Scuola Internazionale Superiore di Studi Avanzati - Trieste

Doctoral Thesis

Developments in Quantum Cohomology and Quantum Integrable Hydrodynamics via Supersymmetric Gauge Theories

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Chapter 1

Introduction

In the last sixty years, theoretical particle physicists dedicated themselves to uncovering the mysteries underlying Quantum Field Theories, in order to help the experimental particle physics colleagues understand the world of elementary particles. The combined efforts finally led to the formulation of the Standard Model, the greatest achievement in particle physics up to now, recently strengthened by the discovery of the Higgs boson at CERN. Nevertheless, although in excellent agreement with experiments, basic and commonly taken for granted properties of the Standard Model such as confinement, existence of a mass gap, and non-perturbative phenomena have not been completely understood yet and are still waiting for a satisfactory explanation at the theoretical level. The basic problem here is that the conventional approach to Quantum Field Theory based on Feynman diagrams relies on a perturbative weak-coupling expansion, which by definition cannot take into account strong-coupling or non-perturbative effects. Since at the moment we are lacking an alternative description of Quantum Field Theory more suitable for tackling these problems, what we can do now is to study those theories on which we have a better analytical control because of the high amount of symmetries they possess: in particular we consider theories which enjoy supersymmetry, conformal symmetry, or both. For these cases, many tools coming from string theory, integrability or geometry appear to be of great help in understanding their properties. In this sense the simplest non-trivial theory in four dimensions is the $U(N)$ $\mathcal{N}=4$ super Yang-Mills theory, being the one which possesses all of the allowed symmetries. By considering theories with fewer number of supersymmetries we get closer to the phenomenological world, but we quickly lose analytical techniques to study them. Nevertheless many remarkable new ideas, such as for example Seiberg-Witten theory and Seiberg duality, arose from considering $\mathcal{N}=2$ and $\mathcal{N}=1$ theories; the hope is that some of these ideas may also be applied to non-supersymmetric cases.

Apart from possible applications in particle physics phenomenology, with time people realized that supersymmetric theories, because of their deep connection to geometrical structures, can also be useful as a different way of approaching problems in mathematics, based on Lagrangians and path integral techniques. Although path integrals were unfamiliar to many mathematicians in the past, especially because of their lack of mathematical rigorousness, they are extremely efficient and led to great discoveries in the theory of topological invariants of manifolds and in the context of integrable systems, as well as in many other different topics. Clearly, this also works in the other direction: common mathematical techniques unfamiliar to physicists can give an important alternative point of view on physics problems and provide hints on how to affront them, thus allowing us to gain a deeper understanding of Quantum Field Theories. Nowadays, supersymmetric theories and string theory are used as an additional source of inspiration in many other different contexts, such as cosmological models, statistical physics systems and condensed matter systems. The interplay and interactions between the various disciplines will lead to many more developments in the future.

In the past, one of the mathematical problems that have been studied with field and string theory methods was the enumerative problem of computing genus zero Gromov-Witten invariants (GW) for Calabi-Yau and Kähler manifolds. Roughly speaking, genus zero GW invariants N_η count the number of holomorphic maps of degree η from a two-dimensional sphere S^2 (a genus zero Riemann surface) to a Kähler manifold M, which is usually denoted as target manifold. From the physics point of view, computing these invariants is especially important when M is a Calabi-Yau three-fold (manifold of complex dimension 3): in fact if one wants to construct supersymmetric generalizations of the Standard Model starting from a ten-dimensional string theory set-up one has to compactify six of the ten dimensions, and the easiest way to preserve some supersymmetry is to consider a Calabi-Yau three-fold as compactification space. In this context, the two-sphere is interpreted as the world-sheet of the strings, and the genus zero GW invariants enter in determining the world-sheet non-perturbative corrections to the Yukawa couplings of the resulting effective four-dimensional theory. On the other hand, from the mathematics point of view this is an interesting problem for any Kähler manifold since it provides a way to distinguish manifolds with different topology, and is related to a quantum deformation of the cohomology ring of M which depends on the Kähler parameters of M.

Originally, computations of these invariants in a physical formalism were performed by considering a particular class of supersymmetric gauge theories in two dimensions: the $\mathcal{N} = (2, 2)$ supersymmetric Gauged Linear Sigma Models (GLSM) on a genus zero Riemann surface Σ_0. The peculiarity of the GLSM is that its space of supersymmetric vacua in the Higgs branch, which is given by a set of equations for the scalar fields of the theory, always defines a Kähler manifold: we can therefore consider the GLSM whose associated manifold is the target M of interest, if such a GLSM exists. Clearly, in order to preserve some supersymmetry on a generic Riemann surface Σ_0, the GLSM has to be topologically twisted: this means that the $U(1)_L$ Lorentz symmetry gets replaced by a combination $U(1)_{L'}$ of $U(1)_L$ and a $U(1)_R$ R-symmetry (vector or axial), in such a way that two out of the four supercharges are scalars under the new Lorentz symmetry and are therefore always preserved. There are two possible ways to twist a GLSM, depending on the choice of R-symmetry used to perform the twist: the A-twist and the B-twist. In the first case, correlators of gauge invariant operators only depend on the Kähler moduli of M and are related to the GW invariants of M: in particular, the three-point functions will provide the Yukawa couplings we mentioned before. On the other hand, correlators in the B-model only depend on the complex structure moduli of M. One is therefore led to consider the A-model GLSM with target M: nevertheless, the presence of world-sheet corrections in the Yukawa couplings of the A-twisted model typically makes the computation rather involved. In order to solve this problem, one can invoke *mirror symmetry*: this is an equivalence between an A-twisted theory with target M and a B-twisted theory with target $\widetilde{M}$ (called mirror manifold). Since B-model correlators are not affected by world-sheet corrections, computations can be performed on the mirror theory and then translated to the A-model. The limitation of this method is that the mirror $\widetilde{M}$ is not always known for the targets of interest.

Nowadays, new gauge theory techniques have been developed in order to study this problem. Very recently it has been shown how to construct supersymmetric gauge theories on compact curved backgrounds without having to perform a topological twist; what is more is that the partition function and other BPS observables can be computed exactly, via the so-called *supersymmetric localization* technique, and are well-behaved thanks to the finite size of the compact background which acts as an IR regulator. In the case of the S^2 untwisted curved background, the partition function Z^{S^2} of a $\mathcal{N} = (2, 2)$ GLSM with target M has been shown to contain all the relevant information about the genus zero GW invariants of M, and these invariants can be extracted without having to know the mirror manifold $\widetilde{M}$.

The first part of this Thesis will be dedicated to the study of GW invariants with this new approach. Chapter 2 contains a short introduction to supersymmetric localization applied to the S^2 case we are interested in, while Chapter 3 explains in more detail how to extract GW invariants from Z^{S^2}. Chapter 3 also provides a large number of examples of both abelian and non-abelian theories, and contains a discussion on an alternative interpretation of Z^{S^2} in terms of Givental's $\mathcal{I}$ and $\mathcal{J}$ functions, mathematical objects entering in the computation of the quantum cohomology ring of M.

The second part of this Thesis, that is Chapter 4, will be dedicated to the analysis of a particular GLSM known as the ADHM GLSM. As we will see, this theory consists of a gauge group $U(k)$, three fields in the adjoint representation, plus fields in the fundamental and anti-fundamental representation charged under a $U(N)$ flavour symmetry. Its associated target manifold M, which we will denote as $\mathcal{M}_{k,N}$, is very special: it is given by the moduli space of k instantons for a pure $U(N)$ Yang-Mills theory. By turning on twisted masses and Fayet-Iliopoulos parameters for this GLSM, we can make $\mathcal{M}_{k,N}$ compact and non-singular; its volume $Z_{k,N} = \mathrm{Vol}(\mathcal{M}_{k,N})$ then coincides with the k-instanton contribution to the instanton partition function $Z_N = \sum_{k\geqslant 0} \Lambda^{2Nk} Z_{k,N}$ for a four-dimensional $\mathcal{N} = 2$ $U(N)$ theory, with Λ energy scale of the 4d theory. If we denote by r the radius of S^2, we can recover this volume from the two-sphere partition function $Z^{S^2}_{k,N}$ of the ADHM GLSM by taking the limit $r \to 0$. For finite r, as we already discussed, $Z^{S^2}_{k,N}$ will in addition contain the genus zero GW invariants of $\mathcal{M}_{k,N}$: these are our original motivation for considering this particular GLSM. In the first half of Chapter 4 we will see in detail how the S^2 partition function for the ADHM GLSM reproduces the known results in the mathematical literature for the $N = 1$ case, and provides an easy way to compute the invariants for any N. A similar analysis can be performed for the moduli space of instantons on ALE spaces: in this cases the associated GLSMs are given by Nakajima quivers. We will briefly comment on ALE spaces of type A and D in Appendix A.

While the Higgs branch of the moduli space of supersymmetric vacua of two-dimensional $\mathcal{N} = (2, 2)$ gauge theories is related to Kähler manifolds and their topological invariants, in recent years the Coulomb branch of these theories has been shown to be deeply connected to quantum integrable systems such as XXX spin chains. In the Coulomb branch, the gauge group G is broken down to $U(1)^{\mathrm{rk}G}$ by the vacuum expectation value of the scalar field in the $\mathcal{N} = (2, 2)$ vector multiplet; in the infra-red we therefore remain with a purely abelian theory. This effective theory can be described in terms of a holomorphic function $\mathcal{W}_{\mathrm{eff}}(\Sigma)$ (known as *effective twisted superpotential*) which only depends on the superfield strength supermultiplets Σ_s containing the field strengths of the various $U(1)$ factors. The Coulomb branch vacua can be determined by solving the equations obtained by extremizing $\mathcal{W}_{\mathrm{eff}}(\Sigma)$.
This has deep connections with the theory of integrable systems: in fact, by the recently proposed Bethe/Gauge correspondence, the Coulomb branch of every $\mathcal{N} = (2, 2)$ GLSM can be associated to a quantum integrable system. Among other things, the correspondence states that the Coulomb branch vacua equations of the gauge theory coincide with the Bethe Ansatz Equations for the associated integrable system: these are equations whose solution determines the free parameters Σ_s in the ansatz formulated by Bethe

for the eigenstates and eigenvalues. Moreover, the correspondence also tells us that the spectrum of the system can naturally be rewritten in terms of gauge theory observables. Since there are many more $\mathcal{N} = (2, 2)$ gauge theories than known integrable systems, this correspondence provides a conjectural way to construct new integrable systems; the problem is that recognizing the associated system is not always an easy task.
Again, the S^2 partition function turns out to be a powerful method to study the Coulomb branch of a general GLSM. In the second half of Chapter 4 we will see how one can extract the twisted effective superpotential $\mathcal{W}_{\text{eff}}$ describing the Coulomb branch of the theory directly from Z^{S^2}, focussing on the example of the ADHM GLSM. In the ADHM case, the Bethe Ansatz Equations for the corresponding integrable system are proposed to be the ones for the periodic $gl(N)$ quantum Intermediate Long Wave (ILW_N) system: this is a system of hydrodynamic type, which can be described in terms of a partial integro-differential equation, and admits an infinite number of conserved quantities $\widehat{I}_l$. Since the quantum ILW_N system has not been completely solved yet (i.e. Hamiltonians $\widehat{I}_l$, eigenstates and spectrum are not completely known), the hope is that our GLSM can provide some information on the solution. In fact we will be able to show that the local observables in the ADHM theory are naturally associated to the eigenvalues of the $\widehat{I}_l$ and can therefore be used to determine the quantum ILW_N spectrum. Moreover, the partition function Z^{S^2} evaluated at a Coulomb branch vacuum can be used to compute the norm of the ILW_N eigenstates. Hydrodynamic systems of similar type are expected to arise by considering the GLSMs associated to Nakajima quivers; we will briefly comment on this in Appendix A.

Apart from the existence of infinite conserved quantities, integrability of the ILW_N system implies the existence of an infinite number of exact solutions known as *solitons*: these are waves whose profile does not change with time. As we will see, in the $N = 1$ case an n-soliton solution can be expressed in terms of a pole ansatz, where the dynamics of the n poles is determined by another quantum integrable system, the n-particles elliptic Calogero-Sutherland model (eCS). Contrary to ILW_1, the eCS system has a finite number of degrees of freedom and conserved quantities: nevertheless it is expected to reduce to ILW_1 in the limit of infinite particles, while keeping the density of particles finite.
It is well-known that the eCS model admits a "relativistic" generalization given by the n-particles elliptic Ruijsenaars-Schneider model (eRS), in which the differential operators corresponding to the eCS Hamiltonians get promoted to finite-difference operators. One can therefore wonder if in the limit of infinite particles the eRS system reduces to a finite-difference version of ILW_1 (ΔILW for short), and if there is a description of this system in gauge theory via Bethe/Gauge correspondence. Chapter 5 of this Thesis is devoted to study these questions.

In order to do this we will first have to understand better the eRS model, since at present this system has not been solved explicitly (that is, we do not know eigenfunctions and eigenvalues of the eRS Hamiltonians). A powerful way to find the solution to eRS, at least perturbatively in the elliptic parameter p, comes from gauge theory. In fact, as we will review, the eRS system admits a gauge theory description in terms of a 5d $\mathcal{N} = 1^*$ $U(n)$ theory on $\mathbb{C}^2 \times S^1$: its instanton partition function in presence of codimension two monodromy defects corresponds to eRS eigenfunctions, while codimension 4 defects give the eRS eigenvalues. Thanks to our good understanding of instanton computations in supersymmetric theories, we can in principle obtain the eRS solution at any order in p. We will then need to study the finite-difference version of ILW. Although this system has received very little attention in the literature, we will review what is known and propose a gauge theory which can be related to it: this is simply the most natural guess, that is the ADHM theory on $S^2 \times S^1$. The proposal is again motivated by the Bethe/Gauge correspondence: assuming that the equations determining the supersymmetric vacua in the Coulomb branch of this 3d theory coincide with the Bethe Ansatz Equations for ΔILW, we can compute the 3d ADHM local observables at these vacua and show that they reproduce the ΔILW spectrum.
Finally, we will need a way to relate eRS to ΔILW. An efficient formalism to do this is the *collective coordinate* description of the eRS system, in which the eRS Hamiltonians given by finite-difference operators are rewritten in terms of operators made out of generators of a Heisenberg algebra. These operators turn out to coincide with the ΔILW quantum Hamiltonians: this is not surprising, since the collective coordinate description is a way to treat the eRS system independently on the number of particles n. At the level of eigenvalues, we will see that there is a very simple relation between the eRS and ΔILW spectra in the $n \to \infty$ limit: while this is expected from the integrable system point of view, it also implies a quite remarkable equivalence between non-local observables (Wilson loops) of the 5d $U(n)$ $\mathcal{N} = 1^*$ theory in the limit of infinite rank n and local observables of the 3d ADHM theory, if we think of the integrable systems in terms of their gauge theory analogues. This hints towards an infra-red duality at $n \to \infty$ between the two theories as a whole, not just at the level of observables. Unfortunately we are not able to prove this proposal at the moment: a more detailed analysis of this problem will have to be postponed to future work.

Chapter 2

Supersymmetric localization

2.1 Supersymmetric localization: an overview

Inspired by earlier mathematical works [1, 2, 3, 4, 5], localization techniques have been introduced in physics in [6, 7], where topologically twisted supersymmetric theories on a compact manifold were considered. In the following years, this idea has been successfully applied to theories on non-compact manifolds with Ω background [8, 9, 10], as well as to non-topologically twisted theories on many different compact manifolds [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] and manifolds with boundaries [33, 34, 35]. Even if in the following we will mainly consider the case of theories on S^2 [27, 28], in this section we want to give some general comments on the idea of supersymmetric localization; more details can be found in [36, 37].

Let δ_Q be a Grassmann-odd symmetry of a quantum field theory with action $S[X]$, where X is the set of fields of the theory; in supersymmetric theories, δ_Q will be a supercharge. We assume that this symmetry is not anomalous (i.e. the path integral measure is δ_Q-invariant) and

$$\delta_Q^2 = \mathcal{L}_B \tag{2.1}$$

with $\mathcal{L}_B$ a Grassmann-even symmetry. What we are interested in is the vacuum expectation value $\langle O_{BPS} \rangle$ of BPS observables, i.e. local or non-local gauge invariant operators preserved by δ_Q:

$$\delta_Q O_{BPS} = 0 \tag{2.2}$$

The localization argument goes as follows. Denote by G the symmetry group associated to δ_Q. If G acts freely on the whole space of field configurations $\mathcal{F}$, then[1]

$$\langle O_{BPS} \rangle = \int_{\mathcal{F}} [dX] O_{BPS}\, e^{-S[X]} = \mathrm{Vol}(G) \int_{\mathcal{F}/G} [dX] O_{BPS}\, e^{-S[X]} \tag{2.3}$$

but for a fermionic symmetry group

$$\mathrm{Vol}(G) = \int d\theta\, 1 = 0 \tag{2.4}$$

This means that the action of G must not be free on the whole $\mathcal{F}$, otherwise even the partition function of the theory would vanish. In fact δ_Q has fixed points, corresponding to the BPS locus $\mathcal{F}_{BPS}$ of δ_Q-invariant field configurations:

$$\mathcal{F}_{BPS} = \{\text{fields } X \in \mathcal{F} \ / \ \delta_Q X = 0\} \tag{2.5}$$

We conclude that our path integral over $\mathcal{F}$ will be non-zero (= localizes) only at the BPS locus $\mathcal{F}_{BPS}$; in many cases the BPS locus is finite-dimensional and therefore the infinite-dimensional path integral reduces to a finite-dimensional one, allowing for an exact computation of the BPS observables.

Another argument for localization, with more content from the computational point of view, is the following one. Consider the perturbed observable

$$\langle O_{BPS} \rangle [t] = \int_{\mathcal{F}} [dX] O_{BPS}\, e^{-S[X] - t\delta_Q V[X]} \tag{2.6}$$

Here V is a Grassmann-odd operator which is invariant under $\mathcal{L}_B$, so that

$$\delta_Q^2 V = \mathcal{L}_B V = 0 \tag{2.7}$$

As long as $V[X]$ does not change the asymptotics at infinity in $\mathcal{F}$ of the integrand, $\langle O_{BPS} \rangle [t]$ does not depend on t (and therefore on $\delta_Q V[X]$) since

$$\begin{aligned} \frac{d}{dt} \langle O_{BPS} \rangle [t] = &- \int_{\mathcal{F}} [dX] O_{BPS}\, \delta_Q V\, e^{-S[X] - t\delta_Q V[X]} = \\ &- \int_{\mathcal{F}} [dX] \delta_Q \left(O_{BPS}\, V\, e^{-S[X] - t\delta_Q V[X]} \right) = 0 \end{aligned} \tag{2.8}$$

The final result is an integral of a total derivative in field space: this gives a boundary term, which vanishes if we assume that the integrand decays fast enough. We therefore

[1] We are ignoring the normalization by the partition function in order to lighten notation.

conclude that

$$\langle O_{BPS}\rangle = \langle O_{BPS}\rangle[t=0] = \langle O_{BPS}\rangle[t] \quad \forall\, t \tag{2.9}$$

This means we can compute $\langle O_{BPS}\rangle$ in the limit $t \to \infty$, in which simplifications typically occur. In particular, one usually chooses V such that the bosonic part of $\delta_Q V$ is positive semi-definite; in this case in the $t \to \infty$ limit the integrand (2.6) localizes to a submanifold $\mathcal{F}_{saddle} \subset \mathcal{F}$ determined by the saddle points of the *localizing action* $S_{loc} = \delta_Q V$:

$$\mathcal{F}_{saddle} = \{X \in \mathcal{F} \;\; / \;\; (\delta_Q V)_{bos} = 0\} \tag{2.10}$$

Still we don't have to forget the previous localization argument, which tells us the path integral is zero outside $\mathcal{F}_{BPS}$; while for certain choices of S_{loc} the two localization loci coincide, in general $\mathcal{F}_{BPS} \neq \mathcal{F}_{saddle}$ and the path integral localizes to

$$\mathcal{F}_{loc} = \mathcal{F}_{BPS} \cap \mathcal{F}_{saddle} \tag{2.11}$$

To evaluate (2.6) we can think of $\hbar_{aux} = 1/t$ as an auxiliary Planck constant (which is <u>not</u> the $\hbar$ of the original action $S[X]$, set to 1) and expand the fields around the saddle point configurations of $\delta_Q V$:

$$X = X_0 + \frac{1}{\sqrt{t}}\delta X \tag{2.12}$$

The semiclassical 1-loop expansion of the total action $S + S_{loc}$

$$S[X_0] + \frac{1}{2}\int\int (\delta X)^2 \frac{\delta^2 S_{loc}[X]}{\delta^2 X}\bigg|_{X=X_0} \tag{2.13}$$

is exact for $t \to \infty$; we can integrate out the fluctuations δX normal to $\mathcal{F}_{loc}$ since the integral is Gaussian, thus obtaining a 1-loop superdeterminant, and we are left with

$$\langle O_{BPS}\rangle = \int_{\mathcal{F}_{loc}} [dX_0]\, O_{BPS}\Big|_{X=X_0} e^{-S[X_0]}\, \mathrm{SDet}^{-1}\left[\frac{\delta^2 S_{loc}[X]}{\delta^2 X}\right]\Bigg|_{X=X_0} \tag{2.14}$$

We can now see why localization is a powerful tool to perform exact computations in supersymmetric theories. The path-integral is often reduced to a finite-dimensional integral, and the integrand is simply given by a ratio of 1-loop fermionic and bosonic determinants. We will see an example of localization in the following section.

A few more comments are in order. First of all, in theories with many Grassmann-odd symmetries $\delta_{Q_1}, \ldots, \delta_{Q_N}$, one can choose any of the δ_{Q_i} to perform the localization, and this choice determines the spectrum of BPS observables one can compute. Moreover, at fixed δ_{Q_i}, we can use different localizing actions S_{loc}; the localization loci $\mathcal{F}_{loc}$ and 1-loop determinants will be different from case to case, but the final answer (2.14) must

be the same for different localization schemes, the result being independent of S_{loc}.
As a final comment, we remark that if we require the path integral to be well-defined, and in particular to be free of infrared divergences, we are naturally led to place the theory on a compact manifold or in an Omega background.

2.2 Supersymmetric localization: the S^2 case

Since in the following we will be working with supersymmetric $\mathcal{N} = (2, 2)$ gauge theories on S^2, in this section we review the main points concerning localization on an euclidean two-sphere of radius r along the lines of [27, 28], to which we refer for further details.
In this setting, the two-sphere S^2 is thought as a conformally flat space; it does not admit Killing spinors, but it admits four complex conformal Killing spinors which realize the $\mathfrak{osp}(2|2, \mathbb{C})$ superconformal algebra on S^2. We take as $\mathcal{N} = (2, 2)$ supersymmetry algebra on S^2 the subalgebra $\mathfrak{su}(2|1) \subset \mathfrak{osp}(2|2, \mathbb{C})$ realized by two out of the four conformal Killing spinors, which does not contain conformal nor superconformal transformations; its bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)_R \subset \mathfrak{su}(2|1)$ generates the isometries of S^2 and an abelian vector R-symmetry, which is now part of the algebra and not an outer isomorphism of it.
We stress that these theories are different from topologically twisted theories on S^2; this latter case has been recently studied in [38, 39].

2.2.1 $\mathcal{N} = (2, 2)$ gauge theories on S^2

The theories we are interested in are $\mathcal{N} = (2, 2)$ gauged linear sigma models (GLSM) on S^2. The basic multiplets of two dimensional $\mathcal{N} = (2, 2)$ supersymmetry are vector and chiral multiplets, which arise by dimensional reduction of four dimensional $\mathcal{N} = 1$ vector and chiral multiplets. In detail

$$\begin{aligned} \text{vector multiplet} &: (A_\mu, \sigma, \eta, \lambda, \bar{\lambda}, D) \\ \text{chiral multiplet} &: (\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F}) \end{aligned} \tag{2.15}$$

with $(\lambda, \bar{\lambda}, \psi, \bar{\psi})$ two component complex Dirac spinors, (σ, η, D) real scalar fields and (ϕ, F) complex scalar fields. A GLSM is specified by the choice of the gauge group G, the representation R of G for the matter fields, and the matter interactions contained in the superpotential $W(\Phi)$, which is an R-charge 2 gauge-invariant holomorphic function of the chiral multiplets Φ. If the gauge group admits an abelian term, we can also add a Fayet-Iliopoulos term ξ and theta-angle θ. All in all, the most general renormalizable

$\mathcal{N} = (2,2)$ Lagrangian density of a GLSM on S^2 can be written down as

$$\mathcal{L} = \mathcal{L}_{\text{vec}} + \mathcal{L}_{\text{chiral}} + \mathcal{L}_W + \mathcal{L}_{FI} \tag{2.16}$$

where

$$\begin{aligned}\mathcal{L}_{\text{vec}} = \frac{1}{g^2}\text{Tr}\Big\{ & \frac{1}{2}\left(F_{12} - \frac{\eta}{r}\right)^2 + \frac{1}{2}\left(D + \frac{\sigma}{r}\right)^2 + \frac{1}{2}D_\mu \sigma D^\mu \sigma + \frac{1}{2}D_\mu \eta D^\mu \eta \\ & - \frac{1}{2}[\sigma,\eta]^2 + \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu \lambda + \frac{i}{2}\bar{\lambda}[\sigma,\lambda] + \frac{1}{2}\bar{\lambda}\gamma_3[\eta,\lambda]\Big\}\end{aligned} \tag{2.17}$$

$$\begin{aligned}\mathcal{L}_{\text{chiral}} = & D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}\sigma^2\phi + \bar{\phi}\eta^2\phi + i\bar{\phi}D\phi + \bar{F}F + \frac{iq}{r}\bar{\phi}\sigma\phi + \frac{q(2-q)}{4r^2}\bar{\phi}\phi \\ & - i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\psi}\sigma\psi - \bar{\psi}\gamma_3\eta\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi - \frac{q}{2r}\bar{\psi}\psi\end{aligned} \tag{2.18}$$

$$\mathcal{L}_W = \sum_j \frac{\partial W}{\partial \phi_j}F_j - \sum_{j,k}\frac{1}{2}\frac{\partial^2 W}{\partial\phi_j\partial\phi_k}\psi_j\psi_k \tag{2.19}$$

$$\mathcal{L}_{FI} = \text{Tr}\left[-i\xi D + i\frac{\theta}{2\pi}F_{12}\right] \tag{2.20}$$

Here we defined q as the R-charge of the chiral multiplet. In addition, if there is a global (flavour) symmetry group G_F it is possible to turn on in a supersymmetric way *twisted masses* for the chiral multiplets. These are obtained by first weakly gauging G_F, then coupling the matter fields to a vector multiplet for G_F, and finally giving a supersymmetric background VEV σ^{ext}, η^{ext} to the scalar fields in that vector multiplet. Supersymmetry on S^2 requires σ^{ext}, η^{ext} being constants and in the Cartan of G_F; in particular η^{ext} should be quantized, and in the following we will only consider $\eta^{\text{ext}} = 0$. The twisted mass terms can simply be obtained by substituting $\sigma \to \sigma + \sigma^{\text{ext}}$ in (2.18).

2.2.2 Localization on S^2 - Coulomb branch

In order to localize the path integral, we consider an $\mathfrak{su}(1|1) \subset \mathfrak{su}(2|1)$ subalgebra generated by two fermionic charges δ_ϵ and $\delta_{\bar{\epsilon}}$. In terms of

$$\delta_{\mathcal{Q}} = \delta_\epsilon + \delta_{\bar{\epsilon}} \tag{2.21}$$

this subalgebra is given by[2]

$$\delta_{\mathcal{Q}}^2 = J_3 + \frac{R_V}{2} \quad , \quad \left[J_3 + \frac{R_V}{2}, \delta_{\mathcal{Q}}\right] = 0 \tag{2.22}$$

[2] $\delta_{\mathcal{Q}}^2$ also generates gauge and flavour transformations.

In particular, we notice that the choice of δ_Q breaks the $SU(2)$ isometry group of S^2 to a $U(1)$ subgroup, thus determining a north and south pole on the two-sphere.
It turns out that $\mathcal{L}_{\text{vec}}$ and $\mathcal{L}_{\text{chiral}}$ are δ_Q-exact terms:

$$\begin{aligned}\bar{\epsilon}\epsilon\mathcal{L}_{\text{vec}} &= \delta_Q\delta_{\bar{\epsilon}}\text{Tr}\left(\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma - \frac{1}{r}\sigma^2\right)\\ \bar{\epsilon}\epsilon\mathcal{L}_{\text{chiral}} &= \delta_Q\delta_{\bar{\epsilon}}\text{Tr}\left(\bar{\psi}\psi - 2i\bar{\phi}\sigma\phi + \frac{q-1}{r}\bar{\phi}\phi\right)\end{aligned} \tag{2.23}$$

This means that we can choose the localizing action as $\mathcal{L}_{\text{vec}}+\mathcal{L}_{\text{chiral}}$; as a consequence, the partition function will not depend on the gauge coupling constant, since it is independent of S_{loc}. For the same reason it will not depend on the superpotential parameters, $\mathcal{L}_W$ being also δ_Q-exact (although the presence of a superpotential constrains the value of the R-charges). This choice of localizing action is referred to as the *Coulomb branch* localization scheme, since the localization locus $\mathcal{F}_{loc}$ mimics a Coulomb branch. In particular, $\mathcal{F}_{loc}$ is given by

$$0 = \phi = \bar{\phi} = F = \bar{F} \tag{2.24}$$

(for generic R-charges) and

$$0 = F_{12} - \frac{\eta}{r} = D + \frac{\sigma}{r} = D_\mu\sigma = D_\mu\eta = [\sigma, \eta] \tag{2.25}$$

These equations imply that σ and η are constant and in the Cartan of the gauge group; moreover, since the gauge flux is GNO quantized on S^2

$$\frac{1}{2\pi}\int F = 2r^2F_{12} = \mathfrak{m} \in \mathbb{Z} \tag{2.26}$$

we remain with

$$F_{12} = \frac{\mathfrak{m}}{2r^2} \quad , \quad \eta = \frac{\mathfrak{m}}{2r} \tag{2.27}$$

One can then compute the one-loop determinants for vector and chiral multiplets around the $\mathcal{F}_{loc}$ field configurations; the final result is

$$Z^{\text{1l}}_{\text{vec}} = \prod_{\alpha>0}\left(\frac{\alpha(\mathfrak{m})^2}{4} + r^2\alpha(\sigma)^2\right) \tag{2.28}$$

$$Z^{\text{1l}}_{\Phi} = \prod_{\rho\in R}\frac{\Gamma\left(\frac{q}{2} - ir\rho(\sigma) - \frac{\rho(\mathfrak{m})}{2}\right)}{\Gamma\left(1 - \frac{q}{2} + ir\rho(\sigma) - \frac{\rho(\mathfrak{m})}{2}\right)} \tag{2.29}$$

with $\alpha > 0$ positive roots of the gauge group G and ρ weights of the representation R of the chiral multiplet. Twisted masses for the chiral multiplet can be added by shifting $\rho(\sigma) \to \rho(\sigma) + \tilde{\rho}(\sigma^{\text{ext}})$ and multiplying over the weights of the representation $\tilde{\rho}$ of the flavour group G_F. The classical part of the action is simply given by the Fayet-Iliopoulos term:

$$S_{FI} = 4\pi i r \xi_{\text{ren}} \text{Tr}(\sigma) + i\theta_{\text{ren}} \text{Tr}(\mathfrak{m}) \tag{2.30}$$

where we are taking into account that in general the Fayet-Iliopoulos parameter runs [28] and the θ-angle gets a shift from integrating out the W-bosons [35], according to

$$\xi_{\text{ren}} = \xi - \frac{1}{2\pi}\sum_l Q_l \log(rM) \quad , \quad \theta_{\text{ren}} = \theta + (s-1)\pi \tag{2.31}$$

Here M is a SUSY-invariant ultraviolet cut-off, s is the rank of the gauge group and Q_l are the charges of the chiral fields with respect to the abelian part of the gauge group. In the Calabi-Yau case the sum of the charges is zero, therefore $\xi_{\text{ren}} = \xi$; on the other hand for Abelian theories there are no W-bosons and $\theta_{\text{ren}} = \theta$.
All in all, the partition function for an $\mathcal{N} = (2,2)$ GLSM on S^2 reads

$$Z_{S^2} = \frac{1}{|\mathcal{W}|}\sum_{\mathfrak{m}\in\mathbb{Z}}\int \left(\prod_{s=1}^{\text{rk}G}\frac{d\sigma_s}{2\pi}\right) e^{-4\pi i r\xi_{\text{ren}}\text{Tr}(\sigma) - i\theta_{\text{ren}}\text{Tr}(\mathfrak{m})} Z_{\text{vec}}^{1l}(\sigma,\mathfrak{m}) \prod_{\Phi} Z_{\Phi}^{1l}(\sigma,\mathfrak{m},\sigma^{\text{ext}}) \tag{2.32}$$

where $|\mathcal{W}|$ is the order of the Weyl group of G. If G has many abelian components, we will have more Fayet-Iliopoulos terms and θ-angles.

2.2.3 Localization on S^2 - Higgs branch

As we saw, equation (2.32) gives a representation of the partition function as an integral over Coulomb branch vacua. For the theories we will consider in this Thesis (i.e. with gauge group $U(N)$ or products thereof) another representation of Z_{S^2} is possible, in which the BPS configurations dominating the path integral are a finite number of points on the Higgs branch, supporting point-like vortices at the north pole and anti-vortices at the south pole of S^2; we will call this *Higgs branch* representation. Its existence has originally been suggested by explicit evaluation of (2.32) for few examples, in which the partition function was shown to reduce to a sum of contributions which can be factorised in terms of a classical part, a 1-loop part, a partition function for vortices and another for antivortices.

Starting from the localization technique, the Higgs branch representation can be obtained by adding another δ_Q-exact term to the action which introduces a parameter χ acting as an auxiliary Fayet-Iliopoulos [27]. Although this implies that the new localization locus is in general different from the one considered in the previous section, we know the final result is independent of the choice of localization action, and this explains why the two representations of the partition function are actually the same. In particular at

$q = 0$ the new localization locus admits a Higgs branch, given by

$$0 = F = D_\mu \phi = \eta\phi = (\sigma + \sigma^{\text{ext}})\phi = \phi\phi^\dagger - \chi \mathbf{1} \tag{2.33}$$

$$0 = F_{12} - \frac{\eta}{r} = D + \frac{\sigma}{r} = D_\mu \sigma = D_\mu \eta = [\sigma, \eta] \tag{2.34}$$

According to the matter content of the theory, this set of equations can have a solution with $\eta = F_{12} = 0$ and $\sigma = -\sigma^{\text{ext}}$, so that for generic twisted masses the Higgs branch consists of a finite number of isolated vacua, which could be different for $\chi \gtrless 0$.

On top of each classical Higgs vacuum there are vortex solutions at the north pole satisfying

$$D + \frac{\sigma}{r} = -i(\phi\phi^\dagger - \chi\mathbf{1}) = iF_{12} \quad , \quad D_- \phi = 0 \tag{2.35}$$

and anti-vortex solutions at the south pole

$$D + \frac{\sigma}{r} = -i(\phi\phi^\dagger - \chi\mathbf{1}) = -iF_{12} \quad , \quad D_+ \phi = 0 \tag{2.36}$$

The size of vortices depends on χ and tends to zero for $|\chi| \to \infty$; in this limit the contribution from the Coulomb branch is suppressed, and we remain with the Higgs branch solutions together with singular point-like vortices and antivortices.

All in all, the partition function Z_{S^2} in the Higgs branch can be schematically written in the form

$$Z_{S^2} = \sum_{\sigma = -\sigma^{\text{ext}}} Z_{\text{cl}} Z_{1\text{l}} Z_{\text{v}} Z_{\text{av}} \tag{2.37}$$

Apart from the classical and 1-loop terms, we have the vortex / anti-vortex partition functions Z_{v}, Z_{av}; they coincide with the ones computed on $\mathbb{R}^2$ with Ω-background, where the Ω-background parameter $\hbar$ depends on the S^2 radius as $\hbar = \frac{1}{r}$. The vortex partition function $Z_{\text{v}}\left(z, \frac{1}{r}\right)$ can be thought of as the two-dimensional analogue of the four-dimensional instanton partition function of $\mathcal{N} = 2$ theories, with $z = e^{-2\pi\xi - i\theta}$ vortex counting parameter. Re-expressing (2.32) in a form similar to (2.37) before performing the integration will be a key ingredient in the next chapters and will reveal a deep connection to the enumerative interpretation of Z_{S^2}.

As a final remark, let us stress once more that although the explicit expressions for Z_{S^2} in the Higgs and Coulomb branch might look very different, they are actually the same because of the localization argument, and in fact the Higgs branch representation (2.37) can be recovered from the Coulomb branch one (2.32) by residue evaluation of the integral.

Chapter 3

Vortex counting and Gromov-Witten invariants

3.1 Gromov-Witten theory from Z_{S^2}

In the previous chapter we introduced a particular class of theories, the two-dimensional $\mathcal{N} = (2,2)$ Gauged Linear Sigma Models on S^2, and we showed how to compute their partition function and BPS observables exactly via supersymmetric localization. As we saw, physical observables can in general receive non-perturbative quantum corrections, which in two dimensions are generated by world-sheet instantons (i.e. vortices).

These GLSMs have been, and still are, of great importance in physics, especially for the study of string theory compactifications. In fact at the classical level, the space X of supersymmetric vacua in the Higgs branch of the theory is given by the set of constant VEVs for the chiral fields minimizing the scalar potential, i.e. solving the F- and D-equations, modulo the action of the gauge group:

$$X = \{\text{constant } \langle\phi\rangle / F = 0, D = 0\}/G \tag{3.1}$$

This space is always a Kähler manifold with Kähler moduli given by the complexified FI parameters $r_l = \xi_l + i\frac{\theta_l}{2\pi}$ and first Chern class $c_1 \geqslant 0$; a very important subcase is when $c_1 = 0$, in which X is a Calabi-Yau manifold. In the following we will refer to X as the *target manifold* of the GLSM. To be more precise, X represents a family of target manifolds, depending on the explicit values of the r_l's; the topological properties of the target space can change while varying the Kähler moduli, and the GLSM is a powerful method to study these changes.

From the physics point of view, the most interesting GLSMs are those whose target is a Calabi-Yau three-fold, since they provide (in the infra-red) a description for a very rich set of four-dimensional vacua of string theory. The study of these sigma models led to great discoveries both in mathematics and in physics such as mirror symmetry [40, 41, 42, 43, 44], which quickly became an extremely important tool to understand world-sheet quantum corrections to the Kähler moduli space of Calabi-Yau three-folds. In fact as we will see shortly, these non-perturbative quantum corrections form a power series whose coefficients, known as Gromov-Witten invariants [45, 46, 47], are related to the mathematical problem of counting holomorphic maps of fixed degree from the world-sheet to the Calabi-Yau target (physically, they give the Yukawa couplings in the four-dimensional effective theory obtained from string theory after compactification on the Calabi-Yau). In general, computing these quantum corrections is highly non-trivial; the problem can be circumvented by invoking mirror symmetry, which allows us to extract these invariants from the mirror geometry, free from quantum corrections. Unfortunately mirror symmetry can only be applied when the Calabi-Yau three-fold under consideration has a known mirror construction; this is the case for complete intersections in a toric variety and few other exceptions, but the whole story is yet to be understood.

When the mirror manifold is not known, we can make use of the exact expressions found in Chapter 2 to compute these non-perturbative corrections; this is why localization computations on S^2 greatly helped making progress in solving this problem. The key point is that, as conjectured in [48] and proved in [49] (the proof being based on [50]), the partition function Z_{S^2} for an $\mathcal{N} = (2, 2)$ GLSM computes the vacuum amplitude of the associated infrared Non-Linear Sigma Model with same target space:

$$Z_{S^2}(t_l, \bar{t}_l) = \langle \bar{0} | 0 \rangle = e^{-\mathcal{K}_K(t_l, \bar{t}_l)} \tag{3.2}$$

Here $\mathcal{K}_K$ is a canonical expression for the exact Kähler potential on the quantum Kähler moduli space $\mathcal{M}_K$ of the Calabi-Yau target X. The Kähler moduli t_l of X are a canonical set of coordinates in $\mathcal{M}_K$, related to the complexified Fayet-Iliopoulos parameters r_l of the GLSM via a change of variables $t_l = t_l(r_m)$ called *mirror map*. The Kähler potential $\mathcal{K}_K(t_a, \bar{t}_a)$ contains all the necessary information about the Gromov-Witten invariants of the target; this allows us to compute them for targets more generic than those whose mirror is known, and in particular for non-abelian quotients.[1]

[1] Of course, a Kähler potential is only defined up to Kähler transformations $\mathcal{K}_K(t_l, \bar{t}_l) \to \mathcal{K}_K(t_l, \bar{t}_l) + f(t_l) + \overline{f}(\bar{t}_l)$ or, if you prefer, to a change of coordinates. The point is that the t_l coordinates are the ones naturally entering in mirror symmetry, and in terms of which the Gromov-Witten invariants are defined.

More in detail, the exact expression reads

$$e^{-\mathcal{K}_K(t,\bar{t})} = -\frac{i}{6}\sum_{l,m,n}\kappa_{lmn}(t^l-\bar{t}^l)(t^m-\bar{t}^m)(t^n-\bar{t}^n)+\frac{\zeta(3)}{4\pi^3}\chi(X)$$
$$+\frac{2i}{(2\pi i)^3}\sum_{\eta}N_\eta\Big(\mathrm{Li}_3(q^\eta)+\mathrm{Li}_3(\bar{q}^\eta)\Big)-\frac{i}{(2\pi i)^2}\sum_{\eta,l}N_\eta\Big(\mathrm{Li}_2(q^\eta)+\mathrm{Li}_2(\bar{q}^\eta)\Big)\eta_l(t^l-\bar{t}^l) \tag{3.3}$$

Here $\chi(X)$ is the Euler characteristic of X, while

$$\mathrm{Li}_k(q)=\sum_{n=1}^{\infty}\frac{q^n}{n^k}\quad,\quad q^\eta=e^{2\pi i\sum_l \eta_l t^l}\quad, \tag{3.4}$$

with η_l an element of the second homology group of the target Calabi-Yau three-fold and N_η genus zero Gromov-Witten invariants.[2]

There is more to this story. Even if Calabi-Yau three-folds are the most relevant targets for physics applications, (3.2) is also valid for generic Calabi-Yau n-folds (even if the standard form for $\mathcal{K}_K$ (3.3) depends on n [51, 52]). Moreover, every compact Kähler target with semi-positive definite first Chern class $c_1 \geq 0$ has Kähler moduli and Gromov-Witten invariants, even if in the $c_1 > 0$ case the Kähler potential computed in (3.2) is not the complete one obtained via tt^* equations [50] (yet, they coincide in a particular holomorphic limit [53]).

In order to also consider these geometries, in [54] we took a different approach to the same problem, by re-interpreting Z_{S^2} in terms of Givental's formalism [55] and its extension to non-abelian quotients in the language of quasi-maps [56]. A good review of Givental's formalism can be found in [57].
What we studied is a large class of both Calabi-Yau ($c_1 = 0$) and Fano ($c_1 > 0$) manifolds, compact and non-compact; in the latter case we must turn on twisted masses to regularize the infinite volume of the target, which corresponds to considering *equivariant* Gromov-Witten invariants. Apart from reproducing the known results for the simplest targets and providing new examples, what we obtained is the possibility of analysing the chamber structure and wall-crossings of the GIT quotient moduli space in terms of integration contour choices of (2.32). In particular we obtained explicit description of the equivariant quantum cohomology and chamber structure of the resolutions of $\mathbb{C}^3/\mathbb{Z}_n$ orbifolds, thus giving a physics proof of the *crepant resolution conjecture* for this case, and of the Uhlembeck partial compactification of the instanton moduli space; this last example will be the main character of the following chapter.

[2] In this Thesis we will only discuss genus zero Gromov-Witten invariants, related to maps from a genus zero surface, since we are studying theories on S^2.

In order to explain the relation between gauge theories on S^2 and Givental's formalism, we will have to follow [58, 59]. Let us introduce the flat sections V_a of the Gauss-Manin connection spanning the vacuum bundle of the theory and satisfying

$$(\hbar D_a \delta_b^c + C_{ab}^c)\, V_c = 0. \tag{3.5}$$

where D_a is the covariant derivative on the vacuum line bundle and C_{ab}^c are the coefficients of the OPE in the chiral ring of observables $\phi_a \phi_b = C_{ab}^c \phi_c$. The observables $\{\phi_a\}$ provide a basis for the vector space of chiral ring operators $H^0(X) \oplus H^2(X)$ with $a = 0, 1, \ldots, b^2(X)$, ϕ_0 being the identity operator. The parameter $\hbar$ is the spectral parameter of the Gauss-Manin connection. Specifying the case $b = 0$ in (3.5), we find that $V_a = -\hbar D_a V_0$ which means that the flat sections are all generated by the fundamental solution $\mathcal{J} := V_0$ of the equation

$$(\hbar D_a D_b + C_{ab}^c D_c)\, \mathcal{J} = 0 \tag{3.6}$$

In order to uniquely fix the solution to (3.6) one needs to supplement some further information about the dependence on the spectral parameter. This is usually done by combining the dimensional analysis of the theory with the the $\hbar$ dependence by fixing

$$(\hbar \partial_\hbar + \mathcal{E})\, \mathcal{J} = 0 \tag{3.7}$$

where the covariantly constant Euler vector field $\mathcal{E} = \delta^a D_a$, δ^a being the vector of scaling dimensions of the coupling constants, scales with weight one the chiral ring structure constants as $\mathcal{E} C_{ab}^c = C_{ab}^c$ to ensure compatibility between (3.6) and (3.7). The metric on the vacuum bundle is given by a symplectic pairing of the flat sections $g_{\bar{a}b} = \langle \bar{a}|b\rangle = V_{\bar{a}}^t E V_b$ and in particular the vacuum-vacuum amplitude, that is the the spherical partition function, can be written as the symplectic pairing

$$\langle \bar{0}|0\rangle = \mathcal{J}^t E \mathcal{J} \tag{3.8}$$

for a suitable symplectic form E [58] that will be specified later.
In the case of non compact targets, the Quantum Field Theory has to be studied in the equivariant sense to regulate its volume divergences already visible in the constant map contribution. This is accomplished by turning on the relevant twisted masses for matter fields which, from the mathematical viewpoint, amounts to work in the context of equivariant cohomology of the target space $H_T^\bullet(X)$ where T is the torus acting on X; the values of the twisted masses assign the weights of the torus action.

The formalism developed by Givental in [55] for the computation of $\mathcal{J}$ is based on the

study of holomorphic maps from S^2 to X, *equivariant* with respect to the maximal torus of the sphere automorphisms $S^1_\hbar \simeq U(1)_\hbar \subset PSL(2,\mathbb{C})$, with $\hbar$ equivariant parameter.
Let us point out immediately that there is a natural correspondence of the results of supersymmetric localization on the two-sphere with Givental's approach: indeed the computation of Z_{S^2} makes use of a supersymmetric charge which closes on a $U(1)$ isometry of the sphere, whose fixed points are the north and south pole. From the string viewpoint it therefore describes the embedding in the target space of a spherical world-sheet *with two marked points*. As an important consequence, the equivariant parameter $\hbar$ of Givental's S^1 action gets identified with the one of the vortex partition functions arising in the localization of the spherical partition function.
Givental's small $\mathcal{J}$-function is given by the $H^0(X) \oplus H^2(X)$ valued generating function [60]

$$\mathcal{J}_X(t_0,\delta,\hbar) = e^{(t_0+\delta)/\hbar}\left(1 + \sum_{\beta\neq 0}\sum_{a=0}^{b^2(X)} Q^\beta \left\langle \frac{\phi_a}{\hbar-\psi_1}, 1\right\rangle_{X_{0,1,d}} \phi^a\right) \tag{3.9}$$

Here $\delta = \sum_{l=1}^{b^2(X)} t_l\phi_l$ with t_l canonical coordinates on $H^2(X)$, while ψ_1 is the first Chern class of the cotangent bundle at one marked point[3] and the sigma model expectation value localizes on the moduli space $X_{0,1,d}$ of holomorphic maps of degree $\beta \in H_2(X,\mathbb{Z})$ from the sphere with one marked point to the target space X. The world-sheet instanton corrections are labelled by the parameter $Q^\beta = e^{\int_\beta \delta}$.
Givental has shown how to reconstruct the $\mathcal{J}$-function from a set of oscillatory integrals, the so called "$\mathcal{I}$-functions" which are generating functions of hypergeometric type in the variables $\hbar$ and $z_l = e^{-r_l}$. Originally this method has been developed for abelian quotients, more precisely for complete intersections in quasi-projective toric varieties; in this case, the $\mathcal{I}$ function is the generating function of solutions of the Picard-Fuchs equations for the mirror manifold $\check{X}$ of X and as such can be expressed in terms of periods on $\check{X}$, with r_l canonical basis of coordinates in the complex structure moduli space of $\check{X}$. Givental's theorem states that for Fano manifolds the $\mathcal{J}$ and $\mathcal{I}$ functions coincide (modulo prefactors in a class of cases) with the identification $t_l = r_l$; on the other hand, for Calabi-Yau manifolds the two functions coincide only after an appropriate change of coordinates $t_l = t_l(r_m)$ (the *mirror map* we already encountered below (3.2)).

Let us pause a moment to describe how this work practically. For simplicity, let us consider an abelian Calabi-Yau three-fold with a single Kähler modulus t and a corresponding cohomology generator $H \in H^2(X)$. Since for a three-fold $b^0(X) = b^6(X) = 1$ while $b^2(X) = b^4(X)$ ($= 1$ in this example) and higher Betti numbers are zero, the

[3]The $\mathcal{J}$ function is a generating function for Gromov-Witten invariants and *gravitational descendant* invariants of X. Gravitational invariants arise from correlators with ψ_1 insertions. Since for genus zero the gravitational descendants can be recovered from the Gromov-Witten invariants, we will often omit them from our discussion.

cohomology generator H is such that $H^4 = 0$. Therefore the expansion in powers of H of the $\mathcal{J}$ function will be[4] (setting $t_0 = 0$)

$$\mathcal{J} = 1 + \frac{H}{\hbar}t + \frac{H^2}{\hbar^2}J^{(2)}(t) + \frac{H^3}{\hbar^3}J^{(3)}(t) \tag{3.10}$$

In particular $J^{(2)}(t) = \eta^{tt}\partial_t F_0$, where η^{tt} is the inverse topological metric and F_0 is the so-called genus zero Gromov-Witten prepotential. On the other hand the expansion for $\mathcal{I}$ (which is written in terms of a typically different coordinate r) reads

$$\mathcal{I} = I^{(0)}(r) + \frac{H}{\hbar}I^{(1)}(r) + \frac{H^2}{\hbar^2}I^{(2)}(r) + \frac{H^3}{\hbar^3}I^{(3)}(r) \tag{3.11}$$

therefore the functions $\mathcal{I}$ and $\mathcal{J}$ are related by

$$\mathcal{J}(t) = \frac{1}{I_0(r(t))}\mathcal{I}(r(t)) \tag{3.12}$$

where the mirror map change of coordinate is given by

$$t(r) = \frac{I^{(1)}}{I^{(0)}}(r) \tag{3.13}$$

with inverse $r(t)$. In the more general case with $b^2(X) > 1$ we will have $b^2(X)$ components t_l, $J_l^{(2)}$ as well as $I_l^{(1)}$, $I_l^{(2)}$, and the mirror maps are still given by (3.13) component by component. If instead we want to work in equivariant cohomology, returning to the $b^2(X) = 1$ example we should also consider the equivariant cohomology generators, say $\widetilde{H}$, in addition to H. Now the expansions will be

$$\mathcal{J} = 1 + \frac{H}{\hbar}t + \dots \quad , \quad \mathcal{I} = I^{(0)}(r) + \frac{H}{\hbar}I^{(1)}(r) + \frac{\widetilde{H}}{\hbar}\widetilde{I}^{(1)}(r) + \dots \tag{3.14}$$

so the mirror map will still be the same, but we will have in addition an *equivariant mirror map*: this is just a normalization factor $e^{-\widetilde{H}\widetilde{I}^{(1)}(r)/\hbar}$ in front of $\mathcal{I}$ which removes the linear term in $\widetilde{H}$. At the end the relation between $\mathcal{I}$ and $\mathcal{J}$ will be

$$\mathcal{J}(t) = \frac{1}{I_0(r(t))}e^{-\widetilde{H}\widetilde{I}^{(1)}(r(t))/\hbar}\mathcal{I}(r(t)) \tag{3.15}$$

This is the function that generates the equivariant Gromov-Witten invariants.

We are now ready to illustrate the relation between Givental's formalism and the spherical partition function. First of all, as shown in many examples in [48, 54] and reviewed in the following sections, we can factorize the expression (2.32) in a form similar to

[4] Notice that this can also been seen as an expansion in $\frac{1}{\hbar}$.

(2.37) even before performing the integral; schematically, we will have

$$Z_{S^2} = \oint d\lambda\, \widetilde{Z}_{1l} \left(z^{-r|\lambda|} \widetilde{Z}_{\rm v} \right) \left(\bar{z}^{-r|\lambda|} \widetilde{Z}_{\rm av} \right) \tag{3.16}$$

with $d\lambda = \prod_{\alpha=1}^{\rm rank} d\lambda_\alpha$ and $|\lambda| = \sum_\alpha \lambda_\alpha$. Here $z = e^{-2\pi\vec{\xi}-i\vec{\theta}}$ labels the different vortex sectors, $(z\bar{z})^{-r\lambda_r}$ is a contribution from the classical action, $\widetilde{Z}_{1l}$ is a one-loop measure and $\widetilde{Z}_{\rm v}$, $\widetilde{Z}_{\rm av}$ are powers series in z, $\bar{z}$ of hypergeometric type.

Our claim is that $\widetilde{Z}_{\rm v}$ coincides with the $\mathcal{I}$-function of the target space X upon identifying the Fayet-Iliopoulos parameters $\xi_l + i\frac{\theta_l}{2\pi}$ with the r_l coordinates, λ_α with the generators of the cohomology and the S^2 radius r with $1/\hbar$ (twisted masses, if present, will be identified with equivariant generators of the cohomology). According to the choice of the FI parameters (and the subsequent choice of integration contours) the target X may change; the integrand in (3.16) will also change, since we factorize it in such a way that $\widetilde{Z}_{\rm v}$ is a convergent series, and convergence depends on the FI's. In particular, in the geometric phase with all the FIs large and positive, the vortex counting parameters are identified with the exponentiated complex Kähler parameters, while in the orbifold phase they label the twisted sectors of the orbifold itself or, in other words, the basis of orbifold cohomology. This is exactly the content of the *crepant resolution* conjecture: the $\mathcal{I}$ function of an orbifold can be recovered from the one of its resolution via analytic continuation in the r_l parameters. We will see an example of this in the following.

The form (3.16) of the spherical partition function has also a very nice direct interpretation by an alternative rewriting of the vacuum amplitude (3.8). Indeed, by mirror symmetry one can rewrite, in the Calabi-Yau case

$$\langle \bar{0} | 0 \rangle = i \int_{\check{X}} \overline{\Omega} \wedge \Omega = \Pi^t S \Pi \tag{3.17}$$

where $\Pi = \int_{\Gamma^i} \Omega$ is the period vector and S is the symplectic pairing. The components of the $\mathcal{I}$-function can be identified with the components of the period vector Π. More in general one can consider an elaboration of the integral form of the spherical partition function worked out in [49], where the integrand is rewritten in a mirror symmetric manifest form, by expressing the ratios of Γ-functions appearing in the Coulomb branch representation (2.32) as

$$\frac{\Gamma(\Sigma)}{\Gamma(1-\bar{\Sigma})} = \int_{Im(Y)\sim Im(Y)+2\pi} \frac{d^2Y}{2\pi i} e^{\left[e^{-Y} - \Sigma Y - c.c.\right]} \tag{3.18}$$

to obtain the right-hand-side of (3.17) and then by applying the Riemann bilinear identity, one gets the left-hand side. The resulting integrals, after the integration over the Coulomb parameters and independently on the fact that the mirror representation is geometric or not, are then of the oscillatory type

$$\Pi_i = \oint_{\Gamma_i} d\vec{Y} e^{r\mathcal{W}_{eff}(\vec{Y})} \tag{3.19}$$

where the effective variables $\vec{Y}$ and potential $\mathcal{W}_{eff}$ are the remnants parametrizing the constraints imposed by the integration over the Coulomb parameters before getting to (3.19). Eq.(3.19) is also the integral representation of Givental's $\mathcal{I}$-function for general Fano manifolds [57].
Now if we want to compute the equivariant Gromov-Witten invariants of X, we have to go to the $\mathcal{J}$-function, which is obtained from the $\mathcal{I}$-function as described before; this in particular implies that we have to normalize (3.16) by $(I_0(z)\overline{I}_0(\overline{z}))^{-1}$ in order to recover the standard form (3.3) for Calabi-Yau three-folds or its analogue for other manifolds. Actually, we will see that a further normalization might be required for the one-loop term in order to reproduce the classical intersection cohomology on the target manifold. Taking into account all normalizations and expressing everything in terms of the canonical coordinates t_l (i.e. going from $\mathcal{I}$ to $\mathcal{J}$ functions), the spherical partition function coincides with the symplectic pairing (3.8)

$$Z_{S^2}^{\text{norm}}(t_l, \bar{t}_l) = \langle \bar{0} | 0 \rangle = \mathcal{J}^t E \mathcal{J} = e^{-\mathcal{K}_K(t_l, \bar{t}_l)} \tag{3.20}$$

which is the correct version of (3.2), and in particular the one-loop part reproduces in the $r \to 0$ limit the (equivariant) volume of the target space. The above statements will be checked for several abelian and non-abelian GIT quotients in the subsequent sections. In fact, our formalism works for both abelian and non-abelian quotients without any complication, while Givental's formalism have been originally developed only for the abelian cases; it has then been extended to non-abelian cases in [61, 62] and expressed in terms of quasi-maps theory in [56]. The Gromov-Witten invariants for the non-abelian quotient $M//G$ are conjectured to be expressible in terms of the ones of the corresponding abelian quotient $M//T$, T being the maximal torus of G, twisted by the Euler class of a vector bundle over it. The corresponding $\mathcal{I}$-function is obtained from the one associated to the abelian quotients multiplied by a suitable factor depending on the Chern roots of the vector bundle. The first example of this kind was the quantum cohomology of the Grassmanian discussed in [63]. This was rigorously proved and extended to flag manifolds in [61]. As we will see, our results give evidence of the above conjecture in full generality, though a rigorous mathematical proof of this result is not available at the moment.[5]

In the rest of this chapter we are going to summarize the results of [54].

[5] A related issue concerning the equivalence of symplectic quotients and GIT quotients via the analysis of vortex moduli space has been also discussed in [64].

3.2 Abelian GLSMs

3.2.1 Projective spaces

Let us start with the basic example, that is $\mathbb{P}^{n-1}$. Its sigma model matter content consists of n chiral fields of charge 1 with respect to the $U(1)$ gauge group, and the renormalized parameters (2.31) in this case are

$$\xi_{\text{ren}} = \xi - \frac{n}{2\pi}\log(rM) \quad , \quad \theta_{\text{ren}} = \theta \tag{3.21}$$

After defining $\tau = -ir\sigma$, the $\mathbb{P}^{n-1}$ partition function (2.32) reads

$$Z_{\mathbb{P}^{n-1}} = \sum_{m\in\mathbb{Z}} \int \frac{\mathrm{d}\tau}{2\pi i} e^{4\pi\xi_{\text{ren}}\tau - i\theta_{\text{ren}} m} \left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1 - \tau - \frac{m}{2}\right)} \right)^n \tag{3.22}$$

With the change of variables [48]

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.23}$$

we are resumming the whole tower of poles coming from the Gamma functions, centered at $\lambda = 0$. Equation (3.22) then becomes

$$Z_{\mathbb{P}^{n-1}} = \oint \frac{d(rM\lambda)}{2\pi i} Z_{11}^{\mathbb{P}^{n-1}} Z_{\text{v}}^{\mathbb{P}^{n-1}} Z_{\text{av}}^{\mathbb{P}^{n-1}} \tag{3.24}$$

where $z = e^{-2\pi\xi + i\theta}$ and

$$\begin{aligned} Z_{11}^{\mathbb{P}^{n-1}} &= (rM)^{-2nrM\lambda} \left(\frac{\Gamma(rM\lambda)}{\Gamma(1 - rM\lambda)} \right)^n \\ Z_{\text{v}}^{\mathbb{P}^{n-1}} &= z^{-rM\lambda} \sum_{l\geq 0} \frac{[(rM)^n z]^l}{(1 - rM\lambda)_l^n} \\ Z_{\text{av}}^{\mathbb{P}^{n-1}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} \frac{[(-rM)^n \bar{z}]^k}{(1 - rM\lambda)_k^n} \end{aligned} \tag{3.25}$$

The Pochhammer symbol $(a)_k$ used in (3.25) is defined as

$$(a)_k = \begin{cases} \prod_{i=0}^{k-1}(a+i) & \text{for } k > 0 \\ 1 & \text{for } k = 0 \\ \prod_{i=1}^{-k} \dfrac{1}{a-i} & \text{for } k < 0 \end{cases} \tag{3.26}$$

Notice that this definition implies the identity

$$(a)_{-d} = \frac{(-1)^d}{(1-a)_d} \tag{3.27}$$

As observed in [65], $Z_{\text{v}}^{\mathbb{P}^{n-1}}$ coincides with the $\mathcal{I}$-function given in the mathematical literature

$$\mathcal{I}_{\mathbb{P}^{n-1}}(H,\hbar;t) = e^{\frac{tH}{\hbar}} \sum_{d\geq 0} \frac{[(\hbar)^{-n}e^{t}]^{d}}{(1+H/\hbar)_{d}^{n}} \tag{3.28}$$

if we identify $\hbar = \frac{1}{rM}$, $H = -\lambda$, $t = \ln z$. The antivortex contribution is the conjugate $\mathcal{I}$-function, with $\hbar = -\frac{1}{rM}$, $H = \lambda$ and $\bar{t} = \ln \bar{z}$. The hyperplane class H satisfies $H^n = 0$; in some sense the integration variable λ satisfies the same relation, because the process of integration will take into account only terms up to λ^{n-1} in Z_{v} and Z_{av}.

We can also add chiral fields of charge $-q_j < 0$ and R-charge $R_j \geqslant 0$; this means that the integrand in (3.22) gets multiplied by

$$\prod_{j=1}^{m} \frac{\Gamma\left(\frac{R_j}{2} - q_j\tau + q_j\frac{m}{2}\right)}{\Gamma\left(1 - \frac{R_j}{2} + q_j\tau + q_j\frac{m}{2}\right)} \tag{3.29}$$

The poles are still as in (3.23), but now

$$\begin{aligned}
Z_{11}^{\mathbb{P}^{n-1}} &= (rM)^{-2rM(n-|q|)\lambda} \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)}\right)^{n} \prod_{j=1}^{m} \frac{\Gamma\left(\frac{R_j}{2} - q_j rM\lambda\right)}{\Gamma\left(1 - \frac{R_j}{2} + q_j rM\lambda\right)} \\
Z_{\text{v}}^{\mathbb{P}^{n-1}} &= z^{-rM\lambda} \sum_{l\geq 0} (-1)^{|q|l} [(rM)^{n-|q|} z]^{l} \frac{\prod_{j=1}^{m} (\frac{R_j}{2} - q_j rM\lambda)_{q_j l}}{(1-rM\lambda)_{l}^{n}} \\
Z_{\text{av}}^{\mathbb{P}^{n-1}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} (-1)^{|q|k} [(-rM)^{n-|q|} \bar{z}]^{k} \frac{\prod_{j=1}^{m} (\frac{R_j}{2} - q_j rM\lambda)_{q_j k}}{(1-rM\lambda)_{k}^{n}}
\end{aligned} \tag{3.30}$$

where we defined $|q| = \sum_{j=1}^{m} q_j$. A very important set of models one can construct in this way is the one of line bundles $\bigoplus_j \mathcal{O}(-q_j)$ over $\mathbb{P}^{n-1}$ (among which we find the local Calabi-Yau's), which can be obtained by setting $R_j = 0$. In order to give meaning to Gromov-Witten invariants in this case, one typically adds twisted masses in the contributions coming from the fibers; we will do this explicitly shortly. Other important models are complete intersections in $\mathbb{P}^{n-1}$, which correspond to GLSM with a superpotential; since the superpotential breaks all flavour symmetries and has R-charge 2, they do not allow twisted masses, and moreover we will need some $R_j \neq 0$ (see the example of the quintic below).

