduce the number of supercharges to two instead of four. However, both these sections are rather independent of the main text of the thesis and might be skipped when desired.

In [15] the $\mathcal{N}=(2,2)$ supersymmetry algebra on S^2 was constructed as follows. As we already mentioned the sphere S^2 is locally conformally flat. So one starts with the global (finite dimensional) part of the two-dimensional superconformal algebra (see [20], p.375). This is generated in the left-moving sector by even generators $\{L_{-1},L_0,L_{+1};\mathcal{J}_0\}$ and odd generators $\{G^+_{-\frac{1}{2}},G^+_{+\frac{1}{2}},G^-_{-\frac{1}{2}},G^-_{+\frac{1}{2}}\}$ [2]. There is also an independent (commuting) right-moving sector, whose generators we denote by the same symbols just marked by tilde. We write out the graded commutation relations for the left-moving generators

$$
\begin{aligned}
&[L_0,G^+_{+\frac{1}{2}}]=-\frac{1}{2}G^+_{+\frac{1}{2}} &\qquad& [L_0,G^-_{+\frac{1}{2}}]=-\frac{1}{2}G^-_{+\frac{1}{2}}\\
&[L_0,G^+_{-\frac{1}{2}}]=+\frac{1}{2}G^+_{-\frac{1}{2}} && [L_0,G^-_{-\frac{1}{2}}]=+\frac{1}{2}G^-_{-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&[L_{+1},G^+_{-\frac{1}{2}}]=+G^+_{+\frac{1}{2}} &\qquad& [L_{-1},G^+_{+\frac{1}{2}}]=-G^+_{-\frac{1}{2}}\\
&[L_{+1},G^-_{-\frac{1}{2}}]=+G^-_{+\frac{1}{2}} && [L_{-1},G^-_{+\frac{1}{2}}]=-G^-_{-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&[\mathcal{J}_0,G^+_{+\frac{1}{2}}]=+G^+_{\frac{1}{2}} &\qquad& [\mathcal{J}_0,G^-_{+\frac{1}{2}}]=-G^-_{\frac{1}{2}}\\
&[\mathcal{J}_0,G^+_{-\frac{1}{2}}]=+G^+_{-\frac{1}{2}} && [\mathcal{J}_0,G^-_{-\frac{1}{2}}]=-G^-_{-\frac{1}{2}}
\end{aligned}
$$

[2] Note, that we are working in the Neveu–Schwartz sector, so the modes of the fermionic supercurrents are labeled by half-integers.

$$
\begin{aligned}
&[L_0, L_{+1}] = -L_{+1} && \{G^+_{+\frac{1}{2}}, G^-_{+\frac{1}{2}}\} = 2L_{+1} \\
&[L_0, L_{-1}] = +L_{-1} && \{G^+_{-\frac{1}{2}}, G^-_{-\frac{1}{2}}\} = 2L_{-1} \\
&[L_{-1}, L_{+1}] = -2L_0 && \{G^+_{+\frac{1}{2}}, G^-_{-\frac{1}{2}}\} = 2L_0 + \mathcal{J}_0 \\
& && \{G^+_{-\frac{1}{2}}, G^-_{+\frac{1}{2}}\} = 2L_0 - \mathcal{J}_0
\end{aligned}
$$

with all other brackets vanishing. This algebra is actually isomorphic to $\mathfrak{sl}(2|1)$. The isomorphism to the Cartan–Weyl basis of $\mathfrak{sl}(2|1)$ (see [21], p.77) is explicitly given as

$$
\begin{aligned}
&L_0 = -H && G^+_{+\frac{1}{2}} = F_+ \\
&L_{+1} = -iE_+ && G^+_{-\frac{1}{2}} = -iF_- \\
&L_{-1} = -iE_- && G^-_{+\frac{1}{2}} = -2i\bar{F}_+ \\
&\mathcal{J}_0 = 2Z && G^-_{-\frac{1}{2}} = 2\bar{F}_-.
\end{aligned}
$$

So we have

$$
\text{global 2D superconformal algebra} = \mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)
$$

and the A-type $\mathcal{N} = (2,2)$ supersymmetry algebra on S^2 was defined as a non-trivial embedding in it[3]. It is generated by even generators $\{J_0, J_+, J_-; R_v\}$, where J_0, J_+, J_- form the standard $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ isometries of S^2 while R_v is the vector R-charge, and by odd supercharges Q_1, Q_2, S_1, S_2. The embedding reads

$$
\begin{aligned}
&J_0 = L_0 - \widetilde{L}_0 && Q_1 = \frac{1}{\sqrt{2}}\left(-iG^-_{+\frac{1}{2}} - \widetilde{G}^-_{-\frac{1}{2}}\right) \\
&J_+ = i\left(L_{-1} + \widetilde{L}_{+1}\right) && Q_2 = \frac{1}{\sqrt{2}}\left(G^-_{-\frac{1}{2}} + i\widetilde{G}^-_{+\frac{1}{2}}\right) \\
&J_- = i\left(L_{+1} + \widetilde{L}_{-1}\right) && S_1 = \frac{1}{\sqrt{2}}\left(G^+_{+\frac{1}{2}} + i\widetilde{G}^+_{-\frac{1}{2}}\right) \\
&R_v = \mathcal{J}_0 + \widetilde{\mathcal{J}}_0 && S_2 = \frac{1}{\sqrt{2}}\left(iG^+_{-\frac{1}{2}} + \widetilde{G}^+_{+\frac{1}{2}}\right).
\end{aligned}
$$

Since the left- and right- moving global superconformal generators form $\mathfrak{sl}(2|1)$ superalgebras and they commute between each other, it is evident that also the $\mathcal{N} = (2,2)$ supersymmetry algebra on S^2 will be isomorphic to $\mathfrak{sl}(2|1)$. Here we really refer to the compact real form, with even algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Explicit relation to the Cartan–Weyl basis of $\mathfrak{sl}(2|1)$ has the form

$$
\{E_+ = J_+, E_- = J_-, H = J_0, Z = \frac{1}{2}R_v\} \cup \{F_+ = cS_2, F_- = cS_1, \bar{F}_+ = \frac{1}{c}Q_2, \bar{F}_- = -\frac{1}{c}Q_1\} \tag{2.2}
$$

[3]The only other non-equivalent possibility, the B-type superalgebra, is just a different embedding.

with $c \in \mathbb{C}$. A required feature of the $\mathcal{N}=(2,2)$ supersymmetry algebra on S^2 is that it reduces to the ordinary Poincaré supersymmetry algebra after taking the flat space limit. This is achieved by the İnönü–Wigner contraction. Indeed, let us define the rescaled generators

$$\begin{aligned} M &= -H \\ P_+ &= -\frac{i}{r}E_+ \\ P_- &= -\frac{i}{r}E_- \\ F_v &= -2Z \end{aligned} \qquad\qquad \begin{aligned} Q_+ &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{r}}F_+ \\ \bar{Q}_+ &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{r}}\bar{F}_+ \\ Q_- &= \frac{e^{+i\frac{\pi}{4}}}{\sqrt{r}}F_- \\ \bar{Q}_- &= \frac{e^{+i\frac{\pi}{4}}}{\sqrt{r}}\bar{F}_-, \end{aligned} \tag{2.3}$$

where r represents the radius of the sphere. Substituting the above dictionary to commutation relations of $\mathfrak{sl}(2|1)$ and taking afterwards the flat space limit $r \to \infty$ reproduces the $\mathcal{N}=(2,2)$ supersymmetry algebra on $\mathbb{R}^2$ without central charges. The notation should be standard; P_+, P_- generate translations in the light-cone directions, M generates $SO(2)$ rotations of $\mathbb{R}^2$ while F_v the vector R-transformations and Q's are the supercharges, two of each chirality.

2.2 Supersymmetric actions on S^2

Theories of prime interest for us will be gauged linear sigma models (GLSMs) on S^2. They are specified by fixing the gauge group G, assigning representations of G to the matter fields and giving a superpotential W determining interactions among chiral multiplets. These models describe coupling of vector and chiral multiplets

$$\text{vector multiplet: } (A_\mu, \sigma, \eta, \lambda, \overline{\lambda}, D) \tag{2.4}$$

$$\text{chiral multiplet: } (\phi, \overline{\phi}, \psi, \overline{\psi}, F, \overline{F}), \tag{2.5}$$

where (σ, η, D) are real scalar fields, $(\phi, \overline{\phi}, F, \overline{F})$ complex scalar fields and $(\lambda, \overline{\lambda}, \psi, \overline{\psi})$ complex Dirac spinors. Whenever the gauge group contains an abelian factor we include a complexified Fayet–Iliopoulos term, i.e. a twisted F-term for the abelian superfield strength Σ.

The most general renormalizable lagrangian density that preserves $\mathcal{N}=(2,2)$ Euclidean supersymmetry reads

$$\mathcal{L} = \mathcal{L}_{\text{vec}} + \mathcal{L}_{\text{chiral}} + \mathcal{L}_W + \mathcal{L}_{FI}, \tag{2.6}$$

where $\mathcal{L}_{\text{vec}}$ describes the pure super Yang–Mills theory

$$\mathcal{L}_{\text{vec}} = \frac{1}{g^2}\text{Tr}\Big\{\frac{1}{2}\Big(F_{12} - \frac{\eta}{r}\Big)^2 + \frac{1}{2}\Big(D + \frac{\sigma}{r}\Big)^2 + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}D_\mu\eta D^\mu\eta - \frac{1}{2}[\sigma,\eta]^2 \\ + \frac{i}{2}\overline{\lambda}\gamma^\mu D_\mu\lambda + \frac{i}{2}\overline{\lambda}[\sigma,\lambda] + \frac{1}{2}\overline{\lambda}\gamma_3[\eta,\lambda]\Big\}. \tag{2.7}$$

$\mathcal{L}_{\text{chiral}}$ includes the kinetic term of a chiral multiplet with R-charge q as well as its minimal coupling to the vector multiplet

$$\mathcal{L}_{\text{chiral}} = D_\mu\overline{\phi}D^\mu\phi + \overline{\phi}\sigma^2\phi + \overline{\phi}\eta^2\phi + i\overline{\phi}D\phi + \overline{F}F + \frac{iq}{r}\overline{\phi}\sigma\phi + \frac{q(2-q)}{4r^2}\overline{\phi}\phi \\ - i\overline{\psi}\gamma^\mu D_\mu\psi + i\overline{\psi}\sigma\psi - \overline{\psi}\gamma_3\eta\psi + i\overline{\psi}\lambda\phi - i\overline{\phi}\,\overline{\lambda}\psi - \frac{q}{2r}\overline{\psi}\psi, \tag{2.8}$$

whereas $\mathcal{L}_W$ provides the matter couplings encoded in the superpotential F-term

$$\mathcal{L}_W = \sum_j \frac{\partial W}{\partial\phi_j}F_j - \sum_{j,k}\frac{1}{2}\frac{\partial^2 W}{\partial\phi_j\partial\phi_k}\psi_j\psi_k \tag{2.9}$$

and $\mathcal{L}_{FI}$ is the Fayet–Iliopoulos term

$$\mathcal{L}_{FI} = \text{Tr}\left[-i\xi D + i\frac{\theta}{2\pi}F_{12}\right]. \tag{2.10}$$

Depending on the choice of matter fields, the Lagrangian might be invariant under a global (flavor) group G_F. In that situation one can introduce *twisted masses* for chiral multiplets by weakly gauging G_F, then minimally coupling the chiral multiplets to the vector multiplet of G_F, and finally giving a vacuum expectation value $\sigma^{\text{ext}}, \eta^{\text{ext}}$ to the two real scalars in the vector multiplet of G_F. Supersymmetry on S^2 requires σ^{ext} and η^{ext} to be constant and in the Cartan subalgebra of G_F. In the following we will set $\eta^{\text{ext}} = 0$. The twisted mass terms can be obtained by substituting $\sigma \to \sigma + \sigma^{\text{ext}}$ in (2.8).

2.3 Localization and the exact partition function

Before we review the full computation in a supersymmetric theory on S^2, we would like to illustrate the idea of equivariant localization on a very simple finite dimensional integral.

2.3.1 Equivariant localization: A toy model example

Idea: reduce multidimensional integration to summation over "fixed points" (similar to residue theorem in complex analysis).

The only ambition of this short section is to build some intuition for what equivariant integration really means. Intuition is gained best by practising with simple examples where one has full control even by just using elementary techniques. In other words we want to uncover the secret why for some class of integrals we can write down the result in a simple way, without really doing the integration (as in the residue theorem).

As a starting point, let us remind the stationary phase method developed for asymptotic expansions of integrals. Consider the integral

$$I(s) = \int_{\mathbb{R}^n} d^n x \; e^{isf(x)} g(x). \tag{2.11}$$

With some mild assumptions on the functions (in this section we are not going to technical details, rather want to emphasize the ideas) the asymptotic expansion for large s reads

$$I(s) \overset{s\to\infty}{\sim} \sum_{\substack{i:\, x_i^* \text{ extremum} \\ \text{of } f(x)}} g(x_i^*) e^{isf(x_i^*)} \left(\frac{2\pi}{s}\right)^{\frac{n}{2}} \frac{e^{i\frac{\pi}{4}\sigma_i}}{|\det\left(\mathrm{Hess}[f](x_i^*)\right)|^{\frac{1}{2}}}, \tag{2.12}$$

where σ_i is the difference between the numbers of positive and negative eigenvalues of $\mathrm{Hess}[f](x_i^*)$. Generally, it is only an asymptotic expansion, therefore not exact. But sometimes it happens to give an exact answer. We want to understand those instances. For now we content ourselves with an example.

Example: Let us focus on the two sphere with unit radius. The function f will be chosen as the height function $f(x,y,z) = z$ while g is set to one. We wish to compute

$$I(s) = \int_{S^2} \omega \; e^{isz} \tag{2.13}$$

with ω the standard volume form on S^2. The critical points of f are the north pole ($z_N^* = 1$) and the south pole ($z_S^* = -1$). Around the north pole f behaves like $f \sim 1 - \frac{1}{2}(x^2+y^2)$ while at the south pole $f \sim -1 + \frac{1}{2}(x^2+y^2)$, which yields for the Hessians

$$\mathrm{Hess}[f](N) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \mathrm{Hess}[f](S) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.14}$$

From the definition of σ follows $\sigma_N = -2, \sigma_S = 2$ and we can write the asymptotic formula for the integral

$$I(s) \overset{s\to\infty}{\sim} \underbrace{e^{is(+1)} \left(\frac{2\pi}{s}\right)^{\frac{2}{2}} \frac{e^{i\frac{\pi}{4}(-2)}}{|1|}}_{N} + \underbrace{e^{is(-1)} \left(\frac{2\pi}{s}\right)^{\frac{2}{2}} \frac{e^{i\frac{\pi}{4}(+2)}}{|1|}}_{S} . \tag{2.15}$$

Direct integration leads to the exact result

$$I(s) = 2\pi \frac{e^{is} - e^{-is}}{is}. \tag{2.16}$$

Comparing the two we see that the leading order term in the asymptotic expansion actually gives the full answer. This is not just a coincidence, as we will see, we secretly integrated a very special differential form.

Equivariant forms

Consider an integration domain D of even dimension $2m$ with a group action $G \curvearrowright D$. Focus on the maximal torus of G

$$T(\xi) = e^{i\sum_{i=1}^{\mathrm{rk}(G)} \xi_i t_i} \tag{2.17}$$

and pick a 1-parameter soubgroup in it $\widetilde{T}(\xi) = e^{i\xi t}$. Act on a point $x \in D$, where we think of x as $X = x_0 + \delta x$ with x_0 a fixed point of this action and δx small. We have

$$\widetilde{T}(\xi) \curvearrowright x = \underbrace{\widetilde{T}(\xi) \curvearrowright x_0}_{x_0} + \underbrace{\widetilde{T}(\xi) \curvearrowright \delta x}_{R^i{}_j(\xi)\delta x^j} . \tag{2.18}$$

Here $R^i{}_j(\xi)$ is a $2m \times 2m$ matrix and by a change of basis we can always bring it to the form

$$R = \begin{pmatrix} \blacksquare & & & \\ & \blacksquare & & \\ & & \blacksquare & \\ & & & \blacksquare \end{pmatrix} ; \quad 2\times 2 \text{ blocks } \blacksquare : \begin{pmatrix} \cos(\nu_i\xi) & \sin(\nu_i\xi) \\ -\sin(\nu_i\xi) & \cos(\nu_i\xi) \end{pmatrix} \tag{2.19}$$

So in the new coordinates (with a slight abuse of notation)

$$\begin{pmatrix} \delta x_i(\xi) \\ \delta y_i(\xi) \end{pmatrix} = \begin{pmatrix} \cos(\nu_i\xi) & \sin(\nu_i\xi) \\ -\sin(\nu_i\xi) & \cos(\nu_i\xi) \end{pmatrix} \begin{pmatrix} \delta x_i \\ \delta y_i \end{pmatrix} ; \ i = 1, \ldots, m \tag{2.20}$$

which defines a vector field generating the action

$$v = \sum_{i=1}^{m} \nu_i \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right). \tag{2.21}$$

The integers ν_i are called indices of the vector field.

We are ready to give the definition of an equivariant differential form. Consider a manifold M of dimension $\dim M = n$. Then an equivariant (with respect to the maximal torus action, here we consider $S^1 \simeq U(1) \simeq SO(2)$) form $\alpha(\xi) \in \bigwedge T^*M \bigotimes \mathcal{C}_\xi$

$$\alpha(\xi) = \sum_{j=0}^{n} \alpha_j(\xi); \quad \alpha_j(\xi) \in \Omega^{(j)}(M) \otimes \mathcal{C}_\xi \tag{2.22}$$

with $\mathcal{C}_\xi$ the space of smooth functions on the maximal torus, satisfies the condition

$$\mathcal{L}_v \alpha_j(\xi) = 0 \quad \forall\, j, \tag{2.23}$$

where $\mathcal{L}_v$ is the Lie derivative with respect to v. It tells us that component forms α_j of all degrees are invariant under the action of the maximal torus generated by the vector field v. The definition can be restated in a more elegant manner after introducing an *equivariant differential*

$$d_{S^1} = d + i\xi\iota_v; \quad d_{S^1}^2 = i\xi\mathcal{L}_v. \tag{2.24}$$

Then an equivariant differential form is such that $d_{S^1}^2 \alpha(\xi) = 0$. Since the equivariant differential on the space of equivariant forms mimics all the properties of a usual external differential we can also here define closed and exact equivariant forms. This naturally leads to the notion of equivariant cohomology, which is just standard de Rham cohomology but with respect to the new equivariant differential.

Let us study the constraints that a closed equivariant form has to satisfy

$$0 = d_{S^1}\alpha(\xi) = i\xi\alpha_1(\xi)[v] + d\alpha_{n-1}(\xi) + \sum_{k=1}^{n-1} d\alpha_{k-1}(\xi) + i\xi\alpha_{k+1}(\xi)[v], \tag{2.25}$$

which imposes the conditions

$$\alpha_1(\xi)[v] = 0 \tag{2.26}$$

$$d\alpha_{n-1}(\xi) = 0 \tag{2.27}$$

$$d\alpha_{k-1}(\xi) + i\xi\alpha_{k+1}(\xi)[v] = 0;\ k = 1, \ldots, n-1. \tag{2.28}$$

For manifolds of even dimension $n = 2m$ they connect the top form α_{2m} with the lowest degree form (function) α_0. This will prove to be the essence of the localization formula.

Localization formula [3, 22]:

For a closed equivariant form $\alpha(\xi)$ on a manifold M of dimension $2m$ holds

$$\int_M \alpha(\xi) = \left(\frac{2\pi}{i\xi}\right)^m \sum_{p:\, v(p)=0} \frac{(\alpha_0(\xi))\,(p)}{\nu_1(p)\cdots\nu_m(p)}. \tag{2.29}$$

To the left hand side clearly contributes just the top form $\int_M \alpha_{2m}(\xi)$ whereas the right hand side contains only $\alpha_0(\xi)$. The sum runs over fixed points of the circle action or equivalently over zeroes of the generating vector field.

Example revisited: Now the reader already suspects what was so special about the differential form appearing in our first example. Indeed, it was a closed equivariant form with respect to the natural $U(1)$ action on the sphere generated by $v = \frac{\partial}{\partial\varphi}$. It is moreover integrated over an even dimensional manifold, thus the localization theorem can be applied. Let us construct the equivariant form. The top form is prescribed

$$\alpha_2(\xi) = e^{i\xi\cos\theta} d(\cos\theta) \wedge d\varphi = \omega e^{i\xi z}. \tag{2.30}$$

The condition $d_{\frac{\partial}{\partial\varphi}}\alpha(\xi) = 0$ yields

$$\alpha(\xi) = -e^{i\xi\cos\theta} + f(\theta)d\theta + e^{i\xi\cos\theta}d(\cos\theta) \wedge d\varphi. \tag{2.31}$$

The top form $\alpha_2(\xi)$ integrates to

$$I(\xi) = \frac{2\pi}{i\xi}\left(\frac{e^{i\xi}}{(+1)} + \frac{e^{-i\xi}}{(-1)}\right) \tag{2.32}$$

while using the localization formula one gets

$$\frac{2\pi}{i\xi}\left(\frac{\alpha_0(\xi)(N)}{\nu_1(N)} + \frac{\alpha_0(\xi)(S)}{\nu_1(S)}\right) \tag{2.33}$$

Remember that the fixed points are the north ($\theta = 0$) and south ($\theta = \pi$) pole, hence $\alpha_0(\xi)(N) = -e^{i\xi}$ and $\alpha_0(\xi)(S) = -e^{-i\xi}$. For the indices of the vector field v we get $\nu_1(N) = -1$ and $\nu_1(S) = +1$ as is sketched in Figure 2.1. Once put all together, it of course matches the left hand side. Now we know why it was sufficient to include the contribution only from the critical points in this case.

Sketch of proof for the localization theorem [4]:

(I) Choose an S^1-invariant metric on M and define a 1-form $\eta = g(v, \cdot)$; v is as usual the vector field generating the S^1-action. This form is S^1-equivariant, $d^2_{S^1}\eta = 0$,

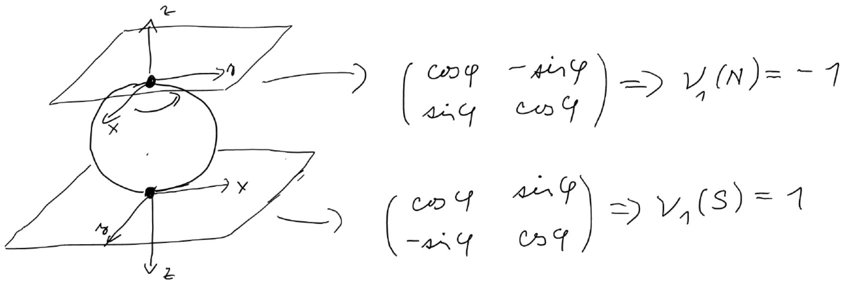

FIGURE 2.1: The integral of a closed equiavriant form receives contributions only from the fixed points: the north and south pole.

since the metric is S^1-invariant. Near a fixed point it has the behavior

$$\eta \approx -\frac{1}{2}\sum_{k=1}^{m} \nu_k \left(x_k dy_k - y_k dx_k\right). \tag{2.34}$$

Then define an equivariantly exact form $\beta(\xi) = d_{S^1}\eta$, near the fixed point it reads

$$\beta(\xi) \approx -\sum_{k=1}^{m} \nu_k dx_k \wedge dy_k + \frac{i\xi}{2}\sum_{k=1}^{m} \nu_k^2 (x_k^2 + y_k^2) \tag{2.35}$$

Further notice that the equivariant form

$$e^{is\beta(\xi)} - 1 = \sum_{n=1}^{\infty} \frac{(is)^n}{n!} \left(d_\xi \eta\right)^n = d_\xi \left(\sum_{n=1}^{\infty} \frac{(is)^n}{n!} \eta \left(d_\xi \eta\right)^{n-1}\right) \tag{2.36}$$

is equivariantly exact. Finally, focus on the original problem; integration of a closed equivariant form $\alpha(\xi)$. We have

$$\int_M \alpha(\xi)\Big(e^{is\beta(\xi)} - 1\Big) = \int_M \alpha(\xi) d_\xi\Big(\cdots\Big) = \int_M d_\xi\Big[\alpha(\xi)\Big(\cdots\Big)\Big] = 0 \tag{2.37}$$

and hence the key relation follows

$$\int_M \alpha(\xi) = \int_M \alpha(\xi) e^{is\beta(\xi)}. \tag{2.38}$$

(II) The integral is obviously independent of s (it only depends on the cohomology class $[\alpha]$). So we can compute the right hand side in the limit $s \to \infty$ for which the leading order term in the asymptotic expansion is exact. The leading order measure coming from the exponential is $\prod_{k=1}^{m}(-is\nu_k)dx_k \wedge dy_k$, which implies that just the α_0 part of $\alpha(\xi)$ contributes, higher degree components give a subleading contribution in s. All in all, we end up with the expression

$$\sum_{p:\text{ fixed points}} [\alpha_0(\xi)]\,(p) \prod_{k=1}^{m}(-is\nu_k) \int dx_k dy_k e^{-\frac{t\xi}{2}\sum_{k=1}^{m}\nu_k^2(x_k^2+y_k^2)}, \tag{2.39}$$

which after straightforward Gaussian integration gives the localization theorem.

2.3.2 Exact partition function on the sphere

Employing the technique of equivariant localization [23] the partition function of a GLSM on S^2 can be computed exactly, see [14, 15]. In order to localize the path integral we focus on the (non-simple) subalgebra of the full $\mathcal{N}=(2,2)$ supersymmetry algebra on the two sphere, generated by Q_2, S_1, which obey the relations [4](up to gauge and flavor transformations)

$$\{Q_2, S_1\} = M + \frac{R}{2}, \quad (Q_2)^2 = (S_1)^2 = 0. \tag{2.40}$$

Here M is the angular momentum that generates $U(1)$ rotations along a Killing vector field that vanishes at two opposite points on the sphere, which we mark as the north and south pole, respectively. R is the R-charge generator. The supercharge with respect to which localization will be performed is then constructed as a sum $\mathcal{Q} = Q_2 + S_1$ and as a consequence of the above equations generates the (non-simple) subalgebra $\mathfrak{su}(1|1) \subset \mathfrak{su}(2|1)$ (again up to gauge and flavor transformations)

$$\mathcal{Q}^2 = M + \frac{R}{2}. \tag{2.41}$$

It turns out that $\mathcal{L}_{\text{vec}}, \mathcal{L}_{\text{chiral}}, \mathcal{L}_W$ are all $\mathcal{Q}$ exact terms. Since the path integral is invariant under deformations by $\mathcal{Q}$ exact terms, we know that the partition function will not depend on any coupling constants included in these terms. However, it is affected by the constraints on R-charges imposed by the superpotential. On the other hand it depends on couplings in the twisted superpotential (in our case this is just the complexified Fayet-Iliopolous parameter t) as well as twisted masses allowed by global symmetries.

Now we want to compute the exact partition function on the sphere

$$Z^{S^2} = \int \mathfrak{D}\varphi\; e^{-S[\varphi]}. \tag{2.42}$$

We know that the action $S[\varphi]$ is $\mathcal{Q}$-closed, since it is supersymmetric by construction. The strategy is to deform the action by a $\mathcal{Q}$-exact term, such that it does not spoil the

[4]For more details about the supersymmetry algebra, the reader should consult the next section.

convergence of the path integral and does not change the asymptotic behavior of the action at infinity in the field space. As we already pointed out the partition function is invariant under such a deformation

$$Z^{S^2}=\int \mathfrak{D}\varphi\, e^{-S[\varphi]}=\int \mathfrak{D}\varphi\, e^{-(S[\varphi]+sS_{\mathrm{def}}[\varphi])} \tag{2.43}$$

We are allowed to take the limit $s\to\infty$, where the exact result is given just by the saddle point analysis of $S_{\mathrm{def}}[\varphi]$. At this point the localization computation splits into two branches, differing by the choice of the deformation action $S_{\mathrm{def}}[\varphi]$.

Localization on the Coulomb branch

The $\mathcal{Q}$-exact deformation term $S_{\mathrm{def}}[\varphi]$ is chosen to be $S_{\mathrm{vec}}+S_{\mathrm{chiral}}$. Since the corresponding Lagrangians $L_{\mathrm{vec}}+L_{\mathrm{chiral}}$ are a sum of squares, the extremization is equivalent to finding field configurations on which $L_{\mathrm{vec}}+L_{\mathrm{chiral}}$ vanishes. This happens on the localization locus

$$\Phi=\overline{\Phi}=F=\overline{F}=0 \tag{2.44}$$

$$F_{12}-\frac{\eta}{r}=D+\frac{\sigma}{r}=D_\mu\sigma=D_\mu\eta=[\sigma,\eta]=0. \tag{2.45}$$

The second line implies that the scalars in the vector multiplet σ,η are constant and in the Cartan subalgebra $\mathfrak{h}$ of the gauge group G. So, the solutions are parametrized by expectation values of fields in the vector multiplet and that is the reason why we denote the space of solutions as a *Coulomb branch.* On account of the quantization condition for the magnetic flux through the sphere

$$\frac{1}{2\pi}\int_{S^2}F=2r^2F_{12}=\mathfrak{m} \tag{2.46}$$

with $\mathfrak{m}$ in the dual weight lattice Λ_W^* corresponding to the gauge group G, in other words $\{\mathfrak{m}\in\mathfrak{h}\mid w(\mathfrak{m})\in\mathbb{Z}\;\forall w\in\Lambda_W\}$. Using this fact one gets

$$F_{12}=\frac{\mathfrak{m}}{2r^2},\quad \eta=\frac{\mathfrak{m}}{2r}. \tag{2.47}$$

To reach the final result for the partition function it remains to evaluate the classical action on the localization locus and compute the one loop determinants around the latter. All $\mathcal{Q}$-exact terms in the classical action vanish on solutions to (2.44),(2.45). The only term that is not $\mathcal{Q}$-exact is the Fayet–Iliopoulos term, which gives the contribution

$$S_{FI}=4\pi i r\xi_{\mathrm{ren}}\mathrm{Tr}(\sigma)+i\theta_{\mathrm{ren}}\mathrm{Tr}(\mathfrak{m}), \tag{2.48}$$

where the parameters undergo renormalization according to

$$\xi_{\text{ren}} = \xi - \frac{1}{2\pi}\sum_l Q_l \log(rM), \quad \theta_{\text{ren}} = \theta + (\text{rk}(G)-1)\pi. \tag{2.49}$$

M is a supersymmetry invariant ultraviolet cutoff and Q_l are charges of chiral fields with respect to the abelian part of the gauge group. Note, that whenever the target space is Calabi–Yau, the sum of abelian charges has to vanish and thus $\xi_{\text{ren}} = \xi$.

The one loop determinants around saddle points given by (2.44),(2.45) were computed for vector and chiral multiplets with the result

$$Z^{1L}_{\text{vec}} = \prod_{\alpha\in\Delta_+}\left(r^2\alpha(\sigma)^2 + \frac{\alpha(\mathfrak{m})^2}{4}\right) \tag{2.50}$$

$$Z^{1L}_{\Phi} = \prod_{w\in R(G)}\prod_{\widetilde{w}\in R(G_F)} \frac{\Gamma\left(\frac{q}{2} - ir\left[w(\sigma)+\widetilde{w}(\sigma^{\text{ext}})\right] - \frac{w(\mathfrak{m})}{2}\right)}{\Gamma\left(1-\frac{q}{2} + ir\left[w(\sigma)+\widetilde{w}(\sigma^{\text{ext}})\right] - \frac{w(\mathfrak{m})}{2}\right)}, \tag{2.51}$$

where Δ_+ is the set of positive roots of $\text{Lie}(G)$, w and $\widetilde{w}$ are weights of representations $R(G)$ and $R(G_F)$ of the gauge and flavor groups in which Φ transforms, while q is the R-charge of the chiral multiplet Φ.

Before we give the final expression for the partition function let us introduce some notations. By $|W_G|$ we mean the order of the Weyl group corresponding to G, then define the integers m_i, $i=1,\dots,\text{rk}(G)$ as $m_i = (\beta_i, \mathfrak{m})$, where $\{\beta_i\}$ is the orthonormal basis of $\mathfrak{h}^\vee$ seen as a vector space, see Appendix A for details. The master formula for the exact partition function of an $\mathcal{N}=(2,2)$ GLSM on S^2 reads

$$Z^{S^2}(G) = \frac{1}{|W_G|}\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_{\text{rk}(G)}\in\mathbb{Z}}\int_{\mathbb{R}^{\text{rk}(G)}}\left(\prod_{s=1}^{\text{rk}(G)}\frac{d(r\sigma_s)}{2\pi}\right) e^{-S_{FI}} Z^{1L}_{\text{vec}}(\sigma,\mathfrak{m})\prod_{\Phi} Z^{1L}_{\Phi}(\sigma,\mathfrak{m};\sigma^{\text{ext}}). \tag{2.52}$$

An immediate fact evident from this form is that the original path integral was reduced to a matrix model defined on the Cartan subalgebra of $\text{Lie}(G)$. Apart from this obvious observation there are many more beautiful and deep connections of this partition function with other areas of mathematics and theoretical physics. They will be explored and (partly) uncovered in the remaining chapters.

Localization on the Higgs branch

When one performs integration in (2.52) for concrete examples, it is always possible to manipulate the partition function to a form of a sum of factorized terms, one factor comes from the north pole while the other from the south pole of S^2. Is this a general pattern and can it be seen already at the path integral level during the procedure of localization? The answer is positive. It was shown in [14, 15] by choosing a different deformation action $S_{\text{def}}[\varphi]$, which changes the localization locus. To the original deformation action $S_{\text{def}}[\varphi]=S_{\text{vec}}+S_{\text{chiral}}$ a new $\mathcal{Q}$-closed and $\mathcal{Q}$-exact term was added

$$S_H=\int \mathcal{Q}\mathrm{Tr}\left[\frac{\epsilon_+^\dagger\lambda-\lambda^\dagger\epsilon_+}{2i}\left(\phi\phi^\dagger-\chi\mathbb{I}\right)\right], \tag{2.53}$$

where χ is a free parameter and ϕ contains all chiral fields. We can evaluate S_{def} and further integrate out the D field. The path integral over D is Gaussian (after completing the square and shifting thus D), it contributes a term $\frac{1}{2}\mathrm{Tr}(\phi\phi^\dagger-\chi\mathbb{I})^2$, while the equation of motion for the shifted D field reads

$$D+\frac{\sigma}{r}+i\left(\phi\phi^\dagger-\chi\mathbb{I}\right)=0. \tag{2.54}$$

The bosonic part of S_{def} becomes after this simple integration

$$\begin{aligned}\mathcal{L}_{\text{def}}\Big|_{\text{bos}}=\mathrm{Tr}\Bigg\{&\frac{1}{2}\left[\sin\theta\left(F_{12}-\frac{\eta}{r}\right)+\cos\theta D_1\eta\right]^2+\frac{1}{2}\left(D_2\eta\right)^2+\frac{1}{2}\left(D_\mu\sigma\right)^2-\frac{1}{2}\left[\sigma,\eta\right]^2\\&+\frac{1}{2}\left[\phi\phi^\dagger-\chi\mathbb{I}+\cos\theta\left(F_{12}-\frac{\eta}{r}\right)-\sin\theta D_1\eta\right]^2\Bigg\}+\mathcal{L}_{\text{chiral}}\Big|_{\text{bos}},\end{aligned} \tag{2.55}$$

where $\theta\in[0,\pi]$ is the latitude coordinate on S^2. Notice that it is a sum of squares ($\mathcal{L}_{\text{chiral}}\big|_{\text{bos}}$ has this property as well), therefore the extrema coincide with configurations where $\mathcal{L}_{\text{def}}\big|_{\text{bos}}=0$. The solutions divide into three categories[5]:

1. *Higgs branch* parametrized by solutions to

$$\begin{array}{ll} F_{12}-\frac{\eta}{r}=0,\quad D_\mu\sigma=0,\quad D_\mu\eta=0,\quad [\sigma,\eta]=0 & \leftarrow \text{coming from } \mathcal{L}_{\text{vec}}\\ \phi\phi^\dagger-\chi\mathbb{I}=0 & \leftarrow \text{coming from } \mathcal{L}_{H}\\ F=0,\quad D_\mu\phi=0,\quad \eta\phi=0,\quad \left(\sigma+\sigma^{\text{ext}}\right)\phi=0 & \leftarrow \text{coming from } \mathcal{L}_{\text{chiral}}\end{array} \tag{2.56}$$

   In [14] it was argued that for general twisted masses the solutions of the above set of equations consist of some number of isolated points (Higgs vacua), depending on the sign of χ.

2. *Coulomb branch.* In the same reference it was shown that the result of path integration is exponentially suppressed either in the limit $\chi\to+\infty$ or $\chi\to-\infty$,

[5] In this whole section we assume that all chiral fields have vanishing R-charge. Since the partition function is holomorphic in the combination (twisted mass) + $\frac{i}{2}$(R-charge), we can obtain non-zero R-charges by analytic continuation.

depending on the matter content of the theory. This is the limit of interest, where Higgs branch configurations dominate.

3. *Singular vortex solutions* existing only at the north pole (θ=0) and *anti-vortex* at the south pole ($\theta = \pi$).

Let us provide some details about the vortex solutions. Focus on the north pole (the derivation for the south pole works along the same lines). Setting $\theta = 0$ in (2.55) we get

$$\mathcal{L}^{(N)}_{\text{def}}\Big|_{\text{bos}} = \frac{1}{2}\left(D_\mu \eta\right)^2 + \frac{1}{2}\left(D_\mu \sigma\right)^2 - \frac{1}{2}[\sigma, \eta]^2 + \frac{1}{2}\left[\phi\phi^\dagger - \chi\mathbb{I} + F_{12} - \frac{\eta}{r}\right]^2 + \mathcal{L}^{(\theta=0)}_{\text{chiral}}\Big|_{\text{bos}}. \quad (2.57)$$

The Lagrangian for chiral fields at the north pole $\mathcal{L}^{(\theta=0)}_{\text{chiral}}\big|_{\text{bos}}$ vanishes on the following configurations

$$\eta\phi = 0, \quad (\sigma + \sigma^{\text{ext}})\phi = 0, \quad D_-\phi = 0, \quad F = 0 \quad (2.58)$$

with $D_- = D_1 + iD_2$. Instead the rest of (2.57) vanishes for

$$D_\mu \eta = 0, \quad D_\mu \sigma = 0, \quad [\sigma, \eta] = 0, \quad F_{12} - \frac{\eta}{r} + \phi\phi^\dagger - \chi\mathbb{I} = 0. \quad (2.59)$$

Now consider the equation of motion for the D-field (2.54) restricted to the localization locus ($D + \frac{\sigma}{r} = 0$ on the localization locus in order to extremize the vector multiplet bosonic Lagrangian), so we have

$$\phi\phi^\dagger = \chi\mathbb{I}. \quad (2.60)$$

Multiplying this equation by η and recalling that $\eta\phi$ is forced to vanish by (2.58), we conclude that $\eta = 0$. Therefore, summarizing the non-trivial equations, we have

$$(\text{NP}): \quad F_{12} + \phi\phi^\dagger - \chi\mathbb{I} = 0, \quad D_-\phi = 0, \quad (\sigma + \sigma^{\text{ext}})\phi = 0. \quad (2.61)$$

These are the vortex equations at the north pole. A similar analysis would reveal the system of anti-vortex equations at the south pole

$$(\text{SP}): \quad F_{12} - \phi\phi^\dagger + \chi\mathbb{I} = 0, \quad D_+\phi = 0, \quad (\sigma + \sigma^{\text{ext}})\phi = 0. \quad (2.62)$$

The conclusion of this analysis is that for each solution to (2.56), i.e. each Higgs vacuum, we have a moduli space of vortices at the plane attached to the north pole

$$p \in \text{Higgs vac.} \ \rightarrow \ \mathcal{M}^{\text{vort}}_p = \bigcup_{k=0}^{\infty} \mathcal{M}^{\text{vort}}_{p,k}; \quad k = \frac{1}{2\pi}\int_{\mathbb{R}^2} \text{Tr}F, \quad (2.63)$$

where k denotes the vorticity number. The same holds at the south pole, just vortices get substituted by anti-vortices. In the localization computation we need to integrate over these moduli spaces. The full moduli space of localization equations on S^2 in the Higgs branch ($\chi \to \pm\infty$) takes the form [15]

$$\mathcal{M}_H = \bigsqcup_{p\in\text{Higgs vac.}} \left(\bigcup_{k=0}^{\infty} \mathcal{M}^{\text{vort}}_{p,k}\right) \oplus \left(\bigcup_{l=0}^{\infty} \mathcal{M}^{\text{anti-vort}}_{p,l}\right) \tag{2.64}$$

The vortex/anti-vortex partition functions at the poles are partition functions of $\mathcal{N}=(2,2)$ theory in the $\mathbb{R}^2$ planes attached to the poles but deformed by an $U(1)_\epsilon$ equivariant action, i.e. living in Ω-background. The reason is that the localizing supercharge on S^2 satisfies $\mathcal{Q}^2 = M + \frac{R}{2}$ and the right-hand side is precisely the generator of $U(1)_\epsilon$ rotations in the Ω-background. The equivariant parameter ϵ gets identified with the radius of the sphere as $\epsilon = \frac{1}{r}$. These partition functions were studied in [24] and admit the representation

$$Z_{\text{vortex}}(p;z;\ldots) = \sum_{k=0}^{\infty} z^k \int_{\mathcal{M}^{\text{vort}}_{p,k}} e^{\omega}, \tag{2.65}$$

where ω is the $U(1)_\epsilon$-equivariant closed form on $\mathcal{M}^{\text{vort}}_{p,k}$, such that the integral computes the equivariant volume of the moduli space $\mathcal{M}^{\text{vort}}_{p,k}$.

The final result obtained for the partition function on S^2 within the Higgs branch localization scheme is therefore given as a sum over the Higgs vacua of contributions from the north/south pole $\mathbb{R}^2_\epsilon$ patches glued together

$$Z^{S^2}(z,\bar{z}) = \sum_{\text{Higgs vacua}} Z^{\text{class}} Z_{1L} Z_{\text{vortex}}(z) Z_{\text{anti-vortex}}(\bar{z}), \tag{2.66}$$

with Z_{1L} being the gluing factor. The crucial new terms are the vortex partition function $Z_{\text{vortex}}(z)$ and the anti-vortex one $Z_{\text{anti-vortex}}(\bar{z})$. In some cases, this factorization of the S^2 partition function can produce expressions for vortex partition functions when they are not known by other methods. We will illustrate the described factorization on many examples in following chapters. The vortex partition function and especially its close cousin Z_{v} introduced at the end of Section 3.1 will turn out to be objects of primary importance.

2.4 Comments on the possibility to have less than $\mathcal{N}=(2,2)$ supersymmetry on S^2

In this section we wish to explore whether a consistent supersymmetric theory with less than four supercharges can be defined on S^2. Namely, we have in mind the generalization of $\mathcal{N}=(0,2)$ theories that exist in flat space. If these theories could be consistently defined on the sphere, then the exact partition function would give a refinement of the superconformal index, providing thus a new tool for studying trialities among $\mathcal{N} = (0,2)$ theories discovered recently [25].

Unfortunately, at least in a particular setting, the answer to this question seems to be negative. However, we are not giving any kind of no-go theorem, just a couple of comments disfavoring a certain scenario that we describe now. Details about the superalgebra for $\mathcal{N} = (2,2)$ supersymmetry on S^2 necessary to understand the following discussion were presented in Section 2.1.1. As we showed the supersymmetry algebra is isomorphic to the compact real form of $\mathfrak{sl}(2|1)$. The strategy will be to look for a sub-superalgebra, where the even part is formed by the full isometry algebra $\mathfrak{so}(3)$ of S^2 while the odd part forms a two dimensional representation of it, i.e. there are two supercharges.

The only non-trivial simple sub-superalgebra of $\mathfrak{sl}(2|1)$ is $\mathfrak{osp}(1|2)$. Again, we are referring to the compact real form with even algebra $\mathfrak{usp}(2) \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3)$ to ensure that it agrees with isometries of S^2.[6] We denote the generators in the Cartan–Weyl basis $\{\hat{H}, \hat{E}_+, \hat{E}_-\}_{\text{even}} \cup \{\hat{F}_+, \hat{F}_-\}_{\text{odd}}$ and their commutation relations are listed in [21], p.76. The embedding of $\mathfrak{osp}(1|2)$ into $\mathfrak{sl}(2|1)$ expressed in terms of generators of the $\mathcal{N} = (2,2)$ supersymmetry algebra on S^2 reads

$$\{\hat{H} = J_0, \hat{E}_+ = J_+, \hat{E}_- = J_-\} \cup \{\hat{F}_+ = \frac{1}{2}\left(Q_2 + S_2\right), \hat{F}_- = \frac{1}{2}\left(Q_1 + S_1\right)\}. \tag{2.67}$$

Let us remark here that a supercharge formed as a general linear combination

$$\mathcal{Q} = a\hat{F}_+ + b\hat{F}_-; \qquad a, b \in \mathbb{C} \tag{2.68}$$

squares to a bosonic symmetry

$$\mathcal{Q}^2 = \frac{a^2}{4}J_+ - \frac{b^2}{4}J_- + \frac{ab}{2}J_0 \tag{2.69}$$

$$= \frac{a^2 - b^2}{4}J_1 + i\frac{a^2 + b^2}{4}J_2 + \frac{ab}{2}J_0, \tag{2.70}$$

which is neither hermitian nor skew-hermitian for any choice of a and b (except for the trivial case $a = b = 0$). This is perhaps also connected with the fact that the vector field generating this symmetry, in the vielbein basis expressed as

$$v^1 = \frac{r}{4}\left(a^2 e^{i\phi} + b^2 e^{-i\phi}\right) \tag{2.71}$$

$$v^2 = \frac{ir}{2}\left(-ab\sin\theta + \frac{a^2}{2}e^{i\phi}\cos\theta - \frac{b^2}{2}e^{-i\phi}\cos\theta\right) \tag{2.72}$$

[6]The other real forms describe spheres in spaces with pseudo-Riemannian signatures, i.e. dS/AdS spaces.

has no zeros, i.e. there are no fixed points. This behavior is in strong contrast with the situation of $\mathcal{N}=(2,2)$ supersymmetry on S^2, where $\mathcal{Q}^2$ was carefully chosen to generate $U(1)$ rotations of the sphere with the north and south pole fixed But this difference has at the end nothing to do with consistency of the theory itself, at most can create problems in the course of a localization computation.

Nevertheless, from our point of view, there is a more severe obstruction for the existence of the theory itself. For the $\mathcal{N}=(2,2)$ supersymmetry algebra on S^2 we have fixed the scaling of the generators (2.3) in order to reproduce the $\mathcal{N}=(2,2)$ supersymmetry algebra on $\mathbb{R}^2$ in the flat space limit $r\to\infty$. The $\mathfrak{osp}(1|2)$ generators are expressed by those of $\mathfrak{sl}(2|1)$ by means of (2.67) and (2.2), hence the scaling remains fixed. Taking now the flat space limit does not reproduce the $\mathcal{N}=(0,2)$ algebra on $\mathbb{R}^2$

$$\begin{array}{ccl} \mathfrak{sl}(2|1) & \xrightarrow[\text{contraction}]{\text{İnönü–Wigner}} & \mathcal{N}=(2,2) \text{ on } \mathbb{R}^2,\ \text{no central charges} \\ \cup & & \\ \mathfrak{osp}(1|2) & \xrightarrow[\text{contraction}]{\text{İnönü–Wigner}} & \text{NOT } \mathcal{N}=(0,2) \text{ on } \mathbb{R}^2 \end{array}$$

or in other words putting to zero supercharges of a given chirality is not consistent with commutation relations of the $\mathfrak{sl}(2|1)$ superalgebra

$$\begin{array}{ccc} \mathfrak{sl}(2|1) & \xrightarrow[\text{contraction}]{\text{İnönü–Wigner}} & \mathcal{N}=(2,2) \text{ on } \mathbb{R}^2 \\ \not\cup & & {\scriptstyle Q_-=0}\big\downarrow{\scriptstyle \bar{Q}_-=0} \\ \text{not a subalgebra} & \longleftarrow & \mathcal{N}=(0,2) \text{ on } \mathbb{R}^2 \end{array}.$$

These are few observations we wanted to mention about the issue of reducing the number of supercharges from four to two on S^2. The arguments are not general enough to claim that it is not possible to define a theory with just two supercharges on the sphere. However, they already highlight some difficulties one encounters on the route towards that objective.

Chapter 3

Quantum cohomology of target spaces from the S^2 partition function

A gauge theory (GLSM) flows in the infra-red to a non-linear sigma model (NLSM) with a corresponding target manifold. The control parameters for such flows are the Fayet-Iliopolous terms. Therefore the space of FI couplings gets divided into chambers, each chamber connected to a different target manifold. We present two main sections below. In the first one, the phase of the model will be fixed and we study the quantum cohomology of the related target manifold using the partition function on S^2. The second part is devoted to the analysis of transitions between distinct phases and how is this aspect encoded at the level of the partition function. Recall, that Z^{S^2} is associated with the UV description of the theory, i.e. the GLSM, and as such has to encompass the geometry of all target manifolds which arise in the infrared. This is certainly true and we show which operation on the partition function is related to crossing a wall in the chamber diagram. All introduced concepts will be clarified by a rather rich list of examples.

3.1 Quantum cohomology of a target manifold

A good starting point is to define a *target manifold* $\mathcal{M}$. We mean by it the manifold of classical supersymmetric vacua, that is the space of scalar fields (modulo the action of a gauge group G) on which the scalar potential vanishes. Since the scalar potential is quadratic in the F- and D-terms we may equivalently write

$$\mathcal{M} = \Big\{ \{\text{scalar fields}\} \mid F = 0,\ D = 0 \Big\} \Big/ G. \tag{3.1}$$

$\mathcal{M}$ is a Kähler manifold as a consequence of working with a theory that has four conserved supercharges. The one-loop beta function of the NLSM is proportional to the Ricci tensor, $\beta^{1L}_{\mu\nu} = \frac{1}{2\pi} R_{\mu\nu};\ \mu, \nu = 1, \ldots, \dim \mathcal{M}$. Moreover on a Kähler manifold the Ricci curvature determines the first Chern class $c_1(\mathcal{M})$ of the manifold. There are three possibilities that can arise classified by the sign of the beta function

- $\beta^{1L}_{\mu\nu} > 0$: The theory is asymptotically free, $c_1(\mathcal{M})$ is positive definite in which case the target space is a *Fano manifold*.
- $\beta^{1L}_{\mu\nu} = 0$: The theory is scale invariant, while the target manifold (Kähler with vanishing Ricci curvature) is *Calabi–Yau*. This will show as quite a distinguished situation on which we comment the most.
- $\beta^{1L}_{\mu\nu} < 0$: The theory is not UV complete, there is an ultraviolet singularity.

