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Non-commutative Integration for Spectral Triples Associated to Quantum Groups

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Foreword

List of Ph.D. publications

This thesis summarizes my research during the Ph.D. in Mathematical Physics at SISSA. The original material that appears here is contained in the following four papers:

- M. Matassa, *A modular spectral triple for κ-Minkowski space*, Journal of Geometry and Physics 76C (2014), pp. 136-157, arXiv:1212.3462,
- M. Matassa, *On the spectral and homological dimension of κ-Minkowski space*, submitted to Journal of Noncommutative Geometry, preprint arXiv:1309.1054,
- M. Matassa, *Non-commutative integration, zeta functions and the Haar state for $SU_q(2)$*, preprint arXiv:1310.7477,
- M. Matassa, *Quantum dimension and quantum projective spaces*, to appear (2014).

Contents

Introduction

Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."
Michael Atiyah

This thesis is dedicated to the study of non-commutative integration, in the sense of spectral triples, for some non-commutative spaces associated to quantum groups. It seems appropriate then, as an introduction, to briefly explain what is the meaning of this statement.

Let us start with non-commutative spaces. One way of thinking about them is as an extension of the duality between spaces and functions to the non-commutative setting. Indeed to any (compact Hausdorff) topological space X we can associate the algebra $C(X)$ of continuous functions from X to $\mathbb{C}$. This algebra comes naturally with some extra structure, which makes $C(X)$ into a *C^*-algebra*. To what extent can we recover the space X from the C^*-algebra $C(X)$? The answer to this question is given by the *Gelfand-Naimark theorem*, which can be phrased as the equivalence of the category of locally compact Hausdorff topological spaces with the category of commutative C^*-algebras. Therefore we can think about these topological spaces in terms of their associated C^*-algebras. In view of this equivalence, we can declare a *non-commutative space* to be a non-commutative C^*-algebra.

Many notions related to topological spaces can be rephrased in algebraic terms, in such a way that they can be generalized to non-commutative spaces. An example is provided by the *Serre-Swan theorem*, which gives an equivalence between the category of vector bundles over a compact space X and the category of finitely generated projective modules over $C(X)$. The latter makes sense even without commutativity, so it can be considered as a definition of vector bundles over non-commutative spaces. For other examples of topological notions that can be generalized to the non-commutative setting we refer to the book [GVF].

In order to go beyond topology, and describe for example the metric properties of a non-commutative space, we can use the approach to non-commutative geometry developed by Connes [Con], which is based on the notion of *spectral triple*. This notion can be motivated by the observation that, for Riemannian manifolds, much geometric information can be retrieved from differential operators naturally defined on them. The definition is as follows.

Definition. A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is given by a unital $*$-algebra $\mathcal{A}$, represented on a Hilbert space $\mathcal{H}$, and by a self-adjoint operator D satisfying the following conditions:

1. the commutator $[D, x] = Dx - xD$ extends to a bounded operator for each $x \in \mathcal{A}$,
2. the operator $(D^2 + 1)^{-1/2}$ is compact.

A similar definition can be given for non-unital algebras, corresponding to non-compact spaces. Thanks to its properties, the operator D allows to define many geometrical notions in an algebraic or analytic fashion. For example, when $\mathcal{A} = C^\infty(M)$ for a compact spin manifold M, and D is a Dirac operator, the distance between two points $p, q \in M$ (which in turn completely characterizes the metric) can be obtained as

$$d(p, q) = \sup\{|f(p) - f(q)| : f \in C^\infty(M), \|[D, f]\| \leq 1\}.$$

This formula can be thought as dual to the usual expression for the distance, where we take the infimum over the lengths of all paths connecting p and q.

Among the many notions that can be defined in terms of D, two of them will play a key role in this thesis: they are the dimension of a manifold and the integral of a function with respect to the volume form associated to a metric. To illustrate how this works, we consider the canonical spectral triple $(C^\infty(M), L^2(M, S), D)$ associated to a compact spin manifold M, where D is the Dirac operator associated to a metric.

One way of obtaining the dimension of the manifold M in consideration, from the spectral point of view, is from the spectrum of the operator $(D^2 + 1)^{-1/2}$, which intuitively can be thought as the line element ds. Indeed, it turns out that the operator $(D^2 + 1)^{-z/2}$, with $z \in \mathbb{C}$, is trace-class for all $\mathrm{Re}(z) > n$, where n is the dimension of M. Similarly, using the operator $(D^2+1)^{-1/2}$ we can recover the integral of a function $f \in C^\infty(M)$. There are many ways of doing so, and one of them is to consider the residue of the zeta function associated to this operator. That is, we consider the linear functional ψ on $C^\infty(M)$ defined by

$$\psi(f) = \operatorname*{Res}_{z=n} \mathrm{Tr}(f(D^2 + 1)^{-z/2}).$$

It turns out that $\psi(f)$ coincides with the integral of f, up to a multiplicative constant.

Generalizing these notions to the case of a not necessarily commutative spectral triple, we will speak of *spectral dimension* and *non-commutative integral.* Clearly, as in the discussion above, in the commutative case these coincide with the dimension and the integral on the space under consideration. The discussion of how these two notions behave for certain non-commutative spaces, related to quantum groups, will be a central aspect of this thesis.

Coming back to the case of manifolds, an important result is the *reconstruction theorem*. It states that, given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with some additional conditions, if $\mathcal{A}$ is commutative then it is of the form $C^\infty(M)$, for some compact spin manifold M, and where D is the Dirac operator determined by the spin structure and the Riemannian metric. This result shows that this notion is able to capture the geometry of this class of manifolds.

Up to this point, our discussion of non-commutative geometry has been centered on reformulating classical notions, in such a way that they make sense also for non-commutative algebras. In principle these can be adopted as axioms, and simply declare that all the interesting non-commutative spaces are those which satisfy them. However this point of view misses an important aspect: there are many new phenomena which appear in the non-commutative world and which have no analogue in the commutative one. It is not difficult to imagine that such new phenomena might be important in the description of non-commutative spaces.

For example, for a commutative C^*-algebra a linear functional obviously satisfies the trace property, simply because the algebra is commutative, while satisfying the trace property for a non-commutative algebra is a non-trivial condition. It turns out that for non-commutative spaces, like for example compact quantum groups, the more natural notion is often a state which is not tracial. This obviously has no analogue in the commutative case.

Therefore it makes sense to look for modifications of the conditions defining a spectral triple, with the understanding that they should reduce to the usual ones for a commutative space. In this thesis we will consider various extensions of the notion of spectral triple, which modify in different ways some of its axioms. We will consider the framework of *twisted spectral triples* [CoMo08], which modifies the commutator condition by requiring the boundedness of a twisted commutator, with the twist being an automorphism of the algebra. On the other hand, the framework of *modular spectral triples* [CPR10, CRT09, CNNR11] modifies the resolvent condition, roughly speaking by replacing the operator trace with a weight having a non-trivial modular group. This in turn is a generalization of *semifinite spectral triples* [BeFa06], where the operator trace is replaced by a semifinite trace.

Another important role in this thesis is played by the theory of quantum groups, for a general reference see for example [KlSc]. They can be seen, as far as we are concerned, as a generalization of the theory of Lie groups. The "quantum" aspect comes from the fact that we do not have a manifold associated to them, but a non-commutative space. Therefore quantum groups and their homogeneous space provide a rich class of examples of non-commutative spaces. It is then very natural to try to describe their geometry using spectral triples.

However, it is fair to say that a general understanding of how quantum groups should fit into the framework of spectral triples is still lacking. This is related to the fact that, in several aspects, they behave differently from the non-commutative spaces that are well understood from the point of view of spectral triples, for example the non-commutative tori. A particular feature of compact quantum groups, which plays a leading role in their theory, is the existence and uniqueness of the Haar state, which is the analogue of the Haar integral for compact Lie groups. However, differently from the commutative case, the Haar state has a non-trivial modular group, so that it is not tracial. A recurring theme throughout this thesis will be the investigation of the compatibility of non-commutative integration, in the sense of spectral triples, with such states.

This thesis is divided in two parts, which have distinct roles in the presentation. The first part, which starts in chapter 1 and ends in chapter 4, provides some background material on non-commutative integration, spectral triples and their extensions. The second part, on the other hand, contains new results obtained by the author.

In chapters 5 and 6 we will consider the case of κ-Minkowski space, which can be seen as a quantum homogeneous space for the κ-Poincaré quantum group. These Hopf algebras are obtained as deformations of the familiar Minkowski space and Poincaré group, which play a forefront role in theoretical physics. For this reason, they have been much studied from the point of view of quantum gravity, with the deformation parameter being related to the Planck length. It is then interesting to describe (the Euclidean version of) this non-commutative space in terms of a spectral triple.

The starting point is an algebra $\mathcal{A}$, originally introduced in [DuSi13], which is naturally associated with the commutation relations of κ-Minkowski space. We use a KMS-weight ω, which is motivated by the κ-Poincaré symmetries, to introduce a Hilbert space $\mathcal{H}$ via the GNS-construction. We emphasize that the choice of this weight is one of the main differences with respect to other approaches. The necessity of a modification to the conditions of a spectral triple, as in our discussion above, appears upon the introduction of a Dirac operator. Indeed, to satisfy a boundedness condition and have the correct classical limit, the use of a twisted commutator turns out to be necessary. Moreover these requests, together with some symmetry conditions, single out a unique Dirac operator D and unique twist.

However the triple $(\mathcal{A}, \mathcal{H}, D)$ is not finitely summable, an outcome which is hinted at by the mismatch in the modular properties of the weight ω and the non-commutative integral. We will repeat and emphasize this argument later, so we do not elaborate further here. This mismatch also hints at the possibility that, by choosing an appropriate weight in the sense of modular spectral triples, we can obtain a finite spectral dimension. This is indeed the case and we find that, in this setting, the spectral dimension is finite and coincides with the classical one. Moreover, by computing the residue at the spectral dimension of the corresponding zeta function, we recover the weight ω up to a constant.

This construction is performed in full detail in chapter 5 for the two-dimensional case, while in chapter 6 it is streamlined for the general n-dimensional case. In chapter 6 we also analyze some additional properties of the zeta function defined in terms of the Dirac operator. We show that, by considering the limit of vanishing deformation parameter, it reduces as it should to the corresponding one of the classical case. Also, as in the commutative setting, this zeta function can be analytically continued to a meromorphic function on the complex plane, with only simple poles. It turns out to have all the poles of the commutative case, but also additional ones due to the presence of the deformation parameter. The significance of these additional poles remains to be investigated.

Another important aspect that we investigate in chapter 6 is the homological dimension of this geometry. In the framework of non-commutative geometry, this notion corresponds to the dimension of the Hochschild homology, which in the commutative case coincides with the classical dimension. However in many examples, coming in particular from quantum groups, one finds that the homological dimension is lower than the classical dimension, a phenomenon known as dimension drop. In many cases it is possible to avoid this drop by introducing a twist in the homology theory, as seen for example in [HaKr05, Had07]. Here we compute the *twisted Hochschild homology* [BrZh08] of the universal enveloping algebra associated to κ-Minkowski space. Similarly to the examples we mentioned above, we show that the dimension drop occurs at the level of Hochschild homology, but can be avoided by introducing a twist. More interestingly, the simplest twist which avoids the drop is the inverse of the modular group of the weight ω, while the other possible twists are given by its positive powers. This is very similar to what happens in [HaKr05, Had07], where the simplest twist is given by the inverse of the modular group of the Haar state, and therefore seems to be a general feature of these non-commutative geometries.

In chapter 7 we leave non-compact spaces and consider instead compact quantum groups. As we mentioned above, in this setting it there is a unique state, the Haar state, which is the non-commutative analogue of the Haar integral for compact Lie groups. We remark that the choice of a state gives a natural notion of non-commutative integration, as it is known from the theory of von Neumann algebras [Tak]. Therefore it would seem natural, from the point of view of spectral triples, to require that the non-commutative integral coincides with the Haar state. However it is clear that this is not possible in the usual setting: indeed, from general properties of spectral triples, it follows that the non-commutative integral is a trace, while the Haar state does not satisfy the trace property. On the other hand, in the extended frameworks we mentioned above, the non-commutative integral need not be a trace. Therefore such a requirement can be in principle satisfied.

In particular, this observation can be used to give a tentative answer to the following question, which naturally arises by considering the framework of modular spectral triples: if we are allowed to replace the operator trace by a weight, are there any preferred choices? In view of the discussion above, a reasonable criterion is to require that the corresponding non-commutative integral should coincide with the Haar state.

Here we will consider in detail this question for the case of the quantum group $SU_q(2)$. More specifically we consider the Dirac operator D_q introduced in [KaSe12], which gives a (twisted) modular spectral triple. We observe that this Dirac operator has an interesting property, namely it implements one of the left covariant differential calculi on $SU_q(2)$.

The non-commutative integral will be defined as the residue at the spectral dimension of a certain zeta function. More precisely, we define a family of zeta functions using the operator

D_q and a family of weights depending on two parameters $a, b \in \mathbb{R}$. These two parameters essentially parametrize the most general diagonal automorphism of $SU_q(2)$, and we remark that the modular group of the Haar state is of this form.

First of all we discuss for which values of the parameters the zeta function is well defined, and determine its spectral dimension. Then we impose the requirement of recovering the Haar state from the non-commutative integral. A necessary condition is that their modular groups coincide. We will show that this condition fixes $b = 1$, but leaves a undetermined. Moreover the non-commutative integral, once properly normalized, turns out to coincide with the Haar state, independently of a.

This result shows that we can partially fix the arbitrariness in the choice of the weight. We still have freedom in the choice of the parameter a, which disappears in the non-commutative integral. On the other hand the spectral dimension depends on a. In particular, after fixing $b = 1$, we have that $n = a+1$. Therefore a preferred choice is $a = 2$, which makes the spectral dimension equal the classical one $n = 3$.

We now argue that there is another requirement, more spectral in nature, that also turns out to fix this value uniquely. Up to this point we have only used the information contained in the residue at $z = n$ of the zeta function, that is the residue at the spectral dimension. But the analytic continuation of the zeta function contains much more information than that. Indeed, from the point of view of the heat kernel expansion on a compact manifold, the residue at $z = n$ corresponds only to the first coefficient of the expansion. Therefore we can look at the next non-trivial coefficient which, in terms of the zeta function, corresponds to computing the residue at a different value. It is easy to see that, for the classical limit of the operator D_q, this coefficent vanishes non-trivially. Therefore we can require an analogue condition for the non-commutative case. It turns out that this condition is satisfied only in the case $a = 2$, which was the value we considered above.

Finally in chapter 8 we consider quantum projective spaces, which are examples of quantum homogeneous spaces. This class of spaces provides an excellent testing ground to study how quantum groups fit into the framework of spectral triples. An important result in this respect is given in [Krä04], where a Dirac operator D is defined on quantized irreducible generalized flag manifolds, which yields a Hilbert space realization of the covariant first-order differential calculus constructed in [HeKo04]. This in particular means that the commutator of D with an element of the coordinate algebra is a bounded operator, which is one of the defining properties of a spectral triple. The other essential property, compactness of the resolvent of D, has not been proven, even though it is expected to hold. In particular it can be checked for the simplest case to which this construction applies, that is the Podleś sphere. In this case the Dirac operator D coincides with the Dirac operator introduced in [DąSi03], which has compact resolvent.

Among the class of q-deformed irreducible flag manifolds are the quantum projective spaces $\mathbb{C}P^\ell_q$, the simplest example of which is again the Podleś sphere, which is obtained for $\ell = 1$.

The case of $\mathbb{C}P_q^2$ has been studied in [DDL08] and then generalized for the case $\ell \geq 2$ in [D'ADą10]. We now briefly recall this construction. The starting point is the introduction of the q-analogue of the module of antiholomorphic differential k-forms Ω^k. More generally the modules Ω_N^k are considered, with $N \in \mathbb{Z}$, corresponding essentially to $\Omega^k = \Omega_0^k$ twisted by certain line bundles. The Hilbert space completion of these is denoted by H_N. For each of these an unbounded self-adjoint operator D_N is introduced, which has bounded commutators with the coordinate algebra $\mathcal{A}(\mathbb{C}P_q^\ell)$. The main result is that $(\mathcal{A}(\mathbb{C}P_q^\ell), H_N, D_N)$ is a family of equivariant spectral triples.

It turns out that these spectral triples are 0^+-summable, in the sense that the operator $(D_N^2 + 1)^{-\epsilon/2}$ is trace-class for every $\varepsilon > 0$. The detailed computation of the spectrum clearly reveals why this is the case: the eigenvalues of this operator grow like a q-number, so exponentially, while their multiplicities grow like a polynomial. We recall that in the classical case it is the balance between the growth of the eigenvalues and their multiplicities that allows to recover the dimension of the manifold in consideration. In this case the eigenvalues grow much faster than their multiplicities, which explains the 0^+-summability.

Here we consider a simple modification of the above construction, which fits into the framework of modular spectral triples. The idea is to consider the action of the element $K_{2\rho}$, which implements the modular group of the Haar state of $\mathcal{A}(\mathbb{C}P_q^\ell)$. In particular we compute the spectral dimension associated to D with respect to the weight $\mathrm{Tr}(K_{2\rho}\cdot)$, with the result that it coincides with the classical dimension. This computation is linked with an important concept in the theory of quantum groups, that of quantum dimension. We also point out that, as a consequence of a property of the quantum dimension, the same result for the spectral dimension is obtained by considering $K_{2\rho}^{-1}$. This in turn is connected with some results from twisted Hochschild (co)homology.

Part I

Background material

Chapter 1

Infinitesimals and Dixmier traces

In this section we introduce the basic notions of a particular theory of integration, formulated in a Hilbert space setting, which is used in the approach to non-commutative geometry introduced by Connes [Con]. It is based on a notion of infinitesimal defined in terms of compact operators, with the integral being defined in terms of linear functionals called Dixmier traces. This theory of integration is different from the more familiar one developed by Segal [Seg53], which provides a natural analogue of the L^p spaces of functional analysis. Before getting to the definitions, we spend some time giving motivations in the form of an informal discussion.

A general reference for this section is the book [Con]. The informal discussion on infinitesimals and integration is based on [Con2]. The examples are taken from [Lan].

1.1 Informal discussion

One of the reasons for developing a theory of non-commutative geometry is to put together the classical notions of geometry with the principles of quantum mechanics. In the latter an *observable* is a self-adjoint operator on a Hilbert space, which is the setting that we want to use. Since all the Hilbert spaces of a given dimension are isomorphic, the setting is canonical at this stage. To seek guidance for a formulation of a theory of integration for operators on a Hilbert space, let us consider the fundamental theorem of calculus in the form

$$\int_a^b df = f(b) - f(a),$$

which clearly relates integral and differential calculus. At the most intuitive level we have a picture of summing over "infinitesimal variations" of the function f. Therefore, if we can formulate a notion of "infinitesimal" for an operator on a Hilbert space, then an integral will be a map (with some properties) that associates a number to such an operator.

What could then be a notion of infinitesimal for an operator on an infinite-dimensional Hilbert space? Since in our naive picture an infinitesimal is something "arbitrarily small", a first guess would be to look for operators which are arbitrarily small with respect to the operator norm. But this is not a good guess, since if an operator T is such that $\|T\| < \epsilon$ for any $\epsilon > 0$, then $T = 0$. Thus we need a weaker notion.

We can slightly weaken it as follows: for any $\epsilon > 0$ there exists a finite dimensional subspace $E \subset H$ such that $\|T \restriction_{E^\perp} \| < \epsilon$. Here the symbol $\restriction$ denotes restriction and $E^\perp$ is the orthogonal complement of E. There are plenty of operators which satisfy this condition. Indeed this condition characterizes the class of compact operators, which can be alternatively considered as the norm limit of finite-rank operators. Therefore before proceeding it is worth reviewing some facts about compact operators on a Hilbert space.

1.2 Compact operators

In the following we consider an infinite-dimensional and separable Hilbert space H. We denote the ideal of compact operators by $K(H)$.

We start by discussing a canonical form for compact operators. Let A be a positive compact operator on H. It can be proven that the spectrum of A consists of the number zero and countably many positive eigenvalues of finite multiplicity, which can be arranged in decreasing order. In particular we can find a complete orthonormal basis $\{u_k\}$ for H, such that $Au_k = \mu_k u_k$ and $\mu_k \to 0$ as $k \to \infty$. So we have a norm convergent expansion

$$A = \sum_{k=0}^{\infty} \mu_k(A)(u_k, \cdot)u_k,$$

where $(\cdot, \cdot)$ denotes the inner product on H, linear in the second variable. Moreover we have arranged the eigenvalues as $\mu_0 \geq \mu_1 \geq \cdots$, with $\mu_k \to 0$.

Now for a generic compact operator $T \in K(H)$, we define its *absolute value* as the compact positive operator $|T| = (T^*T)^{1/2}$. Then $|T|$ admits an expansion of the form

$$|T| = \sum_{k=0}^{\infty} \mu_k(T)(u_k, \cdot)u_k.$$

The *polar decomposition* $T = U|T|$ is obtained by defining $U(|T|\psi) = T\psi$ and $U\phi = 0$ for $\phi \in \ker|T|$, that is U is a partial isometry uniquely determined by T. Then it follows that, by setting $v_k = Uu_k$, we have an expansion of the form

$$T = U|T| = \sum_{k=0}^{\infty} \mu_k(T)(u_k, \cdot)v_k.$$

This is therefore a canonical expansion for a compact operator. We call $\mu_k(T)$ the k-th *singular value* of T, which coincides with the k-th eigenvalue of the compact positive operator $|T| = (T^*T)^{1/2}$. Note that $\mu_0(T) = \|T\|$, since it is the largest eigenvalue. Also if U_1, U_2 are unitary operators then we have

$$U_1 T U_2 = \sum_{k=0}^{\infty} \mu_k(T)(U_2^* u_k, \cdot) U_1 v_k,$$

which shows that $\mu_k(U_1 T U_2) = \mu_k(T)$, that is the singular values are unitary invariant.

Let us discuss now some some ideals of compact operators. An operator T is called trace-class, which we write as $T \in \mathcal{L}^1$, if the following series converges

$$\|T\|_1 = \mathrm{Tr}|T| = \sum_{k=0}^{\infty} \mu_k(T).$$

More generally, for $1 < p < \infty$, the Schatten classes $\mathcal{L}^p$ consist of operators such that

$$\|T\|_p = (\mathrm{Tr}|T|^p)^{1/p} = \left(\sum_{k=0}^{\infty} \mu_k(T)^p \right)^{1/p}$$

is finite. It is possible to prove a Hölder's inequality of the form

$$\mathrm{Tr}|TS| \leq \|T\|_p \|S\|_q, \quad p^{-1} + q^{-1} = 1.$$

The case with $p = 1$ immediately shows that $\mathcal{L}^1$ is an ideal, since $q = \infty$ corresponds to the operator norm. This is true also for the other Schatten classes, and moreover we have the inclusions $\mathcal{L}^1 \subset \mathcal{L}^r \subset \mathcal{L}^p \subset K(H)$ for $1 < r < p < \infty$.

1.3 Order of infinitesimals

We can now refine the notion of infinitesimal by introducing a notion of *order*.

Definition 1.1. We say that a compact operator $T \in K(H)$ is an *infinitesimal of order* α, for $\alpha > 0$, if its singular values are such that $\mu_n(T) = O(n^{-\alpha})$ as $n \to \infty$.

In other words there exists a constant C such that $\mu_n(T) \leq C n^{-\alpha}$. Let us comment on why this is a natural notion. The following inequality holds for the singular values

$$\mu_{n_1+n_2}(T_1 T_2) \leq \mu_{n_1}(T_1) \mu_{n_2}(T_2).$$

Then it immediately follows from the definition that, if T_j is of order α_j, then $T_1 T_2$ is of order $\leq \alpha_1 + \alpha_2$, which is what we would expect from a naive picture of infinitesimals. This inequality also shows that infinitesimals of a given order form an ideal in $B(H)$.

We define the *Dixmier ideal* as the space

$$\mathcal{L}^{1+} = \left\{ T \in K(H) : \sup_{N>1} \frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T) < \infty \right\}.$$

The reason for this definition will be apparent in the next subsection, when we will consider integration. Notice that, as defined, the space $\mathcal{L}^{1+}$ is *not* the space of infinitesimals of order one but is strictly larger. Note also that we have the inclusion $\mathcal{L}^1 \subset \mathcal{L}^{1+}$, which can be seen as a justification for the notation.

Finally we want to warn the reader that in the literature there are many variations on the above theme, both in notation and in definition. A comparison of the notations and definitions used in the main papers in the subject is contained in [CaSu12, subsection 4.1].

1.4 Dixmier traces

We now discuss the introduction of an appropriate notion of integral. With the emphasis put on the notion of infinitesimal, we can now explain why such a theory of integration will be different from that of [Seg53]. Indeed, the operator trace does not satisfy two properties that we would intuitively expect from an integral, namely:

1. infinitesimals of order one *are not* in its domain,

2. infinitesimals of higher order *are not* in its kernel.

This follows from the fact that for an infinitesimal of order one we have $\mu_n(T) = O(n^{-1})$, so that its trace is essentially the (divergent) harmonic series. Since in this case the divergence is logarithmic, we would like to have a way to extract a number from this divergent quantity. One possibility is to use a Dixmier trace, see the discussion in [Con].

To arrive at this notion let us notice that for infinitesimals of order one we have

$$\sum_{n=0}^{N-1} \mu_n(T) \leq C \log N.$$

Then the simplest thing to do is to divide by $\log N$ and take the limit $N \to \infty$, that is

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T).$$

There is a problem with this proposal, however, since not all such sequences will be convergent (although they are bounded). It is also far from clear that such a procedure gives a linear functional, which is clearly a property that we want to have for an integral.

We will now show that when sequences are convergent this is a linear functional. Let us mention in passing that, since the singular values $\mu_n(T)$ are unitary invariant, the same will be true for the result of this procedure.

1.4.1 Linearity

Let us start by defining some useful notation

$$\sigma_N(T)=\sum_{n=0}^{N-1}\mu_n(T), \quad \gamma_N(T)=\frac{\sigma_N(T)}{\log N}.$$

Let us also recall the following characterization of the singular values of an operator T

$$\mu_k(S)=\inf\{\|S(1-P)\| : P \text{ is a projection}, \mathrm{Tr}(P)\leq k\}.$$

From it it is not hard to obtain the following characterization of $\sigma_N(T)$, for the proof see for example [GVF].

Lemma 1.1. *If T is a compact operator then*

$$\sigma_N(T)=\sup\{\|TP\|_1 : P=P^*=P^2, \mathrm{rank}P=N\}\ .$$

If T is also positive then we have

$$\sigma_N(T)=\sup\{\mathrm{Tr}(PTP) : P=P^*=P^2, \mathrm{rank}P=N\}\ .$$

From this lemma it immediately follows that

$$\sigma_N(T_1+T_2)\leq\sigma_N(T_1)+\sigma_N(T_2)\ ,$$

since we have $\|(T_1+T_2)P\|_1\leq\|T_1P\|_1+\|T_2P\|_1$.

The triangle inequality is not good enough for our purposes, because we would like to obtain an *additive* functional. The next step is to prove a "wrong-way triangle inequality".

Lemma 1.2. *For T_1, T_2 positive compact operators we have*

$$\sigma_M(T_1)+\sigma_N(T_2)\leq\sigma_{M+N}(T_1+T_2)\ .$$

Proof. On the left-hand side we have to take the supremum over projections P, P' with $\mathrm{rank}P=M$ and $\mathrm{rank}P'=N$. We have $\mathrm{rank}(P+P')\leq M+N$ so, if we take a projection P'' of rank $M+N$ whose range includes $P\mathcal{H}+P'\mathcal{H}$, then we have the inequalities $P\leq P''$ and $P'\leq P''$ as operators. Therefore

$$\mathrm{Tr}(PT_1P)+\mathrm{Tr}(P'T_2P')\leq\mathrm{Tr}(P''T_1P'')+\mathrm{Tr}(P''T_2P'')=\mathrm{Tr}(P''(T_1+T_2)P'')\ .$$

Using this inequality we obtain

$$\sigma_M(T_1)+\sigma_N(T_2)\leq\sup\{\mathrm{Tr}(P''(T_1+T_2)P'') : \mathrm{rank}P''=N\}=\sigma_{M+N}(T_1+T_2)\ .$$

Notice that this argument requires additivity, so that it works for Tr but not for $\|\cdot\|_1$, hence the restriction to positive operators. □

In particular it follows from this lemma that $\sigma_N(T_1)+\sigma_N(T_2)\le\sigma_{2N}(T_1+T_2)$. Now we show that in some good cases we can obtain an additive functional.

Proposition 1.3. *Let $T = T_1 + T_2$ with T_1, T_2 compact and positive. Suppose the limit $N\to\infty$ of $\gamma_N(T)$ exists and denote it by $\gamma(T)$. Then we have*

$$\gamma(T)=\gamma(T_1)+\gamma(T_2)\ .$$

Proof. Let us consider the two inequalities proven above

$$\begin{aligned}\sigma_N(T_1+T_2)&\le\sigma_N(T_1)+\sigma_N(T_2)\ ,\\ \sigma_N(T_1)+\sigma_N(T_2)&\le\sigma_{2N}(T_1+T_2)\ .\end{aligned}$$

The second property holds for positive operators. From the first one it follows that

$$\gamma_N(T_1+T_2)\le\gamma_N(T_1)+\gamma_N(T_2)\ .$$

If we divide the second one by $\log N$ we get

$$\gamma_N(T_1)+\gamma_N(T_2)\le\frac{\sigma_{2N}(T_1+T_2)}{\log N}=\frac{\sigma_{2N}(T_1+T_2)}{\log 2N}\left(1+\frac{\log 2}{\log N}\right)\ .$$

Therefore we have obtained the inequalities

$$\gamma_N(T_1+T_2)\le\gamma_N(T_1)+\gamma_N(T_2)\le\gamma_{2N}(T_1+T_2)\left(1+\frac{\log 2}{\log N}\right)\ .$$

Now taking the limit $N\to\infty$ we get

$$\gamma(T_1+T_2)\le\gamma(T_1)+\gamma(T_2)\le\gamma(T_1+T_2)\ ,$$

which proves the claimed relation. □

1.4.2 The general case

In general the sequence $\gamma_N(T)$ will not converge, so we need to proceed in a different way. To this end we can consider a linear form $\lim_\omega$ on the space $\ell^\infty(\mathbb{N})$ such that

1. $\lim_\omega\{\gamma_N\}\ge 0$ if $\gamma_N\ge 0$,
2. $\lim_\omega\{\gamma_N\}=\lim_N\{\gamma_N\}$ if $\{\gamma_N\}$ is convergent,
3. $\lim_\omega\{\gamma_1,\gamma_1,\gamma_2,\gamma_2,\cdots\}=\lim_\omega\{\gamma_N\}$.

The first and second conditions are pretty obvious requests, while the third condition, the so-called *scale invariance*, is more obscure but actually crucial for this procedure to work. We do not get into the details of actually defining the limiting procedure, but just mention that that Dixmier proved that there exists an infinity of such scale invariant forms. For more details we refer to the book of Connes [Con].

For a fixed choice of the generalized limit $\lim_\omega$ we define the Dixmier trace as

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T).$$

Notice that in general the value of $\mathrm{Tr}_\omega(T)$ will depend on the choice of $\lim_\omega$, unless of course the sequence $\{\gamma_N\}$ is convergent, in which case all the generalized limits give the same value.

Using the "four parts" argument it is easy to show that $\mathcal{L}^{1+}$ is generated by positive operators. Since the Dixmier trace is additive on positive operators, it can be extended by linearity to the entire $\mathcal{L}^{1+}$. We now list some of its general properties.

Proposition 1.4. *The Dixmier traces satisfy the following properties:*

1. $\mathrm{Tr}_\omega T \geq 0$ *if* $T \geq 0$,
2. $\mathrm{Tr}_\omega(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 \mathrm{Tr}_\omega(T_1) + \lambda_2 \mathrm{Tr}_\omega(T_2)$,
3. $\mathrm{Tr}_\omega(BT) = \mathrm{Tr}_\omega(TB)$ *for any* $B \in \mathcal{B}(\mathcal{H})$,
4. $\mathrm{Tr}_\omega(T) = 0$ *if* T *is of order higher than* 1.

This more general definition is interesting from the theoretical point of view, but not very useful in applications. Indeed such a limiting procedure cannot be exhibited in general, and therefore the trace cannot be computed in practice. However, as we mentioned previously, it is clear that there are operators such that the value of $\mathrm{Tr}_\omega(T)$ does not depend on the choice of the generalized limit, and therefore can be computed using the usual limit.

Definition 1.2. An operator $T \in \mathcal{L}^{1+}$ is called *measurable* if the value $\mathrm{Tr}_\omega(T)$ does not depend on the choice of ω.

This is clearly the case if the sequence $\{\gamma_N\}$ is convergent. It was actually shown in [LSS05] that a *positive* operator $T \in \mathcal{L}^{1+}$ is measurable if *and only if* the sequence $\{\gamma_N\}$ is convergent. In practice this is always the case in non-commutative geometry.

1.5 Some examples

We now compute the Dixmier trace of some operators as examples. We will consider the case of powers of the Laplacian on the n-torus and on the n-sphere. Even though they are elementary examples, the actual computations will take some effort. Later on we will rederive these results in a considerably easier way, but for the moment we will stick to the definition.

This is also a good place to spend a few words on the issue of invertibility of operators. Indeed in the following we will often consider negative powers of a positive operator T, like the Laplacian in these examples, which make sense only if T is invertible. In practice T might not be invertible, but will have a finite-dimensional kernel. We can deal with this annoyance by considering an invertible operator which has essentially the same role as T.

There are at least two ways to proceed. The first one is to consider the operator $(T^2+\varepsilon)^{1/2}$, with $\varepsilon > 0$. The second way is to consider the operator $T+P$, where P projects on the kernel of T. In both cases we obtain the same value for the Dixmier trace, therefore we will just pretend that the operators in consideration are invertible in this sense.

Example 1.1. *The Laplacian Δ on the n-dimensional torus T^n has eigenvalues $\|l_j\|^2$, where l_j is a point of the lattice $\mathbb{Z}^n$ with multiplicity one. We will remove the zero eigenvalue, as discussed above. We want to compute the Dixmier trace of Δ^s, where s is a real number. Since this is a positive operator, the Dixmier trace will exists if and only if the ordinary limit exists, so we want to compute the limit $N \to \infty$ of*

$$\gamma_N = \frac{1}{\log N} \sum_{j=1}^{N-1} \|l_j\|^{2s}.$$

It is easier to rephrase this sum in a different way. Let N_R be the number of lattice points in the ball of radius R centered at the origin of $\mathbb{R}^n$. Then it is not difficult to see that we have $N_R \sim V_n(R)$ for $R \to \infty$, where $V_n(R)$ denotes the volume of the n-ball of radius R. Therefore we can index the sum by N_R and consider the limit $R \to \infty$ of the quantity

$$\gamma_{N_R} = \frac{1}{\log N_R} \sum_{1 \leq \|l\| \leq R} \|l\|^{2s}.$$

Before proceeding we extract the dependence of N_R on R. Let us denote by Ω_n the area of the unit $(n-1)$-sphere, which is given explicitely by $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$. The volume and the area of the n-ball are related by $V_n(R) = n^{-1}\Omega_n R^n$, therefore $N_R \sim n^{-1}\Omega_n R^n$.

Now we need to use some asymptotic formulae. We consider the number of eigenvalues in the thin shell between the radii r and $r + dr$. This can be estimated as

$$N_{r+dr} - N_r \approx \frac{dN_r}{dr} dr = \Omega_n r^{n-1} dr \ .$$

We can replace, in the limit $R \to \infty$, the sum over $\|l\| \leq R$ with an integral over r. Explicitely we have

$$\sum_{\|l\| \leq R} \|l\|^{2s} \sim \int_1^R r^{2s} \Omega_n r^{n-1} dr = \begin{cases} \Omega_n \log R & s = -n/2 \\ \Omega_n \frac{R^{2s+n}-1}{2s+n} & s \neq -n/2 \end{cases} .$$

Now we can finally consider the $R \to \infty$ limit of γ_{N_R}. Let us consider first the case $s = -n/2$. Then, using the fact that $\log N_R \sim n \log R$, we find

$$\gamma_{N_R} \sim \frac{\Omega_n \log R}{n \log R} = \frac{\Omega_n}{n}.$$

Therefore in this case we have $\mathrm{Tr}_\omega \Delta^{-n/2} = \Omega_n/n$. On the other hand it easy to see that for the cases $s > -n/2$ and $s < -n/2$ we get respectively $\gamma_{N_R} \to \infty$ and $\gamma_{N_R} \to 0$.

Example 1.2. *The Laplacian Δ on the n-sphere has eigenvalues $l(l+n-1)$, with $l \in \mathbb{N}$, and with multiplicity*

$$m_l = \begin{pmatrix} l+n \\ n \end{pmatrix} - \begin{pmatrix} l+n-2 \\ n \end{pmatrix} = \frac{(l+n-2)!}{(n-1)!l!}(2l+n-1) \ .$$

As in the previous example, if the Dixmier trace exists, it is given by the limit

$$\mathrm{Tr}_\omega \Delta^{-n/2} = \lim_{N\to\infty} \frac{\sum_{l=1}^N m_l [l(l+n-1)]^{-n/2}}{\log \sum_{l=1}^N m_l} \ .$$

Let us start with the sum of the multiplicities

$$\begin{aligned} \sum_{l=1}^N m_l &= \begin{pmatrix} N+n \\ n \end{pmatrix} + \begin{pmatrix} N+n-1 \\ n \end{pmatrix} - 1 \\ &= \frac{1}{n!N!}(N+n-1)!(2N+n) - 1 \sim \frac{2N^n}{n!} \ , \end{aligned}$$

from which we find that

$$\log \sum_{l=1}^N m_l \sim n \log N \ .$$

For the other sum we have

$$\begin{aligned} \sum_{l=1}^N m_l [l(l+n-1)]^{-n/2} &= \frac{1}{(n-1)!} \sum_{l=1}^N \frac{(l+n-2)!}{l![l(l+n-1)]^{n/2}}(2l+n-1) \\ &\sim \frac{1}{(n-1)!} \sum_{l=1}^N \frac{2l^{n-1}}{[l(l+n-1)]^{n/2}} \\ &\sim \frac{2}{(n-1)!} \sum_{l=1}^N l^{-1} \sim \frac{2}{(n-1)!} \log N \ . \end{aligned}$$

Putting these results together we arrive at

$$\begin{aligned} \mathrm{Tr}_\omega \Delta^{-n/2} &= \lim_{N\to\infty} \frac{\sum_{l=1}^N m_l [l(l+n-1)]^{-n/2}}{\log \sum_{l=1}^N m_l} \\ &= \lim_{N\to\infty} \frac{\frac{2}{(n-1)!} \log N}{n \log N} = \frac{2}{n!} \ . \end{aligned}$$

In this particular examples, the Dixmier trace can be also computed using Weyl's law, which describes the asymptotic behaviour of the eigenvalues of the Laplace-Beltrami operator. Indeed it can be seen as one of the first general results on spectral geometry.