3.2.1.1 Equivariant projective spaces

The same computation can be repeated in the more general equivariant case: since the $\mathbb{P}^{n-1}$ model admits an $SU(n)$ flavour symmetry, we can turn on twisted masses a_i

satisfying $\sum_{i=1}^n a_i = 0$. In this case, the partition function reads (after rescaling the twisted masses as $a_i \to Ma_i$ in order to have dimensionless parameters)

$$Z^{\text{eq}}_{\mathbb{P}^{n-1}} = \sum_{m\in\mathbb{Z}} \int \frac{\mathrm{d}\tau}{2\pi i} e^{4\pi\xi_{\text{ren}}\tau - i\theta_{\text{ren}}m} \prod_{i=1}^n \frac{\Gamma\left(\tau - \frac{m}{2} + irMa_i\right)}{\Gamma\left(1-\tau-\frac{m}{2} - irMa_i\right)} \tag{3.31}$$

Changing variables as

$$\tau = -k + \frac{m}{2} - irMa_j + rM\lambda \tag{3.32}$$

we arrive at

$$Z^{\text{eq}}_{\mathbb{P}^{n-1}} = \sum_{j=1}^n \oint \frac{d(rM\lambda)}{2\pi i} Z^{\mathbb{P}^{n-1}}_{\text{1l, eq}} Z^{\mathbb{P}^{n-1}}_{\text{v, eq}} Z^{\mathbb{P}^{n-1}}_{\text{av, eq}} \tag{3.33}$$

where

$$\begin{aligned}
Z^{\mathbb{P}^{n-1}}_{\text{1l, eq}} &= (z\bar{z})^{irMa_j} (rM)^{-2nrM\lambda} \prod_{i=1}^n \frac{\Gamma(rM\lambda + irMa_{ij})}{\Gamma(1-rM\lambda - irMa_{ij})} \\
Z^{\mathbb{P}^{n-1}}_{\text{v, eq}} &= z^{-rM\lambda} \sum_{l\geq 0} \frac{[(rM)^n z]^l}{\prod_{i=1}^n (1-rM\lambda - irMa_{ij})_l} \\
Z^{\mathbb{P}^{n-1}}_{\text{av, eq}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} \frac{[(-rM)^n \bar{z}]^k}{\prod_{i=1}^n (1-rM\lambda - irMa_{ij})_k}
\end{aligned} \tag{3.34}$$

and $a_{ij} = a_i - a_j$. Since there are just simple poles, the integration can be easily performed:

$$\begin{aligned}
Z^{\text{eq}}_{\mathbb{P}^{n-1}} = \sum_{j=1}^n (z\bar{z})^{irMa_j} \prod_{i\neq j=1}^n \frac{1}{irMa_{ij}} \frac{\Gamma(1+irMa_{ij})}{\Gamma(1-irMa_{ij})} \\
\sum_{l\geq 0} \frac{[(rM)^n z]^l}{\prod_{i=1}^n (1-irMa_{ij})_l} \sum_{k\geq 0} \frac{[(-rM)^n \bar{z}]^k}{\prod_{i=1}^n (1-irMa_{ij})_k}
\end{aligned} \tag{3.35}$$

In the limit $rM \to 0$ the one-loop contribution (i.e. the first line of (3.35)) provides the equivariant volume of the target space:

$$\text{Vol}(\mathbb{P}^{n-1}_{\text{eq}}) = \sum_{j=1}^n (z\bar{z})^{irMa_j} \prod_{i\neq j=1}^n \frac{1}{irMa_{ij}} = \sum_{j=1}^n e^{-4\pi i\xi rMa_j} \prod_{i\neq j=1}^n \frac{1}{irMa_{ij}} \tag{3.36}$$

The non-equivariant volume can be recovered by sending all the twisted masses to zero at the same time, for example by performing the limit $r \to 0$ in which we can use the identity

$$\lim_{r\to 0} \sum_{j=1}^n \frac{e^{-4\pi i\xi rMa_j}}{(4\xi)^{n-1}} \prod_{i\neq j=1}^n \frac{1}{irMa_{ij}} = \frac{\pi^{n-1}}{(n-1)!} \tag{3.37}$$

to obtain

$$\mathrm{Vol}(\mathbb{P}^{n-1}) = \frac{(4\pi\xi)^{n-1}}{(n-1)!} \tag{3.38}$$

3.2.1.2 Weighted projective spaces

Another important generalization consists in studying the target $\mathbb{P}^{\mathbf{w}} = \mathbb{P}(w_0, \ldots, w_n)$, known as the weighted projective space, which has been considered from the mathematical point of view in [66]. This can be obtained from a $U(1)$ gauge theory with $n+1$ fundamentals of (positive) integer charges $w_0, \ldots, w_n$. The partition function reads

$$Z = \sum_m \int \frac{d\tau}{2\pi i} e^{4\pi\xi_{\mathrm{ren}}\tau - i\theta_{\mathrm{ren}} m} \prod_{i=0}^{n} \frac{\Gamma(w_i\tau - w_i\frac{m}{2})}{\Gamma(1 - w_i\tau - w_i\frac{m}{2})} \tag{3.39}$$

so one would expect $n+1$ towers of poles at

$$\tau = \frac{m}{2} - \frac{k}{w_i} + rM\lambda \ , \ \ i = 0 \ldots n \tag{3.40}$$

with integration around $rM\lambda = 0$. Actually, in this way we might be overcounting some poles if the w_i are not relatively prime, and in any case the pole $k = 0$ is always counted $n+1$ times. In order to solve these problems, we will set

$$\tau = \frac{m}{2} - k + rM\lambda - F \tag{3.41}$$

where F is a set of rational numbers defined as

$$F = \left\{ \frac{d}{w_i} \ / \ 0 \leq d < w_i \, , \ \ d \in \mathbb{N} \, , \ \ 0 \leq i \leq n \right\} \tag{3.42}$$

and counted without multiplicity. Let us explain this better with an example: if we consider just $w_0 = 2$ and $w_1 = 3$, we find the numbers $(0, 1/2)$ and $(0, 1/3, 2/3)$, which means $F = (0, 1/3, 1/2, 2/3)$; the multiplicity of these numbers reflects the order of the pole in the integrand, so we will have a double pole (counted by the double multiplicity of $d = 0$) and three simple poles. From the mathematical point of view, the twisted sectors in (3.42) label the base of the orbifold cohomology space.
The partition function then becomes

$$Z = \sum_F \oint \frac{d(rM\lambda)}{2\pi i} Z_{11} \, Z_{\mathrm{v}} \, Z_{\mathrm{av}} \tag{3.43}$$

with integration around $rM\lambda = 0$ and

$$\begin{aligned}
Z_{1l} &= (rM)^{-2|w|rM\lambda - 2\sum_{i=0}^{n}(\omega[w_iF]-\langle w_iF\rangle)} \prod_{i=0}^{n} \frac{\Gamma(\omega[w_iF] + w_irM\lambda - \langle w_iF\rangle)}{\Gamma(1-\omega[w_iF] - w_irM\lambda + \langle w_iF\rangle)} \\
Z_{\mathrm{v}} &= z^{-rM\lambda} \sum_{l\geq 0} \frac{(rM)^{|w|l+\sum_{i=0}^{n}(\omega[w_iF]+[w_iF])} z^{l+F}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_irM\lambda + \langle w_iF\rangle)_{w_il+[w_iF]+\omega[w_iF]}} \\
Z_{\mathrm{av}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} \frac{(-rM)^{|w|k+\sum_{i=0}^{n}(\omega[w_iF]+[w_iF])} \bar{z}^{k+F}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_irM\lambda + \langle w_iF\rangle)_{w_ik+[w_iF]+\omega[w_iF]}}
\end{aligned} \tag{3.44}$$

In the formulae we defined $\langle w_iF\rangle$ and $[w_iF]$ as the fractional and integer part of the number w_iF, so that $w_iF = [w_iF] + \langle w_iF\rangle$, while $|w| = \sum_{i=0}^{n} w_i$. Moreover,

$$\omega[w_iF] = \begin{cases} 0 & \text{for } \langle w_iF\rangle = 0 \\ 1 & \text{for } \langle w_iF\rangle \neq 0 \end{cases} \tag{3.45}$$

This is needed in order for the $\mathcal{J}$ function to start with one in the rM expansion.

As we did earlier, we can also consider adding fields of charge $-q_j < 0$ and R-charge $R_j \geqslant 0$. The integrand in (3.39) has to be multiplied by

$$\prod_{j=1}^{m} \frac{\Gamma\left(\frac{R_j}{2} - q_j\tau + q_j\frac{m}{2}\right)}{\Gamma\left(1-\frac{R_j}{2} + q_j\tau + q_j\frac{m}{2}\right)} \tag{3.46}$$

The positions the of poles do not change, and

$$\begin{aligned}
Z_{1l} &= (rM)^{-2(|w|-|q|)rM\lambda - 2\sum_{i=0}^{n}(\omega[w_iF]-\langle w_iF\rangle) - 2\sum_{j=1}^{m}\langle q_jF\rangle} \\
&\quad \prod_{i=0}^{n} \frac{\Gamma(\omega[w_iF] + w_irM\lambda - \langle w_iF\rangle)}{\Gamma(1-\omega[w_iF] - w_irM\lambda + \langle w_iF\rangle)} \prod_{j=1}^{m} \frac{\Gamma(\frac{R_j}{2} - q_jrM\lambda + \langle q_jF\rangle)}{\Gamma(1-\frac{R_j}{2} + q_jrM\lambda - \langle q_jF\rangle)} \\
Z_{\mathrm{v}} &= z^{-rM\lambda} \sum_{l\geq 0} (-1)^{|q|l+\sum_{j=1}^{m}[q_jF]} (rM)^{(|w|-|q|)l+\sum_{i=0}^{n}(\omega[w_iF]+[w_iF]) - \sum_{j=1}^{m}[q_jF]} z^{l+F} \\
&\quad \frac{\prod_{j=1}^{m}(\frac{R_j}{2} - q_jrM\lambda + \langle q_jF\rangle)_{q_jl+[q_jF]}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_irM\lambda + \langle w_iF\rangle)_{w_il+[w_iF]+\omega[w_iF]}} \\
Z_{\mathrm{av}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} (-1)^{|q|k+\sum_{j=1}^{m}[q_jF]} (-rM)^{(|w|-|q|)k+\sum_{i=0}^{n}(\omega[w_iF]+[w_iF]) - \sum_{j=1}^{m}[q_jF]} \bar{z}^{k+F} \\
&\quad \frac{\prod_{j=1}^{m}(\frac{R_j}{2} - q_jrM\lambda + \langle q_jF\rangle)_{q_jk+[q_jF]}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_irM\lambda + \langle w_iF\rangle)_{w_ik+[w_iF]+\omega[w_iF]}}
\end{aligned} \tag{3.47}$$

As a final comment let us stress that the Non Linear Sigma Model to which the GLSM flows in the IR is well defined only for $|w| \geq |q|$, which means for manifolds with $c_1 \geq 0$.

3.2.2 Quintic

We will now consider in great detail the most famous compact Calabi-Yau threefold, i.e. the quintic hypersurface in $\mathbb{P}^4$. The corresponding GLSM is a $U(1)$ gauge theory with five chiral fields Φ_a of charge $+1$, one chiral field P of charge -5 and a superpotential of the form $W = PG(\Phi_1, \ldots, \Phi_5)$, where G is a homogeneous polynomial of degree five. We choose the vector R-charges to be $2q$ for the Φ fields and $(2 - 5 \cdot 2q)$ for P such that the superpotential has R-charge 2. The quintic threefold is realized in the geometric phase corresponding to $\xi > 0$. For details of the construction see [67] and for the relation to the two-sphere partition function [48]. Here we want to investigate the connection to the Givental formalism. For a Calabi-Yau manifold the sum of gauge charges is zero, which from (2.31) implies $\xi_{\text{ren}} = \xi$, while $\theta_{\text{ren}} = \theta$ because the gauge group is abelian. The spherical partition function is a specialization of the one computed in the previous section:

$$Z = \sum_{m\in\mathbb{Z}} \int_{i\mathbb{R}} \frac{d\tau}{2\pi i} z^{-\tau-\frac{m}{2}} \bar{z}^{-\tau+\frac{m}{2}} \left(\frac{\Gamma\left(q+\tau-\frac{m}{2}\right)}{\Gamma\left(1-q-\tau-\frac{m}{2}\right)} \right)^5 \frac{\Gamma\left(1-5q-5\tau+5\frac{m}{2}\right)}{\Gamma\left(5q+5\tau+5\frac{m}{2}\right)}. \tag{3.48}$$

Since we want to describe the phase $\xi > 0$, we have to close the contour in the left half plane. We use the freedom in q to separate the towers of poles coming from the Φ's and from P. In the range $0 < q < \frac{1}{5}$ the former lie in the left half plane while the latter in the right half plane. So we only pick the poles corresponding to the Φ's, given by

$$\tau_k = -q - k + \frac{m}{2}, \qquad k \geq \max(0, m) \tag{3.49}$$

Then the partition function turns into a sum of residues and we express each residue by the Cauchy contour integral. Finally we arrive at

$$Z = (z\bar{z})^q \oint_{\mathcal{C}(\delta)} \frac{d(rM\lambda)}{2\pi i} Z_{1l}(\lambda, rM) Z_{\text{v}}(\lambda, rM; z) Z_{\text{av}}(\lambda, rM; \bar{z}), \tag{3.50}$$

where the contour $\mathcal{C}(\delta)$ goes around $\lambda = 0$ and

$$\begin{aligned} Z_{1l}(\lambda, rM) &= \frac{\Gamma(1-5rM\lambda)}{\Gamma(5rM\lambda)} \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)} \right)^5 \\ Z_{\text{v}}(\lambda, rM; z) &= z^{-rM\lambda} \sum_{l\geqslant 0} (-z)^l \frac{(1-5rM\lambda)_{5l}}{[(1-rM\lambda)_l]^5} \\ Z_{\text{av}}(\lambda, rM; \bar{z}) &= \bar{z}^{-rM\lambda} \sum_{k\geqslant 0} (-\bar{z})^k \frac{(1-5rM\lambda)_{5k}}{[(1-rM\lambda)_k]^5} \end{aligned} \tag{3.51}$$

The vortex function $Z_{\mathrm{v}}(\lambda, rM; z)$ reproduces the known Givental $\mathcal{I}$-function

$$\mathcal{I}(H, \hbar; t) = \sum_{d \geqslant 0} e^{(H/\hbar + d)t} \frac{(1 + 5H/\hbar)_{5d}}{[(1 + H/\hbar)_d]^5} \tag{3.52}$$

after identifying

$$H = -\lambda \quad , \quad \hbar = \frac{1}{rM} \quad , \quad t = \ln(-z). \tag{3.53}$$

The $\mathcal{I}$-function is valued in cohomology, where $H \in H^2(\mathbb{P}^4)$ is the hyperplane class in the cohomology ring of the embedding space. Because of dimensional reasons we have $H^5 = 0$ and hence the $\mathcal{I}$-function is a polynomial of order four in H

$$\mathcal{I} = I_0 + \frac{H}{\hbar} I_1 + \left(\frac{H}{\hbar}\right)^2 I_2 + \left(\frac{H}{\hbar}\right)^3 I_3 + \left(\frac{H}{\hbar}\right)^4 I_4. \tag{3.54}$$

This is naturally encoded in the explicit residue evaluation of (3.50), see eq.(3.58). Now consider the Picard-Fuchs operator L given by

$$\left(z\frac{d}{dz}\right)^4 - 5^5 \left(z\frac{d}{dz} + \frac{1}{5}\right)\left(z\frac{d}{dz} + \frac{2}{5}\right)\left(z\frac{d}{dz} + \frac{3}{5}\right)\left(z\frac{d}{dz} + \frac{4}{5}\right) \tag{3.55}$$

It can be easily shown that $\{I_0, I_1, I_2, I_3\} \in \mathrm{Ker}(L)$ while $I_4 \notin \mathrm{Ker}(L)$. L is an order four operator and so $\mathbf{I} = (I_0, I_1, I_2, I_3)^T$ form a basis of solutions. There exists another basis formed by the periods of the holomorphic $(3,0)$ form of the mirror manifold. In homogeneous coordinates they are given as $\mathbf{\Pi} = (X^0, X^1, \frac{\partial F}{\partial X^1}, \frac{\partial F}{\partial X^0})^T$ with F the prepotential. Thus there exists a transition matrix $\mathbf{M}$ relating these two bases

$$\mathbf{I} = \mathbf{M} \cdot \mathbf{\Pi} \tag{3.56}$$

There are now two possible ways to proceed. One would be fixing the transition matrix using mirror construction (i.e. knowing explicitly the periods) and then showing that the pairing given by the contour integral in (3.50) after being transformed to the period basis gives the standard formula for the Kähler potential in terms of a symplectic pairing

$$e^{-K} = i\mathbf{\Pi}^\dagger \cdot \mathbf{\Sigma} \cdot \mathbf{\Pi} \tag{3.57}$$

with $\mathbf{\Sigma} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$ being the symplectic form. The other possibility would be to use the fact that the two sphere partition function computes the Kähler potential [48] and then impose equality between (3.50) and (3.57) to fix the transition matrix. We follow this route in the following. The contour integral in (3.50) expresses the Kähler potential as a pairing in the **I** basis. It is governed by Z_{11} which has an expansion

$$Z_{11} = \frac{5}{(rM\lambda)^4} + \frac{400\,\zeta(3)}{rM\lambda} + o(1) \tag{3.58}$$

and so we get after integration (remember that $H/\hbar = -rM\lambda$)

$$\begin{aligned} Z &= -2\chi\zeta(3)I_0\bar{I}_0 - 5(I_0\bar{I}_3 + I_1\bar{I}_2 + I_2\bar{I}_1 + I_3\bar{I}_0) \\ &= \mathbf{I}^\dagger \cdot \mathbf{A} \cdot \mathbf{I}, \end{aligned} \tag{3.59}$$

where

$$\mathbf{A} = \begin{pmatrix} -2\chi\zeta(3) & 0 & 0 & -5 \\ 0 & 0 & -5 & 0 \\ 0 & -5 & 0 & 0 \\ -5 & 0 & 0 & 0 \end{pmatrix} \tag{3.60}$$

gives the pairing in the **I** basis and $\chi = -200$ is the Euler characteristic of the quintic threefold. From the two expressions for the Kähler potential we easily find the transition matrix as

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{5} \\ -\frac{\chi}{5}\zeta(3) & 0 & -\frac{i}{5} & 0 \end{pmatrix}. \tag{3.61}$$

Finally, we know that the mirror map is given by

$$t = \frac{I_1}{2\pi i I_0} \quad , \quad \bar{t} = -\frac{\bar{I}_1}{2\pi i \bar{I}_0} \tag{3.62}$$

so after dividing Z by $(2\pi i)^2 I_0\bar{I}_0$ for the change of coordinates and by a further 2π for the normalization of the $\zeta(3)$ term, we obtain the Kähler potential in terms of t, $\bar{t}$, in a form in which the symplectic product is evident.

3.2.3 Local Calabi–Yau: $\mathcal{O}(p) \oplus \mathcal{O}(-2-p) \to \mathbb{P}^1$

In this section we will study a non-compact (i.e. local) class of Calabi-Yau manifolds: the family of spaces $X_p = \mathcal{O}(p) \oplus \mathcal{O}(-2-p) \to \mathbb{P}^1$ with diagonal equivariant action on the fiber. We will find exact agreement with the $\mathcal{I}$ functions computed in [68], and we will show how the quantum corrected Kähler potential for the Kähler moduli space can be computed when equivariant parameters are turned on.

Here we will restrict only to the phase $\xi > 0$, which is the one related to X_p. The case $\xi < 0$ describes the orbifold phase of the model; this will be studied in the following sections.

3.2.3.1 Case $p = -1$

First of all, we have to write down the partition function; this is given by

$$Z_{-1} = \sum_{m\in\mathbb{Z}} e^{-im\theta} \int \frac{d\tau}{2\pi i} e^{4\pi\xi\tau} \left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1 - \tau - \frac{m}{2}\right)} \right)^2 \left(\frac{\Gamma\left(-\tau - irMa + \frac{m}{2}\right)}{\Gamma\left(1 + \tau + irMa + \frac{m}{2}\right)} \right)^2 \tag{3.63}$$

The poles are located at

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.64}$$

so we can rewrite (3.63) as

$$Z_{-1} = \oint \frac{d(rM\lambda)}{2\pi i} Z_{1l} Z_{\mathrm{v}} Z_{\mathrm{av}} \tag{3.65}$$

where

$$\begin{aligned} Z_{1l} &= \left(\frac{\Gamma(rM\lambda)}{\Gamma(1 - rM\lambda)} \frac{\Gamma(-rM\lambda - irMa)}{\Gamma(1 + rM\lambda + irMa)} \right)^2 \\ Z_{\mathrm{v}} &= z^{-rM\lambda} \sum_{l\geqslant 0} z^l \frac{(-rM\lambda - irMa)_l^2}{(1 - rM\lambda)_l^2} \\ Z_{\mathrm{av}} &= \bar{z}^{-rM\lambda} \sum_{k\geqslant 0} \bar{z}^k \frac{(-rM\lambda - irMa)_k^2}{(1 - rM\lambda)_k^2} \end{aligned} \tag{3.66}$$

Notice that our vortex partition function coincides with the Givental function given in [68]

$$\mathcal{I}^T_{-1}(q) = e^{\frac{H}{\hbar}\ln q} \sum_{d\geqslant 0} \frac{(1 - H/\hbar + \tilde{\lambda}/\hbar - d)_d^2}{(1 + H/\hbar)_d^2} q^d \tag{3.67}$$

after the usual identifications

$$H = -\lambda \quad , \quad \hbar = \frac{1}{rM} \quad , \quad \tilde{\lambda} = ia \quad , \quad q = z \tag{3.68}$$

Now, expanding $\mathcal{I}^T_{-1}$ in $rM = 1/\hbar$ we find

$$\mathcal{I}^T_{-1} = 1 - rM\lambda \log z + o((rM)^2) \tag{3.69}$$

which means the mirror map is trivial and the equivariant mirror map absent, i.e. $\mathcal{I}^T_{-1} = \mathcal{J}^T_{-1}$. What remains to be specified is the normalization of the 1-loop factor. This problem is related to the renormalization scheme used to define the infinite products in the 1-loop determinant in the computation of the spherical partition function. In [27, 28] the ζ-function renormalization scheme is chosen. Indeed this is a reference one, while others can be obtained by a shift in the finite part of the resulting effective action. These determinants appear in the form of ratios of Gamma-functions. The ambiguity amounts to shift the Euler-Mascheroni constant γ appearing in the Weierstrass form of the Gamma-function

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}} \tag{3.70}$$

with a finite function of the parameters. Due to supersymmetry, this function has to be encoded in terms of a holomorphic function $f(z)$, namely $\gamma \to \mathrm{Re} f(z)$. We will fix this normalization by requiring the cancellation of the Euler-Mascheroni constants; moreover we require the normalization to reproduce the correct intersection numbers in classical cohomology, and to start from 1 in the rM expansion in order not to modify the regularized equivariant volume of the target. In our case, the factor

$$(z\bar{z})^{-irMa/2} \left(\frac{\Gamma(1 + irMa)}{\Gamma(1 - irMa)} \right)^2 \tag{3.71}$$

does the job; in general, the normalization factor will be deduced through a case by case analysis. We can now integrate in $rM\lambda$ and expand in rM, obtaining (for $rMa = iq$)

$$\begin{aligned} Z_{-1} = \frac{2}{q^3} - \frac{1}{4q} \ln^2(z\bar{z}) + \Big[- \frac{1}{12} \ln^3(z\bar{z}) - \ln(z\bar{z})(\mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z})) \\ + 2(\mathrm{Li}_3(z) + \mathrm{Li}_3(\bar{z})) + 4\zeta(3) \Big] + o(rM) \end{aligned} \tag{3.72}$$

The terms inside the square brackets reproduce the Kähler potential we are interested in, once we multiply everything by $\frac{1}{2\pi(2\pi i)^2}$ and change variables according to

$$t = \frac{1}{2\pi i} \ln z \quad , \quad \bar{t} = -\frac{1}{2\pi i} \ln \bar{z}. \tag{3.73}$$

3.2.3.2 Case $p = 0$

In this case case the spherical partition function is

$$Z_0 = \sum_{m\in\mathbb{Z}} e^{-im\theta} \int \frac{d\tau}{2\pi i} e^{4\pi\xi\tau} \left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1 - \tau - \frac{m}{2}\right)} \right)^2 \frac{\Gamma\left(-irMa\right)}{\Gamma\left(1 + irMa\right)} \frac{\Gamma\left(-2\tau - irMa + 2\frac{m}{2}\right)}{\Gamma\left(1 + 2\tau + irMa + 2\frac{m}{2}\right)} \tag{3.74}$$

The poles are as in (3.64), and usual manipulations result in

$$\begin{aligned} Z_{1l} &= \left(\frac{\Gamma(rM\lambda)}{\Gamma(1 - rM\lambda)} \right)^2 \frac{\Gamma\left(-irMa\right)}{\Gamma\left(1 + irMa\right)} \frac{\Gamma(-2rM\lambda - irMa)}{\Gamma(1 + 2rM\lambda + irMa)} \\ Z_{\mathrm{v}} &= z^{-rM\lambda} \sum_{l\geqslant 0} z^l \frac{(-2rM\lambda - irMa)_{2l}}{(1 - rM\lambda)_l^2} \\ Z_{\mathrm{av}} &= \bar{z}^{-rM\lambda} \sum_{k\geqslant 0} \bar{z}^k \frac{(-2rM\lambda - irMa)_{2k}}{(1 - rM\lambda)_k^2} \end{aligned} \tag{3.75}$$

Again, we recover the Givental function

$$\mathcal{I}_0^T(q) = e^{\frac{H}{\hbar}\ln q} \sum_{d\geqslant 0} \frac{(1-2H/\hbar+\tilde{\lambda}/\hbar-2d)_{2d}}{(1+H/\hbar)_d^2} q^d \tag{3.76}$$

of [68] under the map (3.68); its expansion in rM

$$\mathcal{I}_0^T = 1 - rM\lambda\left[\log z + 2\sum_{k=1}^{\infty} z^k \frac{\Gamma(2k)}{(k!)^2}\right] - irMa\sum_{k=1}^{\infty} z^k \frac{\Gamma(2k)}{(k!)^2} + o((rM)^2) \tag{3.77}$$

implies that the mirror map is (modulo $(2\pi i)^{-1}$)

$$t = \log z + 2\sum_{k=1}^{\infty} z^k \frac{\Gamma(2k)}{(k!)^2} \tag{3.78}$$

and the equivariant mirror map is

$$\tilde{t} = \frac{1}{2}(t - \log z) = \sum_{k=1}^{\infty} z^k \frac{\Gamma(2k)}{(k!)^2} \tag{3.79}$$

The $\mathcal{J}$ function can be recovered by inverting the equivariant mirror map and changing coordinates accordingly, that is

$$\mathcal{J}_0^T(t) = e^{irMa\tilde{t}(z)}\mathcal{I}_0^T(z) = e^{irMa\tilde{t}(z)} Z_{\text{v}}(z) \tag{3.80}$$

A similar job has to be done for Z_{av}. The normalization for the 1-loop factor is the same as (3.71) but in t coordinates, which means

$$(t\bar{t})^{-irMa/2}\left(\frac{\Gamma(1+irMa)}{\Gamma(1-irMa)}\right)^2; \tag{3.81}$$

Finally, integrating in $rM\lambda$ and expanding in rM we find

$$\begin{aligned} Z_0 = \frac{2}{q^3} - \frac{1}{4q}(t+\bar{t})^2 + \Big[&-\frac{1}{12}(t+\bar{t})^3 - (t+\bar{t})(\text{Li}_2(e^t) + \text{Li}_2(e^{\bar{t}})) \\ &+ 2(\text{Li}_3(e^t) + \text{Li}_3(e^{\bar{t}})) + 4\zeta(3)\Big] + o(rM) \end{aligned} \tag{3.82}$$

As it was shown in [68], this proves that the two Givental functions $\mathcal{J}_{-1}^T$ and $\mathcal{J}_0^T$ are the same, as well as the Kähler potentials; the $\mathcal{I}$ functions look different simply because of the choice of coordinates on the moduli space.

3.2.3.3 Case $p \geq 1$

In the general $p \geq 1$ case we have

$$Z_p = \sum_{m\in\mathbb{Z}} e^{-im\theta} \int \frac{d\tau}{2\pi i} e^{4\pi\xi\tau} \left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1-\tau-\frac{m}{2}\right)} \right)^2 \frac{\Gamma\left(-(p+2)\tau - irMa + (p+2)\frac{m}{2}\right)}{\Gamma\left(1+(p+2)\tau + irMa + (p+2)\frac{m}{2}\right)} \frac{\Gamma\left(p\tau - irMa - p\frac{m}{2}\right)}{\Gamma\left(1-p\tau + irMa - p\frac{m}{2}\right)} \tag{3.83}$$

There are two classes of poles, given by

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.84}$$

$$\tau = -k + \frac{m}{2} + rM\lambda - F + irM\frac{a}{p} \tag{3.85}$$

where $F = \{0, \frac{1}{p}, \ldots, \frac{p-1}{p}\}$ and the integration is around $rM\lambda = 0$. The relevant one for describing the geometry $\mathcal{O}(p) \oplus \mathcal{O}(-2-p) \to \mathbb{P}^1$ is the first one, in which λ can be seen as the cohomology class of the $\mathbb{P}^1$ base and satisfies $\lambda^2 = 0$. In this case

$$Z_p^{(0)} = \oint \frac{d(rM\lambda)}{2\pi i} Z_{11}^{(0)} Z_{\text{v}}^{(0)} Z_{\text{av}}^{(0)} \tag{3.86}$$

with

$$\begin{aligned}
Z_{11}^{(0)} &= \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)}\right)^2 \frac{\Gamma(-(p+2)rM\lambda - irMa)}{\Gamma(1+(p+2)rM\lambda + irMa)} \frac{\Gamma(p\,rM\lambda - irMa)}{\Gamma(1-p\,rM\lambda + irMa)} \\
Z_{\text{v}}^{(0)} &= z^{-rM\lambda} \sum_{l\geqslant 0} (-1)^{(p+2)l} z^l \frac{(-(p+2)rM\lambda - irMa)_{(p+2)l}}{(1-rM\lambda)_l^2 (1-p\,rM\lambda + irMa)_{pl}} \\
Z_{\text{av}}^{(0)} &= \bar{z}^{-rM\lambda} \sum_{k\geqslant 0} (-1)^{(p+2)k} \bar{z}^k \frac{(-(p+2)rM\lambda - irMa)_{(p+2)k}}{(1-rM\lambda)_k^2 (1-p\,rM\lambda + irMa)_{pk}}
\end{aligned} \tag{3.87}$$

Extracting the correct $\mathcal{J}_p^T$ function from the $\mathcal{I}_p^T$ (i.e. form $Z_{\text{v}}^{(0)}$) is quite non-trivial and requires additional techniques such as *Birkhoff factorization*, introduced in [57, 69]. In [68, 70] it is explained how these techniques lead to the correct equivariant Gromov-Witten invariants and Givental functions $\mathcal{J}_p^T$ for $p \geqslant 1$, which coincide with $\mathcal{J}_{-1}^T$ and $\mathcal{J}_0^T$; we refer to these papers for further details.

3.2.4 Orbifold Gromov-Witten invariants

In this section we want to show how the analytic structure of the partition function encodes all the classical phases of an abelian GLSM whose target has $c_1 = 0$ (i.e. a

Calabi-Yau when in the geometric phase). These are given by the secondary fan, which in our conventions is generated by the columns of the charge matrix Q. In terms of the partition function these phases are governed by the choice of integration contours, namely by the structure of poles we are picking up. For example, for a GLSM with $G = U(1)$ the contour can be closed either in the left half plane (for $\xi > 0$) or in the right half plane ($\xi < 0$)[6]. The transition between different phases occurs when some of the integration contours are flipped and the corresponding variables are integrated over. To summarize, a single partition function contains the $\mathcal{I}$-functions of geometries corresponding to all the different phases of the GLSM. These geometries are related by minimally resolving the singularities by blow-up until the complete smoothing of the space takes place (when this is possible). Our procedure consists in considering the GLSM corresponding to the complete resolution and its partition function. Then by flipping contours and doing partial integrations one discovers all other, more singular geometries. In the following we illustrate these ideas on a couple of examples.

3.2.4.1 $K_{\mathbb{P}^{n-1}}$ vs. $\mathbb{C}^n/\mathbb{Z}_n$

Let us consider a $U(1)$ gauge theory with n chiral fields of charge $+1$ and one chiral field of charge $-n$. The secondary fan is generated by two vectors $\{1, -n\}$ and so it has two chambers corresponding to two different phases. For $\xi > 0$ it describes a smooth geometry $K_{\mathbb{P}^{n-1}}$, that is the total space of the canonical bundle over the complex projective space $\mathbb{P}^{n-1}$, while for $\xi < 0$ it describes the orbifold $\mathbb{C}^n/\mathbb{Z}_n$. The case $n = 3$ will reproduce the results of [71, 72, 73]. The partition function reads

$$Z = \sum_m \int_{i\mathbb{R}} \frac{d\tau}{2\pi i} e^{4\pi\xi\tau - i\theta m} \left(\frac{\Gamma(\tau - \frac{m}{2})}{\Gamma(1 - \tau - \frac{m}{2})} \right)^n \frac{\Gamma(-n\tau + n\frac{m}{2} + irMa)}{\Gamma(1 + n\tau + n\frac{m}{2} - irMa)} \tag{3.88}$$

Closing the contour in the left half plane (i.e. for $\xi > 0$) we take poles at

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.89}$$

and obtain

$$\begin{aligned} Z = \oint \frac{d(rM\lambda)}{2\pi i} & \left(\frac{\Gamma(rM\lambda)}{\Gamma(1 - rM\lambda)} \right)^n \frac{\Gamma(-nrM\lambda + irMa)}{\Gamma(1 + nrM\lambda - irMa)} \\ & \sum_{l \geq 0} z^{-rM\lambda} (-1)^{nl} z^{nl} \frac{(-nrM\lambda + irMa)_{nl}}{(1 - rM\lambda)_l^n} \\ & \sum_{k \geq 0} \bar{z}^{-rM\lambda} (-1)^{nk} \bar{z}^{nk} \frac{(-nrM\lambda + irMa)_{nk}}{(1 - rM\lambda)_k^n} \end{aligned} \tag{3.90}$$

[6] This is only true for Calabi-Yau manifolds; for $c_1 > 0$, i.e. $\sum_i Q_i > 0$, the contour is fixed.

We thus find exactly the Givental function for $K_{\mathbb{P}^{n-1}}$. To switch to the singular geometry we flip the contour and do the integration. Closing in the right half plane ($\xi < 0$) we consider

$$\tau = k + \frac{\delta}{n} + \frac{m}{2} + \frac{1}{n} irMa \tag{3.91}$$

with $\delta = 0, 1, 2, \ldots, n-1$. After integration over τ we obtain

$$\begin{aligned} Z = \frac{1}{n} \sum_{\delta=0}^{n-1} & \left(\frac{\Gamma(\frac{\delta}{n} + \frac{1}{n} irMa)}{\Gamma(1 - \frac{\delta}{n} - \frac{1}{n} irMa)} \right)^n \frac{1}{(rM)^{2\delta}} \\ & \sum_{k \geq 0} (-1)^{nk} (\bar{z}^{-1/n})^{nk+\delta+irMa} (rM)^{\delta} \frac{(\frac{\delta}{n} + \frac{1}{n} irMa)_k^n}{(nk+\delta)!} \\ & \sum_{l \geq 0} (-1)^{nl} (z^{-1/n})^{nl+\delta+irMa} (-rM)^{\delta} \frac{(\frac{\delta}{n} + \frac{1}{n} irMa)_l^n}{(nl+\delta)!} \end{aligned} \tag{3.92}$$

as expected from (3.47). Notice that when the contour is closed in the right half plane, vortex and antivortex contributions are exchanged. We can compare the $n = 3$ case corresponding to $\mathbb{C}^3/\mathbb{Z}_3$ with the $\mathcal{I}$-function given in [73]

$$\mathcal{I} = x^{-\lambda/z} \sum_{\substack{d \in \mathbb{N} \\ d \geq 0}} \frac{x^d}{d! z^d} \prod_{\substack{0 \leq b < \frac{d}{3} \\ \langle b \rangle = \langle \frac{d}{3} \rangle}} \left(\frac{\lambda}{3} - bz \right)^3 \mathbf{1}_{\langle \frac{d}{3} \rangle} \tag{3.93}$$

which in a more familiar notation becomes

$$\mathcal{I} = x^{-\lambda/z} \sum_{\substack{d \in \mathbb{N} \\ d \geq 0}} \frac{x^d}{d!} \frac{1}{z^{3\langle \frac{d}{3} \rangle}} (-1)^{3[\frac{d}{3}]} \left(\langle \frac{d}{3} \rangle - \frac{\lambda}{3z} \right)_{[\frac{d}{3}]}^3 \mathbf{1}_{\langle \frac{d}{3} \rangle} \tag{3.94}$$

The necessary identifications are straightforward.

3.2.4.2 Quantum cohomology of $\mathbb{C}^3/\mathbb{Z}_{p+2}$ and crepant resolution

We now consider the orbifold space $\mathbb{C}^3/\mathbb{Z}_{p+2}$ with weights $(1, 1, p)$ and $p > 1$. Its full crepant resolution is provided by a resolved transversal A_{p+1} singularity (namely a local Calabi-Yau threefold obtained by fibering the resolved A_{p+1} singularity over a $\mathbb{P}^1$ base space). The corresponding GLSM contains $p + 2$ abelian gauge groups and $p + 5$ chiral multiplets, with the following charge assignment:

$$\begin{pmatrix} 0 & 1 & 1 & -1 & -1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ -j-1 & j & 0 & 0 & 0 & 0 & \ldots & 0 & \overset{(5+j)\text{th}}{1} & 0 & \ldots & 0 \\ -p-2 & p+1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{3.95}$$

where $1 \leq j \leq p$. In the following we focus on the particular chambers corresponding to the partial resolutions $K_{\mathbb{F}_p}$ and $K_{\mathbb{P}^2(1,1,p)}$. Let us start by discussing the local $\mathbb{F}_p$ chamber: this can be seen by replacing the last row in (3.95) with the linear combination

$$(\text{last row}) \quad \longrightarrow \quad (\text{last row}) - p\,(\text{second row}) - (\text{first row}) \tag{3.96}$$

which corresponds to

$$\begin{pmatrix} -p-2 & p+1 & 1 & 0 & 0 & 0 & \dots \end{pmatrix} \longrightarrow \begin{pmatrix} p-2 & 0 & 0 & 1 & 1 & -p & \dots \end{pmatrix} \tag{3.97}$$

The charge matrix (3.95) now reads $(2 \leq n \leq p)$

$$\begin{pmatrix} 0 & 1 & 1 & -1 & -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ -n-1 & n & 0 & 0 & 0 & 0 & \dots & 0 & \overset{(5+n)\text{th}}{1} & 0 & \dots & 0 \\ p-2 & 0 & 0 & 1 & 1 & -p & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{3.98}$$

and, in a particular sector (i.e. for a particular choice of poles), after turning to infinity p Fayet-Iliopoulos parameters, we remain with the second and the last row:

$$Q = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ p-2 & 0 & 1 & 1 & -p \end{pmatrix} \tag{3.99}$$

which is the charge matrix of $K_{\mathbb{F}_p}$.

Let us see how this happens in detail; since it is easier for our purposes, we will consider the charge matrix (3.98). For generic p, the partition function with the addition of a twisted mass a for the field corresponding to the first column of (3.98) is given by

$$\begin{aligned} Z = &\sum_{m_0,\dots,m_{p+1}} \oint \prod_{i=0}^{p+1} \frac{d\tau_i}{2\pi i} z_i^{-\tau_i - \frac{m_i}{2}} \bar{z}_i^{-\tau_i + \frac{m_i}{2}} \prod_{j=0}^{p} \frac{\Gamma(\tau_j - \frac{m_j}{2})}{\Gamma(1-\tau_j - \frac{m_j}{2})} \\ &\frac{\Gamma(\tau_1 - p\tau_{p+1} - \frac{m_1}{2} + p\frac{m_{p+1}}{2})}{\Gamma(1-\tau_1 + p\tau_{p+1} - \frac{m_1}{2} + p\frac{m_{p+1}}{2})} \left(\frac{\Gamma(-\tau_0 + \tau_{p+1} + \frac{m_0}{2} - \frac{m_{p+1}}{2})}{\Gamma(1+\tau_0 - \tau_{p+1} + \frac{m_0}{2} - \frac{m_{p+1}}{2})} \right)^2 \\ &\frac{\Gamma(\tau_0 + \sum_{j=1}^p j\tau_j - \frac{m_0}{2} - \sum_{j=1}^p j\frac{m_j}{2})}{\Gamma(1-\tau_0 - \sum_{j=1}^p j\tau_j - \frac{m_0}{2} - \sum_{j=1}^p j\frac{m_j}{2})} \\ &\frac{\Gamma(-\sum_{j=1}^p (j+1)\tau_j + (p-2)\tau_{p+1} + \sum_{j=1}^p (j+1)\frac{m_j}{2} - (p-2)\frac{m_{p+1}}{2} + irMa)}{\Gamma(1+\sum_{j=1}^p (j+1)\tau_j - (p-2)\tau_{p+1} + \sum_{j=1}^p (j+1)\frac{m_j}{2} - (p-2)\frac{m_{p+1}}{2} - irMa)} \end{aligned} \tag{3.100}$$

Now, choosing the sector

$$\begin{aligned} \tau_0 &= -k_0 + \frac{m_0}{2} \\ \tau_n &= -k_n + \frac{m_n}{2} \quad , \quad 2 \leq n \leq p \end{aligned} \tag{3.101}$$

and integrating over these variables we arrive at

$$\begin{aligned} Z = & \sum_{k_0,k_n\geq 0} \sum_{l_0,l_n\geq 0} \frac{z_0^{l_0}}{l_0!}\frac{(-1)^{k_0}\bar{z}_i^{k_0}}{k_0!} \prod_{n=2}^{p} \frac{z_i^{l_i}}{l_i!}\frac{(-1)^{k_i}\bar{z}_i^{k_i}}{k_i!} \\ & \sum_{m_1,m_{p+1}} \oint \frac{d\tau_1}{2\pi i}\frac{d\tau_{p+1}}{2\pi i} e^{4\pi\xi_1\tau_1 - i\theta_1 m_1} e^{4\pi\xi_{p+1}\tau_{p+1} - i\theta_{p+1}m_{p+1}} \\ & \frac{\Gamma(\tau_1 - p\tau_{p+1} - \frac{m_1}{2} + p\frac{m_{p+1}}{2})}{\Gamma(1-\tau_1 - \frac{m_1}{2} + p\tau_{p+1} + p\frac{m_{p+1}}{2})} \left(\frac{\Gamma(k_0 + \tau_{p+1} - \frac{m_{p+1}}{2})}{\Gamma(1-l_0-\tau_{p+1}-\frac{m_{p+1}}{2})}\right)^2 \\ & \frac{\Gamma(-k_0 + \tau_1 - \sum_{n=2}^{p} nk_n - \frac{m_1}{2})}{\Gamma(1 + l_0 - \tau_1 + \sum_{n=2}^{p} nl_n - \frac{m_1}{2})} \\ & \frac{\Gamma(-2\tau_1 + \sum_{n=2}^{p}(n+1)k_n + (p-2)\tau_{p+1} + 2\frac{m_1}{2} - (p-2)\frac{m_{p+1}}{2} + irMa)}{\Gamma(1 + 2\tau_2 - \sum_{n=2}^{p}(n+1)l_n - (p-2)\tau_{p+1} + 2\frac{m_2}{2} - (p-2)\frac{m_{p+1}}{2} - irMa)} \end{aligned} \tag{3.102}$$

which defines a linear sigma model with charges (3.99) for $k_0 = k_n = 0$, $l_0 = l_n = 0$ (i.e. when $\xi_0 = \xi_n = \infty$).

The secondary fan of this model has four chambers, but here we concentrate only on three of them, describing $K_{\mathbb{F}_p}$, $K_{\mathbb{P}^2(1,1,p)}$ and $\mathbb{C}^3/\mathbb{Z}_{p+2}$ respectively. Its partition function is given by

$$\begin{aligned} Z = & \sum_{m_1,m_{p+1}} \int \frac{d\tau_1}{2\pi i}\frac{d\tau_{p+1}}{2\pi i} e^{4\pi\xi_1\tau_1 - i\theta_1 m_1} e^{4\pi\xi_{p+1}\tau_{p+1} - i\theta_{p+1}m_{p+1}} \\ & \left(\frac{\Gamma(\tau_{p+1} - \frac{m_{p+1}}{2})}{\Gamma(1-\tau_{p+1}-\frac{m_{p+1}}{2})}\right)^2 \frac{\Gamma(\tau_1 - \frac{m_1}{2})}{\Gamma(1-\tau_1-\frac{m_1}{2})} \frac{\Gamma(-p\tau_{p+1} + \tau_1 + p\frac{m_{p+1}}{2} - \frac{m_1}{2})}{\Gamma(1 + p\tau_{p+1} - \tau_1 + p\frac{m_{p+1}}{2} - \frac{m_1}{2})} \\ & \frac{\Gamma((p-2)\tau_{p+1} - 2\tau_1 - (p-2)\frac{m_{p+1}}{2} + 2\frac{m_1}{2} + irMa)}{\Gamma(1-(p-2)\tau_{p+1} + 2\tau_1 - (p-2)\frac{m_{p+1}}{2} + 2\frac{m_1}{2} - irMa)} \end{aligned} \tag{3.103}$$

If we consider the set of poles

$$\begin{aligned} \tau_{p+1} &= -k_{p+1} + \frac{m_{p+1}}{2} + rM\lambda_{p+1} \\ \tau_1 &= -k_1 + \frac{m_1}{2} + rM\lambda_1 \end{aligned} \tag{3.104}$$

we are describing the canonical bundle over $\mathbb{F}_p$:

$$\begin{aligned} Z_{K_{\mathbb{F}_p}} = & \oint \frac{d(rM\lambda_1)}{2\pi i}\frac{d(rM\lambda_{p+1})}{2\pi i}\left(\frac{\Gamma(rM\lambda_{p+1})}{\Gamma(1-rM\lambda_{p+1})}\right)^2\frac{\Gamma(rM\lambda_1)}{\Gamma(1-rM\lambda_1)} \\ & \frac{\Gamma(-prM\lambda_{p+1}+rM\lambda_1)}{\Gamma(1+prM\lambda_{p+1}-rM\lambda_1)}\frac{\Gamma((p-2)rM\lambda_{p+1}-2rM\lambda_1+irMa)}{\Gamma(1-(p-2)rM\lambda_{p+1}+2rM\lambda_1-irMa)} \\ & \sum_{l_1,l_{p+1}}(-1)^{(p-2)l_{p+1}}z_{p+1}^{l_{p+1}-rM\lambda_{p+1}}z_1^{l_1-rM\lambda_1} \\ & \frac{((p-2)rM\lambda_{p+1}-2rM\lambda_1+irMa)_{2l_1-(p-2)l_{p+1}}}{(1-rM\lambda_{p+1})^2_{l_{p+1}}(1-rM\lambda_1)_{l_1}(1+prM\lambda_{p+1}-rM\lambda_1)_{l_1-pl_{p+1}}} \\ & \sum_{k_1,k_{p+1}}(-1)^{(p-2)k_{p+1}}\bar{z}_{p+1}^{k_{p+1}-rM\lambda_{p+1}}\bar{z}_1^{k_1-rM\lambda_1} \\ & \frac{((p-2)rM\lambda_{p+1}-2rM\lambda_1+irMa)_{2k_1-(p-2)k_{p+1}}}{(1-rM\lambda_{p+1})^2_{k_{p+1}}(1-rM\lambda_1)_{k_1}(1+prM\lambda_{p+1}-rM\lambda_1)_{k_1-pk_{p+1}}} \end{aligned} \tag{3.105}$$

On the other hand, taking poles for

$$\tau_1 = p\tau_{p+1} - p\frac{m_{p+1}}{2} + \frac{m_1}{2} - k_1 \tag{3.106}$$

and integrating over τ_1 we obtain the canonical bundle over $\mathbb{P}^2_{(1,1,p)}$:

$$\begin{aligned} Z_{K_{\mathbb{P}^2_{(1,1,p)}}} = & \sum_{k_1,l_1\geq 0}\frac{z_1^{l_1}}{l_1!}\frac{(-1)^{k_1}\bar{z}_1^{k_1}}{k_1!} \\ & \sum_{m_{p+1}}\int\frac{d\tau_{p+1}}{2\pi i}e^{4\pi(\xi_{p+1}+p\xi_1)\tau_{p+1}-i(\theta_{p+1}+p\theta_1)m_{p+1}}\left(\frac{\Gamma(\tau_{p+1}-\frac{m_{p+1}}{2})}{\Gamma(1-\tau_{p+1}-\frac{m_{p+1}}{2})}\right)^2 \\ & \frac{\Gamma(p\tau_{p+1}-p\frac{m_{p+1}}{2}-k_1)}{\Gamma(1-p\tau_{p+1}-p\frac{m_{p+1}}{2}+l_1)}\frac{\Gamma(-(p+2)\tau_{p+1}+(p+2)\frac{m_{p+1}}{2}+irMa+2k_1)}{\Gamma(1+(p+2)\tau_{p+1}+(p+2)\frac{m_{p+1}}{2}-irMa-2l_1)} \end{aligned} \tag{3.107}$$

with $l_1 = k_1 - m_1 + pm_{p+1}$ and $z_1 = e^{-2\pi\xi_1+i\theta_1}$. In fact, in the limit $\xi_1 \to \infty$ with $\xi_{p+1}+p\xi_1$ finite, only the $k_1 = l_1 = 0$ sector contributes, leaving the linear sigma model of $K_{\mathbb{CP}^2_{(1,1,p)}}$ for $\xi_{p+1}+p\xi_1 > 0$.

From the point of view of the charge matrix, the choice (3.106) corresponds to take linear combinations of the rows, in particular

$$\begin{pmatrix} p-2 & 0 & 1 & 1 & -p \end{pmatrix} \longrightarrow \begin{pmatrix} p-2 & 0 & 1 & 1 & -p \end{pmatrix} + p\begin{pmatrix} -2 & 1 & 0 & 0 & 1 \end{pmatrix} \tag{3.108}$$

which implies $\xi_{p+1} \to \xi_{p+1}+p\xi_1$, $\theta_{p+1} \to \theta_{p+1}+p\theta_1$ and

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ p-2 & 0 & 1 & 1 & -p \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ -p-2 & p & 1 & 1 & 0 \end{pmatrix} \tag{3.109}$$

while the process of integrating in τ_1 is equivalent to the elimination of the second row (notice that we have a simple pole, in this case, i.e. the column $(1 \;\; 0)^T$ appears with multiplicity 1).
The case $p = 2$ appears in [73, 74] and corresponds to a full crepant resolution. So, by one blow down we arrived at $K_{\mathbb{P}^2(1,1,p)}$ whose charge matrix is given by

$$Q = \begin{pmatrix} 1 & 1 & p & -p-2 \end{pmatrix} \tag{3.110}$$

The associated two sphere partition function is correspondingly

$$\begin{aligned} Z = \sum_{m\in\mathbb{Z}} \int \frac{d\tau}{2\pi i} e^{4\pi\xi\tau - i\theta m} \left(\frac{\Gamma(\tau - \frac{m}{2})}{\Gamma(1-\tau-\frac{m}{2})} \right)^2 \frac{\Gamma(p\tau - p\frac{m}{2})}{\Gamma(1-p\tau-p\frac{m}{2})} \\ \frac{\Gamma(-(p+2)\tau + (p+2)\frac{m}{2} + irMa)}{\Gamma(1+(p+2)\tau+(p+2)\frac{m}{2} - irMa)} \end{aligned} \tag{3.111}$$

It has two phases, $K_{\mathbb{P}^2(1,1,p)}$ and a more singular $\mathbb{C}^3/\mathbb{Z}_{p+2}$. The first phase corresponds to closing the integration contour in the left half plane of this effective model; since the result is rather ugly, we will simply state that it can be obtained from (3.47), with the necessary modifications (i.e. twisted masses). For $p = 2$ it matches the formula presented in [73].

The second phase describing $\mathbb{C}^3/\mathbb{Z}_{p+2}$ can be obtained by flipping the contour to the right half plane and doing the integration in the single variable. Finally, we arrive at

$$\begin{aligned} Z = \; & \frac{1}{p+2} \sum_{\delta=0}^{p+1} \left(\frac{\Gamma(\frac{\delta}{p+2} + \frac{1}{p+2} irMa)}{\Gamma(1 - \frac{\delta}{p+2} - \frac{1}{p+2} irMa)} \right)^2 \frac{\Gamma(\langle \frac{p\delta}{p+2} \rangle + \frac{p}{p+2} irMa)}{\Gamma(1 - \langle \frac{p\delta}{p+2} \rangle - \frac{p}{p+2} irMa)} \frac{1}{(rM)^{2\left(\delta - \left[\frac{p\delta}{p+2}\right]\right)}} \\ & \sum_{k\geq 0} (-1)^{(p+2)k} (\bar{z}^{-\frac{1}{p+2}})^{(p+2)k+\delta+irMa} (rM)^{\delta - \left[\frac{p\delta}{p+2}\right]} \\ & \frac{(\frac{\delta}{p+2} + \frac{1}{p+2} irMa)_k^2 (\langle \frac{p\delta}{p+2} \rangle + \frac{p}{p+2} irMa)_{pk + \left[\frac{p\delta}{p+2}\right]}}{((p+2)k+\delta)!} \\ & \sum_{l\geq 0} (-1)^{(p+2)l} (z^{-\frac{1}{p+2}})^{(p+2)l+\delta+irMa} (-rM)^{\delta - \left[\frac{p\delta}{p+2}\right]} \\ & \frac{(\frac{\delta}{p+2} + \frac{1}{p+2} irMa)_l^2 (\langle \frac{p\delta}{p+2} \rangle + \frac{p}{p+2} irMa)_{pl + \left[\frac{p\delta}{p+2}\right]}}{((p+2)l+\delta)!} \end{aligned} \tag{3.112}$$

The $\mathcal{I}$-function of the orbifold case in the δ-sector of the orbifold cohomology is then obtained from the second and third lines of the above formula and for $p = 2$ it matches with [73].

3.3 Non-abelian GLSM

In this section we apply our methods to non-abelian gauged linear sigma models and give new results for some non-abelian GIT quotients. These are also tested against results in the mathematical literature when available.

The first case that we analyse are complex Grassmannians. On the way we also give an alternative proof for the conjecture by Hori and Vafa [63] which can be rephrased stating that the $\mathcal{I}$-function of the Grassmannian can be obtained from the one corresponding to a product of projective spaces, after acting with an appropriate differential operator.

One can also study a more general theory corresponding to holomorphic vector bundles over Grassmannians. These spaces arise in the context of the study of BPS Wilson loop algebra in three dimensional supersymmetric gauge theories. In particular we will discuss the mathematical counterpart of a duality proposed in [75] which extends the standard Grassmannian duality to holomorphic vector bundles over them.

We also study flag manifolds and more general non-abelian quiver gauge theories for which we provide the rules to compute the spherical partition function and the $\mathcal{I}$-function.

3.3.1 Grassmannians

The sigma model for the complex Grassmannian $Gr(N, N_f)$ contains N_f chirals in the fundamental representation of the $U(N)$ gauge group. Its partition function is given by

$$Z_{Gr(N,N_f)} = \frac{1}{N!} \sum_{m_1,\ldots,m_N} \int \prod_{i=1}^{N} \frac{\mathrm{d}\tau_i}{2\pi i} e^{4\pi\xi_{\mathrm{ren}}\tau_i - i\theta_{\mathrm{ren}} m_i} \prod_{i<j}^{N} \left(\frac{m_{ij}^2}{4} - \tau_{ij}^2 \right) \prod_{i=1}^{N} \left(\frac{\Gamma\left(\tau_i - \frac{m_i}{2}\right)}{\Gamma\left(1 - \tau_i - \frac{m_i}{2}\right)} \right)^{N_f} \tag{3.113}$$

As usual, we can write it as

$$\frac{1}{N!} \oint \prod_{i=1}^{N} \frac{d(rM\lambda_i)}{2\pi i} Z_{1l} Z_{\mathrm{v}} Z_{\mathrm{av}} \tag{3.114}$$

where

$$Z_{1l} = \prod_{i=1}^{N} (rM)^{-2nrM\lambda_i} \left(\frac{\Gamma(rM\lambda_i)}{\Gamma(1 - rM\lambda_i)} \right)^{N_f} \prod_{i<j}^{N} (rM\lambda_i - rM\lambda_j)(-rM\lambda_i + rM\lambda_j)$$

$$Z_{\mathrm{v}} = z^{-rM|\lambda|} \sum_{l_1,\ldots,l_N} \frac{[(rM)^{N_f}(-1)^{N-1}z]^{l_1+\ldots+l_N}}{(1 - rM\lambda_1)_{l_1}^{N_f} \ldots (1 - rM\lambda_N)_{l_N}^{N_f}} \prod_{i<j}^{N} \frac{l_i - l_j - rM\lambda_i + rM\lambda_j}{-rM\lambda_i + rM\lambda_j}$$

$$Z_{\text{av}} = \bar{z}^{-rM|\lambda|} \sum_{k_1,\ldots,k_N} \frac{[(-rM)^{N_f}(-1)^{N-1}\bar{z}]^{k_1+\ldots+k_N}}{(1-rM\lambda_1)_{k_1}^{N_f}\ldots(1-rM\lambda_N)_{k_N}^{N_f}} \prod_{i<j}^{N} \frac{k_i-k_j-rM\lambda_i+rM\lambda_j}{-rM\lambda_i+rM\lambda_j}. \tag{3.115}$$

We normalized the vortex and antivortex terms in order to have them starting from one in the rM series expansion and we defined $|\lambda| = \lambda_1 + \ldots + \lambda_N$. The resulting $\mathcal{I}$-function Z_{v} coincides with the one given in [61]

$$\mathcal{I}_{Gr(N,N_f)} = e^{\frac{t\sigma_1}{\hbar}} \sum_{(d_1,\ldots,d_N)} \frac{\hbar^{-N_f(d_1+\ldots+d_N)}[(-1)^{N-1}e^t]^{d_1+\ldots+d_N}}{\prod_{i=1}^N (1+x_i/\hbar)_{d_i}^{N_f}} \prod_{i<j}^{N} \frac{d_i-d_j+x_i/\hbar-x_j/\hbar}{x_i/\hbar-x_j/\hbar} \tag{3.116}$$

if we match the parameters as we did in the previous cases. Here the λ's are interpreted as Chern roots of the tautological bundle.

3.3.1.1 The Hori-Vafa conjecture

Hori and Vafa conjectured [63] that $\mathcal{I}_{Gr(N,N_f)}$ can be obtained by $\mathcal{I}_{\mathbb{P}}$, where $\mathbb{P} = \prod_{i=1}^N \mathbb{P}_{(i)}^{N_f-1}$, by acting with a differential operator. This has been proved in [61]; here we remark that in our formalism this is a simple consequence of the fact that the partition function of non-abelian vortices can be obtained from copies of the abelian ones upon acting with a suitable differential operator [76]. In fact we note that $Z_{Gr(N,N_f)}$ can be obtained from $Z_{\mathbb{P}}$ simply by dividing by $N!$ and identifying

$$\begin{aligned} Z_{11}^{Gr} &= \prod_{i<j}^{N} (rM\lambda_i - rM\lambda_j)(-rM\lambda_i + rM\lambda_j) Z_{11}^{\mathbb{P}} \\ Z_{\text{v}}^{Gr}(z) &= \prod_{i<j}^{N} \frac{\partial_{z_i} - \partial_{z_j}}{-rM\lambda_i + rM\lambda_j} Z_{\text{v}}^{\mathbb{P}}(z_1,\ldots,z_N)\Big|_{z_i=(-1)^{N-1}z} \\ Z_{\text{av}}^{Gr}(\bar{z}) &= \prod_{i<j}^{N} \frac{\partial_{\bar{z}_i} - \partial_{\bar{z}_j}}{-rM\lambda_i + rM\lambda_j} Z_{\text{av}}^{\mathbb{P}}(\bar{z}_1,\ldots,\bar{z}_N)\Big|_{\bar{z}_i=(-1)^{N-1}\bar{z}}. \end{aligned} \tag{3.117}$$

3.3.2 Holomorphic vector bundles over Grassmannians

The $U(N)$ gauge theory with N_f fundamentals and N_a antifundamentals flows in the infra-red to a non-linear sigma model with target space given by a holomorphic vector bundle of rank N_a over the Grassmannian $Gr(N, N_f)$. We adopt the notation $Gr(N, N_f|N_a)$ for this space.

One can prove the equality of the $Gr(N, N_f|N_a)$ and $Gr(N_f - N, N_f|N_a)$ partition functions after a precise duality map in a certain range of parameters, as we will do shortly. At the level of $\mathcal{I}$-functions this proves the isomorphism among the relevant quantum cohomology rings conjectured in [75]. In analysing this duality we follow the approach of [27], where also the main steps of the proof were outlined. However we will detail their calculations and note some differences in the explicit duality map, which we refine in order to get a precise equality of the partition functions.

The $Gr\,(N, N_f|N_a)$ theory

The partition function of the $Gr\,(N, N_f|N_a)$ GLSM is

$$Z = \frac{1}{N!} \sum_{\{m_s \in \mathbb{Z}\}_{s=1}^N} \int_{(i\mathbb{R})^N} \prod_{s=1}^N \frac{d\tau_s}{2\pi i} z_{\text{ren}}^{-\tau_s - \frac{m_s}{2}} \bar{z}_{\text{ren}}^{-\tau_s + \frac{m_s}{2}} \prod_{s<t}^N \left(\frac{m_{st}^2}{4} - \tau_{st}^2 \right)$$
$$\prod_{s=1}^N \prod_{i=1}^{N_f} \frac{\Gamma\left(\tau_s - i\frac{a_i}{\hbar} - \frac{m_s}{2}\right)}{\Gamma\left(1 - \tau_s + i\frac{a_i}{\hbar} - \frac{m_s}{2}\right)} \prod_{s=1}^N \prod_{j=1}^{N_a} \frac{\Gamma\left(-\tau_s + i\frac{\widetilde{a}_j}{\hbar} + \frac{m_s}{2}\right)}{\Gamma\left(1 + \tau_s - i\frac{\widetilde{a}_j}{\hbar} + \frac{m_s}{2}\right)}, \tag{3.118}$$

where as usual $\hbar$ relates to the radius of the sphere and the renormalization scale M as $\hbar = \frac{1}{rM}$ and $a_j, \widetilde{a}_j$ are the dimensionless (rescaled by M^{-1}) equivariant weights for fundamentals and antifundamentals respectively. The renormalized Kahler coordinate z_{ren} is defined as

$$z_{\text{ren}} = e^{-2\pi\xi_{\text{ren}} + i\theta_{\text{ren}}} = \hbar^{N_a - N_f} (-1)^{N-1} z. \tag{3.119}$$

since we have

$$\xi_{\text{ren}} = \xi - \frac{1}{2\pi}(N_f - N_a)\log(rM) \;\;,\;\; \theta_{\text{ren}} = \theta + (N-1)\pi \tag{3.120}$$

From now on we will set $M = 1$. We close the contours in the left half planes, so that we pick only poles coming from the fundamentals. We need to build an N-pole to saturate the integration measure. Hence the partition function becomes a sum over all possible choices of N-poles, i.e. over all combinations how to pick N objects out of N_f. Now the proposal is that duality holds separately for a fixed choice of an N-pole and its corresponding dual. For simplicity of notation let us prove the duality for a particular choice of an N-pole and its $(N_f - N)$-dual

$$(\underbrace{\square, \ldots, \square}_{N}, \underbrace{\bullet, \ldots, \bullet}_{N_f - N}) \overset{\text{dual}}{\longleftrightarrow} (\underbrace{\bullet, \ldots, \bullet}_{N}, \underbrace{\square, \ldots, \square}_{N_f - N}), \tag{3.121}$$

where boxes denote the choice of poles forming the N-pole.