Moduli spaces of Calabi–Yau target manifolds

The most interesting option from our perspective is the Calabi–Yau target manifold, specifically a Calabi–Yau threefold. A fruitful setup in string theory is to factor the spacetime as $M_4 \times CY_3$ and compactify on the threefold. Doing so provides us with an effective four dimensional theory on M_4 that captures the low energy dynamics of string theory. In that case, the NLSM is a superconformal field theory (SCFT). It can be deformed by marginal operators from the chiral and twisted chiral ring, respectively. From the target space point of view this corresponds to metric deformations, which split into two categories:

(i) complexified Kähler class deformations $\longleftrightarrow$ chiral ring operators

(ii) complex structure deformations $\longleftrightarrow$ twisted chiral ring operators

Thus, locally, the moduli space of metric deformations has the product form $\mathfrak{M}_{CY} = \mathfrak{M}^{\text{CS}}_{CY} \times \mathfrak{M}^{\text{K}}_{CY}$. The dimensions of the two components are governed by the non-trivial Hodge numbers of the threefold, $\dim \mathfrak{M}^{\text{CS}}_{CY} = h^{2,1}$ while $\dim \mathfrak{M}^{\text{K}}_{CY} = h^{1,1}$. In addition both of them can be shown to be projective (local) special Kähler manifolds (these kind of spaces appear as target manifolds relevant for theories with eight supercharges, but we are not going to explore this connection any further). As such they certainly admit a Kähler potential for the metric, $g_{m\bar{n}} = \partial_m \partial_{\bar{n}} \mathcal{K}$. For the moduli space $\mathfrak{M}^{\mathrm{K}}_{CY}$ of Kähler class deformations with coordinates t_a, $a = 1, \ldots, h^{1,1}$, the formula around the large volume point reads [17]

$$\begin{aligned} e^{-\mathcal{K}^{\mathrm{K}}(t_a,\bar{t}_a)} &= -\frac{i}{6} \sum_{l,m,n} \kappa_{klm}(t^l - \bar{t}^l)(t^m - \bar{t}^m)(t^n - \bar{t}^n) + \frac{\zeta(3)}{4\pi^3}\chi(\mathcal{M}) \\ &+ \frac{2i}{(2\pi i)^3} \sum_{\substack{\eta \in H_2(\mathcal{M},\mathbb{Z}) \\ \eta \neq 0}} N_\eta \Big(\mathrm{Li}_3(q^\eta) + \mathrm{Li}_3(\bar{q}^\eta) \Big) \\ &- \frac{i}{(2\pi i)^2} \sum_{\eta,l} N_\eta \Big(\mathrm{Li}_2(q^\eta) + \mathrm{Li}_2(\bar{q}^\eta) \Big) \eta_l (t^l - \bar{t}^l), \end{aligned} \tag{3.2}$$

where N_η are the (integral) genus zero Gromov–Witten invariants of the Calabi–Yau threefold $\mathcal{M}$, $\chi(\mathcal{M})$ is its Euler characteristic and the polylogarithms are defined by

$$\mathrm{Li}_k(q) = \sum_{n=1}^{\infty} \frac{q^n}{n^k}, \qquad q^\eta = e^{2\pi i \sum_l \eta_l t^l}. \tag{3.3}$$

Rougly speaking, the *Gromov–Witten invariants* were designed to count the number of pseudo-holomorphic curves from a Riemann surface to the Calabi–Yau manifold. We concentrate only on the genus zero Riemann surface (topology of S^2), that is why we speak about genus zero G–W invariants. The connection with quantum cohomology comes from the *quantum cup product* $\star$

$$\int_{\mathcal{M}} (a \star b)_A \cup c = GW^{\mathcal{M},A}_{0,3}(a,b,c), \tag{3.4}$$

where $a, b, c \in H^*(\mathcal{M})$ and $A \in H_2(\mathcal{M})$. On the right hand side we have a genus zero 3-point Gromov–Witten invariant. Classical cohomology arises as a contribution of constants maps only, that is $A = 0$ sector in quantum cohomology. We have

$$a \star b = \sum_{A \in H_2(\mathcal{M})} (a \star b)_A \, e^A = \underbrace{(a \star b)_0}_{a \cup b} + \sum_{A \neq 0} (a \star b)_A \, e^A \tag{3.5}$$

with e^A a formal exponential often introduced in e.g. theory of characters to deal with convergence issues. From physics point of view, G–W invariants are coefficients of worldsheet instanton corrections to three point functions (Yukawa couplings) in A-twist NLSM.

Changing to the moduli space $\mathfrak{M}^{\mathrm{CS}}_{CY}$ of complex structure deformations, the Kähler potential can be expressed this time in terms of the special projective coordinates X^I and their conjugates $\mathcal{F}_I$, $I = 0, \ldots, h^{2,1}$ as

$$\mathcal{K}^{\mathrm{CS}}(\xi_i, \bar{\xi}_i) = -\log i \left(\overline{X}^I \mathcal{F}_I - X^I \overline{\mathcal{F}}_I \right). \tag{3.6}$$

with $\xi_i,\ i = 1, \ldots, h^{2,1}$ being the coordinates on $\mathfrak{M}^{\mathrm{CS}}_{CY}$. The X^I and $\mathcal{F}_I$ are collectively called periods and can be computed as integrals of the holomorphic 3-form over the basis of 3-cycles and their duals

$$\Pi(\xi) = \left(X^I(\xi), \mathcal{F}_J(\xi)\right) = \left(\int_{A^I} \Omega(\xi), \int_{B_J} \Omega(\xi)\right). \tag{3.7}$$

The conjugate periods $\mathcal{F}_J$ may be written as gradients of a potential function

$$F_J(X) = \frac{\partial \mathcal{F}(X)}{\partial X^J}, \tag{3.8}$$

where $\mathcal{F}(X)$ is called the *prepotential*.

Computing $\mathcal{K}^{\mathrm{K}}$ is a very difficult problem since non-perturbative corrections are involved. However, in some situations, *mirror symmetry* comes to rescue. Having a pair of mirror Calabi–Yau manifolds Y and $\check{Y}$, the mirror theorem states $\mathfrak{M}^{\mathrm{K}}(Y) \simeq \mathfrak{M}^{\mathrm{CS}}(\check{Y})$, where the isomorphism is described by a *mirror map*, see Figure 3.1.

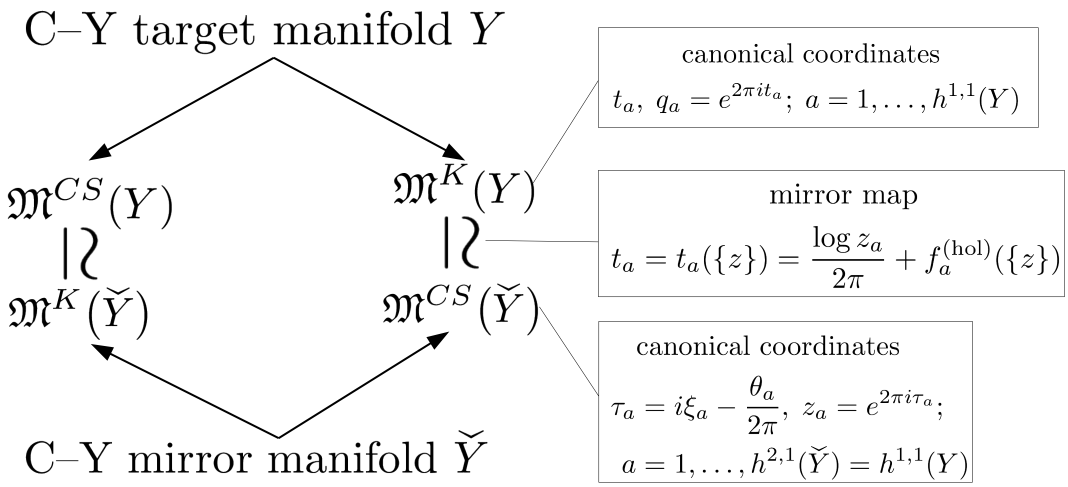

FIGURE 3.1: Isomorphisms between moduli spaces of a mirror pair of Calabi–Yau manifolds Y, $\check{Y}$. The right side shows isomorphism between Kähler moduli space of Y and complex structure moduli space of its mirror $\check{Y}$. Canonical coordinates on both spaces are defined and the isomorphism is expressed by a mirror map. The right side is captured by the partition function on S^2 computed using the $\mathfrak{su}(2|1)_A$ algebra while the left side is related to the $\mathfrak{su}(2|1)_B$ algebra. In this thesis we are dealing with the right side.

This powerful statement evades the hard non-perturbative analysis on $\mathfrak{M}^{\mathrm{K}}(Y)$ by transferring it to a manageable calculation within classical geometry on $\mathfrak{M}^{\mathrm{CS}}(\check{Y})$. The outlined strategy assumes that the mirror manifold $\check{Y}$ as well as the mirror map are known. This is true only for some families of C–Y manifolds like complete intersections in toric manifolds and a handful of other exceptional examples, yet by far not in general.

Kähler potential from partition function on S^2

In [17] a conjecture for computing $\mathcal{K}^{\mathrm{K}}$ directly, without ever referring to mirror symmetry, was put forward. The proposal is based on the sphere partition function of a GLSM with target $\mathcal{M}$. Then for the Kähler potential of $\mathfrak{M}^{\mathrm{K}}(\mathcal{M})$ holds

$$e^{-\mathcal{K}^{\mathrm{K}}(z_l,\bar{z}_l)} = Z^{S^2}(z_l,\bar{z}_l) \tag{3.9}$$

where z_l are related to the FI couplings as

$$z_l = e^{-2\pi\xi_l + i\theta_l} \tag{3.10}$$

with l running over abelian factors in the gauge group. The conjecture does not just give a formula for the Kähler potential. It also allows for extracting the mirror map, giving thus handle on mirror symmetry itself, in situations where standard constructions do not work. A proof was provided later (at the physics level of rigor) in [16] (see also [26]) using result of [27] that determines the Kähler potential in terms of a vacuum to vacuum amplitude in the associated NLSM. So the chain of equalities reads

$$Z^{S^2} \overset{[16]}{=} \langle \overline{0} \,|\, 0 \rangle \overset{[27]}{=} e^{-\mathcal{K}^{\mathrm{K}}}. \tag{3.11}$$

The idea is the following. As we already know, Z^{S^2} is independent of the gauge coupling and thus invariant under the renormalization group flow. This is required as we are matching a quantity in the GLSM with its counterpart in the NLSM. The next crucial input being the independence of the partition function of squashing the sphere to ellipsoids. Thus we can deform the sphere to a cigar geometry, where the path integral selects the vacuum states.

A natural question emerges at this stage. For now we treated the Kähler potential on the moduli space of Kähler class deformations, what about the Kähler potential on moduli space of complex structure deformations? We can not resist to make this small detour. As we shortly commented when building $\mathcal{N}=(2,2)$ supersymmetry algebra on S^2, there was a choice involved. We fixed a particular subalgebra $\mathfrak{su}(2|1)$, but there exists an inequivalent choice, related to the first one by a mirror (outer) automorphism.

Let us distinguish these two inequivalent $\mathcal{N} = (2,2)$ supersymmetry algebras on S^2 by subscripts A, B. In [28] it was demonstrated that the S^2 partition function with respect to the $\mathfrak{su}(2|1)_B$ algebra computes $\mathcal{K}^{\text{CS}}$. As a summary we have

$$\begin{aligned} Z_A^{S^2} &= e^{-\mathcal{K}^{\text{K}}} \\ Z_B^{S^2} &= e^{-\mathcal{K}^{\text{CS}}}. \end{aligned} \tag{3.12}$$

A general picture summing up the ongoing discussion is presented in Figure 3.2. In the

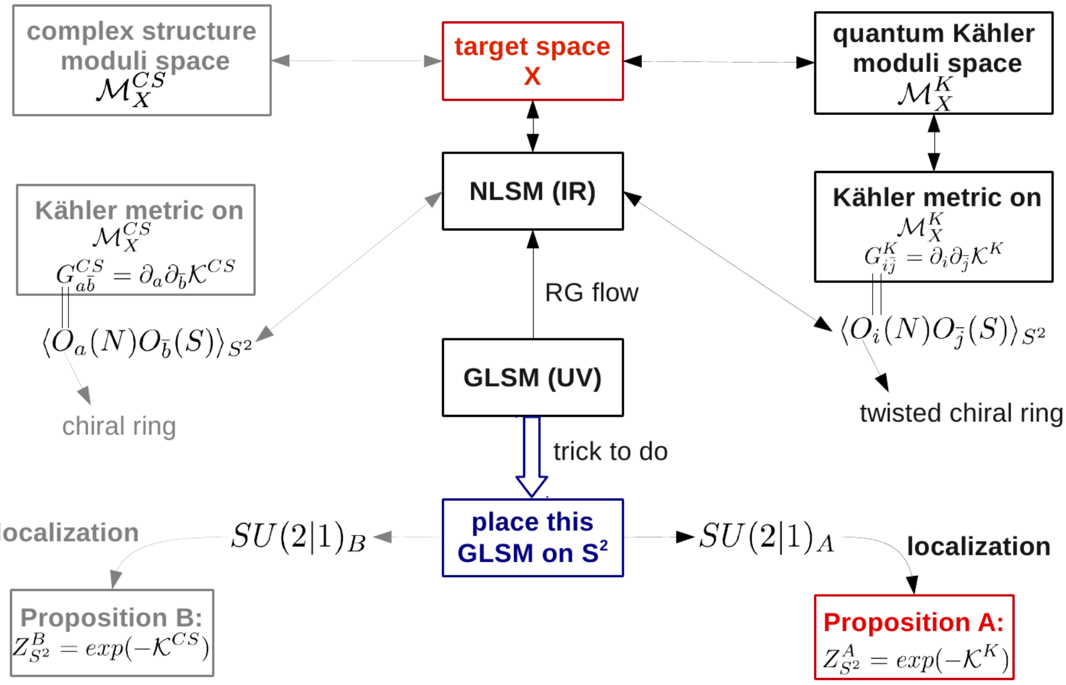

FIGURE 3.2: A scheme for computing exact Kähler potentials for Kähler class and complex structure deformations of a Calabi–Yau manifolds using the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2.

following we will concentrate on $Z_A^{S^2}$, therefore the Kähler potential on the moduli space of Kähler class deformations.

Givental's formalism and partition function on S^2

We developed in [29] a connection between the partition function on S^2 and Givental's approach to mirror symmetry [30] as well as its generalization to non-abelian quotients [31]. Except for the original paper a readable review of Givental's theory can be found in [32] or even with more details in the PhD. thesis of Tom Coates [33], alternatively in the book [34] (Chapters 10, 11 in particular). Here we give just a very brief introduction to this topic before moving to a large number of examples that will hopefully illustrate the quite abstract constructions a bit better.

The correspondence between the partition function on S^2 and Kähler potential introduced in [17] holds for Calabi–Yau target manifolds. Here we generalize to Fano manifolds as well. Our construction enables us to treat compact and non-compact spaces at the same footing, in the non-compact case we work in equivariant cohomology, which is effectively achieved by incorporating twisted masses for global symmetries at the level of the partition function.

Introducing Givental $\mathcal{I}$- and $\mathcal{J}$-functions. These two functions are the fundamental objects in the theory, so we need to mention their basic properties. First we give a rough general picture. They are cohomology valued functions related to each other by normalization (equivariant mirror map) and a change of variables (mirror map). The $\mathcal{I}$-function is a generating function for solutions to the Picard–Fuchs system on the mirror manifold $\widetilde{\mathcal{M}}$ and will be more elementary from the perspective of the partition function. It encodes the mirror and equivariant mirror maps, thus allowing to obtain the $\mathcal{J}$-function, which stores the Gromov–Witten potential.

Let us introduce the flat sections V_a of the Gauss-Manin connection on the vacuum bundle of the theory and satisfying [35, 36]

$$(\hbar D_a \delta_b^c + C_{ab}^c)\, V_c = 0. \tag{3.13}$$

where D_a is the covariant derivative on the vacuum line bundle and C_{ab}^c are the coefficients of the OPE in the chiral ring, $\phi_a \phi_b = C_{ab}^c \phi_c$. The observables $\{\phi_a\}$ provide a basis for the vector space of chiral ring operators $H^0(\mathcal{M}) \oplus H^2(\mathcal{M})$ with $a = 0, 1, \ldots, b^2(\mathcal{M})$, ϕ_0 being the identity operator. The parameter $\hbar$ is the spectral parameter of the Gauss-Manin connection. Setting $b = 0$ in (3.13), we find that $V_a = -\hbar D_a V_0$ which means that the flat sections are all generated by the fundamental solution $\mathcal{J} := V_0$ of the equation

$$(\hbar D_a D_b + C_{ab}^c D_c)\, \mathcal{J} = 0 \tag{3.14}$$

The metric on the vacuum bundle is given by a symplectic pairing of the flat sections $g_{\bar{a}b} = \langle \bar{a}|b\rangle = V_{\bar{a}}^t E V_b$ and in particular the vacuum-vacuum amplitude, that is the the spherical partition function, can be written as the symplectic pairing

$$\langle \bar{0}|0\rangle = \mathcal{J}^t E \mathcal{J} \tag{3.15}$$

for a suitable symplectic form E [35] that will be specified later.

The $\mathcal{J}$-function can be reconstructed from the $\mathcal{I}$-function, which is a function of hypergeometric type. It depends on the spectral parameter $\hbar$, coordinates on the moduli space and cohomology generator(s). Givental's formalism has been developed originally for abelian quotients, more precisely for complete intersections in quasi-projective toric varieties. In this case, the $\mathcal{I}$-function is the generating function of solutions to the Picard-Fuchs equations for the mirror manifold $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. It can be expanded as a polynomial in the cohomology generator(s) with coefficients satisfying the Picard–Fuchs system.

This formalism has been extended to non-abelian GLSM in [37, 38]. The Gromov-Witten invariants for the non-abelian quotient are conjectured to be expressible in terms of the ones of the corresponding abelian quotient twisted by the Euler class of a vector bundle over it. The corresponding $\mathcal{I}$-function is obtained from the one associated to the abelian quotients by multiplying it with a suitable factor depending on the Chern roots of the vector bundle. The first example of this kind was the quantum cohomology of the Grassmanian discussed in [39]. This was rigorously proved and extended to flag manifolds in [37]. As we will see, our results give evidence of the above conjecture in full generality, though a proof is missing.

In order to calculate the equivariant Gromov-Witten invariants from the above functions, one has to consider their asymptotic expansion in $\hbar$. Let us demonstrate the transition between the $\mathcal{I}$ and $\mathcal{J}$ functions on the simplest example. Consider a Calabi–Yau threefold Y with a single coordinate t on $\mathfrak{M}^K(Y)$, i.e. $h^{1,1}(Y) = 1$. This also implies $b^2(Y) = 1$ (remind $h^{2,0} = h^{0,2} = 0$), thus there is a single cohomology generator $H \in H^2(Y)$. The top form is H^3 while all higher powers vanish for dimensional reasons. For the moment assume Y to be compact such that there is no need to introduce equivariant parameters. Then the $\frac{1}{\hbar}$ expansion (in this case equivalent to H expansion) of the $\mathcal{J}$-function terminates (the constant term is set to one due to a particular normalization)

$$\mathcal{J}(H;\hbar;t) = 1 + \left(\frac{H}{\hbar}\right) t + \left(\frac{H}{\hbar}\right)^2 J^{(2)}(t) + \left(\frac{H}{\hbar}\right)^3 J^{(3)}(t). \tag{3.16}$$

The components of the $\mathcal{J}$ function $J^{(2)}$, $J^{(3)}$ are a formal power series in the t-coordinate, which is related to the worldsheet instanton counting parameter q as $q = e^{2\pi i t}$. The coefficient of the $\hbar^{-2}$ term in this expansion is directly related to the genus zero Gromov-Witten prepotential $\mathcal{F}$. In particular $J^{(2)}(t) = \eta^{tt}\partial_t\mathcal{F}$, where η^{tt} is the inverse topological metric. When we have more Kähler moduli with their correposnding cohomology generators, the components of the $\mathcal{J}$ function become vectors. In this situation the above relation gets generalized to $J^{(2)l}(\{t\}) = \eta^{lk}\partial_k F(\{t\})$. Higher order terms in the $\hbar^{-1}$ expansion are related to gravitational descendant insertions.

Now we turn to the problem how to construct the $\mathcal{J}$-function starting from the $\mathcal{I}$-function that is contained in the two-sphere partition function. Again one has to consider the expansion in $\hbar^{-1}$

$$\mathcal{I}(H;\hbar;z) = I^{(0)}(z) + \left(\frac{H}{\hbar}\right) I^{(1)}(z) + \left(\frac{H}{\hbar}\right)^2 I^{(2)}(z) + \left(\frac{H}{\hbar}\right)^3 I^{(3)}(z). \tag{3.17}$$

Comparing coefficients of the two expansions we arrive at the *mirror map*[1]

$$t(z) = \frac{I^{(1)}(z)}{I^{(0(}(z)} = \log z + f^{(\mathrm{hol})}(z) \tag{3.18}$$

and at the same time we see that the $\mathcal{I}$ and $\mathcal{J}$ functions are related as

$$\mathcal{J}(t) = \left.\frac{\mathcal{I}(z)}{I^{(0)}(z)}\right|_{z=z(t)}, \tag{3.19}$$

where $z(t)$ is the inverse mirror map.

Further assume we elaborate our setting and deal with a non-compact Y. This implies presence of equivariant parameters (twisted masses from gauge theory point of view) on top of the cohomology generators. For simplicity assume still a single cohomology generator H and also a single equivariant parameter E. Focus on the $\hbar^{-1}$ order of the expansion for the $\mathcal{I}$ function, it gets changed to

$$\frac{H}{\hbar} I^{(1)}(z) \quad \longrightarrow \quad \frac{H}{\hbar} I^{(1)}(z) + \frac{E}{\hbar} I^{(1)}_{\mathrm{eq}}(z) \tag{3.20}$$

In such a scenario we have to perform an *equivariant mirror map* (normalization of the $\mathcal{I}$ function), which effectively removes the term containing the equivariant parameter at order $\hbar^{-1}$ in addition to the usual mirror map. As a result the $\mathcal{J}$ function is given as

$$\mathcal{J}(t) = \exp\left\{-\frac{E}{\hbar}\frac{I^{(1)}_{\mathrm{eq}}}{I^{(0)}}\right\} \left.\frac{\mathcal{I}}{I^{(0)}}\right|_{z=z(t)}. \tag{3.21}$$

We are now ready to reveal the connection between Givental's formalism and the spherical partition function on S^2. As a first step we can factorize Z^{S^2} in a similar fashion to (2.66) even before integration [17, 29]

$$Z^{S^2} = \oint d\lambda Z_{11} \left(z^{-r|\lambda|} Z_{\mathrm{v}}\right) \left(\bar{z}^{-r|\lambda|} Z_{\mathrm{av}}\right) \tag{3.22}$$

with $d\lambda = \prod_{s=1}^{\mathrm{rk(G)}} d\lambda_s$ and $|\lambda| = \sum_s \lambda_s$, whereas $z = e^{-2\pi\xi + i\theta}$ is the vortex counting parameter. The factors $z\bar{z}^{-r|\lambda|}$ come from the classical action. Z_{v} (resp. Z_{av}) is a close

[1]The $\log z$ term is due to a normalization factor $z^{\frac{H}{\hbar}}$ contained in the $\mathcal{I}$ function multiplying a power series in z, which constant term is one. It is a trivial factor that may or may not be included. On the gauge theory side it corresponds to the (holomorphic piece of) classical action evaluated at the localization locus. We will comment on this later on.

relative to the equivariant vortex partition function Z_{vort} (resp. anti-vortex partition function $Z_{\text{anti-vort}}$), see (2.66), that are localized at the north (resp. south) pole. Z_{1l} is the remaining one-loop measure.

Dictionary. Our claim is that Z_{v} is to be identified with Givental $\mathcal{I}$-function[2] of the target manifold $\mathcal{M}$ upon identifying the vortex counting parameter $z_l = e^{-2\pi\xi_l + i\theta_l}$ with the natural coordinates of the $\mathcal{I}$ function (for a C–Y manifolds Y these are the coordinates on the complex structure moduli space $\mathfrak{M}^{CS}(\check{Y})$ of the mirror manifold $\check{Y}$). Next, λ_s get identified with the cohomology generators, twisted masses with equivariant parameters in cohomology and finally the radius of the sphere with the spectral parameter, $r = \frac{1}{\hbar}$. To extract Gromov-Witten invariants from the spherical partition function one has then to implement the procedure outlined above to compute the $\mathcal{J}$-function. The range of FI parameters determines integration contours corresponding to a given chamber of the GLSM. The one-loop term Z_{1l} has to be properly normalized in order to reproduce classical intersection numbers on the target space. It can also be interpreted as the symplectic pairing introduced in (3.15).

Time has come to expose the theoretical constructions on some examples. We divide them into two categories classified by the gauge group of the underlying gauge theory. First we study abelian models and later we move to more complicated non-abelian theories. For a class of non-abelian models we also comment on certain dualities.

3.2 Abelian GLSMs

3.2.1 Projective spaces

Let us start with the basic example, that is $\mathbb{CP}^{n-1}$. As a first step we need to design a GLSM whose classical vacuum manifold given by (3.1) is isomorphic to $\mathbb{CP}^{n-1}$. For this particular example the construction is easy and we check that the claim we make in a moment is indeed correct. However, in general this can be a hard problem and we do not provide a general recipe.

Consider a sigma model with matter content consisting of n chiral superfields $\Phi_1, \ldots, \Phi_n$ of charge 1 with respect to the $U(1)$ gauge group. The Lagrangian of this model takes the form

$$L = \int d^4\theta \left(\sum_{i=1}^{n} \bar{\Phi}_i e^V \Phi_i - \frac{1}{2e^2}\bar{\Sigma}\Sigma \right) + \frac{1}{2}\left(-\tau \int d^2\widetilde{\theta}\Sigma + c.c \right) \tag{3.23}$$

[2]Or we can include also the term $z^{-r|\lambda|}$ in the definition of the $\mathcal{I}$-function. As we already commented it is responsible for the $\log z$ term in the mirror map. The effect of this factor can be reintroduced at any stage, so we may omit it for simplicity.

with $\tau = i\xi + \frac{\theta}{2\pi}$ the complexified Fayet-Iliopoulos parameter. In general, the FI parameter ξ runs [15]; in our case

$$\xi_{\text{ren}} = \xi - \frac{n}{2\pi}\log(rM) \tag{3.24}$$

with M a SUSY-invariant ultraviolet cut-off[3].

The potential energy computed from this Lagrangian reads

$$U = \sum_{i=1}^{n} |\sigma|^2|\phi_i|^2 + \frac{e^2}{2}\left(\sum_{i=1}^{n}|\phi_i|^2 - \xi\right)^2 \tag{3.25}$$

For $\xi > 0$ the classical vacua ($U = 0$) are achieved for $\sigma = 0$ and

$$\sum_{i=1}^{n}|\phi_i|^2 = \xi \tag{3.26}$$

Therefore the vacuum manifold takes the desired form

$$\left\{\phi_1,\ldots,\phi_n \,\middle|\, \textstyle\sum_{i=1}^{n}|\phi_i|^2 = \xi\right\}\Big/\,U(1) \simeq S^{2n-1}/S^1 \simeq \mathbb{CP}^{n-1}. \tag{3.27}$$

Now that we have showed that our model has a correct target manifold, we can just use equation (2.52) to write down the corresponding S^2 partition function. The gauge group is $U(1)$, that is an abelian rank one group. Therefore, we have a single sum over magnetic fluxes as well as a single integration over the Cartan subalgebra. Moreover the order of the Weyl group is just one. We need to evaluate the on-shell classical action (2.48), contribution from the vector multiplets (2.50) containing the roots drops out and further we are only left with one-loop determinants for the chiral fields (2.51). We assume that R-charges for all chiral fields vanish (we can do that since there is no superpotential), the weights for the representations of the chiral fields in (2.51) are just the abelian charges, so $+1$ for all fields. This finishes the discussion of all needed ingredients, the resulting formula for Z^{S^2} reads

$$Z_{\mathbb{P}^{n-1}} = \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\frac{d(r\sigma)}{2\pi}e^{-4\pi i\xi_{\text{ren}}r\sigma - i\theta m}\left(\frac{\Gamma\left(-ir\sigma-\frac{m}{2}\right)}{\Gamma\left(1+ir\sigma-\frac{m}{2}\right)}\right)^n. \tag{3.28}$$

Further we can perform a change of variables, defining $\tau = -ir\sigma$ the partition function becomes

$$Z_{\mathbb{P}^{n-1}} = \sum_{m\in\mathbb{Z}}\int_{i\mathbb{R}}\frac{\mathrm{d}\tau}{2\pi i}e^{4\pi\xi_{\text{ren}}\tau - i\theta_{\text{ren}}m}\left(\frac{\Gamma\left(\tau-\frac{m}{2}\right)}{\Gamma\left(1-\tau-\frac{m}{2}\right)}\right)^n. \tag{3.29}$$

$\mathbb{P}^{n-1}$ is associated to the phase $\xi_{\text{ren}} > 0$. The term $e^{4\pi\xi_{\text{ren}}\tau}$ governs the asymptotic behavior of the integrand. We see that it gets suppressed in the left τ plane, so we can

[3]Notice that in the Calabi-Yau case the sum of the charges is zero, therefore $\xi_{\text{ren}} = \xi$.

evaluate the integral by closing the contour by a big circle in the left half-plane in order to use residue theorem. After doing it, we need to study the poles of the integrand. Remind that $\Gamma(x)$ has a simple pole for $x = 0, -1, -2, \ldots$. Therefore the factors in the numerator will provide us with poles while those in the denominator with zeroes; their location is

$$\begin{array}{lll} \text{poles:} & \tau - \frac{m}{2} = -k,\ k \in \mathbb{Z}_{\geq 0} & \longrightarrow\ \tau = -k + \frac{m}{2} \\ \text{zeroes:} & 1 - \tau - \frac{m}{2} = -\widetilde{k},\ \widetilde{k} \in \mathbb{Z}_{\geq 0} & \longrightarrow\ \tau = 1 + \widetilde{k} - \frac{m}{2}. \end{array}$$

Clearly they tend to cancel. Setting the two expressions equal we get a range where the zeroes do cancel the poles. This happens for $k \leq m - 1$, then the complementary interval $k \geq m$ specifies the surviving poles. Taking into account the original restriction $k \in \mathbb{Z}_{\geq 0}$ as well, we obtain the true positions of the poles

$$\tau_{\text{pole}} = -k + \frac{m}{2}, \quad k \geq \max(0, m). \tag{3.30}$$

By residue theorem we get a sum over these poles. However, we do not evaluate the residue, rather just rewrite it by Cauchy theorem as a contour integral, the contour being a small circle around the given pole, see Figure 3.3.

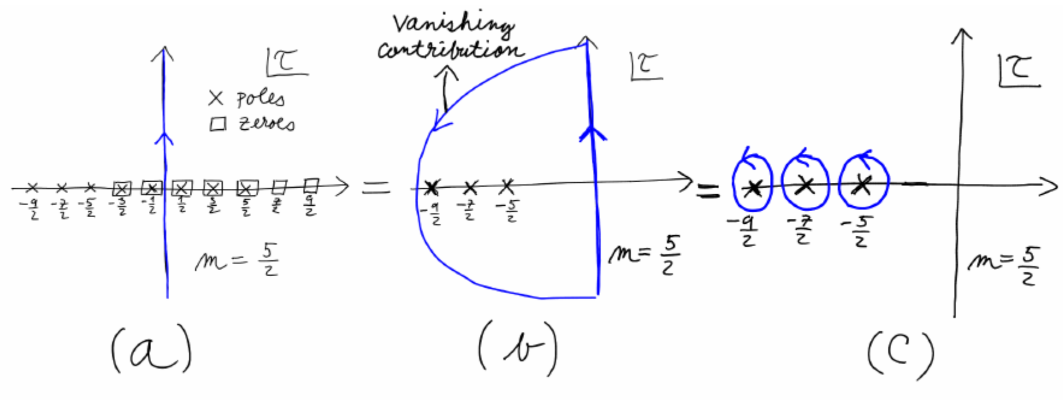

FIGURE 3.3: The figure shows the poles and zeroes of the integrand for a fixed value of $m = \frac{5}{2}$. In (a) the cancellation of poles with zeroes is displayed, together with the original integration contour (blue line). In (b) we see the true poles, enclosed by a closed contour. The contribution of the big circle vanishes due to asymptotic behavior of the integrand selected by the FI term $\xi > 0$. Part (c) shows the reduction to a sum of Cauchy integrals, with small contours around individual poles.

The integration variable on these small circles is denoted by λ. This is the trick how to get the function Z_v in (3.22). Instead the true vortex partition function Z_{vortex}, equation (2.66), does not contain any λ and is obtained after evaluating these Cauchy integrals. However, remind that Z_v gets identified with the $\mathcal{I}$ function and there λ plays an important role as it serves as the cohomology generator. So in summary, we started with an integral over a real line, exploiting the asymptotic behavior of the integrand we noticed that it might be transformed to a contour integral closing it by a big half circle. Further this single contour integral was reduced to a sum of contour integrals, with small contours going around the poles enclosed in the original big contour. The logic of this procedure is summed up in Figure 3.3. The outlined steps are achieved, starting from (3.29), by setting

$$\tau = \tau_{\text{pole}} + rM\lambda, \tag{3.31}$$

where M is just an inverse length scale to keep λ dimensionless. Finally, one ends up with the following formula

$$\sum_{m\in\mathbb{Z}} \sum_{k\geq\max(0,m)} \oint \frac{d(rM\lambda)}{2\pi i} e^{4\pi\xi_{\text{ren}}(\tau_{\text{pole}}+rM\lambda)} \left(\frac{\Gamma\left(\tau_{\text{pole}} - \frac{m}{2} + rM\lambda\right)}{\Gamma\left(1 - \tau_{\text{pole}} - \frac{m}{2} - rM\lambda\right)} \right)^n . \tag{3.32}$$

At this stage we have to handle the double summation. The lattice over which the sum runs is visualized in Figure 3.4, from where it is not difficult to see that the following

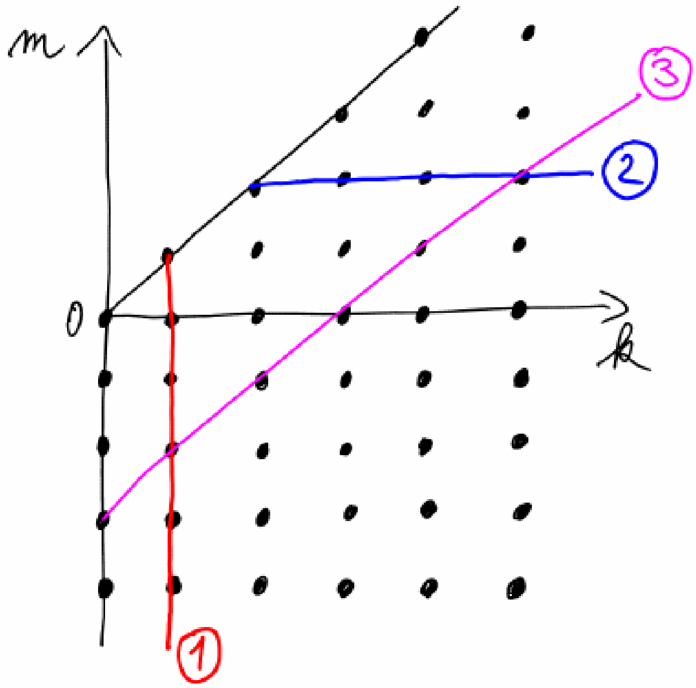

FIGURE 3.4: The figure shows the summation lattice of (3.32). The blue line fixes $m \in \mathbb{Z}$ and sums over $k \geq \max(0, m)$, while the red line fixes $k \geq 0$ and sums over $m \leq k$. Finally, the purple line fixes $L = k - m \geq 0$ and sums over $K \geq 0$.

schematic chain of equalities holds

$$\sum_{m\in\mathbb{Z}}\sum_{k\geq\max(0,m)}(\cdots)=\sum_{k\geq 0}\sum_{m\leq k}(\cdots)=\left|\begin{array}{c} K=k \\ L=k-m \end{array}\right|=\sum_{K\geq 0}\sum_{L\geq 0}(\cdots)\,. \tag{3.33}$$

Plugging to (3.32) the positions of the poles τ_{pole} from (3.30) and the definition of the renormalized FI term (3.24), followed by changing the summation variables to K, L as indicated in (3.33), we arrive at

$$\begin{aligned} Z_{\mathbb{P}^{n-1}} = \oint \frac{d(rM\lambda)}{2\pi i}\,(rM)^{-2nrM\lambda}&\sum_{L\geq 0}[(rM)^n z]^L\left(\frac{1}{\Gamma\left(1+L-rM\lambda\right)}\right)^n \\ &\sum_{K\geq 0}[(rM)^n\bar{z}]^K\left(\Gamma\left(-K+rM\lambda\right)\right)^n. \end{aligned} \tag{3.34}$$

In the above expression a new variable z has been defined as

$$z=e^{-2\pi\xi+i\theta}. \tag{3.35}$$

The ultimate step to be done consists of simplifying the Gamma functions using the following Pochhammer identities

$$\Gamma(a+k)=\Gamma(a)(a)_k,\qquad \Gamma(a-k)=\Gamma(a)\frac{(-1)^k}{(1-a)_k};\quad k\in\mathbb{Z}_{\geq 0}, \tag{3.36}$$

where the Pochhammer symbol $(a)_k$ is defined as

$$(a)_k=\begin{cases} \prod_{i=0}^{k-1}(a+i) & \text{for } k>0 \\ 1 & \text{for } k=0 \\ \prod_{i=1}^{-k}\dfrac{1}{a-i} & \text{for } k<0 \end{cases}. \tag{3.37}$$

The final form of the partition function (3.29) then becomes

$$Z_{\mathbb{P}^{n-1}}=\oint\frac{d(rM\lambda)}{2\pi i}Z_{11}^{\mathbb{P}^{n-1}}Z_{\text{v}}^{\mathbb{P}^{n-1}}Z_{\text{av}}^{\mathbb{P}^{n-1}}, \tag{3.38}$$

where

$$\begin{aligned} Z_{11}^{\mathbb{P}^{n-1}} &= (rM)^{-2nrM\lambda}\left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)}\right)^n \\ Z_{\text{v}}^{\mathbb{P}^{n-1}} &= z^{-rM\lambda}\sum_{l\geq 0}\frac{[(rM)^n z]^l}{(1-rM\lambda)_l^n} \\ Z_{\text{av}}^{\mathbb{P}^{n-1}} &= \bar{z}^{-rM\lambda}\sum_{k\geq 0}\frac{[(-rM)^n\bar{z}]^k}{(1-rM\lambda)_k^n}. \end{aligned} \tag{3.39}$$

The $\mathcal{I}$-function is given by $Z_{\text{v}}^{\mathbb{P}^{n-1}}$, and coincides with the one given in the mathematical literature[4]

$$\mathcal{I}_{\mathbb{P}^{n-1}}(H,\hbar;t) = e^{\frac{tH}{\hbar}} \sum_{d\geq 0} \frac{[(\hbar)^{-n}e^t]^d}{(1+H/\hbar)_d^n} \tag{3.40}$$

if we identify $\hbar = \frac{1}{rM}$, $H = -\lambda$, $t = \ln z$. The antivortex contribution is the conjugate $\mathcal{I}$-function, with $\hbar = -\frac{1}{rM}$, $H = \lambda$ and $\bar{t} = \ln \bar{z}$. The hyperplane class H satisfies $H^n = 0$; in some sense the integration variable λ satisfies the same relation, because the process of integration will take into account only terms up to λ^{n-1} in Z_{v} and Z_{av}.

Complete intersections in $\mathbb{P}^{n-1}$ of type $(q_0,\dots,q_m)$, $q_j > 0$ can be obtained by adding chiral fields of charge $(-q_0,\dots,-q_m)$. This means that the integrand in (3.29) gets multiplied by

$$\prod_{j=0}^{m} \frac{\Gamma\left(\frac{R_j}{2} - q_j\tau + q_j\frac{m}{2}\right)}{\Gamma\left(1-\frac{R_j}{2} + q_j\tau + q_j\frac{m}{2}\right)}. \tag{3.41}$$

The poles are still as in (3.31), but now

$$\begin{aligned}
Z_{11}^{\mathbb{P}^{n-1}} &= (rM)^{-2rM(n-|q|)\lambda}\left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)}\right)^n \prod_{j=0}^{m} \frac{\Gamma\left(\frac{R_j}{2} - q_j rM\lambda\right)}{\Gamma\left(1-\frac{R_j}{2} + q_j rM\lambda\right)} \\
Z_{\text{v}}^{\mathbb{P}^{n-1}} &= z^{-rM\lambda}\sum_{l\geq 0}(-1)^{|q|l}[(rM)^{n-|q|}z]^l \frac{\prod_{j=0}^{m}(\frac{R_j}{2}-q_j rM\lambda)_{q_j l}}{(1-rM\lambda)_l^n} \\
Z_{\text{av}}^{\mathbb{P}^{n-1}} &= \bar{z}^{-rM\lambda}\sum_{k\geq 0}(-1)^{|q|k}[(-rM)^{n-|q|}\bar{z}]^k \frac{\prod_{j=0}^{m}(\frac{R_j}{2}-q_j rM\lambda)_{q_j k}}{(1-rM\lambda)_k^n},
\end{aligned} \tag{3.42}$$

where $|q| = \sum_{j=0}^{n} q_j$ and R_j is the R-charge of the j-th field. Notice that, if we want to describe a bundle over a space, we should set $R_j = 0$ and add twisted masses in the contributions coming from the fibers, since we want to separate the different cohomology generators (i.e. the different integration variables); we will do this explicitly when needed. On the other hand, complete intersections do not require and do not allow twisted masses, because the insertion of the superpotential breaks all flavor symmetry; moreover, since the superpotential must have R-charge 2, we will need some $R_j \neq 0$ (see the example of the quintic below).

3.2.1.1 Equivariant projective spaces

The same computation can be repeated in the more general equivariant case, with twisted masses turned on. In this case, the partition function reads (rescaling the twisted masses

[4]This was already observed in this particular case in [40].

as $a_i \to M a_i$ in order to have dimensionless parameters)

$$Z^{\text{eq}}_{\mathbb{P}^{n-1}} = \sum_{m\in\mathbb{Z}} \int \frac{\mathrm{d}\tau}{2\pi i} e^{4\pi\xi_{\text{ren}}\tau - i\theta_{\text{ren}} m} \prod_{i=1}^{n} \frac{\Gamma\left(\tau - \frac{m}{2} + irMa_i\right)}{\Gamma\left(1-\tau-\frac{m}{2} - irMa_i\right)}. \tag{3.43}$$

We perform a change of variables to shift the poles to $\lambda = 0$

$$\tau = -k + \frac{m}{2} - irMa_j + rM\lambda \tag{3.44}$$

in order to arrive at

$$Z^{\text{eq}}_{\mathbb{P}^{n-1}} = \sum_{j=1}^{n} \oint \frac{d(rM\lambda)}{2\pi i} Z^{\mathbb{P}^{n-1}}_{\text{1l, eq}} Z^{\mathbb{P}^{n-1}}_{\text{v, eq}} Z^{\mathbb{P}^{n-1}}_{\text{av, eq}}, \tag{3.45}$$

where

$$\begin{aligned} Z^{\mathbb{P}^{n-1}}_{\text{1l, eq}} &= (z\bar{z})^{irMa_j} (rM)^{-2nrM\lambda} \prod_{i=1}^{n} \frac{\Gamma(rM\lambda + irMa_{ij})}{\Gamma(1 - rM\lambda - irMa_{ij})} \\ Z^{\mathbb{P}^{n-1}}_{\text{v, eq}} &= z^{-rM\lambda} \sum_{l\geq 0} \frac{[(rM)^n z]^l}{\prod_{i=1}^n (1 - rM\lambda - irMa_{ij})_l} \\ Z^{\mathbb{P}^{n-1}}_{\text{av, eq}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} \frac{[(-rM)^n \bar{z}]^k}{\prod_{i=1}^n (1 - rM\lambda - irMa_{ij})_k} \end{aligned} \tag{3.46}$$

and $a_{ij} := a_i - a_j$. Since there are just simple poles due to the equivariant weights, the integration can be easily performed

$$\begin{aligned} Z^{\text{eq}}_{\mathbb{P}^{n-1}} = \sum_{j=1}^{n} (z\bar{z})^{irMa_j} &\prod_{i\neq j=1}^{n} \frac{1}{irMa_{ij}} \frac{\Gamma(1+irMa_{ij})}{\Gamma(1-irMa_{ij})} \\ &\sum_{l\geq 0} \frac{[(rM)^n z]^l}{\prod_{i=1}^n (1 - irMa_{ij})_l} \sum_{k\geq 0} \frac{[(-rM)^n \bar{z}]^k}{\prod_{i=1}^n (1 - irMa_{ij})_k}. \end{aligned} \tag{3.47}$$

In the limit $rM \to 0$ the one-loop contribution (see the first line of (3.47)) provides the equivariant volume of the target space

$$\text{Vol}(\mathbb{P}^{n-1}_{\text{eq}}) = \sum_{j=1}^{n} (z\bar{z})^{irMa_j} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{1}{irMa_{ij}} = \sum_{j=1}^{n} e^{-4\pi i\xi rMa_j} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{1}{irMa_{ij}}. \tag{3.48}$$

Using the fact that

$$\lim_{r\to 0} \sum_{j=1}^{n} \frac{e^{-4\pi i\xi rMa_j}}{(4\xi)^{n-1}} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{1}{irMa_{ij}} = \frac{\pi^{n-1}}{(n-1)!} \tag{3.49}$$

we find the non-equivariant volume

$$\mathrm{Vol}(\mathbb{P}^{n-1}) = \frac{(4\pi\xi)^{n-1}}{(n-1)!}. \tag{3.50}$$

3.2.1.2 Weighted projective spaces

Another generalization consists in studying the weighted projective space $\mathbb{P}^{\mathbf{w}} = \mathbb{P}(w_0, \ldots, w_n)$, which has been studied from the mathematical point of view in [41]. This can be obtained by considering a $U(1)$ gauge theory with $n+1$ fundamentals of (positive) integer charges $w_0, \ldots, w_n$. The partition function reads

$$Z = \sum_m \int \frac{d\tau}{2\pi i} e^{4\pi\xi_{\mathrm{ren}}\tau - i\theta_{\mathrm{ren}} m} \prod_{i=0}^{n} \frac{\Gamma(w_i\tau - w_i\frac{m}{2})}{\Gamma(1 - w_i\tau - w_i\frac{m}{2})} \tag{3.51}$$

so one would expect $n+1$ towers of poles at

$$\tau = \frac{m}{2} - \frac{k}{w_i} + rM\lambda \ , \ \ i = 0 \ldots n \tag{3.52}$$

with integration around $rM\lambda = 0$. Actually, in this way we might be overcounting some poles if the w_i are not relatively prime, and in any case the pole $\tau = 0$ is always counted $n+1$ times. In order to solve these problems, we will set

$$\tau = \frac{m}{2} - k + rM\lambda - F \tag{3.53}$$

where F is a set of rational numbers defined as

$$F = \Big\{ \frac{d}{w_i} \ / \ 0 \le d < w_i\, , \ d \in \mathbb{N}\, , \ 0 \le i \le n \Big\} \tag{3.54}$$

and every number has to be counted only once. Let us explain this better with an example: if we consider just $w_0 = 2$ and $w_1 = 3$, we find the numbers $(0, \frac{1}{2})$ and $(0, \frac{1}{3}, \frac{2}{3})$, which means $F = (0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3})$; the multiplicity of these numbers reflects the order of the pole in the integrand, so we will have a double pole (counted by the double multiplicity of $d = 0$) and three simple poles.

The partition function then becomes

$$Z = \sum_F \oint \frac{d(rM\lambda)}{2\pi i} Z_{11}\, Z_{\mathrm{v}}\, Z_{\mathrm{av}} \tag{3.55}$$

with integration around $rM\lambda = 0$ and

$$\begin{aligned}
Z_{1l} &= (rM)^{-2|w|rM\lambda - 2\sum_{i=0}^{n}(\omega[w_iF]-\langle w_iF\rangle)} \prod_{i=0}^{n} \frac{\Gamma(\omega[w_iF] + w_i rM\lambda - \langle w_iF\rangle)}{\Gamma(1-\omega[w_iF] - w_i rM\lambda + \langle w_iF\rangle)} \\
Z_{\text{v}} &= z^{-rM\lambda} \sum_{l\geq 0} \frac{(rM)^{|w|l+\sum_{i=0}^{n}(\omega[w_iF]+[w_iF])} z^{l+F}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_i rM\lambda + \langle w_iF\rangle)_{w_il+[w_iF]+\omega[w_iF]}} \\
Z_{\text{av}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} \frac{(-rM)^{|w|k+\sum_{i=0}^{n}(\omega[w_iF]+[w_iF])} \bar{z}^{k+F}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_i rM\lambda + \langle w_iF\rangle)_{w_ik+[w_iF]+\omega[w_iF]}}.
\end{aligned} \tag{3.56}$$

In the formulae we defined $\langle w_iF\rangle$ and $[w_iF]$ as the fractional and integer part of the number w_iF, so that $w_iF = [w_iF] + \langle w_iF\rangle$, while $|w| = \sum_{i=0}^{n} w_i$. Moreover,

$$\omega[w_iF] = \begin{cases} 0 & \text{for } \langle w_iF\rangle = 0 \\ 1 & \text{for } \langle w_iF\rangle \neq 0. \end{cases} \tag{3.57}$$

This is needed in order for the $\mathcal{J}$ function to start with one in the rM expansion. The twisted sectors in (3.54) label the base of the orbifold cohomology space.

Once more, we can also consider complete intersections in $\mathbb{P}^{\mathbf{w}}$ of type $(q_0, \ldots, q_m)$. The integrand in (3.51) has to be multiplied by

$$\prod_{j=0}^{m} \frac{\Gamma\left(\frac{R_j}{2} - q_j\tau + q_j\frac{m}{2}\right)}{\Gamma\left(1-\frac{R_j}{2} + q_j\tau + q_j\frac{m}{2}\right)} \tag{3.58}$$

The poles do not change, and

$$\begin{aligned}
Z_{1l} &= (rM)^{-2(|w|-|q|)rM\lambda - 2\sum_{i=0}^{n}(\omega[w_iF]-\langle w_iF\rangle) - 2\sum_{j=0}^{m}\langle q_jF\rangle} \\
&\quad \prod_{i=0}^{n} \frac{\Gamma(\omega[w_iF] + w_i rM\lambda - \langle w_iF\rangle)}{\Gamma(1-\omega[w_iF] - w_i rM\lambda + \langle w_iF\rangle)} \prod_{j=0}^{m} \frac{\Gamma(\frac{R_j}{2} - q_j rM\lambda + \langle q_jF\rangle)}{\Gamma(1-\frac{R_j}{2} + q_j rM\lambda - \langle q_jF\rangle)} \\
Z_{\text{v}} &= z^{-rM\lambda} \sum_{l\geq 0} (-1)^{|q|l+\sum_{j=0}^{m}[q_jF]} (rM)^{(|w|-|q|)l + \sum_{i=0}^{n}(\omega[w_iF]+[w_iF]) - \sum_{j=0}^{m}[q_jF]} z^{l+F} \\
&\quad \frac{\prod_{j=0}^{m}(\frac{R_j}{2} - q_j rM\lambda + \langle q_jF\rangle)_{q_jl+[q_jF]}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_i rM\lambda + \langle w_iF\rangle)_{w_il+[w_iF]+\omega[w_iF]}} \\
Z_{\text{av}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} (-1)^{|q|k+\sum_{j=0}^{m}[q_jF]} (-rM)^{(|w|-|q|)k + \sum_{i=0}^{n}(\omega[w_iF]+[w_iF]) - \sum_{j=0}^{m}[q_jF]} \bar{z}^{k+F} \\
&\quad \frac{\prod_{j=0}^{m}(\frac{R_j}{2} - q_j rM\lambda + \langle q_jF\rangle)_{q_jk+[q_jF]}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_i rM\lambda + \langle w_iF\rangle)_{w_ik+[w_iF]+\omega[w_iF]}}.
\end{aligned} \tag{3.59}$$

Notice that the non linear sigma model to which the GLSM flows in the IR is well defined only for $|w| \geq |q|$, which means for manifolds with $c_1 \geq 0$.

