SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
Mathematics Area

On degeneracy loci of morphisms between vector bundles

Ph.D. Thesis

Supervisor
Prof. Emilia Mezzetti
Università degli Studi di Trieste

Internal supervisor
Prof. Ugo Bruzzo

Candidate
Fabio Tanturri

Submitted in partial fulfillment of the requirements
for the SISSA degree of "Philosophiæ Doctor"

ACADEMIC YEAR 2012/2013

En güzel deniz: henüz gidilmemiş olanıdır.
En güzel şocuk: henüz büyümedi.
En güzel günlerimiz: henüz yaşamadıklarımız.
Ve sana söylemek istediğim en güzel söz henüz söylememiş olduğum sözdür.

The most beautiful sea hasn't been crossed yet.
The most beautiful child hasn't grown up yet.
The most beautiful days we haven't seen yet.
And the most beautiful words I wanted to tell you I haven't said yet.

Nâzım Hikmet-Ran

You've always been the one
Keeping me forever young
And the best is yet to come.

Scorpions

Contents

Introduction

General background

An algebraic variety is the locus defined by the vanishing of a set of polynomials in several variables, i.e. the set of their common zeroes. Particular algebraic varieties occur when their defining polynomials arise from special situations; for instance, when the polynomials are the minors of fixed order of a matrix M with polynomial entries, the variety is called a *determinantal variety*, and M is a *determinantal representation*. If the minors of M are homogeneous forms, the resulting variety will sit naturally in a projective space.

Many classical algebraic varieties can be seen as determinantal varieties. For instance, rational normal scrolls arise from the maximal minors of $2 \times n$ matrices; other examples are Segre varieties and Veronese varieties. Determinantal varieties are ubiquitous; they are a central topic in both commutative algebra and algebraic geometry, also because of their connections with invariant theory, representation theory and combinatorics. In fact, this topic is the subject of several monographs; let us just mention [Nor76, BV88, Wey03, MR08].

A natural question concerns what kind of properties does a determinantal variety satisfy; for example, one may be interested in its smoothness, its local and global geometry, or the syzygies of the free resolution of its associated ideal. Historically, these problems were faced by means of the Kempf's method, which led to prove that such varieties are in general normal and Cohen-Macaulay; their syzygies were calculated by Lascoux. This method has been recently developed in full generality by Weyman [Wey03], so it is usually referred to as the Kempf-Lascoux-Weyman's method.

If we fix the dimension of a determinantal representation M and we let the coefficients of its entries vary, we obtain families of determinantal varieties. One interesting problem, in this sense, is to parametrize the possible varieties arising this way; one can look at the component $\mathcal{H}$ of the Hilbert scheme and study to what extent such degeneracy loci fill in $\mathcal{H}$. For varieties described by the minors of maximal order, the first contribution in this sense was [Ell75], who proved that determinantal varieties of codimension two and dimension greater than zero are unobstructed and their family is open and dense in $\mathcal{H}$. The series of recent papers [KMMR+01, KMR05, KMR11, Kle11] has shown the same behavior in more determinantal cases, leading to conjecture that this fact should be true in general, for varieties of dimension at least two; the very recent [Kle10, FF10a] both address such a general question.

This determinantal framework can be interpreted as a particular case of morphism between vector bundles on a projective space. Indeed, to give a matrix $M = (m_{ij})$ with $\deg(m_{ij}) = a_j - b_i$, for some sequences of integers $(a_j), (b_i)$, is equivalent to give a map between vector bundles

$$\bigoplus_j \mathcal{O}_{\mathbb{P}^{n-1}}(-a_j) \xrightarrow{M} \bigoplus_i \mathcal{O}_{\mathbb{P}^{n-1}}(-b_i), \tag{1}$$

where n is the number of variables of the polynomials m_{ij}.

More generally, one can consider a morphism between arbitrary vector bundles on an algebraic variety $\mathcal{V}$. By definition, vector bundles trivialize locally, so on an open covering of $\mathcal{V}$ a morphism can be described in terms of matrices of regular functions. The evaluation of one of such matrices in a point x has entries in the base field $\mathbf{k}$, so its rank and corank are perfectly defined. It is natural to consider, then, the set of points in which the rank of the evaluated morphism is not maximal: this is the support of the so-called *degeneracy locus*.

Definition. Given any morphism ϕ between vector bundles on an algebraic variety $\mathcal{V}$, its degeneracy locus is the subscheme in $\mathcal{V}$ locally cut out by the maximal minors of the matrix locally representing ϕ.

One can see that the degeneracy locus of the morphism (1) is exactly the determinantal variety arising from M. In addition to this, degeneracy loci appear in a number of different situations. For example, when one of the two vector bundles is the structure sheaf $\mathcal{O}_{\mathcal{V}}$, the morphism is a (co)section s of the other vector bundle and so its degeneracy locus is the zero locus of s.

Many results are known for general morphisms between vector bundles. For instance, this situation was studied in relation to Schubert varieties, in order to compute the class, in the Chow ring, of a degeneracy locus. This led to the so-called Giambelli-Thom-Porteous formula, which expresses this class as a polynomial in the Chern classes of the vector bundles involved; this formula has been further generalized by Fulton [Ful92]. A comprehensive reference about this topic is Fulton-Pragacz's book [FP98].

General results have also been achieved about the geometry of a degeneracy locus X of a morphism $\mathscr{E} \to \mathscr{F}$, especially regarding the connectedness and non-emptiness of X; for instance, over an algebraically closed field of characteristic zero, a theorem by Fulton and Lazarsfeld [FL81] shows that the ampleness of $\mathscr{E}^* \otimes \mathscr{F}$ and an inequality regarding $\operatorname{rank}(\mathscr{E}), \operatorname{rank}(\mathscr{F}), \dim(\mathcal{V})$ are enough for X to be non-empty and connected. Similar non-emptiness results have been proved also in the cases of symmetric and skew-symmetric morphisms. These kinds of results have been used for example in Brill-Noether Theory, which studies the geometry of the subschemes, in the Jacobian variety of a smooth projective curve, parametrizing special linear series of a fixed degree with dimension bounded below. More on this can be found in [ACGH85].

Families of degeneracy loci can also be investigated. Indeed, one of the main goals of this thesis is to study families of degeneracy loci of morphisms of the form $\mathcal{O}^m_{\mathbb{P}^{n-1}} \to \Omega^1_{\mathbb{P}^{n-1}}(2)$. It is worth noting that these subschemes again include classical objects such as Veronese varieties, Palatini scrolls, and the Segre cubic primal. We refer to the second part of the introduction for an in-depth description of this problem.

So far we have dealt with the properties of degeneracy loci arising from a morphism $\phi : \mathscr{E} \to \mathscr{F}$. It is also interesting to approach an inverse problem, namely to determine sufficient conditions for a subscheme, or a subvariety, to be the degeneracy locus of a suitable ϕ, once fixed $\mathscr{E}$ and $\mathscr{F}$.

The case of determinantal hypersurfaces is the most studied. When the subvariety X has codimension one, a determinantal representation is just a square matrix whose determinant cuts out X, and it is said to be *linear* if its entries are linear forms; linear determinantal representations of curves and surfaces of codimension one and small degree are a classical subject. The case of cubic surfaces was already known in the middle of nineteenth century ([Gra55]); other examples of curves and surfaces were considered in [Sch81]. The general homogeneous forms which can be expressed as linear determinants are determined in [Dic21], where Dickson showed that every plane curve has a determinantal representation. For a detailed historical account, one can refer for instance to [Bea00, Dol12].

Let us mention that, in the case of matrices whose entries are non-necessarily homogeneous, satisfying additional properties (e.g. self-adjoint, symmetric, positive definite matrices), determinantal representations of codimension one affine algebraic varieties have interesting applications in system and control theory, concerning the algebraic and geometric study of linear matrix inequalities (LMIs for short). In Euclidean space, LMIs describe convex semi-algebraic sets; on the Euclidean plane, these sets are proved to be determinantal subvarieties [HV07]. More details about this link between determinantal subvarieties and LMIs can be found, for instance, in [Vin12].

Particular types of determinantal representations are *Pfaffian representations* of hypersurfaces, i.e. (homogeneous) skew-symmetric matrices whose Pfaffian (cfr. Definition 1.4) defines the considered hypersurface. Pfaffian representations are a generalization of determinantal representations, as one can see from the trivial skew-symmetric block matrix having M and $-M^t$ on the antidiagonal and zero on the diagonal.

In linear algebra, Pfaffians have been approached for many purposes; the study of Pfaffian representations is not very developed, though recently it has been strongly reconsidered. We postpone a more detailed historical account to the second part of the introduction.

Main contributions

This thesis is devoted to two problems in the study of degeneracy loci of morphisms between vector bundles on a projective space:

- Hilbert schemes of degeneracy loci of $\mathcal{O}^m_{\mathbb{P}^{n-1}} \to \Omega^1_{\mathbb{P}^{n-1}}(2)$;
- linear Pfaffian representations of cubic surfaces in $\mathbb{P}^3$.

Hilbert schemes of degeneracy loci of $\mathcal{O}^m_{\mathbf{P}(V)} \to \Omega^1_{\mathbf{P}(V)}(2)$

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, let V be a $\mathbf{k}$-vector space of dimension n and $\mathbf{P}(V)$ its projectivization. Degeneracy loci of morphisms of the form $\phi : \mathcal{O}^m_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$ have been considered by several authors, for example Chang [Cha88], Ottaviani [Ott92] and Faenzi, Fania [FF10b].

These varieties were studied already by classical algebraic geometers; see [BM01, FF10b] for a more detailed historical account. For instance, in 1891, Castelnuovo [Cas91] considered the case $(m, n) = (3, 5)$: the degeneracy locus of a general morphism $\phi : \mathcal{O}^3_{\mathbf{P}(V)} \to \Omega_{\mathbb{P}^4}(2)$ is the projected Veronese surface in $\mathbb{P}^4$.

Few years later, Palatini [Pal01, Pal03] focused on $n = 6$. The case $m = 3$ leads to the elliptic scroll surface of degree six, which was further studied by Fano [Fan30]. The case $m = 4$ gives a threefold of degree seven which is a scroll over a cubic surface of $\mathbb{P}^3$, also called Palatini scroll; a conjecture by Peskine states that it is the only smooth threefold in $\mathbb{P}^5$ not to be quadratically normal.

Let us mention also that the case $(m, n) = (4, 5)$ gives rise to the so-called Segre cubic primal, a threefold in $\mathbb{P}^4$ having exactly ten distinct singular points, which has been extensively studied.

As the Hilbert polynomial of X_ϕ is generically fixed, we can define $\mathcal{H}$ as the union of the irreducible components, in the Hilbert scheme of $\mathbf{P}(V)$, containing the degeneracy loci arising from general ϕ's.

Relying on a nice interpretation due to Ottaviani [Ott92, §3.2] (see also Example 2.16), we can identify a morphism of the form above with a skew-symmetric matrix of linear forms in m variables, or with an m-uple of elements in $\Lambda^2 V$; moreover, the natural GL_m-action does not modify its degeneracy locus, so we have a rational map

$$\rho : \mathbf{Gr}(m, \Lambda^2 V) \dashrightarrow \mathcal{H} \tag{2}$$

sending ϕ to X_ϕ.

As an instance of classical results in this direction, let us mention that, if $(m, n) = (3, 5)$, from the results contained in [Cas91] one can prove that the component of $\mathcal{H}$ containing Veronese surfaces in $\mathbb{P}^4$ is birational to $\mathbf{Gr}(3, \Lambda^2 V)$. A similar statement holds for the Palatini scrolls in $\mathbb{P}^5$: the main result of [FM02] states that ρ is birational when $(m, n) = (4, 6)$. In the case $(m, n) = (3, 6)$, however, it was proved in [BM01], and in fact classically known to Fano [Fan30], that ρ is dominant and generically $4 : 1$.

The most recent result has been achieved by Faenzi and Fania [FF10b], who focus on the case in which n is even and the degeneracy locus is smooth, proving the birationality of ρ also in this case.

Our contribution aims for completing the general picture. Our main result is the following.

Theorem. *Let $m, n \in \mathbb{N}$ satisfying $2 \leq m < n - 1$ and let*

$$\rho : \mathbf{Gr}(m, \Lambda^2 V) \dashrightarrow \mathcal{H}$$

be the rational morphism defined above, sending the class of a morphism $\phi : \mathcal{O}^m_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$ to its degeneracy locus X_ϕ in the Hilbert scheme.

i. *If $m \geq 4$ or $(m, n) = (3, 5)$, then ρ is birational; in particular, the Hilbert scheme $\mathcal{H}$ is generically smooth of dimension $m\left(\binom{n}{2} - m\right)$.*

ii. *If $m = 3$ and $n \neq 6$, then ρ is generically injective. Moreover* ii.a. *if n is odd, ρ is dominant on a closed subscheme $\mathcal{H}'$ of $\mathcal{H}$, having* $\operatorname{codim}_{\mathcal{H}} \mathcal{H}' = \frac{1}{8}n(n-3)(n-5)$. *The general element of $\mathcal{H}$ is a general projection in $\mathbf{P}(V)$ of a Veronese surface $v_{\frac{n-1}{2}}(\mathbb{P}^2)$, embedded via the complete linear system of curves of degree $\frac{n-1}{2}$; in particular, $\mathcal{H}$ is irreducible. The general element of $\mathcal{H}'$ is a special projection in $\mathbf{P}(V)$, using as the center of projection the linear space spanned by the partial derivatives of order $\frac{n-5}{2}$ of a non-degenerate polynomial $G \in \mathbf{k}[y_0, y_1, y_2]$ of degree $n-3$;*

ii.b. *if n is even, ρ is dominant on a closed subscheme $\mathcal{H}'$ of $\mathcal{H}$, having* $\operatorname{codim}_{\mathcal{H}} \mathcal{H}' = \frac{3}{8}(n-4)(n-6)$. *The general element of $\mathcal{H}'$ is a projective bundle $\mathbb{P}(\mathscr{G})$ obtained projectivizing a general stable rank-two vector bundle $\mathscr{G}$ on a general plane curve C of degree $\frac{n}{2}$, with determinant* $\det(\mathscr{G}) = \mathcal{O}_C(\frac{n-2}{2})$.

iii. *If $m = 2$ and n is odd, then ρ is dominant but not generically injective. $\mathcal{H}$ is irreducible and its general element is the image in $\mathbf{P}(V)$ of an isomorphism*

$$\mathbb{P}^1 \xrightarrow{[f_1:\ldots:f_n]} \mathbf{P}(V),$$

where $f_1, \ldots, f_n$ are forms of degree $\frac{n-1}{2}$ spanning the whole linear space $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$.

Part i. of the Theorem is the content of Theorem 3.13; the general injectivity of ρ will be proved in Theorem 3.8. In the case $m = 3$, the codimensions of $\mathcal{H}'$ in $\mathcal{H}$ are computed in Proposition 3.15; if n is odd, the characterization of the general element of $\mathcal{H}'$ is performed in Theorem 3.16, while the general element of $\mathcal{H}$ is described in Proposition 3.21. If n is even, this was done in [FF10b]. The case $m = 2$ is entirely discussed in Sect. 3.6.

This theorem gives a complete picture, showing that the case $(m, n) = (3, 6)$ is the unique in which ρ is not generically injective. It shows also that, for $m = 3$, the case $n = 5$ is the only one in which we have birationality. The missing birationality for an odd $n > 6$ can be explained by means of the above description of $\operatorname{Im}(\rho) \subset \mathcal{H}$: the general projection of the Veronese surface is not special in the sense of the Theorem, so it is not in the image of ρ.

The main tool for performing the cohomology computations needed to prove the Theorem is the so-called Kempf-Lascoux-Weyman's method of calculation of syzygies via resolution of singularities; the original idea of Kempf was that the direct image via a map q of a Koszul complex of a resolution of singularities $Y \to X$ can be used to prove results about the defining equations and syzygies of X. This method was successfully used by Lascoux in the case of determinantal varieties, and it is developed in full generality in Weyman's book [Wey03].

The characterization of the general element in $\mathrm{Im}(\rho)$, in the case $m = 3$ and n odd, is proved making use of Macaulay's Theorem on inverse systems [Mac94] and apolarity. We will show that Macaulay correspondence, for plane curves, can be specialized to a correspondence between non-degenerate curves and ideals generated by the Pfaffians of a skew-symmetric matrix.

Linear Pfaffian representations of cubic surfaces

Let $\mathbf{k}$ be a field of characteristic zero, non-necessarily algebraically closed, and let X be the hypersurface in $\mathbb{P}^n_{\mathbf{k}}$ defined by a form $F \in \mathbf{k}[x_0, x_1, \dots, x_n]$ of degree d. One may ask whether the polynomial F^k is the determinant of a matrix M of order kd with linear forms in $\mathbf{k}[x_0, x_1, \dots, x_n]$ as entries, for some integer k.

For $k = 1$, M is a linear determinantal representation, introduced above. Linear determinantal representations of curves and surfaces of small degree are a classical subject and date back to the middle of nineteenth century; we refer again to [Bea00], [Dol12] for a detailed historical account.

A relevant class of matrices with determinant F^2 are Pfaffian representations, that is, skew-symmetric matrices whose Pfaffian (cfr. Definition 1.5) is F, up to constants. As stressed above, Pfaffian representations can be seen as a generalization of determinantal representations.

The references about Pfaffian representations are very recent, even though some general results were probably well-known to the experts before. In [Bea00], Beauville collects many results about determinantal and Pfaffian representations, giving criteria for the existence of linear Pfaffian representations of plane curves, surfaces, threefolds and fourfolds. The fact that a general cubic threefold can be written as a linear Pfaffian had been proved by Adler [AR96, Apx.V], with $\mathbf{k} = \bar{\mathbf{k}}$. With the same method used by Adler, in [IM00] it is proved that a general quartic threefold admits a linear Pfaffian representation. A non-computer-assisted proof of this fact can be found in [BF11].

Again in the case $\mathbf{k} = \bar{\mathbf{k}}$, linear Pfaffian representations of plane curves and their elementary transformations are the subject of [BK11] and [Buc10]; in [Fae07] and [CF09], respectively almost quadratic and almost linear Pfaffian representations of surfaces are considered. In [CKM12] it is proved that every smooth quartic surface admits a linear Pfaffian representation, a result which strengthens the Beauville-Schreyer's one in [Bea00].

We will focus here on the case $d = 3$, i.e. we will deal with cubic surfaces in $\mathbb{P}^3_{\mathbf{k}}$.

Beauville [Bea00] showed that, when $\mathbf{k} = \bar{\mathbf{k}}$, the existence of a linear Pfaffian representation for a smooth cubic surface $\mathbb{S}$ is equivalent to the existence of five points on $\mathbb{S}$ in general position in $\mathbb{P}^3$. In particular, this implies that every smooth cubic surface admits a linear Pfaffian representation.

This result has been generalized by Fania and Mezzetti [FM02], who proved that in fact *any* cubic surface admits a linear Pfaffian representation.

Our contribution is the following. We study how to construct *explicitly* a linear Pfaffian representation, when $\mathbf{k}$ is not necessarily algebraically closed, starting from the least amount of initial data possible.

Let us give the following definition.

Definition. Let $F \in \mathbf{k}[x_0, x_1, \ldots, x_n]$ define the hypersurface X and let $\mathbf{k}'$ be a field containing $\mathbf{k}$. A *linear Pfaffian* $\mathbf{k}'$*-representation* of X is a skew-symmetric matrix whose Pfaffian is F, up to constants, and whose entries are linear forms in $\mathbf{k}'[x_0, x_1, \ldots, x_n]$.
If a point $\mathbf{a} \in \mathbb{P}^n_{\mathbf{k}}$ admits a representative $\underline{a} \in \mathbb{A}^{n+1}_{\mathbf{k}}$, then it will be called a $\mathbf{k}$*-point*.

By convention, a hypersurface X will be considered in $\mathbb{P}^n_{\bar{\mathbf{k}}}$, being $\bar{\mathbf{k}}$ the algebraic closure of $\mathbf{k}$. In this way, X is non-empty even if its defining polynomial $F \in \mathbf{k}[x_0, x_1, \ldots, x_n]$ has no zero in $\mathbb{A}^{n+1}_{\mathbf{k}}$, that is, if X has no $\mathbf{k}$-points.

With these notations, our main theorem is the following.

Theorem.

i. *Every cubic surface $\mathbb{S}$ in $\mathbb{P}^3_{\bar{\mathbf{k}}}$, with equation $F \in \mathbf{k}[x_0, \ldots, x_3]_3$, admits a linear Pfaffian $\mathbf{k}'$-representation, $\mathbf{k}'$ being an algebraic extension of $\mathbf{k}$ of degree $[\mathbf{k}' : \mathbf{k}] \leq 6$. Moreover, it is possible to explicitly realize such a representation.*

ii. *If $\mathbb{S}$ is neither reducible nor a cone, then it is possible to construct explicitly a linear Pfaffian $\mathbf{k}'$-representation of $\mathbb{S}$, where $\mathbf{k}'$ is an algebraic extension of $\mathbf{k}$ of degree $[\mathbf{k}' : \mathbf{k}] \leq 3$. Moreover, if $\mathbf{k} \subseteq \mathbb{R}$, then also $\mathbf{k}'$ can be chosen so.*

iii. *Let $\mathbb{S}$ be neither reducible nor a cone. Given a $\mathbf{k}$-point $\mathbf{a^1}$ which is not a T-point, it is possible to construct explicitly a linear Pfaffian $\mathbf{k}$-representation of $\mathbb{S}$.*

Not to be a T-point is a mild condition which will be defined later (Definition 4.11). It is indeed mild: for instance, if $\mathbb{S}$ is smooth, then all points on $\mathbb{S}$ but possibly a finite number are not T-points.

The first part of the Theorem is contained in Theorem 4.25; the second part is given by Proposition 4.22, while the last part is proved in Theorem 4.19.

On the one hand, these results give a bound for the degree of algebraic extension required to ensure the existence of a linear Pfaffian representation. On the other hand, they are *constructive*: it is possible to implement an algorithm which produces a linear Pfaffian representation, provided the requested inputs.

The proof of the Theorem is based firstly on the tangent plane process, a classical argument (see for instance [Seg51]) which makes us able to produce five points in general position on $\mathbb{S}$. Then we make use of Buchsbaum-Eisenbud Structure Theorem (Theorem 1.8 below) to construct a linear Pfaffian representation.

Structure of the thesis

The structure of the thesis is the following.

In Ch. 1 we provide several preliminaries. In Sect. 1.1 we collect Künneth and Bott formulas for future computations, we recall what a Hilbert scheme is together with its basic properties and we describe projective bundles and projectivizations of sheaves. Sect. 1.2 is pledged to Grothendieck's spectral sequence and to the hypercohomology of the direct image functor, while in Sect. 1.3 we deal with skew-symmetric matrices, Gorenstein ideals and apolarity.

In Ch. 2 we provide some general results about degeneracy loci of morphisms between vector bundles on projective spaces. Most of them in fact are to be used in Ch. 3 and have been inserted for future references; still, they provide a general picture of the subject and are proved in some generality. We believe that most of the contents collected here are known to the experts, even though there seems not to exist a good reference for such a collection of general results. The aim of this chapter is to partially fill this lack, with an eye towards the consecutive chapter.
After defining the degeneracy locus X of a morphism $\phi : \mathscr{E} \to \mathscr{F}$, in Sect. 2.1 we give some results about the dimension and codimension of X and its singularities; then we show that, under some hypotheses, X is a normal, Cohen-Macaulay and reduced. In Sect. 2.2, we show that X is birational to

Y, the (smooth) zero locus of a section of a vector bundle on the projective bundle $\mathbb{P}(\mathscr{F})$. In Sect. 2.3 we give some properties of the sheaf $\mathrm{coker}(\phi)$, showing that it is reflexive, Cohen-Macaulay but not arithmetically Cohen-Macaulay in general. In Sect. 2.4 we analyze the special case of a morphism given by a tritensor, while in Sect. 2.5 we illustrate a way to compute the cohomology groups of some sheaves supported on X. This is performed by means of the direct image of the twisted Koszul complex resolving Y on $\mathbb{P}(\mathscr{F})$: this method is usually referred to as the Kempf-Lascoux-Weyman's method of calculation of syzygies via resolution of singularities (see above). Finally, we give a characterization of the normal sheaf of X in Sect. 2.6.

In Ch. 3 we study the Hilbert scheme $\mathcal{H}$ of degeneracy loci of morphisms of the type $\mathcal{O}^m_{\mathbb{P}^{n-1}} \to \Omega^1_{\mathbb{P}^{n-1}}(2)$. In Sect. 3.1, we introduce some notations and we define explicitly the map ρ in (2); in Sect. 3.2, we provide a geometric interpretation of these degeneracy loci. In Sect. 3.3 we produce an upper bound for the dimension of the space of global sections of the normal sheaf of X in $\mathbf{P}(V)$, performed by means of the Kempf-Lascoux-Weyman's method. In Sect. 3.4 we prove the injectivity (Theorem 3.8) and birationality (Theorem 3.13) of ρ, which are the main results of the chapter. Finally, in Sect. 3.5 we study the cases $m = 3$ and $m = 2$; for $m = 3$, we give in Theorem 3.16 a geometric description of the points of $\mathrm{Im}(\rho)$, and in Proposition 3.21 a geometric description of the general element of $\mathcal{H}$. For $m = 2$, we do the same in Sect. 3.6, showing that ρ is dominant but not generically injective.

In Ch. 4 we focus on Pfaffian representations of cubic surfaces. After giving a detailed introduction of the problem in Sect. 4.1, in Sect. 4.2 we retrace the proof of Buchsbaum-Eisenbud Structure Theorem and we use it to construct a skew-symmetric matrix $\mathbb{T}$ whose Pfaffians generate the ideal of the four fundamental points and the unit point in $\mathbb{P}^3$. This enables us to produce Algorithm 4.6, whose inputs are five points in general position on a surface $\mathbb{S}$ and whose output is a linear Pfaffian representation of $\mathbb{S}$. In Sect. 4.3, we make use of the tangent plane process, a classical argument (see, for example, [Seg51]); starting from a $\mathbf{k}$-point $\mathbf{a^1}$ on an irreducible surface which is not a cone, we show that it is always possible to find four other points on the surface such that all the five points are in general position, provided that $\mathbf{a^1}$ is not a T-point. In Sect. 4.4 we summarize the previous results in Theorem 4.19 and Proposition 4.22, which are our main contributions. Then we discuss the case of reducible surfaces and the case of cones, in order to prove Theorem 4.25. An example of the construction of a Pfaffian representation is finally given.

Acknowledgments

First and foremost, the author wishes to thank Prof. Emilia Mezzetti and Prof. Daniele Faenzi for their guidance during his Ph.D., for suggesting him these problems and for their help regarding this thesis. He would also like to thank Prof. Frank-Olaf Schreyer for interesting remarks about apolarity, and Prof. Ugo Bruzzo for his support. Finally, he is grateful to Giangiacomo Sanna for very useful discussions.

The author is indebted to the International School for Advanced Studies for funding his Ph.D. and to all the programs which supported him in attending many schools, workshops and conferences: the Research Network Program “GDRE-GRIFGA”, the ANR project GeoLMI, the PRIN 2008 “Geometria delle varietà algebriche e dei loro spazi di moduli” and the PRIN 2010/2011 “Geometria delle varietà algebriche”.

Finally, the author is grateful to the Università degli Studi di Trieste and to the Université de Pau et des Pays de l’Adour for their hospitality during these years.

Chapter 1

Preliminaries

In this first chapter we collect some facts which will be used throughout the thesis. Most of them are known or well-known, and are gathered here for future references.

Notations. Throughout the manuscript, $\mathbf{k}$ will be a field. If not otherwise stated, $\mathbf{k}$ is non-necessarily algebraically closed or of characteristic zero.

We will denote by V a $\mathbf{k}$-vector space of positive dimension $n \in \mathbb{N}$; by $\mathbf{P}(V) \cong \mathbb{P}^{n-1}_{\mathbf{k}}$ we will mean the projective space of its one-quotients, i.e. for instance $\mathrm{H}^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1)) \cong V$.

1.1 General facts

1.1.1 Cohomology tools

Let V_1, V_2 be two separated varieties and $\mathcal{F}_1$, $\mathcal{F}_2$ two quasi-coherent sheaves on X_1, X_2 respectively. Let p_1, p_2 be the two natural projections from the product $X_1 \times X_2$ to the factors. When no confusion can arise, we will denote by $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ the tensor product $p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2$.

Künneth formula. *With the notations above, for any $k \in \mathbb{Z}$ one has*

$$\mathrm{H}^k(X_1 \times X_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong \bigoplus_{i+j=k} \mathrm{H}^i(X_1, \mathcal{F}_1) \otimes_{\mathbf{k}} \mathrm{H}^j(X_2, \mathcal{F}_2).$$

For a proof of the Künneth formula in the algebraic geometry framework, one can look for instance at [Gro63, Théorème 6.7.8] or [Kem93, Proposition 9.2.4].