Chapter 2

Non-commutative integration

In the previous section we have introduced a notion of infinitesimal and an abstract notion of integration via the Dixmier traces. However, to prove that this is a sensible notion, we still have to show that in the commutative case we recover the usual integration of functions. We will show that this is the case, at least for smooth functions, by using Connes' trace theorem, which provides a link between Dixmier traces and pseudo-differential operators. In particular using the Wodzicki residue we will be able to easily prove this claim.

We will also introduce the zeta function of an operator and mention how it relates to the Wodzicki residue and the Dixmier traces. In fact the residue of the zeta function provides an alternative way of defining an abstract notion of integration. It is actually defined in greater generality than the Dixmier traces, as we will briefly mention. Moreover this notion can be easily adapted to formulate a notion of integral which is not tracial. For these reasons we will make extensive use of the zeta function approach to integration in the rest of this thesis.

2.1 Pseudo-differential operators

Pseudo-differential operators are a generalization of differential operators born out of the construction of approximate inverses for elliptic differential operators. Here we will consider only the simplest case, that of the so called *classical* pseudo-differential operators.

Let us fix first some notation. We denote by U an open subset of $\mathbb{R}^n$ and by $C_c^\infty(U)$ the smooth (complex-valued) functions with compact support in U. We employ a multi-index notation and for any multi-index $\alpha \in \mathbb{N}^n$ we define the operators

$$D^\alpha = (-i\partial_1)^{\alpha_1} \cdots (-i\partial_n)^{\alpha_n} \ .$$

With these notations we define a differential operator of order d as

$$P = \sum_{|\alpha| \leq d} a_\alpha(x) D^\alpha \ ,$$

where the functions in the expansion are $a_\alpha(x) \in C^\infty(U)$. When acting with P on a function $\psi \in C_c^\infty(U)$ we have, using the Fourier transform, that

$$\begin{aligned} P\psi(x) &= \sum_{|\alpha|\le d} a_\alpha(x) D^\alpha \psi(x) \\ &= \sum_{|\alpha|\le d} a_\alpha(x) D^\alpha \int e^{i\xi x} \mathcal{F}(\psi)(\xi) d^n\xi \\ &= \int e^{i\xi x} \left(\sum_{|\alpha|\le d} a_\alpha(x) \xi^\alpha \right) \mathcal{F}(\psi)(\xi) d^n\xi \ . \end{aligned}$$

Finally using the inverse Fourier transform we can write this as

$$P\psi(x) = \frac{1}{(2\pi)^n} \int\int e^{i\xi(x-y)} p(x,\xi)\psi(y) d^n\xi d^n y \ , \tag{2.1}$$

where we have defined the *symbol* $p(x,\xi)$ of the operator P as

$$p(x,\xi) = \sum_{|\alpha|\le d} a_\alpha(x)\xi^\alpha \ .$$

A pseudo-differential operator is an operator of the general form (2.1), but with a symbol $p(x,\xi)$ belonging to a more general class of functions.

Definition 2.1. A function $p(x,\xi)$ is a *symbol of order* d, written $p \in S^d(U)$, if for any compact subset $K \subset U$ and multi-indices α and β there is a constant $C_{K\alpha\beta}$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{K\alpha\beta}(1+|\xi|)^{d-|\alpha|} \ . \tag{2.2}$$

A *pseudo-differential operator* P of order d is an operator of the form (2.1) with symbol $p \in S^d(U)$. We write $P \in \Psi^d(U)$ for such an operator.

We also write $p \in S^{-\infty}(U)$ if $p \in S^d(U)$ for all $d \in \mathbb{R}$ and write $P \in \Psi^{-\infty}(U)$ for the corresponding operator. Such pseudo-differential operators are called *smoothing*.

It is possible to give an asymptotic expansion of symbols which is unique modulo smoothing operators. If $p_j \in S^{d_j}(U)$, where $\{d_j\}$ is a decreasing sequence of real numbers with $d_j \to -\infty$, then one can find $p \in S^{d_0}(U)$, which is unique modulo $S^{-\infty}$, such that

$$p - \sum_{j=0}^{k-1} p_{d_j} \in S^{d_k}(U)$$

for all $k \in \mathbb{N}$. In this case we write

$$p \sim \sum_{j=0}^{\infty} p_{d_j} \ .$$

At this level of generality it is not always possible to compose two pseudo-differential operators. Here we restrict to the simple class of *classical* pseudo-differential operators, for which composition is always possible. Let us first take another look at ordinary differential operators. If P is a differential operator of order d then the symbol $p(x,\xi)$ is a polynomial of order d in the ξ variable, so that we can isolate its homogeneous parts as follows

$$p(x,\xi) = \sum_{j=0}^{d} p_{d-j}(x,\xi) \ ,$$

where each function $p_{d-j}(x,\xi)$ is homogeneous of degree $d-j$ in ξ, that is $p_{d-j}(x,\lambda\xi) = \lambda^{d-j} p_{d-j}(x,\xi)$. The leading term is called the *principal symbol* and we denote it by $\sigma(P)(x,\xi)$. For a differential operator of order d it is clearly given by

$$\sigma(P)(x,\xi) = p_d(x,\xi) = \sum_{|\alpha|=d} a_\alpha(x)\xi^\alpha \ .$$

After this observation we give the following definition.

Definition 2.2. A symbol p is called a *classical symbol* if the terms p_{d_j} in its asymptotic expansion are homogeneous in ξ of degree d_j and their degrees differ by integers. We call the corresponding operators *classical pseudo-differential operators.*

In this case we can set $d_j = d - j$ and rewrite the expansion as

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_{d-j}(x,\xi) \ .$$

We have $p_{d-j}(x,\lambda\xi) = \lambda^{d-j} p_{d-j}(x,\xi)$ and we call $p_d(x,\xi)$ the *principal symbol* of p.

One can compose operators, therefore this operation induces a composition of the symbols.

Proposition 2.1. *Let P, Q be two classical pseudo-differential operators with symbols $p \in S^{d_1}(U)$ and $q \in S^{d_2}(U)$. Then $P \circ Q$ is again a classical pseudo-differential operator whose symbol $p \circ q$ belongs to $S^{d_1+d_2}(U)$. Moreover we have*

$$(p \circ q)(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} D_x^\alpha p(x,\xi) D_\xi^\alpha q(x,\xi) \ .$$

In particular for the principal symbol we have

$$(p \circ q)_{d_1+d_2}(x,\xi) = p_{d_1}(x,\xi) q_{d_2}(x,\xi) \ .$$

We give one more definition of general nature regarding pseudo-differential operators.

Definition 2.3. A pseudo-differential operator P is called *elliptic* if the principal symbol $\sigma(P)(x,\xi)$ is invertible when $\xi \neq 0$.

Elliptic pseudo-differential operators are characterized by possessing a *parametrix*, that is an inverse modulo smoothing operators, which is a very useful property in applications.

We now briefly discuss the definition of pseudo-differential operators on manifolds. To do this it is important to study the behaviour of $P \in \psi^d(U)$ under the action of a diffeomorphism. Let $\phi : U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Then it can be shown that $\phi_* P(f) = P(\phi^* f) \circ \phi^{-1}$, where $\phi^* f$ is the pullback of the function f, defines an operator $\phi_* P : C_c^\infty(V) \to C^\infty(V)$ that is actually a pseudo-differential operator. In particular $\phi_* P \in \Psi^d(V)$ and if P is classical then $\phi_* P$ is also classical. Therefore, for a compact manifold M, we require that for each coordinate chart (U, ϕ) we have $\phi_* P \in \Psi^d(\phi(U))$.

The operator can recovered from its components via a partition of unity. The symbol will depend on the chosen charts, but it can be shown that the principal symbol transforms as a function on the cotangent bundle T^*M. In particular an operator is elliptic if it is elliptic in each chart. It is also easy to extend this construction to the case of matrix-valued symbols.

As an example, let us look at the Laplace-Beltrami operator. In local coordinates can be written as

$$\Delta f = -g^{\mu\nu} \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} + \text{lower order terms} .$$

Then we immediately find that the principal symbol is given by

$$\sigma(\Delta)(x, \xi) = g^{\mu\nu} \xi_\mu \xi_\nu = \|\xi\|^2 .$$

It is not difficult to show that for the operator $\Delta^{-n/2}$ the principal symbol is given by

$$\sigma(\Delta^{-n/2})(x, \xi) = \|\xi\|^{-n} .$$

These expressions show that they are elliptic pseudo-differential operators.

2.2 Wodzicki residue and examples

An important notion in the theory of pseudo-differential operators is the Wodzicki residue, that we now define. Its relevance to the topic on non-commutative integration will be explained shortly. We use the notation S^*M for the unit "co-sphere", which is defined as $S^*M = \{(x, \xi) \in T^*M : \|\xi\| = 1\}$.

Definition 2.4. Let M be an n-dimensional compact Riemannian manifold. Let P be a pseudo-differential operator of order $-n$, acting on sections of a complex vector bundle $E \to M$. Then we define the *Wodzicki residue* of P as

$$\mathrm{Res}_W(P) = \int_{S^*M} \mathrm{Tr}_E \sigma(P)(x, \xi) dx d\xi .$$

Here $\sigma(P)(x,\xi) = p_{-n}(x,\xi)$ denotes the principal symbol of the operator P: it is a matrix-valued function on T^*M which is homogeneous of degree $-n$ in the fibre coordinates. By Tr_E denotes the trace over the matrix indices. The measure is defined as follows: by dx we denote the volume form on M and the integration over the ellipsoid $\|\xi\| = 1$ is such that

$$\int_{\|\xi\|=1} d\xi = \Omega_n \ ,$$

where $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$ the area of the unit sphere S^{n-1}. There are numerous other possible conventions, for example one can put explicitely the integration over the sphere $|\xi| = 1$, but then one needs a change of variables to make explicit the volume form of M.

Its importance in the theory of pseudo-differential operators is motivated by the following result, due to Wodzicki.

Theorem 2.2. *The Wodzicki residue is a trace on the algebra of classical pseudo-differential operators. If* $\dim(M) > 1$ *it is the only such trace, up to multiplication by a constant.*

Remark 2.1. The condition $\dim(M) > 1$ is related to the fact that for $\dim(M) = 1$ the cotangent bundle $T^*(M)$ is disconnected, so there are actually two residues, which may be linearly combined.

We now compute the Wodzicki residue in a few simple cases.

Example 2.1. *We consider the Laplacian* Δ *on the n-torus* T^n. *Since* Δ *is a second order differential operator, we have that* $\Delta^{-n/2}$ *is a pseudo-differential operator of order* $-n$. *Moreover its principal symbol is* $\sigma_{-n}(\Delta^{-n/2}) = \|\xi\|^{-n}$, *which is the constant function* 1 *on* S^*T^n. *So*

$$\begin{aligned} \mathrm{Res}_W(\Delta^{-n/2}) &= \int_{S^*T^n} \sigma_{-n}(\Delta^{-n/2}) d\mu = \int_{S^*T^n} dx d\xi \\ &= \Omega_n \int_{T^n} dx = \Omega_n (2\pi)^n \ . \end{aligned}$$

Example 2.2. *We repeat the same computation for the Laplacian* Δ *the n-sphere* S^n. *The arguments given for the n-torus hold also in this case. Therefore the computation is*

$$\begin{aligned} \mathrm{Res}_W(\Delta^{-n/2}) &= \int_{S^*S^n} \sigma_{-n}(\Delta^{-n/2}) d\mu = \int_{S^*S^n} dx d\xi \\ &= \Omega_n \int_{S^n} dx = \Omega_n \Omega_{n+1} = \frac{2(2\pi)^n}{(n-1)!} \ . \end{aligned}$$

We can compare these expressions with those obtained by computing the Dixmier trace of the same operators, and observe that the following relation holds

$$\mathrm{Tr}_\omega(\Delta^{-n/2}) = \frac{1}{n(2\pi)^n} \mathrm{Res}_W(\Delta^{-n/2}).$$

2.3 Connes' trace theorem

In the examples we considered above, the Wodzicki residue turned out to be related to the Dixmier trace of the operator in consideration. The precise statement of this correspondence is the content of *Connes' trace theorem*, which is proven in [Con88].

Theorem 2.3. *Let M be an n-dimensional compact Riemannian manifold. Let P be an elliptic pseudo-differential operator of order $-n$, acting on sections of a complex vector bundle $E \to M$. Then P belongs to $\mathcal{L}^{1+}$, is measurable, and we have the equality*

$$\mathrm{Tr}_\omega(P) = \frac{1}{n(2\pi)^n} \mathrm{Res}_W(P) \ .$$

This result greatly simplifies the computation of Dixmier traces on compact manifolds. In particular we can easily show how to recover the usual integration of functions in this setting. We will consider the case corresponding to a canonical spectral triple, which will appear in the next chapter dedicated to spectral triples.

We consider a compact Riemannian spin manifold M of dimension n. We will be concerned with the integration of functions in $C^\infty(M)$, the algebra of smooth functions. We consider the Dirac operator D associated to a certain metric. In this case we need to take in consideration the bundle E appearing in the theorem above, which will be the spinor bundle of dimension $2^{[n/2]}$. The Dirac operator D can be written locally as

$$D = -i\gamma^\mu \partial_\mu + \text{lower order terms} \ .$$

Here $\gamma^\mu = \gamma(dx^\mu)$ and γ is the Clifford multiplication. It is clearly a first order operator whose principal symbol is $\gamma^\mu \xi_\mu$. Using the relations of the Clifford algebra we obtain that the principal symbol of D^2 is given by

$$\sigma(D^2) = \gamma^\mu \gamma^\nu \xi_\mu \xi_\nu = g^{\mu\nu} \xi_\mu \xi_\nu 1_m = \|\xi\|^2 1_m \ .$$

Therefore the principal symbol of $|D|^{-n}$ is given by

$$\sigma_P(|D|^{-n}) = \|\xi\|^{-n} 1_m \ .$$

Proposition 2.4. *For any $f \in C^\infty(M)$ we have*

$$\mathrm{Tr}_\omega(f|D|^{-n}) = c_n \int f(x) dx \ ,$$

where $c_n = 2^{[n/2]} \Omega_n / (n(2\pi)^n)$.

Proof. The Dirac operator D is a first order elliptic operator. Since f is a bounded multiplication operator we have that $f|D|^{-n}$ is a pseudo-differential operator of order $-n$. Its principal symbol is $\sigma(f|D|^{-n}) = f(x)\|\xi\|^{-n}$. Therefore we can apply Connes' trace theorem. Computing the Wodzicki residue we get

$$\begin{aligned}\mathrm{Tr}_\omega(f|D|^{-n}) &= \frac{1}{n(2\pi)^n}\int_{S^*M}\mathrm{Tr}_E\sigma(f|D|^{-n})d\mu = \frac{2^{[n/2]}}{n(2\pi)^n}\int_{S^*M}\sigma(f|D|^{-n})d\mu \\ &= \frac{2^{[n/2]}}{n(2\pi)^n}\int_{S^*M}f(x)dxd\xi = \frac{2^{[n/2]}\Omega_n}{n(2\pi)^n}\int_M f(x)dx = c_n\int_M f\ .\end{aligned}$$

□

Therefore we can recover the integral of f by computing the Dixmier trace of $f|D|^{-n}$. We remark that the same outcome would be obtained, up to the constant, if we had considered instead $f\Delta^{-n/2}$, with Δ the Laplacian. Here we used the Dirac operator since it is the situation one commonly encounters when dealing with spectral triples.

2.4 Zeta functions and integration

We now introduce the notion of zeta function associated to an operator and discuss its role in the theory of non-commutative integration. In particular we consider the case of pseudo-differential operators on compact manifolds, which provides a connection with the Wodzicki residue and justifies the name "residue".

Let us recall that the Riemann zeta function is defined by the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z},$$

which converges for $\mathrm{Re}(z) > 1$. It can be continued analytically to a function which is holomorphic everywhere except at $z = 1$, where it has a simple pole with residue equal to 1.

We now want to define an analogue of the zeta function for certain operators, essentially by replacing the terms n^{-1} in the series with the eigenvalues λ_n of the operator in consideration. In general complex powers of an operator can be defined using the holomorphic functional calculus, but here we can proceed with a more hands-on approach.

Let M be compact manifold of dimension n. We consider a positive elliptic pseudo-differential operator P of order $-n$. Such an operator is compact, see for example [Shu01]. We can assume that $\lambda = 0$ is not an eigenvalue of P, removing its kernel if necessary. The important example to keep in mind is given by $\Delta^{-n/2}$, where Δ is the Laplacian for a fixed Riemannian metric on M. For such an operator P, we define its zeta function as

$$\zeta_P(z) = \mathrm{Tr}(P^z) = \sum_{n=1}^{\infty} \lambda_n^z,$$

where λ_n are the eigenvalues of P. An important result is the following.

Theorem 2.5. *With P as above, the zeta function $\zeta_P(z)$ is holomorphic for* $\mathrm{Re}(z) > 1$*. It has a simple pole at $z = 1$ and moreover*

$$\operatorname*{Res}_{z=1} \zeta_P(z) = \frac{1}{n(2\pi)^n}\mathrm{Res}_W(P).$$

The zeta function $\zeta_P(z)$ can be extended to a meromorphic function, but in general will have additional poles with respect to the Riemann zeta function. These additional poles provide much geometric information about M, and can be also connected with the heat kernel expansion. As a historical remark, this kind of zeta function, for the case of the Laplacian, was investigated for the first by Minakshisundaram and Pleijel in [MiPl49].

Using this theorem we can give another formulation of integration in terms of the zeta function. Indeed it is not difficult to prove that, for $f \in C^\infty(M)$, we have

$$\operatorname*{Res}_{z=n} \mathrm{Tr}(f\Delta^{-z/2}) = c_n \int f(x)dx,$$

where c_n is a constant depending only on the dimension n of M. A similar result can be obtained using $|D|^{-z}$ in the case of a spin manifold, where D is the Dirac operator.

The theorem above also provides a connection between the residue of the zeta function associated to certain operators and their Dixmier traces, since as we have seen the Wodzicki residue is related to the Dixmier trace via Connes' trace theorem. This connection continues to hold more generally, even without making reference to the Wodzicki residue. As we have mentioned when discussing Dixmier traces, a positive operator is measurable if and only if the associated sequence has an ordinary limit. The connection with the zeta function comes from the following Tauberian theorem, see [Con].

Proposition 2.6. *Let T be a positive operator such that $T \in \mathcal{L}^{1+}$. Then the following two conditions are equivalent:*

1. $(s-1)\zeta(s) \to L$ *as* $s \to 1+$,
2. $\frac{1}{\log N}\sum_{n=0}^{N-1} \mu_n(T) \to L$ *for* $N \to \infty$.

Under these conditions, the value of $\mathrm{Tr}_\omega(T)$ is independent of ω and coincides with the residue of $\zeta(s)$ at $s = 1$.

The zeta function approach to integration has several features which make it very useful in non-commutative geometry. For example, the additional poles can be used to extract geometric information about the space in consideration, as for the scalar curvature in [CoMo11]. In this sense it can be used to extend the Dixmier trace to operators which do not lie in the Dixmier ideal, as in the formulation of the local index formula [CoMo95]. Another feature, as we will see in the following, is that it can be easily modified to account for some modular properties of the non-commutative spaces in consideration.

Finally, it is worth mentioning that the zeta function approach to integration is defined in greater generality than the one in terms of the Dixmier trace. Recall that in the previous sections we have shown that, for functions in $C^\infty(M)$ with M compact, the integral can be computed as the Dixmier trace of a certain operator. However, from the point of view of measure theory, integration should be defined for functions in $L^1(M)$. It is then natural to wonder if the result concerning the Dixmier trace can be extended to this class of functions. It can be shown that the correspondence between the Dixmier trace and the integral continues to hold for $L^2(M)$, but fails for $L^1(M)$. On the other hand, the correspondence holds in general for the residue of the zeta function, see [LPS10] for details.

Chapter 3

Spectral triples

In this section we introduce the notion of spectral triple and discuss some of its properties. Spectral triples provide the building blocks for the description of the geometry of non-commutative spaces. We will motivate their definition by reformulating some aspects of Riemannian manifolds in a operator algebra setting, which lends itself to a natural generalization to the non-commutative case. Then we will see how this definition is connected with index theory, which provides a broader perspective for this notion.

General references for this section are [Con] and [GVF].

3.1 Motivation and definition

We begin by discussing briefly how to reformulate the notion of Riemannian manifold in an operator algebra setting. For simplicity we restrict to the case of compact manifolds. The space of continuous functions $C(M)$ on a compact manifold M is a unital C^*-algebra and, on the other hand, from the Gelfand-Naimark theorem it follows that any commutative and unital C^*-algebra is of the form $C(X)$, where X is a compact Hausdorff space. Therefore commutative C^*-algebras provide an algebraic reformulation of topological spaces.

Any C^* algebra A (and therefore any subalgebra of A) can, after the choice of a state, be represented as bounded operators on some Hilbert space H. Therefore A and H provide two natural ingredients for the description of topological spaces.

To go beyond that, and introduce a metric aspect to this reformulation, we need some new element. Since much information about Riemannian manifolds can be retrieved from differential operators naturally defined on them, it is a good idea to include such an operator in this reformulation. This operator, which we denote by D, must have some compatibility with the algebra A, as will emerge from a more detailed analysis. We will consider first order differential operators, since they are easier to handle. That said, the choice of such an operator really depends on the manifold under consideration.

For simplicity, and in view of the many results available in this case, we will consider the case of spin manifolds, where one natural operator that fits these requests is the Dirac operator (this motivates the choice of the symbol D). We now mention a result that allows the reconstruction of the metric from the Dirac operator, see [Con]. Let us recall that in differential geometry the distance between two points $p, q \in M$ is given by

$$d(p,q) = \inf\{\text{length}(\gamma) : \gamma \text{ is a path from } p \text{ to } q\}.$$

It is possible to obtain another expression for the distance, in a certain sense dual to this one, by considering functions on M whose commutator with D satisfies a certain bound.

Proposition 3.1. *The distance between two points $p, q \in M$ can be obtained as*

$$d(p,q) = \sup\{|f(p) - f(q)| : f \in C^\infty(M), \|[D,f]\| \leq 1\}.$$

Since the distance function on a manifold determines its metric (Myers-Steenrood theorem) then this shows that we can recover the metric from a spectral triple. A similar result can be obtained also using the Laplacian, but the formula becomes more complicated.

The definition of spectral triple follows by abstracting some of the essential properties that emerged in this discussion.

Definition 3.1. Let $\mathcal{A}$ be a unital $*$-subalgebra of $B(H)$, where H is a Hilbert space. We call the triple $(\mathcal{A}, H, D)$ a *spectral triple* if

1. D is a self-adjoint operator,
2. $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$,
3. $(D^2 + 1)^{-1/2}$ is compact.

Remark 3.1. By $[D, a]$ extends to a bounded operator we mean the following: for every $a \in \mathcal{A}$ we have $a\,\mathrm{dom}(D) \subset \mathrm{dom}(D)$, so that $[D, a]$ is densely defined. Since $[D, a]$ is bounded on $\mathrm{dom}(D)$ it extends to a bounded operator in $B(H)$.

The definition of spectral triple can be refined in many ways to capture more information about the space under consideration. An example of this is given by the notion of parity.

Definition 3.2. We say that $(\mathcal{A}, H, D)$ is even if there exists a $\mathbb{Z}_2$ grading such that $\mathcal{A}$ is even and D is odd. By this we mean that there exists an operator $\gamma \in B(H)$, with $\gamma^* = \gamma$ and $\gamma^2 = 1$, such that $\gamma a = a\gamma$ for all $a \in \mathcal{A}$ and $D\gamma + \gamma D = 0$. Otherwise $(\mathcal{A}, H, D)$ is odd.

Spin manifolds, which we focused on in this presentation, clearly provide examples of commutative spectral triples. We now give some additional details of this construction.

Example 3.1. *Let M be a compact spin manifold with metric g. We can associate to it the so-called* canonical spectral triple*, which is given by $(C^\infty(M), L^2(M,S), D)$. Here the $*$-algebra of smooth functions $C^\infty(M)$ acts by multiplication on the Hilbert space $L^2(M,S)$*

of square-integrable sections of the spinor bundle. The operator D is the Dirac operator associated to (M, g), which is a self-adjoint operator on this Hilbert space.

It is not difficult to show that all the properties defining a spectral triple are satisfied in this case. Notice that, in order to satisfy the condition of boundedness of $[D, a]$, we have to restrict to the subalgebra $C^\infty(M)$ of the C^*-algebra $C(M)$. Indeed this tension between the continuous and smooth setting is a recurring theme with spectral triples.

Let us note that such a spectral triple is even or odd depending on its dimension n, where the grading is provided by the Clifford algebra grading.

As we briefly discussed, the choice of focusing on spin manifolds at this stage is arbitrary. Indeed it is not difficult to construct spectral triples for more general compact oriented Riemannian manifolds, as in the next example.

Example 3.2. *(Hodge-de Rham) Let M be a compact oriented Riemannian manifold with metric g. Let $L^2(\Lambda^* M, g)$ be the Hilbert space completion of the exterior bundle $\Lambda^* T^*_{\mathbb{C}} M$ with respect to the inner product*

$$(\omega, \rho) = \int_M \omega \wedge *\bar{\rho},$$

where $*$ is the Hodge dual. Let d be the exterior derivative and d^* the adjoint with respect to this inner product. Set $D = d + d^*$. Then $(C^\infty(M), L^2(\Lambda^* M, g), D)$ is a spectral triple.

Thus far we have seen how to construct spectral triples from Riemannian manifolds. This raises the natural question: given a spectral triple $(\mathcal{A}, H, D)$ with $\mathcal{A}$ commutative, can we associate a Riemannian manifold to it? It turns out that it is not possible at this level of generality, so that additional conditions are needed to characterize Riemannian manifolds among commutative spectral triples. What these conditions are really depends on which class of Riemannian manifolds we want to recover. In this sense the cases of spin and spin^c manifolds are the best understood ones.

The most general result in this sense is obtained in [Con13], which is based on previous results contained in [Con96] and [ReVa06]. Given a spectral triple $(\mathcal{A}, H, D)$ with $\mathcal{A}$ commutative, and satisfying some additional conditions, there exists a compact oriented smooth manifold M such that $\mathcal{A}$ is the algebra $C^\infty(M)$ of smooth functions on M. Conversely any such manifold appears in this spectral manner. Furthermore, by refining these conditions to capture the spin^c case, it is also possible to recover the metric from the operator D.

3.2 Summability

Let $(\mathcal{A}, H, D)$ be a spectral triple, with the algebra $\mathcal{A}$ not necessarily commutative. It is useful to have additional analytical control over D, since a priori we only know that it has compact resolvent. For example, when D is the Dirac operator on a spin manifold, we have that $(D^2 + 1)^{-s/2}$ is trace-class for $s > n$, where n is the dimension of the manifold. This easily follows from previous results on pseudo-differential operators and the associated zeta functions. Therefore from the spectrum of D we can extract the dimension of the manifold, which motivates the following general definition.

Definition 3.3. A spectral triple $(\mathcal{A}, H, D)$ is called finitely summable if there exists some $s_0 > 0$ such that $\mathrm{Tr}((D^2+1)^{-s_0/2}) < \infty$. In this case, we define the *spectral dimension* as

$$p = \inf\{s > 0 : \mathrm{Tr}((D^2+1)^{-s/2}) < \infty\}.$$

We have used $(D^2+1)^{-1/2}$ in place of $|D|^{-1}$ since we do not assume invertibility of D. Equivalently we can consider $(D^2+\mu^2)^{-1/2}$ for any non-zero $\mu \in \mathbb{R}$, with the spectral dimension being independent of this choice.

We can also give a notion of summability in terms of the Dixmier ideal. Recall that in a previous chapter the Dixmier ideal was introduced as

$$\mathcal{L}^{1+} = \left\{ T \in K(H) : \sup_{N>1} \frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T) < \infty \right\}.$$

Equivalently, it is the space of all $T \in K(H)$ such that $\sigma_N(T) = O(\log N)$, where σ_N denotes the sum over the first N singular values. Similarly, for $p > 1$ the ideals $\mathcal{L}^{p+}$ can be introduced by real interpolation theory. Concretely, they are the spaces

$$\mathcal{L}^{p+} = \{T \in K(H) : \sigma_N(T) = O(N^{1-1/p})\}.$$

Definition 3.4. A spectral triple $(\mathcal{A}, H, D)$ is called p-summable if $(D^2+1)^{-p/2} \in \mathcal{L}^1(H)$. It is called p^+-summable if $(D^2+1)^{-p/2} \in \mathcal{L}^{1+}(H)$.

Remark 3.2. A variant of the second definition, found for example in [GVF], is given by requiring that $(D^2+1)^{-1/2} \in \mathcal{L}^{p+}(H)$. It is true that if $T \in \mathcal{L}^{p+}$ then $T^p \in \mathcal{L}^{1+}$, while the converse is false, therefore these two definitions are *not equivalent*. It follows, according to our definition, that if a spectral triple is p^+-summable then the spectral dimension is p.

It is always possible to associate, to the unbounded operator D, the bounded operator $F = D(D^2+1)^{-1/2}$, which can be called the *bounded transform* of D. In the case when D is invertible we can consider instead $F = D|D|^{-1}$, which is called the *phase* of D, in clear analogy with the polar decomposition for complex numbers. In both cases the operator F is a Fredholm operator, whose definition we will review in a moment.

We now show that the commutator of F with an element of the algebra A turns out to be compact and, moreover, inherits a certain summability from D. This is crucial for the definition of the Chern character, as we will see. We also take this opportunity to introduce a property of regularity for D, which is useful in many cases.

Definition 3.5. A spectral triple $(\mathcal{A}, H, D)$ is called *Lipschitz regular* if $[|D|, a]$ is bounded for any $a \in \mathcal{A}$.

Proposition 3.2. *Let $(\mathcal{A}, H, D)$ be a spectral triple. Then we have that $[F, a]$ is compact for all $a \in \mathcal{A}$. Moreover if the triple is n^+-summable then $[F, a] \in \mathcal{L}^{n+1}(H)$.*

Proof. We use the extra assumption of Lipschitz regularity to make the proof elementary. This assumption can be removed as in [CP98]. For the same reason we consider the case in which D is invertible. We rewrite the commutator $[F, a]$ as follows

$$\begin{aligned}[F, a] &= D[|D|^{-1}, a] + [D, a]|D|^{-1} \\ &= -F[|D|, a]|D|^{-1} + [D, a]|D|^{-1}.\end{aligned}$$

The compactness immediately follows from the fact that $|D|^{-1}$ is compact. Similarly from the fact that the triple is n^+-summable we get that $[F, a] \in \mathcal{L}^{n+1}(H)$. □

3.3 Spectral triples and index theory

Spectral triples are intimately related to index theory. Even though the latter does not play an important role in this thesis, a brief outline of this connection sets the notion of spectral triple into a broader setting. The material of this section is mainly taken from [CPR].

3.3.1 Fredholm operators

Let us start by giving the definition of Fredholm operators.

Definition 3.6. Let H_1 and H_2 be Hilbert spaces and $F : H_1 \to H_2$ a bounded linear operator. We call F Fredholm if:

1. $\mathrm{ran}(F)$ is closed in H_2,
2. $\ker(F)$ is finite dimensional,
3. $\mathrm{coker}(F) = H_2/\mathrm{ran}(F)$ is finite dimensional.

If F is Fredholm we define its index as

$$\mathrm{Index}(F) = \dim\ker(F) - \dim\mathrm{coker}(F).$$

Example 3.3. *A very simple (and non-trivial) Fredholm operator is given by the shift operator $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$. It is defined by*

$$S\sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^{\infty} a_i e_{i+1}.$$

The range of S *is of codimension* 1*, so it is closed. Moreover the kernel consists only of* 0*, so that* $\mathrm{Index}(S) = \dim\ker(S) - \dim\mathrm{coker}(S) = -1$.

Example 3.4. *If* $F : H \to H$ *is a self-adjoint Fredholm operator then* $\mathrm{Index}(F) = 0$. *This is because in general we have* $\mathrm{coker}(F) = \ker(F^*)$.

There is a useful characterization of Fredholm operators, due to Atkinson.

Proposition 3.3. *Let* $F : H_1 \to H_2$ *and* $S : H_2 \to H_1$ *be bounded linear operators such that* $FS - 1_{H_2}$ *and* $SF - 1_{H_1}$ *are compact operators (on their respective Hilbert spaces). Then* F *and* S *are Fredholm operators. The converse is also true.*

Given F and S as above, we say that S is a *parametrix* or approximate inverse for F, and vice versa. Thus the Fredholm operators are precisely those which are invertible modulo compact operators. This characterization shows that the operators $F = D(D^2+1)^{-1/2}$ or $F = D|D|^{-1}$ for D invertible are Fredholm.

We now summarize the important properties of the index of a Fredholm operator.

Theorem 3.4. *Let* $\mathcal{F}$ *denote the set of Fredholm operators on a Hilbert space* H. *Let* $\pi_0(\mathcal{F})$ *denote the norm connected components of* $\mathcal{F}$. *Then the index is locally constant and induces a bijection* $\mathrm{Index} : \pi_0(\mathcal{F}) \to \mathbb{Z}$. *Moreover the index satisfies*

$$\mathrm{Index}(F^*) = -\mathrm{Index}(F), \qquad \mathrm{Index}(FS) = \mathrm{Index}(F) + \mathrm{Index}(S).$$

Therefore the map Index *gives a group isomorphism.*

It is worth noting that if F is Fredholm and T is compact then $F + T$ is Fredholm and

$$\mathrm{Index}(F + T) = \mathrm{Index}(F).$$

Therefore the index is constant under compact perturbation and sufficiently small norm perturbations. This gives us strong invariance properties for the index. In particular, by considering operators on manifolds which give rise to Fredholm operators, we can construct invariants of the underlying manifolds.

3.3.2 K-homology and K-theory

One of the main techniques employed by Atiyah and Singer to compute the index of elliptic differential operators on manifolds is K-theory. This cohomology theory makes sense also in the non-commutative case, and so does its dual theory called K-homology. The latter is strictly related to spectral triples, as we will see in a moment.

Definition 3.7. Let A be a $*$-algebra. A *Fredholm module* over A is given by a Hilbert space H with $A \subset B(H)$ and an operator $F : H \to H$ such that $(F^2 - 1)a$, $(F - F^*)a$ and $[F, a]$

are compact operators. We say that (H, F) is even if there is an operator $\gamma : H \to H$ such that $\gamma^2 = 1$, $\gamma^* = \gamma$, $\gamma F + F\gamma = 0$ and $\gamma a = a\gamma$. Otherwise we say that (H, F) is odd.

Notice that, if A is unital, then F is Fredholm by Atkinson's theorem. In the even case we can use the grading γ to decompose the Hilbert space as $H = H_+ \oplus H_-$. Similarly, using the fact that F anticommutes with γ, we obtain the operators $F_\pm : H_\pm \to H_\mp$.

By defining an equivalence relation for Fredholm modules it is possible to introduce an abelian group structure, which is called *K-homology* and is denoted by $K^p(A)$, with $p = 0$ in the even case and $p = 1$ in the odd case. It follows from Proposition 3.2 that one can assign a Fredholm module to a spectral triple, and therefore a class in K-homology.

For simplicity in the following we will deal only with normalized Fredholm modules. We say that a Fredholm module (H, F) is *normalized* if $F^2 = 1$ and $F = F^*$. It is not difficult to show that from every Fredholm module one can obtain a normalized Fredholm module belonging to the same class in K-homology.

We also briefly mention the K-theory groups $K_0(A)$ and $K_1(A)$, which are respectively made of projections and unitaries in $M_k(A)$, that is matrix algebras with entries in A, for each k. Also in this case there are appropriate equivalence relations to be required.

3.3.3 Index pairing and Chern character

A pairing between K-theory and K-homology can be defined. To define it we need to handle matrix algebras over A. It is easy to show that if (H, F) is a Fredholm module over the algebra A then $(H^k, F \otimes 1_k)$ is a Fredholm module over the algebra $M_k(A)$.

Let (H, F, γ) be an even Fredholm module and $p \in M_k(A)$ a projection. Then the pairing between $[p] \in K_0(A)$ and $[(H, F, \gamma)] \in K^0(A)$ is given by

$$\langle [p], [(H, F, \gamma)] \rangle = \mathrm{Index}(p(F^+ \otimes 1_k)p).$$

Similarly, when (H, F) is an odd normalized Fredholm module and $u \in M_k(A)$ is a unitary, the pairing between $[u] \in K_1(A)$ and $[(H, F)] \in K^1(A)$ is given by

$$\langle [u], [(H, F)] \rangle = \mathrm{Index}(P_k u P_k).$$

where $P_k = \frac{1}{2}(1 + F) \otimes 1_k$. Notice that P_k is a projection.

It is possible to obtain a simpler formula for the pairing under some assumptions on the Fredholm module. This requires a notion of summability for Fredholm modules.

Definition 3.8. A Fredholm module (H, F) for A is $(p+1)$-summable, with $p \in \mathbb{N}$, if for all $a \in A$ we have $[F, a] \in \mathcal{L}^{p+1}(H)$.

It follows from Proposition 3.2 that, for a spectral triple (A, H, D) with spectral dimension n, the operator F defined in terms of D is $(n+1)$-summable. Therefore finitely summable spectral triples correspond to finitely summable Fredholm modules.

For finitely summable normalized Fredholm modules we can define cyclic cocycles, whose class in periodic cyclic cohomology is called the Chern character.

Definition 3.9. Let (H, F) be a $(p+1)$-summable normalized Fredholm module for A. For any $n \geq p$ with the same parity of the Fredholm module we define

$$\mathrm{Ch}_n(a_0, \cdots, a_n) = \frac{\lambda_n}{2} \mathrm{Tr}(\gamma[F, a_0] \cdots [F, a_n]),$$

for some normalization constants λ_n. We call the class of Ch_* in periodic cyclic cohomology the *Chern character.*

Finally we can compute the pairing between K-theory and K-homology using a representative of the Chern character, as shown in the following theorem of Connes [Con].

Theorem 3.5. *Let (H, F) be a finitely summable normalized Fredholm module over A. Then for any $[e] \in K_0(A)$ we have*

$$\langle [e], [(H, F, \gamma)] \rangle = \frac{1}{(n/2)!} \mathrm{Ch}_n(e, \cdots, e),$$

for n large enough. Similarly, for any $[u] \in K_1(A)$ we have

$$\langle [u], [(H, F)] \rangle = -\frac{1}{\sqrt{2i} 2^n \Gamma(n/2+1)} \mathrm{Ch}_n(e, \cdots, e).$$

It is possible to go further and obtain a more computable form of the pairing. This consists in producing a *local index formula*, which is the analogue of the Atiyah-Singer index theorem. We do not go into this important topic, but just mention that the added flexibility of having an unbounded operator D is crucial to obtain such a formula.

