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Area of Mathematics

Some results on anisotropic mean curvature and other phase transition models

Ph.D. Thesis

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Academic Year 2014/2015

Il presente lavoro costituisce la tesi presentata da Stefano Amato, sotto la direzione del Prof. Giovanni Bellettini, al fine di ottenere l'attestato di ricerca post-universitaria *Doctor Philosophiæ* presso la S.I.S.S.A., *curriculum* in "Mathematical Analysis for Modelling and Applications", Area di Matematica. Ai sensi dell'art. 1, comma 4, dello Statuto della S.I.S.S.A. pubblicato sulla G.U. n.36 del 13.02.2012, il predetto attestato è equipollente al titolo di *Dottore di Ricerca in Matematica*.

Trieste, Anno Accademico 2014-2015.

Ringraziamenti

Trieste, 17 Settembre 2015

Giunto al termine di questo ciclo, desidero rubare poche righe a formule e teoremi per ringraziare la tante persone che hanno contribuito alla stesura di queste pagine.

Ringrazio innanzitutto il mio *supervisor*, prof. Giovanni Bellettini, per i problemi che mi ha proposto come oggetto del mio lavoro di ricerca, e per avermi seguito con dedizione. Non posso dimenticare di ringraziare Giovanni anche per la gentilissima ospitalità che mi ha dimostrato durante i miei viaggi a Roma, e per le tante occasioni di confronto stimolante (non ultime, le volte in cui ci siamo ritrovati a discutere di matematica al tavolo di quell'osteria di Monteporzio).

Ringrazio anche il prof. Maurizio Paolini, per la bella esperienza di collaborazione "a distanza" che abbiamo intrapreso in questi anni, ed il corpo docenti della SISSA, area di Matematica, per avermi formato come ricercatore.

L'avventura del dottorato non avrebbe mai potuto essere così entusiasmante senza la compagnia dei miei colleghi di settore (ed amici), in particolare Lucia, Elisa, Riccardo, Aleks, Gabriele, Luca e Francesca: confido che in futuro le nostre strade abbiano ancora tante occasioni di incrociarsi.

Infine, voglio dedicare un ringraziamento speciale a Silvia, per aver deciso di raggiungermi qui a Trieste ed essermi stata accanto con il suo amore ed il suo supporto.

Contents

Introduction

The present thesis is divided into three parts. In the first part, we analyze a suitable regularization — which we call nonlinear multidomain model — of the motion of a hypersurface under smooth anisotropic mean curvature flow. The second part of the thesis deals with crystalline mean curvature of facets of a solid set of $\mathbb{R}^3$. Finally, in the third part we study a phase-transition model for Plateau's type problems based on the theory of coverings and of BV functions.

Besides the apparent differences between the various parts of the thesis, let us underline what are the connections. An immediate common feature is given by the fact that the first and the second part are both concerned with an anisotropic variational model. In this respect, let us notice that also the last part of the thesis could be generalized to an anisotropic setting, by properly modifying the total variation functional in terms of a convex norm: anyway, in the present thesis we shall be confined to the isotropic case. Moreover, one could recall that solutions of the Plateau's problem are minimal surfaces (*i.e.*, surfaces with zero mean curvature), and minimal surfaces come out as the asymptotic extinction profiles of mean curvature flow [108]. However, what we believe it is the most interesting *trait d'union* between the above mentioned problems is the fact that, in some sense, they are "close" to a phase-transition model: this was already known for anisotropic mean curvature flow (see for instance [31, 41]), while for Plateau's problem it has been observed first in [55].

Let us now turn to the description in deeper details of the three problems that form the core of the thesis.

Anisotropic mean curvature flow and nonlinear multidomain model. Mean curvature flow, namely the motion of a hypersurface having normal velocity equal to its mean curvature, is the gradient flow of the perimeter functional [54, 107, 93, 116, 26]. A natural extension of this evolutive problem, which is of geometric interest and also useful for its applications to material science and crystals growth [60, 150, 44, 41], is the so-called *anisotropic mean curvature flow*. In this case, the ambient space $\mathbb{R}^n$ is endowed with a Finsler metric ϕ, inducing an integral functional P_ϕ (the ϕ-anisotropic perimeter), whose integrand is expressed in terms of ϕ^o (the dual of ϕ). Assume ϕ^2 smooth and uniformly convex, and let T_{ϕ^o} be the gradient of $\frac{1}{2}(\phi^o)^2$. Anisotropic mean curvature flow is nothing but the gradient flow of P_ϕ: as a consequence, the hypersurface evolves with velocity equal to its anisotropic mean curvature along the so-called Cahn-Hoffman direction $T_{\phi^o}(\nu_{\phi^o})$, where we set $\nu_{\phi^o} := \frac{\nu}{\phi^o(\nu)}$, and we let ν denote the outer normal to the evolving hypersurface.

Smooth anisotropic mean curvature flow has been the subject of several papers in the last years [96, 83, 84, 44, 41, 31, 134, 43]. For a well-posed formulation of the problem, we refer the interested reader for instance to [41, Proposition 6.1].

To the aim of the present thesis, it is important to recall the following remarkable result, appeared in [31]: as well as for the isotropic case, the anisotropic perimeter can be approximated in the sense of Γ-convergence [75] by a sequence of elliptic functionals, whose gradient flows converge to anisotropic mean curvature flow. This result is achieved by combining two main ingredients: the first one is the explicit construction of a sequence of sub and supersolutions to the approximating flows, while the second ingredient is the maximum principle. The construction of the sub/supersolutions is in turn suggested by an asymptotic expansion argument, and involves the anisotropic signed distance function d_ϕ from the evolving hypersurface.

A subsequent natural step in the theory could be given by the study of *nonconvex* mean curvature flow. This corresponds to the case when the unit ball of ϕ^o is not anymore convex, but it is just a smooth star-shaped set containing the origin in its interior. Here, the situation becomes immediately much more complicated, since ϕ^o cannot be seen as a dual norm, and so it is not possible to speak about the distance d_ϕ. Moreover, the nonconvexity of ϕ^o leads to the gradient flow of a nonconvex (and nonconcave) functional, which corresponds in general to an ill-posed parabolic problem. Incidentally, let us also mention that, as shown by numerical experiments [87], the motion of a hypersurface having normals in the region where ϕ^o is locally convex should not coincide with the anisotropic mean curvature flow corresponding to the convexified of ϕ^o.

A rather natural strategy to study an ill-posed problem is to regularize it, adding for instance some higher order term, and then passing to the limit as the regularizing parameter goes to zero, see for instance [32] and references therein. In this thesis, instead, we shall deal with a completely different regularization, which is inspired by the so-called *bidomain model*.

The bidomain model, a simplified version of the FitzHugh-Nagumo system, was originally introduced in electrocardiology as an attempt to describe the electric potentials and current flows inside and outside the cardiac cells, see [74, 122, 11, 73] and references therein. In spite of the discrete cellular structure, at a macroscopic level the intra (i) and the extra (e) cellular regions can be thought of as two superimposed and interpenetrating continua, thus coinciding with the domain Ω (the physical region occupied by the heart). Denoting the intra and extra cellular electric potentials respectively by $u_\mathrm{i} = u_{\mathrm{i},\epsilon}$, $u_\mathrm{e} = u_{\mathrm{e},\epsilon}\colon [0,T]\times\Omega\to\mathbb{R}$, the bidomain model can be formulated using the following weakly parabolic system, of variational nature:[(1)]

$$\begin{cases} \epsilon\partial_t(u_\mathrm{i}-u_\mathrm{e}) - \epsilon\mathrm{div}\,(M_\mathrm{i}(x)\nabla u_\mathrm{i}) + \dfrac{1}{\epsilon}f\big(u_\mathrm{i}-u_\mathrm{e}\big) = 0, \\ \\ \epsilon\partial_t(u_\mathrm{i}-u_\mathrm{e}) + \epsilon\mathrm{div}\,(M_\mathrm{e}(x)\nabla u_\mathrm{e}) + \dfrac{1}{\epsilon}f\big(u_\mathrm{i}-u_\mathrm{e}\big) = 0. \end{cases} \tag{0.1}$$

System (0.1) is composed of two singularly perturbed linearly anisotropic reaction-diffusion equations, coupled with suitable initial and boundary conditions. Here $\epsilon\in(0,1)$ is a small positive parameter, f is the derivative of a double-well potential with minima at $s_\pm$, and $M_\mathrm{i}(x), M_\mathrm{e}(x)$ are two symmetric uniformly positive definite matrices.

A crucial role in the description of the electrochemical changes governing the

[(1)]See the functional in (1.28) below.

heart beating is played by the transmembrane potential

$$u = u_\epsilon := u_\mathrm{i} - u_\mathrm{e},$$

which typically exhibits a thin transition region (of order ϵ) which separates the advancing depolarized region where $u_\epsilon \approx s_+$ from the one where $u_\epsilon \approx s_-$, see [31, 43] and references therein. Remarkably, a non-negligible *nonlinear anisotropy* occurs in the limit $\epsilon \to 0^+$, because of the fibered structure of the myocardium. To explain the appearence of the anisotropy, let us introduce the Riemannian norms $\phi_\mathrm{i}, \phi_\mathrm{e}$, defined as

$$(\phi_\mathrm{i}(x,\xi^*))^2 = \alpha_\mathrm{i}(x,\xi^*) := M_\mathrm{i}(x)\xi^*\cdot\xi^*, \qquad (\phi_\mathrm{e}(x,\xi^*))^2 = \alpha_\mathrm{e}(x,\xi^*) := M_\mathrm{e}(x)\xi^*\cdot\xi^*,$$

where ξ^* denotes a generic covector of the dual $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$, $n \geq 2$, and $\cdot$ is the Euclidean scalar product. The squared norms α_i and α_e depend on the spatial variable x, since the fibers orientation changes from point to point, and their hessians $\frac{1}{2}\nabla^2_{\xi^*}\alpha_\mathrm{i}$, $\frac{1}{2}\nabla^2_{\xi^*}\alpha_\mathrm{e}$ (with respect to ξ^*) give M_i and M_e respectively. Then the anisotropy arises, for instance, recalling the following *formal* result [31]. Let Φ be defined as

$$\Phi := \left(\frac{1}{\alpha_\mathrm{i}} + \frac{1}{\alpha_\mathrm{e}}\right)^{-\frac{1}{2}}, \tag{0.2}$$

and assume that Φ^2 is smooth and uniformly *convex*. Then, as $\epsilon \to 0^+$, the zero-level set of u_ϵ approximates a geometric motion of a front, evolving by Φ^o-*anisotropic mean curvature flow*, where, again, Φ^o denotes the dual of Φ. This convergence result is substantiated by a Γ-convergence result (at the level of the corresponding actions) toward a geometric functional, whose integrand is strictly related to (0.2), see [11] and Theorem 1.20 below. Note that Φ is *not Riemannian* anymore (in the language of this thesis, it is a nonlinear anisotropy), and it may also *fail* to be convex (this latter property can be seen through an explicit example described in [43], see also Example 1.18 and Appendix A).

The lack of an underlying scalar product for Φ suggests that it is natural to depart from the Riemannian structure of (0.1) and to consider, more generally, the *nonlinear* bidomain model. This latter is described by

$$\begin{cases} \epsilon\partial_t(u_\mathrm{i} - u_\mathrm{e}) - \epsilon \mathrm{div}\left(T_{\phi_\mathrm{i}}(x,\, \nabla u_\mathrm{i})\right) + \dfrac{1}{\epsilon} f\big(u_\mathrm{i} - u_\mathrm{e}\big) = 0, \\ \epsilon\partial_t(u_\mathrm{i} - u_\mathrm{e}) + \epsilon \mathrm{div}\left(T_{\phi_\mathrm{e}}(x,\, \nabla u_\mathrm{e})\right) + \dfrac{1}{\epsilon} f\big(u_\mathrm{i} - u_\mathrm{e}\big) = 0, \end{cases} \tag{0.3}$$

where now ϕ_i and ϕ_e are two smooth symmetric uniformly convex Finsler metrics, and setting as before $\alpha_\mathrm{i} = \phi_\mathrm{i}^2$, $\alpha_\mathrm{e} = \phi_\mathrm{e}^2$, the maps

$$T_{\phi_\mathrm{i}} := \frac{1}{2}\nabla_{\xi^*}\alpha_\mathrm{i}, \qquad T_{\phi_\mathrm{e}} := \frac{1}{2}\nabla_{\xi^*}\alpha_\mathrm{e}$$

are the so-called duality maps, mapping $(\mathbb{R}^n)^*$ into $\mathbb{R}^n$ (convexity of ϕ_i and ϕ_e is required, in order to ensure well-posedness of (0.3)). Then, a result similar to the previous formal convergence to Φ^o-anisotropic mean curvature flow holds also in this nonlinear setting, still assuming Φ^2 to be uniformly convex, see [43].

Generalizing system (0.3) to an arbitrary number m of Finsler symmetric metrics $\phi_1, \ldots, \phi_m$, leads to rewrite the problem, that we have called the *nonlinear multidomain model*, in a slightly different and more natural way: we seek functions $w^r = w^r_\epsilon$ satisfying the weakly parabolic system

$$\begin{cases} \epsilon\partial_t u - \epsilon \mathrm{div}\left(T_{\phi_r}\left(x, \nabla w^r\right)\right) + \dfrac{1}{\epsilon} f\left(u\right) = 0, \qquad r = 1, \dots, m, \\ u = \displaystyle\sum_{r=1}^{m} w^r, \end{cases} \tag{0.4}$$

where

$$T_{\phi_r} := \frac{1}{2}\nabla_{\xi^*}\alpha_r \quad \text{and} \quad \alpha_r := \phi_r^2, \qquad r = 1, \dots, m.$$

In this respect, our main focus will be to provide an asymptotic analysis of the zero level set of $u = u_\epsilon$ in (0.4): indeed, one of the main results of the thesis will be to show that $\{u_\epsilon(t, \cdot) = 0\}$ converges to the Φ^o-anisotropic mean curvature flow (see (2.97) below), where Φ^2, supposed to be uniformly convex, reads as

$$\Phi^2 := \left(\sum_{r=1}^{m} \frac{1}{\alpha_r}\right)^{-1}, \tag{0.5}$$

as it happens for the linear and nonlinear bidomain models. Our proof, appeared first in [7], remains at a *formal level*, and is based on a new asymptotic expansion for (0.4), rewritten equivalently as a system of one parabolic equation and $(m-1)$ elliptic equations (this shows, among other things, the nonlocality of solutions of (0.4)). The asymptotic expansion we shall perform is simpler, and at the same time carried on at a *higher order* of accuracy, with respect to the one exhibited in [43] for the case $m = 2$.

We stress that confirming rigorously the convergence result for the level sets $\{u_\epsilon(t, \cdot) = 0\}$ is still an open problem, even in the simplest case of (0.1) (see Theorem 1.22 for a precise statement). Here we have to observe that, since we are dealing with systems, we cannot make use of the maximum principle, as it was for the scalar anisotropic Allen-Cahn equation. This, however, could be hopely less hard to prove than a convergence result of the Allen-Cahn's (2×2)-system, to curvature flow of networks (see [59] for a formal result in this direction): indeed, this was among the starting motivations for studying nonlinear multidomain model.

Another open problem, connected with the nonconvex anisotropic mean curvature flow, is given by the analysis of the limit behaviour (if any) as $\epsilon \to 0^+$ of solutions to (0.3) when Φ is nonconvex. The question arises as to whether nonlinear multidomain model could be used in order to provide a notion of solution for nonconvex anisotropic mean curvature flow, at least within the class of anisotropies of the form (0.5). The answer to this question seems, at the moment, out of reach, even at a formal level.

Crystalline mean curvature on facets and connections with capillarity. A sort of limiting case for anisotropic mean curvature flow is when the unit ball B_ϕ of ϕ (sometimes called the Wulff shape) is convex and nonsmooth. Recall that, when ϕ^2 is smooth and uniformly convex, the natural direction for the geometric motion is given by the Cahn-Hoffman vector field $T_{\phi^o}(\nu_{\phi^o})$. However, in the nonsmooth setting, the map T_{ϕ^o} is defined as the subdifferential of the convex function $\frac{1}{2}(\phi^o)^2$, and therefore it is allowed to be multivalued. This implies that there can be infinitely many vector fields X defined on the evolving hypersurface ∂E, and satisfying the constraint

$$X \in T_{\phi^o}(\nu_{\phi^o}). \tag{0.6}$$

As a consequence, there are several open problems concerning the motion. Apart from the planar case,(2) at our best knowledge a "good" definition of flow is still missing, as well as a well-posedness result for short times. It is not even clear how to choose the natural class of sets for studying the motion. Several definitions of "regular sets" (and, hence, of anisotropic mean curvature) have been proposed, like the neighbourhood regularity in [36, 34, 62].(3)

In the present thesis, we shall follow the approach of [38, 39], which, in some sense, naturally fits in the aim of studying anisotropic mean curvature as a localized problem on the facets of a crystal. In particular, we shall require the existence of a *Lipschitz* vector field X, defined just on the boundary of the solid set, and satisfying (0.6). In this framework, anisotropic mean curvature is obtained by minimizing the L^2-norm of the divergence among all vector fields defined on the hypersurface, and satisfying (0.6). Remarkably, anisotropic mean curvature still turns out to correspond to the direction of maximal slope of P_ϕ. However, since there exist several vector fields whose divergence equals the anisotropic mean curvature, it is not clear which is (if any) the natural direction for studying the motion. It is not even clear, in general, if anisotropic mean curvature is attained by a Lipschitz vector field.

In this respect, a first mathematically interesting and challenging case is when $n = 3$, and B_ϕ is a (convex) polyhedron. In this setting, the focus is given by the study of anisotropic mean curvature on facets $F \subset \partial E$ of a solid set $E \subset \mathbb{R}^3$, which are parallel to a facet of the Wulff shape: indeed, under reasonable assumptions on the behaviour of E locally around F (see (3.9)), the anisotropic mean curvature κ_ϕ^E at F can be obtained as a by-product of a minimization problem on divergences of vector fields defined just on the facet (hence, solving a variational problem in one dimension less). We shall call *optimal selection* in F any vector field solving the above mentioned minimization problem. Notice, again, that we cannot guarantee in general the existence of a Lipschitz optimal selection in the facet.

As a first nontrivial step in this analysis, there is the characterization of facets having constant anisotropic mean curvature, also called *ϕ-calibrable* [36]. The notion of calibrability can be given for any convex anisotropy [36, 37], and in any dimension $k \geq 1$ [6, 62] (we shall focus on the case $k = n - 1 = 2$). Actually, the case $k = 1$ is trivial, since all edges contained in the boundary of a planar domain have constant anisotropic mean curvature (see for instance [151], and references therein). Noncalibrable facets allow to construct explicit examples of facet breaking-bending phenomena, see again [36, 37]: indeed, it seems reasonable that, at least at time $t = 0$, the facet breaks in correspondance of the jump set of its curvature and bends if the curvature is continuous and not constant. On the contrary, a calibrable facet is expected to translate parallely to itself with constant velocity, at least for short times.

(2) This will not be the subject of the present thesis. We refer the interested reader for instance to [151], or to the more recent paper [67], and references therein.

(3) This notion has been used in [27] to give a uniqueness result, in general dimension and for convex sets, again combining an approximating argument with the maximum principle.

In our case ($k = 2$), let $\Pi_F \cong \mathbb{R}^2$ be the affine plane spanned by F, and let $\widetilde{B}^F_\phi \subset \Pi_F$ be the facet of B_ϕ which is parallel to F. For simplicity, let us state the problem assuming that E lies, locally around F, in the half-space delimited by Π_F and opposite to the outer normal to ∂E at F (in the language of the present thesis, E is said to be *convex* at F). We say that F is ϕ-calibrable if there exists a vector field $X \in L^\infty(F;\mathbb{R}^2)$ satisfying

$$\begin{cases} X(x) \in \widetilde{B}^F_\phi & \text{for a.e. } x \in F, \\ \mathrm{div} X = h & \text{a.e. in } F, \\ \langle \widetilde{\nu}^F, X\rangle = 1 & \mathcal{H}^1\text{-a.e. on } \partial F, \end{cases} \tag{0.7}$$

where $\widetilde{\nu}^F \in \Pi_F$ is the unit normal vector field to ∂F pointing outside of F, $\langle \widetilde{\nu}^F, X\rangle$ plays the role of a normal trace, and the constant $h > 0$ is determined by an integration by parts (Section 3.4). It is possible to prove [37] that a facet is ϕ-calibrable if and only if its "mean velocity" is less than or equal to the mean velocity of any subset of the facet (Theorem 3.30).[(4)] We say that F is strictly ϕ-calibrable if it is ϕ-calibrable and there is no $B \subset F$, $B \neq \emptyset$, having mean velocity equal to that of F.

Remarkably, the (necessary and sufficient) condition for calibrability provided by [37] turns out to be very similar to the one, given by Giusti in its fundamental paper [101], about the existence of solutions to the capillary problem in the absence of gravity on a bounded connected open set $\Omega \subset \mathbb{R}^n$. For a brief discussion on the action principle for a capillary, we refer the interested reader for instance to [117, 88], or also to Appendix B. Here, we want just to mention that Giusti's result (which we recall in Theorem 3.41) provides a function $u \in \mathcal{C}^2(\Omega)$ such that the subunitary vector field $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ has constant divergence on Ω, and

$$\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \to \nu^\Omega \quad \text{uniformly on } \partial\Omega,$$

if and only if Ω is the unique solution of a prescribed mean curvature problem among its subsets.

In the present thesis, we shall apply Giusti's result as follows. Let ϕ_c be the norm of $\mathbb{R}^3$ induced by the (portion of) Euclidean cylinder

$$B_{\phi_c} := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \ : \max\left(\sqrt{\xi_1^2 + \xi_2^2}, |\xi_3|\right) \leq 1 \right\}.$$

Then, a strictly ϕ_c-calibrable facet F such that E is convex at F is nothing but a set Ω where the problem addressed in Theorem 3.41 has a solution. As a consequence (Proposition 3.43), in such a facet there exists an optimal selection which is induced by a solution u of the capillary problem in the relative interior of F with zero contact angle; moreover, this optimal selection is (disregarding the sign) the horizontal component of the outer unit normal vector to the graph of u.

The link between capillarity and calibrability turns out to be useful also in the case of noncalibrable facets, which corresponds to let the right hand side of the

[(4)] See also [62] for an extension to convex facets in dimension $k \geq 2$.

second equation in (0.7) nonconstant. In particular, by analyzing the sublevel sets of the anisotropic mean curvature, we will show that it is sometimes possible to extend the selection out of the maximal subset of F where the capillary problem is solvable. This extension, in general, may not be induced by a scalar function; nevertheless, it still provides information on the regularity of the anisotropic mean curvature at F, and it could help for a better understanding of the geometric motion.

Constrained BV functions on coverings. In its earliest and simplest formulation, the Plateau's problem consists in finding a surface Σ in the ambient space $\mathbb{R}^3$, spanning a fixed reference smooth loop S, and minimizing the area. As it is well-known, several models have been proposed to solve the mathematical questions related to this problem (and to its generalizations in $\mathbb{R}^n$, for $n \geq 2$), depending on the definition of surface, boundary, and area: parametric and non-parametric solutions, homology classes, integer rectifiable currents, varifolds, just to name a few. General references are for instance [2, 131, 143, 129, 81]; we refer the reader to [76] for a brief overview on the Plateau's problem. In connection with what we are going to discuss, we also mention the recent paper [77], where the authors, extending in a different setting some results of [103], look for a solution of Plateau's problem, minimizing the $(n-1)$-dimensional Hausdorff measure in the class of relatively closed subsets of $\mathbb{R}^n \setminus S$, with nonempty intersection with every loop having unoriented linking number with S equal to 1.

In the present thesis, we link the coverings with the theory of (possibly vector-valued) functions of bounded variation and Γ-convergence, in order to solve the problem of minimal networks in the plane, and to find an embedded solution to Plateau's problem, without fixing a priori the topology of solutions. This idea shares several similarities with the "soap films" covering space model, set up in [55] by Brakke as a new original approach to Plateau's problem in codimension one.

Our model mathematically reproduces the physical structure of an interface separating two (or more) phases. In this respect, for instance in case of two phases, it is useful to merge Plateau's problem in an n-dimensional ($n = 3$ being the physical case) manifold, which is a *covering space* of the open set

$$M := \Omega \setminus S,$$

where $\Omega \subset \mathbb{R}^n$ is usually a bounded connected Lipschitz open set containing the $(n-2)$-dimensional compact embedded Lipschitz manifold S without boundary. In the model of [55], how to choose the covering is part of the model construction, and it can lead to different solutions. Then one has to select some connected components of a *pair covering space* of M in order to pair the sheets and to set up the minimization problem in terms of a suitable notion of current mass. Again, the choice of the pair covering space is part of the model construction.

Here, we approach the problem *without making use of pair covering spaces*, which can be considered as a first simplification of the model. Typical situations that we shall consider are:

- $n = 2$, $S \subset \mathbb{R}^2$ a set of m-distinct points, and an m-sheeted covering space of M; the case $m = 3$ is already interesting, and related to the Steiner graphs (when $m \geq 3$, taking a two-sheeted covering space does not lead to any interesting conclusion);[5]

- $n = 3$, $S \subset \mathbb{R}^3$ a link, and a two-sheeted covering space;[6] this leads to the Plateau's problem.

Our explicit construction of the covering, denoted by $(Y_{\boldsymbol{\Sigma}}, \pi_{\boldsymbol{\Sigma},M})$, requires a suitable pair of cuts $\boldsymbol{\Sigma} = (\Sigma, \Sigma')$, where Σ and Σ' are $(n-1)$-dimensional compact Lipschitz manifolds (not necessarily connected), having S as topological boundary (Definitions 4.2 and 4.3). The construction is made by "cut and paste", with the use of local parametrizations, these latter suggesting the natural way to endow $Y_{\boldsymbol{\Sigma}}$ with the Euclidean metric. The metric aspects here play an important role: as it will be clear from the discussion, we cannot confine ourselves to a purely topological construction of the covering (see Remark 4.4).

Let $m \geq 2$ be the number of sheets of $Y_{\boldsymbol{\Sigma}}$, and let $V \subset \mathbb{R}^{m-1}$ be the set of vertices of a regular simplex. Our idea is to minimize the total variation $|Du|(Y_{\boldsymbol{\Sigma}})$ among all BV functions $u \colon Y_{\boldsymbol{\Sigma}} \to V$, satisfying the following *constraint on the fibers*: for $j = 1, \ldots, m$, denote by $v_j(u)$ the restriction of u to the j-th sheet of the covering (Definition 4.8); then we require that

$$v_j(u) = \tau^{j-1} \circ v_1(u), \qquad j = 1, \ldots, m, \tag{0.8}$$

for a transposition τ of V of order m and independent of j. Roughly speaking, condition (0.8) means that u "behaves" the same way on each covering sheet, the only difference consisting in a fixed transposition of the elements of V having order m. For instance, if $y \in Y_{\boldsymbol{\Sigma}}$ is a jump point of u, then u has to jump at all points of the same fiber of y.[7]

When $m = 2$ and $V = \{\pm 1\}$, condition (0.8) is equivalent to require

$$\sum_{\pi_{\boldsymbol{\Sigma},M}(y)=x} u(y) = 0, \quad \text{for a.e. } x \in M,$$

so that u takes opposite values on (the two) points of the same fiber. To have an idea of the geometric meaning of the total variation we are considering, it is useful to look at the elementary Example 4.10, which refers to the case $m = 3$. The usefulness of constraint (0.8) stands in studying the minimization problem handling with standard BV functions defined on open subsets of $\mathbb{R}^n$. We also remark that the constraint (0.8) forces the boundary datum S to be attained (Corollary 4.26); this represents a difference with the approach of [55], where it may happen that the boundary S is not fully covered by a solution.[8] Perhaps, the most remarkable among its consequences is that all issues about the definition of "boundary" on S are avoided. Finally, the constraint (0.8) plays a crucial role also in forcing the minimum value to be strictly positive (see Lemma 4.25). It seemed to us not immediate to derive the constraint on the fibers from the approach of [55].

What we call a *constrained covering solution with boundary S* is (Definition 4.27) the projection *via* $\pi_{\boldsymbol{\Sigma},M}$ of the jump set of a minimizer. Existence of

[5] See Figures 4.1, 4.3 and 4.6.

[6] See Figure 4.2.

[7] See Figure 4.4 for an example where $m = 3$ and condition (0.8) is violated.

[8] See also Figure 4.5 for an explicit example (in dimension $n = 2$) where the two methods lead to different solutions.

minimizers is proved in Theorem 4.24. Although our construction requires a suitable pair $\mathbf{\Sigma}$ of cuts, constrained covering solutions are actually independent of $\mathbf{\Sigma}$;[(9)] in some sense, this is due to the fact that, working on the covering space, all information about the exact location of the cuts becomes irrelevant, since changing the cuts corresponds just to an isometry on the covering space.

We expect that our model could be generalized in a nontrivial way in various directions; in particular, to more general choices of S (for instance, taking as S the set of all 1-dimensional edges of a polyhedron). In this spirit, we briefly discuss in Section 4.4.3 the case when S is the one-skeleton of a tetrahedron ($n = 3$ and $m = 4$), and, by adapting an argument in [23], we give a regularity result[(10)] (Proposition 4.42) in the sense of Almgren's $(\mathbf{M}, 0, r)$-minimal sets [2, 148].

Plan of the thesis. In **Chapter 1** we fix the basic notation, and collect some preliminary facts on ϕ-anisotropic mean curvature, in the (regular) case ϕ^2 is smooth and uniformly convex. In Section 1.3 we define the star-shaped combination of m anisotropies ($m \geq 2$), which will play a crucial role in the analysis of bidomain and nonlinear multidomain models. We end this chapter recalling some relevant results related to bidomain model (Section 1.4).

Chapter 2 is devoted to nonlinear multidomain model. A well-posedness result is given in Section 2.2, by adapting the original proof in [73] for bidomain model. The remaining of this chapter contains the asymptotic analysis of nonlinear multidomain model, developed up to the second order included, and the already mentioned, formal convergence result to anisotropic mean curvature flow.

In **Chapter 3** we study the problem of finding an explicit optimal selection in facets of a solid set in $\mathbb{R}^3$ with respect to a crystalline norm. In Section 3.1 we briefly collect some results on the anisotropic and Euclidean Cheeger problem, which will be useful in the remaining of the chapter. Then we consider the problem of ϕ-calibrable facets. Let $\widetilde{\phi}$ be the bidimensional metric induced by $\widetilde{B}^F_\phi$, let $P_{\widetilde{\phi}}(F)$ be the $\widetilde{\phi}$-perimeter of F, and denote by $\kappa^F_{\widetilde{\phi}}$ the $\widetilde{\phi}$-mean curvature of ∂F. Then, we show in Theorem 3.36 that

$$\kappa^F_{\widetilde{\phi}} \leq \frac{P_{\widetilde{\phi}}(F)}{|F|} \tag{0.9}$$

is a necessary condition for calibrability when F is $\widetilde{\phi}$-convex (namely, "convex" in the relative geometry induced by $\widetilde{\phi}$). This result was already known for convex facets [37], and in that context the two conditions are actually equivalent. Example 3.35 shows that condition (0.9) is not sufficient anymore for ϕ-calibrability when F is just $\widetilde{\phi}$-convex. In Section 3.4.1, we prove some facts on the calibrability of "annular" facets. Theorems 3.37-3.39 could be considered as a first step towards an extension to the crystalline setting of the study of "oscillating towers" given in [30]. In Section 3.4.2 we generalize to the anisotropic context the case of strips investigated in [113] in the Euclidean setting. The main results of the chapter are contained in Section 3.5, where we link the issue of calibrability with the capillary problem in order to provide some relevant examples of continuous optimal selections in noncalibrable facets.

In **Chapter 4** we set up the explicit "cut and paste" covering construction, and we define the family of constrained BV functions. Then, for any admissible

[(9)] See [55, Proposition 12.1] for a similar result.
[(10)] See also [55, Theorem 10.2] for a similar result.

pair of cuts $\mathbf{\Sigma}$, the minimization problem is set up in Section 4.2. Regularity of constrained covering solutions is based on the well-established regularity theory for isoperimetric sets and minimizing clusters. Then, in Section 4.3, we lift the constraint on the fibers to the class of Sobolev functions on $Y_{\mathbf{\Sigma}}$, showing (Proposition 4.31) that our formulation naturally leads to a Γ-convergence result. In Section 4.4.1 we exploit the case $n = 2$, namely when S consists of $m \geq 2$ distinct points, and we show that a constrained covering solution coincides with the Steiner graph over S. In Section 4.4.2 we test the model in the case of the standard Plateau's problem in $\mathbb{R}^3$: in Theorem 4.36 we show that, at least when $2 < n < 8$, our model is equivalent to solving Plateau's problem using the theory of integral currents modulo 2 [86].

Finally, in **Appendix A** we give an interesting example of nonconvex combined anisotropy, generalizing in some sense that one provided in [43]. **Appendix B** contains a brief discussion on the action principle for a capillary in the absence of gravity, while in **Appendix C** we perform a standard abstract covering construction which is used in Chapter 4.

Bibliographic note. The results of Chapter 2 have been obtained in collaboration with G. Bellettini and M. Paolini, and are published in [7]. The content of Chapter 3 corresponds to a joint work with G. Bellettini and L. Tealdi, appearing in [9]. Finally, Chapter 4 describes the results of [8], obtained in collaboration with G. Bellettini and M. Paolini.

Chapter 1

Preliminaries

Summary. We recall the definition of star-shaped anisotropies, duality maps, anisotropic perimeter (Section 1.1) and anisotropic mean curvature in the regular case (Section 1.2). In Section 1.3, we introduce the operation of star-shaped combination of anisotropies, and we give some examples of star-shaped combination where convexity is not preserved. In Section 1.3.1, we provide a formula for the hessian of the combined anisotropy, which will be useful in Chapter 2. Section 1.4 contains some relevant results on bidomain model, and represents the starting motivation for the formulation of nonlinear multidomain model given in the subsequent chapter.

Basic notation. For $n \in \mathbb{N}$, $n \geq 1$, we denote by $\mathcal{H}^{n-1}$ the Euclidean $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. We let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^n$. For any x, $x' \in \mathbb{R}^n$, we denote by $x \cdot x'$ the scalar product between x and x'. We also let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. The volume of the unit ball of $\mathbb{R}^n$ is denoted by ω_n. For any $E \subseteq \mathbb{R}^n$, we denote by $\overline{E}$ (resp. by $\mathrm{int}(E)$) the closure of E in $\mathbb{R}^n$ (resp. its interior part). When E is a finite perimeter set, we denote by $\partial^* E$ its reduced boundary, and by ν^E the generalized outer normal to $\partial^* E$. Sometimes, whenever no confusion is possible, we shall denote by $|E|$ the volume of E, namely the Euclidean n-dimensional Lebesgue measure $\mathcal{L}^n$ of E.

Let X be a (possibly infinite dimensional) Banach space, with dual X^* and let $g \colon X \to \mathbb{R} \cup \{+\infty\}$ be convex. Let $\langle \cdot, \cdot \rangle$ denote the duality between X^* and X, and let 2^{X^*} be the powerset of X^*. Then, for any $x \in X$, we denote by $\partial g(x) \in 2^{X^*}$ the subdifferential of g at x, namely the subset of X^* defined as

$$\partial g(x) := \{l \in X^* \ : \ g(x) + \langle l, y - x \rangle \leq g(y) \text{ for every } y \in X\}.$$

1.1 Convex anisotropies

Let $n \in \mathbb{N}$, $n \geq 1$. Let V denote either $\mathbb{R}^n$ or its dual $(\mathbb{R}^n)^*$, endowed with the Euclidean norm $|\cdot|$. Clearly, one has the trivial identifications $\mathrm{V} \cong \mathbb{R}^n \cong (\mathbb{R}^n)^*$; nevertheless, in order to distinguish between anisotropies defined on vectors or on covectors, sometimes we shall find more convenient to keep this notation (see Remarks 1.2 and 1.6).

Definition 1.1 (**Star-shaped anisotropies).** *A* star-shaped anisotropy *(or* anisotropy *for short) on* V *is a continuous function* $\phi : \mathrm{V} \to [0, +\infty)$*, positive out of the origin, and positively one-homogeneous. We say that* ϕ *is* symmetric *if* $\phi(-\xi) = \phi(\xi)$ *for any* $\xi \in \mathrm{V}$.

We say that ϕ is linear *if it is the square root of a quadratic form on* V*.*

By Definition 1.1, given any star-shaped anisotropy ϕ there exist $\mathrm{C} \geq \mathrm{c} > 0$ such that

$$\mathrm{c}|\xi| \leq \phi(\xi) \leq \mathrm{C}|\xi|, \qquad \xi \in \mathrm{V}. \tag{1.1}$$

We denote by $B_\phi \subset \mathrm{V}$ the unit ball of a given anisotropy ϕ on V, namely

$$B_\phi := \{\xi \in \mathrm{V} \ : \ \phi(\xi) \leq 1\}.$$

Remark 1.2. The previous definitions of ϕ and B_ϕ can be generalized, by allowing a continuous dependence on the space variable x in some n-dimensional manifold M. This way, $\phi = \phi(x, \xi)$ is defined for $(x, \xi) \in T\mathrm{M}$ (the tangent bundle of M) More precisely, a continuous function $\phi : T\mathrm{M} \to [0, +\infty)$ is called an *inhomogeneous star-shaped anisotropy* on M, provided $\phi(x, \cdot)$ is positively one-homogeneous for any $x \in \mathrm{M}$, and there exist two constants $\mathrm{C} \geq \mathrm{c} > 0$ such that $\mathrm{c}|\xi| \leq \phi(x, \xi) \leq \mathrm{C}|\xi|$ for any $(x, \xi) \in T\mathrm{M}$. Notice that, for any $x \in \mathrm{M}$, $\phi(x, \cdot)$ is defined on $\mathrm{V} := T_x\mathrm{M}$ (the tangent space of M at x). In the present thesis, however, we will be interested in space-independent anisotropies defined on open subsets of $\mathbb{R}^n$.

In this thesis we shall be mainly concerned with the following classes of star-shaped anisotropies.

Definition 1.3 (Convex anisotropies). *We denote by*

$$\mathcal{M}(\mathrm{V})$$

the collection of all symmetric convex anisotropies on V*.*

Definition 1.4 (Regular anisotropies). *We denote by*

$$\mathcal{M}_{\mathrm{reg}}(\mathrm{V}) \subset \mathcal{M}(\mathrm{V})$$

the collection of all symmetric anisotropies ϕ on V*, such that ϕ^2 is of class $\mathcal{C}^2$ and uniformly convex.*

Notice that $\mathcal{M}_{\mathrm{reg}}(\mathrm{V})$ contains all linear anisotropies.

Remark 1.5. Symbols $\mathcal{M}(\mathrm{V})$, $\mathcal{M}_{\mathrm{reg}}(\mathrm{V})$ are meant to remind the word "metric". Indeed, we recall that an inhomogeneous star-shaped anisotropy $\phi\colon T\mathrm{M} \to [0, +\infty)$ such that $\phi(x, \cdot)$ is convex, for every $x \in \mathrm{M}$, is usually called a Finsler metric on M (see for instance [22]). In our case, anyway, a map $\phi \in \mathcal{M}(\mathrm{V})$ will be nothing but a norm in V.

Given an anisotropy ϕ on V, we denote by $\phi^o\colon \mathrm{V}^* \to [0, +\infty)$ the *dual* of ϕ [136], defined as

$$\phi^o(\xi^*) := \sup\{\langle \xi^*, \xi\rangle \ : \ \xi \in B_\phi\}, \qquad \xi^* \in \mathrm{V}^*.$$

It turns out [136] that ϕ^o is an anisotropy on V^*; moreover:

- ϕ^o is convex;

- if $\phi \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V})$, then $\phi^o \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V}^*)$;

- if ϕ is symmetric, then, for any $\nu \in \mathbb{S}^{n-1}$, $\phi^o(\nu)$ is the minimal distance from the origin of all affine hyperplanes which are orthogonal to ν and tangent to B_ϕ;

- ϕ^{oo} coincides with the convexified of ϕ; in particular, $\phi^{oo} = \phi$ if and only if ϕ is convex.

Figure 1.1 below shows some examples of convex anisotropies and dual norms which will be relevant in the present thesis. See also Figure 1.2 for an example of nonconvex anisotropy related to what we shall discuss in Section 1.3.

Remark 1.6. Let M be a n-dimensional manifold, and let ϕ be an inhomogeneous star-shaped anisotropy (recall Remark 1.2). Then, the dual (inhomogeneous) anisotropy ϕ^o is defined as $\phi^o(x,\cdot) = (\phi(x,\cdot))^o$, for every $x \in \mathrm{M}$; in particular, ϕ^o is defined on the cotangent bundle of M.

Definition 1.7 (Duality maps). *Let ϕ be a star-shaped anisotropy on* V*. We define the (maximal monotone possibly multivalued) one-homogeneous map T_{ϕ^o} : $\mathrm{V}^* \to 2^{\mathrm{V}}$ as*

$$T_{\phi^o}(\xi^*) := \frac{1}{2}\partial((\phi^o)^2)(\xi^*), \qquad \xi^* \in \mathrm{V}^*.$$

Similarly, assuming also $\phi \in \mathcal{M}(\mathrm{V})$, we define the map $T_\phi\colon \mathrm{V} \to 2^{\mathrm{V}^*}$ as

$$T_\phi(\xi) := \frac{1}{2}\partial(\phi^2)(\xi), \qquad \xi \in \mathrm{V}.$$

When $\phi \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V})$, both T_ϕ and T_{ϕ^o} are single-valued maps. Then one has [41]

$$T_{\phi^o} \circ T_\phi = \mathrm{id}_{\mathrm{V}}, \qquad T_\phi \circ T_{\phi^o} = \mathrm{id}_{\mathrm{V}^*}.$$

Moreover, the Euler's formula for homogeneous functions implies

$$\langle T_\phi(\xi), \xi\rangle = \phi^2(\xi), \qquad \xi \in \mathrm{V}, \tag{1.2}$$

and similarly for T_{ϕ^o}.

Definition 1.8 (Anisotropic perimeter). *Let ϕ be an anisotropy of* V*. The* ϕ-anisotropic perimeter *of a finite perimeter set $E \subset \mathbb{R}^n$ in the open set $\Omega \subseteq \mathbb{R}^n$ is defined as*

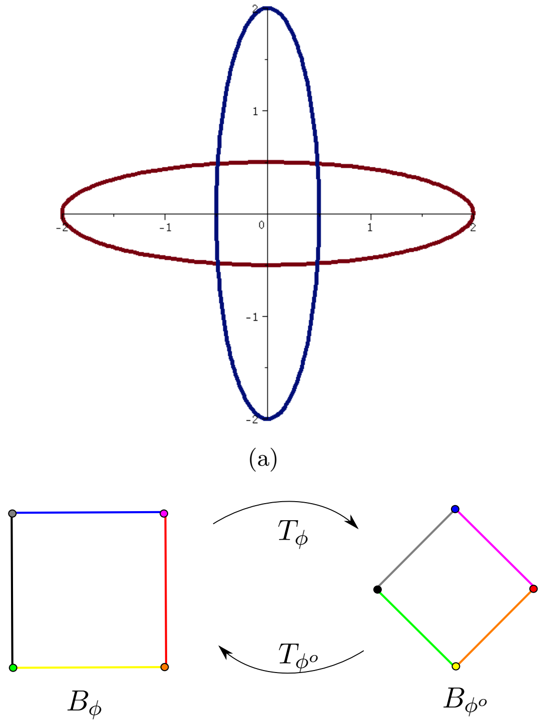
$$P_\phi(E,\Omega) := \omega_n^\phi \int_{\Omega\cap\partial^* E} \phi^o(\nu^E)\ d\mathcal{H}^{n-1},$$

where $\omega_n^\phi := \frac{\omega_n}{|B_\phi|}$.

The constant ω_n^ϕ plays a role in the definition of the ϕ-anisotropic volume $|\cdot|_\phi$, see for instance [44, 41]. We recall that $|\cdot|_\phi = \omega_n^\phi|\cdot|$, so that $|B_\phi|_\phi = \omega_n$ for any ϕ anisotropy of V. It turns out that B_ϕ satisfies the following isoperimetric property: for every set $E \subset \mathbb{R}^n$ of finite perimeter and finite Lebesgue measure, we have

$$P_\phi(E) \geq \left(\frac{|E|}{|B_\phi|}\right)^{\frac{n-1}{n}} P_\phi(B_\phi), \tag{1.3}$$

with equality if and only if E coincides (up to a translation) with B_ϕ. See [146, 147, 149, 90, 91, 85] for a quantitative version of (1.3).



(a)

(b)

Figure 1.1: In (a), an example of unit balls of regular anisotropies ϕ (red line) and ϕ^o (blue line). In (b), an example of unit balls of convex nonregular anisotropies ϕ (on the left) and ϕ^o (on the right). Colours are used in order to represent the action of the duality maps T_ϕ and T_{ϕ^o}. In particular, any edge (resp. any vertex) of B_ϕ is mapped by T_ϕ onto a vertex (resp. an edge) of B_{ϕ^o}, and similarly for T_{ϕ^o}.

1.2 Anisotropic mean curvature in the regular case

Throughout this section, we let $\mathrm{V} := \mathbb{R}^n$, $n \geq 1$, and we let $\phi \in \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^n)$. Let $E \subseteq \mathbb{R}^n$ be a compact set of class $\mathcal{C}^2$, and set

$$\nu_{\phi^o} := \frac{\nu^E}{\phi^o(\nu^E)}.$$

In order to define the ϕ-anisotropic mean curvature of ∂E (Definition 1.9), we shall make use of the ϕ-anisotropic signed distance function from the boundary of E. This is, in some sense, a convenient approach when one looks at the evolving hypersurface as a set of points rather than as the embedding of a reference manifold; moreover, this setting fits in a natural way in the perspective of study anisotropic mean curvature flow as the evolution of a hypersurface separating two phases.[(1)]

For $y, z \in \mathbb{R}^n$, we set $\mathrm{dist}_\phi(y,z) := \phi(z-y)$, $\mathrm{dist}_\phi(z,E) := \inf_{y\in E} \mathrm{dist}_\phi(y,z)$, and we define the *$\phi$-anisotropic signed distance function* d^E_ϕ from ∂E (positive inside E) as

$$d^E_\phi(z) := \mathrm{dist}_\phi(z, \mathbb{R}^n \setminus E) - \mathrm{dist}_\phi(z,E). \tag{1.4}$$

[(1)] For a parametric description of the flow in the isotropic case, we refer the reader for instance to [116].

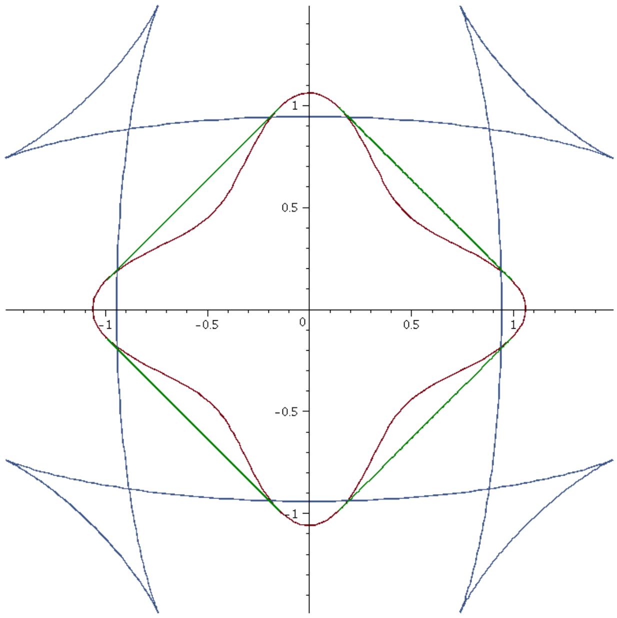


Figure 1.2: Let Φ be the anisotropy whose unit ball B_Φ is bounded by the red line in the picture (compare also Figure 1.3 below). Notice that the set $T_\Phi(B_\Phi)$ (blue line) has self-intersections and cusps. The dual of B_Φ is obtained by removing from $T_\Phi(B_\Phi)$ the four swallow-tails, and, of course, it coincides with the dual of the convexified of B_Φ (green line).

It turns out that there exists a neighbourhood U of ∂E such that d_ϕ^E is Lipschitz in U, and moreover d_ϕ^E satisfies the following *eikonal equation* [44]

$$\phi^o(\nabla d_\phi^E) = 1 \quad \text{in } U. \tag{1.5}$$

Equation (1.5) shows that ∇d_ϕ^E is a ϕ^o-unitary covector field on ∂E, or, equivalently,

$$\nabla d_\phi^E = -\nu_{\phi^o}.$$

Following [41], we dually define the *Cahn-Hoffman vector field* n_ϕ on ∂E as

$$n_\phi := T_{\phi^o}(\nu_{\phi^o}), \quad \text{on } \partial E, \tag{1.6}$$

and, by means of (1.5), we extend the Cahn-Hoffman vector field n_ϕ on the whole of U as

$$N_\phi := -T_{\phi^o}(\nabla d_\phi^E) \quad \text{in } U.$$

Definition 1.9 (Anisotropic mean curvature). *We define the* ϕ-anisotropic mean curvature κ_ϕ^E *of* ∂E *as*

$$\kappa_\phi^E := \mathrm{div} N_\phi = -\mathrm{div}(T_{\phi^o}(\nabla d_\phi^E)) \quad \text{on } \partial E.$$

Anisotropic mean curvature appears in the first variation of the anisotropic perimeter functional. We recall from [41] the following result.(2)

(2) See [42, 33, 25] for an extension of (1.10) to the case of nonconvex smooth anisotropies.

Theorem 1.10 (**First variation of P_ϕ**)**.** *Let $\psi \in \mathcal{C}^1_0(U;\mathbb{R}^n)$, and, for $\delta > 0$ sufficiently small, let $\Psi_\delta \in \mathcal{C}^1_0(U;\mathbb{R}^n)$ be defined as $\Psi_\delta(z) := z + \delta\psi(z) + o(\delta)$. Then*

$$\frac{d}{d\delta} P_\phi(\Psi_\delta(E))\Big|_{\delta=0} = \int_{\partial E} \omega_n^\phi \kappa_\phi^E (\psi \cdot \nu^E)\ d\mathcal{H}^{n-1}.$$

Moreover, a scalar multiple of $\kappa_\phi^E \nu_{\phi^o}$ is the minimizer of

$$\inf\Big\{ \int_{\partial E} \kappa_\phi^E (g \cdot \nu^E)\ d\mathcal{H}^{n-1}\ :\ g \in L^2(\partial E;\mathbb{R}^n),\ ||g||_{L^2(\partial E)} \le 1 \Big\}.$$

Now, it is well known that the (Euclidean) perimeter functional can be approximated, in the sense of Γ-convergence [75], by a sequence of singularly perturbed elliptic functionals whose gradient flows converge in a suitable sense to (Euclidean) mean curvature flow. This result has been extended to the anisotropic context in [31], and can be stated as follows. Let $\Omega \subset \mathbb{R}^n$ a bounded open set, let $W\colon \mathbb{R} \to [0,+\infty)$ the double-well potential $W(s) := (1-s^2)^2$, and set $f := W'$. For $\epsilon \in (0,1)$, let u_ϵ be a solution of the *anisotropic Allen-Cahn* equation (of reaction-diffusion type)

$$\partial_t u = \epsilon \operatorname{div}(T_{\phi^o}(\nabla u)) - \frac{1}{\epsilon} f(u),$$

coupled with an initial condition $u_\epsilon(0,\cdot) = u_{\epsilon,0}(\cdot)$ and a proper boundary condition.

Theorem 1.11 (**Convergence to anisotropic mean curvature flow**)**.** *Let $T > 0$. For any $t \in [0,T)$, let $E(t) \subset \mathbb{R}^n$ be a compact set of class $\mathcal{C}^2$. Assume that $(E(t))_{t\in[0,T)}$ evolves under ϕ-anisotropic mean curvature flow.*(3) *Then, for any $\epsilon \in (0,1)$, it is possible to build $u_{\epsilon,0}$, depending just on ∂E, such that there exist $\epsilon_0 \in (0,1)$ and $C > 0$ such that, for $\epsilon \in (0,\epsilon_0)$,*

$$\{u_\epsilon(t,\cdot) = 0\} \subseteq \{z \in \Omega\ :\ \operatorname{dist}(z,\partial E(t)) \le C\epsilon^3 |\ln \epsilon|^3\},$$

$$\partial E(t) \subseteq \{z \in \Omega\ :\ \operatorname{dist}(z,\{u_\epsilon(t,\cdot) = 0\}) \le C\epsilon^3 |\ln \epsilon|^3\},$$

for all $t \in [0,T)$.

In other words, as $\epsilon \to 0^+$, the Hausdorff distance between $\{u_\epsilon(t,\cdot) = 0\}$ and $\partial E(t)$ is of order less than or equal to $\epsilon^3 |\ln \epsilon|^3$.

1.3 Star-shaped combination of anisotropies

In this section we introduce the operation of star-shaped combination of m anisotropies ($m \ge 2$), which will play a fundamental role in Chapter 2. Even if we shall deal mainly with the regular case, nevertheless it is natural to define this operation for general star-shaped anisotropies (not even convex).

Let $\mathcal{S}$ be the family of *star bodies*, namely

$$\mathcal{S} := \Big\{ K \subset \mathrm{V}\ :\ K = \overline{\operatorname{int}(K)} \text{ is compact, star-shaped with respect to } 0 \in \operatorname{int}(K) \Big\}.$$

(3) For the purposes of the present thesis, it will be sufficient to consider the following definition of anisotropic mean curvature flow: we say that ∂E evolves by *ϕ-anisotropic mean curvature flow* if

$$\text{velocity} = -\kappa_\phi^E \quad \text{in the direction } n_\phi.$$

We refer the reader to the already mentioned [41], and references therein, for a more detailed discussion. Notice that, by virtue of the minus sign in the previous formula, E is expected to shrink locally around those portions of ∂E where κ_ϕ^E is positive.

Remark 1.12. One can check that

$$\mathcal{S} = \{B_\phi \,:\, \phi\colon \mathrm{V} \to [0, +\infty) \text{ anisotropy on } \mathrm{V}\}\,.$$

Indeed, given $K \in \mathcal{S}$, the function

$$\phi_K(\xi) := \inf\{\lambda > 0 \,:\, \xi \in \lambda K\}, \qquad \xi \in \mathrm{V},$$

is the unique star-shaped anisotropy such that $B_{\phi_K} = K$.[(4)]

We now introduce an operation on star-shaped anisotropies. Making use of Remark 1.12, this will be done working on the family $\mathcal{S}$ of star bodies.

For $K \in \mathcal{S}$, let $\varrho_K : \mathbb{S}^{n-1}_{\mathrm{V}} := \{\xi \in \mathrm{V} \,:\, |\xi| = 1\} \to (0, +\infty)$ be the *radial function* of K (see for instance [152]), defined as

$$\varrho_K(\nu) := \sup\big\{\lambda \geq 0 \,:\, \lambda\nu \in K\big\}, \qquad \nu \in \mathbb{S}^{n-1}_{\mathrm{V}}.$$

The function ϱ_K is extended (keeping the same symbol) in a one-homogeneous way on the whole of V, *i.e.*, $\varrho_K(\xi) = |\xi|\varrho_K(\frac{\xi}{|\xi|})$ for any $\xi \in \mathrm{V} \setminus \{0\}$. Notice that

$$\varrho_K(\nu) = \frac{1}{\phi_K(\nu)}, \qquad \nu \in \mathbb{S}^{n-1}_{\mathrm{V}}, \tag{1.7}$$

and

$$K = \{\lambda\nu \,:\, 0 \leq \lambda \leq \varrho_K(\nu),\ \nu \in \mathbb{S}^{n-1}_{\mathrm{V}}\}.$$

Now, consider $K_1, K_2 \in \mathcal{S}$. We let $\varrho_{K_1} \star \varrho_{K_2} : \mathbb{S}^{n-1}_{\mathrm{V}} \to (0, +\infty)$ be defined as follows [43]:

$$\varrho_{K_1} \star \varrho_{K_2}(\nu) := \sqrt{\Big(\varrho_{K_1}(\nu)\Big)^2 + \Big(\varrho_{K_2}(\nu)\Big)^2}, \qquad \nu \in \mathbb{S}^{n-1}_{\mathrm{V}}.$$

Again, $\varrho_{K_1} \star \varrho_{K_2}$ is extended (keeping the same symbol) in a one-homogeneous way on the whole of V.