We will denote by $\Omega^p_{\mathbf{P}(V)}$ the p-th exterior power of the cotangent bundle $\Omega_{\mathbf{P}(V)} = \Omega^1_{\mathbf{P}(V)}$. Let $\mathbf{k} = \bar{\mathbf{k}}$ and $\operatorname{char} \mathbf{k} = 0$. Being $\Omega_{\mathbf{P}(V)}$ a homogeneous vector bundle, its cohomology groups can be computed by means of the Borel-Weil-Bott Theorem. This leads to the following formula.

Bott formula ([Bot57]). *Let* $\mathrm{char}(\mathbf{k}) = 0$ *and* $\mathbf{k} = \bar{\mathbf{k}}$. *Then*

$$\mathrm{h}^i(\mathbf{P}(V), \Omega^p_{\mathbf{P}(V)}(k)) = \begin{cases} \binom{k+n-p-1}{k}\binom{k-1}{p} & \text{for } i = 0, 0 \le p \le n-1, k > p \\ 1 & \text{for } k = 0, 0 \le p = i \le n \\ \binom{-k+p}{-k}\binom{-k-1}{n-p-1} & \text{for} \begin{cases} i = n, 0 \le p \le n-1 \\ k < p-n+1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

1.1.2 Hilbert schemes

In Ch. 3 we will deal with the Hilbert scheme of subschemes of $\mathbf{P}(V)$.

Theorem 1.1. *Let X be a closed subscheme of $\mathbf{P}(V)$. Then*

i. *there exists a projective scheme H, called the* Hilbert scheme*, which parametrizes closed subschemes of $\mathbf{P}(V)$ with the same Hilbert polynomial P as X, and there exists a universal subscheme $W \subseteq \mathbf{P}(V) \times H$, flat over H, such that the fibers of W over closed points $h \in H$ are all closed subschemes of $\mathbf{P}(V)$ with the same Hilbert polynomial P. Furthermore, H is universal in the sense that if T is any other scheme, and if $W' \subseteq \mathbf{P}(V) \times T$ is a closed subscheme, flat over T, all of whose fibers are subschemes of $\mathbf{P}(V)$ with the same Hilbert polynomial P, then there exists a unique morphism $T \to H$ such that $W' = W \times_H T$ as subschemes of $\mathbf{P}(V) \times T$;*

ii. *the Zariski tangent space to H at the point $h \in H$ corresponding to X is given by $\mathrm{H}^0(X, \mathcal{N})$, where $\mathcal{N}$ is the normal sheaf of X in $\mathbf{P}(V)$;*

iii. *if X is a locally complete intersection, and if $\mathrm{h}^1(X, \mathcal{N}) = 0$, then H is non-singular at h, of dimension $\mathrm{h}^0(X, \mathcal{N})$.*

This theorem is due to Grothendieck [Gro62]; for modern references, one can look at [Ser06] or [Har10].

1.1.3 Projectivizations of sheaves

Let us recall some basic facts about projectivizations of sheaves and projective bundles; we are not interested in being exhaustive, so one can refer for instance to [Gro61a, §4] for more details.

Let X be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on X. Its symmetric algebra $\mathrm{Sym}\,\mathcal{F}$ is the sheaf $\bigoplus_{k \ge 0} S^k\mathcal{F}$, where S^k denotes the k-th symmetric power of $\mathcal{F}$. We can define $\mathbb{P}(\mathcal{F})$ as the scheme $\mathrm{Proj}(\mathrm{Sym}\,\mathcal{F})$; when $\mathcal{F}$ is a vector bundle, $\mathbb{P}(\mathcal{F})$ is called a projective bundle. $\mathbb{P}(\mathcal{F})$ is equipped with a relative ample line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$, which is usually denoted by $\mathcal{O}_{\mathcal{F}}(1)$.

There is a natural morphism $\pi : \mathbb{P}(\mathcal{F}) \to X$, which has the property

$$\pi_* \mathcal{O}_{\mathcal{F}}(n) = S^n \mathcal{F}$$

for all $n \in \mathbb{N}$; in particular, $\pi_* \mathcal{O}_{\mathcal{F}}(1) = \mathcal{F}$.

The projectivization reverses inclusion, i.e. if $\mathcal{F} \to \mathcal{F}'$ is a surjection, then we have a closed immersion $\mathbb{P}(\mathcal{F}') \to \mathbb{P}(\mathcal{F})$ and

$$\mathcal{O}_{\mathcal{F}}(1)|_{\mathbb{P}(\mathcal{F}')} = \mathcal{O}_{\mathcal{F}'}(1).$$

There is a short exact sequence

$$0 \longrightarrow \Omega \longrightarrow \pi^* \mathcal{F} \longrightarrow \mathcal{O}_{\mathcal{F}}(1) \longrightarrow 0, \tag{1.1}$$

where Ω is defined as the kernel of the surjection above. This notation is justified by the fact that Ω is the twisted relative cotangent sheaf $\Omega_{\mathbb{P}(\mathcal{F})/X}(1)$, when $\mathcal{F}$ is a vector bundle.

Example 1.2. One important case occurs when $\mathcal{F} = \mathcal{O}_X \otimes V$ for some vector space V. In this case, one has $\mathbb{P}(\mathcal{F}) = X \times \mathbf{P}(V)$. This always happens locally when $\mathcal{F}$ is locally free.

Example 1.3. Another remarkable situation occurs when $\mathcal{F}$ is a sheaf of ideals $\mathcal{I}_{Y/X}$, being Y a subscheme of X. In this case, $\mathbb{P}(\mathcal{I}_{Y/X})$ turns out to be the blow-up of X along Y (see, for example, [EH00, Theorem IV-23]).

1.2 Spectral sequences

1.2.1 Grothendieck's spectral sequence

In [Gro57] Grothendieck introduced a spectral sequence associated to the composition of two functors.

The general setup is the following: let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Suppose we have a left exact functor $G : \mathcal{A} \to \mathcal{B}$ and a left exact functor $F : \mathcal{B} \to \mathcal{C}$, such that G sends injective objects of $\mathcal{A}$ to F-acyclic objects of $\mathcal{B}$. Then there exists a convergent first quadrant cohomological spectral sequence for each object in $\mathcal{A}$:

$$E_2^{i,j} = (R^i G \circ R^j F)(-) \Rightarrow R^{i+j}(G \circ F)(-). \tag{1.2}$$

More details on spectral sequences in general can be found for instance in [Wei94], as well as a proof of the Grothendieck's spectral sequence.

1.2.2 Hypercohomology of the direct image functor

Let $q : \mathscr{X} \to \mathscr{Y}$ be a morphism of schemes and let $K_\bullet$ be a complex of $\mathcal{O}_{\mathscr{X}}$-modules. One can consider the *hypercohomology* of the functor q_* with respect to $K_\bullet$. The two spectral sequences

$$'E_2^{i,j} = \mathscr{H}^j(R^i q_*(K_\bullet)) \text{ with differentials } 'E_2^{i,j} \to {}'E_2^{i-1,j+2}, \tag{1.3}$$

$$''E_2^{i,j} = R^i q_*(\mathscr{H}^j(K_\bullet)) \text{ with differentials } ''E_2^{i,j} \to {}''E_2^{i+2,j-1}, \tag{1.4}$$

abut to the cohomology of the bicomplex of $\mathcal{O}_{\mathscr{Y}}$-modules $q_*(\mathcal{I}^{\bullet\bullet})$, where $\mathcal{I}^{\bullet\bullet}$ is an injective Cartan-Eilenberg resolution of $K_\bullet$ [Gro61b, §12.4]. Here, we denote by $\mathscr{H}^k(-)$ the derived identity functor, taking complexes to complexes (concentrated in only one degree) as

$$\mathscr{H}^k\left(\dots \xrightarrow{d_{i-1}} K_i \xrightarrow{d_i} \dots\right) = \ker(d_k)/\operatorname{Im}(d_{k-1})$$

and by $R^k q_*$ the functor taking complexes to complexes

$$R^k q_*\left(\dots \xrightarrow{d_{i-1}} K_i \xrightarrow{d_i} \dots\right) = \left(\dots \xrightarrow{R^k q_*(d_{i-1})} R^k q_*(K_i) \xrightarrow{R^k q_*(d_i)} \dots\right).$$

We can see their convergence from the following observation. On the one hand, if we replace in (1.2) G with the identity functor in the category of complexes of $\mathcal{O}_{\mathscr{X}}$-modules and F with q_*, we get (1.3) $\Rightarrow R^{j+i}q_*(-)$. On the other hand, if we replace G with q_* and F with the identity functor in the category of complexes of $\mathcal{O}_{\mathscr{Y}}$-modules, we obtain (1.4) $\Rightarrow R^{i+j}q_*(-)$ as well.

1.3 Skew-symmetric matrices and apolarity

In this section we recall some basic algebraic definitions and some known results as Buchsbaum-Eisenbud Structure Theorem and Macaulay's Theorem on inverse systems.

Throughout the section, R will be the polynomial ring $\mathbf{k}[y_0, \dots, y_{m-1}]$ for some integer $m \geq 3$. Its maximal homogeneous ideal $(y_0, \dots, y_{m-1})$ will be denoted by $\mathscr{M}$. Sometimes we will set U to be the $\mathbf{k}$-vector space with basis $\{y_0, \dots, y_{m-1}\}$; in such way, we may identify R with $\mathrm{H}^0(\mathbf{P}(U), \mathcal{O}_{\mathbf{P}(U)}(1))$.

1.3.1 Pfaffians and Gorenstein ideals

Definition 1.4 (Pfaffian)**.** Let $T = (t_{ij})$ be a skew-symmetric matrix of even size $2k$ with entries in a commutative, unitary ring A. Then its determinant is the square of an element in A, called the *Pfaffian* of T.
If we denote by T_{ij} the square matrix of order $(2n-2)$ obtained by deleting from T the i-th and j-th rows and columns, the Pfaffian is defined recursively as

$$\mathrm{Pf}(T) = \begin{cases} \sum_{j<2k} (-1)^j t_{2k,j} \,\mathrm{Pf}(T_{2k,j}) & \text{if } k \geq 2 \\ t_{12} & \text{if } k = 1. \end{cases} \tag{1.5}$$

The submatrices T_{ij} above are skew-symmetric because they are obtained by deleting the rows and columns of the same index, i.e. they are principal submatrices. One can do the same for a skew-symmetric matrix T of odd order $2k-1$: its $(2k-2) \times (2k-2)$ (principal) Pfaffians are the Pfaffians of the skew-symmetric matrices T_i obtained by deleting the i-th row and column.

For the sake of clearness, we recall some basic definitions.

Definition 1.5 (codimension)**.** Let $I \subset A$ be an ideal. If I is prime, then we define $\mathrm{codim}_A(I)$ to be the supremum of lengths of chains of primes descending from I. If I is not prime, then its codimension is defined as the minimum of the codimensions of the primes containing I.

Definition 1.6 (depth, Gorenstein ideal)**.** Let $I \subset A$ be an ideal. Let M be a finitely generated A-module. A sequence of elements $x_1, \dots, x_r \in I$ is called a *regular M-sequence contained in I* if

i. $(x_1, \dots, x_r)M \neq M$;

ii. for i such that $1 \leq i \leq r$, x_i is a non-zero divisor on $M/(x_1, \dots, x_{i-1})M$.

The natural number $\mathrm{depth}(I, M)$ is the length of any maximal regular M-sequence contained in I.
The ideal I is said to be Gorenstein if

$$\mathrm{depth}(I, A) = \mathrm{pd}_A(A/I) = a \qquad \text{and} \qquad \mathrm{Ext}^a_A(A/I, A) \cong A/I \tag{1.6}$$

for some $a \in \mathbb{N}$, where pd denotes the projective dimension.

When $A = R$ and I is homogeneous, $\mathrm{depth}(I, R)$ and $\mathrm{codim}_R(I)$ agree (cfr. for instance [Eis95, Theorem 18.7]).

Proposition 1.7. *I is a Gorenstein ideal if and only if R/I is a Gorenstein ring.*

Proof. If I is Gorenstein, then R/I is Cohen-Macaulay by [Eis95, Exercise 19.9]. Being the canonical module ω_R equal to R [Eis95, §21.3], by [Eis95, Theorem 21.15] $\omega_{R/I} \cong R/I$ and so R/I is Gorenstein.
Conversely, let $c = \operatorname{codim}_R(I)$. A Gorenstein ring is Cohen-Macaulay, so we get $\operatorname{Ext}^c_R(R/I, R) \cong R/I$ by [Eis95, Corollary 21.16] and $\operatorname{pd}_A(A/I) = c$ by [Eis95, Corollary 19.15]. $\square$

The following theorem establishes a link between Gorenstein ideals of depth three and skew-symmetric matrices.

Theorem 1.8 (Buchsbaum-Eisenbud Structure Theorem [BE77])**.**

i. *Let $n \geq 3$ be an odd integer, and let M be a free R-module of rank n. Let $N : M \to M^*$ be an alternating map of rank $n-1$ whose image is contained in $\mathscr{M} \cdot R$ and let $I = \operatorname{Pf}_{n-1}(N)$ be the ideal generated by the $(n-1) \times (n-1)$ Pfaffians of the matrix representing N. If* $\operatorname{depth}(I, R) = 3$*, then I is Gorenstein, and the minimal number of generators of I is n.*

ii. *Every Gorenstein ideal I of R with* $\operatorname{depth}(I, R) = 3$ *arises as in* i..

Remark 1.9. When the entries of the matrix representing N are linear forms in $\mathbf{k}[y_0, \dots, y_{m-1}]$, the hypothesis on the depth is generally satisfied. In other words, if n is odd and $N : \mathcal{O}^n_{\mathbf{P}(U)} \to \mathcal{O}^n_{\mathbf{P}(U)}(1)$ is a general skew-symmetric matrix of linear forms, the subscheme in $\mathbf{P}(U)$ cut out by the $(n-1) \times (n-1)$ Pfaffians of N has codimension three.

1.3.2 Apolarity and Macaulay correspondence

For the rest of this section, let $\mathbf{k}$ be of characteristic zero.

Let $S = \mathbf{k}[\partial_0, \dots, \partial_{m-1}]$ be the ring of differential operators dual to R; i.e., R acts on S (and conversely) by differentiation:

$$y^\alpha(\partial^\beta) = \alpha! \binom{\beta}{\alpha} \partial^{\beta - \alpha} \tag{1.7}$$

if $\beta \geq \alpha$ and 0 otherwise. Here α and β are multi-indices, $\alpha! = \prod \alpha_i!$, $|\alpha| = \sum \alpha_i$, $\binom{\beta}{\alpha} = \prod \binom{\beta_i}{\alpha_i}$ and $\beta \geq \alpha$ if and only if $\beta_i \geq \alpha_i$ for all i. The perfect pairing between forms of degree d and homogeneous differential operators of the same degree is known as *apolarity*.

Definition 1.10 ([Iar84, Syl86])**.** Let $G \in R$ be a homogeneous form of degree $2k$. Having chosen a basis $\{D_i\}$ of S_k, the square matrix $\operatorname{Cat}(G)$ of

order $\binom{k+m-1}{k}$, whose (i,j)-th element is $D_i(D_j(G))$, is called its *catalecticant* matrix.
The form G is said to be *non-degenerate* if $\mathrm{Cat}(G)$ has maximal rank or, equivalently, if the elements $\{\partial^\alpha(G)\}_{|\alpha|=k}$ are linearly independent in the vector space R_k.

We can define, in any degree d, the (dual of the) orthogonal complement of a space of polynomials: given a vector subspace of R_d, its orthogonal complement in S_d is made up by the differential operators which annihilate all the elements in the subspace, and conversely.

Recall that an Artinian ring is a ring satisfying the descending chain condition on ideals. If I is an irrelevant ideal of R, R/I is Artinian. In this case, the following are equivalent:

i. R/I is Gorenstein;

ii. the $\mathbf{k}$-vector space $\{x \in R/I$ such that $\mathscr{M}/I \cdot x = 0\}$ (usually called the *socle* of R/I) is one-dimensional.

The proof of this equivalence can be found, for instance, in [Hun99].

Let us consider now an ideal I of R such that R/I is an Artinian, Gorenstein ring with (one-dimensional) socle in degree k; as $\mathrm{Hilb}(R/I,k) = 1$, there is a homogeneous differential operator $F \in S$ of degree k, determined up to scalar, satisfying $G(F) = 0$ for any $G \in I$. F is usually called the *dual socle generator.*

Conversely, given a form $F \in S$ of degree k, we can define $F^\perp$ as the (homogeneous, irrelevant) ideal in R whose elements G satisfy the property $G(F) = 0$. The ring $R/F^\perp$ is usually denoted by A^F. The ideal $F^\perp$ can be described in terms of the derivatives of F, as follows.

Proposition 1.11. *Let $F \in S$ of degree k. For any $d \leq k$, the homogeneous component $F^\perp \cap R_d$ is the orthogonal complement of the space of partial derivatives of order $k-d$ of F.*

Proof. We have to show that for all $D \in R_d$

$$D(F) = 0 \qquad \Longleftrightarrow \qquad D(y^\alpha(F)) = 0 \qquad \forall\ |\alpha| = k-d.$$

Firstly we remark that by apolarity, for a form $F' \in S$ of degree $k-d$, one has

$$y^\alpha(F') = 0 \qquad \forall\ |\alpha| = k-d \qquad \Longleftrightarrow \qquad F' = 0.$$

Consider now $D \in R$ of degree d. Since $D(y^\alpha(F)) = y^\alpha(D(F))$, it is enough to apply the previous remark to $F' = D(F)$. $\square$

The two correspondences described above are inverse to each other by the following Macaulay's Theorem on inverse systems.

Theorem 1.12 ([Mac94]). *The map $F \mapsto A^F$ gives a bijection between hypersurfaces $\mathrm{V}(F) \subset \mathbf{P}(U)$ of degree k and Artinian graded Gorenstein quotient rings of R with socle in degree k.*

Chapter 2

Morphisms between vector bundles and degeneracy loci

In this chapter we provide some general results about degeneracy loci of morphisms between vector bundles on projective spaces. Most of them in fact are to be used in Ch. 3 and have been inserted for future references; still, they provide a general picture of the subject and are proved in some generality. We believe that most of the contents collected here are known to the experts, even though there seems not to exist a good reference for such a collection of general results. The aim of this chapter is to partially fill this lack, with an eye towards the next chapter.

Throughout this chapter, $\mathbf{k}$ will be supposed to be algebraically closed and of characteristic zero, though Proposition 2.5, the first part of Sect. 2.4 and the contents of Sect. 2.5 are also valid in different characteristics and over fields which are not algebraically closed.

We will denote by $\mathscr{E}$ and $\mathscr{F}$ two vector bundles on the projective space $\mathbf{P}(V)$ and by φ a morphism between them. We will care about distinguishing, from time to time, *any* choice of φ from a *general* choice of φ.

We will set $\operatorname{rank}(\mathscr{E}) = e$ and $\operatorname{rank}(\mathscr{F}) = f$, with $e \geq f$. Since we are interested in degeneracy loci of morphisms, this choice is not restrictive, as shown by the forthcoming Definition 2.1.

Note that we have denoted the morphism by φ instead of ϕ, used so far, to stress that we are supposing $e \geq f$. We will use again ϕ when no assumption on the ranks is taken.

Given any morphism φ, its kernel and cokernel are defined by the following exact sequence of sheaves:

$$0 \longrightarrow \mathscr{K}_\varphi \longrightarrow \mathscr{E} \xrightarrow{\varphi} \mathscr{F} \longrightarrow \mathscr{C}_\varphi \longrightarrow 0 \,. \tag{2.1}$$

When no confusion can arise, we will denote the kernel and the cokernel simply by $\mathscr{K}$, respectively $\mathscr{C}$.

2.1 Degeneracy loci

Definition 2.1. Given any morphism $\varphi : \mathscr{E} \to \mathscr{F}$, its *degeneracy locus* X_φ is the subscheme in $\mathbf{P}(V)$ locally cut out by the maximal minors of the matrix locally representing φ.

As before, we will often drop the subscript and denote the degeneracy locus simply by X.

Let us remark that the scheme X_φ is supported on the set of points $x \in \mathbf{P}(V)$ in which the evaluated morphism $\varphi(x)$ has not maximal rank. We observe that the assumption $e \geq f$ on the ranks of the vector bundles $\mathscr{E}$, $\mathscr{F}$ is not restrictive: indeed, one has clearly $X_\varphi = X_{\varphi^t}$, where φ^t denotes the transposed (dual) morphism.

It is possible to generalize the previous definition to further degeneracy loci in a natural way.

Definition 2.2. For any $k \in \mathbb{N}$, we define $D_k(\varphi)$ as the subscheme in $\mathbf{P}(V)$ cut out by the minors of order $k+1$ of the matrix locally representing φ. One has $D_{f-1}(\varphi) = X_\varphi$. By convention, $D_f(\varphi) = D_{f+1}(\varphi) = \ldots = \mathbf{P}(V)$.

The following theorem provides a relation between D_{k-1} and the singularities of D_k.

Theorem 2.3 ([Băn91, §4.1])**.** *Let $\mathscr{E}$ and $\mathscr{F}$ be as above and let $\mathscr{E}^* \otimes \mathscr{F}$ be globally generated. Then, for a general morphism $\varphi : \mathscr{E} \to \mathscr{F}$, the subschemes $D_k(\varphi)$ either are empty or have pure codimension $(e-k)(f-k)$ in $\mathbf{P}(V)$. Moreover, we have that* $\operatorname{Sing}(D_k(\varphi)) = D_{k-1}(\varphi)$.

From now on, we will suppose X to be non-empty, in particular we will assume $e - f + 1 \leq n - 1$. By the previous theorem, in the case $\mathscr{E}^* \otimes \mathscr{F}$ globally generated, we know dimension and codimension of X_φ for a general φ. Moreover, we have

$$\operatorname{codim}_X(\operatorname{Sing}(X)) = 2(e-f+2) - (e-f+1) \geq 3. \tag{2.2}$$

Furthermore, we know that X_φ is smooth when $2(e-f+2) > n-1$.

So far we have dealt with X_φ as a subscheme; actually, we can identify X_φ with its support, as it has a reduced structure. Moreover, it is normal and Cohen-Macaulay.

Proposition 2.4. *Let $\mathscr{E}^* \otimes \mathscr{F}$ be globally generated. Then, for the general φ, X_φ is normal, Cohen-Macaulay and reduced.*

Proof. Being Cohen-Macaulay is a local property, but X_φ is locally a general determinantal subscheme, and they are known to be Cohen-Macaulay. Alternatively, one can argue as in [ACGH85, (4.1)].
$\mathrm{Sing}(X_\varphi)$ is a proper closed subscheme of X_φ, so the latter is generically smooth; X_φ has no embedded components, so this is enough to show that it is reduced.
It remains to prove that X_φ is normal. This follows at once by Serre's criterion (see for instance [Mat89, Theorem 23.8]): indeed, any local ring of X_φ is Cohen-Macaulay, hence S_2, and by (2.2) it is also regular in codimension (at least) one, hence R_1. □

2.2 $\mathbb{P}(\mathscr{C})$ as the zero locus of a section of a vector bundle

Let φ be general. Consider $\mathbb{P}(\mathscr{C})$, the projectivization of $\mathscr{C}$. The projectivization reverses inclusions, so we have a natural closed embedding $\mathbb{P}(\mathscr{C}) \subset \mathbb{P}(\mathscr{F})$. We call $q : \mathbb{P}(\mathscr{F}) \to \mathbf{P}(V)$ and $\bar{q} : \mathbb{P}(\mathscr{C}) \to \mathbf{P}(V)$ the usual maps arising from the projectivizations of $\mathscr{F}$, $\mathscr{C}$. The following diagram commutes:

$$\begin{tikzcd} \mathbb{P}(\mathscr{F}) \arrow[r, "q"] & \mathbf{P}(V) \\ \mathbb{P}(\mathscr{C}) \arrow[u, hook] \arrow[ur, "\bar{q}"'] & \end{tikzcd}$$

Moreover, we have $\mathcal{O}_{\mathscr{F}}(1)|_{\mathbb{P}(\mathscr{C})} = \mathcal{O}_{\mathscr{C}}(1)$.

We can interpret $\mathbb{P}(\mathscr{C})$ as the zero locus of a section of a vector bundle on $\mathbb{P}(\mathscr{F})$. For this construction, we refer for instance to [Ein93], [Wey03]. From a map $\varphi : \mathscr{E} \to \mathscr{F}$ we get a map $q^*\varphi : q^*\mathscr{E} \to q^*\mathscr{F}$ on $\mathbb{P}(\mathscr{F})$. Composing it with the natural surjection in (1.1) and tensoring with $q^*\mathscr{E}^*$, we get a section $s_\varphi \in \mathrm{H}^0(\mathbb{P}(\mathscr{F}), q^*\mathscr{E}^* \otimes \mathcal{O}_{\mathscr{F}}(1))$. Conversely, given such a section, it is possible to recover a map $\mathscr{E} \to \mathscr{F}$ by tensoring with $q^*\mathscr{E}$ and applying q_*. These correspondences are inverse to each other; in other words, by the adjointness of direct and inverse image functors, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{P}(V)}(\mathscr{E}, \mathscr{F}) \cong \mathrm{H}^0(\mathbb{P}(\mathscr{F}), q^*\mathscr{E}^* \otimes \mathcal{O}_{\mathscr{F}}(1)). \tag{2.3}$$

We may regard φ as a general section s_φ of $q^*\mathscr{E}^* \otimes \mathcal{O}_{\mathscr{F}}(1)$. We denote by Y its zero locus $\mathrm{V}(s_\varphi) \subset \mathbb{P}(\mathscr{F})$.

Proposition 2.5. *For any $\varphi \in \mathrm{Hom}_{\mathbf{P}(V)}(\mathscr{E}, \mathscr{F})$, we have $\mathbb{P}(\mathscr{C}) = Y$.*

Proof. Consider an open subset $\mathcal{U}$ of $\mathbf{P}(V)$, trivializing both $\mathscr{E}$ and $\mathscr{F}$; its preimage $\mathcal{U}' = q^{-1}\mathcal{U}$ is therefore isomorphic to $\mathcal{U} \times \mathbb{P}^{f-1}$. On the one hand, on $\mathcal{U}'$ the morphism φ is represented by a matrix $\varphi_{\mathcal{U}'}$ and the equations describing $\mathbb{P}(\mathscr{C}) \cap \mathcal{U}'$ are determined from the relation

$$\nu \cdot \varphi_{\mathcal{U}'}(\mu) = 0, \tag{2.4}$$

where $\nu \in \mathbb{P}^{f-1}$ and $\mu \in \mathcal{U}$. Indeed, a quotient of $\mathscr{F}$ induces a quotient of $\mathscr{C}$ if and only if its composition with φ is zero.
On the other hand, the sections of $q^*\mathscr{E}^* \otimes \mathcal{O}_{\mathscr{F}}(1)$ are the same of $q^*\mathscr{E}^* \otimes q^*\mathscr{F}$, and imposing the vanishing of s_φ translates on $\mathcal{U}'$ to the same condition (2.4). $\square$

We observe that, to prove the previous proposition, there was no need of the inequality $e \geq f$ on the ranks of $\mathscr{E}$, $\mathscr{F}$. The same is true for the following one.

Proposition 2.6. *Let $\mathscr{E}^* \otimes \mathscr{F}$ be globally generated and φ general. Then $Y = \mathbb{P}(\mathscr{C})$ is a smooth subscheme of $\mathbb{P}(\mathscr{F})$.*

Proof. It follows at once by Theorem 2.3 applied to the morphism

$$\mathcal{O}_{\mathbb{P}(\mathscr{F})} \longrightarrow q^*\mathscr{E}^* \otimes \mathcal{O}_{\mathscr{F}}(1)\ .$$ $\square$

Recall that a vector bundle $\mathscr{G}$ is *ample* if $\mathcal{O}_{\mathbb{P}(\mathscr{G})}(1)$ is an ample line bundle on $\mathbb{P}(\mathscr{G})$.

By [FL81, Theorem II] we are able to provide sufficient conditions for Y to be connected.

Proposition 2.7. *Let $q^*\mathscr{E}^* \otimes \mathcal{O}_{\mathscr{F}}(1)$ be ample on $\mathbb{P}(\mathscr{F})$ and let s be one of its global sections. If $e - f + 1 < n - 1$, then $\mathrm{V}(s)$ is connected.*

Corollary 2.8. Let $\mathscr{E}^* \otimes \mathscr{F}$ be globally generated and φ general; let $q^*\mathscr{E}^* \otimes \mathcal{O}_{\mathscr{F}}(1)$ be ample. If $e - f + 1 < n - 1$, then Y is smooth and connected, hence irreducible.