The poles for the $Gr\,(N, N_f|N_a)$ theory are at positions

$$\tau_s = -k_s + \frac{m_s}{2} + \frac{\lambda_s}{\hbar} \tag{3.122}$$

and we still have to integrate over λ's around $\lambda_s = ia_s$, where s runs from 1 to N. This fully specifies from which fundamental we took the pole. Plugging this into (3.118), the integral reduces to the following form

$$Z = \oint_{\mathcal{M}} \Big\{ \prod_{s=1}^{N} \frac{d\lambda_s}{2\pi i\hbar} \Big\} Z_{11}\left(\frac{\lambda_s}{\hbar}, \frac{a_i}{\hbar}, \frac{\widetilde{a}_j}{\hbar}\right) z^{-\sum_{s=1}^{N}\frac{\lambda_s}{\hbar}} \widetilde{I}\left((-1)^{N_a}\kappa z, \frac{\lambda_s}{\hbar}, \frac{a_i}{\hbar}, \frac{\widetilde{a}_j}{\hbar}\right) \times \bar{z}^{-\sum_{s=1}^{N}\frac{\lambda_s}{\hbar}} \widetilde{I}\left((-1)^{N_a}\bar{\kappa}\bar{z}, \frac{\lambda_s}{\hbar}, \frac{a_i}{\hbar}, \frac{\widetilde{a}_j}{\hbar}\right), \tag{3.123}$$

where we defined $\kappa = \hbar^{N_a - N_f}(-1)^{N-1}$, $\bar{\kappa} = (-\hbar)^{N_a-N_f}(-1)^{N-1}$. Here we are integrating over a product of circles $\mathcal{M} = \bigotimes_{r=1}^{k} S^1(ia_r, \delta)$ with δ small enough such that only the pole at the center of the circle is included. From this form we can read off the I function for $Gr\,(N, N_f|N_a)$ as

$$I = z^{-\sum_{s=1}^{N}\frac{\lambda_s}{\hbar}} \sum_{\{l_s\geq 0\}_{s=1}^{N}} \left((-1)^{N_a}\kappa z\right)^{\sum_{s=1}^{N} l_s} \prod_{s<t}^{N} \frac{\lambda_{st} - \hbar l_{st}}{\lambda_{st}} \prod_{s=1}^{N} \frac{\prod_{j=1}^{N_a}\left(\frac{-\lambda_s + i\widetilde{a}_j}{\hbar}\right)_{l_s}}{\prod_{i=1}^{N_f}\left(1 + \frac{-\lambda_s + ia_i}{\hbar}\right)_{l_s}}, \tag{3.124}$$

where $x_{st} := x_s - x_t$. Now we integrate over λ's in (3.123), which is straightforward since Z_{11} contains only simple poles and the rest is holomorphic in λ's. Finally, we get

$$Z^{(\square,\ldots,\square,\bullet,\ldots,\bullet)} = Z_{\text{class}} Z_{11} Z_{\text{v}} Z_{\text{av}}, \tag{3.125}$$

where the individual pieces are given as follows

$$Z_{\text{class}} = \prod_{s=1}^{N} \left(\hbar^{2(N_a - N_f)} z\bar{z}\right)^{-\frac{ia_s}{\hbar}} \tag{3.126}$$

$$Z_{11} = \prod_{s=1}^{N} \prod_{i=N+1}^{N_f} \frac{\Gamma\left(\frac{ia_{si}}{\hbar}\right)}{\Gamma\left(1 - \frac{ia_{si}}{\hbar}\right)} \prod_{s=1}^{N}\prod_{j=1}^{N_a} \frac{\Gamma\left(-\frac{i(a_s - \widetilde{a}_j)}{\hbar}\right)}{\Gamma\left(1 + \frac{i(a_s-\widetilde{a}_j)}{\hbar}\right)} \tag{3.127}$$

$$Z_{\text{v}} = \sum_{\{l_s\geq 0\}_{s=1}^{N}} \left((-1)^{N_a}\kappa z\right)^{\sum_{s=1}^{N} l_s} \prod_{s<t}^{N}\left(1 - \frac{\hbar l_{st}}{ia_{st}}\right) \prod_{s=1}^{N} \frac{\prod_{j=1}^{N_a}\left(-i\frac{a_s - \widetilde{a}_j}{\hbar}\right)_{l_s}}{\prod_{i=1}^{N_f}\left(1 - i\frac{a_{si}}{\hbar}\right)_{l_s}} \tag{3.128}$$

$$Z_{\text{av}} = Z_{\text{v}}\left[\kappa z \to \bar{\kappa}\bar{z}\right] \tag{3.129}$$

To prove the duality it is actually better to manipulate Z_{v} to a more convenient form (combining the contributions of the vectors and fundamentals by using identities between the Pochhammers)

$$Z_{\text{v}} = \sum_{l=0}^{\infty} \left[(-1)^{N_a + N - N_f}\,\kappa z\right]^{l} Z_l \tag{3.130}$$

with Z_l given by

$$Z_l = \sum_{\{l_s \geq 0 | \sum_{s=1}^N l_s = l\}} \prod_{s=1}^{N} \frac{\prod_{j=1}^{N_a} \left(-i\frac{a_s - \widetilde{a}_j}{\hbar}\right)_{l_s}}{l_s! \prod_{i \neq s}^{N} \left(i\frac{a_{si}}{\hbar} - l_s\right)_{l_i} \prod_{i=N+1}^{N_f} \left(i\frac{a_{si}}{\hbar} - l_s\right)_{l_s}}. \tag{3.131}$$

The dual theory $Gr\,(N_f - N, N_f|N_a)$

Going to the dual theory not only the rank of the gauge group changes to $N_f - N$, but there is a new feature arising. New matter fields $M^i_{\bar{j}}$ appear: they are singlets under the gauge group and couple to the fundamentals and antifundamentals via a superpotential $W^D = \widetilde{\phi}^{\mu\bar{j}} M^i_{\bar{j}} \phi_{\mu i}$. So the partition function (we set $N^D = N_f - N$)

$$Z = \frac{1}{N^D!} \sum_{\{m_s \in \mathbb{Z}\}_{s=1}^{N^D}} \int_{(i\mathbb{R})^{N^D}} \prod_{s=1}^{N^D} \frac{d\tau_s}{2\pi i} (z^D_{ren})^{-\tau_s - \frac{m_s}{2}} (\bar{z}^D_{ren})^{-\tau_s + \frac{m_s}{2}} \prod_{s<t}^{N^D} \left(\frac{m^2_{st}}{4} - \tau^2_{st}\right)$$
$$\prod_{s=1}^{N^D} \prod_{i=1}^{N_f} \frac{\Gamma\left(\tau_s + i\frac{a^D_i}{\hbar} - \frac{m_s}{2}\right)}{\Gamma\left(1 - \tau_s - i\frac{a^D_i}{\hbar} - \frac{m_s}{2}\right)} \prod_{s=1}^{N^D} \prod_{j=1}^{N_a} \frac{\Gamma\left(-\tau_s - i\frac{\widetilde{a}^D_j}{\hbar} + \frac{m_s}{2}\right)}{\Gamma\left(1 + \tau_s + i\frac{\widetilde{a}^D_j}{\hbar} + \frac{m_s}{2}\right)} \prod_{i=1}^{N_f} \prod_{j=1}^{N_a} \frac{\Gamma\left(-i\frac{a_i - \widetilde{a}_j}{\hbar}\right)}{\Gamma\left(1 + i\frac{a_i - \widetilde{a}_j}{\hbar}\right)} \tag{3.132}$$

gets a new contribution from the mesons M, given by the last factor in (3.132) (note that it depends on the original equivariant weights, not on the dual ones). All the computations are analogue to the previous case, so we give the result right after integration

$$Z^{(\bullet,\ldots,\bullet,\square,\ldots,\square)} = Z^D_{\text{class}} Z^D_{11} Z^D_{\text{v}} Z^D_{\text{av}}, \tag{3.133}$$

where the building blocks are

$$Z^D_{\text{class}} = \prod_{s=N+1}^{N_f} \left(\hbar^{2(N_a - N_f)} z^D \bar{z}^D\right)^{-\frac{ia^D_s}{\hbar}} \tag{3.134}$$

$$Z^D_{11} = \prod_{s=N+1}^{N_f} \prod_{i=N+1}^{N_f} \frac{\Gamma\left(\frac{ia^D_{si}}{\hbar}\right)}{\Gamma\left(1 - \frac{ia^D_{si}}{\hbar}\right)} \prod_{j=1}^{N_a} \frac{\Gamma\left(-\frac{i(a^D_s - \widetilde{a}^D_j)}{\hbar}\right)}{\Gamma\left(1 + \frac{i(a^D_s - \widetilde{a}^D_j)}{\hbar}\right)} \prod_{i=1}^{N_f} \prod_{j=1}^{N_a} \frac{\Gamma\left(-i\frac{a_i - \widetilde{a}_j}{\hbar}\right)}{\Gamma\left(1 + i\frac{a_i - \widetilde{a}_j}{\hbar}\right)} \tag{3.135}$$

$$Z^D_{\text{v}} = \sum_{l=0}^{\infty} \left[(-1)^{N_a - N} (\kappa z)^D\right]^l Z^D_l \tag{3.136}$$

$$Z^D_{\text{av}} = \sum_{k=0}^{\infty} \left[(-1)^{N_a - N} (\bar{\kappa}\bar{z})^D\right]^k Z^D_k \tag{3.137}$$

with Z_l^D given by

$$Z_l^D = \sum_{\{l_s \geq 0 | \sum_{s=N+1}^{N_f} l_s = l\}} \prod_{s=N+1}^{N_f} \frac{\prod_{j=1}^{N_a}\left(-i\frac{a_s^D - \widetilde{a}_j^D}{\hbar}\right)_{l_s}}{l_s! \prod_{\substack{i=N+1 \\ i \neq s}}^{N_f}\left(i\frac{a_{si}^D}{\hbar} - l_s\right)_{l_i} \prod_{i=1}^{N}\left(i\frac{a_{si}^D}{\hbar} - l_s\right)_{l_s}}. \tag{3.138}$$

Duality map

We are now ready to discuss the duality between the two theories. The statement is the following. For $N_f \geq N_a + 2$, there exists a duality map $z^D = z^D(z)$ and $a_j^D = a_j^D(a_j)$, $\widetilde{a}_j^D = \widetilde{a}_j^D(\widetilde{a}_j)$ under which the partition functions for $Gr\,(N, N_f|N_a)$ and $Gr\,(N_f - N, N_f|N_a)$ are equal.[7] In the first step we will construct the duality map and then we will show that (3.126–3.131) indeed match with (3.134–3.138). The partition function is a double power series in z and $\bar{z}$ multiplied by Z_{class}. In order to achieve equality of the partition functions, Z_{class} have to be equal after duality map and then the power series have to match term by term. Moreover we can just look at the holomorphic piece Z_{v}, since for the antiholomorphic one everything goes in a similar way. The constant term is Z_{11}, which is a product of gamma functions with arguments linear in the equivariant weights. This implies that the duality map for the equivariant weights is linear. But then the map between the Kahler coordinates can be only a rescaling since a constant term would destroy the matching of Z_{11}. So we arrive at the most general ansatz for the duality map

$$z^D = sz \tag{3.139}$$

$$\frac{a_i^D}{\hbar} = -E\frac{a_i}{\hbar} + C \tag{3.140}$$

$$\frac{\widetilde{a}_j^D}{\hbar} = -F\frac{\widetilde{a}_j}{\hbar} + D \tag{3.141}$$

Matching the constant terms Z_{11} gives the constraints

$$E = F = 1,\ D = -(C + i). \tag{3.142}$$

Imposing further the equivalence of Z_{class} fixes C to be

$$C = \frac{1}{N_f - N}\sum_{i=1}^{N_f}\frac{a_i}{\hbar}. \tag{3.143}$$

which is zero for an $SU(N_f)$ flavour group. We are now at a position where Z_{class} and Z_{11} match, while the only remaining free parameter in the duality map is s. We fix it by

[7] We will see the reason for this range later.

looking at the linear terms in Z_{v} and Z_{v}^D. Of course this does not assure that all higher order terms do match, but we will show that this is the case for $N_f \geq N_a + 2$.So taking only $k = 1$ contributions in Z_{v} and Z_{v}^D we get for s

$$s = (-1)^{N-1}\frac{\mathcal{N}}{\mathcal{D}}, \tag{3.144}$$

where

$$\mathcal{N} = \sum_{s=1}^{N} \frac{\prod_{j=1}^{N_a}\left(-i\frac{a_s-\widetilde{a}_j}{\hbar}\right)}{\prod_{i\neq s}^{N}\left(-i\frac{a_{si}}{\hbar}\right)\prod_{i=N+1}^{N_f}\left(1-i\frac{a_{si}}{\hbar}\right)} \tag{3.145}$$

$$\mathcal{D} = \sum_{s=N+1}^{N_f} \frac{\prod_{j=1}^{N_a}\left(1+i\frac{a_s-\widetilde{a}_j}{\hbar}\right)}{\prod_{i=1}^{N}\left(1+i\frac{a_{si}}{\hbar}\right)\prod_{\substack{i=N+1\\ j\neq s}}^{N_f}\left(-i\frac{a_{si}}{\hbar}\right)}. \tag{3.146}$$

The proposal is that for $N_f \geq N_a + 2$

$$s = (-1)^{N_a}. \tag{3.147}$$

Out of this range s is a complicated rational function in the equivariant parameters. This completes the duality map for $N_f \geq N_a + 2$. In the cases $N_f = N_a$ and $N_f = N_a + 1$ the two partition functions do not match, but differ by a prefactor which depends on the Fayet-Iliopoulos parameter; we refer to [77, 78] for more details.

By construction of the mirror map we know that Z_{class}, Z_{1l} and moreover also the linear terms in Z_{v} match. Now we will prove (d.m. is the shortcut for duality map)

$$Z_{\mathrm{v}} = Z_{\mathrm{v}}^D|_{d.m.} \tag{3.148}$$

for $N_f \geq N_a + 2$. Looking at (3.130) and (3.136) we see that this boils down to

$$Z_l = (-1)^{N_a l} Z_l^D|_{d.m.}. \tag{3.149}$$

The key to prove the above relation is to write Z_l as a contour integral

$$Z_l = \int_{\mathcal{C}_u} \prod_{\alpha=1}^{l} \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right)\Big|_{\epsilon=1}, \tag{3.150}$$

where $\mathcal{C}_u$ is a product of contours having the real axes as base and then are closed in the upper half plane by a semicircle. The integrand has the form

$$f = \frac{1}{\epsilon^l l!}\prod_{\alpha<\beta}^{l}\frac{(\phi_\alpha-\phi_\beta)^2}{(\phi_\alpha-\phi_\beta)^2-\epsilon^2}\prod_{\alpha=1}^{l}\frac{\prod_{j=1}^{N_a}\left(i\frac{\widetilde{a}_j}{\hbar}+\phi_\alpha\right)}{\prod_{i=1}^{N}\left(\phi_\alpha+i\frac{a_i}{\hbar}\right)\prod_{i=N+1}^{N_f}\left(-i\frac{a_i}{\hbar}-\epsilon-\phi_\alpha\right)}. \tag{3.151}$$

It is necessary to add small imaginary parts to ϵ and a_i, $\epsilon \to \epsilon + i\delta$, $-ia_i \to -ia_i + i\hbar\delta'$ with $\delta > \delta'$. The proof of (3.150) goes by direct evaluation. First we have to classify the poles. Due to the imaginary parts assignments, they are at[1]

$$\phi_\alpha = -i\frac{a_i}{\hbar}, \qquad \alpha = 1, \dots, l, \quad i = 1, \dots, N \tag{3.152}$$

$$\phi_\beta = \phi_\alpha + \epsilon, \qquad \beta \geq \alpha \tag{3.153}$$

We have to build an l-pole, which means that the poles are classified by partitions of l into N parts, $l = \sum_{I=1}^N l_I$. The I-th Young tableau $YT(l_I)$ with l_I boxes can only be 1-dimensional (we choose a row) since we have only one ϵ to play with. To illustrate what we have in mind, we show an example of a possible partition:

$$(\underbrace{\square\square\square}_{l_1}, \bullet, \square\square, \square, \dots, \square\square, \underbrace{\bullet}_{l_N}). \tag{3.154}$$

Residue theorem then turns the integral into a sum over all such partitions and the poles corresponding to a given partition are given as

$$\phi^I_{n_I} = -i\frac{a_I}{\hbar} + (n_I - 1)\epsilon + \lambda^I_{n_I}, \tag{3.155}$$

where $I = 1, \dots, N$ labels the position of the Young tableau in the N-vector and $n_I = 1, \dots, l_I$ labels the boxes in $YT(l_I)$. Substituting this in (3.150) we get (the $l!$ gets cancelled by the permutation symmetry of the boxes)

$$\begin{aligned} Z_l = \frac{1}{\epsilon^l} \sum_{\{l_I \geq 0 | \sum_{I=1}^N l_I = l\}} \oint_{\mathcal{M}} \prod_{\substack{I=1\\ l_I \neq 0}}^{N} \prod_{n_I=1}^{l_I} \frac{d\lambda^I_{n_I}}{2\pi i} \\ \times \prod_{\substack{I \neq J\\ l_I \neq 0, l_J \neq 0}}^{N} \prod_{n_I=1}^{l_I} \prod_{n_J=1}^{l_J} \frac{\left(-i\frac{a_{IJ}}{\hbar} + n_{IJ}\epsilon + \lambda^{I,J}_{n_I,n_J}\right)}{\left(-i\frac{a_{IJ}}{\hbar} + (n_{IJ}-1)\epsilon + \lambda^{I,J}_{n_I,n_J}\right)} \prod_{\substack{I=1\\ l_I \neq 0}}^{N} \prod_{n_I \neq n_J}^{l_I} \frac{\left(n_{IJ}\epsilon + \lambda^{I,I}_{n_I,n_J}\right)}{\left((n_{IJ}-1)\epsilon + \lambda^{I,I}_{n_I,n_J}\right)} \\ \times \prod_{\substack{I=1\\ l_I \neq 0}}^{N} \prod_{n_I=1}^{l_I} \frac{\prod_{j=1}^{N_a} \left(i\frac{\tilde{a}_j}{\hbar} - i\frac{a_I}{\hbar} + (n_I - 1)\epsilon + \lambda^I_{n_I}\right)}{\prod_{r=1}^{N} \left(-i\frac{a_{Ir}}{\hbar} + (n_I-1)\epsilon + \lambda^I_{n_I}\right) \prod_{r=N+1}^{N_f} \left(-i\frac{a_{Ir}}{\hbar} - n_I\epsilon - \lambda^I_{n_I}\right)}, \end{aligned} \tag{3.156}$$

where we integrate over $\mathcal{M} = \bigotimes_{r=1}^{l} S^1(0, \delta)$. The computation continues as follows. We separate the poles in λ's (there are only simple poles), the rest is a holomorphic function,

[1] One has to assume a_i to be imaginary at this point. The general result is obtained by analytic continuation after integration.

so we can effectively set the λ's to zero there. Eventually, we obtain

$$
\begin{aligned}
Z_l = \frac{1}{\epsilon^l} \sum_{\{l_I \geq 0 | \sum_{I=1}^N l_I = l\}} & \left[\oint_{\mathcal{M}} \prod_{\substack{I=1 \\ l_I \neq 0}}^N \left\{ \left(\prod_{n_I=1}^{l_I} \frac{d\lambda_{n_I}^I}{2\pi i} \right) \left(\frac{1}{\lambda_1^I} \prod_{n_I=1}^{l_I-1} \frac{1}{\lambda_{n_I+1,n_I}^{I,I}} \right) \right\} \right] \\
& \times \prod_{I \neq J}^N \frac{\left(1 + i\frac{a_{IJ}}{\hbar\epsilon} - l_I\right)_{l_J}}{\left(1 + i\frac{a_{IJ}}{\hbar\epsilon}\right)_{l_J}} \prod_{\substack{I=1 \\ l_I \neq 0}}^N \frac{\epsilon^{l_I - 1}}{l_I} \\
& \times \frac{\prod_{I=1}^N \prod_{j=1}^{N_a} \epsilon^{l_I} \left(\frac{i\frac{\widetilde{a}_j}{\hbar} + a_I}{\epsilon} \right)}{\prod_{I=1}^N \prod_{r \neq I}^N \epsilon^{l_I} \left(-i\frac{a_{Ir}}{\hbar\epsilon}\right) \prod_{\substack{I=1 \\ l_I \neq 0}}^N \epsilon^{l_I-1} (l_I - 1)! \prod_{I=1}^N \prod_{r=N+1}^{N_f} \epsilon^{l_I} \left(-i\frac{a_{rI}}{\hbar\epsilon}\right)},
\end{aligned}
\tag{3.157}
$$

where the integration gives $[\ldots] = 1$. We are left with products of ratios including the equivariant parameters, which we express as Pochhammer symbols and after heavy Pochhammer algebra we finally arrive at (3.131), which proves (3.150).

Now, if the integrand f does not have poles at infinity, which happens exactly for $N_f \geq N_a + 2$, we can write

$$
\int_{\mathcal{C}_u} \prod_{\alpha=1}^l \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right) = (-1)^l \int_{\mathcal{C}_d} \prod_{\alpha=1}^l \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right) \tag{3.158}
$$

with $\mathcal{C}_d$ having the same base as $\mathcal{C}_u$ but is closed in the lower half plane by a semicircle. Both contours are oriented counterclockwise. The lovely fact is that the r.h.s. of the above equation gives the desired result

$$
(-1)^l \int_{\mathcal{C}_d} \prod_{\alpha=1}^l \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right) \Big|_{\epsilon=1} = (-1)^{N_a l} Z_l^D|_{d.m.} \tag{3.159}
$$

after direct evaluation of the integral, completely analogue to that of (3.150).

Example: the $Gr(1,3) \simeq Gr(2,3)$ case

Let us show this isomorphism explicitly in a simple case: we will consider $Gr(1,3)$ and $Gr(2,3)$ in a completely equivariant setting.

Let us first compute the equivariant partition function for $Gr(1,3)$:

$$
\begin{aligned}
Z_{Gr(1,3)} &= \sum_m \int \frac{d\tau}{2\pi i} e^{4\pi\xi_{\mathrm{ren}}\tau - i\theta_{\mathrm{ren}} m} \prod_{j=1}^3 \frac{\Gamma(\tau + irMa_j - \frac{m}{2})}{\Gamma(1 - \tau - irMa_j - \frac{m}{2})} \\
&= \sum_{i=1}^3 ((rM)^6 z\bar{z})^{irMa_i} \prod_{\substack{j=1 \\ j \neq i}}^3 \frac{\Gamma(-irMa_{ij})}{\Gamma(1 + irMa_{ij})} \sum_{l \geq 0} \frac{[(rM)^3 z]^l}{\prod_{j=1}^3 (1 + irMa_{ij})_l} \sum_{k \geq 0} \frac{[(-rM)^3 \bar{z}]^k}{\prod_{j=1}^3 (1 + irMa_{ij})_k}
\end{aligned}
\tag{3.160}
$$

Here we defined $a_{ij} = a_i - a_j$, and the twisted masses have been rescaled according to $a_i \to Ma_i$, so they are now dimensionless. For $Gr(2,3)$ we have (with $\tilde{\theta}_{\text{ren}} = \tilde{\theta} + \pi = \tilde{\theta} + 3\pi$, being $\tilde{\theta} \longrightarrow \tilde{\theta} + 2\pi$ a symmetry of the theory)

$$
\begin{aligned}
Z_{Gr(2,3)} =& \frac{1}{2} \sum_{m_1,m_2} \int \frac{d\tau_1}{2\pi i} \frac{d\tau_2}{2\pi i} e^{4\pi\tilde{\xi}_{\text{ren}}(\tau_1+\tau_2) - i\tilde{\theta}_{\text{ren}}(m_1+m_2)} \\
& \left(-\tau_{12}^2 + \frac{m_{12}^2}{4}\right) \prod_{r=1}^{2} \prod_{j=1}^{3} \frac{\Gamma(\tau_r + irM\tilde{a}_j - \frac{m_r}{2})}{\Gamma(1 - \tau_r - irM\tilde{a}_j - \frac{m_r}{2})} \\
=& \sum_{i<j}^{3} ((rM)^6 \tilde{z}\bar{\tilde{z}})^{irM(\tilde{a}_i+\tilde{a}_j)} \prod_{\substack{k=1 \\ k\neq i,j}}^{3} \frac{\Gamma(-irM\tilde{a}_{ik})}{\Gamma(1+irM\tilde{a}_{ik})} \frac{\Gamma(-irM\tilde{a}_{jk})}{\Gamma(1+irM\tilde{a}_{jk})} \\
& \sum_{l_1,l_2\geq 0} \frac{[(-rM)^3\tilde{z}]^{l_1+l_2}}{\prod_{k=1}^{3}(1+irM\tilde{a}_{ik})_{l_1} \prod_{k=1}^{3}(1+irM\tilde{a}_{jk})_{l_2}} \frac{l_1 - l_2 + irM\tilde{a}_i - irM\tilde{a}_j}{irM\tilde{a}_i - irM\tilde{a}_j} \\
& \sum_{k_1,k_2\geq 0} \frac{[(rM)^3\bar{\tilde{z}}]^{k_1+k_2}}{\prod_{k=1}^{3}(1+irM\tilde{a}_{ik})_{k_1} \prod_{k=1}^{3}(1+irM\tilde{a}_{jk})_{k_2}} \frac{k_1 - k_2 + irM\tilde{a}_i - irM\tilde{a}_j}{irM\tilde{a}_i - irM\tilde{a}_j}
\end{aligned}
\tag{3.161}
$$

In both situations, we are assuming $a_1 + a_2 + a_3 = 0$ and $\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 = 0$. Consider now the partition $(\bullet, \bullet, \square)$ for $Gr(1,3)$ and the dual partition $(\square, \square, \bullet)$ for $Gr(2,3)$; we have respectively

$$
\begin{aligned}
Z_{Gr(1,3)}^{(\bullet,\bullet,\square)} =& ((rM)^6 z\bar{z})^{irMa_3} \frac{\Gamma(-irMa_{31})}{\Gamma(1+irMa_{31})} \frac{\Gamma(-irMa_{32})}{\Gamma(1+irMa_{32})} \\
& \sum_{l\geq 0} \frac{[(rM)^3 z]^l}{l!(1+irMa_{31})_l (1+irMa_{32})_l} \\
& \sum_{k\geq 0} \frac{[(-rM)^3 \bar{z}]^k}{k!(1+irMa_{31})_k (1+irMa_{32})_k} \\
Z_{Gr(2,3)}^{(\square,\square,\bullet)} =& ((rM)^6 \tilde{z}\bar{\tilde{z}})^{irM(\tilde{a}_1+\tilde{a}_2)} \frac{\Gamma(-irM\tilde{a}_{13})}{\Gamma(1+irM\tilde{a}_{13})} \frac{\Gamma(-irM\tilde{a}_{23})}{\Gamma(1+irM\tilde{a}_{23})} \\
& \sum_{l_1,l_2\geq 0} \frac{[(-rM)^3\tilde{z}]^{l_1+l_2}}{\prod_{i=1}^{2} l_i! \prod_{j\neq i}^{3}(1+irM\tilde{a}_{ij})_{l_i}} \frac{l_1 - l_2 + irM\tilde{a}_1 - irM\tilde{a}_2}{irM\tilde{a}_1 - irM\tilde{a}_2} \\
& \sum_{k_1,k_2\geq 0} \frac{[(rM)^3\bar{\tilde{z}}]^{k_1+k_2}}{\prod_{i=1}^{2} k_i! \prod_{j\neq i}^{3}(1+irM\tilde{a}_{ij})_{k_i}} \frac{k_1 - k_2 + irM\tilde{a}_1 - irM\tilde{a}_2}{irM\tilde{a}_1 - irM\tilde{a}_2}
\end{aligned}
\tag{3.162}
$$

Since

$$
\begin{aligned}
& \sum_{l_1,l_2\geq 0} \frac{[(-rM)^3\tilde{z}]^{l_1+l_2}}{\prod_{i=1}^{2} l_i! \prod_{j\neq i}^{3}(1+irM\tilde{a}_{ij})_{l_i}} \frac{l_1 - l_2 + irM\tilde{a}_1 - irM\tilde{a}_2}{irM\tilde{a}_1 - irM\tilde{a}_2} = \\
&= \sum_{l\geq 0} \frac{[(-rM)^3\tilde{z}]^l}{l!(1+irM\tilde{a}_{13})_l (1+irM\tilde{a}_{23})_l} c_l
\end{aligned}
\tag{3.163}
$$

and

$$c_l = \sum_{l_1=0}^{l} \frac{l!}{l_1!(l-l_1)!} \frac{(1+irM\tilde{a}_{23}+l-l_1)_{l_1}(1+irM\tilde{a}_{13}+l_1)_{l-l_1}}{(irM\tilde{a}_{12}-l+l_1)_{l_1}(-irM\tilde{a}_{12}-l_1)_{l-l_1}} = (-1)^l = (-1)^{3l}$$

we can conclude that $Z^{(\bullet,\bullet,\square)}_{Gr(1,3)} = Z^{(\square,\square,\bullet)}_{Gr(2,3)}$ if we identify $a_i = -\tilde{a}_i$ and $\xi = \tilde{\xi}$, $\theta = \tilde{\theta}$ (i.e., $z = \tilde{z}$). It is then easy to prove that $Z_{Gr(1,3)} = Z_{Gr(2,3)}$.

3.3.3 Flag manifolds

Let us now consider a gauged linear sigma model with gauge group $U(s_1)\times\ldots\times U(s_l)$ and matter in the $(s_1,\bar{s}_2)\oplus\ldots\oplus(s_{l-1},\bar{s}_l)\oplus(s_l,n)$ representations, where $s_1<\ldots<s_l<n$. This flows in the infrared to a non-linear sigma model whose target space is the flag manifold $Fl(s_1,\ldots,s_l,n)$. The partition function is given by

$$\begin{aligned} Z_{Fl} &= \frac{1}{s_1!\ldots s_l!} \sum_{\substack{\vec{m}^{(a)} \\ a=1\ldots l}} \int \prod_{a=1}^{l}\prod_{i=1}^{s_a} \frac{\mathrm{d}\tau_i^{(a)}}{2\pi i} e^{4\pi\xi^{(a)}_{\mathrm{ren}}\tau_i^{(a)}-i\theta^{(a)}_{\mathrm{ren}}m_i^{(a)}} Z_{\mathrm{vector}}Z_{\mathrm{bifund}}Z_{\mathrm{fund}} \\ Z_{\mathrm{vector}} &= \prod_{a=1}^{l}\prod_{i<j}^{s_a}\left(\frac{(m_{ij}^{(a)})^2}{4} - (\tau_{ij}^{(a)})^2\right) \\ Z_{\mathrm{bifund}} &= \prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}} \frac{\Gamma\left(\tau_i^{(a)}-\tau_j^{(a+1)}-\frac{m_i^{(a)}}{2}+\frac{m_j^{(a+1)}}{2}\right)}{\Gamma\left(1-\tau_i^{(a)}+\tau_j^{(a+1)}-\frac{m_i^{(a)}}{2}+\frac{m_j^{(a+1)}}{2}\right)} \\ Z_{\mathrm{fund}} &= \prod_{i=1}^{s_l}\left(\frac{\Gamma\left(\tau_i^{(l)}-\frac{m_i^{(l)}}{2}\right)}{\Gamma\left(1-\tau_i^{(l)}-\frac{m_i^{(l)}}{2}\right)}\right)^n \end{aligned} \tag{3.164}$$

This is computed by taking poles at

$$\tau_i^{(a)} = \frac{m_i^{(a)}}{2} - k_i^{(a)} + rM\lambda_i^{(a)} \tag{3.165}$$

which gives

$$Z_{Fl} = \frac{1}{s_1!\ldots s_l!} \oint \prod_{a=1}^{l}\prod_{i=1}^{s_a} \frac{d(rM\lambda_i^{(a)})}{2\pi i} Z_{\text{1-loop}}Z_{\mathrm{v}}Z_{\mathrm{av}} \tag{3.166}$$

where

$$
\begin{aligned}
Z_{\text{1-loop}} =&(rM)^{-2rM\left[\sum_{a=1}^{l-1}(|\lambda^{(a)}|s_{a+1}-|\lambda^{(a+1)}|s_a)+n|\lambda^{(l)}|\right]} \\
&\prod_{a=1}^{l}\prod_{i<j}^{s_a}(rM\lambda_i^{(a)}-rM\lambda_j^{(a)})(rM\lambda_j^{(a)}-rM\lambda_i^{(a)}) \\
&\prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}}\frac{\Gamma\left(rM\lambda_i^{(a)}-rM\lambda_j^{(a+1)}\right)}{\Gamma\left(1-rM\lambda_i^{(a)}+rM\lambda_j^{(a+1)}\right)}\prod_{i=1}^{s_l}\left(\frac{\Gamma\left(rM\lambda_i^{(l)}\right)}{\Gamma\left(1-rM\lambda_i^{(l)}\right)}\right)^n \\
Z_{\text{v}} =&\sum_{\vec{l}^{(a)}}(rM)^{\sum_{a=1}^{l-1}(|l^{(a)}|s_{a+1}-|l^{(a+1)}|s_a)+n|l^{(l)}|}\prod_{a=1}^{l}(-1)^{(s_a-1)|l^{(a)}|}z_a^{|l^{(a)}|-rM|\lambda^{(a)}|} \\
&\prod_{a=1}^{l}\prod_{i<j}^{s_a}\frac{l_i^{(a)}-l_j^{(a)}-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}} \\
&\prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}}\frac{1}{(1-rM\lambda_i^{(a)}+rM\lambda_j^{(a+1)})_{l_i^{(a)}-l_j^{(a+1)}}}\prod_{i=1}^{s_l}\frac{1}{\left[(1-rM\lambda_i^{(l)})_{l_i^{(l)}}\right]^n} \\
Z_{\text{av}} =&\sum_{\vec{k}^{(a)}}(-rM)^{\sum_{a=1}^{l-1}(|k^{(a)}|s_{a+1}-|k^{(a+1)}|s_a)+n|k^{(l)}|}\prod_{a=1}^{l}(-1)^{(s_a-1)|k^{(a)}|}\bar{z}_a^{|k^{(a)}|-rM|\lambda^{(a)}|} \\
&\prod_{a=1}^{l}\prod_{i<j}^{s_a}\frac{k_i^{(a)}-k_j^{(a)}-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}} \\
&\prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}}\frac{1}{(1-rM\lambda_i^{(a)}+rM\lambda_j^{(a+1)})_{k_i^{(a)}-k_j^{(a+1)}}}\prod_{i=1}^{s_l}\frac{1}{\left[(1-rM\lambda_i^{(l)})_{k_i^{(l)}}\right]^n}
\end{aligned}
\tag{3.167}
$$

Here k's and l's are non-negative integers.
This result can be compared with the one in [62]. Indeed our fractions with Pochhammers at the denominator are equivalent to the products appearing there and we find perfect agreement with the Givental $\mathcal{I}$-functions under the by now familiar identification $\hbar = \frac{1}{rM}$, $\lambda = -H$ in Z_{v} and $\hbar = -\frac{1}{rM}$, $\lambda = H$ in Z_{av}.

3.3.4 Quivers

The techniques we used in the flag manifold case can be easily generalized to more general quivers; let us write down the rules to compute their partition functions. Here we will only consider quiver theories with unitary gauge groups and matter fields in the fundamental, antifundamental or bifundamental representation, without introducing twisted masses (they can be inserted straightforwardly). Every node of the quiver, i.e. every gauge group $U(s_a)$, contributes with

- Integral:

$$\frac{1}{s_a!}\oint\prod_{i=1}^{s_a}\frac{d(rM\lambda_i^{(a)})}{2\pi i} \tag{3.168}$$

- One-loop factor:

$$(rM)^{-2rM|\lambda^{(a)}|\sum_i q_{a,i}}\prod_{i<j}^{s_a}(rM\lambda_i^{(a)}-rM\lambda_j^{(a)})(rM\lambda_j^{(a)}-rM\lambda_i^{(a)}) \tag{3.169}$$

- Vortex factor:

$$\sum_{\vec{l}^{(a)}}(rM)^{|l^{(a)}|\sum_i q_{a,i}}(-1)^{(s_a-1)|l^{(a)}|}z_a^{|l^{(a)}|-rM|\lambda^{(a)}|}\prod_{i<j}^{s_a}\frac{l_i^{(a)}-l_j^{(a)}-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}} \tag{3.170}$$

- Anti-vortex factor:

$$\sum_{\vec{k}^{(a)}}(-rM)^{|k^{(a)}|\sum_i q_{a,i}}(-1)^{(s_a-1)|k^{(a)}|}\bar{z}_a^{|k^{(a)}|-rM|\lambda^{(a)}|}\prod_{i<j}^{s_a}\frac{k_i^{(a)}-k_j^{(a)}-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}} \tag{3.171}$$

Here $q_{a,i}$ is the charge of the i-th chiral matter field with respect to the abelian subgroup $U(1)_a\subset U(s_a)$ corresponding to $\xi^{(a)}$ and $\theta^{(a)}$.
Every matter field in a representation of $U(s_a)\times U(s_b)$ and R-charge R contributes with

- One-loop factor:

$$\prod_{i=1}^{s_a}\prod_{j=1}^{s_b}\frac{\Gamma\left(\frac{R}{2}+q_a rM\lambda_i^{(a)}+q_b rM\lambda_j^{(b)}\right)}{\Gamma\left(1-\frac{R}{2}-q_a rM\lambda_i^{(a)}-q_b rM\lambda_j^{(b)}\right)} \tag{3.172}$$

- Vortex factor:

$$\prod_{i=1}^{s_a}\prod_{j=1}^{s_b}\frac{1}{(1-\frac{R}{2}-q_a rM\lambda_i^{(a)}-q_b rM\lambda_j^{(b)})_{q_a l_i^{(a)}+q_b l_j^{(b)}}} \tag{3.173}$$

- Anti-vortex factor:

$$(-1)^{q_a s_b|k^{(a)}|+q_b s_a|k^{(b)}|}\prod_{i=1}^{s_a}\prod_{j=1}^{s_b}\frac{1}{(1-\frac{R}{2}-q_a rM\lambda_i^{(a)}-q_b rM\lambda_j^{(b)})_{q_a k_i^{(a)}+q_b k_j^{(b)}}} \tag{3.174}$$

In particular, the bifundamental $(s_a,\bar{s}_b)$ is given by $q_a=1$, $q_b=-1$. A field in the fundamental can be recovered by setting $q_a=1$, $q_b=0$; for an antifundamental, $q_a=-1$ and $q_b=0$. We can recover the usual formulae if we use (3.26). Multifundamental representations can be obtained by a straightforward generalization: for example, a trifundamental representation gives

$$\prod_{i=1}^{s_a}\prod_{j=1}^{s_b}\prod_{k=1}^{s_c}\frac{1}{(1-\frac{R}{2}-q_a rM\lambda_i^{(a)}-q_b rM\lambda_j^{(b)}-q_c rM\lambda_k^{(c)})_{q_a l_i^{(a)}+q_b l_j^{(b)}+q_c l_k^{(c)}}} \tag{3.175}$$

for the vortex factor.

In principle, these formulae are also valid for adjoint fields, if we set $s_a = s_b$, $q_a = 1$, $q_b = -1$; in practice, the diagonal contribution will give a $\Gamma(0)^{s_a}$ divergence, so the only way we can make sense of adjoint fields is by giving them a twisted mass.

Chapter 4

ADHM quiver and quantum hydrodynamics

4.1 Overview

With an educated use of the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2, in the previous chapter we were able to compute the quantum cohomology (equivariant and not) of many abelian and non-abelian quotients; in particular we discussed how Z^{S^2} is related to Givental's formalism, by identifying the vortex partition function Z_{v} with Givental's $\mathcal{I}$-function.
In this chapter we will dedicate ourselves to the study of a special $\mathcal{N} = (2,2)$ gauge theory: the ADHM quiver, a GLSM whose target space $\mathcal{M}_{k,N}$ describes the moduli space of k instantons for a pure $U(N)$ supersymmetric gauge theory. The associated partition function $Z^{S^2}_{k,N}$ will be a generalization of the Nekrasov instanton partition function which takes into account the corrections associated to the equivariant quantum cohomology of the instanton moduli space.
In the second part of the chapter we will also study the Landau-Ginzburg mirror theory of the ADHM GLSM. Thanks to the Bethe/gauge correspondence, we will see how the mirror is related to quantum integrable systems of hydrodynamic type, and in particular to the so-called $gl(N)$ Periodic Intermediate Long Wave (ILW$_N$) system. This will allow us to compute the spectrum of the ILW system in terms of gauge theory quantities.

4.1.1 6d theories and ADHM equivariant quantum cohomology

The Nekrasov partition function provides an extension of the SW prepotential [79] including an infinite tower of gravitational corrections coupled to the parameters of the so-called Ω-background [9, 10]. By means of the equivariant localization technique, one can reduce the path integration over the infinite-dimensional space of field configurations to a localized sum over the points in the moduli space of BPS configurations which are fixed under the maximal torus of the global symmetries of the theory. In the case of $\mathcal{N}=2$ theories in four dimensions the Nekrasov partition function actually computes the equivariant volume of the instanton moduli space; from a mathematical point of view it encodes the data of the classical equivariant cohomology of the ADHM instanton moduli space.

A D-brane engineering of the pure four-dimensional $\mathcal{N}=2$ $U(N)$ gauge theory is provided by a system of N D3-branes at the singular point of the orbifold geometry $\mathbb{C}^2/\mathbb{Z}_2$. The non-perturbative contributions to this theory are encoded by D(-1)-branes which provide the corresponding instanton contributions [80, 81, 82]. The four-dimensional gauge theory is the effective low energy theory of this system of D3-D(-1) branes on $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, where N D3-branes are located on $\mathbb{C}^2$ and, as the D(-1) branes, are stuck at the singular point of $\mathbb{C}^2/\mathbb{Z}_2$. The Nekrasov partition function can be computed from the D(-1)-branes point of view as a supersymmetric $D=0$ path integral whose fields realize the open string sectors of the D(-1)-D3 system [8, 83]. A particularly relevant point to us is that the open string sectors correspond to the ADHM data and the superpotential of the system imposes the ADHM constraints on the vacua.

A richer description of the construction above, which avoids the introduction of fractional D-brane charges, is obtained by resolving the orbifold A_1 singularity to a smooth ALE space obtained by blowing up the singular point to a two-sphere [84, 85]. The resolution generates a local K3 smooth geometry, namely the Eguchi-Hanson space, given by the total space of the cotangent bundle to the 2-sphere. We remain with a system of D5-D1-D(-1) branes on the minimal resolution of the transversal A_1 singularity $\mathbb{C}^2 \times T^*S^2 \times \mathbb{C}$, which at low energy reduces to a pure six-dimensional $\mathcal{N}=1$ $U(N)$ gauge theory on $\mathbb{C}^2 \times S^2$; the N D5-branes are located on $\mathbb{C}^2 \times S^2$, the k D1 branes are wrapping S^2 and the D(-1) branes are stuck at the North and the South pole of the sphere. From the D1-branes perspective, the theory describing the D(-1)-D1-D5 brane system on the resolved space is a GLSM on the blown-up two-sphere describing the corresponding open string sectors with a superpotential interaction which imposes the ADHM constraints. This is exactly the ADHM GLSM on S^2 we will be analysing in this chapter; the D(-1) branes will be nothing but the vortex/anti-vortex contributions of the spherical partition function describing the effective dynamics of the k D1-branes. The D(-1)-D1-D5 system probes the ADHM geometry from a stringy point of view: the supersymmetric sigma model contains stringy instanton corrections corresponding to the topological sectors with non trivial magnetic flux on the two-sphere[1]. From the mathematical point of view, the stringy instantons are deforming the classical cohomology of the ADHM moduli space to a quantum one; the information about the quantum cohomology is all contained inside $Z^{S^2}_{k,N}$, as explained in chapter 3. The $\mathcal{N}=2$ $D=4$ gauge theory is then obtained by considering the system of D1-D5 branes wrapping the blown-up 2-sphere in the zero radius (i.e. point particle) limit; in this limit $Z^{S^2}_{k,N}$ reproduces the Nekrasov partition function, which only receives contributions from the trivial sector, that is the sector of constant maps.

4.1.2 Quantum hydrodynamics and gauge theories

Connections between supersymmetric theories with eight supercharges and quantum integrable systems of *hydrodynamic* type have been known to exist since a long time. These naturally arise in the context of AGT correspondence. Indeed integrable systems and conformal field theories in two dimensions are intimately related, from several points of view. The link between conformal field theory and quantum KdV was noticed in [86, 87, 88, 89]. In [89] the infinite conserved currents in involutions of the Virasoro algebra Vir have been shown to realize the quantization of the KdV system and the quantum monodromy "T-operators" are shown to act on highest weight Virasoro modules.

More recently an analogous connection between the spectrum of a CFT based on the Heisenberg plus Virasoro algebra $H \oplus Vir$ and the bidirectional Benjamin-Ono (BO_2) system has been shown in the context of a combinatorial proof of AGT correspondence [90], providing a first example of the phenomenon we alluded to before.

In sections 4.5 and 4.6 we study the link between the six dimensional $U(N)$ exact partition function of section 4.2 and quantum integrable systems, finding that the supersymmetric gauge theory provides the quantization of the $gl(N)$ Intermediate Long Wave system (ILW_N). This is a well known one parameter deformation of the BO system. Remarkably, it interpolates between BO and KdV. We identify the deformation parameter with the FI of the S^2 GLSM, by matching the twisted superpotential of the GLSM with the Yang-Yang function of quantum ILW_N as proposed in [91]. Our result shows that the quantum cohomology of the ADHM instanton moduli space is computed by the quantum ILW_N system. In the abelian case $N=1$, when the ADHM moduli space reduces to the Hilbert scheme of points on $\mathbb{C}^2$, this correspondence is discussed in [92, 93, 94].

[1] These are effective stringy instantons in the ADHM moduli space which compute the KK corrections due to the finite size of the blown-up $\mathbb{P}^1$. For the sake of clarity, gravity is decoupled from the D-branes and α' is scaled away as usual.

On top of this we show that the chiral ring observables of the six dimensional gauge theory are related to the commuting quantum Hamiltonians of ILW_N. Let us remark that in the four dimensional limit our results imply that the gauge theory chiral ring provides a basis for the BO_N quantum Hamiltonians. This shows the appearance of the $H \oplus W_N$ algebra in the characterization of the BPS sector of the four dimensional gauge theory as proposed in [95] and is a strong purely gauge theoretic argument in favour of the AGT correspondence.

We also show that classical ILW hydrodynamic equations arise as a collective description of elliptic Calogero-Moser integrable system. Let us notice that the quantum integrability of the BO_N system can be shown by constructing its quantum Hamiltonians in terms of N copies of trigonometric Calogero-Sutherland Hamiltonians with tridiagonal coupling: a general proof in the context of equivariant quantum cohomology of Nakajima quiver varieties can be found in [96]. The relevance of this construction in the study of conformal blocks of W-algebra is discussed in [97]. Our result hints to an analogous rôle of elliptic Calogero system in the problem of the quantization of ILW_N.

It is worth to remark at this point that these quantum systems play a relevant rôle in the description of Fractional Quantum Hall liquids. In particular our results suggest the quantum ILW system to be useful in the theoretical investigation of FQH states on the torus, which are also more amenable to numerical simulations due to the periodic boundary conditions. For a discussion on quiver gauge theories and FQHE in the context of AGT correspondence see [97, 98].

In the first part of this chapter we are going to summarize the results of [99]; the second part will be more focussed on [100].

4.2 The ADHM Gauged Linear Sigma Model

In this section we describe the dynamics of a system of k D1 and N D5-branes wrapping the blown-up sphere of a resolved A_1 singularity. Specifically, we consider the type IIB background $\mathbb{C}^2 \times T^*\mathbb{P}^1 \times \mathbb{C}$ with the D1-branes wrapping the $\mathbb{P}^1$ and space-time filling D5-branes wrapped on $\mathbb{P}^1 \times \mathbb{C}^2$. We focus on the D1-branes, whose dynamics is described by a two-dimensional $\mathcal{N} = (2,2)$ gauged linear sigma model flowing in the infrared to a non-linear sigma model with target space the ADHM moduli space of instantons $\mathcal{M}_{k,N}$. The field content is reported in the table below.

The superpotential of our model is $W = \mathrm{Tr}_k \{\chi([B_1, B_2] + IJ)\}$. It implements as a constraint the fact that an infinitesimal open string plaquette in the D1-D1 sector can be undone as a couple of open strings stretching from the D1 to a D5 and back. We

	χ	B_1	B_2	I	J
D-brane sector	D1/D1	D1/D1	D1/D1	D1/D5	D5/D1
gauge $U(k)$	Adj	Adj	Adj	$\mathbf{k}$	$\bar{\mathbf{k}}$
flavor $U(N)\times U(1)^2$	$\mathbf{1}_{(-1,-1)}$	$\mathbf{1}_{(1,0)}$	$\mathbf{1}_{(0,1)}$	$\bar{\mathbf{N}}_{(0,0)}$	$\mathbf{N}_{(1,1)}$
twisted masses	$\epsilon_1+\epsilon_2$	$-\epsilon_1$	$-\epsilon_2$	$-a_j$	$a_j-\epsilon_1-\epsilon_2$
R-charge	$2-2q$	q	q	$q+p$	$q-p$

TABLE 4.1: ADHM gauged linear sigma model

also consider twisted masses corresponding to the maximal torus in the global symmetry group $U(1)^{N+2}$ acting on $\mathcal{M}_{k,N}$ which we denote as $(a_j,-\epsilon_1,-\epsilon_2)$. The R-charges are assigned as the most general ones which ensure $R(W)=2$ and full Lorentz symmetry at zero twisted masses. These provide an imaginary part to the twisted masses via the redefinition

$$a_i - i\frac{p+q}{2} \longrightarrow a_i \quad , \quad \epsilon_{1,2} - i\frac{q}{2} \longrightarrow \epsilon_{1,2} \tag{4.1}$$

We are interested in computing the partition function $Z^{S^2}_{k,N}$ for this ADHM model. Our computations will be valid for $q>p>0\,,\ q<1$, so that the integration contour in σ is along the real line; the case with negative values for the R-charges can be obtained by analytic continuation, deforming the contour. The S^2 partition function reads

$$Z^{S^2}_{k,N} \;=\; \frac{1}{k!}\sum_{\vec{m}\in\mathbb{Z}^k}\int_{\mathbb{R}^k}\prod_{s=1}^{k}\frac{\mathrm{d}(r\sigma_s)}{2\pi}e^{-4\pi i\xi r\sigma_s - i\theta_{\text{ren}}m_s}Z_{\text{gauge}}Z_{IJ}\,Z_{\text{adj}} \tag{4.2}$$

where

$$Z_{\text{gauge}} = \prod_{s<t}^{k}\left(\frac{m_{st}^2}{4}+r^2\sigma_{st}^2\right) \tag{4.3}$$

and the one-loop determinants of the matter contributions are given by

$$\begin{aligned} Z_{IJ} \;&=\; \prod_{s=1}^{k}\prod_{j=1}^{N}\frac{\Gamma\left(-ir\sigma_s+ira_j-\frac{m_s}{2}\right)}{\Gamma\left(1+ir\sigma_s-ira_j-\frac{m_s}{2}\right)}\frac{\Gamma\left(ir\sigma_s-ir\left(a_j-\epsilon\right)+\frac{m_s}{2}\right)}{\Gamma\left(1-ir\sigma_s+ir\left(a_j-\epsilon\right)+\frac{m_s}{2}\right)} \\ Z_{\text{adj}} \;&=\; \prod_{s,t=1}^{k}\frac{\Gamma\left(1-ir\sigma_{st}-ir\epsilon-\frac{m_{st}}{2}\right)}{\Gamma\left(ir\sigma_{st}+ir\epsilon-\frac{m_{st}}{2}\right)}\frac{\Gamma\left(-ir\sigma_{st}+ir\epsilon_1-\frac{m_{st}}{2}\right)}{\Gamma\left(1+ir\sigma_{st}-ir\epsilon_1-\frac{m_{st}}{2}\right)}\frac{\Gamma\left(-ir\sigma_{st}+ir\epsilon_2-\frac{m_{st}}{2}\right)}{\Gamma\left(1+ir\sigma_{st}-ir\epsilon_2-\frac{m_{st}}{2}\right)} \end{aligned} \tag{4.4}$$

with $\epsilon=\epsilon_1+\epsilon_2$, $\sigma_{st}=\sigma_s-\sigma_t$ and $m_{st}=m_s-m_t$. Z_{IJ} contains the contributions from the chirals in the fundamental and antifundamental I,J, while Z_{adj} the ones corresponding to the adjoint chirals χ,B_1,B_2. The partition function (4.2) is the central character of this chapter and we will refer to it as the stringy instanton partition function.

4.2.1 Reduction to the Nekrasov partition function

A first expected property of $Z^{S^2}_{k,N}$ is its reduction to the Nekrasov partition function in the limit of zero radius of the blown-up sphere. Because of this, in (4.2) we kept explicit the dependence on the radius r. It can be easily shown that in the limit $r \to 0$ our spherical partition function reduces to the integral representation of the k-instanton part $Z^{\text{Nek}}_{k,N}$ of the Nekrasov partition function $Z_N = \sum_k \Lambda^{2Nk} Z^{\text{Nek}}_{k,N}$, where

$$Z^{\text{Nek}}_{k,N} = \frac{1}{k!}\frac{\epsilon^k}{(2\pi i \epsilon_1 \epsilon_2)^k} \oint \prod_{s=1}^{k} \frac{d\sigma_s}{P(\sigma_s)P(\sigma_s+\epsilon)} \prod_{s<t}^{k} \frac{\sigma_{st}^2(\sigma_{st}^2-\epsilon^2)}{(\sigma_{st}^2-\epsilon_1^2)(\sigma_{st}^2-\epsilon_2^2)} \tag{4.5}$$

with $P(\sigma_s) = \prod_{j=1}^{N}(\sigma_s - a_j)$ and Λ the RGE invariant scale.
In order to prove this, let's start by considering (4.4); because of the identity $\Gamma(z) = \Gamma(1+z)/z$, Z_{IJ} and $Z_{\text{gauge}}Z_{\text{adj}}$ can be rewritten as

$$\begin{aligned} Z_{IJ} &= \prod_{s=1}^{k}\prod_{j=1}^{N} \frac{1}{(r\sigma_s - ra_j - i\frac{m_s}{2})(r\sigma_s - ra_j + r\epsilon - i\frac{m_s}{2})} \\ &\prod_{s=1}^{k}\prod_{j=1}^{N} \frac{\Gamma\left(1 - ir\sigma_s + ira_j - \frac{m_s}{2}\right)}{\Gamma\left(1 + ir\sigma_s - ira_j - \frac{m_s}{2}\right)} \frac{\Gamma\left(1 + ir\sigma_s - ir\left(a_j - \epsilon\right) + \frac{m_s}{2}\right)}{\Gamma\left(1 - ir\sigma_s + ir\left(a_j - \epsilon\right) + \frac{m_s}{2}\right)} \end{aligned} \tag{4.6}$$

$$\begin{aligned} Z_{\text{gauge}}Z_{\text{adj}} &= \prod_{s<t}^{k} \frac{\left(r\sigma_{st} + i\frac{m_{st}}{2}\right)\left(r\sigma_{st} - i\frac{m_{st}}{2}\right)\left(r\sigma_{st} + r\epsilon + i\frac{m_{st}}{2}\right)\left(r\sigma_{st} - r\epsilon + i\frac{m_{st}}{2}\right)}{\left(r\sigma_{st} - r\epsilon_1 - i\frac{m_{st}}{2}\right)\left(r\sigma_{st} + r\epsilon_1 - i\frac{m_{st}}{2}\right)\left(r\sigma_{st} - r\epsilon_2 - i\frac{m_{st}}{2}\right)\left(r\sigma_{st} + r\epsilon_2 - i\frac{m_{st}}{2}\right)} \\ &\left(\frac{\epsilon}{ir\epsilon_1\epsilon_2}\right)^k \prod_{s\neq t}^{k} \frac{\Gamma\left(1 - ir\sigma_{st} - ir\epsilon - \frac{m_{st}}{2}\right)}{\Gamma\left(1 + ir\sigma_{st} + ir\epsilon - \frac{m_{st}}{2}\right)} \frac{\Gamma\left(1 - ir\sigma_{st} + ir\epsilon_1 - \frac{m_{st}}{2}\right)}{\Gamma\left(1 + ir\sigma_{st} - ir\epsilon_1 - \frac{m_{st}}{2}\right)} \frac{\Gamma\left(1 - ir\sigma_{st} + ir\epsilon_2 - \frac{m_{st}}{2}\right)}{\Gamma\left(1 + ir\sigma_{st} - ir\epsilon_2 - \frac{m_{st}}{2}\right)} \end{aligned} \tag{4.7}$$

The lowest term in the expansion around $r = 0$ of (4.6) comes from the $\vec{m} = \vec{0}$ sector, and it is given by

$$\frac{1}{r^{2kN}} \prod_{s=1}^{k}\prod_{j=1}^{N} \frac{1}{(\sigma_s - a_j)(\sigma_s - a_j + \epsilon)} \tag{4.8}$$

On the other hand, (4.7) starts as

$$\left(\frac{\epsilon}{ir\epsilon_1\epsilon_2}\right)^k \left(f(\vec{m}) + o(r)\right) \tag{4.9}$$

with $f(\vec{m})$ ratio of Gamma functions independent on r. With this, we can conclude that the first term in the expansion originates from the $\vec{m} = \vec{0}$ contribution, and (4.2) reduces to (4.5), with $\Lambda = r^{-1}$.

4.2.2 Classification of the poles

The explicit evaluation of the partition function (4.2) given above passes by the classification of the poles in the integrand. We now show that these are classified by Young tableaux, just like for the Nekrasov partition function [9]. More precisely, we find a tower of poles for each box of the Young tableaux labelling the tower of Kaluza-Klein modes due to the string corrections.

The geometric phase of the GLSM is encoded in the choice of the contour of integration of (4.2), which implements the suitable stability condition for the hyper-Kähler quotient. In our case the ADHM phase corresponds to take $\xi > 0$ and this imposes to close the contour integral in the lower half plane. Following the discussion of [27], let us summarize the possible poles and zeros of the integrand ($n \geqslant 0$):

	poles ($\sigma^{(p)}$)	zeros ($\sigma^{(z)}$)
I	$\sigma_s^{(p)} = a_j - \frac{i}{r}(n + \frac{\lvert m_s \rvert}{2})$	$\sigma_s^{(z)} = a_j + \frac{i}{r}(1 + n + \frac{\lvert m_s \rvert}{2})$
J	$\sigma_s^{(p)} = a_j - \epsilon + \frac{i}{r}(n + \frac{\lvert m_s \rvert}{2})$	$\sigma_s^{(z)} = a_j - \epsilon - \frac{i}{r}(1 + n + \frac{\lvert m_s \rvert}{2})$
χ	$\sigma_{st}^{(p)} = -\epsilon - \frac{i}{r}(1 + n + \frac{\lvert m_{st} \rvert}{2})$	$\sigma_{st}^{(z)} = -\epsilon + \frac{i}{r}(n + \frac{\lvert m_{st} \rvert}{2})$
B_1	$\sigma_{st}^{(p)} = \epsilon_1 - \frac{i}{r}(n + \frac{\lvert m_{st} \rvert}{2})$	$\sigma_{st}^{(z)} = \epsilon_1 + \frac{i}{r}(1 + n + \frac{\lvert m_{st} \rvert}{2})$
B_2	$\sigma_{st}^{(p)} = \epsilon_2 - \frac{i}{r}(n + \frac{\lvert m_{st} \rvert}{2})$	$\sigma_{st}^{(z)} = \epsilon_2 + \frac{i}{r}(1 + n + \frac{\lvert m_{st} \rvert}{2})$

Poles from J do not contribute, being in the upper half plane. Consider now a pole for I, say $\sigma_1^{(p)}$; the next pole $\sigma_2^{(p)}$ can arise from I, B_1 or B_2, but not from χ, because in this case it would be cancelled by a zero from J. Moreover, if it comes from I, $\sigma_2^{(p)}$ should correspond to a twisted mass a_j different from the one for $\sigma_1^{(p)}$, or the partition function would vanish (as explained in full detail in [27]). In the case $\sigma_2^{(p)}$ comes from B_1, consider $\sigma_3^{(p)}$: again, this can be a pole from I, B_1 or B_2, but not from χ, or it would be cancelled by a zero of B_2. This reasoning takes into account all the possibilities, so we can conclude that the poles are classified by N Young tableaux $\{\vec{Y}\}_k = (Y_1, \ldots, Y_N)$ such that $\sum_{j=1}^N \lvert Y_j \rvert = k$, which describe coloured partitions of the instanton number k. These are the same as the ones used in the pole classification of the Nekrasov partition function, with the difference that to every box is associated not just a pole, but an infinite tower of poles, labelled by a positive integer n; i.e., we are considering three-dimensional Young tableaux.

These towers of poles can be dealt with by rewriting near each pole

$$\sigma_s = -\frac{i}{r}\left(n_s + \frac{\lvert m_s \rvert}{2}\right) + i\lambda_s \tag{4.10}$$

In this way we resum the contributions coming from the "third direction" of the Young tableaux, and the poles for λ_s are now given in terms of usual two-dimensional partitions.

As we will discuss later, this procedure allows for a clearer geometrical interpretation of the spherical partition function. Defining $z = e^{-2\pi\xi + i\theta}$ and $d_s = n_s + \frac{m_s+|m_s|}{2}$, $\tilde{d}_s = d_s - m_s$ so that $\sum_{m_s\in\mathbb{Z}}\sum_{n_s\geqslant 0} = \sum_{\tilde{d}_s\geqslant 0}\sum_{d_s\geqslant 0}$ we obtain the following expression:

$$Z^{S^2}_{k,N} = \frac{1}{k!}\oint\prod_{s=1}^{k}\frac{d(r\lambda_s)}{2\pi i}(z\bar{z})^{-r\lambda_s} Z_{11} Z_{\mathrm{v}} Z_{\mathrm{av}} \tag{4.11}$$

where[2]

$$\begin{aligned} Z_{11} \quad &= \quad \left(\frac{\Gamma(1-ir\epsilon)\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(ir\epsilon)\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)}\right)^k \prod_{s=1}^{k}\prod_{j=1}^{N}\frac{\Gamma(r\lambda_s+ira_j)\Gamma(-r\lambda_s-ira_j+ir\epsilon)}{\Gamma(1-r\lambda_s-ira_j)\Gamma(1+r\lambda_s+ira_j-ir\epsilon)} \\ &\prod_{s\neq t}^{k}(r\lambda_s-r\lambda_t)\frac{\Gamma(1+r\lambda_s-r\lambda_t-ir\epsilon)\Gamma(r\lambda_s-r\lambda_t+ir\epsilon_1)\Gamma(r\lambda_s-r\lambda_t+ir\epsilon_2)}{\Gamma(-r\lambda_s+r\lambda_t+ir\epsilon)\Gamma(1-r\lambda_s+r\lambda_t-ir\epsilon_1)\Gamma(1-r\lambda_s+r\lambda_t-ir\epsilon_2)} \end{aligned} \tag{4.12}$$

$$\begin{aligned} Z_{\mathrm{v}} \quad &= \quad \sum_{\tilde{d}_1,\ldots,\tilde{d}_k\geq 0}((-1)^N z)^{\tilde{d}_1+\ldots+\tilde{d}_k}\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s-ira_j+ir\epsilon)_{\tilde{d}_s}}{(1-r\lambda_s-ira_j)_{\tilde{d}_s}}\prod_{s<t}^{k}\frac{\tilde{d}_t-\tilde{d}_s-r\lambda_t+r\lambda_s}{-r\lambda_t+r\lambda_s} \\ &\frac{(1+r\lambda_s-r\lambda_t-ir\epsilon)_{\tilde{d}_t-\tilde{d}_s}}{(r\lambda_s-r\lambda_t+ir\epsilon)_{\tilde{d}_t-\tilde{d}_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_1)_{\tilde{d}_t-\tilde{d}_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_1)_{\tilde{d}_t-\tilde{d}_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_2)_{\tilde{d}_t-\tilde{d}_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_2)_{\tilde{d}_t-\tilde{d}_s}} \end{aligned} \tag{4.13}$$

$$\begin{aligned} Z_{\mathrm{av}} \quad &= \quad \sum_{d_1,\ldots,d_k\geq 0}((-1)^N \bar{z})^{d_1+\ldots+d_k}\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s-ira_j+ir\epsilon)_{d_s}}{(1-r\lambda_s-ira_j)_{d_s}}\prod_{s<t}^{k}\frac{d_t-d_s-r\lambda_t+r\lambda_s}{-r\lambda_t+r\lambda_s} \\ &\frac{(1+r\lambda_s-r\lambda_t-ir\epsilon)_{d_t-d_s}}{(r\lambda_s-r\lambda_t+ir\epsilon)_{d_t-d_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_1)_{d_t-d_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_1)_{d_t-d_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_2)_{d_t-d_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_2)_{d_t-d_s}} \end{aligned} \tag{4.14}$$

The Pochhammer symbol $(a)_d$ is defined as in (3.26). We observe that the $\frac{1}{k!}$ in (4.11) is cancelled by the $k!$ possible orderings of the λs, so in the rest of this paper we will always choose an ordering and remove the factorial.
Let us remark that Z_{v} appearing in (4.13) is the vortex partition function of the GLSM on equivariant $\mathbb{R}^2$ with equivariant parameter $\hbar = 1/r$. This was originally computed in [101] and recently discussed in the context of AGT correspondence in [28, 65, 76, 102].

As a final comment, let us consider the interesting limit $\epsilon_1 \to -\epsilon_2$, which implies $\epsilon \to 0$. In this limit we can show that all the world-sheet instanton corrections to $Z^{S^2}_{k,N}$ vanish and

[2] Remember that $\theta_{\mathrm{ren}} = \theta + (k-1)\pi$.

this is in agreement with the results of [96] about equivariant Gromov-Witten invariants of the ADHM moduli space.