Definition 1.13 (Star-shaped combination of two sets). *Given $K_1, K_2 \in \mathcal{S}$, we define the* star-shaped combination

$$K_1 \star K_2$$

of K_1 and K_2 as the set whose radial function coincides with $\varrho_{K_1} \star \varrho_{K_2}$:

$$\varrho_{K_1 \star K_2} := \varrho_{K_1} \star \varrho_{K_2}.$$

One checks that $K_1 \star K_2 \in \mathcal{S}$, and that the identity element for $\star$ does not belong to $\mathcal{S}$. Moreover

$$K_1 \star K_2 = K_2 \star K_1.$$

It is clear that the set $K_1 \star K_2$ depends on K_1 and K_2 and not only on $K_1 \cup K_2$.

---

(4) The function ϕ_K is sometimes called *gauge* of K, see [140, 152] and references therein. When $K \in \mathcal{S}$ is convex (we say that K is a *convex body*), ϕ_K is usually called *Minkowski functional* of K, see for instance [136], and it is obviously a convex anisotropy. There exists also the notion of *dual body* K^o of a set $K \in \mathcal{S}$, which turns out to be nothing but $B_{(\phi_K)^o}$. Incidentally, we mention that $(\phi_K)^o$ is sometimes called *support function* of K.

However, it cannot be viewed as the union of an enlargement of K_1 with an enlargement of K_2.

Next formula gives the concrete way to compute the star-shaped combination of two sets $K_1,\, K_2 \in \mathcal{S}$:

$$\partial\,(K_1 \star K_2) := \left\{ \sqrt{\lambda_1^2 + \lambda_2^2}\,\nu \;:\; \nu \in \mathbb{S}_{\mathrm{V}}^{n-1},\; \lambda_j = \varrho_{K_j}(\nu),\; j = 1, 2 \right\}.$$

Remark 1.14. The reason for using star bodies, instead of convex sets, in Definition 1.13 is the following: if K_1 and K_2 are two convex bodies, then $K_1 \star K_2$ is not in general a convex body. An explicit counterexample for $n = 2$ and $\mathrm{V} = \mathbb{R}^2$ is given in [43], and it involves the two ellipses

$$K_1 := \left\{(x, y) \in \mathbb{R}^2 : x^2 + \rho y^2 = 1\right\}, \qquad K_2 := \left\{(x, y) \in \mathbb{R}^2 : \rho x^2 + y^2 = 1\right\},$$

defined for $\rho > 0$. Then:

(i) $K_1 \star K_2$ is (smooth and) strictly convex, for $\rho \in (\frac{1}{3}, 3)$;

(ii) $K_1 \star K_2$ is (smooth and) convex, for $\rho = \frac{1}{3}$ or $\rho = 3$, with zero boundary curvature at the points of intersection with the lines $\{(x, y) \in \mathbb{R}^2 : x \pm y = 0\}$;

(iii) $K_1 \star K_2$ is (smooth and) not convex, for $\rho < \frac{1}{3}$ or $\rho > 3$.

Figure 1.3 shows the sets K_1, K_2, and $K_1 \star K_2$ when $\rho := 8$. Further cases of interest of star-shaped combinations of (2-dimensional) convex sets are given in Example 1.18 below, and in Appendix A.

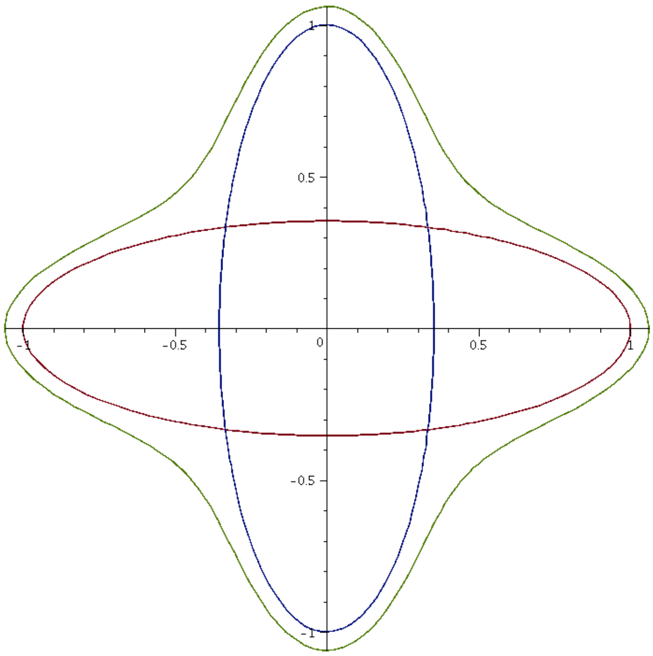


Figure 1.3: The sets K_1 (in red), K_2 (in blue) and their star-shaped combination $K_1 \star K_2$ (in green), defined in Remark 1.14 with the choice $\rho := 8$. Notice that $K_1 \star K_2$ is not convex. The plot has been done using "Maple 16".

Observe that for any K_1, K_2, $K_3 \in \mathcal{S}$ we have:

$$(\varrho_{K_1} \star \varrho_{K_2}) \star \varrho_{K_3} = \varrho_{K_1} \star (\varrho_{K_2} \star \varrho_{K_3}),$$

or equivalently:

$$\varrho_{K_1 \star K_2} \star \varrho_{K_3} = \varrho_{K_1} \star \varrho_{K_2 \star K_3}.$$

This observation leads to the following definition.

Definition 1.15 (Star-shaped combination of m sets). *Given $m \geq 2$ and $K_1, \ldots, K_m \in \mathcal{S}$, we let*

$$\mathop{\star}_{j=1}^{m} \varrho_{K_j}(\nu) := \sqrt{\sum_{j=1}^{m} \big(\varrho_{K_j}(\nu)\big)^2}, \qquad \nu \in \mathbb{S}_{\mathrm{V}}^{n-1}, \tag{1.8}$$

extended (keeping the same symbol) in a one-homogeneous way on the whole of V, and

$$\mathop{\star}_{j=1}^{m} K_j$$

be the set in $\mathcal{S}$ whose radial function is given by $\mathop{\star}_{j=1}^{m} \varrho_{K_j}$.

Again, note that

$$\partial\left(\mathop{\star}_{j=1}^{m} K_j\right) = \left\{\sqrt{\sum_{j=1}^{m} \lambda_j^2}\ \nu \ : \ \nu \in \mathbb{S}^{N-1},\ \lambda_j = \varrho_{K_j}(\nu),\ j = 1, \ldots, m\right\}.$$

Problem 1.16. An open problem is to characterize those sets in $\mathcal{S}$ obtained as star-shaped combination of m symmetric convex bodies, more precisely to characterize the class

$$\left\{\mathop{\star}_{j=1}^{m} K_j \ : \ K_1, \cdots, K_m \text{ smooth symmetric uniformly convex bodies}\right\}.$$

In [43], some necessary conditions are given in the case $m = 2$, such as the impossibility of cusps or re-entrant corners in $\partial(K_1 \star K_2)$.

From (1.7) and (1.8), it follows the formula

$$\left(\phi_{\mathop{\star}_{j=1}^{m} K_j}(\nu)\right)^2 = \left(\sum_{j=1}^{m} \frac{1}{\big(\phi_{K_j}(\nu)\big)^2}\right)^{-1}, \qquad \nu \in \mathbb{S}_{\mathrm{V}}^{n-1}, \tag{1.9}$$

According to (1.9), we now give the following definition.

Definition 1.17 (Combined anisotropy). *Let $m \geq 2$ and let be given m anisotropies $\phi_1, \ldots, \phi_m \colon \mathrm{V} \to [0, +\infty)$. The function*

$$\mathop{\star}_{j=1}^{m} \phi_j := \left(\sum_{j=1}^{m} \frac{1}{\phi_j^2}\right)^{-1/2} \tag{1.10}$$

will be called the star-shaped combination *of $\phi_1, \ldots, \phi_m$, or* combined anisotropy *for short.*

Example 1.18. We give a counterexample showing that, in general, convexity is not preserved under star-shaped combination of a convex anisotropy with the Euclidean one ϕ_{Eucl}.

Let $n = 2$, and let $\mathrm{V} = \mathbb{R}^2$. With slight abuse of notation, for any anisotropy ϕ of $\mathbb{R}^2$, we set $\phi(\theta) := \phi\big((\cos\theta, \sin\theta)\big)$, and $\varrho_{B_\phi}(\theta) := \varrho_{B_\phi}\big((\cos\theta, \sin\theta)\big)$for $\theta \in [0, 2\pi)$. We recall that, if B_ϕ is of class $\mathcal{C}^2$, then it is convex if and only if

$$\phi(\theta) + \phi''(\theta) \geq 0. \tag{1.11}$$

Now, let a, $b > 0$, and let $\varphi \in \mathcal{M}(\mathbb{R}^2)$ be the convex anisotropy such that

$$B_\varphi = \{(x,y) \in \mathbb{R}^2 \ : \ |x| \leq a, \ |y| \leq b\} \cup \{(x,y) \in \mathbb{R}^2 \ : \ (x \pm a)^2 + y^2 \leq b^2\}.$$

Let also A denote the collection of all angles $\theta \in (0, \frac{\pi}{2})$ such that the line $\{(x,y) \in \mathbb{R}^2 \ : \ x\sin\theta = b\cos\theta\}$ intersects the flat portions of the boundary of B_φ. Clearly, A depends on the choice of a and b; in particular, for fixed $b > 0$, A tends to the whole interval $(0, \frac{\pi}{2})$ as $a \to +\infty$.

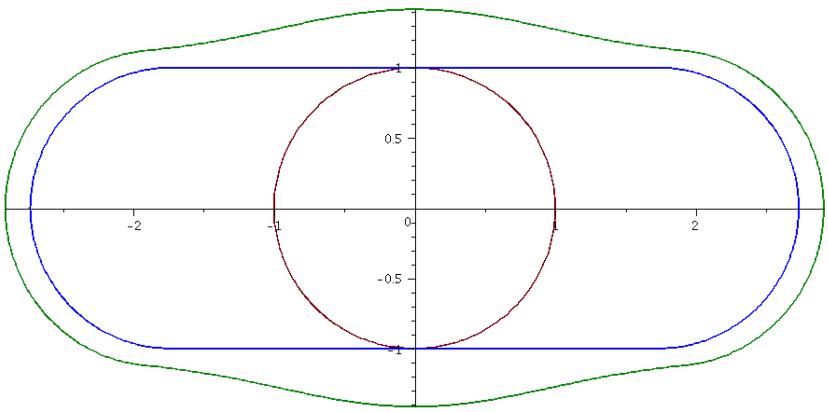


Figure 1.4: The boundaries of $B_{\phi_{\mathrm{Eucl}}}$ (red line), and of B_φ (blue line), defined in Example 1.18, with the choice $a := \sqrt{3}$, $b := 1$. The boundary of the unit ball B_Φ of the combined anisotropy $\Phi = (\varphi \star \phi_{\mathrm{Eucl}})$ is shown in green. Notice that B_Φ is not convex. The plot has been done using “Maple 16”.

We have

$$\varrho_{B_\varphi}(\theta) = \frac{b}{\sin\theta}, \qquad \theta \in A.$$

Let $\Phi := (\varphi \star \phi_{\mathrm{Eucl}})$. Then, by (1.8), we have

$$\varrho_{B_\Phi}(\theta) = \frac{\sqrt{\sin^2\theta + b^2}}{\sin\theta}, \qquad \theta \in A,$$

and so, by (1.7) and (1.9),

$$\Phi(\theta) = \frac{\sin\theta}{\sqrt{\sin^2\theta + b^2}}, \qquad \theta \in A.$$

One can check that

$$\Phi''(\theta) = -\frac{b^2 \sin\theta}{(\sin^2\theta + b^2)^{\frac{5}{2}}}(b^2 + 3 - 2\sin^2\theta), \qquad \theta \in A,$$

so that, in particular,

$$\Phi(\theta) + \Phi''(\theta) = \frac{\sin\theta\big((\sin^2\theta + b^2)^2 - b^2(b^2 + 3 - 2\sin^2\theta)\big)}{(\sin^2\theta + b^2)^{\frac{5}{2}}}, \qquad \theta \in A. \tag{1.12}$$

From (1.12), also recalling (1.11), a necessary condition for B_Φ to be convex is that

$$(\sin^2\theta + b^2)^2 - b^2(b^2 + 3 - 2\sin^2\theta) \geq 0, \qquad \theta \in A,$$

or, equivalently,

$$\sin^4\theta + 4b^2\sin^2\theta - 3b^2 \geq 0, \qquad \theta \in A. \tag{1.13}$$

Notice that the inequality in (1.13) does not depend on the choice of a, and it is violated as $\theta \to 0^+$. Hence, for fixed $b > 0$, it is possible to take $a > 0$ large enough so that (1.13) is not valid for some $\theta \in A$. See Figure 1.18 for the choice $a := \sqrt{3}$ and $b := 1$. In this case, $A = (\frac{\pi}{6}, \frac{\pi}{2})$ and condition (1.13) is violated at all $\theta \in \left(\frac{\pi}{6}, \arcsin\left(\sqrt{\sqrt{7} - 2}\right)\right)$.

1.3.1 On the hessian of the combined anisotropy

Let be given m star-shaped anisotropies $\phi_1, \ldots, \phi_m : \mathrm{V}^* \to [0, +\infty)$,[5] such that ϕ_j^2 is of class $\mathcal{C}^2$ for every $j = 1, \ldots, m$. Let $\Phi := \star_{j=1}^m \phi_j$ be their combined anisotropy. By (1.10), it immediately follows that also Φ^2 is of class $\mathcal{C}^2$. The aim of this short section is to find an appropriate representation of the hessian

$$\frac{1}{2}\nabla^2\Phi^2$$

of Φ^2, which will be useful in Section 2.3.6.

First of all, set for notational convenience

$$\alpha := \Phi^2, \qquad \alpha_j := \phi_j^2, \quad j = 1, \ldots, m.$$

Then formula (1.10) can be rewritten as

$$\alpha = \left(\sum_{j=1}^m \frac{1}{\alpha_j}\right)^{-1}. \tag{1.14}$$

Differentiating (1.14), we get

$$\nabla\alpha = \alpha^2 \sum_{j=1}^m \frac{1}{\alpha_j^2}\nabla\alpha_j.$$

(5) We find more convenient to let the ϕ_j's be defined in the space of covectors, in order to avoid conflicts of notation wih Section 2.3 (where the result of this section wll be applied).

and differentiating again, we end up with

$$\begin{aligned}\frac{1}{2}\nabla^2\alpha =&\alpha^3\left(\sum_{j=1}^{m}\frac{1}{\alpha_j^2}\nabla\alpha_j\right)\otimes\left(\sum_{k=1}^{m}\frac{1}{\alpha_k^2}\nabla\alpha_k\right)\\&-\alpha^2\sum_{j=1}^{m}\frac{1}{\alpha_j^2}\nabla\alpha_j\otimes\nabla\alpha_j+\frac{1}{2}\alpha^2\sum_{j=1}^{m}\frac{1}{\alpha_j^2}\nabla^2\alpha_j,\end{aligned}\tag{1.15}$$

where the tensor product $\eta^*\otimes\zeta^*$ between $\eta^*,\zeta^*\in \mathrm{V}^*\cong(\mathbb{R}^n)^*$ is defined as

$$(\eta^*\otimes\zeta^*)_{j,k}:=\eta_j^*\zeta_k^*,\qquad j,k=1,\dots,n.$$

Set

$$Q:=\frac{1}{2}\alpha^2\sum_{j=1}^{m}\frac{1}{\alpha_j^2}\nabla^2\alpha_j,\tag{1.16}$$

and

$$Q_0:=\frac{1}{2}\nabla^2\alpha-Q.$$

From (1.15) and (1.16), we obtain

$$\begin{aligned}Q_0&=\sum_{j=1}^{m}\left(\frac{\alpha^3}{\alpha_j^4}-\frac{\alpha^2}{\alpha_j^3}\right)\nabla\alpha_j\otimes\nabla\alpha_j+\sum_{\substack{j,k=1,\\ j\neq k}}^{m}\frac{\alpha^3}{\alpha_j^2\alpha_k^2}\ \nabla\alpha_j\otimes\nabla\alpha_k\\&=\alpha^2\sum_{j=1}^{m}\frac{\alpha-\alpha_j}{\alpha_j^4}\ \nabla\alpha_j\otimes\nabla\alpha_j+\alpha^3\sum_{\substack{j,k=1,\\ j\neq k}}^{m}\frac{1}{\alpha_j^2\alpha_k^2}\ \nabla\alpha_j\otimes\nabla\alpha_k.\end{aligned}\tag{1.17}$$

For $m=2$, formulas (1.16) and (1.17) coincide with those given in [43]. Furthermore, we can observe that, as in the case $m=2$, we have

$$Q_0(\xi^*)\xi^*=0,\qquad \xi^*\in\mathrm{V}^*.\tag{1.18}$$

This relation will be used in the asymptotics, see Section 2.3.6. In order to show (1.18) we use Euler's formula $\nabla\alpha_j(\xi^*)\xi^*=2\alpha_j(\xi^*)$. We have

$$\begin{aligned}\frac{1}{2}Q_0(\xi^*)\xi^* =&\alpha^2(\xi^*)\sum_{j=1}^{m}\frac{\alpha(\xi^*)-\alpha_j(\xi^*)}{(\alpha_j(\xi^*))^4}\ \alpha_j(\xi^*)\nabla\alpha_j(\xi^*)\\&+\alpha^3(\xi^*)\sum_{\substack{j,k=1,\\ j\neq k}}^{m}\frac{1}{(\alpha_k(\xi^*))^2(\alpha_j(\xi^*))^2}\ \alpha_k(\xi^*)\nabla\alpha_j(\xi^*)\\=&\sum_{j=1}^{m}\left[\frac{\alpha^2(\xi^*)\Big(\alpha(\xi^*)-\alpha_j(\xi^*)\Big)}{(\alpha_j(\xi^*))^3}+\frac{\alpha^3(\xi^*)}{(\alpha_j(\xi^*))^2}\sum_{\substack{k=1,\\ k\neq j}}^{m}\frac{1}{\alpha_k(\xi^*)}\right]\nabla\alpha_j(\xi^*),\end{aligned}$$

and each terms in the summation leads (recalling (1.14) and omitting the symbol ξ^*) to

$$\frac{\alpha^2}{\alpha_j^2}\left[\frac{\alpha-\alpha_j}{\alpha_j}+\alpha\Big(\frac{1}{\alpha}-\frac{1}{\alpha_j}\Big)\right]=0,\qquad j=1,\dots,m.$$

Using (1.16) and (1.17) we have therefore obtained a representation for

$$\frac{1}{2}\nabla^2\alpha=Q+Q_0.$$

1.4 The bidomain model

We now turn to the formulation of bidomain model. This section aims to provide the starting motivation for the study of nonlinear multidomain model which will be the subject of the next chapter.

The bidomain model is a standard model in electrocardiology, originally introduced in the '70s (see for instance [82, 153], or also the more recent monograph [71]) as an attempt to describe the averaged electric potentials which govern the heart beating. Let $\Omega \subset \mathbb{R}^n$, $n = 3$, be the bounded connected open set which corresponds to the physical region occupied by the heart. Despite its underlying cellular discrete structure, at a macroscopic level it is useful to think of Ω as a continuous superimposed domain. In this scheme, the intra (i) and extra (e) cellular electric potentials $u_{\mathrm{i,e}}$ are defined on the whole of Ω, and they are associated to the current densities

$$-M_{\mathrm{i}}\nabla u_{\mathrm{i}}, \qquad -M_{\mathrm{e}}\nabla u_{\mathrm{e}}, \tag{1.19}$$

where $M_{\mathrm{i,e}}$ are the conductivity tensors, and they are symmetric, positive-definite matrices, continuously depending on the position. The presence of $M_{\mathrm{i,e}}$ is related to the fibered structure of the cardiac tissue, since the resistance of the cellular membrane is significantly higher than at the intracellular connections; at the macroscopic level, these difference are in some sense "averaged", leading to a strong anisotropy factor in the model. Disregarding possible induction effects, the quantities in (1.19) have to satisfy the conservation law

$$\operatorname{div}(M_{\mathrm{i}}\nabla u_{\mathrm{i}}) = -\operatorname{div}(M_{\mathrm{e}}\nabla u_{\mathrm{e}}) = i_{\mathrm{m}} \qquad \text{in } \Omega,$$

where $i_{\mathrm{m}} := C_{\mathrm{m}}\partial_t u + i_{\mathrm{ion}}$ denotes the membrane current density, which consists of a capacitance[(6)] $C_{\mathrm{m}}\partial_t u$ and a ionic term i_{ion}. It is in general quite hard to describe the ionic source density i_{ion}: several addictional gating variables are needed to model the ionic channels' dynamics, each of whom is related to the transmembrane potential $u := u_{\mathrm{i}} - u_{\mathrm{e}}$ through a nonlinear, first order ODE. To gain general insight into the wave propagation in the cardiac excitable media, it is possible to consider the simplified situation (also known as FitzHugh-Nagumo approximation) of a single gating variable $w\colon \Omega \to \mathbb{R}$, which has to satisfy

$$\partial_t w = \beta u - \gamma w,$$

for suitable constants $\beta, \gamma > 0$. Then, the ionic density is given by

$$i_{\mathrm{ion}} = f(u) + \eta w, \tag{1.20}$$

where $\eta > 0$ is a suitable constant, and $f \in \mathcal{C}^1(\mathbb{R})$ is a cubic-like function having only s_- and s_+ as stable zeroes. A standard assumption, which we shall adopt in Section 2.3, is

$$f = W',$$

$W\colon \mathbb{R} \to [0, +\infty)$ being the double-well potential $W(s) := (1 - s^2)^2$.

In the forthcoming discussion, we shall be mainly interested in a large-scale qualitative behaviour of the electric impulses during the depolarization phase.

[(6)]The positive coefficient C_{m} has the physical interpretation of a surface membrane capacitance.

Since the gating variable w plays the main role during the repolarization phase, we simply discard it from (1.20), thus getting

$$\begin{cases} C_{\mathrm{m}}\partial_t u - \operatorname{div}(M_{\mathrm{i}}\nabla u_{\mathrm{i}}) + f(u) = 0, \\ C_{\mathrm{m}}\partial_t u + \operatorname{div}(M_{\mathrm{e}}\nabla u_{\mathrm{e}}) + f(u) = 0, \\ u = u_{\mathrm{i}} - u_{\mathrm{e}}. \end{cases} \tag{1.21}$$

System (1.21) is then rescaled [70] as follows

$$\begin{cases} \epsilon\partial_t u - \epsilon^2\operatorname{div}(M_{\mathrm{i}}\nabla u_{\mathrm{i}}) + f(u) = 0, \\ \epsilon\partial_t u + \epsilon^2\operatorname{div}(M_{\mathrm{e}}\nabla u_{\mathrm{e}}) + f(u) = 0, \\ u = u_{\mathrm{i}} - u_{\mathrm{e}}, \end{cases} \tag{1.22}$$

where $\epsilon > 0$ is a small nondimensional parameter. Roughly speaking, the rescaling procedure is meant to approximate the propagating transition front realizing the depolarization with a real discontinuity surface: indeed, the activation process takes place in a thin layer (also known as the excitation wavefront), typically 1 mm thick, and moving across distances of 1 cm, see [70] and references therein.

Given $T > 0$, system (1.22) is studied for $(t, x) \in (0, T) \times \Omega$, and is coupled with an initial condition

$$u(0, \cdot) = u_0(\cdot) \qquad \text{in } \Omega, \tag{1.23}$$

and two Neumann boundary conditions

$$M_{\mathrm{i,e}}\nabla u_{\mathrm{i,e}} \cdot \nu^{\Omega} = 0 \qquad \text{on } (0, T) \times \partial\Omega. \tag{1.24}$$

Conditions (1.23)-(1.24) are better understood observing that system (1.22) is equivalent to the following parabolic/elliptic system:

$$\begin{cases} \epsilon\partial_t u - \epsilon^2\operatorname{div}(M_{\mathrm{i}}\nabla u_{\mathrm{i}}) + f(u) = 0, \\ \operatorname{div}\big(M_{\mathrm{i}}\nabla u_{\mathrm{i}} + M_{\mathrm{e}}(\nabla u_{\mathrm{i}} - \nabla u)\big) = 0, \end{cases} \tag{1.25}$$

obtained by taking the difference of the two equations in (1.22).

A "degenerate" situation to study system (1.22) corresponds to the so-called *equal anisotropic ratio*, namely when

$$M_{\mathrm{e}} = \lambda M_{\mathrm{i}}, \tag{1.26}$$

for some $\lambda > 0$. In this setting, system (1.22) can be reduced to the following equation, of anisotropic Allen-Cahn type:

$$\epsilon\partial_t u - \epsilon^2\frac{\lambda}{1+\lambda}\operatorname{div}(M_{\mathrm{i}}\nabla u) + f(u) = 0. \tag{1.27}$$

The anisotropy governing (1.27) can be seen as a combined anisotropy (see Remark 2.2 below). Nevertheless, assumption (1.26) seems not to be physiological, as it follows from well-established experimental evidence (for instance, cardiac defribillation cannot be modelled through the single equation (1.27), see [112]).

Coming back to system (1.22), we recall from [73] the following result, which has been proven in a more abstract setting using a degenerate formulation of bidomain model.[(7)]

[(7)] The well-posedness result in [73] is given in a more general statement. Nevertheless, we have preferred to state it as in Theorem 1.19, since this will be the formulation we shall generalize in the next chapter.

Theorem 1.19 (Well-posedness in the linear case). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $T > 0$, and let $u_0 \in H^1(\Omega)$ be such that $u_0 f(u_0) \in L^1(\Omega)$. Then there exists a pair*

$$(u_\mathrm{i}, u_\mathrm{e}) \in \big(L^2\big(0, T; H^1(\Omega)\big)\big)^2,$$

uniquely determined up to a family of additive time-dependent constants, with

$$u := u_\mathrm{i} - u_\mathrm{e} \in H^1\big(0, T; L^2(\Omega)\big) \cap L^\infty\big(0, T; H^1(\Omega)\big),$$

and such that $(u_\mathrm{i}, u_\mathrm{e})$ solves system (1.22) in $(H^1(\Omega))^*$, with initial/boundary conditions (1.23)-(1.24).

To our best knowledge, a well-posedness result for the case of nonlinear anisotropies (even if independent of the position) has not been given so far. In Section 2.2, adapting the original proof in [73], we shall extend Theorem 1.19 to the nonlinear multidomain model.

In the remaining of this section, we list some results concerning the bidomain model. First of all, we notice that bidomain model admits a variational formulation. Indeed, let us consider the family of functionals defined, for $v, w \in H^1(\Omega)$, as

$$\mathcal{F}_\epsilon\big(v, w\big) := \int_\Omega \left\{ \frac{\epsilon}{2}\Big[M_\mathrm{i}\nabla v \cdot \nabla v + M_\mathrm{e}\nabla w \cdot \nabla w\Big] + \frac{1}{\epsilon}W(v - w)\right\}\, dx, \tag{1.28}$$

and extended to $+\infty$ elsewhere in $(L^2(\Omega))^2$; then, system (1.22) is the formal gradient flow of the functionals $\mathcal{F}_\epsilon$ with respect to the degenerate scalar product of $(L^2(\Omega))^2$

$$b\Big((v,\, w), (\tilde{v}, \tilde{w})\Big) := \int_\Omega (v - w)(\tilde{v} - \tilde{w})\; dx,$$

namely

$$b\Big(\partial_t(u_\mathrm{i}, u_\mathrm{e}), (v, w)\Big) + \delta\mathcal{F}_\epsilon\Big((u_\mathrm{i}, u_\mathrm{e}),\, (v, w)\Big) = 0, \quad (v, w) \in (H^1(\Omega))^2.$$

The following result has been obtained in [11].

Theorem 1.20 (Γ-convergence in the linear case). *There exists the $\Gamma\big((L^2(\Omega))^2\big)-\lim_{\epsilon\to 0^+}\mathcal{F}_\epsilon = \mathcal{F}$, and depends only on $\mathrm{u} = v - w$. Moreover, $\mathcal{F}(u)$ is finite if and only if $\mathrm{u} \in BV(\Omega; \{s_\pm\})$; for such functions, we have*

$$\mathcal{F}(v, w) = \int_{J_\mathrm{u}} \sigma(x,\, \nu_\mathrm{u}(x))\; d\mathcal{H}^{n-1}(x),$$

where J_u is the jump set of u, $\nu_\mathrm{u}(x)$ is a unit normal to J_u at $x \in J_\mathrm{u}$, and σ is a convex inhomogeneous symmetric anisotropy.

It is also possible to explicitly characterize $\sigma(x, \cdot)$ as an infimum of an appropriate class of vector-valued functions, see [11] for the details. In particular. when M_i and M_e (and hence σ) are independent of x, we can estimate σ as follows. Let $\phi_{\mathrm{i,e}}$ denote the square root of the quadratic forms associated with $M_{\mathrm{i,e}}$, and let $\Phi := \phi_\mathrm{i} \star \phi_\mathrm{e}$ be their star-shaped combination (Definition 1.17). Recall that, in general, Φ is allowed to be nonconvex. Then:

- $\{\sigma \leq 1\}$ contains the convexified of $\{\Phi \leq 1\}$;

- $\{\sigma \leq 1\}$ is contained in the smallest ellipsoid circumscribing the convexified of $\{\Phi \leq 1\}$ and tangent to it at the intersection with the coordinate axes. Moreover, the strict inclusion holds whenever the two anisotropies are not proportional.

The following problem has been pointed out in [11].

Problem 1.21. Is it true that the unit ball of σ coincides with the convexified of $\{\Phi \leq 1\}$?

Problem 1.21 seems to be related with the next *formal* result, obtained in [31] using an asymptotic expansion argument developed up to the second order included. Again, let us make use of the identification $\mathrm{V} \cong \mathbb{R}^n$ in order to distinguish between anisotropies defined on vectors or on covectors.

Theorem 1.22 (Formal convergence in the linear case). *Let $u_{\mathrm{i}} = u_{\mathrm{i},\epsilon}$, $u_{\mathrm{e}} = u_{\mathrm{i},\epsilon}$ and $u = u_\epsilon$ be given by Theorem 1.19, with initial condition $u_\epsilon(0,\cdot)$ well-prepared*(8) *and possibly depending on ϵ, in particular so that*

$$\{x \in \Omega : u_\epsilon(0,x) = 0\} = \partial E, \qquad \epsilon \in (0,1),$$

where ∂E is smooth and compact in Ω. Suppose furthermore that

$$\Phi \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V}^*). \tag{1.29}$$

Then, for $T > 0$ sufficiently small, the sets $\{u_\epsilon(t,\cdot) = 0\}$ formally converge as $\epsilon \to 0^+$,(9) *to a hypersurface $\partial E(t)$ evolving by anisotropic Φ^o-mean curvature with $\partial E(0) = \partial E$, for any $t \in [0,T]$.*

Theorem 1.22 has been generalized in [43] to the case of $\phi_{\mathrm{i,e}} \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V}^*)$ — namely, dropping the linearity of the original anisotropies. In this case, the current densities in (1.19) have to be replaced by

$$T_{\phi_{\mathrm{i}}}(\nabla u_{\mathrm{i}}), \qquad T_{\phi_{\mathrm{e}}}(\nabla u_{\mathrm{e}}),$$

where, recalling Definition 1.7, the operators $T_{\phi_{\mathrm{i,e}}} = \frac{1}{2}\nabla(\phi_{\mathrm{i,e}})^2$ are allowed to be possibly nonlinear. Theorem 1.23 below has been stated for a different time-scaling, and the asymptotic expansion argument has been developed just up to the first order included.

Theorem 1.23. *Let $u_{\mathrm{i}} = u_{\mathrm{i},\epsilon}$, $u_{\mathrm{e}} = u_{\mathrm{e},\epsilon}$ and $u = u_\epsilon$ solve the following system*

$$\begin{cases} \epsilon^2\, \partial_t u - \epsilon^2 \mathrm{div}(T_{\phi_{\mathrm{i}}}(\nabla u_{\mathrm{i}})) + f(u) = 0, \\ \epsilon^2\, \partial_t u + \epsilon^2 \mathrm{div}(T_{\phi_{\mathrm{e}}}(\nabla u_{\mathrm{e}})) + f(u) = 0, \qquad \text{in } (0,T) \times \Omega. \\ u = u_{\mathrm{i}} - u_{\mathrm{e}}, \end{cases} \tag{1.30}$$

Then, under assumption (1.29)*, the same conclusion of Theorem 1.22 holds.*

We stress that a rigourous mathematical justification of Theorems 1.22-1.23 is still missing, which seems a nontrivial goal to reach due to the lack of maximum principle. Perhaps, a strategy for the proof could try to follow the work in [78],

(8) See [31] for the details.
(9) With an expected speed rate of order ϵ, up to logarithmic corrections.

[79] (see also [1], [139] for a different evolutive problem): here, the authors seek for an approximated solution *via* asymptotic expansion, then showing the convergence to the real solution by means of proper spectral estimates (see also [69]). The relevant efforts stand in the setting up of a refined algorithm for inductively retrieving all terms in the asymptotic expansions just from the evolutive equation. In this respect, it seems nontrivial to repeat this strategy to bidomain model: indeed (Remark 2.15), already determining the 0-order terms in the asymptotic expansion is still an open problem, which deserves further investigation.

We end this section by mentioning the recent paper [127], where stability of the propagating wavefront of bidomain model (here, $n = 2$ and $\Omega = \mathbb{R}^2$) is studied depending on the shape of the combined anisotropy Φ. In particular, if Φ is not convex, then planar fronts are unstable, this giving a possible theoretical explanation of the wrinkling phenomenon appeared in the numerical experiments in [43]. The following conjecture is addressed in the same paper [127].

Problem 1.24. *Suppose that the planar front is stable along all directions where Φ and its convexified coincide. Then the asymptotic shape of the propagating front is given by the unit ball of Φ^o.*

Chapter 2

The nonlinear multidomain model

Summary. In Section 2.1, we introduce nonlinear multidomain model. A well posedness result is given in Section 2.2, while Section 2.3 contains the asymptotic analysis of nonlinear multidomain model, developed up to the second order included, and the formal convergence result of the zero-level set of the solutions to a suitable anisotropic mean curvature flow.

2.1 Formulation of the model

Let $m, n \in \mathbb{N}$, with $m, n \geq 2$. Let $\mathrm{V} := \mathbb{R}^n$, and let $\phi_1, \dots, \phi_m \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V}^*)$. Let also $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary.

Let $W \in \mathcal{C}^2(\mathbb{R})$, $W \geq 0$, and let $f := W'$. We shall assume that $f(0) = 0$, and that there exists $C_f > 0$ such that

$$f' \geq C_f. \tag{2.1}$$

Sometimes we shall also require that

$$0 < \liminf_{|s| \to +\infty} \frac{W(s)}{|s|^4} \leq \liminf_{|s| \to +\infty} \frac{W(s)}{|s|^4} < +\infty, \tag{2.2}$$

and

$$0 < \liminf_{|s| \to +\infty} \frac{f(s)}{|s|^3} \leq \liminf_{|s| \to +\infty} \frac{f(s)}{|s|^3} < +\infty, \tag{2.3}$$

namely W (resp. f) has quartic-like (resp. cubic-like) growth at infinity.

Definition 2.1 (Nonlinear multidomain model). *We call* nonlinear multidomain model *the following degenerate system of parabolic PDE's*

$$\begin{cases} \epsilon^2 \partial_t u - \epsilon^2 \mathrm{div}\left(T_{\phi_r}(\nabla w^r)\right) + f(u) = 0, \qquad r = 1, \dots, m, \\ \\ u = \displaystyle\sum_{r=1}^{m} w^r, \end{cases} \tag{2.4}$$

in the unknown $(w^1, \dots, w^m) \in \left(H^1(0, T; \Omega)\right)^m$, *where* $T_{\phi_r} := \frac{1}{2} \nabla_{\xi^*} \phi_r^2$ *is allowed to be nonlinear, for every* $r = 1, \dots, m$.[(1)]

Notice that bidomain model (1.30) corresponds to the choice $m := 2$, $\phi_1 := \phi_{\mathrm{i}}$, $\phi_2 := \phi_{\mathrm{e}}$, $w^1 := u_{\mathrm{i}}$ and $w^2 := -u_{\mathrm{e}}$.

[(1)] No summation on the index r is obviously understood in (2.4).

Remembering (1.25), system (2.4) is equivalent to

$$\begin{cases} \epsilon^2 \partial_t u - \epsilon^2 \mathrm{div}\left(T_{\phi_1}\left(\nabla w^1\right)\right) + f(u) = 0, \\ \\ \mathrm{div}\left(T_{\phi_1}\left(\nabla w^1\right)\right) = \mathrm{div}\left(T_{\phi_s}\left(\nabla w^s\right)\right), \quad s = 2, \dots, m, \\ \\ u = \displaystyle\sum_{r=1}^{m} w^r, \end{cases}$$

so that we are suggested to couple (2.4) with an initial condition and m Neumann boundary conditions

$$T_{\phi_r}(\nabla w^r) \cdot \nu^\Omega = 0 \quad \text{on } \partial\Omega, \quad r = 1, \dots, m. \tag{2.5}$$

Remark 2.2 (Simplest possible case). Assume that, for $r = 1, \dots, m$, there exist $\lambda_r > 0$ such that $\phi_r = \lambda_r \phi$, for some $\phi \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V}^*)$. Set $T_\phi := \frac{1}{2}\nabla\phi^2$. Then, system (2.4) can be rewritten as

$$\begin{cases} \epsilon^2 \partial_t u - \epsilon^2 \lambda_r^2 \mathrm{div}\Big(T_\phi\big(\nabla w^r\big)\Big) + f(u) = 0, \quad r = 1, \dots, m, \\ \\ u = \displaystyle\sum_{r=1}^{m} w^r. \end{cases} \tag{2.6}$$

Suppose also that ϕ is a linear anisotropy, so that

$$\mathrm{div}\left(T_\phi(\nabla u)\right) = \sum_{r=1}^{m} \mathrm{div}\left(T_\phi(\nabla w^r)\right).$$

Dividing each parabolic equation in (2.4) by λ_r^2, summing over $r = 1, \dots, m$, and dividing by $\sum_{r=1}^{m} \frac{1}{\lambda_r^2}$, we obtain

$$\epsilon^2 \partial_t u - \epsilon^2 \left(\sum_{r=1}^{m} \frac{1}{\lambda_r^2} \right)^{-1} \mathrm{div}\big(T_\phi(\nabla u)\big) + f(u) = 0.$$

Hence, by formula (1.10) u satisfies a scalar anisotropic Allen-Cahn's equation, where we take as anisotropy the star-shaped combination Φ of the original anisotropies, namely

$$\epsilon^2 \partial_t u - \epsilon^2 \mathrm{div}\left(T_\Phi(\nabla u)\right) + \frac{1}{\epsilon} f(u) = 0 \tag{2.7}$$

where as usual $T_\Phi := \frac{1}{2}\nabla\Phi^2$. Under the previous assumptions, we summarize this more precisely as follows. Let u_0 be a suitable function defined on Ω. If $(w^1, \dots, w^m)$ solves (2.6) with an initial condition $\sum_{r=1}^{m} w^r = u_0$ and m Neumann boundary conditions (2.5), then $u := \sum_{r=1}^{m} w^r$ solves (2.7), with initial condition $u = u_0$, and Neumann boundary condition

$$T_\Phi(\nabla u) \cdot \nu^\Omega = 0. \tag{2.8}$$

Conversely, let u solve (2.7) with initial condition $u = u_0$ and Neumann boundary condition (2.8). Then, we get a solution to system (2.6), by taking

$$w^r := \frac{1}{\lambda_r^2} \left(\sum_{s=1}^{m} \frac{1}{\lambda_s^2} \right)^{-1} u, \qquad r = 1, \dots, m.$$

With this choice, conditions (2.5) are automatically satisfied.

2.2 Well-posedness of nonlinear multidomain model

In this section we give an existence result for the nonlinear multidomain model (2.4), for small times and in a suitable weak sense. The author wishes to thank Prof. G. Savaré for the useful advices.

In what follows, we are not interested in considering the parameter ϵ in (2.4), so for simplicity we fix $\epsilon = 1$. We shall consider the Hilbert triple

$$H^1(\Omega) \hookrightarrow L^2(\Omega) = \big(L^2(\Omega)\big)^* \hookrightarrow \big(H^1(\Omega)\big)^*,$$

where previous inclusions are continuous and dense (see for instance [57, p.136]) We will adopt the notation ${}_{(H^1(\Omega))^*}\langle \cdot, \cdot \rangle_{H^1(\Omega)}$ for the duality between $\big(H^1(\Omega)\big)^*$ and $H^1(\Omega)$. Whenever $l \in L^2(\Omega)$ and $v \in H^1(\Omega)$, then duality reduces to

$${}_{(H^1(\Omega))^*}\langle l, v \rangle_{H^1(\Omega)} = \int_\Omega lv\, dx.$$

The aim of this section is to prove the following result.

Theorem 2.3 (Well-posedness of nonlinear multidomain model). *Let $u_0 \in H^1(\Omega)$ be such that $W(u_0) \in L^1(\Omega)$. Then, there exists $\mathbf{w} = (w^1, \ldots, w^m) \in \big(L^2(0,T; H^1(\Omega))\big)^m$ such that, letting $u := \sum_{r=1}^m w^r$, we have*

$$u \in H^1\big(0,T; L^2(\Omega)\big) \cap L^\infty\big(0,T; H^1(\Omega)\big), \qquad u(0,\cdot) = u_0(\cdot), \tag{2.9}$$

and, for $r = 1, \ldots, m$, and for a.e. $t \in (0,T)$,

$$\int_\Omega \partial_t u(t)\, v\, dx + \int_\Omega T_{\phi_r}(\nabla w^r(t)) \cdot \nabla v\, dx + \int_\Omega f(u(t)) v\, dx = 0, \tag{2.10}$$

for all $v \in H^1(\Omega)$. Moreover, $\mathbf{w}$ is uniquely defined up to a map $\mathbf{c} := (c^1, \ldots, c^m) \in (L^2(0,T))^m$ such that $\sum_{r=1}^m c^r = 0$.

Let u be given by Theorem 2.3. Assume also that $f(u(t)) \in L^2(\Omega)$.(2) Then (2.10), (2.9) and (2.1) imply that, for $r = 1, \ldots, m$ and for a.e. $t \in (0,T)$,

$$\operatorname{div}\big(T_{\phi_r}(\nabla w^r(t))\big) = \big(\partial_t u(t) + f(u(t))\big) \in L^2(\Omega). \tag{2.11}$$

In particular (see for instance [92]), there exists the normal trace of T_{ϕ_r}, seen as an element of the dual of $H^1(\partial\Omega)$; moreover, due to (2.11), it has to be zero, this giving a suitable weak sense for the Neumann boundary conditions (2.5).

Theorem 2.3 will be a consequence of Propositions 2.8 and 2.9 below. We recall also [142, Proposition 1.2, p.106] that

$$H^1\big(0,T; L^2(\Omega)\big) \subset \mathcal{C}^0\big(0,T; L^2(\Omega)\big),$$

so that the solution provided by the above mentioned results satisfies the initial condition in (2.9).

Let us start with some preliminary results. First of all, we reformulate (2.4) as a singular nonlinear evolution equation in the unknown u. To this aim, we introduce(3) the functional $\mathcal{G}\colon H^1(\Omega) \to [0, +\infty)$, defined as

$$\mathcal{G}(u) := \inf\left\{ g(\mathbf{w}) \;\middle|\; \mathbf{w} := (w^1, \ldots, w^m) \in \mathrm{Adm}(u) \right\},$$

(2)This happens, for instance, when $f(s) := \frac{d}{ds}(1-s^2)^2$, and $|u_0| \le 1$. In this case, a truncation argument shows that $|u(t)| \le 1$ for every $t \in (0,T)$.

(3)When $m = 2$, and ϕ_1, ϕ_2 are linear anisotropies — namely, in the standard bidomain model — the analog of functional $\mathcal{G}$ has been introduced in [11] in a Γ-convergence framework (see also [73]).

where

$$g(\mathbf{w}) := \frac{1}{2}\int_\Omega \sum_{r=1}^m \phi_r^2(\nabla w^r)\,dx,$$

and, for $u \in H^1(\Omega)$, we set

$$\mathrm{Adm}(u) := \Big\{(w^1,\dots,w^m) \in (H^1(\Omega))^m \ : \ \sum_{r=1}^m w^r = u\Big\}.$$

It is immediate to verify that $\mathcal{G}(u) < +\infty$ for every $u \in H^1(\Omega)$ (see Lemma 2.4 below). We collect in Lemmata 2.4, 2.5 and 2.7 some results concerning $\mathcal{G}$. We shall adopt the notation

$$\overline{u} := \frac{1}{|\Omega|}\int_\Omega u\,dx$$

to denote the mean value of a function $u \in L^1(\Omega)$.

Lemma 2.4. *Let $u \in H^1(\Omega)$. Then there exists $\mathbf{w} = (w^1,\dots,w^m) \in \mathrm{Adm}(u)$ such that $\mathcal{G}(u) = g(\mathbf{w}) < +\infty$. Moreover, $\mathbf{w}$ is uniquely defined up to an element $\mathbf{c} = (c^1,\dots,c^m) \in \mathbb{R}^m$ such that $\sum_{r=1}^m c^r = 0$.*

Proof. Fix $\mathbf{c} \in \mathbb{R}^m$ such that $\sum_{r=1}^m c^r = \overline{u}$, and set

$$\mathrm{Adm}_{\mathbf{c}}(u) := \Big\{\mathbf{w} \in \mathrm{Adm}(u) \ : \ \overline{w^r} = c^r\Big\},$$

which is a closed and convex subset of $\big(H^1(\Omega)\big)^m$. Let also $\mathcal{G}_{\mathbf{c}} \colon H^1(\Omega) \to [0,+\infty)$ be the functional defined as

$$\mathcal{G}_{\mathbf{c}}(u) = \inf_{\mathbf{w}\in \mathrm{Adm}_{\mathbf{c}}(u)} g(\mathbf{w}).$$

Clearly, $\mathcal{G}(u) \le \mathcal{G}_{\mathbf{c}}(u)$. We claim that

$$\mathcal{G}(u) = \mathcal{G}_{\mathbf{c}}(u). \tag{2.12}$$

Indeed, for any $\mathbf{w} \in \mathrm{Adm}(u)$, let $\widehat{\mathbf{w}} = (\widehat{w}^1,\dots,\widehat{w}^m)$ be defined as

$$\widehat{w}^r := w^r + c^r - \overline{w^r}, \qquad r = 1,\dots,m. \tag{2.13}$$

By construction, $\widehat{\mathbf{w}} \in \mathrm{Adm}_{\mathbf{c}}(u)$, so that

$$\mathcal{G}_{\mathbf{c}}(u) \le g(\widehat{\mathbf{w}}) = g(\mathbf{w}). \tag{2.14}$$

Taking the infimum in (2.14) among all $\mathbf{w} \in \mathrm{Adm}(\mathbf{u})$, we get (2.12).

Now, since g is a coercive, sequentially lower semicontinuous, strictly convex functional on $\mathrm{Adm}_{\mathbf{c}}(u)$, it admits a unique minimizer in $\mathrm{Adm}_{\mathbf{c}}(u)$. Recalling (2.12) and (2.13), the statement is proven. □

Lemma 2.5. *The following statements hold:*

1. *there exist positive constants $C_1 \le C_2$ such that*

$$C_1||\nabla u||^2_{L^2(\Omega)} \le \mathcal{G}(u) \le C_2||\nabla u||^2_{L^2(\Omega)}, \qquad u \in H^1(\Omega); \tag{2.15}$$

2. *$\mathcal{G}$ is convex and sequentially weakly lower semicontinuous in $H^1(\Omega)$.*

Proof. (1) Let $u \in H^1(\Omega)$. For any $r = 1,\dots,m$, let c_r (resp. C_r) be the smallest (resp. the biggest) constant appearing in (1.1) with $\phi := \phi_r$, and let $C_2 \ge C_1 > 0$ be such that

$$2mC_1 \le \mathrm{c}_r^2, \quad \mathrm{C}_r^2 \le 2mC_2, \qquad r = 1,\dots,m.$$

Let $\mathbf{w_0} := (u/m, \ldots, u/m)$. Then $\mathbf{w_0} \in \mathrm{Adm}(u)$, and

$$\mathcal{G}(u) \le g(\mathbf{w_0}) = \frac{1}{2}\int_\Omega \sum_{r=1}^m \frac{1}{m^2}\phi_r^2(\nabla u)\,dx \le \frac{1}{2}\int_\Omega \sum_{r=1}^m \frac{\mathrm{C}_r^2}{m^2}|\nabla u|^2\,dx = \int_\Omega C_2|\nabla u|^2\,dx,$$

which proves the inequality on the right hand side of (2.15).

Now, let $\mathbf{w} = (w^1, \ldots, w^m) \in \mathrm{Adm}(u)$. Then

$$g(\mathbf{w}) \ge mC_1\int_\Omega \sum_{r=1}^m |\nabla w^r|^2\,dx \ge \frac{mC_1}{m}\int_\Omega \Big|\sum_{r=1}^m \nabla w^r\Big|^2\,dx = C_1\int_\Omega |\nabla u|^2\,dx. \tag{2.16}$$

The left inequality in (2.15) now follows, by taking the infimum in (2.16) among all $\mathbf{w} \in \mathrm{Adm}(u)$.

(2) Let $u_1,\, u_2 \in H^1(\Omega)$, let $\lambda \in [0,1]$, and set:

$$u := \lambda u_1 + (1-\lambda)u_2.$$

Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathrm{Adm}(u_j)$ be such that $\mathcal{G}(u_j) = g(\mathbf{w}_j)$, for each $j = 1,2$ (Lemma 2.4). Then $\mathbf{w} := \lambda\mathbf{w_1} + (1-\lambda)\mathbf{w_2}$ satisfies $\sum_{r=1}^m w^r = u$. In particular,

$$\mathcal{G}(u) \le g(\mathbf{w}) \le \lambda g(\mathbf{w}_1) + (1-\lambda)g(\mathbf{w}_2) = \lambda\mathcal{G}(u_1) + (1-\lambda)\mathcal{G}(u_2),$$

where convexity of g is used in the last inequality. This proves convexity of $\mathcal{G}$.

Finally, for $k \in \mathbb{N}$ let $u_k, u \in H^1(\Omega)$ be such that $u_k \rightharpoonup u \in H^1(\Omega)$ as $k \to +\infty$. For any $k \in \mathbb{N}$, fix $\mathbf{c_k} \in \mathbb{R}^{\mathbf{m}}$ such that $\sum_{r=1}^m c_k^r = \overline{u_k}$. Clearly, it is not restrictive to assume that $(\mathbf{c}_k)_k$ is bounded. For any $k \in \mathbb{N}$, let $\mathbf{w}_k := (w_k^1, \ldots, w_k^m) \in \mathrm{Adm}_{\mathbf{c}_k}(u_k)$ be such that $\mathcal{G}(u_k) = g(\mathbf{w}_k)$. The sequence $(\mathbf{w}_k)_k$ is bounded in $\big(H^1(\Omega)\big)^m$, so that there exists $\mathbf{w} \in \big(H^1(\Omega)\big)^m$ such that (up to a not-relabelled subsequence) $\mathbf{w}_k \rightharpoonup \mathbf{w}$ in $\big(H^1(\Omega)\big)^m$ as $k \to +\infty$. In particular, $\sum_{r=1}^m w^r = u$ and therefore

$$\mathcal{G}(u) \le g(\mathbf{w}) \le \liminf_{k\to+\infty} g(\mathbf{w}_k) = \liminf_{k\to+\infty} \mathcal{G}(u_k),$$

where the second inequality comes from the sequential weak lower semicontinuity of g. □

Remark 2.6. Let $u \in H^1(\Omega)$, and let $\mathbf{w} \in \mathrm{Adm}(u)$. Then, for any $v \in H^1(\Omega)$, $\mathbf{w}$ has to satisfy the Euler-Lagrange equation

$$\int_\Omega T_{\phi_r}(\nabla w^r)\cdot\nabla v\,dx = \int_\Omega T_{\phi_s}(\nabla w^s)\cdot\nabla v\,dx \qquad r,s = 1\ldots,m. \tag{2.17}$$

Relation (2.17) is in accordance with the results stated in [11, Lemma 4.1] for the (linear) bidomain model. To prove (2.17), we take as admissible variation $\mathbf{w} + \lambda\mathbf{v}$, for $\lambda \in \mathbb{R}$, and $\mathbf{v} \in \big(H^1(\Omega)\big)^m$ such that

$$\begin{cases} v^j = 0, & \text{for } j \ne r, s, \\ v^r = -v^s = v. \end{cases}$$

By the minimality of $\mathbf{w}$, we have $g(\mathbf{w}) \le g(\mathbf{w}+\lambda\mathbf{v})$. In the previous inequality, all terms apart from those in r, s cancel out, so that, letting $\lambda \to 0$, we get (2.17).

For $u \in H^1(\Omega)$, we denote by $\partial\mathcal{G}(u)$ the subdifferential of $\mathcal{G}$ at u, with respect to the strong topology of $H^1(\Omega)$. Recall that $\partial\mathcal{G}(u) \ne \emptyset$, since $\mathcal{G}(u) < +\infty$. Lemma 2.7 below gives a dual characterization of $\partial\mathcal{G}(u)$. Notice that the right hand side of (2.18) is independent of r (recall (2.17)) and also of the choice of $\mathbf{w}$ such that $\mathcal{G}(u) = g(\mathbf{w})$ (Lemma 2.4).

Lemma 2.7 (**Subdifferential of $\mathcal{G}$**). *Let $u \in H^1(\Omega)$, $l \in (H^1(\Omega))^*$, and let $\mathbf{w} = (w^1, \dots, w^m) \in \mathrm{Adm}(u)$ be such that $\mathcal{G}(u) = g(\mathbf{w})$. Then, $l \in \partial\mathcal{G}(u)$ if and only if, for any $v \in H^1(\Omega)$,*

$$ {}_{(H^1(\Omega))^*}\langle l, v\rangle_{H^1(\Omega)} = \int_\Omega T_{\phi_r}(\nabla w^r)\cdot \nabla v\, dx \qquad r = 1, \dots, m. \tag{2.18}$$

Proof. Let us check that, for any $r = 1, \dots, m$,

$$\mathcal{G}(v) \geq \mathcal{G}(u) + \int_\Omega T_{\phi_r}(\nabla w^s)\cdot\nabla(v-u)\, dx, \qquad v \in H^1(\Omega). \tag{2.19}$$

By (2.17), it is sufficient to show (2.19) for $r = 1$. Fix $v \in H^1(\Omega)$, and let $\widetilde{\mathbf{w}} = (\widetilde{w}^1, \dots, \widetilde{w}^m) \in \mathrm{Adm}(v)$. Then

$$\begin{aligned} g(\widetilde{\mathbf{w}}) - g(\mathbf{w}) &= \frac{1}{2}\int_\Omega \Big(\sum_{r=1}^m \phi_r^2(\nabla\widetilde{w}^r) - \phi_r^2(\nabla w^r)\Big)\, dx \\ &\geq \int_\Omega \Big(\sum_{r=1}^m T_{\phi_r}(\nabla w^r)\cdot\nabla(\widetilde{w}^r - w^r)\Big)\, dx \\ &= \int_\Omega \Big(T_{\phi_1}(\nabla w^1)\cdot\sum_{r=1}^m \nabla(\widetilde{w}^r - w^r)\Big)\, dx \\ &= \int_\Omega \Big(T_{\phi_1}(\nabla w^1)\cdot\nabla(v-u)\Big)\, dx \end{aligned} \tag{2.20}$$

where we used the convexity of ϕ_r^2, $r = 1, \dots, m$, together with (2.17). Taking the infimum in (2.20) among all $\widetilde{\mathbf{w}} \in \mathrm{Adm}(v)$, we get (2.19).