Proof. It follows from the last two propositions. $\square$

2.3 The cokernel sheaf

Throughout this section, $\mathscr{E}^* \otimes \mathscr{F}$ is supposed to be globally generated and φ general. Recall the sheaves $\mathscr{K}$, $\mathscr{C}$ defined by the exact sequence (2.1); as $e \geq f$, on a general point $x \in \mathbf{P}(V)$ the map φ_x between the stalks is surjective and $\mathscr{C}_x = 0$. It turns out that $\mathscr{C}_x \neq 0$ if and only $x \in X$, i.e. $\mathscr{C}$ is supported on X.

Proposition 2.9. *For any $k \in \mathbb{N}$ with $k < f$, consider the degeneracy locus $D_k = D_k(\varphi)$ as in Definition 2.2. Then $\mathscr{C}|_{D_k \setminus D_{k-1}}$ is a vector bundle of rank $f - k$ on $D_k \setminus D_{k-1}$.*

Proof. We can argue locally on $\mathbf{P}(V)$, so $\mathscr{C}$ can be seen as the cokernel of a map between free sheaves. In any point $x \in D_k \setminus D_{k-1}$ the map φ_x has constant rank k, so the cokernel is a vector bundle of rank $f - k$. □

We will denote by $\mathscr{L}$ the restriction of $\mathscr{C}$ to X. If i is the natural embedding $X \subset \mathbf{P}(V)$, we have $i_*\mathscr{L} = \mathscr{C}$. By Theorem 2.3, $\mathscr{L}$ turns out to be a line bundle on the smooth locus X^{sm}.

Corollary 2.10. The map $\bar{q} : Y \to X$ is regular and birational; the inverse is defined on X^{sm}, and its image is an open subset in Y whose complement Y' has codimension at least two.

Proof. By Proposition 2.9, $\mathscr{L}$ is a line bundle on X^{sm}, therefore the restriction of $\bar{q}$ to the map $\bar{q}^{-1}(X^{\mathrm{sm}}) \to X^{\mathrm{sm}}$ is an isomorphism. Let us denote the complement of the domain by Y'. It is contracted to $\mathrm{Sing}(X)$ via $\bar{q}$, but the general fiber of this contraction has dimension one, so by (2.2) we have

$$\mathrm{codim}_Y(Y') = \mathrm{codim}_X(\mathrm{Sing}(X)) - 1 \geq 2. \tag{2.5}$$

□

Corollary 2.11. $\mathscr{L}$ is Cohen-Macaulay, hence a torsion-free sheaf on X.

Proof. Let $\mathscr{M}_x$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$. From Proposition 2.9 and from the Auslander-Buchsbaum formula [Eis95, Exercise 19.8], we have $\mathrm{depth}(\mathscr{M}_x, \mathscr{L}_x) = \dim \mathcal{O}_{X,x}$ for any x, hence $\mathscr{L}_x$ is a Cohen-Macaulay module. □

Definition 2.12 ([Bar77])**.** A coherent sheaf $\mathcal{F}$ on X is *normal* if for every open set $U \subseteq X$ and every closed subset $Z \subset U$ of codimension at least two, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U \setminus Z)$ is bijective.

Proposition 2.13. *Suppose that Y is connected; then $\mathscr{L}$ is reflexive.*

Proof. Y is connected by hypothesis and smooth by Proposition 2.6, hence it is irreducible and normal. Being $\bar{q}$ dominant onto X, we get that X is irreducible too. By [Har80, Proposition 1.6], $\mathscr{L}$ is reflexive if and only if it is normal and torsion-free. By Corollary 2.11, we only need to show that $\mathscr{L}$ is normal.

We observe that $\mathcal{O}_{\mathscr{C}}(1)$ is reflexive, hence normal. If U is an open subset of X and Z a closed subset of X of codimension at least two, then $\bar{q}^{-1}(U)$ is open in Y and $\bar{q}^{-1}(Z)$ is closed of codimension at least two, by Corollary 2.10. The normality of $\mathscr{C}$ follows since

$$\mathscr{L}(U) = (\mathcal{O}_{\mathscr{C}}(1))(\bar{q}^{-1}(U)) \longrightarrow (\mathcal{O}_{\mathscr{C}}(1))(\bar{q}^{-1}(U \setminus Z)) = \mathscr{L}(U \setminus Z)$$

is bijective. □

Remark 2.14. By the proof of the last proposition, the irreducibility of X is equivalent to the connectedness of Y, which is ensured for instance by Proposition 2.7.

Suppose then X irreducible; being it normal, we are allowed to consider its divisor class group $\mathrm{Cl}(X)$. Let $t \in \mathbb{Z}$ be the minimum integer such that $\mathrm{h}^0(X, \mathscr{L}(t)) \neq 0$. Taking a global section of $\mathscr{L}(t)$, the class of its zero locus Z can be regarded as an element in $\mathrm{Cl}(X)$. The exact sequence defining Z is

$$0 \longrightarrow \mathscr{L}^*(-t) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

Let us observe that the class of Z determines $\mathscr{L}$, when the latter is reflexive. Indeed, by the previous exact sequence and by the reflexivity of $\mathscr{L}$, we have $\mathscr{L} = \mathcal{I}_Z^*(-t)$. Here, t is uniquely determined for a general morphism between fixed $\mathscr{E}$ and $\mathscr{F}$.

Remark 2.15. We remark that $\mathscr{L}$ is not arithmetically Cohen-Macaulay in general. In Ch. 3 we will see a concrete counterexample (cfr. Lemma 3.3). $\mathscr{L}$ can be proved to be arithmetically Cohen-Macaulay when $\mathscr{E}$ and $\mathscr{F}$ are particular vector bundles, for instance when they are both direct sums of twisted structure sheaves; see for instance [FF10a].

2.4 Tritensors

Suppose we have three $\mathbf{k}$-vector spaces U, V, W and a tritensor $\gamma \in U \otimes V \otimes W$. We may regard γ as a morphism between particular vector bundles.

On the one hand we may consider V as $\mathrm{H}^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1))$, so γ gives rise to a map M with cokernel $\mathscr{C}_M$:

$$W^* \otimes \mathcal{O}_{\mathbf{P}(V)} \xrightarrow{M} U \otimes \mathcal{O}_{\mathbf{P}(V)}(1) \longrightarrow \mathscr{C}_M \longrightarrow 0.$$

On the other hand, considering U as $\mathrm{H}^0(\mathbf{P}(U), \mathcal{O}_{\mathbf{P}(U)}(1))$, the same argument leads to a map N:

$$W^* \otimes \mathcal{O}_{\mathbf{P}(U)} \xrightarrow{N} V \otimes \mathcal{O}_{\mathbf{P}(U)}(1) \longrightarrow \mathscr{C}_N \longrightarrow 0.$$

The two maps N and M are represented by two matrices (still denoted by N, M), whose entries are linear forms. We want to show how N and M are related.

Let $\dim(U) = m$, $\dim(V) = n$, $\dim(W) = s$; let $\{y_i\}_{0 \le i \le m-1}$ be a basis of U and $\{x_i\}_{0 \le i \le n-1}$ a basis of V. In this way, N is an $n \times s$ matrix of linear forms in $\mathbf{k}[y_0, \dots, y_{m-1}]$. Analogously, M is an $m \times s$ matrix of linear forms in $\mathbf{k}[x_0, \dots, x_{n-1}]$. The one can be obtained from the other simply by exchanging the role of variables and rows; namely, if

$$N = \begin{pmatrix} \sum_{k=0}^{m-1} \alpha_{0,0}^k y_k & \dots & \sum_{k=0}^{m-1} \alpha_{0,s-1}^k y_k \\ \vdots & & \vdots \\ \sum_{k=0}^{m-1} \alpha_{n-1,0}^k y_k & \dots & \sum_{k=0}^{m-1} \alpha_{n-1,s-1}^k y_k \end{pmatrix},$$

then

$$M = \begin{pmatrix} \sum_{i=0}^{n-1} \alpha_{i,0}^0 x_i & \dots & \sum_{i=0}^{n-1} \alpha_{i,s-1}^0 x_i \\ \vdots & & \vdots \\ \sum_{i=0}^{n-1} \alpha_{i,0}^{m-1} x_i & \dots & \sum_{i=0}^{n-1} \alpha_{i,s-1}^{m-1} x_i \end{pmatrix}.$$

When the supports of $\mathscr{C}_M$ and $\mathscr{C}_N$ are not empty, we can look at

$$\mathbb{P}(\mathscr{C}_M) \subset \mathbb{P}(U \otimes \mathcal{O}_{\mathbf{P}(V)}(1)) \cong \mathbf{P}(V) \times \mathbf{P}(U)$$

and at

$$\mathbb{P}(\mathscr{C}_N) \subset \mathbb{P}(V \otimes \mathcal{O}_{\mathbf{P}(U)}(1)) \cong \mathbf{P}(U) \times \mathbf{P}(V).$$

Let $\bar{p} : \mathbb{P}(\mathscr{C}_N) \to \mathbf{P}(U)$ be the natural map given by the projectivization of the sheaf $\mathscr{C}_N$ and let $\bar{q} : \mathbb{P}(\mathscr{C}_M) \to \mathbf{P}(V)$ be defined in the same way. The flip automorphism $\mathbf{P}(U) \times \mathbf{P}(V) \cong \mathbf{P}(V) \times \mathbf{P}(U)$ induces isomorphisms

$$\bar{p}^*(W^* \otimes \mathcal{O}_{\mathbf{P}(U)}) \cong \bar{q}^*(W^* \otimes \mathcal{O}_{\mathbf{P}(V)}),$$

$$\mathcal{O}_{V \otimes \mathcal{O}_{\mathbf{P}(U)}(1)}(1) \cong \mathcal{O}_{U \otimes \mathcal{O}_{\mathbf{P}(V)}(1)}(1),$$

hence an isomorphism between

$$\mathrm{Hom}_{\mathbf{P}(U)\times\mathbf{P}(V)}(W^* \otimes \mathcal{O}_{\mathbf{P}(U)\times\mathbf{P}(V)}, \mathcal{O}_{V\otimes\mathcal{O}_{\mathbf{P}(U)}(1)}(1))$$

and

$$\mathrm{Hom}_{\mathbf{P}(V)\times\mathbf{P}(U)}(W^* \otimes \mathcal{O}_{\mathbf{P}(V)\times\mathbf{P}(U)}, \mathcal{O}_{U\otimes\mathcal{O}_{\mathbf{P}(V)}(1)}(1)).$$

Recall from (2.3) that N may be regarded as an element of the former, while M as an element of the latter. It is easy to see that N corresponds to M via the previous isomorphism. From this and Proposition 2.5 it follows that

$$\mathbb{P}(\mathscr{C}_N) \cong \mathbb{P}(\mathscr{C}_M).$$

In this way, given N, we can obtain information about the geometry of $\mathbb{P}(\mathscr{C}_N)$ (and, therefore, of the degeneracy locus of N, since by Corollary 2.10 they are at least birational) by looking at $\mathbb{P}(\mathscr{C}_M)$; and conversely.

This procedure can be sometimes applied also when we have a map which is not given by a matrix of linear forms. Let us show how in a concrete example.

Example 2.16. Suppose we want to study degeneracy loci of a general morphism of the form $\varphi : \mathcal{T}_{\mathbf{P}(V)}(-2) \to U \otimes \mathcal{O}_{\mathbf{P}(V)}$, where $\dim V = n$ and $\dim U = m \leq n-1$. We rewrite the exact sequence (2.1) as

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathcal{T}_{\mathbf{P}(V)}(-2) \xrightarrow{\varphi} U \otimes \mathcal{O}_{\mathbf{P}(V)} \longrightarrow \mathscr{C} \longrightarrow 0\,. \tag{2.6}$$

Note that $\mathrm{rank}(\mathcal{T}_{\mathbf{P}(V)}(-2)) = n-1 \geq m$, so that the usual inequality on the ranks is satisfied and $\mathscr{C}$ is supported on the degeneracy locus X.

We observe that both $\mathscr{C}$ and X do not change if we compose φ with a surjection, so we may consider the (twisted) Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}(V)}(-2) \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}(V)}(-1) \longrightarrow \mathcal{T}_{\mathbf{P}(V)}(-2) \longrightarrow 0 \tag{2.7}$$

and get a map $V \otimes \mathcal{O}_{\mathbf{P}(V)}(-1) \to U \otimes \mathcal{O}_{\mathbf{P}(V)}$. We have a new exact sequence

$$V \otimes \mathcal{O}_{\mathbf{P}(V)}(-1) \xrightarrow{M} U \otimes \mathcal{O}_{\mathbf{P}(V)} \longrightarrow \mathscr{C} \longrightarrow 0\,.$$

The degeneracy loci of φ and M are the same. By Corollary 2.10, $\mathbb{P}(\mathscr{C})$ is at least birational to X.

We can view M as one of the possible realizations of a tritensor in $U \otimes V \otimes V$. In this case the tritensor is not general, since M factors by construction through $\mathcal{T}_{\mathbf{P}(V)}(-2)$. One can see that a necessary and sufficient condition for an $m \times n$ matrix of linear forms

$$M = \begin{pmatrix} \sum_{i=0}^{n-1} \alpha_{i,0}^{0} x_i & \dots & \sum_{i=0}^{n-1} \alpha_{i,n-1}^{0} x_i \\ \vdots & & \vdots \\ \sum_{i=0}^{n-1} \alpha_{i,0}^{m-1} x_i & \dots & \sum_{i=0}^{n-1} \alpha_{i,n-1}^{m-1} x_i \end{pmatrix} \tag{2.8}$$

to factor through $\mathcal{T}_{\mathbf{P}(V)}(-2)$ is

$$\alpha_{i,j}^{k} = -\alpha_{j,i}^{k} \qquad \text{for all } i, j, k. \tag{2.9}$$

The other realization of the tritensor, on $\mathbf{P}(U)$, leads to N, an $n \times n$ matrix which turns out to be skew-symmetric by (2.9). The (i, j)-th element of N is $\sum_{k=0}^{m-1} \alpha_{i,j}^{k} y_k$.

The relation between maps φ of the form (2.6) and skew-symmetric matrices can be seen also by means of the following interpretation, for which we refer to [Ott92, §3.2]. Let φ^t be the dual of φ; a map $U^* \otimes \mathcal{O}_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$ corresponds to m global sections of $\Omega_{\mathbf{P}(V)}(2)$. By considering the global sections of the dual sequence of (2.7)

$$0 \longrightarrow \Omega_{\mathbf{P}(V)}(2) \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}(V)}(1) \longrightarrow \mathcal{O}_{\mathbf{P}(V)}(2) \longrightarrow 0 \tag{2.10}$$

we may identify $\mathrm{H}^0(\mathbf{P}(V), \Omega_{\mathbf{P}(V)}(2))$ with $\Lambda^2 V$, and therefore

$$\varphi^t \in \mathrm{Hom}_{\mathbf{P}(V)}(U^* \otimes \mathcal{O}_{\mathbf{P}(V)}, \Omega_{\mathbf{P}(V)}(2)) \cong U \otimes \Lambda^2 V \subset U \otimes V \otimes V.$$

Considering φ^t as an element

$$N \in \mathrm{Hom}_{\mathbf{P}(U)}(V^* \otimes \mathcal{O}_{\mathbf{P}(U)}, V \otimes \mathcal{O}_{\mathbf{P}(U)}(1))$$

we get a skew-symmetric matrix with linear forms as entries, as above.

With a slight abuse of notation, we have the following commutative diagram

$$\begin{tikzcd} \mathbf{P}(U) & \mathbf{P}(U) \times \mathbf{P}(V) \arrow[l] \arrow[r] & \mathbf{P}(V) \\ & \mathbb{P}(\mathscr{C}) \arrow[ul, "\bar{q}"] \arrow[u, hook] \arrow[ur, "\bar{p}"'] & \end{tikzcd}$$

This point of view allows us to interpret X exploiting its birationality with $\mathbb{P}(\mathrm{coker}(N))$. Making use of these observations, we will carry out the study of the geometry of X in Ch. 3.

2.5 Direct images of the Koszul complex

The description of $\mathbb{P}(\mathscr{C})$ as zero locus of a general global section of $q^*\mathscr{E}^* \otimes \mathcal{O}_{\mathscr{F}}(1)$, given by Proposition 2.5, allows us to write down the Koszul complex resolving $\mathcal{O}_Y$.

$$0 \longrightarrow \Lambda^e(q^*\mathscr{E} \otimes \mathcal{O}_{\mathscr{F}}(-1)) \xrightarrow{\epsilon_e} \dots \xrightarrow{\epsilon_3} \Lambda^2(q^*\mathscr{E} \otimes \mathcal{O}_{\mathscr{F}}(-1)) \xrightarrow{\epsilon_2}$$
$$\xrightarrow{\epsilon_2} q^*\mathscr{E} \otimes \mathcal{O}_{\mathscr{F}}(-1) \xrightarrow{\epsilon_1} \mathcal{O}_{\mathscr{F}} \xrightarrow{\epsilon_0} \mathcal{O}_Y \longrightarrow 0 \quad (2.11)$$

For what follows, we refer to [GP82]. Take $l \in \mathbb{Z}$ such that $-1 \leq l \leq e - f + 1$. Consider the Koszul complex twisted by $\mathcal{O}_{\mathscr{F}}(l)$: we want to describe what its direct image via q is on $\mathbf{P}(V)$. For this sake, recall the two hypercohomology spectral sequences (1.3) and (1.4), with respect to the map $q : \mathbb{P}(\mathscr{F}) \to \mathbf{P}(V)$. Let $K(l)_\bullet$ be the twisted complex (2.11) without the last term, i.e.

$$K(l)_j = \begin{cases} q^*\Lambda^{-j}\mathscr{E} \otimes \mathcal{O}_{\mathscr{F}}(l+j) & \text{if } -e \leq j \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

By construction, $K(l)_\bullet$ is quasi-isomorphic to $\mathcal{O}_Y(l)$ as complexes.

On the one hand, spectral sequence (1.4) applied to $\mathcal{O}_Y(l)$ abuts to the following total complex with zero differentials

$$\mathrm{Tot}_k = \begin{cases} R^k q_*(\mathcal{O}_Y(l)) & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases} \quad (2.12)$$

On the other hand, let us look at spectral sequence (1.3) applied to $K(l)_\bullet$. We have

$$R^i q_*(K(l)_j) =$$
$$\begin{cases} \Lambda^{-j}\mathscr{E} \otimes S^{l+j}\mathscr{F} & \text{if } \begin{cases} l+j \geq 0, i = 0 \\ 0 \geq j \geq -e \end{cases} \\ \Lambda^{-j}\mathscr{E} \otimes S^{-l-j-f}\mathscr{F}^* \otimes \Lambda^f \mathscr{F}^* & \text{if } \begin{cases} -l-j-f \geq 0 \\ 0 \geq j \geq -e, i = f-1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

One can see that, from the second sheet on, all the differentials are zero; the unique exception occurs in the f-th sheet, when we have a map from

$${}'E_f^{f-1,-l-f} = \operatorname{coker}\left(R^{f-1}q_*(K(l)_{-l-f-1}) \to R^{f-1}q_*(K(l)_{-l-f})\right)$$

to

$$'E_f^{0,-l} = \ker\left(q_*(K(l)_{-l}) \to q_*(K(l)_{-l+1})\right).$$

This spectral sequence abuts to a total complex concentrated in non-positive degrees; the comparison between this total complex and (2.12) implies that

- the complex

$$0 \longrightarrow \Lambda^f \mathscr{F}^* \otimes \Lambda^e \mathscr{E} \otimes S^{e-f-l}(\mathscr{F}^*) \longrightarrow \ldots \longrightarrow \Lambda^f \mathscr{F}^* \otimes \Lambda^{f+l} \mathscr{E}$$

  is exact;

- the complex

$$\Lambda^l \mathscr{E} \longrightarrow \ldots \longrightarrow \mathscr{E} \otimes S^{l-1} \mathscr{F} \longrightarrow S^l \mathscr{F}$$

  is exact;

- the two previous complexes fit together into an exact sequence E_l :

$$\begin{aligned} 0 \longrightarrow \Lambda^f \mathscr{F}^* \otimes \Lambda^e \mathscr{E} \otimes S^{e-f-l}(\mathscr{F}^*) \longrightarrow \ldots \longrightarrow \Lambda^f \mathscr{F}^* \otimes \Lambda^{f+l} \mathscr{E} \\ \longrightarrow \Lambda^l \mathscr{E} \longrightarrow \ldots \longrightarrow \mathscr{E} \otimes S^{l-1} \mathscr{F} \longrightarrow S^l \mathscr{F} \ . \end{aligned}$$

- $R^i q_*(\mathcal{O}_Y(l)) = 0$ for any $i > 0$.

E_0 is best known as the Eagon-Northcott complex [EN62], while E_1 is best known as the Buchsbaum-Rim complex [BR64].

Proposition 2.17 ([GP82]).

i. *For any l such that $0 \leq l \leq e - f + 1$, E_l is a resolution of $S^l \mathscr{C}$;*

ii. *the $\mathcal{O}_X$-module $\omega_X = S^{e-f} \mathscr{C} \otimes \Lambda^f \mathscr{F} \otimes \Lambda^e \mathscr{E}^* \otimes \mathcal{O}_{\mathbf{P}(V)}(-n)$ is dualizing;*

iii. *$\mathscr{H}om_X(S^l \mathscr{C}, S^{e-f} \mathscr{C}) \cong S^{e-f-l} \mathscr{C}$ for any l such that $-1 \leq l \leq e-f+1$.*

Note that, from the last proposition, we have

$$\mathscr{H}om_X(\omega_X, \omega_X) \cong \mathcal{O}_X. \tag{2.13}$$

Being the complexes E_l resolutions of $S^l \mathscr{C}$, we may use them to compute the cohomology groups of $\mathcal{O}_X$, $\mathscr{C}$, or their twists by $\mathcal{O}_{\mathbf{P}(V)}(\ell)$. When the vector bundles $\mathscr{E}$, $\mathscr{F}$ involved have "simple" cohomology, for example when they have natural cohomology, a lot of vanishings occur and so the computations become easier.

The computation of the syzygies of X via this technique is usually referred to as the Kempf-Lascoux-Weyman's method (cfr. the Introduction).

Let us also remark that the last terms of E_1 fit in with the exact sequence (2.1) in the following way

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & & \\
& \mathscr{K} \arrow[ur] \arrow[dr] & & & & \\
\Lambda^f \mathscr{F}^* \otimes \Lambda^{f+1}\mathscr{E} \arrow[ur] \arrow[rr] & & \mathscr{E} \arrow[rr, "\varphi"] \arrow[dr] & & \mathscr{F} \arrow[r, two heads] & \mathscr{C} \\
& & & \mathrm{Im}(\varphi) \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd} \tag{2.14}$$

Remark 2.18. We could as well work directly on $\mathbb{P}(\mathscr{F})$, taking the Koszul complex twisted by $\mathcal{O}_{\mathscr{F}}(l)$, and computing cohomology groups. Once we have computed the cohomology groups of, say, $\mathcal{O}_Y(l)$, we have for free the cohomology groups of the twisted structure sheaf of $X = q(Y)$. Indeed, by [Har77, Exercise III.4.1], we only need to show that the higher images $R^{i>0}q_*(\mathcal{O}_Y(l))$ vanish, but this is a consequence of the spectral sequence argument above.

2.6 The normal sheaf

Let us borrow again the notations from the first three sections of this chapter; let $\mathscr{E}^* \otimes \mathscr{F}$ be globally generated and consider a general morphism φ as in (2.1). The aim of this section is to show how can the normal sheaf $\mathcal{N} := \mathcal{N}_{X/\mathbf{P}(V)}$ of X in $\mathbf{P}(V)$ be expressed by means of $\mathscr{C}$.

Lemma 2.19. We have $\mathscr{H}om_X(\mathscr{L}, \mathscr{L}) \cong \mathcal{O}_X$.

Proof. The lemma is trivial when $\mathscr{L}$ is a line bundle, i.e. when X is smooth. For the general case, we look at the map

$$\mathscr{H}om_X(\mathscr{L}, \mathscr{L}) \to \mathscr{H}om_X(S^{n-m-1}\mathscr{L}, S^{n-m-1}\mathscr{L}) \tag{2.15}$$

given by $f \mapsto f^{n-m-1}$. The term on the right is isomorphic to $\mathcal{O}_X$, by (2.13) and Proposition 2.17.

Recall that we showed that $\mathscr{L}$ is torsion-free in Corollary 2.11. The sheaf $\mathscr{H}om_X(\mathscr{L},\mathscr{L})$ turns out to be torsion-free too: indeed, it is a subsheaf of the direct sum of f copies of $\mathscr{L}$, as it results by applying $\mathscr{H}om_X(-,\mathscr{L})$ to sequence (2.1) restricted to X.

The map (2.15) is therefore a non-zero map between two rank-one torsion-free sheaves, so its kernel vanishes. As soon as we consider the chain

$$\mathcal{O}_X \hookrightarrow \mathscr{H}om_X(\mathscr{L},\mathscr{L}) \hookrightarrow \mathscr{H}om_X(S^{n-m-1}\mathscr{L}, S^{n-m-1}\mathscr{L}) \cong \mathcal{O}_X,$$

the lemma is proved. □

Proposition 2.20 ([FF10a, Lemma 3.5])**.** *We have the following cohomological spectral sequence:*

$$E_2^{p,q} = \mathscr{E}xt^p_{\mathbf{P}(V)}(i_*(\mathcal{O}_X), i_*(\mathscr{E}xt^q_X(\mathscr{L},\mathscr{L}))) \Rightarrow \mathscr{E}xt^{p+q}_{\mathbf{P}(V)}(\mathscr{C},\mathscr{C}).$$

Proof. Let $\mathcal{E}$, $\mathcal{F}$ be two coherent sheaves on X and consider the two functors

$$\Psi = \mathscr{H}om_{\mathbf{P}(V)}(i_*(\mathcal{O}_X, i_*(-)) : \operatorname{Coh}(X) \longrightarrow \operatorname{Coh}(\mathbf{P}(V))$$

and

$$\Phi = \mathscr{H}om_X(\mathcal{E}, -) : \operatorname{Coh}(X) \longrightarrow \operatorname{Coh}(X) .$$

Their composition $\Psi \circ \Phi$ sends $\mathcal{F}$ to

$$\mathscr{H}om_{\mathbf{P}(V)}(i_*(\mathcal{O}_X), i_*(\mathscr{H}om_X(\mathcal{E},\mathcal{F}))) \cong \mathscr{H}om_{\mathbf{P}(V)}(i_*(\mathcal{E}), i_*(\mathcal{F})). \qquad (2.16)$$

To see the last isomorphism, we can work locally on $\operatorname{Spec}(A) \subset X$ and $\operatorname{Spec}(B) \subset \mathbf{P}(V)$, replacing i with the closed embedding $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ induced by a surjective map of $\mathbf{k}$-algebras $B \to A$. $\mathcal{E}$ and $\mathcal{F}$ are locally replaced by finitely generated A-modules M, N, which may be regarded as B-modules as well. To prove (2.16) it is sufficient to show the isomorphism

$$\operatorname{Hom}_B(M,N) \cong \operatorname{Hom}_A(A, \operatorname{Hom}_A(M,N));$$

for this sake, we consider the B-morphism taking $u : M \to N$ to the A-morphism taking 1_A to u regarded as an A-morphism. It is straightforward to check that this is indeed an isomorphism.

The spectral sequence in the statement follows from the Grothendieck's spectral sequence (1.2) associated with the composition of the two left-exact functors $\Psi \circ \Phi$, applied after replacing both $\mathcal{E}$ and $\mathcal{F}$ with $\mathscr{L}$. $\square$

Proposition 2.21 ([FF10a, Lemma 3.5]). *Denoting again by i the natural embedding $X \to \mathbf{P}(V)$, we have $i_*\mathcal{N} \cong \mathscr{E}xt^1_{\mathbf{P}(V)}(\mathscr{C}, \mathscr{C})$.*

Proof. The normal sheaf can be characterized also via the isomorphism

$$i_*\mathcal{N} \cong \mathscr{E}xt^1_{\mathbf{P}(V)}\left(i_*(\mathcal{O}_X), i_*(\mathcal{O}_X)\right),$$

so it is sufficient to show that $\mathscr{E}xt^1_{\mathbf{P}(V)}(\mathscr{C}, \mathscr{C})$ is isomorphic to the term on the right. Consider the spectral sequence given by Proposition 2.20. By Lemma 2.19, the conclusion holds if we show that $\mathscr{E}xt^1_X(\mathscr{L}, \mathscr{L}) = 0$. By adjointness we get

$$\mathscr{E}xt^1_X(\mathscr{L}, \mathscr{L}) \cong \mathscr{E}xt^1_Y(\bar{q}^*(\mathscr{L}), \mathcal{O}_Y(1,0)).$$

Recall that $Y \cong \mathbb{P}(\mathscr{L})$, so on Y we have sequence (1.1). Here, the sheaf Ω is supported on Y', as $\bar{q}$ is an isomorphism on the complement $Y \setminus Y'$. If we apply the functor $\mathscr{H}om_Y(-, \mathcal{O}_Y(1,0))$ to (1.1), we get

$$\mathscr{E}xt^1_Y(\Omega, \mathcal{O}_Y(1,0)) \longrightarrow \mathscr{E}xt^1_Y(\bar{q}^*(\mathscr{L}), \mathcal{O}_Y(1,0)) \longrightarrow 0,$$

as $\mathcal{O}_Y(1,0)$ is a line bundle on Y. The first sheaf vanishes since its support, by (2.5), has codimension at least two, so the second one vanishes too. $\square$

Chapter 3

On the Hilbert scheme of degeneracy loci of $\mathcal{O}^m_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$

Within this chapter we focus on a particular type of morphism between vector bundles, namely

$$\phi : \mathcal{O}^m_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2),$$

where $\Omega_{\mathbf{P}(V)} = \Omega^1_{\mathbf{P}(V)}$ is the cotangent bundle. We prove that, for $3 < m < n-1$, the Grassmannian of m-dimensional subspaces of the space of skew-symmetric forms over the n-dimensional $\mathbf{k}$-vector space V is birational to $\mathcal{H}$, the Hilbert scheme of degeneracy loci of m global sections of $\Omega_{\mathbf{P}(V)}(2)$. For $3 = m < n-1$ and n odd, this Grassmannian is proved to be birational to the variety of Veronese surfaces parametrized by the Pfaffians of linear skew-symmetric matrices of order n. For $m = 3$ and for $m = 2$, the description of the general element of $\mathcal{H}$ is given.