Chapter 4

Generalizations of spectral triples

After having defined and discussed the notion of spectral triple in the previous chapter, we now turn to some generalizations. After all, this notion was obtained by considering the case of manifolds, but we expect that the non-commutative world will produce new phenomena that we have to take into account. We will discuss the framework of twisted spectral triples [CoMo08] and that of modular spectral triples [CPR10, CRT09, CNNR11], which in turn is a generalization of semifinite spectral triples [BeFa06].

4.1 Twisted spectral triples

Let us begin with a motivating example. Let M be a compact spin manifold, with Riemannian metric g, and consider the associated canonical spectral triple $(C^\infty(M), L^2(M, S), D)$. Consider now a self-adjoint element $h \in C^\infty(M)$ and define the rescaled metric $g' = e^{-4h} g$, which is conformally equivalent to g. We can construct again a canonical spectral triple out of this metric and it is not difficult to see, after properly identifying the new Hilbert space, that the new Dirac operator is related to the old one by $D' = e^h D e^h$.

We can try to repeat this construction for the case of a general spectral triple $(\mathcal{A}, H, D)$, with $\mathcal{A}$ not necessarily commutative. We consider a self-adjoint element $h = h^* \in \mathcal{A}$ and set $D' = e^h D e^h$. We now wonder if the "perturbed" triple, where we replace D with D', is again a spectral triple. It is easy to see that D' is still self-adjoint and has compact resolvent, but on the other hand $[D', a]$ is not bounded unless h is in the center of $\mathcal{A}$.

Therefore this procedure does not give rise to a spectral triple in general. On the other hand we now show that we can obtain a boundedness condition "with a twist".

Lemma 4.1. *Let $\sigma(a) = e^{2h} a e^{-2h}$. Then $D'a - \sigma(a) D'$ is bounded for any $a \in \mathcal{A}$.*

Proof. This follows immediately from the following computation

$$\begin{aligned}[D', a]_\sigma &= e^h D e^h a - e^{2h} a e^{-2h} e^h D e^h \\ &= e^h (D e^h a e^{-h} - e^h a e^{-h} D) e^h \\ &= e^h [D, e^h a e^{-h}] e^h.\end{aligned} \tag{4.1}$$

Indeed e^h is a bounded operator and the fact that $[D, e^h a e^{-h}]$ is bounded follows from the fact that $(\mathcal{A}, H, D)$ is a spectral triple. □

Motivated by this example we give the following definition [CoMo08].

Definition 4.1. Let $\mathcal{A}$ be a unital $*$-subalgebra of $B(H)$, where H is a Hilbert space. Let σ be a automorphism of $\mathcal{A}$. We call the triple $(\mathcal{A}, H, D)$ a *twisted spectral triple* if

1. D is a self-adjoint operator,
2. $[D, a]_\sigma = Da - \sigma(a)D$ extends to a bounded operator for all $a \in \mathcal{A}$,
3. $(D^2 + 1)^{-1/2}$ is a compact operator.

It is clear that when σ is equal to the identity we obtain the usual definition of spectral triple. The notion of grading can be given as in the case of usual spectral triples. The property of regularity is defined as follows.

Definition 4.2. A twisted spectral triple is called *Lipschitz regular* if moreover $|D|a - \sigma(a)|D|$ is bounded for any $a \in \mathcal{A}$.

The presence of the twist has interesting consequences, for example for the notion of integration defined in terms of the operator D.

Proposition 4.2. *Let $(\mathcal{A}, H, D)$ be a twisted spectral triple, where we denote the twist by σ. Suppose that $D^{-1} \in \mathcal{L}^{n+}$. Then we have the following properties:*

- *the linear functional $\varphi(a) = \mathrm{Tr}_\omega(aD^{-n})$ is a σ^n-trace on $\mathcal{A}$, that is*

$$\varphi(ab) = \varphi(\sigma^n(b)a),$$

- *more generally for any bounded operator $T \in B(H)$ we have*

$$\mathrm{Tr}_\omega(TaD^{-n}) = \mathrm{Tr}_\omega(\sigma^n(a)TD^{-n}),$$

- *if we have Lipschitz regularity then the same is true when D^{-n} is replaced by $|D|^{-n}$.*

Proof. For simplicity we show it for the case $n = 1$, the general case can be similarly proven as in [CoMo08]. We only need to notice that

$$aD^{-1} - D^{-1}\sigma(a) = D^{-1}(Da - \sigma(a)D)D^{-1}.$$

Since the term in parentheses is bounded, it is easy to see that $aD^{-1} - D^{-1}\sigma(a)$ is of trace-class, therefore it vanishes when we take its Dixmier trace.

The other two claims are proven similarly. $\square$

Therefore we see that the presence of the twist σ makes the integral non-tracial. This hints at the fact that such a notion might prove useful in the case of compact quantum groups, where one has to deal with non-tracial states. In practice these results hold independently of the choice of the generalized limit ω, similarly to the untwisted case.

Finally we mention that a twisted spectral triple still defines a class in K-homology, at least if we have Lipschitz regularity. We consider again the case of invertible D.

Proposition 4.3. *Let $(\mathcal{A}, H, D)$ be a twisted spectral triple which is Lipschitz regular. Then (H, F) with $F = D|D|^{-1}$ is a Fredholm module over $\mathcal{A}$. Moreover if $(\mathcal{A}, H, D)$ is finitely summable then so is (H, F).*

Proof. To show it we rewrite the twisted commutator as

$$\begin{aligned}
[D, a]_\sigma = Da - \sigma(a)D &= Da - |D|aF + |D|aF - \sigma(a)D \\
&= |D|(Fa - aF) + (|D|a - \sigma(a)|D|)F \\
&= |D|[F, a] + [|D|, a]_\sigma F.
\end{aligned} \tag{4.2}$$

Therefore we have

$$[F, a] = |D|^{-1}[D, a]_\sigma - |D|^{-1}[|D|, a]_\sigma F.$$

The compactness and summability properties follow at once from those of $|D|^{-1}$. $\square$

4.2 Semifinite spectral triples

The notion of semifinite spectral triple was introduced in [BeFa06], motivated by the study of foliations and by the L^2-index theorem of Atiyah. The idea is to generalize the setting of spectral triples to semifinite von Neumann algebras.

To see what this generalization should entail, let us consider the condition of compactness of the resolvent of the operator D, which requires $(D^2 + 1)^{-1/2}$ to belong to the ideal of compact operators in $B(H)$. From the point of view of von Neumann algebras, the space $B(H)$ is a type I factor. Then we can consider replacing $B(H)$ by any semifinite von Neumann algebra, that is possibly a type II factor. Accordingly, the notion of compact operator should be formulated with respect to this semifinite von Neumann algebra.

Before getting into that, we need to recall some notions relative to traces on von Neumann algebras. A trace τ is called *normal* if, for every bounded increasing net of positive elements $x_\lambda \to x$, we have $\tau(x_\lambda) \to \tau(x)$. It is called *semifinite* if, for all positive $a \in N$, we have that $\tau(a)$ is the supremum of $\tau(b)$ over all $b \leq a$ such that $\tau(b) < \infty$. Semifinite von Neumann algebras can be characterized as those admitting a normal semifinite faithful trace.

In the following we will consider a semifinite von Neumann algebra N, considered as a subspace of $B(H)$, and fix a normal semifinite trace τ on this algebra. We say that an operator is τ-*compact* if it is in the norm closure of the ideal generated by the projections $p \in N$ such that $\tau(p) < \infty$. Finally a closed and densely defined operator is *affiliated* with N if it commutes with every unitary operator in the commutant of N.

Armed with these notions, we can now define semifinite spectral triples.

Definition 4.3. Let $\mathcal{A}$ be a unital $*$-subalgebra of N, where N is a semifinite von Neumann algebra acting on a Hilbert space H. Fix a normal semifinite faithful trace τ on N. We call the triple $(\mathcal{A}, H, D)$ a *semifinite spectral triple* if

1. D is a self-adjoint operator affiliated with N,

2. $[D, a]$ extends to a bounded operator in N for all $a \in \mathcal{A}$,

3. $(D^2 + 1)^{-1/2}$ is compact with respect to the trace τ.

Notice that for $N = B(H)$ this definition reduces to that of spectral triples.

Example 4.1. *One of the motivating example in [BeFa06] is that of measured foliations, that is an application of spectral triples to differential geometry. Non-commutative examples coming from graph algebras are discussed in [PaRe06].*

To define the analogue of the Dixmier ideal $\mathcal{L}^{1+}$ we need a notion of singular values for τ-compact operators. Such a notion is provided in [FaKo86].

Definition 4.4. For $S \in N$ we define, for each $t > 0$, the t-th generalized singular value as

$$\mu_t(S) = \inf\{\|SE\| : E \text{ is a projection in } N, \tau(1 - E) \leq t\}.$$

This definition is motivated by an analogue characterization of the singular values based on the projections in $B(H)$. Indeed, when $N = B(H)$, we have that $\mathrm{Tr}(1 - E)$ is a natural number, and we recover the usual definition of singular values. On the other hand, in the general semifinite case, we have a different singular value for each $t > 0$. This "continuous geometry" is indeed one of the defining characteristics of type II von Neumann algebras. Note also that, using this notion, we can alternatively define the τ-compact operators as the operators $T \in N$ such that $\lim_{t\to\infty} \mu_t(T) = 0$.

The analogue of the Dixmier ideal is defined as

$$\mathcal{L}^{1+}(N) = \big\{T \in N : \|T\|_{\mathcal{L}^{1+}} = \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds < \infty\big\}.$$

It is possible to define Dixmier traces by taking generalized limits as in the type I case.

We now introduce some spaces, denoted by $\mathcal{Z}_p$ for $p \geq 1$, which are strictly related to the Dixmier ideals [CRSS07]. They are interesting because their definition is based on the zeta function, which can be defined in the generality of the semifinite setting. Given a faithful semifinite normal trace τ, the zeta function of a positive τ-compact operator is defined by $\zeta(z) = \tau(T^z)$ for $z \in \mathbb{C}$, under the assumption that there exists some $s_0 > 0$ for which the trace is finite. In this case it is also finite for all $\mathrm{Re}(s) \geq s_0$. Then we define

$$\mathcal{Z}_1 = \{T \in N : \|T\|_{\mathcal{Z}_1} = \limsup_{s \downarrow 1}(s-1)\tau(|T|^s) < \infty\}.$$

Similarly for $p \geq 1$ the spaces $\mathcal{Z}_p$ are defined as

$$\mathcal{Z}_p = \{T \in N : \|T\|_{\mathcal{Z}_p} = \limsup_{s \downarrow p}((s-p)\tau(|T|^s))^{1/s} < \infty\}.$$

The spaces $\mathcal{Z}_p$ can be obtained from $\mathcal{Z}_1$ via the procedure of *p-convexification*. It can be seen as a generalization of the procedure by which the L^p spaces of classical analysis can be obtained from L^1. Several properties of these spaces are proven in [CRSS07], and we summarize some of them in the following theorem.

Theorem 4.1. *(i) The spaces $\mathcal{Z}_1$ and $\mathcal{L}^{1+}$ coincide.*

(ii) For $p \geq 1$, we have $T \in \mathcal{Z}_p$ if and only if $T^p \in \mathcal{Z}_1$.

(ii) For $p > 1$, the space $\mathcal{Z}_p$ is strictly larger than $\mathcal{L}^{p+}$.

Therefore the spaces $\mathcal{Z}_p$ behave well under taking powers. We remark that, in contrast with the Schatten ideals, for the Dixmier ideals it is not true in general that if $T \in \mathcal{L}^{1+}$ then $T^{1/p} \in \mathcal{L}^{p+}$. It is possible to give explicit formulae which link zeta functions, heat kernel expansion and Dixmier traces, see [CRSS07].

4.3 Modular spectral triples

There are several reasons to go beyond semifinite spectral triples. One obvious reason is that there are algebras that do not admit a non-trivial trace, so that a state (or more generally a weight) is needed to study them. This is the motivation that led to the introduction of the concept of *modular spectral triple* in [CPR10]. Here the motivating example is that of the Cuntz algebra, which does not admit a non-trivial trace but has a canonical KMS-state.

Similarly, while an algebra might admit a non-trivial trace, it might be non-faithful and therefore better analyzed using a faithful state. This is the case for the quantum group $SU_q(2)$, which is studied in [CRT09] in the graph algebra picture. In this case a faithful state is given by the Haar state, which is non-tracial, and again one can build a modular spectral triple for this example, which behaves in different way from the semifinite version.

Before getting into the definition of modular spectral triples, we need to review a few facts about weights and the modular theory of von Neumann algebras. Weights are an unbounded version of positive linear functionals (or states, once normalized). The simplest example is given by the operator trace. In general a weight on a von Neumann algebra N is a map

$\omega : N_+ \to [0,\infty]$ satisfying the linearity conditions $\omega(x+y) = \omega(x) + \omega(y)$ for $x, y \in N_+$ and $\omega(\lambda x) = \lambda\omega(x)$ with $\lambda \geq 0$. Here N_+ denotes the positive elements of N and we use the convention $0(+\infty) = 0$. The set $\mathfrak{n}_\omega = \{x \in N : \omega(x^*x) < \infty\}$ is a left ideal of N.

Given a von Neumann algebra N and a normal semifinite faithful weight ω on N, the modular theory allows to create a one-parameter group of $*$-automorphisms of the algebra N, which we call the *modular automorphism group* associated to ω and denote by σ^ω, which assigns to each $t \in \mathbb{R}$ an automorphism of N which we denote by σ_t^ω. Consider the Hilbert space H_ω obtained via the GNS construction for ω, with π_ω the corresponding representation of N on H_ω. Then the modular automorphism group σ^ω is implemented by a unitary one-parameter group $t \mapsto \Delta_\omega^{it} \in B(H_\omega)$. This means that for each $a \in N$ and for all $t \in \mathbb{R}$ we have $\pi_\omega(\sigma_t^\omega(a)) = \Delta_\omega^{it}\pi_\omega(a)\Delta_\omega^{-it}$. We call Δ_ω the *modular operator* associated to ω.

The modular automorphism group σ^ω has a very important property: it is the unique one-parameter automorphism group that satisfies the KMS condition with respect to the weight ω at inverse temperature $\beta = 1$. The KMS condition is defined as follows.

Definition 4.5. Let N be a von Neumann algebra, ω a normal semifinite faithful weight on N and $t \mapsto \alpha_t$ a one-parameter group of automorphisms of N. Then ω satisfies the *KMS condition* at inverse temperature β with respect to α if the following conditions are satisfied:

1. for every $t \in \mathbb{R}$ we have $\omega \circ \alpha_t = \omega$,

2. for every $x, y \in \mathfrak{n}_\omega \cap \mathfrak{n}_\omega^*$ there exists a bounded continuous function $F_{x,y}$ from the horizontal strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Im} z \leq \beta\}$ to $\mathbb{C}$, which is analytic in the interior of the strip and such that for every $t \in \mathbb{R}$ we have
$$F_{x,y}(t) = \omega(x\alpha_t(y)), \quad F_{x,y}(t+i\beta) = \omega(\alpha_t(y)x).$$

The idea is then to choose a weight on N to ensure compatibility with the algebra in consideration. Before giving the definition of modular spectral triple, we need to recall one more notion relative to weights: we say that a semifinite weight ϕ is *strictly* semifinite it its restriction to the fixed point algebra N^{σ^ϕ} is a semifinite trace.

Definition 4.6. Let $\mathcal{A}$ be a unital $*$-subalgebra of N, where N is a semifinite von Neumann algebra acting on a Hilbert space H. Fix a normal strictly semifinite faithful weight ϕ on N with modular group σ^ϕ. We call the triple $(\mathcal{A}, H, D)$ a *modular spectral triple* if

1. $\mathcal{A}$ is invariant under σ^ϕ and consist of analytic vectors for σ^ϕ,

2. D is a self-adjoint operator affiliated with the fixed point algebra N^{σ^ϕ},

3. $[D, a]$ extends to a bounded operator in N for all $a \in \mathcal{A}$,

4. $(D^2+1)^{-1/2}$ is compact with respect to the trace $\tau = \phi|_{N^{\sigma^\phi}}$.

It is worth noting that if $\mathcal{A}$ is pointwise invariant under the modular group σ^ϕ then we reduce to the semifinite case. This observation makes clear the fact that the fixed point algebra plays an important role in this definition. However, in examples it might well be that no element of $\mathcal{A}$ is invariant under the modular group. We will return to this point later on.

Regarding summability, the notion of spectral dimension can be adapted straightforwardly to this case by replacing the trace with the weight under consideration.

Definition 4.7. A modular spectral triple $(\mathcal{A}, H, D)$ is called finitely summable if there exists some $s > 0$ such that

$$\phi((D^2 + 1)^{-s/2}) < \infty \ .$$

We define the *spectral dimension* as the number

$$p = \inf\{s > 0 : \phi((D^2 + 1)^{-s/2}) < \infty\} \ .$$

Example 4.2. *A class of examples, which use in an essential way the presence of a circle action, are given in [CNNR11]. The operator D is the generator of the circle action, and as a consequence these modular spectral triples are one dimensional. A different example is given in [ReSe11] for the case of the Podleś sphere. In this case the Dirac operator is the one previously introduced in [DąSi03], but the use of a weight avoids the dimension drop.*

A modification of this notion has appeared in [Kaa11], by replacing the condition of boundedness of the commutator with the analogue one for a twisted commutator. An interesting example that makes use of this condition is the one given in [KaSe12] for $SU_q(2)$, to which we will return in the second part of this thesis.

Part II

New material

Chapter 5

A modular spectral triple for κ-Minkowski space

In this chapter we consider the problem of describing, in the framework of spectral triples, the geometry of κ-Minkowski space, which is a non-commutative space associated to a quantum deformation of the Poincaré group. Restricting our attention to the two-dimensional case, for simplicity, we will show how this geometry is more naturally described using the notion of modular spectral triple. This is based on the paper [Mat1].

5.1 Introduction

The *κ-Poincaré* algebra was introduced by Lukierski, Ruegg, Nowicki and Tolstoi in [LNRT91, LNR92]. It is a Hopf algebraic deformation of the Poincaré algebra, with a deformation parameter having physical dimension of mass and denoted by κ. A few years later Majid and Ruegg [MaRu94] clarified the bicrossproduct structure of the κ-Poincaré algebra: it consists of a semidirect product of the classical Lorentz algebra, which acts in a deformed way on the translation subalgebra, and a backreaction of the momentum sector on the Lorentz transformations. This allows the introduction of a homogeneous space for the κ-deformed symmetries, as the quotient Hopf algebra of the κ-Poincaré group by the Lorentz group. The result is a non-commutative Hopf algebra, which can be interpreted as the algebra of functions over a non-commutative spacetime, which is called *κ-Minkowski*.

It is interesting to study how (the Euclidean version of) κ-Minkowski space fits into the framework of non-commutative geometry developed by Connes. Despite some attempts [D'An06, IMSS11, IMS12], it is fair to say that a spectral triple that encodes the geometry of this space in a satisfactory way has not yet been constructed. In particular, in none of the spectral triples proposed so far the spectral dimension coincides with the classical one, which makes problematic the status of the classical limit. The aim of this chapter is to provide a new construction of a spectral triple, which uses tools that have been developed to study non-commutative geometries with modular properties.

As a starting point for our construction we consider the $*$-algebra introduced in [DuSi13], which we denote by $\mathcal{A}$, built using the commutation relations associated to κ-Minkowski space. There is an action of the κ-Poincaré algebra on it, which leaves invariant the integral with respect to the Lebesgue measure on $\mathbb{R}^2$, which we denote by ω. Via the GNS construction for this weight we construct a Hilbert space associated to $\mathcal{A}$. We remark that the choice of this weight is an important difference with respect to other approaches. We show that ω is a KMS weight for the algebra $\mathcal{A}$, with the corresponding modular operator playing a major role in the following.

The next step is the introduction of a self-adjoint operator D, which in the classical setting is given by the Dirac operator. We immediately face a difficulty in satisfying the condition of boundedness of the commutator with D, which is related to the structure of the coproduct of the κ-Poincaré algebra. One can relax this condition and consider the framework of *twisted spectral triples*, which requires the boundedness of the twisted commutator defined by $[D, \pi(a)]_\sigma = D\pi(a) - \pi(\sigma(a))D$, where σ is an automorphism of the the algebra $\mathcal{A}$. We prove that, under some assumptions related to symmetry and to the classical limit, there is a unique Dirac operator D and a unique automorphism σ such that the twisted commutator is bounded. We also discuss the relations between D, the Casimir of the κ-Poincaré algebra and the equivariant Dirac operator which has been considered in the literature.

We then study the property of summability of this spectral triple. We show that it is not finitely summable in the usual sense of spectral triples. We argue that this problem can be related to a mismatch in the modular properties of the weight ω, which we defined on our algebra $\mathcal{A}$, and the non-commutative integral defined by the trace on the Hilbert space. This situation can be reconsidered in the framework of *modular spectral triples*. To this end, we consider a specific weight Φ which should correct the mismatch mentioned above.

Strictly speaking, our case does not fit in this framework, because the action of the modular group is not periodic. Indeed it is given by translation in one variable, with the only fixed point under this action being the zero function. Here we do not dwell on how this framework should be modified to treat this case. Nevertheless, we show that we can fruitfully borrow some of its ingredients, and that they give interesting results when applied to our case.

For this reason we are going to refer to this construction, loosely speaking, as a modular spectral triple. In particular, we adapt the notion of spectral dimension in terms of the weight Φ to our case. This weight has the role of fixing the mismatch in the modular properties mentioned above. We find that, in this sense, our spectral triple is finitely summable and its spectral dimension coincides with the classical one. Moreover we show that, by computing the residue at the spectral dimension of an appropriate zeta function defined in terms of D and Φ, we recover the weight ω up to a constant. These results provide some preliminary evidence that these are the right tools to describe the geometry of κ-Minkowski.

Finally we discuss the introduction of a real structure. We define an antilinear isometry $\mathcal{J}$ on the Hilbert space and check the conditions defining a real structure. We show that they are modified, with the most interesting modification being the commutation relation between D and $\mathcal{J}$, which is related to the antipode structure of the κ-Poincaré algebra.

5.2 The $*$-algebra

The aim of this section is to introduce a $*$-algebra, which provides the first ingredient for a spectral triple describing the geometry of κ-Minkowski space. We start by recalling some basic facts about the κ-Poincaré and κ-Minkowski Hopf algebras, and some notions related to the implementation of Hopf algebra symmetries on a Hilbert space. After this short review we describe the $*$-algebra $\mathcal{A}$, which was introduced in [DuSi13], and recall some of its properties which are relevant for the construction of a spectral triple.

5.2.1 The κ-Poincaré and κ-Minkowski algebras

In this subsection we summarize the algebraic properties of the κ-Poincaré algebra $\mathcal{P}_\kappa$ in two dimensions. First we give the usual presentation that appears in the literature, that is as the Hopf algebra generated by the elements P_0, P_1, N satisfying

$$\begin{aligned}
&[P_0, P_1] = 0 \ , \quad [N, P_0] = P_1 \ , \\
&[N, P_1] = \frac{\kappa}{2}(1 - e^{-2P_0/\kappa}) - \frac{1}{2\kappa}P_1^2 \ .
\end{aligned}$$

The coproduct $\Delta : \mathcal{P}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{P}_\kappa$ is defined by the relations

$$\begin{aligned}
&\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \ , \qquad \Delta(P_1) = P_1 \otimes 1 + e^{-P_0/\kappa} \otimes P_1 \ , \\
&\Delta(N) = N \otimes 1 + e^{-P_0/\kappa} \otimes N \ .
\end{aligned}$$

The counit $\varepsilon : \mathcal{P}_\kappa \to \mathbb{C}$ and antipode $S : \mathcal{P}_\kappa \to \mathcal{P}_\kappa$ are defined by

$$\begin{aligned}
&\varepsilon(P_0) = \varepsilon(P_1) = 0 \ , \quad \varepsilon(N) = 0 \ , \\
&S(P_0) = -P_0 \ , \quad S(P_1) = -e^{P_0/\kappa}P_1 \ , \quad S(N) = -e^{P_0/\kappa}N \ .
\end{aligned}$$

An important role is played by the Hopf subalgebra generated by P_0 and P_1, which we denote by $\mathcal{T}_\kappa$, that is the generators of the translations. Indeed the κ-Minkowski space is defined by a Hopf algebra in non-degenerate dual pairing with this subalgebra [MaRu94], which we denote by $\mathcal{M}_\kappa$. If we denote the pairing by $\langle \cdot, \cdot \rangle : \mathcal{T}_\kappa \times \mathcal{M}_\kappa \to \mathbb{C}$, then the structure of $\mathcal{M}_\kappa$ is determined by the duality relations

$$\begin{aligned}
\langle t, xy \rangle &= \langle t^{(1)}, x \rangle \langle t^{(2)}, y \rangle \ , \\
\langle ts, x \rangle &= \langle t, x^{(1)} \rangle \langle s, x^{(2)} \rangle \ .
\end{aligned}$$

Here we have $t, s \in \mathcal{T}_\kappa$, $x, y \in \mathcal{M}_\kappa$ and we use the Sweedler notation for the coproduct

$$\Delta x = \sum_i x_{(i)}^{(1)} \otimes x_{(i)}^{(2)} = x^{(1)} \otimes x^{(2)} .$$

From the pairing we deduce that $\mathcal{M}_\kappa$ is non-commutative, since $\mathcal{T}_\kappa$ is not cocommutative. On the other hand, since $\mathcal{T}_\kappa$ is commutative we have that $\mathcal{M}_\kappa$ is cocommutative. The algebraic relations for the κ-Minkowski Hopf algebra $\mathcal{M}_\kappa$ are then

$$[X_0, X_1] = -\kappa^{-1} X_1 , \qquad \Delta X_\mu = X_\mu \otimes 1 + 1 \otimes X_\mu .$$

This concludes the usual presentation of the κ-Poincaré and κ-Minkowski Hopf algebras. It is important to point out that, using these definitions, κ^{-1} must be considered as a formal parameter in order to make sense of the power series of elements like $e^{-P_0/\kappa}$. As a consequence, the tensor product we have to use is that over the ring $\mathbb{C}[[\kappa^{-1}]]$ of formal power series. On the other hand it is possible to give a different presentation, see for example [DuSi13], where κ^{-1} can be considered as a number, and the tensor product as the usual algebraic tensor product over $\mathbb{C}$. Indeed, instead of considering the exponential $e^{-P_0/\kappa}$ as a power series in P_0, we consider it as an invertible element $\mathcal{E}$ and rewrite the defining relations as

$$\begin{aligned}
&[P_0, P_1] = 0 , \quad [P_0, \mathcal{E}] = [P_1, \mathcal{E}] = 0 , \\
&\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 , \quad \Delta(P_1) = P_1 \otimes 1 + \mathcal{E} \otimes P_1 , \quad \Delta(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E} , \\
&\varepsilon(P_0) = \varepsilon(P_1) = 0 , \quad \varepsilon(\mathcal{E}) = 1 , \\
&S(P_0) = -P_0 , \quad S(P_1) = -\mathcal{E}^{-1} P_1 , \quad S(\mathcal{E}) = \mathcal{E}^{-1} .
\end{aligned}$$

In this presentation we call the subalgebra generated by P_μ and $\mathcal{E}$ the *extended momentum algebra*, and denote again by $\mathcal{T}_\kappa$. An appropriate pairing defining κ-Minkowski space can be easily written in terms of these generators. It can be made into a Hopf $*$-algebra by defining the involution as $P_\mu^* = P_\mu$ and $\mathcal{E}^* = \mathcal{E}$.

In the following we will consider the case of Euclidean signature and so, strictly speaking, we should refer to the Euclidean counterpart of the κ-Poincaré algebra, which is known as the *quantum Euclidean group*. However the boost generator N is not going to play a central role in our discussion, which is going to be based on the extended momentum algebra, and therefore most of our relations do not depend on the signature. Henceforth we only make reference to the κ-Poincaré algebra and make some remarks when needed.

One more remark on the notation: we will write all formulae in terms of the parameter $\lambda := \kappa^{-1}$, instead of κ. The motivation comes from the fact that the Poincaré algebra is obtained in the "classical limit" $\lambda \to 0$, in a similar fashion to the classical limit $\hbar \to 0$ of quantum mechanics. This makes more transparent checking that some formulae reduce, in this limit, to their respective undeformed counterparts.

5.2.2 Definition of the $*$-algebra $\mathcal{A}$

Now let us describe the $*$-algebra introduced in [DuSi13] (but see also [DąPi10]). We summarize some of the main results and fix the notation for the other sections. In two dimensions the underlying algebra of κ-Minkowski is the enveloping algebra of the Lie algebra with generators iX_0 and iX_1 fullfilling $[X_0, X_1] = i\lambda X_1$. It has a faithful representation φ given by

$$\varphi(iX_0) = \begin{pmatrix} -\lambda & 0 \\ 0 & 0 \end{pmatrix} , \qquad \varphi(iX_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

The corresponding simply connected Lie group G consists of 2×2 matrices of the form

$$S(a) = S(a_0, a_1) = \begin{pmatrix} e^{-\lambda a_0} & a_1 \\ 0 & 1 \end{pmatrix} , \tag{5.1}$$

which are obtained by exponentiating φ as follows

$$e^{i\varphi(a_0 X_0 + a_1' X_1)} = \begin{pmatrix} e^{-\lambda a_0} & \frac{1-e^{-\lambda a_0}}{\lambda a_0} a_1' \\ 0 & 1 \end{pmatrix} .$$

The group operations written in the (a_0, a_1) coordinates are given by

$$S(a_0, a_1) S(a_0', a_1') = S(a_0 + a_0', a_1 + e^{-\lambda a_0} a_1') , \qquad S(a_0, a_1)^{-1} = S(-a_0, -e^{\lambda a_0} a_1) .$$

We have that the Lebesgue measure d^2a is right invariant whereas the measure $e^{\lambda a_0} d^2 a$ is left invariant on G, so that G is not unimodular. We denote by $L^1(G)$ the convolution algebra of G with respect to the right invariant measure. We identify functions on G with functions on $\mathbb{R}^2$ by the parametrization (5.1). Then $L^1(G)$ is an involutive Banach algebra consisting of integrable functions on $\mathbb{R}^2$ with product $\hat{\star}$ and involution $\hat{*}$ given by

$$(f \hat{\star} g)(a) = \int f(a_0 - a_0', a_1 - e^{-\lambda(a_0 - a_0')} a_1') g(a_0', a_1') d^2 a' ,$$
$$f^{\hat{*}}(a) = e^{\lambda a_0} \overline{f}(-a_0, -e^{\lambda a_0} a_1) .$$

Any unitary representation $\tilde{\pi}$ of G (assumed to be strongly continuous) gives rise to a representation of $L^1(G)$, denoted with the same symbol, obtained by setting

$$\tilde{\pi}(f) = \int f(a) \tilde{\pi}(S(a)) d^2 a .$$

It is indeed a $*$-representation, since it obeys the relations

$$\tilde{\pi}(f \hat{\star} g) = \tilde{\pi}(f) \tilde{\pi}(g) , \qquad \tilde{\pi}(f^{\hat{*}}) = \tilde{\pi}(f)^{\dagger} ,$$

for any $f, g \in L^1(G)$, where here $\dagger$ denotes the adjoint. Now, following the same procedure as in the Weyl quantization, we can define the map $W_{\tilde{\pi}}(f) := \tilde{\pi}(\mathcal{F}f)$, where $f \in L^1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(L^1(\mathbb{R}^2))$ and $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^2$. It then follows that

$$W_{\tilde{\pi}}(f \star g) = W_{\tilde{\pi}}(f)W_{\tilde{\pi}}(g) \ , \qquad W_{\tilde{\pi}}(f^*) = W_{\tilde{\pi}}(f)^\dagger \ ,$$

where the product $\star$ and the involution $*$ are defined by

$$f \star g = \mathcal{F}^{-1}(\mathcal{F}f \hat{\star} \mathcal{F}g) \ , \qquad f^* = \mathcal{F}^{-1}(\mathcal{F}f)^{\hat{*}} \ . \tag{5.2}$$

Here the formulae are given using the unitary convention for the Fourier transform.

It is important to remark that, although $W_{\tilde{\pi}}$ depends on the choice of $\tilde{\pi}$ (and therefore on the choice of the unitary representation of G), the product $\star$ and the involution $*$ do not depend on such a choice. This is the same strategy which is employed to define the deformed product in quantum mechanics. Indeed, as in that case, one needs to exercise care about the domain of definition of the operations $\star$ and $*$, so in the following we restrict ourselves to a subset of Schwartz functions which have nice compatibility properties with those operations.

Definition 5.1. Denote by $\mathcal{S}_c$ the space of Schwartz functions on $\mathbb{R}^2$ with compact support in the first variable, that is for $f \in \mathcal{S}_c$ we have $\operatorname{supp}(f) \subseteq K \times \mathbb{R}$ for some compact $K \subset \mathbb{R}$. We define $\mathcal{A} = \mathcal{F}(\mathcal{S}_c)$, where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^2$.

Proposition 5.1. *For $f, g \in \mathcal{A}$ we can write (5.2) in the following form*

$$\begin{aligned} (f \star g)(x) &= \int e^{ip_0x_0}(\mathcal{F}_0 f)(p_0, x_1)g(x_0, e^{-\lambda p_0}x_1)\frac{dp_0}{2\pi} \ , \\ f^*(x) &= \int e^{ip_0x_0}(\mathcal{F}_0 \overline{f})(p_0, e^{-\lambda p_0}x_1)\frac{dp_0}{2\pi} \ . \end{aligned} \tag{5.3}$$

We have that $f \star g \in \mathcal{A}$ and $f^* \in \mathcal{A}$, so that $\mathcal{A}$ is a $*$-algebra.

Here $\mathcal{F}_0$ denotes the Fourier transform of f in the first variable, defined as

$$(\mathcal{F}_0 f)(p_0, x_1) = \int e^{-ip_0y_0} f(y_0, x_1) dy_0 \ .$$

Notice that for $\lambda = 0$ we recover the pointwise product $(f \star g)(x) = f(x)g(x)$ and the complex conjugation $f^*(x) = \overline{f}(x)$, giving the correct classical limit for the algebra. The definition of $\mathcal{A}$ implies that any $f \in \mathcal{A}$ is a Schwartz function, but does not have compact support in the first variable. However, by the Paley-Wiener theorem, the compact support of $\mathcal{F}f$ in the first variable implies analiticity of f in the first variable.

The extended momentum algebra $\mathcal{T}_\kappa$ has a natural action on the algebra $\mathcal{A}$. To see this let us first recall some notions related to the implementation of Hopf algebra symmetries [Sit03]. In particular we are interested in the case in which a Hopf algebra H acts on an algebra A. The following definitions are compatibility conditions between the two structures.

Definition 5.2. An algebra A is a *left H-module algebra* if A is a left H-module and the representation respects the algebra structure of A, that is $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$ for all $h \in H$ and $a, b \in A$.

Definition 5.3. A $*$-algebra A is *left H-module $*$-algebra* if A is a left H-module algebra and moreover the action is compatible with the star structure, that is $(h \triangleright a)^* = S(h)^* \triangleright a^*$ for all $h \in H$ and $a \in A$.

With these definitions we can state the following theorem, proven in [DuSi13].

Theorem 5.2. *The algebra $\mathcal{A}$ is a left $\mathcal{T}_\kappa$-module $*$-algebra with respect to the following representation of the extended momentum algebra*

$$(P_\mu \triangleright f)(x) = -i(\partial_\mu f)(x) \ , \qquad (\mathcal{E} \triangleright f)(x) = f(x_0 + i\lambda, x_1) \ .$$

Notice that the action of $\mathcal{E}$ on $f \in \mathcal{A}$ is well-defined, since these functions are analytic in the first variable. This result can actually be extended to the whole κ-Poincaré Hopf algebra $\mathcal{P}_\kappa$ (see again [DuSi13]), but we do not need the explicit formulae here.

5.3 The Hilbert space

Having at our disposal the algebra $\mathcal{A}$, the next step is to introduce a Hilbert space together with a faithful $*$-representation of the algebra on it. In this section we choose a weight ω, which is motivated by symmetry considerations, and use it to obtain a Hilbert space $\mathcal{H}$ via the GNS construction. We show that $\mathcal{H}$ is unitarily equivalent to $L^2(\mathbb{R}^2)$ and determine the corresponding unitary operator U. Then we introduce an unbounded $*$-representation ρ of the extended momentum algebra $\mathcal{T}_\kappa$ on $\mathcal{A}$, and prove that $\rho(P_\mu)$ and $\rho(\mathcal{E})$ are essentially self-adjoint on $\mathcal{H}$. We show that the weight ω satisfies the KMS condition with respect to a certain action α, and determine the corresponding modular operator Δ_ω. Finally we prove some useful formulae that will be used extensively in the rest of this chapter.

5.3.1 Definition of the Hilbert space $\mathcal{H}$

We can introduce a Hilbert space, naturally associated with the $*$-algebra $\mathcal{A}$, via the GNS construction. In our case, which is non-unital, one chooses a finite weight ω on the algebra $\mathcal{A}$. After taking the quotient by the left ideal $\{f \in \mathcal{A} : \omega(f^* \star f) = 0\}$, if non-trivial, one defines an inner product in terms of ω by setting $(f, g) := \omega(f^* \star g)$. The Hilbert space $\mathcal{H}$ is then defined as the completion of the algebra $\mathcal{A}$ in the norm induced by the inner product.

A natural choice for ω is given by the weight used in the commutative case

$$\omega(f) := \int f(x) d^2x \ ,$$

where the integration is with respect to the Lebesgue measure. This is motivated by simplicity, but more importantly by the invariance of ω under the action of the κ-Poincaré algebra [DuSi13], which we recall in the next proposition.

Proposition 5.3. *The weight ω is invariant under the action of $\mathcal{P}_\kappa$, that is for any $f \in \mathcal{A}$ and for any $h \in \mathcal{P}_\kappa$ we have $\omega(h \triangleright f) = \varepsilon(h)\omega(f)$.*

We recall two additional results obtained in [DuSi13], which we need in the following.

Proposition 5.4. *For any $f, g \in \mathcal{A}$ the weight ω satisfies the twisted trace property*

$$\omega(f \star g) = \omega((\mathcal{E} \triangleright g) \star f) \ .$$

Moreover we have

$$\omega(f \star g^*) = \int f(x)\overline{g}(x) d^2x \ .$$

Later the twisted trace property will be rewritten in the language of KMS weights. Now we construct the Hilbert space using the GNS construction for the weight ω.