Assume now that $l \in \partial\mathcal{G}(u)$. Fix $r, s \in \{1, \dots, m\}$, with $r \neq s$, and $v, z \in H^1(\Omega)$. Then, for $\lambda > 0$, define $\mathbf{w}_\lambda = (w_\lambda^1, \dots, w_\lambda^m) \in \big(H^1(\Omega)\big)^m$ as

$$\begin{cases} w_\lambda^j := w^j, & j = 1, \dots, m,\ j \neq r, s \\ w_\lambda^r := w^r + \lambda(v-z), \\ w_\lambda^s := w^s + \lambda z. \end{cases}$$

Since $\sum_{r=1}^m w_\lambda^r = u + \lambda v$, we have $\mathcal{G}(u+\lambda v) \leq g(\widehat{\mathbf{w}})$. By the definition of subdifferential, we have

$$\frac{g(\widehat{\mathbf{w}}) - g(\mathbf{w})}{\lambda} \geq \frac{\mathcal{G}(u+\lambda v) - \mathcal{G}(u)}{\lambda} \geq {}_{(H^1(\Omega))'}\langle l, v\rangle_{H^1(\Omega)}. \tag{2.21}$$

Letting $\lambda \to 0^+$ in (2.21), we end up with:

$$\int_\Omega T_{\phi_r}(\nabla w^r)\cdot\nabla(v-z)\, dx + \int_\Omega T_{\phi_s}(\nabla w^s)\cdot\nabla z\, dx \geq {}_{(H^1(\Omega))'}\langle l, v\rangle_{H^1(\Omega)},$$

or, equivalently,

$$\int_\Omega T_{\phi_r}(\nabla w^r)\cdot\nabla v\, dx + \int_\Omega \big(T_{\phi_s}(\nabla w^s) - T_{\phi_r}(\nabla w^r)\big)\cdot\nabla z\, dx \geq {}_{(H^1(\Omega))'}\langle l, v\rangle_{H^1(\Omega)}.$$

Observe that the second integral in the previous line vanishes, by virtue of (2.17). Arguing similarly for $\lambda < 0$, we get the converse inequality, and hence the equality, which proves our statement. □

Let $\mathfrak{f}\colon \mathrm{Dom}(\mathfrak{f}) \to \big(H^1(\Omega)\big)^*$ be the (nonlinear) operator defined as

$$ {}_{(H^1(\Omega))^*}\langle \mathfrak{f}(u), v\rangle_{H^1(\Omega)} := \int_\Omega f(u)v\, dx, \qquad v \in H^1(\Omega),$$

for every $u \in \mathrm{Dom}(\mathfrak{f}) := \{\mathrm{u} \in H^1(\Omega)\ :\ f(\mathrm{u}) \in (H^1(\Omega))^* \cap L^1(\Omega)\}$.

We postpone the proof of the following result to the end of this section. Let us set, for notational convenience,

$$\mathcal{V} := H^1\big(0, T; L^2(\Omega)\big) \cap L^\infty\big(0, T; H^1(\Omega)\big).$$

Proposition 2.8 (**Reduction to a single evolutive equation**). *Let $u_0 \in H^1(\Omega)$ be such that $W(u_0) \in L^1(\Omega)$. Then, there exists a unique $u \in \mathcal{V}$ such that*

$$\begin{cases} u(t) \in \mathrm{Dom}(\mathfrak{f}), & t \in (0,T), \\ \partial_t u(t) + \partial\mathcal{G}(u(t)) + \mathfrak{f}(u(t)) = 0, & \text{a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{2.22}$$

Observe that, by virtue of Lemma 2.7, we do not need to write the evolutive equation in (2.22) as a differential inclusion. We are now in the position to prove the following result.

Proposition 2.9. *Let $u_0 \in H^1(\Omega)$ be such that $W(u_0) \in L^1(\Omega)$, and let $u \in \mathcal{V}$ be given by Proposition 2.8. Then there exists $\mathbf{w} = (w^1, \dots, w^m) \in \Big(L^2\big(0,T; H^1(\Omega)\big)\Big)^m$, uniquely defined up to a map $\mathbf{c} = (c^1, \dots, c^m) \in \big(L^2(0,T)\big)^m$ with $\sum_{r=1}^m c^r = 0$, satisfying (2.10) in the sense of distributions.*

Proof. By Lemma 2.4, for a.e. $t \in (0,T)$ there exists a unique $\mathbf{w}(t) \in (H^1(\Omega))^m$ such that $g(\mathbf{w}(t)) = \mathcal{G}(u(t))$ and $\overline{w^r(t)} = \frac{\overline{u(t)}}{m}$, for every $r = 1, \dots, m$. Recalling (2.18), we are just left to prove that

$$\mathbf{w} \in \Big(L^2\big(0,T; H^1(\Omega)\big)\Big)^m. \tag{2.23}$$

Indeed, for every $r = 1, \dots, m$, using also (2.15), we have

$$\begin{aligned} \int_0^T ||w^r(t)||^2_{H^1(\Omega)}\, dt \leq & (1 + C^2(\Omega)) \int_0^T \Big(||\nabla w^r(t)||^2_{L^2(\Omega)} + \big(\overline{w^r}(t)\big)^2\Big)\, dt \\ \leq & \frac{(1 + C^2(\Omega))}{C_1} \int_0^T \Big(g(\mathbf{w}(t) + \big(\overline{u(t)}\big)^2\Big)\, dt \\ = & \frac{(1 + C^2(\Omega))}{C_1} \int_0^T \Big(\mathcal{G}(u(t)) + \big(\overline{u(t)}\big)^2\Big)\, dt \\ \leq & \frac{C_2(1 + C^2(\Omega))}{C_1} \int_0^T \Big(||\nabla u(t)||^2_{L^2(\Omega)} + \big(\overline{u(t)}\big)^2\Big)\, dt, \end{aligned} \tag{2.24}$$

where $C(\Omega) > 0$ is the constant given by the Poincaré inequality [57, Corollary 9.19]. Since $u \in \mathcal{V}$, the right hand side of (2.24) is finite. This proves (2.23). $\square$

We have reduced the proof of Theorem 2.3 to proof of Proposition 2.8. In order to conclude, as already mentioned, we set up a standard *minimizing movements* [10] approach.

Fix $\tau_0 > 0$, and let $\tau \in (0, \tau_0)$. Set $v^0_\tau := u_0$, and, for $k \in \mathbb{N}$, $k \geq 1$, let $v^k_\tau \in H^1(\Omega)$ be a minimizer of

$$\frac{1}{2\tau}||u - v^{k-1}_\tau||^2_{L^2(\Omega)} + \mathcal{E}(u), \qquad u \in H^1(\Omega), \tag{2.25}$$

where

$$\mathcal{E}(u) := \mathcal{G}(u) + \int_\Omega W(u)\, dx, \qquad u \in H^1(\Omega).$$

Notice that, by virtue of (2.2)-(2.3), if $W(u) \in L^1(\Omega)$, then $u \in \mathrm{Dom}(\mathfrak{f})$. In particular, $v^k_\tau \in \mathrm{Dom}(\mathfrak{f})$ for every $\tau \in (0, \tau_0)$ and $k \in \mathbb{N}$.

Existence of minimizers for (2.25) follows by direct methods (recall Lemmata 2.4 and 2.5). Let us check, when $\tau_0 > 0$ is sufficiently small, (2.25) admits a unique minimizer: indeed, any minimizer u has to satisfy

$$\frac{u - v^{k-1}_\tau}{\tau} + \partial\mathcal{G}(u) + \mathfrak{f}(u) = 0 \tag{2.26}$$

in the sense of $(H^1(\Omega))^*$. Now, let $v_\tau^{k,1}, v_\tau^{k,2} \in H^1(\Omega)$ be two minimizers of (2.25). From (2.26), we get

$$
\begin{aligned}
-\frac{1}{\tau}||v_\tau^{k,1} - v_\tau^{k,2}||^2_{L^2(\Omega)} =&{}_{(H^1(\Omega))^*}\langle \partial\mathcal{G}(v_\tau^{k,1}) - \partial\mathcal{G}(v_\tau^{k,2}), v_\tau^{k,1} - v_\tau^{k,2}\rangle_{H^1(\Omega)} \\
&+ {}_{(H^1(\Omega))^*}\langle \mathfrak{f}(v_\tau^{k,1}) - \mathfrak{f}(v_\tau^{k,2}), v_\tau^{k,1} - v_\tau^{k,2}\rangle_{H^1(\Omega)} \\
&\geq \int_\Omega \big(f(v_\tau^{k,1}) - f(v_\tau^{k,2})\big)(v_\tau^{k,1} - v_\tau^{k,2})\,dx \\
&\geq - C_f||v_\tau^{k,1} - v_\tau^{k,2}||^2_{L^2(\Omega)},
\end{aligned}
$$

where we used (2.1) and the monotonicity of $\partial\mathcal{G}$. Hence, if $\tau_0 \leq \frac{1}{C_f}$, we end up with $v_\tau^{k,1} = v_\tau^{k,2}$, as claimed.

Observe that, for any $k \geq 1$, the minimality of v_τ^k implies that

$$
\frac{1}{2\tau}||v_\tau^k - v_\tau^{k-1}||^2_{L^2(\Omega)} \leq \mathcal{E}(v_\tau^{k-1}) - \mathcal{E}(v_\tau^k).
$$

In particular,

$$
\mathcal{E}(v_\tau^k) \leq \mathcal{E}(v_\tau^{k-1}), \quad k \geq 1, \tag{2.27}
$$

and

$$
\sum_{k=1}^{\infty} \frac{1}{\tau}||v_\tau^k - v_\tau^{k-1}||^2_{L^2(\Omega)} \leq 2\sum_{k=1}^{\infty} \mathcal{E}(v_\tau^{k-1}) - \mathcal{E}(v_\tau^k) = 2\mathcal{E}(u_0) < +\infty. \tag{2.28}
$$

Now, let us define a map $u_\tau \colon [0,+\infty) \to H^1(\Omega)$ as

$$
u_\tau(t) := v_\tau^{[t/\tau]} + \Big(\frac{t}{\tau} - \Big[\frac{t}{\tau}\Big]\Big)(v_\tau^{[t/\tau+1]} - v_\tau^{[t/\tau]}), \qquad t \geq 0, \tag{2.29}
$$

where, for $t \in \mathbb{R}$, we denote by $[t]$ the smallest integer less or equal than t. Notice that, for every $\tau \in (0,\tau_0)$,

$$
u_\tau(0) = u_0; \tag{2.30}
$$

moreover, $u_\tau(t) \in \mathrm{Dom}(\mathfrak{f})$ for every $t \in (0,T)$, and

$$
u'_\tau(t) = \frac{v_\tau^{[t/\tau+1]} - v_\tau^{[t/\tau]}}{\tau}, \qquad \text{a.e. } t > 0. \tag{2.31}
$$

Coupling (2.31) with (2.26), we get

$$
u'_\tau(t) + \partial\mathcal{G}(v_\tau^{[t/\tau+1]}) + \mathfrak{f}(v_\tau^{[t/\tau+1]}) = 0,
$$

or equivalently

$$
u'_\tau(t) + \partial\mathcal{G}\big(u_\tau(t(\tau)\big) + \mathfrak{f}\big(u_\tau(t(\tau))\big) = 0, \tag{2.32}
$$

where we put for shortness

$$
t(\tau) := \tau\Big[\frac{t}{\tau} + 1\Big].
$$

Lemma 2.10. *There exists $M > 0$ such that, for every $\tau \in (0,\tau_0)$ and $t > 0$,*

$$
||u_\tau||_{L^\infty(0,T;H^1(\Omega))} \leq M, \qquad ||u'_\tau||_{L^2(0,T;L^2(\Omega))} \leq M. \tag{2.33}
$$

Proof. Let us bound the $H^1(\Omega)$-norm of u_τ as follows:

$$
\begin{aligned}
||u_\tau(t)||^2_{H^1(\Omega)} \leq& C(\mathcal{G}(u_\tau(t)) + ||u_\tau(t)||^2_{L^2(\Omega)}) \\
\leq& C \sup_{k\in\mathbb{N}} \Big\{\mathcal{G}(v_\tau^k) + ||v_\tau^k||^2_{L^2(\Omega)}\Big\} \\
=& C \sup_{k\in\mathbb{N}} \Big\{\mathcal{E}(v_\tau^k) + ||v_\tau^k||^2_{L^2(\Omega)}\Big\} \\
\leq& C\big(\mathcal{E}(u_0) + ||u_0||^2_{L^2(\Omega)}\big), \qquad t > 0,
\end{aligned}
$$

where, in order, we used (2.15), convexity of $\mathcal{G}$ and of $||\cdot||_{L^2(\Omega)}$, (2.27), and we have let $C > 1$ be a suitable constant, possibly different from line to line. This proves the first inequality in (2.33). The second estimate follows recalling (2.31) and observing that

$$\begin{aligned}\int_0^T ||u_\tau'(t)||^2_{L^2(\Omega)}\,dt &= \int_0^T \frac{1}{\tau^2}||v_\tau^{[t/\tau+1]} - v_\tau^{[t/\tau]}||^2_{L^2(\Omega)}\,dt \\ &\leq \int_0^{+\infty} \frac{1}{\tau}||v_\tau^{[s+1]} - v_\tau^{[s]}||^2_{L^2(\Omega)}\,ds \\ &\leq \sum_{k=1}^{\infty} \frac{1}{\tau}||v_\tau^k - v_\tau^{k-1}||^2_{L^2(\Omega)} \leq 2\mathcal{E}(u_0),\end{aligned}$$

where the last inequality is due to (2.28). □

Remark 2.11. As a consequence of the second inequality in (2.33), using also Hölder inequality [57], we get for any $t, s > 0$

$$||u_\tau(t) - u_\tau(s)||_{L^2(\Omega)} \leq \Big(\int_0^T ||u_\tau'(r)||^2_{L^2(\Omega)}\,dr\Big)^{\frac{1}{2}} |t-s|^{\frac{1}{2}} \leq M\,|t-s|^{\frac{1}{2}}. \tag{2.34}$$

The proof of the following result is given, in a more general setting, in [14].

Lemma 2.12. *For $\tau \in (0, \tau_0)$, let u_τ be defined as in* (2.29)*. Then there exists $u \in \mathcal{V}$ such that $u(t) \in \mathrm{Dom}(\mathfrak{f})$ for every $t \in (0,T)$, and moreover, up to a subsequence,*

$$u_\tau(t) \rightharpoonup u(t) \text{ in } H^1(\Omega), \quad t > 0. \tag{2.35}$$

Proof. Recalling the first inequality in (2.33), Rellich' Theorem [57, Theorem 9.16], and using a standard diagonalization argument, it is not restrictive to assume that

$$\begin{aligned} u_\tau(q) \text{ weakly converges in } H^1(\Omega), &\qquad q \in \mathbb{Q}^+, \\ u_\tau(q) \text{ strongly converges in } L^2(\Omega), &\qquad q \in \mathbb{Q}^+. \end{aligned}$$

Hence, we are allowed to set

$$u(q) := \lim_{\tau\to 0^+} u_\tau(q), \qquad q \in \mathbb{Q}^+. \tag{2.36}$$

For $t > 0$, we define $u(t)$ as

$$u(t) := \lim_{q\to t} u(q), \qquad q \in \mathbb{Q}^+,\ q \to t. \tag{2.37}$$

This makes sense, since

$$\begin{aligned}||u(q) - u(q')||_{L^2(\Omega)} \leq &||u(q) - u_\tau(q)||_{L^2(\Omega)} + ||u_\tau(q) - u_\tau(q')||_{L^2(\Omega)} \\ &+ ||u(q') - u_\tau(q')||_{L^2(\Omega)}, \qquad q, q' \in \mathbb{Q}^+,\end{aligned} \tag{2.38}$$

and we notice that, by (2.34) and (2.36), all terms in the right hand side of (2.38) cancel as $q, q' \to t$. Finally, let $\big(u_{\tau_k}(t)\big)_{k\in\mathbb{N}}$ be a subsequence weakly convergent in $H^1(\Omega)$ to some w. Then, recalling (2.34), we have for any $q \in \mathbb{Q}^+$

$$||u(q) - w||_{L^2(\Omega)} \leq \liminf_{k\to+\infty} ||u_{\tau_k}(q) - u_{\tau'}(t)||_{L^2(\Omega)} \leq M|q-t|^{\frac{1}{2}}. \tag{2.39}$$

As $q \to t$ in (2.39), using also (2.37), we get $w = u(t)$. This result being independent of the weakly convergent subsequence, we get (2.35).

Notice that (2.35) and the first bound in (2.33) entail

$$u \in L^\infty\big(0, T; H^1(\Omega)\big).$$

Therefore, in order to conclude, we are left to prove that

$$u \in H^1\big(0,T;L^2(\Omega)\big).$$

Recalling the second bound in (2.33), it is not restrictive to assume that there exists $w \in L^2\big(0,T;L^2(\Omega)\big)$ such that $u'_\tau \rightharpoonup w$ in $L^2\big(0,T;L^2(\Omega)\big)$. Letting $\tau \to 0^+$ in

$$u_\tau(t) - u_\tau(s) = \int_s^t u'_\tau(r)\,dr, \qquad t,s > 0,$$

we obtain

$$u(t) - u(s) = \int_s^t w(r)\,dr,$$

namely $u \in AC\big(0,T;L^2(\Omega)\big)$. In particular (see for instance [10, Theorem 2.1]), this implies that $u'(t)$ exists for a.e. $t \in (0,T)$, and moreover

$$u' = w \in L^2\big(0,T;L^2(\Omega)\big). \tag{2.40}$$

The statement is proven. □

Remark 2.13. We notice that, by virtue of (2.34), (2.36) and (2.37), we have

$$u_\tau \to u \;\text{ in } L^2\big(0,T;L^2(\Omega)\big), \quad \textit{as } \tau \to 0^+. \tag{2.41}$$

Moreover, as a byproduct of (2.40), we have

$$u'_\tau \rightharpoonup u' \;\text{ in } L^2\big(0,T;L^2(\Omega)\big), \quad \textit{as } \tau \to 0^+. \tag{2.42}$$

Proof of Proposition 2.8. Let $u \in \mathcal{V}$ be given in Lemma 2.12. We claim that

$$u \text{ satisfies (2.22)}. \tag{2.43}$$

Indeed, by (2.30) and (2.36), we have $u(0) = u_0$, which is the initial condition appearing in (2.22). Now, fix $z \in H^1(\Omega)$. From (2.32), for any $\tau, t > 0$ we have

$$\mathcal{G}(z) \geq \mathcal{G}(u_\tau(t(\tau))) - \int_\Omega \left(u'_\tau(t) + f(u_\tau(t(\tau)))\right)(z - u_\tau(t(\tau)))\,dx. \tag{2.44}$$

Fix $t \in (0,T)$. Integrating (2.44) in the interval $(t, t+h)$, and then dividing by $h > 0$, we get

$$\begin{aligned}\mathcal{G}(z) \geq & \frac{1}{h}\int_t^{t+h} \mathcal{G}(u_\tau(s(\tau)))ds - \frac{1}{h}\int_\Omega (u_\tau(t+h) - u_\tau(t))\,z\,dx \\ & + \frac{1}{h}\int_t^{t+h} ds \int_\Omega dx\, u'_\tau(s) u_\tau(s(\tau)) \\ & - \frac{1}{h}\int_t^{t+h} ds \int_\Omega dx\, f\big(u_\tau(s(\tau))\big)\,(z - u_\tau(s(\tau))\,.\end{aligned}$$

By Lemma 2.12, using also (2.34), we have that

$$u_\tau(s(\tau)) \rightharpoonup u(s) \text{ in } H^1(\Omega) \;\text{ as } \tau \to 0^+, \quad s \in (0,T). \tag{2.45}$$

Letting $\tau \to 0^+$ in (2.2), also recalling (2.41), (2.42), and (2.1), we get

$$\begin{aligned}\mathcal{G}(z) \geq & \frac{1}{h}\int_t^{t+h} \mathcal{G}(u(s))ds - \frac{1}{h}\int_\Omega (u(t+h) - u(t))\,z\,dx \\ & + \frac{1}{h}\int_t^{t+h} ds \int_\Omega dx\, u'(s) u(s) \\ & - \frac{1}{h}\int_t^{t+h} ds \int_\Omega dx\, f(u(s))\,(z - u(s))\,.\end{aligned} \tag{2.46}$$

where we used also Fatou's Lemma, (2.45), and the sequential lower semicontinuity of $\mathcal{G}$ (Lemma 2.5). Letting $h \to 0^+$ in (2.46), we get

$$\mathcal{G}(z) \geq \mathcal{G}(u(t)) - \int_\Omega u'(t)(z - u(t))\, dx - \int_\Omega dx\, f(u(t))\,(z - u(t))\,, \tag{2.47}$$

for a.e. $t \in (0, T)$. By the arbitrariness of $z \in H^1(\Omega)$, (2.47) implies

$$-u'(t) - f(u(t) = \partial\mathcal{G}(u(t)),$$

which proves claim (2.43).

We are left to show that (2.22) admits at most one solution. Indeed, let $u, v \in \mathcal{V}$ solve (2.22), and set $z := u - v$. Then, for a.e. $t \in (0, T)$,

$$\partial_t z(t) + \partial\mathcal{G}(u(t)) - \partial\mathcal{G}(v(t)) + \mathfrak{f}(u(t)) - \mathfrak{f}(v(t)) = 0. \tag{2.48}$$

Taking the duality between (2.48) and $z(t)$, we get:

$$\begin{aligned}\int_\Omega \partial_t z(t) z(t)\, dx + {}_{(H^1(\Omega))'}\langle \partial\mathcal{G}(u(t)) - \partial\mathcal{G}(v(t)), u(t) - v(t)\rangle_{H^1(\Omega)}& \\ + \int_\Omega \big(f(u(t)) - f(v(t))\big)(u(t) - v(t))\, dx = 0,&\end{aligned}$$

which, by the monotonicity of the subdifferential, also recalling (2.1), implies

$$\begin{aligned}\int_\Omega \partial_t z(t) z(t)\, dx \leq& - \int_\Omega \big(f(u(t)) - f(v(t))\big)(u(t) - v(t))\, dx \\ \leq& C_f \int_\Omega |u(t) - v(t)|^2\, dx \\ =& C_f \int_\Omega |z(t)|^2\, dx.\end{aligned}$$

By Gronwall's Lemma [142, Lemma 4.1, p.179], also recalling that $z(0) = 0$, we deduce $z(t) = 0$. □

2.3 Formal asymptotics

In this section we perform a formal asymptotic expansion for the nonlinear multidomain model, assuming for definitiness

$$f(s) = \frac{d}{ds}\left((1 - s^2)^2\right),$$

in particular $s_\pm = \pm 1$. The computations, appeared in [7], will be simpler, and at the same time more general than those made in [43]. This will be apparent particularly in the inner expansion of Section 2.3.2 below. Due to the strong reaction term, we expect the sum $u_\epsilon := \sum_{r=1}^m w_\epsilon^r$ to assume values near to ± 1 in most of the domain with a thin, smooth, transition region where it transversally crosses the unstable zero of f. This motivates the use of matched asymptotics in the outer $\Omega^- \cup \Omega^+$ region (*outer expansion*) and in the transition layer (*inner expansion*).

As a formal consequence (see (2.97) below), the front generated by (2.4) propagates with the same law, up to an error of order $\mathcal{O}(\epsilon)$, as the front generated by a Φ^o-anisotropic mean curvature flow starting from a smooth hypersurface $\partial E \subset \Omega$, where Φ is the star-shaped combination of the m original anisotropies $\phi_1, \ldots, \phi_m \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V}^*)$.

We write the system in the convenient form

$$\begin{cases} \epsilon\, \partial_t u_\epsilon - \epsilon \operatorname{div}\Big(T_{\phi_1}(\nabla w_\epsilon^1)\Big) + \frac{1}{\epsilon} f(u_\epsilon) = 0, \\ \\ \operatorname{div}\Big(T_{\phi_r}(\nabla w_\epsilon^r)\Big) = \operatorname{div}\Big(T_{\phi_s}(\nabla w_\epsilon^s)\Big), \qquad 1 \leq r,\, s \leq m, \\ \\ u_\epsilon = \sum_{r=1}^m w_\epsilon^r. \end{cases} \tag{2.49}$$

This system consists of one parabolic equation and $(m-1)$ elliptic equations, to be coupled with an initial condition at $\{t=0\}$, which in particular is required to satisfy

$$\{u_\epsilon(0,\cdot)=0\}=\partial E, \qquad \epsilon\in(0,1),$$

and m either Neumann boundary conditions at $\cup_{t=0}^{T}(\{t\}\times\partial\Omega)$. We restore in this section the notational dependence on ϵ for $u=u_\epsilon$ and all $w^r=w^r_\epsilon$.

2.3.1 Outer expansion

Given $r=1,\ldots,m$, we formally expand u_ϵ and w^r_ϵ in terms of $\epsilon\in(0,1)$:

$$u_\epsilon=u_0+\epsilon u_1+\epsilon^2u_2+\ldots, \qquad w^r_\epsilon=w^r_0+\epsilon w^r_1+\epsilon^2 w^r_2+\ldots$$

Substituting these expressions into the parabolic equation in (2.49) and using the expansion

$$f(u_\epsilon)=f(u_0)+\epsilon f'(u_0)u_1+\epsilon^2\left(\frac{u_1^2f''(u_0)}{2}+f'(u_0)u_2\right)+\mathcal{O}(\epsilon^3),$$

we get

$$f(u_0)=0, \qquad u_1f'(u_0)=0.$$

Hence, excluding $u_0=0$ (the unstable zero of f), we get in $(0,T)\times\Omega$,

$$u_0\in\{1,-1\}, \tag{2.50}$$

$$u_1\equiv 0. \tag{2.51}$$

We denote by

$$\Sigma_0(t), \qquad t\in(0,T), \tag{2.52}$$

the jump set of $u_0(t,\cdot)$.

Coming back to the elliptic equations in (2.49), we find

$$\begin{cases} \operatorname{div}\Big(T_{\phi_r}(\nabla w_0^r)\Big)=\operatorname{div}\Big(T_{\phi_s}(\nabla w_0^s)\Big), & 1\le r,\, s\le m \\ \displaystyle\sum_{r=1}^m w_0^r=u_0 \implies \sum_{r=1}^m \nabla w_0^r=0, & \end{cases} \tag{2.53}$$

where the last implication is a consequence of (2.50).

Note also that

$$u_2=\frac{1}{f'(u_0)}\operatorname{div}\Big(T_{\phi_r}(\nabla w_0^r)\Big), \qquad r=1,\ldots,m. \tag{2.54}$$

Remark 2.14. System (2.53) consists of $(m-1)$ *nonlinear* elliptic equations in the $(m-1)$ unknown functions w_0^r (for $r=2,\ldots,m$), since we can solve the algebraic equation in (2.53) with respect to w_0^1.

Remark 2.15. It is important to notice that the boundary conditions across the limit interface $\Sigma_0(t)$, to be coupled with (2.53), will arise by matching the outer expansion with the inner expansion, see (2.106) and (2.108) (jump conditions and jump of the normal derivative). We assume the elliptic problem expressed by (2.53), (2.106), (2.108) (and augmented with Neumann or Dirichlet boundary conditions on $\partial\Omega$) to be solvable, thus providing w_0^r for every $r=1,\ldots,m$, and therefore u_2 by (2.54). We stress that, at the author's best knowledge, this is an open question, compare Problem 2.25 at the end of this chapter.

If we now perform a Taylor-expansion for T_{ϕ_r}, we obtain

$$T_{\phi_r}(\eta^* + \epsilon\zeta^*) = T_{\phi_r}(\eta^*) + \epsilon M^r(\eta^*)\zeta^* + \mathcal{O}(\epsilon^2), \qquad \eta^*, \zeta^* \in \mathrm{V}^*,$$

where $M^r = \frac{1}{2}\nabla^2\alpha_r$, which can be used in the elliptic equations of (2.49) to get equations for w_1^r for any $r = 1, \ldots, m$, namely:

$$\mathrm{div}\Big(M^r(\nabla w_0^r)\nabla w_1^r\Big) = \mathrm{div}\Big(M^s(\nabla w_0^s)\nabla w_1^s\Big), \qquad 1 \leq r,\, s \leq m.$$

Moreover, from the relation $\sum_{r=1}^m w_\epsilon^r = u_\epsilon$, and recalling from (2.51) that $u_1 = 0$, we obtain

$$\sum_{r=1}^m w_1^r = 0.$$

By solving this latter equation with respect (for instance) to w_1^1, and substituting it into the previous equation we obtain a system of $(m-1)$ *linear* elliptic equations in the unknowns w_1^r, for $r = 2, \ldots, m$.

Remark 2.16. The outer expansion has been performed without assuming Φ to be convex.

2.3.2 Inner expansion

For any $\epsilon \in (0,1)$ let us consider the set

$$E_\epsilon(t) := \{x \in \Omega \, : \, u_\epsilon(t,x) \geq 0\},$$

the boundary of which will be denoted by

$$\Sigma_\epsilon(t) = \{x \in \Omega \, : \, u_\epsilon(t,x) = 0\}. \tag{2.55}$$

Our aim is to formally identify the geometric evolution law of $\Sigma_\epsilon(t)$ as $\epsilon \to 0^+$.

For $r = 1, \ldots, m$ we seek the shape, in the transition layer, of functions w_ϵ^r satisfying

$$\epsilon^2\partial_t u_\epsilon - \epsilon^2 \mathrm{div}\Big(T_{\phi_r}(\nabla w_\epsilon^r)\Big) + f(u_\epsilon) = 0, \qquad r = 1, \ldots, m, \tag{2.56}$$

with $u_\epsilon = \displaystyle\sum_{r=1}^m w_\epsilon^r$. We put, as usual,

$$\alpha_r := \phi_r^2, \qquad T_{\phi_r} := \frac{1}{2}\nabla\alpha_r, \qquad M^r := \frac{1}{2}\nabla^2\alpha_r, \qquad r = 1, \ldots, m,$$

so that, by Euler's identities for homogeneus functions (recall (1.2)), we have

$$\alpha_r(\xi^*) = T_{\phi_r}(\xi^*) \cdot \xi^* = M^r(\xi^*)\xi^* \cdot \xi^*, \qquad \xi^* \in \mathrm{V}^*. \tag{2.57}$$

Remember that the matrix M^r depends on the covector ξ^*, unless ϕ_r is a linear anisotropy (*i.e.*, unless T_{ϕ_r} is linear).

2.3.3 Main assumptions and basic notation

We assume in this section that

$$\Phi \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V}^*).$$

This allows to look at Φ as the dual of an anisotropy $\varphi \in \mathcal{M}_{\mathrm{reg}}(\mathrm{V})$, namely

$$\varphi = \Phi^o.$$

Keeping the simpler symbol φ instead of Φ^o, we let

$$d_\epsilon^\varphi(t,\, x) := d_\varphi^{E_\epsilon(t)}(x)$$

denote the φ-signed distance function from $\Sigma_\epsilon(t)$, positive in the interior of $E_\epsilon(t)$ (recall the notation in (1.4)).

Following [31], it is convenient to introduce the stretched variable y defined as

$$y = y_\epsilon^\varphi(t,\, x) := \frac{d_\epsilon^\varphi(t,\, x)}{\epsilon}.$$

We parametrize $\Sigma_\epsilon(t)$ with a parameter

$$s \in \Sigma, \tag{2.58}$$

Σ being a fixed reference $(n-1)$-dimensional smooth manifold, and the function $x(s,\, t;\, \epsilon)$ gives the position in Ω of the point s at time t.

We let, for x in a tubular neighbourhood of $\Sigma_\epsilon(t)$,

$$n_\epsilon^\varphi(t,\, x) := -T_\Phi(\nabla d_\epsilon^\varphi(t,\, x)) \tag{2.59}$$

be the (outward) Cahn-Hoffman's vector field (remember (1.6)), for which we suppose the expansion:

$$n_\epsilon^\varphi := n_0^\varphi + \epsilon n_1^\varphi + \ldots$$

Points on the evolving manifold $\Sigma_\epsilon(t)$ are assumed to move in the direction of n_ϵ^φ, *i.e.*

$$\partial_t x(s,\, t;\, \epsilon) = V_\epsilon^\varphi\ n_\epsilon^\varphi,$$

where V_ϵ^φ is positive for an expanding set, and where we assume the validity of the following expansion:

$$V_\epsilon^\varphi = V_0^\varphi + \epsilon V_1^\varphi + \epsilon^2 V_2^\varphi + \ldots$$

The anisotropic projection of a point x on $\Sigma_\epsilon(t)$ will be denoted by $s_\epsilon^\varphi(t,\, x)$, which satisfies

$$\partial_t s_\epsilon^\varphi = 0. \tag{2.60}$$

Hence

$$\partial_t d_\epsilon^\varphi(t,\, x) = V_\epsilon^\varphi\big(s_\epsilon^\varphi(t,\, x),\, t\big). \tag{2.61}$$

We also recall (Definition 1.9) that $\operatorname{div}(T_\Phi(\nabla d_\epsilon^\varphi))$ gives (up to a minus sign) the anisotropic mean curvature of the level hypersurface at that point and [41] it can be approximated by the anisotropic mean curvature κ_ϵ^φ of $\Sigma_\epsilon(t)$ (positive when $E_\epsilon(t)$ is uniformly convex) as follows

$$\operatorname{div}(T_\Phi(\nabla d_\epsilon^\varphi(t,\, x))) = -\kappa_\epsilon^\varphi(s_\epsilon^\varphi(t,\, x), t) - \epsilon y_\epsilon^\varphi h_\epsilon^\varphi(s_\epsilon^\varphi(t,\, x), t) + \mathcal{O}(\epsilon^2 (y_\epsilon^\varphi)^2) \tag{2.62}$$

for a suitable h_ϵ^φ depending on the local shape of $\Sigma_\epsilon(t)$. We assume the expansions

$$\kappa_\epsilon^\varphi = \kappa_0^\varphi + \epsilon\kappa_1^\varphi + \mathcal{O}(\epsilon^2), \qquad h_\epsilon^\varphi = h_0^\varphi + \mathcal{O}(\epsilon).$$

With abuse of notation, for a given ϵ, we let $x(y;\, s,\, t)$ be the point of Ω having signed distance ϵy and projection s on $\Sigma_\epsilon(t)$. We have

$$x(y;\, s,\, t) = x(s,\, t) - \epsilon y n_\epsilon^\varphi + \mathcal{O}(\epsilon^2 y^2). \tag{2.63}$$

For a given ϵ, the triplet $(y;\, s,\, t)$ will parametrize a tubular neighbourhood of $\cup_{t\in(0,T)}(\{t\}\times \Sigma_\epsilon(t))$. We look for functions $U_\epsilon(y;\, s,\, t)$ and $W_\epsilon^r(y;\, s,\, t,\, x)$ $(r = 1, \ldots, m)$ respectively so that

$$u_\epsilon(t,\, x) = U_\epsilon\left(\frac{d_\epsilon^\varphi(t,\, x)}{\epsilon}, s_\epsilon^\varphi(t,\, x), t\right), \tag{2.64}$$

$$w_\epsilon^r(t,\, x) = W_\epsilon^r\left(\frac{d_\epsilon^\varphi(t,\, x)}{\epsilon}, s_\epsilon^\varphi(t,\, x), t, x\right), \qquad r = 1, \ldots, m, \tag{2.65}$$

with

$$\sum_{r=1}^m W_\epsilon^r = U_\epsilon. \tag{2.66}$$

Remark 2.17. Formula (2.64) defines uniquely the function U_ϵ, since to evey (t,x) there corresponds uniquely the triplet (y,s,t). This observation does not apply to (2.65), in view of the explicit dependence of the functions W_ϵ^r on x.

We shall write

$$W_\epsilon^r = W_0^r + \epsilon W_{1,\epsilon}^r = W_0^r + \epsilon W_1^r + \epsilon^2 W_{2,\epsilon}^r, \qquad r = 1,\dots,m, \tag{2.67}$$

where W_0^r and W_1^r are allowed to depend explicitly on x (and hence on ϵ). We suppose the remainders $W_{1,\epsilon}^r$, $W_{2,\epsilon}^r$ to be bounded as $\epsilon \to 0^+$.

We let also

$$S^r := \frac{1}{2}\nabla^3 \alpha_r = \nabla M^r, \qquad r = 1,\dots,m,$$

be the 3-indices, (-1)-homogeneus completely symmetric tensor given by the third derivatives of $\frac{1}{2}\alpha_r$: in components we have

$$S_{ijk}^r := \nabla_k M_{ij}^r, \qquad r = 1,\dots,m,$$

where $\nabla_k = \frac{\partial}{\partial \xi_k^*}$. Finally, for any $j,k = 1,\dots,n$, we introduce the notation

$$M_{\cdot k}^r := \left(M_{1k}^r \dots M_{nk}^r\right), \qquad S_{\cdot jk}^r := \left(S_{1jk}^r \dots S_{njk}^r\right), \qquad r = 1,\dots,m.$$

Warning: We will adopt the convention of summation on repeated indices, *with the exception of the index* r, for which the explicit symbol $\sum_{r=1}^m$ will be always used. For instance, in formulas (2.71), (2.75), (2.76), (2.77) and (2.116) below, no summation on r is understood.

2.3.4 Preliminary expansions

Now we begin to expand all terms in (2.56). We have, using the convention of summation on repeated indices,

$$\epsilon^2 \partial_t u_\epsilon = \epsilon^2 U_{\epsilon s_\beta}\ \partial_t s_{\epsilon\beta}^\varphi + \epsilon U_\epsilon'\ \partial_t d_\epsilon^\varphi + \epsilon^2 U_{\epsilon t} = \epsilon U_\epsilon' V_\epsilon^\varphi + \epsilon^2 U_{\epsilon t}, \tag{2.68}$$

where we used (2.60) and (2.61).

We write

$$U_\epsilon = U_0 + \epsilon U_{1,\epsilon} = U_0 + \epsilon U_1 + \epsilon^2 U_{2,\epsilon}, \tag{2.69}$$

where we require U_0 and U_1 not to depend on ϵ.

Using Taylor's expansion of the nonlinearity f, we get

$$f(U_\epsilon) = f(U_0) + \epsilon U_{1,\epsilon} f'(U_0) + \frac{1}{2}\epsilon^2 (U_{1,\epsilon})^2 f''(U_0) + \mathcal{O}(\epsilon^3). \tag{2.70}$$

To expand the divergence term, we need some additional work. First of all, expanding the operator T_{ϕ_r}, we get

$$T_{\phi_r}(\eta^* + \epsilon\zeta^*) = T_{\phi_r}(\eta^*) + \epsilon M^r(\eta^*)\zeta^* + \frac{1}{2}\epsilon^2 S_{\cdot jk}^r(\eta^*)\zeta_j^*\zeta_k^* + \mathcal{O}(\epsilon^3), \qquad \eta^*, \zeta^* \in \mathrm{V}^*,$$

so that, for any $r = 1,\dots,m$,

$$\begin{aligned}
&\epsilon^2 T_{\phi_r}(\nabla w_\epsilon^r) \\
&= T_{\phi_r}\left(\epsilon W_\epsilon^{r\prime}\ \nabla d_\epsilon^\varphi + \epsilon^2 W_{\epsilon s_\beta}^r \nabla s_{\epsilon\beta}^\varphi + \epsilon^2 \nabla W_\epsilon^r\right) \\
&= \epsilon W_\epsilon^{r\prime}\ T_{\phi_r}(\nabla d_\epsilon^\varphi) + \epsilon^2 W_{\epsilon s_\beta}^r M^r(\nabla d_\epsilon^\varphi)\nabla s_{\epsilon\beta}^\varphi + \epsilon^2 M^r(\nabla d_\epsilon^\varphi)\nabla W_\epsilon^r \\
&\quad + \frac{1}{2W_\epsilon^{r\prime}}\epsilon^3 S_{\cdot jk}^r(\nabla d_\epsilon^\varphi)\left[W_{\epsilon s_\beta}^r \partial_{x_j} s_{\epsilon\beta}^\varphi + \partial_{x_j} W_\epsilon^r\right]\left[W_{\epsilon s_\beta}^r \partial_{x_k} s_{\epsilon\beta}^\varphi + \partial_{x_k} W_\epsilon^r\right] \\
&\quad + \mathcal{O}(\epsilon^4).
\end{aligned} \tag{2.71}$$

Remark 2.18. Since we still have to apply the divergence operator (which produces an extra ϵ^{-1} factor), we need to go through the ϵ^3 term in (2.71). We also observe that the term $\mathcal{O}(\epsilon^4)$ in (2.71) is actually a term of order $\mathcal{O}\left(\epsilon^4 \frac{1}{(W_\epsilon^{r\prime})^2}\right)$ which, *a posteriori*, turns out to be of order $\mathcal{O}(\epsilon^4)$: indeed, from (2.86) below it follows that $W_\epsilon^{r\prime}$ is nonvanishing in the transition layer.

We now recall that by Euler's identities for homogeneous functions we have

$$T_{\phi_r}(\xi^*) = \nabla_i T_{\phi_r}(\xi^*)\xi_i^*, \qquad \xi^* \in \mathrm{V}^*, \tag{2.72}$$

which implies

$$T_{\phi_r}(\nabla d_\epsilon^\varphi) \cdot \nabla s_{\epsilon\beta}^\varphi = M^r(\nabla d_\epsilon^\varphi)\nabla s_{\epsilon\beta}^\varphi \cdot \nabla d_\epsilon^\varphi, \qquad r = 1, \dots, m. \tag{2.73}$$

Differentiating (2.72) with respect to ξ_k^* and using the notation $\nabla_{ik}^2 = \frac{\partial^2}{\partial \xi_k^* \partial \xi_i^*}$, we also have

$$\nabla_{ik}^2 T_{\phi_r}(\xi^*)\xi_i^* = S_{\cdot ik}^r \xi_i^* = 0, \qquad \xi^* \in \mathrm{V}^*, \quad k = 1, \dots, n,$$

which implies

$$S_{ijk}^r(\nabla d_\epsilon^\varphi)\nabla_i d_\epsilon^\varphi = 0, \qquad j,k = 1, \dots, n, \quad r = 1, \dots, m. \tag{2.74}$$

For any $r = 1, \dots, m$, we compute, using (2.73),

$$\begin{aligned}
&\epsilon^2 \mathrm{div}(M^r(\nabla d_\epsilon^\varphi)\nabla W_\epsilon^r) \\
&= \epsilon^2 \partial_{x_i}\left(M_{ij}^r(\nabla d_\epsilon^\varphi) W_{\epsilon x_j}^r\right) \\
&= \epsilon T_{\phi_r}(\nabla d_\epsilon^\varphi) \cdot \nabla W_\epsilon^{r\prime} \\
&\quad + \epsilon^2 W_{\epsilon x_j}^r \mathrm{div}\left(M_{\cdot j}^r(\nabla d_\epsilon^\varphi)\right) + \epsilon^2 M^r(\nabla d_\epsilon^\varphi)\nabla s_{\epsilon\beta}^\varphi \cdot \nabla W_{\epsilon s_\beta}^r \\
&\quad + \epsilon^2 W_{\epsilon x_i x_j}^r M_{ij}^r(\nabla d_\epsilon^\varphi).
\end{aligned} \tag{2.75}$$

By differentiating (2.71) we obtain, using also (2.57),

$$\begin{aligned}
&\epsilon^2 \mathrm{div}\left(T_{\phi_r}(\nabla w_\epsilon^r)\right) \\
&= \alpha_r(\nabla d_\epsilon^\varphi) W_\epsilon^{r\prime\prime} + 2\epsilon W_{\epsilon s_\beta}^{r\;\prime}\, T_{\phi_r}(\nabla d_\epsilon^\varphi) \cdot \nabla s_{\epsilon\beta}^\varphi \\
&\quad + 2\epsilon\, T_{\phi_r}(\nabla d_\epsilon^\varphi) \cdot \nabla W_\epsilon^{r\prime} + \epsilon W_\epsilon^{r\prime}\, \mathrm{div}(T_{\phi_r}(\nabla d_\epsilon^\varphi)) \\
&\quad + \epsilon^2 W_{\epsilon s_\beta s_\delta}^r\, M^r(\nabla d_\epsilon^\varphi)\nabla s_{\epsilon\beta}^\varphi \cdot \nabla s_{\epsilon\delta}^\varphi + \epsilon^2 M^r(\nabla d_\epsilon^\varphi)\nabla s_{\epsilon\beta}^\varphi \cdot \nabla W_{\epsilon s_\beta}^r \\
&\quad + \epsilon^2 W_{\epsilon s_\beta}^r \mathrm{div}\left(M^r(\nabla d_\epsilon^\varphi)\nabla s_{\epsilon\beta}^\varphi\right) \\
&\quad + \epsilon^2 W_{\epsilon x_j}^r \mathrm{div}\left(M_{\cdot j}^r(\nabla d_\epsilon^\varphi)\right) + \epsilon^2\; M^r(\nabla d_\epsilon^\varphi)\nabla s_{\epsilon\beta}^\varphi \cdot \nabla W_{\epsilon s_\beta}^r \\
&\quad + \epsilon^2 W_{\epsilon x_i x_j}^r M_{ij}^r(\nabla d_\epsilon^\varphi) \\
&\quad + \mathcal{O}(\epsilon^3),
\end{aligned} \tag{2.76}$$

where we notice that no contribution of order larger than $\mathcal{O}(\epsilon^3)$ can come from the $\mathcal{O}(\epsilon^3)$ term in (2.71) — because they can only be produced *via* differentiation with respect to y, which in turn gives rise to a scalar product between $\nabla d_\epsilon^\varphi$ and the tensor $S^r(\nabla d_\epsilon^\varphi)$ (which in the end vanishes, due to Euler's identities (2.74)).

Hence, in terms of U_ϵ and W_ϵ^r, the expansion of the r-th parabolic equation in (2.56), for $r = 1, \ldots, m$, reads as, using also (2.68),

$$\begin{aligned}
0 = & -\alpha_r(\nabla d_\epsilon^\varphi) W_\epsilon^{r\,\prime\prime} + f(U_\epsilon) \\
& + \epsilon\Big(V_\epsilon^\varphi U_\epsilon' - 2 W^{r\,\prime}_{\epsilon s_\beta} T_{\phi_r}(\nabla d_\epsilon^\varphi) \cdot \nabla s^\varphi_{\epsilon\beta} - 2 T_{\phi_r}(\nabla d_\epsilon^\varphi) \cdot \nabla W_\epsilon^{r\,\prime} \\
& \qquad - W_\epsilon^{r\,\prime} \operatorname{div}\left(T_{\phi_r}(\nabla d_\epsilon^\varphi)\right)\Big) \\
& + \epsilon^2 \Big(U_{\epsilon t} - W^r_{\epsilon s_\beta s_\delta} M^r(\nabla d_\epsilon^\varphi) \nabla s^\varphi_{\epsilon\beta} \cdot \nabla s^\varphi_{\epsilon\delta} \\
& \qquad - 2 M^r(\nabla d_\epsilon^\varphi) \nabla s^\varphi_{\epsilon\beta} \cdot \nabla W^r_{\epsilon s_\beta} - W^r_{\epsilon s_\beta} \operatorname{div}\Big(M^r(\nabla d_\epsilon^\varphi) \nabla s^\varphi_{\epsilon\beta}\Big) \\
& \qquad - W^r_{\epsilon x_j} \operatorname{div}\left(M^r_{\cdot j}(\nabla d_\epsilon^\varphi)\right) - W^r_{\epsilon x_i x_j} M^r_{ij}(\nabla d_\epsilon^\varphi)\Big) \\
& + \mathcal{O}\big(\epsilon^3\big).
\end{aligned} \tag{2.77}$$

2.3.5 Order 0

Recall [44] that $\nabla d_\epsilon^\varphi$ satisfies the anisotropic eikonal equation

$$(\Phi(\nabla d_\epsilon^\varphi))^2 = 1 \tag{2.78}$$

in the evolving transition layer.

Assuming the formal expansion

$$d_\epsilon^\varphi = d_0^\varphi + \epsilon d_1^\varphi + \epsilon^2 d_2^\varphi + \mathcal{O}(\epsilon^3), \tag{2.79}$$

where $d_0^\varphi(t,\cdot)$ is the φ-signed distance from $\Sigma_0(t)$ (positive in the interior of $\{u_0(t,\cdot) = 1\}$), equation (2.78) leads to

$$\begin{aligned}
1 = & \, \Phi^2(\nabla d_0^\varphi) + 2\epsilon T_\Phi(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi \\
& + \epsilon^2 \Big(2 T_\Phi(\nabla d_0^\varphi) \cdot \nabla d_2^\varphi + \nabla T_\Phi(\nabla d_0^\varphi) \nabla d_1^\varphi \cdot \nabla d_1^\varphi \Big) + \mathcal{O}(\epsilon^3),
\end{aligned}$$

which in particular entails:

$$\Phi^2(\nabla d_0^\varphi) = 1, \tag{2.80}$$

$$T_\Phi(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi = 0, \tag{2.81}$$

$$2 T_\Phi(\nabla d_0^\varphi) \cdot \nabla d_2^\varphi + \nabla T_\Phi(\nabla d_0^\varphi) \nabla d_1^\varphi \cdot \nabla d_1^\varphi = 0. \tag{2.82}$$

Using formula (1.10), equation (2.80) reads as

$$\sum_{r=1}^{m} \frac{1}{\alpha_r(\nabla d_0^\varphi(t,x))} = 1, \tag{2.83}$$

again for all x in a suitable tubular neighbourhood of $\Sigma_\epsilon(t)$.

Remark 2.19 (Weights). The quantities

$$\frac{1}{\alpha_r(\nabla d_0^\varphi)}, \qquad r = 1, \ldots, m$$

can be used as "weights" to obtain a weighted mean of equations (2.77). This observation will be crucial in the sequel.

Collecting all terms of order zero in ϵ from each parabolic equation (2.77), dividing by $\alpha_r(\nabla d_0^\varphi)$, summing $r = 1, \ldots, m$ and using (2.83), we obtain

$$-U_0'' + f(U_0) = 0, \tag{2.84}$$

where we used expansions (2.67), (2.69), (2.70), (2.79) for U_ϵ, W_ϵ^r, $f(U_\epsilon)$, d_ϵ^φ, and we have employed (2.66).

The only admissible solution of (2.84) (see for instance [40, 31]) is the standard standing wave

$$U_0(y,s,t)=\gamma(y), \qquad y\in\mathbb{R},$$

where $\gamma(y)=\mathrm{tgh}(cy)$ (here c is a constant only depending on f); in particular, U_0 does not depend on $(s,\,t)$.

Now we can recover each of the m functions w_0^r, $r=1,\dots,m$, by substituting $f(U_0)=U_0''$ into (2.77):

$$\alpha_r(\nabla d_0^\varphi){w_0^r}''=U_0''=\gamma''.$$

Hence

$$W_0^{r\prime\prime}=\frac{1}{\alpha_r(\nabla d_0^\varphi)}U_0''=\frac{1}{\alpha_r(\nabla d_0^\varphi)}\gamma'', \qquad r=1,\dots,m. \tag{2.85}$$

We also get by integration[(4)]

$$W_0^{r\prime}=\frac{1}{\alpha_r(\nabla d_0^\varphi)}U_0'=\frac{1}{\alpha_r(\nabla d_0^\varphi)}\gamma', \qquad r=1,\dots,m. \tag{2.86}$$

Remark 2.20. The functions $W_0^{r\prime}$ depend explicitly on x (and on t) through the coefficient $\frac{1}{\alpha_r(\nabla d_0^\varphi)}$. They are, on the other hand, independent of s.

2.3.6 Order 1

Let us consider the terms of order ϵ in equations (2.77). To this aim, we use the representation of $\frac{1}{2}\nabla^2\alpha=Q+Q_0$ given in Section 1.3.1 for $\alpha=\Phi^2$, namely

$$Q=\alpha^2\sum_{r=1}^m\frac{1}{\alpha_r^2}M^r,$$

where

$$Q_0(\xi^*)\xi^*=0, \qquad \xi^*\in \mathrm{V}^*. \tag{2.87}$$

Remember that by Euler's identities for homogeneous functions we have

$$T_\Phi(\xi^*)=\frac{1}{2}\nabla^2\alpha(\xi^*)\xi^*=\left(Q(\xi^*)+Q_0(\xi^*)\right)\xi^*, \qquad \xi^*\in \mathrm{V}^*.$$

Hence, using (2.87),

$$\begin{aligned} T_\Phi(\nabla d_0^\varphi)=&\Big(Q(\nabla d_0^\varphi)+Q_0(\nabla d_0^\varphi)\Big)\nabla d_0^\varphi\\ =&Q(\nabla d_0^\varphi)\nabla d_0^\varphi=(\alpha(\nabla d_0^\varphi))^2\sum_{r=1}^m\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}M^r(\nabla d_0^\varphi)\nabla d_0^\varphi\\ =&\sum_{r=1}^m\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}M^r(\nabla d_0^\varphi)\nabla d_0^\varphi, \end{aligned} \tag{2.88}$$

where the last equality follows from (2.78). Therefore

$$\mathrm{div}\,(T_\Phi(\nabla d_0^\varphi))=\sum_{r=1}^m\mathrm{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}T_{\phi_r}(\nabla d_0^\varphi)\right). \tag{2.89}$$

For each $r=1,\dots,\,m$, we now collect all terms of order one in (2.77).

Remembering once more that $U_0=\gamma$ and $w_0^{r\prime}$ do not depend explicitly on s and t so that in particular $W^{r\prime}_{0s_\beta}=0$, we obtain

$$\begin{aligned} &-\alpha_r(\nabla d_0^\varphi)W_1^{r\prime\prime}-2W_0^{r\prime\prime}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi+f'(\gamma)U_1\\ &+\gamma'V_0^\varphi-2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla W_0^{r\prime}-W_0^{r\prime}\mathrm{div}\,(T_{\phi_r}(\nabla d_0^\varphi))=0, \end{aligned} \tag{2.90}$$

[(4)]See Section 2.3.7 below, and in particular equation (2.105).

where we have taken into account that the term

$$-\alpha_r(\nabla d_0^\varphi) W_1^{r\prime\prime} - 2w_0^{r\prime\prime} T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi + f'(\gamma)U_1$$

arises from the expansion at the order ϵ of the first line on the right hand side of (2.77).

Using formula (2.86), equation (2.90) can be rewritten as

$$\begin{aligned}
&-\alpha_r(\nabla d_0^\varphi)W_1^{r\prime\prime} - 2\gamma''\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)} + f'(\gamma)U_1 + \gamma' V_0^\varphi \\
&-\alpha_r(\nabla d_0^\varphi)\gamma'\left[\frac{\mathrm{div}\,(T_{\phi_r}(\nabla d_0^\varphi))}{(\alpha_r(\nabla d_0^\varphi))^2} + 2\,\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla\frac{1}{\alpha_r(\nabla d_0^\varphi)}}{\alpha_r(\nabla d_0^\varphi)}\right] = 0.
\end{aligned} \tag{2.91}$$

Since $\nabla\frac{1}{\alpha_r^2} = \frac{2}{\alpha_r}\nabla\frac{1}{\alpha_r}$, the expression in square brackets is simply

$$\mathrm{div}\Big(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}T_{\phi_r}(\nabla d_0^\varphi)\Big), \qquad r = 1,\dots,m. \tag{2.92}$$

Recalling (2.89), the sum over $r = 1,\dots,m$ of the latter divergences gives $\mathrm{div}\,(T_\Phi(\nabla d_0^\varphi))$. The weighted sum of equations (2.91) finally produces

$$-\mathcal{L}(U_1) = \gamma'\Big[V_0^\varphi - \mathrm{div}\,(T_\Phi(\nabla d_0^\varphi))\Big],$$

where

$$\mathcal{L}(g) := -g'' + f'(\gamma)g,$$

and we make use of (2.81).

Recall now that from (2.62) and the expansions of κ_ϵ^φ it follows

$$\mathrm{div}\,(T_\Phi(\nabla d_\epsilon^\varphi)) = -\kappa_0^\varphi - \epsilon\kappa_1^\varphi - \epsilon y h_0^\varphi + \mathcal{O}(\epsilon^2 y^2), \tag{2.93}$$

in particular

$$\mathrm{div}\,(T_\Phi(\nabla d_0^\varphi)) = -\kappa_0^\varphi. \tag{2.94}$$

We then obtain

$$-\mathcal{L}(U_1) = \gamma'\Big[V_0^\varphi + \kappa_0^\varphi\Big]. \tag{2.95}$$

We recall now from [40, 31, 26] that for equation $-\mathcal{L}(g) = v$ to be solvable, we must enforce the orthogonality condition

$$\int_{\mathbb{R}} \gamma' v\; dy = 0. \tag{2.96}$$

This and (2.95) imply the remarkable fact

$$V_0^\varphi = -\kappa_0^\varphi, \tag{2.97}$$

so that

$$U_1 = 0. \tag{2.98}$$

Remark 2.21 (Convergence to anisotropic mean curvature flow). Note carefully that (2.97) justifies the convergence of solutions of system (2.4) to Φ^o-anisotropic mean curvature flow.

Substituting (2.97) and (2.98) in (2.91), dividing by $\alpha_r(\nabla d_0^\varphi)$ and recalling that the square bracket in (2.91) equals (2.92), we end up with the equation for W_1^r, for any $r = 1,\dots,m$:

$$\begin{aligned}
W_1^{r\prime\prime} &= \frac{1}{\alpha_r(\nabla d_0^\varphi)}\gamma' V_0^\varphi - \gamma'\mathrm{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}T_{\phi_r}(\nabla d_0^\varphi)\right) \\
&\quad - 2\gamma''\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} \\
&= \frac{1}{\alpha_r(\nabla d_0^\varphi)}\gamma'\mathrm{div}\,(T_\Phi(\nabla d_0^\varphi)) - \gamma'\mathrm{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}T_{\phi_r}(\nabla d_0^\varphi)\right) \\
&\quad - 2\gamma''\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2},
\end{aligned} \tag{2.99}$$

since, from (2.94) and (2.97),

$$\operatorname{div}(T_\Phi(\nabla d_0^\varphi)) = V_0^\varphi.$$

As a consequence, recalling (2.83), (2.89) and (2.81), we have

$$\sum_{r=1}^m W_1^{r\prime\prime} = U_1'' = 0, \tag{2.100}$$

where the last equality follows from (2.98).