First of all, consider (4.12). The prefactor gives the usual coefficient $(\frac{\epsilon}{i\epsilon_1\epsilon_2})^k$, while the Gamma functions simplify drastically, and we recover (4.5) with ϵ small, with an additional classical factor $(z\bar{z})^{-r\lambda_r}$ playing the rôle of the usual regulator in the contour integral representation of the Nekrasov partition function. Let us now turn to (4.13); for every Young tableau we have $Z_{\rm v} = 1 + o(\epsilon)$, where the 1 comes from the sector $\tilde{d}_s = 0$. Indeed one can show by explicit computation on the Young tableaux that for any $\tilde{d}_s \neq 0$ $Z_{\rm v}$ gets a positive power of ϵ and therefore does not contribute in the $\epsilon \to 0$ limit.

To clarify this point, let us consider a few examples. We will restrict to $N = 1$ for the sake of simplicity.

- The easiest tableau is $(\square)$; in this case $\lambda = -ia$ and

$$Z_{\rm v} = \sum_{\tilde{d}\geq 0}(-z)^{\tilde{d}}\frac{(ir\epsilon)_{\tilde{d}}}{\tilde{d}!} = 1 + \sum_{\tilde{d}\geq 1}(-z)^{\tilde{d}}\frac{(ir\epsilon)_{\tilde{d}}}{\tilde{d}!} = 1 + o(\epsilon) \qquad (4.15)$$

- Next are the tableaux $(\begin{smallmatrix}\square\\\square\end{smallmatrix})$ and $(\square\square)$. The expression of $Z_{\rm v}$ for $(\begin{smallmatrix}\square\\\square\end{smallmatrix})$ is given in (4.34); there you can easily see from the two Pochhammers $(ir\epsilon)_{\tilde{d}}$ at the numerator that the limit $\epsilon \to 0$ forces the $\tilde{d}_s$ to be zero, leaving $Z_{\rm v} = 1 + o(\epsilon)$; similarly for $(\square\square)$.
- The tableaux for $k = 3$ work as before. A more complicated case is $(\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix})$. One should first consider the Pochhammers of type $(ir\epsilon)_{\tilde{d}}$; in this case, we have

$$(ir\epsilon)_{\tilde{d}_1}(2ir\epsilon)_{\tilde{d}_4}\frac{(ir\epsilon)_{\tilde{d}_2-\tilde{d}_1}(ir\epsilon)_{\tilde{d}_3-\tilde{d}_1}(ir\epsilon)_{\tilde{d}_4-\tilde{d}_2}(ir\epsilon)_{\tilde{d}_4-\tilde{d}_3}(1)_{\tilde{d}_4-\tilde{d}_1}}{(2ir\epsilon)_{\tilde{d}_4-\tilde{d}_1}(1)_{\tilde{d}_2-\tilde{d}_1}(1)_{\tilde{d}_3-\tilde{d}_1}(1)_{\tilde{d}_4-\tilde{d}_2}(1)_{\tilde{d}_4-\tilde{d}_3}}\frac{\tilde{d}_4-\tilde{d}_1+ir\epsilon}{ir\epsilon} \qquad (4.16)$$

  Then one can easily see that this combination always starts with something which is of order ϵ or higher, unless $\tilde{d}_1 = \tilde{d}_2 = \tilde{d}_3 = \tilde{d}_4 = 0$, case in which we get 1.

These examples contain all the possible issues that can arise in the general case.

4.3 Equivariant Gromov-Witten invariants of $\mathcal{M}_{k,N}$

We now turn to discuss the exact partition function (4.11) of the D1-D5 system on the resolved A_1 singularity. As discussed in the previous section, this contains a tower of non-perturbative corrections to the prepotential of the four-dimensional gauge theory corresponding to the effective world-sheet instantons contributions.

From the discussion in chapter 3 we know that these corrections compute the Gromov-Witten invariants and gravitational descendants of the ADHM moduli space; we therefore deduce that the spherical partition function of the D1-D5 GLSM provides conjectural formulae for Givental's $\mathcal{I}$ and $\mathcal{J}$-functions of the ADHM instanton moduli space as follows:

$$\mathcal{I}_{k,N} = \sum_{d_1,\dots,d_k\geq 0}((-1)^N z)^{d_1+\dots+d_k}\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s - ira_j + ir\epsilon)_{d_s}}{(1-r\lambda_s - ira_j)_{d_s}}\prod_{s<t}^{k}\frac{d_t-d_s-r\lambda_t+r\lambda_s}{-r\lambda_t+r\lambda_s}$$
$$\frac{(1+r\lambda_s-r\lambda_t-ir\epsilon)_{d_t-d_s}}{(r\lambda_s-r\lambda_t+ir\epsilon)_{d_t-d_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_1)_{d_t-d_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_1)_{d_t-d_s}}\frac{(r\lambda_s-r\lambda_t+ir\epsilon_2)_{d_t-d_s}}{(1+r\lambda_s-r\lambda_t-ir\epsilon_2)_{d_t-d_s}} \tag{4.17}$$

where λ_s are the Chern roots of the tautological bundle of the ADHM moduli space.

From this expression we find that the asymptotic behaviour in $\hbar$ is

$$\mathcal{I}_{k,N} = 1 + \frac{I^{(N)}}{\hbar^N} + \dots \tag{4.18}$$

Therefore, $I^{(0)} = 1$ for every k, N, while $I^{(1)} = 0$ when $N > 1$; this implies that the equivariant mirror map is trivial, namely $\mathcal{I}_{k,N} = \mathcal{J}_{k,N}$, for $N > 1$. The $N = 1$ case will be discussed in detail in the following subsection. The structure of (4.17) supports the abelian/non-abelian correspondence conjecture of [103]; indeed the first factor in the first line corresponds to the abelian quotient by the Cartan torus $(\mathbb{C}^*)^k$ while the remaining factors express the twisting due to the non-abelian nature of the quotient.

Finally, let us notice[3] that for GIT quotients, and in particular for Nakajima quiver varieties, the notion of quasi-maps and of the corresponding $\mathcal{I}$-function were introduced in [56]. We notice that our $\mathcal{I}_{k,N}$ as in (4.17) should match the quasi-map $\mathcal{I}$-function and therefore, as a consequence of [96], should compute the $\mathcal{J}$-function of the instanton moduli space. Let us underline that the supersymmetric localization approach applies also to other classical groups and can be applied to study the quantum cohomology of general Kähler quotients.

4.3.1 Cotangent bundle of the projective space

As a first example, let us consider the case $\mathcal{M}_{1,N} \simeq \mathbb{C}^2 \times T^*\mathbb{CP}^{N-1}$. The integrated spherical partition function has the form:

$$Z^{S^2}_{1,N} = \sum_{j=1}^{N}(z\bar{z})^{ira_j} Z^{(j)}_{1l} Z^{(j)}_{\mathrm{v}} Z^{(j)}_{\mathrm{av}} \tag{4.19}$$

[3]We thank D.E. Diaconescu, A. Okounkov and D. Maulik for clarifying discussions on this issue.

The j-th contribution comes from the Young tableau $(\bullet, \ldots, \square, \ldots, \bullet)$, where the box is in the j-th position; this means we have to consider the pole $\lambda_1 = -ia_j$. Explicitly:

$$\begin{aligned}
Z_{11}^{(j)} &= \frac{\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)} \prod_{\substack{l=1\\l\neq j}}^{N} \frac{\Gamma(ira_{lj})\Gamma(-ira_{lj}+ir\epsilon)}{\Gamma(1-ira_{lj})\Gamma(1+ira_{lj}-ir\epsilon)} \\
Z_{\mathrm{v}}^{(j)} &= {}_N F_{N-1}\left(\begin{array}{c} \left\{ ir\epsilon, (-ira_{lj}+ir\epsilon)_{\substack{l=1\\l\neq j}}^{N} \right\} \\ \left\{ (1-ira_{lj})_{\substack{l=1\\l\neq j}}^{N} \right\} \end{array} ; (-1)^N z \right) \\
Z_{\mathrm{av}}^{(j)} &= {}_N F_{N-1}\left(\begin{array}{c} \left\{ ir\epsilon, (-ira_{lj}+ir\epsilon)_{\substack{l=1\\l\neq j}}^{N} \right\} \\ \left\{ (1-ira_{lj})_{\substack{l=1\\l\neq j}}^{N} \right\} \end{array} ; (-1)^N \bar{z} \right)
\end{aligned} \tag{4.20}$$

Let us consider in more detail the case $N = 2$. In this case the instanton moduli space reduces to $\mathbb{C}^2 \times T^*\mathbb{P}^1$ and is the same as the moduli space of the Hilbert scheme of two points $\mathcal{M}_{1,2} \simeq \mathcal{M}_{2,1}$. In order to match the equivariant actions on the two moduli spaces, we identify

$$a_1 = \epsilon_1 + 2a \quad , \quad a_2 = \epsilon_2 + 2a \tag{4.21}$$

so that $a_{12} = \epsilon_1 - \epsilon_2$. Then we have

$$Z_{1,2}^{S^2} = (z\bar{z})^{ir(2a+\epsilon_1)} Z_{11}^{(1)} Z_{\mathrm{v}}^{(1)} Z_{\mathrm{av}}^{(1)} + (z\bar{z})^{ir(2a+\epsilon_2)} Z_{11}^{(2)} Z_{\mathrm{v}}^{(2)} Z_{\mathrm{av}}^{(2)} \tag{4.22}$$

where

$$\begin{aligned}
Z_{11}^{(1)} &= \frac{\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)} \frac{\Gamma(-ir\epsilon_1+ir\epsilon_2)\Gamma(2ir\epsilon_1)}{\Gamma(1+ir\epsilon_1-ir\epsilon_2)\Gamma(1-2ir\epsilon_1)} \\
Z_{\mathrm{v}}^{(1)} &= {}_2F_1\left(\begin{array}{c} \{ir\epsilon, 2ir\epsilon_1\} \\ \{1+ir\epsilon_1-ir\epsilon_2\} \end{array} ; z \right) \\
Z_{\mathrm{av}}^{(1)} &= {}_2F_1\left(\begin{array}{c} \{ir\epsilon, 2ir\epsilon_1\} \\ \{1+ir\epsilon_1-ir\epsilon_2\} \end{array} ; \bar{z} \right)
\end{aligned} \tag{4.23}$$

The other contribution is obtained by exchanging $\epsilon_1 \longleftrightarrow \epsilon_2$. By identifying $Z_{\mathrm{v}}^{(1)}$ as the Givental $\mathcal{I}$-function, we expand it in $r = \frac{1}{\hbar}$ in order to find the equivariant mirror map; this gives

$$Z_{\mathrm{v}}^{(1)} = 1 + o(r^2), \tag{4.24}$$

which means there is no equivariant mirror map and $\mathcal{I} = \mathcal{J}$. The same applies to $Z_{\mathrm{v}}^{(2)}$.

Therefore, the only normalization to be dealt with is the one of the symplectic pairing, namely Z_{11}. We already encountered this problem in section 3.2.3; let us see how to solve it also in this example. In (4.22), $Z_{11}^{(1)}$ and $Z_{11}^{(2)}$ contain an excess of $4ir(\epsilon_1 + \epsilon_2)$

in the argument of the Gamma functions ($2ir(\epsilon_1+\epsilon_2)$ from the numerator and another $2ir(\epsilon_1+\epsilon_2)$ from the denominator); this would imply

$$Z_{11}^{(1)} = -\frac{1}{2\epsilon_1^2\epsilon_2(\epsilon_1-\epsilon_2)r^4} + \frac{2i\gamma\epsilon}{\epsilon_1^2\epsilon_2(\epsilon_1-\epsilon_2)r^3} + o(r^{-2}) \tag{4.25}$$

and similarly for $Z_{11}^{(2)}$. To eliminate the Euler-Mascheroni constant, we normalize the partition function multiplying it by[4]

$$(z\bar{z})^{-2ira}\left(\frac{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)}{\Gamma(1+ir\epsilon_1)\Gamma(1+ir\epsilon_2)}\right)^2 \tag{4.26}$$

so that now we have

$$\left(\frac{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)}{\Gamma(1+ir\epsilon_1)\Gamma(1+ir\epsilon_2)}\right)^2 Z_{11}^{(1)} = -\frac{1}{2\epsilon_1^2\epsilon_2(\epsilon_1-\epsilon_2)r^4} + o(r^{-2}) \tag{4.27}$$

Expanding the normalized partition function in r up to order r^{-1}, we obtain[5]

$$\begin{aligned} Z_{1,2}^{\text{norm}} = \frac{1}{r^2\epsilon_1\epsilon_2}\Big[&\frac{1}{2r^2\epsilon_1\epsilon_2} + \frac{1}{4}\ln^2(z\bar{z}) - ir(\epsilon_1+\epsilon_2)\Big(-\frac{1}{12}\ln^3(z\bar{z}) \\ &- \ln(z\bar{z})(\mathrm{Li}_2(z)+\mathrm{Li}_2(\bar{z})) + 2(\mathrm{Li}_3(z)+\mathrm{Li}_3(\bar{z})) + 3\zeta(3)\Big)\Big] \end{aligned} \tag{4.28}$$

The first term in (4.28) correctly reproduces the Nekrasov partition function of $\mathcal{M}_{1,2}$ as expected, while the other terms compute the $H_T^2(X)$ part of the genus zero Gromov-Witten potential in agreement with [104]. We remark that the quantum part of the Gromov-Witten potential turns out to be linear in the equivariant parameter $\epsilon_1+\epsilon_2$ as inferred in section 4.2.2 from general arguments.

We can also compute it with the Givental formalism: expanding the J function up to order r^2, one finds

$$J = 1 + r^2(-\epsilon_1\epsilon_2 - i(\epsilon_1+\epsilon_2)\lambda_1 + \lambda_1^2)\mathrm{Li}_2(z) + o(r^3) \tag{4.29}$$

and the coefficient of $-\lambda_1$ – which is the cohomology generator – at order r^2 will give the first z derivative of the prepotential.

[4]The normalization here has been chosen having in mind the $\mathcal{M}_{2,1}$ case; see the next paragraph.

[5]Notice that the procedure outlined above does not fix a remnant dependence on the coefficient of the $\zeta(3)$ term in Z^{S^2}. In fact, one can always multiply by a ratio of Gamma functions whose overall argument is zero; this will have an effect only on the $\zeta(3)$ coefficient. This ambiguity does not affect the calculation of the Gromov-Witten invariants.

4.3.2 Hilbert scheme of points

Let us now turn to the $\mathcal{M}_{k,1}$ case, which corresponds to the Hilbert scheme of k points. This case was analysed in terms of Givental's formalism in [105]. It is easy to see that (4.17) reduces for $N = 1$ to their results. As remarked after equation (4.17) in the $N = 1$ case there is a non-trivial equivariant mirror map to be implemented. As we will discuss in a moment, this is done by defining the $\mathcal{J}$ function as $\mathcal{J} = (1+z)^{irk\epsilon}\mathcal{I}$, which corresponds to invert the equivariant mirror map; in other words, we have to normalize the vortex part by multiplying it with $(1+z)^{irk\epsilon}$, and similarly for the antivortex. In the following we will describe in detail some examples and extract the relevant Gromov-Witten invariants for them. As we will see, these are in agreement with the results of [106].

For $k = 1$, the only Young tableau ($\square$) corresponds to the pole $\lambda_1 = -ia$. This case is simple enough to be written in a closed form; we find

$$Z_{1,1}^{S^2} = (z\bar{z})^{ira}\frac{\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)}(1+z)^{-ir\epsilon}(1+\bar{z})^{-ir\epsilon} \tag{4.30}$$

From this expression, it is clear that the Gromov-Witten invariants are vanishing. Actually, we should multiply (4.30) by $(1+z)^{ir\epsilon}(1+\bar{z})^{ir\epsilon}$ in order to recover the $\mathcal{J}$-function. Instead of doing this, we propose to use $Z_{1,1}^{S^2}$ as a normalization for $Z_{k,1}^{S^2}$ as

$$Z_{k,1}^{\text{norm}} = \frac{Z_{k,1}^{S^2}}{(-r^2\epsilon_1\epsilon_2 Z_{1,1}^{S^2})^k} \tag{4.31}$$

In this way, we go from $\mathcal{I}$ to $\mathcal{J}$ functions and at the same time we normalize the 1-loop factor in such a way to erase the Euler-Mascheroni constant. The factor $(-r^2\epsilon_1\epsilon_2)^k$ is to make the normalization factor to start with 1 in the r expansion. In summary, we obtain

$$Z_{1,1}^{\text{norm}} = -\frac{1}{r^2\epsilon_1\epsilon_2} \tag{4.32}$$

Let us make a comment on the above normalization procedure. From the general arguments previously discussed we expect the normalization to be independent on λ's. Moreover, from the field theory point of view, the normalization (4.31) is natural since amounts to remove from the free energy the contribution of k free particles. On the other hand, this is non trivial at all from the explicit expression of the $\mathcal{I}$-function (4.17). Actually, a remarkable combinatorial identity proved in [105] ensures that $e^{-I^{(1)}/\hbar} = (1+z)^{ik(\epsilon_1+\epsilon_2)/\hbar}$ and then makes this procedure consistent.

Let us now turn to the $\mathcal{M}_{2,1}$ case. There are two contributions, ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) and ($\square\square$), coming respectively from the poles $\lambda_1 = -ia, \lambda_2 = -ia - i\epsilon_1$ and $\lambda_1 = -ia, \lambda_2 = -ia - i\epsilon_2$.

Notice once more that the permutations of the λ's are cancelled against the $\frac{1}{k!}$ in front of the partition function (4.2). We thus have

$$Z_{2,1}^{S^2} = (z\bar{z})^{ir(2a+\epsilon_1)} Z_{11}^{(\mathrm{col})} Z_{\mathrm{v}}^{(\mathrm{col})} Z_{\mathrm{av}}^{(\mathrm{col})} + (z\bar{z})^{ir(2a+\epsilon_2)} Z_{11}^{(\mathrm{row})} Z_{\mathrm{v}}^{(\mathrm{row})} Z_{\mathrm{av}}^{(\mathrm{row})} \tag{4.33}$$

where, explicitly,

$$\begin{aligned}
Z_{11}^{(\mathrm{col})} &= \frac{\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)} \frac{\Gamma(2ir\epsilon_1)\Gamma(ir\epsilon_2-ir\epsilon_1)}{\Gamma(1-2ir\epsilon_1)\Gamma(1+ir\epsilon_1-ir\epsilon_2)} \\
Z_{\mathrm{v}}^{(\mathrm{col})} &= \sum_{\tilde{d}\geqslant 0}(-z)^{\tilde{d}} \sum_{\tilde{d}_1=0}^{\tilde{d}/2} \frac{(1+ir\epsilon_1)_{\tilde{d}-2\tilde{d}_1}}{(ir\epsilon_1)_{\tilde{d}-2\tilde{d}_1}} \frac{(ir\epsilon)_{\tilde{d}_1}}{\tilde{d}_1!} \frac{(ir\epsilon_1+ir\epsilon)_{\tilde{d}-\tilde{d}_1}}{(1+ir\epsilon_1)_{\tilde{d}-\tilde{d}_1}} \\
&\quad \frac{(2ir\epsilon_1)_{\tilde{d}-2\tilde{d}_1}}{(\tilde{d}-2\tilde{d}_1)!} \frac{(1-ir\epsilon_2)_{\tilde{d}-2\tilde{d}_1}}{(ir\epsilon_1+ir\epsilon)_{\tilde{d}-2\tilde{d}_1}} \frac{(ir\epsilon)_{\tilde{d}-2\tilde{d}_1}}{(1+ir\epsilon_1-ir\epsilon_2)_{\tilde{d}-2\tilde{d}_1}} \\
Z_{\mathrm{av}}^{(\mathrm{col})} &= \sum_{d\geqslant 0}(-\bar{z})^{d} \sum_{d_1=0}^{d/2} \frac{(1+ir\epsilon_1)_{d-2d_1}}{(ir\epsilon_1)_{d-2d_1}} \frac{(ir\epsilon)_{d_1}}{d_1!} \frac{(ir\epsilon_1+ir\epsilon)_{d-d_1}}{(1+ir\epsilon_1)_{d-d_1}} \\
&\quad \frac{(2ir\epsilon_1)_{d-2d_1}}{(d-2d_1)!} \frac{(1-ir\epsilon_2)_{d-2d_1}}{(ir\epsilon_1+ir\epsilon)_{d-2d_1}} \frac{(ir\epsilon)_{d-2d_1}}{(1+ir\epsilon_1-ir\epsilon_2)_{d-2d_1}}
\end{aligned} \tag{4.34}$$

Here we defined $d = d_1 + d_2$ and changed the sums accordingly. The row contribution can be obtained from the column one by exchanging $\epsilon_1 \longleftrightarrow \epsilon_2$. We then have

$$Z_{\mathrm{v}}^{(\mathrm{col,\ row})} = 1 + 2ir\epsilon \mathrm{Li}_1(-z) + o(r^2) \tag{4.35}$$

Finally, we invert the equivariant mirror map by replacing

$$\begin{aligned}
Z_{\mathrm{v}}^{(\mathrm{col,\ row})} &\longrightarrow e^{-2ir\epsilon \mathrm{Li}_1(-z)} Z_{\mathrm{v}}^{(\mathrm{col,\ row})} = (1+z)^{2ir\epsilon} Z_{\mathrm{v}}^{(\mathrm{col,\ row})} \\
Z_{\mathrm{av}}^{(\mathrm{col,\ row})} &\longrightarrow e^{-2ir\epsilon \mathrm{Li}_1(-\bar{z})} Z_{\mathrm{av}}^{(\mathrm{col,\ row})} = (1+\bar{z})^{2ir\epsilon} Z_{\mathrm{av}}^{(\mathrm{col,\ row})}
\end{aligned} \tag{4.36}$$

Now we can prove the equivalence $\mathcal{M}_{1,2} \simeq \mathcal{M}_{2,1}$: by expanding in z, it can be shown that $Z_{\mathrm{v}}^{(1)}(z) = (1+z)^{2ir\epsilon} Z_{\mathrm{v}}^{(\mathrm{col})}(z)$ and similarly for the antivortex part; since $Z_{11}^{(1)} = Z_{11}^{(\mathrm{col})}$ we conclude that $Z^{(1)}(z,\bar{z}) = (1+z)^{2ir\epsilon}(1+\bar{z})^{2ir\epsilon} Z^{(\mathrm{col})}(z,\bar{z})$. The same is valid for $Z^{(2)}$ and $Z^{(\mathrm{row})}$, so in the end we obtain

$$Z_{1,2}^{S^2}(z,\bar{z}) = (1+z)^{2ir\epsilon}(1+\bar{z})^{2ir\epsilon} Z_{2,1}^{S^2}(z,\bar{z}) \tag{4.37}$$

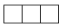
Taking into account the appropriate normalizations, this implies

$$Z_{1,2}^{\mathrm{norm}}(z,\bar{z}) = Z_{2,1}^{\mathrm{norm}}(z,\bar{z}) \quad . \tag{4.38}$$

As further examples, we will briefly comment about the $\mathcal{M}_{3,1}$ and $\mathcal{M}_{4,1}$ cases. For $\mathcal{M}_{3,1}$ there are three contributions to the partition function:

 from the poles $\lambda_1 = -ia,\ \lambda_2 = -ia - i\epsilon_1,\ \lambda_3 = -ia - 2i\epsilon_1$

 from the poles $\lambda_1 = -ia,\ \lambda_2 = -ia - i\epsilon_1,\ \lambda_3 = -ia - i\epsilon_1 - i\epsilon_2$

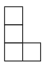
 from the poles $\lambda_1 = -ia,\ \lambda_2 = -ia - i\epsilon_2,\ \lambda_3 = -ia - 2i\epsilon_2$

The study of the vortex contributions tells us that there is an equivariant mirror map, which has to be inverted; however, this is taken into account by the normalization factor. Then, the r expansion gives

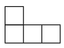
$$Z_{3,1}^{\text{norm}} = \frac{1}{r^4(\epsilon_1\epsilon_2)^2}\Big[-\frac{1}{6r^2\epsilon_1\epsilon_2} - \frac{1}{4}\ln^2(z\bar{z}) + ir(\epsilon_1+\epsilon_2)\Big(-\frac{1}{12}\ln^3(z\bar{z}) \\ - \ln(z\bar{z})(\text{Li}_2(z) + \text{Li}_2(\bar{z})) + 2(\text{Li}_3(z) + \text{Li}_3(\bar{z})) + 3\zeta(3)\Big)\Big] \tag{4.39}$$

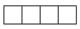
For $\mathcal{M}_{4,1}$ we have five contributions:

 from $\lambda_1 = -ia,\ \lambda_2 = -ia - i\epsilon_1,\ \lambda_3 = -ia - 2i\epsilon_1,\ \lambda_4 = -ia - 3i\epsilon_1$

 from $\lambda_1 = -ia,\ \lambda_2 = -ia - i\epsilon_1,\ \lambda_3 = -ia - 2i\epsilon_1,\ \lambda_4 = -ia - i\epsilon_2$

 from $\lambda_1 = -ia,\ \lambda_2 = -ia - i\epsilon_1,\ \lambda_3 = -ia - i\epsilon_2,\ \lambda_4 = -ia - i\epsilon_1 - i\epsilon_2$

 from $\lambda_1 = -ia,\ \lambda_2 = -ia - i\epsilon_2,\ \lambda_3 = -ia - 2i\epsilon_2,\ \lambda_4 = -ia - i\epsilon_1$

 from $\lambda_1 = -ia,\ \lambda_2 = -ia - i\epsilon_2,\ \lambda_3 = -ia - 2i\epsilon_2,\ \lambda_4 = -ia - 3i\epsilon_2$

Again, we normalize and expand in r to obtain

$$Z_{4,1}^{\text{norm}} = -\frac{1}{r^6(\epsilon_1\epsilon_2)^3}\Big[-\frac{1}{24r^2\epsilon_1\epsilon_2} - \frac{1}{8}\ln^2(z\bar{z}) + ir(\epsilon_1+\epsilon_2)\Big(-\frac{1}{24}\ln^3(z\bar{z}) \\ - \ln(z\bar{z})(\frac{1}{2}\text{Li}_2(z) + \frac{1}{2}\text{Li}_2(\bar{z})) + 2(\frac{1}{2}\text{Li}_3(z) + \frac{1}{2}\text{Li}_3(\bar{z})) + \frac{3}{2}\zeta(3)\Big)\Big] \tag{4.40}$$

4.3.3 A last example

As a last example, let us consider $\mathcal{M}_{2,2}$. In this case, five Young tableaux are contributing:

$$\begin{array}{lll}
\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \bullet\right) & \text{from the poles} & \lambda_1 = -ia_1,\ \lambda_2 = -ia_1 - i\epsilon_1 \\
(\square\square, \bullet) & \text{from the poles} & \lambda_1 = -ia_1,\ \lambda_2 = -ia_1 - i\epsilon_2 \\
\left(\bullet, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) & \text{from the poles} & \lambda_1 = -ia_2,\ \lambda_2 = -ia_2 - i\epsilon_1 \\
(\bullet, \square\square) & \text{from the poles} & \lambda_1 = -ia_2,\ \lambda_2 = -ia_2 - i\epsilon_2 \\
(\square, \square) & \text{from the poles} & \lambda_1 = -ia_1,\ \lambda_2 = -ia_2
\end{array}$$

The order r coefficient in the expansion of the various vortex partition functions is zero, so there is no equivariant mirror map to be inverted. As normalization, we will choose the simplest one, that is we multiply by a factor

$$(z\bar{z})^{ir(\epsilon_1+\epsilon_2-a_1-a_2)} \left(\frac{\Gamma(1-ir\epsilon_1)\Gamma(1-ir\epsilon_2)}{\Gamma(1+ir\epsilon_1)\Gamma(1+ir\epsilon_2)}\right)^4 \tag{4.41}$$

The expansion then gives

$$\begin{aligned}
Z_{2,2}^{\text{norm}} = & \frac{1}{r^6(\epsilon_1\epsilon_2)^2((\epsilon_1+\epsilon_2)^2-(a_1-a_2)^2)} \\
& \Big[\frac{8(\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2-2(a_1-a_2)^2}{r^2((2\epsilon_1+\epsilon_2)^2-(a_1-a_2)^2)((\epsilon_1+2\epsilon_2)^2-(a_1-a_2)^2)} \\
& + \frac{1}{2}\ln^2(z\bar{z}) - ir(\epsilon_1+\epsilon_2)\Big(-\frac{1}{6}\ln^3(z\bar{z}) - \ln(z\bar{z})(2\text{Li}_2(z)+2\text{Li}_2(\bar{z})) \\
& + 2(2\text{Li}_3(z)+2\text{Li}_3(\bar{z})) + c(\epsilon_i,a_i)\zeta(3)\Big)\Big]
\end{aligned} \tag{4.42}$$

where

$$c(\epsilon_i,a_i) = 8 - \frac{4\epsilon_1\epsilon_2(\epsilon_1\epsilon_2+2(\epsilon_1+\epsilon_2)^2+4(a_1-a_2)^2)}{((2\epsilon_1+\epsilon_2)^2-(a_1-a_2)^2)((\epsilon_1+2\epsilon_2)^2-(a_1-a_2)^2)} \tag{4.43}$$

4.3.4 Orbifold cohomology of the ADHM moduli space

We saw in this chapter that the equivariant quantum cohomology of the ADHM moduli space is encoded in the $\mathcal{I}$-function (4.17). The purpose of this section is to use the wallcrossing approach developed in chapter 3 to analyse the equivariant quantum cohomology of the Uhlembeck (partial) compactification of the moduli space of instantons by tuning the FI parameter ξ of the GLSM to zero. Indeed, as we will shortly discuss, in this case there is a reflection symmetry $\xi \to -\xi$ showing that the sign of the FI is not relevant to fix the phase of the GLSM. Actually, fixing $\xi = 0$ allows pointlike instantons. This produces a conjectural formula for the $\mathcal{I}$-function of the ADHM space in the orbifold chamber. In particular for rank one instantons, namely Hilbert schemes of points, our results are in agreement with those in [104].

Let us recall some elementary aspects on the moduli space $\mathcal{M}_{k,N}$ of k $SU(N)$ instantons on $\mathbb{C}^2$. This space is non-compact both because the manifold $\mathbb{C}^2$ is non-compact and because of point-like instantons. The first source of non-compactness is cured by the introduction of the so-called Ω-background which, mathematically speaking, corresponds to working in the equivariant cohomology with respect to the maximal torus of rotations on $\mathbb{C}^2$. The second one can be approached in different ways. A compactification scheme is provided by the Uhlembeck one

$$\mathcal{M}^U_{k,N} = \bigsqcup_{l=0}^{k} \mathcal{M}_{k-l,N} \times S^l\left(\mathbb{C}^2\right) \tag{4.44}$$

Due to the presence of the symmetric product factors this space contains orbifold singularities. A desingularization is provided by the moduli space of torsion free sheaves on $\mathbb{P}^2$ with a framing on the line at infinity. This is described in terms of the ADHM complex linear maps $(B_1, B_2) : \mathbb{C}^k \to \mathbb{C}^k$ and $(I, J^\dagger) : \mathbb{C}^k \to \mathbb{C}^N$ which satisfy the F-term equation

$$[B_1, B_2] + IJ = 0$$

and the D-term equation

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \xi \mathbb{I}$$

where ξ is a parameter that gets identified with the FI parameter of the GLSM and that ensures the stability condition of the sheaf.

Notice that the ADHM equations are symmetric under the reflection $\xi \to -\xi$ and

$$(B_i, I, J) \to (B_i^\dagger, -J^\dagger, I^\dagger)$$

The Uhlembeck compactification is recovered in the $\xi \to 0$ limit. This amounts to set the vortex expansion parameter as

$$(-1)^N z = e^{i\theta} \tag{4.45}$$

giving therefore the orbifold $\mathcal{I}$-function

$$\begin{aligned}\mathcal{I}^U_{k,N} = \sum_{d_1,\ldots,d_k \geq 0} (e^{i\theta})^{d_1+\ldots+d_k} \prod_{r=1}^{k} \prod_{j=1}^{N} \frac{(-r\lambda_r - ira_j + ir\epsilon)_{d_r}}{(1 - r\lambda_r - ira_j)_{d_r}} \prod_{r<s}^{k} \frac{d_s - d_r - r\lambda_s + r\lambda_r}{-r\lambda_s + r\lambda_r} \\ \frac{(1 + r\lambda_r - r\lambda_s - ir\epsilon)_{d_s-d_r}}{(r\lambda_r - r\lambda_s + ir\epsilon)_{d_s-d_r}} \frac{(r\lambda_r - r\lambda_s + ir\epsilon_1)_{d_s-d_r}}{(1 + r\lambda_r - r\lambda_s - ir\epsilon_1)_{d_s-d_r}} \frac{(r\lambda_r - r\lambda_s + ir\epsilon_2)_{d_s-d_r}}{(1 + r\lambda_r - r\lambda_s - ir\epsilon_2)_{d_s-d_r}}\end{aligned} \tag{4.46}$$

In the abelian case, namely for $N = 1$, the above $\mathcal{I}$-function reproduces the results of [104] for the equivariant quantum cohomology of the symmetric product of k points in $\mathbb{C}^2$. Indeed, by using the map to the Fock space formalism for the equivariant quantum cohomology developed in appendix B, it is easy to see that both approaches produce the same small equivariant quantum cohomology. Notice that the map (5.70) reproduces in the $N = 1$ case the one of [104].

4.4 Donaldson-Thomas theory and stringy corrections to the Seiberg-Witten prepotential

It is very interesting to analyse our system also from the D5-brane dynamics point of view. This is a six-dimensional theory which should be related to higher rank equivariant Donaldson-Thomas theory on $\mathbb{C}^2 \times \mathbb{P}^1$. Indeed an interesting and promising aspect is that for $N > 1$ the D1 contributions to the D5 gauge theory dynamics do not factor in abelian $N = 1$ terms and thus keep an intrinsic non-abelian nature, contrary to what happens for the D(-1) contributions in the Coulomb phase [107].

To clarify this connection, let us notice that a suitable framework to compactify the Donaldson-Thomas moduli space was introduced in [108] via ADHM moduli sheaves. In this context one can show that $\mathcal{I}_{k,1} = \mathcal{I}_{DT}$. Moreover the $\mathcal{I}_{k,1}$-function reproduces the 1-legged Pandharipande-Thomas vertex as in [109] for the case of the Hilbert scheme of points of $\mathbb{C}^2$, while the more general ADHM case should follow as the generalization to higher rank. The case of the Hilbert scheme of points is simpler and follows by [110].

The partition function of the D1-branes computed in the previous sections provides non-perturbative corrections to the D5-brane dynamics. It is then natural to resum the D1-brane contributions as

$$Z_N^{DT} = \sum_k q^{2kN} Z_{k,N}^{hol} = \sum_{k,\beta} N_{k,\beta} q^{2kN} z^{\beta} \tag{4.47}$$

where $q = e^{2\pi i \tau}$ and in the second equality we considered the expansion in z of the holomorphic part of the spherical partition function, where $\beta \in H_2(\mathcal{M}_{k,N}, \mathbb{Z})$.

It is interesting to study the free-energy of the above defined partition function and its reduction in the four dimensional blow-down limit $r \to 0$. Indeed, let us observe that the D5 brane theory in this limit is described by an effective four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory at energies below the UV cutoff provided by the inverse radius of the blown-up sphere $1/r$ [111]. Comparing the expansion (4.47) to the results of section (4.2.1), we obtain that the former reduces to the standard Nekrasov instanton partition function upon the identification $q = \Lambda r$. Moreover, keeping into account the limiting behaviour as $\epsilon_i \sim 0$ we have just discussed in the previous subsection, namely that $Z^{S^2}_{k,N}$ has the same divergent behaviour as $Z^{Nek}_{k,N}$ due to the equivariant regularization of the $\mathbb{R}^4$ volume $\frac{1}{\epsilon_1\epsilon_2}$, one can present the resummed partition function (4.47) in the form

$$Z_N^{DT} = \exp\left\{-\frac{1}{\epsilon_1\epsilon_2}\mathcal{E}(a,\epsilon_i,\Lambda;r,z)\right\} \tag{4.48}$$

where $\mathcal{E}$ is the total free energy of the system and is a regular function as $\epsilon_i \sim 0$. The effective geometry arising in the semiclassical limit $\epsilon_1, \epsilon_2 \to 0$ of (4.48) would provide information about the mirror variety encoding the enumerative invariants in (4.47).

In order to pursue this program it is crucial to complement our analysis by including the perturbative sector of the N D5-brane theory in the geometry $\mathbb{C}^2 \times T^*\mathbb{P}^1 \times \mathbb{C}$ whose world-volume theory is described at low-energy by an $\mathcal{N} = 1$ super Yang-Mills theory in six dimensions on $\mathbb{C}^2 \times \mathbb{P}^1$. The perturbative contribution can be computed by considering the dimensional reduction down to the two-sphere. This gives rise to an $\mathcal{N} = (4,4)$ supersymmetric gauge theory, containing three chiral multiplets in the adjoint representation with lowest components (Z_i, Φ), $i = 1,2$, where Z_1, Z_2 and Φ describe the fluctuations along $\mathbb{C}^2$ and $\mathbb{C}$ respectively. Around the flat connection, the vacua are described by covariantly constant fields $D_{\text{adj}(\Phi)}Z_i = 0$ satisfying

$$[Z_1, Z_2] = 0 \tag{4.49}$$

The Cartan torus of the rotation group acts as $(Z_1, Z_2) \to (e^{-\epsilon_1}Z_1, e^{-\epsilon_2}Z_2)$ preserving the above constraints. The one-loop fluctuation determinants for this theory are given by

$$\frac{\det(D_{\text{adj}(\Phi)})\det(D_{\text{adj}(\Phi)}+\epsilon_1+\epsilon_2)}{\det(D_{\text{adj}(\Phi)}+\epsilon_1)\det(D_{\text{adj}(\Phi)}+\epsilon_2)} \quad . \tag{4.50}$$

The zeta function regularization of the above ratio of determinants reads

$$\exp\left[-\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^\infty \frac{dt}{t^{1-s}}\ \text{tr}\ e^{tD_{\text{adj}(\Phi)}}(1-e^{\epsilon_1 t})(1-e^{\epsilon_2 t})\right]_{s=0} \tag{4.51}$$

which can be seen as the regularization of the infinite product

$$\prod_{j,k}^{\infty}\prod_{l\neq m}\left(\frac{\Gamma\left(1-ir(a_{lm}-j\epsilon_1-k\epsilon_2)\right)}{ir\Gamma\left(ir(a_{lm}-j\epsilon_1-k\epsilon_2)\right)}\right)^{-1} \tag{4.52}$$

The above formula is a deformation of the standard formula expressing the perturbative part of the Nekrasov partition function

$$Z^{pert}_{Nek} = \prod_{l\neq m}\prod_{j,k\geq 1} X^{-1}_{lm,j,k} = \prod_{l\neq m}\Gamma_2(a_{lm};\epsilon_1,\epsilon_2) \tag{4.53}$$

with $X_{lm,j,k} = a_{lm} - j\epsilon_1 - k\epsilon_2$, in terms of Barnes double Γ-function [10] (see also [112]). Eq.(4.52) is obtained by resumming the Kaluza-Klein modes on the two-sphere over each four dimensional gauge theory mode organized in spherical harmonics $SU(2)$ multiplets. This can be done by applying the methods in [27] to each tower before boson/fermion cancellation.

More in detail, the one-loop contribution of the D5-D5 partition function on Ω-background can be calculated by making use of the equivariant index theorem for the linearized kinetic operator of the quantum fluctuations in six dimensions. The low-energy field theory on the D5-branes is given by (twisted) maximally supersymmetric Yang-Mills theory on $\mathbb{C}^2 \times S^2$. The relevant complex is the $\bar{\partial}$ Dolbeaux complex [113]

$$0 \to \Omega^{(0,0)} \to \Omega^{(0,1)} \to \Omega^{(0,2)} \to 0 \tag{4.54}$$

The equivariant index of the above complex is given by

$$\frac{(1-t_1^{-1}-t_2^{-1}+t_1^{-1}t_2^{-1})}{(1-t_1^{-1})(1-t_2^{-1})(1-t_1)(1-t_2)}\left(-\frac{t_3}{(1-t_3)}\right)\sum_{l,m} e^{ia_{lm}} \tag{4.55}$$

where we used Künneth decomposition of the cohomology groups of $\mathbb{C}^2 \times S^2$. The first factor computes the equivariant index of the $\bar{\partial}$ operator on $\mathbb{C}^2$, the second that of S^2, while the third factor the twisting by the gauge bundle in the adjoint representation. From (4.55) one can easily compute the ratio of determinants of the one-loop fluctuations via the substitution rûle relating the equivariant index with the equivariant Euler characteristic of the complex:

$$\sum_\alpha c_\alpha e^{w_\alpha} \quad \to \quad \prod_\alpha w_\alpha^{c_\alpha} \tag{4.56}$$

where w_α are the weights of the equivariant action and c_α their multiplicities. Here $t_1 = e^{i\epsilon_1}$, $t_2 = e^{i\epsilon_2}$ and $t_3 = e^{i\epsilon_3}$ with $\epsilon_3 = \sqrt{-1}/r$.

In order to extract the above data from Eq.(4.55), we expand the $\mathbb{C}^2$ factor as

$$\sum_{i,j,\bar{i},\bar{j}=0}^{\infty} \left(1-t_1^{-1}-t_2^{-1}+t_1^{-1}t_2^{-1}\right) t_1^{i-\bar{i}} t_2^{j-\bar{j}} \tag{4.57}$$

and the S^2 factor in the two patches as

$$-\sum_{k=0}^{\infty} t_3^{1+k} \tag{4.58}$$

at the north pole and as

$$\sum_{k=0}^{\infty}\left(t_3^{-1}\right)^k \tag{4.59}$$

at the south pole. Then the product of the eigenvalues is given by

$$\prod_{i,j,\bar{i},\bar{j}=0}^{\infty}\frac{\Gamma(a_{lm}+\epsilon_1(i-\bar{i})+\epsilon_2(j-\bar{j})+\epsilon_3)}{\Gamma(1-a_{lm}-\epsilon_1(i-\bar{i})-\epsilon_2(j-\bar{j})+\epsilon_3)}$$
$$\left(\frac{\Gamma(a_{lm}+\epsilon_1(i-\bar{i}-1)+\epsilon_2(j-\bar{j})+\epsilon_3)}{\Gamma(1-a_{lm}-\epsilon_1(i-\bar{i}-1)-\epsilon_2(j-\bar{j})+\epsilon_3)}\right)^{-1}$$
$$\left(\frac{\Gamma(a_{lm}+\epsilon_1(i-\bar{i})+\epsilon_2(j-\bar{j}-1)+\epsilon_3)}{\Gamma(1-a_{lm}-\epsilon_1(i-\bar{i})-\epsilon_2(j-\bar{j}-1)+\epsilon_3)}\right)^{-1}$$
$$\frac{\Gamma(a_{lm}+\epsilon_1(i-\bar{i}-1)+\epsilon_2(j-\bar{j}-1)+\epsilon_3)}{\Gamma(1-a_{lm}-\epsilon_1(i-\bar{i}-1)-\epsilon_2(j-\bar{j}-1)+\epsilon_3)}$$

where we used the Weierstrass formula for the Γ function performing the product over the index k in (4.58,4.59). The above product simplifies then to

$$Z_{D5-D5}^{S^2}=\prod_{l\neq m}\Gamma_2(a_{lm};\epsilon_1,\epsilon_2)\frac{\Gamma_3\left(a_{lm};\epsilon_1,\epsilon_2,\frac{1}{ir}\right)}{\Gamma_3\left(a_{lm};\epsilon_1,\epsilon_2,-\frac{1}{ir}\right)} \tag{4.60}$$

and implements the finite r corrections to the perturbative Nekrasov partition function. The equality in (4.60) follows by regularizing the infinite set of poles of the ratio of Γ functions. Indeed by using the standard properties of the Γ-function it is easy to see that (4.60) reduces in the $r\to 0$ limit to (4.53) plus corrections expressible in power series in r and ϵ_1,ϵ_2. The leading order term in the small r expansion of (4.60) is (4.53). The first non vanishing correction in the expansion can be computed by expanding

$$\ln\left[\frac{\Gamma(1-irX)}{\Gamma(1+irX)}\right]=2\gamma iXr-\frac{2}{3}iX^3\zeta(3)r^3+O(r^5) \tag{4.61}$$

in (4.52), where γ is the Euler-Mascheroni constant. Carrying the product to a sum at the exponent and using zeta-function regularization for the infinite sums, one gets

$$\ln\left[\frac{Z_{D5-D5}^{S^2}}{Z_{Nek}^{pert}}\right]=-\gamma ir\frac{N(N-1)}{12}\epsilon+\frac{1}{12}i\zeta(3)r^3\left(\sum_{l\neq n}a_{ln}^2-\frac{N(N-1)}{30}(\epsilon_1^2-\epsilon_1\epsilon_2+\epsilon_2^2)\right)\epsilon+O(r^5)$$

where the first term is a regularization scheme dependent constant. We see that the first correction affects the quadratic part of the prepotential implying a modification of the beta function of the theory which keeps into account the contributions of the KK-momenta on the $\mathbb{P}^1$.

We thus conclude that in the limit $r\to 0$, $\mathcal{E}\to\mathcal{F}_{Nek}$ the Nekrasov prepotential of the $\mathcal{N}=2$ gauge theory in the Ω-background. Therefore for $r\to 0$ the effective geometry arising in the semiclassical limit of (4.48) is the Seiberg-Witten curve of pure $\mathcal{N} = 2$ super Yang-Mills [10]. Higher order corrections in r to this geometry encode the effect of stringy corrections. Indeed, the total free energy contains additional world-sheet corrections in z and therefore

$$\mathcal{E} = \mathcal{F}^{Nek}(a, \epsilon_i, \Lambda) + \mathcal{F}^{stringy}(a, \epsilon_i, \Lambda; r, z)$$

These are genuine string corrections to the $\mathcal{N} = 2$ gauge theory in the Ω-background describing the finite radius effects of the blown-up sphere resolving the A_1 orbifold singularity. Let us notice that $\mathcal{F}^{stringy}$ is higher order in the ϵ_i expansion with respect to $\mathcal{F}^{Nek}$, therefore, in this scaling scheme, the resulting Seiberg-Witten limit $\lim_{\epsilon_i \to 0} \mathcal{E} = \mathcal{F}^{SW}$ is unchanged.

As we discussed previously, the stringy contributions are given by a classical term describing the equivariant classical intersection theory in the ADHM moduli space and a world-sheet instanton contribution describing its quantum deformation, that is

$$\mathcal{F}^{stringy}(a, \epsilon_i, \Lambda; r, z) = \mathcal{F}^{stringy}_{cl}(\epsilon_i; r, z) + \epsilon \mathcal{F}^{stringy}_{ws}(a, \epsilon_i, \Lambda; r, z). \tag{4.62}$$

Following [114] we can consider the effect of a partial Ω-background by studying the limit $\epsilon_2 \to 0$ in the complete free energy. Defining

$$\mathcal{V} = \lim_{\epsilon_2 \to 0} \frac{1}{\epsilon_2} \ln Z_N^{DT} \tag{4.63}$$

we find that

$$\mathcal{W} = \mathcal{W}^{NS} + \mathcal{W}^{stringy} \tag{4.64}$$

where $\mathcal{W}^{NS}$ is the Nekrasov-Shatashvili twisted superpotential of the reduced two dimensional gauge theory and $\mathcal{W}^{stringy}$ are its stringy corrections. According to [114], $\mathcal{W}^{NS}$ can be interpreted as the Yang-Yang function of the quantum integrable Hitchin system on the M-theory curve (the sphere with two maximal punctures for the pure $\mathcal{N} = 2$ gauge theory). The superpotential $\mathcal{W}$ should be related to the quantum deformation of the relevant integrable system underlying the classical Seiberg-Witten geometry [96].

4.5 Quantum hydrodynamic systems

As we discussed in section 4.1, the ADHM GLSM we studied in the first part of this chapter is intimately related to a quantum integrable system of hydrodynamic type known as the Intermediate Long Wave system. Here we will describe the basic concepts about hydrodynamic systems which will be needed in the following. In subsection 4.5.1 we recall some basic facts about $gl(N)$ ILW integrable hydrodynamics relevant for the comparison with the six dimensional $U(N)$ gauge theory, focussing on the $N = 1$ case. In the subsequent subsection 4.5.2 we show that the ILW system can be obtained as a hydrodynamic limit of the elliptic Calogero-Moser system.

4.5.1 The Intermediate Long Wave system

One of the most popular integrable systems is the KdV equation

$$u_t = 2uu_x + \frac{\delta}{3}u_{xxx} \tag{4.65}$$

where $u = u(x,t)$ is a real function of two variables. It describes the surface dynamics of shallow water in a channel, δ being the dispersion parameter.

The KdV equation is a particular case of the ILW equation

$$u_t = 2uu_x + \frac{1}{\delta}u_x + \mathcal{T}[u_{xx}] \tag{4.66}$$

where $\mathcal{T}$ is the integral operator

$$\mathcal{T}[f](x) = P.V. \int \coth\left(\frac{\pi(x-y)}{2\delta}\right) f(y)\frac{dy}{2\delta} \tag{4.67}$$

and $P.V. \int$ is the principal value integral.
Equation (4.66) describes the surface dynamics of water in a channel of finite depth. It reduces to (4.65) in the limit of small δ. The opposite limit, that is the infinitely deep channel at $\delta \to \infty$, is called the Benjamin-Ono equation. It reads

$$u_t = 2uu_x + H[u_{xx}] \tag{4.68}$$

where H is the integral operator implementing the Hilbert transform on the real line

$$H[f](x) = P.V. \int \frac{1}{x-y} f(y)\frac{dy}{\pi} \tag{4.69}$$

The equation (4.66) is an integrable deformation of KdV. It has been proved in [115] that the form of the integral kernel in (4.67) is fixed by the requirement of integrability. The version of the ILW system which we will show to be relevant to our case is the periodic one. This is obtained by replacing (4.67) with

$$\mathcal{T}[f](x) = \frac{1}{2\pi} P.V. \int_0^{2\pi} \frac{\theta_1'}{\theta_1}\left(\frac{y-x}{2}, q\right) f(y)dy \tag{4.70}$$

where $q = e^{-\delta}$.

Equation (4.66) is Hamiltonian with respect to the Poisson bracket

$$\{u(x), u(y)\} = \delta'(x - y) \tag{4.71}$$

and reads

$$u_t(x) = \{I_3, u(x)\} \tag{4.72}$$

where $I_3 = \int \frac{1}{3}u^3 + \frac{1}{2}u\mathcal{T}[u_x]$ is the corresponding Hamiltonian. The other flows are generated by $I_2 = \int \frac{1}{2}u^2$ and the further Hamiltonians $I_n = \int \frac{1}{n}u^n + \ldots$, where $n > 3$, which are determined by the condition of being in involution $\{I_n, I_m\} = 0$. These have been computed explicitly in [116]. The more general $gl(N)$ ILW system is described in [117]; explicit formulae for the $gl(2)$ case can be found in Appendix A of [91].

The periodic ILW system can be quantized by introducing creation/annihilation operators corresponding to the Fourier modes of the field u and then by the explicit construction of the quantum analogue of the commuting Hamiltonians I_n above. Explicitly, one introduces the Fourier modes $\{\alpha_k\}_{k\in\mathbb{Z}}$ with commutation relations

$$[\alpha_k, \alpha_l] = k\delta_{k+l}$$

and gets the first Hamiltonians schematically as

$$I_2 = 2\sum_{k>0}\alpha_{-k}\alpha_k - \frac{1}{24},$$

$$I_3 = -\sum_{k>0} k\coth(k\pi t)\alpha_{-k}\alpha_k + \frac{1}{3}\sum_{k+l+m=0}\alpha_k\alpha_l\alpha_m \tag{4.73}$$

where we introduced a *complexified* ILW deformation parameter $2\pi t = \delta - i\theta$. This arises naturally in comparing the Hamiltonian (4.73) with the deformation of the quantum trigonometric Calogero-Sutherland Hamiltonian appearing in the study of the quantum cohomology of $\mathrm{Hilb}^n\left(\mathbb{C}^2\right)$ [106, 118], see Appendix B for details. We are thus led to identify the creation and annihilation operators of the quantum periodic ILW system with the Nakajima operators describing the equivariant cohomology of the instanton moduli space: this is the reason why one has to consider *periodic* ILW to make a comparison with gauge theory results. Moreover, from (4.73) the complexified deformation parameter of the ILW system $2\pi t = \delta - i\theta$ gets identified with the Kähler parameter of the Hilbert scheme of points as $q = e^{-2\pi t}$. In this way the quantum ILW hamiltonian structure reveals to be related to abelian six dimensional gauge theories via BPS/CFT correspondence. In particular the BO limit $t \to \pm\infty$ corresponds to the classical equivariant cohomology of the instanton moduli space described by the four dimensional limit of the abelian gauge theory.

More general quantum integrable systems of similar type arise by considering richer symmetry structures, i.e. the $gl(N)$ quantum ILW systems. These are related to non-abelian gauge theories. A notable example is that of $H \oplus Vir$, where H is the Heisenberg algebra of a single chiral $U(1)$ current. Its integrable quantization depends on a parameter which weights how to couple the generators of the two algebras in the conserved Hamiltonians. The construction of the corresponding quantum ILW system can be found in [91]. This quantum integrable system, in the BO_2 limit, has been shown in [90] to govern the AGT realization of the $SU(2)$ $\mathcal{N} = 2$ $D = 4$ gauge theory with $N_f = 4$. More precisely, the expansion of the conformal blocks proposed in [119] can be proved to be the basis of descendants in CFT which diagonalizes the BO_2 Hamiltonians.

More in general one can consider the algebra $H \oplus W_N$. The main aim of this paper is to show that the partition function of the non-abelian six-dimensional gauge theory on $S^2 \times \mathbb{C}^2$ naturally computes such a quantum generalization. Indeed, as it will be shown in section 4.6, the Yang-Yang function of this system, as it is described in [91], arises as the twisted superpotential of the effective LG model governing the finite volume effects of the two-sphere. In particular, we propose that the Fourier modes of the $gl(N)$ periodic ILW system correspond to the Baranovsky operators acting on the equivariant cohomology of the ADHM instanton moduli space. Evidence for this proposal is given in section 4.6 and in the Appendix B. Moreover in section 4.6 we identify the deformation parameter t in (4.73) with the FI parameter of the gauged linear sigma model on the two sphere.

This generalizes the link between quantum deformed Calogero-Sutherland system and the abelian gauge theory to the $gl(N)$ ILW quantum integrable system and the non-abelian gauge theory in six dimensions.

4.5.2 ILW as hydrodynamic limit of elliptic Calogero-Moser

An important property of the non-periodic ILW system is that its rational solutions are determined by the trigonometric Calogero-Sutherland model (see [120] for details). In this subsection we show a similar result for periodic ILW, namely that the dynamics of the poles of multisoliton solutions for this system is described by elliptic Calogero-Moser. Similar results were obtained in [121, 122]. We proceed by generalizing the approach of [123] where the analogous limit was discussed for trigonometric Calogero-Sutherland versus the BO equation. The strategy is the following: one studies multi-soliton solutions to the ILW system by giving a pole ansatz. The dynamics of the position of the poles turns out to be described by an auxiliary system equivalent to the eCM equations of motion in Hamiltonian formalism.

The Hamiltonian of eCM system for N particles is defined as

$$H_{eCM} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + G^2 \sum_{i<j} \wp(x_i - x_j; \omega_1, \omega_2), \tag{4.74}$$

where $\wp$ is the elliptic Weierstrass $\wp$-function and the periods are chosen as $2\omega_1 = L$ and $2\omega_2 = i\delta$. In the previous section 4.5.1 and in section 4.6 we set $L = 2\pi$. For notational simplicity, from now on we suppress the periods in all elliptic functions. The Hamilton equations read

$$\begin{aligned} \dot{x}_j &= p_j \\ \dot{p}_j &= -G^2 \partial_j \sum_{k\neq j} \wp(x_j - x_k), \end{aligned} \tag{4.75}$$

which can be recast as a second order equation of motion

$$\ddot{x}_j = -G^2 \partial_j \sum_{k\neq j} \wp(x_j - x_k). \tag{4.76}$$

It can be shown (see Appendix C for a detailed derivation) that equation (4.76) is equivalent to the following auxiliary system[6]

$$\begin{aligned} \dot{x}_j &= iG\left\{ \sum_{k=1}^{N} \frac{\theta_1'\left(\frac{\pi}{L}(x_j - y_k)\right)}{\theta_1\left(\frac{\pi}{L}(x_j - y_k)\right)} - \sum_{k\neq j} \frac{\theta_1'\left(\frac{\pi}{L}(x_j - x_k)\right)}{\theta_1\left(\frac{\pi}{L}(x_j - x_k)\right)} \right\} \\ \dot{y}_j &= -iG\left\{ \sum_{k=1}^{N} \frac{\theta_1'\left(\frac{\pi}{L}(y_j - x_k)\right)}{\theta_1\left(\frac{\pi}{L}(y_j - x_k)\right)} - \sum_{k\neq j} \frac{\theta_1'\left(\frac{\pi}{L}(y_j - y_k)\right)}{\theta_1\left(\frac{\pi}{L}(y_j - y_k)\right)} \right\}. \end{aligned} \tag{4.77}$$

In the limit $\delta \to \infty$ $(q \to 0)$, the equation of motion (4.76) reduces to

$$\ddot{x}_j = -G^2 \left(\frac{\pi}{L}\right)^2 \partial_j \sum_{k\neq j} \cot^2\left(\frac{\pi}{L}(x_j - x_k)\right), \tag{4.78}$$

[6] Actually, the requirement that this system should reduce to (4.76) is not sufficient to fix the form of the functions appearing. As will be clear from the derivation below, we could as well substitute $\frac{\theta_1'\left(\frac{\pi}{L}z\right)}{\theta_1\left(\frac{\pi}{L}z\right)}$ by $\zeta(z)$ and the correct equations of motion would still follow. However, we can fix this freedom by taking the trigonometric limit $(\delta \to \infty)$ and requiring that this system reduces to the one in [123].

while the auxiliary system goes to

$$\dot{x}_j = iG\frac{\pi}{L}\left\{\sum_{k=1}^{N}\cot\left(\frac{\pi}{L}(x_j - y_k)\right) - \sum_{k\neq j}\cot\left(\frac{\pi}{L}(x_j - x_k)\right)\right\}$$
$$\dot{y}_j = -iG\frac{\pi}{L}\left\{\sum_{k=1}^{N}\cot\left(\frac{\pi}{L}(y_j - x_k)\right) - \sum_{k\neq j}\cot\left(\frac{\pi}{L}(y_j - y_k)\right)\right\} \tag{4.79}$$

This is precisely the form obtained in [123].

In analogy with [123] we can define a pair of functions which encode particle positions as simple poles

$$u_1(z) = -iG\sum_{j=1}^{N}\frac{\theta_1'\left(\frac{\pi}{L}(z - x_j)\right)}{\theta_1\left(\frac{\pi}{L}(z - x_j)\right)}$$
$$u_0(z) = iG\sum_{j=1}^{N}\frac{\theta_1'\left(\frac{\pi}{L}(z - y_j)\right)}{\theta_1\left(\frac{\pi}{L}(z - y_j)\right)} \tag{4.80}$$

and we also introduce their linear combinations

$$u = u_0 + u_1, \quad \widetilde{u} = u_0 - u_1. \tag{4.81}$$

These satisfy the differential equation

$$u_t + uu_z + i\frac{G}{2}\widetilde{u}_{zz} = 0, \tag{4.82}$$

as long as x_j and y_j are governed by the dynamical equations (4.77). The details of the derivation can be found in Appendix C. Notice that, when the lattice of periodicity is rectangular, (4.82) is nothing but the ILW equation. Indeed, under the condition $x_i = \bar{y}_i$ one can show that $\tilde{u} = -i\mathcal{T}u$ [116]. To recover (4.66) one has to further rescale u, t, x and shift $u \to u + 1/2\delta$. We observe that (4.82) does not explicitly depend on the number of particles N and therefore also holds in the hydrodynamic limit $N, L \to \infty$, with N/L fixed.

4.6 Landau-Ginzburg mirror of the ADHM moduli space and quantum Intermediate Long Wave system

Having discussed in some detail the quantum ILW system in the previous section, it remains to understand how this is related to the ADHM GLSM. Again, mirror symmetry turns out to be of great help in clarifying this connection.

Mirror symmetry for two-dimensional $\mathcal{N}=(2,2)$ gauge theories is a statement about the equivalence of two theories, a GLSM and a twisted Landau-Ginzburg (LG) model (known as *mirror theory*). A twisted LG model is a theory made out of twisted chiral fields Y only (possibly including superfield strengths Σ), and is specified by a holomorphic function $\mathcal{W}(Y,\Sigma)$ which contains all the information about interactions among the fields. As it is well-known [124, 125], The Coulomb branch of a twisted LG model is related to quantum integrable systems via the so-called Bethe/Gauge correspondence. The idea goes as follow. First, we go to the Coulomb branch of the LG model by integrating out the matter fields Y and the massive W-bosons: from

$$\frac{\partial \mathcal{W}}{\partial Y}=0 \tag{4.83}$$

we obtain $Y=Y(\Sigma)$, and substituting back in $\mathcal{W}$ we remain with a purely abelian gauge theory in the infra-red, described in terms of the *twisted effective superpotential*

$$\mathcal{W}_{\text{eff}}(\Sigma)=\mathcal{W}(\Sigma,Y(\Sigma)) \tag{4.84}$$

The effect of integrating out the W-bosons results in a shift of the θ-angle. Now, the Bethe/Gauge correspondence [124, 125] tells us that the twisted effective superpotential of a 2d $\mathcal{N}=(2,2)$ gauge theory coincides with the Yang-Yang function of a quantum integrable system (QIS); this implies that the quantum supersymmetric vacua equations

$$\frac{\partial \mathcal{W}_{\text{eff}}}{\partial \Sigma_s}=2\pi i n_s \tag{4.85}$$

can be identified, after exponentiation, with a set of equations known as Bethe Ansatz Equations (BAE) which determine the spectrum and eigenfunctions of the QIS:

$$\exp\left(\frac{\partial \mathcal{W}_{\text{eff}}}{\partial \Sigma_s}\right)=1 \quad\Longleftrightarrow\quad \text{Bethe Ansatz Equations} \tag{4.86}$$

In particular, to each solution of the BAE is associated an eigenstate of the QIS, and its eigenvalues with respect to the set of quantum Hamiltonians of the system can be expressed as functions of the gauge theory observables $\operatorname{Tr}\Sigma^n$ evaluated at the solution:

$$\text{quantum Hamiltonians QIS} \quad\longleftrightarrow\quad \operatorname{Tr}\Sigma^n\big|_{\text{solution BAE}} \tag{4.87}$$

The Coulomb branch representation of the partition function (2.32) for a GLSM contains all the information about the mirror LG model. We can start by defining

$$\Sigma_s=\sigma_s-i\frac{m_s}{2r} \tag{4.88}$$

which is the twisted chiral superfield corresponding to the superfield strength for the s-th component of the vector supermultiplet in the Cartan of the gauge group G. We can now use the procedure described in [49]: each ratio of Gamma functions can be rewritten as

$$\frac{\Gamma(-ir\Sigma)}{\Gamma(1+ir\overline{\Sigma})} = \int \frac{d^2Y}{2\pi} \exp\Big\{ -e^{-Y} + ir\Sigma Y + e^{-\overline{Y}} + ir\overline{\Sigma Y}\Big\} \tag{4.89}$$

Here Y, $\overline{Y}$ are interpreted as the twisted chiral fields for the matter sector of the mirror Landau-Ginzburg model. The partition function (2.32) then becomes

$$Z_{S^2} = \left| \int d\Sigma\, dY\; e^{-\mathcal{W}(\Sigma,Y)} \right|^2 \tag{4.90}$$

from which we can read $\mathcal{W}(\Sigma, Y)$ of the mirror LG theory; this is a powerful method to recover the twisted superpotential of the mirror theory, when it is not known previously. Here $d\Sigma = \prod_s d\Sigma_s$ and $dY = \prod_j dY_j$ collect all the integration variables.

To recover the IR Coulomb branch of this theory we integrate out the Y, $\overline{Y}$ fields by performing a semiclassical approximation of (4.89), which gives

$$Y = -\ln(-ir\Sigma) \quad , \quad \overline{Y} = -\ln(ir\overline{\Sigma}) \tag{4.91}$$

so that we are left with

$$\frac{\Gamma(-ir\Sigma)}{\Gamma(1+ir\overline{\Sigma})} \sim \exp\Big\{\omega(-ir\Sigma) - \frac{1}{2}\ln(-ir\Sigma) - \omega(ir\overline{\Sigma}) - \frac{1}{2}\ln(ir\overline{\Sigma})\Big\} \tag{4.92}$$

in terms of the function $\omega(x) = x(\ln x - 1)$. The effect of integrating out the W-fields results in having to consider θ_{ren} instead of θ as in (2.31). As discussed in [100, 141] the functions $\omega(\Sigma)$ enter in $\mathcal{W}_{\text{eff}}$, while the logarithmic terms in (4.92) (which modify the effective twisted superpotential with respect to the one on $\mathbb{R}^2$) enter into the integration measure.