Proposition 5.5. *The sesquilinear form $(\cdot, \cdot)$ defined by $(f, g) = \omega(f^* \star g)$ is positive-definite on $\mathcal{A}$. The completion in the induced norm gives a separable Hilbert space $\mathcal{H}$, which is unitarily equivalent to $L^2(\mathbb{R}^2)$ via*

$$(Uf)(x) = \int e^{ip_0x_0}(\mathcal{F}_0 f)(p_0, e^{\lambda p_0}x_1)\frac{dp_0}{2\pi} \ .$$

Proof. From the second part of Proposition 5.4 we have

$$(f, g) = \omega(f^* \star g) = \int f^*(x)\overline{g^*}(x) d^2x \ .$$

In particular if we set $f = g$ we have

$$\|f\|^2 = (f, f) = \int |f^*(x)|^2 d^2x \geq 0 \ . \tag{5.4}$$

We have that $\|f\|$ is clearly finite, since $f^* \in \mathcal{A}$ is in particular a Schwartz function. To show that $(\cdot, \cdot)$ is positive-definite notice that $f^* = 0$ if and only if $f = 0$, due to the properties of the involution, and that the Lebesgue integral is faithful on functions vanishing at infinity on $\mathbb{R}^2$. Therefore $(\cdot, \cdot)$ is an inner product on $\mathcal{A}$ and, by completing with respect to the norm $\|\cdot\|$, we obtain a Hilbert space which we denote by $\mathcal{H}$.

Now we prove that it is unitarily equivalent to $L^2(\mathbb{R}^2)$ and determine the corresponding unitary operator $U : \mathcal{H} \to L^2(\mathbb{R}^2)$. Define $J\psi := \psi^*$, where $*$ is the involution in $\mathcal{A}$, and set $U := J_c J$, where J_c is complex conjugation. From the formula (5.3) we obtain

$$(Uf)(x) = \int e^{ip_0x_0}(\mathcal{F}_0 f)(p_0, e^{\lambda p_0}x_1)\frac{dp_0}{2\pi} \ .$$

From equation (5.4) it follows that U is an isometry from $\mathcal{A} \subset \mathcal{H}$ to $L^2(\mathbb{R}^2)$. Indeed

$$\|f\|^2 = \int |(Jf)(x)|^2 d^2x = \int |(Uf)(x)|^2 d^2x = \|Uf\|^2_{L^2(\mathbb{R}^2)} \ .$$

Since $\mathcal{A}$ is a dense subset of $\mathcal{H}$ it follows that U can be extended by continuity to an isometry from $\mathcal{H}$ to $L^2(\mathbb{R}^2)$. We still need to prove that U is surjective. We have that U is invertible in $\mathcal{A}$ since $J = J_c U$ is invertible in $\mathcal{A}$, therefore the image of U contains $\mathcal{A}$. Notice that $\mathcal{A}$ is also a dense subset of $L^2(\mathbb{R}^2)$. Then, since U is continuous, it contains the closure of $\mathcal{A}$, which is indeed $L^2(\mathbb{R}^2)$. So we have shown that U is isometric and surjective, therefore unitary. □

Corollary 5.6. *The involution J extends to an antiunitary operator from $\mathcal{H}$ to $L^2(\mathbb{R}^2)$.*

Proof. It follows immediately from the previous proposition. Indeed we have that U is unitary from $\mathcal{H}$ to $L^2(\mathbb{R}^2)$ and complex conjugation is an antiunitary operator in $L^2(\mathbb{R}^2)$ so, since $J = J_c U$, it follows that J it is an antiunitary operator from $\mathcal{H}$ to $L^2(\mathbb{R}^2)$. □

For the construction of a spectral triple we need the following condition: for any $f \in \mathcal{A}$ the $*$-representation $\pi(f)$ in $\mathcal{H}$, given by the multiplication $\pi(f)\psi = f \star \psi$ for $\psi \in \mathcal{H}$, should be a bounded operator. This follows from the next proposition.

Proposition 5.7. *For any $f \in \mathcal{A}$ the operator $\pi(f)$ is bounded on $\mathcal{H}$.*

Proof. Recall that the $*$-algebra $\mathcal{A}$ is built from $L^1(G)$, the convolution algebra of the Lie group G associated to the κ-Minkowski space in two dimensions. Any convolution algebra can be completed to a C^*-algebra, for example the group C^*-algebra $C^*(G)$ or the reduced one $C^*_r(G)$. In this case they coincide, since G is amenable, and we denote by $\|\cdot\|_{C^*(G)}$ the associated C^*-norm. Now we can define a C^*-norm on $\mathcal{A}$ by setting $\|f\| := \|\mathcal{F}f\|_{C^*(G)}$ for any $f \in \mathcal{A}$. Indeed using the formulae in (5.2) we have

$$\begin{aligned}\|f \star f^*\| &= \|\mathcal{F}(f \star f^*)\|_{C^*(G)} = \|\mathcal{F}f \hat{\star} \mathcal{F}f^*\|_{C^*(G)} \\ &= \|\mathcal{F}f \hat{\star} (\mathcal{F}f)^{\hat{*}}\|_{C^*(G)} = \|\mathcal{F}f\|^2_{C^*(G)} = \|f\|^2 \ .\end{aligned}$$

In the second line we have used the C^*-property of the norm $\|\cdot\|_{C^*(G)}$. Then, since $\mathcal{H}$ is the Hilbert space associated to $\mathcal{A}$ via the GNS construction, it follows from the general properties of this construction that the operator norm of $\pi(f)$ is bounded by $\|f\|$. □

5.3.2 The representation of $\mathcal{T}_\kappa$

In this subsection we want to extend the representation of the extended momentum algebra $\mathcal{T}_\kappa$, previously defined only on the algebra $\mathcal{A}$, to the Hilbert space $\mathcal{H}$. This representation will be unbounded in general, so we will have to specify the appropriate domains. Let us give some definitions to handle this situation.

Definition 5.4. Let $\mathcal{H}_0$ be a dense linear subspace of a Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$. An *unbounded ∗-representation* of H on $\mathcal{H}_0$ is a homomorphism $\rho : H \to \mathrm{End}(\mathcal{H}_0)$ such that $(\rho(h)\psi, \phi) = (\psi, \rho(h^*)\phi)$ for all $\psi, \phi \in \mathcal{H}_0$ and $h \in H$.

Now suppose that we have a representation π of a $*$-algebra A on $\mathcal{H}$. Moreover suppose that A is a left H-module $*$-algebra, which is the case of interest for a spectral triple with symmetries. We want the representation π to be compatible with the structure of a Hopf algebra H. This leads to the notion of equivariance, given in the next definition.

Definition 5.5. Suppose A is a left H-module $*$-algebra. A $*$-representation π of A on $\mathcal{H}_0$ is called *H-equivariant* (or also covariant) if there exists an unbounded $*$-representation ρ of H on $\mathcal{H}_0$ such that

$$\rho(h)\pi(a)\psi = \pi(h_{(1)} \triangleright a)\rho(h_{(2)})\psi \ , \tag{5.5}$$

for all $h \in H$, $a \in A$ and $\psi \in \mathcal{H}_0$.

Finally we give a definition of equivariance for operators in $\mathcal{H}$.

Definition 5.6. A linear operator T defined on $\mathcal{H}_0$ is said to be *equivariant* if it commutes with $\rho(h)$, that is $T\rho(h)\psi = \rho(h)T\psi$ for all $h \in H$ and $\psi \in \mathcal{H}_0$.

An obvious choice for the dense subspace is $\mathcal{A}$, which by construction is dense in $\mathcal{H}$. Recall that by Proposition 5.2 we have that $\mathcal{A}$ is a left $\mathcal{T}_\kappa$-module $*$-algebra, where the representation $\triangleright$ of $\mathcal{T}_\kappa$ on $\mathcal{A}$ is defined by the formulae

$$(P_\mu \triangleright f)(x) = -i(\partial_\mu f)(x) \ , \qquad (\mathcal{E} \triangleright f)(x) = f(x_0 + i\lambda, x_1) \ . \tag{5.6}$$

Then if we set $\rho(h)\psi := h \triangleright \psi$, for every $h \in \mathcal{T}_k$ and $\psi \in \mathcal{A}$, we get an unbounded representation of $\mathcal{T}_\kappa$ on $\mathcal{A}$. Note that $\mathcal{A}$ is invariant under the action of $\mathcal{T}_\kappa$ and the equivariance property for the representation π is automatic, since π is given by left multiplication and the equivariance property is just a restatement of the fact that $\mathcal{A}$ is a left $\mathcal{T}_\kappa$-module algebra.

To prove that ρ is an unbounded $*$-representation we need to show that the equality $(\rho(h)\phi, \psi) = (\phi, \rho(h^*)\psi)$ holds for all $\phi, \psi \in \mathcal{A}$ and $h \in \mathcal{T}_\kappa$. We only need to check this condition for the generators of $\mathcal{T}_\kappa$, that is for the operators $\rho(P_\mu)$ and $\rho(\mathcal{E})$ with domain $\mathcal{A}$. We are going to prove the stronger statement that these operators are essentially self-adjoint on $\mathcal{H}$, from which the previous equality follows. First we need a simple lemma.

Lemma 5.8. *For any $\psi \in \mathcal{A}$ we have $U\rho(P_0)U^{-1}\psi = \rho(P_0)\psi$ and $U\rho(P_1)U^{-1}\psi = \rho(\mathcal{E})\rho(P_1)\psi$.*

Proof. Recall that, since $\mathcal{A}$ is a left $\mathcal{T}_\kappa$-module $*$-algebra, we have the compatibility property with the involution given by $(h \triangleright \psi)^* = S(h)^* \triangleright \psi^*$. Here S is the antipode map and the equality is valid for any $h \in \mathcal{T}_\kappa$ and $\psi \in \mathcal{A}$. In particular we have the equality $(P_\mu \triangleright \psi^*)^* = S(P_\mu)^* \triangleright \psi$. Then, using the Hopf algebraic rules of $\mathcal{T}_\kappa$, one immediately shows that $S(P_0)^* = -P_0$ and

$S(P_1)^* = -\mathcal{E}^{-1}P_1$. Now for any $\psi \in \mathcal{A}$ we have

$$J\rho(P_\mu)J\psi = (P_\mu \triangleright \psi^*)^* = S(P_\mu)^* \triangleright \psi \ .$$

As a consequence we obtain the relations $J\rho(P_0)J\psi = -\rho(P_0)\psi$ and $J\rho(P_1)J\psi = -\rho(\mathcal{E}^{-1})\rho(P_1)\psi$. Using the definition of the representation $\triangleright$ we obtain the following formulae

$$(J\rho(P_0)J\psi)(x) = i(\partial_0\psi)(x) \ , \qquad (J\rho(P_1)J\psi)(x) = i(\partial_1\psi)(x_0 - i\lambda, x_1) \ .$$

Now recall that we have the relations $J = J_c U$ and $J = U^{-1}J_c$, where J_c stands for complex conjugation. We can use the relations to easily compute the following

$$(U\rho(P_1)U^{-1}\psi)(x) = (\overline{J\rho(P_1)J\overline{\psi}})(x) = -i(\partial_1\psi)(x_0 + i\lambda, x_1) \ .$$

Notice that this can be rewritten as $U\rho(P_1)U^{-1}\psi = \rho(\mathcal{E})\rho(P_1)\psi$. In a similar way one shows that the identity $U\rho(P_0)U^{-1}\psi = \rho(P_0)\psi$ holds. The lemma is proven. □

Before starting the next proof we recall that U is a unitary operator from $\mathcal{H}$ to $L^2(\mathbb{R}^2)$ and that $\mathcal{A}$ is dense in both Hilbert spaces.

Proposition 5.9. *The operators $\rho(P_\mu)$ and $\rho(\mathcal{E})$, with domain $\mathcal{A}$, are essentially self-adjoint on $\mathcal{H}$. Therefore ρ is an unbounded $*$-representation of $\mathcal{T}_\kappa$ on $\mathcal{A}$.*

Proof. We can consider $\rho(P_0)$ and $\rho(\mathcal{E})\rho(P_1)$ as unbounded operators on $L^2(\mathbb{R}^2)$ with domain $\mathcal{A}$. An easy computation shows that they are symmetric operators on this Hilbert space. From the previous lemma it follows that for any $\phi, \psi \in \mathcal{A}$ we have

$$(\phi, \rho(\mathcal{E})\rho(P_1)\psi)_{L^2(\mathbb{R}^2)} = (\phi, U\rho(P_1)U^{-1}\psi)_{L^2(\mathbb{R}^2)} = (U^{-1}\phi, \rho(P_1)U^{-1}\psi) \ .$$

Now, since $\rho(\mathcal{E})\rho(P_1)$ is symmetric on $L^2(\mathbb{R}^2)$ and U is invertible in $\mathcal{A}$, it follows that $\rho(P_1)$ is symmetric in $\mathcal{H}$. The same argument applies to $\rho(P_0)$. Then, using Nelson's analytic vector theorem, it follows that $\rho(P_0)$ and $\rho(P_1)$ are essentially self-adjoint on $\mathcal{H}$, since it easy to show that each element in $\mathcal{A}$ is an analytic vector for them.

For $\rho(\mathcal{E})$ we observe that, for any $\psi \in \mathcal{A}$, we have

$$(\rho(\mathcal{E})\psi)(x) = \psi(x_0 + i\lambda, x_1) = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!}(\partial_0^n\psi)(x) \ ,$$

where we have used the fact that $\psi \in \mathcal{A}$ is analytic in the first variable. This equality can be rewritten in the form $\rho(\mathcal{E})\psi = e^{-\lambda\rho(P_0)}\psi$. It follows then, using the Borel functional calculus for unbounded self-adjoint operators (see for example [Sch12]), that the operator $e^{-\lambda\rho(P_0)}$ is essentially self-adjoint on $\mathcal{H}$. Alternatively one can check, by direct computation, that any $\psi \in \mathcal{A}$ is also an analytic vector for $e^{-\lambda\rho(P_0)}$ and apply Nelson's theorem again. □

In the following we are going to denote the closure of these operators by the same symbols, and we are also going to use the notation $\hat{P}_\mu := \rho(P_\mu)$.

5.3.3 The KMS property of the weight ω

One relevant property of the weight ω is that it satisfies the so-called twisted trace property, given in Proposition 5.4. This property can be recasted in the language of KMS weights [Kus97], which provides great insight into the modular aspects of the spectral triple we are constructing. This is related to the Tomita-Takesaki modular theory, for a review see [BCL10], which moreover contains material on modular spectral triples.

In the next lemma we introduce a one-parameter group of $*$-automorphisms of $\mathcal{A}$, which we denote by σ^ω. This is going to be the modular group of ω, which justifies the notation.

Lemma 5.10. *For any $t \in \mathbb{R}$ and $f \in \mathcal{A}$ define $(\sigma^\omega_t f)(x) := f(x_0 - \lambda t, x_1)$. We have that σ^ω is a one-parameter group of $*$-automorphisms of $\mathcal{A}$.*

Proof. To prove that σ^ω is one-parameter group of automorphisms of $\mathcal{A}$ we have to show that, for any $f, g \in \mathcal{A}$ and $t \in \mathbb{R}$, the property $\sigma^\omega_t(f \star g) = \sigma^\omega_t(f) \star \sigma^\omega_t(g)$ is satisfied. This can be shown by a direct computation

$$\begin{aligned}(\sigma^\omega_t(f) \star \sigma^\omega_t(g))(x) &= \int e^{ip_0x_0}(\mathcal{F}_0\sigma^\omega_t(f))(p_0, x_1)\sigma^\omega_t(g)(x_0, e^{-\lambda p_0}x_1)\frac{dp_0}{2\pi} \\ &= \int e^{ip_0x_0}\int e^{-ip_0q_0} f(q_0 - \lambda t, x_1)g(x_0 - \lambda t, e^{-\lambda p_0}x_1)dq_0\frac{dp_0}{2\pi} \ .\end{aligned}$$

After the change of variable $q_0 \to q_0 + \lambda t$ we obtain

$$\begin{aligned}(\sigma^\omega_t(f) \star \sigma^\omega_t(g))(x) &= \int e^{ip_0(x_0-\lambda t)}\int e^{-ip_0q_0} f(q_0, x_1)g(x_0 - \lambda t, e^{-\lambda p_0}x_1)dq_0\frac{dp_0}{2\pi} \\ &= \int e^{ip_0(x_0-\lambda t)}(\mathcal{F}_0 f)(p_0, x_1)g(x_0 - \lambda t, e^{-\lambda p_0}x_1)\frac{dp_0}{2\pi} = (\sigma^\omega_t(f \star g))(x) \ .\end{aligned}$$

Finally to prove that σ^ω_t is a $*$-automorphism we need to check the additional property $\sigma^\omega_t(f)^* = \sigma^\omega_t(f^*)$. This can again be checked by a direct computation

$$\begin{aligned}\sigma^\omega_t(f)^*(x) &= \int e^{ip_0x_0}(\mathcal{F}_0\overline{\alpha_t(f)})(p_0, e^{-\lambda p_0}x_1)\frac{dp_0}{2\pi} \\ &= \int e^{ip_0x_0}\int e^{-ip_0q_0}\overline{f}(q_0 - \lambda t, e^{-\lambda p_0}x_1)dq_0\frac{dp_0}{2\pi}\end{aligned}$$

Using again the change of variable $q_0 \to q_0 + \lambda t$ we obtain

$$\begin{aligned}\sigma^\omega_t(f)^*(x) &= \int e^{ip_0(x_0-\lambda t)}\int e^{-ip_0q_0}\overline{f}(q_0, e^{-\lambda p_0}x_1)dq_0\frac{dp_0}{2\pi} \\ &= \int e^{ip_0(x_0-\lambda t)}(\mathcal{F}_0\overline{f})(p_0, e^{-\lambda p_0}x_1)\frac{dp_0}{2\pi} = \sigma^\omega_t(f^*)(x) \ .\end{aligned}$$

□

Proposition 5.11. *The weight ω satisfies the KMS condition at inverse temperature $\beta = 1$ with respect to σ^ω. The corresponding modular operator is given by $\Delta_\omega = e^{-\lambda \hat{P}_0}$.*

Proof. We define the function $F_{f,g}(z) := \omega(f \star \sigma^\omega_z g)$. It is bounded continuous and analytic, since σ^ω acts on the first variable and functions in $\mathcal{A}$ are analytic in the first variable. To prove that ω satisfied the KMS condition with respect to σ^ω we need to show that

$$F_{f,g}(t) = \omega(f \star \sigma^\omega_t(g)) \ , \quad F_{f,g}(t + i\beta) = \omega(\sigma^\omega_t(g) \star f) \ .$$

Notice that the action of $\mathcal{E}$ can be rewritten in terms of σ^ω, that is

$$(\mathcal{E} \triangleright f)(x) = f(x_0 + i\lambda, x_1) = (\sigma^\omega_{-i} f)(x) \ . \tag{5.7}$$

Then using the twisted trace property we have

$$\begin{aligned} F_{f,g}(t+i) &= \omega(f \star \sigma^\omega_{t+i}(g)) = \omega((\mathcal{E} \triangleright \sigma^\omega_{t+i}(g)) \star f) \\ &= \omega((\sigma^\omega_{-i}\sigma^\omega_{t+i}(g)) \star f) = \omega(\sigma^\omega_t(g) \star f) \ . \end{aligned}$$

This proves the KMS condition. To determine the modular operator Δ_ω associated with ω consider $f, g \in \mathcal{A}$. Using the fact that σ^ω_t is an automorphism of $\mathcal{A}$ we have

$$\pi(\sigma^\omega_t(f))g = \sigma^\omega_t(f) \star g = \sigma^\omega_t(f \star \sigma^\omega_{-t}(g)) = \sigma^\omega_t(\pi(f)\sigma^\omega_{-t}(g)) \ . \tag{5.8}$$

Now, using the fact that $\hat{P}_0 = -i\partial_0$, we have the following equality

$$(\sigma^\omega_t f)(x) = f(x_0 - \lambda t, x_1) = (e^{-i\lambda t \hat{P}_0} f)(x) \ .$$

Then we see that equation (5.8) can be rewritten as

$$\pi(\sigma^\omega_t(f))g = e^{-i\lambda t \hat{P}_0}\pi(f)e^{i\lambda t \hat{P}_0} g = \Delta^{it}_\omega \pi(f) \Delta^{-it}_\omega g \ .$$

This implies that the modular operator is given by $\Delta_\omega = e^{-\lambda \hat{P}_0}$. □

5.3.4 Some useful formulae

We have seen that the Hilbert space $\mathcal{H}$, constructed via the GNS construction for ω, is unitarily equivalent to $L^2(\mathbb{R}^2)$, where the unitary operator is given by

$$(Uf)(x) = \int e^{ip_0 x_0} (\mathcal{F}_0 f)(p_0, e^{\lambda p_0} x_1) \frac{dp_0}{2\pi} \ .$$

We can associate, to any densely defined operator T on $\mathcal{H}$, the densely defined operator UTU^{-1} on $L^2(\mathbb{R}^2)$. Many properties of operators defined on $\mathcal{H}$ are conserved by this unitary transformation: for example, if T belongs to the p-th Schatten ideal on $\mathcal{H}$, then UTU^{-1}

belongs to the p-th Schatten ideal on $L^2(\mathbb{R}^2)$. This is useful, since we can use results which are formulated for the Hilbert space $L^2(\mathbb{R}^2)$ to establish some properties of operators on $\mathcal{H}$. For example, to prove that an operator is Hilbert-Schmidt in $L^2(\mathbb{R}^2)$, one can write it in the form of an integral operator and check that the kernel belongs to $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$.

Here we collect some useful formulae used extensively in the rest of this chapter.

Lemma 5.12. *For $f \in \mathcal{A}$ and $\psi \in \mathcal{H}$ we have*

$$(U\pi(f)U^{-1}\psi)(x) = \int e^{ipx}(Uf)(x_0, e^{\lambda p_0}x_1)(\mathcal{F}\psi)(p)\frac{d^2p}{(2\pi)^2} \ .$$

Proof. Consider first $\psi \in \mathcal{A}$. From the definition of $\pi(f)$ and the properties of the involution J we obtain that $J\pi(f)J$ acts by multiplication from the right, that is we have $J\pi(f)J\psi = \psi\star Jf$. To compute $U\pi(f)U^{-1}$ we use the fact that U is related to the involution by $J = J_cU$, where J_c denotes complex conjugation. We have the following equality

$$U\pi(f)U^{-1}\psi = J_cJ\pi(f)JJ_c\psi = J_c(J_c\psi \star Jf) = J_c(J_c\psi \star J_cUf) \ .$$

Using the formula (5.3) for the product and after a change of variable we obtain

$$\begin{aligned} J_c(J_c\psi \star J_cUf)(x) &= \int e^{-ip_0x_0}(\overline{\mathcal{F}_0\overline{\psi}})(p_0, x_1)(Uf)(x_0, e^{-\lambda p_0}x_1)\frac{dp_0}{2\pi} \\ &= \int e^{ip_0x_0}(\mathcal{F}_0\psi)(p_0, x_1)(Uf)(x_0, e^{\lambda p_0}x_1)\frac{dp_0}{2\pi} \ . \end{aligned}$$

Finally this expression may be rewritten as

$$(U\pi(f)U^{-1}\psi)(x) = \int e^{ipx}(Uf)(x_0, e^{\lambda p_0}x_1)(\mathcal{F}\psi)(p)\frac{d^2p}{(2\pi)^2} \ .$$

This formula extends by continuity to any $\psi \in L^2(\mathbb{R}^2)$. Indeed since $\pi(f)$ is a bounded operator in $\mathcal{H}$ and since U is a unitary operator from $\mathcal{H}$ to $L^2(\mathbb{R}^2)$ it follows that $U\pi(f)U^{-1}$ is bounded in $L^2(\mathbb{R}^2)$. Therefore $U\pi(f)U^{-1}$ may be extended by continuity to $L^2(\mathbb{R}^2)$. $\square$

The following lemma gives an explicit expression for a function of $\hat{P}_\mu$.

Lemma 5.13. *For g a bounded function, define $g(\hat{P})$ by the functional calculus. We have*

$$(Ug(\hat{P})U^{-1}\psi)(x) = \int e^{ipx}g(p_0, e^{-\lambda p_0}p_1)(\mathcal{F}\psi)(p)\frac{d^2p}{(2\pi)^2} \ .$$

Proof. We know that in $L^2(\mathbb{R}^2)$ the Fourier transform $\mathcal{F}$ is the unitary operator (up to factors of 2π, depending on the normalization) that turns the operators $\tilde{P}_\mu = -i\partial_\mu$ into multiplication operators. This means that for $\psi \in L^2(\mathbb{R}^2)$ we have

$$(g(\tilde{P})\psi)(x) = \int e^{ipx}g(p)(\mathcal{F}\psi)(p)\frac{d^2p}{(2\pi)^2} \ .$$

From the results of the previous section we have that $U\hat{P}_0U^{-1} = \tilde{P}_0$ and $U\hat{P}_1U^{-1} = e^{-\lambda\tilde{P}_0}\tilde{P}_1$, so we obtain $Ug(\hat{P})U^{-1} = g(\tilde{P}_0, e^{-\lambda\tilde{P}_0}\tilde{P}_1)$. The result follows by using the functional calculus for the commuting operators $\tilde{P}_\mu$ in $L^2(\mathbb{R}^2)$. □

Finally we are interested in operators of the form $\pi(f)g(\hat{P})$.

Proposition 5.14. *The Schwartz kernel associated to the operator $U\pi(f)g(\hat{P})U^{-1}$ is given by*

$$K(x,y) = \int e^{ip(x-y)}(Uf)(x_0, e^{\lambda p_0}x_1)g(p_0, e^{-\lambda p_0}p_1)\frac{d^2p}{(2\pi)^2} \ .$$

Proof. Using Lemmata 5.12 and 5.13 we obtain

$$\begin{aligned}(U\pi(f)g(\hat{P})U^{-1}\psi)(x) &= (U\pi(f)U^{-1}Ug(\hat{P})U^{-1}\psi)(x) \\ &= \int e^{ipx}(Uf)(x_0, e^{\lambda p_0}x_1)(\mathcal{F}Ug(\hat{P})U^{-1}\psi)(p)\frac{d^2p}{(2\pi)^2} \\ &= \int e^{ipx}(Uf)(x_0, e^{\lambda p_0}x_1)g(p_0, e^{-\lambda p_0}p_1)(\mathcal{F}\psi)(p)\frac{d^2p}{(2\pi)^2} \ .\end{aligned}$$

The Schwartz kernel corresponding to this operator is given by

$$K(x,y) = \int e^{ip(x-y)}(Uf)(x_0, e^{\lambda p_0}x_1)g(p_0, e^{-\lambda p_0}p_1)\frac{d^2p}{(2\pi)^2} \ .$$

□

5.4 The Dirac operator

The next step in the construction of a spectral triple is the introduction of a self-adjoint unbounded operator D on the Hilbert space, which we call the Dirac operator. In the next subsection we briefly recall the ingredients used in the commutative case, mainly to fix some notation. We show that there is a problem in obtaining a bounded commutator, which is solved by considering instead a twisted commutator with twist σ. Finally we prove that, under some assumptions related to symmetry and to the classical limit, there is a unique pair of a Dirac operator D and automorphism σ such that the twisted commutator is bounded.

5.4.1 A problem with boundedness

Let us briefly recall some facts concerning the Dirac operator D in the two dimensional Euclidean space $\mathbb{R}^2$. Here we consider the algebra of Schwartz functions $\mathcal{A} = \mathcal{S}(\mathbb{R}^2)$ with the $*$-representation π on $\mathcal{H}_r = L^2(\mathbb{R}^2)$ given by pointwise multiplication, that is $(\pi(f)\psi)(x) = f(x)\psi(x)$. We consider the following representation of the Clifford algebra

$$\Gamma^0 := \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \Gamma^1 := \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

These matrices satisfy the anticommutation relations $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$. To consider spinors we define the Hilbert space $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^2$, corresponding to the trivial spinor bundle. The representation π of $\mathcal{A}$ is extended as pointwise multiplication on the two copies of $\mathcal{H}_r$, and we denote it again by π. The inner product is given by

$$(\psi, \phi)_{\mathcal{H}} = \int \left(\overline{\psi_1}(x)\phi_1(x) + \overline{\psi_2}(x)\phi_2(x)\right) d^2x \ ,$$

where ψ_i, ϕ_j are the components of the spinors ψ, ϕ. The Dirac operator for this space is built using the Γ matrices and the operators $\hat{P}_\mu = -i\partial_\mu$ as follows

$$D = \Gamma^\mu \hat{P}_\mu = -\begin{pmatrix} 0 & i\partial_0 + \partial_1 \\ i\partial_0 - \partial_1 & 0 \end{pmatrix} .$$

The Dirac operator D is essentially self-adjoint with respect to the inner product defined above. Since the dimension is even there is a self-adjoint operator χ, called the grading, which satisfies $\chi^2 = 1$ and the following properties: it commutes with the algebra, that is for any $f \in \mathcal{A}$ we have $[\chi, \pi(f)] = 0$, and it anticommutes with the Dirac operator, that is $\{\chi, D\} = 0$. In terms of the Γ matrices it is given by

$$\chi = -i\Gamma^0\Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

One of the requirements in the definition of a spectral triple is that $[D, \pi(f)]$, for $f \in \mathcal{A}$, should extend to a bounded operator in $\mathcal{H}$. From its definition we obtain $[D, \pi(f)] = \Gamma^\mu[\hat{P}_\mu, \pi(f)]$. Using the Leibnitz rule for the derivatives we obtain

$$\hat{P}_\mu \pi(f)\psi = (\hat{P}_\mu f)\psi + f(\hat{P}_\mu \psi) = \pi(\hat{P}_\mu f)\psi + \pi(f)\hat{P}_\mu \psi \ .$$

The previous equation can be rewritten in the form $[D, \pi(f)] = \Gamma^\mu \pi(\hat{P}_\mu f)$. This is a bounded operator, since $\pi(f)$ is bounded for $f \in \mathcal{A}$ and $\hat{P}_\mu f \in \mathcal{A}$, since f is a Schwartz function.

Now let us discuss the deformed case. We refer to the Hilbert space introduced in the previous section as the reduced Hilbert space, and we denote it by $\mathcal{H}_r$. The algebra $\mathcal{A}$ is represented on $\mathcal{H}_r$ by left multiplication, that is $(\pi(f)\psi)(x) = (f \star \psi)(x)$. To consider spinors we define the Hilbert space $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^2$, corresponding to the trivial spinor bundle. The representation π of $\mathcal{A}$ is trivially extended to the two copies of $\mathcal{H}_r$, and we denote it again by π. The inner product of two spinors $\psi, \phi \in \mathcal{H}$ is given by

$$(\psi, \phi)_{\mathcal{H}} = \int \left((\psi_1^* \star \phi_1)(x) + (\psi_2^* \star \phi_2)(x)\right) d^2x \ .$$

As a first attempt, we can try to use the classical Dirac operator $D = \Gamma^\mu \hat{P}_\mu$. Then, as in the commutative case, we have $[D, \pi(f)] = \Gamma^\mu[\hat{P}_\mu, \pi(f)]$. The difference with the commutative case comes from the fact that the representation π is not pointwise multiplication. Indeed, using the fact that the representation π is $\mathcal{T}_\kappa$-equivariant, we obtain

$$\hat{P}_1 \pi(f)\psi = \rho(P_1)\pi(f)\psi = \pi(P_1 \triangleright f)\psi + \pi(\mathcal{E} \triangleright f)\rho(P_1)\psi \ .$$

As a consequence of the non-trivial coproduct of the κ-Poincaré algebra, P_1 does not obey the Leibnitz rule. Then the commutator $[D, \pi(f)]$ is not bounded, since $\rho(P_1)$ is an unbounded operator. Explicitly we have

$$[D, \pi(f)] = \Gamma^\mu \pi(P_\mu \triangleright f) + \Gamma^1 \pi((\mathcal{E} - 1) \triangleright f)\rho(P_1) \ .$$

One can hope to evade this problem by considering a different Dirac operator D. To have a good classical limit we require that, in some technical sense that we will specify later, in the limit $\lambda \to 0$ this Dirac operator D reduces to the classical one. Since we are dealing with a non-commutative space associated to a quantum group, it is natural, as argued for example in [KrWa11, D'An07], to consider this problem in the framework of twisted spectral triples. Then we require the boundedness of

$$[D, \pi(f)]_\sigma = D\pi(f) - \pi(\sigma(f))D \ .$$

Here σ is an automorphism of the algebra $\mathcal{A}$. Since the algebra $\mathcal{A}$ is involutive one also requires the compatibility property $\sigma(f)^* = \sigma^{-1}(f^*)$, for $f \in \mathcal{A}$. In the next subsection we investigate what are the possible choices for the Dirac operator D and the automorphism σ.

5.4.2 The deformed Dirac operator D

Now we state our assumptions for the Dirac operator D and the automorphism σ. The first assumption is of general nature, that is we require that D is self-adjoint on $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^2$ and that it anticommutes with the grading χ. This implies that D is of the form $D = \Gamma^\mu \hat{D}_\mu$, where $\hat{D}_\mu$ are self-adjoint operators on $\mathcal{H}_r$. The second assumption is that, in the limit $\lambda \to 0$, D and σ should reduce respectively to the classical Dirac operator and the identity. The technical statement of this assumption is going to be given later. Next, motivated by the fact that D and σ should be determined by the symmetries, we assume that $\hat{D}_\mu = \rho(D_\mu)$, for some $D_\mu \in \mathcal{T}_\kappa$, and that the automorphism σ is given by $\sigma(f) = \sigma \triangleright f$, for some $\sigma \in \mathcal{T}_\kappa$. The requirement that σ is an automorphism implies that its coproduct is $\Delta(\sigma) = \sigma \otimes \sigma$.

We recall that we do not consider any topology on $\mathcal{T}_\kappa$, therefore any element can be written as a *finite* sum of products of generators. Since the algebra is commutative, any such element is a sum of terms of the form $P_0^i P_1^j \mathcal{E}^k$, with $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$. Notice that, in our definition of $\mathcal{T}_\kappa$, the element $\mathcal{E}$ is not the formal series $e^{-\lambda P_0}$ in P_0, as usually given in the defining relations of κ-Poincaré, but is considered as one of the generators of the algebra.

Lemma 5.15. *Suppose that $A, \sigma \in \mathcal{T}_\kappa$. Then $[\rho(A), \pi(f)]_\sigma = \rho(A)\pi(f) - \pi(\sigma \triangleright f)\rho(A)$ is bounded if and only if the coproduct of A is $\Delta(A) = A' \otimes 1 + \sigma \otimes A$, for some $A' \in \mathcal{T}_\kappa$.*

Proof. Using the fact that the representation π is equivariant we have

$$\begin{aligned}[\rho(A), \pi(f)]_\sigma &= \rho(A)\pi(f) - \pi(\sigma \triangleright f)\rho(A) \\ &= \pi(A_{(1)} \triangleright f)\rho(A_{(2)}) - \pi(\sigma \triangleright f)\rho(A) \ .\end{aligned}$$

For some $B_i, C_j \in \mathcal{T}_\kappa$ this can be rewritten in the form

$$[\rho(A), \pi(f)]_\sigma = \sum_{ij} \pi(B_i \triangleright f)\rho(C_j) \ . \tag{5.9}$$

Now we show that the operator $\rho(C_j)$ is bounded if and only if C_j is a multiple of the unit. Indeed, from their definition, we have that $\rho(P_\mu)$ and $\rho(\mathcal{E})$ are unbounded operators, while the unit corresponds to the identity operator. Using our previous remark on the structure of $\mathcal{T}_\kappa$, we have that a general operator $\rho(h)$, with $h \in \mathcal{T}_\kappa$, can be written in the form

$$\rho(h) = \sum_{ijk} c_{ijk}\rho(P_0^i)\rho(P_1^j)\rho(\mathcal{E}^k) \ .$$

This is unbounded unless all its coefficients are zero, except possibly for c_{000}. Then it follows from equation (5.9) that $[\rho(A), \pi(f)]_\sigma$ is bounded if and only if all the elements C_j are multiples of the unit. Setting $C_j = c_j 1$, with $c_j \in \mathbb{C}$, we can rewrite equation (5.9) as

$$[\rho(A), \pi(f)]_\sigma = \pi(A' \triangleright f) \ , \qquad A' := \sum_{ij} c_j B_i \ .$$

Using the definition of the twisted commutator, we have that $[\rho(A), \pi(f)]_\sigma = \pi(A' \triangleright f)$ implies

$$\rho(A)\pi(f) = \pi(A' \triangleright f) + \pi(\sigma \triangleright f)\rho(A) \ .$$

Finally, using the equivariance of π, this implies that $\Delta(A) = A' \otimes 1 + \sigma \otimes A$. $\square$

The previous lemma tells us that the requirement of boundedness of the twisted commutator severly restricts the form of the coproduct of D_μ. Now we characterize which elements of $\mathcal{T}_\kappa$ have a coproduct of this form.

Lemma 5.16. *Consider an element $A \in \mathcal{T}_\kappa$. Suppose that $\Delta(A) = A' \otimes 1 + \sigma \otimes A$ for some $A', \sigma \in \mathcal{T}_\kappa$, and that $\Delta(\sigma) = \sigma \otimes \sigma$. Then we have $\sigma = \mathcal{E}^m$, for some $m \in \mathbb{Z}$, and for some coefficients $c_j \in \mathbb{C}$ we have that A can be written as*

- $c_1 1 + c_2 \mathcal{E}^m$ *if $m < 0$,*
- $c_1 1 + c_2 P_0$ *if $m = 0$,*

- $c_1 1 + c_2 \mathcal{E} + c_3 P_1$ *if* $m = 1$,
- $c_1 1 + c_2 \mathcal{E}^m$ *if* $m > 1$.

Proof. First of all it is clear that if $\sigma \in \mathcal{T}_\kappa$ is such that $\Delta(\sigma) = \sigma \otimes \sigma$, then we must have $\sigma = \mathcal{E}^m$ for some $m \in \mathbb{Z}$. Any element $A \in \mathcal{T}_\kappa$ is a finite sum of elements of the form $P_0^i P_1^j \mathcal{E}^k$, with $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$, and we can write it as

$$A = \sum_{ijk} c_{ijk} P_0^i P_1^j \mathcal{E}^k = \sum_{ijk} A_{ijk} \ .$$

The request $\Delta(A) = A' \otimes 1 + \sigma \otimes A$ puts a constraint on the possible terms that appear in the sum. A term A_{ijk} is allowed if and only if its coproduct is of the form $\Delta(A_{ijk}) = B_{ijk} \otimes 1 + \sigma \otimes C_{ijk}$, for some $B_{ijk}, C_{ijk} \in \mathcal{T}_\kappa$. Now we discuss which terms are of this form.