Equation (2.99) can be written as[5]

$$\begin{aligned} W_1^{r\prime\prime} =& \gamma' \Bigg[\operatorname{div}\left(\frac{1}{\alpha_r(\nabla d_0^\varphi)} T_\Phi(\nabla d_0^\varphi) \right) - \operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2} T_{\phi_r}(\nabla d_0^\varphi) \right) \\ & - T_\Phi(\nabla d_0^\varphi) \cdot \nabla \frac{1}{\alpha_r(\nabla d_0^\varphi)} \Bigg] - 2\gamma'' \frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}. \end{aligned} \tag{2.101}$$

From (2.100) it follows that $\sum_{r=1}^m W_1^r$ minus a linear function vanishes, namely

$$\sum_{r=1}^m W_1^r - U_1 = C_1 y + C_0.$$

We now claim that $C_0 = C_1 = 0$, and hence

$$\sum_{r=1}^m W_1^r = U_1 (= 0). \tag{2.102}$$

The constant C_0 turns out to be zero for the following argument: as a consequence of (2.66) and (2.55),

$$0 = U_\epsilon(0, t, x) = \sum_{r=1}^m W_\epsilon^r(0, t, x), \qquad \epsilon \in (0, 1),$$

which implies

$$\sum_{r=1}^m W_i^r(0, t, x) = 0, \qquad i \geq 0$$

and hence $C_0 = 0$.

For what concerns the constant C_1, we have, using (2.110) below and (2.81),

$$\begin{aligned} C_1 =& \sum_{r=1}^m W_1^{r\prime} = \sum_{r=1}^m \left\{ (\gamma - 1)\Theta^r - 2\gamma' \, \frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} + w_0^{r\prime} \right\} \\ =& \sum_{r=1}^m \left\{ (\gamma - 1)\Theta^r + w_0^{r\prime} \right\}. \end{aligned}$$

On the other hand, from (2.109) below, it follows $\sum_{r=1}^m w_0^{r\prime} = 0$, so that $C_1 = (\gamma - 1) \sum_{r=1}^m \Theta^r$. In order to conclude the proof of claim (2.102) it is enough to observe that $\sum_{r=1}^m \Theta^r = 0$, as a consequence of the expression of Θ^r in (2.110), and of (2.83) and (2.89), and so $C_1 = 0$.

[5] Although written in a somewhat different form, this result coincides with that of [31], where d_ϵ^φ has not been expanded (hence d_ϵ^φ appears in place of d_0^φ in (2.101), and accordingly the last addendum is not present).

2.3.7 Matching procedure

We are now in a position to recover the first term w_0^r of the outer expansion of w_ϵ^r, by adding to (2.53) a jump condition for w_0^r and a condition for $n_0^\varphi \cdot \nabla w_0^r$ across the interface $\Sigma_0(t)$, defined as the boundary of the external phase $\{u_0(t,\cdot) = 1\}$ (see (2.52)). We set

$$\Sigma_\epsilon(t) = \{x + \epsilon\sigma_1(s,t)n_0^\varphi + \mathcal{O}(\epsilon^2) : x \in \Sigma_0(t)\},$$

for a suitable $\sigma_1 \colon \Sigma \times \mathbb{R} \to \mathbb{R}$, where Σ is the reference manifold in (2.58).

We will make use of the change of variables (2.63), and we will match the two expansions in the region of common validity $|y| \to +\infty$ and x approaching $\Sigma_\epsilon(t)$:

$$w_\epsilon^r\left(t, x(s,t) - \epsilon y n_\epsilon^\varphi + \mathcal{O}(\epsilon^2 y^2)\right) \approx W_\epsilon^r(y; s, t, x(s,t) - \epsilon y n_\epsilon^\varphi + \mathcal{O}(\epsilon^2 y^2)).$$

By expanding the left and right hand sides, understanding that w_ϵ^r is computed at points $x(s,t) \in \Sigma_\epsilon(t)$, we get

$$w_\epsilon^r - \epsilon y n_\epsilon^\varphi \cdot \nabla w_\epsilon^r + \mathcal{O}(\epsilon^2 y^2) \approx W_\epsilon^r - \epsilon y n_\epsilon^\varphi \cdot \nabla W_\epsilon^r + \mathcal{O}(\epsilon^2 y^2), \qquad r = 1, \dots, m.$$

Expanding w_ϵ^r, W_ϵ^r in powers of ϵ, and matching the first two orders, we get in particular

$$\lim_{y \to \pm\infty} W_0^r(y, s(t,\, x), t, x) = w_0^r(t,\, x), \tag{2.103}$$

and

$$\begin{aligned}\lim_{y \to \pm\infty} \Big\{ & W_1^r(y, s(t,\, x), t, x) - w_1^r(t,\, x) \\ & - y\Big(n_0^\varphi \cdot \nabla W_0^r(y, s(t,\, x), t, x) - n_0^\varphi \cdot \nabla w_0^r(t,\, x)\Big)\Big\} = 0,\end{aligned} \tag{2.104}$$

where w_0^r and w_1^r are evaluated at each side of the interface according to when y goes to plus or minus infinity.

Equality (2.103) in particular suggests

$$\lim_{y \to \pm\infty} W_0^{r\prime}(y, s(t,\, x), t, x) = 0, \qquad r = 1, \dots, m, \tag{2.105}$$

and the jump $[\![w_0^r]\!]$ of w_0^r across the interface is given by

$$[\![w_0^r]\!](s(t,\, x), t) = \int_{\mathbb{R}} W_0^{r\prime}(y, s(t,\, x), t, x)\, dy, \qquad r = 1, \dots, m.$$

From (2.86) we get

$$[\![w_0^r]\!] = \frac{c_0}{\alpha_r(\nabla d_0^\varphi)}, \qquad r = 1, \dots, m, \tag{2.106}$$

where

$$c_0 := \int_{\mathbb{R}} \gamma'\, dy \in (0, +\infty).$$

To obtain the equation involving the conormal derivative, we formally differentiate equation (2.104):

$$\begin{aligned}\lim_{y \to \pm\infty} \Big\{ & W_1^{r\prime}(y, s(t,\, x), t, x) \\ & - n_0^\varphi \cdot \nabla W_0^r(y, s(t,\, x), t, x)\Big\} = -n_0^\varphi \cdot \nabla w_0^r(t,\, x),\end{aligned} \tag{2.107}$$

where we used also the fact that

$$\lim_{y \to \pm\infty} y\ n_0^\varphi \cdot \nabla W_0^{r\prime}(y, s(t,\, x), t, x) = 0,$$

since $\nabla W_0^{r\prime} = \gamma' \nabla \frac{1}{\alpha_r(\nabla d_0^\varphi)}$ by (2.86) and γ' decays exponentially to 0 as $y \to \pm\infty$. For the same reason, $W_1^{r\prime}$ is also bounded, thus

$$-[\![n_0^\varphi \cdot \nabla w_0^r]\!](t,\, x) = \int_{\mathbb{R}} \left(W_1^{r\prime\prime}(y, s(t,\, x), t, x) - n_0^\varphi \cdot \nabla W_0^{r\prime}(y, s(t,\, x), t, x)\right)\, dy.$$

Coupling previous line with (2.99) and (2.86), and recalling from (2.59) that $n_0^\varphi = -T_\Phi(\nabla d_0^\varphi)$ we end up with

$$- [\![n_0^\varphi \cdot \nabla w_0^r]\!] = c_0 \operatorname{div} \left[\frac{1}{\alpha_r(\nabla d_0^\varphi)} T_\Phi(\nabla d_0^\varphi) - \frac{1}{\alpha_r^2(\nabla d_0^\varphi)} T_{\phi_r}(\nabla d_0^\varphi) \right]. \tag{2.108}$$

The two jump conditions on w_0 across $\Sigma_0(t)$, together with the *far field* equation (2.53) and appropriate boundary conditions at $\partial\Omega$ allow to retrieve a unique solution w_0.

If we integrate (2.86), and use the matching condition for w_0^r in (2.103), we get for W_0^r the expression

$$W_0^r = \frac{1}{\alpha_r(\nabla d_0^\varphi)}(\gamma - 1) + w_0^{r+}(s,t), \qquad r = 1, \dots, m,$$

where w_0^{r+} is the trace on $\Sigma_0(t)$ of w_0^r from the external phase $\{u_0(t,\cdot) = 1\}$. In particular

$$\sum_{r=1}^m w_0^{r+} = 1. \tag{2.109}$$

Thus, for every $r = 1, \dots, m$,

$$\begin{aligned} &W^r_{0s_\beta} = w^{r+}_{0s_\beta}, \qquad \nabla W_0^r = (\gamma - 1)\nabla \frac{1}{\alpha_r}, \qquad \nabla W^r_{0s_\beta} = 0, \\ &W^r_{0s_\beta s_\delta} = w^{r+}_{0s_\beta s_\delta}, \qquad W^r_{0x_i x_j} = (\gamma - 1)\partial_{x_i x_j} \frac{1}{\alpha_r}. \end{aligned}$$

In a similar fashion we can integrate (2.99), and use the matching condition (2.107), to get, for any $r = 1, \dots, m$,

$$\begin{aligned} W_1^{r\prime} = &(\gamma - 1) \left\{ \frac{1}{\alpha_r(\nabla d_0^\varphi)} \operatorname{div}(T_\Phi(\nabla d_0^\varphi)) - \operatorname{div} \left(\frac{1}{\alpha_r^2(\nabla d_0^\varphi)} T_{\phi_r}(\nabla d_0^\varphi) \right) \right\} \\ &- 2\gamma' \, \frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} + w_0^{r\prime}(s,t) \\ =&(\gamma - 1)\Theta^r(t,\, x) - 2\gamma' \, \frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} + w_0^{r\prime}(s,t), \end{aligned} \tag{2.110}$$

where

$$w_0^{r\prime} := T_\Phi(\nabla d_0^\varphi) \cdot \nabla w_0^{r+},$$

and Θ^r is a shorthand for the expression in braces. Observe that only the last term explicitly depends on s, while the other terms depend on y (by means of γ) and on x (by means of Θ^r). Thus

$$W^{r\prime}_{1s_\beta} = w^{r\,\prime}_{0s_\beta}$$

$$\nabla W_1^{r\prime} = (\gamma - 1)\nabla\Theta^r - 2\gamma'\nabla \left(\frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} \right). \tag{2.111}$$

Remark 2.22. Note that the jump in the *conormal* derivative $[\![n_0^\varphi \cdot \nabla w_0^r]\!]$ vanishes in the special case of equal anisotropic ratio, which, in our context, consists of choosing, for every $r = 1, \dots,\, m$, $\alpha_r := \lambda_r \overline{\alpha}$ with some given smooth symmetric uniformly convex squared anisotropy $\overline{\alpha}$ and positive λ_r (indeed, in this case eikonal equation (2.83) leads to $\overline{\alpha}(\nabla d_0^\varphi) = \sum_{r=1}^m \lambda_r^{-1}$).

Remark 2.23. Given $r = 1, \dots, m$, the function $W_1^r(\cdot, t,\, x)$ is expected to have linear growth at infinity (independent of ϵ), differently with respect to $W_0^r(\cdot, t,\, x)$, which is expected to be bounded at infinity. Observe, however, that $\sum_{r=1}^m W_1^r(\cdot, t,\, x) = 0$, see (2.102).

2.3.8 Order 2

We end our asymptotic analysis considering the $\mathcal{O}(\epsilon^2)$ terms in equation (2.77), which represents an improvement with respect to [43] (in which expansions are performed only up to the order $\mathcal{O}(\epsilon)$ and $m = 2$). Recall that $U_0' = \gamma'$ depends only on y and that $U_1 = 0$. Then the terms of order $\mathcal{O}(\epsilon^2)$ arising from the first line on the right hand side of (2.77) are:

$$\begin{aligned}
&-\alpha_r(\nabla d_0^\varphi)W_2^{r\prime\prime} - 2W_1^{r\prime\prime}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi \\
&-\Big[2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_2^\varphi + M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla d_1^\varphi\Big]W_0^{r\prime\prime} + f'(\gamma)U_2.
\end{aligned} \tag{2.112}$$

The terms of order $\mathcal{O}(\epsilon)$ arising from the terms in the round parentheses in the second and third lines of (2.77) are

$$\begin{aligned}
&\gamma' V_1^\varphi - 2W_{1s_\beta}^{r\prime}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla s_{0\beta}^\varphi - 2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla W_1^{r\prime} \\
&-2M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla W_0^{r\prime} - W_1^{r\prime}\operatorname{div}\big(T_{\phi_r}(\nabla d_0^\varphi)\big) \\
&-W_0^{r\prime}\operatorname{div}\big(M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\big).
\end{aligned} \tag{2.113}$$

Let us shorthand by D^r the sum of the terms of order $\mathcal{O}(1)$ arising from the terms in the round parentheses in the fourth, fifth and sixth lines of (2.77), namely

$$\begin{aligned}
&-W_{0s_\beta s_\delta}^r M^r(\nabla d_0^\varphi)\nabla s_{0\beta}^\varphi\cdot\nabla s_{0\delta}^\varphi - 2M^r(\nabla d_0^\varphi)\nabla s_{0\beta}^\varphi\cdot\nabla W_{0s_\beta}^r \\
&-W_{0s_\beta}^r\operatorname{div}\Big(M^r(\nabla d_0^\varphi)\nabla s_{0\beta}^\varphi\Big) - W_{0x_j}^r\operatorname{div}\big(M_{\cdot j}^r(\nabla d_0^\varphi)\big) \\
&-W_{0x_ix_j}^r M_{ij}^r(\nabla d_0^\varphi).
\end{aligned} \tag{2.114}$$

Collecting together (2.112), (2.113) and (2.114) we get

$$\begin{aligned}
&-\alpha_r(\nabla d_0^\varphi)W_2^{r\prime\prime} - 2W_1^{r\prime\prime}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi \\
&-\Big[2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_2^\varphi + M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla d_1^\varphi\Big]W_0^{r\prime\prime} + f'(\gamma)U_2 \\
&\gamma' V_1^\varphi - 2W_{1s_\beta}^{r\prime}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla s_{0\beta}^\varphi - 2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla W_1^{r\prime} \\
&-2M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla W_0^{r\prime} - W_1^{r\prime}\operatorname{div}\big(T_{\phi_r}(\nabla d_0^\varphi)\big) \\
&-W_0^{r\prime}\operatorname{div}\big(M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\big) + \mathrm{D}^r.
\end{aligned} \tag{2.115}$$

Substituting (2.85), (2.86), (2.101), (2.110), (2.111) into (2.115), also recalling (2.94), and reordering terms, we get, for any $r = 1, \ldots, m$,

$$\begin{aligned}
0 = \; &-\alpha_r(\nabla d_0^\varphi)W_2^{r\prime\prime} + f'(\gamma)U_2 + \gamma' V_1^\varphi \\
&+\gamma'\left\{2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi\left[\frac{\kappa_0^\varphi}{\alpha_r(\nabla d_0^\varphi)} + \operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right)\right]\right\} \\
&+4\gamma' T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right) \\
&-2\gamma' M^r(\nabla d_0)\nabla d_1^\varphi\cdot\nabla\left(\frac{1}{\alpha_r(\nabla d_0^\varphi)}\right) \\
&-\gamma'\frac{1}{\alpha_r(\nabla d_0^\varphi)}\operatorname{div}\big(M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\big) \\
&+2\gamma'\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\operatorname{div}\left(T_{\phi_r}(\nabla d_0^\varphi)\right) \\
&+\gamma''\mathrm{A}^r + \mathrm{B}^r + \mathrm{C}^r + \mathrm{D}^r,
\end{aligned} \tag{2.116}$$

where for brevity we have set

$$\mathrm{A}^r := \left(\frac{2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)}\right)^2 - \frac{\Big[2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_2^\varphi + M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla d_1^\varphi\Big]}{\alpha_r(\nabla d_0^\varphi)},$$

and

$$\mathrm{B}^r := -2W^{r\,\prime}_{1s_\beta} T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla s_{0\beta}^\varphi,$$
$$\mathrm{C}^r := -\big[(\gamma-1)\Theta^r + w_0^{r\,\prime}\big]\mathrm{div}(T_{\phi_r}(\nabla d_0^\varphi)) + (\gamma-1)\nabla\Theta^r\cdot T_{\phi_r}(\nabla d_0^\varphi).$$

Let us now focus the attention to (2.116), where for the moment we neglect the first and the last lines. Dividing by $\alpha_r(\nabla d_0^\varphi)$, we get (again reordering terms, and up to the factor γ')

$$\begin{aligned}
&\frac{2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\kappa_0^\varphi + \frac{2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)}\mathrm{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right)\\
&+4\,\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\cdot\nabla\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi)^2}\right)\\
&+2\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^3}\mathrm{div}\left(T_{\phi_r}(\nabla d_0^\varphi)\right)\\
&-\frac{2}{\alpha_r(\nabla d_0^\varphi)}M^r(\nabla d_0)\nabla d_1^\varphi\cdot\nabla\left(\frac{1}{\alpha_r(\nabla d_0^\varphi)}\right)\\
&-\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}\mathrm{div}\big(M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\big)
\end{aligned}\tag{2.117}$$

Observe now that the first term in (2.117) will disappear when summing up on $r = 1,\ldots,m$, thanks to (2.88) and (2.80). Moreover, the last two terms of (2.117) can be put together giving div $\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right)$ so that, summing up on r, we get

$$\begin{aligned}
&\underbrace{2\sum_{r=1}^m\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)}\mathrm{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right)}_{:=E}\\
&\underbrace{+4\sum_{r=1}^m\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\cdot\nabla\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r^2(\nabla d_0^\varphi)}\right)}_{:=F}\\
&\underbrace{+2\sum_{r=1}^m\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^3}\mathrm{div}T_{\phi_r}(\nabla d_0^\varphi)}_{:=G}\\
&\underbrace{-\sum_{r=1}^m\mathrm{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right)}_{:=H}.
\end{aligned}$$

Now, we claim that

$$\kappa_1^\varphi + yh_0^\varphi = E+F+G+H.\tag{2.118}$$

In order to prove (2.118), we shall make use of the equality[(6)]

$$-\kappa_1^\varphi - yh_0^\varphi = \mathrm{div}\big(\nabla T_\Phi(\nabla d_0^\varphi)\nabla d_1^\varphi\big).$$

Using formulas (1.16), (1.17), and the relations $\nabla\alpha_r = 2T_{\phi_r}$, $\Phi^2(\nabla d_0^\varphi) = 1$, we get

$$\begin{aligned}
-\kappa_1^\varphi - yh_0^\varphi =&\sum_{r=1}^m\mathrm{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right)\\
&+\sum_{r=1}^m\mathrm{div}\left(\frac{1-\alpha_r(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^4}\nabla\alpha_r(\nabla d_0^\varphi)\otimes\nabla\alpha_r(\nabla d_0^\varphi)\nabla d_1^\varphi\right)\\
&+\sum_{\substack{j,r=1,\\ j\neq r}}^m\mathrm{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2(\alpha_j(\nabla d_0^\varphi))^2}\nabla\alpha_r(\nabla d_0^\varphi)\otimes\nabla\alpha_j(\nabla d_0^\varphi)\nabla d_1^\varphi\right).
\end{aligned}$$

(6) Recall (2.93).

Adding and subtracting in the previous line the term

$$4\sum_{r=1}^{m}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^4}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_r}(\nabla d_0^\varphi)\nabla d_1^\varphi\right),$$

it follows

$$\begin{aligned}-\kappa_1^\varphi - yh_0^\varphi =&\sum_{r=1}^{m}\operatorname{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right)\\ &+4\sum_{r=1}^{m}\operatorname{div}\left(\frac{1-\alpha_r(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^4}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right)\\ &+4\sum_{\substack{j,r=1,\\ j\neq r}}^{m}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2(\alpha_j(\nabla d_0^\varphi))^2}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_j}(\nabla d_0^\varphi)\nabla d_1^\varphi\right)\\ &-4\sum_{r=1}^{m}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^4}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_r}(\nabla d_0^\varphi)\nabla d_1^\varphi\right).\end{aligned}$$

Notice that, for every $r=1,\dots,m$, we have

$$\sum_{\substack{j=1,\\ j\neq r}}^{m}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2(\alpha_j(\nabla d_0^\varphi))^2}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_j}(\nabla d_0^\varphi)\nabla d_1^\varphi\right)=0,$$

thanks again to (2.80). Hence

$$\begin{aligned}\kappa_1^\varphi + yh_0^\varphi =&-\sum_{r=1}^{m}\operatorname{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right)\\ &-4\sum_{r=1}^{m}\operatorname{div}\left(\frac{1-\alpha_r(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^4}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right)\\ &+4\sum_{r=1}^{m}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^4}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_r}(\nabla d_0^\varphi)\nabla d_1^\varphi\right)\\ =&-\sum_{r=1}^{m}\operatorname{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right)\\ &+4\sum_{r=1}^{m}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^3}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right),\end{aligned}\tag{2.119}$$

where we used the identity

$$T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_r}(\nabla d_0^\varphi)\nabla d_1^\varphi=(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi).$$

Observe that the term appearing in the fourth line of (2.119) cancels with H, so that, in order to prove claim (2.118), it will be enough to show that

$$E+F+G=4\sum_{r=1}^{m}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right).\tag{2.120}$$

The right hand side of (2.120) can be rewritten as

$$4\sum_{r=1}^{m}\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\right)+4\sum_{r=1}^{m}\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\cdot\nabla\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r^2(\nabla d_0^\varphi)}\right),$$

so that its last addendum cancels with F. Thus, we are left to prove that

$$E+G=4\sum_{r=1}^{m}\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\right),$$

or equivalently

$$\sum_{r=1}^{m} \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)} \left\{ \mathrm{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right) + \frac{\mathrm{div}\,(T_{\phi_r}(\nabla d_0^\varphi))}{(\alpha_r(\nabla d_0^\varphi))^2} - \frac{2}{\alpha_r(\nabla d_0^\varphi)}\mathrm{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\right)\right\} = 0. \tag{2.121}$$

This can be done for instance using the identity

$$\mathrm{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right) = \frac{1}{\alpha_r(\nabla d_0^\varphi)}\mathrm{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\right) + \frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\cdot\nabla\left(\frac{1}{\alpha_r(\nabla d_0^\varphi)}\right);$$

then the quantity in braces in (2.121) equals, for any $r = 1,\dots,m$,

$$\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\cdot\nabla\left(\frac{1}{\alpha_r(\nabla d_0^\varphi)}\right) + \frac{\mathrm{div}\,(T_{\phi_r}(\nabla d_0^\varphi))}{(\alpha_r(\nabla d_0^\varphi))^2} - \frac{1}{\alpha_r(\nabla d_0^\varphi)}\mathrm{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\right),$$

which is identically zero. This proves claim (2.120), and hence (2.118).

Let us come back to (2.116). Dividing by $\alpha_r(\nabla d_0^\varphi)$, summing over $r = 1,\dots,m$, using also (2.83) and (2.118), we get

$$0 = -U_2'' + U_2 f'(\gamma) + \gamma'(V_1^\varphi + \kappa_1^\varphi) + y\gamma' h_0^\varphi + \sum_{r=1}^{m}\frac{1}{\alpha_r(\nabla d_0^\varphi)}\left[\gamma''\mathrm{A}^r + \mathrm{B}^r + \mathrm{C}^r + \mathrm{D}^r\right].$$

Note that we have used $U_2 = \sum_{r=1}^m W_2^r$: in general it may happen that $U_2 - \sum_{r=1}^m W_2^r = \mathcal{O}(\epsilon)$, but we have the freedom[(7)] to redefine the functions W_2^r up to discrepancies of order $\mathcal{O}(\epsilon)$, and put the subsequent errors in the terms U_3 and W_3^r, which we are not interested in.

Incidentally, we notice that

$$\sum_{r=1}^{m}\frac{1}{\alpha_r(\nabla d_0^\varphi)}\mathrm{A}^r = 0. \tag{2.122}$$

Indeed, let us show that

$$\sum_{r=1}^{m}\frac{1}{\alpha_r(\nabla d_0^\varphi)}\mathrm{A}^r = -(2T_\Phi(\nabla d_0^\varphi)\cdot\nabla d_2^\varphi) - \nabla T_\Phi(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla d_1^\varphi. \tag{2.123}$$

Then (2.122) will follow at once recalling (2.82). In order to prove (2.123), we make use of the representation formula for the hessian ∇T_Φ given in Section 1.3.1 (in particular, recall (1.16) and (1.17)). Using also (2.78), we can rewrite $\nabla T_\Phi(\nabla d_0^\varphi)$ as

$$\begin{aligned}\nabla T_\Phi(\nabla d_0^\varphi) = &\sum_{r=1}^{m}\frac{1}{\alpha_r(\nabla d_0^\varphi)}M^r(\nabla d_0^\varphi)\\ &+ 4\sum_{r=1}^{m}\frac{1-\alpha_r(\nabla d_0^\varphi)}{\alpha_r^4(\nabla d_0^\varphi)}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_r}(\nabla d_0^\varphi)\\ &+ 4\sum_{\substack{j,r=1,\\ j\neq r}}^{m}\frac{1}{\alpha_j^2(\nabla d_0^\varphi)\alpha_r^2(\nabla d_0^\varphi)}T_{\phi_j}(\nabla d_0^\varphi)\otimes T_{\phi_r}(\nabla d_0^\varphi);\end{aligned}$$

(7) This is because enforcing the relation between (t,x) and (y,s,t,x) introduces a dependence on ϵ.

as a consequence,

$$\begin{aligned}\nabla T_\Phi(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla d_1^\varphi =&\sum_{r=1}^m \frac{1}{\alpha_r(\nabla d_0^\varphi)} M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla d_1^\varphi\\ &+4\sum_{r=1}^m \frac{1-\alpha_r(\nabla d_0^\varphi)}{\alpha_r^4(\nabla d_0^\varphi)}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)^2\\ &+4\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r^2(\nabla d_0^\varphi)}\sum_{\substack{j=1,\\ j\neq r}}^m \frac{T_{\phi_j}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_j^2(\nabla d_0^\varphi)}.\end{aligned}\tag{2.124}$$

Recalling (2.88), using also (2.81), for any $r=1,\dots,m$ one has

$$\begin{aligned}\sum_{\substack{j=1,\\ j\neq r}}^m \frac{T_{\phi_j}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_j^2(\nabla d_0^\varphi)} =&T_\Phi(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi - \frac{1}{\alpha_r^2(\nabla d_0^\varphi)}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi\\ =&-\frac{1}{\alpha_r^2(\nabla d_0^\varphi)}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi,\end{aligned}\tag{2.125}$$

so that, putting (2.125) into (2.124), we end up with

$$\begin{aligned}\nabla T_\Phi(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla d_1^\varphi =&\sum_{r=1}^m \frac{1}{\alpha_r(\nabla d_0^\varphi)} M^r(\nabla d_0^\varphi)\nabla d_1^\varphi\cdot\nabla d_1^\varphi\\ &-4\sum_{r=1}^m \frac{1}{\alpha_r^3(\nabla d_0^\varphi)}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)^2.\end{aligned}\tag{2.126}$$

Concerning the term $2T_\Phi(\nabla d_0^\varphi)\cdot\nabla d_2^\varphi$ appearing in (2.123), we have

$$2T_\Phi(\nabla d_0^\varphi)\cdot\nabla d_2^\varphi = 2\sum_{r=1}^m \frac{1}{\alpha_r^2(\nabla d_0^\varphi)}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_2^\varphi,\tag{2.127}$$

where we have made use once more of (2.88). Claim (2.123) now follows from(2.126), (2.127), and recalling the definition of A^r.

Observing that

$$\int_{\mathbb{R}} y\gamma'\gamma'\ dy = 0,$$

(so that the orthogonality condition (2.96) leads to drop out the terms with h_0^φ), we end up with the following integrability condition:

$$0 = c_1(V_1^\varphi+\kappa_1^\varphi)+c_0\mathfrak{g},$$

where

$$c_1=\int_{\mathbb{R}}(\gamma')^2\ dy$$

and

$$\mathfrak{g}=\sum_{r=1}^m \frac{1}{\alpha_r(\nabla d_0^\varphi)}\left(\mathrm{B}^r+\mathrm{C}^r+\mathrm{D}^r\right).$$

The term $\mathfrak{g}$ is presumably nonzero, which shows that, in general, $V_1^\varphi\neq\kappa_1^\varphi$. This is a difference with respect to the formal asymptotic analysis of the anisotropic Allen-Cahn's equation [40, 31, 26], and suggests an $\mathcal{O}(\epsilon)$-error estimate between the geometric front and $\Sigma_\epsilon(t)$ (while, in the Allen-Cahn's equation, the estimate can be improved to the order $\mathcal{O}(\epsilon^2)$).

Remark 2.24 (Approximate evolution law and forcing term). The integrability condition for function U_2 relates V_1^φ and κ_1^φ and together with the integrability condition for U_1 leads to the approximate evolution law

$$V_\epsilon = -\kappa_\epsilon^\varphi - \epsilon\frac{c_0}{c_1}\mathfrak{g} + \mathcal{O}(\epsilon^2)$$

for Σ_ϵ. By dropping the $\mathcal{O}(\epsilon^2)$ term we obtain a new approximation Σ_1 of Σ_ϵ which we assume to have an $\mathcal{O}(\epsilon^2)$ error. This allows in turn to recover the $\mathcal{O}(\epsilon)$ term for the signed distance d_1^φ by taking the difference between the signed distance from $\Sigma_1(t)$ and the signed distance from $\Sigma_0(t)$ and dividing by ϵ. Now we can recover the functions w_1^r (which indeed depend on ∇d_1^φ) and solve the differential equation for U_2 (which also depends on ∇d_1^φ) to get U_2. This argument works provided $\mathfrak{g}$ does not depend on d_1^φ, since it is also through $\mathfrak{g}$ that the function U_2 is determined. We see from the definitions of B^r, C^r and D^r, that the function $\mathfrak{g}$ is indeed independent of d_1^φ.

Problem 2.25. Investigate on the existence and regularity of solutions to the elliptic equation (2.53), coupled with (2.106), (2.108), leading to the function w_0^r for any $r = 1, \dots, m$.

We notice that, already for the linear bidomain model, it does not seem trivial to state a variational problem leading to function w_0^1. In this respect, the main obstacle is represented by the jump of the conormal derivative: indeed, the conormal derivative depends on the standard Euclidean normal in a strongly nonlinear way, so that condition (2.108) cannot be obtained from Gauss-Green's formula.

An alternative strategy may be given by the so-called "domain-decomposition method" [135]: in this setting, neglecting for simplicity the boundary condition for w_0^1 at $\partial\Omega$, it is convenient to split w_0^1 as

$$w_0^1 = w + z,$$

where we let $w \in H^1(\Omega \setminus \Sigma)$ be any function solving[(8)]

$$\begin{cases} \operatorname{div}\Big(\big(T_{\phi_1} + T_{\phi_2}\big)(\nabla w)\Big) = 0, & \text{in } \Omega, \\ [\![n_0^\varphi \cdot \nabla w]\!] = h & \text{on } \Sigma, \end{cases}$$

h being the right hand side of (2.108) when $r = 1$. Then, we are reduced to seek for z belonging to a suitable functional space, and such that

$$\begin{cases} \operatorname{div}\Big(\big(T_{\phi_1} + T_{\phi_2}\big)(\nabla z)\Big) = 0, & \text{in } \Omega, \\ [\![z]\!] = \frac{c_0}{\alpha_1(\nabla d_0^\varphi)} - [\![w]\!], & \text{on } \Sigma, \\ [\![n_0^\varphi \cdot \nabla z]\!] = 0 & \text{on } \Sigma. \end{cases} \tag{2.128}$$

Domain decomposition method aims to retrieve a solution of (2.128), by studying the system separately in the two connected open sets Ω_1, $\Omega_2 := \Omega \setminus \overline{\Omega}_1$, where by Ω_1 we denote the region bounded by Σ. Fix a trace function ζ defined on Σ, and let $z_1 := z_1(\zeta)$ be the solution of

$$\begin{cases} \operatorname{div}\Big(\big(T_{\phi_1} + T_{\phi_2}\big)(\nabla v)\Big) = 0, & \text{in } \Omega_1, \\ v = \zeta, & \text{on } \Sigma. \end{cases}$$

Then, let $z_2 := z_2(\zeta)$ be the solution of

$$\begin{cases} \operatorname{div}\Big(\big(T_{\phi_1} + T_{\phi_2}\big)(\nabla v)\Big) = 0, & \text{in } \Omega_2, \\ n_0^\varphi \cdot \nabla v = -n_0^\varphi \cdot \nabla z_1, & \text{on } \Sigma. \end{cases}$$

[(8)]Here, we are assuming $m = 2$ and ϕ_1, ϕ_2 linear anisotropies (namely, the standard bidomain model in Section 1.4).

with proper boundary condition at $\partial\Omega$. Existence of a unique solution for the above elliptic problems can be found for instance in [98]. Let us denote by $\mathrm{tr}(z_2) := \mathrm{tr}(z_2(\zeta))$ the trace of z_2 on Σ. Then, we see that

$$z := \begin{cases} z_1, & \text{in } \Omega_1, \\ z_2, & \text{in } \Omega_2, \end{cases}$$

is a solution of (2.128) if and only if the right hand side of the second equation in (2.128) belongs to the image of the linear operator

$$S \colon \zeta \mapsto \zeta - \mathrm{tr}(z_2(\zeta)).$$

Here we find again a problem of lack of compactness: indeed, it does not seem possible to apply Fredholm's theory in order to deduce surjectivity of S (notice, however, that S is an injective operator).

Chapter 3

Crystalline mean curvature of facets

Summary. In Section 3.1, we collect some results on the ϕ-anisotropic mean curvature problem, and we recall the definition of maximal/minimal anisotropic Cheeger sets, for a convex (possibly nonregular) anisotropy ϕ. Anisotropic mean curvature is defined in Section 3.2, within the class of Lip ϕ-regular sets. In Section 3.3, we give the definition of optimal selection in a facet $F \subset \partial E$ (E being a Lip ϕ-regular solid set) which corresponds to a facet of B_ϕ. In the same section, we also prove that, if E is convex at F, then the maximal Cheeger subset of F is the minimal level set of the anisotropic mean curvature. Section 3.4 is devoted to study calibrability of a facet. We extend the necessary condition for a convex facet to be calibrable to all $\widetilde{\phi}$-convex facets (Definition 3.34), and we present a counterexample showing that the converse implication is not valid in general. In Sections 3.4.1-3.4.2, we prove some facts on the calibrability of "annular" facets and closed strips. The main results of the chapter are contained in Section 3.5, where we link the issue of calibrability with the capillary problem in order to provide some relevant examples of continuous optimal selections in noncalibrable facets.

3.1 Prescribed anisotropic mean curvature problem

Let $m \geq 2$, $\varphi \in \mathcal{M}(\mathbb{R}^m)$, $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lipschitz boundary, and $\beta > 0$. In the following, we shall consider solutions C_β to the *prescribed anisotropic mean curvature problem*, namely solutions to

$$\inf \{P_\varphi(B) - \beta|B| : B \subseteq \Omega, B \neq \emptyset\}. \tag{3.1}$$

Existence of solutions of (3.1) can be proved by direct methods.(1) The following regularity result holds.

Theorem 3.1. *Let φ be the Euclidean norm. Then $\Omega \cap \partial^* C_\beta$ is an analytic hypersurface with constant mean curvature equal to β, and the set $\Omega \cap (\partial C_\beta \setminus \partial^* C_\beta)$ is a closed set with Hausdorff dimension at most $(m-8)$. Moreover, $\partial^* C_\beta$ can meet $\partial^*\Omega$ only tangentially, that is, $\nu^\Omega = \nu^{C_\beta}$ on $\partial^* C_\beta \cap \partial^* \Omega$.*

Proof. The analyticity of $\Omega \cap \partial^* C_\beta$, the closedness and the estimate on the dimension of $\Omega \cap (\partial C_\beta \setminus \partial^* C_\beta)$ follow from classical regularity results, see for instance [145] or [114]. We refer the reader to the latter reference for a proof of the tangentiality condition on $\partial^* C_\beta \cap \partial^* \Omega$. $\square$

For $\varphi \in \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^m)$ of class $\mathcal{C}^{3,\alpha}$ on $\mathbb{R}^m \setminus \{0\}$, and $\alpha \in (0,1)$, solutions of (3.1) are hypersurfaces of class $\mathcal{C}^{1,\alpha}$, out of a closed singular set of zero $\mathcal{H}^{m-1}$-measure, see [4].(2)

(1) We recall that solutions of the prescribed anisotropic mean curvature problem can be approximated by means of a singularly perturbed elliptic PDE of bistable type, see [132].

(2) See also [2, 141] for the case $\varphi \in \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^m)$ of class $\mathcal{C}^{2,1}$.

For $m = 2$, in [15] the authors study the problem for a more general notion of perimeter, and prove that the inner boundary of a solution of (3.1) is a Lipschitz curve out of a closed singular set of zero $\mathcal{H}^1$-measure. The result has been improved in [130, Theorem 4.5], with the following theorem.

Theorem 3.2. *Let $\varphi \in \mathcal{M}(\mathbb{R}^2)$, $\beta > 0$, and let C_β be a solution of* (3.1)*. Then, every connected component of $\Omega \cap \partial C_\beta$ is contained in a translated of $\beta^{-1}\partial B_\varphi$.*

Remark 3.3. In dimension $m > 2$, even with the Euclidean metric, we cannot deduce from Theorem 3.1 that any connected component of $\Omega \cap \partial C_\beta$ is contained in the boundary of a ball of radius β^{-1}, see for instance [109] for an explicit example.

Problem (3.1) is related to the so-called φ-*Cheeger's problem* for Ω, which consists in solving

$$\inf\left\{\frac{P_\varphi(B)}{|B|} : B \subseteq \Omega,\ B \neq \emptyset\right\} =: h_\varphi(\Omega), \tag{3.2}$$

see [62, 65]. Cheeger's problem has been introduced in [68], in the effort to give an estimate from below for the spectrum of the Laplacian operator. A minimizer of (3.2) is sometimes called a φ-Cheeger subset of Ω, while $h_\varphi(\Omega)$ is called the φ-Cheeger constant of Ω. Notice that, when $\beta := h_\varphi(\Omega)$, a nonempty set $B \subseteq \Omega$ solves (3.1) if and only if B is a minimizer of (3.2).

Definition 3.4 (**Cheeger and strict Cheeger sets**)**.** *If Ω is a solution of* (3.2)*, we say that Ω is a* φ-Cheeger set*. If*

$$\frac{P_\varphi(\Omega)}{|\Omega|} < \frac{P_\varphi(B)}{|B|}, \quad B \subset \Omega,\ B \neq \emptyset, \tag{3.3}$$

we say that Ω is a strict *φ-Cheeger set.*

If $B \subseteq \Omega$ is a φ-Cheeger subset of Ω, then B is a φ-Cheeger set (namely, $h_\varphi(B) = h_\varphi(\Omega) = \frac{P_\varphi(B)}{|B|}$). We say that B is a *strict* φ-Cheeger subset of Ω provided that B is a φ-Cheeger subset of Ω, and $\frac{P_\varphi(B)}{|B|} < \frac{P_\varphi(B')}{|B'|}$, for every $B' \subset B$, $B' \neq \emptyset$.

It can be proven [114] that the union of φ-Cheeger subsets of Ω is still a φ-Cheeger subset of Ω.

Definition 3.5 (**Maximal/minimal Cheeger subsets**)**.** *We denote by*

$$\mathrm{Ch}_\varphi(\Omega)$$

the maximal *φ-Cheeger subset of Ω, which is defined as the union of all φ-Cheeger subsets of Ω.*

Moreover, we say that a φ-Cheeger subset C of Ω is minimal *if, for any φ-Cheeger subset $C' \subseteq \Omega$, either $C \subseteq C'$ or $C \cap C' = \emptyset$.*

We observe that any minimal φ-Cheeger subset of Ω is connected. Existence of $\mathrm{Ch}_\varphi(\Omega)$ and of a finite number of minimal φ-Cheeger subsets is proven for example in [65, 63].

When φ is the Euclidean norm, we omit the dependence on φ of the various symbols, thus letting $h(\Omega)$ in place of $h_\varphi(\Omega)$, $\mathrm{Ch}(\Omega)$ in place of $\mathrm{Ch}_\varphi(\Omega)$, and so on.

Concerning uniqueness, examples of planar sets Ω admitting more then one (Euclidean) Cheeger subset, and also an uncountable family of Cheeger subsets, can be found in [110, 114]. Anyway, even when uniqueness fails, it is possible to prove [63] that any connected open set $\Omega \subset \mathbb{R}^m$ with finite volume *generically admits a unique Cheeger subset*, namely it has a unique Cheeger subset up to small perturbations in volume. More precisely, for any compact $K \subset \Omega$, there exists an open set $\Omega_K \subseteq \Omega$ such that $K \subset \Omega_K$, and Ω_k admits a unique Cheeger subset. Further results hold for a convex $\Omega \subset \mathbb{R}^m$, see [5].

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^m$ be convex. Then* $\mathrm{Ch}(\Omega)$ *is the unique Cheeger subset of* Ω*, and it is convex.*

In [61, Remark 3.6] the authors extend the uniqueness result in Theorem 3.6 to the case of an anisotropy $\varphi \in \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^m)$ and a uniformly convex set $\Omega \subset \mathbb{R}^m$ of class $\mathcal{C}^2$. In the anisotropic case $\varphi \in \mathcal{M}(\mathbb{R}^m) \setminus \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^m)$, instead, the uniqueness of the Cheeger subset of a convex set $\Omega \subset \mathbb{R}^m$ is proven, at our best knowledge, only in dimension $m = 2$ (see Theorem 3.8); anyway, when Ω is convex, $\mathrm{Ch}_\varphi(\Omega)$ is also convex [62, Theorem 6.3]. Both in the Euclidean and in the anisotropic case, there is also a necessary and sufficient condition for a smooth enough convex body to be a φ-Cheeger set. It appeared at first in [100] for $m = 2$ and φ Euclidean; in [37] for $m = 2$, $\varphi \in \mathcal{M}(\mathbb{R}^2)$; in [5] for $m \geq 2$ and φ the Euclidean norm; finally in [62] in the whole generality (this latter result is recalled in Theorem 3.7 below).

Given $r > 0$, we say that E *satisfies the rB_ϕ-condition* if, for any $x \in \partial E$, there exists $y \in \mathbb{R}^n$ such that $rB_\phi + y \subseteq E$, and $x \in \partial(rB_\phi + y)$.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^m$ be a convex body satisfying the rB_ϕ-condition for some $r > 0$. Then Ω is a φ-Cheeger set if and only if*

$$\operatorname*{ess\,sup}_{\partial\Omega} \kappa_\varphi^\Omega \leq \frac{P_\varphi(\Omega)}{|\Omega|}.$$

Finally we have a complete characterization of the (unique) Cheeger subset of a planar convex domain, proven in [110] for the Euclidean norm and in [111] for a general anisotropy.

Theorem 3.8 (**Cheeger subset of a planar convex domain**)**.** *If $\Omega \subset \mathbb{R}^2$ is a bounded, open and convex set, then $\mathrm{Ch}_\varphi(\Omega)$ is the union of all φ-balls of radius $r = h_\varphi(\Omega)^{-1}$ that are contained in Ω. Moreover, setting $\Omega_r^- := \{x \in \Omega : \mathrm{dist}_\varphi(x, \partial\Omega) > r\}$, we have*

$$\mathrm{Ch}_\varphi(\Omega) = \Omega_r^- + rB_\varphi,$$

and $|\Omega_r^-| = r^2|B_\phi|$.

In [114] the authors prove that, in the Euclidean case, most of the peculiarities of the planar convex case can be proven also for bidimensional (not necessarily convex) strips.

3.2 Anisotropic mean curvature in the nonregular case

In the remaining of this chapter, we shall denote by

$$\phi \in \mathcal{M}(\mathbb{R}^3) \setminus \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^3)$$

a convex nonregular anisotropy. For simplicity, we shall always assume that(3)

$$\omega_n^\phi = 1.$$

Relevant cases for the present thesis are

- when B_ϕ (and hence B_{ϕ^o}) is a 3-dimensional polyhedron; in this case, we say that ϕ is a *crystalline* anisotropy;
- when $B_\phi = C \times [-1, 1]$, C being a 2-dimensional centrally symmetric convex body; in this case, we say that ϕ is a *cylindrical* anisotropy.

Let $E \subset \mathbb{R}^3$ be a Lipschitz set, and let $\nu_{\phi^o} := \frac{\nu^E}{\phi^o(\nu^E)}$. In order to define the ϕ-anisotropic mean curvature of E, we immediately find a difficulty with respect to the regular case: indeed, when $\phi \in \mathcal{M}(\mathbb{R}^3) \setminus \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^3)$, there can be several possible choices of vector fields $N : \partial E \to \mathbb{R}^3$ satisfying $N(x) \in T_{\phi^o}(\nu_{\phi^o}(x))$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$.

(3) We recall that the constant ω_n^ϕ has been introduced in Definition 1.8.

One possibility to proceed could be to look at the eikonal equation[(4)]

$$\phi^o(\nabla d_\phi^E(z)) = 1, \qquad \text{a.e. in } U,$$

where U is a suitable neighbourhood of ∂E, ϕ^o is the dual of ϕ, and d_ϕ^E denotes the ϕ-signed distance from ∂E, positive in the interior of E. Then, one could require the existence of a bounded vector field $\eta \in \mathrm{Lip}(U;\mathbb{R}^3)$ such that $\eta(z) \in -T_{\phi^o}(\nabla d_\phi^E(z))$ for almost every $z \in U$. In this case, we say that E is *neighbourhood Lip ϕ-regular.* This notion can be generalized to all dimension $n \geq 2$, and it could be weakened, by requiring for instance that $\eta \in L^\infty(U;\mathbb{R}^n)$: in this latter case, the set E is said to be neighbourhood L^∞ ϕ-regular. This notion has been introduced in [36, 34], and has been used in [27] to provide a uniqueness result for a suitable notion of crystalline mean curvature flows of convex sets. In the Euclidean case, E is neighbourhood-Lip ϕ-regular if and only if ∂E is of class $\mathcal{C}^{1,1}$. Neighbourhood regularity of boundaries has some connection with the notion of inner-outer tangent ball: it turns out [28] that, if E is neighbourhood-Lip ϕ-regular, then there exists $r > 0$ such that E and $\mathbb{R}^n \setminus E$ satisfy the rB_ϕ-condition. Moreover, if E is convex, then E is neighbourhood-L^∞ ϕ-regular if and only if E and $\mathbb{R}^n \setminus E$ satisfy the rB_ϕ-condition for some $r > 0$. We also recall that neighbourhood Lipschitz regularity has been used in [62] to give a characterization of convex subsets of $\mathbb{R}^{n-1}$ which are ϕ-calibrable, see Section 3.4.

However, a difficulty related to the definition of neighbourhood regular sets is that the divergence of η does not have a well-defined trace on ∂E. For this reason, in the present thesis we shall adopt a second, stronger notion of regular sets, which has been introduced (in any dimension $n \geq 2$)[(5)] in [38, 39].

Definition 3.9 (Selection). *A* selection *on ∂E is an element of*

$$\mathrm{Nor}_\phi(\partial E) := \{N \colon \partial E \to \mathbb{R}^3 \ : \ N(x) \in T_{\phi^o}(\nu_{\phi^o}(x)) \text{ for } \mathcal{H}^2\text{-a.e. } x \in \partial E\}.$$

Notice that, when $\phi \in \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^3)$, then the unique selection on ∂E is the Cahn-Hoffman vector field $n_\phi = T_{\phi^o}(\nu_{\phi^o})$ (recall (1.6)).

Definition 3.10 (Lip ϕ-regular sets). *We say that E is* Lip ϕ-regular *if there exists a Lipschitz selection on ∂E.*

For instance, assuming that B_ϕ and E are polyhedra, E is Lip ϕ-regular if and only if, at every vertex $v \in \partial E$,

$$\bigcap_{\substack{F \text{ facet of } \partial E,\\ v \in F}} T_{\phi^o}(\nu^F) \neq \emptyset.$$

It turns out that a Lip ϕ-regular set is also neighbourhood Lip ϕ-regular: indeed, for any $N \in \mathrm{Nor}_\phi(\partial E)$ it is possible to exhibit a Lipschitz extension of N inside a tubular neighbourhood U of ∂E, see [38].

Anisotropic mean curvature is defined, in analogy with the regular case (recall Theorem 1.10), by computing the first variation of the perimeter functional. For $\delta > 0$ and $z \in U$, define $\Psi_\delta(z) := z + \delta\psi^e(z)N^e(z)$, where $\psi \in \mathrm{Lip}(U)$ and $N^e \in \mathrm{Lip}(U;\mathbb{R}^3)$ is a Lipschitz extension of N on U. It is convenient to introduce the family

$$\mathscr{H}^2_{\mathrm{div}}(\partial E) := \big\{N \in \mathrm{Nor}_\phi(\partial E) \ : \ \mathrm{div}_\tau N \in L^2(\partial E)\big\}, \tag{3.4}$$

where $\mathrm{div}_\tau N$ is the *tangential divergence* of $N \in \mathrm{Nor}_\phi(\partial E)$ defined as in [38]. Set

$$\mathcal{K}(N) := \int_{\partial E} \phi^o(\nu^E)(\mathrm{div}_\tau N)^2 \, d\mathcal{H}^2, \qquad N \in \mathscr{H}^2_{\mathrm{div}}(\partial E). \tag{3.5}$$

The following result is proven in [38].

(4) Recall (1.5).

(5) In particular, Theorems 3.11-3.13 can be generalized to every dimension $n \geq 2$.

Theorem 3.11 (First variation in the nonregular case). *Suppose that E is Lip ϕ-regular. Then*

$$\inf_{\substack{\psi \in \mathrm{Lip}(\partial E), \\ \int_{\partial E} \phi^o(\nu^E)\psi^2 \, d\mathcal{H}^2 \le 1}} \liminf_{\delta \to 0^+} \frac{P_\phi(\Psi_\delta(E)) - P_\phi(E)}{\delta} = - \inf_{N \in \mathscr{H}^2_{\mathrm{div}}(\partial E)} (\mathcal{K}(N))^{\frac{1}{2}}. \tag{3.6}$$

The minimization problem in (3.6) may admit, in general, more than one solution. Nevertheless, by the strict convexity of $\mathcal{K}$ in the divergence, two minimizers have the same divergence. In the following, we denote by

$$N_{\min} \in \mathscr{H}^2_{\mathrm{div}}(\partial E)$$

any minimizer of (3.5).

Definition 3.12 (Anisotropic mean curvature). *The* ϕ-mean curvature κ^E_ϕ *of* ∂E *is defined as*

$$\kappa^E_\phi := \mathrm{div}_\tau N_{\min}.$$

Actually, Lip ϕ-regular sets have anisotropic mean curvature which is more than just square integrable on ∂E: indeed, the following result holds [39].

Theorem 3.13 (Boundedness of κ^E_ϕ). *We have $\kappa^E_\phi \in L^\infty(\partial E)$.*

Some further regularity properties of κ^E_ϕ are expected for those 2-dimensional portions of ∂E which correspond (via the map T_{ϕ^o}) to 2-dimensional portions of ∂B_ϕ. We shall collect some of these results in Section 3.3.

3.3 Anisotropic mean curvature on facets

Let E be a Lip ϕ-regular set. We say that $F \subset \partial E$ is a (two-dimensional) *facet* of ∂E if F is the closure of a connected component of the relative interior of $\partial E \cap T_x\partial E$, for some $x \in \partial E$ such that the tangent space $T_x\partial E$ of ∂E at x exists. Given a facet $F \subset \partial E$, by $\Pi_F \subset \mathbb{R}^3$ we denote the affine plane spanned by F. Whenever necessary, we identify Π_F with the plane parallel to Π_F and passing through the origin, and F with its orthogonal projection on this latter plane.

Definition 3.14 (Facets of ∂E corresponding to facets of the Wulff shape). *We write*

$$F \in \mathrm{Facets}_\phi(\partial E)$$

if F is parallel to a facet $\widetilde{B}^F_\phi$ of ∂B_ϕ, and $\nu_{\phi^o}(F) = \nu_{\phi^o}(\widetilde{B}^F_\phi)$.

If $F \in \mathrm{Facets}_\phi(\partial E)$, then $\widetilde{B}^F_\phi = T_{\phi^o}(\nu_{\phi^o}(F))$. With a slight abuse of notation, we can see $\widetilde{B}^F_\phi$ as a subset of Π_F. We shall assume, unless otherwise specified, that $\widetilde{B}^F_\phi$ is a convex body which is symmetric with respect to the origin of Π_F. Let $\widetilde{\phi}_F : \Pi_F \to [0, +\infty)$ be the (convex) anisotropy on Π_F such that $\{\widetilde{\phi}_F \le 1\} = \widetilde{B}^F_\phi$. We denote by $\widetilde{\phi}^o_F$ the dual of $\widetilde{\phi}_F$. We denote by $\kappa^B_{\widetilde{\phi}}$ the $\widetilde{\phi}$-curvature of the boundary of a Lip $\widetilde{\phi}$-regular set $B \subset \Pi_F$. If no confusion is possible, we shall omit the dependence on F of $\widetilde{\phi}_F$, thus writing $\widetilde{\phi}$ in place of $\widetilde{\phi}_F$.

The following regularity result is proven in [39].

Theorem 3.15 (Bounded variation of κ^E_ϕ). *Let $F \in \mathrm{Facets}_\phi(\partial E)$. Then*

$$\kappa^E_\phi \in BV(\mathrm{int}(F)).$$

Another result related to $\mathrm{Facets}_\phi(\partial E)$ allows to detect the anisotropic mean curvature of ∂E at a facet F from a minimization problem on F (Proposition 3.22). We need the following definition.

Definition 3.16 (**Convexity at a facet**). *We say that E is* convex *(resp.* concave*) at F if E lies, locally around F, in the half-space obtained as that side of Π_F opposite to (resp. same as) the exterior normal to E at F.*

We recall from [39] a regularity result for the boundary of F, which will be used to give a meaning to the normal trace of a selection (Definition 3.18).

Theorem 3.17. *Let $F \in \mathrm{Facets}_\phi(\partial E)$. Then there exists a finite set $Z_F \subset \partial F$ such that, for any $x \in \partial F \setminus Z_F$, ∂F is a Lipschitz graph locally around x. Moreover, if E is convex (or concave) at F, then F is Lipschitz.*

Now, let $N \in \mathrm{Nor}_\phi(\partial E) \cap \mathrm{Lip}(\partial E; \mathbb{R}^3)$. Notice that the orthogonal component of N with respect to the plane Π_F is constant. Hence,

$$\mathrm{div}_\tau N = \mathrm{div}(\mathrm{proj}_F(N)), \tag{3.7}$$

where $\mathrm{proj}_F(N)\colon F \to \Pi_F$ is the projection of N on F, and its divergence is computed in Π_F. Let $\widetilde{\nu}^F$ be the outer Euclidean unit normal to ∂F (when it exists).

It turns out that

$$\widetilde{\nu}^F \cdot \mathrm{proj}_F(N) = \begin{cases} \widetilde{\phi}^o(\widetilde{\nu}^F(x)) & \text{if } \widetilde{\nu}^F(x) \text{ points outside } E, \\ -\widetilde{\phi}^o(\widetilde{\nu}^F(x)) & \text{if } \widetilde{\nu}^F(x) \text{ points inside } E, \end{cases} \tag{3.8}$$

for any $x \in \partial^* F$ (see [38, 39]).

Definition 3.18 (**Maximal/minimal normal trace c_F^ϕ**). *Let E be a Lip ϕ-regular set, and $F \in \mathrm{Facets}_\phi(\partial E)$. The ϕ-*normal trace at ∂F*,*

$$c_F^\phi \in L^\infty(\partial F),$$

is defined as the right hand side of (3.8).

When E is convex (resp. concave) at the facet F, we have $c_F^\phi = \widetilde{\phi}^o(\widetilde{\nu}^F)$ (resp. $c_F^\phi = -\widetilde{\phi}^o(\widetilde{\nu}^F)$).

We recall [18] that any $N \in \mathscr{H}^2_{\mathrm{div}}(\partial E)$ admits a *normal trace*[(6)] which we shall denote by $\langle \widetilde{\nu}^F, \mathrm{proj}_F(N)\rangle$, and we have $\langle \widetilde{\nu}^F, \mathrm{proj}_F(N)\rangle \in L^\infty(\partial F)$. However, we cannot say in general that $\langle \widetilde{\nu}^F, \mathrm{proj}_F(N)\rangle = c_F^\phi$, for any $N \in \mathscr{H}^2_{\mathrm{div}}(\partial E)$. The result is true under stronger regularity assumptions on the behaviour of ∂E around F. We refer the reader to [37] for a related discussion. To our purposes, we can confine ourselves to the case described in Proposition 3.19 below.