For the whole chapter, we will suppose $\mathbf{k} = \bar{\mathbf{k}}$ and $\operatorname{char} \mathbf{k} = 0$.

The contents of this chapter are mainly included in [Ta13b].

3.1 Introduction

3.1.1 Preliminary constructions

Agreeing with the notations introduced before, let U, V be two $\mathbf{k}$-vector spaces of dimension m, n, with $2 \leq m < n-1$. Let $\phi : U^* \otimes \mathcal{O}_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$. As the degeneracy locus is the same for a map and its transposed, we will rather consider the dual map $\phi^t = \varphi : \mathcal{T}_{\mathbf{P}(V)}(-2) \to U \otimes \mathcal{O}_{\mathbf{P}(V)}$, with kernel and cokernel given by the exact sequence (2.6). Recall that the study of the degeneracy locus of a general morphism φ is equivalent to the study of the degeneracy locus of a morphism given by a general matrix (2.8) satisfying (2.9).

We define $\mathcal{H}$ to be the union of the irreducible components, in the Hilbert scheme, containing the degeneracy loci X's coming from general choices of φ.

We will use the notations and the description introduced in Example 2.16; we will denote by $\mathcal{P}$ the product $\mathbf{P}(U) \times \mathbf{P}(V)$ for short. For any pair of integers a, b, we will denote by $\mathcal{O}_{\mathcal{P}}(a,b)$ the line bundle $p^*\mathcal{O}_{\mathbf{P}(U)}(a) \boxtimes q^*\mathcal{O}_{\mathbf{P}(V)}(b)$.

In Sect. 2.2 we showed that $\mathbb{P}(\mathscr{C}) \subset \mathcal{P}$ can be seen as the zero locus Y of a global section of $\mathcal{O}_{\mathbf{P}(U)}(1) \boxtimes \Omega_{\mathbf{P}(V)}(2)$ (cfr. (2.3)). The Koszul complex (2.11) becomes

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}(U)}(1-n) \boxtimes \mathcal{O}_{\mathbf{P}(V)}(2-n) \xrightarrow{\epsilon_{n-1}} \mathcal{O}_{\mathbf{P}(U)}(2-n) \boxtimes \Omega_{\mathbf{P}(V)}(4-n) \xrightarrow{\epsilon_{n-2}}$$
$$\longrightarrow \dots \xrightarrow{\epsilon_2} \mathcal{O}_{\mathbf{P}(U)}(-1) \boxtimes \Omega^{n-2}_{\mathbf{P}(V)}(n-2) \xrightarrow{\epsilon_1} \mathcal{O}_{\mathcal{P}} \xrightarrow{\epsilon_0} \mathcal{O}_Y \longrightarrow 0, \tag{3.1}$$

where we made use of the isomorphisms $\Lambda^r \mathcal{T}_{\mathbf{P}(V)} \cong \Omega^{n-r-1}_{\mathbf{P}(V)}(n)$ (cfr. for example [Har77, Exercise II.5.16]). Being φ general, this complex is exact.

3.1.2 Hilbert schemes and Grassmannians

As the Hilbert scheme is the same for general choices of φ, we have a rational map

$$\mathrm{Hom}(\mathcal{T}_{\mathbf{P}(V)}(-2), U \otimes \mathcal{O}_{\mathbf{P}(V)}) \dashrightarrow \mathcal{H}, \tag{3.2}$$

sending φ to the point representing its degeneracy locus. We want to study this map. Two natural questions arise:

- if φ and φ' give rise to the same degeneracy locus, is there a relation between them? For instance, are they the same morphism?
- If we consider a deformation X' of a degeneracy locus X, so that $X' \in \mathcal{H}$, is X' still a degeneracy locus of a suitable morphism?

The first question translates into studying whether the map (3.2) is generically injective, the second into studying whether it is dominant.

As for the first question, the map above is clearly not generically injective. Indeed, the group $\mathrm{GL}(U)$ induces an action on $\mathrm{Hom}(\mathcal{T}_{\mathbf{P}(V)}(-2), U \otimes \mathcal{O}_{\mathbf{P}(V)})$, by multiplication on the right of the matrix representing φ. The equations cutting out the degeneracy locus may change, but the ideal described does not and so all the morphisms belonging to a same orbit share the same degeneracy locus; hence, map (3.2) factors through this action.

Recall from Example 2.16 that φ can be seen also as a $(n \times n)$ skew-symmetric matrix N of linear forms in $\mathbf{k}[y_0, \ldots, y_{m-1}]$, or as an m-uple of elements in $\Lambda^2 V$. With this interpretation, an element of $\mathrm{GL}(U)$ acts as a projectivity on these m elements; it does not affect the linear space spanned by them, so the orbit is an element of the Grassmannian $\mathbf{Gr}(m, \Lambda^2 V)$.

We get the following scenario:

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{T}_{\mathbf{P}(V)}(-2), U \otimes \mathcal{O}_{\mathbf{P}(V)}) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{H} \\
\mathbf{Gr}(m, \Lambda^2 V) \arrow[ur, dashed, "\rho"'] &
\end{tikzcd}$$

The behavior of the map ρ is known in the cases

- $(m, n) = (2, 6)$: X is union of three skew lines in $\mathbb{P}^5$. The map ρ is dominant and the general fiber has dimension two [BM01];

- $(m, n) = (3, 5)$: X is a projected Veronese surface in $\mathbb{P}^4$. From the results contained in [Cas91], ρ can be proved to be birational;

- $(m, n) = (3, 6)$: X is an elliptic scroll surface of degree 6. It was proved in [BM01], and in fact classically known to Fano [Fan30], that ρ is dominant and $4 : 1$;

- $(m, n) = (4, 6)$: X is the Palatini scroll, which is, according to conjecture by Peskine, the unique smooth threefold in $\mathbb{P}^5$ not quadratically normal. In this case, ρ turns out to be birational, as shown in [FM02];

- (m, n) such that n is even and $n > 2m - 3 > 1$: X is a scroll over a smooth Pfaffian hypersurface in $\mathbb{P}^{m-1}$ as long as $m \leq 6$, otherwise it is the projectivization of a rank-two sheaf over a singular Pfaffian hypersurface. The map ρ is generically injective for $m = 3$, $n \geq 8$ and birational for $m \geq 4$ [FF10b].

Given this historical account, it is natural to ask whether the map ρ is birational in the missing cases, e.g. when n is odd or when X is singular. We will give an answer, showing that

- If $m \geq 4$ or $(m,n) = (3,5)$, then ρ is birational; in particular, the Hilbert scheme $\mathcal{H}$ is generically smooth of dimension $m\left(\binom{n}{2} - m\right)$. This will be proved in Theorem 3.13;
- If $m = 3$ and $n \neq 6$, then ρ is generically injective (Theorem 3.8). Moreover
  - if n is odd, ρ is dominant on a closed subscheme $\mathcal{H}'$ of $\mathcal{H}$, having $\operatorname{codim}_{\mathcal{H}} \mathcal{H}' = \frac{1}{8}n(n-3)(n-5)$. The general element of $\mathcal{H}$ is a general projection in $\mathbf{P}(V)$ of a Veronese surface $v_{\frac{n-1}{2}}(\mathbb{P}^2)$, embedded via the complete linear system of curves of degree $\frac{n-1}{2}$; in particular, $\mathcal{H}$ is irreducible. The general element of $\mathcal{H}'$ is a special projection in $\mathbf{P}(V)$, using as the center of projection the linear space spanned by the partial derivatives of order $\frac{n-5}{2}$ of a non-degenerate polynomial $G \in \mathbf{k}[y_0, y_1, y_2]$ of degree $n-3$;
  - if n is even, ρ is dominant on a closed subscheme $\mathcal{H}'$ of $\mathcal{H}$, having $\operatorname{codim}_{\mathcal{H}} \mathcal{H}' = \frac{3}{8}(n-4)(n-6)$. The general element of $\mathcal{H}'$ is a projective bundle $\mathbb{P}(\mathscr{G})$ obtained projectivizing a general stable rank-two vector bundle $\mathscr{G}$ on a general plane curve C of degree $\frac{n}{2}$, with determinant $\det(\mathscr{G}) = \mathcal{O}_C(\frac{n-2}{2})$.
- If $m = 2$ and n is odd, then ρ is dominant but not generically injective. $\mathcal{H}$ is irreducible and its general element is the image in $\mathbf{P}(V)$ of an isomorphism
$$\mathbb{P}^1 \xrightarrow{[f_1:\ldots:f_n]} \mathbf{P}(V),$$
where $f_1, \ldots, f_n$ are forms of degree $\frac{n-1}{2}$ spanning the whole linear space $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$.

In the case $m = 3$, the codimensions of $\mathcal{H}'$ in $\mathcal{H}$ are computed in Proposition 3.15; if n is odd, the characterization of the general element of $\mathcal{H}'$ is performed in Theorem 3.16, while the general element of $\mathcal{H}$ is described in Proposition 3.21. If n is even, this was done in [FF10b].

The case $m = 2$ will be entirely discussed in Sect. 3.6, so from now on we will suppose $m \geq 3$.

3.2 Geometric interpretation of X

By Theorem 2.3, for the general φ its degeneracy locus X has dimension $m-1$, regardless of the dimension of the ambient space $\mathbf{P}(V)$. Moreover, X

is singular if and only if $2(n-m+1) > n-1$, so

$$\text{a general } X \text{ is smooth if and only if } n > 2m-3. \tag{3.3}$$

In Example 2.16 we showed how Y can be seen also as $\mathbb{P}(\mathscr{C}_N)$, where $\mathscr{C}_N$ is the cokernel of a skew-symmetric matrix N. We are able to provide a geometric description of $\mathbb{P}(\mathscr{C}_N)$, which depends strongly on the parity of n.

If n is even, then N is a skew-symmetric matrix of even order, whose cokernel $\mathscr{C}_N$ is a rank-two sheaf supported on the hypersurface cut out by the Pfaffian of N (cfr. Definition 1.4); such hypersurface is singular as soon as $m \geq 7$. The projectivization $\mathbb{P}(\mathscr{C}_N)$ is then the closure in $\mathcal{P}$ of a scroll over the smooth locus of this Pfaffian hypersurface. This case has been studied in [FF10b], with the additional hypothesis $n > 2m-3$.

If n is odd, N has odd order and so its determinant is zero; $\mathscr{C}_N$ is a rank-one sheaf on $\mathbf{P}(U)$. The locus where $\mathscr{C}_N$ has higher rank is exactly the subscheme Z defined by the $(n-1) \times (n-1)$ Pfaffians of N. Let I be the ideal of Z; by Theorem 1.8 and by Remark 1.9, for a general N the ideal I is Gorenstein and satisfies

$$\mathrm{pd}_R(R/I) = \mathrm{codim}_R(I) = \mathrm{depth}(I, R) = 3,$$

being $R = \mathbf{k}[y_0, \dots, y_{m-1}]$.

The surjection $V \otimes \mathcal{O}_{\mathbf{P}(U)}(1) \to \mathscr{C}_N$ is given by the Pfaffians of N, as follows from the corresponding map between moduli; so we have

$$\mathscr{C}_N = \mathcal{I}_Z\left(\frac{n+1}{2}\right).$$

Therefore, $\mathbb{P}(\mathscr{C}_N)$ is the blow-up of $\mathbf{P}(U)$ along Z. Viewed as a subscheme of $\mathcal{P}$, $\mathbb{P}(\mathscr{C}_N)$ is the closure of the graph of the rational map given by the $(n-1) \times (n-1)$ Pfaffians of N.

Proposition 3.1. *Y is irreducible, hence so is X; moreover, X is a normal, reduced variety.*

Proof. The irreducibility of Y follows from the geometric description just given; when n is even, we observe that the general Pfaffian hypersurface in $\mathbf{P}(U)$ is irreducible and Y is the closure in $\mathcal{P}$ of a scroll over this hypersurface. When n is odd, Y is a blow-up of $\mathbf{P}(U)$. Being X birational to Y, we deduce the irreducibility of X as well. The other properties are more general and were proved in Proposition 2.4. □

By Proposition 2.17, the dualizing sheaf of X is

$$\omega_X = S^{n-m-1}\mathscr{L} \otimes \mathcal{O}_{\mathbf{P}(V)}(-2). \tag{3.4}$$

Note that, by hypothesis, $n - m - 1 > 0$.

3.3 An upper bound for $\mathrm{h}^0(X, \mathcal{N})$

The aim of this section is to provide an upper bound for $\mathrm{h}^0(X, \mathcal{N})$. Since we have $\mathrm{H}^0(X, \mathcal{N}) \cong \mathrm{H}^0(\mathbf{P}(V), i_*\mathcal{N})$, we can make use of the isomorphism provided by Proposition 2.21.

By Lemma 2.19 and since

$$\mathscr{H}om_{\mathbf{P}(V)}(\mathscr{C}, \mathscr{C}) \cong i_*\mathscr{H}om_X(\mathscr{L}, \mathscr{L}),$$

we have $\mathscr{H}om_{\mathbf{P}(V)}(\mathscr{C}, \mathscr{C}) \cong i_*\mathcal{O}_X$. If we apply $\mathscr{H}om_{\mathbf{P}(V)}(-, \mathscr{C})$ to sequence 2.6, we get the following diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & i_*\mathcal{O}_X \arrow[r] & \mathscr{C}^m \arrow[r] \arrow[d, equal] & \mathscr{H}om_{\mathbf{P}(V)}(\mathrm{Im}(\varphi), \mathscr{C}) \arrow[r] \arrow[d] & i_*\mathcal{N} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & i_*\mathcal{O}_X \arrow[r] & \mathscr{C}^m \arrow[r, "\psi"] & \Omega_{\mathbf{P}(V)}(2) \otimes \mathscr{C} \arrow[r] & \mathscr{Q} \arrow[r] & 0
\end{tikzcd} \tag{3.5}$$

where $\mathscr{Q}$ is defined as the cokernel of ψ and $\mathscr{C}^m$ replaces $U^* \otimes \mathscr{C}$ for short. Via the snake lemma we deduce that the map $i_*\mathcal{N} \to \mathscr{Q}$ is an injection, providing an upper bound

$$\mathrm{h}^0(X, \mathcal{N}) \leq \mathrm{h}^0(\mathbf{P}(V), \mathscr{Q}). \tag{3.6}$$

By computing $\mathrm{h}^0(\mathbf{P}(V), \mathscr{Q})$ and by Theorem 1.1, we will have an upper bound for the dimension of $\mathcal{H}$.

3.3.1 Cohomology computations

The main tool to compute the cohomology groups of the second row of diagram (3.5) is the Koszul complex (3.1). Making use of it, we give the following lemmas.

Lemma 3.2. The cohomology groups of $\mathcal{O}_Y$ are of dimension

$$\mathrm{h}^i(Y,\mathcal{O}_Y)=\begin{cases}1 & \text{if } i=0\\ \binom{\frac{n}{2}-1}{\frac{n}{2}-m} & \text{if } i=m-2, n \text{ even}, n\geq 2m\\ 0 & \text{otherwise}\end{cases}$$

Proof. By means of Künneth and Bott formulas (cfr. Sect. 1.1.1), we are able to compute the cohomology groups of the r-th term in the Koszul complex (3.1). For $1\leq r\leq n-1$ we get

$$\begin{aligned}&\mathrm{h}^i(\mathcal{P},\mathcal{O}_{\mathbf{P}(U)}(-r)\boxtimes\Omega^{n-r-1}_{\mathbf{P}(V)}(n-2r))=\\ &\qquad=\begin{cases}\binom{\frac{n}{2}-1}{\frac{n}{2}-m} & \text{if } n \text{ even}, n\geq 2m, i=\frac{n}{2}+m-2, r=\frac{n}{2}\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

so there is at most one non-vanishing cohomology group. We have

$$\begin{aligned}\mathrm{H}^{\frac{n}{2}+m-2}(\mathcal{P},\mathcal{O}_{\mathbf{P}(U)}(-\frac{n}{2})\boxtimes\Omega^{\frac{n}{2}-1}_{\mathbf{P}(V)}) &\cong \mathrm{H}^{\frac{n}{2}+m-2}(\mathcal{P},\ker(\epsilon_{\frac{n}{2}-1}))\\ &\cong \mathrm{H}^{\frac{n}{2}+m-3}(\mathcal{P},\ker(\epsilon_{\frac{n}{2}-2}))\\ &\cong \dots\\ &\cong \mathrm{H}^{m-1}(\mathcal{P},\ker(\epsilon_0)).\end{aligned}$$

As soon as we consider the long exact sequence coming from the short exact sequence

$$0\longrightarrow\ker(\epsilon_0)\longrightarrow\mathcal{O}_{\mathcal{P}}\longrightarrow\mathcal{O}_Y\longrightarrow 0,$$

we get the result. $\square$

Lemma 3.3. The cohomology groups of $\mathcal{O}_Y^m(1,0)$ are of dimension

$$\mathrm{h}^i(Y,\mathcal{O}_Y^m(1,0))=\begin{cases}m^2 & \text{if } i=0\\ m\binom{\frac{n}{2}-2}{\frac{n}{2}-m-1} & \text{if } i=m-2, n \text{ even}, n\geq 2m+2\\ 0 & \text{otherwise}\end{cases}$$

Proof. The Koszul complex (3.1) twisted by $\mathcal{O}_{\mathcal{P}}(1,0)$ is a locally free resolution of $\mathcal{O}_Y(1,0)$. Again by means of Künneth and Bott formulas, we can compute the cohomology of the r-th term, $1\leq r\leq n-1$, in such resolution:

$$\begin{aligned}&\mathrm{h}^i(\mathcal{P},\mathcal{O}_{\mathbf{P}(U)}(1-r)\boxtimes\Omega^{n-r-1}_{\mathbf{P}(V)}(n-2r))=\\ &\qquad=\begin{cases}\binom{\frac{n}{2}-2}{\frac{n}{2}-m-1} & \text{if } n \text{ even}, n\geq 2m+2, i=\frac{n}{2}+m-2, r=\frac{n}{2}\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

As in the proof of the previous lemma, we obtain

$$\mathrm{H}^{\frac{n}{2}+m-2}(\mathcal{P}, \mathcal{O}_{\mathbf{P}(U)}(1-\frac{n}{2}) \boxtimes \Omega^{\frac{n}{2}-1}_{\mathbf{P}(V)}) \cong \mathrm{H}^{m-1}(\mathcal{P}, \ker(\epsilon_0{}')),$$

where $\epsilon_0{}'$ is the map ϵ_0 in the Koszul complex twisted by $\mathcal{O}_{\mathcal{P}}(1,0)$. The result follows by considering the cohomology groups of the short exact sequence

$$0 \longrightarrow \ker(\epsilon_0{}') \longrightarrow \mathcal{O}_{\mathcal{P}}(1,0) \longrightarrow \mathcal{O}_Y(1,0) \longrightarrow 0. \qquad \square$$

Lemma 3.4. The cohomology groups of $q^*\Omega_{\mathbf{P}(V)}(2) \otimes \mathcal{O}_Y(1,0)$ have dimension

$$\mathrm{h}^i(q^*\Omega_{\mathbf{P}(V)}(2) \otimes \mathcal{O}_Y(1,0)) =$$
$$= \begin{cases} m\binom{n}{2} - 1 & \text{if } i = 0, m > 3 \\ \binom{\frac{n}{2}-1}{\frac{n}{2}-m} & \text{if } i = m-3, n \text{ even}, n \geq 2m > 6 \\ n\binom{\frac{n-3}{2}}{\frac{n-1}{2}-m} & \text{if } i = m-3, n \text{ odd}, n \geq 2m > 6 \\ \frac{1}{8}n(13n-18) & \text{if } i = 0, n \text{ even}, m = 3 \\ \frac{1}{8}(n-1)(n^2+5n+8) & \text{if } i = 0, n \text{ odd}, m = 3 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The Koszul complex (3.1) twisted by $\mathcal{O}_{\mathbf{P}(U)}(1) \boxtimes \Omega_{\mathbf{P}(V)}(2)$ is a resolution of $q^*\Omega_{\mathbf{P}(V)}(2) \otimes \mathcal{O}_Y(1,0)$; let us denote by δ_r its differentials. If

$$\mathcal{G}_r := \Omega^{n-r-1}_{\mathbf{P}(V)}(n-2r) \otimes \Omega_{\mathbf{P}(V)}(2),$$

its r-th term is $\mathcal{O}_{\mathbf{P}(U)}(1-r) \boxtimes \mathcal{G}_r$.

To compute the cohomology groups of $\mathcal{G}_r$, we consider the twisted Euler sequence (2.10), tensored by $\Omega^{n-r-1}_{\mathbf{P}(V)}(n-2r)$:

$$0 \longrightarrow \mathcal{G}_r \longrightarrow V \otimes \Omega^{n-r-1}_{\mathbf{P}(V)}(n-2r+1) \longrightarrow \Omega^{n-r-1}_{\mathbf{P}(V)}(n-2r+2) \longrightarrow 0. \tag{3.7}$$

For any $1 < r < n-1$, by Bott formula we have

$$\mathrm{h}^i(\mathbf{P}(V), V \otimes \Omega^{n-r-1}_{\mathbf{P}(V)}(n-2r+1)) = \begin{cases} n & \text{if } i = r-2, 2r = n+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{h}^i(\mathbf{P}(V), \Omega^{n-r-1}_{\mathbf{P}(V)}(n-2r+2)) = \begin{cases} \binom{n}{2} & \text{if } i = 0, r = 2, n \geq 3 \\ 1 & \text{if } i = r-3 > 0, 2r = n+2 \\ 0 & \text{otherwise} \end{cases}$$

From the long exact sequence induced by (3.7), we get, for any $1<r<n-1$,

$$\mathrm{h}^i\left(\mathbf{P}(V),\mathcal{G}_r\right)=\begin{cases}\binom{n}{2} & \text{if } i=1, r=2\\ n & \text{if } i=r-2\geq 1, 2r=n+1\\ 1 & \text{if } i=r-2\geq 1, 2r=n+2\\ 0 & \text{otherwise}\end{cases}$$

The cohomology groups of $\mathcal{G}_0$ and $\mathcal{G}_{n-1}$ can be computed directly from Bott formula. When $r=1$, one has $\mathcal{G}_1\cong \mathscr{E}nd(\mathcal{T}_{\mathbf{P}(V)})$, for which the only non-vanishing group is $\mathrm{H}^0(\mathbf{P}(V),\mathscr{E}nd(\mathcal{T}_{\mathbf{P}(V)}))\cong\mathbf{k}$.

Again by Künneth formula, we get

$$\mathrm{h}^i(\mathcal{P},\mathcal{O}_{\mathbf{P}(U)}(1-r)\boxtimes\mathcal{G}_r)=\begin{cases}\binom{\frac{n-2}{2}}{\frac{n}{2}-m} & \text{if } r=\frac{n+2}{2}, i=m+\frac{n-4}{2}, m\leq\frac{n}{2}\\ n\binom{\frac{n-3}{2}}{\frac{n-1}{2}-m} & \text{if } r=\frac{n+1}{2}, i=m+\frac{n-5}{2}, m\leq\frac{n-1}{2}\\ 1 & \text{if } r=1, i=0\\ m\binom{n}{2} & \text{if } r=0, i=0\\ 0 & \text{otherwise}\end{cases}$$

Let $\mathrm{par}(n)$ be the parity of n, i.e. $\mathrm{par}(n)=1$ if n odd and 0 otherwise. Fix $\bar{r}:=\frac{n+2-\mathrm{par}(n)}{2}$. Since $\mathcal{G}_r$ has zero cohomology for $r\notin\{0,1,\bar{r}\}$, we have

$$\begin{aligned}\mathrm{H}^{m+\frac{n-4-\mathrm{par}(n)}{2}}(\mathcal{P},\mathcal{O}_{\mathbf{P}(U)}(1-\bar{r})\boxtimes\mathcal{G}_{\bar{r}}) &\cong \mathrm{H}^{m+\frac{n-4-\mathrm{par}(n)}{2}}(\mathcal{P},\ker(\delta_{\bar{r}-1}))\\ &\cong \mathrm{H}^{m+\frac{n-4-\mathrm{par}(n)}{2}-1}(\mathcal{P},\ker(\delta_{\bar{r}-2}))\\ &\cong\dots\\ &\cong \mathrm{H}^{m-1}(\mathcal{P},\ker(\delta_1)).\end{aligned}$$

The next step gives us

$$\mathrm{h}^i(\mathcal{P},\ker(\delta_0))=\begin{cases}1 & \text{if } i=0\\ \binom{\frac{n-2}{2}}{\frac{n}{2}-m} & \text{if } i=m-2, n \text{ even}, n\geq 2m\\ n\binom{\frac{n-3}{2}}{\frac{n-1}{2}-m} & \text{if } i=m-2, n \text{ odd}, n\geq 2m\\ 0 & \text{otherwise}\end{cases}$$

which, taking into account the short exact sequence

$$0\longrightarrow\ker(\delta_0)\longrightarrow\mathcal{G}_0\longrightarrow q^*\Omega_{\mathbf{P}(V)}(2)\otimes\mathcal{O}_Y(1,0)\longrightarrow 0,$$

is enough to conclude. $\square$

Remark 3.5. The previous lemmas are enough to compute the cohomology groups of the sheaves appearing in the second row of (3.5). Indeed, the direct images via q of $\mathcal{O}_Y$, $\mathcal{O}_Y^m(1,0)$, $q^*\Omega_{\mathbf{P}(V)}(2) \otimes \mathcal{O}_Y(1,0)$ are respectively $\mathcal{O}_X$, $\mathscr{C}^m$, and $\mathscr{C} \otimes \Omega_{\mathbf{P}(V)}(2)$. For each of them, we have equalities

$$\mathrm{h}^i(\mathcal{P}, -) = \mathrm{h}^i(\mathbf{P}(V), q_*(-))$$

for any i, as already explained in Remark 2.18.

We are ready to compute the dimension of $\mathrm{H}^0(\mathbf{P}(V), \mathscr{Q})$. Since we want to show that ρ is birational, we compare $\mathrm{h}^0(\mathbf{P}(V), \mathscr{Q})$ with the dimension of $\mathbf{Gr}(m, \Lambda^2 V)$.

Proposition 3.6.

i. *For any $m > 3$ we have* $\mathrm{h}^0(\mathbf{P}(V), \mathscr{Q}) = \dim \mathbf{Gr}(m, \Lambda^2 V)$.

ii. *For $m = 3$ and $n \geq 5$, we have*

$$\mathrm{h}^0(\mathbf{P}(V), \mathscr{Q}) - \dim \mathbf{Gr}(3, \Lambda^2 V) = \begin{cases} \frac{3}{8}(n-4)(n-6) & \textit{if } n \textit{ even} \\ \frac{1}{8}n(n-3)(n-5) & \textit{if } n \textit{ odd} \end{cases}$$

and, in particular, $\mathrm{h}^0(\mathbf{P}(V), \mathscr{Q}) = \dim \mathbf{Gr}(3, \Lambda^2 V)$ *if $n = 5$ or $n = 6$.*

Proof. We can compute $\mathrm{h}^0(\mathbf{P}(V), \mathscr{Q})$ from the second row of diagram (3.5); the cohomology groups are given by Lemmas 3.2, 3.3 and 3.4 (cfr. Remark 3.5). This computation proves the statement in all cases but $n \geq 8$, $m = 4$ and n even. For the remaining cases the argument is the following. By the forthcoming Lemma 3.7, if $n > 2m-3$ we have $\mathrm{h}^0(\mathbf{P}(V), \mathscr{Q}) = \mathrm{h}^0(X, \mathcal{N})$; so to conclude it is sufficient to prove the equality $\mathrm{h}^0(X, \mathcal{N}) = \dim \mathbf{Gr}(m, \Lambda^2 V)$ for $m = 4$, n even and $n \geq 8$, but this has been done in [FF10b, Theorem 1]. □

We conclude this section with the following

Lemma 3.7. If X is smooth, then $\mathrm{h}^k(\mathbf{P}(V), \mathscr{Q}) = \mathrm{h}^k(X, \mathcal{N})$ for any k.