3.2.2 The Quintic threefold

We will now consider the most famous compact Calabi-Yau threefold, i.e. the quintic. The corresponding GLSM is a $U(1)$ gauge theory with five chiral fields Φ_a of charge $+1$, one chiral field P of charge -5 and a superpotential of the form $W = PG(\Phi_1, \ldots, \Phi_5)$, where G is a homogeneous polynomial of degree five. We choose the vector R-charges to be $2q$ for the Φ fields and $(2 - 5 \cdot 2q)$ for P such that the superpotential has R-charge 2. The quintic threefold is realized in the geometric phase corresponding to $\xi > 0$. For details of the construction see [19] and for the relation to the two-sphere partition function [17]. Here we want to investigate the connection to the Givental formalism. For a Calabi-Yau manifold the sum of gauge charges is zero, which implies $\xi_{\text{ren}} = \xi$, and $\theta_{\text{ren}} = \theta$ holds because the gauge group is abelian. The spherical partition function is

$$Z = \sum_{m \in \mathbb{Z}} \int_{i\mathbb{R}} \frac{d\tau}{2\pi i} z^{-\tau - \frac{m}{2}} \bar{z}^{-\tau + \frac{m}{2}} \left(\frac{\Gamma\left(q + \tau - \frac{m}{2}\right)}{\Gamma\left(1 - q - \tau - \frac{m}{2}\right)} \right)^5 \frac{\Gamma\left(1 - 5q - 5\tau + 5\frac{m}{2}\right)}{\Gamma\left(5q + 5\tau + 5\frac{m}{2}\right)}. \tag{3.60}$$

Since we want to describe the phase $\xi > 0$, we have to close the contour in the left half plane. We use the freedom in q to separate the towers of poles coming from Φ's and from P. In the range $0 < q < \frac{1}{5}$ the former lie in the left half plane while the latter in the right half plane. So we pick only the poles corresponding to Φ's given by

$$\tau_k = -q - k + \frac{m}{2}, \qquad k \geq \max(0, m) \tag{3.61}$$

Then the partition function turns into a sum of residues and we express each residue by the Cauchy contour integral. Finally we arrive at

$$Z = (z\bar{z})^q \oint_{\mathcal{C}(\delta)} \frac{d(rM\lambda)}{2\pi i} Z_{11}(\lambda, rM) Z_{\text{v}}(\lambda, rM; z) Z_{\text{av}}(\lambda, rM; \bar{z}), \tag{3.62}$$

where the contour $\mathcal{C}(\delta)$ goes around $\lambda = 0$ and

$$\begin{aligned} Z_{11}(\lambda, rM) &= \frac{\Gamma(1 - 5rM\lambda)}{\Gamma(5rM\lambda)} \left(\frac{\Gamma(rM\lambda)}{\Gamma(1 - rM\lambda)} \right)^5 \\ Z_{\text{v}}(\lambda, rM; z) &= z^{-rM\lambda} \sum_{l \geq 0} (-z)^l \frac{(1 - 5rM\lambda)_{5l}}{[(1 - rM\lambda)_l]^5} \\ Z_{\text{av}}(\lambda, rM; \bar{z}) &= \bar{z}^{-rM\lambda} \sum_{k \geq 0} (-\bar{z})^k \frac{(1 - 5rM\lambda)_{5k}}{[(1 - rM\lambda)_k]^5}. \end{aligned} \tag{3.63}$$

The vortex function $Z_{\text{v}}(\lambda, rM; z)$ reproduces the known Givental $\mathcal{I}$-function

$$\mathcal{I}(H, \hbar; t) = \sum_{d \geq 0} e^{(H/\hbar + d)t} \frac{(1 + 5H/\hbar)_{5d}}{[(1 + H/\hbar)_d]^5} \tag{3.64}$$

after identifying

$$H = -\lambda \quad , \quad \hbar = \frac{1}{rM} \quad , \quad t = \ln(-z). \tag{3.65}$$

The $\mathcal{I}$-function is valued in cohomology, where $H \in H^2(\mathbb{P}^4)$ is the hyperplane class in the cohomology ring of the embedding space. Because of dimensional reasons we have $H^5 = 0$ and hence the $\mathcal{I}$-function is a polynomial of order four in H

$$\mathcal{I} = I_0 + \frac{H}{\hbar} I_1 + \left(\frac{H}{\hbar}\right)^2 I_2 + \left(\frac{H}{\hbar}\right)^3 I_3 + \left(\frac{H}{\hbar}\right)^4 I_4. \tag{3.66}$$

This is naturally encoded in the explicit residue evaluation of (3.62), see eq.(3.69). Now consider the Picard-Fuchs operator L. It can be easily shown that $\{I_0, I_1, I_2, I_3\} \in \mathrm{Ker}(L)$ while $I_4 \notin \mathrm{Ker}(L)$. L is an order four operator and so $\mathbf{I} = (I_0, I_1, I_2, I_3)^T$ form a basis of solutions. There exists another basis formed by the periods of the holomorphic $(3,0)$ form of the mirror manifold. In homogeneous coordinates they are given as $\mathbf{\Pi} = (X^0, X^1, \frac{\partial F}{\partial X^0}, \frac{\partial F}{\partial X^1})^T$ with F the prepotential. Thus there exists a transition matrix $\mathbf{M}$ relating these two bases

$$\mathbf{I} = \mathbf{M} \cdot \mathbf{\Pi} \tag{3.67}$$

There are now two possible ways to proceed. One would be fixing the transition matrix using mirror construction (i.e. knowing explicitly the periods) and then showing that the pairing given by the contour integral in (3.62) after being transformed to the period basis gives the standard formula for the Kähler potential in terms of a symplectic pairing

$$e^{-K} = i\mathbf{\Pi}^\dagger \cdot \mathbf{\Sigma} \cdot \mathbf{\Pi} \tag{3.68}$$

with $\mathbf{\Sigma} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$ being the symplectic form. The other possibility would be to use the fact that the two sphere partition function computes the Kähler potential [17] and then impose equality between (3.62) and (3.68) to fix the transition matrix. We follow this route in the following. The contour integral in (3.62) expresses the Kähler potential as a pairing in the $\mathbf{I}$ basis. It is governed by Z_{11} which has an expansion

$$Z_{11} = \frac{5}{(rM\lambda)^4} + \frac{400\,\zeta(3)}{rM\lambda} + o(1) \tag{3.69}$$

and so we get after integration (remember that $H/\hbar = -rM\lambda$)

$$\begin{aligned} Z &= -2\chi\zeta(3) I_0 \bar{I}_0 - 5(I_0\bar{I}_3 + I_1\bar{I}_2 + I_2\bar{I}_1 + I_3\bar{I}_0) \\ &= \mathbf{I}^\dagger \cdot \mathbf{A} \cdot \mathbf{I}, \end{aligned} \tag{3.70}$$

where

$$\mathbf{A} = \begin{pmatrix} -2\chi\zeta(3) & 0 & 0 & -5 \\ 0 & 0 & -5 & 0 \\ 0 & -5 & 0 & 0 \\ -5 & 0 & 0 & 0 \end{pmatrix} \tag{3.71}$$

gives the pairing in the $\mathbf{I}$ basis and $\chi = -200$ is the Euler characteristic of the quintic threefold. From the two expressions for the Kähler potential we easily find the transition matrix as

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{5} \\ -\frac{\chi}{5}\zeta(3) & 0 & -\frac{i}{5} & 0 \end{pmatrix}. \tag{3.72}$$

Finally, we know that the mirror map is given by

$$t = \frac{I_1}{2\pi i I_0} \quad , \quad \bar{t} = -\frac{\bar{I}_1}{2\pi i \bar{I}_0} \tag{3.73}$$

so after dividing Z by $(2\pi i)^2 I_0 \bar{I}_0$ for the change of coordinates and by a further 2π for the normalization of the $\zeta(3)$ term, we obtain the Kähler potential in terms of t, $\bar{t}$, in a form in which the symplectic product is evident.

3.2.3 Local Calabi–Yau: $\mathcal{O}(p) \oplus \mathcal{O}(-2-p) \to \mathbb{P}^1$

Let us now study the family of spaces $X_p = \mathcal{O}(p) \oplus \mathcal{O}(-2-p) \to \mathbb{P}^1$ with diagonal equivariant action on the fiber. We will find exact agreement with the $\mathcal{I}$ functions computed in [42], and we will show how the quantum corrected Kähler potential for the Kähler moduli space can be computed when equivariant parameters are turned on.
Here we will restrict only to the phase $\xi > 0$, which is the one related to X_p. The case $\xi < 0$ describes the orbifold phase of the model; this will be studied in the following sections.

3.2.3.1 Case $p = -1$

First of all, we have to write down the partition function; this is given by

$$Z_{-1} = \sum_{m\in\mathbb{Z}} e^{-im\theta} \int \frac{d\tau}{2\pi i} e^{4\pi\xi\tau} \left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1 - \tau - \frac{m}{2}\right)} \right)^2 \left(\frac{\Gamma\left(-\tau - irMa + \frac{m}{2}\right)}{\Gamma\left(1 + \tau + irMa + \frac{m}{2}\right)} \right)^2 . \tag{3.74}$$

The poles are located at

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.75}$$

so we can rewrite (3.74) as

$$Z_{-1} = \oint \frac{d(rM\lambda)}{2\pi i} Z_{11} Z_{\mathrm{v}} Z_{\mathrm{av}}, \tag{3.76}$$

where

$$\begin{aligned} Z_{11} &= \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)} \frac{\Gamma(-rM\lambda - irMa)}{\Gamma(1+rM\lambda+irMa)} \right)^2 \\ Z_{\mathrm{v}} &= z^{-rM\lambda} \sum_{l\geq 0} z^l \frac{(-rM\lambda - irMa)_l^2}{(1-rM\lambda)_l^2} \\ Z_{\mathrm{av}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} \bar{z}^k \frac{(-rM\lambda - irMa)_k^2}{(1-rM\lambda)_k^2}. \end{aligned} \tag{3.77}$$

Notice that our vortex partition function coincides with the Givental function given in [42]

$$\mathcal{I}_{-1}^T(q) = e^{\frac{H}{\hbar}\ln q} \sum_{d\geq 0} \frac{(1-H/\hbar+\tilde{\lambda}/\hbar - d)_d^2}{(1+H/\hbar)_d^2} q^d \tag{3.78}$$

after the usual identifications

$$H = -\lambda \quad , \quad \hbar = \frac{1}{rM} \quad , \quad \tilde{\lambda} = ia \quad , \quad q = z. \tag{3.79}$$

Now, expanding $\mathcal{I}_{-1}^T$ in $rM = 1/\hbar$ we find

$$\mathcal{I}_{-1}^T = 1 - rM\lambda \log z + o((rM)^2) \tag{3.80}$$

which means the mirror map is trivial and the equivariant mirror map absent, i.e. $\mathcal{I}_{-1}^T = \mathcal{J}_{-1}^T$. What remains to be specified is the normalization of the 1-loop factor. As explained in [43], this normalization is fixed by requiring the cancellation of the Euler-Mascheroni constants appearing in the Weierstrass form of the Γ-function, further by requiring that it reproduces the classical intersection numbers and starts from 1 in the rM expansion; in our case, the factor

$$(z\bar{z})^{-irMa/2} \left(\frac{\Gamma(1+irMa)}{\Gamma(1-irMa)} \right)^2 \tag{3.81}$$

does the job. We can now integrate in $rM\lambda$ and expand in rM, obtaining (for $rMa = iq$)

$$\begin{aligned} Z_{-1} = \frac{2}{q^3} - \frac{1}{4q}\ln^2(z\bar{z}) + \Big[&- \frac{1}{12}\ln^3(z\bar{z}) - \ln(z\bar{z})(\mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z})) \\ &+ 2(\mathrm{Li}_3(z) + \mathrm{Li}_3(\bar{z})) + 4\zeta(3) \Big] + o(rM). \end{aligned} \tag{3.82}$$

The terms inside the square brackets reproduce the Kähler potential we are interested in, once we multiply everything by $\frac{1}{2\pi(2\pi i)^2}$ and define

$$t = \frac{1}{2\pi i}\ln z \quad , \quad \bar{t} = -\frac{1}{2\pi i}\ln \bar{z}. \tag{3.83}$$

3.2.3.2 Case $p = 0$

In this case case, the spherical partition function reads

$$Z_0 = \sum_{m\in\mathbb{Z}} e^{-im\theta}\int \frac{d\tau}{2\pi i}e^{4\pi\xi\tau}\left(\frac{\Gamma\left(\tau-\frac{m}{2}\right)}{\Gamma\left(1-\tau-\frac{m}{2}\right)}\right)^2 \frac{\Gamma\left(-irMa\right)}{\Gamma\left(1+irMa\right)}\frac{\Gamma\left(-2\tau-irMa+2\frac{m}{2}\right)}{\Gamma\left(1+2\tau+irMa+2\frac{m}{2}\right)}. \tag{3.84}$$

The poles are as in (3.75), and usual manipulations result in

$$\begin{aligned} Z_{11} &= \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)}\right)^2 \frac{\Gamma\left(-irMa\right)}{\Gamma\left(1+irMa\right)}\frac{\Gamma(-2rM\lambda-irMa)}{\Gamma(1+2rM\lambda+irMa)} \\ Z_{\mathrm{v}} &= z^{-rM\lambda}\sum_{l\geq 0} z^l\frac{(-2rM\lambda-irMa)_{2l}}{(1-rM\lambda)_l^2} \\ Z_{\mathrm{av}} &= \bar{z}^{-rM\lambda}\sum_{k\geq 0}\bar{z}^k\frac{(-2rM\lambda-irMa)_{2k}}{(1-rM\lambda)_k^2}. \end{aligned} \tag{3.85}$$

Again, we recover the Givental function

$$\mathcal{I}_0^T(q) = e^{\frac{H}{\hbar}\ln q}\sum_{d\geq 0}\frac{(1-2H/\hbar+\tilde{\lambda}/\hbar-2d)_{2d}}{(1+H/\hbar)_d^2}q^d \tag{3.86}$$

of [42] under the map (3.79); its expansion in rM gives

$$\mathcal{I}_0^T = 1 - rM\lambda\left[\log z + 2\sum_{k=1}^{\infty} z^k\frac{\Gamma(2k)}{(k!)^2}\right] - irMa\sum_{k=1}^{\infty} z^k\frac{\Gamma(2k)}{(k!)^2} + o((rM)^2), \tag{3.87}$$

which implies that the mirror map is (modulo $(2\pi i)^{-1}$)

$$t = \log z + 2\sum_{k=1}^{\infty} z^k\frac{\Gamma(2k)}{(k!)^2} \tag{3.88}$$

while for the equivariant mirror map we get

$$\tilde{t} = \frac{1}{2}(t - \log z) = \sum_{k=1}^{\infty} z^k\frac{\Gamma(2k)}{(k!)^2}. \tag{3.89}$$

The $\mathcal{J}$ function can be recovered by inverting the equivariant mirror map and changing coordinates accordingly, that is

$$\mathcal{J}_0^T(t) = e^{irMa\tilde{t}(z)}\mathcal{I}_0^T(z)\Big|_{z=z(t)} = e^{irMa\tilde{t}(z)}Z_{\mathrm{v}}(z)\Big|_{z=z(t)} . \tag{3.90}$$

A similar job has to be done for Z_{av}. The normalization for the 1-loop factor is the same as (3.81) but in t coordinates, which means

$$(t\bar{t})^{-irMa/2}\left(\frac{\Gamma(1+irMa)}{\Gamma(1-irMa)}\right)^2 . \tag{3.91}$$

Finally, integrating in $rM\lambda$ and expanding in rM we find

$$\begin{aligned} Z_0 = &\frac{2}{q^3} - \frac{1}{4q}(t+\bar{t})^2 + \Big[-\frac{1}{12}(t+\bar{t})^3 - (t+\bar{t})(\mathrm{Li}_2(e^t) + \mathrm{Li}_2(e^{\bar{t}})) \\ &+ 2(\mathrm{Li}_3(e^t) + \mathrm{Li}_3(e^{\bar{t}})) + 4\zeta(3)\Big] + o(rM). \end{aligned} \tag{3.92}$$

As it was shown in [42], this proves that the two Givental functions $\mathcal{J}_{-1}^T$ and $\mathcal{J}_0^T$ are the same, as well as the Kähler potentials; the $\mathcal{I}$ functions look different simply because of the choice of coordinates on the moduli space.

3.2.3.3 Case $p \geq 1$

In the general $p \geq 1$ case, we have

$$\begin{aligned} Z_p = &\sum_{m\in\mathbb{Z}} e^{-im\theta} \int \frac{d\tau}{2\pi i} e^{4\pi\xi\tau} \left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1-\tau-\frac{m}{2}\right)}\right)^2 \\ &\frac{\Gamma\left(-(p+2)\tau - irMa + (p+2)\frac{m}{2}\right)}{\Gamma\left(1+(p+2)\tau + irMa + (p+2)\frac{m}{2}\right)} \frac{\Gamma\left(p\tau - irMa - p\frac{m}{2}\right)}{\Gamma\left(1-p\tau+irMa-p\frac{m}{2}\right)}. \end{aligned} \tag{3.93}$$

There are two classes of poles, given by

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.94}$$

$$\tau = -k + \frac{m}{2} + rM\lambda - F + irM\frac{a}{p}, \tag{3.95}$$

where $F = \{0, \frac{1}{p}, \ldots, \frac{p-1}{p}\}$ and the integration is around $rM\lambda = 0$. This can be understood from the fact that actually the GLSM (3.93) describes the canonical bundle over the weighted projective space $\mathbb{P}_{(1,1,p)}$, which has two chambers. The regular one, associated to the poles (3.94), corresponds to the local $\mathcal{O}(p) \oplus \mathcal{O}(-2-p) \to \mathbb{P}^1$ geometry

$$Z_p^{(0)} = \oint \frac{d(rM\lambda)}{2\pi i} Z_{1l}^{(0)} Z_{\mathrm{v}}^{(0)} Z_{\mathrm{av}}^{(0)} \tag{3.96}$$

with

$$\begin{aligned}
Z_{11}^{(0)} &= \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)}\right)^2 \frac{\Gamma(-(p+2)rM\lambda - irMa)}{\Gamma(1+(p+2)rM\lambda + irMa)} \frac{\Gamma(p\,rM\lambda - irMa)}{\Gamma(1-p\,rM\lambda + irMa)} \\
Z_{\mathrm{v}}^{(0)} &= z^{-rM\lambda} \sum_{l\geqslant 0} (-1)^{(p+2)l} z^l \frac{(-(p+2)rM\lambda - irMa)_{(p+2)l}}{(1-rM\lambda)_l^2 (1-p\,rM\lambda + irMa)_{pl}} \\
Z_{\mathrm{av}}^{(0)} &= \bar{z}^{-rM\lambda} \sum_{k\geqslant 0} (-1)^{(p+2)k} \bar{z}^k \frac{(-(p+2)rM\lambda - irMa)_{(p+2)k}}{(1-rM\lambda)_k^2 (1-p\,rM\lambda + irMa)_{pk}} .
\end{aligned} \tag{3.97}$$

The second chamber, associated to (3.95), is an orbifold one

$$Z_p^{(F)} = \sum_{\delta=0}^{p-1} \oint \frac{d(rM\lambda)}{2\pi i} Z_{11,\delta}^{(F)} Z_{\mathrm{v},\delta}^{(F)} Z_{\mathrm{av},\delta}^{(F)}, \tag{3.98}$$

where $F = \frac{\delta}{p}$. The explicit expression for $Z^{(F)}$ in the above formula can be recovered from (3.59), adding the twisted masses in the appropriate places. Notice that (3.98) can be easily integrated, since there are just simple poles.

3.3 Non-abelian GLSM

In this section we apply our methods to non-abelian gauged linear sigma models and give new results for some non-abelian GIT quotients. These are also tested against results in the mathematical literature when available.

The first case that we analyse are complex Grassmannians. On the way we also give an alternative proof for the conjecture of Hori and Vafa which can be rephrased stating that the $\mathcal{I}$-function of the Grassmannian can be obtained from that corresponding to a product of projective spaces after acting with an appropriate differential operator.

One can also study a more general theory corresponding to holomorphic vector bundles over Grassmannians. These spaces arise in the context of the study of BPS Wilson loop algebra in three dimensional supersymmetric gauge theories. In particular we will discuss the mathematical counterpart of a duality proposed in [44] which extends the standard Grassmannian duality to holomorphic vector bundles over them.

We also study flag manifolds and more general non-abelian quiver gauge theories for which we provide the rules to compute the spherical partition function and the $\mathcal{I}$-function.

3.3.1 Grassmannians

The sigma model for the complex Grassmannian $Gr(s,n)$ (in our notation we mean the set of s-dimensional subspaces in $\mathbb{C}^n$) contains n chirals in the fundamental representation of the $U(s)$ gauge group. Its partition function is given by

$$Z_{Gr(s,n)} = \frac{1}{s!}\sum_{m_1,\dots,m_s}\int\prod_{i=1}^{s}\frac{\mathrm{d}\tau_i}{2\pi i}e^{4\pi\xi_{\mathrm{ren}}\tau_i - i\theta_{\mathrm{ren}}m_i}\prod_{i<j}^{s}\left(\frac{m_{ij}^2}{4}-\tau_{ij}^2\right)\prod_{i=1}^{s}\left(\frac{\Gamma\left(\tau_i-\frac{m_i}{2}\right)}{\Gamma\left(1-\tau_i-\frac{m_i}{2}\right)}\right)^n . \tag{3.99}$$

As usual, we can write it as

$$Z_{Gr(s,n)} = \frac{1}{s!}\oint\prod_{i=1}^{s}\frac{d(rM\lambda_i)}{2\pi i}Z_{11}Z_{\mathrm{v}}Z_{\mathrm{av}}, \tag{3.100}$$

where

$$\begin{aligned}
Z_{11} &= \prod_{i=1}^{s}(rM)^{-2nrM\lambda_i}\left(\frac{\Gamma(rM\lambda_i)}{\Gamma(1-rM\lambda_i)}\right)^n\prod_{i<j}^{s}(rM\lambda_i - rM\lambda_j)(-rM\lambda_i + rM\lambda_j)\\
Z_{\mathrm{v}} &= z^{-rM|\lambda|}\sum_{l_1,\dots,l_s}\frac{[(rM)^n(-1)^{s-1}z]^{l_1+\dots+l_s}}{(1-rM\lambda_1)_{l_1}^n\dots(1-rM\lambda_s)_{l_s}^n}\prod_{i<j}^{s}\frac{l_i-l_j-rM\lambda_i+rM\lambda_j}{-rM\lambda_i+rM\lambda_j}\\
Z_{\mathrm{av}} &= \bar{z}^{-rM|\lambda|}\sum_{k_1,\dots,k_s}\frac{[(-rM)^n(-1)^{s-1}\bar{z}]^{k_1+\dots+k_s}}{(1-rM\lambda_1)_{k_1}^n\dots(1-rM\lambda_s)_{k_s}^n}\prod_{i<j}^{s}\frac{k_i-k_j-rM\lambda_i+rM\lambda_j}{-rM\lambda_i+rM\lambda_j}.
\end{aligned} \tag{3.101}$$

We normalized the vortex and antivortex terms in order to have them starting from one in the rM series expansion and we defined $|\lambda| = \lambda_1 + \dots + \lambda_s$. The resulting $\mathcal{I}$-function Z_{v} coincides with the one given in [37]

$$\mathcal{I}_{Gr(s,n)} = e^{\frac{t\sigma_1}{\hbar}}\sum_{(d_1,\dots,d_s)}\frac{\hbar^{-n(d_1+\dots+d_s)}[(-1)^{s-1}e^t]^{d_1+\dots+d_s}}{\prod_{i=1}^{s}(1+x_i/\hbar)_{d_i}^n}\prod_{i<j}^{s}\frac{d_i-d_j+x_i/\hbar-x_j/\hbar}{x_i/\hbar-x_j/\hbar} \tag{3.102}$$

if we match the parameters as we did in the previous cases. Here the λ's are interpreted as Chern roots of the tautological bundle.

3.3.1.1 The Hori-Vafa conjecture

Hori and Vafa conjectured [39] that $\mathcal{I}_{Gr(s,n)}$ can be obtained by $\mathcal{I}_{\mathbb{P}}$, where $\mathbb{P} = \prod_{i=1}^{s}\mathbb{P}^{n-1}_{(i)}$, by acting with a differential operator. This has been proved in [37]; here we remark that in our formalism this is a simple consequence of the fact that the partition function of non-abelian vortices can be obtained from copies of the abelian ones upon acting with a suitable differential operator [45]. In fact we note that $Z_{Gr(s,n)}$ can be obtained from

$Z_{\mathbb{P}}$ simply by dividing by $s!$ and identifying

$$
\begin{aligned}
Z_{11}^{Gr} &= \prod_{i<j}^{s}(rM\lambda_i - rM\lambda_j)(-rM\lambda_i + rM\lambda_j)Z_{11}^{\mathbb{P}} \\
Z_{\mathrm{v}}^{Gr}(z) &= \prod_{i<j}^{s}\frac{\partial_{z_i}-\partial_{z_j}}{-rM\lambda_i + rM\lambda_j}Z_{\mathrm{v}}^{\mathbb{P}}(z_1,\dots,z_s)\Big|_{z_i=(-1)^{s-1}z} \\
Z_{\mathrm{av}}^{Gr}(\bar{z}) &= \prod_{i<j}^{s}\frac{\partial_{\bar{z}_i}-\partial_{\bar{z}_j}}{-rM\lambda_i + rM\lambda_j}Z_{\mathrm{av}}^{\mathbb{P}}(\bar{z}_1,\dots,\bar{z}_s)\Big|_{\bar{z}_i=(-1)^{s-1}\bar{z}}.
\end{aligned}
\tag{3.103}
$$

3.3.2 Holomorphic vector bundles over Grassmannians

The $U(N)$ gauge theory with N_f fundamentals and N_a antifundamentals flows in the infra-red to a non-linear sigma model with target space given by a holomorphic vector bundle of rank N_a over the Grassmannian $Gr\,(N, N_f)$. We adopt the notation $Gr\,(N, N_f|N_a)$ for this space.

One can prove equality of Z^{S^2} for $Gr\,(N, N_f|N_a)$ and $Gr\,(N_f - N, N_f|N_a)$ after a precise duality map in a certain range of parameters. This will be specified in Appendix B. At the level of $\mathcal{I}$-functions this proves the isomorphism among the relevant quantum cohomology rings conjectured in [44]. In analysing this duality we follow the approach of [14], where also the main steps of the proof were outlined. However we will detail their calculations and note some differences in the explicit duality map, which we refine in order to get a precise equality of the partition functions.

The partition function of the $Gr\,(N, N_f|N_a)$ GLSM is

$$
\begin{aligned}
Z = &\frac{1}{N!}\sum_{\{m_s\in\mathbb{Z}\}_{s=1}^{N}}\int_{(i\mathbb{R})^N}\prod_{s=1}^{N}\frac{d\tau_s}{2\pi i}z_{\mathrm{ren}}^{-\tau_s-\frac{m_s}{2}}\bar{z}_{\mathrm{ren}}^{-\tau_s+\frac{m_s}{2}}\prod_{s<t}^{N}\left(\frac{m_{st}^2}{4}-\tau_{st}^2\right) \\
&\prod_{s=1}^{N}\prod_{i=1}^{N_f}\frac{\Gamma\left(\tau_s - i\frac{a_i}{\hbar}-\frac{m_s}{2}\right)}{\Gamma\left(1-\tau_s + i\frac{a_i}{\hbar}-\frac{m_s}{2}\right)}\prod_{s=1}^{N}\prod_{j=1}^{N_a}\frac{\Gamma\left(-\tau_s + i\frac{\widetilde{a}_j}{\hbar}+\frac{m_s}{2}\right)}{\Gamma\left(1+\tau_s - i\frac{\widetilde{a}_j}{\hbar}+\frac{m_s}{2}\right)},
\end{aligned}
\tag{3.104}
$$

while the one of $Gr\,(N_f - N, N_f|N_a)$ reads

$$
\begin{aligned}
Z = &\frac{1}{N^D!}\sum_{\{m_s\in\mathbb{Z}\}_{s=1}^{N^D}}\int_{(i\mathbb{R})^{N^D}}\prod_{s=1}^{N^D}\frac{d\tau_s}{2\pi i}(z_{ren}^D)^{-\tau_s-\frac{m_s}{2}}(\bar{z}_{ren}^D)^{-\tau_s+\frac{m_s}{2}}\prod_{s<t}^{N^D}\left(\frac{m_{st}^2}{4}-\tau_{st}^2\right) \\
&\prod_{s=1}^{N^D}\prod_{i=1}^{N_f}\frac{\Gamma\left(\tau_s + i\frac{a_i^D}{\hbar}-\frac{m_s}{2}\right)}{\Gamma\left(1-\tau_s - i\frac{a_i^D}{\hbar}-\frac{m_s}{2}\right)}\prod_{s=1}^{N^D}\prod_{j=1}^{N_a}\frac{\Gamma\left(-\tau_s - i\frac{\widetilde{a}_j^D}{\hbar}+\frac{m_s}{2}\right)}{\Gamma\left(1+\tau_s + i\frac{\widetilde{a}_j^D}{\hbar}+\frac{m_s}{2}\right)}\prod_{i=1}^{N_f}\prod_{j=1}^{N_a}\frac{\Gamma\left(-i\frac{a_i-\widetilde{a}_j}{\hbar}\right)}{\Gamma\left(1+i\frac{a_i-\widetilde{a}_j}{\hbar}\right)}.
\end{aligned}
\tag{3.105}
$$

The proof of the equality of the two is shown in detail in Appendix B to hold under the duality map

$$z^D = (-1)^{N_a} z \tag{3.106}$$

$$\frac{a_j^D}{\hbar} = -\frac{a_j}{\hbar} + C \tag{3.107}$$

$$\frac{\widetilde{a}_j^D}{\hbar} = -\frac{\widetilde{a}_j}{\hbar} - (C+i), \tag{3.108}$$

where

$$C = \frac{1}{N_f - N} \sum_{i=1}^{N_f} \frac{a_i}{\hbar}. \tag{3.109}$$

3.3.3 Flag manifolds

Let us consider now a linear sigma model with gauge group $U(s_1)\times\ldots\times U(s_l)$ and with matter in the $(s_1,\bar{s}_2)\oplus\ldots\oplus(s_{l-1},\bar{s}_l)\oplus(s_l,n)$ representations, where $s_1<\ldots<s_l<n$. This flows in the infrared to a non-linear sigma model whose target space is the flag manifold $Fl(s_1,\ldots,s_l,n)$. The partition function is given by

$$\begin{aligned}
Z_{Fl} &= \frac{1}{s_1!\ldots s_l!} \sum_{\substack{\vec{m}^{(a)}\\ a=1\ldots l}} \int \prod_{a=1}^{l}\prod_{i=1}^{s_a} \frac{\mathrm{d}\tau_i^{(a)}}{2\pi i} e^{4\pi\xi_{\mathrm{ren}}^{(a)}\tau_i^{(a)} - i\theta_{\mathrm{ren}}^{(a)} m_i^{(a)}} Z_{\mathrm{vector}} Z_{\mathrm{bifund}} Z_{\mathrm{fund}}\\
Z_{\mathrm{vector}} &= \prod_{a=1}^{l}\prod_{i<j}^{s_a} \left(\frac{(m_{ij}^{(a)})^2}{4} - (\tau_{ij}^{(a)})^2\right)\\
Z_{\mathrm{bifund}} &= \prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}} \frac{\Gamma\left(\tau_i^{(a)} - \tau_j^{(a+1)} - \frac{m_i^{(a)}}{2} + \frac{m_j^{(a+1)}}{2}\right)}{\Gamma\left(1 - \tau_i^{(a)} + \tau_j^{(a+1)} - \frac{m_i^{(a)}}{2} + \frac{m_j^{(a+1)}}{2}\right)}\\
Z_{\mathrm{fund}} &= \prod_{i=1}^{s_l} \left(\frac{\Gamma\left(\tau_i^{(l)} - \frac{m_i^{(l)}}{2}\right)}{\Gamma\left(1 - \tau_i^{(l)} - \frac{m_i^{(l)}}{2}\right)}\right)^n .
\end{aligned} \tag{3.110}$$

After performing the change of variables

$$\tau_i^{(a)} = \frac{m_i^{(a)}}{2} - k_i^{(a)} + rM\lambda_i^{(a)} \tag{3.111}$$

we obtain

$$Z_{Fl} = \frac{1}{s_1!\dots s_l!}\oint \prod_{a=1}^{l}\prod_{i=1}^{s_a}\frac{d(rM\lambda_i^{(a)})}{2\pi i} Z_{\text{1-loop}} Z_{\text{v}} Z_{\text{av}}, \tag{3.112}$$

where

$$\begin{aligned}
Z_{\text{1-loop}} =& (rM)^{-2rM\left[\sum_{a=1}^{l-1}(|\lambda^{(a)}|s_{a+1}-|\lambda^{(a+1)}|s_a)+n|\lambda^{(l)}|\right]} \\
&\prod_{a=1}^{l}\prod_{i<j}^{s_a}(rM\lambda_i^{(a)}-rM\lambda_j^{(a)})(rM\lambda_j^{(a)}-rM\lambda_i^{(a)}) \\
&\prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}}\frac{\Gamma\left(rM\lambda_i^{(a)}-rM\lambda_j^{(a+1)}\right)}{\Gamma\left(1-rM\lambda_i^{(a)}+rM\lambda_j^{(a+1)}\right)}\prod_{i=1}^{s_l}\left(\frac{\Gamma\left(rM\lambda_i^{(l)}\right)}{\Gamma\left(1-rM\lambda_i^{(l)}\right)}\right)^n \\
Z_{\text{v}} =& \sum_{\vec{l}^{(a)}}(rM)^{\sum_{a=1}^{l-1}(|l^{(a)}|s_{a+1}-|l^{(a+1)}|s_a)+n|l^{(l)}|}\prod_{a=1}^{l}(-1)^{(s_a-1)|l^{(a)}|}z_a^{|l^{(a)}|-rM|\lambda^{(a)}|} \\
&\prod_{a=1}^{l}\prod_{i<j}^{s_a}\frac{l_i^{(a)}-l_j^{(a)}-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}} \\
&\prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}}\frac{1}{(1-rM\lambda_i^{(a)}+rM\lambda_j^{(a+1)})_{l_i^{(a)}-l_j^{(a+1)}}}\prod_{i=1}^{s_l}\frac{1}{\left[(1-rM\lambda_i^{(l)})_{l_i^{(l)}}\right]^n} \\
Z_{\text{av}} =& \sum_{\vec{k}^{(a)}}(-rM)^{\sum_{a=1}^{l-1}(|k^{(a)}|s_{a+1}-|k^{(a+1)}|s_a)+n|k^{(l)}|}\prod_{a=1}^{l}(-1)^{(s_a-1)|k^{(a)}|}\bar{z}_a^{|k^{(a)}|-rM|\lambda^{(a)}|} \\
&\prod_{a=1}^{l}\prod_{i<j}^{s_a}\frac{k_i^{(a)}-k_j^{(a)}-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}} \\
&\prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}}\frac{1}{(1-rM\lambda_i^{(a)}+rM\lambda_j^{(a+1)})_{k_i^{(a)}-k_j^{(a+1)}}}\prod_{i=1}^{s_l}\frac{1}{\left[(1-rM\lambda_i^{(l)})_{k_i^{(l)}}\right]^n}.
\end{aligned} \tag{3.113}$$

In the formulae above, k's and l's are non-negative integers. This result can be compared with the one in [38]. Indeed our fractions with Pochhammers at the denominator are equivalent to the products appearing there and we find perfect agreement with the Givental $\mathcal{I}$-function under the, by now familiar, identification $\hbar = \frac{1}{rM}, \lambda = -H$ in Z_{v} and $\hbar = -\frac{1}{rM}, \lambda = H$ in Z_{av}.

3.4 Phase transitions and Gromow–Witten theory

In this section we want to show how the analytic structure of the partition function encodes all the classical phases of an abelian GLSM. These are given by the secondary fan, which in our conventions is generated by the columns of the charge matrix Q. In terms of the partition function these phases are governed by the choice of integration contours, namely by the structure of poles we are picking up. The contour can be closed either in the left half plane (for $\xi > 0$) or in the right half plane ($\xi < 0$)[5]. The transition between different phases occurs when some of the integration contours are flipped and the corresponding variable is integrated. To summarize, a single partition function contains the $\mathcal{I}$-functions of geometries corresponding to all the different phases of the GLSM. These geometries are related by minimally resolving the singularities by blow-up until the complete smoothing of the space takes place (when this is possible). Our procedure consists in considering the GLSM corresponding to the complete resolution and its partition function. Then by flipping contours and doing partial integrations one discovers all other, more singular geometries. In the following we illustrate these ideas on an example.

3.4.1 $K_{\mathbb{P}^{n-1}}$ vs. $\mathbb{C}^n/\mathbb{Z}_n$

Let us consider a $U(1)$ gauge theory with n chiral fields of charge $+1$ and one chiral field of charge $-n$. The secondary fan is generated by two vectors $\{1, -n\}$ and so has two chambers corresponding to two different phases. For $\xi > 0$ it describes a smooth geometry $K_{\mathbb{P}^{n-1}}$, that is the total space of the canonical bundle over the complex projective space $\mathbb{P}^{n-1}$, while for $\xi < 0$ the orbifold $\mathbb{C}^n/\mathbb{Z}_n$. The case $n = 3$ will reproduce the results of [46, 47, 48]. The partition function reads

$$Z = \sum_m \int_{i\mathbb{R}} \frac{d\tau}{2\pi i} e^{4\pi\xi\tau - i\theta m} \left(\frac{\Gamma(\tau - \frac{m}{2})}{\Gamma(1 - \tau - \frac{m}{2})}\right)^n \frac{\Gamma(-n\tau + n\frac{m}{2} + irMa)}{\Gamma(1 + n\tau + n\frac{m}{2} - irMa)}. \tag{3.114}$$

Closing the contour in the left half plane (i.e. for $\xi > 0$) we take poles at

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.115}$$

and obtain

$$\begin{aligned} Z = \oint \frac{d(rM\lambda)}{2\pi i} &\left(\frac{\Gamma(rM\lambda)}{\Gamma(1 - rM\lambda)}\right)^n \frac{\Gamma(-nrM\lambda + irMa)}{\Gamma(1 + nrM\lambda - irMa)} \\ &\sum_{l\geq 0} z^{-rM\lambda} (-1)^{nl} z^{nl} \frac{(-nrM\lambda + irMa)_{nl}}{(1 - rM\lambda)_l^n} \\ &\sum_{k\geq 0} \bar{z}^{-rM\lambda} (-1)^{nk} \bar{z}^{nk} \frac{(-nrM\lambda + irMa)_{nk}}{(1 - rM\lambda)_k^n}. \end{aligned} \tag{3.116}$$

[5]This is only true for Calabi-Yau manifolds; for $c_1 > 0$, i.e. $\sum_i Q_i > 0$, the contour is fixed.

We thus find exactly the Givental function for $K_{\mathbb{P}^{n-1}}$. To switch to the singular geometry we flip the contour and do the integration. Closing in the right half plane ($\xi < 0$) we consider

$$\tau = k + \frac{\delta}{n} + \frac{m}{2} + \frac{1}{n} irMa \tag{3.117}$$

with $\delta = 0, 1, 2, \ldots, n-1$. After integrating over τ, we obtain

$$\begin{aligned} Z = \; & \frac{1}{n} \sum_{\delta=0}^{n-1} \left(\frac{\Gamma(\frac{\delta}{n} + \frac{1}{n} irMa)}{\Gamma(1 - \frac{\delta}{n} - \frac{1}{n} irMa)} \right)^n \frac{1}{(rM)^{2\delta}} \\ & \sum_{k \geq 0} (-1)^{nk} (\bar{z}^{-1/n})^{nk+\delta+irMa} (rM)^{\delta} \frac{(\frac{\delta}{n} + \frac{1}{n} irMa)_k^n}{(nk+\delta)!} \\ & \sum_{l \geq 0} (-1)^{nl} (z^{-1/n})^{nl+\delta+irMa} (-rM)^{\delta} \frac{(\frac{\delta}{n} + \frac{1}{n} irMa)_l^n}{(nl+\delta)!} \end{aligned} \tag{3.118}$$

as expected from (3.59). Notice that when the contour is closed in the right half plane, vortex and antivortex contributions are exchanged. We can compare the $n = 3$ case corresponding to $\mathbb{C}^3/\mathbb{Z}_3$ with [48], given by

$$\mathcal{I} = x^{-\lambda/z} \sum_{\substack{d \in \mathbb{N} \\ d \geq 0}} \frac{x^d}{d! z^d} \prod_{\substack{0 \leq b < \frac{d}{3} \\ \langle b \rangle = \langle \frac{d}{3} \rangle}} \left(\frac{\lambda}{3} - bz \right)^3 \mathbf{1}_{\langle \frac{d}{3} \rangle} \tag{3.119}$$

which in a more familiar notation becomes

$$\mathcal{I} = x^{-\lambda/z} \sum_{\substack{d \in \mathbb{N} \\ d \geq 0}} \frac{x^d}{d!} \frac{1}{z^{3\langle \frac{d}{3} \rangle}} (-1)^{3[\frac{d}{3}]} \left(\langle \frac{d}{3} \rangle - \frac{\lambda}{3z} \right)^3_{[\frac{d}{3}]} \mathbf{1}_{\langle \frac{d}{3} \rangle}. \tag{3.120}$$

The necessary identifications are straightforward.

Chapter 4

Quantum integrable systems from the partition function on S^2

The connection between the partition function Z^{S^2} and quantum integrable models will be established in two steps. First, we introduce the concept of a mirror Landau–Ginzburg model [39], which is described by a twisted superpotential. Once having such an object at our disposal, the Bethe/Gauge correspondence [49, 50] can be applied to obtain the Yang–Yang function and Bethe equations of the associated integrable system.

As was shown in [16], the mirror Landau–Ginzburg theory to a given GLSM can be obtained naturally from the partition function on S^2, even for non-abelian gauged linear sigma models generalizing thus [39]. The GLSM is constructed of chiral and vector multiplets coupled together. Suppose that at low energies it flows to a NLSM with target space a Calabi–Yau manifold. The associated mirror Calabi–Yau manifold is captured by the mirror Landau–Ginzburg model to the GLSM. As we mentioned in Section 1 when introducing $\mathcal{N} = (2,2)$ supersymmetry, mirror symmetry acts at the level of field theories by exchanging chiral and twisted chiral multiplets. So the mirror Landau–Ginzburg model is build out of twisted chiral fields Y. Although twisted chiral fields can not couple directly to vector multiplets, they can couple to the field strength Σ of a vector field, since it is a twisted chiral field. That is how the original vector multiplets reappear on scene. Recall that the lowest component of the superfield strength is the complex scalar field (with real part σ, while imaginary η) from the vector multiplet. Moreover, on the saddle point locus of the vector multiplet, the imaginary part of this scalar is quantized, see (2.47). Thus we have

$$\Sigma_s = \sigma_s - i\eta_s = \sigma_s - i\frac{m_s}{2r}; \quad s = 1, \dots, \mathrm{rk}(G), \tag{4.1}$$

where we slightly abuse notation when denoting by Σ the lowest component of the superfield strength. Then the Landau–Ginzburg theory is fully specified by the twisted superpotential $\widetilde{\mathcal{W}}(Y,\Sigma)$. The procedure how to recover this function from Z^{S^2} was found in [16]; basically one has to rewrite every ratio of Gamma functions appearing in the partition function by an integral identity

$$\frac{\Gamma(-ir\Sigma)}{\Gamma(1+ir\overline{\Sigma})} = \int \frac{dYd\overline{Y}}{2\pi} \exp\left\{-e^{-Y} + ir\Sigma Y + e^{\overline{Y}} + ir\overline{\Sigma}\overline{Y}\right\}. \tag{4.2}$$

The resulting form of the partition function after completing this procedure comes out as

$$Z^{S^2} = \left|\int dYd\Sigma\, e^{-\widetilde{\mathcal{W}}(Y,\Sigma)}\right|^2, \tag{4.3}$$

such that one can read off $\widetilde{\mathcal{W}}(Y,\Sigma)$ easily.

To recover the effective low energy description on the Coulomb branch one has to integrate out massive Y and vector fields (W-bosons). The equations of motion for $Y,\overline{Y}$ are

$$Y = -\ln(-it\Sigma), \quad \overline{Y} = -\ln(ir\overline{\Sigma}), \tag{4.4}$$

so that in the semiclassical approximation we arrive at

$$\frac{\Gamma(-ir\Sigma)}{\Gamma(1+ir\overline{\Sigma})} \sim \exp\left\{\omega(-ir\Sigma) - \frac{1}{2}\ln(-ir\Sigma) - \omega(ir\overline{\Sigma}) - \frac{1}{2}\ln(ir\overline{\Sigma})\right\} \tag{4.5}$$

with $\omega(x) = x(\ln x - 1)$. The outcome of integrating out massive "W-bosons" turns out to be a shift in the theta angle, $\theta \to \theta_{\text{ren}} = \theta + (\text{rk}(G) - 1)\pi$. The same result can be obtained by a slightly different reasoning. Starting from the Coulomb branch expression for the partition function (2.52), we write the integrand as as an exponential of some argument. Subsequently we perform an asymptotic expansion of that argument for large radius of the sphere $r \to \infty$. Using the Stirling formula for logarithms of Gamma functions

$$\begin{aligned}\Gamma(z) &\overset{z\to\infty}{\sim} \omega(z) - \frac{1}{2}\ln z + \frac{1}{2}\ln 2\pi + \mathcal{O}(z^{-1})\\ \Gamma(1+z) &\overset{z\to\infty}{\sim} \omega(z) + \frac{1}{2}\ln z + \frac{1}{2}\ln 2\pi + \mathcal{O}(z^{-1})\end{aligned} \tag{4.6}$$

while keeping leading as well as next to leading order terms gives the desired result. The leading order terms $\omega(\Sigma)$ enter the effective twisted superpotential $\widetilde{\mathcal{W}}_{\text{eff}}$, whereas the next to leading logarithm terms form an integration measure $\mu_{\text{msr}}(\Sigma)$. All in all, we end up with the following expression

$$Z^{S^2} \overset{r\to\infty}{\sim} \left|\int d\Sigma \mu_{\text{msr}}(\Sigma) e^{-\widetilde{\mathcal{W}}_{\text{eff}}(\Sigma)}\right|^2. \tag{4.7}$$

The effective twisted superpotential $\widetilde{\mathcal{W}}_{\text{eff}}$ is given quite in general (having quiver theories in mind) by the following formula

$$\begin{aligned}\widetilde{\mathcal{W}}_{\text{eff}}(\Sigma) = & \sum_{a=1}^{\#\text{nodes}} 2\pi t_a^{\text{ren}} \sum_{I=1}^{k_a} ir\Sigma_I^{(a)} \\ & + \sum_{\Phi} \sum_{w\in R_\Phi^{G_{\text{g}}}} \sum_{\widetilde{w}\in R_\Phi^{G_{\text{f}}}} \omega\Big[- irw(\Sigma) - ir\widetilde{w}(\Sigma^{\text{ext}})\Big]. \end{aligned} \tag{4.8}$$

We focused on a situation where the quiver diagram is composed of nodes corresponding to unitary gauge groups, each having an associated complexified FI parameter t_a^{ren}; the pairing $\widetilde{w}(\Sigma^{\text{ext}})$ is prescribing twisted masses and is fixed by the representation of the flavor group corresponding to the field Φ. Generalization to other situations is fairly straightforward.

Having extracted the effective twisted superpotential from a given gauge theory, we reached the stage when Gauge/Bethe correspondence can be applied. $\widetilde{\mathcal{W}}_{\text{eff}}(\Sigma)$ is to be identified with a Yang–Yang function of an associated quantum integrable system. The equations for supersymmetric vacua

$$\frac{\partial \widetilde{\mathcal{W}}_{\text{eff}}(\Sigma)}{\partial(ir\Sigma_s)} = 2\pi i n_s, \quad n_s \in \mathbb{Z} \tag{4.9}$$

coincide with Bethe ansatz equations (BAE). The spectrum of the commuting integrals of motion (IMs) in the integrable model can be expressed in terms of the gauge observables $\text{Tr}\,(\Sigma^n)$ evaluated at the solution to BAE

$$\text{spectrum of IMs} \longleftrightarrow \text{Tr}\,(\Sigma^n)\Big|_{\text{solution BAE}}, \tag{4.10}$$

however the functional relation is not given by the correspondence and is difficult to establish. Typically it comes as an observation when both sides are known.

We can also study the saddle point approximation to (4.7). The contribution from a given Bethe root $\Sigma_*^{(a)}$, $a = 1, \ldots, (\#\text{solutions to BAE})$ yields

$$Z_{(a)}^{S^2} \approx \left| e^{-\widetilde{\mathcal{W}}_{\text{eff}}(\Sigma)} \mu_{\text{msr}}(\Sigma) \left(\det \frac{\partial^2 \widetilde{\mathcal{W}}_{\text{eff}}(\Sigma)}{\partial\Sigma_s \partial\Sigma_t} \right)^{-\frac{1}{2}} \right|^2_{\Sigma=\Sigma_*^{(a)}}. \tag{4.11}$$

The complete partition function is then the sum of contributions from all saddle points

$$Z^{S^2} \approx \sum_{a=1}^{\#\text{sol. BAE}} Z_{(a)}^{S^2}. \tag{4.12}$$

In many situations it holds true that the norm of the Bethe eigenstate $\Psi^{(a)}$ is expressed (up to to term $|e^{-\widetilde{\mathcal{W}}_{\text{eff}}(\Sigma_*)^{(a)}}|^2$) using the semiclassical approximation

$$\frac{1}{\langle\Psi^{(a)}|\Psi^{(a)}\rangle} = \left|\mu_{\text{msr}}(\Sigma)\left(\det\frac{\partial^2\widetilde{\mathcal{W}}_{\text{eff}}(\Sigma)}{\partial\Sigma_s\partial\Sigma_t}\right)^{-\frac{1}{2}}\right|^2_{\Sigma=\Sigma_*^{(a)}}. \tag{4.13}$$

To get more familiar with the Gauge/Bethe correspondence we first investigate a simple example–the $O(4)$-sigma model. However, the master example of this thesis–*a gauge theory with target manifold the moduli space of instantons for a classical gauge group*–will be introduced in the next chapter. All concepts introduced thus far (quantum cohomology, quantum integrability) will be analyzed and in a certain sense unified for this class of theories. The associated integrable system will turn out to be rather interesting! But simple things first, we finish our small advertisement here, and move to the current example.

Example: $SO(4)$-sigma model

The physical significance of the $SO(4)$-sigma model rests in the fact that it is a toy model for studying integrability of AdS/CFT correspondence. Indeed, it is a reduction of the sigma model in $AdS^5\times S^5$ background to a subsector of strings moving in $\mathbb{R}\times S^3$. This model is formulated in terms of a $SU(2)$ principal chiral field $g=\sum_{a=1}^4 X_a\sigma^a$, where $\sigma^a=(\mathbb{I},i\tau^1,i\tau^2,i\tau^3)$ with τ^i the usual Pauli matrices and the constraint $\sum_{a=1}^4 X_a^2=1$ holds in order to make g $SU(2)$ valued. The action is given as

$$S=const.\int d\sigma d\tau\left[\frac{1}{2}\text{Tr}\left\{\left(g^{-1}\partial_A g\right)^2\right\}+\left(\partial_A X_0\right)^2\right]. \tag{4.14}$$

Here $A=\sigma,\tau$ while X_0 is a coordinate of $\mathbb{R}$ (global time in the original AdS^5 space) and X_a are the embedding coordinates of S^3 in $\mathbb{R}^4$. The action has a global $SU(2)_L\times SU(2)_R\simeq SO(4)$ symmetry.

We start from a formulation of this model as an integrable system specified by its Bethe equations and design a corresponding GLSM. We do not intend to describe this model in detail or attempt to solve it, our goal is only to write down the Yang–Yang function and interpret it from a gauge theory perspective. The quantum state of this model is described by a system of L particles in $SO(4)$ vector representation (or $SU(2)_L\times SU(2)_R$ bi-fundamental) of mass m_0 on a circle of radius $\mathcal{L}$; it is convenient to define a dimensionless parameter $\mu=m_0\mathcal{L}$. The number of $SU(2)_L$ excitations (spin flips) is J_u while the number of $SU(2)_R$ excitations is J_v. The scattering of the particles is governed by the following set of Bethe equations [51]

$$
\begin{aligned}
1 &= e^{i\mu\sinh(\pi\theta_\alpha)}\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{L} S_0^2(\theta_\alpha-\theta_\beta)\prod_{j=1}^{J_u}\frac{\theta_\alpha-u_j+\frac{i}{2}}{\theta_\alpha-u_j-\frac{i}{2}}\prod_{k=1}^{J_v}\frac{\theta_\alpha-v_k+\frac{i}{2}}{\theta_\alpha-v_k-\frac{i}{2}}; \quad \alpha=1,\dots,L\\
1 &= \prod_{\beta=1}^{L}\frac{u_j-\theta_\beta+\frac{i}{2}}{u_j-\theta_\beta-\frac{i}{2}}\prod_{\substack{i=1\\ i\neq j}}^{J_u}\frac{u_j-u_i-i}{u_j-u_i+i}, \quad j=1,\dots,J_u\\
1 &= \prod_{\beta=1}^{L}\frac{v_k-\theta_\beta+\frac{i}{2}}{v_k-\theta_\beta-\frac{i}{2}}\prod_{\substack{l=1\\ l\neq k}}^{J_v}\frac{v_k-v_l-i}{v_k-v_l+i}, \quad k=1,\dots,J_v,
\end{aligned} \tag{4.15}
$$

where the factor $S_0(\theta)$ is given by ratios of Gamma functions

$$
S_0(\theta) = i\frac{\Gamma\left(-\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2i}\right)}{\Gamma\left(\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}-\frac{\theta}{2i}\right)}. \tag{4.16}
$$

Exactly because of this factor the Bethe equations are quite unusual and at the same time interesting. We have three different types of Bethe roots $X := (\theta, u, v)$, so already at this stage we can predict that the quiver diagram of the gauge theory will have three gauge nodes. Taking the logarithm of (4.15) and denoting the resulting right hand sides collectively as R_I, $I = (\alpha, j, k)$ one can show that the associated 1-form $R = R_I dX^I$ is closed, i.e. $\partial_J R_I - \partial_I R_J = 0$, and therefore a potential function $Y(\theta, u, v)$ exists, such that $R = dY$ (at least locally). The function Y is called the Yang–Yang function. However, beware since a small warning is in order. The Bethe equations were presented in a form such that R was closed. But inverting e.g. the second set of Bethe equations for the u-roots leads to a perfectly valid system of BAE, though the condition of closedness breaks down. Indeed the form of Bethe equations in (4.15) differs slightly from the ones presented in [51] by such equivalence operations. The take away message is that in general one has to do some easy manipulations on the BAE before trying to look for the potential Y.