We say that $\partial E \setminus F$ and F *are transversal* if, for $\mathcal{H}^1$-a.e. $x \in \partial F$, the blow-up of ∂E around x is the union of two non parallel planes Π_1 and Π_2, with $\Pi_2 = \Pi_F$, see [37].

Proposition 3.19. *Suppose*

$$F \text{ Lipschitz}, \qquad \partial E \setminus F \text{ and } F \text{ are transversal.} \tag{3.9}$$

Then $\langle \widetilde{\nu}^F, \mathrm{proj}_F(N)\rangle = c_F^\phi$, for any $N \in \mathscr{H}^2_{\mathrm{div}}(\partial E)$.

It is now natural to look at the family

$$\mathscr{H}^2_{\mathrm{div}}(F) := \left\{ \widetilde{N} \in \mathrm{Nor}_\phi(F) : \ \mathrm{div}\widetilde{N} \in L^2(F),\ \langle \widetilde{\nu}^F, \widetilde{N}\rangle = c_F^\phi \ \ \mathcal{H}^1\text{-a.e. on } \partial F \right\},$$

where $\mathrm{Nor}_\phi(F) := \left\{ \widetilde{N} \in L^\infty(F; \Pi_F) \ : \ \widetilde{N}(x) \in \widetilde{B}^F_\phi \text{ for } \mathcal{H}^2\text{-a.e. } x \in F \right\}$. Notice that $\mathscr{H}^2_{\mathrm{div}}(F) \neq \emptyset$, by the Lip ϕ-regularity of E. Set also[(7)]

$$\mathcal{K}_F(\widetilde{N}) := \int_F (\mathrm{div}\widetilde{N})^2 \, dx, \qquad \widetilde{N} \in \mathscr{H}^2_{\mathrm{div}}(F). \tag{3.10}$$

[(6)] See (3.25) below, with $X := \mathrm{proj}_F(N)$.

[(7)] For notational simplicity, hereafter we shall identify the $\mathcal{H}^2$-measure on F with the two-dimensional Lebesgue measure on Π_F.

The minimum problem

$$\inf\left\{\mathcal{K}_F(\widetilde{N}) \ : \ \widetilde{N} \in \mathscr{H}^2_{\rm div}(F)\right\} \tag{3.11}$$

admits a solution, and two minimisers have the same divergence. Notice that the minimum problem (3.11) is *nonlocal*, since it depends on the shape of ∂E around F.

Definition 3.20 (**Optimal selection**). *Given* $F \in {\rm Facets}_\phi(\partial E)$, *we call* optimal selection *in* F, *and we denote by* $\widetilde{N}_{\rm min} \in \mathscr{H}^2_{\rm div}(F)$, *any solution of* (3.11).

Incidentally, we recall [97] that it is not possible for an optimal selection in F to coincide with the gradient of a scalar function, unless the facet is the unit disk.

Remark 3.21 (**Minimality criterion**). Let $\widetilde{N}_0 \in \mathscr{H}^2_{\rm div}(F)$ be such that

$$\int_F {\rm div}(\widetilde{N}_0)\,{\rm div}(\widetilde{N}_0 - \widetilde{N})\;dx \le 0, \qquad \widetilde{N} \in \mathscr{H}^2_{\rm div}(F). \tag{3.12}$$

Then $\widetilde{N}_0$ is an optimal selection in F. In particular, if there exists $\widetilde{N}_0 \in \mathscr{H}^2_{\rm div}(F)$ such that ${\rm div}\widetilde{N}_0$ is constant on F, then $\widetilde{N}_0$ is optimal (condition (3.12) is satisfied with equality instead of the inequality), and necessarily

$${\rm div}\widetilde{N}_0 = \frac{1}{|F|}\int_F {\rm div}\widetilde{N}_0\,dx = \frac{1}{|F|}\int_{\partial F} c_F^\phi\,d\mathcal{H}^1.$$

Let $\widetilde{N}_{\rm min} \in \mathscr{H}^2_{\rm div}(F)$ be an optimal selection in F, and set

$$\kappa_{\phi,F} := {\rm div}(\widetilde{N}_{\rm min}).$$

The following result allows to look at the restriction of κ_ϕ^E at the facet F by studying a problem defined just on the facet. For the sake of completeness, we repeat the proof given in [37, Remark 4.4 and Proposition 4.6].

Proposition 3.22 (**Restriction and localization on facets**). *Assume* (3.9). *Let* $N_{\rm min} \in \mathscr{H}^2_{\rm div}(\partial E)$ *be so that* $\kappa_\phi^E = {\rm div}_\tau N_{\rm min}$. *Then* ${\rm proj}_F(N_{\rm min})$ *is an optimal selection in* F. *In particular,*

$$\kappa_\phi^E = \kappa_{\phi,F} \qquad \mathcal{H}^2\text{-a.e. in } F. \tag{3.13}$$

Proof. Let $N_{\rm min} \in \mathscr{H}^2_{\rm div}(\partial E)$ (resp. $\widetilde{N}_{\rm min} \in \mathscr{H}^2_{\rm div}(F)$) be a minimizer of $\mathcal{K}$ (resp. of $\mathcal{K}_F$), where $\mathcal{K}$ and $\mathcal{K}_F$ are defined respectively in (3.5) and in (3.10). Let $N \in L^\infty(\partial E;\mathbb{R}^3)$ be such that $N = N_{\rm min}$ on $\partial E \setminus F$, and such that ${\rm proj}_F(N) = \widetilde{N}_{\rm min}$. By Proposition 3.19, $N \in \mathscr{H}^2_{\rm div}(\partial E)$. Thus

$$\begin{aligned}
\mathcal{K}(N_{\rm min}) \le \mathcal{K}(N) &= \int_F ({\rm div}\widetilde{N}_{\rm min})^2\;d\mathcal{H}^2 + \int_{\partial E\setminus F} ({\rm div}_\tau N_{\rm min})^2\;d\mathcal{H}^2 \\
&\le \int_F ({\rm div}_\tau N_{\rm min})^2\;d\mathcal{H}^2 + \int_{\partial E\setminus F} ({\rm div}_\tau N_{\rm min})^2\;d\mathcal{H}^2 \\
&= \int_{\partial E} ({\rm div}_\tau N_{\rm min})^2\;d\mathcal{H}^2 = \mathcal{K}(N_{\rm min}),
\end{aligned}$$

which gives the statement. □

Despite its obviousness, the following observation will be used repeatedly in Section 3.5.

Remark 3.23. If there exists $\widetilde{N}_0 \in \mathscr{H}^2_{\rm div}(F)$ such that ${\rm div}\widetilde{N}_0 = \kappa_{\phi,F}$ in ${\rm int}(F)$, then $\widetilde{N}_0$ is an optimal selection in F, since

$$\int_F \left({\rm div}\widetilde{N}_0\right)^2\;dx = \int_F (\kappa_{\phi,F})^2\;dx = \int_F \left({\rm div}\widetilde{N}_{\rm min}\right)^2\;dx \le \int_F \left({\rm div}\widetilde{N}\right)^2\;dx,$$

for any $\widetilde{N} \in \mathscr{H}^2_{\rm div}(F)$.

For notational simplicity, and whenever no confusion is possible, we set

$$\kappa_{\min} := \operatorname{ess\,inf} \kappa_{\phi,F}, \qquad \kappa_{\max} := \operatorname{ess\,sup} \kappa_{\phi,F}. \tag{3.14}$$

Now, we recall from [37, 38] some results on regularity of facets and on the function $\kappa_{\phi,F}$.

Theorem 3.24 (**Regularity of facets**). *Let $F \in \mathrm{Facets}_\phi(\partial E)$, and let E be convex (or concave) at F. Then F is Lip $\widetilde{\phi}$-regular.*

For $\beta \in [\kappa_{\min}, \kappa_{\max}]$, define

$$\begin{aligned} \Omega^F_\beta &:= \{x \in \mathrm{int}(F) \ : \ \kappa_{\phi,F}(x) < \beta\}, \\ \Theta^F_\beta &:= \{x \in \mathrm{int}(F) \ : \ \kappa_{\phi,F}(x) \le \beta\}. \end{aligned} \tag{3.15}$$

Theorem 3.25 (**Sublevels of the anisotropic mean curvature**). *Let $F \in \mathrm{Facets}_\phi(\partial E)$, and suppose that E is convex at F. Then*

$$\kappa_{\min} > 0.$$

Moreover, for any $\beta \in [\kappa_{\min}, \kappa_{\max}]$,

$$\int_{\Omega^F_\beta} \kappa_{\phi,F}\, dx = P_{\widetilde{\phi}}(\Omega^F_\beta), \qquad \int_{\Theta^F_\beta} \kappa_{\phi,F}\, dx = P_{\widetilde{\phi}}(\Theta^F_\beta), \tag{3.16}$$

and Ω^F_β and Θ^F_β are solutions of the variational problem

$$\inf \left\{ P_{\widetilde{\phi}}(B) - \beta|B| \ : \ B \subseteq F \right\}. \tag{3.17}$$

Remark 3.26. In the setting of Theorem 3.24, assume further $\widetilde{\phi} \in \mathcal{M}_{\mathrm{reg}}(\Pi_F)$. Let $\beta \in [\kappa_{\min}, \kappa_{\max}]$. Since Θ^F_β solves (3.17), the $\widetilde{\phi}$-mean curvature of $\partial\Theta^F_\beta$ is less than or equal to β, and equality holds in $\mathrm{int}(F) \cap \partial\Theta^F_\beta$. A similar result holds for Ω^F_β.

In the general case, the following result holds [130]: for any $\beta \in [\kappa_{\min}, \kappa_{\max}]$, $\mathrm{int}(F) \cap \partial\Omega^F_\beta$ and $\mathrm{int}(F) \cap \partial\Theta^F_\beta$ are contained in a translated copy of $\beta^{-1}\partial\widetilde{B}^F_\phi$.

We see from Theorem 3.25 that the sublevel sets of $\kappa_{\phi,F}$ solve a prescribed anisotropic mean curvature problem, recall Section 3.1. Now, we want to show that the minimal level set of the curvature corresponds to the maximal $\widetilde{\phi}$-Cheeger subset of F (Definition 3.5).

Theorem 3.27. *Let $F \in \mathrm{Facets}_\phi(\partial E)$, and assume that E is convex in F. Then*

$$\Theta^F_{\kappa_{\min}} = \mathrm{Ch}_{\widetilde{\phi}}(F). \tag{3.18}$$

Proof. We start with two preliminary steps.

Step 1: $|\Theta^F_{\kappa_{\min}}| > 0$. Essentially, this fact has been observed in [37, Remark 5.3]. We repeat the proof, for the sake of completeness. Let $\beta \in (\kappa_{\min}, \kappa_{\max}]$, so that in particular $|\Theta^F_\beta| > 0$. From Theorem 3.25, using (1.3) (with $\widetilde{\phi}$ replacing ϕ), we get

$$0 = P_{\widetilde{\phi}}(\emptyset) - \beta|\emptyset| \ge P_{\widetilde{\phi}}(\Theta^F_\beta) - \beta|\Theta^F_\beta| \ge \gamma_{\widetilde{\phi}}\sqrt{|\Theta^F_\beta|} - \beta|\Theta^F_\beta|,$$

where $\gamma_{\widetilde{\phi}} := P_{\widetilde{\phi}}(\widetilde{B}^F_\phi)|\widetilde{B}^F_\phi|^{1/2}$. Thus, we deduce the estimate

$$|\Theta^F_\beta| \ge \beta^{-2}\,\gamma^2_{\widetilde{\phi}} \ge \kappa_{\max}^{-2}\,\gamma^2_{\widetilde{\phi}}, \qquad \beta \in (\kappa_{\min}, \kappa_{\max}]. \tag{3.19}$$

By (3.19), and since $\Theta^F_{\kappa_{\min}} = \bigcap_{\beta > \kappa_{\min}} \Theta^F_\beta$, we get Step 1.

Step 2: The $\widetilde{\phi}$-Cheeger constant of F equals $\kappa_{\min}$. By definition of $h_{\widetilde{\phi}}(F)$, using Step 1 and (3.16), we get

$$h_{\widetilde{\phi}}(F) \le \frac{P_{\widetilde{\phi}}(\Theta^F_{\kappa_{\min}})}{|\Theta^F_{\kappa_{\min}}|} = \frac{\int_{\Theta^F_{\kappa_{\min}}} \kappa_{\phi,F}\, dx}{|\Theta^F_{\kappa_{\min}}|} = \kappa_{\min}. \tag{3.20}$$

On the other hand, let C be a $\widetilde{\phi}$-Cheeger subset of F. Then, thanks to Theorem 3.25, we get

$$0 = P_{\widetilde{\phi}}(\Theta^F_{\kappa_{\min}}) - \kappa_{\min}|\Theta^F_{\kappa_{\min}}| \leq P_{\widetilde{\phi}}(C) - \kappa_{\min}|C| = (h_{\widetilde{\phi}}(F) - \kappa_{\min})|C|. \tag{3.21}$$

Coupling (3.20) with (3.21), we get $h_{\widetilde{\phi}}(F) = \kappa_{\min}$. In particular, $\Theta^F_{\kappa_{\min}}$ is a $\widetilde{\phi}$-Cheeger subset of F and $\Theta^F_{\kappa_{\min}} \subseteq \mathrm{Ch}(F)$.

Now, we prove (3.18). Suppose, by contradiction, that there exists a $\widetilde{\phi}$-Cheeger subset $C \subseteq F$ such that $|C \setminus \Theta^F_{\kappa_{\min}}| > 0$. We observe that $\kappa_{\phi,F} > \kappa_{\min}$ on $C \setminus \Theta^F_{\kappa_{\min}}$, hence

$$\kappa_{\min}|C| < \int_C \kappa_{\phi,F}\, dx = \int_C \mathrm{div}\widetilde{N}_{\min}\, dx = \int_{\partial^* C} \langle \widetilde{\nu}^F, \widetilde{N}_{\min}\rangle \, d\mathcal{H}^1 \leq P_{\widetilde{\phi}}(C), \tag{3.22}$$

where $\widetilde{N}_{\min}$ is any optimal selection on F. At the same time, since C is a $\widetilde{\phi}$-Cheeger subset of F, using Step 2 we have $P_{\widetilde{\phi}}(C) = \kappa_{\min}|C|$, which, coupled with (3.22), leads to a contradiction. $\square$

If E is convex at F, and F itself is convex in the Euclidean sense, then a stronger regularity result for $\kappa_{\phi,F}$ has been proven in [37], which we recall in the following Theorem.

Theorem 3.28. *Let $F \in \mathrm{Facets}_\phi(\partial E)$, and assume that E is convex at F. Assume further that F is convex. Then*

$$\kappa_{\phi,F} \text{ is convex.}$$

Moreover,

$$\Omega^F_\beta = \bigcup \left\{ B \subseteq \mathrm{int}(F) \;:\; B \text{ is a translated copy of } \beta^{-1}\widetilde{B}^F_\phi \right\}, \qquad \beta > \kappa_{\min},$$

$$\Theta^F_\beta = \bigcup \left\{ B \subseteq F \;:\; B \text{ is a translated copy of } \beta^{-1}\widetilde{B}^F_\phi \right\}, \qquad \beta \geq \kappa_{\min}.$$

3.4 Calibrability of facets

Let $\phi \in \mathcal{M}(\mathbb{R}^3) \setminus \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^3)$, and let E be a Lip ϕ-regular set. We shall focus on those $F \in \mathrm{Facets}_\phi(\partial E)$ such that $\kappa_{\phi,F}$ is constant, which we shall call ϕ-calibrable. From now on in this chapter, we shall assume (3.9), so that (Proposition 3.22) $\kappa_{\phi,F}$ is the restriction of κ^E_ϕ to F.

Recalling also Remark 3.21, it follows that $\kappa_{\phi,F}$ is constant in $F \in \mathrm{Facets}_\phi(\partial E)$ if and only if there exists $\widetilde{N} \in L^\infty(F; \Pi_F)$ such that

$$\begin{cases} \widetilde{N}(x) \in \widetilde{B}^F_\phi & \mathcal{H}^2\text{-a.e. } x \in F, \\ \mathrm{div}\widetilde{N} = \frac{1}{|F|}\int_{\partial F} c_F \, d\mathcal{H}^1 & \text{in } F, \\ \langle \widetilde{\nu}^F, \widetilde{N}\rangle = c^\phi_F & \mathcal{H}^1\text{-a.e. on } \partial F. \end{cases} \tag{3.23}$$

The following definition has been proposed in [37].

Definition 3.29 (Calibrability). *We say that $F \in \mathrm{Facets}_\phi(\partial E)$ is* ϕ-calibrable *if there exists a solution of* (3.23).

From the view point of crystalline mean curvature flow, the right hand side of the PDE in (3.23), namely

$$v_F := \frac{1}{|F|}\int_{\partial F} c^\phi_F \, d\mathcal{H}^1,$$

can be interpreted as the "mean velocity" of F (in direction normal to $\mathrm{int}(F)$), at time zero. We want to define a similar quantity also for subsets of the facet since, heuristically, subsets of F are expected to move not slower than F, consistently with the comparison principle for crystalline mean curvature flow [34], see Theorem 3.30 below.

Let $B \subseteq F$ be a nonempty set of finite perimeter. We define $c_B^\phi : \partial B \to \mathbb{R}$ as

$$c_B^\phi := \begin{cases} \widetilde{\phi}(\widetilde{\nu}^B) & \text{on } \partial^* B \setminus \partial F, \\ c_F^\phi & \text{otherwise,} \end{cases} \tag{3.24}$$

and we set

$$v_B := \frac{1}{|B|} \int_{\partial^* B} c_B^\phi \, d\mathcal{H}^1.$$

Let us recall [19], [18] that, given a function $u \in BV(\mathrm{int}(F))$ and a vector field $X \in L^\infty(F; \Pi_F)$ with $L^2(F)$-summable divergence, it is possible to define a Radon measure (X, Du) on F by setting

$$(X, Du) \colon w \mapsto -\int_{\mathrm{int}(F)} u\, w \,\mathrm{div} X \, dx - \int_{\mathrm{int}(F)} u\, X \cdot \nabla w \, dx, \quad w \in \mathcal{C}_c^\infty(\mathrm{int}(F));$$

moreover, there exists a function $\langle \widetilde{\nu}^F, X\rangle \in L^\infty(\partial F)$ such that the following generalized Gauss-Green formula holds:

$$\int_{\mathrm{int}(F)} u \,\mathrm{div} X \; dx + \int_{\mathrm{int}(F)} \theta(X, Du) \, d|Du| = \int_{\partial F} \langle \widetilde{\nu}^F, X\rangle \; u \; d\mathcal{H}^1; \tag{3.25}$$

here, $\theta(X, Du) \in L^\infty_{|Du|}(F)$ denotes the density [12] of the measure (X, Du) with respect to $|Du|$. We recall that in [39, Proposition 7.7] it has been shown that, for every optimal selection $\widetilde{N}_{\min}$,

$$-\,\theta(\widetilde{N}_{\min}, D1_{\Omega_\beta^F}) = \widetilde{\phi}^o(\widetilde{\nu}^{\Omega_\beta^F}) = c^\phi_{\Omega_\beta^F}, \qquad \text{for a.e. } \beta \in \mathbb{R}, \tag{3.26}$$

where Ω_β^F is the β-sublevel set of $\kappa_{\phi,F}$ (see (3.15)), and where 1_A denotes the characteristic function of a subset $A \subseteq F$.

Theorem 3.30 ([37], **Characterization of ϕ-calibrable facets**). *Let $F \in \mathrm{Facets}_\phi(\partial E)$. Then, F is ϕ-calibrable if and only if*

$$v_F \le v_B, \qquad B \subseteq F, \; B \neq \emptyset. \tag{3.27}$$

Proof. Assume $\widetilde{N}$ to be a solution of (3.23). In particular, $\mathrm{div}\widetilde{N} = v_F$ in F. Let $B \subseteq F$ be a nonempty set of finite perimeter. Integrating $\mathrm{div}\widetilde{N}$ on B and using (3.25) we get

$$v_F |B| = \int_B \mathrm{div}\widetilde{N} \; dx = \int_{\partial^* B} \langle \widetilde{\nu}^B, \widetilde{N}\rangle \, d\mathcal{H}^1 \le \int_{\partial^* B} c_B^\phi \, d\mathcal{H}^1,$$

where we used (3.24) and (3.23). This gives (3.27).

The converse implication can be proved as follows. Assume that F is not ϕ-calibrable. Let $\widetilde{N}_{\min} \in \mathscr{H}^2_{\mathrm{div}}(F)$ be an optimal selection on F. Recalling that almost every sublevel set of a BV function has finite perimeter, there exists $\beta < v_F$ such that $\Omega_\beta^F \neq \emptyset$, and Ω_β^F has finite perimeter. Applying (3.25) with the choice $u := 1_{\Omega_\beta^F}$ and $X := \widetilde{N}_{\min}$, we have

$$\begin{aligned} \int_{\Omega_\beta^F} \mathrm{div}\widetilde{N}_{\min} \; dx &= -\int_{\mathrm{int}(F)\cap\partial^*\Omega_\beta^F} \theta(\widetilde{N}_{\min}, D1_{\Omega_\beta^F}) \; d\mathcal{H}^1 + \int_{\partial F} \langle \widetilde{\nu}^F, \widetilde{N}_{\min}\rangle \; 1_{\Omega_\beta^F} \; d\mathcal{H}^1 \\ &= -\int_{\mathrm{int}(F)\cap\partial^*\Omega_\beta^F} \theta(\widetilde{N}_{\min}, D1_{\Omega_\beta^F}) \; d\mathcal{H}^1 + \int_{\partial F\cap\partial^*\Omega_\beta^F} \langle \widetilde{\nu}^F, \widetilde{N}_{\min}\rangle \; d\mathcal{H}^1. \end{aligned}$$

Observe that, by definition, $\langle \widetilde{\nu}^F, \widetilde{N}_{\min}\rangle = c_F^\phi = c^\phi_{\Omega^F_\beta}$ on $\partial F \cap \partial^* \Omega^F_\beta$. Therefore, recalling also (3.26), we get

$$\int_{\Omega^F_\beta} \mathrm{div}\widetilde{N}_{\min}\, dx = \int_{\partial^*\Omega^F_\beta} c^\phi_{\Omega^F_\beta}\, d\mathcal{H}^1.$$

Hence,

$$v_F > \beta > \frac{1}{|\Omega^F_\beta|}\int_{\Omega^F_\beta} \mathrm{div}\widetilde{N}_{\min}\, dx = \frac{1}{|\Omega^F_\beta|}\int_{\partial^*\Omega^F_\beta} c^\phi_{\Omega^F_\beta}\, d\mathcal{H}^1 = v_{\Omega^F_\beta},$$

which contradicts (3.27). □

In view of Theorem 3.30, we give the following definition.

Definition 3.31 (**Strict ϕ-calibrability**)**.** *We say that F is* strictly ϕ-calibrable *if*

$$v_F < v_B \qquad \text{for every nonempty } B \subset F.$$

Condition (3.27) is very similar to the definition of $\widetilde{\phi}$-Cheeger set (recall one more Definition 3.4), as clarified in the following remark.

Remark 3.32 (**Calibrability versus Cheeger sets**)**.** Suppose that E *is convex* at F. In this case, the mean velocity of any nonempty finite perimeter set $B \subseteq F$ is

$$v_B = \frac{1}{|B|}\int_{\partial^* B} c^\phi_B\, d\mathcal{H}^1 = \frac{1}{|B|}\int_{\partial^* B} \widetilde{\phi}^o(\widetilde{\nu}^B)\, d\mathcal{H}^1 = \frac{P_{\widetilde{\phi}}(B)}{|B|}. \tag{3.28}$$

Then, using Theorem 3.30, and recalling also Section 3.1, ϕ-calibrability (resp. strict ϕ-calibrability) of F is equivalent to the property that F is a $\widetilde{\phi}$-Cheeger (resp. strict $\widetilde{\phi}$-Cheeger) set.

In the same paper [37], the authors characterize convex ϕ-calibrable facets $F \in \mathrm{Facets}_\phi(\partial E)$ such that E is convex at F.

Theorem 3.33 (**ϕ-calibrability for convex E at F and convex F**)**.** *Suppose that E is convex at $F \in \mathrm{Facets}_\phi(\partial E)$, and that F is convex. Then, F is ϕ-calibrable if and only if*

$$\mathrm{ess}\ \sup_{\partial F} \kappa^F_{\widetilde{\phi}} \le \frac{P_{\widetilde{\phi}}(F)}{|F|}. \tag{3.29}$$

Hence, under the assumptions of Theorem 3.33, problem (3.23) is solvable if and only if the $\widetilde{\phi}$-curvature of ∂F is bounded above by the mean velocity of F; this means, roughly speaking, that the edges of ∂F cannot be too "short". When $\widetilde{\phi}$ is the Euclidean norm of Π_F, (3.29) has been given by Giusti in [101], compare Theorem 3.41.

Definition 3.34 (**$\widetilde{\phi}$-convexity**)**.** *We say that $F \in \mathrm{Facets}_\phi(\partial E)$ is* $\widetilde{\phi}$-convex *if* $\kappa^F_{\widetilde{\phi}} \ge 0$.

One can ask whether the convexity assumption in Theorem 3.33 can be relaxed to just $\widetilde{\phi}$-convexity of F; the next example shows that this can not be expected in general.

Example 3.35. Let $\widetilde{\phi}$ be the two-dimensional crystalline metric having as unit ball the square with side $\ell > 0$, centered at the origin. Let F be as in Figure 3.1, where B^1 and B^2 are two copies of $\{\widetilde{\phi} \le 1\}$, rescaled by a factor L/ℓ, and $R_{\varepsilon,M}$ is a rectangle of height ε and base M. We recall [151] that, for planar crystalline sets, $\kappa^F_{\widetilde{\phi}}$ is the derivative of the vector field obtained as the linear interpolation of the vectors at the vertices represented in the figure. Thus, $\kappa^F_{\widetilde{\phi}}$ equals ℓ/L on the sides a, d, e and h, while $\kappa^F_{\widetilde{\phi}}$ vanishes on the sides b, c, f, and g; hence, F is $\widetilde{\phi}$-convex.

Now, let ϕ be the "cylindrical" norm defined as

$$\phi(\xi) := \phi(\xi_1, \xi_2, \xi_3) := \max\{\widetilde{\phi}(\xi_1, \xi_2), |\xi_3|\},$$

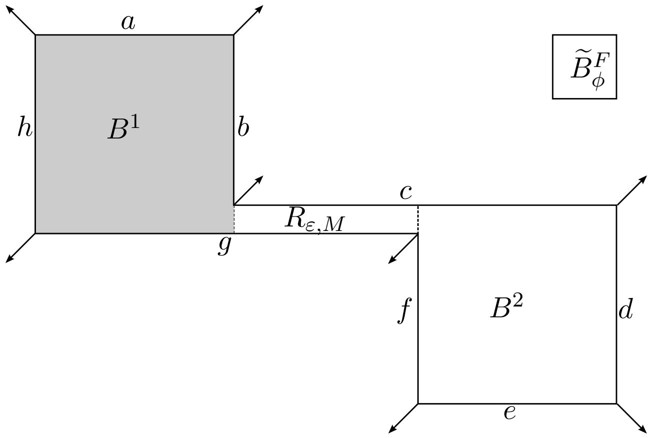


Figure 3.1: An example of $\widetilde{\phi}$-convex facet F satisfying (3.29), and not ϕ-calibrable ($\epsilon > 0$ is sufficiently small and M sufficiently large). Here, $\widetilde{B}^F_\phi$ is the square of length ℓ represented on the top right. In grey, a subset of the facet with mean velocity smaller than the mean velocity of F.

and let $E \subset \mathbb{R}^3$ be any prism with base F, for instance $E = F \times [0,1]$; in particular, $F \in \mathrm{Facets}_\phi(\partial E)$, and E is convex at F.[(8)]

Recalling (3.28), we can compute explicitely the mean velocity of F:

$$v_F = \frac{P_{\widetilde{\phi}}(F)}{|F|} = \frac{2\,(4L - \varepsilon + M)}{2L^2 + \varepsilon M}.$$

Hence $\kappa^F_{\widetilde{\phi}} \leq v_F$ when

$$\varepsilon \leq \frac{2L(-L\ell + 4L + M)}{\ell M + 2L},$$

the right hand side being positive for M large enough. Now, the mean velocity of B^1 is

$$v_{B^1} = \frac{P_{\widetilde{\phi}}(B^1)}{|B^1|} = \frac{4}{L}.$$

Therefore

$$v_{B^1} < v_F \iff \varepsilon < \frac{ML}{2M + L}.$$

Hence, for $\epsilon > 0$ small enough and M large enough, F is not ϕ-calibrable (Theorem 3.33).

However, for $\widetilde{\phi}$-convex facets, it is still possible to recover one implication from Theorem 3.33.

Theorem 3.36. *Assume that E is convex at $F \in \mathrm{Facets}_\phi(\partial E)$. Suppose that $\widetilde{\phi}$ is crystalline, and that F is a $\widetilde{\phi}$-convex, ϕ-calibrable facet. Then* (3.29) *holds.*

Proof. We closely follow the argument in [37, Theorem 8.1]. By contradiction, let $x \in \partial F$ be a point where $\kappa^F_{\widetilde{\phi}}(x) > \frac{P_{\widetilde{\phi}}(F)}{|F|}$. Then, x belongs to the relative interior of an edge L that is parallel to an edge of $\widetilde{B}^F_\phi$, and such that F is convex at L (indeed, $\kappa^F_{\widetilde{\phi}}$ vanishes in all portions of ∂F that do not satisfy the previous requirements, see [151]); with a small abuse of language, we denote by L also the length of this edge, while ℓ is the length of the corresponding edge of $\widetilde{B}^F_\phi$. Since F is $\widetilde{\phi}$-regular, we can deduce that $B_{L/\ell} \cap U \subset F$,

(8) Other choices of $\phi \in \mathcal{M}(\mathbb{R}^3) \setminus \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^3)$ are possible, for which there exists $E \subset \mathbb{R}^3$, E Lip ϕ-regular, such that $F \in \mathrm{Facets}_\phi(\partial E)$ and (3.9) holds.

where U is a neighbourhood of the side L, while $B_{L/\ell}$ denote the rescaled copy of $\widetilde{B}_\phi^F$ having an edge in L, and lying on the same half-plane of F around L. Applying [37, Lemma 8.3], we get

$$\kappa^F_{\widetilde{\phi}}(x) < \frac{\ell}{L}. \tag{3.30}$$

Following [37, Theorem 8.1, Step 3], let us define, for $\epsilon > 0$ sufficiently small, the set F_ϵ of all points of F having Euclidean distance from the line through L greater than or equal to ϵ. Set $\widehat{F}_\epsilon := F_\epsilon \cup (B_{L/\ell} \cap F)$, see Figure 3.2.

It is possible to prove that, for ϵ sufficiently small,(9)

$$|F| = |\widehat{F}_\epsilon| + o(\epsilon^2), \qquad P_{\widetilde{\phi}}(F) = P_{\widetilde{\phi}}(\widehat{F}_\epsilon). \tag{3.31}$$

Moreover, we notice that

$$|\widehat{F}_\epsilon| - |F_\epsilon| = \epsilon L + o(\epsilon), \tag{3.32}$$

and, using [37, Lemma 8.5],

$$P_{\widetilde{\phi}}(\widehat{F}_\epsilon) - P_{\widetilde{\phi}}(F_\epsilon) = \epsilon\ell + o(\epsilon). \tag{3.33}$$

Let $\beta \in \left(\frac{P_{\widetilde{\phi}}(F)}{|F|}, \kappa^F_{\widetilde{\phi}}(x)\right)$. Then, coupling (3.31), (3.32), and (3.33), also recalling (3.30), we get

$$\begin{aligned} P_{\widetilde{\phi}}(F_\epsilon) - \beta|F_\epsilon| =& P_{\widetilde{\phi}}(F) - \epsilon\ell + \beta(\epsilon L - |F|) + o(\epsilon) \\ =& P_{\widetilde{\phi}}(F) - \beta|F| + \epsilon(\beta L - \ell) + o(\epsilon) < P_{\widetilde{\phi}}(F) - \beta|F|, \end{aligned} \tag{3.34}$$

for $\epsilon > 0$ sufficiently small. But then, since F is ϕ-calibrable, $F = \Omega^F_\beta$, so that (3.34) violates Theorem 3.25, a contradiction. □

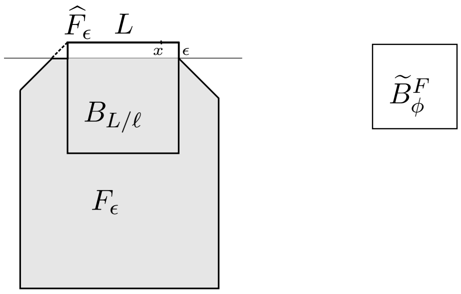


Figure 3.2: The construction used to prove Theorem 3.36: $\widehat{F}_\epsilon$ is obtained by slightly modifying F near the edge L (the original boundary is drawn with a dotted line); $B_{L/\ell}$ is the rescaled copy of $\widetilde{B}_\phi^F$ (represented on the top right) having L as an edge; F_ε is the competitor subset.

3.4.1 Annular facets

In this section we prove some facts about the ϕ-calibrability of "annular facets" $F \in \mathrm{Facets}_\phi(\partial E)$. A more general case with B_ϕ the Euclidean cylinder is covered in [29].

For $x \in \Pi_F$ and $\rho > 0$, we let $B_{\widetilde{\phi}}(x;\rho)$ be the copy of $\rho\widetilde{B}_\phi^F$ centered at x.

(9) Clearly, we just need to justify the second equality in (3.31). Let Γ be a connected component of $\partial F \setminus \partial\widehat{F}_\epsilon$, and let $\epsilon > 0$ be so small that $\widetilde{\nu}^F_{|\Gamma}$ lies between two consecutive vertices ν_1, ν_2 of the unit ball of $\widetilde{\phi}^o$. Then, $\int_\Gamma \widetilde{\phi}^o(\widetilde{\nu}^F)\, d\mathcal{H}^1 = \widetilde{\phi}^o\Big(\int_\Gamma \widetilde{\nu}^F\, d\mathcal{H}^1\Big)$, where we used Jensen's inequality (which holds with equality, since the restriction of $\widetilde{\phi}^o$ on the segment between ν_1, ν_2 is a linear function). Now, a direct computation shows that the right hand side in the previous equality only depends on the ending points of Γ.

Theorem 3.37. *Let* $F \in \mathrm{Facets}_\phi(\partial E)$*. Assume that there exist* $x_1, x_2 \in \mathrm{int}(F)$*, and* $R > r > 0$ *such that*

$$F = B_{\widetilde{\phi}}(x_1; R) \setminus \mathrm{int}(B_{\widetilde{\phi}}(x_2; r)), \qquad B_{\widetilde{\phi}}(x_2; r) \subset\subset B_{\widetilde{\phi}}(x_1; R),$$

and that $\widetilde{\nu}_F$ *points outside of* E *along* $\partial B_{\widetilde{\phi}}(x_1; R)$*, and inside of* E *along* $\partial B_{\widetilde{\phi}}(x_2; r)$.[10] *Then,* F *is* ϕ*-calibrable.*

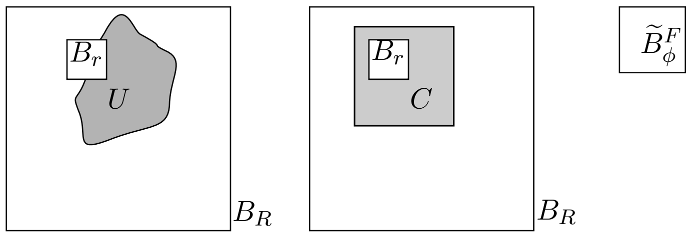


Figure 3.3: On the top right, as an example we take the square as the unit ball $\widetilde{B}^F_\phi$. We shorthand $B_{\widetilde{\phi}}(x_1; R)$ with B_R and $B_{\widetilde{\phi}}(x_2; r)$ with B_r. On the left, the facet F, which can be seen as an "annulus". We assume that $\widetilde{\nu}^F$ points outside (resp. inside) of E on ∂B_R (resp. on ∂B_r). In grey, the sets U and C used in Theorem 3.37 to prove the ϕ-calibrability of F.

Proof. We start by computing the mean normal velocity of F:

$$v_F = \frac{P_{\widetilde{\phi}}(\widetilde{B}^F_\phi)}{|\widetilde{B}^F_\phi|} \frac{R - r}{R^2 - r^2} = \frac{P_{\widetilde{\phi}}(\widetilde{B}^F_\phi)}{|\widetilde{B}^F_\phi|} \frac{1}{R + r}. \tag{3.35}$$

Let C be any subset of F obtained as the difference of two rescaled $\widetilde{\phi}$-balls, namely $C = B_{\widetilde{\phi}}(x; t) \setminus B_{\widetilde{\phi}}(x_2; s)$ for suitable $x \in \mathrm{int}(F)$ and $t \in (r, R]$ such that $B_{\widetilde{\phi}}(x_2; r) \subset\subset B_{\widetilde{\phi}}(x; t)$. Then, recalling (3.24),

$$v_C = \frac{P_{\widetilde{\phi}}(\widetilde{B}^F_\phi)}{|\widetilde{B}^F_\phi|} \frac{t - r}{t^2 - r^2} = \frac{P_{\widetilde{\phi}}(\widetilde{B}^F_\phi)}{|\widetilde{B}^F_\phi|} \frac{1}{t + r}. \tag{3.36}$$

In particular,

$$v_F \leq v_C. \tag{3.37}$$

Now, let $U \subset F$ be a nonempty open finite perimeter set; we have to show that $v_U \geq v_F$. Write

$$\partial^- U := \partial U \cap \partial B_{\widetilde{\phi}}(x_2; r), \qquad \partial^+ U := \partial U \setminus \partial^- U, \qquad \widehat{U} := U \cup B_{\widetilde{\phi}}(x_2; r).$$

Let $t \in (r, R]$ be such that $|\widehat{U}| = |B_{\widetilde{\phi}}(x; t)|$, where $x \in \mathrm{int}(F)$ is such that $B_{\widetilde{\phi}}(x_2; r) \subset\subset B_{\widetilde{\phi}}(x; t) \subset B_{\widetilde{\phi}}(x_1; R)$. By the anisotropic isoperimetric inequality (1.3) (with ϕ replaced by $\widetilde{\phi}$), we get

$$P_{\widetilde{\phi}}(\widehat{U}) \geq P_{\widetilde{\phi}}(B_{\widetilde{\phi}}(x; t)),$$

that is

$$\int_{\partial^+ U} \widetilde{\phi}^o(\widetilde{\nu}^U)\, d\mathcal{H}^1 + \int_{\partial B_{\widetilde{\phi}}(x_2; r)} \widetilde{\phi}^o(\widetilde{\nu}^F)\, d\mathcal{H}^1 - \int_{\partial^- U} \widetilde{\phi}^o(\widetilde{\nu}^F)\, d\mathcal{H}^1 \geq \int_{\partial B_{\widetilde{\phi}}(x; t)} \widetilde{\phi}^o(\widetilde{\nu}^{B_{\widetilde{\phi}}(x;t)})\, d\mathcal{H}^1. \tag{3.38}$$

[10] In particular, E is not convex at F.

Let $C := B_{\widetilde{\phi}}(x;t) \setminus B_{\widetilde{\phi}}(x_2;r)$. Notice that $|C| = |U|$. Then, using also (3.38) and (3.37), we get

$$v_U = \frac{\int_{\partial^+ U} \widetilde{\phi}^o(\widetilde{\nu}^U)\, d\mathcal{H}^1 - \int_{\partial^- U} \widetilde{\phi}^o(\widetilde{\nu}^F)\, d\mathcal{H}^1}{|U|} \geq v_C \geq v_F.$$

□

Remark 3.38. We cannot expect in general to prove ϕ-calibrability of a facet F such that E is convex at F, and which is obtained by removing from a ball a smaller ball. This is a difference compared to what happens when E is not convex at F (Theorem 3.37). To show this fact, let us consider the bidimensional anisotropy having a square of side ℓ as unit ball, and let us consider the facet F in Figure 3.4, obtained by removing from $\frac{S}{\ell}\widetilde{B}^F_\phi$ the ball $B_{\widetilde{\phi}}(x; \frac{s}{\ell})$, where S and s are the Euclidean lengths of the sides of the two squares, and x lies on the diagonal of the bigger one. Let a be the Euclidean distance between the boundaries of the two balls. The mean velocity of the facet is $v_F = \frac{4}{S-s}$. If we consider the set B in Figure 3.4 we get

$$v_B = \frac{4S}{S^2 - (a+s)^2},$$

and the inequality $v_B < v_F$ is verified when $a < -s + \sqrt{Ss}$.

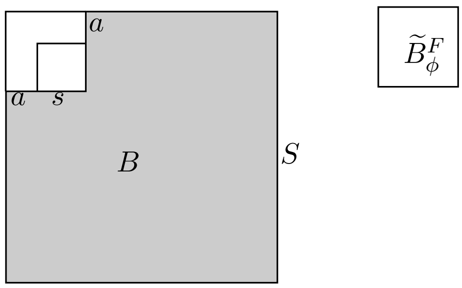


Figure 3.4: If F is a nonconcentric annulus and E is convex at F, then F is non ϕ-calibrable if the distance a between the two connected components of ∂F is small enough.

3.4.2 Closed strips

The case of strips has been investigated in [113] in the Euclidean setting. Our aim is to generalize it to the anisotropic setting.

Assume the facet F to have the following shape. Let $\Gamma := \partial\Omega$ be a closed planar simple curve, where Ω is a $\widetilde{\phi}$-regular and $\widetilde{\phi}$-convex set. For some positive integers $0 < l \leq k$, we denote by Γ_i, $i = 1, .., l$, the relatively open edges of Γ parallel to some edges on the ball $\widetilde{B}^F_\phi$ and with nonzero $\widetilde{\phi}$-mean curvature, and by Γ_j, $j = l+1, ..., k$, each relatively open connected component of Γ when the $\widetilde{\phi}$-mean curvature vanishes (if $k = l$, we mean that there is no such a connected component); κ_i denotes the value of the $\widetilde{\phi}$-curvature of Γ_i. On Γ we take the optimal selection N_Γ, defined as the linear interpolation of the (uniquely determined) vectors on the vertices of Γ; while, on each Γ_j, N_Γ is a constant vector, which we denote by N_{Γ_j}.

For $a > 0$ such that $a \leq \inf_{i=1,...,l} \kappa_i^{-1}$, set

$$F := \{x \in \mathbb{R}^2 \ : \ x = q + tN_\Gamma(q),\ q \in \Gamma,\ |t| \leq a\}. \tag{3.39}$$

Due to the $\widetilde{\phi}$-convexity of Γ and to the bound on a, for any $x \in F$ the $\widetilde{\phi}$-projection $q(x)$ is uniquely determined, and it satisfies $x = q(x) + t(x)N_\Gamma(q(x))$ with $t(x) := -d^\Omega_{\widetilde{\phi}}(x)$.(11)

(11) Recalling the notation in Section 1.2, by $d^\Omega_{\widetilde{\phi}}$ we denote the $\widetilde{\phi}$-signed distance function from $\partial\Omega$, positive in the interior of Ω.

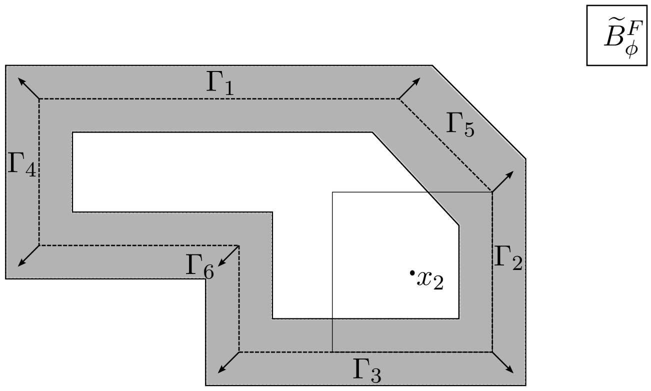


Figure 3.5: The dotted curve Γ is the boundary of a $\widetilde{\phi}$-convex set, where $\widetilde{B}_\phi^F$ is represented in the corner of the picture. Here $l = 4$ and $k = 6$. We represent also the optimal selection N_Γ on the vertices of Γ. In grey we draw the set F defined in (3.39). Finally the point x_2 is the center of the ball $\kappa_i^{-1}\widetilde{B}_\phi^F$ having Γ_i as an edge, lying on the side of Γ opposite to the direction of N_Γ.

Theorem 3.39. *Assume that E is convex at F. Then F is ϕ-calibrable, and $\kappa_{\phi,F} = \frac{1}{a}$.*

Proof. In order to prove the statement, recalling also Remark 3.21, we want to construct a selection with divergence constantly equal to $\frac{1}{a}$. Following [113],[12] we define the vector field $\widetilde{N}$ on F as

$$\widetilde{N}(x) := \begin{cases} \left(1 - \frac{(\kappa_i^{-1}-a)(\kappa_i^{-1}+a)}{(\widetilde{\phi}(x-x_i))^2}\right) \frac{x-x_i}{2a}, & q(x) \in \Gamma_i,\ i = 1, ..., l \\ \\ -\frac{d^\Omega_{\widetilde{\phi}}(x)}{a} N_{\Gamma_j}, & q(x) \in \Gamma_j,\ j = l+1, ..., k, \end{cases}$$

where, for $i = 1, \dots, l$, x_i is the center of the copy of $\kappa_i^{-1} B_{\widetilde{\phi}}$ having Γ_i as an edge, lying in the side of Γ_i opposite to the direction of N_Γ. An immediate computation shows that $\widetilde{\phi}(\widetilde{N}(x)) \le 1$, and $\langle \widetilde{\nu}^F, \widetilde{N}\rangle = 1 = c_F$, so that $\widetilde{N}$ is a selection on F.

Moreover, we notice that

$$\widetilde{N} \in \mathscr{H}^2_{\mathrm{div}}(F). \tag{3.40}$$

Indeed, for every $x \in F$, $\widetilde{N}(x)$ is parallel to $N_\Gamma(q(x))$,[13] which implies that $\mathrm{div}\widetilde{N} \in L^2(F)$, and hence (3.40).

Let us explicitely compute the divergence of $\widetilde{N}$. For any $i = 1, ..., l$ and for any $x \in F$ such that $q(x) \in \Gamma_i$, there holds

$$\begin{aligned} \mathrm{div}\widetilde{N}(x) =& \frac{1}{a}\left(\frac{(\widetilde{\phi}(x-x_i))^2 - (\kappa_i^{-1})^2 + a^2}{(\widetilde{\phi}(x-x_i))^2}\right. \\ &\left. + \frac{((\kappa_i^{-1})^2 - a^2)\big(T_{\widetilde{\phi}}(x-x_i)\cdot(x-x_i)\big)}{a\,(\widetilde{\phi}(x-x_i))^4}\right) = \frac{1}{a}, \end{aligned}$$

where in the last equality we noticed that $T_{\widetilde{\phi}}(x-x_i)\cdot(x-x_i) = (\widetilde{\phi}(x-x_i))^2$. When $x \in F$ is such that $q(x) \in \Gamma_j$, $j = l+1, .., k$ we get:

$$\mathrm{div}\widetilde{N}(x) = -\frac{\nabla d^\Omega_{\widetilde{\phi}}(x)\cdot N_{\Gamma_j}}{a} = \frac{1}{a}.$$

Hence, $\widetilde{N}$ has constant divergence in F. By 3.21, the proof is completed. □

[12] See also [29] for a similar computation.

[13] In general, $\widetilde{N}$ is not continuous in F, since it may jump on $\{x \in F\ :\ q(x) \text{ is a vertex of } \Gamma\}$.

Remembering Remark 3.38, we observe that in Theorem 3.39 we cannot easily drop the symmetry with respect to the curve Γ.

3.5 Optimal selections in facets for the ϕ_c-norm

In this section we shall restrict our attention to the case in which

$$\phi = \phi_c$$

is the Euclidean cylindrical norm in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, *i.e.* the norm of $\mathbb{R}^3$ whose unit ball B_{ϕ_c} is given by

$$B_{\phi_c} := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \ : \max \left(\sqrt{\xi_1^2 + \xi_2^2}, |\xi_3| \right) \leq 1 \right\},$$

see Figure 3.6.

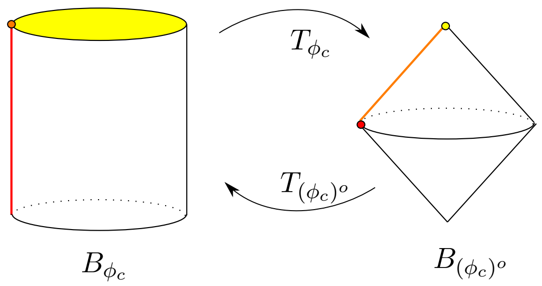


Figure 3.6: The unit ball of ϕ_c (on the left) and its dual (on the right). Colours are used in order to represent the action of the duality maps T_{ϕ_c} and $T_{(\phi_c)^o}$. In particular: the interior of the upper facet of B_{ϕ_c} is mapped by T_{ϕ_c} onto the upper vertex of $B_{(\phi_c)^o}$; each point of the boundary of the facet is mapped onto an edge of the upper half-cone; finally, a vertical edge of B_{ϕ_c} is mapped onto a point of the circle separeting the two half-cones of $B_{(\phi_c)^o}$.

We shall assume that E is a Lip ϕ-regular set, $F \in \text{Facets}_\phi(\partial E)$, and

$$E \text{ is convex at } F.$$

Hence, by Theorems 3.25 and 3.13, we have $\kappa_{\min} > 0$ and $\kappa_{\max} < +\infty$.

Notice that $\widetilde{\phi}_F = \widetilde{\phi}$ is the Euclidean norm in the plane Π_F (identified with the horizontal plane $\mathbb{R}^2$), so that F is of class $\mathcal{C}^{1,1}$ (Theorem 3.24). To avoid possible ambiguity in the notation, in this section we shall restore symbol $\kappa^F_{\widetilde{\phi}}$ in order to denote the (Euclidean) curvature of ∂F. From now on, by $h(F)$ we mean $h(\text{int}(F))$, and by $\text{Ch}(F)$ we mean $\text{Ch}(\text{int}(F))$. It is useful to remember that, by Theorem 3.27, we have $h(F) = \kappa_{\min}$.

Remark 3.40. We recall $\kappa_{\phi_c,F} \in \text{Lip}_{\text{loc}}(\text{int}(F))$, see [64, Theorem 2].

We recall that, by Remark 3.32, F is strictly ϕ_c-calibrable if and only if F is a strict Cheeger set, which in turn is equivalent, when F is convex, to require that $\text{ess sup}_{x \in \partial F} \kappa^F_{\widetilde{\phi}}(x) \leq h(F)$ (Theorem 3.33).

Let us now state the following remarkable result, proven by Giusti in [101], which will play a crucial role in the remaining of this section. We refer the reader for instance to Appendix B for a brief discussion on the action principle for a capillary.

Theorem 3.41. *Let $\Omega \subset \mathbb{R}^k$ be a bounded connected open set with Lipschitz boundary, and let $h := \frac{P(\Omega)}{|\Omega|}$. Then there exists a solution $u \in \mathcal{C}^2(\Omega)$ of*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h \qquad \text{in } \Omega \tag{3.41}$$

if and only if

$$h < \frac{P(B)}{|B|}, \qquad B \subset \Omega,\ B \neq \emptyset. \tag{3.42}$$

Moreover, if Ω is of class $\mathcal{C}^2$, the solution is unique up to an additive constant, bounded from below in Ω, and its graph is vertical at the boundary of Ω, in the sense that

$$\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \to \nu^\Omega \qquad \text{uniformly on } \partial\Omega. \tag{3.43}$$

Finally, if $k = 2$ and Ω is convex, (3.42) *is in turn equivalent to assume that the curvature of $\partial\Omega$, at all points of $\partial\Omega$ where it is defined, is less than or equal to h.*

Remark 3.42. Let u be a solution of (3.41), with $\Omega := \operatorname{int}(F)$ and $h := h(F)$. Repeating the proof in [101, Section 2], which is still valid assuming Ω of class $\mathcal{C}^{1,1}$, one proves that u is bounded from below in $\operatorname{int}(F)$ and satisfies (3.43).

As a corollary of Theorem 3.41, we get the following result.

Proposition 3.43 (Optimal selection in strictly ϕ_c-calibrable facets). *Suppose F to be strictly ϕ_c-calibrable. Then there exists u solving* (3.41) *in $\Omega := \operatorname{int}(F)$. Moreover, the vector field*

$$\widetilde{N} := \begin{cases} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} & \text{in } \operatorname{int}(F), \\ \\ \widetilde{\nu}^F & \text{on } \partial F, \end{cases} \tag{3.44}$$

is an optimal selection in F, continuous in F and analytic in $\operatorname{int}(F)$.

Proof. The first assertion follows recalling Remark 3.32 and Theorem 3.41. By construction, using also Remark 3.42, $\widetilde{N}$ belongs to $\mathscr{H}^2_{\mathrm{div}}(F)$ and satisfies (3.23). Analytic regularity of $\widetilde{N}$ follows from elliptic regularity. $\square$

Clearly, the vector field $\widetilde{N}$ in (3.44) is, up to the sign, the "horizontal" component of the Euclidean outer normal to the subgraph of u.

Remark 3.44 (Lipschitz regularity). Assuming also that $\operatorname{ess\,sup}_{x\in\partial F}\kappa^F_{\widetilde{\phi}}(x) < \frac{P(F)}{|F|} = h(F)$, then (compare [101, p.125])

$$\widetilde{N} \in \operatorname{Lip}(F; \Pi_F).$$

Remark 3.45 (Total variation flow). It is worth to notice that, by virtue of [6, Theorem 17], and for a convex facet F of class $\mathcal{C}^{1,1}$, our construction provides also the solution of the total variation flow in $\mathbb{R}^2$

$$\partial_t u = \operatorname{div}\left(\frac{Du}{|Du|}\right), \tag{3.45}$$

with initial datum the characteristic function of F. Heuristically, if u is a solution of (3.45), and $p(t) = (x, u(t,x))$ is a point of $\operatorname{graph}(u(t)) \subset \mathbb{R}^3$ around which $u(t)$ is sufficiently smooth with nonzero gradient, then the vertical velocity of $p(t)$ equals the mean curvature of the level set of $u(t)$ passing through x; strictly ϕ_c-calibrable flat regions

F of graph$(u(t))$ evolve in vertical direction[14] with velocity equal to $P(F)/|F|$; vertical walls (provided $u(t)$ is discontinuous) of graph$(u(t))$ do not move; finally, isolated points where the gradient of $u(t)$ vanishes, such as local minima or local maxima, may develop instantaneously flat horizontal regions. See also [29, 30, 6, 62], or the last section in [9]. Therefore, there are analogies between the total variation flow in $\mathbb{R}^2$ and the anisotropic mean curvature flow of ϕ_c-calibrable facets; however the two motions differ immediately after the initial time. Indeed, even for ϕ_c-calibrable facets, the graph of $v = 1_F$ decreases its height without distortion of the boundary, while the shape of F is expected in general to change for $t > 0$.

We now give some examples of non ϕ_c-calibrable facets F for which we can exhibit an optimal *continuous* selection.