Proof. If X is smooth, the sheaves $\mathscr{K}$ and $\mathscr{C}$, defined in (2.6), are vector bundles on X. By [GG73, Exercise VI.1(6)], we have $\mathcal{N} \cong (\mathscr{K}|_X)^* \otimes \mathscr{L}$. Applying the functor $\mathscr{H}om_X(-, \mathscr{L})$ to the sequence (2.6) restricted to X, since $\mathscr{H}om_X(\mathscr{L}, \mathscr{L}) = \mathcal{O}_X$ (Lemma 2.19) and $\mathscr{E}xt^1_X(\mathscr{L}, \mathscr{L}) = 0$ ($\mathscr{L}$ is a line bundle), one has

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{L}^m \longrightarrow \mathscr{H}om_X(\mathrm{Im}(\varphi)|_X, \mathscr{L}) \longrightarrow 0;$$

as $\mathscr{E}xt^1_X(\mathrm{Im}(\varphi)|_X, \mathscr{L}) \cong \mathscr{E}xt^2_X(\mathscr{L}, \mathscr{L}) = 0$, one also has

$$0 \longrightarrow \mathscr{H}om_X(\mathrm{Im}(\varphi)|_X, \mathscr{L}) \longrightarrow \mathscr{L} \otimes \Omega_{\mathbf{P}(V)}(2)\big|_X \longrightarrow \mathcal{N} \longrightarrow 0.$$

As these two sequences fit together to the restriction to X of the second row of diagram (3.5), the conclusion follows. □

This lemma shows that, even though $\mathrm{h}^0(\mathbf{P}(V), \mathscr{Q})$ provides only an upper bound for the dimension of $\mathscr{C}$ (inequality (3.6)), when X is smooth the link between $\mathscr{Q}$ and $\mathcal{N}$ is deeper.

3.4 Injectivity and birationality of ρ

This section's purpose is to prove the general injectivity and the birationality of ρ, which are the main results of this paper.

Theorem 3.8. *The map $\rho : \mathbf{Gr}(m, \Lambda^2 V) \dashrightarrow \mathcal{H}$ is injective on its domain of definition for all (m, n) such that $3 \leq m < n - 1$, with the unique exception $(m, n) = (3, 6)$.*

On the one hand, this theorem says that we can identify an open subset of $\mathbf{Gr}(m, \Lambda^2 V)$ with an open subset of a subscheme of $\mathcal{H}$; on the other hand, it gives the lower bound

$$\dim \mathbf{Gr}(m, \Lambda^2 V) \leq \dim \mathcal{H}, \tag{3.8}$$

which will be fundamental in the proof of the birationality of ρ (Theorem 3.13).

The proof of Theorem 3.8 uses an argument analogous to the one used in the proof of [FF10b, Lemma 9]. We need some preliminary results.

Proposition 3.9. *Using the notations of the previous sections, let X_1, X_2 be the degeneracy loci of two morphisms $\varphi_1, \varphi_2 : \mathcal{T}_{\mathbf{P}(V)}(-2) \to U \otimes \mathcal{O}_{\mathbf{P}(V)}$; for $j = 1, 2$, let $\mathscr{C}_j = (i_j)_*(\mathscr{L}_j) = \mathrm{coker}(\varphi_j)$ and let $\bar{q}_j : Y_j \to X_j$ be the projection on $\mathbf{P}(V)$, which is an isomorphism on $Y_j \setminus Y'_j$. Assume that $(m, n) \in$*

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} \text{ such that } 3 \leq m < n - 1\} \setminus \{(3, 6)\}.$$

If $X_1 = X_2$, then $\mathscr{C}_1 \cong \mathscr{C}_2$.

Proof. Being $X := X_1 = X_2$, we deduce by (3.4) that

$$S^{n-m-1}\mathscr{L}_1 \otimes \mathcal{O}_{\mathbf{P}(V)}(-2) \cong S^{n-m-1}\mathscr{L}_2 \otimes \mathcal{O}_{\mathbf{P}(V)}(-2),$$

hence

$$S^{n-m-1}\mathscr{L}_1 \cong S^{n-m-1}\mathscr{L}_2. \tag{3.9}$$

Recall that $\mathscr{L}_j$ is a line bundle on the smooth locus X^{sm}, whose complement has codimension at least three by (2.2). Let D_j be the closure in X of the zero locus of a general element $\eta_j \in \mathrm{H}^0(X^{\mathrm{sm}}, \mathscr{L}_j|_{X^{\mathrm{sm}}})$. Being X, Y normal and irreducible (Proposition 3.1), we are allowed to consider their divisor class groups. Moreover, $\mathscr{L}_j$ is reflexive (Proposition 2.13), so it is determined uniquely by the class of D_j by Remark 2.14. We have

$$\mathscr{C}_1 \cong \mathscr{C}_2 \quad \Leftrightarrow \quad \mathscr{L}_1 \cong \mathscr{L}_2 \quad \Leftrightarrow \quad D_1 \sim D_2,$$

where with $D_1 \sim D_2$ we mean that the two Weil divisors D_j are linearly equivalent, i.e. they represent the same class in $\mathrm{Cl}(X)$. By [Har77, Proposition II.6.5] it follows that $\mathrm{Cl}(X) \cong \mathrm{Cl}(X^{\mathrm{sm}})$; by (2.5), also $\mathrm{Cl}(Y_j \setminus Y_j') \cong \mathrm{Cl}(Y_j)$. As $\bar{q}_1$ is an isomorphism $Y_1 \setminus Y_1' \to X^{\mathrm{sm}}$, we have

$$\mathrm{Cl}(X) \cong \mathrm{Cl}(X^{\mathrm{sm}}) \cong \mathrm{Cl}(Y_1 \setminus Y_1') \cong \mathrm{Cl}(Y_1). \tag{3.10}$$

Consider now the Weil divisor $(n-m-1)D_j$, seen as the closure in X of the zero locus of the section $\eta_j^{n-m-1} \in \mathrm{H}^0(X^{\mathrm{sm}}, (S^{n-m-1}\mathscr{L}_j)\big|_{X^{\mathrm{sm}}})$. From (3.9) we deduce that $(n-m-1)D_1 \sim (n-m-1)D_2$; moreover,

$$\begin{array}{ccc}
(n-m-1)D_1 & =_{\mathrm{Cl}(X)} & (n-m-1)D_2 \\
& \Updownarrow & \\
(n-m-1)\ D_1|_{X^{\mathrm{sm}}} & =_{\mathrm{Cl}(X^{\mathrm{sm}})} & (n-m-1)\ D_2|_{X^{\mathrm{sm}}} \\
& \Updownarrow & \\
(n-m-1)\ (\bar{q}_1^* D_1)|_{Y_1 \setminus Y_1'} & =_{\mathrm{Cl}(Y_1 \setminus Y_1')} & (n-m-1)\ (\bar{q}_1^* D_2)|_{Y_1 \setminus Y_1'} \\
& \Updownarrow & \\
(n-m-1)(\bar{q}_1^* D_1) & =_{\mathrm{Cl}(Y_1)} & (n-m-1)(\bar{q}_1^* D_2).
\end{array}$$

Being Y_1 smooth, one has $\mathrm{Cl}(Y_1) \cong \mathrm{Pic}(Y_1)$. The latter is torsion-free: indeed, if n is odd, Y is a blow-up of $\mathbf{P}(U)$ (cfr. Sect. 3.2). If n is even, this is proved in [FF10b, Lemma 3] making use of the fact that the Pfaffian hypersurface cut out by $\mathrm{Pf}(N_{\varphi_1})$ has torsion-free Picard group, for $(m,n) \neq (3,6)$.

As $\mathrm{Pic}(Y_1)$ has no torsion, we can deduce the equality $\bar{q}_1^* D_1 =_{\mathrm{Cl}(Y_1)} \bar{q}_1^* D_2$, which induces by (3.10) the desired relation $D_1 \sim D_2$. $\square$

Remark 3.10. In the case $(m, n) = (3, 6)$, the last proposition does not guarantee the general injectivity of ρ. Indeed, in this case the Picard group of the hypersurface in $\mathbf{P}(U)$ cut out by $\mathrm{Pf}(N)$ has torsion. In fact, it was proved in [BM01] and classically known to Fano [Fan30] that in this case ρ is $4:1$. As the map is finite and dominant, we have an equality between the dimensions of $\mathbf{Gr}(m, \Lambda^2 V)$ and $\mathcal{H}$, which was confirmed in Proposition 3.6.

Lemma 3.11. For all $3 \leq m < n-1$ we have

$$\mathrm{h}^0(\mathbf{P}(V), \mathrm{Im}(\varphi)) = \mathrm{h}^1(\mathbf{P}(V), \mathrm{Im}(\varphi)) = 0.$$

Proof. Using the notations of the proof of Lemma 3.3, we have $q_* \ker({\epsilon_0}') = \mathrm{Im}(\varphi)$. It is sufficient (cfr. Sect. 2.5 and (2.14)) to check the vanishings

$$\mathrm{h}^0(\mathcal{P}, \ker({\epsilon_0}')) = \mathrm{h}^1(\mathcal{P}, \ker({\epsilon_0}')) = 0.$$

In the proof of Lemma 3.3 we computed that the only possible non-zero cohomology group of $\ker({\epsilon_0}')$ is the $(m-1)$-th, hence the conclusion. $\square$

Lemma 3.12. For all $3 \leq m < n-1$ we have

$$\mathrm{h}^1(\mathbf{P}(V), \mathscr{K} \otimes \Omega_{\mathbf{P}(V)}(2)) = 0,$$

where $\mathscr{K}$ was defined in (2.6).

Proof. Using the notations of the proof of Lemma 3.4, we have $q_* \ker(\delta_1) = \mathscr{K} \otimes \Omega_{\mathbf{P}(V)}(2)$. By the same argument as above, it is sufficient to check the vanishing of $\mathrm{h}^1(\mathcal{P}, \ker(\delta_1))$. In the proof of Lemma 3.4 we computed that the only possible non-zero cohomology group of $\ker(\delta_1)$ is the $(m-1)$-th, hence the conclusion. $\square$

We are now ready to prove Theorem 3.8.

Proof of Theorem 3.8. Fix the notations as in Proposition 3.9 and suppose that $X_1 = X_2$. By Proposition 3.9, this equality induces an isomorphism $\alpha : \mathscr{C}_1 \to \mathscr{C}_2$. We are in the following scenario

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{K}_1 \arrow[r] & \mathcal{T}_{\mathbf{P}(V)}(-2) \arrow[r, "\varphi_1"] \arrow[d, dotted, "\exists\gamma"] & U \otimes \mathcal{O}_{\mathbf{P}(V)} \arrow[r, "\pi_1"] \arrow[d, dotted, "\exists\beta"] & \mathscr{C}_1 \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \mathscr{K}_2 \arrow[r] & \mathcal{T}_{\mathbf{P}(V)}(-2) \arrow[r, "\varphi_2"] & U \otimes \mathcal{O}_{\mathbf{P}(V)} \arrow[r, "\pi_2"] & \mathscr{C}_2 \arrow[r] & 0
\end{tikzcd}$$

We want to show that

- the isomorphism α induces isomorphisms β and γ such that the diagram above commutes;
- up to multiply α by a scalar, we may assume that γ is the identity map.

In this way, we get that φ_1 and φ_2 belong to the same orbit with respect to the action of $\mathrm{GL}(U)$, i.e. they represent the same point in $\mathbf{Gr}(m, \Lambda^2 V)$.

Let us compose π_1 with α. In order to show that such a map can be lifted up to β, we apply the functor $\mathrm{Hom}_{\mathbf{P}(V)}(U \otimes \mathscr{O}_{\mathbf{P}(V)}, -)$ to the sequence

$$0 \longrightarrow \mathrm{Im}(\varphi_2) \longrightarrow U \otimes \mathscr{O}_{\mathbf{P}(V)} \longrightarrow \mathscr{C}_2 \longrightarrow 0.$$

Since the last term is

$$\mathrm{Ext}^1_{\mathbf{P}(V)}(U \otimes \mathscr{O}_{\mathbf{P}(V)}, \mathrm{Im}(\varphi_2)) \cong U^* \otimes \mathrm{H}^1(\mathbf{P}(V), \mathrm{Im}(\varphi_2))$$

and its vanishing is guaranteed by Lemma 3.11, we get

$$\mathrm{End}_{\mathbf{P}(V)}(U \otimes \mathscr{O}_{\mathbf{P}(V)}) \longrightarrow \mathrm{Hom}_{\mathbf{P}(V)}(U \otimes \mathscr{O}_{\mathbf{P}(V)}, \mathscr{C}_2) \longrightarrow 0.$$

Therefore, we can lift up α to β; to check that β is an isomorphism, we observe that $\ker(\beta)$ is free and its image via π_1 is zero by commutativity, so we have a map $\ker(\beta) \to \mathrm{Im}(\varphi)$. By Lemma 3.11, this map has to be zero and so $\ker(\beta)$ is trivial.

To lift up β to γ, we apply the functor $\mathrm{Hom}_{\mathbf{P}(V)}(\mathscr{T}_{\mathbf{P}(V)}(-2), -)$ to the sequence

$$0 \longrightarrow \mathscr{K}_2 \longrightarrow \mathscr{T}_{\mathbf{P}(V)}(-2) \longrightarrow \mathrm{Im}(\varphi_2) \longrightarrow 0$$

to get

$$\mathrm{End}_{\mathbf{P}(V)}(\mathscr{T}_{\mathbf{P}(V)}(-2)) \longrightarrow \mathrm{Hom}_{\mathbf{P}(V)}(\mathscr{T}_{\mathbf{P}(V)}(-2), \mathrm{Im}(\varphi_2)) \longrightarrow 0;$$

indeed, the last term should be

$$\mathrm{Ext}^1_{\mathbf{P}(V)}(\mathscr{T}_{\mathbf{P}(V)}(-2), \mathscr{K}_2) \cong \mathrm{H}^1(\mathbf{P}(V), \mathscr{K}_2 \otimes \Omega_{\mathbf{P}(V)}(2))$$

and its vanishing is guaranteed by Lemma 3.12. Therefore, β can be lifted up to γ.

Let us notice that γ is non-zero and so it is a non-zero multiple $\lambda \mathrm{I}$ of the identity map, as $\mathcal{T}_{\mathbf{P}(V)}(-2)$ is simple. Finally, the conclusion follows as soon as we substitute α, β with their multiples $\lambda^{-1}\alpha, \lambda^{-1}\beta$, so we may take $\gamma = \mathrm{I}$. $\square$

Theorem 3.13. *The map ρ is birational for all (m, n) such that $4 \leq m < n - 1$, and for $(m, n) = (3, 5)$.*

Proof. In the supposed range, we have

$$\begin{aligned} \dim \mathbf{Gr}(m, \Lambda^2 V) &\leq \dim \mathcal{H} && (3.8) \\ &\leq \mathrm{h}^0(\mathbf{P}(V), i_* \mathcal{N}) && \text{Theorem 1.1} \\ &\leq \mathrm{h}^0(\mathbf{P}(V), \mathcal{Q}) && (3.6) \\ &= \dim \mathbf{Gr}(m, \Lambda^2 V). && \text{Proposition 3.6} \end{aligned}$$

It this way we see that ρ is dominant; by Theorem 3.8, ρ is also generically injective, so it is birational. $\square$

Corollary 3.14. In the hypotheses of Theorem 3.13, we obtain that $\mathcal{H}$ is irreducible and generically smooth.

3.5 The case $m = 3$: surfaces

In this section we will discuss the case $m = 3$ and n odd.

By Theorems 3.8 and 3.13, the map ρ is generically injective but not dominant as soon as $n \geq 7$, so we can identify an open subset of $\mathbf{Gr}(3, \Lambda^2 V)$ with an open subset of a subscheme of $\mathcal{H}$. Our aim is to determine its codimension and describe geometrically the points in $\mathrm{Im}(\rho)$ and in $\mathcal{H}$, explaining why a general point of $\mathcal{H}$ cannot be obtained as the degeneracy locus of a morphism $\mathcal{T}_{\mathbf{P}(V)}(-2) \to \mathcal{O}_{\mathbf{P}(V)} \otimes U$.

Proposition 3.15. *If $m = 3$, we have $\operatorname{codim}_{\mathcal{H}} \mathrm{Im}(\rho) = \frac{1}{8} n(n-3)(n-5)$ if n is odd, and $\operatorname{codim}_{\mathcal{H}} \mathrm{Im}(\rho) = \frac{3}{8}(n-4)(n-6)$ if n is even.*

Proof. By Lemma 3.7 and Proposition 3.6, it suffices to show that $\mathcal{H}$ is generically smooth along $\mathrm{Im}(\rho)$. By (3.3) X is smooth; hence, by Theorem 1.1, $\mathcal{H}$ is smooth at X as soon as we prove the vanishing of $\mathrm{h}^1(X, \mathcal{N}) = \mathrm{h}^1(\mathbf{P}(V), \mathcal{Q})$. This can be obtained considering the second row of diagram (3.5) and by means of Lemmas 3.2, 3.3 and 3.4. $\square$

From now on, let us fix $m = 3$ and let us suppose n is odd, satisfying $n \geq 7$. Note that all the following results hold also in the case $n = 5$; see Remark 3.23.

3.5.1 Veronese surfaces in $\mathbf{P}(V)$

Firstly we observe that n is always greater than $2m-3=3$, so by (3.3) X is smooth; therefore, in the settings of the previous sections, Y and X turn out to be isomorphic via $\bar{q}$.

On the one hand, as we saw in Sect. 3.2, Y is the blow-up of $\mathbf{P}(U)$ along the subscheme cut out by the $(n-1)\times(n-1)$ Pfaffians (Pf_i) of N; for the general choice of φ, the ideal generated by these Pfaffians has codimension three and so its associated subscheme is empty.

On the other hand, X is the image of the regular map given by the Pf_i's, the Pfaffians of the matrix obtained by deleting the i-th row and column from N. Being these Pfaffians forms of degree $\frac{n-1}{2}$, linearly independent for the general φ, we can complete them to a basis $\{\mathrm{Pf}_1,\dots,\mathrm{Pf}_n,C_1,\dots,C_{r-n+1}\}$ of $\mathbf{k}[y_0,y_1,y_2]_{\frac{n-1}{2}}$ and use this complete linear system of curves to embed $\mathbf{P}(U)$ in $\mathbb{P}^r$, where

$$r=\dim\left(\mathbf{k}[y_0,y_1,y_2]_{\frac{n-1}{2}}\right)-1=\binom{\frac{n-1}{2}+2}{2}-1.$$

The variety X can be seen as the projection to $\mathbf{P}(V)$ of this Veronese surface in $\mathbb{P}^r$ with respect to the center spanned by the C_i's.

$$\begin{tikzcd}
\mathbf{P}(U) \arrow[r, "{[\mathrm{Pf}_1:\ldots:\mathrm{Pf}_n:C_1:\ldots:C_{r-n+1}]}"] \arrow[dr, "{[\mathrm{Pf}_1:\ldots:\mathrm{Pf}_n]}"'] & \mathbb{P}^r \arrow[d, dashed] \\
& \mathbf{P}(V).
\end{tikzcd} \tag{3.11}$$

However, not every n-uple of forms of degree $\frac{n-1}{2}$ is the set of Pfaffians of a matrix N, and this is the reason why ρ is not dominant: only Veronese surfaces parametrized by Pfaffians are contained in $\mathrm{Im}(\rho)$. In the next section we will explore more this phenomenon.

3.5.2 Apolarity and special projections

When n is even, the general element of $\mathrm{Im}(\rho)$ is a projective bundle $\mathbb{P}(\mathscr{G})$ obtained from a general stable rank-two vector bundle $\mathscr{G}$ on a general plane curve C of degree $\frac{n}{2}$, with determinant $\det(\mathscr{G})=\mathscr{O}_C(\frac{n-2}{2})$; this description was given in [FF10b].

Our aim is to provide a similar description in the odd case. As seen before, the degeneracy locus X is a Veronese surface parametrized by the n Pfaffians of order $n-1$ of the matrix N. X can be thought of as the image of the Veronese surface suitably projected from $\mathbb{P}^r$ to $\mathbf{P}(V)$ (cfr. (3.11)).

Let R be the polynomial ring $\mathrm{H}^0(\mathbf{P}(U), \mathcal{O}_{\mathbf{P}(U)}(1)) = \mathbf{k}[y_0, y_1, y_2]$. Let $S = \mathbf{k}[\partial_0, \partial_1, \partial_2]$; recall that R acts on S (and conversely) by differentiation (1.7).

Theorem 3.16. *Let $G \in R$ be a non-degenerate form of degree $n-3$. Consider a Veronese surface embedded via $|\mathcal{O}_{\mathbf{P}(U)}(\frac{n-1}{2})|$ in $\mathbb{P}^r$, where $r = \binom{\frac{n-1}{2}+2}{2} - 1$; then its projection X in $\mathbf{P}(V)$ with respect to the center spanned by the elements $\{\partial^\alpha(G)\}_{|\alpha|=\frac{n-5}{2}}$ is contained in $\mathrm{Im}(\rho)$.*
Conversely, a general element of $\mathrm{Im}(\rho)$ arises as such a projection.

This characterization is based on Macaulay correspondence (Theorem 1.12). In this case, Macaulay correspondence can be rewritten by means of Theorem 1.8, linking homogeneous polynomials in S with skew-symmetric matrices of linear forms on $\mathbf{P}(U)$. Moreover, if we focus only on non-degenerate polynomials (in the sense of Definition 1.10), the correspondence restricts to linear skew-symmetric matrices.

Proposition 3.17.

i. *The map $F \mapsto F^\perp$ gives a bijection between polynomials $F \in S$ of degree $n-3$, up to scalars, with $n \geq 5$ odd, and Artinian graded (Gorenstein) ideals I of codimension three in R, with socle in degree $n-3$, generated by the Pfaffians of a skew-symmetric matrix of forms of positive degrees in R.*

ii. *This correspondence restricts to a one-to-one correspondence between non-degenerate polynomials $F \in S$ of degree $n-3$, up to scalars, with $n \geq 5$ odd, and Artinian graded (Gorenstein) ideals I of codimension three in R generated in degree $\frac{n-1}{2}$ by the n Pfaffians of a $n \times n$ skew-symmetric matrix of linear forms in R.*

Proof.

i. By Macaulay correspondence, $A^F = R/F^\perp$ is an Artinian graded Gorenstein ring. Being Artinian, $F^\perp$ is irrelevant and so it has codimension three; we can therefore apply Theorem 1.8 and conclude.

   Conversely, an ideal I satisfying the hypotheses has codimension three in $R = \mathbf{k}[y_0, y_1, y_2]$, so it is irrelevant and therefore R/I is an Artinian graded Gorenstein ring with socle in degree $n-3$. We conclude again by Macaulay correspondence.

ii. Let $F \in S$ be a non-degenerate form of degree $n-3$ and let $I = F^{\perp}$ its Gorenstein, codimension-three associated ideal. The partial derivatives of order $\frac{n-3}{2}$ of F span the whole space $S_{\frac{n-3}{2}}$; therefore, by Proposition 1.11, I is zero in degree $\leq \frac{n-3}{2}$. Moreover, a computation shows that $\dim I_{\frac{n-1}{2}} = n$. Let ν be the minimal number of generators of I (hence $\nu \geq n$). By Theorem 1.8, I is generated by the ν Pfaffians of a $\nu \times \nu$ skew-symmetric matrix of homogeneous forms of degree at least one. Therefore, the minimum of the degrees of the generators is $\frac{\nu-1}{2}$, but I is non-zero in degree $\frac{n-1}{2}$, so $\nu = n$ and the entries of the matrix are linear forms.

Conversely, let I satisfy the hypotheses of the statement and let us consider the graded Betti numbers $\beta_{ij}(R/I)$ of the corresponding quotient ring. Being I Gorenstein and minimally generated by n elements of degree $\frac{n-1}{2}$, the Betti numbers are all zero with the exceptions

$$\beta_{0,0}(R/I) = 1, \qquad \beta_{1,\frac{n-1}{2}}(R/I) = n,$$
$$\beta_{2,\frac{n+1}{2}}(R/I) = n, \qquad \beta_{3,n}(R/I) = 1.$$

One can show by computations that

$$\mathrm{Hilb}(R/I, n-3) = 1, \qquad \mathrm{Hilb}(R/I, n-2) = 0,$$

so that the socle is in degree $n-3$. Let F be the dual socle generator; by Macaulay correspondence, $I = F^{\perp}$. If F was degenerate, then by definition its derivatives of order $\frac{n-3}{2}$ would be linearly dependent, i.e. they would not span the whole vector space $S_{\frac{n-3}{2}}$. But this would imply, by Proposition 1.11, that I is non-zero in degree $\frac{n-3}{2}$, hence a contradiction.

□

Remark 3.18. A particular version ($n = 7$) of the second correspondence above was already known and, actually, extensively used. The correspondence between non-degenerate plane quartics and nets of alternating forms on a vector space of dimension seven plays an important role, for instance, in the geometric realizations of prime Fano threefolds of genus twelve [Muk92, Muk95, Sch01].

Remark 3.19. Fixed a Gorenstein, codimension-three ideal I generated by n forms of degree $\frac{n-1}{2}$, Theorem 1.8 guarantees the existence of a $n \times n$ skew-symmetric matrix N of linear forms whose Pfaffians generate I, as we showed in the proof of Proposition 3.17. Actually, *any* minimal system of generators of I arises from a suitable matrix N', congruent to N. Indeed, consider the matrix $A \in \mathrm{GL}_n$ taking the "Pfaffian" system of generators into the new one. Then these new generators are the Pfaffians of the matrix $(A^{-1})^t N A^{-1}$.

Remark 3.20. Let us observe that the correspondence developed in Proposition 3.17 is constructive. On the one hand, it is clear how, from a skew-symmetric matrix, one can get F by apolarity; on the other hand, once given F, it is possible to explicitly realize a skew-symmetric matrix whose Pfaffians generate the ideal $F^{\perp}$. This is possible thanks to the constructive proof of Theorem 1.8; we will see a concrete example of such a construction in Sect. 4.2.1.

We are ready to provide the

Proof of Theorem 3.16. Let $G = \sum c_\beta y^\beta$. Since the projection is a linear map, the composition $\mathbf{P}(U) \to \mathbf{P}(V)$ as in (3.11) is given by n forms of degree $\frac{n-1}{2}$, whose orthogonal complement in $R_{\frac{n-1}{2}}$ is spanned by the elements $(\partial^\alpha(G))_{|\alpha|=\frac{n-5}{2}}$. Let us denote by I the ideal generated by these n forms. By Proposition 3.17 and Proposition 1.11 applied to $F := \sum c_\beta \partial^\beta$, $I = F^{\perp}$ is Gorenstein and has codimension three; by Remark 3.19, any set of generators of I is made up by the Pfaffians of a suitable matrix N, i.e. any possible projection X is in $\mathrm{Im}(\rho)$.

Conversely, consider the image X of $\mathbf{P}(U)$ via the map given by the n Pfaffians (Pf_i) of a general matrix N. Let I be the ideal generated by these Pfaffians. I is generically of codimension three, so Proposition 3.17 applies and we get $I = F^{\perp}$ for some non-degenerate $F = \sum c_\beta \partial^\beta \in S$. By Proposition 1.11 we can complete the set of Pfaffians to a basis $\mathcal{B}$ of $R_{\frac{n-1}{2}}$ with the derivatives of order $\frac{n-5}{2}$ of $G := \sum c_\beta y^\beta$. Consider $\mathbf{P}(U)$ embedded in $\mathbb{P}^r$ via $\mathcal{B}$ and then projected via π to $\mathbf{P}(V)$ with respect to the center spanned by $(\partial^\alpha(G))_{|\alpha|=\frac{n-5}{2}}$. The so-obtained Veronese surface $X' \subset \mathbf{P}(V)$ is in $\mathrm{Im}(\rho)$ for the first part of the statement, so is the image of $\mathbf{P}(U)$ via a map $[f_1 : \ldots : f_n]$ given by the Pfaffians of a suitable matrix.

$$\begin{tikzcd}
& \mathbb{P}^r \arrow[ddl, dashed, "\pi"'] & \\
& \mathbf{P}(U) \arrow[u, "\mathcal{B}"] \arrow[dl, "{[f_1:\ldots:f_n]}"] \arrow[dr, "{[\mathrm{Pf}_1:\ldots:\mathrm{Pf}_n]}"] & \\
\mathbf{P}(V) \arrow[rr, "\exists A"'] & & \mathbf{P}(V)
\end{tikzcd}$$

Since the polynomials (f_i) and (Pf_i) generate the same ideal, there exists $A \in \mathrm{PGL}(V)$ such that the diagram above commutes. It follows that X can be obtained as the projection via $A \circ \pi$ of $\mathbf{P}(U)$ embedded via $\mathcal{B}$ in $\mathbb{P}^r$. $\square$

3.5.3 The general element of $\mathcal{H}$

Theorem 3.16 provided a description of the general point in $\mathrm{Im}(\rho)$; in particular, a general projection in $\mathbf{P}(V)$ of the Veronese surface $v_{\frac{n-1}{2}}(\mathbf{P}(U))$ does not belong to $\mathrm{Im}(\rho)$. Such projections are obviously contained in $\mathcal{H}$, so a natural question is whether they are dense in $\mathcal{H}$.