In order to make contact with gauge theory, specifically to arrive at the particular structure for the Yang–Yang function in (4.15), one has to use a product formula for the Gamma function

$$
\Gamma(x) = \frac{1}{x}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^x}{1+\frac{x}{n}}. \tag{4.17}
$$

Employing this identity we have (remember that we are working with $\log$ BAE)

$$
\sum_{\beta=1}^{L}\log S_0(\theta_\alpha-\theta_\beta) = \frac{3}{2}i\pi L + \sum_{n=1}^{\infty}\sum_{\beta=1}^{L}\log\frac{\theta_{\alpha\beta}-i(2n-1)}{-\theta_{\alpha\beta}-i(2n-1)} + \log\frac{\theta_{\alpha\beta}+i2n}{-\theta_{\alpha\beta}+i2n}, \tag{4.18}
$$

where $\theta_{\alpha\beta} := \theta_\alpha - \theta_\beta$. The explicit expressions for the components of the 1-form R can be stated after minor algebraic manipulations as

$$R_\alpha = i\mu\sinh(\pi\theta_\alpha) + i\pi(1+3L+J_u+J_v) + \sum_{j=1}^{J_u}\log\frac{\theta_\alpha - u_j + \frac{i}{2}}{-(\theta_\alpha - u_j) + \frac{i}{2}} \tag{4.19}$$
$$+ \sum_{k=1}^{J_v}\log\frac{\theta_\alpha - v_k + \frac{i}{2}}{-(\theta_\alpha - v_k) + \frac{i}{2}} + 2\sum_{n=1}^{\infty}\sum_{\beta=1}^{L}\left[\log\frac{\theta_{\alpha\beta} - i(2n-1)}{-\theta_{\alpha\beta} - i(2n-1)} + \log\frac{\theta_{\alpha\beta} + i2n}{-\theta_{\alpha\beta} + i2n}\right]$$

$$R_j = i\pi(1+L+J_u) + \sum_{\beta=1}^{L}\log\frac{u_j - \theta_\beta + \frac{i}{2}}{-(u_j - \theta_\beta) + \frac{i}{2}} + \sum_{l=1}^{J_u}\log\frac{u_{jl} - i}{-u_{jl} - i} \tag{4.20}$$

$$R_k = i\pi(1+L+J_v) + \sum_{\beta=1}^{L}\log\frac{v_k - \theta_\beta + \frac{i}{2}}{-(v_k - \theta_\beta) + \frac{i}{2}} + \sum_{l=1}^{J_v}\log\frac{v_{kl} - i}{-v_{kl} - i}. \tag{4.21}$$

A standard procedure results in the potential function $Y(\theta, u, v)$, i.e. the Yang–Yang function of the integrable model or the effective twisted superpotential for a related gauge theory

$$\begin{aligned}
Y(\theta,u,v) = {} & i\pi(1+3L+J_u+J_v)\sum_{\alpha=1}^{L}\theta_\alpha + i\pi(1+L+J_u)\sum_{j=1}^{J_u}u_j \\
& + i\pi(1+L+J_v)\sum_{k=1}^{J_v}v_k + \frac{i}{\pi}\mu\sum_{\alpha=1}^{L}\cosh(\pi\theta_\alpha) \\
& + \sum_{\alpha=1}^{L}\sum_{j=1}^{J_u}\left[\omega\left(\theta_\alpha - u_j + \frac{i}{2}\right) + \omega\left(-(\theta_\alpha - u_j) + \frac{i}{2}\right)\right] \\
& + \sum_{\alpha=1}^{L}\sum_{k=1}^{J_v}\left[\omega\left(\theta_\alpha - v_k + \frac{i}{2}\right) + \omega\left(-(\theta_\alpha - v_k) + \frac{i}{2}\right)\right] \\
& + 2\sum_{n=1}^{\infty}\sum_{\alpha,\beta=1}^{L}\left[\omega\left(\theta_{\alpha\beta} - i(2n-1)\right) + \omega\left(\theta_{\alpha\beta} + i2n\right)\right] \\
& + \sum_{j,m=1}^{J_u}\omega(u_{jm} - i) + \sum_{k,l=1}^{J_v}\omega(v_{kl} - i) + \text{const.}
\end{aligned} \tag{4.22}$$

Let us extract from this function the gauge theory content. As anticipated, the gauge group has three factors $G = U(J_u)\times U(J_v)\times U(L)$. The first two lines fix the FI terms associated to the individual unitary groups. Actually, the $\cosh(\cdot)$ term does not have the desired structure, but for the purpose of this example let us just assume the particles to be massless ($m_0 = 0$), such that this term vanishes. The third line represents a contribution from a bifundamental field with respect to $U(J_u)\times U(L)$ with twisted mass $\frac{i}{2}$, while the fourth is a bifundamental for $U(J_v)\times U(L)$ and the same twisted mass. The fifth line corresponds to an infinite number of adjoint fields for $U(L)$, there are two fields with a twisted mass $-i(2n-1)$ and two fields with $i(2n)$, $n = 1,\ldots,\infty$. Finally the last line realizes contributions of $U(J_u)$ (resp. $U(J_v)$) adjoint fields with twisted mass $-i$. The quiver diagram of this theory is shown in Figure 4.1.

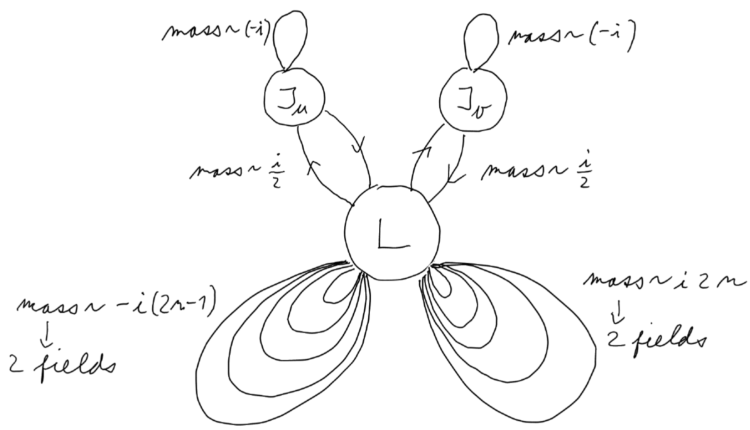

FIGURE 4.1: Quiver diagram for a gauge theory associated to the $O(4)$-sigma model.

Chapter 5

$\mathcal{N} = (2,2)$ GLSMs with target spaces the k-instanton moduli spaces for classical gauge groups

ADHM construction of instanton moduli spaces

We do not attempt to give here a thorough overview of constructions of instanton moduli spaces. Merely, this introduction should serve as a basic summary of important facts. The interested reader is encouraged to check other references, which are good in our opinion [52, 53, 54]. The elementary data entering the ADHM construction can be found in Table 5.1. We have two auxiliary vector spaces V and W together with maps

G	G_D	V	W
$U(N)$	$U(k)$	$\mathbb{C}^k$	$\mathbb{C}^N$
$O(N)$	$Sp(k)$	$\mathbb{C}^{2k}$	$\mathbb{R}^N$
$Sp(N)$	$O(k)$	$\mathbb{R}^k$	$\mathbb{C}^{2N}$

TABLE 5.1: Summary of ADHM data for classical gauge groups G.

$(B_1,\ B_1,\ I,\ J)$ between them defined as

$$x = (B_1,\ B_2,\ I,\ J) \in \mathcal{S}_{k,N} := \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W).$$

The group action of G and G_D on a point $x \in \mathcal{S}_{k,N}$ is as follows

$$(B_1,\ B_2,\ I,\ J) \mapsto \left(g_D \cdot B_1 \cdot g_D^{-1},\ g_D \cdot B_2 \cdot g_D^{-1},\ g \cdot I \cdot g_D^{-1},\ g_D \cdot J \cdot g^{-1}\right).$$

The scheme of vector spaces together with maps between them as well as natural group actions is visualized in Figure 5.1

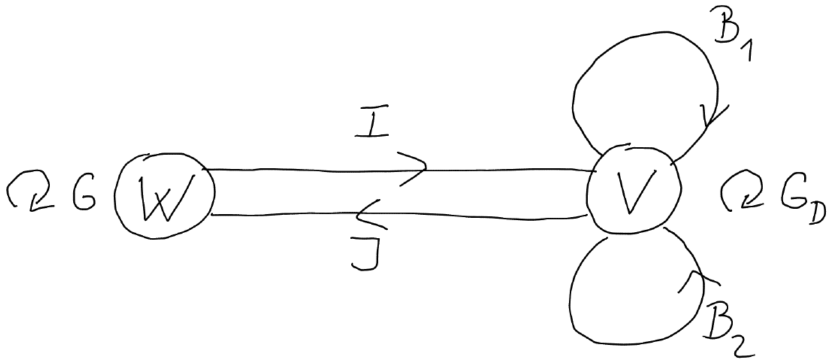

FIGURE 5.1: Quiver diagram for the ADHM construction encoding maps between vector spaces.

The essential part of the construction is the introduction of moment maps

$$\mu_{\mathbb{C}} = IJ + [B_1, B_2] \tag{5.1}$$

$$\mu_{\mathbb{R}} = \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J, \tag{5.2}$$

which are covariant under the group action. This presentation is valid for unitary groups. However, other classical groups can be embedded into this formalism choosing suitably the rank and as a next step imposing additional constraints on the maps. We will list these when handling various cases. Then the moduli space of k G-instantons can be represented as a quotient of the space of solutions to the moment maps by the dual group G_D

$$\mathcal{M}_{k,N} = \{x \in \mathcal{S}_{k,N} \mid (\mu_{\mathbb{C}} = 0) \wedge (\mu_{\mathbb{R}} = 0)\}/G_D$$

GLSM point of view

Now, we would like to construct a gauged linear sigma model such that $\mathcal{M}_{k,N}$ is interpreted as a vacuum manifold of this model. Hence, it is necessary to give a definition of the vacuum manifold, possibly of the same nature as in the construction of $\mathcal{M}_{k,N}$. Luckily, there is one precisely fitting our purpose. The space $\mathcal{X}$ of supersymmetric vacua on the Higgs branch is given by the constant vacuum expectation values (VEVs) for bosonic fields minimizing the scalar potential, i.e. solving the F- and D- term equations, modulo the action of the gauge group $G_{\text{gauge}}^{\text{GLSM}}$ (we will relate it to G and G_D very soon). So, when we denote dynamic scalar fields collectively as Φ, the definition reads

$$\mathcal{X} = \{\langle\Phi\rangle \mid F = 0 \wedge D = 0\}/G_{\text{gauge}}^{\text{GLSM}}.$$

At this point the relation of these two constructions should be pretty clear. To sum up, let us provide a dictionary in Table 5.2.

ADHM	**GLSM**
space of fields Φ	space of maps $\mathcal{S}$
G	flavor group G_F^{GLSM}
G_D	gauge group $G_{\mathrm{gauge}}^{\mathrm{GLSM}}$
$\mu_{\mathbb{C}}$	F-term
$\mu_{\mathbb{R}}$	D-term

TABLE 5.2: Dictionary between ADHM and GLSM constructions.

In particular, we see that the constraint coming from the moment map $\mu_{\mathbb{C}}$ will be imposed by a superpotential in the Lagrangian of the GLSM. Technically, this is done by introducing a Lagrange multiplier χ, i.e. a non-dynamical field that does not occur in the space of maps in the ADHM construction. Concretely, we add a superpotential term to the Lagrangian of the form

$$\mathcal{W} = \mathrm{Tr}_V\left(\chi\mu_{\mathbb{C}}\right). \tag{5.3}$$

The relation between D-terms and the real moment map still needs to be discussed. The claim is

$$\mu_{\mathbb{R}} = D^c T_c^{[L(\omega_1)]}, \tag{5.4}$$

where

- D^c are the D-terms, $c = 1, \ldots, \dim \mathrm{Lie}(G_D)$
- $T_c^{[L(\omega_1)]}$ are generators of $\mathrm{Lie}(G_D)$ in the standard representation $L(\omega_1) \simeq V$ corresponding to the fundamental weight ω_1

Once constructed the two dimensional $\mathcal{N} = (2,2)$ auxiliary GLSMs, corresponding to G-instanton moduli spaces, the ultimate goal is to write down the S^2 partition function (2.52) for them. This is a fairly easy task, since it is determined purely by the group theory data just indicated. Now, we move to individual classical groups and make explicit the program outlined above.

5.1 GLSM with target the $U(N)$ k-instanton moduli space

There is no need to impose constraints in this case, the construction goes through precisely as described. However, it is worth mentioning that we work with a deformed real moment map. Instead of forcing it to vanish we require

$$\mu_{\mathbb{R}} = \xi \mathbb{I}. \tag{5.5}$$

Hermiticity of $\mu_{\mathbb{R}}$ implies $\xi \in \mathbb{R}$. On the GLSM side ξ corresponds to turning on a Fayet-Iliopoulos term[1]. The purpose of this deformation is to cure non-compactness of the moduli space caused by so called point-like instantons (region of topological instanton charge shrinking to zero size). In the original four dimensional gauge theory, it means to consider instantons on a non-commutative spacetime $\mathbb{R}^4$, where the amount of non-commutativity is measured by ξ [55].

5.1.1 Brane construction in type II string theory

The analysis of the instanton moduli space $\mathcal{M}_{k,N}$ for unitary gauge groups is based on [43]. It can be described by a system of k Dp – N D$(p+4)$ branes in type II string theory on $\mathbb{C}^2 \times T^*S^2 \times \mathbb{C}$ [56, 57]. The cotangent bundle of the two-sphere T^*S^2 comes equipped with an asymptotically lovally Eucledian metric, forming thus the Eguchi–Hanson space, which can bee also seen as resolution of the singularity at the fixed point of $\mathbb{C}^2/\mathbb{Z}_2$. Specifically, the Higgs branch of the moduli space of classical supersymmetric vacua for such a theory coincides with $\mathcal{M}_{k,N}$ as a manifold. We will focus on the case when $p = 1$, i.e. we work with a system of D1-D5 branes.

The D5 branes are extended along $\mathbb{C}^2$ and they also wrap the sphere S^2 while the D1 branes are wrapping S^2. We will (almost exclusively) study the low energy effective theory on the D1 branes, which naturally leads to a sigma model on S^2. The main tool for investigating this GLSM will be the partition function on S^2 (2.52). As we showed in previous chapters, it captures the equivariant quantum cohomology of the target space and also provides a connection to quantum integrable systems.

Let us work out the matter content of the low energy $U(k)$ gauge theory on the D1 branes. The fields in the GLSM on S^2 arise from open strings stretching between D1 – D1 and D1 – D5 branes. From the dynamics of D1 branes arises a vector multiplet and adjoint chiral multiplets B_1, B_2. In addition there are N fundamental and N antifundamental

[1] This indeed implies that we are dealing with a semisimple Lie algebra $A_{N-1} \oplus \mathfrak{u}(1)$, where ξ corresponds to the $\mathfrak{u}(1)$ part.

chiral multiplets emerging from strings connecting D1 with D5 branes. In order to impose the complex moment map condition (5.1), we need to introduce a superpotential

$$\mathcal{W} = \mathrm{Tr}_V \big\{ \chi \left([B_1, B_2] + IJ \right) \big\}, \tag{5.6}$$

where χ is a non-dynamical field in the adjoint and serves merely as a Lagrange multiplier. The field content is conveniently summarized in Table 5.3. The R-charge as-

	χ	B_1	B_2	I	J
D-brane sector	D1/D1	D1/D1	D1/D1	D1/D5	D5/D1
$G^{\text{gauge}}_{\text{GLSM}} = G_D = U(k)$	Ad	Ad	Ad	$\mathbf{k}$	$\overline{\mathbf{k}}$
$G_F = G \times U(1)^2 = U(N) \times U(1)^2$	$\mathbf{1}_{(-1,-1)}$	$\mathbf{1}_{(1,0)}$	$\mathbf{1}_{(0,1)}$	$\overline{\mathbf{N}}_{(\frac{1}{2},\frac{1}{2})}$	$\mathbf{N}_{(\frac{1}{2},\frac{1}{2})}$
twisted masses	$\epsilon_1 + \epsilon_2$	$-\epsilon_1$	$-\epsilon_2$	$-a_j - \frac{\epsilon}{2}$	$a_j - \frac{\epsilon}{2}$
R-charge	$2 - 2q$	q	q	$q + p$	$q - p$

TABLE 5.3: Matter content of the ADHM GLSM with target space the instanton moduli space of $U(N)$ instantons.

signment clearly satisfies the requirement for the superpotential to have total charge 2 and is chosen in order to move poles away from the real axes in the expression for the partition function. For the purpose of separation of poles we impose $1 > q > p > 0$. Later on we will be quite sloppy about this technical detail, but one should keep in mind that it is needed to appropriately define the contour integrals.

5.1.2 S^2 partition function for $U(N)$-ADHM GLSM

The partition function for this model is given by

$$Z^{S^2}_{U(k)} = \frac{1}{k!} \sum_{\{m_1,\ldots,m_k\} \in \mathbb{Z}^k} \int_{\mathbb{R}^k} \prod_{s=1}^{k} \frac{d(r\sigma_s)}{2\pi} e^{-4\pi i \xi r \sigma_s - i\theta_{\text{ren}} m_s} Z_{\text{VM}} Z_J Z_I Z_\chi Z_{B_1} Z_{B_2} \tag{5.7}$$

and the contributions from individual fields can be easily worked using Appendix A. Using the notation $\epsilon = \epsilon_1 + \epsilon_2$, $\sigma_{st} = \sigma_s - \sigma_t$ and similarly $m_{st} = m_s - m_t$ the expressions are

$$Z_{\text{VM}} = \prod_{s<t}^{k} \left(\frac{m_{st}^2}{4} + r^2 \sigma_{st}^2 \right) \tag{5.8}$$

$$Z_I = \prod_{s=1}^{k} \prod_{j=1}^{N} \frac{\Gamma\left(-ir\sigma_s + ir(a_j + \frac{\epsilon}{2}) - \frac{m_s}{2}\right)}{\Gamma\left(1 + ir\sigma_s - ir(a_j + \frac{\epsilon}{2}) - \frac{m_s}{2}\right)} \tag{5.9}$$

$$Z_J = \prod_{s=1}^{k}\prod_{j=1}^{N} \frac{\Gamma\left(ir\sigma_s + ir(-a_j+\frac{\epsilon}{2}) + \frac{m_s}{2}\right)}{\Gamma\left(1 - ir\sigma_s - ir(-a_j+\frac{\epsilon}{2}) + \frac{m_s}{2}\right)} \tag{5.10}$$

$$Z_\chi = \prod_{s,t=1}^{k} \frac{\Gamma\left(1 - ir\sigma_{st} - ir\epsilon - \frac{m_{st}}{2}\right)}{\Gamma\left(ir\sigma_{st} + ir\epsilon - \frac{m_{st}}{2}\right)} \tag{5.11}$$

$$Z_{B_1} = \prod_{s,t=1}^{k} \frac{\Gamma\left(-ir\sigma_{st} + ir\epsilon_1 - \frac{m_{st}}{2}\right)}{\Gamma\left(1 + ir\sigma_{st} - ir\epsilon_1 - \frac{m_{st}}{2}\right)} \tag{5.12}$$

$$Z_{B_2} = Z_{B_1}\Big|_{\epsilon_1\to\epsilon_2}. \tag{5.13}$$

Let us study now the *point like limit* $r \to 0$, when the sphere shrinks to zero size. In this situation a D(-1) – D3 brane system remains, which is a setting considered in [10, 11]. Therefore our expectation is that in the point like limit the partition function Z^{S^2} reduces to the instanton partition function for pure super Yang–Mills (SYM) theory based on unitary gauge group

$$Z_k^{\text{inst}}(U(N)) = \frac{1}{k!}\frac{\epsilon^k}{(2\pi i\epsilon_1\epsilon_2)^k}\oint\prod_{s=1}^{k}\frac{d\sigma_s}{P(\sigma_s)P(\sigma_s+\epsilon)}\prod_{s<t}^{k}\frac{\sigma_{st}^2(\sigma_{st}^2-\epsilon^2)}{(\sigma_{st}^2-\epsilon_1^2)(\sigma_{st}^2-\epsilon_2^2)}, \tag{5.14}$$

where we have defined $P(\sigma_s) = \prod_{j=1}^N(\sigma_s - a_j - \frac{\epsilon}{2})$. Analyzing the Laurent expansion of Z^{S^2} around $r = 0$ one discovers that the leading order term really gives the wanted result

$$Z_{U(k)}^{S^2} \overset{r\to 0}{\sim} \frac{1}{r^{2kN}} Z_k^{\text{inst}}(U(N)) + \text{higher order terms in } r \tag{5.15}$$

We will reveal that for an arbitrary classical group, the power of the radius for the reduced partition function is $r^{-2h^\vee k}$, where $h^\vee$ is the dual Coxeter number of the group G. For unitary, symplectic and orthogonal groups these numbers take the following values: $h^\vee(U(N)) = N, h^\vee(Sp(N)) = N+1, h^\vee(O(N)) = N-2$. So if we introduce a formal counting parameter Λ_6 and define $Z_{\text{tot}}^{S^2} = \sum_k \Lambda_6^k Z_k^{S^2}$ ($Z_k^{S^2}$ was denoted $Z_{G_D}^{S^2}$) we have

$$Z_{\text{tot}}^{S^2} \overset{r\to 0}{\sim} \sum_k \Lambda_6^k r^{-2h^\vee k} Z_k^{\text{inst}} = \sum_k \Lambda_4^{2h^\vee k} Z_k^{\text{inst}} := Z^{\text{inst}}. \tag{5.16}$$

For this to hold the scaling $\Lambda_6 = (r\Lambda_4)^{2h^\vee}$ has to be imposed.

Next we move to study the opposite region, the *flat space limit*. As was discussed in Chapter 3, the asymptotic expansion around $r \to \infty$ provides us in particular with the effective twisted superpotential of the associated mirror Landau–Ginzburg model. Keeping also the next-to-leading term yields

$$Z_{U(N)}^{S^2} \overset{r\to\infty}{\sim} \frac{1}{k!}\left(\frac{\epsilon}{r\epsilon_1\epsilon_2}\right)^k \left|\int\prod_{s=1}^{k}\frac{d(r\Sigma_s)}{2\pi}\left(\frac{\prod_{t\neq s}^k D(\Sigma_{st})}{\prod_{s=1}^k Q(\Sigma_s)}\right)^{\frac{1}{2}} e^{-\widetilde{\mathcal{W}}_{\text{eff}}(\Sigma)}\right|^2 \tag{5.17}$$

with the integration measure

$$Q(\Sigma_s) = r^{2N} \prod_{j=1}^{N} \left(\Sigma_s - a_j - \frac{\epsilon}{2}\right)\left(-\Sigma_s + a_j - \frac{\epsilon}{2}\right), \quad D(\Sigma_{st}) = \frac{(\Sigma_{st})(\Sigma_{st}+\epsilon)}{(\Sigma_{st}-\epsilon_1)(\Sigma_{st}-\epsilon_2)}. \tag{5.18}$$

The formula for the effective twisted superpotential reads

$$\begin{aligned}\widetilde{\mathcal{W}}_{\text{eff}}(\Sigma) = &\Big(2\pi t - i(k-1)\pi\Big)\sum_{s=1}^{k} ir\Sigma_s \\ &+ \sum_{s=1}^{k}\sum_{j=1}^{N}\left[\omega\left(ir\Sigma_s - ira_j - ir\frac{\epsilon}{2}\right) + \omega\left(-ir\Sigma_s + ira_j - ir\frac{\epsilon}{2}\right)\right] \\ &+ \sum_{s,t=1}^{k}\Big[\omega\Big(ir\Sigma_{st} + ir\epsilon\Big) + \omega\Big(ir\Sigma_{st} - ir\epsilon_1\Big) + \omega\Big(ir\Sigma_{st} - ir\epsilon_2\Big)\Big].\end{aligned} \tag{5.19}$$

Identifying this potential with the Yang–Yang function one arrives to the Bethe ansatz equations (4.9)

$$\begin{aligned}&\prod_{j=1}^{N}\left(\Sigma_s - a_j - \frac{\epsilon}{2}\right)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\Sigma_{st}-\epsilon_1)(\Sigma_{st}-\epsilon_2)}{(\Sigma_{st})(\Sigma_{st}-\epsilon)} \\ &= e^{-2\pi t}\prod_{j=1}^{N}\left(-\Sigma_s + a_j - \frac{\epsilon}{2}\right)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(-\Sigma_{st}-\epsilon_1)(-\Sigma_{st}-\epsilon_2)}{(-\Sigma_{st})(-\Sigma_{st}-\epsilon)}.\end{aligned} \tag{5.20}$$

The proposal of [58] states that the above Yang–Yang function and Bethe equations describe the $gl(N)$ *periodic Intermediate Long Wave system.* Here we expose just few basic features, much more will be said in the next chapter. It is a system of hydrodynamic type and has two very interesting limits governed by the control parameter t (the exponential of it usually stands for a twist parameter in the integrability vocabulary). For $t \to \pm\infty$ we get the *Benjamin–Ono limit* (BO). In this limit the BAE are actually easy to solve as will be explained later. The other branch is when $t \to 0$, then one finishes in the *Korteweg–de Vries limit* (KdV). The KdV point in the parameter space is much harder to analyze.

5.2 GLSM with target the $Sp(N)$ k-instanton moduli space

The ADHM construction for symplectic groups of rank N can be embedded into the one for unitary groups for rank $2N$. Further, we have to impose some constraints. In this case they are given as

$$J = \begin{pmatrix} K \\ \\ \widetilde{K} \end{pmatrix}; \quad I^\dagger = \begin{pmatrix} -\widetilde{K}^* \\ \\ K^* \end{pmatrix}; \quad \begin{array}{l} K: \;\; N \times k \text{ matrix} \\ \\ \widetilde{K}: \;\; N \times k \text{ matrix} \end{array}$$
$$B_1,\ B_2: \; k \times k \text{ symmetric matrices,}$$

where we mean complex conjugation by star. Inserting these expressions into (5.1) and (5.2) yields

$$\mu_{\mathbb{C}} = K^T \widetilde{K} - \widetilde{K}^T K + [B_1, B_2] \tag{5.21}$$
$$\mu_{\mathbb{R}} = K^T K^* - K^\dagger K + \widetilde{K}^T \widetilde{K}^* - \widetilde{K}^\dagger \widetilde{K} + [B_1, B_1^*] + [B_2, B_2^*] \tag{5.22}$$

The GLSM is defined by the following matter content summarized in Table 5.4 together

	$G^{\text{GLSM}}_{\text{gauge}} = G_D = O(k)$	$G_F = G \times U(1)^2 = Sp(N) \times U(1)^2$
χ	$Ad = \bigwedge^2 L(\omega_1)$	$\mathbf{1}_{(-1,-1)}$
B_1	$\text{Sym}^2 L(\omega_1) = L(2\omega_1) \oplus \bigwedge^2 L(\omega_1)$	$\mathbf{1}_{(1,0)}$
B_2	$\text{Sym}^2 L(\omega_1) = L(2\omega_1) \oplus \bigwedge^2 L(\omega_1)$	$\mathbf{1}_{(0,1)}$
I	$L(\omega_1)$	$[L(\omega_1) \cap L(\omega_1)]_{\left(\frac{1}{2},\frac{1}{2}\right)}$
J	$L(\omega_1)$	$[L(\omega_1) \cap L(\omega_{2N-1})]_{\left(\frac{1}{2},\frac{1}{2}\right)}$

Table 5.4: Field content forming a GLSM with target space the $Sp(N)$-instanton moduli space.

with a superpotential of the form

$$\mathcal{W} = \text{Tr}_V \Big\{ \chi \left(K^T \widetilde{K} - \widetilde{K}^T K + [B_1, B_2] \right) \Big\}. \tag{5.23}$$

The second column for I and J in Table 5.4 needs perhaps some comments. We are looking at $Sp(N)$ as $Sp(N) \simeq Sp(2N, \mathbb{C}) \cap U(2N, \mathbb{C})$. Hence the first entry in the square bracket is the standard representation for C_N. For I, also the second one is the standard (fundamental) representation $L(\omega_1) \simeq W$ for $A_{2N-1} \oplus \mathfrak{u}(1)$ while for J it is the dual (anti-fundamental) representation $W^* \simeq \bigwedge^{2N-1} W \simeq L(\omega_{2N-1})$. The weights for $L(\omega_1)$ of C_N are $\{\underbrace{\beta_1, \ldots, \beta_N}, \underline{-\beta_1, \ldots, -\beta_N}\}$. On the other hand for W they are given as $\{\underbrace{\beta_1, \ldots, \beta_N}, \beta_{N+1}, \ldots, \beta_{2N}\}$ and for W^* as $\{\underline{-\beta_1, \ldots, -\beta_N}, -\beta_{N+1}, \ldots, -\beta_{2N}\}$. So in summary, as a result of the intersection, we are using the weights $\{\beta_1, \ldots, \beta_N\}$ for I and $\{-\beta_1, \ldots, -\beta_N\}$ for J, respectively.

To make sure that the proposed field content works fine we should check whether the resulting D-terms are really correct, i.e. we want to test (5.4). Clearly, $\mu_{\mathbb{R}}$ is a $k \times k$

matrix, moreover skew symmetric and purely imaginary, hence Hermitian. This implies $\mu_{\mathbb{R}} \in \mathrm{Lie}(O(k))$ with respect to the symmetric bilinear form that defines the canonical scalar product on V. Thus, it can be indeed expanded in a basis of $O(k)$ generators as claimed in (5.4). The generators are skew symmetric and we choose them to be purely imaginary and so the coordinates D^c are real, accordingly. In the first step the check will be provided for I, J and then for $B_{1,2}$. First of all, the completeness relation tells us

$$D^c = \frac{1}{\lambda}\mathrm{Tr}_V\left(\mu_{\mathbb{R}} T^c_{[V]}\right) \tag{5.24}$$

with $\mathrm{Tr}_V\left(T_a^{[V]}T_b^{[V]}\right) = \lambda\delta_{ab}$. λ will be fixed by computing the index of the standard representation $L(\omega_1)$, i.e. a relative normalization of quadratic bilinear products taken in different representations. Normalizing $\mathrm{Tr}\left(T_a^{(Ad)}T_b^{(Ad)}\right) = \delta_{ab}$ as is customary, the index formula gives us the answer

$$\mathrm{Tr}\left(T_a^{[L(\omega_1)]}T_b^{[L(\omega_1)]}\right) = \underbrace{\frac{\dim L(\omega_1)}{\dim\mathfrak{g}}(\omega_1,\omega_1+2\rho)}_{=2 \text{ for } \mathfrak{g}=B_l,\, D_l}\delta_{ab} \quad\Rightarrow\quad \lambda = 2. \tag{5.25}$$

In the above formula ρ is the Weyl vector. Applying these formulae to D-terms corresponding to I, J fields yields

$$D^c(I) = \frac{1}{2}\mathrm{Tr}_V\left(II^\dagger T^c_{[V]}\right) = \frac{1}{2}\mathrm{Tr}_W\left(I^\dagger T^c_{[V]}I\right) \tag{5.26}$$

$$D^c(J) = -\frac{1}{2}\mathrm{Tr}_V\left(J^\dagger J T^c_{[V]}\right) = -\frac{1}{2}\mathrm{Tr}_W\left(JT^c_{[V]}J^\dagger\right). \tag{5.27}$$

Thinking of K (the same for $\widetilde{K}$) as

$$K = \underbrace{\begin{pmatrix} — & K_1^T & — \\ — & K_2^T & — \\ & \vdots & \\ — & K_N^T & — \end{pmatrix}}_{k} \tag{5.28}$$

the above D-term equations can be simplified further

$$D^c(I) = \frac{1}{2}\sum_{A=1}^{N} \widetilde{K}_A^\dagger T^c_{[V]}\widetilde{K}_A + K_A^\dagger T^c_{[V]}K_A \tag{5.29}$$

$$D^c(J) = -\frac{1}{2}\sum_{A=1}^{N} \widetilde{K}_A^T T^c_{[V]}\widetilde{K}_A^* + K_A^T T^c_{[V]}K_A^* = \left[D^c(I)\right]^*, \tag{5.30}$$

$$\tag{5.31}$$

where in the second equality for $D^c(J)$ we used that $T^c_{[V]}$ are purely imaginary. This is the expected form of D-terms for $I,\ J$ as we wanted to show.

At this moment it is time to analyse the contribution of $B_{1,2}$. The derivation is the same for both, therefore we drop the index for neater notation. It was already emphasized that B has to be symmetric, thus we can expand it in a basis of $k \times k$ symmetric matrices S^I as $B = B_I S^I$, $I = 1, \ldots, \frac{k(k+1)}{2}$. Moreover, we choose S^I to be real, so the coordinates B_I are complex. Then the real moment map can be expressed as

$$\mu_{\mathbb{R}} = [B, B^*] = B_I \left(B^J\right)^* \left[S^I, S_J\right] \tag{5.32}$$

while the D-term contribution is of the form

$$D^c \left(T_c^{[L(\omega_1)]}\right)_i^{\ j} = \underbrace{\left(B^J\right)^* \left(T^c_{[\mathrm{Sym}^2 L(\omega_1)]}\right)_J^{\ I} B_I}_{D^c} \left(T_c^{[L(\omega_1)]}\right)_i^{\ j}. \tag{5.33}$$

To proceed further it is essential to simplify the object $\left(T^c_{[\mathrm{Sym}^2 L(\omega_1)]}\right)_J^{\ I} \left(T_c^{[L(\omega_1)]}\right)_i^{\ j}$. This expression is real and skew symmetric in the pairs of indices (I, J) and (i, j), respectively. Consequently, it is fixed uniquely by these symmetries (up to normalization)

$$\left(T^c_{[\mathrm{Sym}^2 L(\omega_1)]}\right)_J^{\ I} \left(T_c^{[L(\omega_1)]}\right)_i^{\ j} = \left(S^I\right)_i^{\ k} \left(S_J\right)_k^{\ j} - \left(S_J\right)_i^{\ k} \left(S^I\right)_k^{\ j} = \left[S^I, S_J\right]_i^{\ j}, \tag{5.34}$$

which leads us to the conclusion

$$D^c \left(T_c^{[L(\omega_1)]}\right)_i^{\ j} = B_I \left(B^J\right)^* \left[S^I, S_J\right]_i^{\ j} = \left(\mu_{\mathbb{R}}\right)_i^{\ j}. \tag{5.35}$$

This finishes the proof.

The above discussion justified the specification of the model given in Table 5.4 together with (5.23). We explicitly validated that the F- and D-terms agree with (5.1) and (5.2). This implies we have constructed our model in a correct way, specifically with a vacuum manifold the moduli space of k $Sp(N)$-instantons.

5.2.1 S^2 partition function for $Sp(N)$-ADHM GLSM

In this subsection we are going to study the S^2 partition function for this model. It naturally splits into two cases

$$Z^{S^2}_{O(k)} = \begin{cases} Z^{S^2}_{D_l} & \text{for } k = 2l \\ Z^{S^2}_{B_l} & \text{for } k = 2l+1 \end{cases}$$

Note, that we are marking the partition function by $O(k)$, i.e. the gauge group of the two dimensional gauged linear sigma model. As one can see from equation (2.52) it is fully specified by data contained in Table 5.4 together with basic input on Lie algebra theory summarized in Appendix A. The twisted masses (equivariant weights) of the maximal torus of $G_F = G \times U(1)^2$ will be denoted as $(a_j;\ \epsilon_1,\ \epsilon_2)$. Their values for individual fields can be read off from the second column of Table 5.4. We have $\epsilon = \epsilon_1 + \epsilon_2$ as usual and further define $\epsilon_+ = \frac{\epsilon_1+\epsilon_2}{2}$. Regarding the R-charges we have, roughly speaking, the assignment $(I, J, \chi, B_1, B_2) = (0, 0, 2, 0, 0)$. With these information at hand, it is easy to write down the S^2 partition function.

Even orthogonal gauge group

In this case the form of the partition function is governed by the D_l series

$$Z^{S^2}_{D_l} = \frac{1}{2^{l-1}l!} \sum_{\{m_1,\ldots,m_l\}\in\mathbb{Z}^l} \int_{\mathbb{R}^l} \prod_{s=1}^{l} \frac{d(r\sigma_s)}{2\pi} Z_{\text{VM}} Z_I Z_J Z_\chi Z_{B_1} Z_{B_2}, \tag{5.36}$$

where the contributions from various fields are listed below

$$Z_{\text{VM}} = \prod_{s<t}^{l} \left[(r\sigma_s - r\sigma_t)^2 + \frac{(m_s - m_t)^2}{4} \right] \left[(r\sigma_s + r\sigma_t)^2 + \frac{(m_s + m_t)^2}{4} \right] \tag{5.37}$$

$$Z_I = \prod_{j=1}^{N} \prod_{s=1}^{l} \frac{\Gamma\left(-ir\sigma_s - ir\left(a_j + \epsilon_+\right) - \frac{m_s}{2}\right)}{\Gamma\left(1 + ir\sigma_s + ir\left(a_j + \epsilon_+\right) - \frac{m_s}{2}\right)} \frac{\Gamma\left(ir\sigma_s - ir\left(a_j + \epsilon_+\right) + \frac{m_s}{2}\right)}{\Gamma\left(1 - ir\sigma_s + ir\left(a_j + \epsilon_+\right) + \frac{m_s}{2}\right)} \tag{5.38}$$

$$Z_J = \prod_{j=1}^{N} \prod_{s=1}^{l} \frac{\Gamma\left(-ir\sigma_s - ir\left(-a_j + \epsilon_+\right) - \frac{m_s}{2}\right)}{\Gamma\left(1 + ir\sigma_s + ir\left(-a_j + \epsilon_+\right) - \frac{m_s}{2}\right)} \frac{\Gamma\left(ir\sigma_s - ir\left(-a_j + \epsilon_+\right) + \frac{m_s}{2}\right)}{\Gamma\left(1 - ir\sigma_s + ir\left(-a_j + \epsilon_+\right) + \frac{m_s}{2}\right)} \tag{5.39}$$

$$\begin{aligned} Z_\chi = \prod_{s=1}^{l} \frac{\Gamma(1+ir\epsilon)}{\Gamma(-ir\epsilon)} \prod_{s<t}^{l} &\left\{ \frac{\Gamma\left(1 - ir(\sigma_s - \sigma_t) + ir\epsilon - \frac{m_s - m_t}{2}\right)}{\Gamma\left(ir(\sigma_s - \sigma_t) - ir\epsilon - \frac{m_s - m_t}{2}\right)} \right. \\ &\times \frac{\Gamma\left(1 + ir(\sigma_s - \sigma_t) + ir\epsilon + \frac{m_s - m_t}{2}\right)}{\Gamma\left(-ir(\sigma_s - \sigma_t) - ir\epsilon + \frac{m_s - m_t}{2}\right)} \frac{\Gamma\left(1 - ir(\sigma_s + \sigma_t) + ir\epsilon - \frac{m_s + m_t}{2}\right)}{\Gamma\left(ir(\sigma_s + \sigma_t) - ir\epsilon - \frac{m_s + m_t}{2}\right)} \\ &\left. \times \frac{\Gamma\left(1 + ir(\sigma_s + \sigma_t) + ir\epsilon + \frac{m_s + m_t}{2}\right)}{\Gamma\left(-ir(\sigma_s + \sigma_t) - ir\epsilon + \frac{m_s + m_t}{2}\right)} \right\} \end{aligned} \tag{5.40}$$

$$\begin{aligned} Z_{B_1} = \prod_{s=1}^{l} \frac{\Gamma(-ir\epsilon_1)}{\Gamma(1+ir\epsilon_1)} \prod_{s<t}^{l} &\left\{ \frac{\Gamma\left(-ir(\sigma_s - \sigma_t) - ir\epsilon_1 - \frac{m_s - m_t}{2}\right)}{\Gamma(\left(1 + ir(\sigma_s - \sigma_t) + ir\epsilon_1 - \frac{m_s - m_t}{2}\right)} \right. \\ &\times \frac{\Gamma\left(ir(\sigma_s - \sigma_t) - ir\epsilon_1 + \frac{m_s - m_t}{2}\right)}{\Gamma(\left(1 - ir(\sigma_s - \sigma_t) + ir\epsilon_1 + \frac{m_s - m_t}{2}\right)} \frac{\Gamma\left(-ir(\sigma_s + \sigma_t) - ir\epsilon_1 - \frac{m_s + m_t}{2}\right)}{\Gamma(\left(1 + ir(\sigma_s + \sigma_t) + ir\epsilon_1 - \frac{m_s + m_t}{2}\right)} \\ &\left. \times \frac{\Gamma\left(ir(\sigma_s + \sigma_t) - ir\epsilon_1 + \frac{m_s + m_t}{2}\right)}{\Gamma(\left(1 - ir(\sigma_s + \sigma_t) + ir\epsilon_1 + \frac{m_s + m_t}{2}\right)} \right\} \end{aligned}$$

$$\times \prod_{s=1}^{l} \frac{\Gamma(-ir2\sigma_s - ir\epsilon_1 - m_s)}{\Gamma(1 + ir2\sigma_s + ir\epsilon_1 - m_s)} \frac{\Gamma(ir2\sigma_s - ir\epsilon_1 + m_s)}{\Gamma(1 - ir2\sigma_s + ir\epsilon_1 + m_s)} \tag{5.41}$$

$$Z_{B_2} = Z_{B_1}|_{\epsilon_1 \to \epsilon_2} . \tag{5.42}$$

Odd orthogonal gauge group

As opposed to the previous situation, now we are dealing with the B_l algebra, which amounts just to take into account extra contributions arising from additional roots ($\pm\beta_i$) and a zero weight for the standard representation $L(\omega_1)$. If we write

$$Z^{S^2}_{O(2l)} = \sum_{\vec{m} \in \mathbb{Z}^l} \int_{\mathbb{R}^l} \prod_{s=1}^{l} \frac{d(r\sigma_s)}{2\pi} z_{O(2l)}\left(\sigma|\vec{a}, \epsilon_1, \epsilon_2\right) \tag{5.43}$$

then

$$z_{O(2l+1)}\left(\sigma|\vec{a}, \epsilon_1, \epsilon_2\right) = \left\{\text{extra}\right\} \cdot z_{O(2l)}\left(\sigma|\vec{a}, \epsilon_1, \epsilon_2\right) . \tag{5.44}$$

Contributions from individual fields to $\left\{\text{extra}\right\}$ are listed hereafter

$$\text{VM}: \quad \prod_{s=1}^{l} \left[(r\sigma_s)^2 + \frac{m_s^2}{4}\right] \quad \text{extra positive roots } \beta_1, \ldots, \beta_l$$

$$I: \quad \prod_{j=1}^{N} \frac{\Gamma\left(-ir(a_j + \epsilon_+)\right)}{\Gamma\left(1 + ir(a_j + \epsilon_+)\right)} \quad \text{extra zero weight for } L(\omega_1)$$

$$J: \quad \frac{\Gamma\left(-ir(-a_j + \epsilon_+)\right)}{\Gamma\left(1 + ir(-a_j + \epsilon_+)\right)} \quad \text{extra zero weight for } L(\omega_1)$$

$$\chi: \quad \prod_{s=1}^{l} \frac{\Gamma\left(1 - ir\sigma_s + ir\epsilon - \frac{m_s}{2}\right)}{\Gamma(\left(ir\sigma_s - ir\epsilon - \frac{m_s}{2}\right)} \frac{\Gamma\left(1 + ir\sigma_s + ir\epsilon + \frac{m_s}{2}\right)}{\Gamma(\left(-ir\sigma_s - ir\epsilon + \frac{m_s}{2}\right)} \quad \text{extra roots } \pm\beta_1, \ldots, \pm\beta_l$$

$$B_1: \quad \underbrace{\frac{\Gamma(-ir\epsilon_1)}{\Gamma(1 + ir\epsilon_1)}}_{\text{extra zero weight for } L(\omega_1)} \quad \underbrace{\prod_{s=1}^{l} \frac{\Gamma(\left(-ir\sigma_s - ir\epsilon_1 - \frac{m_s}{2}\right)}{\Gamma\left(1 + ir\sigma_s + ir\epsilon_1 - \frac{m_s}{2}\right)} \frac{\Gamma(\left(ir\sigma_s - ir\epsilon_1 + \frac{m_s}{2}\right)}{\Gamma\left(1 - ir\sigma_s + ir\epsilon_1 + \frac{m_s}{2}\right)}}_{\text{extra roots } \pm\beta_1, \ldots, \pm\beta_l}$$

$$B_2: \quad B_1|_{\epsilon_1 \to \epsilon_2}$$

$$|W|: \quad \frac{1}{2} \quad \text{extra factor in the order of the Weyl group}$$

In summary, we can certainly combine the two cases explored above and write a compact form for $O(k)$

$$Z^{S^2}_{O(k)} = \frac{1}{2^{\lfloor\frac{k}{2}\rfloor-1}\lfloor\frac{k}{2}\rfloor!}\sum_{\vec{m}\in\mathbb{Z}^{\lfloor\frac{k}{2}\rfloor}}\int_{\mathbb{R}^{\lfloor\frac{k}{2}\rfloor}}\prod_{s=1}^{\lfloor\frac{k}{2}\rfloor}\frac{d(r\sigma_s)}{2\pi}\{\text{extra}\}^{k_{\text{mod }2}}\, z_{O(2l)}\left(\sigma|\vec{a},\epsilon_1,\epsilon_2\right)\Big|_{l=\lfloor\frac{k}{2}\rfloor}. \tag{5.45}$$

The important thing to explain next is the limit when the radius of S^2 approaches zero. By the same arguments given for unitary groups, we expect the S^2 partition function to reduce to the instanton part of Nekrasov partition function for four dimensional $\mathcal{N}=2$ pure SYM theory based on $Sp(N)$ gauge group [59]. Performing the asymptotic expansion around $r\to 0$, the leading order term in fact provides the predicted observation

$$Z^{S^2}_{O(k)} \overset{r\to 0}{\sim} \frac{1}{r^{2k(N+1)}}Z^{\text{inst}}_k\left(Sp(N)\right) + \text{higher order terms in } r \tag{5.46}$$

Note that the factor $r^{-2k(N+1)}$ is the same as in the 4D limit of 5D instanton partition function on $\mathbb{C}^2\times S^1_r$ compactified on the circle.

On the other hand, investigating the $r\to\infty$ asymptotic expansion allows us to obtain the associated mirror Landau–Ginzburg model or more precisely its effective IR description, which is encoded in the effective twisted superpotential. Using the Stirling formula for Gamma functions it can be easily computed with the result (we remind the definition $\omega(x)=x(\log x-1)$)

$$\begin{aligned}
\widetilde{\mathcal{W}}_{eff} = k_{\text{mod }2}&\Bigg[\sum_{s=1}^{\lfloor\frac{k}{2}\rfloor}\omega(-ir\Sigma_s+ir\epsilon)+\omega(ir\Sigma_s+ir\epsilon)+\omega(-ir\Sigma_s-ir\epsilon_1)\\
&+\omega(ir\Sigma_s-ir\epsilon_1)+\omega(-ir\Sigma_s-ir\epsilon_2)+\omega(ir\Sigma_s-ir\epsilon_2)\Bigg]\\
&+\sum_{s=1}^{\lfloor\frac{k}{2}\rfloor}\sum_{j=1}^{N}\Big\{\omega(-ir\Sigma_s-ir(a_j+\epsilon_+))+\omega(ir\Sigma_s-ir(a_j+\epsilon_+))\\
&+\omega(-ir\Sigma_s-ir(-a_j+\epsilon_+))+\omega(ir\Sigma_s-ir(-a_j+\epsilon_+))\Big\}\\
&+\sum_{s<t}^{\lfloor\frac{k}{2}\rfloor}\Big\{\omega(-ir(\Sigma_s-\Sigma_t)+ir\epsilon)+\omega(ir(\Sigma_s-\Sigma_t)+ir\epsilon)\\
&+\omega(-ir(\Sigma_s+\Sigma_t)+ir\epsilon)+\omega(ir(\Sigma_s+\Sigma_t)+ir\epsilon)\Big\}\\
&+\sum_{s<t}^{\lfloor\frac{k}{2}\rfloor}\Big\{\omega(-ir(\Sigma_s-\Sigma_t)-ir\epsilon_1)+\omega(ir(\Sigma_s-\Sigma_t)-ir\epsilon_1)\\
&+\omega(-ir(\Sigma_s+\Sigma_t)-ir\epsilon_1)+\omega(ir(\Sigma_s+\Sigma_t)-ir\epsilon_1)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s=1}^{\lfloor\frac{k}{2}\rfloor}\Big\{\omega(-ir2\Sigma_s-ir\epsilon_1)+\omega(ir2\Sigma_s-ir\epsilon_1)\Big\}\\
&+\sum_{s<t}^{\lfloor\frac{k}{2}\rfloor}\Big\{\omega(-ir(\Sigma_s-\Sigma_t)-ir\epsilon_2)+\omega(ir(\Sigma_s-\Sigma_t)-ir\epsilon_2)\\
&+\omega(-ir(\Sigma_s+\Sigma_t)-ir\epsilon_2)+\omega(ir(\Sigma_s+\Sigma_t)-ir\epsilon_2)\Big\}\\
&+\sum_{s=1}^{\lfloor\frac{k}{2}\rfloor}\Big\{\omega(-ir2\Sigma_s-ir\epsilon_2)+\omega(ir2\Sigma_s-ir\epsilon_2)\Big\}.
\end{aligned}\tag{5.47}
$$

If we assume that the interpretation of the effective twisted superpotential as a Yang–Yang function of some integrable system still holds, the corresponding Bethe equations derived from $\widetilde{\mathcal{W}}_{eff}$ as

$$
\frac{\partial\widetilde{\mathcal{W}}_{eff}}{\partial ir\Sigma_s}=2\pi i n_s;\quad n_s\in\mathbb{Z}\tag{5.48}
$$

have the form ($s=1,\dots,\lfloor\frac{k}{2}\rfloor$ being the free index labeling individual equations of the system)

$$
\begin{aligned}
&\left[\frac{\Sigma_s+\epsilon}{\Sigma_s-\epsilon}\frac{\Sigma_s-\epsilon_1}{\Sigma_s+\epsilon_1}\frac{\Sigma_s-\epsilon_2}{\Sigma_s+\epsilon_2}\right]^{k_{\mathrm{mod}\ 2}}\frac{2\Sigma_s-\epsilon_1}{2\Sigma_s+\epsilon_1}\frac{2\Sigma_s-\epsilon_2}{2\Sigma_s+\epsilon_2}\prod_{j=1}^{N}\frac{\Sigma_s-(a_j+\epsilon_+)}{\Sigma_s+(a_j+\epsilon_+)}\frac{\Sigma_s-(-a_j+\epsilon_+)}{\Sigma_s+(-a_j+\epsilon_+)}\\
&=\prod_{t=1}^{\lfloor\frac{k}{2}\rfloor}\frac{\Sigma_s-\Sigma_t-\epsilon}{\Sigma_s-\Sigma_t+\epsilon}\frac{\Sigma_s-\Sigma_t+\epsilon_1}{\Sigma_s-\Sigma_t-\epsilon_1}\frac{\Sigma_s-\Sigma_t+\epsilon_2}{\Sigma_s-\Sigma_t-\epsilon_2}\prod_{\substack{t=1\\t\neq s}}^{\lfloor\frac{k}{2}\rfloor}\frac{\Sigma_s+\Sigma_t-\epsilon}{\Sigma_s+\Sigma_t+\epsilon}\frac{\Sigma_s+\Sigma_t+\epsilon_1}{\Sigma_s+\Sigma_t-\epsilon_1}\frac{\Sigma_s+\Sigma_t+\epsilon_2}{\Sigma_s+\Sigma_t-\epsilon_2}.
\end{aligned}\tag{5.49}
$$

Remark. There is no twist parameter present in the Bethe equations. It is so because in the GLSM we are dealing with a gauge group that has a simple Lie algebra, therefore does not admit a Fayet-Iliopoulus term. In the language of the ILW hydrodynamical integrable system studied before this means the model is frozen in the “KdV like phase”.