Consider first the generator P_0. Then P_0^i is allowed for $i = 0, 1$, since we have $\Delta(1) = 1 \otimes 1$ and $\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0$. For higher powers we get cross terms which cannot be of the allowed form, for example if we compute the coproduct for $i = 2$ we find

$$\Delta(P_0^2) = P_0^2 \otimes 1 + 2 P_0 \otimes P_0 + 1 \otimes P_0^2 \ .$$

The same argument applies to P_1, for which we have $\Delta(P_1) = P_1 \otimes 1 + \mathcal{E} \otimes P_1$. Higher powers are not allowed as in the case of P_0. For $\mathcal{E}$ on the other hand any power is acceptable, since $\Delta(\mathcal{E}^k) = \mathcal{E}^k \otimes \mathcal{E}^k$ is an automorphism. Now we consider the mixed terms. Any term of the form $P_0^i P_1^j$ for $i, j \geq 1$ is not allowed because of the cross terms. For example for $P_0 P_1$ we have

$$\Delta(P_0 P_1) = P_0 P_1 \otimes 1 + \mathcal{E} P_0 \otimes P_1 + P_1 \otimes P_0 + \mathcal{E} \otimes P_0 P_1 \ .$$

Similarly mixed terms like $P_0^i \mathcal{E}^k$ or $P_1^j \mathcal{E}^k$ are not allowed for $i, j, k \geq 1$. For example we have

$$\begin{aligned} \Delta(P_0 \mathcal{E}^k) &= \mathcal{E}^k P_0 \otimes \mathcal{E}^k + \mathcal{E}^k \otimes \mathcal{E}^k P_0 \ , \\ \Delta(P_1 \mathcal{E}^k) &= \mathcal{E}^k P_1 \otimes \mathcal{E}^k + \mathcal{E}^k \otimes \mathcal{E}^k P_1 \ . \end{aligned}$$

Now we discuss which terms are compatible when we fix $\sigma = \mathcal{E}^m$, for some $m \in \mathbb{Z}$. The coproduct of A_{ijk} must have the form $\Delta(A_{ijk}) = B_{ijk} \otimes 1 + \mathcal{E}^m \otimes C_{ijk}$. The unit and $\mathcal{E}^m$ satisfy this requirement for any $m \in \mathbb{Z}$. They are the only possible terms for $m < 0$ and $m > 1$. On the other hand for $m = 0$ we can also have P_0, while for $m = 1$ we can also have P_1. □

Now it is time to discuss the requirement of the classical limit for D. First of all we need to recall that the parameter λ is a *physical constant* of the model, which has the physical dimension of a length. Since also the coordinates x_μ have the dimensions of a length, it follows that the Dirac operator must have dimension $[D] = -1$, where by $[A]$ we denote the physical dimension of A in units of length. From this observation it follows that the generators of $\mathcal{T}_\kappa$

have physical dimensions $[P_\mu] = -1$, $[\mathcal{E}] = 0$ and the unit has $[1] = 0$. Now we can give the technical statement of the assumption of the classical limit.

Definition 5.7. We say that the Dirac operator $D = \Gamma^\mu \hat{D}_\mu$ obeys the *classical limit* if for all $\psi \in \mathcal{A}$ we have $\lim \hat{D}_\mu \psi = \hat{P}_\mu \psi$ for $\lambda \to 0$, and moreover $[\hat{D}_\mu] = -1$.

Now that we have stated all the assumptions we can prove the following theorem.

Theorem 5.17. *Suppose that $D_\mu, \sigma \in \mathcal{T}_\kappa$ and that the Dirac operator $D = \Gamma^\mu \rho(D_\mu)$ obeys the classical limit. Then the twisted commutator*

$$[D, \pi(f)]_\sigma = \Gamma^\mu \left(\rho(D_\mu)\pi(f) - \pi(\sigma \triangleright f)\rho(D_\mu) \right)$$

is bounded if and only if we have $D_0 = \frac{1}{\lambda}(1 - \mathcal{E})$, $D_1 = P_1$ and $\sigma = \mathcal{E}$.

Proof. Using Lemma 5.15 we have that $[D, \pi(f)]_\sigma$ is bounded if and only if $\Delta(D_\mu) = D'_\mu \otimes 1 + \sigma \otimes D_\mu$, for some $D'_\mu \in \mathcal{T}_\kappa$. Such elements are classified by Lemma 5.16, depending on $m \in \mathbb{Z}$. The condition that the operators $\hat{D}_\mu$ obey the classical limit imposes $m = 1$. Indeed only for this choice we have the element P_1, which corresponds to the operator $\rho(P_1) = -i\partial_1$.

Now we need to consider the restriction on the coefficients, imposed again by the classical limit, for $m = 1$. Let us start with D_0, for which we have

$$(\rho(D_0)\psi)(x) = \lambda^{-1} \left(c_1 \psi(x) + c_2 \psi(x_0 - i\lambda, x_1) \right) - i c_3 (\partial_1 \psi)(x) \ .$$

We have rescaled the coefficients in such a way that $c_k \in \mathbb{C}$ are dimensionless. The operator $\rho(D_0)$ should reduce to $-i\partial_0$ in the limit $\lambda \to 0$. We see immediately that we must have $c_3 = 0$, since this term is not affected by the limit. To see what are the requirements on the remaining two coefficients, we expand ψ in Taylor series in λ, which is allowed since $\psi \in \mathcal{A}$ is analytic in the first variable. We have

$$(\rho(D_0)\psi)(x) = \lambda^{-1}(c_1 + c_2)\psi(x) + i c_2 (\partial_0 \psi)(x) + \lambda^{-1} c_2 \sum_{n=2}^{\infty} (\partial_0^n \psi)(x) \frac{(i\lambda)^n}{n!} \ .$$

The first term diverges in the limit $\lambda \to 0$ unless $c_1 = -c_2$, which we must require. The second term on the other hand is not affected by this limit, and forces us to choose $c_2 = -1$. For the third term, after exchanging the limit with the series, we find that it vanishes for $\lambda \to 0$. Therefore the classical limit fixes $D_0 = \lambda^{-1}(1 - \mathcal{E})$.

We can repeat the same argument for D_1, for which we write

$$(\rho(D_1)\psi)(x) = \lambda^{-1} \left(c'_1 \psi(x) + c'_2 \psi(x_0 - i\lambda, x_1) \right) - i c'_3 (\partial_1 \psi)(x) \ .$$

This operator must reduce to $-i\partial_1$ in the limit $\lambda \to 0$. From the previous discussion it is obvious that we must have $c'_1 = c'_2 = 0$. Finally the third coefficient is the same as in the commutative case, that is $c'_3 = 1$, therefore $D_1 = P_1$. ∎

The uniqueness of D and σ, under the symmetry and classical limit assumptions, is the main result of this section. We have a nice compatibility between the twisting of the commutator and the modular properties of the algebra. Indeed the modular operator Δ_ω, associated to the weight ω, implements the twist in the Hilbert space $\mathcal{H}$, that is $\pi(\sigma(f)) = \Delta_\omega \pi(f) \Delta_\omega^{-1}$. Moreover σ is the analytic extension at $t = -i$ of σ_t^ω, the modular group of ω, compare with the discussion on the index map given in [CoMo08].

Another interesting feature is the simple relation between D^2, the square of the Dirac operator, and C, the *first Casimir* of the κ-Poincaré algebra (more precisely of the quantum Euclidean group, since we are in Euclidean signature). The latter is given by

$$C = \frac{4}{\lambda^2} \sinh^2 \left(\frac{\lambda \hat{P}_0}{2} \right) + e^{\lambda \hat{P}_0} \hat{P}_1^2 \ . \tag{5.10}$$

Then an elementary computation shows that we have the relation $D^2 = \Delta_\omega C$, where Δ_ω is the modular operator of the weight ω. Apart from the presence of the modular operator, this is the same property that one has in the commutative case. This connection is made more suggestive if, following [D'An07], we rewrite the twisted commutator in the form

$$K^{-1} \left(D' \pi(a) - (K^{-1} \pi(a) K) D' \right) \ .$$

The usual definition of the twisted commutator is obtained by setting $D = K^{-1} D'$ and $\pi(\sigma(a)) = K^{-2} \pi(a) K^2$. From the previous remark it follows that $K = \Delta_\omega^{-1/2}$, so we have $(D')^2 = C$ and D' is exactly the square root of the Casimir operator.

We remark that the operator D, which we introduced in this section, serves the purpose of describing the geometry of κ-Minkowski space from the spectral point of view. This is in principle distinct from the operator one introduces to describe the physical properties of fermions, which is the one that deserves to be called the Dirac operator. This has been studied in the literature, see for example [NST93, AAA04] and also [D'An06]. Mathematically such an operator is required to be equivariant (or covariant, in more physical terms) under the κ-Poincaré algebra (the quantum Euclidean group, in Euclidean signature). It is given by

$$D^{eq} = \Gamma^0 \left(\frac{1}{\lambda} \sinh(\lambda \hat{P}_0) - \frac{\lambda}{2} e^{\lambda \hat{P}_0} \hat{P}_1^2 \right) + \Gamma^1 e^{\lambda \hat{P}_0} \hat{P}_1 \ .$$

In our notations it can be written as $D^{eq} = \Gamma^\mu \rho(D_\mu^{eq})$, where

$$D_0^{eq} = \frac{1}{2\lambda} (\mathcal{E}^{-1} - \mathcal{E}) - \frac{\lambda}{2} \mathcal{E}^{-1} P_1^2 \ , \quad D_1^{eq} = \mathcal{E}^{-1} P_1 \ .$$

By our previous results it follows that such an operator does not have a twisted bounded commutator. A relevant algebraic property that D^{eq} satisfies is $(D^{eq})^2 = C + \frac{\lambda^2}{4} C^2$, where C is again the Casimir of the κ-Poincaré algebra. Differently from the commutative case, we do not have that the operator D^{eq} is the square root of the Casimir operator C. On the other hand, as we remarked above, this role is essentially played by the operator D.

The most obvious difference between D and D^{eq} is that the former is equivariant only under the extended momentum algebra $\mathcal{T}_\kappa$, while the latter is equivariant under the full κ-Poincaré algebra $\mathcal{P}_\kappa$. However we point out that only the subalgebra $\mathcal{T}_\kappa$ is relevant for the introduction of κ-Minkowski space, so at least the minimal requirement of equivariance under $\mathcal{T}_\kappa$ is satisfied. Moreover, for any $h \in \mathcal{P}_\kappa$, we have the interesting property that the twisted commutator of D^2 with $\rho(h)$ is zero. This follows from a one-line computation

$$[D^2, \rho(h)]_\sigma = D^2 \rho(h) - \Delta_\omega \rho(h) \Delta_\omega^{-1} D^2 = \Delta_\omega (C\rho(h) - \rho(h)C) = 0 \ .$$

5.5 The spectral dimension

In this section we study the property of summability of our spectral triple. We will show that it is not finitely summable in usual sense of spectral triples, but it is finitely summable if we adapt some definitions from the framework of *modular spectral triples*. This is an extension of the concept of spectral triple, introduced among other reasons to handle algebras having a KMS state. Since, as we have seen in the previous sections, there is a natural KMS weight on the algebra $\mathcal{A}$, it seems appropriate to use these tools in this case.

The main result of this section is that the spectral dimension, computed using the weight Φ, exists and is equal to the classical dimension two. Moreover the residue at $s = 2$ of the function $\Phi\left(\pi(f)(D^2 + \mu^2)^{-s/2}\right)$, for $f \in \mathcal{A}$ and $\mu \neq 0$, exists and gives $\omega(f)$ up to a constant, which shows that we recover the notion of integration given by ω using the operator D.

5.5.1 A problem with finite summability

The concept of finite summability for a non-unital spectral triple is far more subtle than in the unital case, see [CGRS12] for a detailed discussion of some of the issues arising. We just point out that, while in the unital case the definition of the operator D is enough to characterize the spectral dimension, in the non-unital case one needs a delicate interplay between D and the algebra $\mathcal{A}$. We consider the notions of summability given in [CGRS12].

Definition 5.8. Let $(\mathcal{A}, \mathcal{H}, D)$ be a non-compact spectral triple. We say that it is finitely summable and call p the spectral dimension if the following quantity exists

$$p := \inf\{s > 0 : \forall a \in \mathcal{A}, a \geq 0,\ \mathrm{Tr}\left(\pi(a)(D^2 + 1)^{-s/2}\right) < \infty\} \ .$$

In addition we say that $(\mathcal{A}, \mathcal{H}, D)$ is $\mathcal{Z}_p$-summable if for all $a \in \mathcal{A}$ we have

$$\limsup_{s \downarrow p} \left| (s - p) \mathrm{Tr}\left(\pi(a)(D^2 + 1)^{-s/2}\right) \right| < \infty \ .$$

Now we show that our spectral triple is not finitely summable in this sense.

Proposition 5.18. *Let $h \in \mathcal{A}$ such that $h = f \star f$ with $f > 0$. Then the operator $\pi(h)(D^2 + 1)^{-s/2}$ is not trace class for any $s > 0$. In other words, the spectral triple is not finitely summable.*

Proof. We have that if $\pi(h)(D^2+1)^{-s/2}$ is trace class then also $\pi(f)(D^2+1)^{-s/2}\pi(f)$ is trace class, while the converse statement is not true in general, see the discussion in [CGRS12]. Proving that $\pi(f)(D^2+1)^{-s/2}\pi(f)$ is trace class is the same as proving that $\pi(f)(D^2+1)^{-s/4}$ is Hilbert-Schmidt, which is easy to check using the integral formula for the kernel. Now we show that $\pi(f)(D^2+1)^{-s/4}$ is not Hilbert-Schmidt for any $s > 0$, from which the proposition follows.

We have that the Hilbert-Schmidt norm of $\pi(f)(D^2 + 1)^{-s/4}$, as an operator on $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^2$, is equal to the Hilbert-Schmidt norm of $A := U\pi(f)(D^2+1)^{-s/4}U^{-1}$ as an operator on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$. Using Proposition 5.14 we find that the Schwartz kernel of A is given by

$$K_A(x, y) = \int e^{ip(x-y)}(Uf)(x_0, e^{\lambda p_0}x_1)G_s(p_0, e^{-\lambda p_0}p_1)\frac{d^2p}{(2\pi)^2} \ ,$$

where the function G_s is defined by

$$G_s(p) := \Big(\lambda^{-2}(1 - e^{-\lambda p_0})^2 + p_1^2 + 1\Big)^{-s/4} \ .$$

For fixed x define the function $h_x(p) := (Uf)(x_0, e^{\lambda p_0}x_1)G_s(p_0, e^{-\lambda p_0}p_1)$. With this definition we can write the kernel K_A as an inverse Fourier transform

$$K_A(x, y) = \int e^{ip(x-y)}h_x(p)\frac{d^2p}{(2\pi)^2} = (\mathcal{F}^{-1}h_x)(x - y) \ .$$

Now it is easy to compute the Hilbert-Schmidt norm of A. We have

$$\begin{aligned}\|A\|_2^2 &= 2\int\int |K_A(x, y)|^2 d^2x d^2y = 2\int\int |(\mathcal{F}^{-1}h_x)(x - y)|^2 d^2x d^2y \\ &= 2\int\int |(\mathcal{F}^{-1}h_x)(y)|^2 d^2x d^2y \ .\end{aligned}$$

The factor 2 comes from the dimension of the spinor bundle, since $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^2$. Now, using the fact that the Fourier transform is a unitary operator in $L^2(\mathbb{R}^2)$ (up to the factor $(2\pi)^2$, in our conventions), we can rewrite the previous expression as

$$\|A\|_2^2 = 2\int\int |h_x(p)|^2 d^2x\frac{d^2p}{(2\pi)^2} = 2\int\int |(Uf)(x_0, e^{\lambda p_0}x_1)G_s(p_0, e^{-\lambda p_0}p_1)|^2 d^2x\frac{d^2p}{(2\pi)^2} \ .$$

After the change of variables $x_1 \to e^{-\lambda p_0} x_1$, $p_1 \to e^{\lambda p_0} p_1$ we find

$$\|A\|_2^2 = 2 \int |(Uf)(x)|^2 d^2x \int |g_s(p)|^2 \frac{d^2 p}{(2\pi)^2} = \frac{2}{(2\pi)^2} \|Uf\|_2^2 \|G_s\|_2^2 \,. \tag{5.11}$$

Now consider the norm $\|G_s\|_2$, which is given by the expression

$$\|G_s\|_2^2 = \int \left(\lambda^{-2} (1 - e^{-\lambda p_0})^2 + p_1^2 + 1 \right)^{-s/2} d^2 p \,.$$

This function is not integrable for any $s > 0$ (the integrand does not go to zero for $p_0 \to \infty$). Therefore the operator $\pi(f)(1 + D^2)^{-s/4}$ is not Hilbert-Schmidt for any $s > 0$. □

Now we argue that using the framework of twisted spectral triples is not enough to describe the non-commutative geometry of κ-Minkowski space, and that some more refined notion is needed to capture the modular properties associated to this geometry. First we recall a result obtained in [CoMo08]: consider a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with twist σ and such that $D^{-1} \in \mathcal{L}^{n+}$. Define the linear functional $\varphi(a) = \mathrm{Tr}_\omega(\pi(a) D^{-n})$, where Tr_ω is the Dixmier trace. Then for any $a, b \in \mathcal{A}$ we have $\varphi(ab) = \varphi(\sigma^n(b) a)$. Putting aside the issues of the non-compact case, which are not needed for this heuristic argument, having a spectral dimension $n = 2$ in our case would imply $\varphi(f \star g) = \varphi(\sigma^2(g) \star f)$. The KMS condition for ω, on the other hand, can be rewritten in terms of the twist σ and reads $\omega(f \star g) = \omega(\sigma(g) \star f)$. Since we expect the Dixmier trace to be linked to the weight ω, we see that there is a discrepancy between the two notions of integration, which are due to their different modular properties.

5.5.2 Modular spectral triples

As we have seen, the spectral triple we constructed is not finitely summable in the usual sense. We now want to reconsider it in the framework of modular spectral triples. The main point of the modular version of a spectral triple is to enable the use of a weight, instead of the operator trace, to measure the growth of the resolvent of the operator D. In particular the spectral triple could be finitely summable in this sense. We also recall that a modification of this idea involving twisted commutators has been considered in [KaSe12, Kaa11].

As we argued at the end of the previous subsection, there is a mismatch between the modular properties of the non-commutative integral and the weight ω. Therefore we cannot recover the weight ω, which gives a basic notion of integration on the algebra. We can try to correct this mismatch by choosing an appropriate weight in the sense of modular spectral triples. Since the twist σ is implemented by Δ_ω, the modular operator of ω, this fact hints to the possibility of correcting it by considering the weight $\Phi(\cdot) := \mathrm{Tr}(\Delta_\omega \cdot)$.

We now discuss an issue that arises by considering non-unital algebras. In Chapter 4 we have given the definition of modular spectral triples which is relevant for the unital case. This definition can be, in principle, modified as in the case of spectral triples, which is discussed in Appendix A. However, if we want to mantain the resolvent condition in terms of the semifinite trace, we need to consider multiplication only by elements which are invariant under the action of the modular group, that is which belong to the fixed point algebra.

In the case we are considering, the action of the modular group of the weight ω is given by $(\sigma^\omega_t f)(x) = f(x_0 - \lambda t, x_1)$, that is translation in the first variable. Therefore the fixed points of this action are functions which are constant in the first variable. Then, since the functions in $\mathcal{A}$ vanish at infinity, the only such fixed point is given by the zero function.

We recall that the notion of modular spectral triple has been formalized on the basis of examples where the action of the modular group is periodic. In this situation, since we can define a conditional expectation, it makes sense to consider the resolvent condition relative to the fixed point algebra under this action. This fact implies that, strictly speaking, our construction does not fit into this framework.

Here we do not dwell on what should be the correct condition in this case. Instead, we simply adapt the corresponding notion of spectral dimension to our needs.

Definition 5.9. Let $(\mathcal{A}, \mathcal{H}, D)$ be a non-compact modular spectral triple with weight Φ. We say that it is finitely summable and call p the spectral dimension if the following quantity exists

$$p := \inf\{s > 0 : \forall a \in \mathcal{A}, a \geq 0,\ \Phi\left(\pi(a)(D^2+1)^{-s/2}\right) < \infty\} \ .$$

Notice that we do not require the elements $a \in \mathcal{A}$ to be in the fixed point algebra.

The main result of this section is that the spectral dimension, computed according to this definition, exists and is equal to the classical dimension two. Moreover the residue at $s = 2$ of the function $\Phi\left(\pi(f)(D^2+\mu^2)^{-s/2}\right)$, for $f \in \mathcal{A}$ and $\mu \neq 0$, exists and gives $\omega(f)$ up to a constant, which shows that we recover the notion of integration given by ω using the operator D. This gives an analogue of the $\mathcal{Z}_2$-summability condition, and is in line with similar results obtained for the modular spectral triples studied so far.

Before starting the computation we note the following easy but useful lemma.

Lemma 5.19. *For all $f \in \mathcal{A}$ we have $\Phi\left(\pi(f)(D^2+1)^{-s/2}\right) = \mathrm{Tr}\left(\pi(\sigma^\omega_{-i} f)\Delta_\omega(D^2+1)^{-s/2}\right)$.*

Proof. It is easily proven by the following computation

$$\begin{aligned}\Phi\left(\pi(f)(D^2+1)^{-s/2}\right) &= \mathrm{Tr}\left(\Delta_\omega \pi(f)\Delta_\omega^{-1}\Delta_\omega(D^2+1)^{-s/2}\right)\\ &= \mathrm{Tr}\left(\pi(\sigma^\omega_{-i} f)\Delta_\omega(D^2+1)^{-s/2}\right) \ .\end{aligned}$$

In the last line we have used the fact that σ^ω is implemented by the modular operator Δ_ω, that is $\pi(\sigma^\omega_t f) = \Delta^{it}_\omega \pi(f)\Delta^{-it}_\omega$ for any $f \in \mathcal{A}$. □

The next subsection is devoted to proving the results announced above.

5.5.3 The spectral dimension with the weight Φ

We can restrict our attention to the operator $\pi(f)\Delta_\omega(D^2+1)^{-s/2}$ on $\mathcal{H}$ and, via the unitary operator U, to the operator $U\pi(f)\Delta_\omega(D^2+1)^{-s/2}U^{-1}$ on $L^2(\mathbb{R}^2)\otimes\mathbb{C}^2$. Now we want to see if this operator is trace class for some $s>0$ and compute its trace. To prove that an operator A on $L^2(\mathbb{R}^n)$ is trace class, a possible strategy is to show that it is a pseudo-differential operator of sufficiently negative order (see for example [Shu01, Chapter IV]). We say that A is a pseudo-differential operator of order m if its symbol a satisfies the condition $|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\leq c_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}$, where $c_{\alpha\beta}$ are constants and we use the multi-index notation. However this class of symbols is not well adapted to the present situation, as we now argue.

Using Proposition 5.14 we know that the symbol of $U\pi(f)\Delta_\omega(D^2+1)^{-s/2}U^{-1}$ is of the form $(Uf)(x_0,e^{\lambda\xi_0}x_1)g(\xi)$, for some function g. Now consider the derivative with respect to x_1, which is given by $e^{\lambda\xi_0}(\partial_1 Uf)(x_0,e^{\lambda\xi_0}x_1)g(\xi)$. By examining the behaviour at $\xi_0\to\infty$ we see that we can not bound this function uniformly in x_1. Consider first the case $x_1\neq 0$: by defining $y_1=e^{\lambda\xi_0}x_1$, we have $x_1^{-1}y_1(\partial_1 Uf)(x_0,y_1)$ and this goes to zero for $y_1\to\pm\infty$, since Uf is a Schwartz function. For $x_1=0$, however, we get $e^{\lambda\xi_0}(\partial_1 Uf)(x_0,0)$, which grows exponentially in ξ_0. This implies that at $x_1=0$ we can not satisfy the condition for a pseudo-differential operator of negative order, and so we can not use the related results.

Since we have this problem only at $x_1=0$, which is a set of measure zero in $\mathbb{R}^2$, we can expect to be able to overcome this problem by using a criterion which involves an L^1 condition on the symbol. To this end we are going to use the following theorem given in [Ars08]. Let A be an operator in $L^2(\mathbb{R}^n)$ with symbol $a(x,\xi)$. If the symbol satisfies the condition $\partial_x^\alpha\partial_\xi^\beta a\in L^p(\mathbb{R}^n\times\mathbb{R}^n)$ for $|\alpha|,|\beta|\leq[n/2]+1$, where $1\leq p<\infty$, then A belongs to the p-th Schatten ideal in $L^2(\mathbb{R}^n)$. Here we are using the multi-index notation for α,β and $[n]$ denotes the integer part of n. Using this result we can prove the following.

Theorem 5.20. *Let $f\in\mathcal{A}$ and $\mu\neq 0$. Then the operator $\pi(f)\Delta_\omega(D^2+\mu^2)^{-s/2}$ is trace class for $s>2$. In particular we have spectral dimension $p=2$ according to Definition 5.9.*

Proof. Using Proposition 5.14 we have that the symbol of the operator $U\pi(f)\Delta_\omega(D^2+\mu^2)^{-s/2}U^{-1}$ is given by $a(x,\xi):=(Uf)(x_0,e^{\lambda\xi_0}x_1)G_s^\Delta(\xi_0,e^{-\lambda\xi_0}\xi_1)$, where

$$G_s^\Delta(\xi):=e^{-\lambda\xi_0}\left(\lambda^{-2}(1-e^{-\lambda\xi_0})^2+\xi_1^2+\mu^2\right)^{-s/2}.$$

As we remarked above, to prove that this operator is trace class we are going to show that the symbol satisfies the condition $\partial_x^\alpha\partial_\xi^\beta a\in L^1(\mathbb{R}^2\times\mathbb{R}^2)$, for $|\alpha|,|\beta|\leq 2$. Let us start by showing that a is integrable. Using a change of variables we get

$$\begin{aligned}\int\int|a(x,\xi)|d^2xd^2\xi&=\int\int|(Uf)(x_0,e^{\lambda\xi_0}x_1)|G_s^\Delta(\xi_0,e^{-\lambda\xi_0}\xi_1)d^2xd^2\xi\\&=\int|(Uf)(x)|d^2x\int G_s^\Delta(\xi)d^2\xi=\|Uf\|_1\|G_s^\Delta\|_1 .\end{aligned}$$

We have that $\|Uf\|_1$ is finite since Uf is a Schwartz function. To prove that $\|G_s^\Delta\|_1$ is finite we need to consider the asymptotic behaviour of the function $G_s^\Delta(\xi)$. This is given by

$$\begin{aligned} G_s^\Delta(\xi) &\sim e^{-\lambda\xi_0}|\xi_1|^{-s} , & \xi_0 \to \infty \ , \ |\xi_1| \to \infty \ , \\ G_s^\Delta(\xi) &\sim e^{-\lambda\xi_0}\left(e^{-2\lambda\xi_0} + \xi_1^2\right)^{-s/2} \ , & \xi_0 \to -\infty \ , \ |\xi_1| \to \infty \ . \end{aligned}$$

For $\xi_0 \to \infty$ this function is integrable if $s > 1$. To study the other case we use the integral

$$\int (c^2 + x^2)^{-s/2} dx = \sqrt{\pi} \frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} (c^2)^{-\frac{s-1}{2}} \ .$$

Then the function G_s^Δ is integrable for $\xi_0 \to -\infty$ is $s > 2$, as we have

$$\int G_s^\Delta(\xi) d\xi_1 \sim e^{-\lambda\xi_0} \int \left(e^{-2\lambda\xi_0} + \xi_1^2\right)^{-s/2} d\xi_1 \sim e^{-\lambda\xi_0} e^{(s-1)\lambda\xi_0} \ .$$

Now we consider the derivatives in x and show that the integral of $\partial_x^\alpha a$ vanishes, since Uf is a Schwartz function. In the following we use the notation ∂_0 and ∂_1 to denote derivatives with respect to the first and second variable. For the derivatives in x_0 we have

$$\int \partial_{x_0}^n (Uf)(x_0, e^{\lambda\xi_0}x_1) dx_0 = (\partial_0^{n-1}Uf)(\infty, e^{\lambda\xi_0}x_1) - (\partial_0^{n-1}Uf)(-\infty, e^{\lambda\xi_0}x_1) = 0 \ .$$

Similarly for the derivatives in x_1 we obtain

$$\begin{aligned} \int \partial_{x_1}^n (Uf)(x_0, e^{\lambda\xi_0}x_1) dx_1 &= e^{n\lambda\xi_0} \int (\partial_1^n Uf)(x_0, e^{\lambda\xi_0}x_1) dx_1 = e^{(n-1)\lambda\xi_0} \int (\partial_1^n Uf)(x) dx_1 \\ &= e^{(n-1)\lambda\xi_0} \left((\partial_1^{n-1}Uf)(x_0, \infty) - (\partial_1^{n-1}Uf)(x_0, -\infty)\right) = 0 \ . \end{aligned}$$

Now we consider the derivatives with respect to ξ_0. There are two contributions, one coming from $(Uf)(x_0, e^{\lambda\xi_0}x_1)$ and the other from $G_s^\Delta(\xi_0, e^{-\lambda\xi_0}\xi_1)$. First we will consider derivatives acting on the term $(Uf)(x_0, e^{\lambda\xi_0}x_1)$. Taking one derivative we obtain

$$\begin{aligned} \partial_{\xi_0}(Uf)(x_0, e^{\lambda\xi_0}x_1) &= e^{\lambda\xi_0}\lambda x_1 (\partial_1 Uf)(x_0, e^{\lambda\xi_0}x_1) \\ &= \lambda x_1 \partial_{x_1}(Uf)(x_0, e^{\lambda\xi_0}x_1) \\ &= \lambda\partial_{x_1}\left(x_1 (Uf)(x_0, e^{\lambda\xi_0}x_1)\right) - \lambda(Uf)(x_0, e^{\lambda\xi_0}x_1) \ . \end{aligned} \tag{5.12}$$

The second term corresponds to the case with zero derivatives, so we have already proven that it gives a finite contribution. The first term on the other hand vanishes upon integration in x_1, since Uf is a Schwartz function. For the second derivative using (5.12) we obtain

$$\begin{aligned} \partial_{\xi_0}^2 (Uf)(x_0, e^{\lambda\xi_0}x_1) &= \lambda x_1 \partial_{x_1}\partial_{\xi_0}(Uf)(x_0, e^{\lambda\xi_0}x_1) \\ &= \lambda\partial_{x_1}\left(x_1 \partial_{\xi_0}(Uf)(x_0, e^{\lambda\xi_0}x_1)\right) - \lambda\partial_{\xi_0}(Uf)(x_0, e^{\lambda\xi_0}x_1) \end{aligned}$$

The second term corresponds to the case of one derivative, which we have proven to be finite. For the first term we just need to notice that, for fixed ξ_0, equation (5.12) implies that $\partial_{\xi_0}(Uf)(x_0, e^{\lambda\xi_0}x_1)$ is a Schwartz function. Then the first term vanishes by the previous argument. □

Since we have established that the operator $\pi(f)\Delta_\omega(D^2+\mu^2)^{-s/2}$ is trace class we can now compute its trace. We are going to show that the residue at $s=2$ recovers, up to a constant, the weight ω on the algebra $\mathcal{A}$.

Proposition 5.21. *Let $f \in \mathcal{A}$ and $\mu \neq 0$. Then we have*

$$\lim_{s\to 2}(s-2)\Phi\left(\pi(f)(D^2+\mu^2)^{-s/2}\right) = \frac{1}{2\pi}\omega(f) \ .$$

Proof. Using Lemma 5.19 it suffices to prove the analogue result with $\pi(f)\Delta_\omega(D^2+\mu^2)^{-s/2}$. In the previous theorem we have shown that the operator $A := U\pi(f)\Delta_\omega(D^2+\mu^2)^{-s/2}U^{-1}$, with $\mu \neq 0$, is trace class for $s > 2$. We can compute its trace by integrating the kernel, that is

$$\mathrm{Tr}\left(\pi(f)\Delta_\omega(D^2+\mu^2)^{-s/2}\right) = 2\int\int (Uf)(x_0, e^{\lambda\xi_0}x_1)G_s^\Delta(\xi_0, e^{-\lambda\xi_0}\xi_1)d^2x\frac{d^2\xi}{(2\pi)^2} \ .$$

Here the factor 2 comes from the dimension of the spinor bundle, since $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^2$. As shown in the previous theorem this is actually equal to

$$\mathrm{Tr}\left(\pi(f)\Delta_\omega(D^2+\mu^2)^{-s/2}\right) = 2\int (Uf)(x)d^2x\int G_s^\Delta(\xi)\frac{d^2\xi}{(2\pi)^2} \ .$$

It is then easy to see that, for any $f \in \mathcal{A}$, we have $\int (Uf)(x)d^2x = \int f(x)d^2x$.

Now we need to compute the integral in ξ, which is given by

$$c(s) := \int G_s^\Delta(\xi)\frac{d^2\xi}{(2\pi)^2} = \int e^{-\lambda\xi_0}\left(\lambda^{-2}\left(1-e^{-\lambda\xi_0}\right)^2 + \xi_1^2 + \mu^2\right)^{-s/2}\frac{d^2\xi}{(2\pi)^2} \ .$$

The integral over ξ_1 can be easily computed using the standard formula

$$\int (x^2+a^2)^{-s/2}dx = \sqrt{\pi}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}(a^2)^{-\frac{s-1}{2}} \ , \qquad s > 1 \ .$$

Using this result we have

$$c(s) = \frac{\sqrt{\pi}}{(2\pi)^2}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\int e^{-\lambda\xi_0}\left(\lambda^{-2}\left(1-e^{-\lambda\xi_0}\right)^2+\mu^2\right)^{-\frac{s-1}{2}}d\xi_0 \ .$$

Now, using the change of variable $r = e^{-\lambda\xi_0}$, we rewrite the integral in ξ_0 as

$$\int e^{-\lambda\xi_0}\left(\lambda^{-2}\left(1-e^{-\lambda\xi_0}\right)^2+\mu^2\right)^{-\frac{s-1}{2}}d\xi_0 = \lambda^{s-2}\int\left((1-r)^2+\lambda^2\mu^2\right)^{-\frac{s-1}{2}}dr \ .$$

This integral can be solved analytically for $s > 2$

$$\int_0^\infty \left((1-r)^2 + a^2\right)^{-\frac{s-1}{2}} dr = a^{1-s} \left(a \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{s}{2}-1\right)}{\Gamma\left(\frac{s-1}{2}\right)} + {}_2F_1\left(\frac{1}{2}, \frac{s-1}{2}, \frac{3}{2}, -\frac{1}{a^2}\right) \right) ,$$

where ${}_2F_1(a,b,c,z)$ is the ordinary hypergeometric function. The result is then

$$c(s) = \frac{\sqrt{\pi}}{(2\pi)^2} \frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \lambda^{s-2} (\lambda\mu)^{1-s} \left(\lambda\mu \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{s}{2}-1\right)}{\Gamma\left(\frac{s-1}{2}\right)} + {}_2F_1\left(\frac{1}{2}, \frac{s-1}{2}, \frac{3}{2}, -\frac{1}{\lambda^2\mu^2}\right) \right) .$$

Now we can easily show the analogue of $\mathcal{Z}_2$-summability, that is

$$\limsup_{s \downarrow 2} \left| (s-2) \mathrm{Tr}\left(\pi(f) \Delta_\omega (D^2 + \mu^2)^{-s/2} \right) \right| < \infty .$$

We need to study the behaviour of the function $c(s)$ around $s = 2$. Notice that the second term, the one involving the hypergeometric function, is regular at $s = 2$, while the first term has a simple pole at $s = 2$, which comes from the function $\Gamma\left(\frac{s}{2}-1\right)$. Indeed the Laurent expansion of this function at $s = 2$ is given by

$$\Gamma\left(\frac{s}{2}-1\right) = \frac{2}{s-2} - \gamma + O(s-2) ,$$

where γ is the Euler-Mascheroni constant. Using this fact we have

$$\begin{aligned} \limsup_{s \downarrow 2} (s-2) c(s) &= \limsup_{s \downarrow 2} (s-2) \frac{\sqrt{\pi}}{(2\pi)^2} \frac{1}{\Gamma\left(\frac{s}{2}\right)} \lambda^{s-2} (\lambda\mu)^{1-s} \lambda\mu \frac{\sqrt{\pi}}{2} \Gamma\left(\frac{s}{2}-1\right) \\ &= \frac{\sqrt{\pi}}{(2\pi)^2} (\lambda\mu)^{-1} \lambda\mu \frac{\sqrt{\pi}}{2} 2 = \frac{1}{4\pi} . \end{aligned}$$

Notice that the result of this limit does not depend on μ, as expected. Finally

$$\lim_{s \to 2} (s-2) \Phi\left(\pi(f) (D^2 + \mu^2)^{-s/2} \right) = \frac{1}{2\pi} \int \sigma^\omega_{-i}(f)(x) d^2x .$$

Using the properties of the functions in $\mathcal{A}$ and the Cauchy theorem we have that

$$\int f(x_0 + z, x_1) d^2x = \int f(x) d^2x ,$$

for any $z \in \mathbb{C}$. Then the result follows, since $\sigma^\omega_{-i}(f)(x) = f(x_0 + i\lambda, x_1)$. □

5.6 The real structure

In this section we discuss the possibility of introducing a real structure on the spectral triple. We briefly review the commutative case, to set the notation and also since some computations will be identical in the non-commutative case. Consider a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ of spectral dimension n. It is even if there exists a $\mathbb{Z}_2$-grading χ on $\mathcal{H}$.

Definition 5.10. A real structure for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an antilinear isometry $\mathcal{J} : \mathcal{H} \to \mathcal{H}$ with the following properties:

1. $\mathcal{J}^2 = \varepsilon(n)$,
2. $\mathcal{J} D = \varepsilon'(n) D \mathcal{J}$,
3. $[\pi(f), \mathcal{J}\pi(g^*)\mathcal{J}^{-1}] = 0$,
4. $\mathcal{J}\chi = i^n \chi \mathcal{J}$ if it is even,
5. $[[D, \pi(f)], \mathcal{J}\pi(g)\mathcal{J}^{-1}] = 0$.