For the case of the ADHM GLSM we have to start from (4.2); defining $t = \xi - i\frac{\theta}{2\pi}$ as the complexified Fayet-Iliopoulos[7], equation (4.2) becomes

$$Z^{S^2}_{k,N} = \frac{1}{k!}\left(\frac{\epsilon}{r\epsilon_1\epsilon_2}\right)^k \int \prod_{s=1}^{k} \frac{\mathrm{d}^2(r\Sigma_s)}{2\pi} \left| \left(\frac{\prod_{s=1}^k \prod_{t\neq s=1}^k D(\Sigma_{st})}{\prod_{s=1}^k Q(\Sigma_s)} \right)^{\frac{1}{2}} e^{-\mathcal{W}_{\text{eff}}} \right|^2 \tag{4.93}$$

[7]The sign of θ is different from the choice made in section 4.2.

where the logarithmic terms in (4.92) give the integration measure in terms of the functions

$$\begin{aligned} Q(\Sigma_s) &= r^{2N}\prod_{j=1}^{N}(\Sigma_s - a_j - \frac{\epsilon}{2})(-\Sigma_s + a_j - \frac{\epsilon}{2}) \\ D(\Sigma_{st}) &= \frac{(\Sigma_{st})(\Sigma_{st}+\epsilon)}{(\Sigma_{st}-\epsilon_1)(\Sigma_{st}-\epsilon_2)} \end{aligned} \tag{4.94}$$

$\mathcal{W}_{\text{eff}}$ is the effective twisted superpotential of the mirror LG model in the Coulomb branch:

$$\begin{aligned} \mathcal{W}_{\text{eff}} &= (2\pi t - i(k-1)\pi)\sum_{s=1}^{k} ir\Sigma_s + \sum_{s=1}^{k}\sum_{j=1}^{N}\Big[\omega(ir\Sigma_s - ira_j - ir\frac{\epsilon}{2}) + \omega(-ir\Sigma_s + ira_j - ir\frac{\epsilon}{2})\Big] \\ &+ \sum_{s,t=1}^{k}\left[\omega(ir\Sigma_{st}+ir\epsilon) + \omega(ir\Sigma_{st} - ir\epsilon_1) + \omega(ir\Sigma_{st} - ir\epsilon_2)\right] \end{aligned} \tag{4.95}$$

The complex conjugation refers to Σ and t; in particular, we have

$$\overline{\mathcal{W}_{\text{eff}}(ir\Sigma, t)} = \mathcal{W}_{\text{eff}}(-ir\overline{\Sigma}, \overline{t}) = -\mathcal{W}_{\text{eff}}(ir\overline{\Sigma}, \overline{t}) \ . \tag{4.96}$$

The claim is that the function $\mathcal{W}_{\text{eff}}$ of (4.95) coincides with the Yang-Yang function of the $gl(N)$ Intermediate Long Wave system, as proposed in [91].

Let us now perform a semiclassical analysis around the saddle points of (4.95). As we will see shortly, this provides the Bethe-ansatz equations for the quantum integrable system at hand. By definition, the saddle points are solutions of the equations

$$\frac{\partial \mathcal{W}_{\text{eff}}}{\partial(ir\Sigma_s)} = 0 \tag{4.97}$$

This implies

$$\begin{aligned} &2\pi t - i(k-1)\pi + \sum_{j=1}^{N}\ln\frac{\Sigma_s - a_j - \frac{\epsilon}{2}}{-\Sigma_s + a_j - \frac{\epsilon}{2}} \\ &+ \sum_{\substack{t=1 \\ t\neq s}}^{k}\ln\frac{(\Sigma_{st}+\epsilon)(\Sigma_{st}-\epsilon_1)(\Sigma_{st}-\epsilon_2)}{(-\Sigma_{st}+\epsilon)(-\Sigma_{st}-\epsilon_1)(-\Sigma_{st}-\epsilon_2)} = 0 \end{aligned} \tag{4.98}$$

or, by exponentiating and using $(-1)^{k-1} = \prod_{\substack{t=1\\t\neq s}}^{k} \frac{(\Sigma_{st})}{(-\Sigma_{st})}$,

$$\begin{aligned}
&\prod_{j=1}^{N}(\Sigma_s - a_j - \frac{\epsilon}{2})\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\Sigma_{st}-\epsilon_1)(\Sigma_{st}-\epsilon_2)}{(\Sigma_{st})(\Sigma_{st}-\epsilon)}\\
&= e^{-2\pi t}\prod_{j=1}^{N}(-\Sigma_s + a_j - \frac{\epsilon}{2})\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(-\Sigma_{st}-\epsilon_1)(-\Sigma_{st}-\epsilon_2)}{(-\Sigma_{st})(-\Sigma_{st}-\epsilon)}
\end{aligned} \tag{4.99}$$

These are the Bethe ansatz equations governing the spectrum of the integrable system for generic t as appeared also in [91, 94]. To be more precise, remember that $\theta \to \theta + 2\pi n$ is a symmetry of the theory; the saddle points will be solutions to

$$\frac{\partial \mathcal{W}_{\text{eff}}}{\partial(ir\Sigma_s)} = 2\pi i n_s \tag{4.100}$$

but this leaves the Bethe ansatz equations (4.99) unchanged.
Around the BO point $t \to \infty$, the solutions to (4.99) can be labelled by colored partitions of N, $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N)$ such that the total number of boxes $\sum_{l=1}^{N}|\lambda_l|$ is equal to k. In the limit $t \to \infty$, the roots of the Bethe equations are given by

$$\Sigma_m^{(l)} = a_l + \frac{\epsilon}{2} + (i-1)\epsilon_1 + (j-1)\epsilon_2 \quad , \quad m = 1, \ldots, |\lambda_l| \tag{4.101}$$

with i, j running over all possible rows and columns of the tableau λ_l; those are exactly the poles appearing in the contour integral representation for the 4d Nekrasov partition function [9]. In the large t case, the roots will be given in terms of a series expansion in powers of $e^{-2\pi t}$.

4.6.1 Derivation via large r limit and norm of the ILW wave-functions

The previous results can also (and maybe better) be understood in terms of a large r limit of (4.2). In other words this amounts to set $\epsilon_3 \sim r^{-1} \sim 0$ with ϵ_1, ϵ_2 finite and as such is a six-dimensional analogue of the Nekrasov-Shatashvili limit [114]. We can use Stirling's approximation:

$$\begin{aligned}
\Gamma(z) &\sim \sqrt{2\pi}\, z^{z-\frac{1}{2}}\, e^{-z}\, (1 + o(z^{-1})) \quad , \quad z \to \infty\\
\Gamma(1+z) &\sim \sqrt{2\pi}\, z^{z+\frac{1}{2}}\, e^{-z}\, (1 + o(z^{-1})) \quad , \quad z \to \infty
\end{aligned} \tag{4.102}$$

which implies

$$\begin{aligned} \ln\Gamma(z) &\sim \omega(z) - \frac{1}{2}\ln z + \frac{1}{2}\ln 2\pi + o(z^{-1}) \quad , \quad z\to\infty \\ \ln\Gamma(1+z) &\sim \omega(z) + \frac{1}{2}\ln z + \frac{1}{2}\ln 2\pi + o(z^{-1}) \quad , \quad z\to\infty \end{aligned} \tag{4.103}$$

Consider for example the contribution from the I field; we have

$$\begin{aligned} \ln\Gamma(-ir\Sigma_s + ira_j + ir\frac{\epsilon}{2}) &\sim \omega(-ir\Sigma_s + ira_j + ir\frac{\epsilon}{2}) - \frac{1}{2}\ln(-ir\Sigma_s + ira_j + ir\frac{\epsilon}{2}) + \frac{1}{2}\ln 2\pi \\ \ln\Gamma(1 + ir\overline{\Sigma}_s - ira_j - ir\frac{\epsilon}{2}) &\sim \omega(ir\overline{\Sigma}_s - ira_j - ir\frac{\epsilon}{2}) + \frac{1}{2}\ln(ir\overline{\Sigma}_s - ira_j - ir\frac{\epsilon}{2}) + \frac{1}{2}\ln 2\pi \end{aligned} \tag{4.104}$$

Doing the limit for all of the fields, we find again

$$Z^{S^2}_{k,N} = \frac{1}{k!}\left(\frac{\epsilon}{r\epsilon_1\epsilon_2}\right)^k \int \prod_{s=1}^{k} \frac{\mathrm{d}^2(r\Sigma_s)}{2\pi} \left| \left(\frac{\prod_{s=1}^k \prod_{t\neq s=1}^k D(\Sigma_{st})}{\prod_{s=1}^k Q(\Sigma_s)}\right)^{\frac{1}{2}} e^{-\mathcal{W}_{\text{eff}}} \right|^2 \tag{4.105}$$

Refining the semiclassical approximation around the saddle points of $\mathcal{W}_{\text{eff}}$ up to quadratic fluctuations, we obtain (eliminating the $k!$ by choosing an order for the saddle points)

$$Z^{S^2}_{k,N} = \left| e^{-\mathcal{W}_{\text{cr}}} \left(\frac{\epsilon}{r\epsilon_1\epsilon_2}\right)^{\frac{k}{2}} \left(\frac{\prod_{s=1}^k \prod_{t\neq s=1}^k D(\Sigma_{st})}{\prod_{s=1}^k Q(\Sigma_s)}\right)^{\frac{1}{2}} \left(\text{Det}\frac{\partial^2\mathcal{W}_{\text{eff}}}{r^2\partial\Sigma_s\partial\Sigma_t}\right)^{-\frac{1}{2}} \right|^2 \tag{4.106}$$

Apart from the classical term $|e^{-\mathcal{W}_{\text{cr}}}|^2$, this can be seen as the inverse norm square of the eigenstates of the infinite set of integrals of motion for the ILW system, where each eigenstate corresponds to an $N-$partition $\vec{\lambda}$ of k and as such we can denote it by $|\vec{\lambda}\rangle$:

$$Z^{S^2}_{k,N} = \frac{|e^{-\mathcal{W}_{\text{cr}}}|^2}{\langle\vec{\lambda}|\vec{\lambda}\rangle} \tag{4.107}$$

Comparing with (4.106), we find

$$\frac{1}{\langle\vec{\lambda}|\vec{\lambda}\rangle} = \left| \left(\frac{\epsilon}{r\epsilon_1\epsilon_2}\right)^{\frac{k}{2}} \left(\frac{\prod_{s=1}^k \prod_{t\neq s=1}^k D(\Sigma_{st})}{\prod_{s=1}^k Q(\Sigma_s)}\right)^{\frac{1}{2}} \left(\text{Det}\frac{\partial^2\mathcal{W}_{\text{eff}}}{r^2\partial\Sigma_s\partial\Sigma_t}\right)^{-\frac{1}{2}} \right|^2 \tag{4.108}$$

For real parameters (for example when $t\to\infty$), this formula agrees with the expression for the norm of the ILW eigenstates given in [91].

4.6.2 Quantum ILW Hamiltonians

In this subsection we propose that the chiral ring observables of the $U(N)$ six-dimensional gauge theory correspond to the set of commuting quantum Hamiltonians of the $gl(N)$

ILW system. Due to R-symmetry selection rules, the chiral ring observables vanish in the perturbative sector and are therefore completely determined by their non-perturbative contributions. These are computed by the effective two-dimensional GLSM describing D1-branes dynamics in presence of D(-1)s. More precisely, chiral observables of the GLSM provide a basis for the quantum Hamiltonians of the corresponding integrable system [114, 124, 125]. This implies that in our case the quantum Hamiltonians for the ILW system are given by linear combinations of $\mathrm{Tr}\,\Sigma^n$ operators, for generic values of t:

$$\text{ILW quantum Hamiltonians} \quad \longleftrightarrow \quad \mathrm{Tr}\,\Sigma^n(t) \tag{4.109}$$

which is just a particular case of (4.87), consequence of the Bethe/Gauge correspondence. The calculation of the local chiral ring observables of $U(N)$ gauge theory on $\mathbb{C}^2 \times S^2$ is analogous to the one on $\mathbb{C}^2$, the crucial difference being that in the six dimensional case the bosonic and fermionic zero-modes in the instanton background acquire an extra dependence on the two-sphere coordinates. As a consequence, the sum over the fixed points is replaced by the sum over the vacua of the effective GLSM giving

$$\mathrm{tr}\; e^{\Phi} = \sum_{l=1}^{N} \left(e^{a_l} - e^{-\frac{\epsilon_1+\epsilon_2}{2}} (1-e^{\epsilon_1})(1-e^{\epsilon_2}) \sum_m e^{\Sigma_m^{(l)}(t)} \right) \tag{4.110}$$

where $\Sigma_m^{(l)}(t)$ are the solutions of the Bethe equations (4.99). We expect the above formula can be proved in a rigorous mathematical setting in the context of ADHM moduli sheaves introduced in [126]. In the $N = 2$ case the first few terms read

$$\begin{aligned}
\frac{\mathrm{Tr}\Phi^2}{2} &= a^2 - \epsilon_1\epsilon_2 \left(\sum_{m=1}^{|\lambda|} 1 + \sum_{n=1}^{|\mu|} 1 \right) \\
\frac{\mathrm{Tr}\Phi^3}{3} &= -2\epsilon_1\epsilon_2 \left(\sum_{m=1}^{|\lambda|} \Sigma_m + \sum_{n=1}^{|\mu|} \Sigma_n \right) \\
\frac{\mathrm{Tr}\Phi^4}{4} &= \frac{a^4}{2} - 3\epsilon_1\epsilon_2 \left(\sum_{m=1}^{|\lambda|} \Sigma_m^2 + \sum_{n=1}^{|\mu|} \Sigma_n^2 \right) - \epsilon_1\epsilon_2 \frac{\epsilon_1^2+\epsilon_2^2}{4} \left(\sum_{m=1}^{|\lambda|} 1 + \sum_{n=1}^{|\mu|} 1 \right) \\
\frac{\mathrm{Tr}\Phi^5}{5} &= -4\epsilon_1\epsilon_2 \left(\sum_{m=1}^{|\lambda|} \Sigma_m^3 + \sum_{n=1}^{|\mu|} \Sigma_n^3 \right) - \epsilon_1\epsilon_2(\epsilon_1^2+\epsilon_2^2) \left(\sum_{m=1}^{|\lambda|} \Sigma_m + \sum_{n=1}^{|\mu|} \Sigma_n \right) \quad .
\end{aligned} \tag{4.111}$$

A check the proposal (4.109) can be obtained by considering the four dimensional limit where explicit formulae are already known. Indeed in the four dimensional limit $t \to \pm\infty$

the roots of the Bethe equations reduces to (4.101) [91]

$$\Sigma_m^{(l)} \;=\; a+\frac{\epsilon}{2}+(i-1)\epsilon_1+(j-1)\epsilon_2 \;=\; a-\frac{\epsilon}{2}+i\epsilon_1+j\epsilon_2 \quad , \quad i,j\geqslant 1 \quad , \quad m=1,\ldots,|\lambda| \ . \tag{4.112}$$

Consequently, (4.110) reduces to the known formula for the chiral ring observables of four-dimensional $U(N)$ SYM [112, 127]:

$$\begin{aligned}\mathrm{Tr}\Phi^{n+1} = \sum_{l=1}^{N} a_l^{n+1} + \sum_{l=1}^{N}\sum_{j=1}^{k_1^{(l)}} \Big[\Big(a_l+\epsilon_1\lambda_j^{(l)}+\epsilon_2(j-1)\Big)^{n+1} - \Big(a_l+\epsilon_1\lambda_j^{(l)}+\epsilon_2 j\Big)^{n+1} \\ - (a_l+\epsilon_2(j-1))^{n+1} + (a_l+\epsilon_2 j)^{n+1}\Big]\end{aligned} \tag{4.113}$$

where $\lambda^{(l)} = \{\lambda_1^{(l)} \geq \lambda_2^{(l)} \geq \ldots \geq \lambda_{k_1^{(l)}}^{(l)}\}$, $l=1,\ldots,N$ indicate colored partitions of the instanton number $k=\sum_{l,j}\lambda_j^{(l)}$. Since the four-dimensional limit corresponds to the $t\to\infty$ limit, we expect that the above chiral observables are related to the quantum Hamiltonians of the BO system. For definiteness, let us consider the case $N=2$. For $N=2$ the Young tableaux correspond to bipartitions $(\lambda,\,\mu)=(\lambda_1\geqslant\lambda_2\geqslant\ldots,\mu_1\geqslant\mu_2\geqslant\ldots)$ such that $|\lambda|+|\mu|=k$. For Benjamin-Ono, the eigenvalues of the Hamiltonian operators $\mathbf{I}_n$ are given by linear combinations of the eigenvalues of two copies of trigonometric Calogero-Sutherland system [91, 90] as

$$h_{\lambda,\mu}^{(n)} = h_\lambda^{(n)}(a) + h_\mu^{(n)}(-a) \tag{4.114}$$

with

$$h_\lambda^{(n)}(a) = \epsilon_2 \sum_{j=1}^{k_1^{(\lambda)}} \left[\left(a+\epsilon_1\lambda_j+\epsilon_2\left(j-\frac{1}{2}\right)\right)^n - \left(a+\epsilon_2\left(j-\frac{1}{2}\right)\right)^n\right] \tag{4.115}$$

where $k_1^{(\lambda)}$ is the number of boxes in the first row of the partition λ, and λ_j is the number of boxes in the j-th column. In particular, $h_{\lambda,\mu}^{(1)}=\epsilon_1\epsilon_2 k$. In terms of (4.115), the $N=2$ chiral observables (4.113) read

$$\frac{\mathrm{Tr}\Phi^{n+1}}{n+1} = \frac{a^{n+1}+(-a)^{n+1}}{n+1} - \sum_{i=1}^{n}\frac{1+(-1)^{n-i}}{2}\frac{n!}{i!(n+1-i)!}\left(\frac{\epsilon_2}{2}\right)^{n-i} h_{\lambda,\mu}^{(i)} \tag{4.116}$$

The contributions from $i = 0$, $i = n+1$ are zero, so they were not considered in the sum. The first few cases are:

$$\begin{array}{rclcrcl} & & \frac{\mathrm{Tr}\Phi^2}{2} = a^2 - \epsilon_1\epsilon_2 k & , & \frac{\mathrm{Tr}\Phi^3}{3} & = & -h^{(2)}_{\lambda,\mu} \\ \frac{\mathrm{Tr}\Phi^4}{4} & = & \frac{a^4}{2} - h^{(3)}_{\lambda,\mu} - \frac{\epsilon_2^2}{4}\epsilon_1\epsilon_2 k & , & \frac{\mathrm{Tr}\Phi^5}{5} & = & -h^{(4)}_{\lambda,\mu} - \frac{\epsilon_2^2}{2}h^{(2)}_{\lambda,\mu} \end{array} \tag{4.117}$$

We now rewrite the above formulae in terms of the BO Bethe roots (4.112) so that

$$\begin{aligned} h^{(1)}_\lambda &= \epsilon_1\epsilon_2 \sum_{m=1}^{|\lambda|} 1 \\ h^{(2)}_\lambda &= 2\epsilon_1\epsilon_2 \sum_{m=1}^{|\lambda|} \Sigma_m \\ h^{(3)}_\lambda &= 3\epsilon_1\epsilon_2 \sum_{m=1}^{|\lambda|} \Sigma_m^2 + \epsilon_1\epsilon_2\frac{\epsilon_1^2}{4}\sum_{n=1}^{|\lambda|} 1 \\ h^{(4)}_\lambda &= 4\epsilon_1\epsilon_2 \sum_{m=1}^{|\lambda|} \Sigma_m^3 + \epsilon_1\epsilon_2\epsilon_1^2\sum_{n=1}^{|\lambda|} \Sigma_m \,. \end{aligned} \tag{4.118}$$

4.6.3 Quantum KdV

Another very interesting limit to analyse is the $\delta \to 0$ limit which provides a connection with quantum KdV system. Let us recall that KdV is a bi-Hamiltonian system, displaying a further Poisson bracket structure behind the standard one (4.71), namely

$$\{U(x), U(y)\} = 2\left(U(x) + U(y)\right)\delta'(x-y) + \delta'''(x-y) \tag{4.119}$$

The mapping between the Hamiltonians of the integrable hierarchy with respect to the first and second Hamiltonian structure can be obtained via the Miura transform

$$U(x) = u_x(x) - u(x)^2 \tag{4.120}$$

A quantization scheme for KdV system starting from the second Hamiltonian structure was presented in [89] where it was shown that the quantum Hamiltonians correspond to the Casimir operators in the enveloping algebra UVir. In particular, the profile function $U(x)$ is the semiclassical limit of the energy-momentum tensor of the two-dimensional conformal field theory.

It is interesting to observe that the chiral ring observables of the abelian six-dimensional gauge theory provide an alternative quantization of the same system, obtained starting from the first Poisson bracket structure. Indeed the quantum ILW Hamiltonian $\mathrm{tr}\Phi^3$

reads in the $U(1)$ case

$$
\begin{aligned}
H_{ILW} = & (\epsilon_1+\epsilon_2)\sum_{p>0}\frac{p}{2}\frac{q^p+1}{q^p-1}\alpha_{-p}\alpha_p + \sum_{p,q>0}\left[\epsilon_1\epsilon_2\alpha_{p+q}\alpha_{-p}\alpha_{-q} - \alpha_{-p-q}\alpha_p\alpha_q\right] \\
& - \frac{\epsilon_1+\epsilon_2}{2}\frac{q+1}{q-1}\sum_{p>0}\alpha_{-p}\alpha_p
\end{aligned} \tag{4.121}
$$

where the free field is $\partial\phi = iQ\sum_{k>0} z^k\alpha_k - iQ\epsilon_1\epsilon_2\sum_{k>0} z^{-k}\alpha_{-k}$ and $Q = b+1/b$, $b = \sqrt{\epsilon_1/\epsilon_2}$. This reproduces in the semiclassical limit $b \to 0$ the hydrodynamic profile $\partial\phi \to iQu$ and from (4.121) the ILW Hamiltonian up to and overall factor $-(\epsilon_1+\epsilon_2)$. Let us notice that due to the twisting with the equivariant canonical bundle of $\mathbb{C}^2$, the Hermitian conjugation for the oscillators reads $\alpha_k^\dagger = \epsilon_1\epsilon_2\alpha_{-k}$, $\alpha_{-k}^\dagger = \alpha_k/\epsilon_1\epsilon_2$. By setting $\theta = 0$ and in the $2\pi t = \delta \to 0$ limit (4.121) reduces to

$$
H_{qKdV} = \delta\,(\epsilon_1+\epsilon_2)\sum_{p>0}\frac{(1-p^2)}{12}\alpha_{-p}\alpha_p + \sum_{p,q>0}\left[\epsilon_1\epsilon_2\alpha_{p+q}\alpha_{-p}\alpha_{-q} - \alpha_{-p-q}\alpha_p\alpha_q\right] \tag{4.122}
$$

which in turn corresponds to the quantum KdV Hamiltonian. Notice that the extra term in $\mathrm{tr}\Phi^2$ in (4.121), which is crucial in order to get a finite $t \to 0$ limit, is the counterpart of the shift in u_x/δ in the ILW equation (4.66). We expect that the spectrum of the higher quantum KdV Hamiltonians can be obtained by substituting into (4.111) the solutions of the $N=1$ Bethe equations (4.99) expanded around $t=0$; nevertheless the $N=1$ equations seem to be incomplete in this limit [128, 129].

The alternative expansion in an imaginary dispersion parameter θ around the dispersionless KdV point $q=1$ of the quantum Hamiltonian has a nice interpretation in terms of the orbifold quantum cohomology of the symmetric product of points $S^k(\mathbb{C}^2)$. Indeed when $\delta = 0$, namely $q = e^{i\theta}$, the Hamiltonian of the six dimensional abelian gauge theory can be shown to reduce to that describing the orbifold quantum cohomology of the symmetric product of points: see section 4.3.4 and Appendix B.

Let us finally remark that also the BLZ quantization scheme can be recovered in the context of gauge theory. To this end, one has to consider the $U(2)$ case, whose relevant algebra is precisely $H\oplus Vir$. In this case, the $t\to 0$ limit of $gl(2)$ quantum ILW reduces to a decoupled $U(1)$ current and the BLZ system of quantum Hamiltonians [91].

Chapter 5

ΔILW and elliptic Ruijsenaars: a gauge theory perspective

5.1 Introduction

In the last part of the previous chapter we discussed the relation between the ADHM theory on S^2 and the quantum periodic ILW system, based on the Bethe/gauge correspondence. Among other things, this correspondence allowed us to compute the spectrum of the ILW system in terms of gauge theory quantities; this spectrum is then conjectured to be given by the eigenvalues of N coupled copies of the elliptic Calogero-Sutherland (eCS) system. All the results are obtained as a perturbative series expansion around the known solutions of the Benjamin-Ono and trigonometric Calogero-Sutherland systems.

In this chapter we will see how these results can be reorganized in a more elegant way, by considering the ADHM theory on $S^2 \times S^1_\gamma$ with γ radius of the extra circle, focussing on the case $N = 1$. The generating function for the ILW spectrum (4.110) turns out to coincide with the first gauge theory observable $\langle \mathrm{Tr}\, \sigma \rangle$ in three dimensions; moreover, this can be thought as the eigenvalue of the first quantum Hamiltonian $\widehat{\mathcal{H}}_1$ of a *finite-difference version* of the ILW system (we will refer to it as ΔILW) [130], which is therefore expected to be a generating function for the whole set of quantum ILW Hamiltonians $\widehat{I}_l$.
From the mathematical point of view, the field $\eta(x)$ satisfying the ΔILW equation and the quantum ΔILW Hamiltonians $\widehat{\mathcal{H}}_l$ enter in the Heisenberg Fock space representation of the so-called elliptic Ding-Iohara algebra, whose detailed analysis was performed in [131]. This algebra is deeply connected with the free field representation of a quantum integrable system known as the n-particles elliptic Ruijsenaars-Schneider (eRS) model.

When this system is considered in the limit $n \to \infty$, the elliptic Ding-Iohara algebra provides a precise way to relate ΔILW and eRS at the level of eigenvalues; this generalizes and clarifies the connection we discovered between ILW and eCS spectra.
This connection can be translated in gauge theoretical terms. While the ΔILW system corresponds to the ADHM quiver on $S^2 \times S^1_\gamma$, the n-particles eRS system has a gauge theory analogue as a 5d $N = 1^*$ $U(n)$ theory on $\mathbb{C}^2 \times S^1_\gamma$ in Omega background coupled to a 3d $T[U(n)]$ defect on $\mathbb{C} \times S^1_\gamma$; eigenfunctions and eigenvalues of eRS correspond to the coupled 3d-5d instanton partition function Z^{inst}_{3d-5d} and the VEV of the $U(n)$ Wilson loop in the fundamental representation $\langle W^{U(n)}_\square \rangle$ respectively, in the so-called Nekrasov-Shatashvili limit [114]. We therefore expect, and we will show, that in the $n \to \infty$ limit the Wilson loop VEV $\langle W^{U(n)}_\square \rangle$ coming from this coupled 5d-3d theory reduces to the $\langle \operatorname{Tr} \sigma \rangle$ observable of the ADHM quiver on $S^2 \times S^1_\gamma$, thus providing a remarkable connection between these two very different supersymmetric gauge theories.

The rest of the chapter is organized as follows. In the first part we will discuss the trigonometric and elliptic quantum Ruijsenaars-Schneider models, as well as their analogue in supersymmetric gauge theory language. In the second part we briefly review the basic notions about the trigonometric and elliptic Ding-Iohara algebrae that we will need for our purposes, together with their relation to the Ruijsenaars-Schneider quantum systems. The last part concerns the correspondence between the ADHM theory on $S^2 \times S^1_\gamma$, the ΔILW system and the Ding-Iohara algebra. Once all these ingredients are understood, we will conclude by stating and giving computational evidence for the proposed correspondence between ΔILW and eRS in the large number of particles limit.

5.2 The quantum Ruijsenaars-Schneider integrable systems

The n-particles trigonometric quantum Ruijsenaars-Schneider system (tRS) is an integrable system which describes a set of n particles on a circle of radius γ, subject to the interaction determined by the Hamiltonian

$$D^{(1)}_{n,\vec{\tau}}(q,t) = \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{t\tau_i - \tau_j}{\tau_i - \tau_j} T_{q,i} \tag{5.1}$$

Here τ_l are the positions of the particles, t is a parameter determining the strength of the interaction, and $T_{q,i}$ is a shift operator acting as

$$T_{q,i} f(\tau_1, \dots, \tau_i, \dots, \tau_n) = f(\tau_1, \dots, q\tau_i, \dots, \tau_n) \tag{5.2}$$

on functions of the τ_l variables; we can think of it as $T_{q,i} = e^{i\gamma\hbar\,\tau_i\partial_{\tau_i}} = q^{\tau_i\partial_{\tau_i}}$ with $q = e^{i\gamma\hbar}$ and $\hbar$ quantization parameter. In the limit $\gamma \to 0$, $D^{(1)}_{n,\vec{\tau}}$ reduces to the n-particles trigonometric Calogero-Sutherland Hamiltonian. For completeness, let us mention that the operator (5.1) is the first of a set of n commuting operators, given by

$$D^{(r)}_{n,\vec{\tau}}(q,t) = t^{r(r-1)/2} \sum_{\substack{I\subset\{1,2,\ldots,n\}\\ \#I=r}} \prod_{\substack{i\in I\\ j\notin I}} \frac{t\tau_i - \tau_j}{\tau_i - \tau_j} \prod_{i\in I} T_{q,i} \quad \text{for} \quad r = 1,\ldots,n \tag{5.3}$$

From the mathematical point of view, the operator $D^{(1)}_{n,\vec{\tau}}$ coincides with the first Macdonald difference operator; its eigenfunctions, known as Macdonald polynomials, are given by symmetric polynomials in n variables τ_l of total degree $k \leqslant n$, and are in one-to-one correspondence with partitions $\lambda = (\lambda_1,\ldots,\lambda_n)$ of k of length n. Building blocks for these polynomials are the power sum symmetric polynomials $p_m = \sum_{l=1}^n \tau_l^m$. The Macdonald operator acts on the Macdonald polynomial $P_\lambda(\vec{\tau};q,t)$ corresponding to the partition λ as

$$D^{(1)}_{n,\vec{\tau}}(q,t) P_\lambda(\vec{\tau};q,t) = E^{(\lambda;n)}_{tRS} P_\lambda(\vec{\tau};q,t) \tag{5.4}$$

with an eigenvalue given by

$$E^{(\lambda;n)}_{tRS} = \sum_{j=1}^{n} q^{\lambda_j} t^{n-j} \tag{5.5}$$

Let us consider an example. Take $k = 2$; in this case we have two partitions $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, corresponding to the Macdonald polynomials

$$\frac{1}{2}(p_1^2 - p_2) \;\; \text{for} \;\; \square\square \quad , \quad \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)} p_2 \;\; \text{for} \;\; \begin{smallmatrix}\square\\\square\end{smallmatrix} \tag{5.6}$$

This is the expression in terms of power sum symmetric polynomials, which is the same for any n. Now, we can fix n; then we will have:

- For $n = 2$ the eigenfunction for the partition $(1,1)$ and its eigenvalue are

$$P_{(1,1)}(\tau_1,\tau_2;q,t) = \tau_1\tau_2 \quad , \quad E^{((1,1);2)}_{tRS} = qt + q \tag{5.7}$$

  while for the partition $(2,0)$ we have

$$P_{(2,0)}(\tau_1,\tau_2;q,t) = \tau_1\tau_2 + \frac{1-qt}{(1+q)(1-t)}(\tau_1^2 + \tau_2^2) \quad , \quad E^{((2,0);2)}_{tRS} = q^2 t + 1 \tag{5.8}$$

- For $n = 3$ the partition $(1,1,0)$ has eigenfunction

$$P_{(1,1,0)}(\tau_1,\tau_2,\tau_3;q,t) = \tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3 \tag{5.9}$$

and eigenvalue

$$E_{tRS}^{((1,1,0);2)} = qt^2 + qt + 1 \tag{5.10}$$

while the partition $(2,0,0)$ has eigenfunction

$$P_{(2,0,0)}(\tau_1,\tau_2,\tau_3;q,t) = \tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3 + \frac{1-qt}{(1+q)(1-t)}(\tau_1^2+\tau_2^2+\tau_3^2) \tag{5.11}$$

and eigenvalue

$$E_{tRS}^{((2,0,0);2)} = q^2t^2 + t + 1 \tag{5.12}$$

The generic n case follows along these lines.

There exists a generalization of the tRS model, known as the elliptic Ruijsenaars-Schneider model (eRS), defined by the elliptic version of the Hamiltonian (5.1), that is

$$D_{n,\vec{\tau}}^{(1)}(q,t;p) = \sum_{i=1}^{n}\prod_{j\neq i}^{n}\frac{\Theta_p(t\tau_i/\tau_j)}{\Theta_p(\tau_i/\tau_j)}T_{q,i} \tag{5.13}$$

where

$$\Theta_p(x) = (p;p)_\infty(x;p)_\infty(p/x;p)_\infty \quad , \quad (x;p)_\infty = \prod_{s=0}^{\infty}(1-xp^s) \tag{5.14}$$

For $p=0$, the Hamiltonian (5.13) reduces to (5.1). The solution to this model, i.e. eigenfunctions and eigenvalues of (5.13), is not known yet; nevertheless, one could try to solve this system perturbatively around the known tRS solution by expanding (5.13) around $p\sim 0$. It turns out that eigenfunctions can still be labelled by partitions of k of length n, although this time the eigenfunctions are symmetric polynomials in the τ_l/τ_m variables.
It turns out that supersymmetric gauge theories can help in determining the solution to the eRS system. In fact, recent results [132] have shown that eigenfunctions and eigenvalues of the tRS and eRS systems can be obtained by an instanton counting computation for a $U(n)$ $\mathcal{N}=1^*$ theory in 5d, coupled to a 3d $T[U(n)]$ defect; these computations, although lengthy, are well understood and allow us to obtain the desired solution to the eRS system, perturbatively in p. This is the subject of the next section.

5.3 Ruijsenaars systems from gauge theory

As discussed in [132], the tRS and eRS models have an alternative description in gauge theoretical terms. Let us start by considering the n-particles tRS system. This is related to the so-called $\mathcal{N}=2^*$ $T[U(n)]$ quiver on $\mathbb{C}_{\widetilde{\epsilon}_1}\times S^1_\gamma$, with $\widetilde{\epsilon}_1$ parameter of Omega background on $\mathbb{C}$.

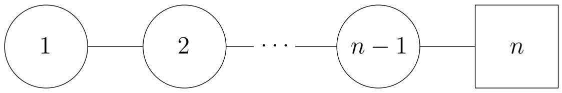


FIGURE 5.1: The $T[U(n)]$ quiver

The $T[U(n)]$ theory has gauge group $G = \otimes_{s=1}^{n-1} U(s)$, with an associated $\mathcal{N} = 4$ vector multiplet for each factor in G, and $\mathcal{N} = 4$ hypermultiplets in the bifundamental of $U(s) \otimes U(s+1)$ with $s = 1, \ldots, n-1$, where the last group $U(n)$ is intended as a flavour group. This theory depends on two sets of (exponentiated) parameters: first of all we have the twisted masses μ_a, $a = 1, \ldots, n$ for the $U(n)$ flavour group; then there are the Fayet-Iliopoulos parameters τ_i with $i =, 1 \ldots, n$[1]. Moreover, we turn on the canonical $\mathcal{N} = 2^*$ deformation parameter t, which corresponds to a twisted mass parameter for the adjoint $\mathcal{N} = 2$ chiral multiplets contained inside the $\mathcal{N} = 4$ vector multiplets.

To understand how this gauge theory is related to the tRS system, we have to analyse the supersymmetric vacua in the Coulomb branch. The theory in the Coulomb branch is described by the twisted effective superpotential $\widetilde{\mathcal{W}}_{\text{eff}}(\vec{\mu}, \vec{\tau}, t, \vec{\sigma})$, with $\sigma_\alpha^{(s)}$ scalars in the vector multiplets of the Cartan of G. When the Omega background parameter is turned off, one can show that the equations

$$\exp\left(\frac{\partial \widetilde{\mathcal{W}}_{\text{eff}}}{\partial \sigma_\alpha^{(s)}}\right) = 1 \tag{5.15}$$

determining the supersymmetric vacua, i.e. the twisted chiral ring relations, reduce to a classical version of the Hamiltonian (5.1), in which the operator $T_{q,i}$ is replaced by the classical momentum $p_{\tau_i} \sim \exp(\partial \widetilde{\mathcal{W}}_{\text{eff}} / \partial \tau_i)$. The Omega background quantizes the system by turning p_{τ_i} to $T_{q,i}$, with $q \sim e^{i\gamma\widetilde{\epsilon}_1}$; the vacua equation now becomes an operator equation annihilating the partition function of the $T[U(n)]$ theory on $\mathbb{C}_{\widetilde{\epsilon}_1} \times S^1_\gamma$, which therefore corresponds to the eigenfunction of the quantum Hamiltonian (5.1). This partition function coincides with the holomorphic blocks B_l of the $T[U(n)]$ theory ($l = 1 \ldots, n!$), which can be obtained from the partition function of our theory on the squashed three-sphere S_b^3 as

$$Z_{S_b^3}(\vec{\mu}, \vec{\tau}, t, q) = \sum_{l=1}^{n!} |B_l(\vec{\mu}, \vec{\tau}, t, q)|^2 \tag{5.16}$$

after an appropriate identification of $\widetilde{\epsilon}_1$ with the squashing parameter b. The corresponding eigenvalues are given by VEV $\langle W_\square^{SU(n)} \rangle = \mu_1 + \ldots + \mu_n$ of a flavour Wilson

[1]Here we introduced an additional topological $U(1)$ as in [132], so that the physical FI parameter at the s-th gauge node is τ_{j+1}/τ_j.

loop wrapping S^1_γ in the fundamental representation of $U(n)$.

Clearly, the situation is still incomplete. In our gauge theory version of tRS we have additional parameters μ_a which were not appearing in the original system discusses in the previous section, both in the eigenfunction and in the eigenvalue. Moreover, we notice that the holomorphic blocks B_l are infinite series in the FI parameters and have to be thought as formal eigenstates, as they might not be normalizable. So, in a sense, we found an *off-shell* solution for tRS.

This issue was already clarified in [114]. The point is that the gauge theory related to the tRS and eRS systems is not really $T[U(n)]$, but the 5d $\mathcal{N} = 1^*$ $U(n)$ theory on $\mathbb{C}^2_{\tilde{\epsilon}_1,\tilde{\epsilon}_2} \times S^1_\gamma$ in the so-called Nekrasov-Shatashvili (NS) limit $\tilde{\epsilon}_2 \to 0$ in presence of codimension 2 and 4 defects; these correspond respectively to eigenfunctions and eigenvalues for tRS or eRS. In fact, the $T[U(n)]$ theory can be thought of as a different way of describing a codimension two monodromy defect for this 5d theory; 3d and 5d theories can be coupled by gauging the $U(n)$ flavour group of $T[U(n)]$, and in the decoupling limit we remain with just the 3d theory. The mass m for the adjoint field in the 5d $\mathcal{N} = 2$ vector multiplet breaks supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1^*$ and coincides with the parameter t of 3d $\mathcal{N} = 2^*$ deformation as $t \sim e^{-i\gamma m}$, while the 3d twisted masses μ_a correspond to the 5d Coulomb branch moduli. These moduli are fixed by the Coulomb branch vacua equations for the 5d theory in the NS limit given in [114]; considering the solutions in which the Coulomb branch meets the Higgs branch of the theory (the special loci known as the *Higgs branch root* [133]) ensures normalizability of the eigenstates. These solutions are labelled by partitions λ of length n of an integer $k \leqslant n$; explicitly, we obtain

$$\mu_a = q^{\lambda_a} t^{n-a} \quad , \quad a = 1, \dots, n \tag{5.17}$$

At these values the series B_l truncate to the Macdonald polynomial corresponding to λ, and $\langle W^{SU(n)}_\square \rangle$ reduces to (5.5). We are therefore able to recover the complete tRS solution in purely gauge theoretical terms, in the limit in which the 5d theory is decoupled from the 3d one; if we consider instead the coupled system we reproduce the eRS system, where the elliptic deformation parameter p is given by $Q = e^{-8\pi^2\gamma/g^2_{YM}}$ with g_{YM} 5d Yang-Mills coupling and the μ_a are still given by (5.17).

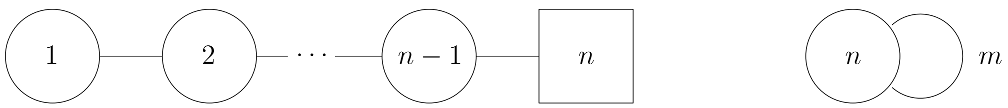


FIGURE 5.2: The 3d $T[U(n)]$ theory as a defect for the 5d $U(n)$ $\mathcal{N} = 1^*$ theory

To sum up, the tRS/gauge theory dictionary can be summarized as follows:

quantum tRS	**3d-5d gauge theory (decoupled)**
number of particles n	rank 3d flavour group
particle positions τ_j	3d Fayet-Iliopoulos parameters
interaction coupling t	3d $\mathcal{N} = 2^*$ deformation parameter
shift parameter q	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
partitions λ of $k \leqslant n$	5d Coulomb-Higgs vacua (fix μ_a)
eigenvalue $E_{tRS}^{(\lambda;n)}$	$\langle W_{\square}^{SU(n)}\rangle$ for flavour $U(n)$ at fixed μ_a
eigenfunctions $P_\lambda(\vec{\tau}; q, t)$	holomorphic blocks B_l at fixed μ_a

while for the eRS/gauge theory dictionary we have

quantum eRS	**3d-5d gauge theory (coupled)**
number of particles n	rank 3d flavour group / 5d gauge group
particle positions τ_j	3d Fayet-Iliopoulos parameters
interaction coupling t	3d $\mathcal{N} = 2^*$ / 5d $\mathcal{N} = 1^*$ deformation $e^{-i\gamma m}$
shift parameter q	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic deformation p	3d-5d coupling parameter Q
partitions λ of $k \leqslant n$	5d Coulomb-Higgs vacua (fix μ_a)
eigenvalue $E_{tRS}^{(\lambda;n)}$	$\langle W_{\square}^{SU(n)}\rangle$ for 5d $U(n)$ in NS limit at fixed μ_a
eigenfunctions	$Z_{\rm inst}^{3d-5d}$ in NS limit at fixed μ_a

Reinterpreting the eRS system in terms of a supersymmetric gauge theory can help us in explicitly computing eigenvalues and eigenfunctions of this integrable system perturbatively in Q, as done in [132], because of our good understanding of instanton computations. For example, in [132] the authors computed the first correction in Q to the eigenvalue

$$\langle W_{\square}^{SU(n)}\rangle = \langle W_{\square}^{U(n)}\rangle / \langle W_{\square}^{U(1)}\rangle \tag{5.18}$$

that is

$$\langle W_{\square}^{U(n)}\rangle = \sum_{a=1}^{n}\mu_a - Q\frac{(q-t)(1-t)}{qt^n}\sum_{a=1}^{n}\mu_a\prod_{\substack{b=1\\ b\neq a}}^{n}\frac{(\mu_a - t\mu_b)(t\mu_a - q\mu_b)}{(\mu_a-\mu_b)(\mu_a - q\mu_b)} + o(Q^2) \tag{5.19}$$

$$\langle W_{\square}^{U(1)}\rangle = \frac{(Qt^{-1};Q)_\infty (Qtq^{-1};Q)_\infty}{(Q;Q)_\infty (Qq^{-1};Q)_\infty} \tag{5.20}$$

This formula will become important later in this chapter.

5.4 Free field realization of Ruijsenaars systems

As we saw in section 4.5.2, the trigonometric and elliptic Calogero-Sutherland models determine the pole dynamics of multi-soliton solutions of the Benjamin-Ono or ILW systems. If we think of the trigonometric and elliptic Ruijsenaars-Schneider models as finite-difference versions of Calogero-Sutherland, we can therefore expect them to determine the pole dynamics of solitons for the finite-difference version of BO or ILW studied in [130, 134, 135], although to the best of our knowledge this correspondence has not been studied in the literature in detail.
Fortunately, the same problem has been discussed in mathematical terms in [131]. There the authors considered the collective coordinate description of quantum tRS in terms of a deformed Heisenberg algebra, finding a relation with the so-called Ding-Iohara algebra. This collective coordinate description allows one to consider tRS independently of n and reduces to a quantum integrable system with an infinite number of commuting Hamiltonians, which has later been interpreted as the finite-difference BO system [134, 135]. Similarly, the twisted elliptic deformation of the Ding-Iohara algebra studied in [131] has been recognized as the finite-difference version of ILW in [130]. Here we will briefly review the results of [131] which are relevant for our discussion; the finite-difference versions of BO and ILW will be introduced in the next section.

5.4.1 The trigonometric case

Let us start by considering the collective coordinate description of tRS. In order to do so, we will first need to introduce the Macdonald symmetric functions; we will follow the conventions of [136, 137, 138]. Let $\Lambda_n(q,t) = \mathbb{Q}(q,t)[\tau_1,\ldots,\tau_n]^{\mathfrak{S}_n}$ be the space of n-variables symmetric polynomials over $\mathbb{Q}(q,t)$, with $\mathfrak{S}_n$ the n-th symmetric group. As in section 5.2, let us introduce the power sum symmetric polynomials $p_m = \sum_{l=1}^n \tau_l^m$ and define $p_\lambda = p_{\lambda_1}\cdots p_{\lambda_{l(\lambda)}}$ for a partition of size $|\lambda| = k$ and length $l(\lambda) = \#\{i : \lambda_i \neq 0\}$. Now, let $\rho_n^{n+1} : \Lambda_{n+1}(q,t) \to \Lambda_n(q,t)$ be the homomorphism given by

$$(\rho_n^{n+1} f)(\tau_1,\ldots,\tau_n) = f(\tau_1,\ldots,\tau_n,0) \quad \text{for} \quad f \in \Lambda_{n+1}(q,t) \tag{5.21}$$

and define the ring of symmetric functions $\Lambda(q,t)$ as the projective limit defined by $\{\rho_n^{n+1}\}_{n\geqslant 1}$

$$\Lambda(q,t) = \lim_{\longleftarrow n} \Lambda_n(q,t) \tag{5.22}$$

The set $\{p_\lambda\}$ forms a basis of $\Lambda(q,t)$. By defining $n_\lambda(a) = \#\{i : \lambda_i = a\}$ and

$$z_\lambda = \prod_{a\geqslant 1} a^{n_\lambda(a)} n_\lambda(a)! \quad , \quad z_\lambda(q,t) = z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \tag{5.23}$$

we can introduce the inner product

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda(q,t) \tag{5.24}$$

The set $\{\tilde{p}_\lambda\} = \{z_\lambda^{-1}(q,t) p_\lambda\}$ will therefore be a dual basis with respect to $\{p_\lambda\}$ under the inner product (5.24); moreover we have

$$\sum_\lambda p_\lambda(\tau)\tilde{p}_\lambda(\tilde{\tau}) = \prod(q,t)(\tau,\tilde{\tau}) \tag{5.25}$$

in terms of the so-called reproduction kernel

$$\prod(q,t)(\tau,\tilde{\tau}) = \prod_{i,j\geqslant 1} \frac{(t\tau_i\tilde{\tau}_j;q)_\infty}{(\tau_i\tilde{\tau}_j;q)_\infty} \quad , \quad (a;q)_\infty = \prod_{s\geqslant 0}(1-aq^s) \tag{5.26}$$

This is true more in general: given two basis $\{u_\lambda\}$, $\{v_\lambda\}$ of $\Lambda(q,t)$, they are dual under (5.24) if and only if $\sum_\lambda u_\lambda(\tau)v_\lambda(\tilde{\tau}) = \prod(q,t)(\tau,\tilde{\tau})$; in this sense, the form of the inner product is determined by the form of the kernel function. For our discussion, the relevant basis of symmetric functions is given by the Macdonald basis $\{P_\lambda(\tau;q,t)\}$, uniquely determined by the conditions

$$\begin{aligned} &(1) \quad P_\lambda(\tau;q,t) = m_\lambda(\tau) + \sum_{\mu<\lambda} u_{\lambda\mu}(q,t) m_\mu(\tau) \quad \text{with} \quad u_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t) \\ &(2) \quad \langle P_\lambda(\tau;q,t), P_\mu(\tau;q,t)\rangle_{q,t} = 0 \quad \text{for} \quad \lambda \neq \mu \end{aligned} \tag{5.27}$$

where $m_\lambda(\tau)$ are monomial symmetric functions and $\lambda > \mu \Longleftrightarrow |\lambda| = |\mu|$ with $\lambda_1 + \ldots + \lambda_i \geqslant \mu_1 + \ldots + \mu_i$ for all i. From the functions $P_\lambda(\tau;q,t)$ we recover the n-variables Macdonald polynomials as $P_\lambda(\tau_1,\ldots,\tau_n;q,t) = P_\lambda(\tau_1,\ldots,\tau_n,0,0,\ldots;q,t)$; these are eigenstates of the Hamiltonians (5.1), (5.3) and satisfy (5.4).

We are now ready to discuss the free field realization of the tRS Hamiltonian (5.1). The idea here is to introduce a (q,t)-deformed version of the Heisenberg algebra $\mathcal{H}(q,t)$, with generators a_m ($m \in \mathbb{Z}$) and commutation relations

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0} \tag{5.28}$$

A canonical basis in the Fock space of $\mathcal{H}(q,t)$ is given by the set of states $a_{-\lambda}|0\rangle =$

$a_{-\lambda_1} \cdots a_{-\lambda_{l(\lambda)}}|0\rangle$ depending on a partition λ; a generic state will be a linear combination of the basis ones, with coefficients in $\mathbb{Q}(q,t)$. Let us notice that the bra-ket product among basis states is such that

$$\langle 0|0\rangle = 1 \quad , \quad \langle 0|a_\lambda a_{-\mu}|0\rangle = \delta_{\lambda,\mu} z_\lambda(q,t) \tag{5.29}$$

and therefore coincides with the inner product (5.24). This is in agreement with the natural isomorphism between this Fock space and $\Lambda(q,t)$, simply given by

$$a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda \tag{5.30}$$

for a fixed partition λ. Now, in order to reproduce $D^{(1)}_{n,\vec{\tau}}$ in terms of bosonic operators, we follow [131] (see also [136, 137, 138]) and introduce the vertex operators

$$\begin{aligned} \eta(z) &= \exp\left(\sum_{n>0} \frac{1-t^{-n}}{n} a_{-n} z^n\right) \exp\left(-\sum_{n>0} \frac{1-t^n}{n} a_n z^{-n}\right) \\ &=: \exp\left(-\sum_{n\neq 0} \frac{1-t^n}{n} a_n z^{-n}\right) := \sum_{n\in\mathbb{Z}} \eta_n z^{-n} \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} \xi(z) &= \exp\left(-\sum_{n>0} \frac{1-t^{-n}}{n} (tq^{-1})^{n/2} a_{-n} z^n\right) \exp\left(\sum_{n>0} \frac{1-t^n}{n} (tq^{-1})^{n/2} a_n z^{-n}\right) \\ &=: \exp\left(\sum_{n\neq 0} \frac{1-t^n}{n} (tq^{-1})^{|n|/2} a_n z^{-n}\right) := \sum_{n\in\mathbb{Z}} \xi_n z^{-n} \end{aligned} \tag{5.32}$$

together with

$$\phi(z) = \exp\left(\sum_{n>0} \frac{1-t^n}{1-q^n} a_{-n} \frac{z^n}{n}\right) \quad , \quad \phi^*(z) = \exp\left(\sum_{n>0} \frac{1-t^n}{1-q^n} a_n \frac{z^n}{n}\right) \tag{5.33}$$

By defining $\phi_n(\tau) = \prod_{i=1}^n \phi(\tau_i)$ one can show that the kernel function is reproduced by the operators $\phi_n(\tau)$, $\phi_n^*(\tau)$ as

$$\langle 0|\phi_n^*(\tau)\phi_n(\tilde{\tau})|0\rangle = \prod(q,t)(\tau,\tilde{\tau}) \tag{5.34}$$

while the action of $D^{(1)}_{n,\vec{\tau}}$ in terms of a_n oscillators can be expressed by the formulae

$$\begin{aligned} [\eta(z)]_1 \phi_n(\tau)|0\rangle &= \left[t^{-n} + t^{-n+1}(1-t^{-1}) D^{(1)}_{n,\vec{\tau}}(q,t)\right] \phi_n(\tau)|0\rangle \\ [\xi(z)]_1 \phi_n(\tau)|0\rangle &= \left[t^n + t^{n-1}(1-t) D^{(1)}_{n,\vec{\tau}}(q^{-1},t^{-1})\right] \phi_n(\tau)|0\rangle \end{aligned} \tag{5.35}$$

where $[\ \ ]_1$ means the constant term in z, so that for example $[\eta(z)]_1 = \eta_0$. For completeness, let us mention here that the action of the higher order Hamiltonians $D^{(r)}_{n,\vec{\tau}}$ in terms of bosonic fields is given by the operators

$$\mathcal{O}_r(q,t) = \left[\frac{\epsilon_r(z_1,\dots,z_r)}{\prod_{1\leqslant i<j\leqslant r}\omega(z_i,z_j)} \eta(z_1)\dots\eta(z_r) \right]_1 \tag{5.36}$$

where

$$\begin{aligned} \omega(z_i,z_j) &= \frac{(z_i - q^{-1}z_j)(z_i - tz_j)(z_i - qt^{-1}z_j)}{(z_i - z_j)^3} \\ \epsilon_r(z_1,\dots,z_r) &= \prod_{1\leqslant i<j\leqslant r} \frac{(z_i - tz_j)(z_i - t^{-1}z_j)}{(z_i - z_j)^2} \end{aligned} \tag{5.37}$$

It is easy to see that by taking the normal order product these operators reduce to

$$\mathcal{O}_r(q,t) = \left[\prod_{1\leqslant i<j\leqslant r} \frac{(z_i - z_j)^2}{(z_i - qz_j)(z_i - q^{-1}z_j)} : \eta(z_1)\dots\eta(z_r) : \right]_1 \tag{5.38}$$

For $r=1$ we immediately recover $\mathcal{O}_1 = [\eta(z)]_1 = \eta_0$.

As a final comment, let us discuss the relation between these vertex operators and the free field realization of the quantum Ding-Iohara algebra $\mathcal{U}(q,t)$. Set

$$g(z) = \frac{G^+(z)}{G^-(z)} \quad , \quad G^\pm(z) = (1 - q^{\pm 1}z)(1 - t^{\mp 1}z)(1 - q^{\mp 1}t^{\pm 1}z) \tag{5.39}$$

Notice that $g(z) = g(z^{-1})^{-1}$. By definition, $\mathcal{U}(q,t)$ is the unital associative algebra generated by the currents

$$x^\pm(z) = \sum_{n\in\mathbb{Z}} x^\pm_n z^{-n} \quad , \quad \psi^\pm(z) = \sum_{\pm n\in\mathbb{N}} \psi^\pm_n z^{-n} \tag{5.40}$$

and the central element $\gamma^{\pm 1/2}$ satisfying

$$\begin{aligned} &[x^+(z), x^-(w)] = \frac{(1-q)(1-t^{-1})}{1-qt^{-1}} \Big(\delta(\gamma^{-1}z/w)\psi^+(\gamma^{1/2}w) - \delta(\gamma z/w)\psi^-(\gamma^{-1/2}w) \Big) \\ &x^\pm(z)x^\pm(w) = g(z/w)^{\pm 1} x^\pm(w)x^\pm(z) \\ &\psi^\pm(z)\psi^\pm(w) = \psi^\pm(w)\psi^\pm(z) \\ &\psi^+(z)\psi^-(w) = \frac{g(\gamma w/z)}{g(\gamma^{-1}w/z)} \psi^-(w)\psi^+(z) \\ &\psi^+(z)x^\pm(w) = g(\gamma^{\mp 1/2}w/z)^{\mp 1} x^\pm(w)\psi^+(z) \\ &\psi^-(z)x^\pm(w) = g(\gamma^{\mp 1/2}z/w)^{\pm 1} x^\pm(w)\psi^-(z) \end{aligned} \tag{5.41}$$

where we used the formal expression $\delta(z) = \sum_{m\in\mathbb{Z}} z^m$ for the delta function. The claim, shown in [131, 136, 137, 138], is that there is a representation ρ of $\mathcal{U}(q,t)$ on the Fock space of our Heisenberg algebra, given by

$$\rho(\gamma) = \left(tq^{-1}\right)^{1/2} \quad , \quad \rho(x^+(z)) = \eta(z) \quad , \quad \rho(x^-(z)) = \xi(z) \quad , \quad \rho(\psi^\pm(z)) = \varphi^\pm(z) \tag{5.42}$$

with

$$\begin{aligned} \varphi^+(z) &=: \eta(\gamma^{1/2}z)\xi(\gamma^{-1/2}z) := \\ &= \exp\left(-\sum_{n>0}\frac{1-t^n}{n}(tq^{-1})^{-n/4}(1-(tq^{-1})^n)a_n z^{-n}\right) = \sum_{n\in\mathbb{N}}\varphi_n^+ z^{-n} \\ \varphi^-(z) &=: \eta(\gamma^{-1/2}z)\xi(\gamma^{1/2}z) : \\ &= \exp\left(\sum_{n>0}\frac{1-t^{-n}}{n}(tq^{-1})^{-n/4}(1-(tq^{-1})^n)a_{-n} z^{n}\right) = \sum_{n\in\mathbb{N}}\varphi_{-n}^- z^{n} \end{aligned} \tag{5.43}$$

An important point to notice is that since $[\varphi^\pm(z)]_1 = 1$ we get $[\eta_0, \xi_0] = 0$, which corresponds to the commutativity $[D_n^{(1)}(q,t), D_n^{(1)}(q^{-1},t^{-1})] = 0$ of the Macdonald operators.

5.4.2 The elliptic case

We can now turn to the collective coordinates description of the eRS model. The goal would be to find an elliptic analogue of the family of commuting operators (5.38) containing (5.13), and an associated elliptic version $\mathcal{U}(q,t,pq^{-1}t)$ of the Ding-Iohara algebra. It turns out that there are many ways to introduce an elliptic deformation of this algebra: for example, the one in [131] differs by construction from the one in [136, 137, 138]; for what we are interested in, the version of [131] is the most relevant one. In this section we just recollect the main formulas we will need for the upcoming discussion.

In the elliptic case, the vertex operator gets modified as

$$\eta(z;pq^{-1}t) = \exp\left(\sum_{n>0}\frac{1-t^{-n}}{n}\frac{1-(pq^{-1}t)^n}{1-p^n}a_{-n}z^n\right)\exp\left(-\sum_{n>0}\frac{1-t^n}{n}a_n z^{-n}\right) \tag{5.44}$$

with p parameter of elliptic deformation. The elliptic commuting operators $\mathcal{O}_r(q,t;p)$ are constructed from (5.44) as in (5.36), with the ω and ϵ_r functions replaced by

$$\omega(z_i,z_j;p) = \frac{\Theta_p(q^{-1}z_j/z_i)\Theta_p(tz_j/z_i)\Theta_p(qt^{-1}z_j/z_i)}{\Theta_p(z_j/z_i)^3} \tag{5.45}$$

$$\epsilon_r(z_1, \ldots, z_r; p) = \prod_{1 \leqslant i < j \leqslant r} \frac{\Theta_p(tz_j/z_i)\Theta_p(t^{-1}z_j/z_i)}{\Theta_p(z_j/z_i)^2} \tag{5.46}$$

where

$$\Theta_p(z) = (p;p)_\infty (z;p)_\infty (pz^{-1};p)_\infty \tag{5.47}$$

The analogue of equation (5.35), now relating the eRS Hamiltonian to its bosonic operator version, reads

$$\left[\eta(z;pq^{-1}t)\right]_1 \phi_n(\tau;p) = \phi_n(\tau;p) \left[t^{-n} \prod_{i=1}^{n} \frac{\Theta_p(qt^{-1}z/\tau_i)}{\Theta_p(qz/\tau_i)} \frac{\Theta_p(tz/\tau_i)}{\Theta_p(z/\tau_i)} \eta(z;pq^{-1}t)\right]_1$$

$$+ t^{-n+1}(1-t^{-1}) \frac{(pt^{-1};p)_\infty (ptq^{-1};p)_\infty}{(p;p)_\infty (pq^{-1};p)_\infty} D^{(1)}_{n,\vec{\tau}}(q,t;p)\phi_n(\tau;p) \tag{5.48}$$

with $\phi_n(\tau;p) = \phi(\tau_1, \ldots, \tau_n;p)$ the opportune elliptic generalization of (5.33); see [131] for further details. The interesting conjecture of [131], which we will verify in few cases in the following sections, is that

$$\lim_{n\to\infty} \left[t^{-n} \prod_{i=1}^{n} \frac{\Theta_p(qt^{-1}z/\tau_i)}{\Theta_p(qz/\tau_i)} \frac{\Theta_p(tz/\tau_i)}{\Theta_p(z/\tau_i)} \eta(z;pq^{-1}t)\right]_1 |0\rangle = 0 \tag{5.49}$$

As we will see, the limit $n \to \infty$ allows us to recover information about the finite-difference version of ILW starting from the eRS system, and can be intuitively understood as a hydrodynamic limit of eRS. From the gauge theory point of view, this limit will lead to a remarkable relation between the 3d-5d coupled system of section 5.3 and the ADHM theory on $S^2 \times S^1$.

5.5 The finite-difference ILW system

In the previous section, we saw how the tRS and eRS systems can be described in terms of bosonic Heisenberg operators. In [130, 134, 135] this free field representation has been interpreted as a realization of the finite-difference version of the Benjamin-Ono and ILW systems respectively (ΔBO and ΔILW for short). Scope of this section is to introduce the main properties of these new hydrodynamic systems. The discussion will necessarily be incomplete, since to the best of our knowledge these equations have received extremely little attention in the literature; we refer the reader to [130, 134, 135] for further details.

The finite-difference version of the ILW equation has been studied in the classical limit in [130] and reads

$$\frac{\partial}{\partial t_0}\eta(z,t_0) = \frac{i}{2}\eta(z,t_0)P.V.\int_{-1/2}^{1/2}(\Delta_\gamma\zeta)(\pi(w-z))\cdot\eta(w,t_0)dw \tag{5.50}$$

where the discrete Laplacian Δ_γ is defined as $(\Delta_\gamma f)(x) = f(x+\gamma) - 2f(x) + f(x-\gamma)$ and γ is a complex number. It is easy to show that in the limit $\gamma \to 0$ (5.50) reduces to (4.66), after an appropriate Galilean transformation on $\eta(z,t_0)$. The finite-difference Benjamin-Ono limit of this equation has been studied in greater detail in [134, 135].

The ΔILW system has a deep connection to the eRS and the elliptic deformation of the Ding-Iohara algebra we discussed in the previous section. In fact, the Hamiltonians $\mathcal{H}_r$ for classical ΔILW given in [130] are exactly reproduced by a certain classical limit of the commuting operators $\mathcal{O}_r$ introduced in section 5.4.2; we therefore propose our $\mathcal{O}_r$ to be the quantum Hamiltonians $\widehat{\mathcal{H}}_r$ for quantum ΔILW. Moreover, the $\eta(z;pq^{-1}t)$ field of (5.44) can be shown to satisfy (5.50) in the sense of (4.72), where this time the Hamiltonian generating the time evolution of the system is $\mathcal{H}_1$.