5.3 GLSM with target the $O(N)$ k-instanton moduli space

As opposed to the previous case of symplectic groups, now we are doing the embedding for G_D, concretely $Sp(k)\subset U(2k)$. Constraints needed to be imposed on the maps take the following form

$$
J=\Big(K\quad,\quad\widetilde{K}\Big);\quad I^\dagger=\Big(-\widetilde{K}^*\quad,\quad K^*\Big);\quad K,\ \widetilde{K}:\quad N\times k\ \text{matrices}
$$

$$B_{1,2} = \left(\begin{array}{c|c} P_{1,2} & \widetilde{Q}_{1,2} \\ \hline Q_{1,2} & P_{1,2}^T \end{array}\right); \quad Q_{1,2},\ \widetilde{Q}_{1,2} : \ k \times k \text{ skew symmetric matrices.}$$

Writing the moment maps in block form

$$\mu_{\mathbb{C},\mathbb{R}} = \left(\begin{array}{c|c} R_{\mathbb{C},\mathbb{R}} & \widetilde{S}_{\mathbb{C},\mathbb{R}} \\ \hline S_{\mathbb{C},\mathbb{R}} & -R_{\mathbb{C},\mathbb{R}}^T \end{array}\right)$$

and inserting the above expressions into (5.1) and (5.2) results in

$$R_{\mathbb{C}} = [P_1, P_2] + \widetilde{Q}_1 Q_2 - \widetilde{Q}_2 Q_1 - \widetilde{K}^T K \tag{5.50}$$

$$S_{\mathbb{C}} = Q_1 P_2 - P_2^T Q_1 + P_1^T Q_2 - Q_2 P_1 + K^T K \tag{5.51}$$

$$\widetilde{S}_{\mathbb{C}} = \widetilde{Q}_1 P_2^T - P_2 \widetilde{Q}_1 + P_1 \widetilde{Q}_2 - \widetilde{Q}_2 P_1^T - \widetilde{K}^T \widetilde{K} \tag{5.52}$$

for the complex moment map, while

$$R_{\mathbb{R}} = \sum_{a=1}^{2} \left(\left[P_a, P_a^\dagger \right] + Q_a^* Q_a - \widetilde{Q}_a \widetilde{Q}_a^* \right) + \widetilde{K}^T \widetilde{K}^* - K^\dagger K \tag{5.53}$$

$$S_{\mathbb{R}} = \sum_{a=1}^{2} \left(Q_a P_a^\dagger - P_a^* Q_a + \widetilde{Q}_a^* P_a - P_a^T \widetilde{Q}_a^* \right) - K^T \widetilde{K}^* - \widetilde{K}^\dagger K \tag{5.54}$$

$$\widetilde{S}_{\mathbb{R}} = \sum_{a=1}^{2} \left(\widetilde{Q}_a P_a^* - P_a^\dagger \widetilde{Q}_a + Q_a^* P_a^T - P_a Q_a^* \right) - \widetilde{K}^T K^* - K^\dagger \widetilde{K} \tag{5.55}$$

for the real. Note that $S_{\mathbb{C},\mathbb{R}}, \widetilde{S}_{\mathbb{C},\mathbb{R}}$ are symmetric $k \times k$ matrices. Therefore, $\mu_{\mathbb{C},\mathbb{R}} \in \mathrm{Lie}(Sp(2k,\mathbb{C}))$ with respect to the skew symmetric form $\left(\begin{array}{c|c} 0 & +\mathbb{I} \\ \hline -\mathbb{I} & 0 \end{array}\right)$ on V.

Let us motivate the field content, which will be summarized shortly. For instance, focus on B_1. Looking at the block form given above, we can count the number of independent components; it is $k^2 + \frac{k(k-1)}{2} + \frac{k(k-1)}{2} = k(2k-1)$. So, if we want to figure out the representation of $Sp(k)$ in which B_1 transforms, we know that its dimension must be equal to this number. Further, for $Sp(k)$ all irreducible representations are included in the tensor algebra of the standard representation V (unlike for orthogonal groups, where we have also spin representations). It is not hard to guess that the appropriate representation should be $\bigwedge^2 V$, indeed the dimension is $\frac{2k(2k-1)}{2} = k(2k-1)$ as was required. Now we are ready to define the gauged linear sigma model, its matter fields are encoded in Table 5.5. The notation for the weights of I and J with respect to G_D in the first column is analogous to that given in the symplectic case for G. One just needs to replace N by k in those arguments, so we will not repeat them here. Except for the

	$G^{\text{GLSM}}_{\text{gauge}} = G_D = Sp(k)$	$G_F = G \times U(1)^2 = O(N) \times U(1)^2$
χ	$Ad = \text{Sym}^2 L(\omega_1)$	$\mathbf{1}_{(-1,-1)}$
B_1	$\bigwedge^2 L(\omega_1) = L(\omega_2) \oplus \mathbb{C}$	$\mathbf{1}_{(1,0)}$
B_2	$\bigwedge^2 L(\omega_1) = L(\omega_2) \oplus \mathbb{C}$	$\mathbf{1}_{(0,1)}$
J	$[L(\omega_1) \cap L(\omega_1)]$	$L(\omega_1)_{(\frac{1}{2},\frac{1}{2})}$
I	$[L(\omega_1) \cap L(\omega_{2k-1})]$	$L(\omega_1)_{(\frac{1}{2},\frac{1}{2})}$

TABLE 5.5: Field content forming a GLSM with target space the $O(N)$-instanton moduli space.

specified matter content also a superpotential term has to be included

$$\mathcal{W} = \text{Tr}_V \big\{ \chi\, (IJ + [B_1, B_2]) \,\big\} \tag{5.56}$$

in order to impose the complex moment map condition $\mu_{\mathbb{C}} = 0$.

To dispel any doubts a strict reader still might have about choosing the right field content, let us work out the D-terms. Using just the cyclicity of trace we arrive promptly to equations for I and J (note that the trace is over different vector spaces)

$$D^c(I) = \frac{1}{2}\text{Tr}_V \left(I I^\dagger T^c_{[V]} \right) = \frac{1}{2}\text{Tr}_W \left(I^\dagger T^c_{[V]} I \right) \tag{5.57}$$

$$D^c(J) = -\frac{1}{2}\text{Tr}_V \left(J^\dagger J T^c_{[V]} \right) = -\frac{1}{2}\text{Tr}_W \left(J T^c_{[V]} J^\dagger \right). \tag{5.58}$$

Writing K, $\widetilde{K}$ as in (5.28) while $T^c_{[V]}$ in the block form

$$T^c_{[V]} = \left(\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array} \right); \quad B,\ C:\ \text{symmetric } k \times k \text{ matrices}$$

these can be expressed explicitly as

$$D^c(I) = (D^c(I))^T = \frac{1}{2}\sum_{a=1}^{N} \{ K_a^T A K_a^* + K_a^T B \widetilde{K}_a^* - \widetilde{K}_a^T C K_a^* + \widetilde{K}_a^T A^T \widetilde{K}_a^* \}$$

$$D^c(J) = -\frac{1}{2}\sum_{a=1}^{N} \{ K_a^T A K_a^* + K_a^T B \widetilde{K}_a^* + \widetilde{K}_a^T C K_a^* - \widetilde{K}_a^T A^T \widetilde{K}_a^* \}. \tag{5.59}$$

The more interesting part of the argument appears to be the contribution from B_1, B_2. They are the same, so we drop the index and expand B in a basis of $2k \times 2k$ matrices M^I as $B = B_I M^I$, $I = 1, \ldots, k(2k-1)$. Recall that we already did the counting of independent entries in B, therefore the range of the index I should be no surprise.

Directly from the definition of the real moment maps follows

$$\mu_{\mathbb{R}}(B) = B_I \left(B^J\right)^* \left[M^I, M_J^\dagger\right] \quad \Longrightarrow \quad \left[M^I, M_J^\dagger\right] \in \mathrm{Lie}(Sp(2k,\mathbb{C})) \tag{5.60}$$

since $\mu_{\mathbb{R}}$ is in the Lie algebra of the symplectic group as we already saw. On the other hand, the canonical form of the D-term contribution corresponding to B looks like

$$D^c(B)\left(T_c^{[V]}\right)_i^{\ j} = \underbrace{\left(B^J\right)^* \left(T^c_{[\bigwedge^2 V]}\right)_J^{\ I} B_I}_{D^c(B)} \left(T_c^{[V]}\right)_i^{\ j}. \tag{5.61}$$

Consequently, we need to figure out what $\left(T^c_{[\bigwedge^2 V]}\right)_J^{\ I}\left(T_c^{[V]}\right)_i^{\ j}$ is. This is not so hard, think of it as a set of matrices with (i,j)-indices labeled by (I,J). Certainly we know that each matrix of this set is in the Lie algebra of the symplectic group since $\left(T_c^{[V]}\right)_i^{\ j}$ is. Hence we have to build the right hand side out of the set $\{M_I, M_I^\dagger\}$ keeping this condition. However, there is only one possible combination of the M matrices being in $\mathrm{Lie}(Sp(2k,\mathbb{C}))$, namely $\left[M^I, M_J^\dagger\right]_i^{\ j}$! We could have proceeded also in a different way to show this relation. Contracting by $T^a_{[V]}$ and taking the trace yields

$$\left(T^a_{[\bigwedge^2 V]}\right)_J^{\ I} = \frac{1}{2}\mathrm{Tr}_V\left(M_J^\dagger T^a_{[V]} M^I - M^I T^a_{[V]} M_J^\dagger\right). \tag{5.62}$$

Then it is enough to check that this expression indeed furnishes a representation, $T^a_{[\bigwedge^2 V]} T^b_{[\bigwedge^2 V]} = f^{ab}{}_c T^c_{[\bigwedge^2 V]}$. We are leaving this as an exercise for a persistent reader. In either case, we are lead to conclude

$$D^c(B)\left(T_c^{[V]}\right)_i^{\ j} = \left[\mu_{\mathbb{R}}(B)\right]_i^{\ j}, \tag{5.63}$$

which finishes the discussion of D-terms and shows that the field content of the model produces the desired equations for classical vacua.

5.3.1 S^2 partition function for $O(N)$-ADHM GLSM

Employing the basic facts about Lie algebras given in Appendix A, we can easily write down the partition function on S^2 for this model

$$Z^{S^2}_{Sp(k)} = \frac{1}{2^k k!} \sum_{\{m_1,\ldots,m_k\}\in\mathbb{Z}^k} \int_{\mathbb{R}^k} \prod_{s=1}^{k} \frac{d(r\sigma_s)}{2\pi} Z_{\mathrm{VM}} Z_J Z_I Z_\chi Z_{B_1} Z_{B_2}. \tag{5.64}$$

As an example let us work out the weights for B_1. It transforms in the representation $\bigwedge^2 V$, where $V \simeq \mathbb{C}^{2k}$ is the standard representation of C_k with weights $\pm\beta_i$, $i = 1,\ldots,k$.

Hence the weights of $\bigwedge^2 V$ are the pairwise sums of distinct weights of V, explicitly

$$\text{wights of } \bigwedge^2 V = \{\beta_i - \beta_j, -(\beta_i - \beta_j), \beta_i + \beta_j, -(\beta_i + \beta_j)|\ 1 \le i < j \le k\} \cup \{\underbrace{0, \ldots, 0}_{k-\text{times}}\}$$

and so the contribution from B_1 is precisely as given below in the list of contributions of individual fields

$$Z_{\text{VM}} = \prod_{s=1}^{k}\left[(2r\sigma_s)^2 + \frac{(2m_s)^2}{4}\right]\prod_{s<t}^{k}\left[(r\sigma_s - r\sigma_t)^2 + \frac{(m_s-m_t)^2}{4}\right]\left[(r\sigma_s + r\sigma_t)^2 + \frac{(m_s+m_t)^2}{4}\right] \tag{5.65}$$

$$\begin{aligned} Z_J = &\prod_{s=1}^{k}\left[\frac{\Gamma\left(-ir\sigma_s - ir\epsilon_+ - \frac{m_s}{2}\right)}{\Gamma\left(1+ir\sigma_s + ir\epsilon_+ - \frac{m_s}{2}\right)}\right]^{N_{\text{mod }2}}\prod_{j=1}^{\lfloor\frac{N}{2}\rfloor}\left\{\frac{\Gamma\left(-ir\sigma_s - ir\left(a_j+\epsilon_+\right) - \frac{m_s}{2}\right)}{\Gamma\left(1+ir\sigma_s + ir\left(a_j+\epsilon_+\right) - \frac{m_s}{2}\right)}\right.\\ &\left.\times\frac{\Gamma\left(-ir\sigma_s - ir\left(-a_j+\epsilon_+\right) - \frac{m_s}{2}\right)}{\Gamma\left(1+ir\sigma_s + ir\left(-a_j+\epsilon_+\right) - \frac{m_s}{2}\right)}\right\} \end{aligned} \tag{5.66}$$

$$\begin{aligned} Z_I = &\prod_{s=1}^{k}\left[\frac{\Gamma\left(ir\sigma_s - ir\epsilon_+ + \frac{m_s}{2}\right)}{\Gamma\left(1-ir\sigma_s + ir\epsilon_+ + \frac{m_s}{2}\right)}\right]^{N_{\text{mod }2}}\prod_{j=1}^{\lfloor\frac{N}{2}\rfloor}\left\{\frac{\Gamma\left(ir\sigma_s - ir\left(a_j+\epsilon_+\right) + \frac{m_s}{2}\right)}{\Gamma\left(1-ir\sigma_s + ir\left(a_j+\epsilon_+\right) + \frac{m_s}{2}\right)}\right.\\ &\left.\times\frac{\Gamma\left(ir\sigma_s - ir\left(-a_j+\epsilon_+\right) + \frac{m_s}{2}\right)}{\Gamma\left(1-ir\sigma_s + ir\left(-a_j+\epsilon_+\right) + \frac{m_s}{2}\right)}\right\} \end{aligned} \tag{5.67}$$

$$\begin{aligned} Z_\chi = &\prod_{s=1}^{k}\frac{\Gamma(1+ir\epsilon)}{\Gamma(-ir\epsilon)}\prod_{s=1}^{k}\frac{\Gamma\left(1-ir2\sigma_s + ir\epsilon - m_s\right)}{\Gamma\left(ir2\sigma_s - ir\epsilon - m_s\right)}\frac{\Gamma\left(1+ir2\sigma_s + ir\epsilon + m_s\right)}{\Gamma\left(-ir2\sigma_s - ir\epsilon + m_s\right)}\\ &\times\prod_{s<t}^{k}\left\{\frac{\Gamma\left(1-ir(\sigma_s-\sigma_t) + ir\epsilon - \frac{m_s-m_t}{2}\right)}{\Gamma\left(ir(\sigma_s-\sigma_t) - ir\epsilon - \frac{m_s-m_t}{2}\right)}\frac{\Gamma\left(1+ir(\sigma_s-\sigma_t) + ir\epsilon + \frac{m_s-m_t}{2}\right)}{\Gamma\left(-ir(\sigma_s-\sigma_t) - ir\epsilon + \frac{m_s-m_t}{2}\right)}\right.\\ &\left.\times\frac{\Gamma\left(1-ir(\sigma_s+\sigma_t) + ir\epsilon - \frac{m_s+m_t}{2}\right)}{\Gamma\left(ir(\sigma_s+\sigma_t) - ir\epsilon - \frac{m_s+m_t}{2}\right)}\frac{\Gamma\left(1+ir(\sigma_s+\sigma_t) + ir\epsilon + \frac{m_s+m_t}{2}\right)}{\Gamma\left(-ir(\sigma_s+\sigma_t) - ir\epsilon + \frac{m_s+m_t}{2}\right)}\right\} \end{aligned} \tag{5.68}$$

$$\begin{aligned} Z_{B_1} = &\prod_{s=1}^{k}\frac{\Gamma(-ir\epsilon_1)}{\Gamma(1+ir\epsilon_1)}\prod_{s<t}^{k}\left\{\frac{\Gamma\left(-ir(\sigma_s-\sigma_t) - ir\epsilon_1 - \frac{m_s-m_t}{2}\right)}{\Gamma(\left(1+ir(\sigma_s-\sigma_t) + ir\epsilon_1 - \frac{m_s-m_t}{2}\right)}\right.\\ &\times\frac{\Gamma\left(ir(\sigma_s-\sigma_t) - ir\epsilon_1 + \frac{m_s-m_t}{2}\right)}{\Gamma(\left(1-ir(\sigma_s-\sigma_t) + ir\epsilon_1 + \frac{m_s-m_t}{2}\right)}\frac{\Gamma\left(-ir(\sigma_s+\sigma_t) - ir\epsilon_1 - \frac{m_s+m_t}{2}\right)}{\Gamma(\left(1+ir(\sigma_s+\sigma_t) + ir\epsilon_1 - \frac{m_s+m_t}{2}\right)}\\ &\left.\times\frac{\Gamma\left(ir(\sigma_s+\sigma_t) - ir\epsilon_1 + \frac{m_s+m_t}{2}\right)}{\Gamma(\left(1-ir(\sigma_s+\sigma_t) + ir\epsilon_1 + \frac{m_s+m_t}{2}\right)}\right\} \end{aligned} \tag{5.69}$$

$$Z_{B_2} = Z_{B_1}\big|_{\epsilon_1\to\epsilon_2}\,. \tag{5.70}$$

It is interesting to study the two important limits as before, i.e. the *point like limit* when the sphere shrinks to zero size or the *flat space limit* when the radius is sent to infinity. At first we do the point like limit, where the leading order term in the asymptotic expansion around $r \to 0$ is supposed to capture the instanton partition function for pure

SYM with orthogonal group as a gauge group. This is indeed what happens, we obtain[2]

$$Z^{S^2}_{Sp(k)} \overset{r\to 0}{\sim} \frac{1}{r^{2k(N-2)}} \left[(-1)^{kN} 2^{4k}\right] Z^{\text{inst}}_k\left(O(N)\right) + \text{higher order terms in } r. \tag{5.71}$$

The power of the radius is the same as would come from compactification of a 5D instanton partition function to 4D on a circle of radius r.

Next, let us concentrate on the flat space limit. The by now familiar argument tells us that we are thus obtaining the mirror LG description in the infra-red, which is encoded in the effective twisted superpotential $\widetilde{\mathcal{W}}_{eff}$. We write the integrand of the S^2 partition function as an exponential of some argument, subsequently perform the leading asymptotic expansion for $r \to \infty$ of this argument, which results in a sum of holomorphic and anti-holomorphic piece. The holomorphic part is $\widetilde{\mathcal{W}}_{eff}$ that we were looking for. Applying the Stirling formula for Gamma functions the outlined computation yields

$$\begin{aligned}
\widetilde{\mathcal{W}}_{eff} = N_{\text{mod } 2}&\left[\sum_{s=1}^{k} \omega(-ir\Sigma_s - ir\epsilon_+) + \omega(ir\Sigma_s - ir\epsilon_+)\right] \\
&+ \sum_{s=1}^{k} \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} \Big\{\omega(-ir\Sigma_s - ir(a_j+\epsilon_+)) + \omega(-ir\Sigma_s - ir(-a_j+\epsilon_+)) \\
&+ \omega(ir\Sigma_s - ir(a_j+\epsilon_+)) + \omega(ir\Sigma_s - ir(-a_j+\epsilon_+))\Big\} \\
&+ \sum_{s=1}^{k} \Big\{\omega(-ir2\Sigma_s + ir\epsilon) + \omega(ir2\Sigma_s + ir\epsilon)\Big\} \\
&+ \sum_{s<t}^{k} \Big\{\omega(-ir(\Sigma_s - \Sigma_t) + ir\epsilon) + \omega(ir(\Sigma_s - \Sigma_t) + ir\epsilon) \\
&+ \omega(-ir(\Sigma_s + \Sigma_t) + ir\epsilon) + \omega(ir(\Sigma_s + \Sigma_t) + ir\epsilon)\Big\} \\
&+ \sum_{s<t}^{k} \Big\{\omega(-ir(\Sigma_s - \Sigma_t) - ir\epsilon_1) + \omega(ir(\Sigma_s - \Sigma_t) - ir\epsilon_1) \\
&+ \omega(-ir(\Sigma_s + \Sigma_t) - ir\epsilon_1) + \omega(ir(\Sigma_s + \Sigma_t) - ir\epsilon_1)\Big\} \\
&+ \sum_{s<t}^{k} \Big\{\omega(-ir(\Sigma_s - \Sigma_t) - ir\epsilon_2) + \omega(ir(\Sigma_s - \Sigma_t) - ir\epsilon_2) \\
&+ \omega(-ir(\Sigma_s + \Sigma_t) - ir\epsilon_2) + \omega(ir(\Sigma_s + \Sigma_t) - ir\epsilon_2)\Big\}.
\end{aligned} \tag{5.72}$$

By the Gauge/Bethe correspondence $\widetilde{\mathcal{W}}_{eff}$ is supposed to give the Yang–Yang function of some integrable system. It is natural to claim that this will be a deformation of the ILW integrable system, a version that is based on orthogonal groups instead of unitary.

[2]The factor in the square brackets depends to which reference one compares. See e.g. [59], [54]. The factor 2^{4k} comes essentially from the roots $2\beta_i$ of the C_k Lie algebra and seems to be missing in [59] while present in [54]. Nevertheless, we compared with [59].

The Bethe equations derived in a standard way read this time

$$
\begin{aligned}
&\left[\frac{\Sigma_s-\epsilon_+}{\Sigma_s+\epsilon_+}\right]^{N_{\mathrm{mod}\ 2}} \frac{2\Sigma_s+\epsilon}{2\Sigma_s-\epsilon} \prod_{j=1}^{\lfloor \frac{N}{2} \rfloor} \frac{\Sigma_s-(a_j+\epsilon_+)}{\Sigma_s+(a_j+\epsilon_+)} \frac{\Sigma_s-(-a_j+\epsilon_+)}{\Sigma_s+(-a_j+\epsilon_+)} \\
&= \prod_{t=1}^{k} \frac{\Sigma_s-\Sigma_t-\epsilon}{\Sigma_s-\Sigma_t+\epsilon} \frac{\Sigma_s-\Sigma_t+\epsilon_1}{\Sigma_s-\Sigma_t-\epsilon_1} \frac{\Sigma_s-\Sigma_t+\epsilon_2}{\Sigma_s-\Sigma_t-\epsilon_2} \prod_{\substack{t=1 \\ t\neq s}}^{k} \frac{\Sigma_s+\Sigma_t-\epsilon}{\Sigma_s+\Sigma_t+\epsilon} \frac{\Sigma_s+\Sigma_t+\epsilon_1}{\Sigma_s+\Sigma_t-\epsilon_1} \frac{\Sigma_s+\Sigma_t+\epsilon_2}{\Sigma_s+\Sigma_t-\epsilon_2}.
\end{aligned}
\tag{5.73}
$$

Once again, there is no twist parameter and thus the model is settled in the “KdV-like regime”.

Chapter 6

Unitary ADHM Gauged Linear Sigma Model unveiled

6.1 Quantum cohomology and equivariant Gromov–Witten invariants

We set up the route towards finding the $\mathcal{I}$-function for the instanton moduli space $\mathcal{M}_{k,N}$ (in this chapter we always mean for a unitary group). In order to do that one needs to examine the structure of the integral defining the partition function (5.7). The parameter that controls the behavior of the integrals is the FI parameter ξ. Notice that the constraints on the maps in the ADHM construction

$$\mu_{\mathbb{C}} = IJ + [B_1, B_2] = 0 \tag{6.1}$$

$$\mu_{\mathbb{R}} = \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J = \xi 1 \tag{6.2}$$

are invariant under

$$(\xi \to -\xi) \text{ and } \begin{cases} (B_1, B_2, I, J) \to \left(B_1^\dagger, B_2^\dagger, -J^\dagger, I^\dagger\right) \\ (B_1, B_2, I, J) \to \left(B_2^\dagger, B_1^\dagger, J^\dagger, I^\dagger\right) \end{cases} . \tag{6.3}$$

So we are free to choose the phase $\xi > 0$. Then the defining integrals over real lines can be thought of as contour integrals closed by big circles in the lower half planes. The contribution from these big circles is guaranteed to vanish because of an exponential damping factor precisely when $\xi > 0$. Consequently, the computation will be performed using the residue theorem.

The explicit evaluation of the partition function (5.7) relies on classification of poles in the integrand. We now show that these are labeled by Young tableaux, just like for the Nekrasov partition function [10]. More precisely, we find a tower of poles above each box of the Young tableaux labeled by a positive integer n. Following the discussion of [14, 17], let us summarize the possible poles and zeros of the integrand ($n \geq 0$):

	poles ($\sigma^{(p)}$)	zeros ($\sigma^{(z)}$)
I	$\sigma_s^{(p)} = a_j - \frac{i}{r}(n + \frac{\lvert m_s\rvert}{2})$	$\sigma_s^{(z)} = a_j + \frac{i}{r}(1 + n + \frac{\lvert m_s\rvert}{2})$
J	$\sigma_s^{(p)} = a_j - \epsilon + \frac{i}{r}(n + \frac{\lvert m_s\rvert}{2})$	$\sigma_s^{(z)} = a_j - \epsilon - \frac{i}{r}(1 + n + \frac{\lvert m_s\rvert}{2})$
χ	$\sigma_{st}^{(p)} = -\epsilon - \frac{i}{r}(1 + n + \frac{\lvert m_{st}\rvert}{2})$	$\sigma_{st}^{(z)} = -\epsilon + \frac{i}{r}(n + \frac{\lvert m_{st}\rvert}{2})$
B_1	$\sigma_{st}^{(p)} = \epsilon_1 - \frac{i}{r}(n + \frac{\lvert m_{st}\rvert}{2})$	$\sigma_{st}^{(z)} = \epsilon_1 + \frac{i}{r}(1 + n + \frac{\lvert m_{st}\rvert}{2})$
B_2	$\sigma_{st}^{(p)} = \epsilon_2 - \frac{i}{r}(n + \frac{\lvert m_{st}\rvert}{2})$	$\sigma_{st}^{(z)} = \epsilon_2 + \frac{i}{r}(1 + n + \frac{\lvert m_{st}\rvert}{2})$

Poles from J do not contribute, being in the upper half plane. Consider now a pole for I, say $\sigma_1^{(p)}$; the next pole $\sigma_2^{(p)}$ can arise from I, B_1 or B_2, but not from χ, because in this case it would be cancelled by a zero from J. Moreover, if it comes from I, $\sigma_2^{(p)}$ should correspond to a twisted mass a_j different from the one for $\sigma_1^{(p)}$, or the partition function would vanish (as explained in full detail in [14]). In the case $\sigma_2^{(p)}$ comes from B_1, consider $\sigma_3^{(p)}$: again, this can be a pole from I, B_1 or B_2, but not from χ, or it would be cancelled by a zero of B_2. This reasoning takes into account all the possibilities, so we can conclude that the poles are classified by N Young tableaux $\{\vec{Y}\}_k = (Y_1, \ldots, Y_N)$ such that $\sum_{j=1}^N |Y_j| = k$, which describe colored partitions of the instanton number k. These are the same as the ones used in the pole classification of the Nekrasov partition function, with the difference that to every box is associated not just a pole, but an infinite tower of poles, labeled by a positive integer n; i.e., we are dealing with three-dimensional Young tableaux.

These towers of poles can be dealt with by rewriting near each pole

$$\sigma_s = -\frac{i}{r}\left(n_s + \frac{|m_s|}{2}\right) + i\lambda_s \tag{6.4}$$

In this way we resum the contributions coming from the "third direction" of the Young tableaux, and the poles for λ_s are now given in terms of usual two-dimensional partitions. The change of variables allows us to show the factorization of the partition function before performing the integral over λs, see (3.22).

Defining $z = e^{-2\pi\xi + i\theta}$ and $d_s = n_s + \frac{m_s + |m_s|}{2}$, $\tilde{d}_s = d_s - m_s$ brings the double sum over magnetic fluxes m_s and residues in the tower n_s $\sum_{m_s \in \mathbb{Z}} \sum_{n_s \geq 0}$ to the form $\sum_{\tilde{d}_s \geq 0} \sum_{d_s \geq 0}$. Finally, we obtain the following expression

$$Z_{k,N}^{S^2} = \frac{1}{k!} \oint \prod_{s=1}^{k} \frac{d(r\lambda_s)}{2\pi i} (z\bar{z})^{-r\lambda_s} Z_{11} Z_{\mathrm{v}} Z_{\mathrm{av}} \tag{6.5}$$

where

$$Z_{1l} = \left(\frac{\Gamma(1-ir\epsilon)\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(ir\epsilon)\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)}\right)^k \prod_{s=1}^{k}\prod_{j=1}^{N}\frac{\Gamma(r\lambda_s+ira_j)\Gamma(-r\lambda_s-ira_j+ir\epsilon)}{\Gamma(1-r\lambda_s-ira_j)\Gamma(1+r\lambda_s+ira_j-ir\epsilon)}$$
$$\prod_{s\neq t}^{k}(r\lambda_s-r\lambda_t)\frac{\Gamma(1+r\lambda_s-r\lambda_t-ir\epsilon)\Gamma(r\lambda_s-r\lambda_t+ir\epsilon_1)\Gamma(r\lambda_s-r\lambda_t+ir\epsilon_2)}{\Gamma(-r\lambda_s+r\lambda_t+ir\epsilon)\Gamma(1-r\lambda_s+r\lambda_t-ir\epsilon_1)\Gamma(1-r\lambda_s+r\lambda_t-ir\epsilon_2)} \tag{6.6}$$

$$Z_{\mathrm{v}} = \sum_{\tilde{d}_1,\ldots,\tilde{d}_k\geq 0}((-1)^N z)^{\tilde{d}_1+\ldots+\tilde{d}_k}\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s-ira_j+ir\epsilon)_{\tilde{d}_s}}{(1-r\lambda_s-ira_j)_{\tilde{d}_s}}\prod_{s<t}^{k}\frac{\tilde{d}_t-\tilde{d}_s-r\lambda_t+r\lambda_s}{-r\lambda_t+r\lambda_s}$$
$$\frac{(1+r\lambda_s-r\lambda_t-ir\epsilon)_{\tilde{d}_t-\tilde{d}_s}}{(r\lambda_s-r\lambda_t+ir\epsilon)_{\tilde{d}_t-\tilde{d}_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_1)_{\tilde{d}_t-\tilde{d}_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_1)_{\tilde{d}_t-\tilde{d}_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_2)_{\tilde{d}_t-\tilde{d}_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_2)_{\tilde{d}_t-\tilde{d}_s}} \tag{6.7}$$

$$Z_{\mathrm{av}} = \sum_{d_1,\ldots,d_k\geq 0}((-1)^N \bar{z})^{d_1+\ldots+d_k}\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s-ira_j+ir\epsilon)_{d_s}}{(1-r\lambda_s-ira_j)_{d_s}}\prod_{s<t}^{k}\frac{d_t-d_s-r\lambda_t+r\lambda_s}{-r\lambda_t+r\lambda_s}$$
$$\frac{(1+r\lambda_s-r\lambda_t-ir\epsilon)_{d_t-d_s}}{(r\lambda_s-r\lambda_t+ir\epsilon)_{d_t-d_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_1)_{d_t-d_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_1)_{d_t-d_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_2)_{d_t-d_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_2)_{d_t-d_s}} \tag{6.8}$$

The Pochhammer symbol $(a)_d$ is defined as

$$(a)_d = \begin{cases} \prod_{i=0}^{d-1}(a+i) & \text{for } d>0 \\ 1 & \text{for } d=0 \\ \prod_{i=1}^{|d|}\dfrac{1}{a-i} & \text{for } d<0 \end{cases} \tag{6.9}$$

Notice that this definition implies the identity

$$(a)_{-d} = \frac{(-1)^d}{(1-a)_d} \tag{6.10}$$

We observe that the $\frac{1}{k!}$ in (6.5) is cancelled by the $k!$ possible orderings of the λs, so in the rest of this paper we will always choose an ordering and remove the factorial.

As was discussed in Chapter 3, the function Z_{v} given in (6.7) provides us with a conjectural expression for the $\mathcal{I}$-function of the instanton moduli space $\mathcal{M}_{k,N}$

$$\mathcal{I}_{k,N} = \sum_{d_1,\ldots,d_k\geq 0}((-1)^N z)^{d_1+\ldots+d_k}\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s-ira_j+ir\epsilon)_{d_s}}{(1-r\lambda_s-ira_j)_{d_s}}\prod_{s<t}^{k}\frac{d_t-d_s-r\lambda_t+r\lambda_s}{-r\lambda_t+r\lambda_s}$$

$$\frac{(1+r\lambda_s-r\lambda_t-ir\epsilon)_{d_t-d_s}}{(r\lambda_s-r\lambda_t+ir\epsilon)_{d_t-d_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_1)_{d_t-d_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_1)_{d_t-d_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_2)_{d_t-d_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_2)_{d_t-d_s}}. \tag{6.11}$$

The λ_s should be interpreted as Chern roots of the tautological bundle over $\mathcal{M}_{k,N}$. From the above formula we find the expansion in $\hbar$

$$\mathcal{I}_{k,N}=1+\frac{I^{(N)}}{\hbar^N}+\dots \tag{6.12}$$

Therefore, $I^{(0)}=1$ for every k,N, while $I^{(1)}=0$ when $N>1$; this implies that the equivariant mirror map is trivial, namely $\mathcal{I}_{k,N}=\mathcal{J}_{k,N}$, for $N>1$. The $N=1$ case will be discussed in detail in the following subsections.

A final remark is that in the limit $\epsilon\to 0$, all world-sheet instanton corrections to $Z^{S^2}_{k,N}$ vanish (i.e. $Z_{\mathrm{v}}=1+\mathcal{O}(\epsilon)$) [43], which is in agreement with general results on equivariant Gromov–Witten theory for the instanton moduli space [60].

6.1.1 Cotangent bundle of the projective space

As a first example, let us consider the case $\mathcal{M}_{1,N}\simeq\mathbb{C}^2\times T^*\mathbb{CP}^{N-1}$. The integrated spherical partition function has the form

$$Z_{1,N}=\sum_{j=1}^{N}(z\bar{z})^{ira_j}Z^{(j)}_{11}Z^{(j)}_{\mathrm{v}}Z^{(j)}_{\mathrm{av}}. \tag{6.13}$$

The j-th contribution comes from the Young tableau $(\bullet,\dots,\square,\dots,\bullet)$, where the box is in the j-th position; this means we have to consider the pole $\lambda_1=-ia_j$. Explicitly one arrives at

$$\begin{aligned}
Z^{(j)}_{11}&=\frac{\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)}\prod_{\substack{l=1\\l\neq j}}^{N}\frac{\Gamma(ira_{lj})\Gamma(-ira_{lj}+ir\epsilon)}{\Gamma(1-ira_{lj})\Gamma(1+ira_{lj}-ir\epsilon)}\\
Z^{(j)}_{\mathrm{v}}&={}_NF_{N-1}\left(\begin{matrix}\left\{ir\epsilon,(-ira_{lj}+ir\epsilon)_{\substack{l=1\\l\neq j}}^{N}\right\}\\ \left\{(1-ira_{lj})_{\substack{l=1\\l\neq j}}^{N}\right\}\end{matrix};(-1)^N z\right)\\
Z^{(j)}_{\mathrm{av}}&={}_NF_{N-1}\left(\begin{matrix}\left\{ir\epsilon,(-ira_{lj}+ir\epsilon)_{\substack{l=1\\l\neq j}}^{N}\right\}\\ \left\{(1-ira_{lj})_{\substack{l=1\\l\neq j}}^{N}\right\}\end{matrix};(-1)^N \bar{z}\right)
\end{aligned} \tag{6.14}$$

with ${}_pF_q$ the generalized hypergeometric function

$$ {}_pF_q\left(\begin{array}{c}\{a_1,\ldots,a_p\}\\ \{b_1,\ldots,b_q\}\end{array};z\right)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}. \tag{6.15}$$

Let us set $N=2$ to focus in more detail on $\mathcal{M}_{1,2}$. In this case the instanton moduli space reduces to $\mathbb{C}^2\times T^*\mathbb{P}^1$ and is isomorphic to the moduli space of the Hilbert scheme of two points $\mathcal{M}_{1,2}\simeq\mathcal{M}_{2,1}$. In order to match the equivariant actions on the two moduli spaces, we identify

$$ a_1=\epsilon_1+2a \qquad , \qquad a_2=\epsilon_2+2a \tag{6.16}$$

so that $a_{12}=\epsilon_1-\epsilon_2$. Then we have

$$ Z_{1,2}=(z\bar{z})^{ir(2a+\epsilon_1)}Z_{11}^{(1)}Z_{\mathrm{v}}^{(1)}Z_{\mathrm{av}}^{(1)}+(z\bar{z})^{ir(2a+\epsilon_2)}Z_{11}^{(2)}Z_{\mathrm{v}}^{(2)}Z_{\mathrm{av}}^{(2)}, \tag{6.17}$$

where

$$ Z_{11}^{(1)}=\frac{\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)}\frac{\Gamma(-ir\epsilon_1+ir\epsilon_2)\Gamma(2ir\epsilon_1)}{\Gamma(1+ir\epsilon_1-ir\epsilon_2)\Gamma(1-2ir\epsilon_1)} \tag{6.18}$$

$$ Z_{\mathrm{v}}^{(1)}={}_2F_1\left(\begin{array}{c}\{ir\epsilon,2ir\epsilon_1\}\\ \{1+ir\epsilon_1-ir\epsilon_2\}\end{array};z\right) \tag{6.19}$$

$$ Z_{\mathrm{av}}^{(1)}={}_2F_1\left(\begin{array}{c}\{ir\epsilon,2ir\epsilon_1\}\\ \{1+ir\epsilon_1-ir\epsilon_2\}\end{array};\bar{z}\right). \tag{6.20}$$

The other contribution is obtained by exchanging $\epsilon_1\leftrightarrow\epsilon_2$. Identifying $Z_{\mathrm{v}}^{(1)}$ as the Givental $\mathcal{I}$-function, we expand it in $r=\frac{1}{\hbar}$ in order to find the equivariant mirror map. This gives

$$ Z_{\mathrm{v}}^{(1)}=1+o(r^2), \tag{6.21}$$

which means there is no equivariant mirror map and $\mathcal{I}=\mathcal{J}$. The same reasoning applies to $Z_{\mathrm{v}}^{(2)}$.

Thus it only remains to properly normalize the symplectic pairing given by Z_{11}. This issue is related to the regularization scheme for the 1-loop determinants (2.50) and (2.51). To compute them the ζ-regularization scheme was used. We will fix the properly normalized pairing Z_{11}^{norm} by requiring

- vanishing coefficient of the Euler–Mascheroni constant γ in Z_{11}^{norm}, referring here to the Weierstrass form of the Gamma function

$$ \frac{1}{\Gamma(x)}=xe^{\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}} \tag{6.22}$$

- correct intersection numbers in classical cohomology
- the constant term in the r expansion of Z_{11}^{norm} is 1, so that the equivariant volume of the target space is not changed

Clearly it is rather a case by case analysis, nevertheless can be done with some practice. Looking at (6.18) we notice that the coefficient of γ is $4ir(\epsilon_1+\epsilon_2)$ and the normalization must cancel it. In this case the normalization that satisfies all the three criteria turns out to be

$$(z\bar{z})^{-2ira}\left(\frac{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)}{\Gamma(1+ir\epsilon_1)\Gamma(1+ir\epsilon_2)}\right)^2. \tag{6.23}$$

Expanding the normalized partition function in r up to order r^{-1}, we obtain [1]

$$\begin{aligned} Z_{1,2}^{\text{norm}} &= \frac{1}{r^2\epsilon_1\epsilon_2}\Big[\frac{1}{2r^2\epsilon_1\epsilon_2}+\frac{1}{4}\ln^2(z\bar{z})-ir(\epsilon_1+\epsilon_2)\Big(-\frac{1}{12}\ln^3(z\bar{z})-\ln(z\bar{z})(\mathrm{Li}_2(z)+\mathrm{Li}_2(\bar{z})) \\ &+ 2(\mathrm{Li}_3(z)+\mathrm{Li}_3(\bar{z}))+3\zeta(3)\Big)\Big] \end{aligned} \tag{6.24}$$

The first term in (6.24) correctly reproduces the Nekrasov partition function of $\mathcal{M}_{1,2}$ as expected, while the other terms compute the $H_T^2(\mathcal{M}_{1,2})$ part of the genus zero Gromov-Witten potential in agreement with [61]. We remark that the quantum part of the Gromov-Witten potential turns out to be linear in the equivariant parameter $\epsilon_1+\epsilon_2$ in accord with general results.

6.1.2 Hilbert scheme of points

Let us now turn to the $\mathcal{M}_{k,1}$ case, which corresponds to the Hilbert scheme of k points. This case was analysed in terms of Givental formalism in [62]. It is easy to see that (6.11) reduces for $N=1$ to their results.

As remarked after equation (6.11) in the $N=1$ case there is a non-trivial equivariant mirror map to be implemented. As we will discuss in a moment, this is done by defining the $\mathcal{J}$ function as $\mathcal{J}=(1+z)^{irk\epsilon}\mathcal{I}$, which corresponds to normalizing by the equivariant mirror map; in other words, we have to normalize the vortex part by multiplying it with $(1+z)^{irk\epsilon}$, and similarly for the antivortex. In the following we will describe in detail some examples and extract the relevant Gromov-Witten invariants for them. As we will see, these are in agreement with the results of [63].

[1] Notice that the procedure outlined above does not fix a remnant dependence on the coefficient of the $\zeta(3)$ term in Z^{S^2}. In fact, one can always multiply by a ratio of Gamma functions whose overall argument is zero; this will have an effect only on the $\zeta(3)$ coefficient. This ambiguity does not affect the calculation of the Gromov-Witten invariants.

For $k=1$, the only Young tableau ($\square$) corresponds to the pole $\lambda_1 = -ia$. This case is simple enough to be written in a closed form; we find

$$Z_{1,1}^{S^2} = (z\bar{z})^{ira} \frac{\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)} (1+z)^{-ir\epsilon}(1+\bar{z})^{-ir\epsilon}. \tag{6.25}$$

From this expression, it is clear that the Gromov-Witten invariants are vanishing. Actually, we should multiply (6.25) by $(1+z)^{ir\epsilon}(1+\bar{z})^{ir\epsilon}$ in order to recover the $\mathcal{J}$-function. Instead of doing this, we propose to use $Z_{1,1}$ as a normalization for $Z_{k,1}$ as follows

$$Z_{k,1}^{\text{norm}} = \frac{Z_{k,1}^{S^2}}{(-r^2\epsilon_1\epsilon_2 Z_{1,1}^{S^2})^k} \tag{6.26}$$

In this way, we go from $\mathcal{I}$ to $\mathcal{J}$ functions and at the same time we normalize the 1-loop factor in such a way to erase the Euler-Mascheroni constant. The factor $(-r^2\epsilon_1\epsilon_2)^k$ is to make the normalization factor to start with 1 in the r expansion. In summary, we obtain

$$Z_{1,1}^{\text{norm}} = -\frac{1}{r^2\epsilon_1\epsilon_2} \tag{6.27}$$

Let us make a comment on the above normalization procedure. The z dependent part of the normalization (except for the trivial factor z^{ira}), which corresponds to the equivariant mirror map is $(1+z)^{irk\epsilon}$. Actually a remarkable combinatorial identity proved in [62] ensures that $e^{-\frac{I^{(1)}}{\hbar}} = (1+z)^{\frac{ik\epsilon}{\hbar}}$ making thus this procedure consistent.

Let us now turn to the $\mathcal{M}_{2,1}$ case. There are two contributions, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ (col) and $\square\square$ (row), coming respectively from the poles $\lambda_1=-ia, \lambda_2=-ia-i\epsilon_1$ and $\lambda_1=-ia, \lambda_2=-ia-i\epsilon_2$. Notice once more that the permutations of the λs are cancelled against the $\frac{1}{2!}$ in front of the partition function (5.7). We thus have

$$Z_{2,1}^{S^2} = (z\bar{z})^{ir(2a+\epsilon_1)} Z_{1l}^{(\text{col})} Z_{\text{v}}^{(\text{col})} Z_{\text{av}}^{(\text{col})} + (z\bar{z})^{ir(2a+\epsilon_2)} Z_{1l}^{(\text{row})} Z_{\text{v}}^{(\text{row})} Z_{\text{av}}^{(\text{row})}, \tag{6.28}$$

where

$$\begin{aligned}
Z_{1l}^{(\text{col})} &= \frac{\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)} \frac{\Gamma(2ir\epsilon_1)\Gamma(ir\epsilon_2-ir\epsilon_1)}{\Gamma(1-2ir\epsilon_1)\Gamma(1+ir\epsilon_1-ir\epsilon_2)} \\
Z_{\text{v}}^{(\text{col})} &= \sum_{\tilde{d}\geq 0} (-z)^{\tilde{d}} \sum_{\tilde{d}_1=0}^{\tilde{d}/2} \frac{(1+ir\epsilon_1)_{\tilde{d}-2\tilde{d}_1}}{(ir\epsilon_1)_{\tilde{d}-2\tilde{d}_1}} \frac{(ir\epsilon)_{\tilde{d}_1}}{\tilde{d}_1!} \frac{(ir\epsilon_1+ir\epsilon)_{\tilde{d}-\tilde{d}_1}}{(1+ir\epsilon_1)_{\tilde{d}-\tilde{d}_1}} \\
&\quad \frac{(2ir\epsilon_1)_{\tilde{d}-2\tilde{d}_1}}{(\tilde{d}-2\tilde{d}_1)!} \frac{(1-ir\epsilon_2)_{\tilde{d}-2\tilde{d}_1}}{(ir\epsilon_1+ir\epsilon)_{\tilde{d}-2\tilde{d}_1}} \frac{(ir\epsilon)_{\tilde{d}-2\tilde{d}_1}}{(1+ir\epsilon_1-ir\epsilon_2)_{\tilde{d}-2\tilde{d}_1}} \\
Z_{\text{av}}^{(\text{col})} &= \sum_{d\geq 0} (-\bar{z})^{d} \sum_{d_1=0}^{d/2} \frac{(1+ir\epsilon_1)_{d-2d_1}}{(ir\epsilon_1)_{d-2d_1}} \frac{(ir\epsilon)_{d_1}}{d_1!} \frac{(ir\epsilon_1+ir\epsilon)_{d-d_1}}{(1+ir\epsilon_1)_{d-d_1}} \\
&\quad \frac{(2ir\epsilon_1)_{d-2d_1}}{(d-2d_1)!} \frac{(1-ir\epsilon_2)_{d-2d_1}}{(ir\epsilon_1+ir\epsilon)_{d-2d_1}} \frac{(ir\epsilon)_{d-2d_1}}{(1+ir\epsilon_1-ir\epsilon_2)_{d-2d_1}}.
\end{aligned} \tag{6.29}$$

Here we defined $d = d_1 + d_2$ and changed the sums accordingly. The row contribution can be obtained from the column one by exchanging $\epsilon_1 \leftrightarrow \epsilon_2$. We can then expand $Z_{\mathrm{v}}^{(\text{col, row})}$ in r as

$$Z_{\mathrm{v}}^{(\text{col, row})} = 1 + 2ir\epsilon \mathrm{Li}_1(-z) + \mathcal{O}(r^2). \tag{6.30}$$

Finally, we invert the equivariant mirror map by replacing

$$\begin{aligned} Z_{\mathrm{v}}^{(\text{col, row})} &\longrightarrow e^{-2ir\epsilon\mathrm{Li}_1(-z)} Z_{\mathrm{v}}^{(\text{col, row})} = (1+z)^{2ir\epsilon} Z_{\mathrm{v}}^{(\text{col, row})} \\ Z_{\mathrm{av}}^{(\text{col, row})} &\longrightarrow e^{-2ir\epsilon\mathrm{Li}_1(-\bar{z})} Z_{\mathrm{av}}^{(\text{col, row})} = (1+\bar{z})^{2ir\epsilon} Z_{\mathrm{av}}^{(\text{col, row})}. \end{aligned} \tag{6.31}$$

Now we can prove the equivalence $\mathcal{M}_{1,2} \simeq \mathcal{M}_{2,1}$: by expanding in z, it can be shown that $Z_{\mathrm{v}}^{(1)}(z) = (1+z)^{2ir\epsilon} Z_{\mathrm{v}}^{(\text{col})}(z)$ and similarly for the antivortex part; since $Z_{1l}^{(1)} = Z_{1l}^{(\text{col})}$ we conclude that $Z^{(1)}(z,\bar{z}) = (1+z)^{2ir\epsilon}(1+\bar{z})^{2ir\epsilon} Z^{(\text{col})}(z,\bar{z})$. The same is valid for $Z^{(2)}$ and $Z^{(\text{row})}$, so in the end we obtain

$$Z_{1,2}^{S^2}(z,\bar{z}) = (1+z)^{2ir\epsilon}(1+\bar{z})^{2ir\epsilon} Z_{2,1}^{S^2}(z,\bar{z}) \tag{6.32}$$

Taking into account the appropriate normalizations, this implies

$$Z_{1,2}^{\text{norm}}(z,\bar{z}) = Z_{2,1}^{\text{norm}}(z,\bar{z}) \quad . \tag{6.33}$$

Some further examples for higher k, N illustrating the outlined procedure can be found in [43].

6.2 The Intermediate Long Wave system

In the previous chapter we derived Bethe equations associated to the mirror Landau–Ginzburg model of the unitary ADHM GLSM. It turned out that those Bethe equations correspond to the $gl(N)$ periodic Intermediate Long Wave system (ILW for $N = 1$; otherwise ILW$_N$). Now we shall review some properties of this integrable model of hydrodynamic type and later on we provide more details about the correspondence with gauge theory.

The (non-periodic) ILW equation [64]

$$u_t = 2uu_x + \frac{1}{\delta}u_x + \mathcal{T}[u_{xx}] \tag{6.34}$$

is an integro-differential equation for $u(x,t)$ that describes dynamics of a thin layer of fluid on top of a thick layer of fluid that flows through a channel in a constant gravitational field; the total height of the fluids is $h := h_1 + h_2$, $\frac{h_1}{h_2} \ll 1$. The amplitude

A of the waves is assumed to be small ($A \ll h_1$), while their wavelength λ is large ($h_1 \ll \lambda$). The parameter δ entering the ILW equation is effectively $\delta = \frac{h}{\lambda}$ and the integral operator $\mathcal{T}$ is defined as

$$\mathcal{T}[f](x) = P.V. \int \coth\left(\frac{\pi(x-y)}{2\delta}\right) f(y) \frac{dy}{2\delta}, \tag{6.35}$$

where $P.V.$ means the principal value prescription. For a more solid set up of the model see for instance the book [65] and references therein.

This equation has two essential limits (or a historically more correct statement is that it was designed to interpolate between the two already known integrable systems)[2]. For $\delta \to 0$ (shallow water with respect to the wavelength) one recovers the Korteweg–de Vries (KdV) equation

$$u_t = 2uu_x + \frac{\delta}{3} u_{xxx}, \tag{6.36}$$

while for $\delta \to \infty$ (deep water) ILW reduces to the Benjamin–Ono (BO) equation

$$u_t = 2uu_x + H[u_{xx}], \tag{6.37}$$

where H is the integral operator of Hilbert transform on the real linear

$$H[f](x) = P.V. \int \frac{1}{x-y} f(y) \frac{dy}{\pi}. \tag{6.38}$$

We will be actually interested in the periodic version of ILW, where one imposes the identification $x \sim x + 2\pi$. It is obtained by modifying the integral kernel $\mathcal{T}$ to

$$\mathcal{T}[f](x) = \frac{1}{2\pi} P.V. \int_0^{2\pi} \frac{\theta_1'}{\theta_1}\left(\frac{y-x}{2}, q\right) f(y) dy, \tag{6.39}$$

where $q = e^{-\delta}$ and θ_1 denotes the Jacobi theta function. A prime on it means a derivative with respect to the argument, not the nome q.