Proposition 3.21. *$\mathcal{H}$ is irreducible; its general element is a general projection in $\mathbf{P}(V)$ of the Veronese surface $v_{\frac{n-1}{2}}(\mathbf{P}(U)) \subset \mathbb{P}^r$, where $r = \binom{\frac{n-1}{2}+2}{2} - 1$.*

To prove this Proposition, we consider a parametrization of such general projections. The linear space $\mathbf{k}[y_0, y_1, y_2]_{\frac{n-1}{2}}$ has dimension $r+1$, so we have a rational map

$$\mathbb{A}^{(r+1)n} \overset{\xi}{\dashrightarrow} \mathcal{H} \tag{3.12}$$

sending n linearly independent forms $f_1, \ldots, f_n$ of degree $\frac{n-1}{2}$ to the point representing the image of the map

$$\mathbf{P}(U) \xrightarrow{[f_1:\ldots:f_n]} \mathbf{P}(V). \tag{3.13}$$

From the irreducibility of $\mathbb{A}^{(r+1)n}$ we deduce that $\mathrm{Im}(\xi)$ is irreducible.

Lemma 3.22. We have $\dim(\mathrm{Im}(\xi)) = \dim(\mathcal{H})$.

Proof. On the one hand, there is a natural GL_3-action on $\mathbf{k}[y_0, y_1, y_2]_1$, acting as a change of basis on U; this induces an action on $\mathbf{k}[y_0, y_1, y_2]_{\frac{n-1}{2}}$ and therefore on $\mathbb{A}^{(r+1)n}$, and one can see that ξ factors through this action. On the other hand, take two points V_1, V_2 in $\mathrm{Im}(\xi)$ such that $V_1 = V_2$. By the commutativity of the diagram

$$\begin{tikzcd}
 & V_1 \arrow[dd, equal] \\
\mathbf{P}(U) \arrow[ur, "{[f_1:\ldots:f_n]}", "\sim"'] \arrow[dr, "{[g_1:\ldots:g_n]}"', "\sim"] & \\
 & V_2
\end{tikzcd}$$

we get an automorphism of $\mathbf{P}(U)$, i.e. the two maps $[f_1 : \ldots : f_n]$ and $[g_1 : \ldots : g_n]$ belong to the same class modulo GL_3. Hence

$$\begin{aligned}\dim(\mathrm{Im}(\xi)) &= \dim(\mathbb{A}^{(r+1)n}) - \dim(\mathrm{GL}_3) \\ &= n\binom{\frac{n-1}{2}+2}{2} - 9 \\ &= \frac{1}{8}n(n+3)(n+1) - 9.\end{aligned}$$

By Proposition 3.15,

$$\begin{aligned}\dim(\mathcal{H}) &= \dim(\mathrm{Im}(\rho)) + \mathrm{codim}_{\mathcal{H}}(\mathrm{Im}(\rho)) \\ &= 3\binom{n}{2} - 9 + \frac{1}{8}n(n-3)(n-5) \\ &= \frac{1}{8}n(n+3)(n+1) - 9\end{aligned}$$

and therefore the conclusion follows. □

Proof of Proposition 3.21. From Lemma 3.22 we deduce that the closure of $\mathrm{Im}(\xi)$ in $\mathcal{H}$ is an irreducible component of $\mathcal{H}$. As $\mathcal{H}$ is generically smooth along $\mathrm{Im}(\rho)$ (cfr. Proposition 3.15), $\mathrm{Im}(\rho)$ is contained in only one irreducible component of the Hilbert scheme, namely $\overline{\mathrm{Im}(\xi)}$. But $\mathcal{H}$ was defined as the union of the irreducible components containing $\mathrm{Im}(\rho)$, so it turns out that $\mathcal{H} = \overline{\mathrm{Im}(\xi)}$ and this concludes the proof. □

Remark 3.23. Let us remark that the statement of Theorem 3.16 makes perfectly sense also when $n = 5$. In this case, a general element of $\mathrm{Im}(\rho)$ is a projection in $\mathbb{P}^4$ of a Veronese surface in $\mathbb{P}^5$, and there is no distinction between general projections and special projections as those arising in the statement. In other words, any general projection of the Veronese surface in $\mathbb{P}^4$ is in $\mathrm{Im}(\rho)$.
In the proof of Proposition 3.21 we saw that $\mathcal{H} = \overline{\mathrm{Im}(\xi)}$, so we get that ρ is dominant. This, together with the general injectivity, confirms once again the birationality of ρ proved in Theorem 3.13.

3.6 The case $m = 2$: curves

Throughout the chapter we have supposed $m \geq 3$, so we have not dealt so far with degeneracy loci of dimension one. When $m = 2$ and n is even, the general degeneracy locus is union of $\frac{n}{2}$ skew lines in $\mathbf{P}(V)$; in the case

$n = 6$, ρ is proved to be dominant, and the general fiber has dimension two [BM01].

When $m = 2$ and $n \geq 5$ is odd, the degeneracy locus is $\mathbb{P}^1$ embedded in $\mathbf{P}(V)$. Our aim is to give a nice description of the general point of $\mathrm{Im}(\rho)$ and $\mathcal{H}$, so from now on we fix $m = 2$ and n odd, with $n \geq 5$.

We can redo all the constructions performed in the previous sections; we can define $\mathscr{Q}$ as in (3.5), which gives an upper bound for the dimension of the Hilbert scheme $\mathcal{H}$ as in (3.6). With the same technique adopted in Sect. 3.3.1, we are able to compute

$$\mathrm{h}^0(\mathbf{P}(V), \mathscr{Q}) = \frac{1}{2}(n^2 + n - 8). \tag{3.14}$$

This implies

$$\dim \mathbf{Gr}(2, \Lambda^2 V) > \dim \mathcal{H}$$

and so ρ fails to be generically injective.

The general degeneracy locus X is still easy to describe: similarly to the case $m = 3$, the elements in $\mathrm{Im}(\rho)$ are the images of maps

$$\mathbf{P}(U) \xrightarrow{[f_1:\ldots:f_n]} \mathbf{P}(V),$$

where $f_1, \ldots, f_n$ are forms of degree $\frac{n-1}{2}$ in $\mathbf{k}[y_0, y_1]$, obtained as Pfaffians of a general $n \times n$ skew-symmetric matrix with entries in $\mathbf{k}[y_0, y_1]_1$.

Lemma 3.24. Let $n \geq 3$ be odd. For a general $n \times n$ skew-symmetric matrix N with entries in $\mathbf{k}[y_0, y_1]_1$, its $(n-1) \times (n-1)$ Pfaffians span the whole $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$.

Proof. For the $(n-1) \times (n-1)$ Pfaffians of a general N, not to span $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$ is a closed condition, so it is sufficient to exhibit, for any odd k, a matrix N_k not satisfying it. For this sake, we consider the $k \times k$ matrix

$$N_k = \begin{pmatrix} 0 & y_0 & & & & & \\ -y_0 & 0 & y_1 & & & & \\ & -y_1 & 0 & y_0 & & & \\ & & -y_0 & 0 & y_1 & & \\ & & & & \ddots & \ddots & \\ & & & & & 0 & y_1 \\ & & & & & -y_1 & 0 \end{pmatrix}.$$

If we denote by $\mathrm{Pf}_i(N_k)$ the $(k-1) \times (k-1)$ Pfaffian obtained from N_k by deleting the i-th row and column, it is easy to check that

$$\begin{aligned}
&\mathrm{Pf}_{2i+1}(N_k) = y_0^i y_1^{k-i} && \text{for any } 0 \leq i \leq \frac{k-1}{2}, \\
&\mathrm{Pf}_{2i}(N_k) = 0 && \text{for any } 1 \leq i \leq \frac{k-1}{2},
\end{aligned}$$

and this concludes the proof. □

Proposition 3.25. *If $m = 2$, $n \geq 5$ and n is odd, then ρ is dominant. The general element of $\mathcal{H}$ is the image in $\mathbf{P}(V)$ of a map*

$$\mathbf{P}(U) \xrightarrow{[f_1:\ldots:f_n]} \mathbf{P}(V),$$

where $f_1, \ldots, f_n$ are forms of degree $\frac{n-1}{2}$ spanning the whole linear space $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$.

Proof. Let $r = \dim(\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}) - 1 = \frac{n-1}{2}$. We can define a rational map ξ, as in (3.12), sending an n-uple of forms $f_1, \ldots, f_n$ to the image of the map (3.13). ξ is defined on the n-uples which span the whole linear space $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$; its image $\mathrm{Im}(\xi)$ is irreducible.
By Lemma 3.24, $\mathrm{Im}(\rho) = \mathrm{Im}(\xi)$. If we prove that

$$\dim(\mathrm{Im}(\xi)) \geq \mathrm{h}^0(\mathbf{P}(V), \mathscr{Q}), \tag{3.15}$$

then $\overline{\mathrm{Im}(\rho)}$ is the unique irreducible component of $\mathcal{H}$, and so we are done.
Similarly to Lemma 3.22, we have

$$\begin{aligned}
\dim(\mathrm{Im}(\xi)) &= \dim(\mathbb{A}^{(r+1)n}) - \dim(\mathrm{GL}_2) \\
&= n\frac{n+1}{2} - 4;
\end{aligned}$$

this coincides with (3.14), so (3.15) holds. □

Chapter 4

Pfaffian representations of cubic surfaces

In the previous chapter we have dealt with a fixed pair of vector bundles $\mathcal{T}_{\mathbf{P}(V)}(-2), U \otimes \mathcal{O}_{\mathbf{P}(V)}$ on the projective space $\mathbf{P}(V)$ and we have focused on the properties of the degeneracy locus of a morphism φ, as φ varies. In this chapter we analyze an inverse problem: fixed a scheme X in $\mathbf{P}(V)$ and a pair of vector bundles on $\mathbf{P}(V)$, is there a morphism φ such that its degeneracy locus is X? And if so, how can we obtain such morphism?

In particular, we will focus on the case of cubic surfaces $\mathbb{S}$ in $\mathbb{P}^3_{\mathbf{k}}$, where $\mathbf{k}$ is a field of characteristic zero, non-necessarily algebraically closed. As for the vector bundles, we are interested in alternating morphisms of the form

$$M : \mathcal{O}^6_{\mathbb{P}^3_{\mathbf{k}}}(-1) \to \mathcal{O}^6_{\mathbb{P}^3_{\mathbf{k}}},$$

i.e. 6×6 skew-symmetric matrices whose entries are linear polynomials. We look for matrices M whose Pfaffian (cfr. Definition 1.5) identifies the surface $\mathbb{S}$. Such matrices are called *(linear) Pfaffian representations*.

In this line of thought, in this chapter we describe an algorithm which requires a homogeneous polynomial F of degree three in $\mathbf{k}[x_0, x_1, x_2, x_3]$ and a zero $\mathbf{a}$ of F in $\mathbb{P}^3_{\mathbf{k}}$ and ensures a linear Pfaffian representation of $\mathrm{V}(F)$ with entries in $\mathbf{k}[x_0, \ldots, x_3]$, under mild assumptions on F and $\mathbf{a}$. We will use this result to give an explicit construction of (and to prove the existence of) a linear Pfaffian representation of $\mathrm{V}(F)$, with entries in $\mathbf{k}'[x_0, \ldots, x_3]$, being $\mathbf{k}'$ an algebraic extension of $\mathbf{k}$ of degree at most six.

The content of this chapter constitutes the paper [Ta13a].

4.1 Introduction

We will use the following two definitions.

Definition 4.1. Let $F \in \mathbf{k}[x_0, \ldots, x_n]$ define the hypersurface X and let $\mathbf{k}'$ be a field containing $\mathbf{k}$. A linear Pfaffian $\mathbf{k}'$-representation of X is a skew-symmetric matrix whose Pfaffian is F, up to constants, and whose entries are linear forms in $\mathbf{k}'[x_0, \ldots, x_n]$.

Definition 4.2 ($\mathbf{k}$-point)**.** If a point $\mathbf{a} \in \mathbb{P}^n_{\bar{\mathbf{k}}}$ admits a representative $\underline{a} \in \mathbb{A}^{n+1}_{\mathbf{k}}$, then it will be called a $\mathbf{k}$-point.

By convention, a hypersurface X will be considered in $\mathbb{P}^n_{\bar{\mathbf{k}}}$, being $\bar{\mathbf{k}}$ the algebraic closure of $\mathbf{k}$. In this way, X is non-empty even if its defining polynomial $F \in \mathbf{k}[x_0, \ldots, x_n]$ has no zero in $\mathbb{A}^{n+1}_{\mathbf{k}}$, that is, if X has no $\mathbf{k}$-points.

When no confusion can arise, we will denote $\mathbb{P}^n_{\mathbf{k}}$ simply by $\mathbb{P}^n$.

According to these notations, in [Bea00] Beauville provided a proof of the following theorem:

Theorem 4.3. *Let $\mathbb{S}$ be a surface of degree d in $\mathbb{P}^3_{\bar{\mathbf{k}}}$, without singular* $\mathbf{k}$-*points. The following conditions are equivalent:*

i. $\mathbb{S}$ *admits a linear Pfaffian* $\mathbf{k}$-*representation;*

ii. $\mathbb{S} \cap \mathbb{P}^3_{\mathbf{k}}$ *contains a finite, reduced, arithmetically Gorenstein subscheme Z of index $2d-5$, not contained in any surface of degree $d-2$.*

Moreover, the degree of Z is $\frac{1}{6}d(d-1)(2d-1)$.

Here a finite, reduced subscheme Z of degree c in $\mathbb{P}^n_{\mathbf{k}}$, with homogeneous ideal $I_Z \subset \mathbf{k}[x_0, \ldots, x_n]$, is said to be arithmetically Gorenstein (AG for short) if $\mathbf{k}[x_0, \ldots, x_n]/I_Z$ is a Gorenstein ring. For such a scheme, its *index* is the (unique) integer N such that

$$\dim\left(\mathbf{k}[x_0, \ldots, x_n]/I_Z\right)_p + \dim\left(\mathbf{k}[x_0, \ldots, x_n]/I_Z\right)_{N-p} = c \quad \text{for all } p \in Z.$$

The proof of Theorem 4.3 is based on considering the rank-two vector bundle coker(M) and its scheme Z associated via the Hartshorne-Serre correspondence.

As remarked by Beauville, another way to prove the existence of a Pfaffian representation is via Theorem 1.8. Indeed, taking $\mathbf{k}[x_0, \ldots, x_3]_{(x_0,\ldots,x_3)}$ as R, an AG subscheme Z as those arising in Theorem 4.3 satisfies the hypotheses of ii. in Theorem 1.8: R/I_Z is Gorenstein by definition, and Proposition 1.7 shows that this implies I_Z Gorenstein. The fact that depth$(I_Z, R) =$

3 follows from (1.6) and $\mathrm{pd}_R(R/I_Z) = 3$, which is true since Gorenstein ideals are Cohen-Macaulay and from the Auslander-Buchsbaum formula [Eis95, Exercise 19.8].

Given Z as in Theorem 4.3, one can apply Theorem 1.8: I_Z is generated by the $(2d-2)\times(2d-2)$ principal Pfaffians extracted from a skew-symmetric $(2d-1)\times(2d-1)$ matrix T with linear forms as entries. Then the surface admits a Pfaffian $\mathbf{k}$-representation

$$\left(\begin{array}{c|c} T & -C^t \\ \hline C & 0 \end{array}\right), \tag{4.1}$$

where C is a suitable $1\times(2d-1)$ matrix with linear forms as entries, which can be found by formula (1.5) (see Sect. 4.2.2 for more details).

Here we focus on the case $d = 3$. If $\mathbf{k} = \bar{\mathbf{k}}$, then by [DGO85] a set of five points in $\mathbb{P}^3_{\mathbf{k}}$ is an AG scheme if and only if they are in general position, i.e. no four of them are on a plane. This fact, together with Theorem 1.8, implies

Corollary 4.4. If $\mathbf{k} = \bar{\mathbf{k}}$, every smooth cubic surface in $\mathbb{P}^3_{\mathbf{k}}$ admits a linear Pfaffian representation [Bea00].

This result has been generalized in [FM02] as follows.

Proposition 4.5. *If $\mathbf{k} = \bar{\mathbf{k}}$, every cubic surface in $\mathbb{P}^3_{\mathbf{k}}$ admits a linear Pfaffian representation.*

We study how to construct *explicitly* a linear Pfaffian $\mathbf{k}$-representation, where $\mathbf{k}$ is not necessarily algebraically closed, starting from the least amount of initial data possible. We will show that, in general, it is sufficient to know a $\mathbf{k}$-point on $\mathbb{S}$.

We will prove the following

Theorem (Theorem 4.19)**.** *Let $\mathbb{S}$ be a cubic surface, neither reducible nor a cone, whose equation is $F \in \mathbf{k}[x_0,\dots,x_3]_3$. Given a $\mathbf{k}$-point $\mathbf{a^1}$, which is not a T-point — in the sense of Definition 4.11 — it is possible to construct explicitly a linear Pfaffian $\mathbf{k}$-representation of $\mathbb{S}$.*

The same method can be used to prove a weaker result, if $\mathbf{a^1}$ is not given:

Proposition (Proposition 4.22)**.** *Let $\mathbb{S}$ be a cubic surface, neither reducible nor a cone, whose equation is $F \in \mathbf{k}[x_0,\dots,x_3]_3$. Then it is possible to construct explicitly a linear Pfaffian $\mathbf{k}'$-representation of $\mathbb{S}$, where $\mathbf{k}'$ is an algebraic extension of degree $[\mathbf{k}' : \mathbf{k}] \leq 3$.*
Moreover, if $\mathbf{k} \subseteq \mathbb{R}$, then also $\mathbf{k}'$ can be chosen so.

On the one hand, these results strengthen one implication of Theorem 4.3 and give a bound for the degree of algebraic extension required to produce a linear Pfaffian representation. On the other hand, they are constructive: it is possible to implement an algorithm which produces a linear Pfaffian representation, provided the requested inputs.

After discussing the cases of reducible surfaces and cones, we will be able to prove the next theorem, which strengthens Proposition 4.5.

Theorem (Theorem 4.25)**.** *Every cubic surface in $\mathbb{P}^3_{\mathbf{k}}$, with equation $F \in \mathbf{k}[x_0, \ldots, x_3]_3$, admits a linear Pfaffian $\mathbf{k}'$-representation, $\mathbf{k}'$ being an algebraic extension of $\mathbf{k}$ of degree $[\mathbf{k}' : \mathbf{k}] \leq 6$.*
Moreover, it is possible to explicitly realize such a representation.

4.2 From five points to a Pfaffian representation

In this section, we make explicit the construction of the proof of Theorem 1.8, in the particular case of the ideal I of the four fundamental points and the unit point

$$[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0], [0 : 0 : 0 : 1], [1 : 1 : 1 : 1] \tag{4.2}$$

in $\mathbb{P}^3_{\mathbb{Q}}$. This produces the skew-symmetric matrix $\mathbb{T}$ in (4.5), whose Pfaffians generate I; we will make use of $\mathbb{T}$ to implement Algorithm 4.6, which produces a linear Pfaffian $\mathbf{k}$-representation of a cubic surface $\mathbb{S}$ starting from five $\mathbf{k}$-points in general position on $\mathbb{S}$.

From now on, we will consider only *linear* Pfaffian representations.

4.2.1 An explicit construction

For the sake of completeness, we recall briefly the constructions made in [BE82] in the proof of Theorem 1.8.

Let R be the ring of polynomials $\mathbf{k}[x_0, \ldots, x_3]$ and let I be a Gorenstein ideal with $\operatorname{depth}(I, R) = 3$. From a minimal free resolution of I

$$\underline{F}: \qquad 0 \longrightarrow F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow R/I \longrightarrow 0, \tag{4.3}$$

where $F_0 \cong R \cong F_3$, it is possible to make a change of basis in F_1 such that the map $F_2 \to F_1$ is alternating. This can be found by equipping this resolution with a graded commutative algebra, the symmetric square of $\underline{F}$

$$s_2(\underline{F}) = (\underline{F} \otimes \underline{F})/M,$$

where M is the graded submodule of $\underline{F} \otimes \underline{F}$ generated by the elements of the set

$$\left\{ f \otimes g - (-1)^{(\deg f)(\deg g)} g \otimes f \mid f, g \text{ homogeneous elements of } \underline{\mathrm{F}} \right\}.$$

By convention, an element f has degree i if and only if it belongs to F_i; the degree of $(f \otimes g)$ is simply $\deg(f) + \deg(g)$. The differential is inherited from $\underline{F}$ as follows:

$$\partial(f \otimes g) = \partial f \otimes g + (-1)^{\deg f} f \otimes \partial g.$$

The symmetric square $s_2(\underline{F})$ is a complex of projective R-modules, canonically isomorphic to $\underline{F}$ in degree 0 and 1. Therefore, there exists a map of complexes $\Phi : s_2(\underline{F}) \to \underline{F}$ which lifts up these two isomorphisms and it can be chosen so that the restrictions of Φ to $F_0 \otimes F_k$ are the isomorphisms $F_0 \otimes F_k \cong F_k$.

The multiplication in $s_2(\underline{F})$ is given by $f \cdot g = \Phi(\overline{f \otimes g})$, where $\overline{f \otimes g}$ is the class of $f \otimes g$ modulo M. Since $F_3 \cong R$, this multiplication induces a map $F_k \otimes F_{3-k} \to R$, which turns to be a perfect pairing.

This can be viewed as an isomorphism between F_1 and ${F_2}^*$, which makes the composition $F_2 \longrightarrow F_1 \longrightarrow F_2^*$ an alternating map.

Let us consider the special case where I is the ideal of the points (4.2). We have the free resolution (4.3), with $F_1 \cong R^5 \cong F_2$. We have to develop $\Phi_3 : s_2(\underline{F})_3 \to F_3$ in the diagram

$$\begin{tikzcd}
\dots \arrow[r] & s_2(\underline{F})_3 \arrow[r, "d'_3"] \arrow[d, "\Phi_3"] & s_2(\underline{F})_2 \arrow[r, "d'_2"] \arrow[d, "\Phi_2"] & s_2(\underline{F})_1 \arrow[r, "d'_1"] \arrow[d, "\Phi_1"] & s_2(\underline{F})_0 \arrow[r, "\pi"] \arrow[d, "\Phi_0"] & R/I \arrow[r] & 0 \\
0 \arrow[r] & F_3 \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r, "\pi"] & R/I \arrow[r] & 0
\end{tikzcd} \tag{4.4}$$

We choose the ordered basis of $s_2(\underline{F})_2 \cong (F_0 \otimes F_2) \oplus (\wedge^2 F_1)$ to be formed by the classes modulo M of $1 \otimes f_1^2, 1 \otimes f_2^2, \dots, 1 \otimes f_5^2, f_1^1 \otimes f_2^1, f_1^1 \otimes f_3^1, \dots, f_4^1 \otimes f_5^1$, where the f_i^1s are a basis of F_1 and the f_j^2s are a basis of F_2. A similar convention is fixed for $s_2(\underline{F})_3 \cong (F_0 \otimes F_3) \oplus (F_1 \otimes F_2)$.

After a computation with [CoCoA], we consider the maps of diagram (4.4) to be

$$d_3 = \begin{pmatrix} x_0x_1 - x_1x_3 \\ x_1x_2 - x_2x_3 \\ -x_0x_2 + x_1x_2 \\ -x_1x_3 + x_2x_3 \\ x_0x_3 - x_1x_3 \end{pmatrix}, \qquad {d_1}^t = {d'_1}^t = \begin{pmatrix} x_1x_3 - x_2x_3 \\ x_0x_3 - x_2x_3 \\ x_1x_2 - x_2x_3 \\ x_0x_2 - x_2x_3 \\ x_0x_1 - x_2x_3 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} -x_2 & x_0 & 0 & 0 & x_2 \\ x_2 & -x_1 & x_1 & 0 & 0 \\ x_3 & -x_3 & x_3 & x_0 - x_3 & 0 \\ -x_3 & x_3 & 0 & -x_1 + x_3 & x_1 \\ 0 & 0 & -x_3 & 0 & -x_2 \end{pmatrix}.$$

The isomorphisms Φ_0 and Φ_1 are represented by identity matrices. With straightforward computations we get the matrices d_2' and d_3'. By trials, we can lift up Φ_1 by finding matrices Φ_2 and Φ_3 such that the diagrams

$$\begin{tikzcd} & s_2(\underline{F})_2 \arrow[dl, "\Phi_2"'] \arrow[d, "\Phi_1 \circ d_2'"] & \\ F_2 \arrow[r, "d_2"] & \mathrm{Im}(d_2) \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} & s_2(\underline{F})_3 \arrow[dl, "\Phi_3"'] \arrow[d, "\Phi_2 \circ d_3'"] & \\ F_3 \arrow[r, "d_3"] & \mathrm{Im}(d_3) \arrow[r] & 0 \end{tikzcd}$$

commute. A possible choice for Φ_2 is

$$\left(\begin{array}{c|cccccccccc} & -x_3 & x_1 & 0 & 0 & x_3 & x_3 - x_0 & x_3 & 0 & -x_1 & 0 \\ & -x_3 & 0 & -x_2 & -x_1 & 0 & -x_2 & 0 & 0 & 0 & x_2 \\ I_5 & 0 & -x_2 & -x_2 & -x_1 & -x_2 & -x_2 & -x_0 & 0 & x_2 & x_2 \\ & 0 & 0 & 0 & 0 & x_3 & x_3 & x_3 & -x_2 & -x_1 & 0 \\ & 0 & x_3 & x_3 & x_3 & x_3 & x_3 & x_3 & 0 & -x_1 & -x_0 \end{array}\right).$$

This choice is indeed the unique with linear forms as entries in the right block, since the syzygies are of degree two. The map Φ_3 turns to be

$$\left(I_1 \,\middle|\, 0\ 0\ 0\ -1\ 0\ 0\ 0\ 0\ -1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ -1\ 0\ 0\ 1\ 0\ 0\ -1\ 0 \right).$$

The isomorphism resulting from Φ_3 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} : F_1 \longrightarrow {F_2}^*$$

and, with respect to this change of basis, the map d_2 turns to be alternating, represented by the skew-symmetric matrix

$$\mathbb{T} = \begin{pmatrix} 0 & 0 & -x_3 & 0 & -x_2 \\ 0 & 0 & x_3 & x_0 - x_1 & x_1 \\ x_3 & -x_3 & 0 & x_1 - x_3 & -x_1 \\ 0 & -x_0 + x_1 & -x_1 + x_3 & 0 & 0 \\ x_2 & -x_1 & x_1 & 0 & 0 \end{pmatrix}. \tag{4.5}$$

It is easy to verify that the 4×4 principal Pfaffians of $\mathbb{T}$ — listed in (4.6) — are exactly the five generators of I, that is, the entries of d_1.

4.2.2 From five points to Pfaffian representations: an algorithm

The procedure just shown can be applied as long as we have the ideal of a set X of five points in general position on a cubic surface $\mathbb{S}$. Due to the classical fact that two sets of five points in general position in $\mathbb{P}^3$ are projectively equivalent, instead of repeating the previous construction it is also possible to realize a Pfaffian representation in the following way.

By solving a linear system, we can find the matrix A of the projectivity which maps X to the five points (4.2). Replacing x_0, x_1, x_2, x_3 in (4.5) with the columns of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} \cdot A^t,$$

we get a matrix T whose Pfaffians P_i generate the ideal of X. Finding a Pfaffian representation is then straightforward: if $\mathbb{S} = \mathrm{V}(F)$, then F belongs to the ideal of X. Therefore, one can find five linear forms L_i such that $F = \sum_{i=1}^{5}(-1)^{i+1} L_i P_i$. Setting

$$C = \begin{pmatrix} L_1 & L_2 & L_3 & L_4 & L_5 \end{pmatrix}$$

and by (1.5), we get a Pfaffian representation of the form (4.1).