Example 3.46 (Non ϕ_c-calibrable convex facets). Let F be convex and not ϕ_c-calibrable (see Theorem 3.33). By virtue of Theorem 3.6, the maximal Cheeger subset Ch(F) of F is strictly Cheeger, and (Theorem 3.1) it is of class $\mathcal{C}^{1,1}$. Moreover (Theorem 3.27) Ch$(F) = \Theta^F_{\kappa_{\min}}$. Applying Proposition 3.43 with Ch(F) in place of F, we get a function $u \in \mathcal{C}^2(\mathrm{int}(\mathrm{Ch}(F)))$ solving (3.41) in int(Ch(F)) with $h := h(F)$. Set

$$\widetilde{N} := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \qquad \text{in } \ \mathrm{int}(\mathrm{Ch}(F)).$$

By Theorem 3.28, $\kappa_{\phi,F}$ is convex in F, so that there cannot be subsets of F with positive Lebesgue measure where $\kappa_{\phi,F}$ is constant, except for $\Theta^F_{\kappa_{\min}}$. Hence, for every $\beta \in [\kappa_{\min}, \kappa_{\max})$, $\mathrm{int}(F) \cap \partial\Theta^F_\beta = \{x \in \mathrm{int}(F) \, : \, \kappa_{\phi,F}(x) = \beta\}$. From Theorems 3.25 and 3.28, each connected component of $\mathrm{int}(F) \cap \partial\Theta^F_\beta$ is contained in a circumference of radius β^{-1}. Thus, we extend $\widetilde{N}$ in $\mathrm{int}(F) \setminus \Theta^F_{\kappa_{\min}}$ as the outward normal unit vector to the level curves of $\kappa_{\phi,F}$ — namely, $\widetilde{N} := \widetilde{\nu}^{\partial\Theta^F_\beta}$ on $\{\kappa_{\phi,F} = \beta\}$. By construction, recalling also Remark 3.42, $\mathrm{div}\widetilde{N} = \kappa_{\phi,F}$ in int(F), and $\widetilde{N}$ verifies the third equation in (3.23). Hence, $\widetilde{N} \in \mathscr{H}^2_{\mathrm{div}}(F)$, and $\widetilde{N}$ is an optimal selection in F (Remark 3.23). Moreover, $\widetilde{N}$ is continuous in F, analytic in $\mathrm{int}(\Theta^F_{\kappa_{\min}})$, and $\widetilde{N}(x) \in \partial\widetilde{B}^F_\phi$ for any $x \in \mathrm{int}(F) \setminus \Theta^F_{\kappa_{\min}}$.

The following examples have been inspired by [88, 114]. For $r > 0$ and $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$, we set $B_r(\bar{x}_1, \bar{x}_2) := \{x = (x_1, x_2) \in \mathbb{R}^2 \, : \, (x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2 \le r^2\}$.

Example 3.47 (Rounded two circle facets). Let $\theta \in (0, \frac{\pi}{2})$, and

$$\mathfrak{P}_\theta := B_1(0,0) \cup B_{\sin\theta}(\cos\theta, 0).$$

One can prove [114] the following facts: $\mathfrak{P}_\theta$ admits a unique (hence maximal) Cheeger subset Ch$(\mathfrak{P}_\theta)$ (as in Figure 3.7(a)); moreover, there exists a unique $\theta_0 \in (0, \frac{\pi}{2})$ such that $\mathfrak{P}_{\theta_0}$ is Cheeger. Our idea is to construct an optimal selection, solving (3.41) in Ch$(\mathfrak{P}_\theta)$ (for $\theta \neq \theta_0$), and then foliate the remaining part of $\mathfrak{P}_\theta$ with arcs of circles, taking as vector field the outward unit normal to the arcs. Fix $\theta \neq \theta_0$, so that

$$h(\mathfrak{P}_\theta) = \frac{P(\mathrm{Ch}(\mathfrak{P}_\theta))}{|\mathrm{Ch}(\mathfrak{P}_\theta)|} < \frac{P(\mathfrak{P}_\theta)}{|\mathfrak{P}_\theta|}. \tag{3.46}$$

Notice that

$$h(\mathfrak{P}_\theta) < \frac{1}{\sin\theta}, \tag{3.47}$$

since $h(\mathfrak{P}_\theta)$ equals the curvature of $\mathrm{int}(\mathfrak{P}_\theta) \cap \partial\mathrm{Ch}(\mathfrak{P}_\theta)$, which is strictly less than $\frac{1}{\sin\theta}$ by the geometry of $\mathfrak{P}_\theta$.

Even if $\mathfrak{P}_\theta$ is regarded as a facet of a three-dimensional set E convex at $\mathfrak{P}_\theta$, the set E cannot be Lip ϕ_c-regular, since $\mathfrak{P}_\theta$ is not of class $\mathcal{C}^{1,1}$.[15] Thus we perform the following

[14] Upwards or downwards depending on whether F consists of points of local minimum or local maximum of $u(t)$.

[15] Therefore, strictly speaking, we cannot apply Theorem 3.11 in order to define κ^E_ϕ on $\mathfrak{P}_\theta$.

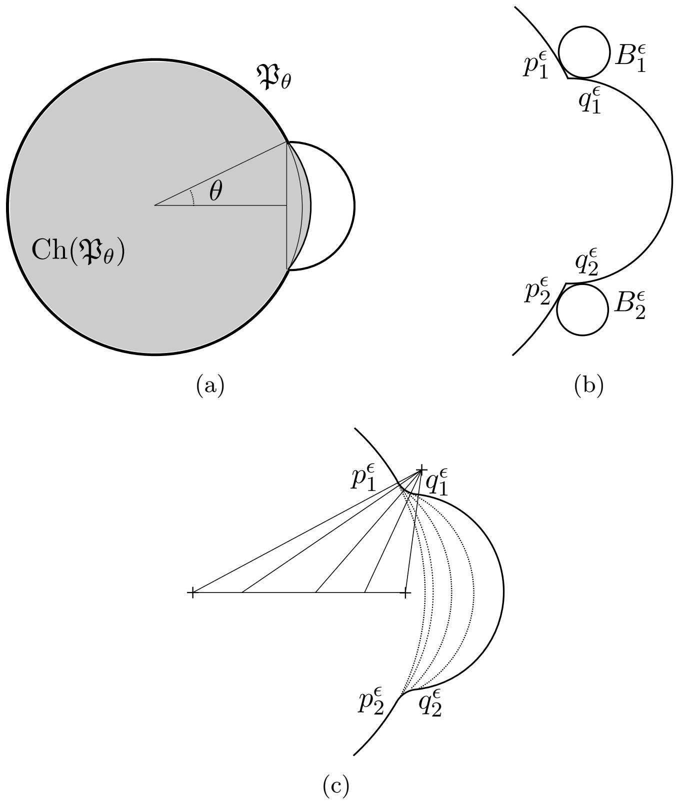


Figure 3.7: In (a), the set $\mathfrak{P}_\theta$ and its maximal Cheeger subset $\mathrm{Ch}(\mathfrak{P}_\theta)$ (in grey). In (b), the construction of the facet $F = F_\epsilon$ in Example 3.47. In (c), some sublevel sets Θ_β^F of $\kappa_{\phi_c,F}$ are represented. For every $\beta \in (\kappa_{\min}, \kappa_{\max})$, $\mathrm{int}(F) \cap \partial\Theta_\beta^F$ is an arc of circumference with radius β^{-1}, and tangent to ∂F. For any $\beta \in (1, \frac{1}{\sin\theta})$, such an arc is unique, and its terminal points belong to the arcs bounded by p_j^ϵ and q_j^ϵ, for $j = 1, 2$.

smoothing construction near the non-differentiability points of $\partial\mathfrak{P}_\theta$. For $\epsilon > 0$, let B_1^ϵ, B_2^ϵ be the two closed disks satisfying the following properties: for $j = 1, 2$, B_j^ϵ is externally tangent to $\mathfrak{P}_\theta$, and $\mathfrak{P}_\theta \cap B_j^\epsilon = \{p_j^\epsilon, q_j^\epsilon\}$, for some $p_j^\epsilon \in \partial B_1(0,0)$, and $q_j^\epsilon \in \partial B_{\sin\theta}(\cos\theta, 0)$. According to Figure 3.7(b), we define $F = F_\epsilon$ as the union of $\mathfrak{P}_\theta$ with the curved triangles having vertices p_j^ϵ, q_j^ϵ, and $(\cos\theta, (-1)^j \sin\theta)$, for $j = 1, 2$.

By construction F is of class $\mathcal{C}^{1,1}$ (and it is not convex). Recalling also (3.46), we choose $\epsilon > 0$ so small that

$$\left| \frac{P(\mathfrak{P}_\theta)}{|\mathfrak{P}_\theta|} - \frac{P(F)}{|F|} \right| = O(\epsilon) < \frac{P(\mathfrak{P}_\theta)}{|\mathfrak{P}_\theta|} - \frac{P(\mathrm{Ch}(\mathfrak{P}_\theta))}{|\mathrm{Ch}(\mathfrak{P}_\theta)|}. \tag{3.48}$$

In particular,

$$\frac{P(\mathrm{Ch}(\mathfrak{P}_\theta))}{|\mathrm{Ch}(\mathfrak{P}_\theta)|} < \frac{P(F)}{|F|},$$

which implies that F is not Cheeger, or equivalently (Remark 3.32) that F is not ϕ_c-calibrable. It must be underlined that our argument neither provides nor excludes the ϕ_c-calibrability of $F := \mathfrak{P}_{\theta_0}^\epsilon$. We observe that, for any $\beta \in (1, \frac{1}{\sin\theta})$, there is a *unique* circumference $\widehat{\Gamma}_\beta \subset F$, with curvature β, and tangent to ∂F at two points, lying on the arcs of ∂F bounded by p_j^ϵ, q_j^ϵ, for $j = 1, 2$: see Figure 3.7(c). We denote by Γ_β the shortest connected component of $\mathrm{int}(F) \cap \widehat{\Gamma}_\beta$. Then $\mathrm{Ch}(F)$ is determined as the subset of F containing $B_1(0,0)$,[16] and such that $\mathrm{int}(F) \cap \partial\mathrm{Ch}(F) = \Gamma_{h(F)}$. In particular, $\mathrm{Ch}(F)$

[16] Actually, we have $B_1(0,0) \subset \mathrm{Ch}(F)$, since it can be proven that $B_1(0,0) \subset \mathrm{Ch}(\mathfrak{P}_\theta)$.

is strictly Cheeger and of class $\mathcal{C}^{1,1}$. Furthermore, recalling Remark 3.26, and taking into account the geometry of F, we have

$$\mathrm{int}(F) \cap \partial\Theta^F_\beta = \Gamma_\beta = \mathrm{int}(F) \cap \partial\Omega^F_\beta, \qquad \beta \in (\kappa_{\min}, \kappa_{\max}). \tag{3.49}$$

Now, we exclude the presence of regions in $\mathrm{int}(F) \setminus \mathrm{Ch}(F)$ where $\kappa_{\phi_c,F}$ is constant. Suppose by contradiction that there exists $\bar\beta \in (\kappa_{\min}, \kappa_{\max}]$ such that $\{\kappa_{\phi_c,F} = \bar\beta\}$ has positive Lebesgue measure. If $\bar\beta < \kappa_{\max}$, then

$$\mathrm{int}(F) \cap \partial\Theta^F_{\bar\beta} \neq \mathrm{int}(F) \cap \partial\Omega^F_{\bar\beta}, \tag{3.50}$$

which contradicts (3.49). If $\bar\beta = \kappa_{\max}$, then $\partial\Theta^F_{\kappa_{\max}} = \partial F$, and so (Remark 3.26) $\operatorname{ess\,sup} \kappa^F_{\widetilde\phi} = \frac{1}{\sin\theta} \leq \kappa_{\max}$. On the other hand, since we are assuming $\mathrm{int}(F) \cap \partial\Omega^F_{\kappa_{\max}} \neq \emptyset$, $\mathrm{int}(F) \cap \partial\Omega^F_{\kappa_{\max}}$ should be an arc of circumference with curvature $\kappa_{\max}$, and tangent to ∂F. In particular, by the geometry of F, $\kappa_{\max} < \frac{1}{\sin\theta}$, a contradiction.

As a consequence, we have

$$\kappa_{\max} = \frac{1}{\sin\theta},$$

otherwise $\kappa_{\phi_c,F}$ would be constantly equal to $\kappa_{\max}$ in the full-measure subset of F bounded by $\Gamma_{\kappa_{\max}}$, and not containing $B_1(0,0)$ — again a contradiction.

We define $\widetilde N$ in F as follows: $\widetilde N := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ in $\mathrm{int}(\mathrm{Ch}(F))$, where u is given by Theorem 3.41, with $\Omega = \mathrm{int}(\mathrm{Ch}(F))$ and $h = h(F)$; while, for $\beta \in (h(F), \frac{1}{\sin\theta})$ and $x \in \Gamma_\beta$, $\widetilde N(x)$ is the outward unit normal to Θ^F_β at x. Notice that $\widetilde N \in \mathscr{H}^2_{\mathrm{div}}(F)$ (Remark 3.42), and it is an optimal selection in F (Remark 3.23). Concerning the regularity of $\widetilde N$, we notice that $\widetilde N$ is continuous in F, and analytic in $\mathrm{int}(\mathrm{Ch}(F))$. Moreover, $\widetilde N(x) \in \partial \widetilde B^F_\phi$ for any $x \in \mathrm{int}(F) \setminus \mathrm{Ch}(F)$.

By modifying Example 3.47, we now build an optimal selection for a facet F admitting an open region outside of $\Theta^F_{\kappa_{\min}}$ where $\kappa_{\phi_c,F}$ is constant (and equal to $\kappa_{\max}$).

Example 3.48 (Rounded proboscis). Let $M > 0$, and let θ, θ_0 and $\mathfrak{P}_\theta$ be as in Example 3.47. Set

$$\mathfrak{P}_{\theta,M} := B_1(0,0) \cup \{x \in \mathbb{R}^2 : \ x = y + (c,0), \ y \in B_{\sin\theta}(\cos\theta, 0), \ c \in [0,M]\},$$

see Figure 3.8(a).

We claim that, for any $M > 0$ and any $\theta < \theta_0$,

$$\frac{P(\mathfrak{P}_\theta)}{|\mathfrak{P}_\theta|} < \frac{P(\mathfrak{P}_{\theta,M})}{|\mathfrak{P}_{\theta,M}|}. \tag{3.51}$$

Indeed, since $P(\mathfrak{P}_\theta) = 2(\pi-\theta) + \pi\sin\theta$, $|\mathfrak{P}_\theta| = \pi + \frac{\pi}{2}\sin^2\theta - (\theta - \sin\theta\,\cos\theta)$, $P(\mathfrak{P}_{\theta,M}) = P(\mathfrak{P}_\theta) + 2M$, and $|\mathfrak{P}_{\theta,M}| = |\mathfrak{P}_\theta| + 2M\sin\theta$, (3.51) is equivalent to $P(\mathfrak{P}_\theta)\sin\theta < |\mathfrak{P}_\theta|$, *i.e.*

$$(\pi-\theta)(2\sin\theta - 1) - \sin\theta\,\cos\theta + \frac{\pi}{2}\sin^2\theta < 0. \tag{3.52}$$

Let us show that the left hand side of (3.52) is strictly increasing in $(0, \frac{\pi}{2})$. Indeed, computing the first derivative (w.r.t. θ), we get

$$2(\pi-\theta)\cos\theta + \pi\cos\theta\sin\theta + 2\sin\theta(\sin\theta - 1). \tag{3.53}$$

Notice that, since $\theta \in (0, \frac{\pi}{2})$, the first term in (3.53) is greater than $\pi\cos\theta$. Now, using for instance the identities $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, since $t \in (0,1)$, it is easy to show that

$$\begin{aligned}\pi\cos\theta(1+\sin\theta) + 2\sin\theta(\sin\theta - 1) &= \frac{1-t}{(1+t^2)^2}\left[\pi(1+t)(t^2+2t+1) - 2t(1-t)\right] \\ &\geq \frac{1-t}{(1+t^2)^2}\left[\pi(1+t)^3 - \frac{1}{2}\right] > 0,\end{aligned}$$

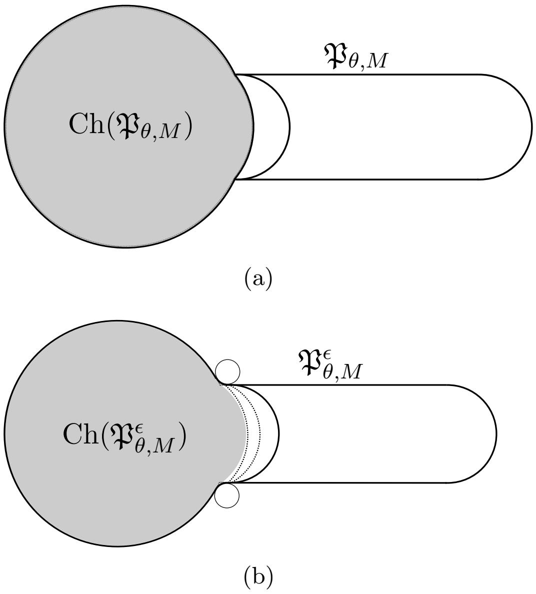


Figure 3.8: In (a), the set $\mathfrak{P}_{\theta,M}$ and its Cheeger subset $\mathrm{Ch}(\mathfrak{P}_{\theta,M})$. In (b), the facet $F = F_\epsilon$ described in Example 3.48. In this case, there are two full-measure subsets where $\kappa_{\phi_c,F}$ is constant.

for every $\theta \in (0, \frac{\pi}{2})$. As a consequence, the left hand side of (3.52) is strictly increasing in $[0, \frac{\pi}{2}]$, and it is zero just at one value of $\theta \in (0, \frac{\pi}{2})$, which must coincide with θ_0.

Fix $\theta \in (0, \theta_0)$. For $\epsilon > 0$, let $\mathfrak{P}^{\epsilon}_{\theta,M}$ be the set of class $\mathcal{C}^{1,1}$ obtained by taking the union of $\mathfrak{P}_{\theta,M}$ with the curved triangles, bounded by $\mathfrak{P}_{\theta,M}$ and a disk with radius ϵ and externally tangent to $\mathfrak{P}_{\theta,M}$: see Figure 3.8(b). Similarly to Example 3.47, we choose $\epsilon > 0$ so small (depending on the difference between the two terms in (3.51)) that $F = F_\epsilon := \mathfrak{P}^{\epsilon}_{\theta,M}$ is not Cheeger.

By construction, F is neither convex nor ϕ_c-calibrable. Moreover, for any $\beta \in (1, \frac{1}{\sin\theta})$, there still exists a *unique* circumference $\widehat{\Gamma}_\beta \subset F$ with curvature β, and tangent to ∂F at two points; again, referring to Figure 3.8(b), these points must lie on the arcs bounded by p^ϵ_j, q^ϵ_j, where $j = 1, 2$. We denote by Γ_β the shortest connected component of $\mathrm{int}(F) \cap \widehat{\Gamma}_\beta$.

Similarly to Example 3.47, we can still determine $\mathrm{Ch}(F)$ as the unique subset of F (strictly) containing $B_1(0,0)$, and such that $\mathrm{int}(F) \cap \partial\mathrm{Ch}(F) = \Gamma_{h(F)}$. In particular, $\mathrm{Ch}(F)$ is strictly Cheeger. Moreover, reasoning as in Example 3.47,(17) there is no $\bar\beta \in (h(F), \frac{1}{\sin\theta})$ such that $\kappa_{\phi_c,F} = \bar\beta$ in some subset of F with positive Lebesgue measure.

Therefore:

- for any $\beta \in (h(F), \frac{1}{\sin\theta})$, Θ^F_β is the closed subset of F containing $B_1(0,0)$, and such that $\mathrm{int}(F) \cap \partial\Theta^F_\beta = \Gamma_\beta$;
- $\kappa_{\max} = \frac{1}{\sin\theta}$, and $\kappa_{\phi_c,F} = \kappa_{\max}$ in $\mathrm{int}(F) \setminus \bigcup_{\beta < \frac{1}{\sin\theta}} \Theta^F_\beta$.

Also in this case, we can exhibit an optimal selection $\widetilde{N}$ (Remarks 3.23-3.42) which is continuous in F, and analytic in $\mathrm{int}(\mathrm{Ch}(F))$. More precisely, $\widetilde{N}$ is defined as follows: $\widetilde{N} := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ in $\mathrm{int}(\mathrm{Ch}(F))$, where u is given in Theorem 3.41 (with the choice $\Omega := \mathrm{int}(\mathrm{Ch}(F))$, and $h := h(F)$); for $\beta \in (h(F), \frac{1}{\sin\theta})$ and $x \in \Gamma_\beta$, $\widetilde{N}(x)$ is the outer normal to Θ^F_β at x; finally, if $\kappa_{\phi_c,F}(x) = \kappa_{\max}$, we set $\widetilde{N}(x) := \widetilde{N}(\tilde{x})$, where $\tilde{x} \in \mathrm{int}(F) \cap \partial\Omega^F_{\frac{1}{\sin\theta}}$ is such that $\tilde{x}_2 = x_2$.

(17) Recall in particular the proof of (3.50).

We notice the presence of a full-measure subset of F, unrelated to the maximal Cheeger subset of F, and where it is possible to construct an optimal selection without making use of Theorem 3.41.

We conclude this section with an example in which we are not able to provide an explicit optimal selection, even if we determine the ϕ_c-mean curvature of F.

Example 3.49 ("Dumbbell-like" facet). Let θ and θ_0 be as in Example 3.47, and suppose $\theta \in (0, \theta_0)$. Let $M > 2\sin\theta + 1$, and let $\mathfrak{D}_{\theta,M}$ be the set obtained as the union of $\mathfrak{P}_\theta \cup \mathfrak{P}'_\theta$, and the strip $[\cos\theta, \cos\theta + M] \times [-\sin\theta, \sin\theta]$, where $\mathfrak{P}_\theta$ is the set in Example 3.47, and $\mathfrak{P}'_\theta$ is its symmetric with respect to the straight line $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \cos\theta + \frac{M}{2}\}$.

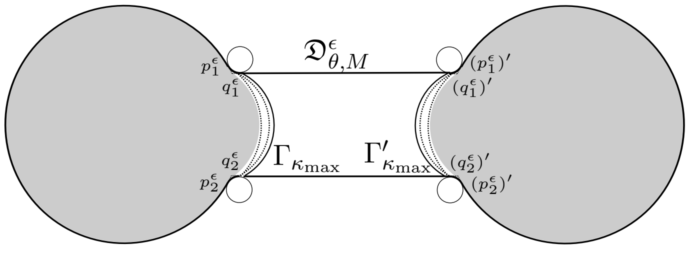


Figure 3.9: The dumbbell facet $\mathfrak{D}^\epsilon_{\theta,M}$ in Example 3.49. In grey, its maximal Cheeger subset, some level sets of $\kappa_{\phi_c,F}$, and the set $\{\kappa_{\phi_c,F} = \kappa_{\max}\}$ bounded by the arcs $\Gamma_{\kappa_{\max}}$ and $\Gamma'_{\kappa_{\max}}$. Notice that in this case $\kappa_{\max} < \frac{1}{\sin\theta}$.

We observe that

$$\frac{P(\mathfrak{D}_{\theta,M})}{|\mathfrak{D}_{\theta,M}|} = \frac{M + 2(\pi - \theta)}{\cos\theta\,\sin\theta + \pi - \theta + M\sin\theta}, \tag{3.54}$$

which, as $M \to +\infty$, tends to $\frac{1}{\sin\theta}$. In particular, since $\theta < \theta_0$, recalling also (3.46) and (3.47),

$$\frac{P(\mathrm{Ch}(\mathfrak{P}_\theta))}{|\mathrm{Ch}(\mathfrak{P}_\theta)|} < \frac{P(\mathfrak{D}_{\theta,M})}{|\mathfrak{D}_{\theta,M}|}, \tag{3.55}$$

for $M > 0$ sufficiently large.

For $\epsilon > 0$, let B_j^ϵ and $(B_j^\epsilon)'$ for $j = 1, 2$, be the four balls of radius ϵ externally tangent to $\mathfrak{D}_{\theta,M}$ in p_j^ϵ and q_j^ϵ, and in $(p_j^\epsilon)'$ and $(q_j^\epsilon)'$ respectively. For M such that (3.55) holds, let $F = F_\epsilon := \mathfrak{D}^\epsilon_{\theta,M}$ be the set of class $\mathcal{C}^{1,1}$ obtained by taking the union of $\mathfrak{D}_{\theta,M}$ with the four curved triangles, bounded by p_j^ϵ, q_j^ϵ and $(\cos\theta, (-1)^j \sin(\theta))$ and $(p_j^\epsilon)'$, $(q_j^\epsilon)'$ and $(\cos\theta + M, (-1)^j \sin(\theta))$ respectively, see Figure 3.9. Then, we choose $\epsilon > 0$ so small that (3.55) holds with F replacing $\mathfrak{D}_{\theta,M}$; hence, F is not ϕ_c-calibrable.

For any $\beta \in (1, \frac{1}{\sin\theta})$ let Γ_β (resp. Γ'_β) be the arc of minimal length of the circumference of radius $\frac{1}{\beta}$, which is internally tangent to ∂F in two points, belonging to the arcs bounded by p_j^ϵ and q_j^ϵ (resp. $(p_j^\epsilon)'$ and $(q_j^\epsilon)'$), for $j = 1, 2$. Let $C_\beta \subset F$ be the disconnected set bounded by $\Gamma_\beta \cup \Gamma'_\beta$, let $C_{\frac{1}{\sin\theta}} := \cup_{\beta \in (1, \frac{1}{\sin\theta})} C_\beta$, and let $\Gamma_{\frac{1}{\sin\theta}}$ and $\Gamma'_{\frac{1}{\sin\theta}}$ be the two connected components of $\mathrm{int}(F) \cap \partial C_{\frac{1}{\sin\theta}}$.

Reasoning as in Example 3.47, $\mathrm{Ch}(F)$ is the disconnected subset of F bounded by $\Gamma_{h(F)}$ and $\Gamma'_{h(F)}$ (see again Figure 3.9). Moreover,(18) for all $\bar\beta \in (\kappa_{\min}, \kappa_{\max})$, we can still exclude the presence of regions of the form $\{\kappa_{\phi_c,F} = \bar\beta\}$ with positive Lebesgue measure. As a consequence,

$$\mathrm{int}(F) \cap \partial\Theta^F_\beta = \Gamma_\beta \cup \Gamma'_\beta = \mathrm{int}(F) \cap \partial\Omega^F_\beta, \quad \beta \in (\kappa_{\min}, \kappa_{\max}).$$

(18) Recall once again the proof of (3.50).

By the geometry of F, $\kappa_{\max} \le \frac{1}{\sin\theta}$. Therefore, we have $|F \setminus \Omega^F_{\kappa_{\max}}| > 0$: indeed, if $Q \subset F$ is the connected (full-measure) set bounded by $\Gamma_{\frac{1}{\sin\theta}} \cup \Gamma'_{\frac{1}{\sin\theta}}$, then $Q \subseteq F \setminus \Omega^F_{\kappa_{\max}}$.

It is interesting to show now that, differently from Example 3.48, the maximal value $\kappa_{\max}$ of $\kappa_{\phi_c,F}$ depends on M, and

$$\kappa_{\max} < \frac{1}{\sin\theta}. \tag{3.56}$$

Indeed, recalling (3.16) and the equality $F = \Theta^F_{\kappa_{\max}}$, the value $\kappa_{\max}$ must verify

$$P(F) - P(\Omega^F_{\kappa_{\max}}) = \kappa_{\max}|F \setminus \Omega^F_{\kappa_{\max}}|. \tag{3.57}$$

Notice that

$$P(F) - P(\Omega^F_{\kappa_{\max}}) = 2M - 2\mathcal{H}^1(\Gamma_{\kappa_{\max}}) + O(\epsilon). \tag{3.58}$$

We estimate $\mathcal{H}^1(\Gamma_{\kappa_{\max}})$ with the length of the arc of circumference of curvature $\kappa_{\max}$ contained in $B_{\sin\theta}(\cos\theta, 0)$, and passing through the points $(\cos\theta, \pm\sin\theta)$. We denote by $\omega := \omega(\kappa_{\max})$ the angle such that

$$\sin\omega = \kappa_{\max}\sin\theta. \tag{3.59}$$

Notice that proving (3.56) is in turn equivalent to show that $\omega \neq \frac{\pi}{2}$. From (3.58), we get

$$P(F) - P(\Omega^F_{\kappa_{\max}}) = 2\left[M - 2\omega\frac{\sin\theta}{\sin\omega}\right] + O(\epsilon). \tag{3.60}$$

Similarly, we estimate $|F \setminus \Omega^F_{\kappa_{\max}}|$ with the area of the connected subset of the strip $[\cos\theta, \cos\theta + M] \times [-\sin\theta, \sin\theta]$ bounded by two arcs of circumference of curvature $\kappa_{\max}$, and passing through the vertices of the strip. Thus

$$|F \setminus \Omega^F_{\kappa_{\max}}| = 2\left[M\sin\theta - \omega\frac{\sin^2\theta}{\sin^2\omega} + \sin^2\theta\,\frac{\cos\omega}{\sin\omega}\right] + O(\epsilon). \tag{3.61}$$

Combining (3.57), (3.60), and (3.61) we get

$$M\,(1 - \sin\omega) = \omega\,\frac{\sin\theta}{\sin\omega} + \sin\theta\,\cos\omega + O(\epsilon), \tag{3.62}$$

which does not admit $\omega = \frac{\pi}{2}$ as a solution, for $\epsilon > 0$ sufficiently small. This proves (3.56).

Remark 3.50. Referring to Example 3.49, we notice that we can still apply Theorem 3.41 separately in each connected component of $\mathrm{Ch}(F)$, thus obtaining a subunitary vector field X satisfying $\mathrm{div}X = h(F)$ in $\mathrm{Ch}(F)$, and (3.43) on $\partial F \cap \partial\mathrm{Ch}(F)$.

If we extend X following the normal direction of the curvature level lines in $\Omega^F_{\kappa_{\max}} \setminus \mathrm{Ch}(F)$, and then transporting the field parallelly to itself in $\Theta^F_{\kappa_{\max}} \setminus \Omega^F_{\kappa_{\max}}$, we end up with a field not belonging to $\mathscr{H}^2_{\mathrm{div}}(F)$. Indeed we cannot avoid the field to jump in the normal direction of some vertical discontinuity segment.

We observe that the difficulty for building an optimal selection seems to be related to the presence of two minimal Cheeger subsets of F. We are not aware whether there exists an optimal selection equal to X in $\mathrm{Ch}(F)$.

As we have already said, we are not able to find an optimal selection $\widetilde{N}_{\min}$ in F: we notice that [101, Theorem 1.1] cannot be applied with the choice of $h = \kappa_{\phi_c,F}$, since any Ω^F_β violates [101, formula (1.3)].

Chapter 4

Constrained BV functions on coverings

Summary. In Section 4.1, we define the family of admissible cuts, the space $BV(Y_{\mathbf{\Sigma}}; V)$, and we give a handy formula for the total variation. In Section 4.2, we introduce the family of constrained BV functions, and we set up the minimization problem. In Section 4.2.1, we show that, actually, the model does not depend on the admissible cuts, while in Section 4.2.2 we give a compactness result which allows to prove existence of minimizers, and we give our definition of constrained covering solutions. Another key result is a "nonconstancy lemma", showing that the jump set of any constrained function turns out to have strictly positive $\mathcal{H}^{n-1}$-measure in the fibers over any open set containing a loop around the boundary frame S. Then, in Section 4.3, we lift the constraint on the fibers to the class of Sobolev functions on $Y_{\mathbf{\Sigma}}$, showing that our formulation naturally leads to a Γ-convergence result. Section 4.4.1 is devoted to the case when S consists of $m \geq 2$ distinct points, and we show that a constrained covering solution coincides with the Steiner graph over S. In Section 4.4.2 we test the model in the case of the standard Plateau's problem in $\mathbb{R}^3$, showing that our model is equivalent to solving Plateau's problem using the theory of integral currents modulo 2. Finally (Section 4.4.3), in the aim to extend our model to more general choices of S, we consider the case when the boundary frame is the one-dimensional skeleton of a regular tetrahedron.

4.1 BV functions on coverings

Let $n \in \mathbb{N}$, $n \geq 2$. For $r > 0$ and $x \in \mathbb{R}^n$, we set $B_r(x) := \{x' \in \mathbb{R}^n : |x' - x| < r\}$, and $B_r := B_r(0)$. Throughout this chapter, $\Omega \subseteq \mathbb{R}^n$ denotes a nonempty connected open set. Unless otherwise specified, we let $S \subset \Omega$ be a boundaryless, compact, embedded, smooth submanifold of dimension $n-2$, not necessarily connected nor oriented.

We define the base set as

$$M := \Omega \setminus S, \tag{4.1}$$

which is path connected.

Example 4.1. Typical choices will be:

- $n = 2$, and S a finite number m of distinct points;
- $n = 3$, and S a tame link (that is, a finite number of disjoint closed embedded smooth space curves).

We shall perform a different covering construction depending on the dimension n. Indeed, apart from the construction in Section 4.4.3, our covering space will consist of $m := m(n)$ sheets, where

$$m := \begin{cases} \text{cardinality of } S & \text{if } n = 2, \\ 2 & \text{if } n > 2. \end{cases} \tag{4.2}$$

The family of admissible cuts is defined distinguishing between the following two alternatives. We say that an $(n-1)$-dimensional submanifold $\Sigma \subset \Omega$ is Lipschitz provided that, locally around any of its points, Σ is the graph of a Lipschitz function defined on a suitable $(n-1)$-orthonormal frame.

Definition 4.2 (**Admissible cuts,** $n=2$)**.** *Let* $n=2$, *and* $S := \{p_1, \dots, p_m\}$. *We denote by* $\mathrm{Cuts}(\Omega, S)$ *the set of all* $\Sigma := \cup_{i=1}^{m-1} \Sigma_i \subset \Omega$ *where:*

- *for* $i = 1, \dots, m-1$, Σ_i *is a Lipschitz simple curve, starting at* p_i *and ending at* p_{i+1};
- *if* $m > 2$, *then* $\Sigma_i \cap \Sigma_{i+1} = \{p_{i+1}\}$ *for* $i = 1, \dots, m-2$;
- $\Sigma_i \cap \Sigma_l = \emptyset$ *for any* $1 \le i < l \le m-1$, *with* $l \neq i+1$.

We also denote by $\mathbf{Cuts}(\Omega, S)$ *the set of all pairs* $\mathbf{\Sigma} := (\Sigma, \Sigma')$ *such that:*

(i) $\Sigma, \Sigma' \in \mathrm{Cuts}(\Omega, S)$, *and* $\Sigma \cap \Sigma' = S$;

(ii) for $m > 2$, *and for any* $i = 2, \dots, m-1$, *let* C_i *be a sufficiently small disk centered at* p_i, *and denote by* x_i *(resp.* y_i*) the intersection of* C_i *with* Σ_{i-1} *(resp. with* Σ_i*). Then, there exists an arc of* C_i *connecting* x_i *and* y_i, *and not intersecting* $\Sigma' = \cup_{j=1}^{m-1} \Sigma'_j$.

Roughly speaking, condition (*ii*) in Definition 4.2 means that Σ lies from one side of Σ' locally around S.

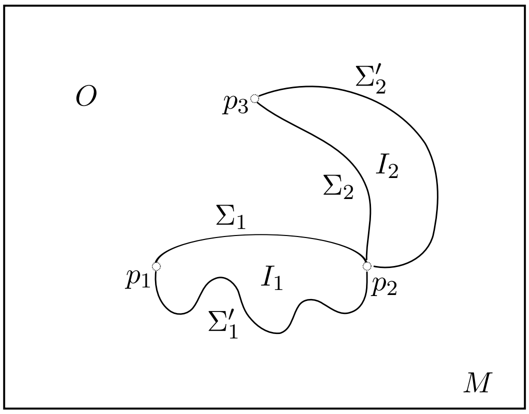


Figure 4.1: The base set $M = \Omega \setminus S$, when $n=2$, $m=3$, Ω is a rectangle, and $S = \{p_1, p_2, p_3\}$. In the figure, an example of admissible pair of cuts is shown.

Definition 4.3 (**Admissible cuts,** $n>2$)**.** *Let* $n > 2$. *We denote by* $\mathrm{Cuts}(\Omega, S)$ *the set of all* $(n-1)$*-dimensional compact embedded Lipschitz submanifolds* $\Sigma \subset \Omega$ *having* S *as topological boundary.*

We also let $\mathbf{Cuts}(\Omega, S)$ *be the set of all pairs* $\mathbf{\Sigma} := (\Sigma, \Sigma')$ *such that* $\Sigma, \Sigma' \in \mathrm{Cuts}(\Omega, S)$, *and* $\Sigma \cap \Sigma' = S$.

Referring to Definitions 4.2-4.3, we call the elements of $\mathrm{Cuts}(\Omega, S)$ (resp. of $\mathbf{Cuts}(\Omega, S)$) *admissible cuts* (resp. *admissible pairs of cuts*). When $n > 2$, we shall always suppose that $\mathrm{Cuts}(\Omega, S)$ and $\mathbf{Cuts}(\Omega, S)$ are nonempty. A typical situation is when S is the topological boundary of some $(n-1)$-dimensional, compact, embedded, orientable, smooth submanifold $\Sigma \subset \Omega$. Indeed, the orientability of Σ gives a unit normal vector field on $\Sigma \setminus S$ — hence, in particular, a direction to follow in order to "enlarge" the cut, separating its two faces. The construction is standard (in the case $n=3$, it is given for instance in [118, p.147]). Necessary and sufficient conditions for the existence of this $(n-1)$-dimensional orientable submanifold can be found in [154]. When $n=3$, and S is a tame link, there exists [137, Theorem 4, p.120] an embedded orientable surface, called *Seifert surface*, whose boundary is S.

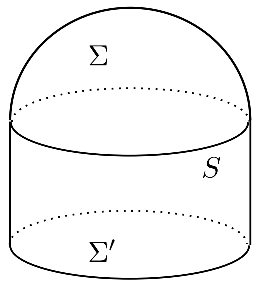


Figure 4.2: An example of admissible pair of cuts in the case $S \subset \mathbb{R}^3$ is a circle. In the figure, Σ is a closed half-sphere, while Σ' is a portion of cylinder under S, with the addiction of the lower base.

Remark 4.4. An m-sheeted covering of M can be constructed in a standard way [118, p.147] using a single orientable cut $\Sigma \in \mathrm{Cuts}(\Omega, S)$, by suitably identifying m copies of $\Omega \setminus \Sigma$. This construction is perhaps more intuitive than the one based on (4.5) and corresponds, essentially, to the case in which Σ and Σ' coincide. However, in order to rigorously define the covering, one needs to slightly separate the "faces" of Σ. Since our minimization problem (see (4.19) below) depends on the metric on the covering space, we find more convenient to use the construction via admissible pairs of cuts. However, it is worth noticing that, concretely, it will be enough to deal with only one of the two cuts of the pair $\mathbf{\Sigma}$.

4.1.1 "Cut and paste" construction of the covering

In this section we explicitly construct the covering $(Y_{\mathbf{\Sigma}}, \pi_{\mathbf{\Sigma},M})$. As a consequence, we shall end up with local parametrizations which naturally bring the Euclidean metric on $Y_{\mathbf{\Sigma}}$.

Let $n \geq 2$ and m be as in (4.2). Let $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(\Omega, S)$. We consider m disjoint copies of the open sets

$$D := \Omega \setminus \Sigma, \quad D' := \Omega \setminus \Sigma', \tag{4.3}$$

which we denote respectively by

$$(D, j), \quad j = 1, \ldots, m, \qquad (D', j'), \quad j' = m+1, \ldots, 2m. \tag{4.4}$$

Points in the space

$$\mathcal{X} := \bigcup_{j=1}^{m} (D, j) \cup \bigcup_{j'=m+1}^{2m} (D', j')$$

are identified as follows. For $i = 1, \ldots, m-1$, let I_i be the bounded open set enclosed by Σ_i and Σ'_i; set also $O := \Omega \setminus \cup_{i=1}^{m-1} \overline{I_i}$. Let $x, x' \in M$, $j \in \{1, \ldots, m\}$, and $j' \in \{m+1, \ldots, 2m\}$; then $(x, j) \sim (x', j')$ if and only if $x = x'$, and one of the following conditions holds:

$$\begin{cases} j \equiv j' \ (\mathrm{mod}\ m), & x = x' \in O, \\ j \equiv j' - i \ (\mathrm{mod}\ m), & x = x' \in I_i, \quad i = 1, \ldots, m-1. \end{cases} \tag{4.5}$$

Of course, any point is also identified with itself. See Figures 4.1-4.3 for an example in the case $n = 2$, $m = 3$.

Then $\sim$ is an equivalence relation, and the quotient space

$$Y_{\mathbf{\Sigma}} := \mathcal{X} / \sim$$

is endowed with the quotient topology given by the projection $\widetilde{\pi} : \mathcal{X} \to Y_{\mathbf{\Sigma}}$ induced by $\sim$. The covering space $Y_{\mathbf{\Sigma}}$ will depend on the choice of Ω; for notational simplicity we shall not indicate such a dependence.

We set $\pi \colon (x,j) \in \mathcal{X} \mapsto x \in M$, and we denote by

$$\pi_{\mathbf{\Sigma},M} \colon Y_{\mathbf{\Sigma}} \to M \tag{4.6}$$

the projection $\pi_{\mathbf{\Sigma},M}(\widetilde{\pi}(x,j)) := x$, for any $(x,j) \in \mathcal{X}$. This latter map is well defined, since if $(x,j) \sim (x',j')$, then $\pi_{\mathbf{\Sigma},M}(\widetilde{\pi}(x,j)) = x = x' = \pi_{\mathbf{\Sigma},M}(\widetilde{\pi}(x',j'))$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\widetilde{\pi}"] \arrow[dr, "\pi"'] & Y_{\mathbf{\Sigma}} \arrow[d, "\pi_{\mathbf{\Sigma},M}"] \\ & M \end{tikzcd} \tag{4.7}$$

Definition 4.5 (**Local parametrizations**). *We set*

$$\begin{aligned} &\Psi_j \colon D \to \widetilde{\pi}\big((D,j)\big), \qquad \Psi_j := \widetilde{\pi} \circ \big(\pi_{|_{(D,j)}}\big)^{-1}, \quad j = 1,\dots,m, \\ &\Psi_{j'} \colon D' \to \widetilde{\pi}\big((D',j')\big), \qquad \Psi_{j'} := \widetilde{\pi} \circ \big(\pi_{|_{(D',j')}}\big)^{-1}, \quad j' = m+1,\dots,2m. \end{aligned} \tag{4.8}$$

The covering space[(1)] $Y_{\mathbf{\Sigma}}$ admits a natural structure of differentiable manifold, with $2m$ local parametrizations Ψ_j, $\Psi_{j'}$ given by (4.8).

Remark 4.6. For $j \in \{1,\dots,m\}$ and $j' \in \{m+1,\dots,2m\}$, we have

$$\Psi_{j'}^{-1} \circ \Psi_j = \mathrm{id} = \Psi_j^{-1} \circ \Psi_{j'} \quad \text{on } D \cap D',$$

where id is the identity map on $D \cap D'$. The pair $(Y_{\mathbf{\Sigma}}, \pi_{\mathbf{\Sigma},M})$ is an m-sheeted covering of M. Notice that $\cup_{j=1}^m \Psi_j(D) = Y_{\mathbf{\Sigma}} \setminus \pi_{\mathbf{\Sigma},M}^{-1}(\Sigma \setminus S)$.

Remark 4.7 (**Non-zero thickness wires**). Our covering construction applies without modifications to the (simpler) case of a base domain $M := \Omega \setminus \overline{C}$, where $C \subset \Omega$ is a thin open neighbourhood of S.

4.1.2 Total variation on the m-sheeted covering

The covering space $Y_{\mathbf{\Sigma}}$ is an n-dimensional connected orientable smooth non complete manifold; it is endowed with a natural volume measure μ, which is the push-forward $\mathcal{L}^n_{\#}$ of the n-dimensional Lebesgue measure $\mathcal{L}^n$ in M via the maps (4.8). More specifically, let $E \subseteq Y_{\mathbf{\Sigma}}$ be a Borel set. Then we can write E as the union of the following $2m$ disjoint Borel sets

$$E \cap \widetilde{\pi}((D,j)),\ j = 1,\dots,m, \qquad E \cap \widetilde{\pi}((\Sigma \setminus S, j')),\ j' = m+1,\dots,2m, \tag{4.9}$$

and we set

$$\mu(E) := \sum_{j=1}^m \Psi_{j\#}\mathcal{L}^n(E \cap \widetilde{\pi}((D,j))) = \sum_{j=1}^m \mathcal{L}^n(\pi_{\mathbf{\Sigma},M}(E \cap \widetilde{\pi}((D,j)))).$$

Notice that Σ' does not appear in (4.9). Choosing D' in place of D amounts in considering Σ' in place of Σ and does not change the subsequent discussion.

For $k \in \mathbb{N}$, $k \geq 1$, we set $L^1(Y_{\mathbf{\Sigma}};\mathbb{R}^k) := L^1_\mu(Y_{\mathbf{\Sigma}};\mathbb{R}^k)$ and $L^1_{\mathrm{loc}}(Y_{\mathbf{\Sigma}};\mathbb{R}^k) := L^1_{\mu_{\mathrm{loc}}}(Y_{\mathbf{\Sigma}};\mathbb{R}^k)$. The relevant case in this paper will be

$$k := m - 1,$$

where we recall that m is defined in (4.2).

(1) Since S has been removed, $Y_{\mathbf{\Sigma}}$ is not branched.

Definition 4.8 (**The functions** $v_h(u)$)**.** *Let* $u : Y_{\boldsymbol{\Sigma}} \to \mathbb{R}^k$. *For* $j = 1, \ldots, m$ *and* $j' = m+1, \ldots, 2m$, *we let* $v_j(u) : D \to \mathbb{R}^k$, $v_{j'}(u) : D' \to \mathbb{R}^k$ *be the maps defined by*

$$v_j(u) := u \circ \Psi_j, \qquad v_{j'}(u) := u \circ \Psi_{j'}. \tag{4.10}$$

Clearly, if $u \in L^1(Y_{\boldsymbol{\Sigma}}; \mathbb{R}^k)$ then $v_j(u) \in L^1(D; \mathbb{R}^k)$, $v_{j'}(u) \in L^1(D'; \mathbb{R}^k)$.
By construction (recall (4.5)), we have

$$v_j(u) = v_{j'}(u) \ \text{ in } O \quad \text{if } j \equiv j' \ (\text{mod } m), \tag{4.11}$$

$$v_j(u) = v_{j'}(u) \ \text{ in } I_i \quad \text{if } j \equiv j' - i \ (\text{mod } m), \quad i = 1, \ldots, m-1. \tag{4.12}$$

Let Ω be bounded. Our aim is to define the total variation of a function $u \in L^1(Y_{\boldsymbol{\Sigma}}; \mathbb{R}^k)$. We say that u is in $BV_\mu(Y_{\boldsymbol{\Sigma}}; \mathbb{R}^k) =: BV(Y_{\boldsymbol{\Sigma}}; \mathbb{R}^k)$ if its distributional gradient[(2)] $Du\colon \eta \in (C^1_c(Y_{\boldsymbol{\Sigma}}))^k \mapsto -\int_{Y_{\boldsymbol{\Sigma}}} \sum_{l=1}^k u_l \nabla \eta_l \, d\mu \in \mathbb{R}^n$ is a bounded $(k \times n)$-matrix of Radon measures on $Y_{\boldsymbol{\Sigma}}$. Let us denote by $|Du|$ the *total variation* measure of Du [12]; we recall [12, Proposition 1.47] that, for any open subset $E \subseteq Y_{\boldsymbol{\Sigma}}$, we have

$$|Du|(E) = \sup \left\{ \sum_{l=1}^k \int_E u_l \ \text{div}\eta_l \, d\mu \ : \ \eta \in \left(C^1_c(E; \mathbb{R}^n)\right)^k, \ ||\eta||_\infty \le 1 \right\}, \tag{4.13}$$

which is $L^1(Y_{\boldsymbol{\Sigma}})$-lower semicontinuous.

Remark 4.9 (**Representation of the total variation, I**)**.** Let $u \in BV(Y_{\boldsymbol{\Sigma}}; \mathbb{R}^k)$ and $E \subseteq Y_{\boldsymbol{\Sigma}}$ be a Borel set. Then

$$\begin{aligned} |Du|(E) = &\sum_{j=1}^m |Dv_j(u)| \Big(\pi_{\boldsymbol{\Sigma},M} \big(E \cap \widetilde{\pi}((D,j)) \big) \Big) \\ &+ \sum_{j'=m+1}^{2m} |Dv_{j'}(u)| \Big(\pi_{\boldsymbol{\Sigma},M} \big(E \cap \widetilde{\pi}((\Sigma \setminus S, j')) \big) \Big). \end{aligned} \tag{4.14}$$

In order to prove (4.14), let us first assume $E \subseteq \widetilde{\phi}((D,1))$ is open. Then, recalling (4.13), we have

$$\begin{aligned} |Du|(E) &= \sup \left\{ \sum_{l=1}^k \int_{\Psi_1^{-1}(E)} (v_1(u))_l \ \text{div}\eta_l \, d\mathcal{L}^n \ : \ \eta \in \left(C^1_c(\Psi_1^{-1}(E); \mathbb{R}^n)\right)^k, \ ||\eta||_\infty \le 1 \right\} \\ &= |Dv_1(u)|(\Psi_1^{-1}(E)) = |Dv_1(u)|(\pi_{\boldsymbol{\Sigma},M}(E)), \end{aligned} \tag{4.15}$$

which gives (4.14). From (4.15) and [12, Proposition 1.43], we get (4.14) for every Borel set $E \subseteq Y_{\boldsymbol{\Sigma}}$ contained in a single chart. The general case follows by the splitting in (4.9).

Example 4.10. Let $n = 2$, $m = 3$, $S = \{p_1, p_2, p_3\}$, $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Sigma' = \Sigma'_1 \cup \Sigma'_2$ be as in Figure 4.1. For $j = 1, 2, 3$, fix $\alpha_j, \beta_j \in \mathbb{R}^2$, and let $u \in BV(Y_{\boldsymbol{\Sigma}}; \mathbb{R}^2)$ be such that, for every $j = 1, 2, 3$, $v_j(u)$ is equal to α_j inside a disk $B \subset M$ of radius $r > 0$ compactly contained in O (or in I_1, or in I_2) and β_j outside. Then, from (4.14), it follows

$$\begin{aligned} |Du|(Y_{\boldsymbol{\Sigma}}) = &\sum_{j=1}^3 |Dv_j(u)|(B \cap D) + \sum_{j'=4}^6 |Dv_{j'}(u)|(\Sigma \setminus S) \\ = &2\pi r \sum_{j=1}^3 |\beta_j - \alpha_j| + \mathcal{H}^1(\Sigma) \sum_{\substack{j,l=1 \\ j<l}}^3 |\beta_l - \beta_j|. \end{aligned} \tag{4.16}$$

[(2)]Let $\phi \in C^1_c(Y_{\boldsymbol{\Sigma}})$. For $i = 1, \ldots, n$, let e_i be the i-th element of the canonical basis of $\mathbb{R}^n$. Then $\nabla_i \phi(y) := \lim_{h \to 0} h^{-1} \big(\phi(\Psi_1(\pi_{\boldsymbol{\Sigma},M}(y) + h e_i)) - \phi(y) \big)$ is well-defined for every $y \in \widetilde{\pi}((D,1))$. Similarly for other points in $Y_{\boldsymbol{\Sigma}}$. We set $\nabla \phi := (\nabla_1 \phi, \ldots, \nabla_n \phi)$. For $\Phi := (\phi_1, \ldots, \phi_n) \in C^1_c(Y_{\boldsymbol{\Sigma}}; \mathbb{R}^n)$, we set $\text{div}\Phi := \sum_{i=1}^n \nabla_i \phi_i$.

On the other hand, if B is centered at a point of $\Sigma \setminus S$, and $B \cap \Sigma' = \emptyset$, then

$$\begin{aligned} |Du|(Y_{\mathbf{\Sigma}}) =& 2\pi r \sum_{j=1}^{3} |\beta_j - \alpha_j| + \mathcal{H}^1(\Sigma \cap B) \sum_{\substack{j,l=1 \\ j<l}}^{3} |\alpha_l - \alpha_j| \\ &+ \big(\mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B)\big) \sum_{\substack{j,l=1 \\ j<l}}^{3} |\beta_l - \beta_j|. \end{aligned} \tag{4.17}$$

In particular, if $\alpha_1, \alpha_2, \alpha_3$ are the vertices of an equilateral triangle in $\mathbb{R}^2$ having side of length ℓ, and if $\beta_1 := \alpha_2$, $\beta_2 := \alpha_3$, $\beta_3 := \alpha_1$,(3) both (4.16) and (4.17) reduce to

$$3\ell\Big(2\pi r + \mathcal{H}^1(\Sigma)\Big).$$

4.2 The constrained minimum problem

Let $\ell > 0$, and let $V := \{\alpha_1, \dots, \alpha_m\} \subset \mathbb{R}^{m-1}$ be such that

$$|\alpha_j - \alpha_l| = \ell, \qquad j, l = 1, \dots, m, \quad j \neq l.$$

We define

$$BV(Y_{\mathbf{\Sigma}}; V) := \Big\{ u \in BV(Y_{\mathbf{\Sigma}}; \mathbb{R}^{m-1}) \ : \ u(x) \in V \ \mu\text{-a.e. in } Y_{\mathbf{\Sigma}} \Big\}.$$

We denote by

$$\mathcal{T}(V)$$

the set of all maps $\tau : V \to V$ such that, for $h \in \{1, \dots, m-1\}$ coprime with m,

$$\tau(\alpha_j) = \alpha_l \ \text{ where } \ l \equiv j + h \ (\text{mod } m), \ j \in \{1, \dots, m\}.$$

For $\tau \in \mathcal{T}(V)$, define $\tau^0 := \text{id}$ in V, and $\tau^l := \tau \circ (\tau)^{l-1}$, for any $l \in \mathbb{N}$, $l \geq 1$. Notice that m coincides with the smallest positive integer κ such that $\tau^\kappa = \text{id}$ (we call τ a *transposition* of V of *order* m).

Definition 4.11 (**Constrained** BV **functions on coverings**)**.** *We denote by*

$$BV_{\text{constr}}(Y_{\mathbf{\Sigma}}; V)$$

the set of all $u \in BV(Y_{\mathbf{\Sigma}}; V)$ *for which there exists* $\tau \in \mathcal{T}(V)$ *such that*

$$v_j(u) = \tau^{j-1} \circ v_1(u), \qquad j = 1, \dots, m. \tag{4.18}$$

Remark 4.12. In view of (4.11) and (4.12), the constraint (4.18) is equivalent to require $v_{j'}(u) = \tau^{j'-1} \circ v_{m+1}(u)$, for $j' = m+1, \dots, 2m$.

To have an idea of the meaning of the constraint (4.18) in the case $m = 3$, the reader may refer to Figure 4.4.

Our constrained minimization problem, which in principle could depend on the choice of $\mathbf{\Sigma}$, can be now stated as follows:

$$\mathscr{A}^{\Omega}_{\text{constr}}(S, \mathbf{\Sigma}) := \inf \Big\{ |Du|\big(Y_{\mathbf{\Sigma}}\big) \ : \ u \in BV_{\text{constr}}(Y_{\mathbf{\Sigma}}; V) \Big\}. \tag{4.19}$$

The independence of $\mathscr{A}^{\Omega}_{\text{constr}}(S, \mathbf{\Sigma})$ of Σ will be shown in Corollary 4.18.

(3) With this choice, and letting $V = \{\alpha_1, \alpha_2, \alpha_3\}$, u belongs to $BV_{\text{constr}}(Y_{\mathbf{\Sigma}}; V)$, see Definition 4.11) in Section 4.2.