The ILW equation (6.34) is Hamiltonian with respect to the Poisson structure

$$\{u(x), u(y)\} = \delta'(x-y) \tag{6.40}$$

and can be writen as

$$u_t(x) = \{I_2, u(x)\}, \tag{6.41}$$

where $I_2 = \int \frac{1}{3}u^3 + \frac{1}{2}u\mathcal{T}[u_x]$ is the corresponding Hamiltonian. The other conserved quantities (integrals of motion) have the form $I_1 = \int \frac{1}{2}u^2$ and the higher Hamiltonians

[2] ILW can be seen as an integrable deformation of KdV and in [66] it was shown that the requirement of integrability fixes the integration kernel $\mathcal{T}$ (6.35).

$I_n = \int \frac{1}{n} u^n + \ldots$ with $n > 3$ can be fixed by the involution condition $\{I_n, I_m\} = 0$. They have been computed explicitly in [67].

The generalization to ILW$_N$ is described in [68] by a system of N coupled integrable integro-differential equations in N fields; more explicit formulae for the $gl(2)$ case can be found in [58].

Solitons. They form an important class of solutions to non-linear partial differential equations describing physical systems. A solitonic wave has a time independent profile localized in space while traveling at constant velocity. Scattering of solitons does not change them, just introduces a phase shift. The stability of such solutions is strongly related to itegrability of the underlying PDEs.

A N-soliton for the BO system can be described by a rational function whose poles evolve in time according to the N-particle trigonometric Calogero–Sutherland (tCS) system [69]. In [70] it has been generalized to N-soliton solutions of ILW; in that case the dynamics of poles is governed by the elliptic Calogero–Sutherland (eCS) model with N particles. Let us review the reasoning here.

The Hamiltonian of eCS system for N particles is defined as

$$H_{eCS} = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + G^2 \sum_{i<j} \wp(x_i - x_j; \omega_1, \omega_2), \tag{6.42}$$

where $\wp$ is the elliptic Weierstrass $\wp$-function and the periods are chosen as $2\omega_1 = L$ and $2\omega_2 = i\delta$; sometimes we set $L = 2\pi$ for convenience. For notational simplicity, from now on we suppress the periods in all elliptic functions. The Hamilton equations read

$$\begin{aligned} \dot{x}_j &= p_j \\ \dot{p}_j &= -G^2 \partial_j \sum_{k \neq j} \wp(x_j - x_k), \end{aligned} \tag{6.43}$$

which can be recast as a second order equation of motion

$$\ddot{x}_j = -G^2 \partial_j \sum_{k \neq j} \wp(x_j - x_k). \tag{6.44}$$

It can be shown (see Appendix C for detailed derivation) that equation (6.44) is equivalent to the following auxiliary system [3]

$$\begin{aligned} \dot{x}_j &= iG\left\{ \sum_{k=1}^{N} \frac{\theta_1'\left(\frac{\pi}{L}(x_j - y_k)\right)}{\theta_1\left(\frac{\pi}{L}(x_j - y_k)\right)} - \sum_{k\neq j} \frac{\theta_1'\left(\frac{\pi}{L}(x_j - x_k)\right)}{\theta_1\left(\frac{\pi}{L}(x_j - x_k)\right)} \right\} \\ \dot{y}_j &= -iG\left\{ \sum_{k=1}^{N} \frac{\theta_1'\left(\frac{\pi}{L}(y_j - x_k)\right)}{\theta_1\left(\frac{\pi}{L}(y_j - x_k)\right)} - \sum_{k\neq j} \frac{\theta_1'\left(\frac{\pi}{L}(y_j - y_k)\right)}{\theta_1\left(\frac{\pi}{L}(y_j - y_k)\right)} \right\}. \end{aligned} \tag{6.45}$$

In the limit $\delta \to \infty$ ($q \to 0$), the equation of motion (6.44) reduces to

$$\ddot{x}_j = -G^2 \left(\frac{\pi}{L}\right)^2 \partial_j \sum_{k\neq j} \cot^2\left(\frac{\pi}{L}(x_j - x_k)\right), \tag{6.46}$$

while the auxiliary system goes to

$$\begin{aligned} \dot{x}_j &= iG\frac{\pi}{L}\left\{ \sum_{k=1}^{N} \cot\left(\frac{\pi}{L}(x_j - y_k)\right) - \sum_{k\neq j} \cot\left(\frac{\pi}{L}(x_j - x_k)\right) \right\} \\ \dot{y}_j &= -iG\frac{\pi}{L}\left\{ \sum_{k=1}^{N} \cot\left(\frac{\pi}{L}(y_j - x_k)\right) - \sum_{k\neq j} \cot\left(\frac{\pi}{L}(y_j - y_k)\right) \right\}. \end{aligned} \tag{6.47}$$

Thus one regains the BO soliton solutions derived in [69]. In analogy with [69] we can define a pair of functions which encode particle positions as simple poles

$$\begin{aligned} u_1(z) &= -iG \sum_{j=1}^{N} \frac{\theta_1'\left(\frac{\pi}{L}(z - x_j)\right)}{\theta_1\left(\frac{\pi}{L}(z - x_j)\right)} \\ u_0(z) &= iG \sum_{j=1}^{N} \frac{\theta_1'\left(\frac{\pi}{L}(z - y_j)\right)}{\theta_1\left(\frac{\pi}{L}(z - y_j)\right)} \end{aligned} \tag{6.48}$$

and also introduce their linear combinations

$$u = u_0 + u_1, \quad \widetilde{u} = u_0 - u_1. \tag{6.49}$$

These satisfy the differential equation

$$u_t + u u_z + i\frac{G}{2}\widetilde{u}_{zz} = 0, \tag{6.50}$$

as long as x_j and y_j are governed by the dynamical equations (6.45). The details of the derivation can be found in the Appendix C. Notice that, when the lattice of periodicity

[3] Actually, the requirement that this system should reduce to (6.44) is not sufficient to fix the form of the functions appearing. As will be clear from the derivation below, we could as well substitute $\frac{\theta_1'\left(\frac{\pi}{L}z\right)}{\theta_1\left(\frac{\pi}{L}z\right)}$ by $\zeta(z)$ and the correct equation of motion would still follow. However, we can fix this freedom by taking the trigonometric limit ($\delta \to \infty$) and requiring that this system reduces to the one in [69].

is rectangular, (6.50) is nothing but the ILW equation. Indeed, under the condition $x_i = \bar{y}_i$ one can show that $\tilde{u} = -i\mathcal{T}u$ [67]. To recover (6.34) one has to further rescale $u \to Gu$ and $t \to -t/G$ and shift $u \to u + 1/2\delta$. We observe that (6.50) does not explicitly depend on the number of particles N and holds also in the hydrodynamical limit $N, L \to \infty$, with N/L fixed.

Quantization. The periodic ILW can be canonically quantized; this is done by expanding the periodic function $u(x)$ into Fourier modes $\{\alpha_k\}_{k\in\mathbb{Z}}$ and subsequently promoting them to creation/annihilation operators. From (6.40) one can deduce that the obey the Heisenberg algebra

$$[\alpha_k, \alpha_l] = k\delta_{k+l,0}. \tag{6.51}$$

One can get the quantum Hamiltonians $\hat{I}_n$ by employing an appropriate quantization procedure which has to deal with normal ordering ambiguities [58]. For the lowest Hamiltonians one gets (the oscillators are rescaled with respect to [58]; we will comment on it momentarily)

$$\hat{I}_1 = 2\sum_{k=1}^{\infty} \alpha_{-k}\alpha_k - \frac{1}{24} \tag{6.52}$$

$$\hat{I}_2 = \frac{Q}{2}\sum_{k=1}^{\infty} k\frac{(-q)^k+1}{(-q)^k-1}\alpha_{-k}\alpha_k + \sum_{k,l=1}^{\infty}\Big[\epsilon_1\epsilon_2\alpha_{k+l}\alpha_{-k}\alpha_{-l}\Big] - \frac{Q}{2}\frac{(-q)+1}{(-q)-1}\sum_{k=1}^{\infty}\alpha_{-k}\alpha_k, \tag{6.53}$$

where we introduced a complexification of the δ parameter as $2\pi t = \delta - i\theta$; the relation to q is given by $q = e^{-2\pi t}$.

Quantization of ILW$_N$ is based on the algebra $H \oplus W_N$ with H the Heisenberg algebra and W_2 the Virasoro algebra while W_N is a generalization of it for $N > 3$. The case $N = 2$ corresponding to $H \oplus Vir$ was studied in [58]. For instance one has the Hamiltonian

$$\hat{I}_2 = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} L_{-k}a_k + 2iQ\sum_{k=1}^{\infty} k\frac{1+q^k}{1-q^k}a_{-k}a_k + \frac{1}{3}\sum_{\substack{n,m,k\in\mathbb{Z}\\ n+m+k=0}} a_n a_m a_k; \qquad \begin{array}{l} a_k \in H \\ L_k \in Vir \end{array}. \tag{6.54}$$

The Virasoro generators L_k can be rewritten in terms of a second set of Heisenberg generators c_k, $[c_k, c_l] = \frac{k}{2}\delta_{k+l,0}$, as

$$L_k = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0,k}} c_{k-n}c_n + i(kQ - 2P)c_k. \tag{6.55}$$

Once this is done we can further relate a_k and c_k to the Baranovsky operators $\alpha_k^{(1)}, \alpha_k^{(2)}$ [71] by a change of basis

$$\begin{cases} a_k = -\dfrac{i}{\sqrt{\epsilon_1\epsilon_2}}\dfrac{\alpha_k^{(1)}+\alpha_k^{(2)}}{2}, & c_k = -\dfrac{i}{\sqrt{\epsilon_1\epsilon_2}}\dfrac{\alpha_k^{(1)}-\alpha_k^{(2)}}{2} \\ a_{-k} = i\sqrt{\epsilon_1\epsilon_2}\,\dfrac{\alpha_{-k}^{(1)}+\alpha_{-k}^{(2)}}{2}, & c_{-k} = i\sqrt{\epsilon_1\epsilon_2}\,\dfrac{\alpha_{-k}^{(1)}-\alpha_{-k}^{(2)}}{2} \end{cases} ; \quad k \in \mathbb{Z}_+. \tag{6.56}$$

Plugging these relations into (6.54) ones arrives at (the momentum P is set to zero here)

$$\begin{aligned}\hat{I}_2 =& \frac{i}{2\sqrt{\epsilon_1\epsilon_2}}\sum_{n,k>0}[\epsilon_1\epsilon_2\alpha_{-n}^{(1)}\alpha_{-k}^{(1)}\alpha_{n+k}^{(1)} - \alpha_{-n-k}^{(1)}\alpha_n^{(1)}\alpha_k^{(1)} + \epsilon_1\epsilon_2\alpha_{-n}^{(2)}\alpha_{-k}^{(2)}\alpha_{n+k}^{(2)} - \alpha_{-n-k}^{(2)}\alpha_n^{(2)}\alpha_k^{(2)}] \\ &+ \frac{iQ}{2}\sum_{k>0}k[\alpha_{-k}^{(1)}\alpha_k^{(1)} + \alpha_{-k}^{(2)}\alpha_k^{(2)} + 2\alpha_{-k}^{(2)}\alpha_k^{(1)}] \\ &+ iQ\sum_{k>0}k\frac{q^k}{1-q^k}[\alpha_{-k}^{(1)}\alpha_k^{(1)} + \alpha_{-k}^{(2)}\alpha_k^{(2)} + \alpha_{-k}^{(1)}\alpha_k^{(2)} + \alpha_{-k}^{(2)}\alpha_k^{(1)}].\end{aligned} \tag{6.57}$$

6.3 Quantum cohomology for $\mathcal{M}_{k,1}$ in oscillator formalism and connection to ILW

In the previous two sections we studied quantum cohomology of the instanton moduli space on one side and the Intermediate Long Wave integrable system on the other. One can already suspect a link between these two concepts since both of them arose from a single unitary ADHM gauge theory on S^2. The purpose of this section is to even strengthen the bridge between these topics. In order to do it we need to introduce a Fock space formalism for multiplication in quantum cohomology of the instanton moduli space $\mathcal{M}_{k,1}$ that was developed in [63]. We want to show two things: first of all that the Fock space formalism correctly reproduces the Gromov–Witten potential for $\mathcal{M}_{2,1}$ computed in (6.24) and after that we observe that the operators on the Fock space responsible for quantum multiplication in cohomology are the Hamiltonians of quantized ILW.

In [61, 63] the quantum cohomology of the Hilbert scheme of points on $\mathbb{C}^2$, i.e. $\mathcal{M}_{k,1}$, was described using oscillator formalism. One introduces creation-annihilation operators α_k, $k \in \mathbb{Z}$ obeying the Heisenberg algebra

$$[\alpha_p, \alpha_q] = p\delta_{p+q}. \tag{6.58}$$

Positive modes annihilate the vacuum

$$\alpha_p|\emptyset\rangle = 0 \quad , p > 0 \tag{6.59}$$

and the natural basis of the Fock space is given by

$$|Y\rangle = \frac{1}{|Aut(Y)| \prod_i Y_i} \prod_i \alpha_{-Y_i} |\emptyset\rangle, \tag{6.60}$$

where $|Aut(Y)|$ is the order of the automorphism group of the partition $Y = \sum_{i=1}^{\ell(Y)} Y_i$. The number of boxes of the Young tableau is counted by the eigenvalue of the energy operator

$$K = \sum_{p>0} \alpha_{-p}\alpha_p. \tag{6.61}$$

Fix now the subspace $\mathrm{Ker}(K-k)$ with $k \in \mathbb{Z}_+$ and allow linear combinations with coefficients being rational functions of the equivariant weights. This space is identified with the equivariant cohomology $H_T^*(\mathcal{M}_{k,1}, \mathbb{Q})$. Explicitly

$$|Y\rangle \in H_T^{2k-2\ell(Y)}(\mathcal{M}_{k,1}, \mathbb{Q}), \tag{6.62}$$

where $\ell(Y)$ denotes the number of parts of the partition Y.

According to [63], the generator of the small quantum cohomology is given by the state $|D\rangle = -|2, 1^{k-2}\rangle$ describing the divisor which corresponds to the collision of two point-like instantons. The operator generating the quantum product by $|D\rangle$ is given by the quantum Hamiltonian

$$\begin{aligned} H_D := (\epsilon_1+\epsilon_2) \sum_{p>0} \frac{p}{2} \frac{(-q)^p+1}{(-q)^p-1} \alpha_{-p}\alpha_p + \sum_{p,q>0} \Big[\epsilon_1\epsilon_2 \alpha_{p+q}\alpha_{-p}\alpha_{-q} - \alpha_{-p-q}\alpha_p\alpha_q\Big] \\ - \frac{\epsilon_1+\epsilon_2}{2} \frac{(-q)+1}{(-q)-1} K. \end{aligned} \tag{6.63}$$

The basic three-point function $\langle D|H_D|D\rangle$ can be obtained once the scalar product on the Fock space is fixed. We define it by

$$\langle Y|Y'\rangle = \frac{(-1)^{K-\ell(Y)}}{(\epsilon_1\epsilon_2)^{\ell(Y)} |Aut(Y)| \prod_i Y_i} \delta_{YY'}. \tag{6.64}$$

The computation of $\langle D|H_D|D\rangle$ then yields

$$\langle D|H_D|D\rangle = (\epsilon_1+\epsilon_2) \left(\frac{(-q)^2+1}{(-q)^2-1} - \frac{1}{2} \frac{(-q)+1}{(-q)-1} \right) \langle D|\alpha_{-2}\alpha_2|D\rangle = (-1)(\epsilon_1+\epsilon_2) \frac{1+q}{1-q} \langle D|D\rangle, \tag{6.65}$$

where we have used $\langle D|\alpha_{-2}\alpha_2|D\rangle = 2\langle D|D\rangle$. By (6.64), we finally get

$$\langle D|H_D|D\rangle = \frac{\epsilon_1+\epsilon_2}{(\epsilon_1\epsilon_2)^{k-1}} \frac{1}{2(k-2)!} \left(1 + 2\frac{q}{1-q}\right). \tag{6.66}$$

Rewriting $1 + 2\frac{q}{1-q} = (q\partial_q)^3 \left[\frac{(\ln q)^3}{3!} + 2\mathrm{Li}_3(q)\right]$, we obtain the genus zero prepotential

$$F^0 = F^0_{cl} + \frac{\epsilon_1 + \epsilon_2}{(\epsilon_1\epsilon_2)^{k-1}} \frac{1}{2(k-2)!} \left[\frac{(\ln q)^3}{3!} + 2\mathrm{Li}_3(q)\right]. \tag{6.67}$$

The above formula matches the prepotential one can extract from (6.24). In [43] it was extended also for $k = 3, 4$.

Now we can make the link between quantum cohomology of the Hilbert scheme of points $\mathcal{M}_{k,1}$ and the quantized ILW explicit. We observe that the operator of multiplication in quantum cohomology H_D (6.63) agrees with the quantum Hamiltonian $\hat{I}_2$ of ILW (6.53) once the identification $Q = \epsilon_1 + \epsilon_2$ is imposed; the number of points k is given by the eigenvalue of the energy operator (6.61), which corresponds to $\hat{I}_1$ (6.52). Notice that the complexified parameter $2\pi t = \delta - i\theta$ represents the Kähler parameter $2\pi t = \xi - i\theta$ of the Hilbert scheme of points $\mathcal{M}_{k,1}$. The BO limit $t \to \pm\infty$ is translated to cohomology of the instanton moduli space as a reduction from the quantum to classical equivariant cohomology.

The generalization of the Fock space formalism to the rank N ADHM instanton moduli space was given by Baranovsky in [71] in terms of N copies of Nakajima operators as $\beta_k = \sum_{i=1}^N \alpha_k^{(i)}$. For example, in the $N = 2$ case the quantum Hamiltonian becomes (modulo terms proportional to the quantum momentum) [60]

$$\begin{aligned} H_D =& \frac{1}{2}\sum_{i=1}^{2}\sum_{n,k>0}[\epsilon_1\epsilon_2\alpha_{-n}^{(i)}\alpha_{-k}^{(i)}\alpha_{n+k}^{(i)} - \alpha_{-n-k}^{(i)}\alpha_n^{(i)}\alpha_k^{(i)}] \\ &- \frac{\epsilon_1+\epsilon_2}{2}\sum_{k>0} k[\alpha_{-k}^{(1)}\alpha_k^{(1)} + \alpha_{-k}^{(2)}\alpha_k^{(2)} + 2\alpha_{-k}^{(2)}\alpha_k^{(1)}] \\ &- (\epsilon_1+\epsilon_2)\sum_{k>0} k\frac{q^k}{1-q^k}[\alpha_{-k}^{(1)}\alpha_k^{(1)} + \alpha_{-k}^{(2)}\alpha_k^{(2)} + \alpha_{-k}^{(2)}\alpha_k^{(1)} + \alpha_{-k}^{(1)}\alpha_k^{(2)}]. \end{aligned} \tag{6.68}$$

Comparing this expression with the Hamiltonian $\hat{I}_2$ for ILW_2 (6.57), we conclude that they match.

6.4 Correspondence between ILW and ADHM gauge theory: details

In Chapter 4 we derived the mirror LG description of the ADHM gauge theory. It is described by the effective twisted superpotential, which is identified as a Yang–Yang function of an integrable model by the Gauge/Bethe correspondence. The corresponding Bethe equations were obtained in (5.20). They appeared in [58] and were claimed to describe the ILW_N integrable system. In the previous sections we have prepared the ground to motivate this correspondence. One can explicitly compute the spectrum for a couple of lowest quantum Hamiltonians $\hat{I}_n$ of ILW_N at the BO point of parameter space $q = 0$. The crucial conclusion is that the eigenvalues are expressed as symmetric functions of solutions to the Bethe equations (5.20), as we will review in the next subsection.

The BO point is chosen here for a simple reason. It is the only point in the q space where we know how to solve the BAE exactly. However, one can build a perturbation theory around $q = 0$ and therefore get the Bethe roots in a power expansion in q. Consequently also the eigenvalues of the Hamiltonians as well as the eigenvectors are supposed to have a form of a series in q. This expansion together with some other properties of the ILW Bethe equations is summarized in Appendix D. Here we present just the solutions to BAE at the BO point. They are classified by N-tuples of partitions $\vec{Y} = (Y^{(1)}, \ldots, Y^{(N)})$ such that $\sum_{l=1}^{N} |Y^{(l)}| = k$. Then the Bethe roots are given as

$$\Sigma_m^{(l)} = a_l - \frac{\epsilon}{2} - (I-1)\epsilon_1 - (J-1)\epsilon_2, \quad m = 1, \ldots, |Y^{(l)}|, \tag{6.69}$$

where I and J run over columns and rows of the Young diagram $Y^{(l)}$. These solutions exactly match the poles appearing in the instanton partition function of Nekrasov [10].

We can as well provide information about the norm of the eigenstates $|\Psi(q)\rangle$. The formula for the norm was proposed in [58]

$$\frac{1}{\langle\Psi(q)|\Psi(q)\rangle} = \left| \left(\frac{\epsilon}{r\epsilon_1\epsilon_2}\right)^{\frac{k}{2}} \left(\frac{\prod_{s=1}^{k}\prod_{t\neq s}^{k} D(\Sigma_{st})}{\prod_{s=1}^{k} Q(\Sigma_s)}\right)^{\frac{1}{2}} \left(\det \frac{\partial^2 \widetilde{\mathcal{W}}_{\text{eff}}}{r^2 \partial\Sigma_s \partial\Sigma_t}\right)^{-\frac{1}{2}} \right|^2_{\Sigma=\Sigma^*(q)} . \tag{6.70}$$

By $\Sigma^*(q)$ we mean the solutions to Bethe equations (5.20), further recall the definitions: $\widetilde{\mathcal{W}}_{\text{eff}}$ is given in (5.19) while D and Q are defined in (5.18). From the gauge theory point of view this is derived as a saddle point approximation to (5.17) as was explained in Chapter 3 around equation (4.13).

6.4.1 Quantum ILW Hamiltonians

As was noted in (4.10), we expect a relation between quantum Hamiltonians of ILW and the observables $\mathrm{Tr}(\Sigma^n)$ [4]

$$\text{spectrum of ILW quantum Hamiltonians} \quad \longleftrightarrow \quad \mathrm{Tr}\,\Sigma^n(q)\Big|_{\text{solution BAE}} . \tag{6.71}$$

[4]For a more detailed exposition of these topics as well as further generalizations, especially regarding the AGT correspondence, the reader is suggested to check [72].

Consider the following generating function for $\mathrm{Tr}\Phi^n$ [73]

$$\mathrm{Tr}\; e^{\beta\Phi} = \sum_{l=1}^{N} \left(e^{\beta a_l} - e^{-\beta\frac{\epsilon_1+\epsilon_2}{2}}(1-e^{\beta\epsilon_1})(1-e^{\beta\epsilon_2}) \sum_{m=1}^{|Y^{(l)}|} e^{\beta\Sigma_m(q)} \right). \tag{6.72}$$

$\Sigma_m(q)$ are solutions of the Bethe equations (5.20) and β is just a formal counting parameter. Setting $N=2$, expanding in β and collecting terms with common powers gives the few lowest terms (here $a := a_1 = -a_2$ and the two partitions $Y^{(1)}, Y^{(2)}$ are denoted as λ, μ)

$$\begin{aligned}
\frac{\mathrm{Tr}\Phi^2}{2} &= a^2 - \epsilon_1\epsilon_2 \left(\sum_{m=1}^{|\lambda|} 1 + \sum_{n=1}^{|\mu|} 1 \right) \\
\frac{\mathrm{Tr}\Phi^3}{3} &= -2\epsilon_1\epsilon_2 \left(\sum_{m=1}^{|\lambda|} \Sigma_m + \sum_{n=1}^{|\mu|} \Sigma_n \right) \\
\frac{\mathrm{Tr}\Phi^4}{4} &= \frac{a^4}{2} - 3\epsilon_1\epsilon_2 \left(\sum_{m=1}^{|\lambda|} \Sigma_m^2 + \sum_{n=1}^{|\mu|} \Sigma_n^2 \right) - \epsilon_1\epsilon_2 \frac{\epsilon_1^2+\epsilon_2^2}{4} \left(\sum_{m=1}^{|\lambda|} 1 + \sum_{n=1}^{|\mu|} 1 \right) \\
\frac{\mathrm{Tr}\Phi^5}{5} &= -4\epsilon_1\epsilon_2 \left(\sum_{m=1}^{|\lambda|} \Sigma_m^3 + \sum_{n=1}^{|\mu|} \Sigma_n^3 \right) - \epsilon_1\epsilon_2(\epsilon_1^2+\epsilon_2^2) \left(\sum_{m=1}^{|\lambda|} \Sigma_m + \sum_{n=1}^{|\mu|} \Sigma_n \right) \quad .
\end{aligned} \tag{6.73}$$

Note that only very simple symmetric polynomials of the Σs appear in the relation. If we think of Σ as a $k \times k$ matrix with $N=2$ blocks corresponding to the pair of partitions (λ, μ), we can say that the right hand side depends only on $\mathrm{Tr}(\Sigma^n)$ restricted to a subspace of a given partition.

In the Benjamin–Ono limit $q \to 0$ (or $q \to \infty$) we know that the solutions of BAE are given by (6.69), so in that case the generating function (6.72) reduces to the generating function of chiral ring observables in a four dimensional $U(N)$ SYM [74, 75]

$$\begin{aligned}
\mathrm{Tr}\Phi^{n+1} = \sum_{l=1}^{N} a_l^{n+1} + \sum_{l=1}^{N} \sum_{j=1}^{\ell(Y^{(l)})} \Big[\Big(a_l + \epsilon_1 Y_j^{(l)} + \epsilon_2(j-1)\Big)^{n+1} - \Big(a_l + \epsilon_1 Y_j^{(l)} + \epsilon_2 j\Big)^{n+1} \\
- (a_l + \epsilon_2(j-1))^{n+1} + (a_l+\epsilon_2 j)^{n+1} \Big],
\end{aligned} \tag{6.74}$$

where $\ell(Y^{(l)})$ is the length of the partition $Y^{(l)}$ (in our conventions the number of boxes in the first column) while $Y_j^{(l)}$ is the number of boxes in the j-th row. At the BO point we can explicitly show that the chiral observables (6.74) are related to the spectrum of quantum BO Hamiltonians. Let us still focus on $N=2$, therefore we have a pair of Young diagrams (λ, μ) such that $|\lambda| + |\mu| = k$. The eigenvalues for the BO Hamiltonians $\hat{I}_n$ can be expressed as linear combinations of eigenvalues for Hamiltonians of two coupled trigonometric Calogero–Sutherland models [58, 76]

$$h^{(n)}_{\lambda,\mu} = h^{(n)}_{\lambda}(a) + h^{(n)}_{\mu}(-a), \tag{6.75}$$

where the function $h^{(n)}_{\lambda}(a)$ is defined as

$$h^{(n)}_{\lambda}(a) = \epsilon_2 \sum_{j=1}^{\ell(\lambda)} \left[\left(a + \epsilon_1\lambda_j + \epsilon_2 \left(j - \frac{1}{2} \right) \right)^n - \left(a + \epsilon_2 \left(j - \frac{1}{2} \right) \right)^n \right]. \tag{6.76}$$

Then the following relation between the $U(2)$ chiral observables (6.74) and the above eigenvalues holds

$$\frac{\mathrm{Tr}\Phi^{n+1}}{n+1} = \frac{a^{n+1} + (-a)^{n+1}}{n+1} - \sum_{i=1}^{n} \frac{1 + (-1)^{n-i}}{2} \frac{n!}{i!(n+1-i)!} \left(\frac{\epsilon_2}{2} \right)^{n-i} h^{(i)}_{\lambda,\mu}. \tag{6.77}$$

For illustration we list a couple of examples for low n

$$\begin{aligned} \frac{\mathrm{Tr}\Phi^2}{2} &= a^2 - \epsilon_1\epsilon_2 k \quad , \quad & \frac{\mathrm{Tr}\Phi^3}{3} &= -h^{(2)}_{\lambda,\mu} \\ \frac{\mathrm{Tr}\Phi^4}{4} &= \frac{a^4}{2} - h^{(3)}_{\lambda,\mu} - \frac{\epsilon_2^2}{4}\epsilon_1\epsilon_2 k \quad , \quad & \frac{\mathrm{Tr}\Phi^5}{5} &= -h^{(4)}_{\lambda,\mu} - \frac{\epsilon_2^2}{2} h^{(2)}_{\lambda,\mu}. \end{aligned} \tag{6.78}$$

Notice that the term $\epsilon_1\epsilon_2 k$ is nothing but the lowest tCS eigenvalue $h^{(1)}_{\lambda,\mu}$.

Finally, relations (6.73) allow us to express the functions $h^{(n)}_{\lambda}$ in terms of Bethe roots Σ_m corresponding to the partition λ

$$\begin{aligned} h^{(1)}_{\lambda} &= \epsilon_1\epsilon_2 \sum_{m=1}^{|\lambda|} 1 \\ h^{(2)}_{\lambda} &= 2\epsilon_1\epsilon_2 \sum_{m=1}^{|\lambda|} \Sigma_m \\ h^{(3)}_{\lambda} &= 3\epsilon_1\epsilon_2 \sum_{m=1}^{|\lambda|} \Sigma_m^2 + \epsilon_1\epsilon_2 \frac{\epsilon_1^2}{4} \sum_{n=1}^{|\lambda|} 1 \\ h^{(4)}_{\lambda} &= 4\epsilon_1\epsilon_2 \sum_{m=1}^{|\lambda|} \Sigma_m^3 + \epsilon_1\epsilon_2\epsilon_1^2 \sum_{n=1}^{|\lambda|} \Sigma_m\,. \end{aligned} \tag{6.79}$$

All the above discussion was made at the BO point since this is the only situation when we know how to solve the Bethe equations exactly. Nevertheless, presumably the formulae remain valid also for ILW provided one replaces the Bethe roots $\Sigma(q = 0)$ by $\Sigma(q)$. However, to get them is rather a hard task. A first step in this direction was accomplished in Appendix D, where the first order correction in q is given.

In this section we treated just the $N = 2$ case. A generalization to higher rank case can be found in Appendix E.

Chapter 7

Generalizations of ADHM GLSM for unitary groups

In previous chapters we discussed in detail some properties of the auxiliary GLSM on S^2 realizing the k-instanton moduli space of a $U(N)$ gauge theory on $\mathbb{C}^2$ as its target space. A possible generalization of this setting consists in replacing the Euclidean space $\mathbb{C}^2$ by an asymptotically locally Euclidean (ALE) space $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$ [77]. Such theories are described by Nakajima quiver varieties and the quiver defines for us a quiver gauge theory on S^2. One can write down the partition function Z^{S^2} and the tools developed precedently for studying Gromov-Witten theory or making connection with integrable systems can be applied step by step.

In the following we will focus only on the situation $\Gamma = \mathbb{Z}_p$, i.e. on the ALE space of type A_{p-1}. Moreover, we discuss just the integrability part of the story, for results about quantum cohomology as well as comments about the ALE space of type D_{p-1} see [29**?**
].

7.1 The A_{p-1} type ALE space: GLSM on S^2

We want to study the k-instanton moduli space $\mathcal{M}(\vec{k},\vec{N},p)$ of a $U(N)$ gauge theory on A_{p-1} ALE space. The data for the ADHM-like construction are neatly summarized in an affine quiver diagram of type $\hat{A}_{p-1}$ with framing at all nodes, see Figure 7.1. The vector $\vec{k} = (k_0, k_1, \dots, k_{p-1})$ (resp. $\vec{N} = (N_0, N_1, \dots, N_{p-1})$) prescribes the dimensions of vector spaces corresponding to the nodes (resp. the framing vector spaces); the extra node of the affine diagram is marked by the subscript zero. These two vectors are not

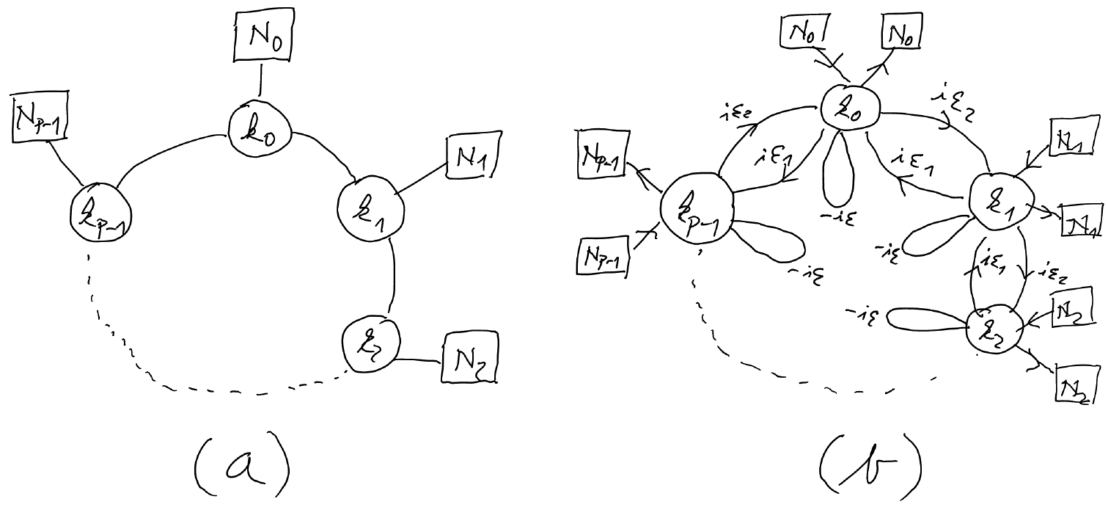

Figure 7.1: (a) $\hat{A}_{p-1}$ affine Dynkin diagram with framing encoding data for ADHM-like construction on ALE spaces of type A_{p-1}. (b) The corresponding decorated quiver defining the GLSM on S^2.

really independent since they are linked by the equation for the first Chern class of the gauge bundle, which we assume to be fixed (usually vanishing).

From the defining quiver diagram we can extract the characteristics of the corresponding GLSM on S^2. It is a gauge theory with gauge group $G = \prod_{\alpha=0}^{p-1} U(k_\alpha)$, flavor group $G_F = U(1)^2 \times \prod_{\alpha=0}^{p-1} U(N_\alpha)$ and a matter content summarized in Table 7.1. To get

	$\chi^{(\alpha)}$	$B^{(\alpha,\alpha+1)}$	$B^{(\alpha,\alpha-1)}$	$I^{(\alpha)}$	$J^{(\alpha)}$
gauge G	$Ad^{(\alpha)}$	$(\overline{\mathbf{k}}^{(\alpha)}, \mathbf{k}^{(\alpha+1)})$	$(\overline{\mathbf{k}}^{(\alpha)}, \mathbf{k}^{(\alpha-1)})$	$\mathbf{k}^{(\alpha)}$	$\overline{\mathbf{k}}^{(\alpha)}$
flavor G_F	$\mathbf{1}_{(-1,-1)}$	$\mathbf{1}_{(1,0)}$	$\mathbf{1}_{(0,1)}$	$\overline{\mathbf{N}}^{(\alpha)}_{\left(\frac{1}{2},\frac{1}{2}\right)}$	$\mathbf{N}^{(\alpha)}_{\left(\frac{1}{2},\frac{1}{2}\right)}$
twisted mass	$\epsilon = \epsilon_1 + \epsilon_2$	ϵ_1	ϵ_2	$-a_j^{(\alpha)} - \frac{\epsilon}{2}$	$a_j^{(\alpha)} - \frac{\epsilon}{2}$
R-charge	2	0	0	0	0

Table 7.1: Matter content of a GLSM with target space $\mathcal{M}(\vec{k}, \vec{N}, p)$.

the correct ADHM-like equations defining $\mathcal{M}(\vec{k}, \vec{N}, p)$ as equations of classical vacua we need to include the following superpotential (labeling of nodes is modulo p)

$$\mathcal{W} = \sum_{\alpha=0}^{p-1} \mathrm{Tr}_{V^{(\alpha)}} \Big[B^{(\alpha,\alpha+1)} B^{(\alpha+1,\alpha)} - B^{(\alpha,\alpha-1)} B^{(\alpha-1,\alpha)} + I^{(\alpha)} J^{(\alpha)} \Big]. \tag{7.1}$$

This fully specifies all data needed in equation (2.52) to write down the partition function

$$
\begin{aligned}
Z^{S^2}_{\vec{k},\vec{N},p} =& \frac{1}{k_0! \cdots k_{p-1}!} \sum_{(\vec{m}^{(0)},\ldots,\vec{m}^{(p-1)})\in\mathbb{Z}^{k_0+\ldots+k_{p-1}}} \\
&\times \int_{\mathbb{R}^{|\vec{k}|}} \prod_{\alpha=0}^{p-1}\prod_{s=1}^{k_\alpha} \frac{d(r\sigma_s^{(\alpha)})}{2\pi} e^{-4\pi i \xi^{(\alpha)} r\sigma_s^{(\alpha)} - i\theta^{(\alpha)} m_s^{(\alpha)}} Z_{\text{vec}} Z_{\text{adj}} Z_{\text{bif}} Z_{\text{f/af}}.
\end{aligned} \tag{7.2}
$$

We included the shifts in the θ-angles to Z_{vec}. Individual contributions to the integrand take the form

$$
Z_{\text{vec}} = \prod_{\alpha=1}^{p-1}\prod_{s\neq t}^{k_\alpha} \frac{\Gamma\left(1 - ir(\sigma_s^{(\alpha)} - \sigma_t^{(\alpha)}) - \frac{m_s^{(\alpha)} - m_t^{(\alpha)}}{2}\right)}{\Gamma\left(ir(\sigma_s^{(\alpha)} - \sigma_t^{(\alpha)}) - \frac{m_s^{(\alpha)} - m_t^{(\alpha)}}{2}\right)} \tag{7.3}
$$

$$
Z_{\text{adj}} = \prod_{\alpha=1}^{p-1}\prod_{s,t=1}^{k_\alpha} \frac{\Gamma\left(1 - ir(\sigma_s^{(\alpha)} - \sigma_t^{(\alpha)}) - ir\epsilon - \frac{m_s^{(\alpha)} - m_t^{(\alpha)}}{2}\right)}{\Gamma\left(ir(\sigma_s^{(\alpha)} - \sigma_t^{(\alpha)}) + ir\epsilon - \frac{m_s^{(\alpha)} - m_t^{(\alpha)}}{2}\right)} \tag{7.4}
$$

$$
\begin{aligned}
Z_{\text{bif}} = \prod_{\alpha=0}^{p-1}\prod_{s=1}^{k_\alpha}\prod_{t=1}^{k_{\alpha-1}} & \frac{\Gamma\left(-ir\sigma_s^{(\alpha)} + ir\sigma_s^{(\alpha-1)} + ir\epsilon_1 - \frac{m_s^{(\alpha)}}{2} + \frac{m_t^{(\alpha-1)}}{2}\right)}{\Gamma\left(1 + ir\sigma_s^{(\alpha)} - ir\sigma_s^{(\alpha-1)} - ir\epsilon_1 - \frac{m_s^{(\alpha)}}{2} + \frac{m_t^{(\alpha-1)}}{2}\right)} \\
& \frac{\Gamma\left(ir\sigma_s^{(\alpha)} - ir\sigma_s^{(\alpha-1)} + ir\epsilon_2 + \frac{m_s^{(\alpha)}}{2} - \frac{m_t^{(\alpha-1)}}{2}\right)}{\Gamma\left(1 - ir\sigma_s^{(\alpha)} + ir\sigma_s^{(\alpha-1)} - ir\epsilon_2 + \frac{m_s^{(\alpha)}}{2} - \frac{m_t^{(\alpha-1)}}{2}\right)}
\end{aligned} \tag{7.5}
$$

$$
\begin{aligned}
Z_{\text{f/af}} = \prod_{\alpha=0}^{p-1}\prod_{s=1}^{k_\alpha}\prod_{j=1}^{N_\alpha} & \frac{\Gamma\left(-ir\sigma_s^{(\alpha)} + ir(a_j^{(\alpha)} + \frac{\epsilon}{2}) - \frac{m_s^{(\alpha)}}{2}\right)}{\Gamma\left(1 + ir\sigma_s^{(\alpha)} - ir(a_j^{(\alpha)} + \frac{\epsilon}{2}) - \frac{m_s^{(\alpha)}}{2}\right)} \\
& \frac{\Gamma\left(ir\sigma_s^{(\alpha)} + ir(-a_j^{(\alpha)} + \frac{\epsilon}{2}) + \frac{m_s^{(\alpha)}}{2}\right)}{\Gamma\left(1 - ir\sigma_s^{(\alpha)} - ir(-a_j^{(\alpha)} + \frac{\epsilon}{2}) + \frac{m_s^{(\alpha)}}{2}\right)}.
\end{aligned} \tag{7.6}
$$

7.2 Connection between generalized qILW integrable system, instanton counting on ALE spaces and spin CS model

In [78] the quantum Intermediate Long Wave integrable system of type (N,p), qILW(N,p) for short, was introduced (see discussion around (1.7) there). The central object of this model are the Bethe equations, which govern the common spectrum of commuting integrals of motion $\mathbf{I}_k$, $k = 1, 2 \ldots$. A couple of important observations was made by the authors. We summarize them schematically in Figure 7.2.

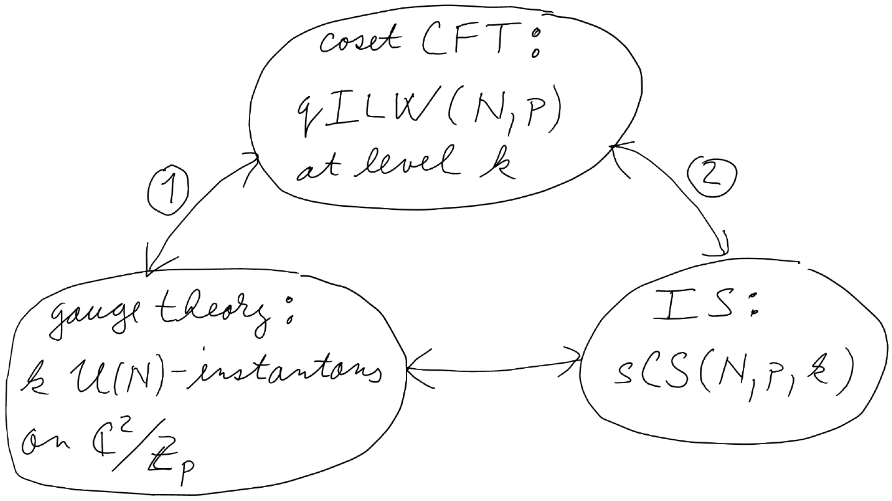

FIGURE 7.2: Diagram showing connections among qILW, spin CS model and instanton counting on ALE spaces

7.2.1 Correspondence between qILW(N, p) and instanton counting on ALE spaces

Let us comment about the first arrow. For a study in a similar spirit, i.e. relating instanton counting problem on ALE spaces $\mathbb{C}^2/\mathbb{Z}_p$ for unitary groups with conformal coset theories see also [79]. As a first step on the gauge theory side, we perform the $r \to \infty$ expansion of the partition function (7.2), obtaining thus the effective twisted superpotential. A computation leads to (recall the definition $\Sigma_s^{(\alpha)} = \sigma_s^{(\alpha)} - \frac{i}{2r} m_s^{(\alpha)}$)

$$Z^{S^2}_{\vec{k},\vec{N},p} \overset{r\to\infty}{\sim} \prod_{\alpha=0}^{p-1} \frac{(r\epsilon)^{k_\alpha}}{k_\alpha!} \left| \int \prod_{\alpha=0}^{p-1}\prod_{s=1}^{k_\alpha} \frac{d(r\Sigma_s^{(\alpha)})}{2\pi} \underbrace{\left(\prod_{\alpha=0}^{p-1}\prod_{s=1}^{k_\alpha} \frac{\prod_{t\neq s}^{k_\alpha} D(\Sigma_s^{(\alpha)} - \Sigma_t^{(\alpha)})}{Q(\Sigma_s^{(\alpha)}) \prod_{t=1}^{k_{\alpha-1}} F(\Sigma_s^{(\alpha)} - \Sigma_t^{(\alpha-1)})} \right)^{\frac{1}{2}}}_{\mu_{\mathrm{msr}}(\Sigma)} e^{-\widetilde{\mathcal{W}}_{\mathrm{eff}}(\Sigma)} \right|^2 . \tag{7.7}$$

The functions forming the integration measure $\mu_{\mathrm{msr}}(\Sigma)$ read

$$D(\Sigma_s^{(\alpha)} - \Sigma_t^{(\alpha)}) = r^2 \left(\Sigma_s^{(\alpha)} - \Sigma_t^{(\alpha)}\right)\left(\Sigma_s^{(\alpha)} - \Sigma_t^{(\alpha)} + \epsilon\right) \tag{7.8}$$

$$F(\Sigma_s^{(\alpha)} - \Sigma_t^{(\alpha-1)}) = r^2 \left(\Sigma_s^{(\alpha)} - \Sigma_t^{(\alpha-1)} - \epsilon_1\right)\left(\Sigma_s^{(\alpha)} - \Sigma_t^{(\alpha-1)} - \epsilon_2\right) \tag{7.9}$$

$$Q(\Sigma_s^{(\alpha)}) = \prod_{j=1}^{N_\alpha} r^2 \left(\Sigma_s^{(\alpha)} - a_j^{(\alpha)} - \frac{\epsilon}{2}\right)\left(\Sigma_s^{(\alpha)} - a_j^{(\alpha)} + \frac{\epsilon}{2}\right) \tag{7.10}$$

and the expression for the effective twisted superpotential is

$$\widetilde{\mathcal{W}}_{\mathrm{eff}}(\Sigma) = 2\pi \sum_{\alpha=0}^{p-1}\sum_{s=1}^{k_\alpha} irt^{(\alpha)}\Sigma_s^{(\alpha)} + \sum_{\alpha=1}^{p-1}\sum_{\substack{s,t=1\\ s\neq t}}^{k_\alpha} \left[\omega\left(ir\Sigma_s^{(\alpha)} - ir\Sigma_t^{(\alpha)}\right) + \omega\left(ir\Sigma_s^{(\alpha)} - ir\Sigma_t^{(\alpha)} + ir\epsilon\right)\right]$$

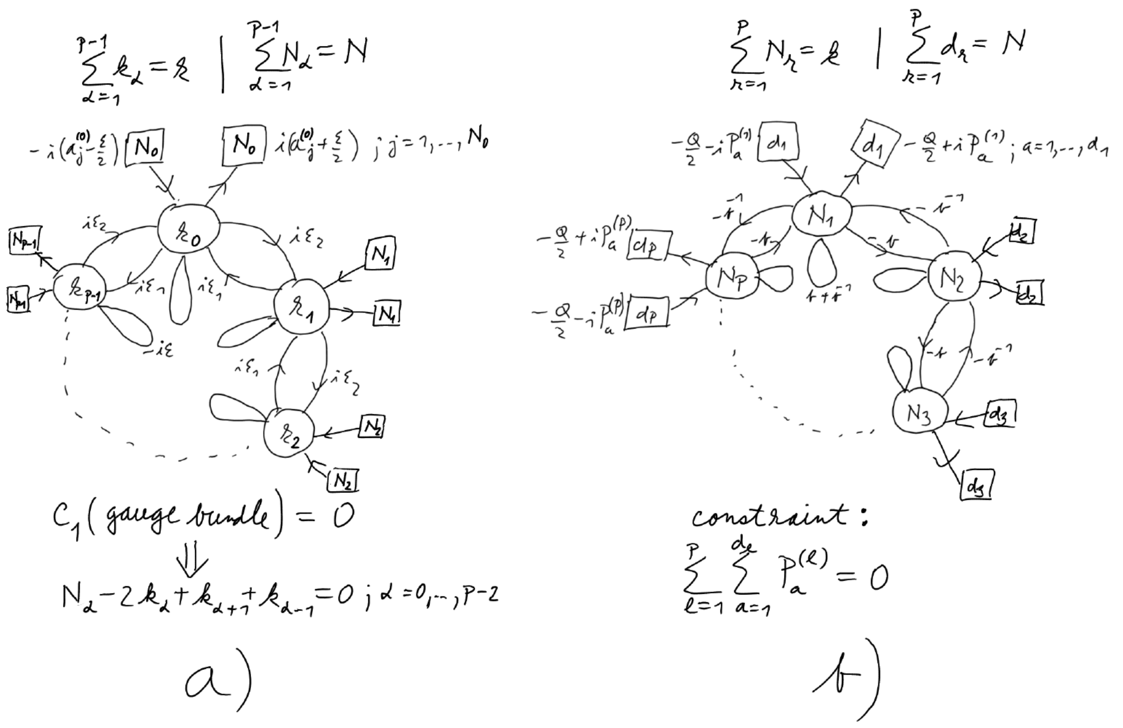

FIGURE 7.3: a) Quiver diagram corresponding to instanton counting on ALE spaces: k $U(N)$-instantons on $\mathbb{C}^2/\mathbb{Z}_p$. b) Auxiliary quiver which leads to qILW(N,p) Bethe equations.

$$+\sum_{\alpha=0}^{p-1}\sum_{s=1}^{k_\alpha}\sum_{j=1}^{N_\alpha}\left[\omega\left(ir\Sigma_s^{(\alpha)}-ira_j^{(\alpha)}-ir\frac{\epsilon}{2}\right)+\omega\left(-ir\Sigma_s^{(\alpha)}+ira_j^{(\alpha)}-ir\frac{\epsilon}{2}\right)\right]$$
$$+\sum_{\alpha=0}^{p-1}\sum_{s=1}^{k_\alpha}\sum_{t=1}^{k_{\alpha-1}}\left[\omega\left(ir\Sigma_s^{(\alpha)}-ir\Sigma_t^{(\alpha-1)}-ir\epsilon_1\right)+\omega\left(-ir\Sigma_s^{(\alpha)}+ir\Sigma_t^{(\alpha-1)}-ir\epsilon_2\right)\right]. \tag{7.11}$$

Then one can apply (4.9) and derive the Bethe equations from $\widetilde{\mathcal{W}}_{\text{eff}}(\Sigma)$. The statement is that Bethe equations determined from quiver gauge theory associated to instanton counting on A_{p-1} ALE spaces precisely match those of qILW(N,p) in a certain region of parameter space on both sides of the correspondence. Actually, at this point we adopt the CFT notation used in [78], where the BAE appeared for the first time. The dictionary is established comparing Figure 7.3(a) with Figure 7.3(b). We can write these equations in a rather elegant form

$$\prod_{a=1}^{d_k}\frac{x_j^{(k)}+iP_a^{(k)}+\frac{i}{2}V^{(k)}}{x_j^{(k)}+iP_a^{(k)}-\frac{i}{2}V^{(k)}}=e^{i\theta^{(k)}}\prod_{l=1}^{p}\prod_{\substack{i=1\\(l,i)\neq(k,j)}}^{N_l}\frac{x_j^{(k)}-x_i^{(l)}+\mathbf{C}_{kl}}{x_j^{(k)}-x_i^{(l)}-\mathbf{C}_{lk}};\qquad \begin{array}{l}k=1,\dots,p\\ j=1,\dots,N_k\end{array} \tag{7.12}$$

with $V^{(k)} = -iQ$, $e^{i\theta^{(k)}} = -(q_k)^{-1}$ and $\mathbf{C}$ the adjacency matrix of the quiver graph

$$\mathbf{C} = \begin{bmatrix} Q & -b & 0 & \dots & 0 & -b^{-1} \\ -b^{-1} & Q & -b & \dots & 0 & 0 \\ 0 & -b^{-1} & Q & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & -b^{-1} & Q & -b \\ -b & \dots & \dots & \dots & -b^{-1} & Q \end{bmatrix} \tag{7.13}$$

In general, this system of equations is extremely hard to solve. However, significant simplification appears in the limit $(q_1, \dots, q_p) = (0, \dots, 0)$. Note that this is a generalization of the BO limit for ILW. Though, it is very subtle to take this limit, one has to redistribute[1] terms first in a precise way. The resulting form of the equations becomes

$$\begin{aligned} 0 = &\prod_{l=1}^{d_k} \left(x_j^{(k)} - \frac{Q}{2} + iP_l^{(k)} \right) \prod_{\substack{i=1 \\ i \neq j}}^{N_k} \frac{1}{\left(x_j^{(k)} - x_i^{(k)} \right) \left(x_j^{(k)} - x_i^{(k)} - Q \right)} \\ \times &\prod_{i=1}^{N_{k+1}} \left(x_j^{(k)} - x_i^{(k+1)} - b \right) \prod_{i=1}^{N_{k-1}} \left(x_j^{(k)} - x_i^{(k-1)} - b^{-1} \right). \end{aligned} \tag{7.14}$$

In this case solutions can be actually expressed explicitly in terms of N-tuples of Young diagrams whose boxes are colored by p colors, the total number of boxes being $k = \sum_{r=1}^{p} N_r$. To give an example we set $N = p = 2$ for simplicity (here we associate white color with the first node and black with the second one). As a next step one has to specify the parameters of qILW(N, p) to match instanton counting. This depends on

[1] And even multiply the equation by one in a special form $1 = \prod_{\substack{i=1 \\ i \neq j}}^{N_k} \frac{x_j^{(k)} - x_i^{(k)}}{x_j^{(k)} - x_i^{(k)}}$. The denominator stays, while the numerator is brought to the other side and killed by the limit.

the level k and splits into two branches, integer and half-integer level, respectively

$$\begin{cases} k \in \mathbb{Z}_{\geq 0}: & (d_1, d_2) = (2, 0) \\ & (N_1, N_2) = (k, k) \\ k \in \mathbb{Z}_{\geq 0} + \frac{1}{2}: & (d_1, d_2) = (0, 2) \\ & (N_1, N_2) = \left(k - \frac{1}{2}, k + \frac{1}{2}\right) \end{cases}$$

At each level we have a corresponding system of equations (7.14). We list the first four non-trivial contributions in Figure 7.4. Exactly in the same way are labeled fixed points on the moduli space of $U(2)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$.