We summarize the whole procedure in Algorithm 4.6 (p. 78), presented in pseudocode, where $\mathbb{T} = \mathbb{T}(x_0, x_1, x_2, x_3)$ in (4.5) is seen as a matrix depending on four variables, the Pfaffians of which are

$$\begin{aligned}
\mathrm{Pf}_1(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_1(x_0 - x_3)\\
\mathrm{Pf}_2(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_2(x_3 - x_1)\\
\mathrm{Pf}_3(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_2(x_1 - x_0)\\
\mathrm{Pf}_4(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_3(x_1 - x_2)\\
\mathrm{Pf}_5(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_3(x_0 - x_1).
\end{aligned} \tag{4.6}$$

Remark 4.7. Algorithm 4.6 involves only linear equations. If the five given points are $\mathbf{k}$-points, as well as $F \in \mathbf{k}[x_0, \ldots, x_3]_3$, then the output Pfaffian representation of $\mathbb{S} = \mathrm{V}(F)$ is a $\mathbf{k}$-representation too, for a suitable choice of the representatives of the points.

Remark 4.8. The matrix associated to the (non-homogeneous) linear system in line 12 of the algorithm is 20×20; it depends only on the projectivity

---

Algorithm 4.6 from five points in general position to a Pfaffian representation

---

Require: $F \in \mathbf{k}[x_0, \dots, x_3]_3$ and $\mathbf{a^1}, \mathbf{a^2}, \mathbf{a^3}, \mathbf{a^4}, \mathbf{a^5}$ $\mathbf{k}$-points in general position on $\mathbb{S} = \mathrm{V}(F)$

Ensure: M, a Pfaffian $\mathbf{k}$-representation of $\mathbb{S}$ depending on some arbitrary parameters $\alpha_{i,j}$

1: **choose** a representative $\underline{a^i} = (a_0^i, a_1^i, a_2^i, a_3^i) \in \mathbb{A}^4_{\mathbf{k}}$ of $\mathbf{a^i}$ for every $1 \le i \le 5$

2: **compute** the solution $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of the linear system

$$\begin{pmatrix} a_0^1 & a_0^2 & a_0^3 & a_0^4 \\ a_1^1 & a_1^2 & a_1^3 & a_1^4 \\ a_2^1 & a_2^2 & a_2^3 & a_2^4 \\ a_3^1 & a_3^2 & a_3^3 & a_3^4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} a_0^5 \\ a_1^5 \\ a_2^5 \\ a_3^5 \end{pmatrix}$$

3: **compute** the change of basis matrix A from $(\lambda_i a^i)_{1 \le i \le 4}$ to the standard basis of $\mathbb{A}^4_{\mathbf{k}}$, so that

$$\lambda_i A \begin{pmatrix} a_0^i \\ a_1^i \\ a_2^i \\ a_3^i \end{pmatrix} = \begin{pmatrix} \delta_i^1 \\ \delta_i^2 \\ \delta_i^3 \\ \delta_i^4 \end{pmatrix} \qquad \text{for every } 1 \le i \le 4$$

4: **for** $i = 1$ **to** 4 **do**

5: **set** z_{i-1} as the i-th row of the column vector $A \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$

6: **set** $\mathbb{T}(x_0, x_1, x_2, x_3)$ as in (4.5)

7: **set** $T = \mathbb{T}(z_0, z_1, z_2, z_3)$

8: **for** $i = 1$ **to** 5 **do**

9: **set** $P_i = \mathrm{Pf}_i(\mathbb{T})(z_0, z_1, z_2, z_3)$ as in (4.6)

10: **set** $L_i = \sum_{j=0}^{3} \alpha_{i,j} x_j$

11: **set** $G = F - \sum_{i=1}^{5} (-1)^{i+1} L_i P_i$

12: **compute** solutions of the linear system given by equaling the coefficients of G to zero, $\alpha_{i,j}$ as unknowns

13: **substitute** the solutions in L_i

14: **set** M as the matrix

$$\left(\begin{array}{ccccc|c} & & & & & L_1 \\ & & & & & L_2 \\ & & T & & & L_3 \\ & & & & & L_4 \\ & & & & & L_5 \\ \hline -L_1 & -L_2 & -L_3 & -L_4 & -L_5 & 0 \end{array}\right)$$

---

applied in line 9, not on the choice of F. Regardless to this projectivity, its rank is 15.

Since for any choice of $F \supset \{\mathbf{a^1}, \mathbf{a^2}, \mathbf{a^3}, \mathbf{a^4}, \mathbf{a^5}\}$ a solution of this linear system does exist, the "Pfaffian representation depending on some parameters" ensured by Algorithm 4.6 turns to be a five-dimensional linear space of Pfaffian representations.

Classes of equivalent representations

We recall that two Pfaffian $\mathbf{k}$-representations M and M' are said to be equivalent if and only if there exists $X \in \mathrm{GL}_{\mathbf{k}}(6)$ such that $M' = XMX^t$. Let $\mathrm{coker}(M)$, $\mathrm{coker}(M')$ be the cokernel sheaves of M, M', seen as maps $\mathcal{O}_{\mathbb{P}^3}^6(-1) \to \mathcal{O}_{\mathbb{P}^3}^6$. Then from [Bea00, (2.3)] it follows that M, M' are equivalent if and only if $\mathrm{coker}(M) \cong \mathrm{coker}(M')$.

In this way the study of equivalence classes of Pfaffian representations of a cubic surface $\mathbb{S}$ is strongly linked to the study of certain sheaves on $\mathbb{S}$.

Remark 4.9. Let Z be a fixed set of five points in general position on a surface $\mathbb{S}$ without singular $\mathbf{k}$-points and consider the Pfaffian representations given by Algorithm 4.6, which are a five-dimensional linear space by Remark 4.8. It turns out that all these representations are equivalent. Indeed, by [Bea00, (7.1)], up to automorphism there exists only one pair (E, s), with E rank-two vector bundle on $\mathbb{S}$ and $s \in \mathrm{H}^0(\mathbb{S}, E)$, such that Z is the zero locus of s. In addition, these classes of pairs $[(E, s)]$ are in bijection with the equivalence classes of the pairs $[(M, \bar{s})]$, where $E = \mathrm{coker}(M)$ and $\bar{s} \in \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}^6)$ corresponds to s via the isomorphism $\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}^6) \cong \mathrm{H}^0(\mathbb{S}, E)$. It follows that all the representations produced in the algorithm belong to a unique equivalence class.

It is worth noting that, as Z varies among the possible sets of five $\mathbf{k}$-points in general position on a surface $\mathbb{S}$ without singular $\mathbf{k}$-points, Algorithm 4.6 is surjective onto the possible Pfaffian $\mathbf{k}$-representations of $\mathbb{S}$, and therefore onto their equivalence classes. Indeed, as shown in [Bea00, (7.2)], a general global section of $E = \mathrm{coker}(M)$ has five points in general position as its zero locus Z and therefore M can be produced via the algorithm with input Z.

We remark that, in [Buc10], elementary transformations were used to construct non-equivalent Pfaffian representations of curves starting from a given one. This technique can be used in the case of surfaces as well.

Remark 4.10. The bijection between Pfaffian representations M and cokernel sheaves $E = \operatorname{coker}(M)$ tells us more, when dealing with the algebraic closure $\bar{\mathbf{k}}$.

Let $\mathscr{S}^3/\operatorname{GL}(6)$ be the set of equivalence classes of the 6×6 skew-symmetric matrices of linear forms in $\mathbb{P}^3_{\bar{\mathbf{k}}}$; let $\operatorname{pf} : \mathscr{S}^3/\operatorname{GL}(6) \to |\mathscr{O}_{\mathbb{P}^3_{\bar{\mathbf{k}}}}(3)|$ be the map which associates to a class $[M]$ the cubic surface in $\mathbb{P}^3_{\bar{\mathbf{k}}}$ with equation $\operatorname{Pf}(M)$. As noticed in [FM02], for the general $\mathbb{S}$ the fiber $\operatorname{pf}^{-1}(\mathbb{S})$ can be identified with an open subset of the moduli space of simple rank-two vector bundles E on $\mathbb{S}$ with $c_1(E) = \mathscr{O}_{\mathbb{S}}(2)$, $c_2(E) = 5$.

Since the (projective) dimension of $\mathscr{S}^3/\operatorname{GL}(6)$ is $59 - 35 = 24$ and the dimension of $|\mathscr{O}_{\mathbb{P}^3_{\bar{\mathbf{k}}}}(3)|$ is 19, then for the general $\mathbb{S}$ we have a five-dimensional space of essentially different Pfaffian $\bar{\mathbf{k}}$-representations of $\mathbb{S}$.

The space $\mathscr{S}^3/\operatorname{GL}(6)$ has been recently considered in [Han12], in relation to the space of pairs $(\mathbb{S}, \Pi)$, being Π a complete pentahedron inscribed in $\mathbb{S}$.

4.3 Constructing five points on a surface

Given an equation $F \in \mathbf{k}[x_0, \ldots, x_3]_3$, in general it is not easy to find a zero of F in $\mathbb{A}^4_{\mathbf{k}}$. For example, if $\mathbf{k} = \mathbb{Q}$, the problem of the existence of rational points on cubic surfaces, reliable to diophantine equations, has been strongly faced in the last century (see, for example, [Mor69], [Seg51] and the more recent [Man86]).

Our next aim is to weaken the required inputs of Algorithm 4.6.

4.3.1 From one point to five points

It is well known that from a *general* choice of a $\mathbf{k}$-point on a *general* cubic surface with equation in $\mathbf{k}[x_0, \ldots, x_3]_3$ it is possible to find infinitely many others $\mathbf{k}$-points on the surface; this can be performed by using the tangent plane process, a classical argument (for example, see [Seg51]). It starts by taking the tangent plane to the cubic surface $\mathbb{S}$ at a smooth point P. $\mathrm{T}_P\,\mathbb{S}$ cuts $\mathbb{S}$ in a curve of degree three, for which P is a singular point. A line through P, lying on the tangent plane, intersects $\mathbb{S}$ twice in P, while the third intersection is *generically* different and gives us another $\mathbf{k}$-point on $\mathbb{S}$.

We want to get rid of this "generality". Theorem 4.13 will show how, under reasonable hypotheses, the tangent plane process applied to a starting $\mathbf{k}$-point can be repeated to produce four other $\mathbf{k}$-points on $\mathbb{S}$, such that the five points are all together in general position. This will prove, under these hypotheses, that we only need a $\mathbf{k}$-point on $\mathbb{S}$ to construct an explicit Pfaffian $\mathbf{k}$-representation.

Definition 4.11. A point $P \in \mathbb{S}$ will be called a T-point for $\mathbb{S}$ if P is smooth for $\mathbb{S}$ and $\mathrm{T}_P\,\mathbb{S} \cap \mathbb{S}$ is set-theoretically union of lines.

Let us observe that the so-called Eckardt points, i.e. smooth points P with $\mathrm{T}_P\,\mathbb{S} \cap \mathbb{S}$ made up of three lines through P, are T-points. Moreover, a smooth points P is a T-point if and only if $\mathrm{T}_P\,\mathbb{S}$ is a tritangent plane.

In general, for a T-point P one expects $\mathrm{T}_P\,\mathbb{S} \cap \mathbb{S}$ to be union of three distinct lines, but it is possible to have one line with multiplicity three or two lines, one of them with multiplicity two.

The role of T-points will be clear in a while. Let us remark that, for a smooth point P which is not a T-point, $\mathrm{T}_P\,\mathbb{S} \cap \mathbb{S}$ is either an irreducible cubic curve with P as a singular point, or union of a line through P and a smooth conic passing through P.

Remark 4.12. Let P be a T-point for $\mathbb{S}$. If $\mathrm{T}_P\,\mathbb{S} \cap \mathbb{S}$ is a line r with multiplicity three, or union of a line r with multiplicity two and another line, then r is union of singular points for $\mathbb{S}$ and T-points for $\mathbb{S}$ sharing the same tangent plane.

Theorem 4.13. *Let $\mathbb{S}$ be an irreducible cubic surface which is not a cone, whose equation is $F \in \mathbf{k}[x_0, \dots, x_3]_3$. Given a $\mathbf{k}$-point $\mathbf{a^1}$ on $\mathbb{S}$ which is not a T-point — in the sense of Definition 4.11 — it is possible to explicitly construct four other $\mathbf{k}$-points on $\mathbb{S}$ such that the five points together are in general position.*

The constructive proof, which requires some steps and preliminary lemmas, will be the subject of next subsection. In Sect. 4.4.1 we will see how this construction can be adapted if some of the hypotheses are missing.

4.3.2 The tangent plane process

Let us consider $F = F(x_0, x_1, x_2, x_3) \in \mathbf{k}[x_0, \dots, x_3]_3$. Then we set, for every $\underline{a} = (a_0, a_1, a_2, a_3) \in \mathbb{A}^4_{\mathbf{k}}$:

- $P_{1,\underline{a}}(x_0, x_1, x_2, x_3) = \sum_{i=0}^{3} a_i \frac{\partial F}{\partial x_i}$;
- $P_{2,\underline{a}}(x_0, x_1, x_2, x_3) = \sum_{i=0}^{3} x_i \frac{\partial F}{\partial x_i}(\underline{a})$.

They are the equations of the first and the second polar of the point $\mathbf{a} = [a_0 : a_1 : a_2 : a_3]$ with respect to the surface $\mathbb{S} = \mathrm{V}(F)$. If $\mathbf{a}$ is smooth, $P_{2,\underline{a}}$ defines $\mathrm{T}_{\mathbf{a}}\,\mathbb{S}$.

If $\underline{x} = (x_0, x_1, x_2, x_3)$, for every $\underline{a} \in \mathbb{A}^4_{\mathbf{k}}$ we have:

$$F(\underline{a} + t\underline{x}) = F(\underline{a}) + tP_{2,\underline{a}}(\underline{x}) + t^2 P_{1,\underline{a}}(\underline{x}) + t^3 F(\underline{x}). \tag{4.7}$$

We will consider the first and the second polar $\mathrm{V}(P_{1,\mathbf{a}})$ and $\mathrm{V}(P_{2,\mathbf{a}})$, for $\mathbf{a} \in \mathbb{P}^3_{\mathbf{k}}$, as hypersurfaces in $\mathbb{P}^3_{\mathbf{k}}$.

Lemma 4.14. Let $\mathbf{a}$ be a singular point on a cubic surface $\mathbb{S}$, whose equation is $F \in \mathbf{k}[x_0, \dots, x_3]_3$. Let us assume that $\mathbb{S}$ is neither reducible, nor a cone. Then there are at most six lines through $\mathbf{a}$ lying on $\mathbb{S}$.

Proof. By (4.7), if a point $\mathbf{x} \in \mathbb{S} \cap \mathrm{V}(P_{1,\mathbf{a}})$, also the whole line through $\mathbf{a}$ and $\mathbf{x}$ does. $P_{1,\mathbf{a}}$ is not the zero polynomial since $\mathbb{S}$ is not a cone, moreover F is irreducible: this means that the intersection $\mathbb{S} \cap \mathrm{V}(P_{1,\mathbf{a}})$ is transversal. It is therefore a curve of degree six, union of lines through $\mathbf{a}$. □

Lemma 4.15. Let $\mathbb{S}$ be an irreducible, cubic surface which is not a cone and let us assume $\mathbf{a} \in \mathbb{S}$ is not a T-point.

i. If $\mathbf{a}$ is smooth, then on $\mathrm{T}_{\mathbf{a}}\,\mathbb{S}$ there are only finitely many T-points for $\mathbb{S}$. Moreover $\mathrm{V}(P_{1,\mathbf{a}}) \cap \mathrm{T}_{\mathbf{a}}\,\mathbb{S}$ is union of at most two lines through $\mathbf{a}$ and any line through $\mathbf{a}$ lying on $\mathbb{S}$ lies also on $\mathrm{V}(P_{1,\mathbf{a}}) \cap \mathrm{T}_{\mathbf{a}}\,\mathbb{S}$.

ii. If $\mathbf{a}$ is singular, then point i. still holds if we replace $\mathrm{T}_{\mathbf{a}}\,\mathbb{S}$ with a plane π through $\mathbf{a}$, for all but finitely many choices of π.

Proof. We distinguish two classes of T-points: let us call $\mathcal{A}$ the set of T-points P for which $\mathrm{T}_P\,\mathbb{S} \cap \mathbb{S}$ is union of three distinct lines, $\mathcal{A}'$ the set of T-points not in $\mathcal{A}$.

Either $\mathbb{S}$ contains finitely many lines or infinitely many ones. In the first case, note that $\mathcal{A}$ is a finite set, since mutual intersections of lines on $\mathbb{S}$ are finite in number; $\mathcal{A}'$ is contained in a union of lines on $\mathbb{S}$, by Remark 4.12.

If $\mathbb{S}$ contains infinitely many lines, then it is well-known (for example, see [Con39]) that $\mathbb{S}$ is either reducible, an irreducible cone or a ruled cubic with a double line. By hypotheses the first two cases cannot occur. Moreover, a cubic surface with a double line which is not a cone is projectively equivalent to either $\mathrm{V}(x_0^2x_3 + x_0x_1x_2 + x_1^3)$ or $\mathrm{V}(x_0^2x_2 + x_1^2x_3)$ (see, for example, [Abh60]). The study of these two cases leads to Table 4.1 and Table 4.2.

If $\mathbb{S}$ is projectively equivalent to $\mathrm{V}(x_0^2x_3 + x_0x_1x_2 + x_1^3)$, then Table 4.1 shows that there are no T-points at all. If $\mathbb{S}$ is projectively equivalent to $\mathrm{V}(x_0^2x_2 + x_1^2x_3)$, then $\mathcal{A}$ is contained in the line $[s : t : 0 : 0]$ and $\mathcal{A}'$ is contained in the union of the lines $[s : 0 : 0 : t]$ and $[0 : s : t : 0]$, as shown in Table 4.2.

coordinates of $\mathbf{a}$	$\mathrm{T}_{\mathbf{a}}\,\mathbb{S} \cap \mathbb{S}$ (if smooth)
$[1 : s : t : -s^3 - st]$	$\begin{cases} x_0(-2s^3 - st) + x_1(3s^2 + t) + sx_2 + x_3 = 0 \\ (x_0 s - x_1)(2x_0^2 s^2 - x_0 x_1 s + t x_0^2 - x_0 x_2 - x_1^2) = 0 \end{cases}$
$[0 : 0 : s : t]$	singular

Table 4.1: points on $\mathbb{S} = \mathrm{V}(x_0^2 x_3 + x_0 x_1 x_2 + x_1^3)$.

coordinates of $\mathbf{a}$	restrictions	$\mathrm{T}_{\mathbf{a}}\,\mathbb{S} \cap \mathbb{S}$ (if smooth)
$[1 : t : -t^2 s : s]$	$s \neq 0 \neq t$	line and irreducible conic
$[1 : t : 0 : 0]$	$t \neq 0$	$\begin{cases} x_2 + x_3 t^2 = 0 \\ x_3(x_0 t \pm x_1) = 0 \end{cases}$
$[1 : 0 : 0 : s]$		$\begin{cases} x_2 = 0 \\ x_1^2 x_3 = 0 \end{cases}$
$[0 : 1 : t : 0]$		$\begin{cases} x_3 = 0 \\ x_0^2 x_2 = 0 \end{cases}$
$[0 : 0 : s : t]$		singular

Table 4.2: points on $\mathbb{S} = \mathrm{V}(x_0^2 x_2 + x_1^2 x_3)$.

Now, let us assume $\mathbf{a}$ is smooth. Since it is not a T-point, $\mathrm{T}_{\mathbf{a}}\,\mathbb{S}$ cannot contain lines made up of T-points, so every such a line intersects $\mathrm{T}_{\mathbf{a}}\,\mathbb{S}$ in one and only one point. Since they are finite in number, the first statement of i. is proved.

For the second statement, let $\mathbf{x} \neq \mathbf{a}$ be a point in $\mathbb{P}^3_{\mathbf{k}}$ and let $Y = \mathrm{V}(P_{1,\mathbf{a}}) \cap \mathrm{T}_{\mathbf{a}}\,\mathbb{S}$. By (4.7), the point $\mathbf{x} \in Y$ if and only if either $F(\mathbf{a} + t\mathbf{x})$ is the zero polynomial or the line through $\mathbf{a}$ and $\mathbf{x}$ intersects $\mathbb{S}$ only in $\mathbf{a}$. This means that, if $\mathbf{x} \in Y$, also the whole line through it and $\mathbf{a}$ is contained in Y; the conclusion then follows as soon as we prove that Y is a curve, that is, $\mathrm{V}(P_{1,\mathbf{a}}) \not\supseteq \mathrm{T}_{\mathbf{a}}\,\mathbb{S}$. For this sake, note that $\mathbf{a}$ is not a T-point and so there exists a point $\mathbf{y}$ on $\mathbb{S} \cap \mathrm{T}_{\mathbf{a}}\,\mathbb{S}$ such that the line r through $\mathbf{y}$ and $\mathbf{a}$ does not lie on $\mathbb{S}$. The line r intersects $\mathbb{S}$ in $\mathbf{a}$ with multiplicity two and in $\mathbf{y}$ with multiplicity one: this implies $\mathbf{y} \notin \mathrm{V}(P_{1,\mathbf{a}})$. Part i. of the lemma is proved.

If $\mathbf{a}$ is singular, then by Lemma 4.14 only finitely many planes through $\mathbf{a}$ contain a line on $\mathbb{S}$ through $\mathbf{a}$. For any other choice π, the same argument of the smooth case holds, if we replace $\mathrm{T}_{\mathbf{a}}\,\mathbb{S}$ with π. This proves part ii. of the lemma. $\square$

Proof of Theorem 4.13. We divide the proof into four steps.

Step 1: looking for the second point.

Either $\mathbf{a^1}$ is smooth or it is singular.

If $\mathbf{a^1}$ is smooth, then by hypotheses $\mathbb{S} \cap \mathrm{T}_{\mathbf{a^1}} \mathbb{S}$ is a cubic curve, neither set-theoretically union of lines ($\mathbf{a^1}$ is not a T-point), nor the whole tangent plane ($\mathbb{S}$ is irreducible).

Every line ℓ on $\mathrm{T}_{\mathbf{a^1}} \mathbb{S}$ through $\mathbf{a^1}$, but those contained in $\mathrm{T}_{\mathbf{a^1}} \mathbb{S} \cap \mathrm{V}(P_{1,\mathbf{a^1}})$ as in Lemma 4.15, has one and only one intersection with $\mathbb{S}$ different from $\mathbf{a^1}$. Here we do not care about any line on $\mathrm{T}_{\mathbf{a^1}} \mathbb{S} \cap \mathbb{S}$ through $\mathbf{a^1}$, since by Lemma 4.15 it would be contained in $\mathrm{T}_{\mathbf{a^1}} \mathbb{S} \cap \mathrm{V}(P_{1,\mathbf{a^1}})$ as well.

Fix a line ℓ; the so-obtained $\mathbf{a^2}$ is smooth. Otherwise, ℓ would have multiplicity of intersection at least four with $\mathbb{S}$, and therefore $\ell \subset \mathbb{S}$, which is not.

Moreover, by Lemma 4.15, $\mathbf{a^2}$ can be a T-point only for finitely many choices of ℓ, and so these choices can be avoided.

By (4.7), in coordinates we have, having chosen a representative $\underline{a^1}$ for $\mathbf{a^1}$,

$$\underline{a^2} = F(\underline{y}) \cdot \underline{a^1} - P_{1,\underline{a^1}}(\underline{y}) \cdot \underline{y},$$

for any choice of $\underline{y} = (y_0, y_1, y_2, y_3)$ representing the class $\mathbf{y} \in \mathrm{T}_{\mathbf{a^1}} \mathbb{S}$. Let us observe that $P_{1,\underline{a^1}}(\underline{y}) \neq 0$ and that $\underline{a^2}$ has coordinates in $\mathbf{k}$.

If $\mathbf{a^1}$ is singular, the previous argument can be repeated by replacing the role of $\mathrm{T}_{\mathbf{a^1}} \mathbb{S}$ above with a plane π satisfying Lemma 4.15.

In both cases, we have constructed a smooth point $\mathbf{a^2}$ on $\mathbb{S}$, which is not a T-point.

Step 2: looking for the third point.

The tangent plane process can be repeated as in step 1 — smooth case — starting from $\mathbf{a^2}$ to construct next point $\mathbf{a^3}$. Summarizing, every line on $\mathrm{T}_{\mathbf{a^2}} \mathbb{S}$ through $\mathbf{a^2}$ with the exception of

- finitely many (by Lemma 4.15) lines through T-points,
- at most two lines in $\mathrm{T}_{\mathbf{a^2}} \mathbb{S} \cap \mathrm{V}(P_{1,\mathbf{a^2}})$ as in Lemma 4.15

has exactly one intersection with $\mathbb{S}$ different from $\mathbf{a^2}$, say $\mathbf{a^3}$. It is smooth and not a T-point.

To state that $\mathbf{a^3}$ is in general position with $\mathbf{a^1}$ and $\mathbf{a^2}$, we only need to verify that it does not lie on the line ℓ' through them. This is for free, since $\mathbf{a^3}$ belongs to $\mathrm{T}_{\mathbf{a^2}}\,\mathbb{S}$ but $\mathbf{a^1}$ does not, otherwise $\ell' \subseteq \mathbb{S}$, which is not by construction.

Step 3: looking for the fourth point.

The tangent plane process can be repeated as in step 1 — smooth case — starting from $\mathbf{a^3}$ to construct next point $\mathbf{a^4}$. We need to choose it not on the plane π_{123} containing $\mathbf{a^1}$, $\mathbf{a^2}$ and $\mathbf{a^3}$.

The planes $\mathrm{T}_{\mathbf{a^3}}\,\mathbb{S}$ and π_{123} are distinct — for example, the first one does not contain $\mathbf{a^2}$ by construction — so their intersection is a line through $\mathbf{a^3}$, say ℓ''.

Claim. The system

$$\begin{cases} \mathbf{y} \in \mathbb{S} \\ \mathbf{y} \in \mathrm{T}_{\mathbf{a^3}}\,\mathbb{S} \\ \mathrm{T}_{\mathbf{y}}\,\mathbb{S} \ni \mathbf{a^2} \end{cases} \tag{4.8}$$

which can be translated in homogeneous equations of degree $3, 1, 2$ respectively, has finitely many solutions $\mathbf{y} \in \mathbb{P}^3_{\overline{\mathbf{k}}}$.

Indeed, the system represents the intersection on the plane $\mathrm{T}_{\mathbf{a^3}}\,\mathbb{S}$ between the cubic curve $\mathcal{C} = \mathbb{S} \cap \mathrm{T}_{\mathbf{a^3}}\,\mathbb{S}$ and the conic $\mathcal{Q}$ defined on $\mathrm{T}_{\mathbf{a^3}}\,\mathbb{S}$ by the condition $\mathrm{T}_{\mathbf{y}}\,\mathbb{S} \ni \mathbf{a^2}$. By construction, $\mathbf{a^3}$ is not a T-point and therefore $\mathcal{C}$ is either irreducible or union of a line and an irreducible conic containing $\mathbf{a^3}$; $\mathcal{Q}$ does not pass through $\mathbf{a^3}$ and so it cannot be contained in $\mathcal{C}$. This proves the claim.

The finitely many solutions of system (4.8) correspond to finitely many lines on $\mathrm{T}_{\mathbf{a^3}}\,\mathbb{S}$ through $\mathbf{a^3}$. Since we want $\mathbf{a^2} \notin \mathrm{T}_{\mathbf{a^4}}\,\mathbb{S}$, we will avoid them.

Summarizing, every line on $\mathrm{T}_{\mathbf{a^3}}\,\mathbb{S}$ through $\mathbf{a^3}$ with the exception of

- finitely many lines through the solutions $\mathbf{y}$ of system (4.8),
- ℓ'',
- finitely many (by Lemma 4.15) lines through T-points,
- at most two lines in $\mathrm{T}_{\mathbf{a^3}}\,\mathbb{S} \cap \mathrm{V}(P_{1,\mathbf{a^3}})$ as in Lemma 4.15

has exactly one intersection with $\mathbb{S}$ different from $\mathbf{a^3}$, say $\mathbf{a^4}$. It is smooth and not a T-point, moreover $\mathbf{a^2} \notin \mathrm{T}_{\mathbf{a^4}}\,\mathbb{S}$.

Step 4: looking for the fifth point.

We can apply the usual tangent plane process to find $\mathbf{a^5}$ in general position with $\mathbf{a^1}, \mathbf{a^2}, \mathbf{a^3}$ and $\mathbf{a^4}$. Let us call π_{ijk} the plane through different $\mathbf{a^i}, \mathbf{a^j}, \mathbf{a^k}$.

The planes π_{134}, π_{234} and π_{124} intersect $\mathrm{T}_{\mathbf{a^4}}\,\mathbb{S}$ into three lines through $\mathbf{a^4}$: in fact they are four different planes, since $\mathbf{a^2}, \mathbf{a^3} \notin \mathrm{T}_{\mathbf{a^4}}\,\mathbb{S}$.

The line $\pi_{123} \cap \mathrm{T}_{\mathbf{a^4}}\,\mathbb{S}$ cannot be contained in $\mathrm{T}_{\mathbf{a^4}}\,\mathbb{S} \cap \mathbb{S}$, since $\mathbf{a^4} \notin \pi_{123}$ and by construction $\mathbf{a^4}$ is not a T-point. This means that $\pi_{123} \cap \mathrm{T}_{\mathbf{a^4}}\,\mathbb{S} \cap \mathbb{S}$ contains at most three points.