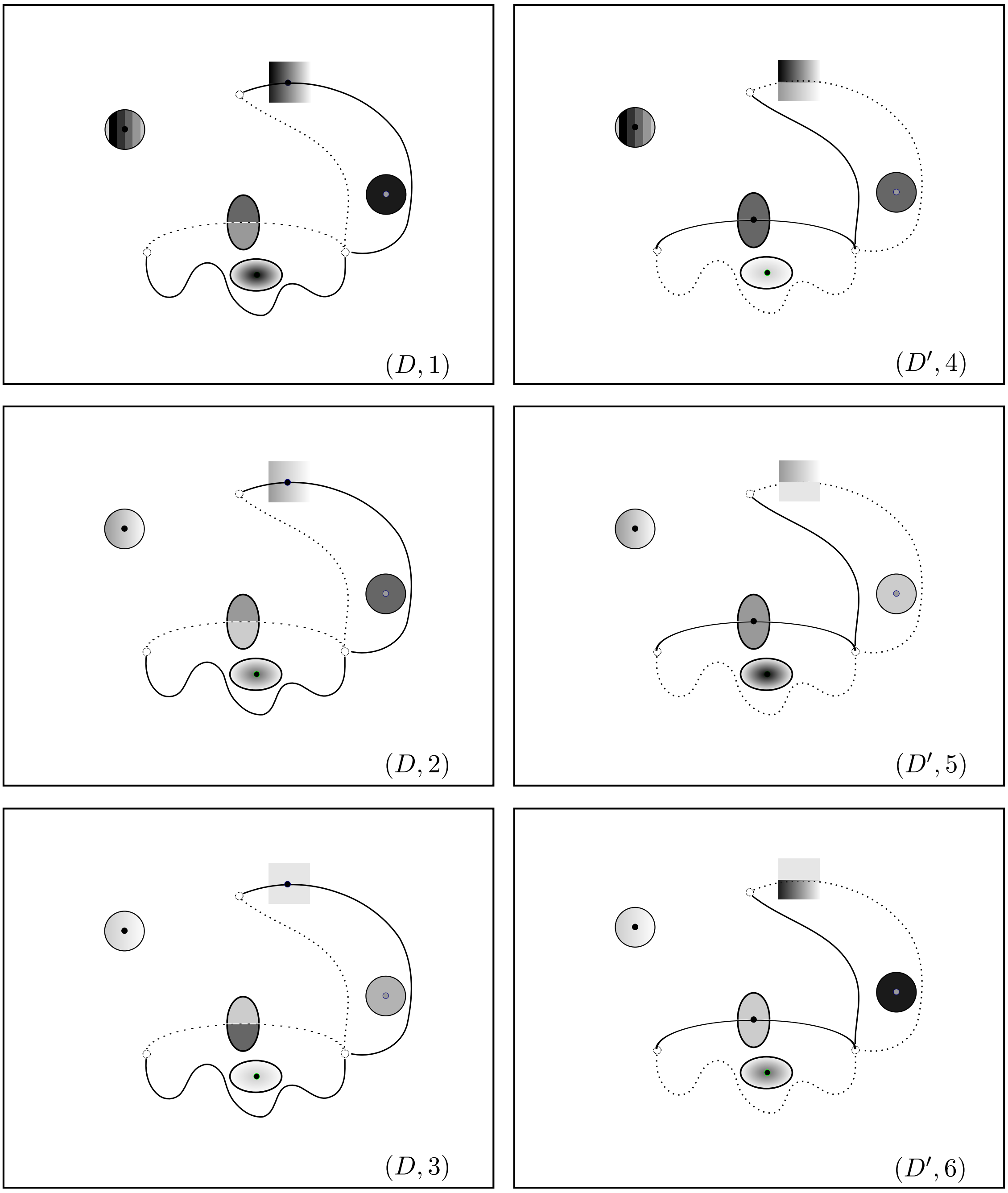


Figure 4.3: The triple covering space $Y_{\mathbf{\Sigma}}$, for M as in Figure 4.1. A dashed curve denotes that an admissible cut has been removed. In the picture, some examples of admissible neighbourhoods are shown. Identifications are meant by using the same grey level and shape. Note that a complete counterclockwise (small) turn around any point of S corresponds to move one sheet forward in $Y_{\mathbf{\Sigma}}$. Moreover, $m = 3$ turns around a point of S correspond to a single turn in $Y_{\mathbf{\Sigma}}$.

Remark 4.13. When $m = 2$, we fix the choice $\ell := 2$ and $V := \{\pm 1\}$, so that

$$BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\}) = \Big\{ u \in BV(Y_{\mathbf{\Sigma}}) \ : \ |u| = 1, \ v_1(u) = -v_2(u) \Big\}.$$

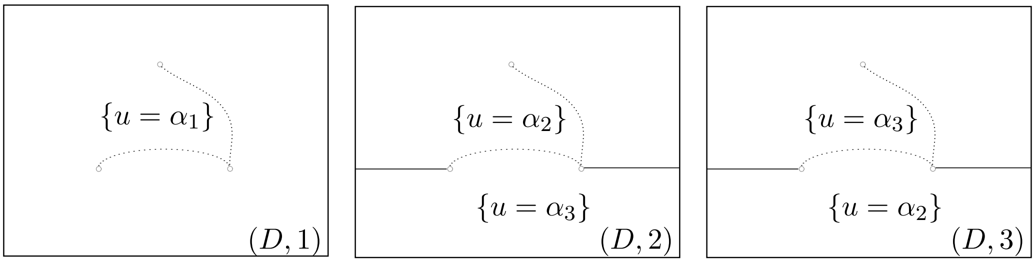


Figure 4.4: A function $u \in BV(Y_{\mathbf{\Sigma}}; V) \setminus BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, where $Y_{\mathbf{\Sigma}}$ is the covering space in Figure 4.3. Notice that we need to specify the values of u just on the three charts drawn in the picture.

Clearly, $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ if and only if $u \in BV(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ and

$$\sum_{\pi_{\mathbf{\Sigma},M}(y)=x} u(y) = 0. \tag{4.20}$$

Notice that the sum in (4.20) contains only 2 terms.

The functional in (4.19) attains the same value when evaluated at u and at $\tau \circ u$, for any $\tau \in \mathcal{T}(V)$. By virtue of the constraint, $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma})$ will turn out to be strictly positive (see Theorem 4.24 below).

Remark 4.14 (**Unbounded open sets**)**.** Let Ω be unbounded. Then, instead of (4.19), we shall consider the minimization problem

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}) := \inf \left\{ |Du|(Y_{\mathbf{\Sigma}}) \ : \ u \in BV^{\mathrm{loc}}_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V) \right\}, \tag{4.21}$$

where

$$\begin{aligned} BV^{\mathrm{loc}}_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V) := \Big\{ & u \in L^1_{\mathrm{loc}}(Y_{\mathbf{\Sigma}}; V) \ : \ |Du|(E) < \infty, \ E \subset Y_{\mathbf{\Sigma}} \text{ open rel. compact}, \\ & \exists\, \tau \in \mathcal{T}(V) \text{ s.t. } v_j(u) = \tau^{j-1} \circ v_1(u), \ j = 1, \dots, m \Big\}. \end{aligned}$$

We notice that the previous discussion (in particular, formula (4.14)) still holds true when Ω is unbounded.

The next observation shows a difference between our model and the model in [55], while Remark 4.16 seems to suggest a model closer to the one in [55].

Remark 4.15 (**Monotonicity with respect to the base domain**)**.** Let Ω, $\Omega' \subseteq \mathbb{R}^n$ be connected open sets, such that $\Omega \subseteq \Omega'$. Then

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}) \leq \mathscr{A}^{\Omega'}_{\mathrm{constr}}(S, \mathbf{\Sigma}). \tag{4.22}$$

Indeed, let us assume that Ω' is bounded (the case in which Ω or Ω' are unbounded being similar). For $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S) \subseteq \mathbf{Cuts}(\Omega', S)$, let us denote by $Y'_{\mathbf{\Sigma}}$ the covering space of $M' := \Omega' \setminus S$. It is natural to see $Y_{\mathbf{\Sigma}}$ as a subset of $Y'_{\mathbf{\Sigma}}$, so that, for any $u \in BV_{\mathrm{constr}}(Y'_{\mathbf{\Sigma}}; V)$, we have $u_{|Y_{\mathbf{\Sigma}}} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$. In particular, $|Du|(Y_{\mathbf{\Sigma}}) \leq |Du|(Y'_{\mathbf{\Sigma}})$, which gives (4.22).

Remark 4.16 (**A Dirichlet-type formulation**)**.** By slightly modifying our construction, it is possible to set up a minimization problem such that the minimum value decreases when the base domain becomes larger (the opposite of (4.22)). Let $\Omega, \Omega' \subset \mathbb{R}^n$ be connected open sets, such that $\Omega \subset \Omega'$ and $\Omega' \setminus \overline{\Omega} \neq \emptyset$. Fix $\alpha \in V$, and let $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S) \subseteq \mathbf{Cuts}(\Omega', S)$. Let us consider the following Dirichlet-type problem:

$$\mathscr{B}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}, \Omega') := \inf\{|Du|(Y'_{\mathbf{\Sigma}}) : u \in BV_{\mathrm{constr}}(Y'_{\mathbf{\Sigma}}; V), \ v_1(u) = \alpha \text{ in } \Omega' \setminus \overline{\Omega}\}.$$

Then, the larger is Ω, the smaller is the value of $\mathscr{B}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}, \Omega')$.

4.2.1 Independence of the admissible pair of cuts

In this section we show that constrained – covering solutions are independent of admissible cuts. Our proof of Theorem 4.17 relies on general facts in coverings' theory, which we recall in Appendix C. Nevertheless, at least when $m = 2$, it is possibile to give a different proof which is independent of the abstract covering construction performed at the end of this paper.

For any $u \in BV_{\text{constr}}(Y_{\mathbf{\Sigma}}; V)$, let

$$J_u \subset Y_{\mathbf{\Sigma}}$$

be the *set of approximate jump points* of u in $Y_{\mathbf{\Sigma}}$, which is defined as follows (here, we adapt [12, Definition 3.67, p.163] to our setting): let $y \in \Psi_1(D, 1)$, $x := \pi_{\mathbf{\Sigma},M}(y)$, and let $r > 0$ be such that $B_r(x)$ is contained in D. Given a unit vector $\nu \in \mathbb{R}^n$, set $B_r(y) := \Psi_1(B_r(x))$, $B^+_{r,\nu}(y) := \{y' \in B_r(y) \ : \ (\pi_{\mathbf{\Sigma},M}(y') - x) \cdot \nu > 0\}$, $B^-_{r,\nu}(y) := \{y' \in B_r(y) \ : \ (\pi_{\mathbf{\Sigma},M}(y') - x) \cdot \nu < 0\}$. Now, we say that y is an approximate jump point of u if there exist a unit vector $\nu \in \mathbb{R}^n$, and two distinct $\alpha, \beta \in V$ satisfying $\lim_{r\to 0^+} r^{-n} \int_{B^+_{r,\nu}(y)} |u-\alpha| \, d\mu = 0 = \lim_{r\to 0^+} r^{-n} \int_{B^-_{r,\nu}(y)} |u-\beta| \, d\mu$. Similarly we proceed when y belongs to the other covering sheets.

Theorem 4.17. *Let Ω be bounded. Let $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$, and let $u \in BV_{\text{constr}}(Y_{\mathbf{\Sigma}}; V)$. Then, for any $\widehat{\mathbf{\Sigma}} \in \mathbf{Cuts}(\Omega, S)$, there exists $\widehat{u} \in BV_{\text{constr}}(Y_{\widehat{\mathbf{\Sigma}}}; V)$ such that*

$$\pi_{\mathbf{\Sigma},M}(J_u) = \pi_{\widehat{\mathbf{\Sigma}},M}(J_{\widehat{u}}). \tag{4.23}$$

Proof. Let $\mathrm{f}\colon Y_{\mathbf{\Sigma}} \to Y_{\widehat{\mathbf{\Sigma}}}$ be the homeomorphism defined in (C.5) below. We set $\widehat{u}\colon Y_{\widehat{\mathbf{\Sigma}}} \to V$ as

$$\widehat{u} := u \circ \mathrm{f}^{-1}.$$

By definition of f, it follows that $u \in BV_{\text{constr}}(Y_{\widehat{\mathbf{\Sigma}}}; V)$, and $J_{\widehat{u}} = \mathrm{f}(J_u)$. Hence

$$\pi_{\widehat{\mathbf{\Sigma}},M}(J_{\widehat{u}}) = \pi_{\widehat{\mathbf{\Sigma}},M}(\mathrm{f}(J_u)) = \pi_{\mathbf{\Sigma},M}(J_u),$$

where in the last equality we have made use of (C.5). □

Corollary 4.18 (Independence). *The value $\mathscr{A}^{\Omega}_{\text{constr}}(S, \mathbf{\Sigma})$ in (4.19) is independent of $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$.*

Proof. We consider the case in which Ω is bounded, the unbounded case being similar. Let $\mathbf{\Sigma}$, $\widehat{\mathbf{\Sigma}} \in \mathbf{Cuts}(\Omega, S)$. Let $u_{\min} \in BV_{\text{constr}}(Y_{\mathbf{\Sigma}}; V)$ be such that $\mathscr{A}^{\Omega}_{\text{constr}}(S, \mathbf{\Sigma}) = m\ell\, \mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}}))$. Let $\widehat{u} \in BV_{\text{constr}}(Y_{\widehat{\mathbf{\Sigma}}}; V)$ be the function given by Theorem 4.17, applied with $u = u_{\min}$. Then, by (4.29) below and (4.23), we have

$$\mathscr{A}^{\Omega}_{\text{constr}}(S, \widehat{\mathbf{\Sigma}}) \le m\ell\mathcal{H}^{n-1}(\pi_{\widehat{\mathbf{\Sigma}},M}(J_{\widehat{u}})) = m\ell\, \mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})) = \mathscr{A}^{\Omega}_{\text{constr}}(S, \mathbf{\Sigma}).$$

Arguing similarly for the converse inequality, we get $\mathscr{A}^{\Omega}_{\text{constr}}(S, \widehat{\mathbf{\Sigma}}) = \mathscr{A}^{\Omega}_{\text{constr}}(S, \mathbf{\Sigma})$. □

In accordance with Corollary 4.18, we set

$$\mathscr{A}^{\Omega}_{\text{constr}}(S) := \mathscr{A}^{\Omega}_{\text{constr}}(S, \mathbf{\Sigma}).$$

Corollary 4.19 (Upper bound). *We have*

$$\mathscr{A}^{\Omega}_{\text{constr}}(S) \le m\ell \inf \Big\{ \mathcal{H}^{n-1}(\Sigma) \ : \ \Sigma \in \mathrm{Cuts}(\Omega, S) \Big\}. \tag{4.24}$$

Proof. Let $\tau \in \mathcal{T}(V)$. Let u be the τ-constrained lift of v (Definition 4.20 below), with v identically equal to some $\alpha \in V$. Then (4.26) holds, and (4.24) follows. □

In Sections 4.4.1 and 4.4.2 we shall prove that, when $m = 2$, $n \le 7$, and $\Omega = \mathbb{R}^n$, (4.24) holds as an equality (see Corollary 4.35 and Theorem 4.39). Notice that, by the regularity of area minimizing currents modulo 2 [144, Theorem 6.2.1], the infimum on the right hand side of (4.24) is a minimum, provided $n \le 7$.

4.2.2 Existence of minimizers

Let $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(\Omega, S)$. Concerning functions defined on the base set, clearly $BV(M; V) = BV(\Omega; V)$, and moreover[(4)]

$$BV(\Omega; V) = BV(D; V),$$

so that any $v \in BV(D; V)$ (or more generally any $v \in BV(\Omega \setminus C; V)$, with C a finite union of cuts) can be considered also as a BV function in Ω, whose total variation in general may increase by a contribution due to the two traces of v on Σ (more generally on C). In the following, we denote by

$$J_v \subset \Omega$$

the set of approximate jump points of v *considered as a function in* $BV(\Omega; V)$.

The next definition will be frequently used in the remaining of the chapter.

Definition 4.20 (**Constrained lift**). *Let $v \in BV(D; V)$, and let $\tau \in \mathcal{T}(V)$. Then the function defined as*

$$u := \tau^{j-1} \circ v \circ \Psi_j^{-1} \text{ in } \Psi_j(D), \qquad j = 1, \dots, m, \tag{4.25}$$

*is in $BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, and $v_1(u) = v$. We call u the τ-*constrained lift *of v.*

In particular, when v is identically equal to some $\alpha \in V$, we have

$$\pi_{\mathbf{\Sigma},M}(J_u) = \Sigma \setminus S, \tag{4.26}$$

for every $\tau \in \mathcal{T}(V)$.

Lemma 4.21 (**Splitting of the projection of the jump**). *Let $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$, and let $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$. Then*

$$\pi_{\mathbf{\Sigma},M}(J_u) = \Big(J_{v_1(u)} \setminus (\Sigma \setminus S)\Big) \cup \Big(J_{v_{m+1}(u)} \cap (\Sigma \setminus S)\Big). \tag{4.27}$$

Proof. Let us split J_u as the union of the following $2m$ disjoint sets:

$$J_u \cap \widetilde{\pi}((D, j)),\ j = 1, \dots, m, \qquad J_u \cap \widetilde{\pi}((\Sigma \setminus S, j')),\ j' = m+1, \dots, 2m. \tag{4.28}$$

By the constraint (4.18), for each $j = 2, \dots, m$ (resp. for each $j' = m+2, \dots, 2m$), to each point in $J_u \cap \widetilde{\pi}((D, j))$ (resp. in $J_u \cap \widetilde{\pi}((\Sigma \setminus S, j')))$ there corresponds a unique point in $J_u \cap \widetilde{\pi}((D, 1))$ (resp. in $J_u \cap \widetilde{\pi}((\Sigma \setminus S, m+1)))$, belonging to the same fiber, and *viceversa.* Hence

$$\pi_{\mathbf{\Sigma},M}(J_u) = \pi_{\mathbf{\Sigma},M}\Big(J_u \cap \widetilde{\pi}((D, 1))\Big) \cup \pi_{\mathbf{\Sigma},M}\Big(J_u \cap \widetilde{\pi}((\Sigma \setminus S, m+1))\Big).$$

By definition of J_u, $J_{v_1(u)}$, $J_{v_{m+1}(u)}$, using also the local parametrizations Ψ_1, Ψ_{m+1}, it follows that $\pi_{\mathbf{\Sigma},M}\big(J_u \cap \widetilde{\pi}((D, 1))\big) = J_{v_1(u)} \setminus (\Sigma \setminus S)$, and $\pi_{\mathbf{\Sigma},M}\big(J_u \cap \widetilde{\pi}((\Sigma \setminus S, m+1))\big) = J_{v_{m+1}(u)} \cap (\Sigma \setminus S)$, and (4.27) follows. $\square$

Next lemma seems to be consistent with [55, Lemma 10.1]. From formula (4.29) below, we see that $|Du|(Y_{\mathbf{\Sigma}})$ is indeed independent of the orientation of Σ.

Lemma 4.22 (**Representation of the total variation on the covering, II**). *Let $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(\Omega, S)$, and let $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$. Then*

$$\begin{aligned} |Du|(Y_{\mathbf{\Sigma}}) =& m\ell \Big(\mathcal{H}^{n-1}(J_{v_1(u)} \setminus \Sigma) + \mathcal{H}^{n-1}(J_{v_{m+1}(u)} \cap \Sigma)\Big) \\ =& m\ell\, \mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_u)). \end{aligned} \tag{4.29}$$

(4) We recall from (4.3) that $D = \Omega \setminus \Sigma$.

Proof. Recall the splitting in (4.14), with the choice $E := Y_{\mathbf{\Sigma}}$. By (4.11), we have

$$\begin{aligned} |Dv_j(u)|(D) &= |Dv_1(u)|(D), \qquad j = 1, \dots, m, \\ |Dv'_j(u)|(\Sigma) &= |Dv_{m+1}(u)|(\Sigma), \qquad j' = m+1, \dots, 2m. \end{aligned} \tag{4.30}$$

By [12, Theorem 3.84], we have

$$|Dv_1(u)|(D) = \ell\, \mathcal{H}^{n-1}(J_{v_1(u)} \setminus \Sigma), \quad |Dv_{m+1}(u)|(\Sigma) = \ell\, \mathcal{H}^{n-1}(J_{v_{m+1}(u)} \cap \Sigma). \tag{4.31}$$

Substituting (4.31) into (4.14), and recalling (4.30), we get the first equality in (4.29). The second equality is now a consequence of (4.27). □

We are now in the position to prove the following compactness result.

Corollary 4.23 (**Compactness**). *Let Ω be bounded with Lipschitz boundary. Let $(u_h)_{h\in\mathbb{N}} \subset BV_{\rm constr}(Y_{\mathbf{\Sigma}}; V)$ be such that $\sup_{h\in\mathbb{N}} |Du_h|(Y_{\mathbf{\Sigma}}) < +\infty$. Then there exist $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; V)$ and a subsequence of $(u_h)_{h\in\mathbb{N}}$ converging to u in $L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^{m-1})$.*

Proof. For $h \in \mathbb{N}$, define $v_1(u_h) \in BV(D; V) = BV(\Omega; V)$ as in (4.10). From (4.14) and (4.30) we have

$$\begin{aligned} \sup_{h\in\mathbb{N}} |Dv_1(u_h)|(\Omega) &= \sup_{h\in\mathbb{N}} \Big[|Dv_1(u_h)|(D) + |Dv_1(u_h)|(\Sigma) \Big] \\ &\le \frac{1}{m} \sup_{h\in\mathbb{N}} |Du_h|(Y_{\mathbf{\Sigma}}) + \ell \mathcal{H}^{n-1}(\Sigma) < +\infty. \end{aligned}$$

Since Ω is a bounded Lipschitz domain, there exists $v \in BV(\Omega; V)$ such that, up to a not relabelled subsequence, $v_1(u_h) \to v$ in $L^1(\Omega; \mathbb{R}^k)$. The proof is completed, letting u be defined as in (4.25). □

We are now in the position to show that problem (4.19) has a solution; a key result is represented by Lemma 4.25 below.

Theorem 4.24 (**Existence of minimizers**). *Let Ω be a bounded connected open set with Lipschitz boundary. Let $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$. Then $\mathscr{A}^{\Omega}_{\rm constr}(S)$ is a minimum, and $\mathscr{A}^{\Omega}_{\rm constr}(S) > 0$.*

Proof. By the lower semicontinuity of the total variation, also recalling Corollary 4.23, existence of minimizers for problem (4.19) follows by direct methods. Positivity of $\mathscr{A}^{\Omega}_{\rm constr}(S)$ follows from (4.33) below, with the choice $A := \Omega$. □

Next lemma shows, in particular, that the jump set of any function in $BV_{\rm constr}(Y_{\mathbf{\Sigma}}; V)$ has strictly positive $\mathcal{H}^{n-1}$−measure in the fibers over any open subset of Ω containing a loop around a point of S. We stress that this is due just to the constraint (4.18).

Lemma 4.25 (**Non-constancy**). *Let $A \subseteq \Omega$ be a nonempty connected open set such that $\pi^{-1}_{\mathbf{\Sigma},M}(A \setminus S)$ does not consist of m connected components. Then, for every $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; V)$,*

$$\mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_u)\big) > 0. \tag{4.32}$$

Moreover, if A is bounded with Lipschitz boundary, then

$$\inf \big\{ \mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_u)\big) \ :\ u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; V) \big\} > 0. \tag{4.33}$$

Proof. In order to show (4.32), suppose by contradiction that there exists $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; V)$ such that

$$\mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_u)\big) = 0. \tag{4.34}$$

Applying (4.27) to (4.34), we get

$$0 = \mathcal{H}^{n-1}(A \cap (J_{v_1(u)} \setminus \Sigma)) + \mathcal{H}^{n-1}(A \cap J_{v_{m+1}(u)} \cap \Sigma). \tag{4.35}$$

Now, consider the (connected open) set $A^S := A \setminus S$. Applying (4.14) with the choice $E := \pi^{-1}_{\mathbf{\Sigma},M}(A^S)$, we get

$$\begin{aligned} |Du|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) =& m\,|Dv_1(u)|\left(\pi_{\mathbf{\Sigma},M}(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)\cap\widetilde{\pi}((D,1)))\right)\\ &+ m\,|Dv_{m+1}(u)|\left(\pi_{\mathbf{\Sigma},M}(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)\cap\widetilde{\pi}(\Sigma\setminus S,m+1)))\right)\\ =& m\,|Dv_1(u)|\left(A^S\setminus\Sigma\right) + m\,|Dv_{m+1}(u)|\left(A^S\cap\Sigma\right)\\ =& m\ell\left(\mathcal{H}^{n-1}(A\cap(J_{v_1(u)}\setminus\Sigma)) + \mathcal{H}^{n-1}(A\cap J_{v_{m+1}(u)}\cap\Sigma)\right), \end{aligned} \tag{4.36}$$

which, coupled with (4.35), implies $|Du|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) = 0$. Then[(5)] u is constant on each connected component of $\pi^{-1}_{\mathbf{\Sigma},M}(A^S)$. By the assumption on A, there exists at least one connected component of $\pi^{-1}_{\mathbf{\Sigma},M}(A^S)$, not contained in a single covering sheet. This contradicts the validity of the constraint (4.18), proving (4.32).

Now, let us suppose, still by contradiction, that there exists a sequence $(u_h)_h \subset BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}};V)$ such that $\lim_{h\to+\infty}\mathcal{H}^{n-1}\big(A\cap\pi_{\mathbf{\Sigma},M}(J_{u_h})\big) = 0$. For $h\in\mathbb{N}$, set $\hat{u}_h := u_h|_{\pi^{-1}_{\mathbf{\Sigma},M}(A^S)}$. In particular, reasoning as above, $|D\hat{u}_h|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) = m\ell\,\mathcal{H}^{n-1}(A\cap\pi_{\mathbf{\Sigma},M}(J_{u_h}))$. Let us apply Corollary 4.23, replacing Ω with A. Then, up to a not relabelled subsequence, there exists $\hat{u}\in BV_{\mathrm{constr}}(\pi^{-1}_{\mathbf{\Sigma},M}(A^S);V)$ such that $\hat{u}_h\to u$ in $L^1(\pi^{-1}_{\mathbf{\Sigma},M}(A^S);V)$, and by lower semicontinuity,

$$|D\hat{u}|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) \le \liminf_{h\to+\infty}|D\hat{u}_h|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) = m\ell\lim_{h\to+\infty}\mathcal{H}^{n-1}\big(A\cap\pi_{\mathbf{\Sigma},M}(J_{u_h})\big) = 0.$$

Hence $\hat{u}$ is constant on $\pi^{-1}_{\mathbf{\Sigma},M}(A^S)$, a contradiction with (4.32). □

As a further consequence of Lemma 4.25, the boundary datum S is covered by any constrained function in the covering space. In Theorem 4.36, using also (4.38) below, we shall prove that equality holds in (4.37) when $2 < n \le 7$ and u is a minimizer.

Corollary 4.26. *Let Ω be bounded (resp. unbounded), and let $u\in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}};V)$ (resp. $u\in BV^{\mathrm{loc}}_{\mathrm{constr}}(Y_{\mathbf{\Sigma}};V)$). Then*

$$S \subseteq \overline{\pi_{\mathbf{\Sigma},M}(J_u)}\setminus\pi_{\mathbf{\Sigma},M}(J_u). \tag{4.37}$$

Proof. The relation $S\cap\pi_{\mathbf{\Sigma},M}(J_u) = \emptyset$ is trivial, recall also (4.27). Now, suppose by contradiction that there exists a point $p\in S\setminus\overline{\pi_{\mathbf{\Sigma},M}(J_u)}$. Take an open ball B centered at p, with $B\subset\Omega\setminus\overline{\pi_{\mathbf{\Sigma},M}(J_u)}$, and apply Lemma 4.25 with the choice $A := B$. Then, since $A\cap\pi_{\mathbf{\Sigma},M}(J_u) = \emptyset$, we end up with a contradiction with (4.32). □

In view of Lemma 4.22, we give the following definition.

Definition 4.27 (Constrained-covering solutions). *Let Ω be bounded with Lipschitz boundary and let $u_{\min}$ be a minimizer of problem (4.19). We call*

$$\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$$

a constrained-covering solution *(in Ω) with boundary S.*

A similar definition is given when Ω is unbounded, assuming existence of $u_{\min}$ minimizing (4.21).

Remark 4.28. No topological restrictions on $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ are required.

(5) See [12, Proposition 3.2]; this constancy result can be generalized to our setting, considering first the case in which a connected open set $E\subsetneq Y_{\mathbf{\Sigma}}$ is contained in a single chart, and then reasoning for each connected component of $E\cap\widetilde{\pi}((D,1))$, $E\cap\widetilde{\pi}((D',m+1))$.

Recalling Remark 4.6, we observe that the proof of analytic regularity for the reduced boundary of minimizing clusters [115] applies in our setting. Indeed, since the classical arguments (such as monotonicity formula, excess decay, tilt lemma) are *local*, they can be symmetrically reproduced on the m sheets of the covering space, thus respecting the constraint on the fibers. In particular, the following results hold.

Theorem 4.29 (**Regularity,** $n = 2$)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with Lipschitz boundary (resp. an unbounded connected open set) , and let $u_{\min}$ be a minimizer of* (4.19) *(resp. of* (4.21)*). Then $J_{u_{\min}}$, and hence, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$, is the union of finitely many segments. Moreover, for each singular point x of $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ there exist exactly three segments of $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ having x as one of their endpoints, and meeting at x at $\frac{2\pi}{3}$-angles. Moreover,*

$$\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}}) \subseteq S \cup \partial\Omega. \tag{4.38}$$

Proof. We can confine ourselves to the proof of (4.38). Recalling Lemma 4.21, we have

$$\Big(\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})\Big) \cap D = \Big(\overline{J_{v_1(u_{\min})}} \setminus J_{v_1(u_{\min})}\Big) \cap D. \tag{4.39}$$

By the regularity of local minimizing clusters, $J_{v_1(u_{\min})} \cap D$ coincides with the relative boundary in D of the set $\cup_{\alpha \in V}\{v_1(u_{\min}) = \alpha\}$. In particular, $J_{v_1(u_{\min})} \cap D$ is relatively closed in D, which by (4.39) implies

$$\Big(\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})\Big) \cap D = \emptyset.$$

Similarly we argue on D', and (4.38) follows. □

The proof of the regularity result in the case $m = 2$ is analogous, so that we omit the details.

Theorem 4.30 (**Regularity,** $m = 2$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary (resp. an unbounded connected open set), and let $u_{\min}$ be a minimizer of* (4.19) *(resp. of* (4.21)*). Then $J_{u_{\min}}$, and hence $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$, is an analytic submanifold, possibly excepting for a set of Hausdorff dimension at most $n-8$. Moreover,* (4.38) *holds.*

4.3 Regularization

The interest in using V-valued BV functions in the context of covering spaces is substantiated by a Γ-,convergence result.

Let us first consider the case $m = 2$. Let Ω be bounded with Lipschitz boundary. The main idea is to lift the constraint (4.18) onto the Sobolev space $H^1(Y_{\mathbf{\Sigma}}) := \{u \in L^2_\mu(Y_{\mathbf{\Sigma}}) \ : Du \in L^2_\mu(Y_{\mathbf{\Sigma}}; \mathbb{R}^n)\}$. Recalling Remark 4.13, we set

$$H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}) := \Big\{u \in H^1(Y_{\mathbf{\Sigma}}) \ : \sum_{\pi_{\mathbf{\Sigma},M}(y)=x} u(y) = 0 \text{ for a.e. } x \in M\Big\}. \tag{4.40}$$

For $\epsilon \in (0,1)$, let us consider the functionals $F_\epsilon \colon L^1(Y_{\mathbf{\Sigma}}) \to [0,+\infty]$, defined as

$$F_\epsilon(u) := \int_{Y_{\mathbf{\Sigma}}} \left[\epsilon|\nabla u|^2 + \frac{1}{\epsilon}(1-u^2)^2\right] d\mu \qquad \text{if } u \in H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}),$$

and extended to $+\infty$ in $L^1(Y_{\mathbf{\Sigma}}) \setminus H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}})$.

Proposition 4.31 (**Γ-convergence,** $m = 2$)**.** *Assume $n \geq 2$ and $m = 2$. If $(u_{\epsilon_h})_h \subset L^1(Y_{\mathbf{\Sigma}})$ is such that $\sup_h F_{\epsilon_h}(u_{\epsilon_h}) < +\infty$, then there exist $u \in L^1(Y_{\mathbf{\Sigma}})$ and a subsequence of $(u_{\epsilon_h})_h$ converging to u in $L^1(Y_{\mathbf{\Sigma}})$. Moreover,*

$$\big(\Gamma(L^1(Y_{\mathbf{\Sigma}})) - \lim_{\epsilon \to 0^+} F_\epsilon\big)(u) = \begin{cases} \frac{c_0}{2}|Du|(Y_{\mathbf{\Sigma}}), & \text{if } u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\}), \\ +\infty, & \text{otherwise in } L^1(Y_{\mathbf{\Sigma}}), \end{cases}$$

where $c_0 := \xi(1) - \xi(-1)$, and $\xi(t) := 2\int_0^t |1-s^2|\, ds$.

Proof. The proof of the equicoerciveness statement is standard (see, *e.g.*, [123]). The Γ-lim inf inequality follows using the lower semicontinuity of the total variation, and the fact that the constraint (4.20) is closed under almost everywhere convergence in $Y_{\mathbf{\Sigma}}$. The Γ-lim sup construction follows by recalling that the local parametrizations of $Y_{\mathbf{\Sigma}}$ are the identity (Remark 4.6); in order to get the validity of the constraint in (4.40), it is sufficient to use the standard construction, since the optimal one-dimensional profile is odd (hence, the corresponding recovering sequence is in $H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}})$). See [123] for the details. □

Now, let us conclude this section with the case $n = 2$ and $m = 3$. Let $V := \{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{R}^2$ be the set of vertices of an equilateral triangle, centered at the origin. With a slight abuse of notation, it is natural to identify $\mathcal{T}(V)$ with $\{\frac{2\pi}{3}, \frac{4\pi}{3}\}$, see (4.42) below. The idea is now to lift the constraint (4.18) onto the Sobolev space $H^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2) := \{u \in L^2_\mu(Y_{\mathbf{\Sigma}}; \mathbb{R}^2) \ : \ Du \in L^2_\mu(Y_{\mathbf{\Sigma}}; \mathbb{R}^2 \times \mathbb{R}^2)\}$, by asking that

$$\exists\, \theta \in \Big\{\frac{2\pi}{3}, \frac{4\pi}{3}\Big\} \quad \text{s.t.} \quad v_j(u) = e^{i(j-1)\theta} \circ v_1(u), \ j = 1, 2, 3, \tag{4.41}$$

and then setting

$$H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \mathbb{R}^2) := \Big\{u \in H^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2) \ : \ (4.41) \text{ holds}\Big\}. \tag{4.42}$$

where, for $\theta \in [0, 2\pi)$, $e^{i\theta} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of angle θ.

Let $\widetilde{W} \colon \mathbb{R}^2 \to [0, +\infty)$ be a triple-well potential with superlinear growth at infinity, and such that $\widetilde{W}^{-1}(0) = V$. We assume also that

$$\widetilde{W}(e^{i\theta} x) = \widetilde{W}(x), \qquad x \in \mathbb{R}^2, \ \theta \in \Big\{\frac{2\pi}{3}, \frac{4\pi}{3}\Big\}. \tag{4.43}$$

For instance, one could consider the choice $\widetilde{W}(x) := \prod_{j=1}^3 |x - \alpha_j|^2$. For $\epsilon \in (0, 1)$, let us consider the functionals $G_\epsilon \colon L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2) \to [0, +\infty]$, defined as

$$G_\epsilon(u) := \int_{Y_{\mathbf{\Sigma}}} \Big[\epsilon |\nabla u|^2 + \frac{1}{\epsilon} \widetilde{W}(u)\Big]\, d\mu \qquad \text{if } u \in H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \mathbb{R}^2),$$

and extended to $+\infty$ in $L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2) \setminus H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$.

Proposition 4.32 ($\Gamma-$**convergence,** $m = 3$)**.** *Assume $n = 2$ and $m = 3$. If $(u_{\epsilon_h})_h \subset L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$ is such that $\sup_h G_{\epsilon_h}(u_{\epsilon_h}) < +\infty$, then there exist $u \in L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$ and a subsequence of $(u_{\epsilon_h})_h$ converging to u in $L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$. Moreover, there exists the $\Gamma(L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2))$-limit of $(G_\epsilon)_\epsilon$ as $\epsilon \to 0^+$, which is finite just on functions $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, and it equals $|Du|(Y_{\mathbf{\Sigma}})$ up to a positive multiplicative constant depending only on $\widetilde{W}$.*

Proof. Again, the proof of the equicoerciveness statement is standard (see, *e.g.*, [20]). Let $(u_\epsilon)_\epsilon \subset H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$ be such that $(G_\epsilon(u_\epsilon))_\epsilon$ is equibounded, and $u_\epsilon \to u$ in $L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$ for some $u \in L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$. Then $\widetilde{W}(u) = 0$ a.e. in $Y_{\mathbf{\Sigma}}$, or equivalently $u(x) \in V$ a.e. in $Y_{\mathbf{\Sigma}}$. The fact that u satisfies (4.18) for some $\tau \in \mathcal{T}(V)$ follows at once by the constraint in (4.42). The $\Gamma - \liminf$ inequality is now a consequence of the lower semicontinuity of the total variation.

Let us sketch the proof of the $\Gamma-\limsup$ construction, which is a slight modification of the one provided in [20]. Without loss of generality, we can assume $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, and $\pi_{\mathbf{\Sigma},M}(J_u)$ contained in the union of a finite number of segments. For small $\epsilon > 0$, consider an ϵ-tubular neighbourhood $T_\epsilon \subset \Omega$ of $\pi_{\mathbf{\Sigma},M}(J_u)$; let also $Z_\epsilon \subset T_\epsilon$ be the Lipschitz open set containing the triple junctions, such that $\pi_{\mathbf{\Sigma},M}(Z_\epsilon) = \bigcap_{j=1}^3 \{|d_j| < \epsilon\}$, where, for every $j = 1, 2, 3$, d_j denotes a signed distance from $\{u = \alpha_j\}$. Then, we construct a map $u_\epsilon \in H^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$ so that:

- $u_\epsilon = u$ in $Y_{\mathbf{\Sigma}} \setminus \pi^{-1}_{\mathbf{\Sigma},M}(T_\epsilon)$,

- in $\pi^{-1}_{\mathbf{\Sigma},M}(T_\epsilon \setminus Z_\epsilon)$, u_ϵ represents the transition between the two corresponding zeroes of W, along suitable optimal profiles which depend only on W (see [20]);
- u_ϵ in $\pi^{-1}_{\mathbf{\Sigma},M}(Z_\epsilon)$ is defined by interpolating the trace of u_ϵ on $\partial\pi^{-1}_{\mathbf{\Sigma},M}(Z_\epsilon)$ with zero (the barycenter of V) along the segments starting at the triple junction.

Here we notice that, since $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, and thanks to the simmetry assumption (4.43) on $\widetilde{W}$, u_ϵ satisfies (4.41), and therefore $u_\epsilon \in H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$. Moreover, the contribution to $G_\epsilon(u_\epsilon)$ on $\pi^{-1}_{\mathbf{\Sigma},M}(Z_\epsilon)$ is of order ϵ. Then the statement follows. □

Remark 4.33. Proposition 4.32 can be extended to the case $m \geq 3$, combining the standard tools in [20] (which actually hold for every $m \geq 2$).

4.4 Constrained covering solutions when $n = 2, 3$

4.4.1 Minimal networks in the plane

In this section we exploit the case $n = 2$, $m \geq 2$, and $S := \{p_1, \dots, p_m\} \subset \Omega$, with $p_j \neq p_l$ for any $j, l = 1, \dots, m$, $j \neq l$.

Theorem 4.34. *Assume that*

$$\mathrm{dist}(S, \partial\Omega) > \inf\{\mathcal{H}^1(\Sigma) \ : \ \Sigma \in \mathrm{Cuts}(\Omega, S)\}. \tag{4.44}$$

Then $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is connected.

Proof. By contradiction, suppose that there exist two disjoint nonempty sets C_1, C_2, relatively closed in $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$, and such that $C_1 \cup C_2 = \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$. By Theorem 4.29, for each $j = 1, 2$, C_j consists of segments (possibly meeting at triple junctions); moreover, by virtue of (4.44), also recalling (4.29) and (4.24), we have $\overline{C_j} \cap \partial\Omega = \emptyset$. Set $S_j := \overline{C_j} \cap S$, for $j = 1, 2$. Note that

$$S_j \neq \emptyset, \qquad j = 1, 2. \tag{4.45}$$

Indeed, suppose by contradiction that (for example) $S_1 = \emptyset$; then, by (4.38), $\overline{C_1} \setminus C_1 \subset \partial\Omega$, and therefore there exists a connected open set $A \subset \Omega$ such that $\Omega \cap \partial A \subseteq C_1$, and $A \cap S = \emptyset$. Thanks to Theorem 4.17, it is not restrictive to assume also that $A \cap \Sigma = \emptyset$. Now, it is immediate to modify $v_1(u_{\min})$ inside A so that it does not jump anymore on $\Omega \cap \partial A$. Taking any constrained lift of the modified function, minimality of $u_{\min}$ is contradicted, proving (4.45).

Let us choose now two tubular neighborhoods T, U of C_1, so that $T \subset\subset U$,

$$(U \setminus \overline{T}) \cap \pi_{\mathbf{\Sigma},M}(J_{u_{\min}}) = \emptyset, \tag{4.46}$$

and $T \cap C_2 = \emptyset$. In particular, there must be $j \in \{1, \dots, m\}$ such that Σ_j connects a point of S_1 with a point of S_2. Therefore, $\pi^{-1}_{\mathbf{\Sigma},M}(U \setminus \overline{T})$ does not consist of m distinct connected components, so that, applying Lemma 4.25 with the choice $A := U \setminus \overline{T}$, we get a contradiction with (4.46). □

Corollary 4.35. *Assume* (4.44). *Then $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is a Steiner graph*(6) *connecting the points of S.*

Proof. Let $C \subset \Omega$ be a Steiner graph connecting the points of S. By Theorems 4.29-4.34, $\mathcal{H}^1(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})) \leq \mathcal{H}^1(C)$. On the other hand, fix any $\Sigma \in \mathrm{Cuts}(\Omega, S)$ such that $\Sigma \cap C = S$. Then, define $v \in BV(\Omega; V)$ so that: for $j = 1, \dots, m-1$, $v := \alpha_j$ on the connected open set whose boundary contains Σ_j, and is contained in $\Sigma_j \cup C$; $v := \alpha_m$ elsewhere in Ω. Finally, consider the τ-contrained lift $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ of v, where $\tau(j) := j - 1 \pmod m$. By construction, $\pi_{\mathbf{\Sigma},M}(J_u) = C$, and the statement follows. □

(6) See, *e.g.*, [99].

In order to get Corollary 4.35, we cannot avoid condition (4.44), see Figure 4.5 for a counterexample when $m = 2$. This is another difference with respect to the model proposed in [55]: in our model the boundary of Ω is "wettable" in principle, and therefore, in order to avoid a minimizer to touch $\partial\Omega$, we need a condition of the form (4.44) (see also Remark 4.16).

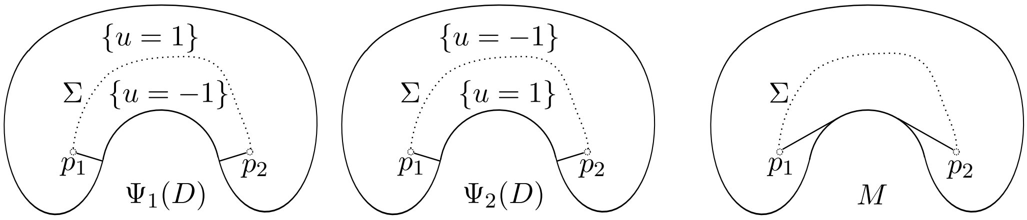


Figure 4.5: Let Ω be the "bean-shaped" domain in the picture, let $S := \{p_1, p_2\}$, and let $\Sigma \in \mathrm{Cuts}(\Omega, S)$ be the dashed curve. The two pictures on the left show the constrained covering solution, while the right picture shows the solution of [55].

4.4.2 Plateau's problem

In this section we exploit the case $n = 3$, hence $m = 2$, so that $Y_{\mathbf{\Sigma}}$ is a double-covering space of M.

Let $S \subset \mathbb{R}^3$ be a tame link. Let $\Omega \subset \mathbb{R}^3$ be bounded with Lipschitz boundary, and $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (4.19). By Theorem 4.30, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is an embedded analytic surface in M. We ask now whether $\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ coincides with S (compare with (4.38)). To this aim, we need an assumption, analogous to (4.44), in order to avoid components of $\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})}$ touching $\partial\Omega$; roughly speaking, we have to show that "long thin" hairs reaching the boundary of Ω cannot occur in a constrained double−covering solution.

Theorem 4.36 (**Attaining the boundary condition**). *Let $2 \leq n \leq 7$. Let $\bar{r} > 0$ be such that $S \subset B_{\bar{r}}$. There exists $R > \bar{r}$ such that, if $\Omega \supset B_R$, then any minimizer $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ of problem* (4.19) *satisfies*

$$\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}}) = S. \tag{4.47}$$

Proof. Fix $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(B_{\bar{r}}, S) \subset \mathbf{Cuts}(\Omega, S)$. Set

$$\mathscr{A}(r) := \mathscr{A}^{B_r}_{\mathrm{constr}}(S), \qquad r \geq \bar{r},$$

and let $u_r \in BV_{\mathrm{constr}}(Y^r_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (4.19) for $\Omega = B_r$; here, $Y^r_{\mathbf{\Sigma}}$ denotes the double covering space of the base set $B_r \setminus S$.

By (4.22), $\mathscr{A}(\cdot)$ is nondecreasing; in addition, it is bounded (see (4.24)). Set $\epsilon := 4\,\mathcal{H}^{n-1}(\Sigma) - \mathscr{A}(\bar{r}) \geq 0$, so that by (4.24),

$$\mathscr{A}(r) - \mathscr{A}(\bar{r}) \leq \epsilon, \qquad r \geq \bar{r}. \tag{4.48}$$

Write $\mathscr{A}(r) = 4\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar{r}}) + 4t(r, \bar{r})$, where $t(r, \bar{r}) := \mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \setminus B_{\bar{r}})$. Since $\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar{r}}$ is a competitor for the computation of $\mathscr{A}(\bar{r})$, we have

$$\begin{aligned}\mathscr{A}(\bar{r}) \leq& 4\,\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar{r}})\\ \leq& 4\,\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar{r}}) + 4\,t(r, \bar{r}) = \mathscr{A}(r).\end{aligned} \tag{4.49}$$

Coupling (4.48) and (4.49), we get

$$4t(r, \bar{r}) \leq \mathscr{A}(r) - \mathscr{A}(\bar{r}) \leq \epsilon, \qquad r \geq \bar{r}. \tag{4.50}$$

Suppose $\epsilon = 0$. Then, by (4.50), we have $\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \setminus B_{\bar r}) = 0$, which, by the assumption $2 \le n \le 7$ and Theorem 4.30, implies that the constrained double−covering solution does not reach ∂B_r, for any $r > \bar r$. Then the statement follows, taking an arbitrary $R > \bar r$.

Suppose $\epsilon > 0$, and let $r > \bar r$ be such that $\left(\overline{\pi_{\mathbf{\Sigma},M}(J_{u_r})} \setminus B_{\bar r}\right) \cap \partial B_r \neq \emptyset$. By the assumption $2 \le n \le 7$ and Theorem 4.30, there exists $x \in (\pi_{\mathbf{\Sigma},M}(J_{u_r}) \setminus B_{\bar r}) \cap \partial B_{(r+\bar r)/2}$. Take $\delta \in (0, (r - \bar r)/2)$. By the lower density estimate for local minimizers of the perimeter functional (see for instance [115, Theorem 21.11]), we have

$$c_n \delta^{n-1} \le 4\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_\delta(x)) \le 4t(r, \bar r) \le \epsilon, \tag{4.51}$$

for some positive constant c_n depending only on n. Inequalities (4.51) hold for each $\delta \in (0, (r - \bar r)/2)$; this is possible only if $r \le r_\epsilon := \bar r + 2(\epsilon/c_n)^{\frac{1}{n-1}}$. Hence, taking $R > r_\epsilon$, the assertion follows. □

Now, we compare the constrained double−covering solutions with other classical notions of solutions to Plateau's problem.

Remark 4.37 (**Area-minimizing currents**). Let $n = 3$, and assume that Ω contains the closed convex envelope of S. Let $T_{\min}$ be a rectifiable two-current [86] solving Plateau's problem with boundary S in the sense of currents. By [125, Theorem 5.6], the support of $T_{\min}$ is contained in Ω; moreover, by [105], it is an embedded, orientable smooth surface $\Sigma_{\min} \subset \Omega$ up to the boundary S. In particular, $\Sigma_{\min} \in \mathrm{Cuts}(\Omega, S)$. Hence, by (4.24)

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \le 4\,\mathbf{M}(T_{\min}), \tag{4.52}$$

where $\mathbf{M}(T_{\min})$ is the mass of $T_{\min}$.

It is worth noticing that there is not an absolute positive constant $c \in (0, 4]$, satisfying

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \ge c\,\mathbf{M}(T_{\min}) \tag{4.53}$$

for any S. As a counterexample, let $\widehat{B_1} := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, x_3 = 0\}$, let $S := \partial \widehat{B_1}$, and, for $\epsilon > 0$, let $\Omega := (1 + \epsilon)\widehat{B_1} \times (-2, 2)$. As admissible pair of cuts, we take as Σ the closure of $\widehat{B_1}$, and $\Sigma' := \{x \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 \le 1, \ x_3 = -\sqrt{1 - x_1^2 - x_2^2}\}$. Now, let $v \in BV(\Omega; \{\pm 1\})$ be defined as $v(x_1, x_2, x_3) := 1$ if $x_3 > 0$, and -1 elsewhere. Finally, let $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be the constrained lift of v. Then, recalling (4.29), it is immediate to verify that

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \le \frac{|Du|(Y_{\mathbf{\Sigma}})}{4} = \pi\big((1+\epsilon)^2 - 1\big) \to 0 \quad \text{as } \epsilon \to 0^+.$$

At the same time, the minimal mass in the sense of currents is π (the area of $\widehat{B}_1$), independently of ϵ.

Another (not rigorous but more intuitive) example of the failure of inequality (4.53) can be obtained taking as S the boundary of a very thin Möbius band: in this case a surface similar to the Möbius band is expected to be the double covering solution with boundary S, while the support of the minimal current is expected to be approximately a double disk.

Remark 4.38 (**Disk-type area-minimizers**). Let $n = 3$ and suppose that S is connected. Recalling (4.52) and the results in [129], [81], we have

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \le 4 \min\{\mathrm{area}(X) \ : \ X \in H^1(\mathrm{D}; \mathbb{R}^3), \ X \text{ spans } S\}, \tag{4.54}$$

where $\mathrm{D} \subset \mathbb{R}^2$ is the unit disk, $\mathrm{area}(X) := \int_{\mathrm{D}} |\partial_{x_1} X \wedge \partial_{x_2} X| \, dx_1 \, dx_2$, and the meaning of "$X$ spans S" is given for instance in [81]. We observe that (4.54) can be obtained independently of (4.52), by reproducing the proof of Theorems 4.17 and 4.39.

Now, we show that, when $n < 8$, constrained double−covering solutions give an equivalent way to solve Plateau's problem in the sense of integral currents modulo 2 [86].

Theorem 4.39 (**Area-minimizing integral currents mod** 2)**.** *Let $2 \le n \le 7$, and let Ω be as in Theorem 4.36. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\boldsymbol{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (4.19). Then $\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}})$ can be seen as an integral current modulo 2 with boundary S, and $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S)$ coincides with $4\mathbf{M}_2(T_{2,\min})$, where $\mathbf{M}_2$ is the mass and $T_{2,\min}$ is a mass-minimizing integral current modulo 2 having boundary S.*

Proof. By Theorems 4.30 and 4.36, $\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}})$ is an embedded analytic hypersurface satisfying (4.47). In particular, $\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}})$ can be considered as the support of an integral current modulo 2 having S as boundary support. This gives

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) = 4\mathcal{H}^{n-1}(\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}})) \ge 4\,\mathbf{M}_2(T_{2,\min}).$$

The converse inequality follows from the interior regularity of minimal integral currents modulo 2 [144, Theorem 6.2.1] and Corollary 4.19, since the area-minimizing current mod 2 with boundary S belongs to $\mathrm{Cuts}(\Omega, S)$. $\square$

Remark 4.40. Let $n \ge 2$. Recalling Theorem 4.30 and Lemma 4.25, we have

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \ge 4 \inf \Big\{ \mathcal{H}^{n-1}(K) \ : \ K \subset M \text{ rel. closed}, \ K \cap \rho(\mathbb{S}^1) \neq \emptyset \\ \text{for every } S\text{-simple link } \rho \in C(\mathbb{S}^1; M) \Big\}, \tag{4.55}$$

where, according to [103, p.4], ρ is said an *S-simple link* if $\mathrm{link}(\rho; C) = 1$ for some connected component C of S, and $\mathrm{link}(\rho; C') = 0$ for all connected components C' of $S \setminus C$.(7) The right hand side of (4.55) has been recently investigated in [103] and [77], for more general choices of S.

We notice that, in general, we cannot expect the inequality in (4.55) to be an equality. A counterexample, with $n = 2$ and $m = 6$, is obtained taking S as the set of (six) vertices of two triangles, as in Figure 4.6. Then the right hand side of (4.55) is attained by the union of G_1 and G_2, the two Steiner graphs corresponding to the triangles. On the other hand, by Theorem 4.34, $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S)$ is strictly larger than $\mathcal{H}^1(G_1) + \mathcal{H}^1(G_2)$.

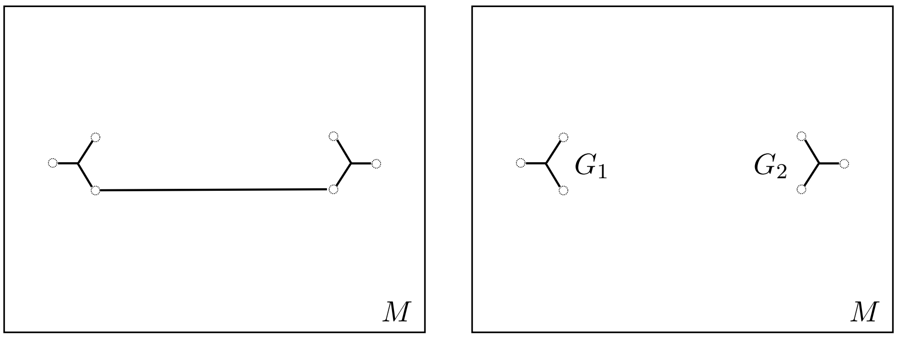


Figure 4.6: Let S be the set of vertices of two triangles, which are sufficiently far one from the other. In the left picture, the constrained covering solution is shown, in the case $\Omega = \mathbb{R}^2$. Notice that $\mathscr{A}^{\mathbb{R}^2}_{\mathrm{constr}}(S)$ is strictly larger than the length of the two Steiner graphs drawn in the right picture.

4.4.3 The tetrahedron

We end this section coming back to the m-sheeted covering construction given in Section 4.1, for a possible interesting extension in dimension $n = 3$. As for the case of minimal networks, Example 4.41 below shows that the covering construction has essentially to be

(7) Here, $\mathrm{link}(\rho; C)$ denotes the linking number [106] between $\rho \in C(\mathbb{S}^1; M)$ and a boundaryless compact embedded Lipschitz $(n-1)$-dimensional submanifold $C \subset M$.

chosen depending on the solution that one would like to obtain. In our present case, we aim to design a covering construction giving, possibly, the solution obtained by J. Taylor in [148].

Example 4.41. Let $\mathrm{S} \subset \mathbb{R}^3$ be the one-skeleton of a regular tetrahedron T centered at 0 (here, Ω can be thought of as a large ball containing S). Referring to Figure 4.7, let us denote by F_j the (closed) facet of T opposite to the vertex p_j, for $j = 1, 2, 3$. We now aim to define a 4-sheeted, "cut and paste" covering of $\mathrm{M} := \Omega \setminus \mathrm{S}$ following the procedure described in Section 4.1.1. To this aim, we take as family $\mathrm{Cuts}(\Omega, \mathrm{S})$ of admissible cuts the collection of all $\Sigma = \cup_{j=1}^3 \Sigma_j \subset \Omega$ such that:

- for $j = 1, 2, 3$, Σ_j is a 2-dimensional compact embedded Lipschitz submanifold, having the edges of F_j as topological boundary;
- for $j, l = 1, 2, 3$, $j \neq l$, $\Sigma_j \cap \Sigma_l$ equals the intersection of the topological boundaries of F_j and F_l.

Clearly, the easiest example of an element of $\mathrm{Cuts}(\Omega, \mathrm{S})$ is given by $\cup_{j=1}^3 F_j$. Then, we select the family $\mathbf{Cuts}(\Omega, \mathrm{S})$ of admissible pairs of cuts as the collection of all pairs $\mathbf{\Sigma} := (\Sigma, \Sigma')$ such that $\Sigma, \Sigma' \in \mathrm{Cuts}(\Omega, \mathrm{S})$, and $\Sigma \cap \Sigma' = \mathrm{S}$; moreover, as in Definition 4.2-(*ii*), we require Σ to "lie on one side" of Σ' locally around S.

Fix now $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(\Omega, \mathrm{S})$. Then, the covering $(Y_{\mathbf{\Sigma}}, \pi_{\mathbf{\Sigma},\mathrm{M}})$ is obtained identifying four copies of the open sets $D := \Omega \setminus \Sigma$, $D' := \Omega \setminus \Sigma'$ as in (4.5) (with the choice $m = 4$). Namely, assuming for simplicity $\Sigma = \cup_{j=1}^3 F_j$, crossing the facet F_j coming from $\Omega \setminus T$ (resp. from T) corresponds to moving j-sheets forward (resp. backward) in the covering, for $j = 1, 2, 3$. Finally, the minimization problem can be set up as in Section 4.2; here, $V := \{\alpha_1, \ldots, \alpha_4\} \subset \mathbb{R}^3$ is the set of vertices of a regular tetrahedron centered at 0 (not necessarily equal to T). Existence of minimizers in the class $BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ follows by adapting the arguments in Theorem 4.24.(8) Concerning regularity of minimizers, and referring to [2] for the notion of $(1, \delta)$-restricted sets, we can state the following result.