The fifth condition is usually called the first order condition. Here $\varepsilon(n)$ and $\varepsilon'(n)$ are mod 8 periodic functions which are given by

$$\begin{aligned} \varepsilon(n) &= (1, 1, -1, -1, -1, -1, 1, 1) \ , \\ \varepsilon'(n) &= (1, -1, 1, 1, 1, -1, 1, 1) \ . \end{aligned}$$

We shortly review the case $n = 2$. Consider $\mathcal{J} = C J_c$, where J_c is complex conjugation and C is a 2×2 matrix, which in our conventions is given by $C = i\Gamma^0$. The grading is given by the matrix $\chi = -i\Gamma^0\Gamma^1$, which satisfies $\chi^2 = 1$. The Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ and acting with the operator $\mathcal{J}$ on a spinor ψ we get

$$\mathcal{J} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} -i\overline{\psi_2}(x) \\ -i\overline{\psi_1}(x) \end{pmatrix} .$$

The fact that $\mathcal{J}$ is an antilinear isometry follows from a simple computation

$$\begin{aligned} (\mathcal{J}\phi, \mathcal{J}\psi)_{\mathcal{H}} &= \sum_{k=1}^{2} ((\mathcal{J}\phi)_k, (\mathcal{J}\psi)_k)_{L^2(\mathbb{R}^2)} = \sum_{k=1}^{2} (-i\overline{\phi_k}, -i\overline{\psi_k})_{L^2(\mathbb{R}^2)} \\ &= \sum_{k=1}^{2} (\psi_k, \phi_k)_{L^2(\mathbb{R}^2)} = (\psi, \phi)_{\mathcal{H}} \ . \end{aligned}$$

The first condition is easily verified using the properties of the Γ matrices. To verify the second condition write the Dirac operator as $D = \Gamma^\mu \hat{P}_\mu$. We have

$$\mathcal{J} D \mathcal{J} = i\Gamma^0 J_c \Gamma^\mu \hat{P}_\mu i\Gamma^0 J_c = \Gamma^\mu J_c \hat{P}_\mu J_c \ .$$

Using the definition of $\hat{P}_\mu$ we get $\Gamma^\mu J_c \hat{P}_\mu J_c = -\Gamma^\mu \hat{P}_\mu = -D$. Applying $\mathcal{J}$ and using $\mathcal{J}^2 = 1$ we obtain $\mathcal{J} D = -D\mathcal{J}$. For the third condition we note that $\mathcal{J}\pi(g^*)\mathcal{J}^{-1} = i\Gamma^0 J_c \overline{g} i\Gamma^0 J_c = g$.

The fourth condition follows from the computation

$$\mathcal{J}\chi = i\Gamma^0 J_c(-i)\Gamma^0\Gamma^1 = i\Gamma^0(-i)\Gamma^0\Gamma^1 J_c = -(-i\Gamma^0\Gamma^1)(i\Gamma^0 J_c) = -\chi\mathcal{J} \ .$$

Finally the fifth condition follows since $[D,\pi(f)] = \Gamma^\mu\pi(D_\mu f)$ and therefore

$$[[D,\pi(f)],\mathcal{J}\pi(g)\mathcal{J}^{-1}] = \Gamma^\mu[\pi(D_\mu f),\mathcal{J}\pi(g)\mathcal{J}^{-1}] = 0 \ .$$

Now we consider the non-commutative case. For $f \in \mathcal{A}$ define the operator $\tilde{J}f := \sigma^\omega_{i/2}(f^*)$, see [Kus97]. Since ω satisfies the KMS condition with respect to σ^ω, the term $\sigma^\omega_{i/2}$ compensates for the lack of the trace property, as shown in the next lemma.

Lemma 5.22. *The operator $\tilde{J}$ is an antilinear isometry on $\mathcal{H}_r$.*

Proof. Recall that $\sigma^\omega_t(f)(x) = f(x_0 - \lambda t, x_1)$ and that for $f, g \in \mathcal{A}$ we have the KMS property $\omega(f \star g) = \omega(\sigma^\omega_{-i}(g) \star f)$. Then we have the following

$$\begin{aligned}
(\tilde{J}f,\tilde{J}g) &= \omega((\sigma^\omega_{i/2}(f^*))^* \star \sigma^\omega_{i/2}(g^*)) = \omega(\sigma^\omega_{-i/2}(f) \star \sigma^\omega_{i/2}(g^*)) \\
&= \omega(\sigma^\omega_{-i}\sigma^\omega_{i/2}(g^*) \star \sigma^\omega_{-i/2}(f)) = \omega(\sigma^\omega_{-i/2}(g^*) \star \sigma^\omega_{-i/2}(f)) \\
&= \omega(\sigma^\omega_{-i/2}(g^* \star f)) = \omega(g^* \star f) = (g, f) \ .
\end{aligned}$$

The property $\omega(\sigma^\omega_{-i/2}(f)) = \omega(f)$ holds for any $f \in \mathcal{A}$, as discussed in the previous sections. In particular we have that $\|\sigma^\omega_{i/2} f^*\| = \|f\|$, so it is an antilinear isometry on $\mathcal{A}$. Since $\mathcal{A}$ is dense in $\mathcal{H}_r$ this operator can be extended by continuity to the whole Hilbert space. □

To extend $\tilde{J}$ from $\mathcal{H}_r$ to $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^2$ we introduce the operator $\mathcal{J} = C\tilde{J}$, where $C = i\Gamma^0$ is the same matrix as in the commutative case. Now we only have to check the various properties satisfied by this operator, which are given in the following.

Proposition 5.23. *The operator $\mathcal{J}$ is an antilinear isometry on $\mathcal{H}$. Moreover it satisfies the following properties:*

1. $\mathcal{J}^2 = 1$,
2. $\mathcal{J}D = -\Delta_\omega^{-1}D\mathcal{J}$,
3. $[\pi(f),\mathcal{J}\pi(g^*)\mathcal{J}^{-1}] = 0$,
4. $\mathcal{J}\chi = -\chi\mathcal{J}$,
5. $[[D,\pi(f)]_\sigma,\mathcal{J}\pi(g)\mathcal{J}^{-1}] = 0$.

Proof. First we show that $\mathcal{J}$ is an antilinear isometry. When acting on a spinor ψ we get

$$\mathcal{J}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \begin{pmatrix}-i\sigma^\omega_{i/2}(\psi_2^*)(x)\\ -i\sigma^\omega_{i/2}(\psi_1^*)(x)\end{pmatrix} .$$

Then the result follows from the previous lemma and the following computation

$$(\mathcal{J}\phi,\mathcal{J}\psi)_{\mathcal{H}}=\sum_{k=1}^{2}\left(-i\sigma^{\omega}_{i/2}(\phi_k^*),-i\sigma^{\omega}_{i/2}(\psi_k^*)\right)_{\mathcal{H}_r}=\sum_{k=1}^{2}(\psi_k,\phi_k)_{\mathcal{H}_r}=(\psi,\phi)_{\mathcal{H}}\ .$$

For any $f\in\mathcal{A}$ and $z\in\mathbb{C}$ we have $\sigma^{\omega}_{z}(f)^*=\sigma^{\omega}_{\bar{z}}(f^*)$. Writing $\tilde{J}\psi=\sigma^{\omega}_{-i/2}(\psi)^*$ we have

$$\tilde{J}^2\psi=(\sigma^{\omega}_{-i/2}(\sigma^{\omega}_{-i/2}\psi)^*)^*=\sigma^{\omega}_{i/2}(\sigma^{\omega}_{-i/2}\psi)=\psi\ .$$

Using this relation the first property is proven as in the commutative case

$$\mathcal{J}^2=i\Gamma^0\tilde{J}i\Gamma^0\tilde{J}=(\Gamma^0)^2\tilde{J}^2=1\ .$$

For the second property recall that the Dirac operator is given by $D=\Gamma^{\mu}\hat{D}_{\mu}$, where $\hat{D}_0=\frac{1}{\lambda}(1-e^{-\lambda\hat{P}_0})$ and $\hat{D}_1=\hat{P}_1$. Using the properties of the Γ matrices we obtain $\mathcal{J}D\mathcal{J}=\Gamma^{\mu}\tilde{J}\hat{D}_{\mu}\tilde{J}$, as in the commutative case. To compute $\tilde{J}\hat{D}_{\mu}\tilde{J}$ notice that, for any $h\in\mathcal{T}_{\kappa}$, we have

$$\tilde{J}\rho(h)\tilde{J}\psi=\tilde{J}\rho(h)\sigma^{\omega}_{i/2}(\psi^*)=(\sigma^{\omega}_{-i/2}\rho(h)\sigma^{\omega}_{i/2}(\psi^*))^*\ .$$

But $\sigma^{\omega}_{i/2}$ commutes with $\rho(h)$ for any $h\in\mathcal{T}_{\kappa}$. Then, using the property of compatibility of the representation with the star structure $h\triangleright a^*=(S(h)^*\triangleright a)^*$, we obtain

$$\tilde{J}\rho(h)\tilde{J}\psi=(\rho(h)\psi^*)^*=(h\triangleright\psi^*)^*=S(h)^*\triangleright\psi=\rho(S(h)^*)\psi\ .$$

If we apply this result to $\hat{D}_{\mu}=\rho(D_{\mu})$ we obtain

$$\begin{aligned}\tilde{J}\hat{D}_0\tilde{J}&=\frac{1}{\lambda}\rho(1-\mathcal{E}^{-1})=-\frac{1}{\lambda}\rho(\mathcal{E}^{-1})\rho(1-\mathcal{E})=-\Delta_{\omega}^{-1}\hat{D}_0\ ,\\ \tilde{J}\hat{D}_1\tilde{J}&=\rho(-\mathcal{E}^{-1}P_1)=-\rho(\mathcal{E}^{-1})\rho(P_1)=-\Delta_{\omega}^{-1}\hat{D}_1\ .\end{aligned}$$

Then we obtain $\mathcal{J}D\mathcal{J}=-\Delta_{\omega}^{-1}D$, from which the second property follows. For the third one we notice that $\mathcal{J}\pi(f^*)\mathcal{J}=\tilde{J}\pi(f^*)\tilde{J}$. We can easily show that $\tilde{J}\pi(f^*)\tilde{J}$ corresponds to right multiplication by $\sigma^{\omega}_{i/2}(f)$. Indeed we have

$$\begin{aligned}\tilde{J}\pi(f^*)\tilde{J}\psi&=\sigma^{\omega}_{i/2}(f^*\star\sigma^{\omega}_{i/2}(\psi^*))^*=\sigma^{\omega}_{i/2}((\sigma^{\omega}_{i/2}(\psi^*))^*\star f)\\ &=\sigma^{\omega}_{i/2}(\sigma^{\omega}_{-i/2}(\psi)\star f)=\psi\star\sigma^{\omega}_{i/2}(f)\ .\end{aligned}$$

Then the property follows from the general fact that right multiplication commutes with left multiplication. The proof of the fourth property is identical to the classical case. For the fifth property recall that $[D,\pi(f)]_{\sigma}=\Gamma^{\mu}\pi(D_{\mu}\triangleright f)$. Then from the previous property

$$[[D,\pi(f)]_{\sigma},\mathcal{J}\pi(g)\mathcal{J}^{-1}]=\Gamma^{\mu}[\pi(D_{\mu}\triangleright f),\mathcal{J}\pi(g)\mathcal{J}^{-1}]=0\ .$$

The proof is complete. □

Several conditions are identical to the commutative case, due to the Clifford structure being the same. The first order condition requires the use of the twisted commutator $[D, \pi(f)]_\sigma$, which is argued to be the natural choice for quantum groups in [D'An07].

Also the property involving the commutator of $\mathcal{J}$ and D is modified by the presence of the modular operator Δ_ω, in particular we have $\mathcal{J}D = -\Delta_\omega^{-1}D\mathcal{J}$. We have seen that, in checking this property, the role of the antipode was crucial. Since the antipode of P_1 is not trivial, this implies that we cannot have $\mathcal{J}D = -D\mathcal{J}$.

Chapter 6

On the spectral and homological dimension of κ-Minkowski space

In this chapter we extend our construction of a modular spectral triple for κ-Minkowski space, previously given in two dimensions, to the general n-dimensional case. We consider in some detail the properties of the zeta function associated to the Dirac operator, and initiate a study of the homological properties of this geometry. This is based on the paper [Mat2].

6.1 Introduction

The κ-Poincaré and κ-Minkowski Hopf algebras can be defined in any dimension, so it is interesting to try extending the construction given in the previous chapter to the general n-dimensional case. It turns out that there are no major difficulties in doing so, therefore we will not provide the full details regarding the computations, as they are almost identical to the two-dimensional case. This simplicity is connected to one of the physical requirements of the κ-Poincaré algebra, that is leaving undeformed the Lorentz subalgebra, which essentially makes all the "space" directions behave in the same way.

On the other hand, we want to provide further evidence that, although this construction is still not well understood as part of a general framework, it should be relevant for the description of the non-commutative geometry of κ-Minkowski space. Using the same ingredients of the two-dimensional case, we find that the spectral dimension according to our definition is in general equal to the classical one. Moreover, by computing the residue at the spectral dimension of the associated zeta function, we recover the weight ω as in the two-dimensional case. These results confirm the intuition that moving from the two-dimensional case to the general one does not change much.

Next we analyze some properties of the introduced zeta function. We show that, by considering the limit of vanishing deformation parameter, it reduces as it should to the classical setting. Also, as in the commutative setting, this zeta function can be analytically continued to a meromorphic function on the complex plane, with only simple poles. We find all the poles of the commutative case, but also additional ones due to the presence of the deformation parameter. The significance of these additional poles remains to be investigated.

Another important issue we analyze is the homological dimension of this geometry. In the framework of non-commutative geometry this notion is given by the dimension of the Hochschild homology, which in the commutative case coincides with the spectral dimension. However in many examples, coming in particular from quantum groups, one finds that the homological dimension is lower that the spectral dimension, a phenomenon known as dimension drop. In many cases it is possible to avoid this drop by introducing a twist in the homology theory, as seen for example in [HaKr05, Had07]. Here we compute the *twisted Hochschild homology* [BrZh08] of the universal enveloping algebra associated to κ-Minkowski space. Similarly to the examples we mentioned above, we show that the dimension drop occurs at the level of Hochschild homology, but can be avoided by introducing a twist. More interestingly, the simplest twist which avoids the drop is the inverse of the modular group of the weight ω, while the other possible twists are given by its positive powers. This should be compared to the case of [HaKr05, Had07] and other examples, where the twist is the inverse of the modular group of the Haar state, and therefore seems to be a general feature of these non-commutative geometries.

6.2 The spectral triple

6.2.1 The κ-Poincaré and κ-Minkowski algebras

In this subsection we summarize some algebraic properties of the κ-Poincaré Hopf algebra, which we denote by $\mathcal{P}_\kappa$, and the associated κ-Minkowski space. For our purposes we do not need to present the full algebra, but only a certain Hopf subalgebra which is used to define κ-Minkowski space. For details we refer to the original paper [MaRu94] and for the construction in any number of dimension see [LuRu94]. We denote by $\mathcal{T}_\kappa$ the algebra generated by P_μ, with μ ranging from 0 to $n-1$, which satisfy $[P_\mu, P_\nu] = 0$. We turn it into a Hopf algebra by defining the coproduct $\Delta : \mathcal{T}_\kappa \to \mathcal{T}_\kappa \otimes \mathcal{T}_\kappa$, the counit $\varepsilon : \mathcal{T}_\kappa \to \mathbb{C}$ and antipode $S : \mathcal{T}_\kappa \to \mathcal{T}_\kappa$ as

$$\begin{aligned} &\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 , \quad \Delta(P_j) = P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j , \\ &\varepsilon(P_\mu) = 0 , \quad S(P_0) = -P_0 , \quad S(P_j) = -e^{P_0/\kappa} P_j . \end{aligned}$$

We adopt the usual general relativistic convention of greek indices going from 0 to $n-1$, while latin indices go from 1 to $n-1$. Notice the asymmetry in the coproduct, which distinguishes the "time direction" from the "space directions".

κ-Minkowski space is defined via a non-degenerate dual pairing with $\mathcal{T}_\kappa$, see [MaRu94]. From the pairing we deduce that as an algebra it is non-commutative, since $\mathcal{T}_\kappa$ is not cocommutative, that is the coproduct in $\mathcal{T}_\kappa$ is not trivial. In particular the algebraic relations for the generators X^μ of the κ-Minkowski Hopf algebra take the form

$$[X^0, X^j] = -\kappa^{-1} X^j \ , \qquad \Delta X^\mu = X^\mu \otimes 1 + 1 \otimes X^\mu \ .$$

As in the previous chapter, we avoid the use of the formal element $e^{-P_0/\kappa}$ by introducing the invertible element $\mathcal{E}$, and rewrite the defining relations as

$$\begin{aligned}
&[P_\mu, P_\nu] = 0 \ , \quad [P_\mu, \mathcal{E}] = 0 \ , \\
&\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \ , \quad \Delta(P_j) = P_j \otimes 1 + \mathcal{E} \otimes P_j \ , \quad \Delta(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E} \ , \\
&\varepsilon(P_\mu) = 0 \ , \quad \varepsilon(\mathcal{E}) = 1 \ , \\
&S(P_0) = -P_0 \ , \quad S(P_j) = -\mathcal{E}^{-1} P_j \ , \quad S(\mathcal{E}) = \mathcal{E}^{-1} \ .
\end{aligned}$$

The remarks made in the previous chapter for the two-dimensional case, regarding the extended momentum algebra and the Euclidean signature, also apply to this case.

6.2.2 The algebraic construction

We begin by generalizing to n dimensions the construction of the $*$-algebra given in [DuSi13]. We will skip most of the computations, since they are completely analogous to the two-dimensional case, but we will provide some details regarding the modular aspects of the construction.

The underlying algebra of the n-dimensional κ-Minkowski space is the enveloping algebra of the Lie algebra with generators ix^0 and ix^k (with $k = 1, \dots, n-1$), fullfilling the commutation relations $[x^0, x^k] = i\lambda x^k$. It has a faithful $n \times n$ matrix representation φ given by

$$\varphi(ix^0) = \begin{pmatrix} -\lambda & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & 0 \end{pmatrix} , \qquad \varphi(ix^k) = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} .$$

The matrix $\varphi(ix^k)$ has non-zero values only in the $(k+1)$-th column. An element of the associated group G can be presented in the form

$$S(a) = \begin{pmatrix} e^{-\lambda a_0} & a_1 & \cdots & a_{n-1} \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} . \tag{6.1}$$

Here we use the notation $a = (a_0, \vec{a})$, where $\vec{a} = (a_1, \cdots, a_{n-1})$. The group operations written in the $(a_0, \cdots, a_{n-1})$ coordinates are given by

$$S(a)S(b) = S(a_0 + b_0, \vec{a} + e^{-\lambda a_0}\vec{b}) \ , \qquad S(a)^{-1} = S(-a_0, -e^{\lambda a_0}\vec{a}) \ . \tag{6.2}$$

Proposition 6.1. *The left and right invariant measures on G are given respectively by $d\mu_L(a) = e^{\lambda(n-1)a_0} d^n a$ and $d\mu_R(a) = d^n a$, where $d^n a$ is the Lebesgue measure on $\mathbb{R}^n$.*

Proof. We do the computation for the left invariant case. Using the $(a_0, \cdots, a_{n-1})$ coordinates and the group operations given in (6.2) we easily find

$$\begin{aligned}\int f(a\cdot b)d\mu_L(b) &= \int f(a_0+b_0, \vec{a}+e^{-\lambda a_0}\vec{b})e^{\lambda(n-1)b_0}d^n b \\ &= \int f(a_0+b_0, \vec{a}+\vec{b})e^{\lambda(n-1)(a_0+b_0)}d^n b \\ &= \int f(b_0, \vec{b})e^{\lambda(n-1)b_0}d^n b = \int f(b)d\mu_L(b) \ .\end{aligned}$$

The right invariant case is treated similarly. □

Thus G is not a unimodular group, with the modular function $e^{-\lambda(n-1)a_0}$ playing a central role in the following. We consider the convolution algebra of G with respect to the right invariant measure, and we identify functions on G with functions on $\mathbb{R}^n$ by the parametrization (6.1). The convolution algebra is an involutive Banach algebra consisting of integrable functions on $\mathbb{R}^n$ with product $\hat{\star}$ and involution $\hat{*}$ given by

$$\begin{aligned}(f\hat{\star}g)(a) &= \int f(a_0-a'_0, \vec{a}-e^{-\lambda(a_0-a'_0)}\vec{a}')g(a'_0, \vec{a}')d^n a' \ , \\ f^{\hat{*}}(a) &= e^{\lambda(n-1)a_0}\overline{f}(-a_0, -e^{\lambda a_0}\vec{a}) \ .\end{aligned}$$

We pass from momentum space to configuration space via the Fourier transform. The star-product and involution are defined, in terms of the convolution algebra operations, as

$$f\star g := \mathcal{F}^{-1}\left(\mathcal{F}(f)\hat{\star}\mathcal{F}(g)\right) \ , \qquad f^* := \mathcal{F}^{-1}\left(\mathcal{F}(f)^{\hat{*}}\right) \ .$$

These formulae are written for clarity using the unitary convention for the Fourier transform, but in the following we will use the physicists convention with the $(2\pi)^n$ in momentum space. We restrict our attention to the following space of functions.

Definition 6.1. Denote by $\mathcal{S}_c$ the space of Schwartz functions on $\mathbb{R}^n$ with compact support in the first variable, that is for $f \in \mathcal{S}_c$ we have $\mathrm{supp}(f) \subseteq K \times \mathbb{R}^{n-1}$ for some compact $K \subset \mathbb{R}$. We define $\mathcal{A} = \mathcal{F}(\mathcal{S}_c)$, where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^n$.

On this space we can safely perform all the operations we need. The next proposition shows that $\mathcal{A}$ is a $*$-algebra and gives explicit formulae for the star product and the involution. We use the notation $x = (x_0, \vec{x})$ and $\vec{x} = (x_1, \dots, x_{n-1})$ for the coordinates on $\mathcal{A}$.

Proposition 6.2. *For $f, g \in \mathcal{A}$ we have*

$$
\begin{aligned}
(f \star g)(x) &= \int e^{ip_0 x_0} (\mathcal{F}_0 f)(p_0, \vec{x}) g(x_0, e^{-\lambda p_0} \vec{x}) \frac{dp_0}{2\pi} \ , \\
f^*(x) &= \int e^{ip_0 x_0} (\mathcal{F}_0 \overline{f})(p_0, e^{-\lambda p_0} \vec{x}) \frac{dp_0}{2\pi} \ .
\end{aligned}
$$

We have that $f \star g \in \mathcal{A}$ and $f^* \in \mathcal{A}$, so that $\mathcal{A}$ is a $*$-algebra.

Proof. Let us show how this works for the involution. From the definitions we get

$$
\begin{aligned}
f^*(x) = \mathcal{F}^{-1}(\mathcal{F}(f)^{\hat{*}})(x) &= \int e^{ipx} (\mathcal{F}(f)^{\hat{*}})(p) \frac{d^n p}{(2\pi)^n} \\
&= \int e^{ipx} e^{(n-1)\lambda p_0} (\overline{\mathcal{F} f})(-p_0, -e^{\lambda p_0} \vec{p}) \frac{d^n p}{(2\pi)^n} \ .
\end{aligned}
$$

Now using the change of variables $\vec{p} \to e^{-\lambda p_0} \vec{p}$ we find

$$
f^*(x) = \int e^{ip_0 x_0} e^{ie^{-\lambda p_0} \vec{p} \cdot \vec{x}} (\overline{\mathcal{F} f})(-p) \frac{d^n p}{(2\pi)^n} = \int e^{ip_0 x_0} e^{ie^{-\lambda p_0} \vec{p} \cdot \vec{x}} (\mathcal{F} \bar{f})(p) \frac{d^n p}{(2\pi)^n} \ .
$$

Finally performing the Fourier transform in the $\vec{p}$ variables we find

$$
f^*(x) = \int e^{ip_0 x_0} (\mathcal{F}_0 \bar{f})(p_0, e^{-\lambda p_0} \vec{x}) \frac{dp_0}{2\pi}
$$

The product can be computed in a similar way. For more details see [DuSi13]. □

A nice property of this algebra is that it comes naturally with an action of the κ-Poincaré algebra $\mathcal{P}_\kappa$ on it. In [DuSi13] it was proven that $\mathcal{A}$ is a left $\mathcal{P}_\kappa$-module $*$-algebra, which means that the action of the κ-Poincaré symmetries on $\mathcal{A}$ preserves the Hopf algebraic structure. In particular the action of the translations sector is elementary, with $(P_\mu \triangleright f)(x) = -i(\partial_\mu f)(x)$ and $(\mathcal{E} \triangleright f)(x) = f(x_0 + i\lambda, \vec{x})$. This remains true for the n-dimensional case.

Now we can introduce a Hilbert space by using the GNS-construction for $\mathcal{A}$, after the choice of some weight ω. There is a natural choice which respects the symmetries of the κ-Poincaré Hopf algebra, see [DuSi13, Mer11]. It is simply given by the integral of a function $f \in \mathcal{A}$ with respect to the Lebesgue measure over $\mathbb{R}^n$, and we denote it by ω. However, differently from the commutative case, it does not satisfy the trace property.

Proposition 6.3. *For $f, g \in \mathcal{A}$ we have the twisted trace property*

$$
\int (f \star g)(x) d^n x = \int (\sigma^{n-1}(g) \star f)(x) d^n x \ ,
$$

where we define $\sigma(g)(x) := g(x_0 + i\lambda, \vec{x})$.

Proof. First we use the change of variables $\vec{x} \to e^{\lambda p_0}\vec{x}$ and obtain

$$\int (f \star g)(x) d^n x = \int\int e^{ip_0 x_0} (\mathcal{F}_0 f)(p_0, \vec{x}) g(x_0, e^{-\lambda p_0}\vec{x}) \frac{dp_0}{2\pi} d^n x$$
$$= \int\int e^{ip_0 x_0} e^{(n-1)\lambda p_0} (\mathcal{F}_0 f)(p_0, e^{\lambda p_0}\vec{x}) g(x_0, \vec{x}) d^n x \frac{dp_0}{2\pi} \ .$$

Now, using the analiticity of the functions of $\mathcal{A}$ in the first variable, we can shift $x_0 \to x_0 + i(n-1)\lambda$ to obtain the action of σ^{n-1} on g, that is

$$\int (f \star g)(x) d^n x = \int\int e^{ip_0 x_0} (\mathcal{F}_0 f)(p_0, e^{\lambda p_0}\vec{x}) g(x_0 + i(n-1)\lambda, \vec{x}) d^n x \frac{dp_0}{2\pi}$$
$$= \int\int e^{ip_0 x_0} (\mathcal{F}_0 f)(p_0, e^{\lambda p_0}\vec{x}) \sigma^{n-1}(g)(x_0, \vec{x}) d^n x \frac{dp_0}{2\pi} \ .$$

It only remains to rewrite this expression in terms of the $\star$-product. Writing explicitely the Fourier transform $\mathcal{F}_0 f$ we have

$$\int (f \star g)(x) d^n x = \int\int e^{ip_0 x_0} \int e^{-ip_0 y_0} f(y_0, e^{\lambda p_0}\vec{x}) \sigma^{n-1}(g)(x_0, \vec{x}) dy_0 d^n x \frac{dp_0}{2\pi} \ .$$

We need to do some rearranging: change $p_0 \to -p_0$, relabel $y_0 \leftrightarrow x_0$ and exchange the order of the x_0 and y_0 integral. The result of these operations is

$$\int (f \star g)(x) d^n x = \int\int e^{ip_0 x_0} f(x_0, e^{-\lambda p_0}\vec{x}) \int e^{-ip_0 y_0} \sigma^{n-1}(g)(y_0, \vec{x}) dy_0 d^n x \frac{dp_0}{2\pi} \ .$$

But now the last integral is just the Fourier transform of $\sigma^{n-1}(g)$ in the y_0 variable, so

$$\int (f \star g)(x) d^n x = \int\int e^{ip_0 x_0} (\mathcal{F}_0 \sigma^{n-1}(g))(p_0, \vec{x}) f(x_0, e^{-\lambda p_0}\vec{x}) \frac{dp_0}{2\pi} d^n x \ .$$

Finally we observe that the right hand side is just the integral of the function $(\sigma^{n-1}(g) \star f)(x)$, which proves the result. □

As in the previous chapter, this property can be rephrased as a KMS condition for ω.

Proposition 6.4. *The weight ω satisfies the KMS condition with respect to the modular group σ^ω, defined by $(\sigma^\omega_t f)(x_0, \vec{x}) := f(x_0 - t(n-1)\lambda, \vec{x})$. The associated modular operator is $\Delta_\omega = e^{-(n-1)\lambda P_0}$, where $P_0 = -i\partial_0$.*

On the Hilbert space $\mathcal{H}$, obtained by the GNS-construction for ω, the algebra $\mathcal{A}$ acts via left multiplication, that is $\pi(f)\psi := f \star \psi$. In the following we omit the representation symbol π and just write f for the operator of left multiplication by this function.

It is important to point out that the Hilbert space $\mathcal{H}$ is not $L^2(\mathbb{R}^n)$. On the other hand, using the fact that $\mathcal{A}$ is dense in both Hilbert spaces, one can easily find a unitary operator between the two, as for the two-dimensional case. One can also find, using this unitary operator, the Schwartz kernel of a certain class of operators which will be of interest to us in the following. These results are the content of the next proposition.

Proposition 6.5. *The Hilbert space $\mathcal{H}$ obtained by the GNS-construction for ω is unitarily equivalent to $L^2(\mathbb{R}^n)$, via the unitary operator given by*

$$(Uf)(x) = \int e^{ip_0 x_0} (\mathcal{F}_0 \overline{f})(p_0, e^{-\lambda p_0} \vec{x}) \frac{dp_0}{2\pi} \ .$$

Consider now the operator $U\pi(f)g(P)U^{-1}$ acting on $L^2(\mathbb{R}^n)$, where $f \in \mathcal{A}$ and $P_\mu = -i\partial_\mu$. Then its Schwartz kernel is given by

$$K(x,y) = \int e^{ip(x-y)} (Uf)(x_0, e^{\lambda p_0} \vec{x}) g(p_0, e^{-\lambda p_0} \vec{p}) \frac{d^n p}{(2\pi)^n} \ .$$

6.2.3 Dirac operator and differential calculus

The next step is the introduction of a self-adjoint operator D satisfying certain conditions, the so-called Dirac operator. From the analysis given in the two-dimensional case, we know that to obtain a boundedness condition for D we need to use a twisted commutator. This amounts to introducing an automorphism σ of the algebra $\mathcal{A}$, the twist. Then for each $f \in \mathcal{A}$ the operator $[D, f]_\sigma = Df - \sigma(f)D$ should be bounded. We need to find what are the possible choices for D and the automorphism σ such that this condition is fulfilled. We consider some additional assumptions which are related to the symmetries and the classical limit, which we state precisely below. The analysis for the general n-dimensional case is essentially identical to the two-dimensional case, so we skip the details of the computations.

As in the classical case we enlarge the Hilbert space to accomodate for spinors. Therefore we consider $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^{[n/2]}$, where $\mathcal{H}_r$ is the Hilbert space previously introduced. Here $[n/2]$ is the dimension of the spinor bundle on $\mathbb{R}^n$, and we use the notation Γ^μ for the matrices representing the Clifford algebra, which satisfy $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$. Then we can write D in the form $D = \Gamma^\mu \hat{D}_\mu$, where $\hat{D}_\mu$ are self-adjoint operators on $\mathcal{H}_r$.

Now we state our assumptions for the Dirac operator D and the automorphism σ. We denote by ρ the map from the extended momentum algebra $\mathcal{T}_\kappa$ to (possibly unbounded) operators on $\mathcal{H}$, which is constructed similarly to the two-dimensional case. Since D should be determined by the symmetries, we assume that $\hat{D}_\mu = \rho(D_\mu)$ for some $D_\mu \in \mathcal{T}_\kappa$, which is basically the requirement of equivariance. Similarly we assume that σ is given by $\sigma(f) = \sigma \triangleright f$ for some $\sigma \in \mathcal{T}_\kappa$, which to be an automorphism must have a coproduct of the form $\Delta(\sigma) = \sigma \otimes \sigma$. Since the parameter λ is a *physical constant* of the model, which has the dimension of a length, the Dirac operator must have the dimension of an inverse length. Moreover we require that D reduces to the classical Dirac operator in the limit $\lambda \to 0$, by which we mean that for all $\psi \in \mathcal{A}$ we should have $\lim \hat{D}_\mu \psi = \hat{P}_\mu \psi$.

Proposition 6.6. *Under the assumptions given above, we have that there is a unique operator D and a unique automorphism σ such that $[D, f]_\sigma$ is bounded for every $f \in \mathcal{A}$. They are given by $D = \Gamma^\mu D_\mu$, with $D_0 = \lambda^{-1}(1 - e^{-\lambda P_0})$ and $D_j = P_j$, while $\sigma = e^{-\lambda P_0}$.*

Notice that formally for $\lambda \to 0$ we obtain the usual Dirac operator on $\mathbb{R}^n$. We have the interesting relation $D^2 = \Delta_\omega^{(n-1)^{-1}} C$, where Δ_ω is the modular operator of the weight ω and C is the first Casimir of the κ-Poincaré algebra, which is given by

$$C = \frac{4}{\lambda^2} \sinh^2 \left(\frac{\lambda P_0}{2} \right) + \sum_{j=1}^{n-1} e^{\lambda P_0} P_j^2 \ .$$

Now we discuss some aspects of the differential calculus associated with the operator D. For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ one defines the $\mathcal{A}$-bimodule Ω_D^1 of one-forms as the linear span of operators of the form $a[D, b]$, with $a, b \in \mathcal{A}$. Then $d(a) = [D, a]$ is a derivation of $\mathcal{A}$ with values in Ω_D^1, that is $d(ab) = d(a)b + ad(b)$, which immediately follows from the properties of the commutator. In the twisted case this definition must be modified, since we have

$$[D, ab]_\sigma = [D, a]_\sigma b + \sigma(a)[D, b]_\sigma \ .$$

The necessary modification is very simple [CoMo08]. One simply defines Ω_D^1 to be the linear span of operators of the form $a[D, b]_\sigma$, with the bimodule structure given by $a \cdot [D, b]_\sigma \cdot c = \sigma(a)[D, b]_\sigma c$. Then it is obvious that $d_\sigma(a) = [D, a]_\sigma$ is a derivation of $\mathcal{A}$ with values in Ω_D^1.

In the non-compact case, already at the untwisted level, it is not completely clear how one should generalize this notion. One can replace the algebra $\mathcal{A}$ with some unitization, as done in [GGISV04]. However in this case there is no analogue of the one-form dx^μ, since the function x^μ does not belong to $\mathcal{A}$ or some unitization of it. Nevertheless, it is clear that in the commutative case $[D, x^\mu]$ extends to a bounded operator, in particular it is equal to $-i\Gamma^\mu$. This is also the case in this non-commutative setting: indeed notice that for any $f \in \mathcal{A}$ we have the equality

$$[D, f]_\sigma = \Gamma^\mu (D_\mu \triangleright f) \ ,$$

where $D_0 = \lambda^{-1}(1 - e^{-\lambda P_0})$ and $D_j = P_j$. Now it is easy to see that the twisted commutator $[D, x^\mu]_\sigma$ extends to a bounded operator and in particular $[D, x^\mu]_\sigma = -i\Gamma^\mu$, as in the commutative case. Adopting the natural notation $df = [D, f]_\sigma$, we can write $df = dx^\mu (iD_\mu \triangleright f)$. Then from the bimodule structure on Ω_D^1 it then follows that

$$df = dx^\mu \cdot (iD_\mu \triangleright f) = \sigma^{-1}(iD_\mu \triangleright f) \cdot dx^\mu \ .$$

The introduction of bicovariant differential calculi on κ-Minkowski space was investigated in [Sit95]. It follows from our construction that the differential calculus defined by the operator

D is an example of such a bicovariant differential calculus. We have the relation

$$x^\mu \cdot dx^\nu - dx^\nu \cdot x^\mu = \sigma(x^\mu)dx^\nu - dx^\nu x^\mu = i\lambda\delta_0^\mu dx^\nu \ .$$

Therefore in the notation of [Sit95] we obtain $[x^\mu, dx^\nu] = iA_\rho^{\mu\nu} dx^\rho$ with $A_\rho^{\mu\nu} = \lambda\delta_0^\mu\delta_\rho^\nu$.

6.3 Spectral dimension

6.3.1 The spectral dimension

In the previous chapter we have shown that the analogue of this construction, for the two-dimensional case, does not give a finitely summable (twisted) spectral triple. This is true also for the general n-dimensional case, and we can try to interpret this result with the following heuristic argument. Suppose we did find a spectral dimension n, coinciding with the classical dimension. Then, from the general properties of twisted spectral triples, it would follow that $\varphi(ab) = \varphi(\sigma^n(b)a)$, where $\sigma(a) = e^{-\lambda P_0} a e^{\lambda P_0}$ and φ is the non-commutative integral (defined, for example, in terms of the Dixmier trace). The weight ω, on the other hand, satisfies $\omega(f \star g) = \omega(\sigma_i^\omega(g) \star f)$, where $\sigma_i^\omega(a) = e^{-(n-1)\lambda P_0} a e^{(n-1)\lambda P_0}$. Therefore we have a mismatch between the modular properties of the weight ω and the integral φ, which shows that we can not recover the weight ω from the non-commutative integral.

In the previous chapter we have argued that we need to use a weight to obtain finite summability, as in the framework of modular spectral triples. We recall that the relevant definition for us is the following.

Definition 6.2. Let $(\mathcal{A}, \mathcal{H}, D)$ be a non-compact modular spectral triple with weight Φ. We say that it is finitely summable and call p the *spectral dimension* if the following quantity exists

$$p := \inf\{s > 0 : \forall a \in \mathcal{A}, a \geq 0,\ \Phi\left(a(D^2+1)^{-s/2}\right) < \infty\} \ .$$

We can choose our weight to be of the form $\Phi(\cdot) = \mathrm{Tr}(\Delta_\Phi \cdot)$, where Δ_Φ is a positive and invertible operator. We call it the *modular operator* associated to the weight Φ, and denote the corresponding modular group by σ^Φ. As we discussed above, the mismatch of one power of $e^{-\lambda P_0}$ suggests setting $\Delta_\Phi = e^{-\lambda P_0}$ as the modular element. It is instructive to consider a slightly more general situation, which we discuss in the following proposition.