Now, since ΔILW reduces to ILW as $\gamma \to 0$, and taking into account that the time evolution for quantum ΔILW will be given by $\widehat{\mathcal{H}}_1 = \eta_0$, we expect η_0 to be a *generating function* for the ILW quantum Hamiltonians $\widehat{I}_l$ of section 4.5. This is also in agreement with an observation made in [131], which relates the γ expansion of η_0 to the operator of quantum multiplication in the small quantum cohomology ring of the instanton moduli space $\mathcal{M}_{k,1}$ [106]; this is now not surprising, since we already discussed in Chapter 4 how this operator of quantum multiplication is identified with the quantum ILW Hamiltonian $\widehat{I}_3$. Let us show how this works in practice. In order to avoid confusion with the notation, we will rename the Kähler modulus of $\mathcal{M}_{k,1}$ as $\tilde{t}$ instead of t which was used in Chapter 4; the quantum cohomology parameter will be denoted as $\tilde{p} = e^{-2\pi\tilde{t}}$. For reasons which will be clear in the next sections, the elliptic deformation parameter p and the quantum cohomology parameter $\tilde{p}$ have to be identified as

$$p = -\tilde{p}\sqrt{qt^{-1}} \tag{5.51}$$

Moreover, let us reparametrize q and t as $q = e^{i\gamma\epsilon_1}$ and $t = e^{-i\gamma\epsilon_2}$ in order to compare with the gauge theory results of the next section. We can now rewrite (5.44) as

$$\eta(z;pq^{-1}t) = \exp\left(\sum_{n>0}\lambda_{-n}z^n\right)\exp\left(\sum_{n>0}\lambda_n z^{-n}\right) \tag{5.52}$$

with commutation relations for the λ_m

$$[\lambda_m,\lambda_n] = -\frac{1}{m}\frac{(1-q^m)(1-t^{-m})(1-(pq^{-1}t)^m)}{1-p^m}\delta_{m+n,0} \tag{5.53}$$

It is actually more convenient to go to the standard normalization for the oscillators, by defining

$$\lambda_m = \frac{1}{|m|}\sqrt{-\frac{(1-q^{|m|})(1-t^{-|m|})(1-(pq^{-1}t)^{|m|})}{1-p^{|m|}}}\overline{a}_m \tag{5.54}$$

with commutation relations

$$[\overline{a}_m, \overline{a}_n] = m\delta_{m+n,0} \tag{5.55}$$

After substituting $p = -\tilde{p}\sqrt{qt^{-1}}$ we arrive at

$$\begin{aligned}\lambda_m &= \frac{1}{|m|}\sqrt{-\frac{(1-q^{|m|})(1-t^{-|m|})(1-(-\tilde{p}q^{-1/2}t^{1/2})^{|m|})}{1-(-\tilde{p}q^{1/2}t^{-1/2})^{|m|}}}\overline{a}_m = \\ &= \gamma^2\sqrt{\epsilon_1\epsilon_2}\Bigg[1 + i\gamma\frac{\epsilon_1+\epsilon_2}{4}m\frac{1+(-\tilde{p})^m}{1-(-\tilde{p})^m} + \\ &\quad + \gamma^2\left(-\frac{(\epsilon_1+\epsilon_2)^2}{8}m^2\frac{(-\tilde{p})^m}{(1-(-\tilde{p})^m)^2} - m^2\frac{5(\epsilon_1+\epsilon_2)^2-4\epsilon_1\epsilon_2}{96}\right) + \dots\Bigg]\overline{a}_m\end{aligned} \tag{5.56}$$

We therefore end up with the generating function for the ILW Hamiltonians $\widehat{I}_l$

$$\eta_0 = [\eta(z; -\tilde{p}q^{-1/2}t^{1/2})]_1 = 1 + \gamma^2\widehat{I}_2 + \gamma^3\widehat{I}_3 + \gamma^4\widehat{I}_4 + \dots \tag{5.57}$$

where the first few Hamiltonians are given by

$$\widehat{I}_2 = \sum_{m>0}\overline{a}_{-m}\overline{a}_m \tag{5.58}$$

$$\widehat{I}_3 = i\frac{\epsilon_1+\epsilon_2}{2}\sum_{m>0} m\frac{1+(-\tilde{p})^m}{1-(-\tilde{p})^m}\,\overline{a}_{-m}\overline{a}_m + \frac{1}{2}\sum_{m,n>0}(\overline{a}_{-m-n}\overline{a}_m\overline{a}_n + \overline{a}_{-m}\overline{a}_{-n}\overline{a}_{m+n}) \tag{5.59}$$

and

$$\begin{aligned}\widehat{I}_4 &= \frac{1}{6}\sum_{m,n,l>0}(\overline{a}_{-m-n-l}\overline{a}_m\overline{a}_n\overline{a}_l + \overline{a}_{-m}\overline{a}_{-n}\overline{a}_{-l}\overline{a}_{m+n+l}) + \frac{1}{4}\sum_{\substack{m,n,l,k>0\\ m+n=l+k}}\overline{a}_{-m}\overline{a}_{-n}\overline{a}_l\overline{a}_k \\ &+ i\frac{\epsilon_1+\epsilon_2}{8}\sum_{m,n>0}\left[m\frac{1+(-\tilde{p})^m}{1-(-\tilde{p})^m} + n\frac{1+(-\tilde{p})^n}{1-(-\tilde{p})^n} + (m+n)\frac{1+(-\tilde{p})^{m+n}}{1-(-\tilde{p})^{m+n}}\right](\overline{a}_{-m-n}\overline{a}_m\overline{a}_n + \overline{a}_{-m}\overline{a}_{-n}\overline{a}_{m+n}) \\ &- \frac{2(\epsilon_1+\epsilon_2)^2-\epsilon_1\epsilon_2}{12}\sum_{m>0}m^2\overline{a}_{-m}\overline{a}_m - \frac{(\epsilon_1+\epsilon_2)^2}{2}\sum_{m>0}m^2\frac{(-\tilde{p})^m}{(1-(-\tilde{p})^m)^2}\overline{a}_{-m}\overline{a}_m\end{aligned} \tag{5.60}$$

Let us study the eigenvalue problem for these Hamiltonians; this will be needed for comparison with gauge theory results. Denoting by k the eigenvalue of $\widehat{I}_2$, we restrict ourselves to the cases $k = 2$ and $k = 3$ in the following.

The $k = 2$ case

A state with $k = 2$ can generically be written as

$$(c_1 \overline{a}_{-1}^2 + c_2 \overline{a}_{-2})|0\rangle \tag{5.61}$$

in terms of two constants c_1, c_2 to be determined. The eigenvalue equation for the $\widehat{I}_3$ Hamiltonian

$$\begin{aligned} \widehat{I}_3(c_1 \overline{a}_{-1}^2 + c_2 \overline{a}_{-2})|0\rangle &= E_3(c_1 \overline{a}_{-1}^2 + c_2 \overline{a}_{-2})|0\rangle = \\ = \left[\left(c_2 + i(\epsilon_1+\epsilon_2)\frac{1-\tilde{p}}{1+\tilde{p}}c_1\right)\overline{a}_{-1}^2 \right. &+ \left. \left(c_1 + 2i(\epsilon_1+\epsilon_2)\frac{1+\tilde{p}^2}{1-\tilde{p}^2}c_2\right)\overline{a}_{-2}\right]|0\rangle \end{aligned} \tag{5.62}$$

results in an equation for the energy

$$\left(E_3 - i(\epsilon_1+\epsilon_2)\frac{1-\tilde{p}}{1+\tilde{p}}\right)\left(E_3 - 2i(\epsilon_1+\epsilon_2)\frac{1+\tilde{p}^2}{1-\tilde{p}^2}\right) = 1 \tag{5.63}$$

which has the two solutions

$$\begin{aligned} E_3^{(1)} &= i(2\epsilon_1+\epsilon_2) + \tilde{p}\frac{2i(\epsilon_1+\epsilon_2)\epsilon_2}{\epsilon_1-\epsilon_2} + \tilde{p}^2\frac{2i(\epsilon_1+\epsilon_2)(2\epsilon_1^3 - 7\epsilon_1^2\epsilon_2 + 2\epsilon_1\epsilon_2^2 - \epsilon_2^3)}{(\epsilon_1-\epsilon_2)^3} + o(\tilde{p}^3) \\ E_3^{(2)} &= i(\epsilon_1+2\epsilon_2) + \tilde{p}\frac{2i(\epsilon_1+\epsilon_2)\epsilon_1}{\epsilon_2-\epsilon_1} + \tilde{p}^2\frac{2i(\epsilon_1+\epsilon_2)(2\epsilon_2^3 - 7\epsilon_2^2\epsilon_1 + 2\epsilon_2\epsilon_1^2 - \epsilon_1^3)}{(\epsilon_2-\epsilon_1)^3} + o(\tilde{p}^3) \end{aligned} \tag{5.64}$$

Similarly, the eigenvalue equation for $\widehat{I}_4$

$$\widehat{I}_4(c_1 \overline{a}_{-1}^2 + c_2 \overline{a}_{-2})|0\rangle = E_4(c_1 \overline{a}_{-1}^2 + c_2 \overline{a}_{-2})|0\rangle \tag{5.65}$$

results in the equation

$$\begin{aligned} &\left[E_4 + (\epsilon_1+\epsilon_2)^2\left(\frac{1}{3} + \frac{\tilde{p}^2}{(1-\tilde{p}^2)^2}\right) - \frac{2\epsilon_1\epsilon_2}{3}\right]\left[E_4 + 4(\epsilon_1+\epsilon_2)^2\left(\frac{1}{3} + \frac{\tilde{p}^2}{(1-\tilde{p}^2)^2}\right) - \frac{2\epsilon_1\epsilon_2}{3}\right] = \\ &= -\frac{(\epsilon_1+\epsilon_2)^2}{4}\left(\frac{1-\tilde{p}}{1+\tilde{p}} + \frac{1+\tilde{p}^2}{1-\tilde{p}^2}\right)^2 \end{aligned} \tag{5.66}$$

with solutions

$$\begin{aligned} E_4^{(1)} = &-\left(\frac{\epsilon_2^2}{3} + \epsilon_1\epsilon_2 + \frac{4\epsilon_1^2}{3}\right) - \tilde{p}\frac{(\epsilon_1+\epsilon_2)\epsilon_2(3\epsilon_1+\epsilon_2)}{\epsilon_1-\epsilon_2} \\ &+ \tilde{p}^2\frac{2(\epsilon_1+\epsilon_2)(-2\epsilon_1^4 + 7\epsilon_1^3\epsilon_2 + \epsilon_1^2\epsilon_2^2 + \epsilon_1\epsilon_2^3 + \epsilon_2^4)}{(\epsilon_1-\epsilon_2)^3} + o(\tilde{p}^3) \end{aligned}$$

$$
\begin{aligned}
E_4^{(2)} = & -\left(\frac{\epsilon_1^2}{3}+\epsilon_1\epsilon_2+\frac{4\epsilon_2^2}{3}\right)-\tilde{p}\frac{(\epsilon_1+\epsilon_2)\epsilon_1(3\epsilon_2+\epsilon_1)}{\epsilon_2-\epsilon_1} \\
& +\tilde{p}^2\frac{2(\epsilon_1+\epsilon_2)(-2\epsilon_2^4+7\epsilon_2^3\epsilon_1+\epsilon_1^2\epsilon_2^2+\epsilon_2\epsilon_1^3+\epsilon_1^4)}{(\epsilon_2-\epsilon_1)^3}+o(\tilde{p}^3)
\end{aligned} \tag{5.67}
$$

We therefore have two eigenstates, whose constants c_1, c_2 have to satisfy the relations

$$
\begin{aligned}
c_2 &= \left(i\epsilon_1+\tilde{p}\frac{2i\epsilon_1(\epsilon_1+\epsilon_2)}{\epsilon_1-\epsilon_2}+\tilde{p}^2\frac{2i\epsilon_1(\epsilon_1+\epsilon_2)(\epsilon_1^2-4\epsilon_1\epsilon_2-\epsilon_2^2)}{(\epsilon_1-\epsilon_2)^3}+o(\tilde{p}^3)\right)c_1 \\
c_2 &= \left(i\epsilon_2+\tilde{p}\frac{2i\epsilon_2(\epsilon_1+\epsilon_2)}{\epsilon_2-\epsilon_1}+\tilde{p}^2\frac{2i\epsilon_2(\epsilon_1+\epsilon_2)(\epsilon_2^2-4\epsilon_1\epsilon_2-\epsilon_1^2)}{(\epsilon_2-\epsilon_1)^3}+o(\tilde{p}^3)\right)c_1
\end{aligned} \tag{5.68}
$$

The remaining constant c_1 enters only in the normalization of the eigenstates, and will be of no importance for our discussion.

As a final remark, let us notice here that in the Benjamin-Ono limit $\tilde{p}\to 0$ the eigenstates become

$$
\begin{aligned}
&(\overline{a}_{-1}^2+i\epsilon_1\overline{a}_2)|0\rangle \\
&(\overline{a}_{-1}^2+i\epsilon_2\overline{a}_2)|0\rangle
\end{aligned} \tag{5.69}
$$

These can be compared with the $\gamma\to 0$ limit of the Macdonald polynomials of (5.6), given by the Jack polynomials $p_1^2-\frac{\epsilon_1}{\epsilon_2}p_2$ and $p_1^2-p_2$ (eigenfunctions of the trigonometric Calogero-Sutherland system) for the partitions (2,0) and (1,1) respectively. It is easy to see that these Jack polynomials coincide with (5.69) under the isomorphism

$$
\overline{a}_{-m}|0\rangle \longleftrightarrow -i\epsilon_2 p_m \tag{5.70}
$$

The $k=3$ case

A generic state with $k=3$ can be written as

$$
(c_1\overline{a}_{-1}^3+c_2\overline{a}_{-2}\overline{a}_{-1}+c_3\overline{a}_{-3})|0\rangle \tag{5.71}
$$

The eigenvalue equation for $\widehat{I}_3$

$$
\widehat{I}_3(c_1\overline{a}_{-1}^3+c_2\overline{a}_{-2}\overline{a}_{-1}+c_3\overline{a}_{-3})|0\rangle = E_3(c_1\overline{a}_{-1}^3+c_2\overline{a}_{-2}\overline{a}_{-1}+c_3\overline{a}_{-3})|0\rangle \tag{5.72}
$$

leads to an equation for the eigenvalue E_3 with three solutions

$$
\begin{aligned}
E_3^{(1)} &= i\frac{3}{2}(\epsilon_1+\epsilon_2)+3i\epsilon_1+\tilde{p}\frac{3i\epsilon_2(\epsilon_1+\epsilon_2)}{2\epsilon_1-\epsilon_2} \\
&\quad -\tilde{p}^2\frac{3i\epsilon_2(22\epsilon_1^3+18\epsilon_1^2\epsilon_2-3\epsilon_1\epsilon_2^2+\epsilon_2^3)}{(2\epsilon_1-\epsilon_2)^3}+o(\tilde{p}^3) \\
E_3^{(2)} &= i\frac{5}{2}(\epsilon_1+\epsilon_2)-\tilde{p}\frac{2i(\epsilon_1+\epsilon_2)(\epsilon_1^2-7\epsilon_1\epsilon_2+\epsilon_2^2)}{2\epsilon_1^2-5\epsilon_1\epsilon_2+2\epsilon_2^2} \\
&\quad +\tilde{p}^2\frac{2i(20\epsilon_1^7-121\epsilon_1^6\epsilon_2+6\epsilon_1^5\epsilon_2^2+34\epsilon_1^4\epsilon_2^3+34\epsilon_1^3\epsilon_2^4+6\epsilon_1^2\epsilon_2^5-121\epsilon_1\epsilon_2^6+20\epsilon_2^7)}{(2\epsilon_1^2-5\epsilon_1\epsilon_2+2\epsilon_2^2)^3}+o(\tilde{p}^3) \\
E_3^{(3)} &= i\frac{3}{2}(\epsilon_1+\epsilon_2)+3i\epsilon_2+\tilde{p}\frac{3i\epsilon_1(\epsilon_1+\epsilon_2)}{2\epsilon_2-\epsilon_1} \\
&\quad -\tilde{p}^2\frac{3i\epsilon_1(22\epsilon_2^3+18\epsilon_2^2\epsilon_1-3\epsilon_2\epsilon_1^2+\epsilon_1^3)}{(2\epsilon_2-\epsilon_1)^3}+o(\tilde{p}^3)
\end{aligned}
\tag{5.73}
$$

Similarly, the equation for $\widehat{I}_4$

$$
\widehat{I}_4(c_1\overline{a}_{-1}^3+c_2\overline{a}_{-2}\overline{a}_{-1}+c_3\overline{a}_{-3})|0\rangle = E_4(c_1\overline{a}_{-1}^3+c_2\overline{a}_{-2}\overline{a}_{-1}+c_3\overline{a}_{-3})|0\rangle \tag{5.74}
$$

admits non-trivial solutions only for the E_4 energies

$$
\begin{aligned}
E_4^{(1)} &= -\left(\frac{\epsilon_2^2}{2}+\frac{9\epsilon_1\epsilon_2}{4}+\frac{9\epsilon_1^2}{2}\right)-\tilde{p}\frac{3\epsilon_2(\epsilon_1+\epsilon_2)(5\epsilon_1+\epsilon_2)}{2(2\epsilon_1-\epsilon_2)} \\
&\quad +\tilde{p}^2\frac{3\epsilon_2(\epsilon_1+\epsilon_2)(47\epsilon_1^3+2\epsilon_1^2\epsilon_2+\epsilon_1\epsilon_2^2+\epsilon_2^3)}{(2\epsilon_1-\epsilon_2)^3}+o(\tilde{p}^3) \\
E_4^{(2)} &= -\left(\frac{3\epsilon_1^2}{2}+\frac{7\epsilon_1\epsilon_2}{4}+\frac{3\epsilon_2^2}{2}\right)+\tilde{p}\frac{(\epsilon_1+\epsilon_2)^2(\epsilon_1^2-13\epsilon_1\epsilon_2+\epsilon_2^2)}{2\epsilon_1^2-5\epsilon_1\epsilon_2+2\epsilon_2^2} \\
&\quad -\tilde{p}^2\frac{(\epsilon_1+\epsilon_2)^2(40\epsilon_1^6-303\epsilon_1^5\epsilon_2+345\epsilon_1^4\epsilon_2^2-325\epsilon_1^3\epsilon_2^3+345\epsilon_1^2\epsilon_2^4-303\epsilon_1\epsilon_2^5+40\epsilon_2^6)}{(2\epsilon_1^2-5\epsilon_1\epsilon_2+2\epsilon_2^2)^3}+o(\tilde{p}^3) \\
E_4^{(3)} &= -\left(\frac{\epsilon_1^2}{2}+\frac{9\epsilon_1\epsilon_2}{4}+\frac{9\epsilon_2^2}{2}\right)-\tilde{p}\frac{3\epsilon_1(\epsilon_1+\epsilon_2)(5\epsilon_2+\epsilon_1)}{2(2\epsilon_2-\epsilon_1)} \\
&\quad +\tilde{p}^2\frac{3\epsilon_1(\epsilon_1+\epsilon_2)(47\epsilon_2^3+2\epsilon_2^2\epsilon_1+\epsilon_2\epsilon_1^2+\epsilon_1^3)}{(2\epsilon_2-\epsilon_1)^3}+o(\tilde{p}^3)
\end{aligned}
\tag{5.75}
$$

We conclude that there are three eigenstates, labelled by the three partitions (3,0,0), (2,1,0), (1,1,1) of $k=3$. The eigenvalue equations fix the values of c_2 and c_3 in terms of the overall normalization c_1; again, in the limit $\tilde{p}\to 0$ the eigenstates are mapped to Jack polynomials under (5.70).

5.6 Finite-difference ILW from ADHM theory on $S^2 \times S^1$

We discussed in Chapter 4 how the ILW system is related to the ADHM GLSM on S^2 with $N = 1$; in particular, the equations determining the supersymmetric vacua in the Coulomb branch correspond to the Bethe Ansatz Equations for ILW, and the local gauge theory observables $\langle \mathrm{Tr}\, \Sigma^l \rangle$ evaluated at the solutions of these equations give the ILW spectrum. We might therefore expect the finite-difference version of ILW introduced in the previous section to have an analogue in gauge theory; here we propose this gauge theory to be the ADHM quiver on $S^2 \times S^1_\gamma$. Let us see how this works.
First of all, let us consider the case in which $\gamma \ll r$ radius of S^2. Then the IR theory will be effectively two-dimensional. The supersymmetric Coulomb branch vacua equations (4.99) for $N = 1$ will be modified to

$$\begin{aligned} &\sin[\tfrac{\gamma}{2}(\Sigma_s - a)] \prod_{\substack{t=1\\t\neq s}}^k \frac{\sin[\frac{\gamma}{2}(\Sigma_{st} - \epsilon_1)] \sin[\frac{\gamma}{2}(\Sigma_{st} - \epsilon_2)]}{\sin[\frac{\gamma}{2}(\Sigma_{st})] \sin[\frac{\gamma}{2}(\Sigma_{st} - \epsilon)]} = \\ &\tilde{p} \sin[\tfrac{\gamma}{2}(-\Sigma_s + a - \epsilon)] \prod_{\substack{t=1\\t\neq s}}^k \frac{\sin[\frac{\gamma}{2}(\Sigma_{st} + \epsilon_1)] \sin[\frac{\gamma}{2}(\Sigma_{st} + \epsilon_2)]}{\sin[\frac{\gamma}{2}(\Sigma_{st})] \sin[\frac{\gamma}{2}(\Sigma_{st} + \epsilon)]} \end{aligned} \tag{5.76}$$

because of the 1-loop contributions coming from the KK tower of chiral multiplets. Here $\epsilon = \epsilon_1 + \epsilon_2$ and $\tilde{p} = e^{-2\pi \tilde{t}}$ with $\tilde{t}$ Fayet-Iliopoulos parameter. For simplicity, from now on we will set $a = 0$. When $\tilde{t} \to \infty$ (i.e. $\tilde{p} \to 0$), the solutions are labelled by partitions λ of k, and are given by

$$\Sigma_s = (i-1)\epsilon_1 + (j-1)\epsilon_2 \mod 2\pi i \tag{5.77}$$

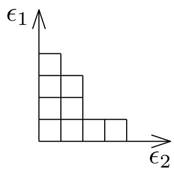


FIGURE 5.3: The partition (4,3,1,1) of $k = 9$

For $\tilde{t}$ finite we can change variables to $\sigma_s = e^{i\gamma\Sigma_s}$, $q = e^{i\gamma\epsilon_1}$, $t = e^{-i\gamma\epsilon_2}$ and rewrite (5.76) as

$$(\sigma_s - 1) \prod_{\substack{t=1\\t\neq s}}^k \frac{(\sigma_s - q\sigma_t)(\sigma_s - t^{-1}\sigma_t)}{(\sigma_s - \sigma_t)(\sigma_s - qt^{-1}\sigma_t)} = \frac{\tilde{p}}{\sqrt{qt^{-1}}} (1 - qt^{-1}\sigma_s) \prod_{\substack{t=1\\t\neq s}}^k \frac{(\sigma_s - q^{-1}\sigma_t)(\sigma_s - t\sigma_t)}{(\sigma_s - \sigma_t)(\sigma_s - q^{-1}t\sigma_t)} \tag{5.78}$$

These are supposed to be the Bethe Ansatz Equations for the finite-difference ILW system. Perturbatively in $\tilde{p}$ small the eigenfunctions are still labelled by partitions of k, and the eigenvalues of the ΔILW Hamiltonians $\widehat{\mathcal{H}}_r$ will be related to $\langle \mathrm{Tr}\,\sigma^r \rangle$ evaluated at the solutions λ of (5.78). In particular, from what we noticed in section 4.5, we expect the combination

$$\mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_\lambda \tag{5.79}$$

(which is just the equivariant Chern character of the $U(1)$ instanton moduli space) to be the eigenvalue of $\widehat{\mathcal{H}}_1$. But now, since $\widehat{\mathcal{H}}_1$ is a generating function for the ILW Hamiltonians $\widehat{I}_l$, our $\mathcal{E}_1$ will be a generating function for the ILW eigenvalues E_l according to

$$\mathcal{E}_1^{(\lambda)} = 1 + \gamma^2 \epsilon_1 \epsilon_2 k + \gamma^3 \epsilon_1 \epsilon_2 E_3^{(\lambda)} + \gamma^4 \epsilon_1 \epsilon_2 E_4^{(\lambda)} + \dots \tag{5.80}$$

This can be verified immediately. Let us list here the eigenvalue $\mathcal{E}_1^{(\lambda)}$ for the solutions of (5.78) at low k:

- Case $k = 0$

$$\mathcal{E}_1^{(\varnothing)} = 1 \tag{5.81}$$

- Case $k = 1$

$$\begin{aligned} \mathcal{E}_1^{(1)} &= (q + t^{-1} - qt^{-1}) - \tilde{p}\sqrt{qt^{-1}} \frac{(1-q)(1-t)(q-t)}{qt} \\ &\quad + \tilde{p}^2 qt^{-1} \frac{(1-q)(1-t)(q-t)}{qt} + o(\tilde{p}^3) \end{aligned} \tag{5.82}$$

- Case $k = 2$, partition $(2,0)$

$$\begin{aligned} \mathcal{E}_1^{(2,0)} &= (q^2 + t^{-1} - q^2 t^{-1}) - \tilde{p}\sqrt{qt^{-1}} \frac{(1-q^2)(1-t)^2(q-t)}{t(1-qt)} \\ &\quad + \tilde{p}^2 \frac{(1-q^2)(1-t)(q-t)}{qt^2(1-qt)^3} [q^3 + t + qt + q^2t^2 + 3q^3t^2 + q^4t^2 + 2q^2t^3 \\ &\quad - 3q^2t - 2q^3t - 2qt^2 - qt^3 - 2q^4t^3] + o(\tilde{p}^3) \end{aligned} \tag{5.83}$$

Expanded in γ as in (5.80), this expression reproduces $E_3^{(1)}$ of (5.64) and $E_4^{(1)}$ of (5.67).

- Case $k = 2$, partition $(1, 1)$

$$\begin{aligned}\mathcal{E}_1^{(1,1)} &= (q + t^{-2} - qt^{-2}) - \tilde{p}\sqrt{qt^{-1}}\frac{(1-q)^2(1-t^2)(q-t)}{qt^2(1-qt)} \\ &+ \tilde{p}^2\frac{(1-q)(1-t^2)(q-t)}{t^3(1-qt)^3}[2 + 2q^2t + 3q^2t^2 + t^3 + 2qt^3 - q - 3qt \\ &- q^3t - 2t^2 - qt^2 - q^2t^3 - q^2t^4] + o(\tilde{p}^3)\end{aligned} \tag{5.84}$$

The expansion in γ reproduces $E_3^{(2)}$ of (5.64) and $E_4^{(2)}$ of (5.67).

- Case $k = 3$, partition $(3, 0, 0)$

$$\begin{aligned}\mathcal{E}_1^{(3,0,0)} &= (q^3 + t^{-1} - q^3t^{-1}) - \tilde{p}\sqrt{qt^{-1}}\frac{q(1-t)^2(1-q^3)(q-t)}{t(1-q^2t)} \\ &+ \tilde{p}^2\frac{(1-t)^2(1-q^3)(q-t)}{t^2(1-q^2t)^3}[q^4 + t + 2qt + q^5t + qt^2 + q^5t^2 + 2q^6t^2 \\ &- q^2t - 3q^3t - 2q^4t - 2q^3t^2 - q^4t^2] + o(\tilde{p}^3)\end{aligned} \tag{5.85}$$

The expansion in γ reproduces $E_3^{(1)}$ of (5.73) and $E_4^{(1)}$ of (5.75).

- Case $k = 3$, partition $(2, 1, 0)$

$$\begin{aligned}\mathcal{E}_1^{(2,1,0)} &= (q^2 + qt^{-1} + t^{-2} - qt^{-2} - q^2t^{-1}) \\ &- \tilde{p}\sqrt{qt^{-1}}\frac{(1-q)(1-t)(q-t)}{qt^2(1-qt^2)(1-q^2t)}[1 + 2qt + 2q^2t^2 + 2q^3t^3 + q^4t^4 \\ &- q^2 - q^3t - 2qt^2 - q^4t^2 - qt^3 - 2q^2t^3] + o(\tilde{p}^2)\end{aligned} \tag{5.86}$$

The expansion in γ reproduces $E_3^{(2)}$ of (5.73) and $E_4^{(2)}$ of (5.75).

- Case $k = 3$, partition $(1, 1, 1)$

$$\begin{aligned}\mathcal{E}_1^{(1,1,1)} &= (q + t^{-3} - qt^{-3}) - \tilde{p}\sqrt{qt^{-1}}\frac{(1-q)^2(1-t^3)(q-t)}{qt^3(1-qt^2)} \\ &+ \tilde{p}^2\frac{(1-q)^2(1-t^3)(q-t)}{t^4(1-qt^2)^3}[2 + t + qt + q^2t^2 + t^5 + 2qt^5 + qt^6 \\ &- t^2 - 2qt^2 - 2t^3 - 3qt^3 - qt^4] + o(\tilde{p}^3)\end{aligned} \tag{5.87}$$

The expansion in γ reproduces $E_3^{(3)}$ of (5.73) and $E_4^{(3)}$ of (5.75).

We can therefore conclude that the ADHM theory on $S^2 \times S^1_\gamma$ is the gauge theory whose underlying integrable system corresponds to ΔILW, as expected from the S^2 case.

5.7 ΔILW as free field Ruijsenaars: the gauge theory side

Let us summarise what we have been doing until now. First of all, in section 5.2 we introduced the n-particles quantum trigonometric and elliptic Ruijsenaars-Schneider models, and in section 5.3 we reformulated them in terms of a 5d $\mathcal{N} = 1^* \; U(n)$ gauge theory in presence of codimension 2 and 4 defects, which correspond respectively to eigenfunctions and eigenvalues of tRS or eRS. This reformulation allows us to perform explicit computations for the eRS system, thanks to our understanding of instantons in supersymmetric gauge theories. In section 5.4 we reviewed the collective coordinate realization of tRS and eRS in terms of free bosons; in section 5.5 this realization has been given an interpretation in terms of a finite-difference version of the Benjamin-Ono and ILW systems, which from the gauge theory point of view are related to the ADHM theory on $S^2 \times S^1_\gamma$ as discussed in section 5.6.
As we have seen, the free boson formalism is a powerful way to relate tRS to ΔBO and eRS to ΔILW. Intuitively, one would expect ΔILW to arise as a hydrodynamic limit of eRS, in which the number of particles n is sent to infinity while keeping the density of particles finite. This can be nicely seen from (5.48) (or its trigonometric version (5.35)), as this equation implies a relation between eRS and ΔILW eigenvalues, which simplifies greatly in the limit $n \to \infty$ if we believe in the conjecture (5.49). Actually, thanks to the gauge theory computations, we will be able to show explicitly the validity of (5.49) at first order in the elliptic deformation p. This would hint to an unexpected equivalence at large n between our 5d theory with defects and the 3d ADHM theory: although we are not able to give a justification in gauge theory of this equivalence at the moment, in this section we will state the correspondence and give computational evidence of its validity.

5.7.1 The trigonometric case: ΔBO from tRS

Let us first consider the equation (5.35) for the trigonometric case, i.e.

$$[\eta(z)]_1 \phi_n(\tau)|0\rangle = \left[t^{-n} + t^{-n+1}(1-t^{-1}) D^{(1)}_{n,\bar{\tau}}(q,t)\right] \phi_n(\tau)|0\rangle \tag{5.88}$$

Here we are taking $t^{-1} < 1$; in the opposite case, we'll just have to consider the second equation in (5.35). We already know that eigenstates and eigenvalues of $[\eta(z)]_1$ are labelled by partitions λ of k and are independent of the length of the partition. In particular, from (5.77) we know that the eigenvalue is given by

$$\mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_{(i,j)\in\lambda} q^{i-1} t^{1-j} = 1 + (1-t^{-1}) \sum_{j=1}^{k} (q^{\lambda_j} - 1) t^{1-j} \tag{5.89}$$

From this expression it is clear that the λ_j which are zero do not contribute to the final result. On the other hand, eigenfunctions and eigenvalues of tRS are also labelled by the same partitions λ of k, but both of them depend on the length n of λ, i.e. on the number of particles. Explicitly, the tRS eigenvalue is given by (5.5)

$$E_{tRS}^{(\lambda;n)} = \sum_{j=1}^{n} q^{\lambda_j} t^{n-j} \tag{5.90}$$

Equation (5.88) is telling us that there is a relation between the ΔBO and tRS eigenvalues: at fixed λ (eigenstate) we have

$$\mathcal{E}_1^{(\lambda)} = t^{-n} + t^{-n+1}(1-t^{-1})E_{tRS}^{(\lambda;n)} \tag{5.91}$$

This equality can be easily shown to be true for all n. In fact

$$\begin{aligned} E_{tRS}^{(\lambda;n)} &= t^{n-1}\sum_{j=1}^{k} q^{\lambda_j} t^{1-j} + t^{n-1}\sum_{j=1}^{n} t^{1-j} - t^{n-1}\sum_{j=1}^{k} t^{1-j} \\ &= t^{n-1}\sum_{j=1}^{k}(q^{\lambda_j}-1)t^{1-j} + t^{n-1}\frac{1-t^{-n}}{1-t^{-1}} \end{aligned} \tag{5.92}$$

which, inserted in (5.91), reproduces (5.89).

Let us now study what happens the limit $n \to \infty$: even if this is not really relevant for the discussion at the trigonometric level, it will become very important when we discuss the elliptic case. First of all, we notice that $\mathcal{E}_1^{(\lambda)}$ and $E_{tRS}^{(\lambda;n)}$ fail to be proportional to each other because of the constant term t^{-n}, which however disappears when $n \to \infty$: this is in agreement with the conjecture (5.49) of [131] considered in the trigonometric limit. Then the right hand side of (5.91) becomes

$$\lim_{n\to\infty}\left[t^{-n} + t^{-n+1}(1-t^{-1})E_{tRS}^{(\lambda;n)}\right] = 1 + (1-t^{-1})\sum_{j=1}^{k}(q^{\lambda_j}-1)t^{1-j} \tag{5.93}$$

and coincides with $\mathcal{E}_1^{(\lambda)}$ of (5.89). Therefore, we can conclude that there are two ways to recover the ΔBO eigenvalue from the tRS one at fixed λ. The first possibility is to use (5.91) as it is: this works for all n, but requires the knowledge of the constant term, which in this case is just t^{-n}. The second possibility consists in taking the limit $n \to \infty$ on the right hand side of (5.91): this method is the most suitable one if one does not know the explicit expression for the constant term, since this is conjectured to vanish in the limit, but requires the knowledge of the tRS eigenvalue for generic n. As we are going to discuss now, for the elliptic case the second way is the only one available to us.

5.7.2 The elliptic case: ΔILW from eRS

At the elliptic level, the equation we have to consider is (5.48)

$$\left[\eta(z;-\tilde{p}q^{-1/2}t^{1/2})\right]_1 \phi_n(\tau;p) = \phi_n(\tau;p)\left[t^{-n}\prod_{i=1}^{n}\frac{\Theta_p(qt^{-1}z/\tau_i)}{\Theta_p(qz/\tau_i)}\frac{\Theta_p(tz/\tau_i)}{\Theta_p(z/\tau_i)}\eta(z;pq^{-1}t)\right]_1$$

$$+\,t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}D^{(1)}_{n,\vec{\tau}}(q,t;p)\phi_n(\tau;p) \tag{5.94}$$

or better its analogue for the eigenvalues

$$\mathcal{E}_1^{(\lambda)}(\tilde{p}) = \left[t^{-n}\prod_{i=1}^{n}\frac{\Theta_p(qt^{-1}z/\tau_i)}{\Theta_p(qz/\tau_i)}\frac{\Theta_p(tz/\tau_i)}{\Theta_p(z/\tau_i)}\eta(z;pq^{-1}t)\right]_1$$

$$+\,t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}E^{(\lambda;n)}_{eRS}(p) \tag{5.95}$$

Unlike the trigonometric case, here we no longer know the constant term in (5.95); therefore, if we want to recover $\mathcal{E}_1^{(\lambda)}(\tilde{p})$ from $E^{(\lambda;n)}_{eRS}(p)$ we should take the large n limit of this equation, which under the conjecture (5.49) reads

$$\mathcal{E}_1^{(\lambda)}(\tilde{p}) = \lim_{n\to\infty}\left[t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}E^{(\lambda;n)}_{eRS}(p)\right] \tag{5.96}$$

Another problem is that we do not have closed form expressions for the eigenvalues; we can only recover them perturbatively around the trigonometric values, thanks to computations in gauge theory. In particular, as we have seen the eigenvalue $\mathcal{E}_1^{(\lambda)}(\tilde{p})$ for ΔILW can be obtained from the ADHM theory on $S^2\times S^1_\gamma$, with parameters identified as $q=e^{i\gamma\epsilon_1}$, $t=e^{-i\gamma\epsilon_2}$, $\tilde{p}=e^{-2\pi\tilde{t}}$, and it is given by (5.79). On the other hand, the eigenvalue $E^{(\lambda;n)}_{eRS}(p)$ for eRS coincides with the Wilson loop (5.18) for the 5d $\mathcal{N}=1^*$ $U(n)$ theory on $\mathbb{C}^2_{\tilde{\epsilon}_1,\tilde{\epsilon}_2}\times S^1_\gamma$ in the NS limit $\tilde{\epsilon}_2\to 0$, with Coulomb branch parameters μ_a fixed by (5.17); in this case $q=e^{i\gamma\tilde{\epsilon}_1}$, $t=e^{-i\gamma m}$ and $p=Q=e^{-8\pi^2\gamma/g^2_{YM}}$. With these results we can verify the conjecture (5.49) by proving the validity of (5.96), at least at order p in the elliptic deformation parameter. Let us show this for the lowest values of k.

- Case $k=0$

  The general strategy is as follows. At fixed n, we consider the $E^{(\lambda;n)}_{eRS}(p)$ eigenvalue (5.18) and evaluate it at the values of μ_a (5.17) corresponding to the length n partition $\lambda=(0,0,\ldots,0)$. After doing this for the lowest values of n, we are able to recognize how the eigenvalue depends on n; with this result we can then study the behaviour at large n. In the case at hand, this procedure gives us

$$\begin{aligned} & t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}E_{eRS}^{((0,0,\dots,0);n)}(p) = \\ & = (1-t^{-n})\left[1+p\frac{(1-q)(1-t)(q-t)}{q^2t(1-q^{-1}t^{1-n})}t^{1-n}+o(p^2)\right] \end{aligned} \tag{5.97}$$

which in the limit $n\to\infty$ is just $1+o(p^2)$, in agreement with (5.81) at order $o(p^2)$.

- Case $k=1$

Here the relevant partition is $\lambda=(1,0,\dots,0)$; the eigenvalue depends on n as

$$\begin{aligned} & t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}E_{eRS}^{((1,0,\dots,0);n)}(p) = \\ & = \left[1-t^{-n}+(q-1)(1-t^{-1})\right] \\ & + p\frac{(1-q)(1-t)(q-t)(1+q^{-1}t^{1-n})}{q^3(1-q^{-1}t^{2-n})(1-q^{-2}t^{1-n})}\left[(1-q)(1-t^{-1})t^{1-n}+q^2t^{-1}(1-t^{-n})(1-q^{-2}t^{2-n})\right] \\ & + o(p^2) \end{aligned} \tag{5.98}$$

which in the limit $n\to\infty$ reduces to

$$(q+t^{-1}-qt^{-1})+p\frac{(1-q)(1-t)(q-t)}{qt}+o(p^2) \tag{5.99}$$

Comparison with (5.82) tells us that we have to identify $p=-\tilde{p}\sqrt{qt^{-1}}$ as we anticipated in (5.51).

- Case $k=2$, partition (2,0)

For the partition $\lambda=(2,0,\dots,0)$ we obtain

$$\begin{aligned} & t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}E_{eRS}^{((2,0,\dots,0);n)}(p) = \\ & = \left[1-t^{-n}+(q^2-1)(1-t^{-1})\right] \\ & + p\frac{(1-q^2)(1-t)^2(q-t)(1-q^{-1}t^{-n})}{t(1-qt)(1-q^{-2}t^{1-n})} \\ & + p\frac{(1-q)(1-t)(q-t)(1-q^{-2}t^{-n})(1-q^{-3}t^{2-n})(1-t^{1-n})}{q^2(1-q^{-1}t^{2-n})(1-q^{-2}t^{1-n})(1-q^{-3}t^{1-n})}t^{-n} \\ & + o(p^2) \end{aligned} \tag{5.100}$$

which in the limit $n \to \infty$ reduces to

$$(q^2 + t^{-1} - q^2 t^{-1}) + p\frac{(1-q^2)(1-t)^2(q-t)}{t(1-qt)} + o(p^2) \tag{5.101}$$

This matches (5.83) for $p = -\tilde{p}\sqrt{qt^{-1}}$ as expected.

- Case $k = 2$, partition (1,1)

For the partition $\lambda = (1, 1, 0, \ldots, 0)$ we have

$$\begin{aligned} &t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty (ptq^{-1};p)_\infty}{(p;p)_\infty (pq^{-1};p)_\infty} E_{eRS}^{((1,1,0,\ldots,0);n)}(p) = \\ &= \left[1 - t^{-n} + (q-1)(1-t^{-2})\right] \\ &+ p\frac{(1-q)^2(1-t^2)(q-t)(1-t^{1-n})}{qt^2(1-qt)(1-q^{-1}t^{2-n})} \\ &+ p\frac{(1-q)(1-t)(q-t)(1-q^{-1}t^{-n})(1-q^{-2}t^{3-n})(1-t^{2-n})}{q^2(1-q^{-1}t^{3-n})(1-q^{-1}t^{2-n})(1-q^{-2}t^{1-n})}t^{-n} \\ &+ o(p^2) \end{aligned} \tag{5.102}$$

which in the limit $n \to \infty$ becomes

$$(q + t^{-2} - qt^{-2}) + p\frac{(1-q)^2(1-t^2)(q-t)}{qt^2(1-qt)} + o(p^2) \tag{5.103}$$

This matches (5.84) for $p = -\tilde{p}\sqrt{qt^{-1}}$.

- Case $k = 3$, partition (3,0,0)

For the partition $\lambda = (3, 0, 0, \ldots, 0)$ we have

$$\begin{aligned} &t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty (ptq^{-1};p)_\infty}{(p;p)_\infty (pq^{-1};p)_\infty} E_{eRS}^{((3,0,0,\ldots,0);n)}(p) = \\ &= \left[1 - t^{-n} + (q^3-1)(1-t^{-1})\right] \\ &+ p\frac{q(1-q^3)(1-t)^2(q-t)(1-q^{-2}t^{-n})}{t(1-q^2t)(1-q^{-3}t^{1-n})} \\ &+ p\frac{(1-q)(1-t)(q-t)(1-q^{-3}t^{-n})(1-q^{-4}t^{2-n})(1-t^{1-n})}{q^2(1-q^{-1}t^{2-n})(1-q^{-3}t^{1-n})(1-q^{-4}t^{1-n})}t^{-n} + o(p^2) \end{aligned} \tag{5.104}$$

which in the limit $n \to \infty$ becomes

$$(q^3 + t^{-1} - q^3 t^{-1}) + p\frac{q(1-q^3)(1-t)^2(q-t)}{t(1-q^2t)} + o(p^2) \tag{5.105}$$

This matches (5.85) for $p = -\tilde{p}\sqrt{qt^{-1}}$.

- Case $k=3$, partition (2,1,0)

For the partition $\lambda=(2,1,0,\dots,0)$ we have

$$\begin{aligned}
&t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}E_{eRS}^{((2,1,0,\dots,0);n)}(p)=\\
&=\left[1-t^{-n}+(q-1)(1-t^{-1})(1+q+t^{-1})\right]\\
&+p\frac{(1-q)(1-t)(q-t)(1-q^2)(1-qt^2)(1-t^{1-n})}{qt^2(1-qt)(1-q^2t)(1-q^{-1}t^{2-n})}\\
&+p\frac{(1-q)(1-t)(q-t)(1-t^2)(1-q^2t)(1-q^{-1}t^{-n})}{t(1-qt)(1-qt^2)(1-q^{-2}t^{1-n})}\\
&+p\frac{(1-q)(1-t)(q-t)(1-q^{-1}t^{-n+1})(1-q^{-2}t^{-n+3})}{q^2(1-q^{-1}t^{-n+3})(1-q^{-1}t^{-n+2})(1-q^{-2}t^{-n+2})}\\
&\quad\frac{(1-q^{-2}t^{-n})(1-q^{-3}t^{-n+2})(1-t^{-n+2})}{(1-q^{-2}t^{-n+1})(1-q^{-3}t^{-n+1})}t^{-n}+o(p^2)
\end{aligned}\tag{5.106}$$

which in the limit $n\to\infty$ becomes

$$\begin{aligned}
&(q^2+t^{-2}+qt^{-1}-qt^{-2}-q^2t^{-1})\\
&+p\frac{(1-q)(1-t)(q-t)}{qt^2(1-qt^2)(1-q^2t)}\frac{\left[(1-q^2)(1-qt^2)^2+qt(1-t^2)(1-q^2t)^2\right]}{(1-qt)}+o(p^2)
\end{aligned}\tag{5.107}$$

This matches (5.86) for $p=-\tilde{p}\sqrt{qt^{-1}}$.

- Case $k=3$, partition (1,1,1)

For the partition $\lambda=(1,1,1,0,\dots,0)$ we have

$$\begin{aligned}
&t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}E_{eRS}^{((1,1,1,0,\dots,0);n)}(p)=\\
&=\left[1-t^{-n}+(q-1)(1-t^{-1})(1+t^{-1}+t^{-2})\right]\\
&+p\frac{(1-q)^2(1-t^3)(q-t)(1-t^{2-n})}{qt^3(1-qt^2)(1-q^{-1}t^{3-n})}\\
&+p\frac{(1-q)(1-t)(q-t)(1-q^{-1}t^{-n})(1-q^{-2}t^{4-n})(1-t^{3-n})}{q^2(1-q^{-1}t^{4-n})(1-q^{-1}t^{3-n})(1-q^{-2}t^{1-n})}t^{-n}\\
&+o(p^2)
\end{aligned}\tag{5.108}$$

which in the limit $n\to\infty$ becomes

$$(q+t^{-3}-qt^{-3})+p\frac{(1-q)^2(1-t^3)(q-t)}{qt^3(1-qt^2)}+o(p^2)\tag{5.109}$$

This matches (5.87) for $p=-\tilde{p}\sqrt{qt^{-1}}$.

5.7.3 The gauge theory correspondence

The above computations suggest the validity of conjecture (5.96): it is therefore possible to recover the ΔILW eigenvalues starting from the eRS ones, by taking the limit $n \to \infty$. This is not surprising from the integrable systems point of view, since ΔILW is expected to arise as a hydrodynamic limit of eRS; nevertheless, this correspondence looks quite non-trivial from the gauge theory point of view, in which (5.96) is rewritten as

$$1 - (1-q)(1-t^{-1})\mathrm{Tr}\,\sigma\big|_\lambda = \lim_{n\to\infty}\left[t^{-n+1}(1-t^{-1})\langle W_\square^{U(n)}\rangle\right]\Big|_\lambda \tag{5.110}$$

Here we are proposing an equivalence between a local observable in the 3d ADHM theory and a non-local observable (Wilson loop) in the 5d $\mathcal{N}=1^*$ $U(n)$ theory when $n \to \infty$. This might indicate an infra-red duality of some sort which relates the two theories in this limit; for clarity, let us introduce here the corresponding dictionary:

	3d ADHM theory	**3d-5d theory (coupled), $n \to \infty$**
lives on	$S^2_r \times S^1_\gamma$	$\mathbb{C}^2_{\tilde\epsilon_1,\tilde\epsilon_2} \times S^1_\gamma$
coupling t	twisted mass $e^{-i\gamma\epsilon_2}$	5d $\mathcal{N}=1^*$ mass deformation $e^{-i\gamma m}$
shift q	twisted mass $e^{i\gamma\epsilon_1}$	Omega background $e^{i\gamma\tilde\epsilon_1}$
elliptic parameter p	Fayet-Iliopoulos $\tilde p = -p/\sqrt{qt^{-1}}$	3d-5d coupling Q
partitions λ of k	ADHM Coulomb vacua	5d Coulomb-Higgs vacua
observable	$\langle \mathrm{Tr}\,\sigma\rangle$	$\langle W_\square^{U(\infty)}\rangle$ in NS limit $\tilde\epsilon_2 \to 0$

More in general, we expect the ADHM local observable $\langle \mathrm{Tr}\,\sigma^r\rangle$ to be related to the $n \to \infty$ limit of the 5d Wilson loop $\langle W^{U(n)}\rangle$ in the rank r antisymmetric representation. Although at the moment we do not have a good explanation for this duality, we notice that a similar 3d/5d duality appeared in [133]. There the 3d theory arises as the world-volume theory of vortex strings probing the Higgs branch of the 5d theory; in our context, the 3d theory is more related to instanton counting for a 7d $U(1)$ pure super Yang-Mills theory. It is possible that by considering brane constructions of these theories a natural interpretation for this duality will arise: further investigation on this point is needed.

Appendix A

A and D-type singularities

The k-instanton moduli space for $U(N)$ gauge theories on ALE spaces $\mathbb{C}^2/\Gamma$ with Γ finite subgroup of $SU(2)$ has been described in [139] in terms of quiver representation theory. We can therefore apply the same procedure we used in the main text: we consider a system of D1-D5 branes on $\mathbb{C}^2/\Gamma \times T^*S^2 \times \mathbb{C}$ and think of Nakajima quivers as GLSM on S^2, whose partition function will give us information about the quantum cohomology of the corresponding target ALE space. Similar results were discussed in [140]. We will focus on A and D-type singularities and consider the Hilbert scheme of points on their resolutions, as well as the orbifold phase given by the symmetric product of points.

A.1 A_{p-1} singularities

Let us start by considering the A_{p-1} case, i.e. $\Gamma = \mathbb{Z}_p$ with $p \geqslant 2$. The moduli space $\mathcal{M}(\vec{k}, \vec{N}, p)$ of instantons on this space can be obtained via an ADHM-like construction, whose data are encoded in the associated Nakajima quiver, which in this case is the affine quiver $\widehat{A}_{p-1}$ with framing at all nodes (figure A.1). The vector $\vec{k} = (k_0, \ldots, k_{p-1})$ parametrizes the dimensions of the vector spaces at the nodes of the quiver, while the vector $\vec{N} = (N_0, \ldots, N_{p-1})$ gives the dimensions of the framing vector spaces; the extra node on the affine Dynkin diagram corresponds to k_0. The choice of $\vec{N}$ determines $\vec{k}$ once the Chern class of the gauge vector bundle has been fixed [139].
The Nakajima quiver can be easily transposed to a GLSM on S^2. This theory will have gauge group $G = \prod_{b=0}^{p-1} U(k_b)$, flavour group $G_F = \prod_{b=0}^{p-1} U(N_b) \times U(1)^2$ and the matter content summarized in the following table:

	$\chi^{(b)}$	$B^{(b,b+1)}$	$B^{(b,b-1)}$	$I^{(b)}$	$J^{(b)}$
gauge G	$Adj^{(b)}$	$(\overline{\mathbf{k}}^{(b)}, \mathbf{k}^{(b+1)})$	$(\overline{\mathbf{k}}^{(b)}, \mathbf{k}^{(b-1)})$	$\mathbf{k}^{(b)}$	$\overline{\mathbf{k}}^{(b)}$
flavor G_F	$\mathbf{1}_{(-1,-1)}$	$\mathbf{1}_{(1,0)}$	$\mathbf{1}_{(0,1)}$	$\overline{\mathbf{N}}^{(b)}_{(1/2,1/2)}$	$\mathbf{N}^{(b)}_{(1/2,1/2)}$
twisted masses	$\epsilon_+ = \epsilon_1 + \epsilon_2$	$-\epsilon_1$	$-\epsilon_2$	$-a^{(b)}_j - \frac{\epsilon_+}{2}$	$a^{(b)}_j - \frac{\epsilon_+}{2}$
R-charge	2	0	0	0	0

With the superpotential

$$W = \sum_{b=0}^{p-1} \mathrm{Tr}_b[\chi^{(b)}(B^{(b,b+1)}B^{(b+1,b)} - B^{(b,b-1)}B^{(b-1,b)} + I^{(b)}J^{(b)})]$$

(assuming the identification $b \sim b+p$), the F and D-term equations describing the classical space of supersymmetric vacua in the Higgs branch coincide with the ADHM-like equations characterizing $\mathcal{M}(\vec{k}, \vec{N}, p)$.

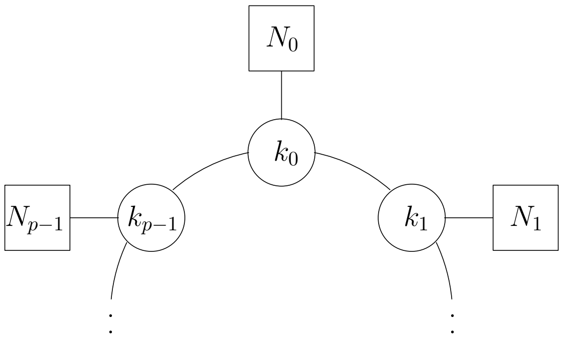


FIGURE A.1: The affine $\widehat{A}_{p-1}$ quiver.

We can now compute the partition function on S^2 for this GLSM by applying the prescription described in this Thesis. Defining $z_b = e^{-2\pi\xi_b - i\theta_b} = e^{-2\pi t_b}$, with $t_b = \xi_b + i\theta_b/2\pi$ complexified Fayet-Iliopoulos parameter, the partition function can be written as

$$Z_{\vec{k},\vec{N},p} = \frac{1}{k_0! \dots k_{p-1}!} \sum_{\vec{m}_0,\dots,\vec{m}_{p-1} \in \mathbb{Z}} \int \prod_{b=0}^{p-1} \prod_{s=1}^{k_b} \frac{d(r\sigma^{(b)}_s)}{2\pi} z_b^{ir\sigma^{(b)}_s + m^{(b)}_s/2} \bar{z}_b^{ir\sigma^{(b)}_s - m^{(b)}_s/2} Z_{\mathrm{vec}} Z_{\mathrm{adj}} Z_{\mathrm{bif}} Z_{\mathrm{f+af}} \tag{A.1}$$

where the various pieces of the integrand are given by

$$
\begin{aligned}
Z_{\text{vec}} &= \prod_{b=0}^{p-1}\prod_{s<t}^{k_b}(-1)^{m_s^{(b)}-m_t^{(b)}}\left(\left(r\sigma_s^{(b)}-r\sigma_t^{(b)}\right)^2+\left(\frac{m_s^{(b)}}{2}-\frac{m_t^{(b)}}{2}\right)^2\right)\\
&= \prod_{b=0}^{p-1}\prod_{s\neq t}^{k_b}\frac{\Gamma\left(1-ir\sigma_s^{(b)}+ir\sigma_t^{(b)}-\frac{m_s^{(b)}}{2}+\frac{m_t^{(b)}}{2}\right)}{\Gamma\left(ir\sigma_s^{(b)}-ir\sigma_t^{(b)}-\frac{m_s^{(b)}}{2}+\frac{m_t^{(b)}}{2}\right)}\\
Z_{\text{adj}} &= \prod_{b=0}^{p-1}\prod_{s,t=1}^{k_b}\frac{\Gamma\left(1-ir\sigma_s^{(b)}+ir\sigma_t^{(b)}-ir\epsilon_+-\frac{m_s^{(b)}}{2}+\frac{m_t^{(b)}}{2}\right)}{\Gamma\left(ir\sigma_s^{(b)}-ir\sigma_t^{(b)}+ir\epsilon_+-\frac{m_s^{(b)}}{2}+\frac{m_t^{(b)}}{2}\right)}
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
Z_{\text{bif}} &= \prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\prod_{t=1}^{k_{b-1}}\frac{\Gamma\left(-ir\sigma_s^{(b)}+ir\sigma_t^{(b-1)}+ir\epsilon_1-\frac{m_s^{(b)}}{2}+\frac{m_t^{(b-1)}}{2}\right)}{\Gamma\left(1+ir\sigma_s^{(b)}-ir\sigma_t^{(b-1)}-ir\epsilon_1-\frac{m_s^{(b)}}{2}+\frac{m_t^{(b-1)}}{2}\right)}\\
&\qquad\frac{\Gamma\left(ir\sigma_s^{(b)}-ir\sigma_t^{(b-1)}+ir\epsilon_2+\frac{m_s^{(b)}}{2}-\frac{m_t^{(b-1)}}{2}\right)}{\Gamma\left(1-ir\sigma_s^{(b)}+ir\sigma_t^{(b-1)}-ir\epsilon_2+\frac{m_s^{(b)}}{2}-\frac{m_t^{(b-1)}}{2}\right)}\\
Z_{\text{f+af}} &= \prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\prod_{j=1}^{N_b}\frac{\Gamma\left(-ir\sigma_s^{(b)}+ira_j^{(b)}+ir\frac{\epsilon_+}{2}-\frac{m_s^{(b)}}{2}\right)}{\Gamma\left(1+ir\sigma_s^{(b)}-ira_j^{(b)}-ir\frac{\epsilon_+}{2}-\frac{m_s^{(b)}}{2}\right)}\\
&\qquad\frac{\Gamma\left(ir\sigma_s^{(b)}-ira_j^{(b)}+ir\frac{\epsilon_+}{2}+\frac{m_s^{(b)}}{2}\right)}{\Gamma\left(1-ir\sigma_s^{(b)}+ira_j^{(b)}-ir\frac{\epsilon_+}{2}+\frac{m_s^{(b)}}{2}\right)}
\end{aligned}
\tag{A.3}
$$

We are not interested to explicitly evaluate the partition function at the moment. Instead, we want to study the IR Coulomb branch effective field theory; this is a purely abelian gauge theory with the Cartan of G as gauge group, and can be described by a function $\mathcal{W}_{\text{eff}}$, known as the effective twisted superpotential, which is a function of the scalar components of the vector superfields of our effective theory. By the Bethe/ gauge correspondence [124, 125], $\mathcal{W}_{\text{eff}}$ corresponds to the Yang-Yang function of some quantum integrable system; in our case, we will see that the associated integrable system is the generalization of the periodic Intermediate Long Wave introduced in [128].

A.1.1 Analysis of the Coulomb branch

Here we will follow the procedure described in Chapter 4 for the case $p = 1$. Defining $ir\Sigma_s^{(b)} = ir\sigma_s^{(b)} + \frac{m_s^{(b)}}{2}$, we can take the large radius limit $r \to \infty$ of (A.1); by using

Stirling's approximation we have

$$\frac{\Gamma(-ir\Sigma)}{\Gamma(1+ir\overline{\Sigma})} \sim \exp\left\{\omega(-ir\Sigma) - \frac{1}{2}\ln(-ir\Sigma) - \omega(ir\overline{\Sigma}) - \frac{1}{2}\ln(ir\overline{\Sigma})\right\} \tag{A.4}$$

with $\omega(x) = x(\ln x - 1)$. Therefore we can rewrite the partition function at large radius as

$$Z_{\vec{k},\vec{N},p} = \prod_{b=0}^{p-1} \frac{(r\epsilon_+)^{k_b}}{k_b!} \int \prod_{b=0}^{p-1}\prod_{s=1}^{k_b} \frac{d(r\Sigma_s^{(b)})}{2\pi} \left| \left(\prod_{b=0}^{p-1}\prod_{s=1}^{k_b} \frac{\prod_{t\neq s}^{k_b} D(\Sigma_s^{(b)} - \Sigma_t^{(b)})}{Q_b(\Sigma_s^{(b)}) \prod_{t=1}^{k_{b-1}} F(\Sigma_s^{(b)} - \Sigma_t^{(b-1)})} \right)^{\frac{1}{2}} e^{-\mathcal{W}_{\text{eff}}} \right|^2 \tag{A.5}$$

where the functions entering the integration measure are

$$\begin{aligned} D(\Sigma_s^{(b)} - \Sigma_t^{(b)}) &= r^2(\Sigma_s^{(b)} - \Sigma_t^{(b)})(\Sigma_s^{(b)} - \Sigma_t^{(b)} + \epsilon_+) \\ F(\Sigma_s^{(b)} - \Sigma_t^{(b-1)}) &= r^2(\Sigma_s^{(b)} - \Sigma_t^{(b-1)} - \epsilon_1)(\Sigma_s^{(b)} - \Sigma_t^{(b-1)} + \epsilon_2) \\ Q_b(\Sigma_s^{(b)}) &= \prod_{j=1}^{N_b} r^2 \left(\Sigma_s^{(b)} - a_j^{(b)} - \frac{\epsilon_+}{2}\right)\left(\Sigma_s^{(b)} - a_j^{(b)} + \frac{\epsilon_+}{2}\right) \end{aligned} \tag{A.6}$$

while the twisted effective superpotential reads

$$\begin{aligned} \mathcal{W}_{\text{eff}} = \; & 2\pi \sum_{b=0}^{p-1}\sum_{s=1}^{k_b} irt_b\Sigma_s^{(b)} + \sum_{b=0}^{p-1}\sum_{s=1}^{k_b}\sum_{j=1}^{N_b} \left[\omega(ir\Sigma_s^{(b)} - ira_j^{(b)} - ir\frac{\epsilon_+}{2}) + \omega(-ir\Sigma_s^{(b)} + ira_j^{(b)} - ir\frac{\epsilon_+}{2})\right] \\ & + \sum_{b=0}^{p-1}\sum_{s,t\neq s}^{k_b} \left[\omega(ir\Sigma_s^{(b)} - ir\Sigma_t^{(b)}) + \omega(ir\Sigma_s^{(b)} - ir\Sigma_t^{(b)} + ir\epsilon_+)\right] \\ & + \sum_{b=0}^{p-1}\sum_{s=1}^{k_b}\sum_{s=1}^{k_{b-1}} \left[\omega(ir\Sigma_s^{(b)} - ir\Sigma_t^{(b-1)} - ir\epsilon_1) + \omega(-ir\Sigma_s^{(b)} + ir\Sigma_t^{(b-1)} - ir\epsilon_2)\right] \end{aligned} \tag{A.7}$$

From the Bethe/gauge correspondence, the equations determining the supersymmetric vacua in the Coulomb branch (saddle points of $\mathcal{W}_{\text{eff}}$)

$$\exp\left(\frac{\partial \mathcal{W}_{\text{eff}}}{\partial(ir\Sigma_s^{(b)})}\right) = 1 \tag{A.8}$$

correspond to Bethe Ansatz Equations for a quantum integrable system. For our theory, the equations are

$$\prod_{j=1}^{N_b} \frac{\Sigma_s^{(b)} - a_j^{(b)} - \frac{\epsilon_+}{2}}{-\Sigma_s^{(b)} + a_j^{(b)} - \frac{\epsilon_+}{2}} \prod_{\substack{t=1 \\ t\neq s}}^{k_b} \frac{\Sigma_s^{(b)} - \Sigma_t^{(b)} + \epsilon_+}{\Sigma_s^{(b)} - \Sigma_t^{(b)} - \epsilon_+} \prod_{t=1}^{k_{b-1}} \frac{\Sigma_s^{(b)} - \Sigma_t^{(b-1)} - \epsilon_1}{\Sigma_s^{(b)} - \Sigma_t^{(b-1)} + \epsilon_2} \prod_{t=1}^{k_{b+1}} \frac{\Sigma_s^{(b)} - \Sigma_t^{(b+1)} - \epsilon_2}{\Sigma_s^{(b)} - \Sigma_t^{(b+1)} + \epsilon_1} = e^{-2\pi t_b} \tag{A.9}$$

These are exactly the Bethe Ansatz Equations for the generalization of the periodic Intermediate Long Wave quantum system proposed in [128]. They can be rewritten in a form which generalizes to any quiver:

$$\prod_{j=1}^{N_b} \frac{\Sigma_s^{(b)} - a_j^{(b)} - \frac{\epsilon_+}{2}}{-\Sigma_s^{(b)} + a_j^{(b)} - \frac{\epsilon_+}{2}} \prod_{c=0}^{p-1} \prod_{\substack{t=1 \\ (c,t)\neq(b,s)}}^{k_c} \frac{\Sigma_s^{(b)} - \Sigma_t^{(c)} + \mathbf{C}_{bc}^T}{\Sigma_s^{(b)} - \Sigma_t^{(c)} - \mathbf{C}_{bc}} = e^{-2\pi t_b} \tag{A.10}$$

where

$$\mathbf{C}_{bc} = \begin{bmatrix} \epsilon_+ & -\epsilon_1 & 0 & \dots & 0 & -\epsilon_2 \\ -\epsilon_2 & \epsilon_+ & -\epsilon_1 & \dots & 0 & 0 \\ 0 & -\epsilon_2 & \epsilon_+ & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -\epsilon_1 & 0 \\ 0 & 0 & \vdots & -\epsilon_2 & \epsilon_+ & -\epsilon_1 \\ -\epsilon_1 & 0 & \dots & 0 & -\epsilon_2 & \epsilon_+ \end{bmatrix} \tag{A.11}$$

is the adjacency matrix of the quiver graph. Let us remark that when $\epsilon_1 = \epsilon_2$, (A.11) reduces to the Cartan matrix of the affine $\widehat{A}_{p-1}$ algebra. A similar observation has been made for XXX spin chains with higher rank spin group in [124], relatively to Cartan matrices of non-affine Lie algebras.

The solutions to (A.10) are in one to one correspondence with the supersymmetric vacua in the Coulomb branch and with the eigenstates of the infinite set of integrals of motion for the generalized PILW system. We can now perform a semiclassical analysis of the partition function around a vacuum α to obtain a formula for the inverse norm of the eigenstates, along the lines of (4.108). The semiclassical approximation gives

$$Z_{\vec{k},\vec{N},p} = \left| e^{-\mathcal{W}_{\text{eff}}} \prod_{b=0}^{p-1} (r\epsilon_+)^{\frac{k_b}{2}} \left(\prod_{b=0}^{p-1} \prod_{s=1}^{k_b} \frac{\prod_{t\neq s}^{k_b} D(\Sigma_s^{(b)} - \Sigma_t^{(b)})}{Q_b(\Sigma_s^{(b)}) \prod_{t=1}^{k_{b-1}} F(\Sigma_s^{(b)} - \Sigma_t^{(b-1)})} \right)^{\frac{1}{2}} \left(\text{Det} \frac{\partial^2 \mathcal{W}_{\text{eff}}}{r^2 \partial\Sigma_s^{(a)} \partial\Sigma_t^{(b)}} \right)^{-\frac{1}{2}} \right|^2 \tag{A.12}$$

where we chose an ordering for the saddle points in order to eliminate the factorials; here the Σ's are the solutions corresponding to the vacuum α. The expression for the norm of the state $|\alpha\rangle$ is then

$$\frac{1}{\langle\alpha|\alpha\rangle} = \left| \prod_{b=0}^{p-1} (r\epsilon_+)^{\frac{k_b}{2}} \left(\prod_{b=0}^{p-1} \prod_{s=1}^{k_b} \frac{\prod_{t\neq s}^{k_b} D(\Sigma_s^{(b)} - \Sigma_t^{(b)})}{Q_b(\Sigma_s^{(b)}) \prod_{t=1}^{k_{b-1}} F(\Sigma_s^{(b)} - \Sigma_t^{(b-1)})} \right)^{\frac{1}{2}} \left(\text{Det} \frac{\partial^2 \mathcal{W}_{\text{eff}}}{r^2 \partial\Sigma_s^{(a)} \partial\Sigma_t^{(b)}} \right)^{-\frac{1}{2}} \right|^2 \tag{A.13}$$

where we removed the prefactor $|e^{-\mathcal{W}_{\text{eff}}}|^2$. This is the usual expression for the norm of the eigenstates provided in other cases by Gaudin and Korepin, see also [141].