Summarizing, every line on $\mathrm{T}_{\mathbf{a^4}}\,\mathbb{S}$ through $\mathbf{a^4}$ with the exception of

- three lines lying on the planes π_{134}, π_{234} and π_{124},
- at most three lines through the points in $\pi_{123} \cap \mathrm{T}_{\mathbf{a^4}}\,\mathbb{S} \cap \mathbb{S}$,
- at most two lines in $\mathrm{T}_{\mathbf{a^4}}\,\mathbb{S} \cap \mathrm{V}(P_{1,\mathbf{a^4}})$ as in Lemma 4.15

has exactly one intersection with $\mathbb{S}$ different from $\mathbf{a^4}$, say $\mathbf{a^5}$, in general position with $\mathbf{a^1}$, $\mathbf{a^2}$, $\mathbf{a^3}$, $\mathbf{a^4}$.

□

Remark 4.16. Following the proof of Theorem 4.13, it is possible to implement an algorithm which requires a **k**-point on $\mathbb{S}$, not a T-point, and ensures five **k**-points in general position on $\mathbb{S}$. To test if a given point is a T-point or not, it is sufficient to check the reducibility of a polynomial of degree three in three variables, a task which can be easily performed by means of a software computation.

Remark 4.17. If $\mathbb{S}$ is a smooth cubic surface, then any T-point P has $\mathrm{T}_P\,\mathbb{S} \cap \mathbb{S}$ made up of three distinct lines. In such a situation, Theorem 4.13 can be proved with the weaker hypothesis: the starting point $\mathbf{a^1}$ is not an Eckardt point.

Remark 4.18. In the statement of Theorem 4.13 we require that $\mathbf{a^1}$ is not a T-point. Indeed, if $\mathbf{a^1}$ is Eckardt, then the tangent plane process fails at the very first step. If $\mathbf{a^1}$ is a non-Eckardt T-point, then the tangent plane process could give rise to either singular or other T-points, which can make one loose control in subsequent steps.

In fact, this does happen in the following example: take $\mathbb{S} = \mathrm{V}(x_0x_1x_3 + x_2^3 + x_2x_3^2)$ and $\mathbf{a^1} = [0:0:0:1]$. The tangent plane process gives rise to points on the line $[s:t:0:0]$, which are either singular or Eckardt points. The process then stops at the second step.

Codimension three AG subschemes have been considered also in [MP97], where they are obtained as zero loci of sections of certain rank-three sheaves. In the case of five points in general position in $\mathbb{P}^3_{\mathbf{k}}$, it turns out that all such sets are the zero loci of appropriate sections of the vector bundle $\Omega_{\mathbb{P}^3}(3)$, which can be interpreted as four-tuple of quadrics, that is, linear combinations (using linear forms as coefficients) of the syzygies of the map $(x_0\ x_1\ x_2\ x_3)$. The membership of such a zero locus to a surface $\mathbb{S}$ imposes conditions to the linear combination.

4.4 Main results and further generalizations

In this last section, we firstly make use of Theorem 4.13 and Algorithm 4.6 to prove Theorem 4.19; if we drop the requirement of the starting point, then a weaker result holds (Proposition 4.22). After discussing the cases of reducible surfaces and cones, we state Theorem 4.25. A concrete example is finally given.

Theorem 4.19. *Let $\mathbb{S}$ be a cubic surface, neither reducible nor a cone, whose equation is $F \in \mathbf{k}[x_0, \ldots, x_3]_3$. Given a $\mathbf{k}$-point $\mathbf{a^1}$, which is not a T-point — in the sense of Definition 4.11 — it is possible to construct explicitly a linear Pfaffian $\mathbf{k}$-representation of $\mathbb{S}$.*

Proof. Given $\mathbf{a^1}$, one can apply Theorem 4.13 and construct four other $\mathbf{k}$-points $\mathbf{a^2}$, $\mathbf{a^3}$, $\mathbf{a^4}$, $\mathbf{a^5}$ on $\mathbb{S}$ such that they are all together in general position. With these initial data, Algorithm 4.6 ensures a Pfaffian $\mathbf{k}$-representation of $\mathbb{S}$. □

Remark 4.20. Let us work on $\bar{\mathbf{k}}$ and let $\mathbb{S}$ be general. In Remark 4.9 we saw that the Pfaffian representations produced by Algorithm 4.6 are all equivalent, once fixed the inputs $\mathbf{a^1}, \mathbf{a^2}, \mathbf{a^3}, \mathbf{a^4}, \mathbf{a^5}$. The constructive proof of Theorem 4.19 provides a new algorithm to construct many Pfaffian representations starting from just one point $\mathbf{a^1}$: we claim that neither this algorithm is surjective onto the possible Pfaffian representations of $\mathbb{S}$, once fixed $\mathbf{a^1}$. Indeed, by Remark 4.10, the space of essentially different Pfaffian representations of $\mathbb{S}$ is five-dimensional. Since we can suppose $\mathbb{S}$ is smooth, $\mathbf{a^1}$ is not singular. The procedure described in the proof of Theorem 4.13 consists in taking a point on a plane cubic curve in each step, and so the space of sets of five points obtained starting from $\mathbf{a^1}$ is four-dimensional. The conclusion follows again by Remark 4.9.

Remark 4.21. The procedure lying beneath the proof of Theorem 4.19 involves only linear equations and can be implemented in a deterministic algorithm.

4.4.1 Weakening hypotheses

No starting points

One of the hypotheses of Theorem 4.19 was a $\mathbf{k}$-point on $\mathbb{S}$. If this is not given, then one can manage to find a $\mathbf{k}'$-point $\mathbf{a}$, being $\mathbf{k}'$ an algebraic extension of degree at most three, simply by solving a polynomial equation of degree three (given by intersecting $\mathbb{S}$ with two arbitrary planes). For the general choice of these two planes, $\mathbf{a}$ is not a T-point and so Theorem 4.19 applies. This proves the following proposition.

Proposition 4.22. *Let $\mathbb{S}$ be a cubic surface, neither reducible nor a cone, whose equation is $F \in \mathbf{k}[x_0, \ldots, x_3]_3$. Then it is possible to construct explicitly a Pfaffian $\mathbf{k}'$-representation of $\mathbb{S}$, where $\mathbf{k}'$ is an algebraic extension of $\mathbf{k}$ of degree $[\mathbf{k}' : \mathbf{k}] \leq 3$.*

Moreover, if $\mathbf{k} \subseteq \mathbb{R}$, then also $\mathbf{k}'$ can be chosen so.

Reducible surfaces

Let $\mathbb{S}$ be a reducible cubic surface. Then $\mathbb{S}$ is either union of three planes with equation π_1, π_2, π_3 or union of a plane π and a quadratic irreducible surface $\mathcal{S}$. In both cases, simple Pfaffian representations can be constructed, as we will show.

In the first case, a Pfaffian representation is given by

$$\left(\begin{array}{c|c} 0 & M \\ \hline -M & 0 \end{array}\right),$$

where

$$M = \begin{pmatrix} \pi_1 & 0 & 0 \\ 0 & \pi_2 & 0 \\ 0 & 0 & \pi_3 \end{pmatrix}.$$

In the second case, let us consider the matrix

$$\mathbb{T}' = \begin{pmatrix} 0 & -x_3 & -x_2 \\ x_3 & 0 & -x_1 \\ x_2 & x_1 & 0 \end{pmatrix}.$$

If $\mathcal{S} \ni [1 : 0 : 0 : 0]$, then we can find three linear forms L_1, L_2, L_3 such that an equation for $\mathcal{S}$ is $\sum_{i=1}^{3} (-1)^{i+1} L_i x_i$. A Pfaffian representation of $\mathcal{S}$

is then given by

$$P = \left(\begin{array}{ccc|c} & & & L_1 \\ & \mathbb{T}' & & L_2 \\ & & & L_3 \\ \hline -L_1 & -L_2 & -L_3 & 0 \end{array} \right)$$

by formula (1.5).

If $[1 : 0 : 0 : 0] \notin \mathcal{S}$, then it is sufficient to apply to x_1, x_2, x_3 in $\mathbb{T}'$ the projectivity which maps a given point $\mathbf{a}$ on $\mathcal{S}$ to $[1 : 0 : 0 : 0]$, as described in Sect. 4.2.2. Again by formula (1.5) one finds three linear forms and a Pfaffian representation P of $\mathcal{S}$ as above.

A Pfaffian representation of $\mathbb{S}$ is then given by

$$\left(\begin{array}{c|c|c} 0 & 0 & \pi \\ \hline 0 & P & 0 \\ \hline -\pi & 0 & 0 \end{array} \right).$$

Remark 4.23. Let $F \in \mathbf{k}[x_0, \ldots, x_3]_3$ be an equation for the reducible surface $\mathbb{S}$. The representations just constructed are not $\mathbf{k}$-representations, in general. This is due to the fact that the splitting field of a polynomial of degree three is generally an algebraic extension of $\mathbf{k}$ of degree six.

However, for such reducible surfaces we can state: it is possible to construct explicitly a Pfaffian $\mathbf{k}'$-representation, being $\mathbf{k}'$ an algebraic extension of $\mathbf{k}$ of degree at most six.

Cones

Let $\mathbb{S}$ be an irreducible cone. If we suppose non-restrictively that its vertex is $[1 : 0 : 0 : 0]$, then $\mathbb{S}$ is defined by an equation $F \in \mathbf{k}[x_1, x_2, x_3]$. Let us call $\mathcal{C}$ the plane cubic curve defined by F in $\mathbb{P}^3_{\mathbf{k}} \cap \mathrm{V}(x_0)$.

As previously done, we can find a $\mathbf{k}'$-point $\mathbf{a}$ on $\mathcal{C}$, being $\mathbf{k}'$ an algebraic extension of $\mathbf{k}$, simply by solving a polynomial equation of degree three.

The construction of $\mathbf{k}'$-points on a plane cubic curve is a widely studied subject in literature (see for example [ST92]). Starting from a set X of $\mathbf{k}'$-points, it consists in considering tangent lines to the curve in each point of X, and secant lines through each pair of points of X; the third intersection of such lines with $\mathcal{C}$ is then set as a new element in X.

This process fails for particular choices of $X = \{\mathbf{a}\}$: for example, if $\mathbf{a}$ is an inflection point of the curve. For a general choice of $\mathbf{a}$, this process produces a lot of $\mathbf{k}'$-points on $\mathcal{C}$, and we can manage to find five points among them such that no three are collinear. Then the following proposition applies.

Proposition 4.24. *Let $\mathbb{S}$ be a cone over a plane cubic curve $\mathcal{C}$, with equation $F \in \mathbf{k}'[x_0, \dots, x_3]_3$. If there exist five $\mathbf{k}'$-points on $\mathcal{C}$ such that no three of them are on a line, then there exist five $\mathbf{k}'$-points in general position on $\mathbb{S}$.*

Proof. We can suppose the vertex is $[1 : 0 : 0 : 0]$, so that the equation of the plane curve (and the cone) is $C = C(x_1, x_2, x_3)$. Let $\underline{a^i} = (a_0^i, a_1^i, a_2^i, a_3^i)$ represent the five points. The vanishing of each of the 4×4 minors of the matrix

$$\begin{pmatrix} y_1 & a_1^1 & a_2^1 & a_3^1 \\ y_2 & a_1^2 & a_2^2 & a_3^2 \\ y_3 & a_1^3 & a_2^3 & a_3^3 \\ y_4 & a_1^4 & a_2^4 & a_3^4 \\ y_5 & a_1^5 & a_2^5 & a_3^5 \end{pmatrix} \tag{4.9}$$

imposes a non-trivial close condition to $\underline{y} \in \mathbb{A}^5_{\mathbf{k}'}$, since the 3×3 minors of the matrix obtained by deleting the first column in (4.9) are non-vanishing by hypotheses. So there exists $\underline{y}$ satisfying none of these conditions and we get five points in general positions on $\mathbb{S}$. □

Let us remark that also in the case of cones it is possible to implement an algorithm which requires an equation $F \in \mathbf{k}[x_0, \dots, x_3]$ for the surface $\mathbb{S}$ and ensures a Pfaffian $\mathbf{k}'$-representation of $\mathbb{S}$, being $[\mathbf{k}' : \mathbf{k}] \leq 3$.

Summarizing, we can prove the following theorem, which is a generalization of Proposition 4.5.

Theorem 4.25. *Every cubic surface in $\mathbb{P}^3_{\mathbf{k}}$, with equation $F \in \mathbf{k}[x_0, \dots, x_3]_3$, admits a Pfaffian $\mathbf{k}'$-representation, $\mathbf{k}'$ being an algebraic extension of $\mathbf{k}$ of degree $[\mathbf{k}' : \mathbf{k}] \leq 6$.*
Moreover, it is possible to explicitly realize such a representation.

Proof. It follows from Proposition 4.22, Remark 4.23 and from the discussion about cones made above. □

4.4.2 An example

Let $F = x_0 x_1^2 + x_1 x_3^2 + x_2^3$ be the equation of $\mathbb{S}$, the unique cubic surface which does not admit a linear determinantal representation by [BL98], up to projectivity. Let us consider the point $\mathbf{a^1} = [1 : 0 : 0 : 0]$, which is singular and therefore not a T-point. Then Theorem 4.19 applies, and we can construct explicitly a Pfaffian $\mathbb{Q}$-representation of $\mathbb{S}$.

According to the proof of Theorem 4.13, we choose the plane $x_3 = 0$, which does not cut $\mathbb{S}$ in three lines. Considering the point $[1 : 1 : 0 : 0]$, the line through it and $\mathbf{a^1}$ intersects $\mathbb{S}$ in $\mathbf{a^2} = [0 : 1 : 0 : 0]$.
We have

$$\mathrm{T}_{\mathbf{a^2}}\,\mathbb{S} \cap \mathbb{S} : \begin{cases} x_0 = 0 \\ x_1x_3^2 + x_2^3 = 0 \end{cases}$$

and so we choose a point on $x_0 = 0$, say $[0 : 0 : 1 : 1]$. The line through it and $\mathbf{a^2}$ intersects $\mathbb{S}$ in $\mathbf{a^3} = [0 : -1 : 1 : 1]$.
We have

$$\mathrm{T}_{\mathbf{a^3}}\,\mathbb{S} \cap \mathbb{S} : \begin{cases} x_0 + x_1 + 3x_2 - 2x_3 = 0 \\ -x_1^3 - 3x_1^2x_2 + 2x_1^2x_3 + x_1x_3^2 + x_2^3 = 0 \end{cases}$$

and so we choose a point satisfying the first equation, say $[5 : 0 : -1 : 1]$. The line through it and $\mathbf{a^3}$ intersects $\mathbb{S}$ in $\mathbf{a^4} = [-10 : 1 : 1 : -3]$.
We have

$$\mathrm{T}_{\mathbf{a^4}}\,\mathbb{S} \cap \mathbb{S} : \begin{cases} x_0 - 11x_1 + 3x_2 - 6x_3 = 0 \\ 11x_1^3 - 3x_1^2x_2 + 6x_1^2x_3 + x_1x_3^2 + x_2^3 = 0 \end{cases}$$

and so we choose a point satisfying the first equation, say $[40 : 2 : -2 : 2]$. The line through it and $\mathbf{a^4}$ intersects $\mathbb{S}$ in $\mathbf{a^5} = [95 : 1 : -6 : 11]$.
A Pfaffian $\mathbb{Q}$-representation can be obtained via Algorithm 4.6. For example, simplifying denominators, we have $P = (p_{ij})$ with the following entries:

$p_{12} = 0,$ $p_{13} = x_2 - x_3,$
$p_{14} = 0,$ $p_{15} = 3x_2 + x_3,$
$p_{16} = 1470x_1 + 686x_2 + 588x_3,$ $p_{23} = -x_2 + x_3,$
$p_{24} = 34x_0 - 510x_1 - 170x_2 - 340x_3,$ $p_{25} = 2x_1 + x_2 + x_3,$
$p_{26} = 1372x_1 + 588x_3,$ $p_{34} = 8670x_1 + 6120x_2 + 2550x_3,$
$p_{35} = -34x_1 - 17x_2 - 17x_3,$ $p_{36} = -23324x_1 - 10829x_3,$
$p_{45} = 0,$ $p_{46} = 774690x_1 - 624750x_2,$
$p_{56} = -21658x_1 + 11662x_2 + 833x_3.$

References

[Abh60] Shreeram Abhyankar. Cubic surfaces with a double line. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.*, 32:455–511, 1960.

[ACGH85] Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths, and Joseph D. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[AR96] Allan Adler and Sundararaman Ramanan. *Moduli of abelian varieties*, volume 1644 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[Băn91] Constantin Bănică. Smooth reflexive sheaves. In *Proceedings of the Colloquium on Complex Analysis and the Sixth Romanian-Finnish Seminar*, volume 36, pages 571–593, 1991.

[Bar77] Wolf Barth. Some properties of stable rank-2 vector bundles on $\mathbf{P}_n$. *Math. Ann.*, 226(2):125–150, 1977.

[BE77] David A. Buchsbaum and David Eisenbud. Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. *Amer. J. Math.*, 99(3):447–485, 1977.

[BE82] David A. Buchsbaum and David Eisenbud. Gorenstein ideals of height 3. In *Seminar D. Eisenbud/B. Singh/W. Vogel, Vol. 2*, volume 48 of *Teubner-Texte zur Math.*, pages 30–48. Teubner, Leipzig, 1982.

[Bea00] Arnaud Beauville. Determinantal hypersurfaces. *Michigan Math. J.*, 48:39–64, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[BF11] Maria Chiara Brambilla and Daniele Faenzi. Moduli spaces of rank-2 ACM bundles on prime Fano threefolds. *Michigan Math. J.*, 60(1):113–148, 2011.

[BK11] Anita Buckley and Tomaž Košir. Plane curves as Pfaffians. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 10(2):363–388, 2011.

[BL98] Michela Brundu and Alessandro Logar. Parametrization of the orbits of cubic surfaces. *Transform. Groups*, 3(3):209–239, 1998.

[BM01] Dolores Bazan and Emilia Mezzetti. On the construction of some Buchsbaum varieties and the Hilbert scheme of elliptic scrolls in $\mathbb{P}^5$. *Geom. Dedicata*, 86(1-3):191–204, 2001.

[Bot57] Raoul Bott. Homogeneous vector bundles. *Ann. of Math. (2)*, 66:203–248, 1957.

[BR64] David A. Buchsbaum and Dock S. Rim. A generalized Koszul complex. II. Depth and multiplicity. *Trans. Amer. Math. Soc.*, 111:197–224, 1964.

[Buc10] Anita Buckley. Elementary transformations of Pfaffian representations of plane curves. *Linear Algebra Appl.*, 433(4):758–780, 2010.

[BV88] Winfried Bruns and Udo Vetter. *Determinantal rings*, volume 45 of *Monografías de Matemática [Mathematical Monographs]*. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1988.

[Cas91] Guido Castelnuovo. Ricerche di geometria della retta nello spazio a quattro dimensioni. *Atti R. Ist. Veneto Sc.*, 2:855–901, 1891.

[CF09] Luca Chiantini and Daniele Faenzi. On general surfaces defined by an almost linear Pfaffian. *Geom. Dedicata*, 142:91–107, 2009.

[Cha88] Mei-Chu Chang. Buchsbaum subvarieties of codimension 2 in $\mathbf{P}^n$. *Bull. Amer. Math. Soc. (N.S.)*, 19(1):269–272, 1988.

[CKM12] Emre Coskun, Rajesh S. Kulkarni, and Yusuf Mustopa. Pfaffian quartic surfaces and representations of Clifford algebras. *Doc. Math.*, 17:1003–1028, 2012.

[Con39] Fabio Conforto. *Le superficie razionali.* Zanichelli Editore, Bologna, 1939.

[DGO85] Edward D. Davis, Anthony V. Geramita, and Ferruccio Orecchia. Gorenstein algebras and the Cayley-Bacharach theorem. *Proc. Amer. Math. Soc.*, 93(4):593–597, 1985.

[Dic21] Leonard E. Dickson. Determination of all general homogeneous polynomials expressible as determinants with linear elements. *Trans. Amer. Math. Soc.*, 22(2):167–179, 1921.

[Dol12] Igor V. Dolgachev. *Classical algebraic geometry: a modern view.* Cambridge University Press, 2012.

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[Ein93] Lawrence Ein. On the cohomology of projectively Cohen-Macaulay determinantal subvarieties of $\mathbf{P}^n$. In *Geometry of complex projective varieties (Cetraro, 1990)*, volume 9 of *Sem. Conf.*, pages 143–152. Mediterranean, Rende, 1993.

[Eis95] David Eisenbud. *Commutative algebra with a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Ell75] Geir Ellingsrud. Sur le schéma de Hilbert des variétés de codimension 2 dans $\mathbf{P}^e$ à cône de Cohen-Macaulay. *Ann. Sci. École Norm. Sup. (4)*, 8(4):423–431, 1975.

[EN62] John A. Eagon and Douglas G. Northcott. Ideals defined by matrices and a certain complex associated with them. *Proc. Roy. Soc. Ser. A*, 269:188–204, 1962.

[Fae07] Daniele Faenzi. A remark on Pfaffian surfaces and aCM bundles. In *Vector bundles and low codimensional subvarieties: state of the art and recent developments*, volume 21 of *Quad. Mat.*, pages 209–217. Dept. Math., Seconda Univ. Napoli, Caserta, 2007.

[Fan30] Gino Fano. Reti di complessi lineari dello spazio S_5 aventi una rigata assegnata di rette-centri. *Rend. Reale Accad. Naz. Lincei*, XI:227–232, 1930.

[FF10a] Daniele Faenzi and Maria Lucia Fania. On the Hilbert scheme of determinantal subvarieties. *ArXiv e-prints* 1012.4692, December 2010.

[FF10b] Daniele Faenzi and Maria Lucia Fania. Skew-symmetric matrices and Palatini scrolls. *Math. Ann.*, 347(4):859–883, 2010.

[FL81] William Fulton and Robert Lazarsfeld. On the connectedness of degeneracy loci and special divisors. *Acta Math.*, 146(3-4):271–283, 1981.

[FM02] Maria Lucia Fania and Emilia Mezzetti. On the Hilbert scheme of Palatini threefolds. *Adv. Geom.*, 2(4):371–389, 2002.

[FP98] William Fulton and Piotr Pragacz. *Schubert varieties and degeneracy loci*, volume 1689 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998. Appendix J by the authors in collaboration with I. Ciocan-Fontanine.

[Ful92] William Fulton. Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. *Duke Math. J.*, 65(3):381–420, 1992.

[GG73] Martin Golubitsky and Victor Guillemin. *Stable mappings and their singularities*. Springer-Verlag, New York, 1973. Graduate Texts in Mathematics, Vol. 14.

[GP82] Laurent Gruson and Christian Peskine. Courbes de l'espace projectif: variétés de sécantes. In *Enumerative geometry and classical algebraic geometry (Nice, 1981)*, volume 24 of *Progr. Math.*, pages 1–31. Birkhäuser Boston, Mass., 1982.

[Gra55] Hermann Grassmann. Die stereometrische Gleichungen dritten Grades und die dadurch erzeugten Oberflächen. *J. Reine Angew. Math.*, (49):21–36, 1855.

[Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[Gro61a] Alexander Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.

[Gro61b] Alexander Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

[Gro62] Alexander Grothendieck. *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*. Secrétariat mathématique, Paris, 1962.

[Gro63] Alexander Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II. *Inst. Hautes Études Sci. Publ. Math.*, (17):91, 1963.

[Han12] Frédéric Han. Pfaffian bundles on cubic surfaces and configurations of planes. E-print arXiv:1210.1763. 10/2012.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Har80] Robin Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.

[Har10] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[Hun99] Craig Huneke. Hyman Bass and ubiquity: Gorenstein rings. In *Algebra, K-theory, groups, and education (New York, 1997)*, volume 243 of *Contemp. Math.*, pages 55–78. Amer. Math. Soc., Providence, RI, 1999.

[HV07] J. William Helton and Victor Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, 60(5):654–674, 2007.

[Iar84] Anthony Iarrobino. Compressed algebras: Artin algebras having given socle degrees and maximal length. *Trans. Amer. Math. Soc.*, 285(1):337–378, 1984.

[IM00] Atanas Iliev and Dimitri Markushevich. Quartic 3-fold: Pfaffians, vector bundles, and half-canonical curves. *Michigan Math. J.*, 47(2):385–394, 2000.

[Kem93] George R. Kempf. *Algebraic varieties*, volume 172 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.

[Kle10] Jan O. Kleppe. Deformations of modules of maximal grade and the Hilbert scheme at determinantal schemes. *ArXiv e-prints* 1012.1976, December 2010.

[Kle11] Jan O. Kleppe. Families of low dimensional determinantal schemes. *J. Pure Appl. Algebra*, 215(7):1711–1725, 2011.

[KMMR$^+$01] Jan O. Kleppe, Juan C. Migliore, Rosa Miró-Roig, Uwe Nagel, and Chris Peterson. Gorenstein liaison, complete intersection liaison invariants and unobstructedness. *Mem. Amer. Math. Soc.*, 154(732):viii+116, 2001.

[KMR05] Jan O. Kleppe and Rosa M. Miró-Roig. Dimension of families of determinantal schemes. *Trans. Amer. Math. Soc.*, 357(7):2871–2907, 2005.

[KMR11] Jan O. Kleppe and Rosa M. Miró-Roig. Families of determinantal schemes. *Proc. Amer. Math. Soc.*, 139(11):3831–3843, 2011.

[Mac94] Francis S. Macaulay. *The algebraic theory of modular systems.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1994. Revised reprint of the 1916 original, with an introduction by Paul Roberts.

[Man86] Yuri I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, translated from the Russian by M. Hazewinkel.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[Mor69] Louis J. Mordell. *Diophantine equations.* Pure and Applied Mathematics, Vol. 30. Academic Press, London, 1969.

[MP97] Juan C. Migliore and Chris Peterson. A construction of codimension three arithmetically Gorenstein subschemes of projective space. *Trans. Amer. Math. Soc.*, 349(9):3803–3821, 1997.

[MR08] Rosa M. Miró-Roig. *Determinantal ideals*, volume 264 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 2008.

[Muk92] Shigeru Mukai. Fano 3-folds. In *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, volume 179 of *London Math. Soc. Lecture Note Ser.*, pages 255–263. Cambridge Univ. Press, Cambridge, 1992.

[Muk95] Shigeru Mukai. Curves and symmetric spaces. I. *Amer. J. Math.*, 117(6):1627–1644, 1995.

[Nor76] Douglas G. Northcott. *Finite free resolutions.* Cambridge University Press, Cambridge, 1976. Cambridge Tracts in Mathematics, No. 71.

[Ott92] Giorgio Ottaviani. On 3-folds in $\mathbf{P}^5$ which are scrolls. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(3):451–471, 1992.

[Pal01] Francesco Palatini. Sui sistemi lineari di complessi lineari di rette nello spazio a cinque dimensioni. *Atti R. Ist. Veneto Sc.*, (60):371–383, 1901.

[Pal03] Francesco Palatini. Sui complessi lineari di rette negli iperspazi. *Giorn. di Mat.*, (41):85–96, 1903.

[Sch81] Friedrich Schur. Ueber die durch collineare Grundgebilde erzeugten Curven und Flächen. *Math. Ann.*, 18(1):1–32, 1881.

[Sch01] Frank-Olaf Schreyer. Geometry and algebra of prime Fano 3-folds of genus 12. *Compositio Math.*, 127(3):297–319, 2001.

[Seg51] Beniamino Segre. On the rational solutions of homogeneous cubic equations in four variables. *Math. Notae*, 11:1–68, 1951.

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2006.

[ST92] Joseph H. Silverman and John Tate. *Rational points on elliptic curves.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1992.

[Syl86] James J. Sylvester. Sur une extension d'un théorème de Clebsch relatif aux courbes de quartième degré. *C. R. Acad. Sci.*, (102):1532–1534, 1886.

[Ta13a] Fabio Tanturri. Pfaffian representations of cubic surfaces. *Geom. Dedicata*, 2013, DOI 10.1007/s10711-012-9818-x.

[Ta13b] Fabio Tanturri. On the Hilbert scheme of degeneracy loci of sections of the twisted cotangent bundle. SISSA preprint 44/2013/MATE, October 2013.

[Vin12] Victor Vinnikov. LMI representations of convex semialgebraic sets and determinantal representations of algebraic hypersurfaces: past, present, and future. In *Mathematical methods in systems, optimization, and control*, volume 222 of *Oper. Theory Adv. Appl.*, pages 325–349. Birkhäuser/Springer Basel AG, Basel, 2012.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.

[Wey03] Jerzy Weyman. *Cohomology of vector bundles and syzygies*, volume 149 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2003.

[CoCoA] CoCoATeam. *CoCoA: a system for doing Computations in Commutative Algebra.* Available at http://cocoa.dima.unige.it.