Proposition 6.7. *Let $\Phi_t(\cdot) = \mathrm{Tr}(\Delta_\Phi^t \cdot)$ be the weight with modular operator $\Delta_\Phi^t = e^{-t\lambda P_0}$, with $t \in \mathbb{R}$. Then, for any $f \in \mathcal{A}$, we have $\Phi_t\left(f(D^2+\mu^2)^{-s/2}\right) < \infty$ if and only if $t > 0$ and $s > n-1+t$. In this case the spectral dimension is given by $p = n-1+t$.*

Proof. First of all notice that we have $\Delta_\Phi^t f = \sigma^t(f)\Delta_\Phi^t$, so that we can consider without loss of generality the operator $A := f\Delta_\Phi^t(D^2+\mu^2)^{-s/2}$. Using the unitary operator U we can consider A as an operator on $L^2(\mathbb{R}^n \otimes \mathbb{C}^{2^{[n/2]}})$ whose symbol, thanks to Proposition 6.5, is given by $a(x,\xi) := (Uf)(x_0, e^{\lambda\xi_0}\vec{x})G^{\Delta}_{s,t}(\xi_0, e^{-\lambda\xi_0}\vec{\xi})$, where we have defined

$$G^{\Delta}_{s,t}(\xi) = e^{-t\lambda\xi_0}\left(\lambda^{-2}\left(1-e^{-\lambda\xi_0}\right)^2 + \vec{\xi}^2 + \mu^2\right)^{-s/2} .$$

To prove that the operator A is trace-class it suffices to show that its symbol and certain number of its derivatives are integrable [Ars08]. We now show that the symbol $a(x,\xi)$ is integrable. With a simple change of variables we can factorize the integral as

$$\int |a(x,\xi)| d^n x d^n \xi = \int G^{\Delta}_{s,t}(\xi) d^n \xi \int |(Uf)(x)| d^n x .$$

The integral of $(Uf)(x)$ is clearly finite. We can now perform the integral in the variables $(\xi_1, \cdots, \xi_{n-1})$ using the well known formula

$$\int (\xi^2 + a^2)^{-z/2} d^N \xi = \pi^{N/2} \frac{\Gamma\left(\frac{z-N}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} a^{-(z-N)} ,$$

which is valid for $\mathrm{Re}(z) > N$. Then we have

$$\int G^{\Delta}_{s,t}(\xi) d^n \xi = \pi^{(n-1)/2} \frac{\Gamma\left(\frac{s-(n-1)}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \int e^{-t\lambda\xi_0}\left(\lambda^{-2}\left(1-e^{-\lambda\xi_0}\right)^2 + \mu^2\right)^{-\frac{s-(n-1)}{2}} d\xi_0 ,$$

provided that $s > n-1$. To proceed further we consider the asymptotics of the integrand

$$\tilde{I}_t(s) := e^{-t\lambda\xi_0}\left(\lambda^{-2}\left(1-e^{-\lambda\xi_0}\right)^2 + \mu^2\right)^{-\frac{s-(n-1)}{2}} .$$

For $\xi_0 \to +\infty$ we have $\tilde{I}_t \sim e^{-t\lambda|\xi_0|}$, so it integrable provided that $t > 0$, independently of s. In the other regime $\xi_0 \to -\infty$ we have instead

$$\tilde{I}_t \sim e^{t\lambda|\xi_0|} e^{-(s-(n-1))\lambda|\xi_0|} = e^{-(s-(n-1)-t)\lambda|\xi_0|} ,$$

which is integrable when $s > n-1+t$. It is easy to see that the various derivatives of the symbol $a(x,\xi)$ are integrable under these conditions. Finally taking the infimum over s we obtain that the spectral dimension is $p = n-1+t$. □

We see that by introducing the weight Φ_t we are able to obtain a finite spectral dimension. But what about the free parameter t? Of course the most natural choice is to fix $t = 1$, in such a way that the spectral dimension coincides with the classical dimension n. We now want to argue that this ambiguity arises because different values of t give rise to the same modular group for the non-commutative integral. The argument that we give here is somewhat heuristic because of the difficulties involved with treating a non-compact geometry, but can be easily justified in the compact setting under some mild assumptions [Mat3].

Let us say that we define our non-commutative integral in terms of Φ_t, as we will do later as a residue of a certain zeta function. Then on general grounds we expect to have

$$\varphi(ab) = \varphi\left(\sigma_i^{\Phi_t}\left(\sigma^p(b)\right)a\right) \ .$$

Here σ is the twist coming from the twisted commutator and σ^Φ is the modular group of Φ. With respect to the case of twisted spectral triples there is an additional twisting, coming from the modular operator of Φ_t. Now we have that the twist of the commutator is $\sigma(a) = e^{-\lambda P_0} a e^{\lambda P_0}$ while the modular group of the weight is $\sigma_i^{\Phi_t}(a) = e^{t\lambda P_0} a e^{-t\lambda P_0}$. Finally for $t > 0$ the spectral dimension is $p = n - 1 + t$. Using these formulae we can compute

$$\begin{aligned} \sigma_i^{\Phi_t}\left(\sigma^p(b)\right)a &= e^{t\lambda P_0} e^{-(n-1+t)\lambda P_0} b e^{(n-1+t)\lambda P_0} e^{-t\lambda P_0} a \\ &= e^{-(n-1)\lambda P_0} b e^{(n-1)\lambda P_0} a = \sigma_i^\omega(b) a \ . \end{aligned}$$

We learn, from this short computation, that this specific combination allows us to recover the KMS condition for the weight ω. This provides strength to the argument that recovering the weight ω from the non-commutative integral provides the right guidance in this setting. At the same time it shows that this is not enough to fix the free parameter t, since it disappears in the combination $\sigma^{\Phi_t} \circ \sigma^p$. We do not know at the moment what kind of condition could select the value $t = 1$ uniquely (apart from recovering the classical dimension, of course).

6.3.2 Poles of the zeta function

In the following we fix $t = 1$. Then the function $\Phi\left(f(D^2 + \mu^2)^{-s/2}\right)$ has a singularity at $s = n$, whose nature we now want to investigate, along with its analytic continuation to the complex plane. The singularities of this kind of "zeta function" play an important role in the local index formula of Connes and Moscovici [CoMo95]. Before starting the analysis, let us briefly review the commutative case of $\mathbb{R}^n$. The Dirac operator is given by $D = -i\Gamma^\mu \partial_\mu$, where Γ^μ are the gamma matrices satisfying the relations $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$ and the dimension of the spinor bundle is $2^{[n/2]}$. We consider the zeta function defined by

$$\zeta_f(z) = \mathrm{Tr}\left(f(D^2 + \mu^2)^{-z/2}\right) \ .$$

Here μ is, as usual, a non-zero real number needed to compensate for the lack of invertibility of D. An immediate computation shows that

$$\zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n} \int \left(\xi^2 + \mu^2\right)^{-z/2} d^n\xi \int f(x) d^n x \ ,$$

where the coefficient $2^{[n/2]}$ comes from the trace over the spinor bundle. The integral over ξ is finite for $\mathrm{Re}(z) > n$ and we get

$$I_c(z) := \int \left(\xi^2 + \mu^2\right)^{-z/2} d^n\xi = \pi^{n/2}\mu^{n-z}\frac{\Gamma\left(\frac{z-n}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} . \tag{6.3}$$

We obtain an analytic continuation using well-known properties of the gamma function, and we find that the only singularities of $\xi_f(z)$ are simple poles. Indeed $\Gamma(z)$ has poles on the negative real axis at $z = 0, -1, -2, \cdots$, so that the function $\Gamma\left(\frac{z-n}{2}\right)$ has poles at $z = n - 2m$, where $m \in \mathbb{N}_0$. When n is even the poles at $z = 0, -2, -4, \cdots$ are canceled by the zeroes of $\Gamma\left(\frac{z}{2}\right)$. Then the result is that $\zeta_f(z)$ has simple poles at $z = n, n-2, \cdots, 2$ when n is even, and has simple poles at $z = n, n-2, \cdots, 1, -1, -3, \cdots$ when n is odd.

For compact Riemannian manifolds this kind of zeta function has been studied by Minakshisundaram and Pleijel [MiPl49], and here we have the analogous result for $\mathbb{R}^n$. We can easily compute the residue at $z = n$ of $\zeta_f(z)$, which is given by

$$\mathrm{Res}_{z=n}\zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n}\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int f(x)d^n x .$$

Now we are ready to study the singularities and the analytic continuation in the case of κ-Minkowski space, where the relevant zeta function is defined by

$$\zeta_f(z) := \Phi\left(f(D^2+\mu^2)^{-z/2}\right) ,$$

where we recall that $\Phi(\cdot) = \mathrm{Tr}(\Delta_\Phi \cdot)$ and we omit the representation symbol π.

Proposition 6.8. *Let $f \in \mathcal{A}$ and $\mathrm{Re}(z) > n$. Then we have*

$$\zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n} I(z) \int f(x)d^n x ,$$

where $I(z) = \frac{1}{2}(I_c(z) + I_\lambda(z))$, with the function $I_c(z)$ being the classical result given in (6.3), which is independent of λ, and the function $I_\lambda(z)$ being given by

$$I_\lambda(z) = \pi^{(n-1)/2}\mu^{(n-1)-z}\frac{\Gamma\left(\frac{z-(n-1)}{2}\right)}{\Gamma\left(\frac{z}{2}\right)}\lambda^{-1}{}_2F_1\left(\frac{1}{2}, \frac{z-(n-1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda\mu)^2}\right) .$$

The function $I(z)$ reduces to the classical one $I_c(z)$ in the limit $\lambda \to 0$.

Proof. From the proof of Proposition 6.7 we have

$$\mathrm{Tr}\left(f\Delta_\Phi(D^2+\mu^2)^{-z/2}\right) = \frac{2^{[n/2]}}{(2\pi)^n}\int G_s^\Delta(\xi)d^n\xi \int (Uf)(x)d^n x ,$$

where we recall that we have set $t = 1$. Similarly to the classical case we set $I(z) := \int G_s^{\Delta}(\xi) d^n\xi$. We already partially computed this integral, and the result was

$$I(z) = \pi^{(n-1)/2} \frac{\Gamma\left(\frac{z-(n-1)}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} \int e^{-t\lambda\xi_0} \left(\lambda^{-2}\left(1-e^{-\lambda\xi_0}\right)^2 + \mu^2\right)^{-\frac{z-(n-1)}{2}} d\xi_0 \ .$$

We need to compute the last integral. First we do the change of variable $r = e^{-\lambda\xi_0}$ and obtain

$$I(z) = \pi^{(n-1)/2} \frac{\Gamma\left(\frac{z-(n-1)}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} \lambda^{z-n} \int_0^{\infty} \left((1-r)^2 + (\lambda\mu)^2\right)^{-\frac{z-(n-1)}{2}} dr \ .$$

This integral can be computed analytically. We use the formula

$$\int_0^{\infty} \left((1-r)^2 + a^2\right)^{-z} dr = a^{-2z} \left[\frac{a\sqrt{\pi}}{2} \frac{\Gamma\left(z-\frac{1}{2}\right)}{\Gamma(z)} + {}_2F_1\left(\frac{1}{2}, z; \frac{3}{2}; -\frac{1}{a^2}\right)\right] ,$$

which is valid for $\mathrm{Re}(z) > 1/2$. Here ${}_2F_1(a,b;c;z)$ is the ordinary hypergeometric function. Therefore the integral in $I(z)$ is finite for $\mathrm{Re}(z) > n$ and we have

$$I(z) = \frac{1}{2} \pi^{n/2} \mu^{n-z} \frac{\Gamma\left(\frac{z-n}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} + \frac{1}{2} I_\lambda(z) \ ,$$

where we have defined the function

$$I_\lambda(z) := 2\pi^{(n-1)/2} \mu^{(n-1)-z} \frac{\Gamma\left(\frac{z-(n-1)}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} \lambda^{-1} {}_2F_1\left(\frac{1}{2}, \frac{z-(n-1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda\mu)^2}\right) \ .$$

Notice that we have $I(z) = \frac{1}{2}(I_c(z) + I_\lambda(z))$. Finally we have $\int Uf = \int f$, which is valid for $f \in \mathcal{A}$, from which the first part of the proposition follows.

Now we want to consider the classical limit of $I(z)$, in the case $\mathrm{Re}(z) > n$. Using the linear transformation formulae for the hypergeometric function ${}_2F_1(a,b;c;z)$ it is easy to obtain an asymptotic expansion for large negative z, see [AbSt70]. This expansion takes the form

$${}_2F_1(a,b;c;-z) \sim \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} z^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} z^{-b} \ .$$

With this result it is easy to compute the limit

$$\lim_{\lambda\to 0} \lambda^{-1} {}_2F_1\left(\frac{1}{2}, \frac{z-(n-1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda\mu)^2}\right) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{z-n}{2}\right)}{\Gamma\left(\frac{z-(n-1)}{2}\right)} \mu \ .$$

Then we see that $I(z)$ reduces to $I_c(z)$ in the classical limit $\lambda \to 0$. □

Corollary 6.9. *For $f \in \mathcal{A}$ we have*

$$\mathrm{Res}_{z=n}\zeta_f(z) = c_n\omega(f) \ ,$$

where the constant is defined as $c_n = \frac{2^{[n/2]}}{(2\pi)^n}\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$.

Proof. The function $I_\lambda(z)$ is regular at $z = n$, so we get one-half of the classical residue, that is we have $\mathrm{Res}_{z=n}I(z) = \pi^{n/2}/\Gamma\left(\frac{n}{2}\right)$. The result follows immediately. □

Proposition 6.10. *Let $f \in \mathcal{A}$. Then the zeta function*

$$\zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n} I(z) \int f(x) d^n x$$

has a meromorphic extension to the complex plane with only simple poles.

Proof. Since $I(z) = \frac{1}{2}(I_c(z) + I_\lambda(z))$, where $I_c(z)$ is the integral (6.3) arising in the commutative case, the zeta function $\zeta_f(z)$ has the poles of the commutative case plus additional poles coming from the function $I_\lambda(z)$. To study them consider the hypergeometric function ${}_2F_1(a, b; c; z)$, with the assumption that c does not belong to $\{0, -1, -2, \cdots\}$. The series defining ${}_2F_1(a, b; c; z)$ is convergent in the open disk $|z| < 1$, but can be analytically continued to the entire complex plane with a branch cut from $z = 1$ to $z = \infty$. Therefore the function

$${}_2F_1\left(\frac{1}{2}, \frac{z-(n-1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda\mu)^2}\right)$$

does not have any poles in z. Now recall that the function $I_\lambda(z)$ is defined by

$$I_\lambda(z) = 2\pi^{(n-1)/2}\lambda^{-1}\mu^{(n-1)-z}\frac{\Gamma\left(\frac{z-(n-1)}{2}\right)}{\Gamma\left(\frac{z}{2}\right)}{}_2F_1\left(\frac{1}{2}, \frac{z-(n-1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda\mu)^2}\right) \ .$$

Therefore the only poles of this function come from the ratio of the two gamma functions. These are simple poles, from which the claim follows. □

It is interesting to note that the poles of the function $I_c(z)$ come from the ratio $\Gamma\left(\frac{z-n}{2}\right)/\Gamma\left(\frac{z}{2}\right)$, while the poles of the function $I_\lambda(z)$ come from the ratio $\Gamma\left(\frac{z-(n-1)}{2}\right)/\Gamma\left(\frac{z}{2}\right)$: the latter are therefore the poles of the $(n-1)$-dimensional case. If we think of the Lorentzian version of κ-Minkowski space, we can relate this result to the different properties of the time direction from the space directions, evident already from the commutation relations.

6.4 Twisted homology

6.4.1 Motivation and preliminaries

In this section we want to study the homological properties of κ-Minkowski space. In the non-compact setting it is not completely clear, at least as far as we understand, which algebra should be considered in this respect. A possibility is to consider a certain unitization of the algebra $\mathcal{A}$ in consideration, as done in [GGISV04]. On the other hand, already at the commutative level, if we consider $\mathbb{R}^n$ we would like to have an analog of the volume form $dx^1 \wedge \cdots \wedge dx^n$, but is clear that the functions x^μ do not belong to a unital algebra.

Our aim is to investigate the homological properties of the enveloping algebra $U(\mathfrak{g}_\kappa)$, where $\mathfrak{g}_\kappa$ is the Lie algebra underlying κ-Minkowski space. General results for the twisted homology of an enveloping algebra are given in [BrZh08], where the twist is called the *Nakayama automorphism*. Here we choose a more elementary approach, which involves the explicit computation using the Chevalley-Eilenberg complex for the Lie algebra $\mathfrak{g}_\kappa$. This choice also allows us to do a more detailed comparison with other examples coming from quantum groups.

Let us start by recalling some notions from homological algebra, following the exposition given in [Kha]. Let $\mathfrak{g}$ be a Lie algebra and M be a left $\mathfrak{g}$-module. The *Lie algebra homology* of $\mathfrak{g}$ with coefficients in M is, by definition, the homology of the *Chevalley-Eilenberg complex*

$$M \xleftarrow{\delta} M \otimes \Lambda^1\mathfrak{g} \xleftarrow{\delta} M \otimes \Lambda^2\mathfrak{g} \xleftarrow{\delta} \cdots \ ,$$

where $\Lambda^k\mathfrak{g}$ denotes the k-th exterior power of $\mathfrak{g}$ and the differential δ is defined by

$$\begin{aligned} \delta(m \otimes X_1 \wedge \cdots \wedge X_n) = &\sum_{i<j}^{n} (-1)^{i+j} m \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n \\ &+ \sum_{i=1}^{n} (-1)^i X_i(m) \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n \ , \end{aligned} \tag{6.4}$$

where the hat denotes omission. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Given a $U(\mathfrak{g})$-bimodule M, we define the left $\mathfrak{g}$-module M^{ad}, where $M^{ad} = M$ as vector spaces and the left module structure is defined for all $X \in \mathfrak{g}$ and $m \in M$ by

$$X(m) = Xm - mX \ .$$

We can define a map

$$\varepsilon : M^{ad} \otimes \Lambda^n\mathfrak{g} \to M \otimes U(\mathfrak{g})^{\otimes n}$$

from the Lie algebra complex to the Hochschild complex by

$$\varepsilon(m \otimes X_1 \wedge \cdots \wedge X_n) = \sum_{s \in S_n} \mathrm{sgn}(s) m \otimes X_{s(1)} \otimes \cdots \otimes X_{s(n)} \ .$$

One can prove that $\varepsilon : C(\mathfrak{g}, M^{ad}) \to C(U(\mathfrak{g}), M)$ is a quasi-isomorphism, so it induces an isomorphism between the corresponding homology groups

$$H_*(\mathfrak{g}, M^{ad}) \cong H_*(U(\mathfrak{g}), M) \ .$$

In particular if we choose $M = {}_\sigma U(\mathfrak{g})$, that is $U(\mathfrak{g})$ with the bimodule structure $a \cdot b \cdot c = \sigma(a)bc$, then on the right we have the *twisted Hochschild homology* $H_*(U(\mathfrak{g}), {}_\sigma U(\mathfrak{g}))$. The *twisted Hochschild dimension* is defined, according to [BrZh08], as the maximum of the homological dimension of $H_*(U(\mathfrak{g}), {}_\sigma U(\mathfrak{g}))$ over all the automorphisms σ of $U(\mathfrak{g})$. The case $\sigma = \mathrm{id}$ gives the usual Hochschild homology. Interest in this twisted homology theory comes from the fact that, in several examples coming from quantum groups, it allows to avoid the phenomenon of dimension drop. We will see that this is the case also here.

6.4.2 The two dimensional case

To make the general result more transparent, it is useful to show first the relevant computations for the two dimensional case. We also find convenient, for notational clarity, to write x_1, x_2 instead of x_0, x_1 as in the previous sections. Since the Lie algebra is two dimensional the complex is given by

$$M \overset{\delta}{\leftarrow} M \otimes \Lambda^1 g \overset{\delta}{\leftarrow} M \otimes \Lambda^2 g \overset{\delta}{\leftarrow} 0 \ .$$

The differential δ acting on $M \otimes \Lambda^2 g$ takes the form

$$\delta(m \otimes X_1 \wedge X_2) = -m \otimes [X_1, X_2] - X_1(m) \otimes X_2 + X_2(m) \otimes X_1 \ .$$

We write X_1 and X_2 in the x_1, x_2 basis as $X_1 = c_1^1 x_1 + c_1^2 x_2$ and $X_2 = c_2^1 x_1 + c_2^2 x_2$, for some coefficients. Their commutator is given by

$$[X_1, X_2] = (c_1^1 c_2^2 - c_2^1 c_1^2) i\lambda x_2 \ .$$

Notice that for $m \otimes X_1 \wedge X_2$ to be non-trivial we need $c_1^1 c_2^2 - c_2^1 c_1^2 \neq 0$. Indeed we have

$$m \otimes X_1 \wedge X_2 = (c_1^1 c_2^2 - c_2^1 c_1^2) m \otimes x_1 \wedge x_2 \ .$$

Proposition 6.11. *The twisted homological dimension of $U(\mathfrak{g}_\kappa)$ is equal to two.*

Proof. Since $\Lambda^3 \mathfrak{g}$ is trivial we only have to show that there exists a non-trivial element $m \otimes X_1 \wedge X_2$ such that $\delta(m \otimes X_1 \wedge X_2) = 0$. Computing the differential we get

$$\delta(m \otimes X_1 \wedge X_2) = -(c_1^1 c_2^2 - c_2^1 c_1^2) \left((i\lambda m + x_1(m)) \otimes x_2 - x_2(m) \otimes x_1 \right) \ .$$

Since $c_1^1 c_2^2 - c_2^1 c_1^2 \neq 0$, the condition $\delta(m \otimes X_1 \wedge X_2) = 0$ implies

$$(i\lambda m + x_1(m)) \otimes x_2 - x_2(m) \otimes x_1 = 0 \ .$$

This in turn implies the conditions $x_2(m) = 0$ and $i\lambda m + x_1(m) = 0$.

We have $X(m) = \sigma(X)m - mX$, where σ is an automorphism of the form

$$\sigma(x_1) = x_1 + i\lambda\mu \ , \qquad \sigma(x_2) = x_2 \ ,$$

with $\mu \in \mathbb{C}$. By the Poincaré–Birkhoff–Witt theorem we can write $m \in U(\mathfrak{g}_\kappa)$ as

$$m = \sum_{a,b} f_{a,b} x_1^a x_2^b \ ,$$

where the sum is finite, $f_{a,b}$ are numerical coefficients and the exponents are non-negative integers. Since the automorphism σ acts trivially on x_2, the condition $x_2(m) = 0$ implies that m commutes with x_2, that is m should not depend on x_1.

The second condition, on the other hand, can be rewritten as

$$i\lambda m + x_1(m) = i\lambda(1+\mu)m + [x_1, m] = 0 \ .$$

An easy computation then shows that

$$[x_1, m] = \sum_b f_{0,b}[x_1, x_2^b] = i\lambda \sum_b f_{0,b} b x_2^b \ .$$

Plugging this result into $i\lambda m + x_1(m) = 0$ we obtain

$$\begin{aligned} i\lambda m + x_1(m) &= i\lambda(1+\mu)\sum_b f_{0,b} x_2^b + i\lambda \sum_b f_{0,b} b x_2^b \\ &= \sum_b f_{0,b} i\lambda(1+b+\mu) x_2^b = 0 \ . \end{aligned}$$

For each $b \in \mathbb{N}_0$, the corresponding term in the sum vanishes if $f_{0,b} = 0$ or $\mu = -(1+b)$. Since at least one of the coefficients must be non-zero, we must have $f_{0,b} \neq 0$ and $\mu = -(1+b)$ for some $b \in \mathbb{N}_0$, which in turn implies that all the other coefficients must be zero. For any such choice we obtain an element m such that $\delta(m \otimes X_1 \wedge X_2) = 0$, which concludes the proof. □

In particular, for the case $b = 0$ the automorphism $\sigma(x_1) = x_1 - i\lambda$, $\sigma(x_2) = x_2$ corresponds to the inverse of the modular group considered in the previous chapter.

6.4.3 The n-dimensional case

Let us write $\delta = \delta_1 + \delta_2$, where δ_1 and δ_2 are given respectively by the first and second line of equation (6.4). To study the case of a general dimension we start by proving two lemmata, which allows to rewrite the differential in a easier form. The first one is valid for any Lie algebra $\mathfrak{g}$, and simply requires some gymnastics with differential forms, while the second one is related to the simple structure of the commutation relations of the Lie algebra $\mathfrak{g}_\kappa$.

Lemma 6.12. *Let $X_i \in \mathfrak{g}$ be given by $X_i = c_i^j x_j$, where c_i^j are numerical coefficients and $\{x_j\}$ is a basis of the Lie algebra $\mathfrak{g}$. Then we have*

$$\delta_2(m \otimes X_1 \wedge \cdots \wedge X_n) = \det C \sum_{j=1}^{n} x_j(m) \otimes x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_n \ ,$$

where C is the matrix formed by the coefficients c_i^j.

Proof. Denoting by $C_{i,j}$ the (i,j)-minor of the matrix C we can write

$$X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_n = \sum_{j=1}^{n} C_{i,j} x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_n \ .$$

If we expand X_i in the basis of the generators we can write

$$X_i(m) = \sum_{k=1}^{n} c_i^k x_k(m) \ .$$

Then the second line of the differential δ given by (6.4) becomes

$$\delta_2 = \sum_{j=1}^{n} \sum_{k=1}^{n} x_k(m) \otimes \sum_{i=1}^{n} (-1)^i c_i^k C_{i,j} x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_n \ .$$

The sum over i of $(-1)^i c_i^k C_{i,j}$ looks like a Laplace expansion of the determinant of some matrix. Indeed, it is the determinant of the matrix obtained from C by replacing the j-th column, given by c_a^j with $a = 1, \cdots, n$, with the column c_a^k. If $k \neq j$ then, after this replacement, we obviously have two linearly dependent columns, so the determinant vanishes. On the other hand if $k = j$ we obtain $\det C$, independent of j. So we can write

$$\sum_{i=1}^{n} (-1)^i c_i^k C_{i,j} = \det C \delta_j^k \ .$$

Plugging this result into the previous formula we find the result. □

Lemma 6.13. *With the same notation as above, consider the Lie algebra $\mathfrak{g}_\kappa$ with commutation relations $[x_1, x_j] = i\lambda x_j$, where $j > 1$. Then we have*

$$\delta_1(m \otimes X_1 \wedge \cdots \wedge X_n) = \det C i\lambda(n-1) m \otimes \widehat{x}_1 \wedge x_2 \wedge \cdots \wedge x_n \ .$$

Proof. We start by computing the commutator of two elements X_i

$$[X_i, X_j] = i\lambda \sum_{k=1}^{n} (c_i^1 c_j^k - c_i^k c_j^1) x_k = i\lambda (c_i^1 X_j - c_j^1 X_i) \ .$$

Then the first line of the differential δ given by (6.4) becomes

$$\delta_1 = \sum_{i<j} i\lambda c_i^1 (-1)^{i+j} m \otimes X_j \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n$$
$$- \sum_{i<j} i\lambda c_j^1 (-1)^{i+j} m \otimes X_i \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n \ .$$

Now we can bring X_i and X_j to their missing spots, picking up some signs. When we move X_i we have to go across $i-1$ terms, so we pick a $(-1)^{i-1}$, while when we move X_j we have to go across $j-2$ terms, since also X_i is missing, so we pick a $(-1)^{j-2}$. Then we have

$$\delta_1 = \sum_{i<j} i\lambda c_i^1 (-1)^i m \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n$$
$$+ \sum_{i<j} i\lambda c_j^1 (-1)^j m \otimes \wedge X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n \ .$$

Notice that now j and i do not appear anymore respectively in the first and the second sum. It is not difficult to see that we can rewrite them as

$$\delta_1 = \sum_{i=1}^{n} i\lambda (n-i) c_i^1 (-1)^i m \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n$$
$$+ \sum_{j=1}^{n} i\lambda (j-1) c_j^1 (-1)^j m \otimes \wedge X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n \ .$$

Summing the two contributions we get

$$\delta_1 = i\lambda(n-1) \sum_{i=1}^{n} (-1)^i c_i^1 m \otimes \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n \ .$$

Writing the wedge products in terms of the minors of C we obtain

$$\delta_1 = i\lambda(n-1) \sum_{j=1}^{n} \sum_{i=1}^{n} (-1)^i c_i^1 C_{i,j} m \otimes \wedge x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_n \ .$$

Finally, using the same arguments of the previous lemma, we obtain

$$\delta_1 = \det C i\lambda (n-1) m \otimes \widehat{x}_1 \wedge x_2 \wedge \cdots \wedge x_n \ .$$

This concludes the proof of the lemma. □

Theorem 6.14. *Let $\mathfrak{g}_\kappa$ be the Lie algebra associated with κ-Minkowski space in n-dimensions, which is characterized by the commutation relations $[x_1, x_j] = i\lambda x_j$, where $j > 1$. Then the twisted homological dimension of $U(\mathfrak{g}_\kappa)$ is equal to n.*

Proof. As in the two dimensional case, we only need to show that there is an element $m \otimes X_1 \wedge \cdots \wedge X_n$ such that $\delta(m \otimes X_1 \wedge \cdots \wedge X_n) = 0$. Putting together the two previous lemmata we have the following expression for the differential

$$\begin{aligned}\delta(m \otimes X_1 \wedge \cdots \wedge X_n) &= \det C i\lambda(n-1) m \otimes \widehat{x}_1 \wedge x_2 \wedge \cdots \wedge x_n \\ &\quad + \det C \sum_{j=1}^{n} x_j(m) \otimes x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_n \ .\end{aligned}$$

Since $\det C$ is different from zero, we need to impose the conditions $x_j(m) = 0$ for $j = 2, \cdots, n$. Again we have to consider automorphisms which are of the form

$$\sigma(x_1) = x_1 + i\lambda\mu \ , \qquad \sigma(x_j) = x_j \ ,$$

with $\mu \in \mathbb{C}$. Then $x_j(m) = [x_j, m] = 0$ implies that m does not depend on x_1. The other condition we need to impose is $i\lambda(n-1)m + x_1(m) = 0$, which can be rewritten in the form

$$i\lambda(n - 1 + \mu)m + [x_1, m] = 0 \ .$$

By the Poincaré–Birkhoff–Witt theorem we can write $m \in U(\mathfrak{g}_\kappa)$ as

$$m = \sum_{a_2,\cdots,a_n} f_{0,a_2,\cdots,a_n} x_2^{a_2} \cdots x_n^{a_n} \ ,$$

where the sum is finite, $f_{0,a_2,\cdots,a_n}$ are numerical coefficients and the exponents are non-negative integers. We have already imposed the condition that m does not depend on x_1. Now we compute the commutator of m with x_1

$$\begin{aligned}[x_1, m] &= \sum_{a_2,\cdots,a_n} f_{0,a_2,\cdots,a_n} [x_1, x_2^{a_2} \cdots x_n^{a_n}] \\ &= \sum_{a_2,\cdots,a_n} f_{0,a_2,\cdots,a_n} \left([x_1, x_2^{a_2}] x_3^{a_3} \cdots x_n^{a_n} + \cdots + x_2^{a_2} \cdots x_{n-1}^{a_{n-1}} [x_1, x_n^{a_n}]\right) \\ &= i\lambda \sum_{a_2,\cdots,a_n} f_{0,a_2,\cdots,a_n} (a_2 + \cdots + a_n) x_2^{a_2} \cdots x_n^{a_n} \ .\end{aligned}$$

Using this result we finally obtain

$$i\lambda m + x_1(m) = \sum_{a_2,\cdots,a_n} f_{0,a_2,\cdots,a_n} i\lambda\big(n - 1 + \sum_{k=2}^{n} a_k + \mu\big) x_2^{a_2} \cdots x_n^{a_n} = 0 \ .$$

For each $(a_2, \cdots, a_n) \in \mathbb{N}_0^{n-1}$, the corresponding term in the sum vanishes if $f_{0,a_2,\cdots,a_n} = 0$ or $\mu = -(n - 1 + \sum_{k=2}^{n} a_k)$. Since at least one of the coefficients must be non-zero, we must have $f_{0,a_2,\cdots,a_n} \neq 0$ and $\mu = -(n-1+\sum_{k=2}^{n} a_k)$ for some $(a_2, \cdots, a_n) \in \mathbb{N}_0^{n-1}$, which in turn implies that all the other coefficients must be zero. For any such choice we obtain an element

m such that $\delta(m \otimes X_1 \wedge \cdots \wedge X_n) = 0$, which concludes the proof. □

In particular, by choosing $a_k = 0$ for all $k = 2, \cdots, n$, we obtain the automorphism given by $\sigma(x_1) = x_1 - i(n-1)\lambda$ and $\sigma(x_j) = x_j$. This is exactly the inverse of the modular group σ^{ω}_{-i} of the weight ω we introduced in the first part, which was the starting point for the construction. Other choices give negative powers of σ^{ω}_{-i}. This is very similar to what happens for the twisted homology of $SL_q(2)$ [HaKr05] and for the Podleś spheres [Had07].

There is another feature of this result which is worth mentioning. We fix now the simplest non-trivial cycle. Passing from from the Lie algebra complex to the Hochschild complex via

$$\varepsilon(m \otimes X_1 \wedge \cdots \wedge X_n) = \sum_{s \in S_n} \mathrm{sgn}(s) m \otimes X_{s(1)} \otimes \cdots \otimes X_{s(n)} ,$$

we see that this cycle corresponds to

$$c = \det C \sum_{s \in S_n} \mathrm{sgn}(s) 1 \otimes x_{s(1)} \otimes \cdots \otimes x_{s(n)} .$$

We notice that it has the same form as in commutative case. Indeed the analogy goes further since, as we discussed in the section on the Dirac operator and the differential calculus, we have that $[D, x^{\mu}]_{\sigma} = -i\Gamma^{\mu}$. Therefore, if we represent this cycle on the Hilbert space by $a_0[D, a_1]_{\sigma} \cdots [D, a_n]_{\sigma}$, we get exactly the orientation cycle of the commutative case, which corresponds to the volume form $dx^1 \wedge \cdots \wedge dx^n$.

Chapter 7

Non-commutative integration, zeta functions and the Haar state for $SU_q(2)$

In this chapter we study a notion of non-commutative integration, in the spirit of modular spectral triples, for the quantum group $SU_q(2)$. We define the non-commutative integral as the residue at the spectral dimension of a zeta function, which is constructed using a Dirac operator and a weight. We consider the Dirac operator introduced in [KaSe12] and a family of weights depending on two parameters, which are related to the diagonal automorphisms of $SU_q(2)$. Requiring the non-commutative integral to coincide with the Haar state fixes one of the parameters. Moreover, by imposing an additional condition on the zeta function, also the second parameter can be fixed. For this unique choice the spectral dimension coincides with the classical one. This is based on the paper [Mat3].

7.1 Introduction

Many works have been devoted to studying how quantum groups and their homogeneous spaces fit into the framework of spectral triples. In the last years, in particular, there have been several proposals of extensions of this notion, which could be used to accomodate these classes of non-commutative geometries. This is not unexpected, since the axioms of a spectral triple are tailored on the case of manifolds, and one expects that new features, which appear only in the non-commutative world, should play a role in the description of non-commutative geometries. In Chapter 4 we have presented the approaches of twisted and modular spectral triples, with the first one modifying the commutator condition and the second one, roughly speaking, modifying the resolvent condition. Also, more recently, there has been an attempt to merge these two approaches [Kaa11], motivated by a construction for the quantum group $SU_q(2)$ which appeared in [KaSe12]. However there is a natural question that arises by considering the framework of modular spectral triples: if we are allowed to replace the operator trace by a weight, are there any preferred choices?

Let us consider this question for the case of compact quantum groups, where we can be guided by symmetry. In this setting it is well known that there is a unique state, the Haar state, which is the non-commutative analogue of the Haar integral for compact Lie groups. But the choice of a state gives a notion of non-commutative integration, as it is known from the theory of von Neumann algebras [Tak]. Therefore it would seem natural, from the point of view of spectral triples, to require that the non-commutative integral coincides with the Haar state. However it is clear that this is not possible in the usual setting: indeed, from the general properties of spectral triples, it follows that the non-commutative integral is a trace, while the Haar state does not satisfy the trace property. On the other hand, in the extended frameworks we mentioned above the non-commutative integral need not be a trace, so that such a requirement can be in principle satisfied. This could be used as a reasonable criterion to choose a weight in the context of modular spectral triples.

Here we will consider this question in detail for the case of the quantum group $SU_q(2)$. More specifically we consider the Dirac operator D_q introduced in [KaSe12], which gives a (twisted) modular spectral triple. We observe that this Dirac operator has an interesting property, namely it implements a left covariant differential calculus on $SU_q(2)$.

The non-commutative integral will be defined as the residue at the spectral dimension of a certain zeta function. More precisely, we define a family of zeta functions using the operator D_q and a family of weights depending on two parameters $a, b \in \mathbb{R}$. These two parameters essentially parametrize the most general diagonal automorphism of $SU_q(2)$, and we remark that the modular group of the Haar state is of this form.

First of all we determine for which values of the parameters the zeta function is well defined, and determine its spectral dimension. Then we impose the requirement of recovering the Haar state from the non-commutative integral. A necessary condition is that their modular groups coincide. We will show that this condition fixes $b = 1$, but leaves a undetermined. Moreover the non-commutative integral, once properly normalized, turns out to coincide with the Haar state, independently of the value of a.

This result shows that we can partially fix the arbitrariness in the choice of the weight. We still have freedom in the choice of the parameter a, which disappears in the non-commutative integral. On the other hand the spectral dimension depends on a. In particular, after fixing $b = 1$, we have that $n = a+1$. Therefore a preferred choice is $a = 2$, which makes the spectral dimension equal the classical one $n = 3$.

We now argue that there is another requirement, more spectral in nature, that also turns out to fix this value uniquely. Up to this point we have only used the information contained in the residue at $z = n$ of the zeta function, that is the residue at the spectral dimension. But the analytic continuation of the zeta function contains much more information than that. Indeed, from the point of view of the heat kernel expansion on a compact manifold, the residue at $z = n$ corresponds only to the first coefficient of the expansion. Therefore we can look at the next non-trivial coefficient which, in terms of the zeta function, corresponds to computing the residue at a different value. It is easy to see that, for the classical limit of the operator D_q, this coefficent vanishes non-trivially. Therefore we can require an analogue condition for the non-commutative case. It turns out that this condition is satisfied only in the case $a = 2$, which was the value we considered above.

From these results we see that the parameters a and b control the behaviour of the heat kernel coefficients. This is by no means obvious from their definition. The parameter b controls the first coefficient and can be fixed by the request of recovering the Haar state. Instead the parameter a controls the second heat kernel coefficient, and can be fixed by a non-trivial vanishing condition. In particular, for this unique choice the spectral dimension turns out to coincide with the classical one.

7.2 Non-commutative integration

In the description of non-commutative geometry via spectral triples a fundamental role is played by the Dirac operator D, which is used to formulate many geometrical notions at the level of operator algebra. For example the notion of integration, which is our main focus, can be expressed as the Dixmier trace of a certain power of this operator. Indeed, in the case of a compact manifold of dimension n, for any continuous function f the Dixmier trace of $f|D|^{-n}$ coincides with the usual integral, up to a normalization constant. However computing the Dixmier trace is not an easy task in general, and for this reason it is useful to reformulate this notion of integration in a way which is easier to handle.