7.2.2 Correspondence between qILW(N, p) and spin Calogero–Sutherland model

Next, we focus on the second arrow in Figure 7.2. To begin with, let us explain the notations on the Integrable System side. We are considering a generalized spin Calogero–Sutherland model sCS(N, p, k), where N denotes the number of copies of ordinary spin Calogero–Sutherland models, k the number of particles on the circle and p the number of spin degrees of freedom for each particle. The spin projections are labeled by $r = 1, \ldots, p$ and the number of particles in a given projection r is N_r, so that $k = \sum_{r=1}^{p} N_r$ is the total number of particles. The proposal put forward in [78] is that the integral of motion $\mathbf{I}_2$ for qILW$(1, p)$ coincides in the limit $(q_1, \ldots, q_p) = (0, \ldots, 0)$ with the Hamiltonian of sCS$(1, p, k)$. Moreover, the spectrum of $\mathbf{I}_2$ can be written using the roots of the Bethe equations in the following form

$$\mathbf{I}_2 \sim \frac{2i}{p} \sum_{j=1}^{N_1} x_j^{(1)}. \tag{7.15}$$

In other words the sum runs just over Bethe roots corresponding to the first node of the quiver. In this limit, as we discussed, the roots are labeled by colored Young diagrams, where the i-th node of the quiver is colored by $(i-1)$-th color. In the language of colored Young diagrams we sum only over boxes colored by 0 (it is customary to choose the p different colors as $0, \ldots, p-1$; rules for coloring the diagram can be found e.g. in [79]). In the following we do not show the correspondence in full generality, rather concentrate on a special case $p = 2$ which we treat in full detail. The difficulty of generalizing to p arbitrary will be explained in the course of upcoming discussion. So, the plan is to

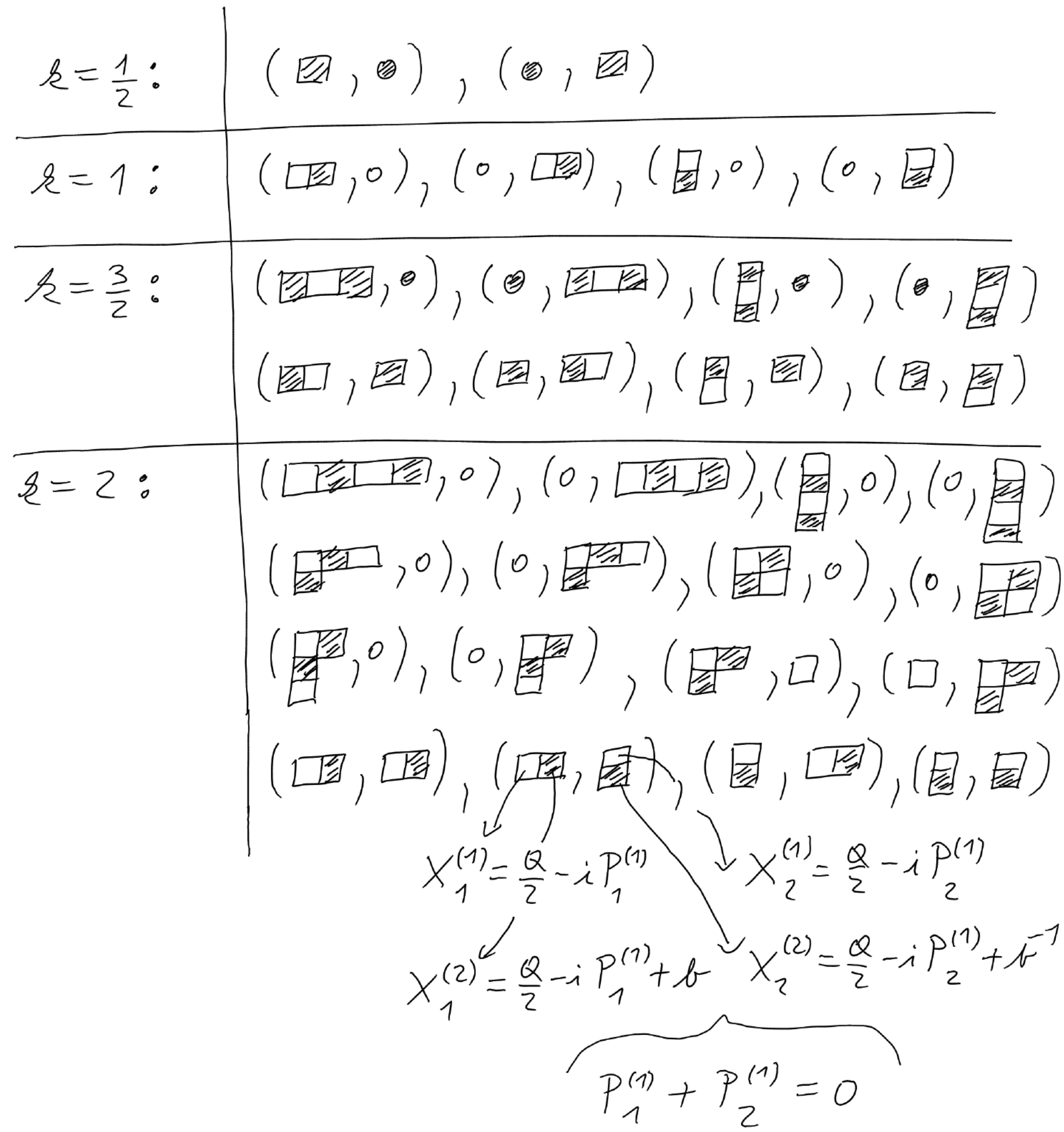

FIGURE 7.4: Solutions for qILW$(2,2)$ up to level $k=2$. The level equals the power of the instanton counting parameter in the instanton partition function.

take equation (7.15) and compare it to eigenvalues of the sCS$(1,p,k)$ Hamiltonian. The spectrum of sCS$(1,p,k)$ was computed in [80]. For convenience we quote just the results that will be needed in a moment. First of all recall the normalized Hamiltonian

$$H_{\beta,p} = W^{-\beta}\overline{H}_{\beta,p}W^{\beta}, \tag{7.16}$$

where $W = \prod_{i<j}^{k} \sin\frac{\pi}{L}(y_i - y_j)$ and $\overline{H}_{\beta,p}$ equals

$$\overline{H}_{\beta,p} = -\frac{1}{2}\sum_{i=1}^{k}\frac{\partial^2}{\partial y_i^2} + \frac{\pi^2}{2L^2}\sum_{i\neq j}^{k}\frac{\beta\left(\beta+\mathbf{P}_{ij}\right)}{\sin^2\frac{\pi}{L}\left(y_i-y_j\right)}. \tag{7.17}$$

In the above formulae y_i are the coordinates of k particles placed on a circle of length L, β is the coupling constant and $\mathbf{P}_{ij}$ the spin exchange operator for particles i and j. Then the spectrum of $H_{\beta,p}$ reads

$$E_{\beta,p} = \sum_{i=1}^{k} \bar{K}_i^2 + \beta \sum_{i=1}^{k} (2i-k-1)\,\bar{K}_i + \frac{\beta^2 k\left(k^2-1\right)}{12}. \tag{7.18}$$

We have to pause a bit to explain the meaning of $\bar{K}_i$. Clearly, they label the state whose eigenvalue we are computing. Pick a strictly decreasing sequence $\mathbf{K} = (K_1, \ldots, K_k)$, $K_i > K_{i+1}$. This object labels the eigenstates. In a next step decompose it in a unique way as $\mathbf{K} = \underline{\mathbf{K}} - p\bar{\mathbf{K}}$, where $\underline{\mathbf{K}} \in \{1, \ldots, p\}^k$ and $\bar{\mathbf{K}} \in \mathbb{Z}^k$. For some readers the following form might be more illuminating

$$\underline{K}_i = 1 + (K_i - 1)_{\mathrm{mod}\, p} \tag{7.19}$$

$$\bar{K}_i = -\left\lfloor \frac{K_i - 1}{p} \right\rfloor. \tag{7.20}$$

The crucial step in the construction is the introduction of a vacuum state $\mathbf{K}^0$. At the same time it is also the obstacle we mentioned above. This vacuum was given only for $p = 2$ in [80] and has the form

$$\mathbf{K}^0 = (M, M-1, \ldots, M-k+1)\,, \quad M = \frac{k}{2} + 1. \tag{7.21}$$

By the integrality requirement, this makes sense only for k even. Moreover the solution to the minimization problem (with constraints stated above) is unique only for $k = 4l+2$ while for $k = 4l$ it can be chosen consistently in this form. For k odd the vacuum state is never unique, nevertheless by practicing with examples we collected evidence that there is always a choice supporting the results derived below. Once we have the vacuum, we define $\mathbf{K} = \sigma + \mathbf{K}^0$. From the definitions given above it follows that σ is a non-increasing sequence. By restricting σ to $\mathbb{Z}^k_{\geq 0}$ we obtain a partition λ. In the rest we are going to focus only on states which are labelled by partitions. The coloring of the partition (0-coloring when the box in the first row and first column is colored by 0 and 1-coloring when it is colored by 1) is dependent on k. For $k = 4l+1$ and $k = 4l+2$ we have to apply 0-coloring while $k = 4l$ and $k = 4l+3$ requires 1-coloring. In the following we focus on $k = 4l+2$, where we have a unique vacuum and a 0-coloring. However, the conclusions remain valid for k general, one just needs to do appropriate changes in the derivation. We will study the normalized energy eigenvalue for states corresponding to partitions

$$\mathcal{E}_{\beta.p}(\lambda) = E_{\beta,p}(\mathbf{K}) - E_{\beta,p}(\mathbf{K}^0) = \sum_{i=1}^{k} \left(\bar{K}_i - \bar{K}_i^0\right)\left(\bar{K}_i + \bar{K}_i^0\right) + \beta \sum_{i=1}^{k} (2i-k-1)\left(\bar{K}_i - \bar{K}_i^0\right) \tag{7.22}$$

and show that it can be matched with the spectrum of $\mathbf{I}_2$. At this point we need to introduce some characteristics of colored Young diagrams. First, consider the number of boxes colored by 0 in the i-th row. We denote this as $C_i^{(0)}(\lambda)$. Drawing a colored diagram and looking at it for sufficient time, we can write a formula

$$C_i^{(0)}(\lambda) = 1 + \left\lfloor \frac{\lambda_i - 1 - (i-1)_{\text{mod } p}}{p} \right\rfloor. \tag{7.23}$$

On the other hand, using (7.20), we have an expression for $\bar{K}_i - \bar{K}_i^{(0)}$

$$\bar{K}_i - \bar{K}_i^{(0)} = -\left\lfloor \frac{\lambda_i + K_i^{(0)} - 1}{p} \right\rfloor + \left\lfloor \frac{K_i^{(0)} - 1}{p} \right\rfloor \tag{7.24}$$

and plugging in (7.21) while setting $p = 2$ at the same time yields a simple relation

$$\bar{K}_i - \bar{K}_i^{(0)} = -C_i^{(0)}(\lambda). \tag{7.25}$$

Still, we need to build three more quantities out of $C_i^{(0)}(\lambda)$

$$|C^{(0)}(\lambda)| = \sum_{i=1}^{\#\text{rows}(\lambda)} C_i^{(0)}(\lambda) \tag{7.26}$$

$$n^{(0)}(\lambda) = \sum_{i=1}^{\#\text{rows}(\lambda)} (i-1)\, C_i^{(0)}(\lambda) \tag{7.27}$$

$$n^{(0)}(\lambda^t) = \sum_{i=1}^{\#\text{rows}(\lambda^t)} (i-1)\, C_i^{(0)}(\lambda^t), \tag{7.28}$$

where λ^t is the transposed Young diagram. It will be useful to have a formula for $n^{(0)}(\lambda^t)$ just in terms of data related to the original partition λ

$$\begin{aligned} n^{(0)}(\lambda^t) &= \sum_{i=1}^{\#\text{rows}(\lambda)} \sum_{j=1}^{C_i^{(0)}(\lambda)} \left[(i-1)_{\text{mod } p} + (j-1)\, p\right] \\ &= \sum_{i=1}^{\#\text{rows}(\lambda)} C_i^{(0)}(\lambda) \left[(i-1)_{\text{mod } p} + \frac{p}{2}\left(C_i^{(0)}(\lambda) - 1\right)\right]. \end{aligned} \tag{7.29}$$

Equipped with these information we can rewrite the normalized energy eigenvalue (7.22) just using characteristics of a colored Young diagram. The essential ingredient is equation (7.25) which implies $p = 2$. After some algebra, combining (7.25)–(7.29), we finally arrive at[2]

$$\mathcal{E}_{\beta.p=2}(\lambda) = n^{(0)}(\lambda^t) - (2\beta+1)\, n^{(0)}(\lambda) + \left[\frac{k}{2}(2\beta+1) - \beta\right] |C^{(0)}(\lambda)|. \tag{7.30}$$

To accomplish the comparison we just have to write the spectrum of $\mathbf{I}_2$ (7.15) in terms of (7.26)–(7.28). Remind that all the above discussion assumes $N = 1$, so only one of the nodes in the quiver contains a single fundamental/antifundamental pair. We mark this node by a star. Then we have (we freely change between the gauge theory notation and CFT notation: $Q \leftrightarrow \epsilon$, $b \leftrightarrow \epsilon_1$, $b^{-1} \leftrightarrow \epsilon_2$)

contribution from $\frac{\epsilon}{2} - iP_1^{(*)}$: $\quad \left(\frac{\epsilon}{2} - iP_1^{(*)}\right) |C^{(0)}(\lambda)|$

contribution from ϵ_2 : $\quad 0 \cdot C_1^{(0)}(\lambda) + 1 \cdot C_2^{(0)}(\lambda) + \cdots + (\#\text{rows}(\lambda) - 1) \cdot C^{(0)}_{\#\text{rows}(\lambda)}(\lambda)$

contribution from ϵ_1 : $\quad 0 \cdot C_1^{(0)}(\lambda^t) + 1 \cdot C_2^{(0)}(\lambda^t) + \cdots + \left(\#\text{rows}(\lambda^t) - 1\right) \cdot C^{(0)}_{\#\text{rows}(\lambda^t)}(\lambda^t)$

Consequently, it is straightforward to conclude

$$\mathbf{I}_2 \sim \frac{2i}{p}\left[\left(\frac{\epsilon}{2} - iP_1^{(*)}\right) |C^{(0)}(\lambda)| + \epsilon_2 n^{(0)}(\lambda) + \epsilon_1 n^{(0)}(\lambda^t)\right]. \tag{7.31}$$

Note that this equation holds for general p. Now one has to decide in which sense to match (7.30) with (7.31). Recall that the BO Hamiltonian of rank n was given by linear combinations of (scalar) CS Hamiltonians up to this rank. Transferring the same reasoning to the current situation means that we can not securely compare the term proportional to $|C^{(0)}(\lambda)|$ since its coefficient gets shifted by some multiple of the lower rank Hamiltonian $\mathbf{I}_1$ (whose eigenvalue is proportional to $|C^{(0)}(\lambda)|$). Therefore we can compare only the relative normalization of $n^{(0)}(\lambda)$ and $n^{(0)}(\lambda^t)$ in (7.30) versus (7.31), which leads to a map among parameters

$$\frac{\epsilon_2}{\epsilon_1} = -(2\beta+1)\,. \tag{7.32}$$

As in the relation between eigenvalues of Hamiltonians of BO and ILW, also here we expect equation (7.15) to remain valid in the ILW(N,p) case. Just the Bethe roots will not be given by a simple expression anymore, instead as a series in the q parameters.

[2]This formula appears in [80], but there are typos present.

Appendix A

Lie algebra basics: classical series

For convenience, positive roots, weights for the standard representation $L(\omega_1)$ corresponding to the fundamental weight ω_1 as well as the order of the Weyl group are listed. This will allow us to build the weights of all representations entering the ADHM GLSM constructions for classical gauge groups and thus express the S^2 partition function. Write β_i for the orthonormal basis, $(\beta_i, \beta_j) = \delta_{ij}$. The material can be found in any book on Lie algebra theory, in particular [81].

A.1 A_l series

Order of the Weyl group: $|W| = (l+1)!$

Positive roots: $\Delta_+ = \{\beta_i - \beta_j \mid i, j = 1, \ldots, l+1,\ i < j\}$

Weights of the standard representation[1]: $\boldsymbol{\mu}_{L(\omega_1)} = \{\beta_i,\ i = 1, \ldots, l+1\}$

A.2 B_l series

Order of the Weyl group: $|W| = 2^l l!$

Positive roots: $\Delta_+ = \{\beta_i,\ \beta_i - \beta_j,\ \beta_i + \beta_j \mid i, j = 1, \ldots, l,\ i < j\}$

Weights of the standard representation: $\boldsymbol{\mu}_{L(\omega_1)} = \{0,\ \pm\beta_i,\ i = 1, \ldots, l\}$

A.3 C_l series

Order of the Weyl group: $|W| = 2^l l!$

[1] Actually, we have in mind $\mathfrak{u}(l+1)$, so the constraint $\sum_{i=1}^{l+1} \mu_i = 0$ is not imposed.

Positive roots: $\Delta_+ = \{2\beta_i,\ \beta_i - \beta_j,\ \beta_i + \beta_j \mid i, j = 1, \ldots, l,\ i < j\}$

Weights of the standard representation: $\boldsymbol{\mu}_{L(\omega_1)} = \{\pm\beta_i,\ i = 1, \ldots, l\}$

A.4 D_l series

Order of the Weyl group: $|W| = 2^{l-1} l!$

Positive roots: $\Delta_+ = \{\beta_i - \beta_j,\ \beta_i + \beta_j \mid i, j = 1, \ldots, l,\ i < j\}$

Weights of the standard representation: $\boldsymbol{\mu}_{L(\omega_1)} = \{\pm\beta_i,\ i = 1, \ldots, l\}$

Appendix B

Duality $Gr\left(N, N_f|N_a\right) \simeq Gr\left(N_f - N, N_f|N_a\right)$

The Grassmannian $Gr\left(N, N_f|N_a\right)$ is defined as a $U(N)$ gauge theory with N_f fundamentals and N_a antifundamentals, so we can write the partition function in the form

$$\begin{aligned} Z = \frac{1}{N!} \sum_{\{m_s \in \mathbb{Z}\}_{s=1}^N} \int_{(i\mathbb{R})^N} \prod_{s=1}^N \frac{d\tau_s}{2\pi i} z_{\text{ren}}^{-\tau_s - \frac{m_s}{2}} \bar{z}_{\text{ren}}^{-\tau_s + \frac{m_s}{2}} \prod_{s<t}^N \left(\frac{m_{st}^2}{4} - \tau_{st}^2 \right) \\ \prod_{s=1}^N \prod_{i=1}^{N_f} \frac{\Gamma\left(\tau_s - i\frac{a_i}{\hbar} - \frac{m_s}{2}\right)}{\Gamma\left(1 - \tau_s + i\frac{a_i}{\hbar} - \frac{m_s}{2}\right)} \prod_{s=1}^N \prod_{j=1}^{N_a} \frac{\Gamma\left(-\tau_s + i\frac{\widetilde{a}_j}{\hbar} + \frac{m_s}{2}\right)}{\Gamma\left(1 + \tau_s - i\frac{\widetilde{a}_j}{\hbar} + \frac{m_s}{2}\right)}, \end{aligned} \tag{B.1}$$

where $\hbar$ relates to the radius of the sphere and the renormalization scale M as $\hbar = \frac{1}{rM}$ and $a_j, \widetilde{a}_j$ are the dimensionless (rescaled by M^{-1}) equivariant weights for fundamentals and antifundamentals respectively. The renormalized Kahler coordinate z_{ren} is defined as

$$z_{\text{ren}} = e^{-2\pi\xi_{\text{ren}} + i\theta_{\text{ren}}} = \hbar^{N_a - N_f} (-1)^{N-1} z. \tag{B.2}$$

since we have

$$\xi_{\text{ren}} = \xi - \frac{1}{2\pi}(N_f - N_a)\log(rM) \ , \ \ \theta_{\text{ren}} = \theta + (N-1)\pi \tag{B.3}$$

From now on we are setting $M = 1$. We close the contours in the left half planes, so that we pick only poles coming from the fundamentals. We need to build an N-pole to saturate the integration measure. Hence the partition function becomes a sum over all possible choices of N-poles, i.e. over all combinations how to pick N objects out of N_f. Now the proposal is that duality holds separately for a fixed choice of an N-pole and its corresponding dual. For simplicity of notation let us prove the duality for a particular choice of an N-pole and its (N_f-N)-dual

$$(\underbrace{\square,\dots,\square}_{N},\underbrace{\bullet,\dots,\bullet}_{N_f-N}) \overset{\text{dual}}{\longleftrightarrow} (\underbrace{\bullet,\dots,\bullet}_{N},\underbrace{\square,\dots,\square}_{N_f-N}), \tag{B.4}$$

where boxes denote the choice of poles forming the N-pole.

B.0.1 $Gr\,(N, N_f|N_a)$

The poles are at positions

$$\tau_s = -k_s + \frac{m_s}{2} + \frac{\lambda_s}{\hbar} \tag{B.5}$$

and it still remains to be integrated over λ's around $\lambda_s = ia_s$, where s runs from 1 to N. This fully specifies from which fundamental we took the pole. Plugging this into (B.1), the integral reduces to the following form

$$\begin{aligned} Z = \oint_{\mathcal{M}} \Big\{\prod_{s=1}^{N}\frac{d\lambda_s}{2\pi i\hbar}\Big\} Z_{1l}\left(\frac{\lambda_s}{\hbar},\frac{a_i}{\hbar},\frac{\widetilde{a}_j}{\hbar}\right) z^{-\sum_{s=1}^{N}\frac{\lambda_s}{\hbar}}\,\widetilde{I}\left((-1)^{N_a}\kappa z,\frac{\lambda_s}{\hbar},\frac{a_i}{\hbar},\frac{\widetilde{a}_j}{\hbar}\right) \\ \times \bar{z}^{-\sum_{s=1}^{N}\frac{\lambda_s}{\hbar}}\,\widetilde{I}\left((-1)^{N_a}\bar{\kappa}\bar{z},\frac{\lambda_s}{\hbar},\frac{a_i}{\hbar},\frac{\widetilde{a}_j}{\hbar}\right), \end{aligned} \tag{B.6}$$

where we defined $\kappa = \hbar^{N_a-N_f}(-1)^{N-1}$, $\bar{\kappa} = (-\hbar)^{N_a-N_f}(-1)^{N-1}$. Here we are integrating over a product of circles $\mathcal{M} = \bigotimes_{r=1}^{k} S^1(ia_r,\delta)$ with δ small enough such that only the pole at the center of the circle is included. From this form we can read of the I function for $Gr\,(N, N_f|N_a)$ as

$$I = z^{-\sum_{s=1}^{N}\frac{\lambda_s}{\hbar}} \sum_{\{l_s\geq 0\}_{s=1}^{N}} \left((-1)^{N_a}\kappa z\right)^{\sum_{s=1}^{N} l_s} \prod_{s<t}^{N}\frac{\lambda_{st}-\hbar l_{st}}{\lambda_{st}} \prod_{s=1}^{N}\frac{\prod_{j=1}^{N_a}\left(\frac{-\lambda_s+i\widetilde{a}_j}{\hbar}\right)_{l_s}}{\prod_{i=1}^{N_f}\left(1+\frac{-\lambda_s+ia_i}{\hbar}\right)_{l_s}}, \tag{B.7}$$

where $x_{st} := x_s - x_t$. Now we integrate over λ's in (B.6), which is straightforward since Z_{1l} contains only simple poles and the rest is holomorphic in λ's. Finally, we get

$$Z^{(\square,\dots,\square,\bullet,\dots,\bullet)} = Z_{\text{class}} Z_{1l} Z_{\text{v}} Z_{\text{av}}, \tag{B.8}$$

where the individual pieces are given as follows

$$Z_{\text{class}} = \prod_{s=1}^{N}\left(\hbar^{2(N_a-N_f)} z\bar{z}\right)^{-\frac{ia_s}{\hbar}} \tag{B.9}$$

$$Z_{1l} = \prod_{s=1}^{N}\prod_{i=N+1}^{N_f}\frac{\Gamma\left(\frac{ia_{si}}{\hbar}\right)}{\Gamma\left(1-\frac{ia_{si}}{\hbar}\right)} \prod_{s=1}^{N}\prod_{j=1}^{N_a}\frac{\Gamma\left(-\frac{i(a_s-\widetilde{a}_j)}{\hbar}\right)}{\Gamma\left(1+\frac{i(a_s-\widetilde{a}_j)}{\hbar}\right)} \tag{B.10}$$

$$Z_{\rm v} = \sum_{\{l_s \geq 0\}_{s=1}^N} \left((-1)^{N_a} \kappa z\right)^{\sum_{s=1}^N l_s} \prod_{s<t}^N \left(1 - \frac{\hbar l_{st}}{i a_{st}}\right) \prod_{s=1}^N \frac{\prod_{j=1}^{N_a} \left(-i \frac{a_s - \widetilde{a}_j}{\hbar}\right)_{l_s}}{\prod_{i=1}^{N_f} \left(1 - i \frac{a_{si}}{\hbar}\right)_{l_s}} \tag{B.11}$$

$$Z_{\rm av} = Z_{\rm v}\left[\kappa z \to \bar{\kappa} \bar{z}\right] \tag{B.12}$$

To prove the duality it is actually better to manipulate $Z_{\rm v}$ to a more convenient form (combining the contributions of the vectors and fundamentals by using identities between the Pochhammers)

$$Z_{\rm v} = \sum_{l=0}^{\infty} \left[(-1)^{N_a + N - N_f} \kappa z\right]^l Z_l \tag{B.13}$$

with Z_l given by

$$Z_l = \sum_{\{l_s \geq 0 | \sum_{s=1}^N l_s = l\}} \prod_{s=1}^N \frac{\prod_{j=1}^{N_a} \left(-i \frac{a_s - \widetilde{a}_j}{\hbar}\right)_{l_s}}{l_s! \prod_{i \neq s}^N \left(i \frac{a_{si}}{\hbar} - l_s\right)_{l_i} \prod_{i=N+1}^{N_f} \left(i \frac{a_{si}}{\hbar} - l_s\right)_{l_s}}. \tag{B.14}$$

B.0.2 The dual theory $Gr\left(N_f - N, N_f | N_a\right)$

Going to the dual theory not only the rank of the gauge group changes to $N_f - N$, but there is a new feature arising. New matter fields $M^i_{\bar{j}}$ appear, they are singlets under the gauge group and couple to the fundamentals and antifundamentals via a superpotential $W^D = \widetilde{\phi}^{\mu \bar{j}} M^i_{\bar{j}} \phi_{\mu i}$. So the partition function gets a new contribution from the mesons M (we set $N^D = N_f - N$)

$$\begin{aligned} Z = \frac{1}{N^D!} \sum_{\{m_s \in \mathbb{Z}\}_{s=1}^{N^D}} \int_{(i\mathbb{R})^{N^D}} \prod_{s=1}^{N^D} \frac{d\tau_s}{2\pi i} \left(z^D_{ren}\right)^{-\tau_s - \frac{m_s}{2}} \left(\bar{z}^D_{ren}\right)^{-\tau_s + \frac{m_s}{2}} \prod_{s<t}^{N^D} \left(\frac{m_{st}^2}{4} - \tau_{st}^2\right) \\ \prod_{s=1}^{N^D} \prod_{i=1}^{N_f} \frac{\Gamma\left(\tau_s + i \frac{a^D_i}{\hbar} - \frac{m_s}{2}\right)}{\Gamma\left(1 - \tau_s - i \frac{a^D_i}{\hbar} - \frac{m_s}{2}\right)} \prod_{s=1}^{N^D} \prod_{j=1}^{N_a} \frac{\Gamma\left(-\tau_s - i \frac{\widetilde{a}^D_j}{\hbar} + \frac{m_s}{2}\right)}{\Gamma\left(1 + \tau_s + i \frac{\widetilde{a}^D_j}{\hbar} + \frac{m_s}{2}\right)} \prod_{i=1}^{N_f} \prod_{j=1}^{N_a} \frac{\Gamma\left(-i \frac{a_i - \widetilde{a}_j}{\hbar}\right)}{\Gamma\left(1 + i \frac{a_i - \widetilde{a}_j}{\hbar}\right)}, \end{aligned} \tag{B.15}$$

where the last factor is the new contribution of the mesons (note that it depends on the original equivariant weights, not on the dual ones). All the computations are analogue to the previous case, so we give the result right after integration

$$Z^{(\bullet, \dots, \bullet, \square, \dots, \square)} = Z^D_{\rm class} Z^D_{11} Z^D_{\rm v} Z^D_{\rm av}, \tag{B.16}$$

where the building blocks are

$$Z^D_{\text{class}} = \prod_{s=N+1}^{N_f} \left(\hbar^{2(N_a-N_f)} z^D \bar{z}^D\right)^{-\frac{ia^D_s}{\hbar}} \tag{B.17}$$

$$Z^D_{11} = \prod_{s=N+1}^{N_f} \prod_{i=N+1}^{N_f} \frac{\Gamma\left(\frac{ia^D_{si}}{\hbar}\right)}{\Gamma\left(1-\frac{ia^D_{si}}{\hbar}\right)} \prod_{j=1}^{N_a} \frac{\Gamma\left(-\frac{i(a^D_s-\widetilde{a}^D_j)}{\hbar}\right)}{\Gamma\left(1+\frac{i(a^D_s-\widetilde{a}^D_j)}{\hbar}\right)} \prod_{i=1}^{N_f}\prod_{j=1}^{N_a} \frac{\Gamma\left(-i\frac{a_i-\widetilde{a}_j}{\hbar}\right)}{\Gamma\left(1+i\frac{a_i-\widetilde{a}_j}{\hbar}\right)} \tag{B.18}$$

$$Z^D_{\text{v}} = \sum_{l=0}^{\infty} \left[(-1)^{N_a-N}\left(\kappa z\right)^D\right]^l Z^D_l \tag{B.19}$$

$$Z^D_{\text{av}} = \sum_{k=0}^{\infty} \left[(-1)^{N_a-N}\left(\bar{\kappa} \bar{z}\right)^D\right]^k Z^D_k \tag{B.20}$$

with Z^D_l given by

$$Z^D_l = \sum_{\{l_s\geq 0 | \sum_{s=N+1}^{N_f} l_s = l\}} \prod_{s=N+1}^{N_f} \frac{\prod_{j=1}^{N_a}\left(-i\frac{a^D_s-\widetilde{a}^D_j}{\hbar}\right)_{l_s}}{l_s! \prod_{\substack{i=N+1\\ i\neq s}}^{N_f}\left(i\frac{a^D_{si}}{\hbar}-l_s\right)_{l_i} \prod_{i=1}^{N}\left(i\frac{a^D_{si}}{\hbar}-l_s\right)_{l_s}}. \tag{B.21}$$

B.0.3 Duality map

We are now ready to discuss the duality between the two theories. The statement is the following. For $N_f \geq N_a + 2$, there exists a duality map $z^D = z^D(z)$ and $a^D_j = a^D_j(a_j)$, $\widetilde{a}^D_j = \widetilde{a}^D_j(\widetilde{a}_j)$ under which the partition functions for $Gr\left(N, N_f|N_a\right)$ and $Gr\left(N_f-N, N_f|N_a\right)$ are equal.[1] In the first step we will construct the duality map and then we will show that (B.9–B.14) indeed match with (B.17–B.21). The partition function is a double power series in z and $\bar{z}$ multiplied by Z_{class}. In order to achieve equality of the partition functions, Z_{class} have to be equal after duality map and then the power series have to match term by term. Moreover we can look only at the holomorphic piece Z_{v}, for the antiholomorphic everything goes in a similar way. The constant term is Z_{11}, which is a product of gamma functions with arguments linear in the equivariant weights. This implies that the duality map for the equivariant weights is linear. But then the map between the Kahler coordinates can be only a rescaling since a constant term would destroy the matching of Z_{11}. So we arrive at the most general ansatz for the duality map

$$z^D = sz \tag{B.22}$$

$$\frac{a^D_i}{\hbar} = -E\frac{a_i}{\hbar} + C \tag{B.23}$$

[1] We will see the reason for this range later.

$$\frac{\widetilde{a}_j^D}{\hbar} = -F\frac{\widetilde{a}_j}{\hbar} + D \tag{B.24}$$

Matching the constant terms Z_{11} gives the constraints

$$E = F = 1,\ D = -(C + i). \tag{B.25}$$

Imposing further the equivalence of Z_{class} fixes C to be

$$C = \frac{1}{N_f - N}\sum_{i=1}^{N_f}\frac{a_i}{\hbar}. \tag{B.26}$$

We are now at a position where Z_{class} and Z_{11} match, while the only remaining free parameter in the duality map is s. We fix it by looking at the linear terms in Z_{v} and Z_{v}^D. Of course this does not assure that all higher order terms do match, but we will show that this is the case for $N_f \geq N_a + 2$.[2] So taking only $k = 1$ contributions in Z_{v} and Z_{v}^D we get for s

$$s = (-1)^{N-1}\frac{\mathcal{N}}{\mathcal{D}}, \tag{B.27}$$

where

$$\mathcal{N} = \sum_{s=1}^{N}\frac{\prod_{j=1}^{N_a}\left(-i\frac{a_s-\widetilde{a}_j}{\hbar}\right)}{\prod_{i\neq s}^{N}\left(-i\frac{a_{si}}{\hbar}\right)\prod_{i=N+1}^{N_f}\left(1-i\frac{a_{si}}{\hbar}\right)} \tag{B.28}$$

$$\mathcal{D} = \sum_{s=N+1}^{N_f}\frac{\prod_{j=1}^{N_a}\left(1+i\frac{a_s-\widetilde{a}_j}{\hbar}\right)}{\prod_{i=1}^{N}\left(1+i\frac{a_{si}}{\hbar}\right)\prod_{\substack{i=N+1\\ j\neq s}}^{N_f}\left(-i\frac{a_{si}}{\hbar}\right)}. \tag{B.29}$$

The proposal is that for $N_f \geq N_a + 2$

$$s = (-1)^{N_a}. \tag{B.30}$$

Out of this range s is a complicated rational function in the equivariant parameters. This completes the duality map for $N_f \geq N_a + 2$ and suggests that there is no duality map for $N_f < N_a + 2$.

[2]A direct computation for a handful of examples suggests that higher order terms do not match for s obtained as just outlined if $N_f < N_a + 2$.

B.0.4 Proof of equivalence of the partition functions

By construction of the duality map we know that Z_{class}, Z_{1l} and moreover also the linear terms in Z_{v} match. Now we will prove (d.m. is the shortcut for duality map)

$$Z_{\text{v}} = Z_{\text{v}}^D|_{d.m.} \tag{B.31}$$

for $N_f \geq N_a + 2$. Looking at (B.13) and (B.19) we see that this boils down to

$$Z_l = (-1)^{N_a l} Z_l^D|_{d.m.}. \tag{B.32}$$

The key to prove the above relation is to write Z_l as a contour integral

$$Z_l = \int_{\mathcal{C}_u} \prod_{\alpha=1}^{l} \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right)\Big|_{\epsilon=1}, \tag{B.33}$$

where $\mathcal{C}_u$ is a product of contours having the real axes as base and then are closed in the upper half plane by a semicircle. The integrand has the form

$$f = \frac{1}{\epsilon^l l!} \prod_{\alpha<\beta}^{l} \frac{(\phi_\alpha - \phi_\beta)^2}{(\phi_\alpha - \phi_\beta)^2 - \epsilon^2} \prod_{\alpha=1}^{l} \frac{\prod_{j=1}^{N_a}\left(i\frac{\widetilde{a}_j}{\hbar} + \phi_\alpha\right)}{\prod_{i=1}^{N}\left(\phi_\alpha + i\frac{a_i}{\hbar}\right)\prod_{i=N+1}^{N_f}\left(-i\frac{a_i}{\hbar} - \epsilon - \phi_\alpha\right)}. \tag{B.34}$$

It is necessary to add small imaginary parts to ϵ and a_i, $\epsilon \to \epsilon + i\delta$, $-ia_i \to -ia_i + i\hbar\delta'$ with $\delta > \delta'$. The proof of (B.33) goes by direct evaluation. First we have to classify the poles. Due to the imaginary parts assignments, they are at [1]

$$\phi_\alpha = -i\frac{a_i}{\hbar}, \qquad \alpha = 1, \ldots, l, \quad i = 1, \ldots, N \tag{B.35}$$

$$\phi_\beta = \phi_\alpha + \epsilon, \qquad \beta \geq \alpha \tag{B.36}$$

We have to build an l-pole, which means that the poles are classified by partitions of l into N parts, $l = \sum_{I=1}^{N} l_I$. The I-th Young tableau $YT(l_I)$ with l_I boxes can be only 1-dimensional (we choose a row) since we have only one ϵ to play with. To illustrate what we have in mind, we show an example of a possible partition

$$(\underbrace{\square\square\square}_{l_1}, \bullet, \square\square, \square, \ldots, \square\square, \underbrace{\bullet}_{l_N}). \tag{B.37}$$

[1] One has to assume a_i to be imaginary at this point. The general result is obtained by analytic continuation after integration.

Residue theorem then turns the integral into a sum over all such partitions and the poles corresponding to a given partition are given as

$$\phi^I_{n_I} = -i\frac{a_I}{\hbar} + (n_I - 1)\epsilon + \lambda^I_{n_I}, \tag{B.38}$$

where $I = 1, \ldots, N$ labels the position of the Young tableau in the N-vector and $n_I = 1, \ldots, l_I$ labels the boxes in $YT(l_I)$. Substituting this in (B.33) we get (the $l!$ gets cancelled by the permutation symmetry of the boxes)

$$\begin{aligned}
Z_l = &\frac{1}{\epsilon^l} \sum_{\{l_I \geq 0 | \sum_{I=1}^N l_I = l\}} \oint_{\mathcal{M}} \prod_{\substack{I=1\\ l_I\neq 0}}^N \prod_{n_I=1}^{l_I} \frac{d\lambda^I_{n_I}}{2\pi i} \\
&\times \prod_{\substack{I\neq J\\ l_I\neq 0, l_J\neq 0}}^N \prod_{n_I=1}^{l_I}\prod_{n_J=1}^{l_J} \frac{\left(-i\frac{a_{IJ}}{\hbar} + n_{IJ}\epsilon + \lambda^{I,J}_{n_I,n_J}\right)}{\left(-i\frac{a_{IJ}}{\hbar} + (n_{IJ}-1)\epsilon + \lambda^{I,J}_{n_I,n_J}\right)} \prod_{\substack{I=1\\ l_I\neq 0}}^N \prod_{n_I\neq n_J}^{l_I} \frac{\left(n_{IJ}\epsilon + \lambda^{I,I}_{n_I,n_J}\right)}{\left((n_{IJ}-1)\epsilon + \lambda^{I,I}_{n_I,n_J}\right)} \\
&\times \prod_{\substack{I=1\\ l_I\neq 0}}^N \prod_{n_I=1}^{l_I} \frac{\prod_{j=1}^{N_a}\left(i\frac{\tilde{a}_j}{\hbar} - i\frac{a_I}{\hbar} + (n_I-1)\epsilon + \lambda^I_{n_I}\right)}{\prod_{r=1}^N \left(-i\frac{a_{Ir}}{\hbar} + (n_I-1)\epsilon + \lambda^I_{n_I}\right)\prod_{r=N+1}^{N_f}\left(-i\frac{a_{Ir}}{\hbar} - n_I\epsilon - \lambda^I_{n_I}\right)},
\end{aligned} \tag{B.39}$$

where we integrate over $\mathcal{M} = \bigotimes_{r=1}^l S^1(0,\delta)$. The computation continues as follows. We separate the poles in λ's (there are only simple poles), the rest is a holomorphic function, so we can effectively set the λ's to zero there. Eventually, we obtain

$$\begin{aligned}
Z_l = &\frac{1}{\epsilon^l} \sum_{\{l_I \geq 0 | \sum_{I=1}^N l_I = l\}} \left[\oint_{\mathcal{M}} \prod_{\substack{I=1\\ l_I\neq 0}}^N \left\{ \left(\prod_{n_I=1}^{l_I} \frac{d\lambda^I_{n_I}}{2\pi i}\right)\left(\frac{1}{\lambda^I_1}\prod_{n_I=1}^{l_I-1}\frac{1}{\lambda^{I,I}_{n_I+1,n_I}}\right)\right\}\right] \\
&\times \prod_{I\neq J}^N \frac{\left(1 + i\frac{a_{IJ}}{\hbar\epsilon} - l_I\right)_{l_J}}{\left(1 + i\frac{a_{IJ}}{\hbar\epsilon}\right)_{l_J}} \prod_{\substack{I=1\\ l_I\neq 0}}^N \frac{\epsilon^{l_I-1}}{l_I} \\
&\times \frac{\prod_{I=1}^N \prod_{j=1}^{N_a} \epsilon^{l_I}\left(\frac{i\frac{\tilde{a}_j}{\hbar} + a_I}{\epsilon}\right)}{\prod_{I=1}^N\prod_{r\neq I}^N \epsilon^{l_I}\left(-i\frac{a_{Ir}}{\hbar\epsilon}\right)\prod_{\substack{I=1\\ l_I\neq 0}}^N \epsilon^{l_I-1}(l_I-1)!\prod_{I=1}^N\prod_{r=N+1}^{N_f}\epsilon^{l_I}\left(-i\frac{a_{rI}}{\hbar\epsilon}\right)},
\end{aligned} \tag{B.40}$$

where the integration gives $[\ldots] = 1$. We are left with products of ratios including the equivariant parameters, which we express as Pochhammer symbols and after heavy Pochhammer algebra we finally arrive at (B.14), which proves (B.33).

Now, if the integrand f does not have poles at infinity, which happens exactly for $N_f \geq N_a + 2$, we can write

$$\int_{\mathcal{C}_u} \prod_{\alpha=1}^{l} \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right) = (-1)^l \int_{\mathcal{C}_d} \prod_{\alpha=1}^{l} \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right) \tag{B.41}$$

with $\mathcal{C}_d$ having the same base as $\mathcal{C}_u$ but is closed in the lower half plane by a semicircle. Both contours are oriented counterclockwise. The lovely fact is that the r.h.s. of the above equation gives the desired result

$$(-1)^l \int_{\mathcal{C}_d} \prod_{\alpha=1}^{l} \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right) \Big|_{\epsilon=1} = (-1)^{N_a l} Z_l^D|_{d.m.} \tag{B.42}$$

after direct evaluation of the integral, completely analogue to that of (B.33).

B.0.5 Example: the $Gr(1,3) \simeq Gr(2,3)$ case

Let us show this isomorphism explicitly in a simple case: we will consider $Gr(1,3)$ and $Gr(2,3)$ in a completely equivariant setting.
Let us first compute the equivariant partition function for $Gr(1,3)$:

$$\begin{aligned} Z_{Gr(1,3)} &= \sum_m \int \frac{d\tau}{2\pi i} e^{4\pi\xi_{\text{ren}}\tau - i\theta_{\text{ren}} m} \prod_{j=1}^{3} \frac{\Gamma(\tau + irMa_j - \frac{m}{2})}{\Gamma(1 - \tau - irMa_j - \frac{m}{2})} \\ &= \sum_{i=1}^{3} ((rM)^6 z\bar{z})^{irMa_i} \prod_{\substack{j=1 \\ j\neq i}}^{3} \frac{\Gamma(-irMa_{ij})}{\Gamma(1+irMa_{ij})} \sum_{l\geq 0} \frac{[(rM)^3 z]^l}{\prod_{j=1}^{3}(1+irMa_{ij})_l} \sum_{k\geq 0} \frac{[(-rM)^3 \bar{z}]^k}{\prod_{j=1}^{3}(1+irMa_{ij})_k} \end{aligned} \tag{B.43}$$

Here we defined $a_{ij} = a_i - a_j$, and the twisted masses have been rescaled according to $a_i \to Ma_i$, so they are now dimensionless. For $Gr(2,3)$ we have (with $\tilde{\theta}_{\text{ren}} = \tilde{\theta} + \pi =$

$\tilde{\theta}+3\pi$, being $\tilde{\theta}\longrightarrow\tilde{\theta}+2\pi$ a symmetry of the theory)

$$\begin{aligned}
Z_{Gr(2,3)} =& \frac{1}{2}\sum_{m_1,m_2}\int\frac{d\tau_1}{2\pi i}\frac{d\tau_2}{2\pi i}e^{4\pi\tilde{\xi}_{\text{ren}}(\tau_1+\tau_2)-i\tilde{\theta}_{\text{ren}}(m_1+m_2)}\\
&\left(-\tau_{12}^2+\frac{m_{12}^2}{4}\right)\prod_{r=1}^{2}\prod_{j=1}^{3}\frac{\Gamma(\tau_r+irM\tilde{a}_j-\frac{m_r}{2})}{\Gamma(1-\tau_r-irM\tilde{a}_j-\frac{m_r}{2})}\\
=&\sum_{i<j}^{3}((rM)^6\tilde{z}\bar{\tilde{z}})^{irM(\tilde{a}_i+\tilde{a}_j)}\prod_{\substack{k=1\\k\neq i,j}}^{3}\frac{\Gamma(-irM\tilde{a}_{ik})}{\Gamma(1+irM\tilde{a}_{ik})}\frac{\Gamma(-irM\tilde{a}_{jk})}{\Gamma(1+irM\tilde{a}_{jk})}\\
&\sum_{l_1,l_2\geq 0}\frac{[(-rM)^3\tilde{z}]^{l_1+l_2}}{\prod_{k=1}^{3}(1+irM\tilde{a}_{ik})_{l_1}\prod_{k=1}^{3}(1+irM\tilde{a}_{jk})_{l_2}}\frac{l_1-l_2+irM\tilde{a}_i-irM\tilde{a}_j}{irM\tilde{a}_i-irM\tilde{a}_j}\\
&\sum_{k_1,k_2\geq 0}\frac{[(rM)^3\bar{\tilde{z}}]^{k_1+k_2}}{\prod_{k=1}^{3}(1+irM\tilde{a}_{ik})_{k_1}\prod_{k=1}^{3}(1+irM\tilde{a}_{jk})_{k_2}}\frac{k_1-k_2+irM\tilde{a}_i-irM\tilde{a}_j}{irM\tilde{a}_i-irM\tilde{a}_j}
\end{aligned}\tag{B.44}$$

In both situations, we are assuming $a_1+a_2+a_3=0$ and $\tilde{a}_1+\tilde{a}_2+\tilde{a}_3=0$. Consider now the partition $(\bullet,\bullet,\square)$ for $Gr(1,3)$ and the dual partition $(\square,\square,\bullet)$ for $Gr(2,3)$; we have respectively

$$\begin{aligned}
Z^{(\bullet,\bullet,\square)}_{Gr(1,3)} =& ((rM)^6 z\bar{z})^{irMa_3}\frac{\Gamma(-irMa_{31})}{\Gamma(1+irMa_{31})}\frac{\Gamma(-irMa_{32})}{\Gamma(1+irMa_{32})}\\
&\sum_{l\geq 0}\frac{[(rM)^3 z]^l}{l!(1+irMa_{31})_l(1+irMa_{32})_l}\\
&\sum_{k\geq 0}\frac{[(-rM)^3\bar{z}]^k}{k!(1+irMa_{31})_k(1+irMa_{32})_k}\\
Z^{(\square,\square,\bullet)}_{Gr(2,3)} =& ((rM)^6\tilde{z}\bar{\tilde{z}})^{irM(\tilde{a}_1+\tilde{a}_2)}\frac{\Gamma(-irM\tilde{a}_{13})}{\Gamma(1+irM\tilde{a}_{13})}\frac{\Gamma(-irM\tilde{a}_{23})}{\Gamma(1+irM\tilde{a}_{23})}\\
&\sum_{l_1,l_2\geq 0}\frac{[(-rM)^3\tilde{z}]^{l_1+l_2}}{\prod_{i=1}^{2}l_i!\prod_{j\neq i}^{3}(1+irM\tilde{a}_{ij})_{l_i}}\frac{l_1-l_2+irM\tilde{a}_1-irM\tilde{a}_2}{irM\tilde{a}_1-irM\tilde{a}_2}\\
&\sum_{k_1,k_2\geq 0}\frac{[(rM)^3\bar{\tilde{z}}]^{k_1+k_2}}{\prod_{i=1}^{2}k_i!\prod_{j\neq i}^{3}(1+irM\tilde{a}_{ij})_{k_i}}\frac{k_1-k_2+irM\tilde{a}_1-irM\tilde{a}_2}{irM\tilde{a}_1-irM\tilde{a}_2}
\end{aligned}\tag{B.45}$$

Since

$$\begin{aligned}
&\sum_{l_1,l_2\geq 0}\frac{[(-rM)^3\tilde{z}]^{l_1+l_2}}{\prod_{i=1}^{2}l_i!\prod_{j\neq i}^{3}(1+irM\tilde{a}_{ij})_{l_i}}\frac{l_1-l_2+irM\tilde{a}_1-irM\tilde{a}_2}{irM\tilde{a}_1-irM\tilde{a}_2} =\\
&=\sum_{l\geq 0}\frac{[(-rM)^3\tilde{z}]^l}{l!(1+irM\tilde{a}_{13})_l(1+irM\tilde{a}_{23})_l}c_l
\end{aligned}\tag{B.46}$$

and

$$c_l = \sum_{l_1=0}^{l} \frac{l!}{l_1!(l-l_1)!} \frac{(1+irM\tilde{a}_{23}+l-l_1)_{l_1}(1+irM\tilde{a}_{13}+l_1)_{l-l_1}}{(irM\tilde{a}_{12}-l+l_1)_{l_1}(-irM\tilde{a}_{12}-l_1)_{l-l_1}} = (-1)^l = (-1)^{3l}$$

we can conclude that $Z_{Gr(1,3)}^{(\bullet,\bullet,\square)} = Z_{Gr(2,3)}^{(\square,\square,\bullet)}$ if we identify $a_i = -\tilde{a}_i$ and $\xi = \tilde{\xi}$, $\theta = \tilde{\theta}$ (i.e., $z = \tilde{z}$). It is then easy to prove that $Z_{Gr(1,3)} = Z_{Gr(2,3)}$.