A.1.2 Equivariant quantum cohomology of $\mathcal{M}(\vec{k},\vec{N},p)$

In this section we will explicitly evaluate the partition function (A.1) at fixed $\vec{k}$, $\vec{N}$ and p in the simplest cases. The goal is to compute the equivariant quantum Gromov-Witten potential for the moduli space $\mathcal{M}(\vec{k},\vec{N},p)$.

In the following we follow the same procedure described in the main text. We start by performing the change of variables $ir\sigma_s^{(b)} = -r\lambda_s^{(b)} + l_s^{(b)} - \frac{m_s^{(b)}}{2}$, and define $k_s^{(b)} = l_s^{(b)} - m_s^{(b)}$. Then (A.1) becomes

$$Z_{\vec{k},\vec{N},p} = \frac{1}{k_0!\dots k_{p-1}!}\oint \prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\frac{d(r\lambda_s^{(b)})}{2\pi i} Z_{1l} Z_{\mathrm{v}} Z_{\mathrm{av}} \tag{A.14}$$

where

$$\begin{aligned}
Z_{1l} &= \prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\left(\frac{\Gamma(1-ir\epsilon_+)}{\Gamma(ir\epsilon_+)}(z_b\bar{z}_b)^{-r\lambda_s^{(b)}}\right)\prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\prod_{t\neq s}^{k_b}(r\lambda_s^{(b)}-r\lambda_t^{(b)})\frac{\Gamma(1+r\lambda_s^{(b)}-r\lambda_t^{(b)}-ir\epsilon_+)}{\Gamma(-r\lambda_s^{(b)}+r\lambda_t^{(b)}+ir\epsilon_+)}\\
&\prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\prod_{t=1}^{k_{b-1}}\frac{\Gamma(r\lambda_s^{(b)}-r\lambda_t^{(b-1)}+ir\epsilon_1)}{\Gamma(1-r\lambda_s^{(b)}+r\lambda_t^{(b-1)}-ir\epsilon_1)}\frac{\Gamma(-r\lambda_s^{(b)}+r\lambda_t^{(b-1)}+ir\epsilon_2)}{\Gamma(1+r\lambda_s^{(b)}-r\lambda_t^{(b-1)}-ir\epsilon_2)}\\
&\prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\prod_{j=1}^{N_b}\frac{\Gamma(r\lambda_s^{(b)}+ira_j^{(b)}+ir\frac{\epsilon_+}{2})}{\Gamma(1-r\lambda_s^{(b)}-ira_j^{(b)}-ir\frac{\epsilon_+}{2})}\frac{\Gamma(-r\lambda_s^{(b)}-ira_j^{(b)}+ir\frac{\epsilon_+}{2})}{\Gamma(1+r\lambda_s^{(b)}+ira_j^{(b)}-ir\frac{\epsilon_+}{2})}\\
Z_{\mathrm{v}} &= \sum_{\{\vec{l}\}}\prod_{b=0}^{p-1}\prod_{s=1}^{k_b}(-1)^{N_b l_s^{(b)}}\prod_{b=0}^{p-1} z_b^{l_s^{(b)}}\prod_{b=0}^{p-1}\prod_{s<t}^{k_b}\frac{l_t^{(b)}-l_s^{(b)}-r\lambda_t^{(b)}+r\lambda_s^{(b)}}{-r\lambda_t^{(b)}+r\lambda_s^{(b)}}\frac{(1+r\lambda_s^{(b)}-r\lambda_t^{(b)}-ir\epsilon_+)_{l_t^{(b)}-l_s^{(b)}}}{(r\lambda_s^{(b)}-r\lambda_t^{(b)}+ir\epsilon_+)_{l_t^{(b)}-l_s^{(b)}}}\\
&\prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\prod_{t=1}^{k_{b-1}}\frac{1}{(1-r\lambda_s^{(b)}+r\lambda_t^{(b-1)}-ir\epsilon_1)_{l_s^{(b)}-l_t^{(b-1)}}}\frac{1}{(1+r\lambda_s^{(b)}-r\lambda_t^{(b-1)}-ir\epsilon_2)_{l_t^{(b-1)}-l_s^{(b)}}}\\
&\prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\prod_{j=1}^{N_b}\frac{(-r\lambda_s^{(b)}-ira_j^{(b)}+ir\frac{\epsilon_+}{2})_{l_s^{(b)}}}{(1-r\lambda_s^{(b)}-ira_j^{(b)}-ir\frac{\epsilon_+}{2})_{l_s^{(b)}}}\\
Z_{\mathrm{av}} &= \sum_{\{\vec{k}\}}\prod_{b=0}^{p-1}\prod_{s=1}^{k_b}(-1)^{N_b k_s^{(b)}}\prod_{b=0}^{p-1} \bar{z}_b^{k_s^{(b)}}\prod_{b=0}^{p-1}\prod_{s<t}^{k_b}\frac{k_t^{(b)}-k_s^{(b)}-r\lambda_t^{(b)}+r\lambda_s^{(b)}}{-r\lambda_t^{(b)}+r\lambda_s^{(b)}}\frac{(1+r\lambda_s^{(b)}-r\lambda_t^{(b)}-ir\epsilon_+)_{k_t^{(b)}-k_s^{(b)}}}{(r\lambda_s^{(b)}-r\lambda_t^{(b)}+ir\epsilon_+)_{k_t^{(b)}-k_s^{(b)}}}\\
&\prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\prod_{t=1}^{k_{b-1}}\frac{1}{(1-r\lambda_s^{(b)}+r\lambda_t^{(b-1)}-ir\epsilon_1)_{k_s^{(b)}-k_t^{(b-1)}}}\frac{1}{(1+r\lambda_s^{(b)}-r\lambda_t^{(b-1)}-ir\epsilon_2)_{k_t^{(b-1)}-k_s^{(b)}}}\\
&\prod_{b=0}^{p-1}\prod_{s=1}^{k_b}\prod_{j=1}^{N_b}\frac{(-r\lambda_s^{(b)}-ira_j^{(b)}+ir\frac{\epsilon_+}{2})_{k_s^{(b)}}}{(1-r\lambda_s^{(b)}-ira_j^{(b)}-ir\frac{\epsilon_+}{2})_{k_s^{(b)}}}
\end{aligned} \tag{A.15}$$

As we saw, the vortex partition function Z_{v} is interpreted in quantum cohomology as Givental's $\mathcal{I}$ function, and in order to extract the Gromov-Witten prepotential we have to normalize in an appropriate way Z_{11} and invert the equivariant mirror map in $Z_{\rm v}$. For ALE spaces the equivariant mirror map is known explicitly. It appears only when $N = \sum_{b=0}^{p-1} N_b = 1$, in which case the construction in [139] forces the vectors $\vec{N}$, $\vec{k}$ to be $\vec{N} = (1, 0, \ldots, 0)$ and $\vec{k} = (k, k, \ldots, k)$, and it consists in multiplying $Z_{\rm v}$ by $(1 + \prod_{b=0}^{p-1} z_b)^{ikr\epsilon_+}$ (and similarly for $Z_{\rm av}$). On the contrary the normalization factor for Z_{11} is not known, and we will find it case by case, by requiring a particular coefficient in the partition function to vanish (this corresponds, from the mathematical point of view, to the requirement that the intersection $\langle 1, 1, \ln z\rangle = 0$, with $\ln z$ a Kähler moduli of the target space).

A.1.2.1 The $N = 1$, $k = 1$ sector

Since for $N = 1$ the vectors $\vec{N}$, $\vec{k}$ are fixed as written above, we will refer to the $N = 1$ instanton moduli space as $\mathcal{M}(k, 1, p)$. For $k = 1$, this space is known in the mathematical literature as $\mathcal{M}(1, 1, p) = \mathbb{Z}_p\text{-Hilb}(\mathbb{C}^2)$. The equivariant quantum Gromov-Witten potential $F_{(1,1,p)}$ for $\mathcal{M}(1, 1, p)$ has been computed explicitly for ϵ_1, ϵ_2 generic in [104] $(p = 2)$ and [142] $(p = 3)$; in the special limit $\epsilon_1 = \epsilon_2 = \epsilon$ explicit computations are provided in [143] in terms of the (inverse) Cartan matrix and root system of the non-affine algebra A_{p-1} for generic p. More in detail, let C_i^j be the A_{p-1} Cartan matrix, $i, j = 1 \ldots p-1$, let α_i be the basis of fundamental weights for the A_{p-1} algebra, and define R^+ as the set of $p(p-1)/2$ positive roots. Then we have

$$
\begin{aligned}
F_{(1,1,p)} = {} & \frac{1}{p\,\epsilon^2} - \frac{1}{2}\sum_{i,j=1}^{p-1} \langle \alpha_i, \alpha_j\rangle \ln z_i \ln z_j + \frac{\epsilon}{6} \sum_{i,j,k=1}^{p-1} \sum_{\beta\in R^+} \langle \alpha_i, \beta\rangle\langle \alpha_j, \beta\rangle\langle \alpha_k, \beta\rangle \ln z_i \ln z_j \ln z_k \\
& + 2\epsilon \sum_{\beta\in R^+} \mathrm{Li}_3\left(\prod_{i=1}^{p-1} z_i^{\langle \alpha_i, \beta\rangle}\right)
\end{aligned}
\tag{A.16}
$$

with the product $\langle \alpha_i, \alpha_j\rangle = \alpha_i^T C^{-1} \alpha_j$ expressed in terms of the inverse Cartan matrix.

Case $p = 2$

The A_1 Cartan matrix is just $C = 2$, with inverse $C^{-1} = \frac{1}{2}$, while $\alpha_1 = 1$, therefore $\langle \alpha_1, \alpha_1\rangle = \frac{1}{2}$. The only positive root corresponds to $\beta = C\alpha_1 = 2$, which implies $\langle \alpha_1, \beta\rangle = 1$. All in all, we get

$$
F_{(1,1,2)} = \frac{1}{2\,\epsilon^2} - \frac{1}{4}\ln^2 z_1 + \frac{\epsilon}{6}\ln^3 z_1 + 2\epsilon\,\mathrm{Li}_3(z_1)
\tag{A.17}
$$

We can compare this expression with what we obtain from the evaluation of the partition function $Z_{1,1,2}$. The poles of (A.14) are labelled by partitions of $\widehat{k} = \sum_{b=0}^{p-1} k_b = pk$; in particular, for positive Fayet-Iliopoulos parameters, in our case the poles are located at

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \end{cases}$$

Inverting the mirror map consists in replacing $Z_{\text{v}} \to (1 + z_0 z_1)^{ir\epsilon_+} Z_{\text{v}}$ and $Z_{\text{av}} \to (1 + \overline{z}_0 \overline{z}_1)^{ir\epsilon_+} Z_{\text{av}}$. After trials and errors, we also found a good normalization

$$Z_{11} \to (z_0 z_1 \overline{z}_0 \overline{z}_1)^{-ira_1^{(0)} - ir\frac{\epsilon_+}{2}} \frac{\Gamma(1 - ir\epsilon_+)}{\Gamma(1 + ir\epsilon_+)} Z_{11} \tag{A.18}$$

for the 1-loop part. All it remains to do is to evaluate the partition function at the two poles, sum the two contributions, and expand in small r. At the end we obtain

$$\begin{aligned} Z_{1,1,2}^{\text{norm}} = &- \frac{1}{2\epsilon_1\epsilon_2} - \frac{1}{4}\ln^2(z_1\overline{z}_1) + i\epsilon_+ \Bigg(- \frac{1}{12}\ln^3(z_1\overline{z}_1) + 4\zeta(3) \\ &+ 2(\text{Li}_3(z_1) + \text{Li}_3(\overline{z}_1)) - \ln(z_1\overline{z}_1)(\text{Li}_2(z_1) + \text{Li}_2(\overline{z}_1)) \Bigg) \end{aligned} \tag{A.19}$$

From this expression we can extract the genus zero Gromov-Witten prepotential (see for example [48]); for the sake of comparison we redefine $\epsilon_1 \to i\epsilon_1$, $\epsilon_2 \to i\epsilon_2$, so that now

$$F_{(1,1,2)} = \frac{1}{2\,\epsilon_1\epsilon_2} - \frac{1}{4}\ln^2 z_1 + \frac{\epsilon_+}{12}\ln^3 z_1 + \epsilon_+ \,\text{Li}_3(z_1) \tag{A.20}$$

This coincide with the expression given in [104] for generic ϵ_1, ϵ_2 and reduces to (A.17) in the special limit $\epsilon_1 = \epsilon_2 = \epsilon$.

Case $p = 3$

The A_2 data are the following:

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad , \quad C^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad , \quad \alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A.21}$$

The three positive roots are $\beta_1 = C\alpha_1$, $\beta_2 = C\alpha_2$ and $\beta_3 = C(\alpha_1+\alpha_2)$, therefore

$$\begin{aligned} F_{(1,1,3)} = & \frac{1}{3\,\epsilon^2} - \frac{1}{6}\left(\ln^2 z_1 + \ln^2 z_1 z_2 + \ln^2 z_2\right) \\ & + \frac{\epsilon}{6}\left(\ln^3 z_1 + \ln^3 z_1 z_2 + \ln^3 z_2\right) \\ & + 2\epsilon\,\left(\mathrm{Li}_3(z_1) + \mathrm{Li}_3(z_2) + \mathrm{Li}_3(z_1 z_2)\right) \end{aligned} \tag{A.22}$$

The relevant poles for the partition function $Z_{1,1,3}$ are at

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_2 \end{cases}$$

Inverting the mirror map by $Z_{\mathrm{v}} \to (1+z_0 z_1 z_2)^{ir\epsilon_+} Z_{\mathrm{v}}$ and $Z_{\mathrm{av}} \to (1+\overline{z}_0\overline{z}_1\overline{z}_2)^{ir\epsilon_+} Z_{\mathrm{av}}$, and normalizing the 1-loop part as

$$Z_{1l} \to (z_0 z_1 z_2 \overline{z}_0 \overline{z}_1 \overline{z}_2)^{-ira_1^{(0)} - ir\frac{\epsilon_+}{2}} \frac{\Gamma(1-ir\epsilon_+)}{\Gamma(1+ir\epsilon_+)} Z_{1l} \tag{A.23}$$

we obtain

$$\begin{aligned} Z_{1,1,3}^{\mathrm{norm}} = & -\frac{1}{3\epsilon_1\epsilon_2} - \frac{1}{6}\left(\ln^2(z_1\overline{z}_1) + \ln^2(z_1\overline{z}_1 z_2\overline{z}_2) + \ln^2(z_2\overline{z}_2)\right) \\ & + i\Bigg(-\frac{\epsilon_1+2\epsilon_2}{9}\ln^3(z_1\overline{z}_1) - \frac{\epsilon_1+2\epsilon_2}{6}\ln^2(z_1\overline{z}_1)\ln(z_2\overline{z}_2) \\ & \qquad - \frac{2\epsilon_1+\epsilon_2}{6}\ln(z_1\overline{z}_1)\ln^2(z_2\overline{z}_2) - \frac{2\epsilon_1+\epsilon_2}{9}\ln^3(z_2\overline{z}_2)\Bigg) \\ & + i\epsilon_+\Bigg(6\zeta(3) + 2(\mathrm{Li}_3(z_1) + \mathrm{Li}_3(z_2) + \mathrm{Li}_3(z_1z_2) + \mathrm{Li}_3(\overline{z}_1) + \mathrm{Li}_3(\overline{z}_2) + \mathrm{Li}_3(\overline{z}_1\overline{z}_2)) \\ & \qquad - \ln(z_1\overline{z}_1)(\mathrm{Li}_2(z_1) + \mathrm{Li}_2(\overline{z}_1)) - \ln(z_2\overline{z}_2)(\mathrm{Li}_2(z_2) + \mathrm{Li}_2(\overline{z}_2)) \\ & \qquad - \ln(z_1z_2\overline{z}_1\overline{z}_2)(\mathrm{Li}_2(z_1z_2) + \mathrm{Li}_2(\overline{z}_1\overline{z}_2))\Bigg) \end{aligned} \tag{A.24}$$

The corresponding genus zero Gromov-Witten prepotential after the redefinition $\epsilon_1 \to i\epsilon_1$, $\epsilon_2 \to i\epsilon_2$ reads

$$\begin{aligned}F_{(1,1,3)} = & \frac{1}{3\,\epsilon_1\epsilon_2} - \frac{1}{6}\left(\ln^2 z_1 + \ln^2 z_1 z_2 + \ln^2 z_2\right) \\ & + \left(\frac{\epsilon_1+2\epsilon_2}{9}\ln^3 z_1 + \frac{\epsilon_1+2\epsilon_2}{6}\ln^2 z_1 \ln z_2 + \frac{2\epsilon_1+\epsilon_2}{6}\ln z_1 \ln^2 z_2 + \frac{2\epsilon_1+\epsilon_2}{9}\ln^3 z_2\right) \\ & + \epsilon_+ \left(\mathrm{Li}_3(z_1) + \mathrm{Li}_3(z_2) + \mathrm{Li}_3(z_1 z_2)\right)\end{aligned} \tag{A.25}$$

and coincides with the expression given in [142] for generic ϵ_1, ϵ_2, or with (A.22) when $\epsilon_1 = \epsilon_2 = \epsilon$.

Case $p = 4$

In this case the A_3 data are

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad , \quad C^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix} \tag{A.26}$$

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{A.27}$$

The six positive roots are given by $\beta_1 = C\alpha_1$, $\beta_2 = C\alpha_2$, $\beta_3 = C\alpha_3$, $\beta_4 = C(\alpha_1+\alpha_2)$, $\beta_5 = C(\alpha_2+\alpha_3)$, $\beta_6 = C(\alpha_1+\alpha_2+\alpha_3)$; we thus obtain

$$\begin{aligned}F_{(1,1,4)} = & \frac{1}{4\,\epsilon^2} - \frac{1}{8}\left(\ln^2 z_1 + \ln^2 z_2 + \ln^2 z_3 + \ln^2 z_1 z_2 + \ln^2 z_2 z_3 + \ln^2 z_1 z_2 z_3\right) \\ & + \frac{\epsilon}{6}\left(\ln^3 z_1 + \ln^3 z_2 + \ln^3 z_3 + \ln^3 z_1 z_2 + \ln^3 z_2 z_3 + \ln^3 z_1 z_2 z_3\right) \\ & + 2\epsilon\left(\mathrm{Li}_3(z_1) + \mathrm{Li}_3(z_2) + \mathrm{Li}_3(z_3) + \mathrm{Li}_3(z_1 z_2) + \mathrm{Li}_3(z_2 z_3) + \mathrm{Li}_3(z_1 z_2 z_3)\right)\end{aligned} \tag{A.28}$$

On the other hand, we can compute the partition function $Z_{1,1,4}$. This time we have four poles at

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\ \lambda_1^{(3)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 3i\epsilon_1 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\ \lambda_1^{(3)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \end{cases}$$

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(3)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} - 2i\epsilon_2 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} \\ \lambda_1^{(3)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} - 2i\epsilon_2 \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} - 3i\epsilon_2 \end{cases}$$

The mirror map is inverted by $Z_{\rm v} \to (1+z_0z_1z_2z_3)^{ir\epsilon_+}Z_{\rm v}$ and $Z_{\rm av} \to (1+\overline{z}_0\overline{z}_1\overline{z}_2\overline{z}_3)^{ir\epsilon_+}Z_{\rm av}$, while we normalize the 1-loop part with

$$Z_{1l} \to (z_0z_1z_2z_3\overline{z}_0\overline{z}_1\overline{z}_2\overline{z}_3)^{-ira_1^{(0)}-ir\frac{\epsilon_+}{2}}\frac{\Gamma(1-ir\epsilon_+)}{\Gamma(1+ir\epsilon_+)}Z_{1l} \tag{A.29}$$

At the end we get

$$\begin{aligned} Z^{\rm norm}_{1,1,4} = &-\frac{1}{4\epsilon_1\epsilon_2} - \frac{1}{8}\Bigg(\ln^2(z_1\overline{z}_1) + \ln^2(z_2\overline{z}_2) + \ln^2(z_3\overline{z}_3) \\ &+ \ln^2(z_1\overline{z}_1z_2\overline{z}_2) + \ln^2(z_2\overline{z}_2z_3\overline{z}_3) + \ln^2(z_1\overline{z}_1z_2\overline{z}_2z_3\overline{z}_3)\Bigg) \\ &+ i\Bigg(-\frac{\epsilon_1+3\epsilon_2}{8}\ln^3(z_1\overline{z}_1) - \frac{\epsilon_1+3\epsilon_2}{4}\ln^2(z_1\overline{z}_1)\ln(z_2\overline{z}_2) - \frac{\epsilon_1+3\epsilon_2}{8}\ln^2(z_1\overline{z}_1)\ln(z_3\overline{z}_3) \\ &\quad - \frac{\epsilon_1+\epsilon_2}{3}\ln^3(z_2\overline{z}_2) - \frac{\epsilon_1+\epsilon_2}{2}\ln(z_1\overline{z}_1)\ln^2(z_2\overline{z}_2) - \frac{\epsilon_1+\epsilon_2}{2}\ln(z_1\overline{z}_1)\ln(z_2\overline{z}_2)\ln(z_3\overline{z}_3) \\ &\quad - \frac{\epsilon_1+\epsilon_2}{2}\ln^2(z_2\overline{z}_2)\ln(z_3\overline{z}_3) - \frac{3\epsilon_1+\epsilon_2}{8}\ln^3(z_3\overline{z}_3) - \frac{3\epsilon_1+\epsilon_2}{8}\ln(z_1\overline{z}_1)\ln^2(z_3\overline{z}_3) \\ &\quad - \frac{3\epsilon_1+\epsilon_2}{4}\ln(z_2\overline{z}_2)\ln^2(z_3\overline{z}_3)\Bigg) \\ &+ i\epsilon_+\Bigg(8\zeta(3) + 2(\text{Li}_3(z_1) + \text{Li}_3(z_2) + \text{Li}_3(z_3) + \text{Li}_3(z_1z_2) + \text{Li}_3(z_2z_3) + \text{Li}_3(z_1z_2z_3)) \\ &\quad + 2(\text{Li}_3(\overline{z}_1) + \text{Li}_3(\overline{z}_2) + \text{Li}_3(\overline{z}_3) + \text{Li}_3(\overline{z}_1\overline{z}_2)) + \text{Li}_3(\overline{z}_2\overline{z}_3)) + \text{Li}_3(\overline{z}_1\overline{z}_2\overline{z}_3)) \\ &\quad - \ln(z_1\overline{z}_1)(\text{Li}_2(z_1) + \text{Li}_2(\overline{z}_1)) - \ln(z_2\overline{z}_2)(\text{Li}_2(z_2) + \text{Li}_2(\overline{z}_2)) - \ln(z_3\overline{z}_3)(\text{Li}_2(z_3) + \text{Li}_2(\overline{z}_3)) \\ &\quad - \ln(z_1z_2\overline{z}_1\overline{z}_2)(\text{Li}_2(z_1z_2) + \text{Li}_2(\overline{z}_1\overline{z}_2)) - \ln(z_2z_3\overline{z}_2\overline{z}_3)(\text{Li}_2(z_2z_3) + \text{Li}_2(\overline{z}_2\overline{z}_3)) \\ &\quad - \ln(z_1z_2z_3\overline{z}_1\overline{z}_2\overline{z}_3)(\text{Li}_2(z_1z_2z_3) + \text{Li}_2(\overline{z}_1\overline{z}_2\overline{z}_3))\Bigg) \end{aligned} \tag{A.30}$$

which corresponds to a prepotential

$$
\begin{aligned}
F_{(1,1,4)} = & \frac{1}{4\,\epsilon_1\epsilon_2} - \frac{1}{8}\left(\ln^2 z_1 + \ln^2 z_2 + \ln^2 z_3 + \ln^2 z_1 z_2 + \ln^2 z_2 z_3 + \ln^2 z_1 z_2 z_3\right) \\
& + \left(\frac{\epsilon_1 + 3\epsilon_2}{8}\ln^3 z_1 + \frac{\epsilon_1 + 3\epsilon_2}{4}\ln^2 z_1 \ln z_2 + \frac{\epsilon_1 + 3\epsilon_2}{8}\ln^2 z_1 \ln z_3 + \frac{\epsilon_1 + \epsilon_2}{3}\ln^3 z_2 \right. \\
& \quad + \frac{\epsilon_1 + \epsilon_2}{2}\ln z_1 \ln^2 z_2 + \frac{\epsilon_1 + \epsilon_2}{2}\ln z_1 \ln z_2 \ln z_3 + \frac{\epsilon_1 + \epsilon_2}{2}\ln^2 z_2 \ln z_3 \\
& \quad \left. + \frac{3\epsilon_1 + \epsilon_2}{8}\ln^3 z_3 + \frac{3\epsilon_1 + \epsilon_2}{8}\ln z_1 \ln^2 z_3 + \frac{3\epsilon_1 + \epsilon_2}{4}\ln z_2 \ln^2 z_3\right) \\
& + \epsilon_+\left(\mathrm{Li}_3(z_1) + \mathrm{Li}_3(z_2) + \mathrm{Li}_3(z_3) + \mathrm{Li}_3(z_1 z_2) + \mathrm{Li}_3(z_2 z_3) + \mathrm{Li}_3(z_1 z_2 z_3)\right)
\end{aligned}
\tag{A.31}
$$

A.1.2.2 The $N = 1$, $k = 2$ sector

When $N = 1$ but $k \geqslant 2$ there no longer is a general expression for the Gromov-Witten prepotential in terms of the Cartan matrix and positive roots of the algebra A_{p-1}, since also $\ln z_0$ enters in the prepotential. We will therefore make good use of our partition function and provide such results, in the simplest cases; certainly this procedure can be pursued further, the only difficulty being an integral which becomes more and more complicated. The results of this case should be compared with [144].

Case $p = 2$

As usual, we start by listing the poles of the partition function, which in this case read

$$
\begin{cases}
\lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\
\lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\
\lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\
\lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 3i\epsilon_1
\end{cases}
\qquad
\begin{cases}
\lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\
\lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\
\lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\
\lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2
\end{cases}
$$

$$
\begin{cases}
\lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\
\lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\
\lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\
\lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 - i\epsilon_2
\end{cases}
\qquad
\begin{cases}
\lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\
\lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\
\lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\
\lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_2
\end{cases}
$$

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_2 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 3i\epsilon_2 \end{cases}$$

After inverting the mirror map according to $Z_{\rm v} \to (1 + z_0 z_1)^{2ir\epsilon_+} Z_{\rm v}$ and $Z_{\rm av} \to (1 + \overline{z}_0 \overline{z}_1)^{2ir\epsilon_+} Z_{\rm av}$, and normalizing the 1-loop part as

$$Z_{11} \to (z_0 z_1 \overline{z}_0 \overline{z}_1)^{-2ira_1^{(0)} - ir\epsilon_+} \left(\frac{\Gamma(1 - ir\epsilon_+)}{\Gamma(1 + ir\epsilon_+)} \right)^2 Z_{11} \tag{A.32}$$

we obtain

$$\begin{aligned} Z^{\rm norm}_{2,1,2} = & \frac{1}{8\epsilon_1^2\epsilon_2^2} + \frac{1}{8\epsilon_1\epsilon_2} \left(\ln^2(z_0\overline{z}_0 z_1 \overline{z}_1) + \ln^2(z_1\overline{z}_1) \right) \\ & - i\frac{\epsilon_+}{2\epsilon_1\epsilon_2} \left(-\frac{1}{12}\ln^3(z_0\overline{z}_0 z_1\overline{z}_1) - \frac{1}{12}\ln^3(z_1\overline{z}_1) + 7\zeta(3) \right) \\ & - i\frac{\epsilon_+}{2\epsilon_1\epsilon_2} \Big(2(\mathrm{Li}_3(z_1) + \mathrm{Li}_3(z_0 z_1) + \mathrm{Li}_3(\overline{z}_1) + \mathrm{Li}_3(\overline{z}_0\overline{z}_1)) \\ & - \ln(z_1\overline{z}_1)(\mathrm{Li}_2(z_1) + \mathrm{Li}_2(\overline{z}_1)) - \ln(z_0 z_1 \overline{z}_0 \overline{z}_1)(\mathrm{Li}_2(z_0 z_1) + \mathrm{Li}_2(\overline{z}_0\overline{z}_1)) \Big) \end{aligned} \tag{A.33}$$

From this we can extract (after the usual redefinition $\epsilon_1 \to i\epsilon_1$, $\epsilon_2 \to i\epsilon_2$)

$$\begin{aligned} F_{(2,1,2)} = & \frac{1}{8\epsilon_1^2\epsilon_2^2} - \frac{1}{8\epsilon_1\epsilon_2} \left(\ln^2 z_0 z_1 + \ln^2 z_1 \right) \\ & + \frac{\epsilon_+}{2\epsilon_1\epsilon_2} \left(\frac{1}{12}\ln^3 z_0 z_1 + \frac{1}{12}\ln^3 z_1 + \mathrm{Li}_3(z_1) + \mathrm{Li}_3(z_0 z_1) \right) \end{aligned} \tag{A.34}$$

Case $p = 3$

This time we have the nine poles

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 3i\epsilon_1 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 4i\epsilon_1 \\ \lambda_2^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 5i\epsilon_1 \end{cases} \quad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 3i\epsilon_1 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 4i\epsilon_1 \\ \lambda_2^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \end{cases} \quad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 3i\epsilon_1 \\ \lambda_2^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_2 \end{cases}$$

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\ \lambda_2^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 - i\epsilon_2 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 - i\epsilon_2 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\ \lambda_2^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 - i\epsilon_2 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_2 \end{cases}$$

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_1 \\ \lambda_2^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_2 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 3i\epsilon_2 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 - i\epsilon_2 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_2 \\ \lambda_2^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 - 2i\epsilon_2 \end{cases}$$

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_2 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 3i\epsilon_2 \\ \lambda_2^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 4i\epsilon_2 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 2i\epsilon_2 \\ \lambda_2^{(0)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 3i\epsilon_2 \\ \lambda_2^{(2)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 4i\epsilon_2 \\ \lambda_2^{(1)} = -ia_1^{(0)} - i\frac{\epsilon_+}{2} - 5i\epsilon_2 \end{cases}$$

The mirror map and the normalization factor are given by $Z_{\mathrm{v}} \to (1 + z_0 z_1 z_2)^{2ir\epsilon_+} Z_{\mathrm{v}}$, $Z_{\mathrm{av}} \to (1 + \overline{z}_0\overline{z}_1\overline{z}_2)^{2ir\epsilon_+} Z_{\mathrm{av}}$ and

$$Z_{1\mathrm{l}} \to (z_0 z_1 z_2 \overline{z}_0\overline{z}_1\overline{z}_2)^{-2ira_1^{(0)} - ir\epsilon_+} \left(\frac{\Gamma(1 - ir\epsilon_+)}{\Gamma(1 + ir\epsilon_+)}\right)^2 Z_{1\mathrm{l}} \tag{A.35}$$

The partition function is therefore

$$
\begin{aligned}
Z^{\text{norm}}_{2,1,3} = & \frac{1}{18\epsilon_1^2\epsilon_2^2} + \frac{1}{3\epsilon_1\epsilon_2}\Bigg(\frac{1}{4}\ln^2(z_0\overline{z}_0) + \frac{1}{2}\ln(z_0\overline{z}_0)\ln(z_1\overline{z}_1) + \frac{1}{2}\ln(z_0\overline{z}_0)\ln(z_2\overline{z}_2) \\
& + \frac{5}{6}\ln(z_1\overline{z}_1)\ln(z_2\overline{z}_2) + \frac{7}{12}\ln^2(z_1\overline{z}_1) + \frac{7}{12}\ln^2(z_2\overline{z}_2)\Bigg) \\
& - i\frac{1}{3\epsilon_1\epsilon_2}\Bigg(- \frac{7\epsilon_1 + 11\epsilon_2}{36}\ln^3(z_1\overline{z}_1) - \frac{11\epsilon_1 + 7\epsilon_2}{36}\ln^3(z_2\overline{z}_2) \\
& - \frac{5\epsilon_1 + 7\epsilon_2}{12}\ln^2(z_1\overline{z}_1)\ln(z_2\overline{z}_2) - \frac{7\epsilon_1 + 5\epsilon_2}{12}\ln(z_1\overline{z}_1)\ln^2(z_2\overline{z}_2)\Bigg) \\
& - i\frac{\epsilon_+}{3\epsilon_1\epsilon_2}\Bigg(9\zeta(3) - \frac{1}{12}\ln^3(z_0\overline{z}_0) - \frac{1}{4}\ln^2(z_0\overline{z}_0)\ln(z_1\overline{z}_1) - \frac{1}{4}\ln(z_0\overline{z}_0)\ln^2(z_1\overline{z}_1) \\
& - \frac{1}{4}\ln^2(z_0\overline{z}_0)\ln(z_2\overline{z}_2) - \frac{1}{4}\ln(z_0\overline{z}_0)\ln^2(z_2\overline{z}_2) - \frac{1}{2}\ln(z_0\overline{z}_0)\ln(z_1\overline{z}_1)\ln(z_2\overline{z}_2) \\
& + 2(\text{Li}_3(z_1) + \text{Li}_3(z_2) + \text{Li}_3(z_1z_2) + \text{Li}_3(z_0z_1z_2)) \\
& + 2(\text{Li}_3(\overline{z}_1) + \text{Li}_3(\overline{z}_2) + \text{Li}_3(\overline{z}_1\overline{z}_2) + \text{Li}_3(\overline{z}_0\overline{z}_1\overline{z}_2)) \\
& - \ln(z_1\overline{z}_1)(\text{Li}_2(z_1) + \text{Li}_2(\overline{z}_1)) - \ln(z_2\overline{z}_2)(\text{Li}_2(z_2) + \text{Li}_2(\overline{z}_2)) \\
& - \ln(z_1z_2\overline{z}_1\overline{z}_2)(\text{Li}_2(z_1z_2) + \text{Li}_2(\overline{z}_1\overline{z}_2)) \\
& - \ln(z_0z_1z_2\overline{z}_0\overline{z}_1\overline{z}_2)(\text{Li}_2(z_0z_1z_2) + \text{Li}_2(\overline{z}_0\overline{z}_1\overline{z}_2))\Bigg)
\end{aligned}
\tag{A.36}
$$

from which

$$
\begin{aligned}
F_{(3,1,2)} = & \frac{1}{18\epsilon_1^2\epsilon_2^2} - \frac{1}{3\epsilon_1\epsilon_2}\Bigg(\frac{1}{4}\ln^2 z_0 + \frac{1}{2}\ln z_0 \ln z_1 + \frac{1}{2}\ln z_0 \ln z_2 \\
& + \frac{5}{6}\ln z_1 \ln z_2 + \frac{7}{12}\ln^2 z_1 + \frac{7}{12}\ln^2 z_2\Bigg) \\
& + \frac{1}{3\epsilon_1\epsilon_2}\Bigg(\frac{7\epsilon_1 + 11\epsilon_2}{36}\ln^3 z_1 + \frac{11\epsilon_1 + 7\epsilon_2}{36}\ln^3 z_2 \\
& + \frac{5\epsilon_1 + 7\epsilon_2}{12}\ln^2 z_1 \ln z_2 + \frac{7\epsilon_1 + 5\epsilon_2}{12}\ln z_1 \ln^2 z_2\Bigg) \\
& + \frac{\epsilon_+}{3\epsilon_1\epsilon_2}\Bigg(\frac{1}{12}\ln^3 z_0 + \frac{1}{4}\ln^2 z_0 \ln z_1 + \frac{1}{4}\ln z_0 \ln^2 z_1 \\
& + \frac{1}{4}\ln^2 z_0 \ln z_2 + \frac{1}{4}\ln z_0 \ln^2 z_2 + \frac{1}{2}\ln z_0 \ln z_1 \ln z_2\Bigg) \\
& + \frac{\epsilon_+}{3\epsilon_1\epsilon_2}(\text{Li}_3(z_1) + \text{Li}_3(z_2) + \text{Li}_3(z_1z_2) + \text{Li}_3(z_0z_1z_2))
\end{aligned}
\tag{A.37}
$$

A.1.2.3 The $N = 2$ sector, $p = 2$

For the cases $N \geqslant 2$ we do not know of any computation of the Gromov-Witten prepotential, so we will have to rely on our partition function. Here we will consider the case $p = 2$; by [139], the vectors $\vec{N}$, $\vec{k}$ are constrained at the values $\vec{N} = (0, 2)$, $\vec{k} = (k-1, k)$ or $\vec{N} = (2, 0)$, $\vec{k} = (k, k)$, corresponding respectively to fractional or integral instanton number $\frac{k_0+k_1}{2}$. We can compute the Gromov-Witten prepotential for small values of k as we did for in the previous examples, the main difference being the absence of equivariant mirror map; let us present here the final results.

Case $\vec{N} = (0, 2)$, $\vec{k} = (0, 1)$

The two poles are given by $\lambda_1^{(1)} = -ia_1^{(1)} - i\frac{\epsilon_+}{2}$ and $\lambda_1^{(1)} = -ia_2^{(1)} - i\frac{\epsilon_+}{2}$ respectively.

$$Z_{11} \to (z_1\bar{z}_1)^{-ir\frac{a_1^{(1)}+a_2^{(1)}}{2}} \frac{\Gamma(1 - ir\epsilon_+)}{\Gamma(1 + ir\epsilon_+)} Z_{11} \tag{A.38}$$

$$\begin{aligned} Z^{\text{norm}}_{(0,1),(0,2),2} = & \frac{2}{(a_1^{(1)} - a_2^{(1)})^2 - \epsilon_+^2} - \frac{1}{4}\ln^2(z_1\bar{z}_1) \\ & + i\epsilon_+\left(4\zeta(3) - \frac{1}{12}\ln^3(z_1\bar{z}_1) + 2(\text{Li}_3(z_1) + \text{Li}_3(\bar{z}_1)) - \ln(z_1\bar{z}_1)(\text{Li}_2(z_1) + \text{Li}_2(\bar{z}_1))\right) \end{aligned} \tag{A.39}$$

We notice that this coincides with (A.19) if we identify $a_1^{(1)} \longleftrightarrow \epsilon_1$, $a_2^{(1)} \longleftrightarrow \epsilon_2$. After the redefinition $\epsilon_1 \to i\epsilon_1$, $\epsilon_2 \to i\epsilon_2$, $a_1^{(1)} \to ia_1^{(1)}$, $a_2^{(1)} \to ia_2^{(1)}$ we obtain

$$F_{(0,1),(0,2),2} = \frac{2}{\epsilon_+^2 - (a_1^{(1)} - a_2^{(1)})^2} - \frac{1}{4}\ln^2 z_1 + \frac{\epsilon_+}{12}\ln^3 z_1 + \epsilon_+\text{Li}_3(z_1) \tag{A.40}$$

Case $\vec{N} = (2, 0)$, $\vec{k} = (1, 1)$

The four poles are

$$\begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} - i\epsilon_1 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_1^{(0)} - i\dfrac{\epsilon_+}{2} - i\epsilon_2 \end{cases}$$

$$\begin{cases} \lambda_1^{(0)} = -ia_2^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_2^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_1 \end{cases} \qquad \begin{cases} \lambda_1^{(0)} = -ia_2^{(0)} - i\frac{\epsilon_+}{2} \\ \lambda_1^{(1)} = -ia_2^{(0)} - i\frac{\epsilon_+}{2} - i\epsilon_2 \end{cases}$$

$$Z_{1l} \to (z_1 z_2 \overline{z}_1 \overline{z}_2)^{-ir\frac{a_1^{(0)}+a_2^{(0)}}{2}} \left(\frac{\Gamma(1-ir\epsilon_+)}{\Gamma(1+ir\epsilon_+)}\right)^2 Z_{1l} \tag{A.41}$$

$$\begin{aligned} Z^{\text{norm}}_{(1,1),(2,0),2} = & \frac{1}{2\epsilon_1\epsilon_2} \frac{2}{\left(\epsilon_+^2 - (a_1^{(0)} - a_2^{(0)})^2\right)} \\ & + \frac{1}{8\epsilon_1\epsilon_2} \ln^2(z_0\overline{z}_0 z_1 \overline{z}_1) + \frac{1}{2\left(\epsilon_+^2 - (a_1^{(0)} - a_2^{(0)})^2\right)} \ln^2(z_1\overline{z}_1) \\ & - i\frac{\epsilon_+}{2\epsilon_1\epsilon_2}\Bigg(- \frac{1}{12} \ln^3(z_0\overline{z}_0 z_1\overline{z}_1) + 4\zeta(3) \\ & + 2(\text{Li}_3(z_0 z_1) + \text{Li}_3(\overline{z}_0\overline{z}_1)) - \ln(z_0 z_1 \overline{z}_0 \overline{z}_1)(\text{Li}_2(z_0 z_1) + \text{Li}_2(\overline{z}_0\overline{z}_1))\Bigg) \\ & - i\frac{2\epsilon_+}{\left(\epsilon_+^2 - (a_1^{(0)} - a_2^{(0)})^2\right)}\Bigg(- \frac{1}{12}\ln^3(z_1\overline{z}_1) + 4\zeta(3) + 2(\text{Li}_3(z_1) + \text{Li}_3(\overline{z}_1)) \\ & - \ln(z_1\overline{z}_1)(\text{Li}_2(z_1) + \text{Li}_2(\overline{z}_1))\Bigg) \end{aligned} \tag{A.42}$$

After the usual redefinition of the twisted masses we have

$$\begin{aligned} F_{(1,1),(2,0),2} = & \frac{1}{2\epsilon_1\epsilon_2} \frac{2}{\left(\epsilon_+^2 - (a_1^{(0)} - a_2^{(0)})^2\right)} \\ & - \frac{1}{8\epsilon_1\epsilon_2}\left(\ln^2 z_0 z_1\right) - \frac{1}{2\left(\epsilon_+^2 - (a_1^{(0)} - a_2^{(0)})^2\right)} \ln^2 z_1 \\ & + \frac{\epsilon_+}{2\epsilon_1\epsilon_2}\left(\frac{1}{12}\ln^3 z_0 z_1 + \text{Li}_3(z_0 z_1)\right) \\ & + \frac{2\epsilon_+}{\left(\epsilon_+^2 - (a_1^{(0)} - a_2^{(0)})^2\right)}\left(\frac{1}{12}\ln^3 z_1 + \text{Li}_3(z_1)\right) \end{aligned} \tag{A.43}$$

A.2 D_p singularities

We now consider the quiver associated to a D_p singularity ($p \geqslant 4$), which corresponds to $\Gamma = BD_{4(p-2)}$ binary dihedral group. This discrete group has the presentation

$$\langle g, \tau \mid g^{2(p-2)} = \tau^4 = 1,\ g^{p-2} = \tau^2,\ \tau g \tau^{-1} = g^{-1} \rangle \tag{A.44}$$

and order $4(p-2)$. A possible realization is given by

$$g = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.45}$$

with α a primitive $2(p-2)$-th root of unity. The k-instanton moduli space for $U(N)$ gauge theories on ALE spaces of type D_p has been described by [139] in terms of the quiver corresponding to the Dynkin diagram of the affine $\widehat{D}_p$ algebra. In this section we will only study the case $N = 1$; for the generic N case, see [145].

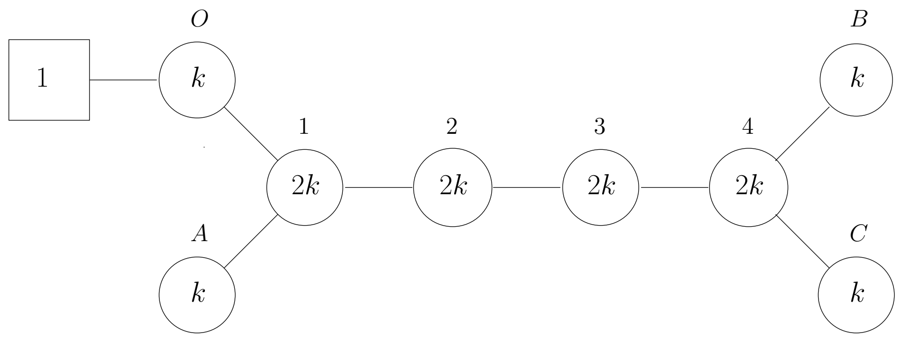


FIGURE A.2: The affine $\widehat{D}_p$ Dynkin diagram, in the case $p = 7$.

We can regard this as an $\mathcal{N} = (2,2)$ theory, whose Higgs branch coincides with the instanton moduli space we are considering. In this language, every circular node corresponds to a $U(k)$ or $U(2k)$ gauge group together with a matter field χ_b in the adjoint representation; the squared node represents a $U(1)$ flavour group for the matter fields I, J in the fundamental, antifundamental representation of $U(k)_O$; the lines are two matter fields $B_{b,b\pm1}$ in the bifundamental representation of two consecutive gauge groups. The superpotential of the theory is given by

$$\begin{aligned} W =& \mathrm{Tr}_O[\chi_O(B_{O,1}B_{1,O} + IJ)] + \mathrm{Tr}_A[\chi_A(B_{A,1}B_{1,A})] \\ &+ \mathrm{Tr}_1[\chi_1(B_{1,2}B_{2,1} - B_{1,O}B_{O,1} - B_{1,A}B_{A,1})] \\ &+ \sum_{b=2}^{p-4} \mathrm{Tr}_b[\chi_b(B_{b,b+1}B_{b+1,b} - B_{b,b-1}B_{b-1,b})] \\ &+ \mathrm{Tr}_{p-3}[\chi_{p-3}(-B_{p-3,p-4}B_{p-4,p-3} + B_{p-3,B}B_{B,p-3} + B_{p-3,C}B_{C,p-3})] \\ &+ \mathrm{Tr}_B[\chi_B(-B_{B,p-3}B_{p-3,B})] + \mathrm{Tr}_C[\chi_C(-B_{C,p-3}B_{p-3,C})] \end{aligned} \tag{A.46}$$

for $p \geqslant 5$, while in the special case $p = 4$ it reduces to

$$\begin{aligned} W =& \mathrm{Tr}_O[\chi_O(B_{O,1}B_{1,O} + IJ)] + \mathrm{Tr}_A[\chi_A(B_{A,1}B_{1,A})] + \mathrm{Tr}_B[\chi_B(-B_{B,1}B_{1,B})] \\ &+ \mathrm{Tr}_C[\chi_C(-B_{C,1}B_{1,C})] + \mathrm{Tr}_1[\chi_1(B_{1,B}B_{B,1} + B_{1,C}B_{C,1} - B_{1,O}B_{O,1} - B_{1,A}B_{A,1})] \end{aligned} \tag{A.47}$$

This last case is symmetric under exchange of A, B, C, as expected from the associated quiver. To describe completely the theory, we must also specify the R-charges and the twisted masses for the matter fields; these are summarized in the following table.

	χ_b	I	J	$B_{b,b+1}$	$B_{b,b-1}$
R-charge	2	0	0	0	0
twisted mass	$\epsilon_+ = \epsilon_1 + \epsilon_2$	$-a - \frac{\epsilon_+}{2}$	$a - \frac{\epsilon_+}{2}$	$-\epsilon_1$	$-\epsilon_2$

Here a is the twisted mass corresponding to the flavour group $U(1)$.

We can now compute the partition function on S^2 for this quiver theory; this will give us information about the quantum cohomology of these ALE spaces. Defining $z = e^{-2\pi\xi - i\theta}$, with ξ, θ Fayet-Iliopoulos and theta-angle parameters, the partition function reads

$$Z_{p,k,N} = \frac{1}{(k!)^4(2k!)^{p-3}} \sum_{\vec{m}\in\mathbb{Z}} \int \prod_{J=O,A,B,C} \prod_{s=1}^{k} \frac{d(r\sigma_s^{(J)})}{2\pi} \prod_{I=1}^{p-3} \prod_{s=1}^{2k} \frac{d(r\sigma_s^{(I)})}{2\pi} Z_{\mathrm{cl}} Z_{\mathrm{g,ad}} Z_{\mathrm{f,af}} Z_{\mathrm{bf}} \tag{A.48}$$

where the various pieces in the integrand are given by

$$\begin{aligned} Z_{\mathrm{cl}} =& \prod_{I=1}^{p-3} \prod_{s=1}^{2k} z_I^{ir\sigma_s^{(I)} + \frac{m_s^{(I)}}{2}} \bar{z}_I^{ir\sigma_s^{(I)} - \frac{m_s^{(I)}}{2}} \prod_{J=O,A,B,C} \prod_{s=1}^{k} z_J^{ir\sigma_s^{(J)} + \frac{m_s^{(J)}}{2}} \bar{z}_J^{ir\sigma_s^{(J)} - \frac{m_s^{(J)}}{2}} \\ Z_{\mathrm{g,ad}} =& \prod_{I=1}^{p-3} \prod_{s<t=1}^{2k} \left(r^2(\sigma_{s,t}^{(I)})^2 + \frac{(m_{s,t}^{(I)})^2}{4} \right) \prod_{J=O,A,B,C} \prod_{s<t=1}^{k} \left(r^2(\sigma_{s,t}^{(J)})^2 + \frac{(m_{s,t}^{(J)})^2}{4} \right) \\ & \prod_{I=1}^{p-3} \prod_{s,t=1}^{2k} \frac{\Gamma(1 - ir\sigma_{s,t}^{(I)} - \frac{m_{s,t}^{(I)}}{2} - ir\epsilon_+)}{\Gamma(ir\sigma_{s,t}^{(I)} - \frac{m_{s,t}^{(I)}}{2} + ir\epsilon_+)} \prod_{J=O,A,B,C} \prod_{s,t=1}^{k} \frac{\Gamma(1 - ir\sigma_{s,t}^{(J)} - \frac{m_{s,t}^{(J)}}{2} - ir\epsilon_+)}{\Gamma(ir\sigma_{s,t}^{(J)} - \frac{m_{s,t}^{(J)}}{2} + ir\epsilon_+)} \\ Z_{\mathrm{f,af}} =& \prod_{s=1}^{k} \frac{\Gamma(-ir\sigma_s^{(O)} - \frac{m_s^{(O)}}{2} + ira + ir\frac{\epsilon_+}{2})}{\Gamma(1 + ir\sigma_s^{(O)} - \frac{m_s^{(O)}}{2} - ira - ir\frac{\epsilon_+}{2})} \frac{\Gamma(ir\sigma_s^{(O)} + \frac{m_s^{(O)}}{2} - ira + ir\frac{\epsilon_+}{2})}{\Gamma(1 - ir\sigma_s^{(O)} + \frac{m_s^{(O)}}{2} + ira - ir\frac{\epsilon_+}{2})} \end{aligned}$$

$$Z_{\text{bf}} = \prod_{I=1}^{p-4} \prod_{s,t=1}^{2k} \frac{\Gamma(-ir\sigma_{s,t}^{(I+1,I)} - \frac{m_{s,t}^{(I+1,I)}}{2} + ir\epsilon_1)}{\Gamma(1 + ir\sigma_{s,t}^{(I+1,I)} - \frac{m_{s,t}^{(I+1,I)}}{2} - ir\epsilon_1)} \frac{\Gamma(ir\sigma_{s,t}^{(I+1,I)} + \frac{m_{s,t}^{(I+1,I)}}{2} + ir\epsilon_2)}{\Gamma(1 - ir\sigma_{s,t}^{(I+1,I)} + \frac{m_{s,t}^{(I+1,I)}}{2} - ir\epsilon_2)}$$
$$\prod_{J=O,A} \prod_{s=1}^{2k} \prod_{t=1}^{k} \frac{\Gamma(-ir\sigma_{s,t}^{(1,J)} - \frac{m_{s,t}^{(1,J)}}{2} + ir\epsilon_1)}{\Gamma(1 + ir\sigma_{s,t}^{(1,J)} - \frac{m_{s,t}^{(1,J)}}{2} - ir\epsilon_1)} \frac{\Gamma(ir\sigma_{s,t}^{(1,J)} + \frac{m_{s,t}^{(1,J)}}{2} + ir\epsilon_2)}{\Gamma(1 - ir\sigma_{s,t}^{(1,J)} + \frac{m_{s,t}^{(1,J)}}{2} - ir\epsilon_2)}$$
$$\prod_{J=B,C} \prod_{s=1}^{k} \prod_{t=1}^{2k} \frac{\Gamma(-ir\sigma_{s,t}^{(J,p-3)} - \frac{m_{s,t}^{(J,p-3)}}{2} + ir\epsilon_1)}{\Gamma(1 + ir\sigma_{s,t}^{(J,p-3)} - \frac{m_{s,t}^{(J,p-3)}}{2} - ir\epsilon_1)} \frac{\Gamma(ir\sigma_{s,t}^{(J,p-3)} + \frac{m_{s,t}^{(J,p-3)}}{2} + ir\epsilon_2)}{\Gamma(1 - ir\sigma_{s,t}^{(J,p-3)} + \frac{m_{s,t}^{(J,p-3)}}{2} - ir\epsilon_2)} \tag{A.49}$$

Here we used the compact notation $\sigma_{s,t}^{(I,J)} = \sigma_s^{(I)} - \sigma_t^{(J)}$ and $\sigma_{s,t}^{(I)} = \sigma_s^{(I)} - \sigma_t^{(I)}$.

A.2.1 Instanton partiton function

As explained in the main text, the small radius limit $r \to 0$ produces a contour integral representation for the instanton part of Nekrasov partition function at fixed k. In this case, we obtain

$$Z_{p,k,N}^{\text{inst}} = \frac{\epsilon^{2k(p-1)}}{(ir)^{2Nk}} \oint \prod_{J=O,A,B,C} \prod_{s=1}^{k} \frac{d\sigma_s^{(J)}}{2\pi i} \prod_{I=1}^{p-3} \prod_{s=1}^{2k} \frac{d\sigma_s^{(I)}}{2\pi i}$$
$$\prod_{s=1}^{k} \frac{1}{(\sigma_s^{(O)} - a - \frac{\epsilon_+}{2})(-\sigma_s^{(O)} + a - \frac{\epsilon_+}{2})} \prod_{I=1}^{p-3} \prod_{\substack{s,t=1 \\ s\neq t}}^{2k} (\sigma_{s,t}^{(I)})(\sigma_{s,t}^{(I)} - \epsilon_+)$$
$$\prod_{J=O,A,B,C} \prod_{\substack{s,t=1 \\ s\neq t}}^{k} (\sigma_{s,t}^{(J)})(\sigma_{s,t}^{(J)} - \epsilon_+) \prod_{I=1}^{p-4} \prod_{s,t=1}^{2k} \frac{1}{(\sigma_{s,t}^{(I+1,I)} - \epsilon_1)(-\sigma_{s,t}^{(I+1,I)} - \epsilon_2)}$$
$$\prod_{J=O,A} \prod_{s=1}^{2k} \prod_{t=1}^{k} \frac{1}{(\sigma_{s,t}^{(1,J)} - \epsilon_1)(-\sigma_{s,t}^{(1,J)} - \epsilon_2)} \prod_{J=B,C} \prod_{s=1}^{k} \prod_{t=1}^{2k} \frac{1}{(\sigma_{s,t}^{(J,p-3)} - \epsilon_1)(-\sigma_{s,t}^{(J,p-3)} - \epsilon_2)} \tag{A.50}$$

The factorials have been omitted, since they are cancelled by the possible orderings of the integration variables.

A.2.2 Equivariant quantum cohomology

For r finite, the partition function computes the equivariant quantum cohomology of the moduli space of instantons on the ALE space. In particular, after factorizing (A.48) as

$$Z_{p,k,N} = \frac{1}{(k!)^4(2k!)^{p-3}} \oint \prod_{J=O,A,B,C} \prod_{s=1}^{k} \frac{d(r\lambda_s^{(J)})}{2\pi i} \prod_{I=1}^{p-3} \prod_{s=1}^{2k} \frac{d(r\lambda_s^{(I)})}{2\pi i} Z_{1l} Z_{\mathrm{v}} Z_{\mathrm{av}} \tag{A.51}$$

$$\begin{aligned}
Z_{1l} = & \left(\frac{\Gamma(1-ir\epsilon_+)}{\Gamma(ir\epsilon_+)}\right)^{2k(p-1)} \prod_{I=1}^{p-3}\prod_{s=1}^{2k} (z_I \bar{z}_I)^{-r\lambda_s^{(I)}} \prod_{J=O,A,B,C}\prod_{s=1}^{k} (z_J \bar{z}_J)^{-r\lambda_s^{(J)}} \\
& \prod_{I=1}^{p-3}\prod_{s=1}^{2k}\prod_{t\neq s}^{2k} (r\lambda_{s,t}^{(I)}) \frac{\Gamma(1+r\lambda_{s,t}^{(I)}-ir\epsilon_+)}{\Gamma(-r\lambda_{s,t}^{(I)}+ir\epsilon_+)} \prod_{J=O,A,B,C}\prod_{s=1}^{k}\prod_{t\neq s}^{k} (r\lambda_{s,t}^{(J)}) \frac{\Gamma(1+r\lambda_{s,t}^{(J)}-ir\epsilon_+)}{\Gamma(-r\lambda_{s,t}^{(J)}+ir\epsilon_+)} \\
& \prod_{I=1}^{p-4}\prod_{s,t=1}^{2k} \frac{\Gamma(r\lambda_{s,t}^{(I+1,I)}+ir\epsilon_1)}{\Gamma(1-r\lambda_{s,t}^{(I+1,I)}-ir\epsilon_1)} \frac{\Gamma(-r\lambda_{s,t}^{(I+1,I)}+ir\epsilon_2)}{\Gamma(1+r\lambda_{s,t}^{(I+1,I)}-ir\epsilon_2)} \\
& \prod_{J=O,A}\prod_{s=1}^{2k}\prod_{t=1}^{k} \frac{\Gamma(r\lambda_{s,t}^{(1,J)}+ir\epsilon_1)}{\Gamma(1-r\lambda_{s,t}^{(1,J)}-ir\epsilon_1)} \frac{\Gamma(-r\lambda_{s,t}^{(1,J)}+ir\epsilon_2)}{\Gamma(1+r\lambda_{s,t}^{(1,J)}-ir\epsilon_2)} \\
& \prod_{J=B,C}\prod_{s=1}^{k}\prod_{t=1}^{2k} \frac{\Gamma(r\lambda_{s,t}^{(J,p-3)}+ir\epsilon_1)}{\Gamma(1-r\lambda_{s,t}^{(J,p-3)}-ir\epsilon_1)} \frac{\Gamma(-r\lambda_{s,t}^{(J,p-3)}+ir\epsilon_2)}{\Gamma(1+r\lambda_{s,t}^{(J,p-3)}-ir\epsilon_2)} \\
& \prod_{s=1}^{k} \frac{\Gamma(r\lambda_s^{(O)}+ira+ir\frac{\epsilon_+}{2})}{\Gamma(1-r\lambda_s^{(O)}-ira-ir\frac{\epsilon_+}{2})} \frac{\Gamma(-r\lambda_s^{(O)}-ira+ir\frac{\epsilon_+}{2})}{\Gamma(1+r\lambda_s^{(O)}+ira-ir\frac{\epsilon_+}{2})}
\end{aligned} \tag{A.52}$$

$$\begin{aligned}
Z_{\mathrm{v}} = & \sum_{\{\vec{l}\}\in\mathbb{N}} \prod_{s=1}^{k} (-1)^{Nl_s^{(O)}} \prod_{I=1}^{p-3}\prod_{s=1}^{2k} z_I^{l_s^{(I)}} \prod_{J=O,A,B,C}\prod_{s=1}^{k} z_J^{l_s^{(J)}} \\
& \prod_{I=1}^{p-3}\prod_{s<t}^{2k} \frac{l_{t,s}^{(I)}-r\lambda_{t,s}^{(I)}}{-r\lambda_{t,s}^{(I)}} \frac{(1+r\lambda_{s,t}^{(I)}-ir\epsilon_+)_{l_{t,s}^{(I)}}}{(r\lambda_{s,t}^{(I)}+ir\epsilon_+)_{l_{t,s}^{(I)}}} \prod_{J=O,A,B,C}\prod_{s<t}^{k} \frac{l_{t,s}^{(J)}-r\lambda_{t,s}^{(J)}}{-r\lambda_{t,s}^{(J)}} \frac{(1+r\lambda_{s,t}^{(J)}-ir\epsilon_+)_{l_{t,s}^{(J)}}}{(r\lambda_{s,t}^{(J)}+ir\epsilon_+)_{l_{t,s}^{(J)}}} \\
& \prod_{I=1}^{p-4}\prod_{s=1}^{2k}\prod_{t=1}^{2k} \frac{1}{(1-r\lambda_{s,t}^{(I+1,I)}-ir\epsilon_1)_{l_{s,t}^{(I+1,I)}}} \frac{1}{(1+r\lambda_{s,t}^{(I+1,I)}-ir\epsilon_2)_{l_{t,s}^{(I,I+1)}}} \\
& \prod_{J=O,A}\prod_{s=1}^{2k}\prod_{t=1}^{k} \frac{1}{(1-r\lambda_{s,t}^{(1,J)}-ir\epsilon_1)_{l_{s,t}^{(1,J)}}} \frac{1}{(1+r\lambda_{s,t}^{(1,J)}-ir\epsilon_2)_{l_{t,s}^{(J,1)}}} \\
& \prod_{J=B,C}\prod_{s=1}^{k}\prod_{t=1}^{2k} \frac{1}{(1-r\lambda_{s,t}^{(J,p-3)}-ir\epsilon_1)_{l_{s,t}^{(J,p-3)}}} \frac{1}{(1+r\lambda_{s,t}^{(J,p-3)}-ir\epsilon_2)_{l_{t,s}^{(p-3,J)}}} \\
& \prod_{s=1}^{k} \frac{(-r\lambda_s^{(O)}-ira+ir\frac{\epsilon_+}{2})_{l_s^{(O)}}}{(1-r\lambda_s^{(O)}-ira-ir\frac{\epsilon_+}{2})_{l_s^{(O)}}}
\end{aligned} \tag{A.53}$$

$$
\begin{aligned}
Z_{\mathrm{av}} = & \sum_{\{\vec{k}\}\in\mathbb{N}} \prod_{s=1}^{k} (-1)^{Nk_s^{(O)}} \prod_{I=1}^{p-3}\prod_{s=1}^{2k} \bar{z}_I^{k_s^{(I)}} \prod_{J=O,A,B,C}\prod_{s=1}^{k} \bar{z}_J^{k_s^{(J)}} \\
& \prod_{I=1}^{p-3}\prod_{s<t}^{2k} \frac{k_{t,s}^{(I)} - r\lambda_{t,s}^{(I)}}{-r\lambda_{t,s}^{(I)}} \frac{(1+r\lambda_{s,t}^{(I)} - ir\epsilon_+)_{k_{t,s}^{(I)}}}{(r\lambda_{s,t}^{(I)} + ir\epsilon_+)_{k_{t,s}^{(I)}}} \prod_{J=O,A,B,C}\prod_{s<t}^{k} \frac{k_{t,s}^{(J)} - r\lambda_{t,s}^{(J)}}{-r\lambda_{t,s}^{(J)}} \frac{(1+r\lambda_{s,t}^{(J)} - ir\epsilon_+)_{k_{t,s}^{(J)}}}{(r\lambda_{s,t}^{(J)} + ir\epsilon_+)_{k_{t,s}^{(J)}}} \\
& \prod_{I=1}^{p-4}\prod_{s=1}^{2k}\prod_{t=1}^{2k} \frac{1}{(1-r\lambda_{s,t}^{(I+1,I)} - ir\epsilon_1)_{k_{s,t}^{(I+1,I)}}} \frac{1}{(1+r\lambda_{s,t}^{(I+1,I)} - ir\epsilon_2)_{k_{t,s}^{(I,I+1)}}} \\
& \prod_{J=O,A}\prod_{s=1}^{2k}\prod_{t=1}^{k} \frac{1}{(1-r\lambda_{s,t}^{(1,J)} - ir\epsilon_1)_{k_{s,t}^{(1,J)}}} \frac{1}{(1+r\lambda_{s,t}^{(1,J)} - ir\epsilon_2)_{k_{t,s}^{(J,1)}}} \\
& \prod_{J=B,C}\prod_{s=1}^{k}\prod_{t=1}^{2k} \frac{1}{(1-r\lambda_{s,t}^{(J,p-3)} - ir\epsilon_1)_{k_{s,t}^{(J,p-3)}}} \frac{1}{(1+r\lambda_{s,t}^{(J,p-3)} - ir\epsilon_2)_{k_{t,s}^{(p-3,J)}}} \\
& \prod_{s=1}^{k} \frac{(-r\lambda_s^{(O)} - ira + ir\frac{\epsilon_+}{2})_{k_s^{(O)}}}{(1-r\lambda_s^{(O)} - ira - ir\frac{\epsilon_+}{2})_{k_s^{(O)}}}
\end{aligned}
\tag{A.54}
$$

we can identify Z_{v} with Givental's $\mathcal{I}$-function for our target space.