Proposition 4.42. *Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ be a minimizer of problem* (4.19). *Let $x \in \pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}})$, and let $r > 0$ be such that $B_r(x) \subseteq \Omega$. Then $\pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}}) \cap B_r(x)$ is $(1, \delta)$-restricted with respect to $\overline{\Omega} \setminus B_r(x)$, for any $\delta \in (0, r)$.*

Proof. Fix a perturbation $\varphi \in \mathrm{Lip}(\Omega; \Omega)$ of the identity, compactly supported in $B_r(x)$. Using the same construction as in [23, Theorem 2], we define a function $v^* \in BV(\Omega; V)$ such that

$$v^* = v_1(u_{\min}) \quad \text{outside } B_r(x), \qquad J_{v^*} \cap B_r(x) = \varphi(J_{v_1(u_{\min})} \cap B_r(x)). \tag{4.56}$$

Let $\tau \in \mathcal{T}(V)$ be such that $v_j(u_{\min}) = \tau^{j-1} \circ v_1(u_{\min})$, for $j = 2, 3, 4$. Then, we define $u^* \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ as the τ-constrained lift of v^*. The statement now follows recalling [23, Corollary 1], (4.56), and using the minimality of $u_{\min}$. $\square$

Assume that there exists $r > 0$ such that $\mathrm{dist}(\pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}}), \partial\Omega) > r$. Then, as a consequence of Proposition 4.42, and by the general theory of Almgren's minimal sets [2], we get that $\pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}})$ is $(\mathbf{M}, 0, r)$-minimal. Figure 4.7 represents a minimizer $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ of problem (4.19), where $\Sigma := \cup_{j=1}^3 F_j$, and Σ' lies in $\Omega \setminus T$ (the cut Σ' is not drawn in the picture). More precisely, let $v \in BV(\Omega; V)$ be such that: $v := \alpha_1$ in $\Omega \setminus T$ and in the tetrahedron with vertices $0, p_1, p_2, p_3$; $v := \alpha_2$ in the tetrahedron with vertices $0, p_1, p_2, p_4$; $v := \alpha_3$ in the tetrahedron with vertices $0, p_1, p_3, p_4$; $v := \alpha_4$, in the tetrahedron with vertices $0, p_2, p_3, p_4$. Let also $\tau \in \mathcal{T}(V)$ be the transposition such that $\tau(j) := j + 1$, for $j = 1, 2, 3$. Then, $u_{\min}$ is defined as the τ-constrained lift of v, recall Definition 4.20. By construction, u does not jump on the fiber of the facets F_j's. Notice that $\pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}}) = \mathrm{int}(T) \cap C$, where C is the (infinite) cone over S, and it coincides with the solution provided by [148].

(8) It is possible to check that, up to a homeomorphism, the covering construction is independent of the chosen admissible pair of cuts.

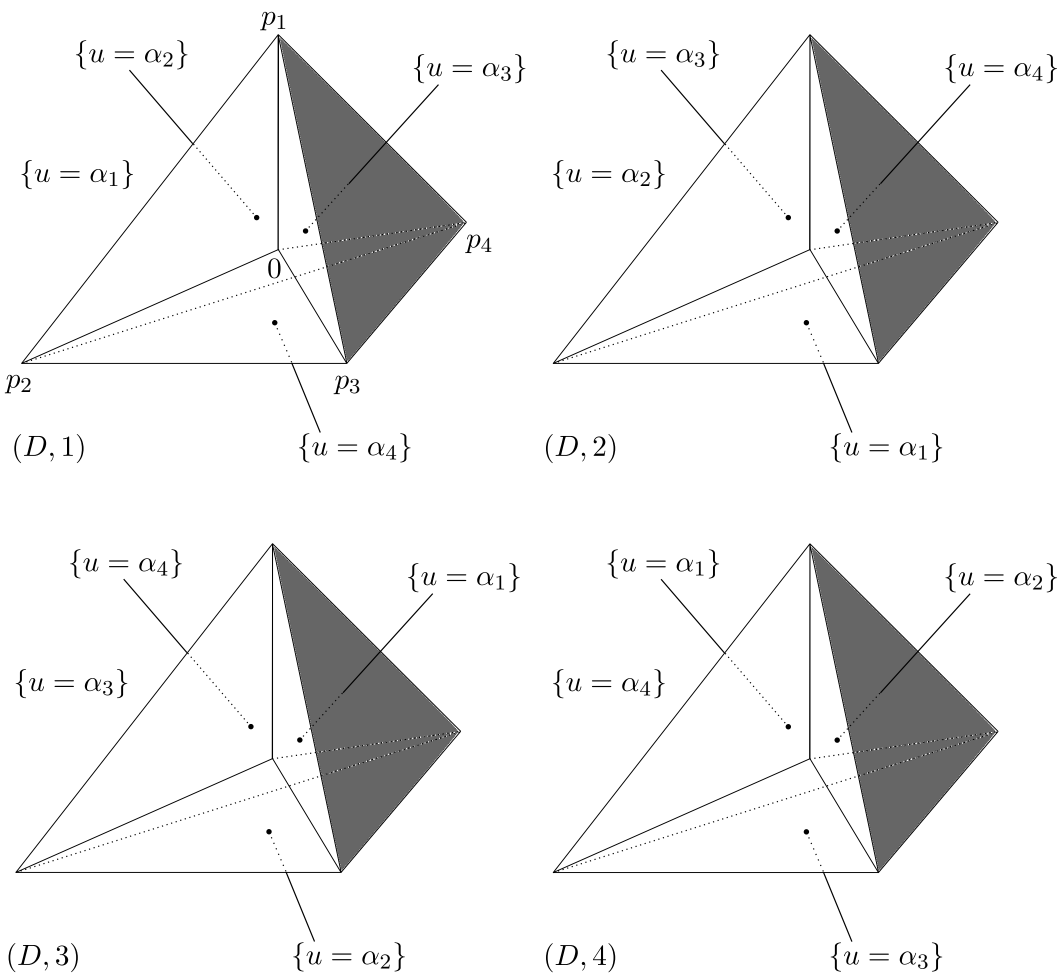


Figure 4.7: A minimizer $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, when S is the one-skeleton of a regular tetrahedron centered at 0. The picture refers to the choice $\Sigma := \cup_{j=1}^{3} F_j$. The copies of the facet F_2 have been coloured in grey to denote that they have been removed from the covering sheets drawn in the figure.

Appendix A

Remarks on the generalized inverted anisotropic ratio

In Section 1.3 we have seen that, if K_1 and K_2 are two convex bodies, then their star-shaped combination $K_1 \star K_2$ (Definition 1.13) is not in general a convex body. An explicit counterexample has been given in [43], involving the two ellipses

$$K_1 := \left\{(x,y) \in \mathbb{R}^2 : x^2 + \rho y^2 = 1\right\}, \qquad K_2 := \left\{(x,y) \in \mathbb{R}^2 : \rho x^2 + y^2 = 1\right\}, \tag{A.1}$$

defined for $\rho > 0$. In this case, we say that K_1 and K_2 satisfy the *inverted anisotropic ratio*. Then, we recall from Remark 1.14 that $K_1 \star K_2$ is (smooth and) not convex if and only if $\rho < \frac{1}{3}$ or $\rho > 3$.

In this section we want to consider a slightly more general situation. For $n = 2$ and $\mathrm{V} = \mathbb{R}^2$, let ϕ_1, ϕ_2 be the linear anisotropies on $\mathbb{R}^2$ such that

$$B_{\phi_1} = \{(x, y \in \mathbb{R}^2 \ : \ x^2 + ay^2 = 1\}, \quad B_{\phi_2} = \{(x, y \in \mathbb{R}^2 \ : \ bx^2 + cy^2 = 1\},$$

for given $a, b, c > 0$. Notice that the usual inverted anisotropic ratio (A.1) corresponds to the choice

$$a := b := \rho, \qquad c := 1. \tag{A.2}$$

Consistently with the notation in Example 1.18, set $\phi_j(\theta) := \phi_j((\cos\theta, \sin\theta))$, for $j = 1, 2$ and $\theta \in [0, 2\pi)$, and set $\Phi(\theta) := \phi_1(\theta) \star \phi_2(\theta)$. By (1.9), we have

$$(\phi_1^2 + \phi_2^2)\Phi^2 = \phi_1^2\phi_2^2. \tag{A.3}$$

After some computation, (A.3) can be rewritten as

$$2(p + q\cos 2\theta)\Phi^2 = u\cos 4\theta + v\cos 2\theta + w, \tag{A.4}$$

with $p := 1 + a + b + c$, $q := 1 - a + b - c$, $u := \frac{(a-1)(c-b)}{2}$, $v := 2(b - ac)$, and $w := \frac{3ac+3b+ab+c}{2}$.

We now want to study whether Φ is convex. Since we need to twice differentiate Φ (recall condition (1.11)), we see immediately from (A.4), that a special case of combined anisotropy corresponds to the choice $q = 0$, *i.e.*

$$1 + b = a + c, \tag{A.5}$$

so that (A.4) reduces to

$$2p\Phi^2(\theta) = u\cos 4\theta + v\cos 2\theta + w. \tag{A.6}$$

Notice that (A.2) can be seen as a particular case of (A.5). For this reason, and in analogy with the standard terminology, we say that ϕ_1 and ϕ_2 realize a *generalized inverted anisotropic ratio* provided (A.5) holds.

Differentiating (A.6) and dividing by 2, we get

$$2p\Phi(\theta)\Phi'(\theta) = -2u\sin 2\theta - v\sin 2\theta,$$

which entails

$$\Phi'(\theta) = -\frac{2u\sin 4\theta + v\sin 2\theta}{2p\Phi(\theta)}. \tag{A.7}$$

Differentiating again, and dividing by 2, we end up with

$$u(\Phi'(\theta))^2 + \Phi(\theta)\Phi''(\theta) = -4u\cos 4\theta - v\cos 2\theta. \tag{A.8}$$

Now, notice that $\Phi + \Phi'' \geq 0$ if and only if

$$2p(\Phi^2 + \Phi\Phi'') \geq 0. \tag{A.9}$$

Substituting (A.6), (A.8) and (A.7) into (A.9), after some computation we obtain

$$7u\cos 4\theta + v\cos 2\theta + \frac{(2u\sin 4\theta + v\sin 2\theta)^2}{u\cos 4\theta + v\cos 2\theta + w} - w \leq 0.$$

Multiplying previous line by the positive(1) quantity $(u\cos 4\theta + v\cos 2\theta + w)$, and re-ordering terms, we can rephrase (A.9) as

$$\begin{aligned}&3u^2\cos^2 4\theta + 8uv\cos 4\theta\cos 2\theta - 8uv\cos^3 2\theta\\ &+ 6uw\cos 4\theta + 8uv\cos 2\theta + 4u^2 + v^2 - w^2 \leq 0.\end{aligned} \tag{A.10}$$

Set $z := \cos 2\theta$, $z \in [-1,1]$. Then, after some computation, we can rewrite the convexity condition (A.10) as

$$12u^2z^4 + 8uvz^3 + 12u(w-u)z^2 + 7u^2 + v^2 - z^2 - 6uw \leq 0. \tag{A.11}$$

Let $F(z)$ denote the left hand side of (A.11). We now claim that F attains its maximum on the interval $[-1,1]$ at $z = 0$ (*i.e.*, $\theta = \frac{\pi}{4}$). Indeed, one can easily see that $F'(z) \geq 0$ if and only if $uz \geq 0$; by (A.5), we have $2u = -(a-1)^2 \leq 0$, and so $F'(z) \geq 0$ if and only if $z \leq 0$, which proves our claim.

As a consequence,

$$F(z) \leq F(0) = 7u^2 + v^2 - w^2 - 6uw, \qquad z \in [-1,1]. \tag{A.12}$$

From (A.12), we deduce that (A.11) holds (and hence, Φ is a convex anisotropy) if and only if $F(0) \leq 0$; on the contrary, if $F(0) > 0$, there is some range of directions around $\theta = \frac{\pi}{4}$ where B_Φ is nonconvex.

Next Lemma gives an explicit formula in order to represent the values of $a, c > 0$ such that $F(0) = 0$, see Figure A.1.

Lemma A.1. *We have* $7u^2 + v^2 - w(w+6u) = g(a,c)$, *where*

$$g(a,c) := 3a^4 + 4a^3c - 8a^3 - 16ac^2 - 20a^2c + 12ac + 6a^2 + 4c - 1. \tag{A.13}$$

Proof. Let us separately compute the two terms $I := 7u^2 + v^2$, and $II := -w(6u+w)$. We have

$$\begin{aligned}I =&7\frac{(a-1)^4}{4} + 4(a-1)^2(1-c)^2 = \frac{(a-1)^2}{4}[7(a-1)^2 + 16(1-c)^2]\\ =&\frac{(a-1)^2}{4}[7a^2 - 14a + 16c^2 - 32c + 23],\end{aligned} \tag{A.14}$$

(1) Indeed, this quantity corresponds to the right hand side of (A.3).

while

$$
\begin{aligned}
II =& \frac{w}{2}\left(-6(a-1)^2 + 4c(a+1) + (a-1)(c+3)\right) \\
=& \frac{w}{2}\left((a-1)(-5a+9) + 4c(a+1)\right) \\
=& \frac{4c(a+1)+(a-1)(c+3)}{4}[(a-1)(-5a+9)+4c(a+1)] \\
=& \frac{1}{4}\left[16c^2(a+1)^2 + 4c(a-1)(a+1)(9-5a) + (a-1)^2(a+3)(9-5a)\right. \\
& \left. + 4c(a+1)(a-1)(a+3)\right] \\
=& \frac{1}{4}\left[16c^2(a+1)^2 + 4c(a-1)(12-4a) + (a-1)^2(-5a^2-6a+27)\right].
\end{aligned}
\tag{A.15}
$$

Putting together (A.14) and (A.15), we get:

$$
\begin{aligned}
4(I+II) =& (a-1)^2[7a^2 - 14a + 16c^2 - 32c + 23 + 5a^2 + 6a - 27] \\
& - 16c^2(a+1)^2 - 4c(a-1)(12-4a) \\
=& (a-1)^2[12a^2 + 16c^2 - 8a - 32c - 4] - 16c(a+1)[ac - a^2 + 4a + c - 3] \\
=& (a^2 - 2a + 1)[12a^2 + 16c^2 - 8a - 32c - 4] \\
& - 16(ac + c)[ac - a^2 + 4a + c - 3] \\
=& 12a^4 + 16a^2c^2 - 8a^3 - 32a^2c - 4a^2 - 24a^3 - 32ac^2 \\
& + 16a^2 + 64ac + 8a + 12a^2 + 16c^2 - 8a - 32c - 4 \\
& - 16(a^2c^2 - a^3c + 4a^2c + ac^2 - 3ac + ac^2 - a^2c + 4ac + c^2 - 3c).
\end{aligned}
$$

Reordering terms, after some cancellations we finally get

$$
4(I+II) = 12a^3 + 16a^3c - 32a^3 - 64ac^2 - 80a^2c + 48ac + 24a^2 + 16c - 4,
$$

which gives (A.13). □

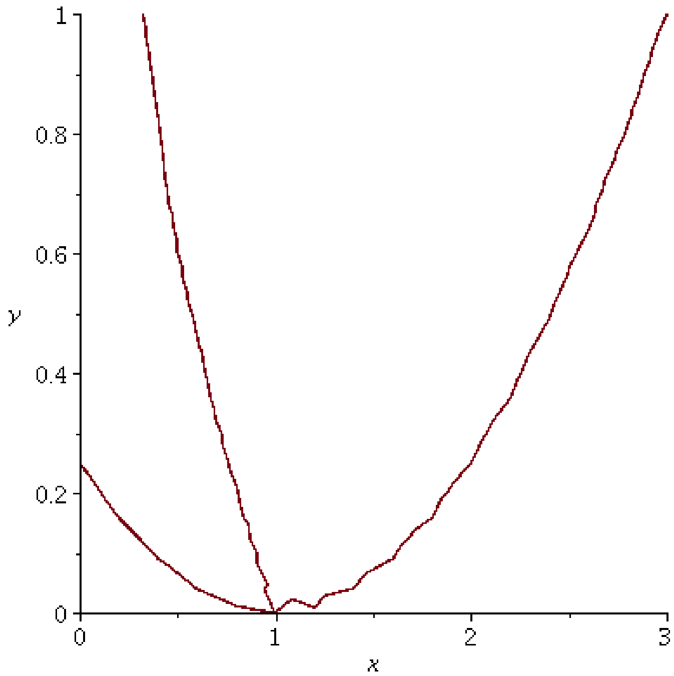


Figure A.1: The plot of the curve $\Gamma := \{(a,c) \in \mathbb{R}^2 \ : \ a, c > 0, \ g(a,c) = 0\}$ (using "Maple 16"). The regions of nonconvexity of Φ corresponds to the unbounded sets coloured in grey. When $c = 1$ (inverted anisotropic ratio), we retrieve the values $a = \frac{1}{3}$ and $a = 3$ (consistently with the result in [43]). Notice that Γ does not intersect the vertical line $\{a = 1\}$, since in this case ϕ_1 is the Euclidean norm and Φ is a multiple of ϕ_1.

Appendix B

The capillary problem in the absence of gravity

We give here a brief overview of the action principle for a capillary, referring the reader for instance to [117, 88, 51], and references therein, for a more complete discussion on this topic.

In the absence of gravity, the *capillary problem* on a bounded connected Lipschitz open set $\Omega \subset \mathbb{R}^k$ ($k = 2$ being the physical case) can be stated as follows: given $b, \mu \in \mathbb{R}$, solve

$$\inf \left\{ \mathscr{G}_\mu(u) \ : \ u \in BV(\Omega), \ \int_\Omega u \, dx = b \right\}, \tag{B.1}$$

where $BV(\Omega)$ is the space of functions with bounded variation in Ω, and $\mathscr{G}_\mu$ is the strictly convex functional

$$\mathscr{G}_\mu(u) := \int_\Omega \sqrt{1 + |Du|^2} - \int_{\partial\Omega} \mu u \, d\mathcal{H}^{k-1}. \tag{B.2}$$

Here, $\int_\Omega \sqrt{1 + |Du|^2}$ is the area of the (generalised) graph of u [117, 102], u can be thought of as the height of the liquid, and the last term in (B.2) involves the trace of u on $\partial\Omega$. Let $\mu \geq 0$ (up to a change of sign of b, this is not restrictive, since $\mathscr{G}_\mu(u) = \mathscr{G}_{-\mu}(-u)$.) Then, one can show [117] that, when $\mu > 1$, the functional $\mathscr{G}_\mu$ is unbounded from below, while, if $\mu = 0$, then problem (B.1) is trivially solved by a suitable constant. In what follows, we shall confine ourselves to the case

$$\mu \in (0, 1].$$

We then set $\mu = \cos\gamma$, where γ represents, for $m = 2$, the (assigned) contact angle between the liquid and the bounding walls of the capillary tube $\Omega \times \mathbb{R}$. From the first variation computation of $\mathscr{G}_\mu$, supposing for simplicity that $\partial\Omega$ is of class $\mathcal{C}^1$, it turns out that if $\mu \in (0, 1)$, then solving (B.1) is equivalent to find

$$u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega}) \tag{B.3}$$

such that

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = h \qquad \text{in } \Omega \tag{B.4}$$

for a suitable constant $h \in \mathbb{R}$ independent of b. The *prescribed mean curvature equation* (B.4) is coupled with the Neumann-type boundary condition

$$\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nu^\Omega = \mu \qquad \text{on } \partial\Omega. \tag{B.5}$$

The constant h is identified integrating by parts, since

$$h = \frac{1}{|\Omega|} \int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) dx = \frac{1}{|\Omega|} \int_{\partial\Omega} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nu^\Omega \, d\mathcal{H}^{k-1} = \frac{\mu P(\Omega)}{|\Omega|}. \tag{B.6}$$

From (B.5), it follows that solutions of (B.1) can be expected only when $\mu < 1$. Once μ has been chosen, the problem becomes to find necessary and sufficient conditions on the set Ω ensuring existence of solutions of (B.3), (B.4) and (B.5). In this respect, it is convenient to introduce the *prescribed mean curvature functionals* defined, for $\lambda \in \mathbb{R}$, and $\mu \in [-1, 1]$, as

$$\mathscr{F}_{\lambda,\mu}(B) := P(B, \Omega) + \mu\mathcal{H}^{m-1}(\partial^* B \cap \partial\Omega) - \lambda|B|, \qquad B \subseteq \Omega,$$

where $\partial^* B$ denotes the reduced boundary [12] of the finite perimeter set B, and $P(\cdot, \Omega)$ is the perimeter in Ω (if $\mu = 1$ we have $\mathscr{F}_{\lambda,1}(B) = P(B) - \lambda|B|$ for any $B \subseteq \Omega$). The problem

$$\inf\{\mathscr{F}_{\lambda,\mu}(B) : B \text{ of finite perimeter}, B \subseteq \Omega\} \tag{B.7}$$

has been studied by several authors, see for instance [145, 88, 42, 65], (see also [24, 45, 46, 47, 48, 49, 50]) and references therein. By direct methods, it turns out that there exists a solution of (B.7) and, again, if such a solution is sufficiently regular, its boundary inside Ω has mean curvature equal to λ, and contact angle with $\partial\Omega$ equal to $\arccos \mu$.

Now, let $\mu \in (0, 1)$ and h be as in (B.6).[(1)] Then [88, Chapter 7] there exists a solution of (B.3), (B.4) and (B.5) if and only if

$$0 = \mathscr{F}_{h,\mu}(\emptyset) = \mathscr{F}_{h,\mu}(\Omega) < \mathscr{F}_{h,\mu}(B), \qquad B \subset \Omega,\ B \neq \emptyset; \tag{B.8}$$

moreover, the solution is unique up to an additive constant, and it is bounded from below in Ω. On the other hand [89], if (B.8) is violated, still (B.4) admits a solution in some nonempty set $B^* \subset \Omega$, and such a solution becomes unbounded on $\Omega \cap \partial B^*$. In this situation, the expected physical phenomenon is that the height of the fluid increases unboundedly on $\Omega \setminus B^*$, until part of the base in B^* remains uncovered.

In connection with the case $\mu = 1$, and for taking into account unbounded functions u, we mention that problem (B.1) can be generalised into a minimization over subsets which are not necessarily subgraphs of a function. This formulation is originally due to M. Miranda [120, 121], and has led to the notion of *generalised solution*.

Theorem B.1 shows that (B.8) is a necessary and sufficient condition also in the case $\mu = 1$, thus identifying a "maximal" set Ω where the elliptic equation (B.4) has a solution. The result in [101] is given, more generally, for a right hand side of (B.4) belonging to $\mathrm{Lip}(\Omega) \cap L^\infty(\Omega)$.

Theorem B.1 ([101]). *Let $\Omega \subset \mathbb{R}^k$ be a bounded connected open set with Lipschitz boundary, and let $h := \frac{P(\Omega)}{|\Omega|}$. Then there exists a solution $u \in \mathcal{C}^2(\Omega)$ of*

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = h \qquad \text{in } \Omega \tag{B.9}$$

if and only if

$$h < \frac{P(B)}{|B|}, \qquad B \subset \Omega,\ B \neq \emptyset. \tag{B.10}$$

Moreover, if Ω is of class $\mathcal{C}^2$, the solution is unique up to an additive constant, bounded from below in Ω, and its graph is vertical at the boundary of Ω, in the sense that

$$\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \to \nu^\Omega \qquad \text{uniformly on } \partial\Omega. \tag{B.11}$$

Finally, if $k = 2$ and Ω is convex, (B.10) *is in turn equivalent to assume that the curvature of $\partial\Omega$, at all points of $\partial\Omega$ where it is defined, is less than or equal to h.*

Similarly to the case $\mu \in (0, 1)$, if Ω does not satisfy (B.10), the fluid height is expected to become unbounded in correspondence of the complement of some nonempty regular set $B^\star \subset \Omega$, such that $\Omega \cap \partial B^\star$ has mean curvature equal to h. Moreover, it is proven in [101, Theorem 3.2] that u is unbounded from above around a relatively open region (if any) of $\partial\Omega$ where the maximum of the mean curvature of $\partial\Omega$ equals $P(\Omega)/|\Omega|$.

[(1)] Note that, for any $B \subseteq \Omega$, there holds $\mathscr{F}_{h,\mu}(\Omega \setminus B) = \tilde{\mathscr{F}}(B)$, where we set $\tilde{\mathscr{F}}(B) := P(B, \Omega) - \mu\mathcal{H}^{m-1}(\partial^* B \cap \partial\Omega) + h|B|$.

Appendix C

An abstract covering construction

In this appendix we give an alternative construction of the covering of M built up in Section 4.1. The construction is standard (see, *e.g.*, [104, 118]), and has the advantage to avoid all issues about the definition of admissible cuts. Setting up the minimization problem on the covering space M_H below could have an independent interest; we have preferred to use the "cut and paste" construction (and next proving independence of the cuts) in order to deal with a more "handy" formula (like (4.29)) for the total variation of a BV function defined on the covering space.

Let Ω, S, M, and m be as in Section 4.1. Fix $x_0 \in M$, and set $C_{x_0}([0,1];M) := \{\gamma \in C\big([0,1];M\big) \ : \ \gamma(0) = x_0\}$. For $\gamma \in C_{x_0}([0,1];M)$, let $[\gamma]$ be the class of paths in $C_{x_0}([0,1];M)$ which are homotopic to γ with fixed endpoints. We recall that the universal covering of M is the pair $(\widetilde{M}, \mathfrak{p})$, where $\widetilde{M} := \big\{[\gamma] \ : \ \gamma \in C_{x_0}([0,1];M)\big\}$ and $\mathfrak{p} \colon [\gamma] \in \widetilde{M} \mapsto \mathfrak{p}([\gamma]) := \gamma(1) \in M$. A basis for the topology of $\widetilde{M}$ is given by the family $\{[\gamma\lambda] : [\gamma] \in \widetilde{M},\ \gamma(1) \in B \text{ open ball},\ \lambda \in C([0,1];B),\ \lambda(0) = \gamma(1)\}$.

Let $\pi_1(M, x_0)$ be the first fundamental group of M with base point $x_0 \in M$, and let

$$H := \{[\rho] \in \pi_1(M, x_0) \ : \ \mathrm{link}(\rho; S) \equiv 0 \pmod m\}.$$

Remark C.1. *H is a (normal) subgroup of $\pi_1(M, x_0)$ of index m.*

For $\gamma \in C_{x_0}([0,1];M)$, set $\bar{\gamma}(t) := \gamma(1-t)$ for all $t \in [0,1]$. Associated with H, we can consider the following equivalence relation $\sim_H$ on $\widetilde{M}$: for $[\gamma], [\lambda] \in \widetilde{M}$,

$$[\gamma] \sim_H [\lambda] \iff \gamma(1) = \lambda(1), \quad \mathrm{link}(\gamma\bar{\lambda}; S) \equiv 0 \pmod m.$$

We denote by $[\gamma]_H$ the equivalence class of $[\gamma] \in \widetilde{M}$ induced by $\sim_H$, and we set

$$M_H := \widetilde{M} / \sim_H .$$

Letting $\widetilde{\mathfrak{p}}_H \colon \widetilde{M} \to M_H$ be the projection induced by $\sim_H$, we endow M_H with the corresponding quotient topology. We set $\mathfrak{p}_{H,M} \colon [\gamma]_H \in M_H \mapsto \gamma(1) \in M$, so that we have the following commutative diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{\mathfrak{p}}_H"] \arrow[dr, "\mathfrak{p}"'] & M_H \arrow[d, "\mathfrak{p}_{H,M}"] \\ & M \end{tikzcd} \tag{C.1}$$

and the pair $(M_H, \mathfrak{p}_{H,M})$ is a covering of M, see [104, Proposition 1.36].

Let (Y, π_Y) be a covering of M, and let $y_0 \in \pi_Y^{-1}(x_0)$. By $(\pi_Y)_* \colon \pi_1(Y, y_0) \to \pi_1(M, x_0)$ we denote the homomorphism defined as $(\pi_Y)_*([\varrho]) := [\pi_Y \circ \varrho]$. By [104, Proposition 1.36], we have

$$(\mathfrak{p}_{H,M})_*(\pi_1(M_H, [x_0]_H)) = H. \tag{C.2}$$

Proposition C.2. *Let* $\boldsymbol{\Sigma} \in \mathrm{Cuts}(\Omega, S)$*. Then* $Y_{\boldsymbol{\Sigma}}$ *and* M_H *are homeomorphic.*

Proof. Recall the notation in Section 4.1.1. By [104, p. 28], it is not restrictive to assume that $x_0 \in O$. Let $y_0 \in \pi_{\boldsymbol{\Sigma},M}^{-1}(x_0)$, and let $[\varrho] \in \pi_1(Y_{\boldsymbol{\Sigma}}, y_0)$. By [104, Proposition 1.36], since H and $(\pi_{\boldsymbol{\Sigma},M})_*\big(\pi_1(Y_{\boldsymbol{\Sigma}}, y_0)\big)$ have the same index, the statement follows if we are able to prove that $(\pi_{\boldsymbol{\Sigma},M})_*([\varrho]) \in H$, or equivalently that

$$\mathrm{link}(\pi_{\boldsymbol{\Sigma},M} \circ \varrho; S) \equiv 0 \pmod{m}, \tag{C.3}$$

where m is given in (4.2).

Let us first consider the case $n = 2$. Notice that

$$\mathrm{link}(\pi_{\boldsymbol{\Sigma},M} \circ \varrho; S) = \sum_{j=1}^{m} \mathrm{link}(\pi_{\boldsymbol{\Sigma},M} \circ \varrho; p_j), \tag{C.4}$$

and, for any $j = 1, \dots, m$, $\mathrm{link}(\pi_{\boldsymbol{\Sigma},M} \circ \varrho; p_j)$ equals the number of times that $\pi_{\boldsymbol{\Sigma},M} \circ \varrho$ turns around p_j, a counterclockwise (resp. clockwise) turn around p_j being counted with positive (resp. negative) sign. By construction (see for instance Figure 4.3 when $m = 3$), any counterclockwise (resp. clockwise) turn of $\pi_{\boldsymbol{\Sigma},M} \circ \varrho$ around a point in S corresponds to moving one sheet forward (resp. backward) in $Y_{\boldsymbol{\Sigma}}$. Thus, the sum in the right hand side of (C.4) is equal to the number of sheets visited by the loop ϱ until it comes back to y_0. It is now clear that this number can be only a multiple of m, proving (C.3).

The case $n > 2$ is even simpler, since we have $m = 2$, and (C.3) follows noticing that $[\varrho]$ can change sheet in $Y_{\boldsymbol{\Sigma}}$ just an even number of times. □

Let $\boldsymbol{\Sigma}$, $\widehat{\boldsymbol{\Sigma}} \in \mathbf{Cuts}(\Omega, S)$. By Proposition C.2, and by general results in coverings theory [104], there exists a homeomorphism $\mathrm{f} \colon Y_{\boldsymbol{\Sigma}} \to Y_{\widehat{\boldsymbol{\Sigma}}}$ such that

$$\pi_{\boldsymbol{\Sigma},M} = \pi_{\widehat{\boldsymbol{\Sigma}},M} \circ \mathrm{f}. \tag{C.5}$$

The map f is defined by path-lifting. More precisely, fix $x_0 \in M$, and let $y_0 \in Y_{\boldsymbol{\Sigma}}$, $\widehat{y}_0 \in Y_{\widehat{\boldsymbol{\Sigma}}}$ be such that $\pi_{\boldsymbol{\Sigma},M}(y_0) = x_0 = \pi_{\widehat{\boldsymbol{\Sigma}},M}(\widehat{y}_0)$. Let $y \in Y_{\boldsymbol{\Sigma}}$, and let $\gamma \in C([0,1]; Y_{\boldsymbol{\Sigma}})$ be such that $\gamma(0) = y_0$, $\gamma(1) = y$. Then, $\mathrm{f}(y) \in Y_{\widehat{\boldsymbol{\Sigma}}}$ is defined as the ending point of the lift of $\pi_{\boldsymbol{\Sigma},M} \circ \gamma$ to $Y_{\widehat{\boldsymbol{\Sigma}}}$, starting at $\widehat{y}_0$.

Bibliography

[1] N. D. Alikakos, P. W. Bates, and X. Chen, *Convergence of the Cahn-Hiliiard equation to the Hele-Shaw model*, Arch. Rational Mech. Anal. **128** (1994), 165-205.

[2] F. J. Almgren, *Existence and Regularity Almost Everywhere of Solutions to Elliptic Variational Problems with Constraints*, Mem. Amer. Math. Soc. **165**, Providence (RI), 1976.

[3] F. J. Almgren, and J. E. Taylor, *Flat flow is motion by crystalline curvature for curves with crystalline energies* , J. Differential Geom. **42** (1995), 1-22.

[4] F. J. Almgren, J. E. Taylor, and L. Wang, *Curvature-driven flows: a variational approach,* SIAM J. Control Optim. **31** (1993), 387-437.

[5] F. Alter, and V. Caselles, *Uniqueness of the Cheeger set of a convex body*, Nonlinear Anal. **70** (2009), 32-44.

[6] F. Alter, V. Caselles, and A. Chambolle, *A characterization of convex calibrable sets in* $\mathbb{R}^n$, Math. Ann. **332** (2005), 329-366.

[7] S. Amato, G. Bellettini, and M. Paolini, *The nonlinear multidomain model: a new formal asymptotic analysis*, "Proceedings of Geometric Partial Differential Equations", CRM Series **15** (2013), (A. Chambolle, M. Novaga, and E. Valdinoci, eds.), Pisa, 33-74.

[8] S. Amato, G. Bellettini, and M. Paolini, *Constrained BV functions on covering spaces for minimal networks and Plateau's type problem*, accepted paper (Adv. Calc. Var.), 2015. Preprint SISSA `http://urania.sissa.it/xmlui/handle/1963/34452`.

[9] S. Amato, G. Bellettini, and L. Tealdi, *Anisotropic mean curvature on facets and relations with capillarity*, Geometric Flows **1** (2015), 80-110.

[10] L. Ambrosio, *Minimizing movements*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **19** (1995), 191-246.

[11] L. Ambrosio, P. Colli Franzone, and G. Savaré, *On the asymptotic behaviour of anisotropic energies arising in the cardiac bidomain model*, Interfaces Free Bound. **2** (2000), 213-266.

[12] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Univ. Press, Oxford, 2000.

[13] L. Ambrosio, R. Ghezzi, and V. Magnani, *BV functions and sets of finite perimeter in sub-Riemannian manifolds*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), 489-517.

[14] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Birkhäuser, Basel, 2008.

[15] L. Ambrosio, M. Novaga, and E. Paolini, *Some regularity results for minimal crystals.* ESAIM Control Optim. Calc. Var. **8** (2002), 69-103.

[16] F. Andreu, V. Caselles, J. I. Diaz, and J. M. Mazon, *Some qualitative properties of the total variation flow*, J. Funct. Anal. **188** (2002), 516-547.

[17] F. Andreu-Vaillo, V. Caselles, and J. M. Mazon, *Parabolic quasilinear equations minimizing linear growth functionals*, Birkäuser, Basel, 2004.

[18] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. **135** (1983), 293-318.

[19] G. Anzellotti, *Traces of bounded vector fields and the divergence theorem*, preprint Dipartimento di Matematica Univ. Trento, 1983.

[20] S. Baldo, *Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (1990), 37-65.

[21] D. Bao, S.-S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, New York, 2000.

[22] A. Bejancu, *Finsler Geometry and Applications*, Ellis Horwood Limited, Chichester, 1990.

[23] G. Bellettini, *An almost everywhere regularity result for minimal partitions*, Boll. Unione Mat. Ital. **4-A** (1990), 57-63.

[24] G. Bellettini, *A numerical approach to a minimum problem with applications in image segmentations*, Ann. Univ. Ferrara **36** (1990), 99-111.

[25] G. Bellettini, *An introduction to anisotropic and crystalline mean curvature flow*, Hokkaido Univ. Tech. Rep. Ser. Math. **145** (2010), 102-162.

[26] G. Bellettini, *Lecture Notes on Mean Curvature Flow: Barriers and Singular Perturbations*, Edizioni Sc. Norm. Sup. Pisa, Pisa, 2014.

[27] G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga, *Crystalline mean curvature flow of convex sets*, Arch. Ration. Mech. Anal. **179** (2006), 109-152.

[28] G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga, *The volume preserving crystalline mean curvature flow of convex sets in* $\mathbb{R}^n$, J. Math. Pures Appliquée **92** (2009), 499-527.

[29] G. Bellettini, V. Caselles, and M. Novaga, *The total variation flow in* $\mathbb{R}^n$, J. Differential Equations **184** (2002), 475-525.

[30] G. Bellettini, V. Caselles, and M. Novaga, *Explicit solutions of the eigenvalue problem* $-\mathrm{div}(Du/|Du|) = u$, SIAM J. Math. Anal. **36** (2005), 1095-1129.

[31] G. Bellettini, P. Colli Franzone, and M. Paolini, *Convergence of front propagation for anisotropic bistable reaction–diffusion equations*, Asymptotic Anal. **15** (1997), 325-358.

[32] G. Bellettini, G. Fusco, and N. Guglielmi, *A concept of solution for forward-backward equations of the form* $u_t = \frac{1}{2}(\phi'(u_x))_x$ *and numerical experiments for the singular perturbation* $u_t = -\epsilon^2 u_{xxxx} + \frac{1}{2}(\phi'(u_x))_x$, Discrete Contin. Dyn. Syst. **16** (2006), 783-842.

[33] G. Bellettini, and L. Mugnai, *Anisotropic geometric functionals and gradient flows*, Banach Cent. Publ. **86** (2009), 21-43.

[34] G. Bellettini, and M. Novaga, *Approximation and comparison for non-smooth anisotropic motion by mean curvature in* $\mathbb{R}^n$, Math. Mod. Meth. Appl. Sc. **10** (2000), 1-10.

[35] G. Bellettini, M. Novaga, and G. Orlandi, *Eventual regularity for the parabolic minimal surface equation*, Discrete Contin. Dyn. Syst. **35** (2015), 5711-5723.

[36] G. Bellettini, M. Novaga, and M. Paolini, *Facet-breaking for three-dimensional crystals evolving by mean curvature*, Interfaces Free Bound. **1** (1999), 39-55.

[37] G. Bellettini, M. Novaga, and M. Paolini, *Characterization of facet-breaking for nonsmooth mean curvature flow in the convex case*, Interfaces Free Bound. **3** (2001), 415-446.

[38] G. Bellettini, M. Novaga, and M. Paolini, *On a crystalline variational problem, part I: first variation and global L^∞-regularity*, Arch. Ration. Mech. Anal. **157** (2001), 165-191.

[39] G. Bellettini, M. Novaga, and M. Paolini, *On a crystalline variational problem, part II: BV-regularity and structure of minimizers on facets*, Arch. Ration. Mech. Anal. **157** (2001), 193-217.

[40] G. Bellettini, and M. Paolini, *Quasi-optimal error estimates for the mean curvature flow with a forcing term*, Differential Integral Equations **8** (1995), 735-752.

[41] G. Bellettini, and M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. J. **25** (1996), 537-566.

[42] G. Bellettini, and M. Paolini, *Numerical simulations of measurements of capillary contact angles*, IMA J. Numer. Anal. **16** (1996), 165-178.

[43] G. Bellettini, M. Paolini, and F. Pasquarelli, *Non convex mean curvature flow as a formal singular limit of the non linear bidomain model*, Adv. Differential Equations **18** (2013), 895-934.

[44] G. Bellettini, M. Paolini, and S. Venturini, *Some results on surface measures in Calculus of Variations*, Ann. Mat. Pura Appl. **170** (1996), 329-359.

[45] G. Bellettini, M. Paolini, and C. Verdi, *Γ-convergence of discrete approximations to interfaces with prescribed mean curvature*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **1** (1990), 317-328.

[46] G. Bellettini, M. Paolini, and C. Verdi, *Numerical minimization of geometrical type problems related to calculus of variations*, Calcolo **27** (1990), 251-278.

[47] G. Bellettini, M. Paolini, and C. Verdi, *Convex approximations of functionals with curvature*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **2** (1991), 297-306.

[48] G. Bellettini, M. Paolini, and C. Verdi, *Front-tracking and variational methods to approximate interfaces with prescribed mean curvature*, "Proc. Numerical Methods for Free Boundary Problems", Jyväskylä, 1990, (P. Neittaanmäki, ed.), Birkhäuser (1991), 83-92.

[49] G. Bellettini, M. Paolini, and C. Verdi, *Numerical minimization of functionals with curvature by convex approximations*, "Progress in partial differential equations: calculus of variations, applications", Pitman Research Notes in Mathematics Series **267** (1992), (C. Bandle, J. Bemelmans, M. Chipot, M. Grüter, and J. Saint Jean Paulin, eds.), Longman Scientific & Technical Harlow, 124-138.

[50] G. Bellettini, M. Paolini, and C. Verdi, *Convergence of discrete approximations to sets of prescribed mean curvature*, "Free boundary problems involving solids", Pitman Research Notes in Mathematics Series **281** (1993), (J.M. Chadam, and H. Rasmussen, eds.), Longman Scientific & Technical Harlow, 164-169.

[51] J. Berthier, and K. A. Brakke, *The Physics of Microdroplets*, Wiley, Hoboken (NJ), 2012.

[52] E. Bonnetier, E. Bretin and A. Chambolle, *Consistency result for a non monotone scheme for anisotropic mean curvature flow*, Interfaces Free Bound. **14** (2012), 1-35.

[53] R. Bott, and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer – Verlag, New York, 1982.

[54] K. A. Brakke, *The Motion of a Surface by its Mean Curvature*, Mathematical Notes **20**, Princeton Univ. Press, Princeton (NJ), 1978.

[55] K. Brakke, *Soap films and covering spaces*, J. Geom. Anal. **5** (1995), 445-514.

[56] H. Brezis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert.* North-Holland Mathematics Studies, Amsterdam-London: North-Holland Publishing Comp., 1973.

[57] H. Brezis, *Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.

[58] A. Briani, A. Chambolle, M. Novaga, and G. Orlandi, *On the gradient flow of a one-homogeneous functional*, Confluentes Math. **3** (2011), 617-635.

[59] L. Bronsard, and F. Reitich, *On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation*, Arch. Ration. Mech. Anal. **124** (1993), 355-379.

[60] J.W. Cahn, C. A. Handwerker, and J. E. Taylor, *Geometric models of crystal growth*, Acta Metall. **40** (1992), 1443-1474.

[61] V. Caselles, A. Chambolle, and M. Novaga, *Uniqueness of the Cheeger set of a convex body*, Pacific J. Math. **232** (2007), 77-90.

[62] V. Caselles, A. Chambolle, S. Moll, and M. Novaga, *A characterization of convex calibrable sets in* $\mathbb{R}^n$ *with respect to anisotropic norms*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 803-832.

[63] V. Caselles, A. Chambolle, and M. Novaga, *Some remarks on uniqueness and regularity of Cheeger sets*, Rend. Semin. Mat. Univ. Padova **123** (2010), 191-201.

[64] V. Caselles, A. Chambolle, and M. Novaga, *Regularity for solutions of the total variation denoising problem*, Rev. Mat. Iberoamericana **27** (2011), 233-252.

[65] V. Caselles, G. Facciolo, and E. Meinhardt, *Anisotropic Cheeger sets and applications*, SIAM J. Imaging Sci. **2** (2009), 1211-1254.

[66] A. Chambolle, and M. Novaga, *Convergence of an algorithm for the anisotropic and crystalline mean curvature flow*, SIAM J. Math. Anal. **37** (2006), 1878-1987.

[67] A. Chambolle, and M. Novaga, *Existence and uniqueness for planar anisotropic and crystalline curvature flow*, Adv. Studies Pure Math. **67** (2015), 87-113.

[68] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, “Problems in Analysis, A Symposium in Honor of Salomon Bochner” (R. C. Gunning, ed.), Princeton Univ. Press **625** (1970), 195-199.

[69] X. Chen, *Spectrums for the Allen-Cahn, Cahn-Hilliard, and phase-field equations for generic interface*, Comm. Partial Diff. Eqns. **19** (1994), 1371-1395.

[70] P. Colli Franzone, L. Guerri, and S. Rovida, *Wavefront propagation in an activation model of the anisotropic cardiac tissue: asymptotic analysis and numerical simulations*, J. Math. Biol. **28** (1990), 121-176.

[71] P. Colli Franzone, L. F. Pavarino, and S. Scacchi, *Mathematical Cardiac Electrophysiology*, Springer, Berlin, 2014.

[72] P. Colli Franzone, M. Pennacchio, and G. Savaré, *Multiscale modeling for the bioelectric activity of the heart*, SIAM J. Math. Anal. **37** (2005), 1333-1370.

[73] P. Colli Franzone, and G. Savaré, *Degenerate evolution systems modeling the cardiac electric field ad micro and macroscopic level*, “Evolution equations, semigroups and functional analysis” **50** (2002), (A. Lorenzi, and B. Ruf, eds.), Birkhäuser, Basel, 49-78.

[74] J. Coromilas, A. L. Dillon, and S. M. Wit, *Anisotropy reentry as a cause of ventricular tachyarrhytmias*, "Cardiac Electrophysiology: From Cell to Bedside", ch.49, W. B. Saunders Co., Philadelphia (1994), 511-526.

[75] G. Dal Maso, *An Introduction to* Γ – *Convergence*, Birkhäuser, Boston, 1993.

[76] G. David, *Should we solve Plateau's problem again?*, "Advances in Analysis: The Legacy of Elias M. Stein", (C. Fefferman, A. D. Ionescu, D. H. Phong, and S. Wainger, eds.), Princeton Univ. Press, Princeton (NJ), 2014.

[77] C. De Lellis, F. Ghilardin, and F. Maggi, *A direct approach to Plateau's problem*, ArXiv e-prints, 2014.

[78] P. De Mottoni, and M. Schatzman, *Evolution géométrique d'interfaces*, C. R. Acad. Sci. Sér. I Math **309** (1989), 453-458.

[79] P. De Mottoni, and M. Schatzman, *Geometrical evolution of developed interfaces*, Trans. Amer. Math. Soc. **347** (1995), 1533-1589.

[80] U. Dierkes, S. Hildebrandt, A. Kuster, and O. Wohlrab, *Minimal Surfaces* (three vol.), Springer – Verlag, Berlin, 1992.

[81] U. Dierkes, S. Hildebrandt, and F. Sauvigny, *Minimal Surfaces*, Springer – Verlag, Berlin, 2010.

[82] R. S. Eisenberg, and E. A. Johnson, *Three-dimensional electric field problems in physiology*, Prog. Biophys. Mol. Biol. **20** (1970), 1-65.

[83] C. M. Elliott, M. Paolini, and R. Schätzle, *Interface estimates for the fully anisotropic Allen-Cahn equation and anisotropic mean-curvature flow*, Math. Models Methods Appl. Sci. **6** (1996), 1103-1118.

[84] C. M. Elliott, and R. Schätzle, *The limit of the anisotropic double-obstacle AllenCahn equation*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), 1217-1234.

[85] L. Esposito, N. Fusco, and C. Trombetti, *A quantitative version of the isoperimetric inequality: the anisotropic case*, Ann. Scuola Norm. Sup. Pisa **5** (2005), 619-652.

[86] H. Federer, *Geometric Measure Theory*, Springer – Verlag, Berlin, 1968.

[87] F. Fierro, R. Goglione, M. Paolini, *Numerical simulations of mean curvature flow in presence of a nonconvex anisotropy*, Math. Models Methods Appl. Sci. **8** (1998), 573-601.

[88] R. Finn, *Equilibrium Capillary Surfaces*, Springer – Verlag, New York, 1986.

[89] B. S. Fischer, and R. Finn, *Existence theorems and measurement of the capillary contact angle*, Zeit. Anal. Anwend. **12** (1993), 405-423.

[90] I. Fonseca, *The Wulff theorem revisited*, Proc. Roy. London Soc. **432** (1991), 125-145.

[91] I. Fonseca, and S Müller, *A uniqueness proof for the Wulff theorem*, Proc. Roy. Soc. Edinburgh **119** (1991), 125-136.

[92] G. P. Galdi, *An introduction to the Mathematical Theory of the Navier-Stokes Equations*, Springer, 2011.

[93] Y. Giga, *Surface Evolution Equations. A level set approach.*, Monographs in Mathematics **99**, Birkhäuser, Basel, 2006.

[94] M. H. Giga, and Y. Giga, *Evolving graphs by singular weighted curvature*, Arch. Ration. Mach. Anal. **141** (1998), 117-198.

[95] M. H. Giga, and Y. Giga, *Generalized motion by nonlocal curvature in the plane*, Arch. Ration. Mach. Anal. **159** (2001), 295-333.

[96] Y. Giga, and S. Goto, *Geometric evolutions of phase-boundaries*, "On the Evolution of Phase-Boundaries" (M. Gurtin, and Mc Fadden, eds.), IMA Vols Math. Appl. **43** (1992), Springer, Berlin, 51-66.

[97] Y. Giga, M. Paolini, and P. Rybka, *On the motion by singular interfacial energy*, Japan J. Indust. Appl. Math. **18** (2001), 231-248.

[98] D. Gilbarg, and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2015.

[99] E. N. Gilbert, and H.O. Pollak, *Steiner minimal trees*, SIAM J. Appl. Math. **16** (1969), 1–29.

[100] E. Giusti, *Boundary value problems for non-parametric surfaces of prescribed mean curvature*, Ann. Scuola Norm. Sup. Pisa **3** (1976), 501-548

[101] E. Giusti, *On the equation of surfaces of prescribed mean curvature*, Invent. Math. **46** (1978), 111-137.

[102] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, Birkhäuser, Basel, 1984.

[103] J. Harrison, *Soap film solutions to Plateau's problem*, J. Geom. Anal. **24** (2014), 271-297.

[104] A. E. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2001.

[105] R. Hardt, L. Simon, *Boundary Regularity and Embedded Solutions for the Oriented Plateau Problem*, Ann. of Math. **110** (1979), 439-486.

[106] M. W. Hirsch, *Differential Topology*, Springer – Verlag, New York, 1976.

[107] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), 237-266.

[108] G. Huisken, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equations, **77** (1989), 369-378.

[109] M. T. Hussain, *Cheeger sets for unit cube: analytical and numerical solutions for L^∞ and L^2 norms*, Master Degree's Thesis, Massachusetts Institute of Technology, 2008.

[110] B. Kawohl, and T. Lachand-Robert, *Characterization of Cheeger sets for convex subsets of the plane*, Pacific J. Math. **225** (2006), 103-118.

[111] B. Kawohl, and M. Novaga. *The p-Laplace eigenvalue problem as $p \to 1$ and Cheeger sets in a Finsler metric*, J. Convex Anal. **15** (2008), 623-634.

[112] J. P. Keener, and J. Sneyd, *Mathematical Physiology*, Springer-Verlag, New York, 1998.

[113] D. Krejčiřík, and A. Pratelli, *The Cheeger constant of curved strips*, Pacific J. Math. **254** (2011), 309-333.

[114] G. P. Leonardi, and A. Pratelli, *On the Cheeger sets in strips and non-convex domains*, preprint (2014), available for download at `http://arxiv.org/abs/1409.1376`.

[115] F. Maggi, *Sets of Finite Perimeter and Geometrical Variational Problems: an Introduction to Geometric Measure Theory*, Cambridge Stud. Adv. Math. **135**, Cambridge, 2012.

[116] C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Progr. Math. **290**, Birkhäuser Springer, Basel, 2011.

[117] U. Massari, and M. Miranda, *Minimal Surfaces of Codimension One*, North-Holland Math. Studies, North-Holland, Amsterdam, 1984.

[118] W. S. Massey, *Algebraic Topology, an Introduction* (corr. repr.), Springer – Verlag, New York, 1987.

[119] B. Merriman, J. Bence, and S. Osher, *Diffusion generated motion by mean curvature*, "$\mathcal{AMS}$ Selected Lectures in Mathematics: The Computational Crystal Growers Workshop", Amer. Math. Soc. (1993), 73-83.

[120] M. Miranda, *Superfici cartesiane generalizzate ed insiemi di perimetro localmente finito sui prodotti cartesiani*, Ann. Scuola Norm. Sup. Pisa **3** (1964), 515-542.

[121] M. Miranda, *Superfici minime illimitate*, Ann. Scuola Norm. Sup. Pisa **4** (1977), 313-322.

[122] R. M. Miura, *Accurate computation of the stable solitary wave for the FitzHugh-Nagumo equations*, J. Math. Biol. **13** (1981/82), 247-269.

[123] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Ration. Mech. Anal. **98** (1987), 123-142.

[124] S. Moll, *The anisotropic total variation flow*, Math. Ann. **332** (2005), 177-218.

[125] F. Morgan, *Geometric Measure Theory: a beginner's guide*, Academic Press, Burlington (MA), 2009.

[126] F. Morgan, *Regularity of isoperimetric hypersurfaces in Riemannian manifolds*, Trans. Amer. Math. Soc. **355** (2003), 5041-5052.

[127] Y. Mori, and H. Matano, *Stability of front solutions of the bidomain equation*, preprint.

[128] J. C. Neu, and W. Krassowska, *Homogenization of syncytial tissues*, Crit. Rev. Biomed. Eng. **21** (1993), 137-199.

[129] J. C. C. Nitsche, *Lectures on Minimal Surfaces, vol.1: Introduction, Fundamentals, Geometry and Basic Boundary Problems*, Cambridge Univ. Press, Cambridge, 1989.

[130] M. Novaga, and E. Paolini, *Regularity results for boundaries in $\mathbb{R}^2$ with prescribed anisotropic curvature.* Ann. Mat. Pura Appl. **184** (2005), 239-261.

[131] R. Osserman, *A Survey on Minimal Surfaces*, Van Nostrand, New York, 1969.

[132] M. Paolini, *A quasi-optimal error estimate for a discrete singularly perturbed approximation to the prescribed curvature problem*, Math. Comp. **66** (1997), 45-67.

[133] M. Paolini, *Capillary and calibrability of sets in crystalline mean curvature flow*, Oberwolfach Reports **2** (2005), 560-562.

[134] P. Pozzi, *On the gradient flow of the anisotropic area functional*, Math. Nachr. **285** (2012), 707-726.

[135] A. Quarteroni, and A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, Oxford Univ. Press, Oxford, 1999.

[136] R.T. Rockafellar, *Convex analysis*, Princeton Landmarks in Math., Princeton University Press, Princeton (NJ), 1997.

[137] D. Rolfsen, *Knots and Links*, Mathematics Lecture Series, Houston, 1990.

[138] L. Rudin, S. Osher, and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica D **60** (1992), 259-268.

[139] S. Schaubeck, *Sharp Interface Limits for Diffuse Interface Models*, PhD Thesis, Universität Regensburg, 2013.

[140] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications **44**, Cambridge Univ. Press, Cambridge, 1993.

[141] R. Schoen, L. Simon, and F. J. Almgren, *Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals. I, II*, Acta Math. **139** (1977), 217-265.

[142] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations* American Mathematical Soc., Providence (RI), 1997.

[143] L. Simon, *Survey lectures on minimal submanifolds*, “Seminar on Minimal Submanifolds” (E. Bombieri ed.), Ann. Math. Studies **103** (1083), Princeton Univ. Press, Princeton (NJ), 3-52.

[144] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. **88** (1968), 62-105.

[145] I. Tamanini, *Regularity results for almost minimal oriented hypersurfaces in* $\mathbb{R}^N$, Quad. Dip. Mat. Univ. Salento **1** (1984), 1-92.

[146] J. E. Taylor, *Existence and structure of solutions to a class of nonelliptic variational problems*, Symposia Mathematica **14** (1974), 499-508.

[147] J. E. Taylor, *Unique structure of solutions to a class of nonelliptic variational problems*, Proc. Symp. Pure Math. **27** (1975), 419-427.

[148] J. E. Taylor, *The structure of singularities in soap – bubble – like and soap – film – like minimal surfaces*, Ann. of Math. **103** (1976), 489-539.

[149] J. E. Taylor, *Crystalline variational problems*, Bull. Amer. Math. Soc. **84** (1978), 568-588.

[150] J.E. Taylor, *II-Mean curvature and weighted mean curvature*, Acta Metall. Mater. **40** (1992), 1475–1485.

[151] J. E. Taylor, *Motion of curves by crystalline curvature, including triple junctions and boundary points*, Proc. Symp. Pure Math. **54** (1993), 417-438.

[152] A. C. Thompson, *Minkowski Geometry*, Encyclopedia of Mathematics and its Applications **63**, Cambridge Univ. Press, Cambridge, 1996.

[153] L. Tung, *A bi-domain model for describing ischemic myocardial dc potentials*, PhD Thesis, Massachussets Institue of Technology, 1978.

[154] C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. **72** (1960), 292-311.