Appendix C

Details on the proof of (6.45) and (6.50)

C.1 Proof of (6.45)

First of all we pass to the ζ-function representation of (6.45) by employing the identity

$$\frac{\theta_1'\left(\frac{\pi}{L}z\right)}{\theta_1\left(\frac{\pi}{L}z\right)} = \zeta(z) - \frac{2\eta_1}{L}z. \tag{C.1}$$

The dependence on η_1 is immaterial as it drops out in the resulting equations of motion. After doing so and computing $\ddot{x}_j$ from (6.45) we get

$$\ddot{x}_j = -G^2\left(L_1 + L_2 + L_3\right), \tag{C.2}$$

where

$$\begin{aligned} L_1 = &-\sum_{k=1}^{N}\wp(x_j - y_k)\Big[\sum_{l=1}^{N}\zeta(x_j - y_l) - \sum_{l\neq j}\zeta(x_j - x_l) + \sum_{l=1}^{N}\zeta(y_k - x_l) - \sum_{l\neq k}\zeta(y_k - y_l)\Big] \\ &+ \sum_{k\neq j}\wp(x_j - x_k)\Big[\sum_{l=1}^{N}\zeta(x_j - y_l) - \sum_{l\neq j}\zeta(x_j - x_l) - \sum_{l=1}^{N}\zeta(x_k - y_l) + \sum_{l\neq k}\zeta(x_k - x_l)\Big] \end{aligned} \tag{C.3}$$

$$\begin{aligned} L_2 = \frac{2\eta_1}{L}\Bigg\{ &-\sum_{k\neq j}\left(\wp(x_j - x_k) + \frac{2\eta_1}{L}\right)\Big[\sum_{l}(x_j - y_l) - \sum_{l\neq j}(x_j - x_l) - \sum_{l}(x_k - y_l) + \sum_{l\neq k}(x_k - x_l)\Big] \\ &+ \sum_{k}\left(\wp(x_j - y_k) + \frac{2\eta_1}{L}\right)\Big[\sum_{l}(x_j - y_l) - \sum_{l\neq j}(x_j - x_l) + \sum_{l}(y_k - x_l) - \sum_{l\neq k}(y_k - y_l)\Big]\Bigg\} \end{aligned} \tag{C.4}$$

$$L_3 = \frac{2\eta_1}{L}\Bigg\{ -\sum_k \Big[\sum_l \zeta(x_j - y_l) - \sum_{l\neq j}\zeta(x_j - x_l) + \sum_l \zeta(y_k - x_l) - \sum_{l\neq k}\zeta(y_k - y_l)\Big]$$
$$+\sum_{k\neq j}\Big[\sum_l \zeta(x_j - y_l) - \sum_{l\neq j}\zeta(x_j - x_l) - \sum_l \zeta(x_k - y_l) + \sum_{l\neq k}\zeta(x_k - x_l)\Big]\Bigg\}. \tag{C.5}$$

Now we show that L_2 and L_3 actually vanish. For L_2 it is straightforward, since $[\ldots]$ in the first row vanishes for all $k \neq j$ and $[\ldots]$ in the second row vanishes for all k, respectively. Both facts follow easily just by writing the sums as $\sum_{l\neq k,j}(\ldots) + [\text{rest}]$. Slightly more involved is vanishing of L_3. Collecting sums with common range as above, we finally arrive at a relation

$$L_3 = \frac{2\eta_1}{L}\Bigg\{\Big[\sum_{k\neq j}\{\zeta(x_j - x_k) + \sum_{l\neq k}\zeta(x_k - x_l)\}\Big] + \Big[(y_j - y_k)\Big] - \Big[(x_j - y_k)\Big] - \Big[(y_j - x_k)\Big]\Bigg\}. \tag{C.6}$$

which vanishes term by term since

$$\begin{aligned}
&\sum_{k\neq j}\{\zeta(u_j - v_k) + \sum_{l\neq k}\zeta(v_k - u_l)\} = \sum_{k\neq j}\{\zeta(u_j - v_k) + \zeta(v_k - u_j) + \sum_{l\neq k,j}\zeta(v_k - u_l)\}\\
&= \sum_{k\neq j}\sum_{l\neq k,j}\zeta(v_k - u_l) = \sum_{\substack{\text{pairs}(m,n), m\neq n\\ (m,n)\neq j}}\Big[\zeta(v_m - u_n) + \zeta(u_n - v_m)\Big] = 0,
\end{aligned} \tag{C.7}$$

where we used that ζ is odd. Summarizing, we have $\ddot{x}_j = -G^2 L_1$ which matches (6.44) in force of the following identity between Weierstrass $\wp$ and ζ functions

$$\begin{aligned}
0 &= \sum_{k\neq j}\wp'(x_j - x_k)\\
&+\sum_{k=1}^N \wp(x_j - y_k)\Big[\sum_{l=1}^N \zeta(x_j - y_l) - \sum_{l\neq j}\zeta(x_j - x_l) + \sum_{l=1}^N \zeta(y_k - x_l) - \sum_{l\neq k}\zeta(y_k - y_l)\Big]\\
&-\sum_{k\neq j}\wp(x_j - x_k)\Big[\sum_{l=1}^N \zeta(x_j - y_l) - \sum_{l\neq j}\zeta(x_j - x_l) - \sum_{l=1}^N \zeta(x_k - y_l) + \sum_{l\neq k}\zeta(x_k - x_l)\Big].
\end{aligned} \tag{C.8}$$

We prove this identity using Liouville's theorem. Let us denote the right hand side by $R\left(x_j; \{x_k\}_{k\neq j}, \{y_k\}_{k=1}^N\right)$. R is a symmetric function under independent permutations of $\{x_k\}_{k\neq j}$ and $\{y_k\}_{k=1}^N$, respectively. Next, we show double periodicity in all variables. Although the ζ's introduce shifts, these cancel each other[1], so double periodicity follows immediately. The non-trivial step is to show holomorphicity. First, the relation should

[1]All ζ's appear in pairs, where a given variable appears with positive and negative signs in the argument.

hold for all j. In particular we can choose $j = 1$, other cases are obtained just by relabeling. By double periodicity we can focus only on poles at the origin, so there will be poles in $x_j - y_k$ and $x_j - x_l$, $l \neq j$. By the symmetries described above we have to check only three cases: $x_1 - y_1$, $x_2 - y_1$ and $x_1 - x_2$. To do so, we use the Laurent series for $\wp$ and ζ

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \wp^R(z), \;\; \wp^R(z) = \sum_{n=1}^{\infty} c_{n+1} z^{2n} \\
\zeta(z) &= \frac{1}{z} + \zeta^R(z), \quad \zeta^R(z) = -\sum_{n=1}^{\infty} \frac{c_{n+1}}{2n+1} z^{2n+1}
\end{aligned} \tag{C.9}$$

Let us now show the vanishing of the residues at each pole.

Pole in $x_2 - y_1$

There are only two terms in (C.8) contributing

$$\begin{aligned}
&\zeta(x_2 - y_1)\Big[\wp(x_1 - x_2) - \wp(x_1 - y_1)\Big] \\
&\sim \frac{1}{x_2 - y_1}\Big[\frac{1}{(x_1 - x_2)^2} - \frac{1}{(x_1 - y_1)^2} + \sum_{n\geq 1} c_{n+1}\left((x_1 - x_2)^{2n} - (x_1 - y_1)^{2n}\right)\Big] \\
&= \frac{x_2 - y_1}{x_2 - y_1}\Big[\frac{1}{(x_1 - x_2)^2 (x_1 - y_1)} + \sum_{n\geq 1} c_{n+1} \sum_{k=1}^{2n} \binom{2n}{k} (-1)^k x_1^{2n-k} \sum_{l=0}^{k-1} x_2^{k-1-l} y_1^l\Big].
\end{aligned} \tag{C.10}$$

So indeed the residue vanishes.

Pole in $x_1 - y_1$

The terms contributing to this pole read

$$\begin{aligned}
&\wp(x_1 - y_1)\sum_{k\neq 1}\Big\{\big[\zeta(x_1 - y_k) - \zeta(y_1 - y_k)\big] - \big[\zeta(x_1 - x_k) - \zeta(y_1 - x_k)\big]\Big\} \\
&+ \zeta(x_1 - y_1)\sum_{k\neq 1}\Big[\;\wp(x_1 - y_k) - \wp(x_1 - x_k)\Big] \\
&\sim \frac{1}{(x_1 - y_1)^2}\sum_{k\neq 1}\Big\{\Big[\frac{1}{x_1 - y_k} - \frac{1}{y_1 - y_k}\Big] - \Big[\frac{1}{x_1 - x_k} - \frac{1}{y_1 - x_k}\Big] \\
&\qquad\qquad + \Big[\zeta^R(x_1 - y_k) - \zeta^R(y_1 - y_k)\Big] - \Big[\zeta^R(x_1 - x_k) - \zeta^R(y_1 - x_k)\Big]\Big\} \\
&+ \frac{1}{x_1 - y_1}\sum_{k\neq 1}\Big[\wp^R(x_1 - y_k) - \wp^R(x_1 - x_k) + \frac{1}{(x_1 - y_k)^2} - \frac{1}{(x_1 - x_k)^2}\Big].
\end{aligned} \tag{C.11}$$

Collecting all the rational terms gives a regular term

$$\sum_{k\neq 1}\Big[\frac{1}{(x_1-x_k)^2(y_1-x_k)}-\frac{1}{(x_1-y_k)^2(y_1-y_k)}\Big] \tag{C.12}$$

and we stay with the rest

$$\sum_{k\neq 1}\frac{1}{x_1-y_1}\Bigg\{\wp^R(x_1-y_k)-\wp^R(x_1-x_k)+\frac{1}{x_1-y_1}\Big[\big(\zeta^R(x_1-y_k)-\zeta^R(y_1-y_k)\big) \\ -\big(\zeta^R(x_1-x_k)-\zeta^R(y_1-x_k)\big)\Big]\Bigg\}. \tag{C.13}$$

In the following we show that the terms in the square parenthesis in the above formula factorizes a term (x_1-y_1) which, after combining with the rest, cancels the pole completely. Indeed, we just use (C.9) and binomial theorem to get

$$\begin{aligned}\Big[\dots\Big] &= -(x_1-y_1)\sum_{n\geq 1}\frac{c_{n+1}}{2n+1}\sum_{l=1}^{2n}\binom{2n+1}{l}(-1)^l\left(y_k^{2n+1-l}-x_k^{2n+1-l}\right)\sum_{m=0}^{l-1}y_1^{l-1-m}x_1^m \\ \wp^R(x_1-y_k)-\wp^R(x_1-x_k) &= \sum_{n\geq 1}c_{n+1}\sum_{l=1}^{2n}\binom{2n}{l-1}(-1)^l x_1^{l-1}\left(y_k^{2n+1-l}-x_k^{2n+1-l}\right)\end{aligned} \tag{C.14}$$

and after combining these two terms we get

$$\Big\{\dots\Big\} = \sum_{n\geq 1}c_{n+1}\sum_{l=1}^{2n}\binom{2n}{l-1}(-1)^l\left(y_k^{2n+1-l}-x_k^{2n+1-l}\right)\Big[x_1^{l-1}-\frac{1}{l}\sum_{m=0}^{l-1}y_1^{l-1-m}x_1^m\Big], \tag{C.15}$$

however the terms in the square brackets of (C.15) factorizes once more a term (x_1-y_1)

$$\Big[\dots\Big] = (x_1-y_1)\frac{1}{l}\sum_{m=1}^{l-1}(l-m)x_1^{l-1-m}y_1^{m-1} \tag{C.16}$$

so that we end up with a regular term

$$\sum_{k\neq 1}\sum_{n\geq 1}c_{n+1}\sum_{l=1}^{2n}\binom{2n}{l-1}\frac{(-1)^l}{l}\left(y_k^{2n+1-l}-x_k^{2n+1-l}\right)\sum_{m=1}^{l-1}(l-m)x_1^{l-1-m}y_1^{m-1}. \tag{C.17}$$

Summarizing, we have shown the vanishing of the residue at the pole in (x_1-y_1) and we now move on to the last one.

Pole in $x_1 - x_2$

Analysis of (C.8) gives the following terms contributing to this pole

$$\begin{aligned}&\wp'(x_1-x_2)+\zeta(x_1-x_2)\Big[\sum_{k\neq 1,2}\wp(x_1-x_k)-\sum_k\wp(x_1-y_k)\Big]\\&-\wp(x_1-x_2)\Big[\sum_k\zeta(x_1-y_k)-\sum_{k\neq 1}\zeta(x_1-x_k)-\sum_k\zeta(x2-y_k)+\sum_{k\neq 2}\zeta(x_2-x_k)\Big].\end{aligned}\tag{C.18}$$

In analogy with the previous case let us first deal with the rational terms

$$\begin{aligned}&\frac{-2}{(x_1-x_2)^3}+\frac{1}{x_1-x_2}\Big[\sum_{k\neq 1,2}\frac{1}{(x_1-x_k)^2}-\sum_k\frac{1}{(x_1-y_k)^2}\Big]\\&-\frac{1}{(x_1-x_2)^2}\Big[\frac{-2}{x_1-x_2}+\sum_k\left(\frac{1}{x_1-y_k}-\frac{1}{x_2-y_k}\right)-\sum_{k\neq 1,2}\left(\frac{1}{x_1-x_k}-\frac{1}{x_2-x_k}\right)\Big]\\&=\sum_k\frac{1}{(x_1-y_k)^2(x_2-y_k)}-\sum_{k\neq 1,2}\frac{1}{(x_1-x_k)^2(x_2-x_k)},\end{aligned}\tag{C.19}$$

which give a regular contribution as we wanted. For the remaining terms we can write, using the same methods as above

$$\begin{aligned}&\frac{1}{x_1-x_2}\Bigg\{\sum_{k\neq 1,2}\wp^R(x_1-x_k)-\sum_k\wp^R(x_1-y_k)-\frac{1}{x_1-x_2}\Big[\sum_k\left(\zeta(x_1-y_k)-\zeta(x_2-y_k)\right)\\&\qquad\qquad-\sum_{k\neq 1,2}\left(\zeta(x_1-x_k)-\zeta(x_2-x_k)\right)\Big]\Bigg\}\\&=\sum_{n\geq 1}c_{n+1}\sum_{l=1}^{2n+1}\binom{2n}{l-1}\frac{(-1)^l}{l}\sum_{m=1}^{l-1}(l-m)x_1^{l-1-m}x_2^{m-1}\Big[\sum_{k\neq 1,2}x_k^{2n+1-l}-\sum_k y_k^{2n+1-l}\Big],\end{aligned}\tag{C.20}$$

which explicitly shows the vanishing of the residue of this last pole.

We just showed that $R\left(x_j;\{x_k\}_{k\neq j},\{y_k\}_{k=1}^N\right)$ is holomorphic in the whole complex plane for all variables. Liouville's theorem then implies it must be a constant. Hence we can set any convenient values for the variables to show this constant to be zero. Taking the limit $y_k\to 0$ for all k we get

$$\begin{aligned}&-\lim_{y_k\to 0}\sum_k\wp(x_1-y_k)\sum_{l\neq k}\frac{1}{y_k-y_l}+\sum_{k\neq 1}\wp'(x_1-x_k)+N\wp(x_1)\Big[N\zeta(x_1)-\sum_{k\neq 1}\zeta(x_1-x_k)-\sum_k\zeta(x_k)\Big]\\&-\sum_{k\neq 1}\wp(x_1-x_k)\Big[N\zeta(x_1)-\sum_{l\neq 1}\zeta(x_1-x_l)-N\zeta(x_k)+\sum_{l\neq k}\zeta(x_k-x_l)\Big]\end{aligned}\tag{C.21}$$

The first term can be written as

$$\lim_{y_k\to 0}\sum_{\substack{\text{pairs}(m,n),m\neq n\\ m,n\in\{1,\dots,N\}}}\frac{1}{y_n-y_m}\Big[\wp'(x_1)(y_n-y_m)+\mathcal{O}\left((y_n-y_m)^2\right)\Big]=\frac{N(N-1)}{2}\wp'(x_1) \tag{C.22}$$

Sending $x_k\to 0$, $k\neq 1$ simplifies R further

$$\begin{aligned}&(N-1)\left(\frac{N}{2}+1\right)\wp'(x_1)-(N-1)\wp(x_1)\zeta(x_1)\\&+\lim_{\substack{x_k\to 0\\ k\neq 1}}\left\{\sum_{k\neq 1}\wp(x_1-x_k)\Big[N\zeta(x_k)-\sum_{l\neq k}\zeta(x_k-x_l)\Big]-N\wp(x_1)\sum_{k\neq 1}\zeta(x_k)\right\},\end{aligned} \tag{C.23}$$

where the second line yields

$$\lim_{\substack{x_k\to 0\\ k\neq 1}}\left\{\underbrace{N\sum_{k\neq 1}\frac{1}{x_k}\Big[\wp(x_1-x_k)-\wp(x_1)\Big]}_{-N(N-1)\wp'(x_1)}\underbrace{-\sum_{k\neq 1}\wp(x_1-x_k)\sum_{l\neq k}\zeta(x_k-x_l)}_{(N-1)\wp(x_1)\zeta(x_1)+\frac{(N-1)(N-2)}{2}\wp'(x_1)}\right\}.$$

Putting everything together we finally obtain

$$\text{const}=\lim_{\substack{y_k\to 0\\ x_l\to 0,l\neq 1}}R(\dots)=0\implies R(\dots)=0,$$

which concludes the proof of (C.8).

C.2 Proof of (6.50)

By simplifying the left hand side of (6.50) one gets

$$\begin{aligned}\sum_{j=1}^{N}\Bigg\{&G\Big[\wp(z-x_j)\zeta(z-x_j)+\frac{1}{2}\wp'(z-x_j)\Big]+G\Big[\wp(z-y_j)\zeta(z-y_j)+\frac{1}{2}\wp'(z-y_j)\Big]\\&+\wp(z-x_j)\Big[-i\dot{x}_j-G\sum_{k=1}^{N}\zeta(z-y_k)+G\sum_{k\neq j}\zeta(z-x_k)\Big]\\&+\wp(z-y_j)\Big[i\dot{y}_j-G\sum_{k=1}^{N}\zeta(z-x_k)+G\sum_{k\neq j}\zeta(z-y_k)\Big]\\&+G\frac{2\eta_1}{L}\Big[i\dot{y}_j-i\dot{x}_j+G\left(\wp(z-y_j)-\wp(z-x_j)\right)\sum_k(y_k-x_k)\Big]\Bigg\}.\end{aligned} \tag{C.24}$$

Going on-shell w.r.t. auxiliary system (6.45), we arrive at

$$\mathrm{LHS} = X_1 + X_2, \tag{C.25}$$

where

$$\begin{aligned}
X_1 = \sum_{j=1}^{N}\Bigg\{ & \frac{1}{2}\wp'(z-x_j) + \wp(z-x_j)\Big[\sum_{k=1}^{N}\left(\zeta(z-x_k) - \zeta(z-y_k) + \zeta(x_j-y_k)\right) - \sum_{k\neq j}\zeta(x_j-x_k)\Big] \\
& + \frac{1}{2}\wp'(z-y_j) + \wp(z-y_j)\Big[\sum_{k=1}^{N}\left(\zeta(z-y_k) - \zeta(z-x_k) + \zeta(y_j-x_k)\right) - \sum_{k\neq j}\zeta(y_j-y_k)\Big]\Bigg\} \\
X_2 = G^2\,\frac{2\eta_1}{L} & \sum_{j=1}^{N}\sum_{k\neq j}\Big\{\zeta(y_j-x_k) + \zeta(x_j-y_k) - \zeta(y_j-y_k) - \zeta(x_j-x_k)\Big\}.
\end{aligned} \tag{C.26}$$

It is easy to see that X_2 vanishes, since we can rearrange the sum to pairs of ζ's with positive and negative arguments respectively

$$\begin{aligned}
X_2 = G^2\frac{2\eta_1}{L}\sum_{\substack{\mathrm{pairs}(m,n), m\neq n \\ m,n\in\{1,\dots,N\}}}\Bigg\{ & \Big[\zeta(y_m-x_n) + \zeta(x_n-y_m)\Big] + \Big[\zeta(x_m-y_n) + \zeta(y_n-x_m)\Big] \\
& - \Big[\zeta(x_m-x_n) + \zeta(x_n-x_m)\Big] - \Big[\zeta(y_m-y_n) + \zeta(y_n-y_m)\Big]\Bigg\} \\
= 0. &
\end{aligned} \tag{C.27}$$

The vanishing of X_1 looks more intriguing, but actually reduces to the already proven relation (C.8). Indeed, we can write X_1 as

$$X_1 = \frac{1}{2(N-1)}\sum_{j=1}^{N}\Big[R\left(\{x\},\{y\}\right)\Big|_{x_j=z} + R\left(\{x\}\leftrightarrow\{y\}\right)\Big|_{y_j=z}\Big] = 0,$$

which concludes the proof of (6.50).

Appendix D

Expansion in the twist parameter q and some other properties of ILW BAE

Let us define the twist parameter as $q = (-1)^N e^{2\pi t}$. Then we can recast the Bethe equations for the ILW integrable system (5.20) in a more suitable form (with $\lambda_{st} := \lambda_s - \lambda_t$)

$$\begin{aligned}
&\prod_{j=1}^{N}\left(\lambda_s - a_j + \frac{\epsilon}{2}\right)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\lambda_{st}+\epsilon_1)(\lambda_{st}+\epsilon_2)}{\lambda_{st}(\lambda_{st}+\epsilon)}\\
&= q\prod_{j=1}^{N}\left(\lambda_s - a_j - \frac{\epsilon}{2}\right)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\lambda_{st}-\epsilon_1)(\lambda_{st}-\epsilon_2)}{\lambda_{st}(\lambda_{st}-\epsilon)}, \quad s = 1, \ldots, k. \qquad \text{(D.1)}
\end{aligned}$$

D.1 Perturbation theory around the B–O points $q = 0$ and $q = \infty$

The goal is to perform an asymptotic expansion for solutions of (D.1) around $q = 0$. Hence we expand λ_s in a powers series in q as

$$\lambda_s = \sum_{n=0}^{\infty} q^n \lambda_s^{(n)}, \qquad \text{(D.2)}$$

plug it into the Bethe equations and work at a fixed order in q. In the following, analytical results will be provided only up to first order. However, we wrote an algorithm in `Mathematica` working to any order. The disadvantage of the computer algebra approach is pretty clear, one has to keep k and N reasonably small [1]. Various cross-checks were completed, all confirming the results presented below.

Remark. The Bethe equations (D.1) are invariant under a simultaneous transformation $q \to q^{-1}$, $\epsilon_1 \to -\epsilon_1$ and $\epsilon_2 \to -\epsilon_2$. This means that once an asymptotic expansion around $q = 0$ is established, we know it also for $q = \infty$.

D.1.1 Solutions at leading order in q

Substituting (D.2) into the BAE (D.1) and keeping only terms of order q^0, we arrive at

$$\prod_{j=1}^{N} \left(\lambda_s^{(0)} - a_j + \frac{\epsilon}{2}\right) \prod_{\substack{t=1 \\ t \neq s}}^{k} \frac{\left(\lambda_s^{(0)} - \lambda_t^{(0)} + \epsilon_1\right)\left(\lambda_s^{(0)} - \lambda_t^{(0)} + \epsilon_2\right)}{\lambda_s^{(0)} - \lambda_t^{(0)}\left(\lambda_s^{(0)} - \lambda_t^{(0)} + \epsilon\right)} = 0. \tag{D.3}$$

This is just the B–O limit of ILW Bethe equations, so we can readily classify the solutions. They are labeled by colored partitions of the instanton number k, i.e. N-tuples of Young diagrams as in Figure D.1. Each box in a colored partition is given by three coordinates

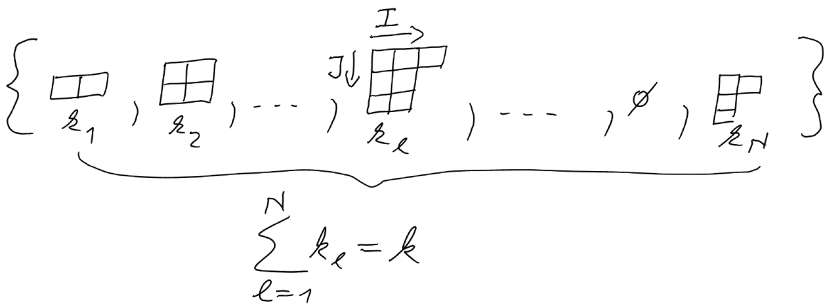

FIGURE D.1: An example of a colored partition of the instanton number k. Implicitly, it defines also our conventions for Young diagrams.

$(l, \{J, I\})$. Moreover, precisely one Bethe root is associated to every box. Therefore, Bethe roots at leading order in the q-expansion can be expressed in terms of colored partition data

$$\begin{aligned} \lambda_s^{(0)} := \lambda_{(l,\{J,I\})}^{(0)} = a_l - \frac{\epsilon}{2} - (I-1)\,\epsilon_1 - (J-1)\,\epsilon_2, \quad & l = 1, \dots, N \\ & J = 1, \dots, \#\text{rows in } Y_l \\ & I = 1, \dots, \#\text{columns in row}_J \text{ of } Y_l. \end{aligned} \tag{D.4}$$

[1]The order of the q-expansion that can be achieved in practice depends crucially on the choice of k and N.

D.1.2 Solutions at first order in q

The structure of Bethe equations at first order in the q-expansion is

$$A_s^{(0)}(N)\lambda_s^{(1)} + \sum_{\substack{t=1\\t\neq s}}^{k} B_{st}^{(0)}(N)\left(\lambda_s^{(1)} - \lambda_t^{(1)}\right) = C_s^{(0)}(N), \tag{D.5}$$

where $A,\ B,\ C$ are known functions of $\lambda_s^{(0)}$ and remaining parameters of the model as well

$$A_s^{(0)}(N) = \left[\sum_{j=1}^{N}\prod_{\substack{l=1\\l\neq j}}^{N}\left(\lambda_s^{(0)} - a_j + \frac{\epsilon}{2}\right)\right]\prod_{\substack{t=1\\t\neq s}}^{k}\frac{\left(\lambda_{st}^{(0)}+\epsilon_1\right)\left(\lambda_{st}^{(0)}+\epsilon_2\right)}{\lambda_{st}^{(0)}\left(\lambda_{st}^{(0)}+\epsilon\right)} \tag{D.6}$$

$$B_{st}^{(0)}(N) = -\epsilon_1\epsilon_2\prod_{j=1}^{N}\left(\lambda_s^{(0)} - a_j + \frac{\epsilon}{2}\right)\frac{2\lambda_{st}^{(0)}+\epsilon}{\left(\lambda_{st}^{(0)}\right)^2\left(\lambda_{st}^{(0)}+\epsilon\right)^2}\prod_{\substack{u=1\\u\neq s,\,t}}^{k}\frac{\left(\lambda_{su}^{(0)}+\epsilon_1\right)\left(\lambda_{su}^{(0)}+\epsilon_2\right)}{\lambda_{su}^{(0)}\left(\lambda_{su}^{(0)}+\epsilon\right)} \tag{D.7}$$

$$C_s^{(0)}(N) = \prod_{j=1}^{N}\left(\lambda_s^{(0)} - a_j - \frac{\epsilon}{2}\right)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{\left(\lambda_{st}^{(0)}-\epsilon_1\right)\left(\lambda_{st}^{(0)}-\epsilon_2\right)}{\lambda_{st}^{(0)}\left(\lambda_{st}^{(0)}-\epsilon\right)}. \tag{D.8}$$

A straightforward manipulation brings the linear system (D.5) to a matrix form[2]

$$\mathbf{M}^{(0)}(N)\cdot\boldsymbol{\lambda}^{(1)} = \mathbf{C}^{(0)}(N) \tag{D.9}$$

with the matrix $\mathbf{M}$ given as

$$\mathbf{M}^{(0)}(N) = \begin{bmatrix} A_1 + \sum_{t\neq 1}^{k} B_{1t} & -B_{12} & \dots & -B_{1k} \\ -B_{21} & A_2 + \sum_{t\neq 2}^{k} B_{2t} & \dots & -B_{2k} \\ \vdots & \vdots & \dots & \vdots \\ -B_{k1} & -B_{k2} & \dots & A_k + \sum_{t\neq k}^{k} B_{kt} \end{bmatrix}. \tag{D.10}$$

[2]This structure remains true at any order in q. Just the matrix elements and the right hand side become complicated functions of Bethe roots in all lower orders. It also suggests that even exact solutions for any q are still labeled by colored partitions. We checked this for the exact results that we got by `Mathematica`. However, square roots appear in the analytic solutions and one has to combine choices of different branches properly to make the combinatorics work.

Actually, it is not really convenient to try to invert this matrix directly. A better option turns out to be a division of the index $s \equiv (l, \{J, I\})$ into cases

$$(l,\{J,I\}) = \begin{cases} (l,\{1,1\}) \\ (l,\{1,I\})\,; \quad I = 2,\dots,C_{l,1} \\ (l,\{J,1\})\,; \quad J = 2,\dots,R_l \\ (l,\{J,I\})\,; \quad J = J = 2,\dots,R_l \mid I = 2,\dots,C_{l,J} \end{cases}$$

and then solving recursively, as we will sketch now. For cleaner notations we introduced $R_l = \#\text{rows in } Y_l$ and $C_{l,J} = \#\text{columns in row}_J$ of Y_l, respectively.

Case I: $(l, \{1, 1\})$**.** Still, we need to subdivide into two branches; a pivot (upper-left) box in a bigger Young diagram or a single box diagram, respectively.

1. $k_l > 1$: In this case we get an easy equation for $\lambda^{(1)}_{(l,\{1,1\})}$

$$A_l \lambda^{(1)}_{(l,\{1,1\})} = 0. \tag{D.11}$$

The form of A_l is not important at this point as long as it does not vanish. The solution is trivial

$$\lambda^{(1)}_{(l,\{1,1\})} = 0. \tag{D.12}$$

2. $k_l = 1$: The right hand side does not vanish as previously. A little bit of algebra leads us to a result for a Bethe root attached to a single box diagram

$$\lambda^{(1)}_{(l,\{1,1\})} = -\epsilon \prod_{\substack{m=1\\ m\neq l}}^{N} \left\{ \frac{a_{lm} + (R_m - 1)\epsilon_2 - \epsilon_1}{a_{lm} + R_m\epsilon_2} \prod_{\widetilde{J}=1}^{R_m} \left[\frac{a_{lm} + (\widetilde{J}-2)\epsilon_2 + (C_{m\widetilde{J}} - 1)\epsilon 1}{a_{lm} + (\widetilde{J}-1)\epsilon_2 + (C_{m\widetilde{J}} - 1)\epsilon 1} \times \frac{a_{lm} + \widetilde{J}\epsilon_2 + C_{m\widetilde{J}}\epsilon_1}{a_{lm} + (\widetilde{J}-1)\epsilon_2 + C_{m\widetilde{J}}\epsilon_1} \right] \right\}. \tag{D.13}$$

Case II: $(l, \{1, I\})$**.** We are focusing on the first row of the l-th Young diagram. The analysis we are just about to show will uncover that only the last box in the row can get a non-vanishing contribution at first order of the q-expansion; moreover, just provided it is a "corner" box (i.e. the second row is shorter than the first one). For a more accessible presentation, we want to anticipate that this will prove to be a general feature. Corner boxes get corrections while Bethe roots attached to inner boxes do not. To demonstrate our statements in equations, in this event

(D.5) yields

$$A_{l,I}\left(\lambda^{(1)}_{(l,\{1,I\})}-\lambda^{(1)}_{(l,\{1,I-1\})}\right)=\mathrm{RHS}_{l,I}. \tag{D.14}$$

This can be solved recursively with the result

$$\lambda^{(1)}_{(l,\{1,I\})}=\lambda^{(1)}_{(l,\{1,1\})}+\sum_{\widetilde{I}=2}^{I}\frac{\mathrm{RHS}_{l,\widetilde{I}}}{A_{l,\widetilde{I}}};\quad I=2,\ldots,C_{l1}. \tag{D.15}$$

However, we already derived that $\lambda^{(1)}_{(l,\{1,1\})}=0$ and the further essential piece of information is

$$\mathrm{RHS}_{l,\widetilde{I}}=\begin{cases}\neq 0 & \left(\widetilde{I}=C_{l1}\right)\wedge(C_{l1}>C_{l2})\\ =0 & \text{otherwise}\end{cases},$$

which brings us to conclude (we do not show intermediate results since they are quite nasty, whereas in the final formulas some simplifications occur)

$$\begin{aligned}\lambda^{(1)}_{(l,\{1,I\})}&=\delta_{I,C_{l1}}\theta(C_{l1}-C_{l2})\frac{\epsilon\epsilon_2}{\epsilon_1}\frac{(R_l-1)\,\epsilon_2-C_{l1}\epsilon_1}{R_l\epsilon_2+(1-C_{l1})\,\epsilon_1}\\
&\times\prod_{\widetilde{J}=2}^{R_l}\frac{\left[(\widetilde{J}-2)\epsilon_2+(C_{l\widetilde{J}}-C_{l1})\epsilon_1\right]\left[\widetilde{J}\epsilon_2+(C_{l\widetilde{J}}-C_{l1}+1)\epsilon_1\right]}{\left[(\widetilde{J}-1)\epsilon_2+(C_{l\widetilde{J}}-C_{l1})\epsilon_1\right]\left[(\widetilde{J}-1)\epsilon_2+(C_{l\widetilde{J}}-C_{l1}+1)\epsilon_1\right]}\\
&\times\prod_{\substack{m=1\\m\neq l}}^{N}\left\{\frac{a_{lm}+(R_m-1)\epsilon_2-C_{l1}\epsilon_1}{a_{lm}+R_m\epsilon_2+(1-C_{l1})\epsilon_1}\prod_{\widetilde{J}=1}^{R_m}\left[\frac{a_{lm}+(\widetilde{J}-2)\epsilon_2+(C_{m\widetilde{J}}-C_{l1})\epsilon_1}{a_{lm}+(\widetilde{J}-1)\epsilon_2+(C_{m\widetilde{J}}-C_{l1})\epsilon_1}\right.\right.\\
&\left.\left.\times\frac{a_{lm}+\widetilde{J}\epsilon_2+(C_{m\widetilde{J}}-C_{l1}+1)\epsilon_1}{a_{lm}+(\widetilde{J}-1)\epsilon_2+(C_{m\widetilde{J}}-C_{l1}+1)\epsilon_1}\right]\right\}\end{aligned} \tag{D.16}$$

with $\theta(\cdot)$ being the step function.

Case III: $(l,\{J,1\})$**.** Here we concentrate on the first column. The conclusion is the same, non-vanishing contribution gets only the last box provided it is a corner one (i.e. the second column is shorter than the first one). All the derivations go along the same lines, so we write just the result

$$\begin{aligned}\lambda^{(1)}_{(l,\{J,1\})}&=\delta_{J,R_l}\delta_{1,C_{l,R_l}}(-\epsilon_1)\frac{\left[C_{l,R_l-1}(C_{l,R_l}\epsilon_1+\epsilon_2)\right]\left[-2\epsilon_2+(C_{l,R_l-1}-1)\epsilon_1\right]}{\left[C_{l,R_l}(C_{l,R_l-1}\epsilon_1-\epsilon_2)\right]\left[-\epsilon_2+(C_{l,R_l-1}-1)\epsilon_1\right]}\\
&\times\prod_{\widetilde{J}=1}^{R_l-2}\frac{\left[(\widetilde{J}-R_l-1)\epsilon_2+(C_{l\widetilde{J}}-1)\epsilon_1\right]\left[(\widetilde{J}-R_l+1)\epsilon_2+C_{l\widetilde{J}}\epsilon_1\right]}{\left[(\widetilde{J}-R_l)\epsilon_2+(C_{l\widetilde{J}}-1)\epsilon_1\right]\left[(\widetilde{J}-R_l)\epsilon_2+C_{l\widetilde{J}}\epsilon_1\right]}\\
&\times\prod_{\substack{m=1\\m\neq l}}^{N}\left\{\frac{a_{lm}+(R_m-R_l)\epsilon_2-\epsilon_1}{a_{lm}+(R_m-R_l+1)\epsilon_2}\prod_{\widetilde{J}=1}^{R_m}\left[\frac{a_{lm}+(\widetilde{J}-R_l-1)\epsilon_2+(C_{m\widetilde{J}}-1)\epsilon_1}{a_{lm}+(\widetilde{J}-R_l)\epsilon_2+(C_{m\widetilde{J}}-1)\epsilon_1}\right.\right.\end{aligned}$$

$$\times \frac{a_{lm} + (\widetilde{J} - R_l + 1)\epsilon_2 + C_{m\widetilde{J}}\epsilon_1}{a_{lm} + (\widetilde{J} - R_l)\epsilon_2 + C_{m\widetilde{J}}\epsilon_1}\Bigg]\Bigg\} \tag{D.17}$$

Case IV: $(l, \{J, I\})$**.** It remains to treat the rest of the boxes that were omitted in I–III, namely $J = 2, \ldots, R_l$ and $I = 2, \ldots, C_{lJ}$. As we already stressed many times, non-trivial contributions get only boxes which form corners in the profile. Finally, we arrive at

$$\begin{aligned}\lambda_{(l,\{J,I\})} &= \delta_{I,C_{lJ}}\theta(C_{lJ} - C_{l,J+1})\epsilon\epsilon_2 \frac{C_{l,J-1} - C_{lJ} + 1}{-\epsilon_2 + (C_{l,J-1} - C_{lJ} + 1)\epsilon_1} \\ &\times \frac{[(R_l - J)\epsilon_2 - C_{lJ}\epsilon_1]\,[-2\epsilon_2 + (C_{l,J-1} - C_{l,J})\epsilon_1]}{[(1 - C_{lJ})\epsilon_1 + (R_l - J + 1)\epsilon_2]\,[-\epsilon_2 + (C_{l,J-1} - C_{lJ})\epsilon_1]} \\ &\times \prod_{\substack{\widetilde{J}=1 \\ \widetilde{J}\neq J, J-1}}^{R_l} \frac{\left[(\widetilde{J} - J - 1)\epsilon_2 + (C_{l\widetilde{J}} - C_{lJ})\epsilon_1\right]\left[(\widetilde{J} - J + 1)\epsilon_2 + (C_{l\widetilde{J}} - C_{lJ} + 1)\epsilon_1\right]}{\left[(\widetilde{J} - J)\epsilon_2 + (C_{l\widetilde{J}} - C_{lJ})\epsilon_1\right]\left[(\widetilde{J} - J)\epsilon_2 + (C_{l\widetilde{J}} - C_{lJ} + 1)\epsilon_1\right]} \\ &\times \prod_{\substack{m=1 \\ m\neq l}}^{N} \left\{\frac{a_{lm} + (R_m - J)\epsilon_2 - C_{lJ}\epsilon_1}{a_{lm} + (R_m - J + 1)\epsilon_2 + (1 - C_{lJ})\epsilon_1} \prod_{\widetilde{J}=1}^{R_m}\left[\frac{a_{lm} + (\widetilde{J} - J - 1)\epsilon_2 + (C_{m\widetilde{J}} - C_{lJ})\epsilon_1}{a_{lm} + (\widetilde{J} - J)\epsilon_2 + (C_{m\widetilde{J}} - C_{lJ})\epsilon_1}\right.\right. \\ &\times \left.\left.\frac{a_{lm} + (\widetilde{J} - J + 1)\epsilon_2 + (C_{m\widetilde{J}} - C_{lJ} + 1)\epsilon_1}{a_{lm} + (\widetilde{J} - J)\epsilon_2 + (C_{m\widetilde{J}} - C_{lJ} + 1)\epsilon_1}\right]\right\}.\end{aligned} \tag{D.18}$$

In the above paragraphs we fully described the asymptotic q-expansion of the Bethe roots around the B–O point $q = 0$ (and $q = \infty$ as well by the symmetry we mentioned) up to first order. The formulae for next-to-leading order corrections might not be very enlightening, however what is important, is the structure of the boxes that receive non-vanishing corrections.

Summary. Most of the Bethe roots do not receive corrections in the q-expansion; they are just given by the leading-order solution (D.4). The only roots that get corrected are associated to boxes that form a corner in the profile of the Young diagram as is shown in Figure D.2. The correction for a single box diagram is given in equation

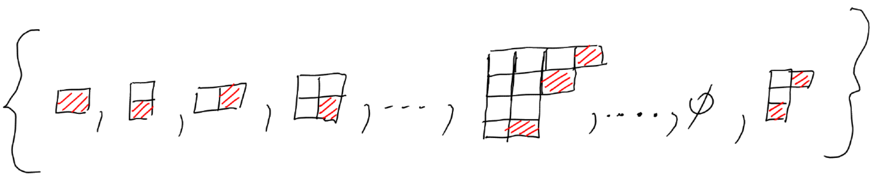

FIGURE D.2: Boxes that receive corrections in the q-expansion are marked in red. They are referred to as "corner" boxes.

(D.13), for a corner box in the first row in (D.16), for a corner box in first column in (D.17) while for the remaining corners in (D.18)

D.2 Perturbation theory around the KdV point $q = (-1)^N$

This kind of expansion might shed new light on the quantization of KdV integrable system (with dispersion), which is a long-standing problem of Mathematical Physics [3]. Unfortunately, we do not have much to say about it. The perturbation theory around this point is singular and pretty hard to analyze. We do not pursue this direction further here.

D.3 Some properties of ILW BAE

D.3.1 Exact sum rule for Bethe roots

By a standard technique for Bethe equations we can derive a sum rule. As a first step manipulate the Bethe equations to the form

$$\prod_{j=1}^{N} \frac{\lambda_s - a_j + \frac{\epsilon}{2}}{\lambda_s - a_j - \frac{\epsilon}{2}} = q(-1)^{(k-1)} \prod_{\substack{t=1\\ t\neq s}}^{k} \frac{\lambda_{st}+\epsilon}{-\lambda_{st}+\epsilon}\frac{\lambda_{st}-\epsilon_1}{-\lambda_{st}-\epsilon_1}\frac{\lambda_{st}-\epsilon_2}{-\lambda_{st}-\epsilon_2}. \tag{D.19}$$

Notice, that the double product over t can be extended to the whole range without changing anything. The point of this rewriting was to factor some signs of this product, such that when we exchange s and t it goes to its inverse. Subsequently, taking the logarithm produces an antisymmetric function (in (st) indices), which is just what we wanted

$$\sum_{j=1}^{N} \log \frac{\lambda_s - a_j + \frac{\epsilon}{2}}{\lambda_s - a_j - \frac{\epsilon}{2}} - \log q - (k-1)i\pi - \sum_{t=1}^{k} \underbrace{\log \frac{\lambda_{st}+\epsilon}{-\lambda_{st}+\epsilon}\frac{\lambda_{st}-\epsilon_1}{-\lambda_{st}-\epsilon_1}\frac{\lambda_{st}-\epsilon_2}{-\lambda_{st}-\epsilon_2}}_{-i\chi(\lambda_{st})} = 2\pi i\widetilde{n}_s. \tag{D.20}$$

As we anticipated $\chi(\lambda_{st})$ is antisymmetric and $\widetilde{n}_s \in \mathbb{Z}$. Bringing the $i\pi$ term to the right

$$\sum_{j=1}^{N} -i\log \frac{\lambda_s - a_j + \frac{\epsilon}{2}}{\lambda_s - a_j - \frac{\epsilon}{2}} + i\log q + \sum_{t=1}^{k}\chi(\lambda_{st}) = 2\pi n_s \tag{D.21}$$

defines a new mode number n_s

$$n_s = \left(\widetilde{n}_s + \frac{k-1}{2}\right) \in \begin{cases} \mathbb{Z},\ k \text{ odd} \\ \mathbb{Z}+\frac{1}{2},\ k \text{ even} \end{cases}.$$

[3]The dispersionless KdV hierarchy is solved. The generating function for the quantized Hamiltonians (integrals of motion) was first discovered by Eliashberg (see also [82]) while their spectrum was computed in [83].

Summing over s kills the most complicated term $\chi(\lambda_{st})$ (since it was carefully constructed to be anti-symmetric), which was the true motivation behind all this. Hence, we get a constraint for the Bethe roots that we call a sum rule

$$\sum_{s=1}^{k}\sum_{j=1}^{N} -i\log\frac{\lambda_s - a_j + \frac{\epsilon}{2}}{\lambda_s - a_j - \frac{\epsilon}{2}} + ik\log q = 2\pi\sum_{s=1}^{k} n_s. \tag{D.22}$$

D.3.2 Two possible limits

Next, we want to discuss two possible limits of the ILW BAE. The first one is inspired by gauge theory while the second one rather by the integrable system world. Since we know that our Bethe equations are connected to instanton counting, it is natural to study the Nekrasov-Shatashvili limit, $\epsilon_2 \to 0$. In this situation the equations simplify drastically and we obtain

$$\prod_{j=1}^{N}\frac{\lambda_s - a_j + \frac{\epsilon_1}{2}}{\lambda_s - a_j - \frac{\epsilon_1}{2}} = q, \quad s = 1\ldots,k. \tag{D.23}$$

The second available limit is to send $\epsilon_1 \to \pm\infty$, $\epsilon_2 \to \mp\infty$ while keeping $\epsilon_1+\epsilon_2 = \epsilon$ fixed. This reduces to the Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ spin chain with twist. Indeed, setting $u_s = \frac{i}{\epsilon}\lambda_s$, $\nu_j = \frac{i}{\epsilon}a_j$ and $\theta = -i\log q$ we get the Heisenberg Bethe equations in standard form

$$\prod_{j=1}^{N}\frac{u_s - \nu_j + \frac{i}{2}}{u_s - \nu_j - \frac{i}{2}} = e^{i\theta}\prod_{\substack{t=1\\ t\neq s}}^{k}\frac{u_s - u_t + i}{u_s - u_t - i}. \tag{D.24}$$

A feasible idea could be to write $\epsilon_1 = \frac{1}{\delta}$, $\epsilon_2 = -\frac{1}{\delta} + \epsilon$ and build a perturbation theory in $\delta \to 0$.

Appendix E

$\mathbf{BO}_N$ Hamiltonians versus $\mathbf{tCS}_N$

In Section 6.4.1 we observed that the spectrum of the chiral operator $\mathrm{Tr}\Phi^{n+1}$ can be expressed as a linear combination of the eigenvalues of the integrals of motion (IMs) of the Benjamin-Ono integrable system[1]. We showed explicitly the connection between $SU(2)$ $\mathcal{N}=2$ supersymmetric Yang-Mills theory and Vir $\oplus$ H CFT. In this Appendix we consider the $SU(N)$ gauge theory versus $\mathrm{W}_N \oplus \mathrm{H}$ algebra, focusing mainly on I_3, which we identify as the basic Hamiltonian, whose spectrum was computed in [84]. As a preliminary check and also to build the dictionary between [84] and [58] we can specialize to the Vir$\oplus$H case[2]. The dictionary is obtained by direct comparison of explicit expressions for IMs and their eigenvalues and can be found in Table E.1. Comparing

Litvinov	Estienne *et al.*	gauge theory
b	$i\sqrt{g}$	ϵ_2
a_k	$\sqrt{2}a_k$	
P_*	special eigenstates	a_*
$b\left(h_\lambda^{(2)}(P) - 2P\lvert\lambda\rvert\right)$	$e_\lambda^{(3),+}(g)$	

Table E.1: Dictionary between [84] and [58]

the expressions for $I_3^+(g)$ in [84] and I_2 in [58] (the labeling is unfortunately shifted) we

[1]This is was checked up to $n=4$, where explicit results for the eigenvalues of the IMs are available.

[2]In [84] the eigenvalues were computed for a special class of eigenstates. In general, the eigenvalues depend on the momentum P, which characterizes the eigenstates, i.e. does not enter into the IMs. So picking a special class of eigenstates translates into setting a given value of the momentum $P = P_*$.

get

$$I_3^+(g) = 2ibI_2 \implies E^{(3),+}_{\overrightarrow{\lambda}}(g) = 2ib\left(-\frac{i}{2}\, h^{(2)}_{\overrightarrow{\lambda}}\Big|_{P=P_*}\right)$$
$$= b\, h^{(2)}_{\overrightarrow{\lambda}}\Big|_{P=P_*}. \tag{E.1}$$

To highlight how one picks the special value P_* let us still concentrate only on the Vir$\oplus$H case. Taking the result for $E^{(3),+}_{\overrightarrow{\lambda}}(g)$ from [84] and using the third row of table E.1 we can write

$$\begin{aligned} E^{(3),+}_{(\lambda,\mu)}(g) &= e^{(3),+}_{\lambda}(g) + e^{(3),+}_{\mu}(g) - \sqrt{2g}(q-\alpha_0)(|\lambda|-|\mu|) \\ &= bh^{(2)}_{(\lambda,\mu)}(P) + b\left[\sqrt{2}i(q-\alpha_0) - 2P\right](|\lambda|-|\mu|), \end{aligned} \tag{E.2}$$

where $\alpha_0 = \frac{i}{\sqrt{2}}Q$ and q is a charge for the zero mode b_0 of an auxiliary bosonic field, $b_0|q\rangle = q|q\rangle$. By imposing (E.1) the bracket $[\ldots]$ is forced to vanish, which leads to

$$P_* = \frac{i}{\sqrt{2}}(q-\alpha_0). \tag{E.3}$$

Finally, concluding the Vir $\oplus$ H CFT or $SU(2)$ gauge theory respectively, we get for $\overrightarrow{\lambda} = (\lambda,\mu)$

$$E^{(3),+}_{\overrightarrow{\lambda}}(g) = b\, h^{(2)}_{\overrightarrow{\lambda}}\Big|_{P=P_*} = -\epsilon_2\, \frac{\mathrm{Tr}\Phi^3_{\overrightarrow{\lambda}}}{3}\Bigg|_{a=a_*}. \tag{E.4}$$

At this point we are ready to make connection between the $\mathrm{W}_{N-1}\oplus$H CFT and $SU(N)$ gauge theory for $I_3^+(g)$ and $\mathrm{Tr}\Phi^3$. First, we write the result for $E^{(3),+}_{\overrightarrow{\lambda}}(g)$ [84] and manipulate it to a more convenient form for us

$$\begin{aligned} E^{(3),+}_{\overrightarrow{\lambda}}(g) =& \sum_{l=1}^{N} e^{(3),+}_{\lambda_l} + (1-g)\sum_{l=1}^{N}(N+1-2l)|\lambda_l| \\ =& \epsilon_2^2 \sum_{l=1}^{N}\sum_{j=1}^{\#\mathrm{rows}(\lambda_l)}\left\{\left(a_l + \epsilon_1|\mathrm{row}_j(\lambda_l)| + \epsilon_2\left(j-\frac{1}{2}\right)\right)^2 - \left(a_l+\epsilon_2\left(j-\frac{1}{2}\right)\right)^2\right\} \\ & - 2\epsilon_2\sum_{l=1}^{N} a_l\lambda_l + (1+\epsilon_2^2)\sum_{l=1}^{N}(N+1-2l)|\lambda_l|. \end{aligned} \tag{E.5}$$

Then we need also to rewrite the expression for $\mathrm{Tr}\Phi^{n+1}$ (6.74)

$$\mathrm{Tr}\Phi^{n+1}_{\overrightarrow{\lambda}} = \sum_{l=1}^{N} a_l^{n+1} + \sum_{l=1}^{N}\sum_{j=1}^{\#\mathrm{rows}(\lambda_l)}(-\epsilon_2)\sum_{i=1}^{n}\binom{n+1}{i}\left(\frac{\epsilon_2}{2}\right)^{n-i}\frac{1+(-1)^{n-i}}{2}$$

$$\times\left[\left(a_l+\epsilon_1|\mathrm{row}_j(\lambda_l)|+\epsilon_2\left(j-\frac{1}{2}\right)\right)^i-\left(a_l+\epsilon_2\left(j-\frac{1}{2}\right)\right)^i\right]. \tag{E.6}$$

In particular, setting $n=2$ in (E.6) and comparing with (E.5) leads to the desired relation

$$\mathrm{Tr}\Phi^3_{\vec{\lambda}}=\sum_{l=1}^{N}a_l^3-\frac{3}{\epsilon_2}E^{(3),+}_{\vec{\lambda}}(g)+3\sum_{l=1}^{N}|\lambda_l|\Big[\frac{1+\epsilon_2^2}{\epsilon_2}(N+1-2l)-2a_l\Big]. \tag{E.7}$$

The last piece has to vanish, thus fixing the special value a_l^*

$$a_l^*=\frac{1+\epsilon_2^2}{\epsilon_2}\frac{1}{2}(N+1-2l)=Q\rho_l, \tag{E.8}$$

where ρ_l are the components[2] of the Weyl vector for $SU(N)$.

Finally, the key relation connecting the operator $\mathrm{Tr}\Phi^3$ and the energy of BO_3 integrable system is

$$\mathrm{Tr}\Phi^3_{\vec{\lambda}}\Big|_{a_l^*}=\sum_{l=1}^{N}(a_l^*)^3-\frac{3}{\epsilon_2}E^{(3),+}_{\vec{\lambda}}(g). \tag{E.9}$$

[2]In the orthonormal basis $\{\beta_l\}_{l=1}^N$ of $\mathbb{R}^N\supset\mathfrak{h}^\vee$.

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