A.2.3 Analysis of the Coulomb branch

Let us conclude with a few comments on the integrable system side of the D_p ALE quiver. As familiar by now, the mirror LG model in the Coulomb branch can be recovered by taking the large radius limit $r \to \infty$ of (A.48). We obtain

$$
Z_{k,1,p}^{S^2} = \frac{(r\epsilon)^{2k(p-1)}}{(k!)^4(2k!)^{p-3}} \left| \int \prod_{J=O,A,B,C}\prod_{s=1}^{k} \frac{d(r\Sigma_s^{(J)})}{2\pi} \prod_{I=0}^{p-3}\prod_{s=1}^{2k} \frac{d(r\Sigma_s^{(I)})}{2\pi} Z_{\mathrm{meas}}(\Sigma) e^{-\mathcal{W}_{\mathrm{eff}}(\Sigma)} \right|^2
\tag{A.55}
$$

Here the integration measure is given by

$$
\begin{aligned}
Z_{\mathrm{meas}}(\Sigma) = & \left(\frac{\prod_{I=1}^{p-3}\prod_{s,t\neq s}^{2k} D(\Sigma_s^{(I)} - \Sigma_t^{(I)}) \prod_{J=O,A,B,C}\prod_{s,t\neq s}^{k} D(\Sigma_s^{(J)} - \Sigma_t^{(J)})}{\prod_{s=1}^{k} Q(\Sigma_s^{(O)}) \prod_{I=1}^{p-4}\prod_{s=1}^{2k}\prod_{t=1}^{2k} F(\Sigma_s^{(I+1)} - \Sigma_t^{(I)})} \right)^{\frac{1}{2}} \\
& \left(\frac{1}{\prod_{J=O,A}\prod_{s=1}^{2k}\prod_{t=1}^{k} F(\Sigma_s^{(1)} - \Sigma_t^{(J)}) \prod_{J=B,C}\prod_{s=1}^{k}\prod_{t=1}^{2k} F(\Sigma_s^{(J)} - \Sigma_t^{(p-3)})} \right)^{\frac{1}{2}}
\end{aligned}
\tag{A.56}
$$

with

$$
\begin{aligned}
D(\Sigma_s^{(I)} - \Sigma_t^{(I)}) &= r^2(\Sigma_s^{(I)} - \Sigma_t^{(I)})(\Sigma_s^{(I)} - \Sigma_t^{(I)} + \epsilon_+) \\
F(\Sigma_s^{(I+1)} - \Sigma_t^{(I)}) &= r^2(\Sigma_s^{(I+1)} - \Sigma_t^{(I)} - \epsilon_1)(\Sigma_s^{(I+1)} - \Sigma_t^{(I)} + \epsilon_2) \\
Q(\Sigma_s^{(O)}) &= r^2\left(\Sigma_s^{(O)} - a - \frac{\epsilon_+}{2}\right)\left(\Sigma_s^{(O)} - a + \frac{\epsilon_+}{2}\right)
\end{aligned}
\tag{A.57}
$$

The twisted effective superpotential has the form

$$
\begin{aligned}
\mathcal{W}_{\text{eff}}(\Sigma) &= 2\pi\sum_{I=1}^{p-3}\sum_{s=1}^{2k} irt_I\Sigma_s^{(I)} + 2\pi\sum_{J=O,A,B,C}\sum_{s=1}^{k} irt_J\Sigma_s^{(J)} \\
&+ \sum_{s=1}^{k}\left[\omega(ir\Sigma_s^{(O)} - ira - ir\frac{\epsilon_+}{2}) + \omega(-ir\Sigma_s^{(O)} + ira - ir\frac{\epsilon_+}{2})\right] \\
&+ \sum_{I=1}^{p-3}\sum_{s,t\neq s}^{2k}\left[\omega(ir\Sigma_s^{(I)} - ir\Sigma_t^{(I)}) + \omega(ir\Sigma_s^{(I)} - ir\Sigma_t^{(I)} + ir\epsilon_+)\right] \\
&+ \sum_{J=O,A,B,C}\sum_{s,t\neq s}^{k}\left[\omega(ir\Sigma_s^{(J)} - ir\Sigma_t^{(J)}) + \omega(ir\Sigma_s^{(J)} - ir\Sigma_t^{(J)} + ir\epsilon_+)\right] \\
&+ \sum_{I=1}^{p-4}\sum_{s=1}^{2k}\sum_{t=1}^{2k}\left[\omega(ir\Sigma_s^{(I+1)} - ir\Sigma_t^{(I)} - ir\epsilon_1) + \omega(-ir\Sigma_s^{(I+1)} + ir\Sigma_t^{(I)} - ir\epsilon_2)\right] \\
&+ \sum_{J=O,A}\sum_{s=1}^{2k}\sum_{t=1}^{k}\left[\omega(ir\Sigma_s^{(1)} - ir\Sigma_t^{(J)} - ir\epsilon_1) + \omega(-ir\Sigma_s^{(1)} + ir\Sigma_t^{(J)} - ir\epsilon_2)\right] \\
&+ \sum_{J=B,C}\sum_{s=1}^{k}\sum_{t=1}^{2k}\left[\omega(ir\Sigma_s^{(J)} - ir\Sigma_t^{(p-3)} - ir\epsilon_1) + \omega(-ir\Sigma_s^{(J)} + ir\Sigma_t^{(p-3)} - ir\epsilon_2)\right]
\end{aligned}
\tag{A.58}
$$

From (A.58) we recover a set of Bethe Ansatz Equations, which can be written as

$$
\prod_{j=1}^{N_b}\frac{\Sigma_s^{(b)} - a_j^{(b)} - \frac{\epsilon_+}{2}}{-\Sigma_s^{(b)} + a_j^{(b)} - \frac{\epsilon_+}{2}}\prod_c\prod_{\substack{t=1\\(c,t)\neq(b,s)}}^{k_c}\frac{\Sigma_s^{(b)} - \Sigma_t^{(c)} + \mathbf{C}_{bc}^T}{\Sigma_s^{(b)} - \Sigma_t^{(c)} - \mathbf{C}_{bc}} = e^{-2\pi t_b}
\tag{A.59}
$$

Here $c = O, A, 1, \ldots, p-3, B, C$, while $\vec{N} = (1, 0, \ldots, 0)$ and $\vec{k} = (k, k, , 2k, \ldots, 2k, k, k)$ as discusses earlier ($a_1^{(O)} = a$). The matrix

$$\mathbf{C}_{bc} = \begin{bmatrix} \epsilon_+ & 0 & -\epsilon_1 & 0 & 0 & \cdots & 0 \\ 0 & \epsilon_+ & -\epsilon_1 & 0 & \ddots & \ddots & 0 \\ -\epsilon_2 & -\epsilon_2 & \epsilon_+ & -\epsilon_1 & \ddots & \ddots & 0 \\ 0 & 0 & -\epsilon_2 & \epsilon_+ & -\epsilon_1 & \ddots & 0 \\ 0 & 0 & \ddots & -\epsilon_2 & \epsilon_+ & -\epsilon_1 & -\epsilon_1 \\ \vdots & \vdots & \ddots & 0 & -\epsilon_2 & \epsilon_+ & 0 \\ 0 & 0 & \cdots & 0 & -\epsilon_2 & 0 & \epsilon_+ \end{bmatrix} \tag{A.60}$$

is again the adjacency matrix of the quiver graph, and reduces to the Cartan matrix of the affine $\widehat{D}_p$ algebra for $\epsilon_1 = \epsilon_2$. We expect (A.59) to be related to a quantum hydrodynamical integrable system, a sort of D_p-type generalization of ILW. Solutions to (A.59) will correspond to eigenstates of the QIS; expressions for the norm of the eigenstates can be obtained by performing a semiclassical approximation of the partition function around the corresponding vacua, as we already discussed in the previous sections.

Appendix B

Equivariant quantum cohomology in oscillator formalism

Let us show here that the Gromov-Witten potentials computed for $\mathcal{M}_{k,1}$ in section 4.3.2 are in agreement with the results on quantum multiplication for the Hilbert scheme of points obtained in [106].

Following the notation of [104] and [106], the Fock space description of the equivariant cohomology of the Hilbert scheme of points of $\mathbb{C}^2$ is given in terms of creation-annihilation operators α_k, $k \in \mathbb{Z}$ obeying the Heisenberg algebra

$$[\alpha_p, \alpha_q] = p\delta_{p+q} \tag{B.1}$$

The vacuum is annihilated by the positive modes

$$\alpha_p|\emptyset\rangle = 0 \quad , \quad p > 0 \tag{B.2}$$

and the natural basis on the Fock space is given by

$$|Y\rangle = \frac{1}{|Aut(Y)| \prod_i Y_i} \prod_i \alpha_{Y_i} |\emptyset\rangle \tag{B.3}$$

where $|Aut(Y)|$ is the order of the automorphism group of the partition and Y_i are the lengths of the columns of the Young tableau Y. The total number of boxes of the Young tableau is counted by the eigenvalue of the energy $K = \sum_{p>0} \alpha_{-p}\alpha_p$. Fix now the subspace $\mathrm{Ker}(K - k)$ for $k \in \mathbb{Z}_+$ and allow linear combinations with coefficients being rational functions of the equivariant weights. This space is then identified with the equivariant cohomology $H_T^*(\mathcal{M}_{k,1},\mathbb{Q})$. More specifically

$$|Y\rangle \in H_T^{2n-2\ell(Y)}(\mathcal{M}_{k,1},\mathbb{Q})\,, \tag{B.4}$$

where $\ell(Y)$ denotes the number of parts of the partition Y.

According to [106], the generator of the small quantum cohomology is then given by the state $|D\rangle = -|2,1^{k-2}\rangle$ which describes the divisor corresponding to the collision of two point-like instantons.

The operator generating the quantum product by $|D\rangle$ can be recognized as the fundamental quantum Hamiltonian of the ILW system (or, equivalently, as the quantum deformed Calogero-Sutherland Hamiltonian)

$$\begin{aligned} H_D \equiv\ & (\epsilon_1+\epsilon_2)\sum_{p>0}\frac{p}{2}\frac{(-q)^p+1}{(-q)^p-1}\alpha_{-p}\alpha_p \\ & + \sum_{p,q>0}\left[\epsilon_1\epsilon_2\alpha_{p+q}\alpha_{-p}\alpha_{-q}-\alpha_{-p-q}\alpha_p\alpha_q\right]-\frac{\epsilon_1+\epsilon_2}{2}\frac{(-q)+1}{(-q)-1}K \end{aligned} \tag{B.5}$$

We can then compute the basic three point function as $\langle D|H_D|D\rangle$, where the inner product is normalized to be

$$\langle Y|Y'\rangle = \frac{(-1)^{K-\ell(Y)}}{(\epsilon_1\epsilon_2)^{\ell(Y)}\,|Aut(Y)|\prod_i Y_i}\delta_{YY'} \tag{B.6}$$

The computation gives

$$\langle D|H_D|D\rangle = (\epsilon_1{+}\epsilon_2)\left(\frac{(-q)^2+1}{(-q)^2-1}-\frac{1}{2}\frac{(-q)+1}{(-q)-1}\right)\langle D|\alpha_{-2}\alpha_2|D\rangle = (-1)(\epsilon_1{+}\epsilon_2)\frac{1+q}{1-q}\langle D|D\rangle,$$

where we have used $\langle D|\alpha_{-2}\alpha_2|D\rangle = 2\langle D|D\rangle$. By (B.6), we finally get

$$\langle D|H_D|D\rangle = \frac{\epsilon_1+\epsilon_2}{(\epsilon_1\epsilon_2)^{k-1}}\frac{1}{2(k-2)!}\left(1+2\frac{q}{1-q}\right) \tag{B.7}$$

Rewriting $1+2\frac{q}{1-q} = (q\partial_q)^3\left[\frac{(\ln q)^3}{3!}+2\mathrm{Li}_3(q)\right]$, we obtain that the genus zero prepotential is

$$F^0 = F^0_{cl} + \frac{\epsilon_1+\epsilon_2}{(\epsilon_1\epsilon_2)^{k-1}}\frac{1}{2(k-2)!}\left[\frac{(\ln q)^3}{3!}+2\mathrm{Li}_3(q)\right] \tag{B.8}$$

The above formula precisely agrees with the results of Chapter 4, see (4.39) and (4.40) for the cases $k=3,4$ respectively.

The generalization of the Fock space formalism to the rank N ADHM instanton moduli space was given by Baranovsky in [146] in terms of N copies of Nakajima operators as

$\beta_k = \sum_{i=1}^{N} \alpha_k^{(i)}$. For example, in the $N = 2$ case the quantum Hamiltonian becomes (modulo terms proportional to the quantum momentum) [96]

$$\begin{aligned} H_D =& \frac{1}{2}\sum_{i=1}^{2}\sum_{n,k>0}[\epsilon_1\epsilon_2\alpha_{-n}^{(i)}\alpha_{-k}^{(i)}\alpha_{n+k}^{(i)} - \alpha_{-n-k}^{(i)}\alpha_n^{(i)}\alpha_k^{(i)}] \\ & - \frac{\epsilon_1+\epsilon_2}{2}\sum_{k>0} k[\alpha_{-k}^{(1)}\alpha_k^{(1)} + \alpha_{-k}^{(2)}\alpha_k^{(2)} + 2\alpha_{-k}^{(2)}\alpha_k^{(1)}] \\ & - (\epsilon_1+\epsilon_2)\sum_{k>0} k\frac{q^k}{1-q^k}[\alpha_{-k}^{(1)}\alpha_k^{(1)} + \alpha_{-k}^{(2)}\alpha_k^{(2)} + \alpha_{-k}^{(2)}\alpha_k^{(1)} + \alpha_{-k}^{(1)}\alpha_k^{(2)}] \end{aligned} \tag{B.9}$$

This is the same as the I_3 Hamiltonian for $gl(2)$ ILW given in [91]:

$$I_3 = \sum_{k\neq 0} L_{-k}a_k + 2iQ\sum_{k>0} ka_{-k}a_k\frac{1+q^k}{1-q^k} + \frac{1}{3}\sum_{n+m+k=0} a_n a_m a_k \tag{B.10}$$

In fact, after rewriting the Virasoro generators in terms of Heisenberg generators according to

$$L_n = \sum_{k\neq\{0,n\}} c_{n-k}c_k + i(nQ-2P)c_n \quad , \quad [c_m, c_n] = \frac{m}{2}\delta_{m+n,0} \tag{B.11}$$

and ignoring terms proportional to the momentum, we arrive at

$$\begin{aligned} I_3 =& \sum_{n,k>0}[a_{-n-k}c_n c_k + 2a_{-n}c_{-k}c_{n+k} + 2c_{-n-k}c_n a_k + c_{-n}c_{-k}a_{n+k}] \\ & + 2iQ\sum_{k>0} k[a_{-k}a_k - \frac{1}{2}(c_{-k}a_k - a_{-k}c_k)] \\ & + 4iQ\sum_{k>0} ka_{-k}a_k\frac{q^k}{1-q^k} + \sum_{n,k>0} a_{-n-k}a_n a_k + \sum_{n,k>0} a_{-n}a_{-k}a_{n+k} \end{aligned} \tag{B.12}$$

where we used

$$\sum_{k\neq 0}\sum_{n\neq\{0,-k\}} c_{-n-k}c_n a_k = \sum_{n,k>0}[a_{-n-k}c_n c_k + 2a_{-n}c_{-k}c_{n+k} + 2c_{-n-k}c_n a_k + c_{-n}c_{-k}a_{n+k}] \tag{B.13}$$

These a_k modes are the ones related to the Baranovsky operators. Finally, by making the substitution

$$a_k = -\frac{i}{\sqrt{\epsilon_1\epsilon_2}}\frac{\alpha_k^{(1)}+\alpha_k^{(2)}}{2} \quad , \quad c_k = -\frac{i}{\sqrt{\epsilon_1\epsilon_2}}\frac{\alpha_k^{(1)}-\alpha_k^{(2)}}{2} \tag{B.14}$$

for positive modes and

$$a_{-k} = i\sqrt{\epsilon_1\epsilon_2}\,\frac{\alpha_{-k}^{(1)}+\alpha_{-k}^{(2)}}{2} \quad , \quad c_{-k} = i\sqrt{\epsilon_1\epsilon_2}\,\frac{\alpha_{-k}^{(1)}-\alpha_{-k}^{(2)}}{2} \tag{B.15}$$

for the negative ones, we obtain

$$\begin{aligned}
I_3 =& \frac{i}{2\sqrt{\epsilon_1\epsilon_2}} \sum_{n,k>0} [\epsilon_1\epsilon_2 \alpha^{(1)}_{-n}\alpha^{(1)}_{-k}\alpha^{(1)}_{n+k} - \alpha^{(1)}_{-n-k}\alpha^{(1)}_{n}\alpha^{(1)}_{k} + \epsilon_1\epsilon_2 \alpha^{(2)}_{-n}\alpha^{(2)}_{-k}\alpha^{(2)}_{n+k} - \alpha^{(2)}_{-n-k}\alpha^{(2)}_{n}\alpha^{(2)}_{k}] \\
&+ \frac{iQ}{2} \sum_{k>0} k[\alpha^{(1)}_{-k}\alpha^{(1)}_{k} + \alpha^{(2)}_{-k}\alpha^{(2)}_{k} + 2\alpha^{(2)}_{-k}\alpha^{(1)}_{k}] \\
&+ iQ \sum_{k>0} k \frac{q^k}{1-q^k} [\alpha^{(1)}_{-k}\alpha^{(1)}_{k} + \alpha^{(2)}_{-k}\alpha^{(2)}_{k} + \alpha^{(1)}_{-k}\alpha^{(2)}_{k} + \alpha^{(2)}_{-k}\alpha^{(1)}_{k}]
\end{aligned} \tag{B.16}$$

in agreement with (B.9).

Appendix C

Hydrodynamic limit of elliptic Calogero-Moser

C.1 Details on the proof of (4.77) and (4.82)

C.1.1 Proof of (4.77)

First of all we pass to the ζ-function representation of (4.77) by employing the identity

$$\frac{\theta_1'\left(\frac{\pi}{L}z\right)}{\theta_1\left(\frac{\pi}{L}z\right)} = \zeta(z) - \frac{2\eta_1}{L}z. \tag{C.1}$$

As was mentioned all the dependence on η_1 drops out in the result. After doing so and computing $\ddot{x}_j$ from (4.77) we get

$$\ddot{x}_j = -G^2\left(L_1 + L_2 + L_3\right), \tag{C.2}$$

where

$$\begin{aligned} L_1 = &-\sum_{k=1}^{N}\wp(x_j - y_k)\Big[\sum_{l=1}^{N}\zeta(x_j - y_l) - \sum_{l\neq j}\zeta(x_j - x_l) + \sum_{l=1}^{N}\zeta(y_k - x_l) - \sum_{l\neq k}\zeta(y_k - y_l)\Big] \\ &+\sum_{k\neq j}\wp(x_j - x_k)\Big[\sum_{l=1}^{N}\zeta(x_j - y_l) - \sum_{l\neq j}\zeta(x_j - x_l) - \sum_{l=1}^{N}\zeta(x_k - y_l) + \sum_{l\neq k}\zeta(x_k - x_l)\Big] \end{aligned} \tag{C.3}$$

$$L_2 = \frac{2\eta_1}{L}\Bigg\{ -\sum_{k\neq j}\left(\wp(x_j-x_k)+\frac{2\eta_1}{L}\right)\Big[\sum_l (x_j-y_l)-\sum_{l\neq j}(x_j-x_l)-\sum_l (x_k-y_l)+\sum_{l\neq k}(x_k-x_l)\Big]$$
$$+\sum_k\left(\wp(x_j-y_k)+\frac{2\eta_1}{L}\right)\Big[\sum_l (x_j-y_l)-\sum_{l\neq j}(x_j-x_l)+\sum_l (y_k-x_l)-\sum_{l\neq k}(y_k-y_l)\Big]\Bigg\} \tag{C.4}$$

$$L_3 = \frac{2\eta_1}{L}\Bigg\{ -\sum_k\Big[\sum_l \zeta(x_j-y_l)-\sum_{l\neq j}\zeta(x_j-x_l)+\sum_l \zeta(y_k-x_l)-\sum_{l\neq k}\zeta(y_k-y_l)\Big]$$
$$+\sum_{k\neq j}\Big[\sum_l \zeta(x_j-y_l)-\sum_{l\neq j}\zeta(x_j-x_l)-\sum_l \zeta(x_k-y_l)+\sum_{l\neq k}\zeta(x_k-x_l)\Big]\Bigg\} \tag{C.5}$$

The terms L_2 and L_3 are manifestly vanishing, although showing the vanishing of L_3 is slightly involved. By collecting sums with common range, we have the relation

$$L_3 = \frac{2\eta_1}{L}\Bigg\{\Big[\sum_{k\neq j}\{\zeta(x_j-x_k)+\sum_{l\neq k}\zeta(x_k-x_l)\}\Big]+\Big[(y_j-y_k)\Big]-\Big[(x_j-y_k)\Big]-\Big[(y_j-x_k)\Big]\Bigg\}. \tag{C.6}$$

which vanishes term by term since

$$\sum_{k\neq j}\{\zeta(u_j-v_k)+\sum_{l\neq k}\zeta(v_k-u_l)\} = \sum_{k\neq j}\{\zeta(u_j-v_k)+\zeta(v_k-u_j)+\sum_{l\neq k,j}\zeta(v_k-u_l)\}$$
$$=\sum_{k\neq j}\sum_{l\neq k,j}\zeta(v_k-u_l) = \sum_{\substack{\text{pairs}(m,n),m\neq n\\(m,n)\neq j}}\Big[\zeta(v_m-u_n)+\zeta(u_n-v_m)\Big]=0, \tag{C.7}$$

where we used that ζ is odd. Summarizing, we have $\ddot{x}_j = -G^2L_1$ which matches (4.76) in force of the following identity between Weierstrass $\wp$ and ζ functions:

$$0=\sum_{k\neq j}\wp'(x_j-x_k)$$
$$+\sum_{k=1}^N\wp(x_j-y_k)\Big[\sum_{l=1}^N\zeta(x_j-y_l)-\sum_{l\neq j}\zeta(x_j-x_l)+\sum_{l=1}^N\zeta(y_k-x_l)-\sum_{l\neq k}\zeta(y_k-y_l)\Big]$$
$$-\sum_{k\neq j}\wp(x_j-x_k)\Big[\sum_{l=1}^N\zeta(x_j-y_l)-\sum_{l\neq j}\zeta(x_j-x_l)-\sum_{l=1}^N\zeta(x_k-y_l)+\sum_{l\neq k}\zeta(x_k-x_l)\Big]. \tag{C.8}$$

We prove this identity using Liouville's theorem. Let us denote the right hand side by $R\left(x_j;\{x_k\}_{k\neq j},\{y_k\}_{k=1}^N\right)$. R is a symmetric function under independent permutations of $\{x_k\}_{k\neq j}$ and $\{y_k\}_{k=1}^N$, respectively. Next, we show double periodicity in all variables.

Although the ζ's introduce shifts, these cancel each other[1], so double periodicity follows immediately. The non-trivial step is to show holomorphicity. First, the relation should hold for all j. In particular we can choose $j = 1$, other cases are obtained just by relabelling. By double periodicity we can focus only on poles at the origin, so there will be poles in $x_j - y_k$ and $x_j - x_l$, $l \neq j$. By the symmetries described above we have to check only three cases: $x_1 - y_1$, $x_2 - y_1$ and $x_1 - x_2$. To do so, we use the Laurent series for $\wp$ and ζ

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \wp^R(z), \;\; \wp^R(z) = \sum_{n=1}^{\infty} c_{n+1} z^{2n} \\ \zeta(z) &= \frac{1}{z} + \zeta^R(z), \quad \zeta^R(z) = -\sum_{n=1}^{\infty} \frac{c_{n+1}}{2n+1} z^{2n+1}\end{aligned} \tag{C.9}$$

Let us now show the vanishing of the residues at each pole.

Pole in $x_2 - y_1$

There are only two terms in (C.8) contributing

$$\begin{aligned}&\zeta(x_2 - y_1)\Big[\wp(x_1 - x_2) - \wp(x_1 - y_1)\Big] \\ &\sim \frac{1}{x_2 - y_1}\Big[\frac{1}{(x_1 - x_2)^2} - \frac{1}{(x_1 - y_1)^2} + \sum_{n\geq 1} c_{n+1}\left((x_1 - x_2)^{2n} - (x_1 - y_1)^{2n}\right)\Big] \\ &= \frac{x_2 - y_1}{x_2 - y_1}\Big[\frac{1}{(x_1 - x_2)^2(x_1 - y_1)} + \sum_{n\geq 1} c_{n+1} \sum_{k=1}^{2n} \binom{2n}{k} (-1)^k x_1^{2n-k} \sum_{l=0}^{k-1} x_2^{k-1-l} y_1^l\Big].\end{aligned} \tag{C.10}$$

So indeed the residue vanishes.

Pole in $x_1 - y_1$

The terms contributing to this pole read

$$\begin{aligned}&\wp(x_1 - y_1)\sum_{k\neq 1}\Big\{\big[\zeta(x_1 - y_k) - \zeta(y_1 - y_k)\big] - \big[\zeta(x_1 - x_k) - \zeta(y_1 - x_k)\big]\Big\} \\ &+ \zeta(x_1 - y_1)\sum_{k\neq 1}\Big[\wp(x_1 - y_k) - \wp(x_1 - x_k)\Big]\end{aligned} \tag{C.11}$$

[1] All ζ's appear in pairs, where a given variable appears with positive and negative signs in the argument.

$$
\begin{aligned}
&\sim \frac{1}{(x_1-y_1)^2}\sum_{k\neq 1}\Big\{\Big[\frac{1}{x_1-y_k}-\frac{1}{y_1-y_k}\Big]-\Big[\frac{1}{x_1-x_k}-\frac{1}{y_1-x_k}\Big] \\
&+\Big[\zeta^R(x_1-y_k)-\zeta^R(y_1-y_k)\Big]-\Big[\zeta^R(x_1-x_k)-\zeta^R(y_1-x_k)\Big]\Big\} \\
&+\frac{1}{x_1-y_1}\sum_{k\neq 1}\Big[\wp^R(x_1-y_k)-\wp^R(x_1-x_k)+\frac{1}{(x_1-y_k)^2}-\frac{1}{(x_1-x_k)^2}\Big].
\end{aligned}
\tag{C.12}
$$

Collecting all the rational terms gives a regular term

$$
\sum_{k\neq 1}\Big[\frac{1}{(x_1-x_k)^2(y_1-x_k)}-\frac{1}{(x_1-y_k)^2(y_1-y_k)}\Big] \tag{C.13}
$$

and we stay with the rest

$$
\begin{aligned}
\sum_{k\neq 1}\frac{1}{x_1-y_1}\Big\{&\wp^R(x_1-y_k)-\wp^R(x_1-x_k)+\frac{1}{x_1-y_1}\Big[\left(\zeta^R(x_1-y_k)-\zeta^R(y_1-y_k)\right) \\
&-\left(\zeta^R(x_1-x_k)-\zeta^R(y_1-x_k)\right)\Big]\Big\}.
\end{aligned}
\tag{C.14}
$$

In the following we show that the terms in the square parenthesis in the above formula factorizes a term (x_1-y_1) which, after combining with the rest, cancels the pole completely. Indeed, we just use (C.9) and binomial theorem to get

$$
\begin{aligned}
\Big[\dots\Big] &= -(x_1-y_1)\sum_{n\geq 1}\frac{c_{n+1}}{2n+1}\sum_{l=1}^{2n}\binom{2n+1}{l}(-1)^l\left(y_k^{2n+1-l}-x_k^{2n+1-l}\right)\sum_{m=0}^{l-1}y_1^{l-1-m}x_1^m \\
\wp^R(x_1-y_k)-\wp^R(x_1-x_k) &= \sum_{n\geq 1}c_{n+1}\sum_{l=1}^{2n}\binom{2n}{l-1}(-1)^l x_1^{l-1}\left(y_k^{2n+1-l}-x_k^{2n+1-l}\right)
\end{aligned}
\tag{C.15}
$$

and after combining these two terms we get

$$
\Big\{\dots\Big\} = \sum_{n\geq 1}c_{n+1}\sum_{l=1}^{2n}\binom{2n}{l-1}(-1)^l\left(y_k^{2n+1-l}-x_k^{2n+1-l}\right)\Big[x_1^{l-1}-\frac{1}{l}\sum_{m=0}^{l-1}y_1^{l-1-m}x_1^m\Big],
\tag{C.16}
$$

however the terms in the square brackets of (C.16) factorizes once more a term (x_1-y_1)

$$
\Big[\dots\Big] = (x_1-y_1)\frac{1}{l}\sum_{m=1}^{l-1}(l-m)x_1^{l-1-m}y_1^{m-1} \tag{C.17}
$$

so that we end up with a regular term

$$\sum_{k\neq 1}\sum_{n\geq 1} c_{n+1}\sum_{l=1}^{2n}\binom{2n}{l-1}\frac{(-1)^l}{l}\left(y_k^{2n+1-l}-x_k^{2n+1-l}\right)\sum_{m=1}^{l-1}(l-m)x_1^{l-1-m}y_1^{m-1}. \quad \text{(C.18)}$$

Summarizing, we have shown the vanishing of the residue at the pole in $(x_1 - y_1)$ and we now move on to the last one.

Pole in $x_1 - x_2$

Analysis of (C.8) gives the following terms contributing to this pole

$$\begin{aligned}&\wp'(x_1-x_2)+\zeta(x_1-x_2)\Big[\sum_{k\neq 1,2}\wp(x_1-x_k)-\sum_k\wp(x_1-y_k)\Big]\\ &-\wp(x_1-x_2)\Big[\sum_k\zeta(x_1-y_k)-\sum_{k\neq 1}\zeta(x_1-x_k)-\sum_k\zeta(x2-y_k)+\sum_{k\neq 2}\zeta(x_2-x_k)\Big].\end{aligned} \quad \text{(C.19)}$$

In analogy with the previous case let us first deal with the rational terms

$$\begin{aligned}&\frac{-2}{(x_1-x_2)^3}+\frac{1}{x_1-x_2}\Big[\sum_{k\neq 1,2}\frac{1}{(x_1-x_k)^2}-\sum_k\frac{1}{(x_1-y_k)^2}\Big]\\ &-\frac{1}{(x_1-x_2)^2}\Big[\frac{-2}{x_1-x_2}+\sum_k\left(\frac{1}{x_1-y_k}-\frac{1}{x_2-y_k}\right)-\sum_{k\neq 1,2}\left(\frac{1}{x_1-x_k}-\frac{1}{x_2-x_k}\right)\Big]\\ &=\sum_k\frac{1}{(x_1-y_k)^2(x_2-y_k)}-\sum_{k\neq 1,2}\frac{1}{(x_1-x_k)^2(x_2-x_k)},\end{aligned} \quad \text{(C.20)}$$

which give a regular contribution as we wanted. For the remaining terms we can write, using the same methods as above

$$\begin{aligned}&\frac{1}{x_1-x_2}\Bigg\{\sum_{k\neq 1,2}\wp^R(x_1-x_k)-\sum_k\wp^R(x_1-y_k)-\frac{1}{x_1-x_2}\Big[\sum_k(\zeta(x_1-y_k)-\zeta(x_2-y_k))\\ &\qquad -\sum_{k\neq 1,2}(\zeta(x_1-x_k)-\zeta(x_2-x_k))\Big]\Bigg\}\\ &=\sum_{n\geq 1}c_{n+1}\sum_{l=1}^{2n+1}\binom{2n}{l-1}\frac{(-1)^l}{l}\sum_{m=1}^{l-1}(l-m)x_1^{l-1-m}x_2^{m-1}\Big[\sum_{k\neq 1,2}x_k^{2n+1-l}-\sum_k y_k^{2n+1-l}\Big],\end{aligned} \quad \text{(C.21)}$$

which explicitly shows the vanishing of the residue of this last pole.

We just showed that $R\left(x_j;\{x_k\}_{k\neq j},\{y_k\}_{k=1}^N\right)$ is holomorphic in the whole complex plane for all variables. Liouville's theorem then implies it must be a constant. Hence we can set any convenient values for the variables to show this constant to be zero. Taking the limit $y_k \to 0$ for all k we get

$$-\lim_{y_k\to 0}\sum_k \wp(x_1-y_k)\sum_{l\neq k}\frac{1}{y_k-y_l}+\sum_{k\neq 1}\wp'(x_1-x_k)+N\wp(x_1)\Big[N\zeta(x_1)-\sum_{k\neq 1}\zeta(x_1-x_k)-\sum_k\zeta(x_k)\Big]$$
$$-\sum_{k\neq 1}\wp(x_1-x_k)\Big[N\zeta(x_1)-\sum_{l\neq 1}\zeta(x_1-x_l)-N\zeta(x_k)+\sum_{l\neq k}\zeta(x_k-x_l)\Big] \tag{C.22}$$

The first term can be written as

$$\lim_{y_k\to 0}\sum_{\substack{\text{pairs}(m,n),m\neq n\\ m,n\in\{1,\dots,N\}}}\frac{1}{y_n-y_m}\Big[\wp'(x_1)(y_n-y_m)+\mathcal{O}\left((y_n-y_m)^2\right)\Big]=\frac{N(N-1)}{2}\wp'(x_1) \tag{C.23}$$

Sending $x_k\to 0$, $k\neq 1$ simplifies R further

$$(N-1)\left(\frac{N}{2}+1\right)\wp'(x_1)-(N-1)\wp(x_1)\zeta(x_1)$$
$$+\lim_{\substack{x_k\to 0\\ k\neq 1}}\left\{\sum_{k\neq 1}\wp(x_1-x_k)\Big[N\zeta(x_k)-\sum_{l\neq k}\zeta(x_k-x_l)\Big]-N\wp(x_1)\sum_{k\neq 1}\zeta(x_k)\right\}, \tag{C.24}$$

where the second line yields

$$\lim_{\substack{x_k\to 0\\ k\neq 1}}\Bigg\{\underbrace{N\sum_{k\neq 1}\frac{1}{x_k}\Big[\wp(x_1-x_k)-\wp(x_1)\Big]}_{-N(N-1)\wp'(x_1)}\underbrace{-\sum_{k\neq 1}\wp(x_1-x_k)\sum_{l\neq k}\zeta(x_k-x_l)}_{(N-1)\wp(x_1)\zeta(x_1)+\frac{(N-1)(N-2)}{2}\wp'(x_1)}\Bigg\}.$$

Putting everything together we finally obtain

$$\text{const}=\lim_{\substack{y_k\to 0\\ x_l\to 0,l\neq 1}}R(\dots)=0\implies R(\dots)=0,$$

which concludes the proof of (C.8).

C.1.2 Proof of (4.82)

By simplifying the left hand side of (4.82) one gets

$$\sum_{j=1}^{N}\Bigg\{G\Big[\wp(z-x_j)\zeta(z-x_j)+\frac{1}{2}\wp'(z-x_j)\Big]+G\Big[\wp(z-y_j)\zeta(z-y_j)+\frac{1}{2}\wp'(z-y_j)\Big]$$
$$+\wp(z-x_j)\Big[-i\dot{x}_j-G\sum_{k=1}^{N}\zeta(z-y_k)+G\sum_{k\neq j}\zeta(z-x_k)\Big]$$
$$+\wp(z-y_j)\Big[i\dot{y}_j-G\sum_{k=1}^{N}\zeta(z-x_k)+G\sum_{k\neq j}\zeta(z-y_k)\Big]$$
$$+G\frac{2\eta_1}{L}\Big[i\dot{y}_j-i\dot{x}_j+G\left(\wp(z-y_j)-\wp(z-x_j)\right)\sum_{k}(y_k-x_k)\Big]\Bigg\}. \tag{C.25}$$

Going on-shell w.r.t. auxiliary system (4.77), we arrive at

$$\text{LHS}=X_1+X_2, \tag{C.26}$$

where

$$X_1=\sum_{j=1}^{N}\Bigg\{\frac{1}{2}\wp'(z-x_j)+\wp(z-x_j)\Big[\sum_{k=1}^{N}(\zeta(z-x_k)-\zeta(z-y_k)+\zeta(x_j-y_k))-\sum_{k\neq j}\zeta(x_j-x_k)\Big]$$
$$+\frac{1}{2}\wp'(z-y_j)+\wp(z-y_j)\Big[\sum_{k=1}^{N}(\zeta(z-y_k)-\zeta(z-x_k)+\zeta(y_j-x_k))-\sum_{k\neq j}\zeta(y_j-y_k)\Big]\Bigg\}$$
$$X_2=G^2\,\frac{2\eta_1}{L}\sum_{j=1}^{N}\sum_{k\neq j}\Big\{\zeta(y_j-x_k)+\zeta(x_j-y_k)-\zeta(y_j-y_k)-\zeta(x_j-x_k)\Big\}. \tag{C.27}$$

It is easy to see that X_2 vanishes, since we can rearrange the sum to pairs of ζ's with positive and negative arguments respectively

$$X_2=G^2\frac{2\eta_1}{L}\sum_{\substack{\text{pairs}(m,n),m\neq n\\ m,n\in\{1,\dots,N\}}}\Bigg\{\Big[\zeta(y_m-x_n)+\zeta(x_n-y_m)\Big]+\Big[\zeta(x_m-y_n)+\zeta(y_n-x_m)\Big]$$
$$-\Big[\zeta(x_m-x_n)+\zeta(x_n-x_m)\Big]-\Big[\zeta(y_m-y_n)+\zeta(y_n-y_m)\Big]\Bigg\}$$
$$=0. \tag{C.28}$$

The vanishing of X_1 looks more intriguing, but actually reduces to the already proven relation (C.8). Indeed, we can write X_1 as

$$X_1=\frac{1}{2(N-1)}\sum_{j=1}^{N}\Big[R\left(\{x\},\{y\}\right)\Big|_{x_j=z}+R\left(\{x\}\leftrightarrow\{y\}\right)\Big|_{y_j=z}\Big]=0,$$

which concludes the proof of (4.82).

Bibliography

[1] J. Duistermaat and G. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, *Inventiones mathematicae* **69** (1982), no. 2 259–268.

[2] E. Witten, *Supersymmetry and morse theory*, *J. Differential Geom.* **17** (1982), no. 4 661–692.

[3] N. Berline and M. Vergne, *Classes caractéristiques équivariantes. formule de localisation en cohomologie équivariante*, *CR Acad. Sci. Paris* **295** (1982) 539–541.

[4] N. Berline and M. Vergne, *Zeros d'un champ de vecteurs et classes caracteristiques equivariantes*, *Duke Math. J.* **50** (06, 1983) 539–549.

[5] M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, *Topology* **23** (1984), no. 1 1 – 28.

[6] E. Witten, *Topological Quantum Field Theory*, *Commun.Math.Phys.* **117** (1988) 353.

[7] E. Witten, *Mirror Manifolds And Topological Field Theory*, *ArXiv High Energy Physics - Theory e-prints* (Dec., 1991) [hep-th/9112056].

[8] G. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs Branches*, *Communications in Mathematical Physics* **209** (2000) 97–121 [hep-th/9712241].

[9] N. A. Nekrasov, *Seiberg-Witten Prepotential From Instanton Counting*, *ArXiv High Energy Physics - Theory e-prints* (June, 2002) [hep-th/0206161].

[10] N. Nekrasov and A. Okounkov, *Seiberg-Witten Theory and Random Partitions*, *ArXiv High Energy Physics - Theory e-prints* (June, 2003) [hep-th/0306238].

[11] V. Pestun, *Localization of Gauge Theory on a Four-Sphere and Supersymmetric Wilson Loops*, *Communications in Mathematical Physics* **313** (July, 2012) 71–129 [0712.2824].

[12] A. Kapustin, B. Willett and I. Yaakov, *Exact results for Wilson loops in superconformal Chern-Simons theories with matter*, *Journal of High Energy Physics* **3** (Mar., 2010) 89 [0909.4559].

[13] D. L. Jafferis, *The exact superconformal R-symmetry extremizes Z*, *Journal of High Energy Physics* **5** (May, 2012) 159 [1012.3210].

[14] N. Hama, K. Hosomichi and S. Lee, *Notes on SUSY gauge theories on three-sphere*, *Journal of High Energy Physics* **3** (Mar., 2011) 127 [1012.3512].

[15] K. Hosomichi, R.-K. Seong and S. Terashima, *Supersymmetric gauge theories on the five-sphere*, *Nuclear Physics B* **865** (Dec., 2012) 376–396 [1203.0371].

[16] S. Kim, *The complete superconformal index for N=6 Chern-Simons theory*, *Nuclear Physics B* **821** (Nov., 2009) 241–284 [0903.4172].

[17] V. Pestun, *Localization of the four-dimensional N=4 SYM to a two-sphere and 1/8 BPS Wilson loops*, *Journal of High Energy Physics* **12** (Dec., 2012) 67 [0906.0638].

[18] Y. Imamura and S. Yokoyama, *Index for three dimensional superconformal field theories with general R-charge assignments*, *Journal of High Energy Physics* **4** (Apr., 2011) 7 [1101.0557].

[19] N. Hama, K. Hosomichi and S. Lee, *SUSY gauge theories on squashed three-spheres*, *Journal of High Energy Physics* **5** (May, 2011) 14 [1102.4716].

[20] J. Gomis, T. Okuda and V. Pestun, *Exact results for 't Hooft loops in Gauge theories on S^4*, *Journal of High Energy Physics* **5** (May, 2012) 141 [1105.2568].

[21] F. Benini, T. Nishioka and M. Yamazaki, *4d index to 3d index and 2d topological quantum field theory*, *Phys. Rev. D* **86** (Sept., 2012) 065015 [1109.0283].

[22] Y. Ito, T. Okuda and M. Taki, *Line operators on $S^1 \times R^3$ and quantization of the Hitchin moduli space*, *Journal of High Energy Physics* **4** (Apr., 2012) 10 [1111.4221].

[23] L. F. Alday, M. Fluder and J. Sparks, *The large N limit of M2-branes on Lens spaces*, *Journal of High Energy Physics* **10** (Oct., 2012) 57 [1204.1280].

[24] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic Genera of Two-Dimensional Gauge Theories with Rank-One Gauge Groups*, *Letters in Mathematical Physics* **104** (Apr., 2014) 465–493 [1305.0533].

[25] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic Genera of 2d $N = 2$ Gauge Theories*, *Communications in Mathematical Physics* (Nov., 2014) [1308.4896].

[26] K. Hori, H. Kim and P. Yi, *Witten Index and Wall Crossing*, *ArXiv e-prints* (July, 2014) [1407.2567].

[27] F. Benini and S. Cremonesi, *Partition Functions of N=(2,2) Gauge Theories on S^2 and Vortices*, *Communications in Mathematical Physics* (July, 2014) [1206.2356].

[28] N. Doroud, J. Gomis, B. Le Floch and S. Lee, *Exact results in $D = 2$ supersymmetric gauge theories*, *Journal of High Energy Physics* **5** (May, 2013) 93 [1206.2606].

[29] A. Bawane, G. Bonelli, M. Ronzani and A. Tanzini, *N=2 supersymmetric gauge theories on $S^2 \times S^2$ and Liouville Gravity*, *ArXiv e-prints* (Nov., 2014) [1411.2762].

[30] M. Sinamuli, *On N=2 supersymmetric gauge theories on $S^2 \times S^2$, ArXiv e-prints* (Nov., 2014) [1411.4918].

[31] D. Rodriguez-Gomez and J. Schmude, *Partition functions for equivariantly twisted N=2 gauge theories on toric Kahler manifolds*, *ArXiv e-prints* (Dec., 2014) [1412.4407].

[32] J. Qiu, L. Tizzano, J. Winding and M. Zabzine, *Gluing Nekrasov partition functions*, *ArXiv e-prints* (Mar., 2014) [1403.2945].

[33] S. Sugishita and S. Terashima, *Exact results in supersymmetric field theories on manifolds with boundaries*, *Journal of High Energy Physics* **11** (Nov., 2013) 21 [1308.1973].

[34] D. Honda and T. Okuda, *Exact results for boundaries and domain walls in 2d supersymmetric theories*, *ArXiv e-prints* (Aug., 2013) [1308.2217].

[35] K. Hori and M. Romo, *Exact Results In Two-Dimensional (2,2) Supersymmetric Gauge Theories With Boundary*, *ArXiv e-prints* (Aug., 2013) [1308.2438].

[36] M. Mariño, *Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories*, *Journal of Physics A Mathematical General* **44** (Nov., 2011) 3001 [1104.0783].

[37] S. Cremonesi, *An Introduction to Localisation and Supersymmetry in Curved Space*, *PoS* **Modave2013** (2013) 002.

[38] C. Closset, S. Cremonesi and D. S. Park, *The equivariant A-twist and gauged linear sigma models on the two-sphere*, *ArXiv e-prints* (Apr., 2015) [1504.06308].

[39] F. Benini and A. Zaffaroni, *A topologically twisted index for three-dimensional supersymmetric theories*, *ArXiv e-prints* (Apr., 2015) [1504.03698].

[40] L. J. Dixon, *SOME WORLD SHEET PROPERTIES OF SUPERSTRING COMPACTIFICATIONS, ON ORBIFOLDS AND OTHERWISE*, .

[41] W. Lerche, C. Vafa and N. P. Warner, *Chiral Rings in N=2 Superconformal Theories*, *Nucl.Phys.* **B324** (1989) 427.

[42] P. Candelas, M. Lynker and R. Schimmrigk, *Calabi-yau manifolds in weighted* $\{P4\}$, *Nuclear Physics B* **341** (1990), no. 2 383 – 402.

[43] P. S. Aspinwall, C. Lutken and G. G. Ross, *Construction and Couplings of Mirror Manifolds*, *Phys.Lett.* **B241** (1990) 373–380.

[44] B. R. Greene and M. Plesser, *Duality in Calabi-Yau Moduli Space*, *Nucl.Phys.* **B338** (1990) 15–37.

[45] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, *Inventiones mathematicae* **82** (1985), no. 2 307–347.

[46] M. Dine, N. Seiberg, X. Wen and E. Witten, *Nonperturbative Effects on the String World Sheet*, *Nucl.Phys.* **B278** (1986) 769.

[47] E. Witten, *Topological sigma models*, *Comm. Math. Phys.* **118** (1988), no. 3 411–449.

[48] H. Jockers, V. Kumar, J. M. Lapan, D. R. Morrison and M. Romo, *Two-Sphere Partition Functions and Gromov-Witten Invariants*, *ArXiv e-prints* (Aug., 2012) [1208.6244].

[49] J. Gomis and S. Lee, *Exact Kähler potential from gauge theory and mirror symmetry*, *Journal of High Energy Physics* **4** (Apr., 2013) 19 [1210.6022].

[50] S. Cecotti and C. Vafa, *Topological anti-topological fusion*, *Nuclear Physics B* **367** (Dec., 1991) 359–461.

[51] Y. Honma and M. Manabe, *Exact Kähler potential for Calabi-Yau fourfolds*, *Journal of High Energy Physics* **5** (May, 2013) 102 [1302.3760].

[52] H. Jockers and M. Romo , *private communication.*

[53] S. Cecotti, D. Gaiotto and C. Vafa, *tt^* geometry in 3 and 4 dimensions*, *Journal of High Energy Physics* **5** (May, 2014) 55 [1312.1008].

[54] G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, *Vortex partition functions, wall crossing and equivariant Gromov-Witten invariants*, *ArXiv e-prints* (July, 2013) [1307.5997].

[55] A. B. Givental, *Equivariant Gromov - Witten Invariants*, in *eprint arXiv:alg-geom/9603021*, p. 3021, Mar., 1996.

[56] I. Ciocan-Fontanine, B. Kim and D. Maulik, *Stable quasimaps to GIT quotients*, *ArXiv e-prints* (June, 2011) [1106.3724].

[57] T. Coates and A. Givental, *Quantum Riemann - Roch, Lefschetz and Serre*, *ArXiv Mathematics e-prints* (Oct., 2001) [math/0110142].

[58] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nuclear Physics B* **405** (Sept., 1993) 279–304 [hep-th/9302103].

[59] B. Dubrovin, *Geometry of 2d topological field theories*, *ArXiv High Energy Physics - Theory e-prints* (July, 1994) [hep-th/9407018].

[60] D. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, .

[61] A. Bertram, I. Ciocan-Fontanine and B. Kim, *Two Proofs of a Conjecture of Hori and Vafa*, *ArXiv Mathematics e-prints* (Apr., 2003) [math/0304403].

[62] A. Bertram, I. Ciocan-Fontanine and B. Kim, *Gromov-Witten Invariants for Abelian and Nonabelian Quotients*, *ArXiv Mathematics e-prints* (July, 2004) [math/0407254].

[63] K. Hori and C. Vafa, *Mirror Symmetry*, *ArXiv High Energy Physics - Theory e-prints* (Feb., 2000) [hep-th/0002222].

[64] E. Gonzalez and C. T. Woodward, *A wall-crossing formula for Gromov-Witten invariants under variation of git quotient*, *ArXiv e-prints* (Aug., 2012) [1208.1727].

[65] T. Dimofte, S. Gukov and L. Hollands, *Vortex Counting and Lagrangian 3-Manifolds*, *Letters in Mathematical Physics* **98** (Dec., 2011) 225–287 [1006.0977].

[66] T. Coates, A. Corti, Y.-P. Lee and H.-H. Tseng, *The Quantum Orbifold Cohomology of Weighted Projective Spaces*, *ArXiv Mathematics e-prints* (Aug., 2006) [math/0608481].

[67] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nuclear Physics B* **403** (Aug., 1993) 159–222 [hep-th/9301042].

[68] B. Forbes and M. Jinzenji, *J functions, non-nef toric varieties and equivariant local mirror symmetry of curves*, *ArXiv Mathematics e-prints* (Mar., 2006) [math/0603728].

[69] M. A. Guest, *Quantum cohomology via D-modules*, *ArXiv Mathematics e-prints* (June, 2002) [math/0206212].

[70] B. Forbes and M. Jinzenji, *On equivariant mirror symmetry for local P^2*, *ArXiv e-prints* (Sept., 2007) [0710.0049].

[71] M. Aganagic, V. Bouchard and A. Klemm, *Topological Strings and (Almost) Modular Forms*, *Communications in Mathematical Physics* **277** (Feb., 2008) 771–819 [hep-th/0607100].

[72] T. Coates, A. Corti, H. Iritani and H.-H. Tseng, *Computing Genus-Zero Twisted Gromov-Witten Invariants*, *ArXiv Mathematics e-prints* (Feb., 2007) [math/0702234].

[73] T. Coates, *Wall-Crossings in Toric Gromov-Witten Theory II: Local Examples*, *ArXiv e-prints* (Apr., 2008) [0804.2592].

[74] A. Brini and A. Tanzini, *Exact Results for Topological Strings on Resolved $Y^{p,q}$ Singularities*, *Communications in Mathematical Physics* **289** (July, 2009) 205–252 [0804.2598].

[75] A. Kapustin and B. Willett, *Wilson loops in supersymmetric Chern-Simons-matter theories and duality*, *ArXiv e-prints* (Feb., 2013) [1302.2164].

[76] G. Bonelli, A. Tanzini and J. Zhao, *Vertices, vortices and interacting surface operators*, *Journal of High Energy Physics* **6** (June, 2012) 178 [1102.0184].

[77] F. Benini, D. S. Park and P. Zhao, *Cluster algebras from dualities of 2d N=(2,2) quiver gauge theories*, *ArXiv e-prints* (June, 2014) [1406.2699].

[78] J. Gomis and B. Le Floch, *M2-brane surface operators and gauge theory dualities in Toda*, *ArXiv e-prints* (July, 2014) [1407.1852].

[79] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, *Nuclear Physics B* **426** (Sept., 1994) 19–52 [hep-th/9407087].

[80] E. Witten, *Small instantons in string theory*, *Nuclear Physics B* **460** (Feb., 1996) 541–559 [hep-th/9511030].

[81] M. R. Douglas, *Gauge fields and D-branes*, *Journal of Geometry and Physics* **28** (Dec., 1998) 255–262 [hep-th/9604198].

[82] M. Billó, M. Frau, I. Pesando, F. Fucito, A. Lerda and A. Liccardo, *Classical gauge instantons from open strings*, *Journal of High Energy Physics* **2** (Feb., 2003) 45 [hep-th/0211250].

[83] U. Bruzzo, F. Fucito, J. F. Morales and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, *Journal of High Energy Physics* **5** (May, 2003) 54 [hep-th/0211108].

[84] D.-E. Diaconescu, J. Gomis and M. R. Douglas, *Fractional branes and wrapped branes*, *Journal of High Energy Physics* **2** (Feb., 1998) 13 [hep-th/9712230].

[85] D.-E. Diaconescu and J. Gomis, *Fractional branes and boundary states in orbifold theories*, *Journal of High Energy Physics* **10** (Oct., 2000) 1 [hep-th/9906242].

[86] R. Sasaki and I. Yamanaka, *Virasoro Algebra, Vertex Operators, Quantum Sine-Gordon and Solvable Quantum Field Theories*, *Adv.Stud.Pure Math.* **16** (1988) 271–296.

[87] T. Eguchi and S.-K. Yang, *Deformations of Conformal Field Theories and Soliton Equations*, *Phys.Lett.* **B224** (1989) 373.

[88] B. Kupershmidt and P. Mathieu, *Quantum Korteweg-de Vries Like Equations and Perturbed Conformal Field Theories*, *Phys.Lett.* **B227** (1989) 245.

[89] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory, quantum KdV theory and Thermodynamic Bethe Ansatz*, *Communications in Mathematical Physics* **177** (Apr., 1996) 381–398 [hep-th/9412229].

[90] V. A. Alba, V. A. Fateev, A. V. Litvinov and G. M. Tarnopolskiy, *On combinatorial expansion of the conformal blocks arising from AGT conjecture*, *Lett.Math.Phys.* **98** (2011) 33–64 [1012.1312].

[91] A. V. Litvinov, *On spectrum of ILW hierarchy in conformal field theory*, *Journal of High Energy Physics* **11** (Nov., 2013) 155 [1307.8094].

[92] A. Okounkov, *Math.Coll. at Simons Center of Geometry and Physics*, *http://media.scgp.stonybrook.edu/video/video.php?f=20130124_4_qtp.mp4*.

[93] N. Nekrasov, *Talk at Pomeranchuk-100*, *http://www.itep.ru/rus/docs/09_Nekrasov.pdf*.

[94] N. Nekrasov and A. Okounkov, *Quantum cohomology and quantum intermediate long wave equation*, *in preparation*.

[95] G. Bonelli and A. Tanzini, *Hitchin systems, N=2 gauge theories and W-gravity*, *Physics Letters B* **691** (July, 2010) 111–115 [0909.4031].

[96] D. Maulik and A. Okounkov, *Quantum Groups and Quantum Cohomology*, *ArXiv e-prints* (Nov., 2012) [1211.1287].

[97] B. Estienne, V. Pasquier, R. Santachiara and D. Serban, *Conformal blocks in Virasoro and W theories: Duality and the Calogero-Sutherland model*, *Nucl.Phys.* **B860** (2012) 377–420 [1110.1101].

[98] R. Santachiara and A. Tanzini, *Moore-Read Fractional Quantum Hall wavefunctions and SU(2) quiver gauge theories*, *Phys.Rev.* **D82** (2010) 126006 [1002.5017].

[99] G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, *The stringy instanton partition function*, *Journal of High Energy Physics* **1** (Jan., 2014) 38 [1306.0432].

[100] G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, *Six-dimensional supersymmetric gauge theories, quantum cohomology of instanton moduli spaces and gl(N) Quantum Intermediate Long Wave Hydrodynamics*, *Journal of High Energy Physics* **7** (July, 2014) 141 [1403.6454].

[101] S. Shadchin, *On F-term contribution to effective action*, *Journal of High Energy Physics* **8** (Aug., 2007) 52 [hep-th/0611278].

[102] G. Bonelli, A. Tanzini and J. Zhao, *The Liouville side of the vortex*, *Journal of High Energy Physics* **9** (Sept., 2011) 96 [1107.2787].

[103] I. Ciocan-Fontanine, B. Kim and C. Sabbah, *The Abelian/Nonabelian correspondence and Frobenius manifolds*, *Inventiones Mathematicae* **171** (Oct., 2007) 301–343 [math/0610265].

[104] J. Bryan and T. Graber, *The Crepant Resolution Conjecture*, *ArXiv Mathematics e-prints* (Oct., 2006) [math/0610129].

[105] I. Ciocan-Fontanine, M. Konvalinka and I. Pak, *Quantum cohomology of $Hilb_n(\mathbb{C}^2)$ and the weighted hood walk on the young diagrams*, .

[106] A. Okounkov and R. Pandharipande, *Quantum cohomology of the Hilbert scheme of points in the plane*, *ArXiv Mathematics e-prints* (Nov., 2004) [math/0411210].

[107] M. Cirafici, A. Sinkovics and R. J. Szabo, *Instantons, quivers and noncommutative Donaldson-Thomas theory*, *Nuclear Physics B* **853** (Dec., 2011) 508–605 [1012.2725].

[108] D.-E. Diaconescu, *Moduli of ADHM sheaves and the local Donaldson-Thomas theory*, *Journal of Geometry and Physics* **62** (Apr., 2012) 763–799 [0801.0820].

[109] R. Pandharipande and R. P. Thomas, *The 3-fold vertex via stable pairs*, *ArXiv e-prints* (Sept., 2007) [0709.3823].

[110] A. Okounkov and R. Pandharipande, *The local Donaldson-Thomas theory of curves*, *ArXiv Mathematics e-prints* (Dec., 2005) [math/0512573].

[111] P. Di Vecchia, A. Lerda and P. Merlatti, */N=1 and /N=2 super-Yang-Mills theories from wrapped branes*, *Nuclear Physics B* **646** (Dec., 2002) 43–68 [hep-th/0205204].

[112] A. S. Losev, A. Marshakov and N. A. Nekrasov, *Small Instantons, Little Strings and Free Fermions*, *ArXiv High Energy Physics - Theory e-prints* (Feb., 2003) [hep-th/0302191].

[113] A. Iqbal, C. Vafa, N. Nekrasov and A. Okounkov, *Quantum foam and topological strings*, *Journal of High Energy Physics* **4** (Apr., 2008) 11 [hep-th/0312022].

[114] N. A. Nekrasov and S. L. Shatashvili, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, in *XVITH INTERNATIONAL CONGRESS ON MATHEMATICAL PHYSICS. Held 3-8 August 2009 in Prague, Czech Republic. Edited by Pavel Exner (Doppler Institute, Prague, Czech Republic). Published by World Scientific Publishing Co. Pte. Ltd., 2010. ISBN #9789814304634, pp. 265-289* (P. Exner, ed.), pp. 265–289, Mar., 2010. 0908.4052.

[115] A. N. W. Hone and V. S. Novikov, *On a functional equation related to the intermediate long wave equation*, *Journal of Physics A: Mathematical and General* **37** (2004), no. 32 L399.

[116] D. Lebedev and A. Radul, *Periodic Intermediate Long Wave equation: the undressingmethod*, *Theor.Mat.Fyz.* **70/2** (1987) 202–210.

[117] D. Lebedev and A. Radul, *GENERALIZED INTERNAL LONG WAVES EQUATIONS: CONSTRUCTION, HAMILTONIAN STRUCTURE AND CONSERVATION LAWS*, *Commun.Math.Phys.* **91** (1983) 543.

[118] A. Okounkov and R. Pandharipande, *The quantum differential equation of the Hilbert scheme of points in the plane*, *ArXiv e-prints* (June, 2009) [0906.3587].

[119] L. F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett.Math.Phys.* **91** (2010) 167–197 [0906.3219].

[120] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*. SIAM studies in applied mathematics. SIAM, Philadelphia, 1981. Autre tirage : 1985.

[121] A. Abanov, *private communication* , .

[122] A. Hone, *private communication*, .

[123] A. G. Abanov, E. Bettelheim and P. Wiegmann, *Integrable hydrodynamics of Calogero-Sutherland model: Bidirectional Benjamin-Ono equation*, *J.Phys.* **A42** (2009) 135201 [0810.5327].

[124] N. A. Nekrasov and S. L. Shatashvili, *Supersymmetric Vacua and Bethe Ansatz*, *Nuclear Physics B Proceedings Supplements* **192** (July, 2009) 91–112 [0901.4744].

[125] N. Nekrasov and S. Shatashvili, *Quantum Integrability and Supersymmetric Vacua*, *Progress of Theoretical Physics Supplement* **177** (2009) 105–119 [0901.4748].

[126] D.-E. Diaconescu, *Moduli of ADHM sheaves and the local Donaldson-Thomas theory*, *Journal of Geometry and Physics* **62** (Apr., 2012) 763–799 [0801.0820].

[127] R. Flume, F. Fucito, J. F. Morales and R. Poghossian, *Matone's relation in the presence of gravitational couplings*, *JHEP* **0404** (2004) 008 [hep-th/0403057].

[128] M. N. Alfimov and A. V. Litvinov, *On spectrum of ILW hierarchy in conformal field theory II: coset CFT's*, *ArXiv e-prints* (Nov., 2014) [1411.3313].

[129] N. Nekrasov, *private communication*, .

[130] J. Shiraishi and Y. Tutiya, *Periodic ILW equation with discrete Laplacian*, *Journal of Physics A Mathematical General* **42** (Oct., 2009) 4018 [0904.2644].

[131] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi and S. Yanagida, *A commutative algebra on degenerate CP^1 and Macdonald polynomials*, *Journal of Mathematical Physics* **50** (Sept., 2009) 095215 [0904.2291].

[132] M. Bullimore, H.-C. Kim and P. Koroteev, *Defects and Quantum Seiberg-Witten Geometry*, *ArXiv e-prints* (Dec., 2014) [1412.6081].

[133] H.-Y. Chen, T. J. Hollowood and P. Zhao, *A 5d/3d duality from relativistic integrable system*, *Journal of High Energy Physics* **7** (July, 2012) 139 [1205.4230].

[134] Y. Tutiya and J. Shiraishi, *On some special solutions to periodic Benjamin-Ono equation with discrete Laplacian*, *ArXiv e-prints* (Nov., 2009) [0911.5005].

[135] J. Shiraishi and Y. Tutiya, *Periodic Benjamin-Ono Equation with Discrete Laplacian and 2D-TODA Hierarchy*, in *New Trends in Quantum Integrable Systems* (B. Feigin, M. Jimbo and M. Okado, eds.), pp. 357–371, Oct., 2011. 1004.1455.

[136] Y. Saito, *Elliptic Ding-Iohara Algebra and the Free Field Realization of the Elliptic Macdonald Operator*, *ArXiv e-prints* (Jan., 2013) [1301.4912].

[137] Y. Saito, *Commutative Families of the Elliptic Macdonald Operator*, *SIGMA* **10** (Mar., 2014) 21 [1305.7097].

[138] Y. Saito, *Elliptic Ding-Iohara Algebra and Commutative Families of the Elliptic Macdonald Operator*, *ArXiv e-prints* (Sept., 2013) [1309.7094].

[139] P. Kronheimer and H. Nakajima, *Yang-mills instantons on ale gravitational instantons*, *Mathematische Annalen* **288** (1990), no. 1 263–307.

[140] I. Ciocan-Fontanine, E. Diaconescu and B. Kim, *From I to J in two dimensional (4, 4) quiver gauge theories*, *preprint*.

[141] N. Nekrasov and S. Shatashvili, *Bethe/gauge correspondence on curved spaces*, *Journal of High Energy Physics* **1** (Jan., 2015) 100 [1405.6046].

[142] J. Bryan, T. Graber and R. Pandharipande, *The orbifold quantum cohomology of C^2/Z_3 and Hurwitz-Hodge integrals*, *ArXiv Mathematics e-prints* (Oct., 2005) [math/0510335].

[143] J. Bryan and A. Gholampour, *Root Systems and the Quantum Cohomology of ADE resolutions*, *ArXiv e-prints* (July, 2007) [0707.1337].

[144] D. Maulik and A. Oblomkov, *Quantum cohomology of the Hilbert scheme of points on {A}_n -resolutions*, *Journal of the American Mathematical Society* **22** (Oct., 2009) 1055–1091 [0802.2737].

[145] G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, *Quantum Cohomology and Quantum Hydrodynamics from Supersymmetric Quiver Gauge Theories*, *ArXiv e-prints* (May, 2015) [1505.07116].

[146] V. Baranovsky, *Moduli of Sheaves on Surfaces and Action of the Oscillator Algebra*, *ArXiv Mathematics e-prints* (Nov., 1998) [math/9811092].