One such reformulation is achieved by defining the non-commutative integral as a residue of a zeta function involving D, as is done for example in the case of the local index formula [CoMo95]. Going back to the manifold case, we can define this zeta function as $\zeta_f(z) = \mathrm{Tr}(f|D|^{-z})$, for $z \in \mathbb{C}$. Then $\zeta_f(z)$ is holomorphic for all $\mathrm{Re}(z) > n$ and the residue at $z = n$ coincides with the integral of f, again up to a normalization constant. For general results that relate the Dixmier trace, the asymptotics of the zeta function and the heat kernel expansion see [CRSS07] (see also [CGRS12] for the non-unital case). The results of these papers are moreover valid in the semifinite setting, which is of interest to us.

The aim of this section is to define the non-commutative integral as the residue at the spectral dimension [CPRS06] of a certain zeta function. However, differently from the usual case, our zeta function is not defined solely in terms of the spectrum of D, but involves also the choice of a weight ϕ. The usual setting is recovered by taking ϕ equal to the operator trace. The motivation comes from the approach of modular spectral triples, which, as we mentioned in the introduction, allows the choice of such a weight. Although we have in mind the specific example of $SU_q(2)$ the content of this section is more general in nature.

7.2.1 The weight

The two definitions of non-commutative integration that we mentioned, that is the Dixmier trace and the residue of the zeta function, rely crucially on the behaviour of the spectrum of D. But in the non-commutative world, especially in the case of quantum groups, the spectra of naturally defined operators can be very different from their commutative counterparts. This usually spoils summability conditions, or even the compactness condition. One way out of these problems is to consider a Dirac operator which has the same spectrum as the classical one, which is the idea of isospectral deformations [CoLa01]. However it is clear that in this case, by remaining in the realm of usual spectral triples, one obtains a non-commutative integral which has the trace property. Therefore, if we consider the case of quantum groups, we do not recover the Haar state, since it is a non-tracial state.

Another possibility is to modify the definition of spectral triple to account for such features. As we have discussed in previous chapters, one framework which goes in this direction is that of modular spectral triples, where the idea is to define the notions of compactness and summability with respect to a weight ϕ.

For our purposes, that is for the definition of a notion of non-commutative integration, it is sufficient to consider a stripped-down version of this framework. We consider a $*$-algebra $\mathcal{A}$, which is represented as bounded operators on a Hilbert space $\mathcal{H}$, and a self-adjoint operator D acting this space. We take a weight of the form $\phi(\cdot) = \mathrm{Tr}(\Delta_\phi \cdot)$, where Δ_ϕ is a positive and invertible operator. This is essentially the statement of the Radon–Nikodym theorem for semifinite weights on von Neumann algebras.

We now define a zeta function in terms of D and ϕ, and the corresponding notion of spectral dimension. In the following we will provide definitions which are appropriate to the case of compact spaces. We also assume, for simplicity, that D is invertible.

Definition 7.1. The *zeta function* associated to D and ϕ is defined by

$$\zeta(z) := \phi(|D|^{-z}) = \mathrm{Tr}(\Delta_\phi |D|^{-z}).$$

If it exists, we define the *spectral dimension* to be the number

$$n := \inf\{s > 0 : \zeta(s) < \infty\}.$$

In the following we will assume that $\zeta(z)$ has a simple pole at $z = n$. This condition is related to the ideals $\mathcal{Z}_p$ introduced in [CRSS07].

To define the non-commutative integral, we first define a zeta function depending on a fixed element $x \in \mathcal{A}$. As a compatibility requirement between $\mathcal{A}$ and ϕ we ask that $\sigma^\phi(x) \in \mathcal{A}$, where $\sigma^\phi(x) = \Delta_\phi^{-1} x \Delta_\phi$ is the modular group of ϕ. Notice that this condition is part of the requirements for a modular spectral triple.

Definition 7.2. For any $x \in \mathcal{A}$ we define

$$\zeta_x(z) := \phi(x|D|^{-z}) = \mathrm{Tr}(\Delta_\phi x|D|^{-z}).$$

Then the *non-commutative integral* is the linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ defined by

$$\varphi(x) := \operatorname*{Res}_{z=n} \zeta_x(z) = \operatorname*{Res}_{z=n} \mathrm{Tr}(\Delta_\phi x|D|^{-z}),$$

where n is the spectral dimension.

We remark that, thanks to the condition $\sigma^\phi(x) \in \mathcal{A}$, the zeta function $\zeta_x(z)$ exists for $\mathrm{Re}(z) > n$ and has at most a simple pole. Indeed $\Delta_\phi x|D|^{-z} = \sigma^\phi(x)\Delta_\phi|D|^{-z}$ and the statement easily follows by applying Hölder's inequality for the trace.

7.2.2 The modular group

We now want to determine the modular group of the non-commutative integral φ. By this we mean the automorphism θ such that $\varphi(xy) = \varphi(\theta(y)x)$ for $x, y \in \mathcal{A}$. In the commutative case θ is the identity, simply because the pointwise product of functions is commutative. This is also true, although not trivially, if D satisfies the conditions for a spectral triple.

On the other hand, by considering the setting of twisted spectral triples, where we require the twisted commutator to be bounded for some automorphism σ, we find under suitable conditions that $\varphi(xy) = \varphi(\sigma^n(y)x)$. An additional twisting appears, in the setting of modular spectral triples, via the modular operator Δ_ϕ associated to a weight ϕ.

Here we want to take both these twistings into account and prove a theorem that, under some simplifying assumptions, determines the modular group of the non-commutative integral. We assume that $[D, x]_\sigma = Dx - \sigma(x)D$ is bounded for every $x \in \mathcal{A}$, for a fixed automorphism σ, and that it satisfies a regularity property specified below. We also consider an algebra $\mathcal{A}$ which is defined in terms of generators and relations, as for compact quantum groups, and require that σ acts diagonally on its generators. These conditions can be clearly weakened, but they suffice for the examples that we have in mind.

Theorem 7.1. *With the assumptions above, let φ be the non-commutative integral with spectral dimension n. Assume furthermore that D satisfies the following regularity property:*

- *there exists some $0 < r \leq 1$ such that $|D|^r[|D|^s, x]_{\sigma^s}|D|^{-s}$ is a bounded operator, for every element $x \in \mathcal{A}$ and for all $s \geq n$.*

Then the modular group of φ is given by $\theta = \sigma^\phi \circ \sigma^n$.

Proof. To prove the result we show that the following chain of equalities holds

$$\begin{aligned}
\varphi(xy) &= \operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi xy|D|^{-s}) \\
&= \operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi x|D|^{-s}\sigma^s(y)) \\
&= \operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi x|D|^{-s}\sigma^n(y)) \\
&= \operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi \sigma^\phi(\sigma^n(y))x|D|^{-s}) = \varphi(\theta(y)x).
\end{aligned}$$

Let us start with the first one. Consider the expression

$$\operatorname{Tr}(\Delta_\phi xy|D|^{-s}) - \operatorname{Tr}(\Delta_\phi x|D|^{-s}\sigma^s(y)) = \operatorname{Tr}(\Delta_\phi x|D|^{-s}[|D|^s, y]_{\sigma^s}|D|^{-s}),$$

where we have used the identity

$$y|D|^{-s} - |D|^{-s}\sigma^s(y) = |D|^{-s}[|D|^s, y]_{\sigma^s}|D|^{-s}.$$

Now consider $0 < r \leq 1$ as in the statement. Using Hölder's inequality we have

$$\begin{aligned}
\left|\operatorname{Tr}\left(\Delta_\phi x|D|^{-s}[|D|^s, y]_{\sigma^s}|D|^{-s}\right)\right| &= \left|\operatorname{Tr}\left(\Delta_\phi x|D|^{-(s+r)}|D|^r[|D|^s, y]_{\sigma^s}|D|^{-s}\right)\right| \\
&\leq \operatorname{Tr}\left(|\Delta_\phi x|D|^{-(s+r)}|\right) \left\||D|^r[|D|^s, y]_{\sigma^s}|D|^{-s}\right\|.
\end{aligned}$$

By assumption $\||D|^r[|D|^s, y]_{\sigma^s}|D|^{-s}\|$ is finite for all $s > n$. Therefore, since

$$\operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi x|D|^{-(s+r)}) = 0,$$

we find that the residue at $s = n$ of this term vanishes, hence we have

$$\varphi(xy) = \operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi xy|D|^{-s}) = \operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi x|D|^{-s}\sigma^s(y)).$$

The second step consists in proving that

$$\operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi x|D|^{-s}\sigma^s(y)) = \operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi x|D|^{-s}\sigma^n(y)).$$

Subtracting these quantities and using Hölder's inequality we find

$$\left|\operatorname{Tr}\left(\Delta_\phi x|D|^{-s}(\sigma^s(y) - \sigma^n(y))\right)\right| \leq \operatorname{Tr}\left(|\Delta_\phi x|D|^{-s}|\right) \|\sigma^s(y) - \sigma^n(y)\|.$$

Since σ acts diagonally on the generators of $\mathcal{A}$, the element $y \in \mathcal{A}$ can be written as a finite sum of homogeneous elements with respect to σ. Therefore we can consider, without loss of generality, that $\sigma(y) = \lambda y$ for some λ. Therefore we have $\|\sigma^s(y) - \sigma^n(y)\| = |\lambda^s - \lambda^n|\|y\|$.

The residue at $s = n$ of this quantity vanishes, so we get

$$\operatorname*{Res}_{s=n} \left|\operatorname{Tr}\left(\Delta_\phi x|D|^{-s}(\sigma^s(y) - \sigma^n(y)\right)\right| \leq \operatorname*{Res}_{s=n} \operatorname{Tr}\left(|\Delta_\phi x|D|^{-s}|\right)|\lambda^s - \lambda^n|\|y\| = 0.$$

For the last step have to show that

$$\operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi x|D|^{-s}\sigma^n(y)) = \operatorname*{Res}_{s=n} \operatorname{Tr}(\Delta_\phi \sigma^\phi(\sigma^n(y))x|D|^{-s}).$$

But this immediately follows from the trace property and the property of the modular operator $\sigma^n(y)\Delta_\phi = \Delta_\phi\sigma^\phi(\sigma^n(y))$. Finally, putting all the steps together and denoting $\theta = \sigma^\phi \circ \sigma^n$, we have that $\varphi(xy) = \varphi(\theta(y)x)$. The proof is complete. □

Let us now consider the case in which $\mathcal{A}$ is the coordinate algebra of a compact quantum group, with its Haar state h having the modular group ϑ. We can use this theorem as a criterion to check if the non-commutative integral φ coincides with the Haar state h. Indeed a necessary condition for this to happen is that the modular group θ of φ coincides with ϑ.

The strategy is the following: given an operator D and a weight ϕ we consider the associated zeta function and compute its spectral dimension which, if it exists, we denote by n. Then if D satisfies the assumptions of Theorem 7.1, we check if the modular group $\theta = \sigma^\phi \circ \sigma^n$ coincides with the modular group ϑ of the Haar state. In the rest of this chapter we will perform this analysis for the case of the quantum group $SU_q(2)$, using the Dirac operator D_q introduced in [KaSe12]. We will also briefly mention the case of the Podleś sphere.

7.3 Background on $SU_q(2)$

In this section we provide some background on the quantum group $SU_q(2)$, which we will be our focus in the rest of the chapter. We use the notations and conventions of the book by Klimyk and Schmüdgen [KlSc]. For $0 < q < 1$ we denote by $\mathcal{A} := \mathcal{O}(SU_q(2))$ the unital Hopf $*$-algebra with generators a, b, c, d satisfying the relations

$$ab = qba\ , \quad ac = qca\ , \quad bd = qdb\ , \quad cd = qdc\ , \quad bc = cb\ ,$$
$$ad = 1 + qcb\ , \quad da = 1 + q^{-1}bc\ ,$$

with the usual Hopf algebra structure and the involution given by

$$a^* = d\ , \quad b^* = -qc\ , \quad c^* = -q^{-1}b\ , \quad d^* = a\ .$$

For each $l \in \frac{1}{2}\mathbb{N}_0$ there is a unique (up to unitary equivalence) irreducible corepresentation V_l of the coalgebra $\mathcal{A}$ of dimension $2l+1$. If we fix a vector space basis in each V_l, and denote by $t^l_{i,j} \in \mathcal{A}$ the corresponding matrix coefficients, then we have the following analogue of the Peter-Weyl theorem: the set $\{t^l_{i,j} \in \mathcal{A} : l \in \frac{1}{2}\mathbb{N}_0,\ -l \leq i, j \leq l\}$ is a vector space basis of $\mathcal{A}$.

With a suitable choice of basis in $V_{1/2}$ one has

$$a = t^{1/2}_{-1/2,-1/2} \ , \quad b = t^{1/2}_{-1/2,1/2} \ , \quad c = t^{1/2}_{1/2,-1/2} \ , \quad d = t^{1/2}_{1/2,1/2} \ .$$

We also need to consider the quantized enveloping algebra $U_q(\mathfrak{sl}(2))$. This is a Hopf algebra generated by k, k^{-1}, e, f with relations

$$kk^{-1} = k^{-1}k = 1 \ , \quad ke = qek \ , \quad kf = q^{-1}ek \ , \quad [e,f] = \frac{k^2 - k^{-2}}{q - q^{-1}} \ .$$

It carries the Hopf algebra structure

$$\Delta(k) = k \otimes k \ , \quad \Delta(e) = e \otimes k + k^{-1} \otimes e \ , \quad \Delta(f) = f \otimes k + k^{-1} \otimes f \ ,$$
$$S(k) = k^{-1} \ , \quad S(e) = -qe \ , \quad S(f) = -q^{-1} f \ ,$$
$$\varepsilon(k) = 1 \ , \quad \varepsilon(e) = \varepsilon(f) = 0 \ .$$

It becomes a Hopf $*$-algebra, which we denote by $U_q(\mathfrak{su}(2))$, by adding the involution

$$k^* = k \ , \quad e^* = f \ , \quad f^* = e \ .$$

There is a dual pairing between the Hopf algebras $U_q(\mathfrak{sl}(2))$ and $\mathcal{A}$, which we denote by $\langle \cdot, \cdot \rangle$. This pairing is used to define the left and right actions of $U_q(\mathfrak{sl}(2))$ on $\mathcal{A}$ by the formulae

$$g \triangleright x := x_{(1)} \langle g, x_{(2)} \rangle \ , \quad g \triangleleft x := \langle g, x_{(1)} \rangle x_{(2)} \ , \qquad x \in \mathcal{A} \ , \ g \in U_q(\mathfrak{sl}(2)) \ ,$$

where we used Sweedler's notation for the coproduct. These actions make $\mathcal{A}$ into a $U_q(\mathfrak{sl}(2))$-bimodule. For the actions of the generators on the basis $t^l_{i,j}$ we have

$$k \triangleright t^l_{i,j} = q^j t^l_{i,j} \ , \qquad t^l_{i,j} \triangleleft k = q^i t^l_{i,j} \ ,$$
$$e \triangleright t^l_{i,j} = \sqrt{[l+1/2]^2 - [j+1/2]^2} t^l_{i,j+1} \ , \quad f \triangleright t^l_{i,j} = \sqrt{[l+1/2]^2 - [j-1/2]^2} t^l_{i,j-1} \ .$$

In the previous formulae we have used the q-numbers $[x]_q$, which are defined as $[x]_q := (q^{-x} - q^x)/(q^{-1} - q)$. In the following we will also use the notation

$$\partial_k := k \triangleright \ , \quad \partial_e := e \triangleright \ , \quad \partial_f := f \triangleright \ ,$$

for the operators acting on a suitable completion of $\mathcal{A}$. Observe that, since $\Delta(k^n) = k^n \otimes k^n$ for every $n \in \mathbb{Z}$, we have that $k^n \triangleright$ and $\triangleleft k^n$ are algebra automorphisms on $\mathcal{A}$.

7.3.1 The Haar state

We denote by $A := C^*(SU_q(2))$ the universal C^*-completion of the $*$-algebra $\mathcal{A}$. Let h be the Haar state of A whose values on the basis elements are

$$h(a^i b^j c^k) = h(d^i b^j c^k) = \delta_{i0}\delta_{jk}(-1)^k[k+1]^{-1} \ , \quad h(t^l_{i,j}) = \delta_{l0} \ .$$

Let $\mathcal{H}_h$ denote the GNS space $L^2(A,h)$, where the inner product is defined by $(x,y) := h(x^*y)$. The representation of A on $\mathcal{H}_h$ is induced by left multiplication in A. The set $\{t^l_{i,j} \in \mathcal{A} : l \in \frac{1}{2}\mathbb{N}_0, \ -l \leq i,j \leq l\}$ of matrix coefficients is an orthogonal basis for $\mathcal{H}_h$, with

$$(t^l_{i,j}, t^{l'}_{i',j'}) = \delta_{l,l'}\delta_{i,i'}\delta_{j,j'}q^{-2i}[2l+1]^{-1} \ .$$

We also introduce the orthonormal basis $\xi^l_{i,j} := t^l_{i,j}/\sqrt{q^{-2i}[2l+1]^{-1}}$.

The Haar state does not satisfy the trace property, but instead we have $h(xy) = h(\vartheta(y)x)$ where $\vartheta(x) = k^{-2} \triangleright x \triangleleft k^{-2}$. In particular on the generators we have

$$\vartheta(a) = q^2 a \ , \quad \vartheta(b) = b \ , \quad \vartheta(c) = c \ , \quad \vartheta(d) = q^{-2} d \ .$$

It follows from the theory of compact quantum groups that the Haar state extends to a KMS state on the C^*-algebra A for the strongly continuous one-parameter group ϑ_t, given by

$$\vartheta_t(a) = q^{-2it} a \ , \quad \vartheta_t(b) = b \ , \quad \vartheta_t(c) = c \ , \quad \vartheta_t(d) = q^{2it} d \ .$$

This action can be analytically extended, and we recover the modular group ϑ of the Haar state as $\vartheta = \vartheta_i$. In particular, the associated modular operator Δ_F can be written as $\Delta_F = \Delta_L \Delta_R$, where Δ_L and Δ_R are the left and right modular operators defined by

$$\Delta_L(t^l_{i,j}) = q^{2j} t^l_{i,j} \ , \qquad \Delta_R(t^l_{i,j}) = q^{2i} t^l_{i,j} \ .$$

These modular operators also implement one parameters groups of automorphisms, which are given by $\sigma_{L,t}(x) = \Delta_L^{it} x \Delta_L^{-it}$ and $\sigma_{R,t}(x) = \Delta_R^{it} x \Delta_R^{-it}$. They can be extended to complex actions, and we denote their extensions at $t = i$ by σ_L and σ_R. Restricted to $\mathcal{A}$, they coincide with the left and right action of k^{-2}, that is we have

$$\sigma_L(x) = k^{-2} \triangleright x \ , \qquad \sigma_R(x) = x \triangleleft k^{-2} \ .$$

Finally we note that the modular group ϑ of the Haar state can be rewritten in terms of these automorphisms as $\vartheta = \sigma_L \circ \sigma_R$.

7.3.2 A decomposition

We now consider a decomposition of the algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}_h$ which has a particular geometrical significance [NeTu05]. For $n \in \mathbb{Z}$ define

$$\mathcal{A}_n = \{x \in \mathcal{A} : \sigma_{L,t}(x) = q^{int} x\} \ .$$

Then we have the decomposition $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n$. The norm closure of $\mathcal{A}_n$, which we denote by A_n, is the analogue of the space of continuous sections of the line bundle over the sphere with winding number n. In particular the fixed point algebra under the left action on $\mathcal{A}$, that is the space $\mathcal{A}_0$, is isomorphic to the standard Podleś sphere. This algebra decomposition can be extended to a Hilbert space decomposition. If we denote by $\mathcal{H}_n = L^2(A_n, h)$ the GNS space corresponding to A_n, then we have

$$\mathcal{H}_h = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n \ .$$

7.4 The Dirac operator D_q

We now turn our attention to the implementation of the quantum group $SU_q(2)$ in the framework of spectral triples. In particular we will consider the spectral triple introduced in [KaSe12], which is an example which fits into the framework of modular spectral triples [Kaa11]. We will focus our attention mainly on the Dirac operator D_q, which is defined as

$$D_q = \begin{pmatrix} (q^{-1}-q)^{-1}(q\partial_{k^{-2}} - 1) & q^{-1/2}\partial_e \partial_{k^{-1}} \\ q^{1/2}\partial_f \partial_{k^{-1}} & (q^{-1}-q)^{-1}(1 - q^{-1}\partial_{k^{-2}}) \end{pmatrix} .$$

It acts on the Hilbert space $\mathcal{H} = \mathcal{H}_h \oplus \mathcal{H}_h$, where $\mathcal{H}_h$ is the GNS space constructed using the Haar state in the previous section. The Dirac operator D_q satisfies some interesting properties, which we summarize in the following proposition [KaSe12].

Proposition 7.2. *Let D_q be the Dirac operator given above. Then:*

1. *the twisted commutator $[D_q, x]_{\sigma_L} = D_q x - \sigma_L(x) D_q$ is bounded,*
2. *the twisted commutator is Lipschitz regular, that is $[|D_q|, x]_{\sigma_L}$ is bounded,*
3. *we have $D_q^2 = \chi^{-1}\Delta_L^{-1} C_q$, where C_q is the Casimir of $SU_q(2)$ and $\chi = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$.*

It is interesting to point out the relation with the Dirac operator for the Podleś sphere, which has been introduced in [DąSi03], see also [NeTu05]. In our notation, the Hilbert space for the spectral triple associated to the Podleś sphere is given by $\mathcal{H}_1 \oplus \mathcal{H}_{-1}$. This Hilbert space sits inside $\mathcal{H}_h \oplus \mathcal{H}_h$, since we have the decomposition $\mathcal{H}_h = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n$. If we restrict the Dirac operator D_q to this subspace we obtain

$$D_q|_{\mathcal{H}_1 \oplus \mathcal{H}_{-1}} = \begin{pmatrix} (q^{-1}-q)^{-1}(qq^{-1}-1) & q^{-1/2}q^{1/2}\partial_e \\ q^{1/2}q^{-1/2}\partial_f & (q^{-1}-q)^{-1}(1-q^{-1}q) \end{pmatrix} = \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix}.$$

Therefore it reproduces the usual Dirac operator for the Podleś sphere, which makes it a natural object to consider. We also point out that, since the Podleś sphere corresponds to the fixed point algebra of $\mathcal{A}$ under the left action, it follows that the twisted commutator condition $[D_q, x]_{\sigma_L}$ reduces to the usual one.

7.4.1 The left-covariant differential calculus

Here we point out an interesting feature of the operator D_q, namely that it implements a left-covariant differential calculus on $SU_q(2)$. In particular it is isomorphic to the number 10 of the list given in [Hec01], where a complete classification of left-covariant differential calculi on $SU_q(2)$ is obtained. In the context of twisted spectral triples, this particular calculus has been considered previously in the paper [KrWa11], where it is given as an example of a more general framework. The operator D_q that we consider here, however, is slightly different from the one that appears in that paper.

Proposition 7.3. *The operator D_q implements a left covariant differential calculus on $SU_q(2)$.*

Proof. To prove this statement recall that two first-order differential calculi (Ω^1_1, d_1) and (Ω^1_2, d_2) are isomorphic if whenever $\sum_j a_j d_1 b_j = 0$ we have $\sum_j a_j d_2 b_j = 0$ and vice versa. For a twisted spectral triple we can realize a differential calculus in the following way: we define Ω^1_D to be the span of operators of the form $a \cdot [D, b]_\sigma$ with bimodule structure given by $a \cdot [D, b]_\sigma \cdot c = \sigma(a)[D, b]_\sigma c$, where $a, b, c \in \mathcal{A}$. Then it is easy to check that $d_\sigma(a) = [D, a]_\sigma$ defines a derivation with values in Ω^1_D, see also [KrWa11].

To proceed we compute the twisted commutator of D_q with $x \in \mathcal{A}$, using the coproduct structure of $U_q(\mathfrak{sl}(2))$. We find the result

$$\begin{aligned} [D_q, x]_{\sigma_L} &= (q^{-1}-q)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\partial_{k^{-2}}(x) - x) \\ &\quad + q^{-1/2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \partial_e(\partial_{k^{-1}}(x)) + q^{1/2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \partial_f(\partial_{k^{-1}}(x)). \end{aligned}$$

We note in passing that this expression shows that it is a bounded operator. Using this formula it is easy to see that the calculus defined by D_q is isomorphic to the one given in [KrWa11]. This one in turn is, by construction, isomorphic to the differential calculus number 10 in the list appearing in [Hec01], from which the claim follows. □

7.4.2 A regularity property

The Dirac operator D_q satisfies the Lipschitz regularity property, that is $[|D_q|, x]_{\sigma_L}$ is bounded for every $x \in \mathcal{A}$, see [KaSe12, Lemma 3.5]. Here we prove a similar regularity property, namely the one that appears as a requirement in Theorem 7.1 for $r = 1$. This property is similar to the one of strong Lipschitz-regularity which appears in [Mos10, Proposition 3.3].

Lemma 7.4. *The operator $|D_q|[|D_q|^s, x]_{\sigma_L^s}|D_q|^{-s}$ is bounded for every $x \in \mathcal{A}$ and for all $s \in \mathbb{R}$.*

Proof. We start by noting that, since the Dirac operator satisfies $D_q^2 = \chi^{-1}\Delta_L^{-1}C_q$, the action of $|D_q|$ on the two components of the Hilbert space $\mathcal{H}_h \oplus \mathcal{H}_h$ is the same up to a constant. Therefore we can restrict our attention to one of them, let us say the first one, on which we have $|D_q|\xi^l_{i,j} = q^{1/2}q^{-j}[l+1/2]\xi^l_{i,j}$. Moreover, since the twisted commutator is well-behaved with respect to products and adjoints, we can restrict to the case $x = a$ or $x = c$. We can decompose the action of these operators on an element $\xi^l_{i,j}$ of the Hilbert space as

$$\begin{aligned} a\xi^l_{i,j} &= \alpha^{l+}_{i,j}\xi^{l+1/2}_{i-1/2,j-1/2} + \alpha^{l-}_{i,j}\xi^{l-1/2}_{i-1/2,j-1/2} , \\ c\xi^l_{i,j} &= \gamma^{l-}_{i,j}\xi^{l+1/2}_{i+1/2,j-1/2} + \gamma^{l-}_{i,j}\xi^{l-1/2}_{i+1/2,j-1/2} . \end{aligned}$$

We have the following bounds on the coefficients $\alpha^{l+}_{i,j}, \gamma^{l+}_{i,j} \leq C_1 q^{l+j}$ and $\alpha^{l-}_{i,j}, \gamma^{l-}_{i,j} \leq C_2$, see [KaSe12, Lemma 3.5]. We start by considering the case $x = a$. Then we immediately obtain

$$\begin{aligned} [|D_q|^s, a]_{\sigma_L^s}\xi^l_{i,j} &= \alpha^{l+}_{i,j}q^{s/2}(q^{-s(j-1/2)}[l+1]^s - q^{-s(j-1)}[l+1/2]^s)\xi^{l+1/2}_{i-1/2,j-1/2} \\ &\quad + \alpha^{l-}_{i,j}q^{s/2}(q^{-s(j-1/2)}[l]^s - q^{-s(j-1)}[l+1/2]^s)\xi^{l-1/2}_{i-1/2,j-1/2} . \end{aligned}$$

Now we want to show that $|D_q|[|D_q|^s, a]_{\sigma_L^s}|D_q|^{-s}$ is a bounded operator. To do this we apply it to $\xi^l_{i,j}$, compute the inner product with $\xi^{l\pm 1/2}_{i-1/2,j-1/2}$ and then show that both terms are bounded by a constant, which does not depend on l. For the first one we have

$$\begin{aligned} &\left(|D_q|[|D_q|^s, a]_{\sigma_L^s}|D_q|^{-s}\xi^l_{i,j}, \xi^{l+1/2}_{i-1/2,j-1/2}\right) \\ &= q\alpha^{l+}_{i,j}q^{-j}(q^{s/2}[l+1]^s - q^s[l+1/2]^s)[l+1/2]^{-s}[l+1] . \end{aligned}$$

Using the inequality $\alpha^{l+}_{i,j} \leq C_1 q^{l+j}$ and $[l] \sim q^{-l}$, valid for large l, we obtain

$$\left(|D_q|[|D_q|^s, a]_{\sigma_L^s}|D_q|^{-s}\xi^l_{i,j}, \xi^{l+1/2}_{i-1/2,j-1/2}\right) \leq C_1' q^{l+j}q^{-j}(q^{s/2} - q^s)q^{-sl}q^{sl}q^{-l} \leq C_1'' .$$

Computing the other inner product we get

$$\begin{aligned} &(|D_q|[|D_q|^s, a]_{\sigma_L^s}|D_q|^{-s}\xi^l_{i,j}, \xi^{l-1/2}_{i-1/2,j-1/2}) \\ &= q\alpha^{l-}_{i,j}q^{-j}(q^{s/2}[l]^s - q^s[l+1/2]^s)[l+1/2]^{-s}[l] . \end{aligned}$$

To bound this term we first observe that

$$q^{s/2}[l]^s - q^s[l+1/2]^s = \frac{q^{s/2}}{(q^{-1}-q)^s}\left((q^{-l}-q^l)^s - (q^{-l}-q^{l+1})^s\right) .$$

Then for large l we find $q^{s/2}[l]^s - q^s[l+1/2]^s \sim q^{-sl}q^{2l}$. Using this result and $\alpha^{l+}_{i,j} \leq C_2$ we find

$$(|D_q|[|D_q|^s, a]_{\sigma^s_L}|D_q|^{-s}\xi^l_{i,j}, \xi^{l-}_{i-,j-}) \leq C_2' q^{-j}q^{-sl}q^{2l}q^{sl}q^{-l} \leq C_2'' q^{l-j}$$

Since $-l \leq j \leq l$ we have that this term is bounded. The proof for the case $x = c$ is completely analogous, since $\gamma^{\pm l}_{i,j}$ satisfies the same bounds as $\alpha^{\pm l}_{i,j}$, therefore we skip it. □

7.5 The zeta function

In this section we define a family of zeta functions, depending on the Dirac operator D_q and a family of weights, with the aim of studying the corresponding notion of non-commutative integration. We point out that in this case it is not possible to use the operator trace, since it is known that the associated spectral dimension does not exist [KaSe12]. In view of the requirement that $\sigma^\phi(x) \in \mathcal{A}$ for every $x \in \mathcal{A}$, we can restrict the freedom in the choice of the weight ϕ by taking Δ_ϕ such that it implements an automorphism of $SU_q(2)$. The complete list of automorphisms for $SL_q(2)$ can be found in [HaKr05]: there are two families, one of which acts diagonally, which depend on two parameters. For simplicity we consider only the diagonal case, which takes the following form on the generators

$$\sigma_{\lambda,\mu}(a) = \lambda a , \quad \sigma_{\lambda,\mu}(b) = \mu b , \quad \sigma_{\lambda,\mu}(c) = \mu^{-1} c , \quad \sigma_{\lambda,\mu}(d) = \lambda^{-1} d .$$

We point out that the modular group ϑ of the Haar state is of this form, with $\lambda = q^{-2}$ and $\mu = 1$. Therefore we can parametrize our weight by two real number $a, b \in \mathbb{R}$ as

$$\phi^{(a,b)}(\cdot) := \mathrm{Tr}(\Delta_L^{-a}\Delta_R^b \cdot) .$$

The minus sign is chosen for later convenience.

7.5.1 The spectral dimension

We start by computing the spectral dimension associated to the zeta function constructed with D_q and $\phi^{(a,b)}$. This imposes some restrictions on the values of the parameters a, b. Moreover we discuss the meromorphic extension of this function.

Proposition 7.5. *Let* $\zeta^{(a,b)}(z) := \mathrm{Tr}(\Delta_L^{-a}\Delta_R^b|D_q|^{-z})$. *Then*

1. *if* $a \pm b > 0$ *then* $\zeta^{(a,b)}(z)$ *is holomorphic for all* $z \in \mathbb{C}$ *such that* $\mathrm{Re}(z) > a + |b|$,

2. *in this case the corresponding spectral dimension is* $n = a + |b|$,

3. *$\zeta^{(a,b)}(z)$ has a meromorphic extension to the complex plane, with only simple poles if $b \neq 0$ and with only double poles if $b = 0$.*

Proof. From Proposition 7.2 we have $D_q^2 = \chi^{-1}\Delta_L^{-1}C_q$, where C_q is the Casimir and

$$\chi = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} .$$

Therefore we can write $|D_q|^{-z} = \chi^{z/2}\Delta_L^{z/2}C_q^{-z/2}$. The Hilbert space is $\mathcal{H} = \mathcal{H}_h \oplus \mathcal{H}_h$, where $\mathcal{H}_h$ is the GNS space constructed using the Haar state. An orthonormal basis for this space is given by $\{\xi^l_{i,j} \in \mathcal{A} : l \in \frac{1}{2}\mathbb{N}_0,\ -l \leq i,j \leq l\}$. Then we have

$$\zeta^{(a,b)}(z) = (q^{-z/2} + q^{z/2}) \sum_{2l=0}^{\infty} \sum_{i,j=-l}^{l} (\xi^l_{i,j}, \Delta_L^{-a}\Delta_R^{b}\Delta_L^{z/2}C_q^{-z/2}\xi^l_{i,j}) .$$

The modular operators act as $\Delta_L\xi^l_{i,j} = q^{2j}\xi^l_{i,j}$, $\Delta_R\xi^l_{i,j} = q^{2i}\xi^l_{i,j}$, while for Casimir we have $C_q\xi^l_{i,j} = [l+1/2]_q^2\xi^l_{i,j}$. Therefore we get

$$(\xi^l_{i,j}, \Delta_L^{-a}\Delta_R^{b}\Delta_L^{z/2}C_q^{-z/2}\xi^l_{i,j}) = q^{(z-2a)j}q^{2bi}[l+1/2]_q^{-z} .$$

To proceed we use the following trick [KrWa10]. For every $z \in \mathbb{C}$ we have the absolutely convergent series expansion

$$\begin{aligned} [l+1/2]_q^{-z} &= (q^{-1}-q)^z q^{(l+1/2)z}(1-q^{2l+1})^{-z} \\ &= (q^{-1}-q)^z q^{(l+1/2)z} \sum_{k=0}^{\infty} \begin{pmatrix} z+k-1 \\ k \end{pmatrix} q^{(2l+1)k} . \end{aligned}$$

Therefore we can rewrite our zeta function as

$$\zeta^{(a,b)}(z) = \frac{q^{-z/2}+q^{z/2}}{(q^{-1}-q)^{-z}} \sum_{2l=0}^{\infty}\sum_{i,j=-l}^{l}\sum_{k=0}^{\infty} \begin{pmatrix} z+k-1 \\ k \end{pmatrix} q^{(z-2a)j}q^{2bi}q^{(l+1/2)z}q^{(2l+1)k} .$$

Now we consider the sum

$$S_k^{(a,b)}(z) := \sum_{2l=0}^{\infty}\sum_{i,j=-l}^{l} q^{(z-2a)j}q^{2bi}q^{(l+1/2)z}q^{(2l+1)k} .$$

The sums over i and j can be easily performed and we get

$$S_k^{(a,b)}(z) = \sum_{2l=0}^{\infty} \frac{q^{(z-2a)}q^{(z-2a)l} - q^{-(z-2a)l}}{q^{(z-2a)}-1} \frac{q^{2b}q^{2bl}-q^{-2bl}}{q^{2b}-1} q^{(l+1/2)z}q^{(2l+1)k} .$$

We can break this sum into four terms

$$S_k^{(a,b)}(z) = \frac{q^{k+z/2}}{(1-q^{z-2a})(1-q^{2b})} \left(q^{z-2(a-b)} S_1 - q^{z-2a} S_2 - q^{2b} S_3 + S_4 \right) ,$$

where we have defined

$$S_1 := \sum_{2l=0}^{\infty} q^{2l(z-a+b+k)} , \quad S_2 := \sum_{2l=0}^{\infty} q^{2l(z-a-b+k)} ,$$
$$S_3 := \sum_{2l=0}^{\infty} q^{2l(a+b+k)} , \quad S_4 := \sum_{2l=0}^{\infty} q^{2l(a-b+k)} .$$

Since $0 < q < 1$, the series $\sum_{l=0}^{\infty} q^{cl}$ is absolutely convergent when $\mathrm{Re}(c) > 0$. We want this to be the case for any $k \geq 0$. From S_3 and S_4 we see that this imposes $a+b > 0$ and $a-b > 0$. For S_1 and S_2, that depend on z, instead we have to require

$$\mathrm{Re}(z) > a - b , \quad \mathrm{Re}(z) > a + b .$$

We can then easily sum the geometric series and, after some rearranging, we arrive at

$$S_k^{(a,b)}(z) = \frac{q^{k+z/2} \left(1 - q^{2k+z}\right)}{\left(1 - q^{z-(a+b)+k}\right) \left(1 - q^{z-(a-b)+k}\right) \left(1 - q^{a+b+k}\right) \left(1 - q^{a-b+k}\right)} .$$

Now, going back to the expression for $\zeta^{(a,b)}(z)$, we see that we can safely exchange the sum over k with the other sums. The result is then

$$\zeta^{(a,b)}(z) = \frac{q^{-z/2} + q^{z/2}}{(q^{-1} - q)^{-z}} \sum_{k=0}^{\infty} \begin{pmatrix} z + k - 1 \\ k \end{pmatrix} S_k^{(a,b)}(z) .$$

The statement about the meromorphic extension is clear from the form of $S_k^{(a,b)}(z)$. □

In the following we will consider the case $a \pm b > 0$, so that the spectral dimension exists according to the proposition above. Moreover we exclude $b = 0$, since in this case the zeta function has a double pole at the spectral dimension.

7.5.2 The modular property

We now consider the non-commutative integral φ associated to the zeta function and determine its modular group θ. In particular we can investigate the connection with the Haar state of $SU_q(2)$, which satisfies the property $h(xy) = h(\vartheta(y)x)$, where $\vartheta = \sigma_L \circ \sigma_R$. Therefore, to recover the Haar state from the non-commutative integral, a necessary condition is that $\theta = \vartheta$. We now show that this condition fixes the parameter $b = 1$.

Proposition 7.6. *Let $\zeta_x^{(a,b)}(z) = \mathrm{Tr}(\Delta_L^{-a} \Delta_R^{b} x |D_q|^{-z})$, with $n = a + |b|$ the associated spectral dimension. Let $\varphi(x) = \operatorname*{Res}_{z=n} \zeta_x^{(a,b)}(z)$ be the non-commutative integral, as in Definition 7.2.*

Finally let θ be the modular group of φ and $\vartheta = \sigma_L \circ \sigma_R$ the modular group of the Haar state. Then we have $\theta = \vartheta$ if and only if $b = 1$.

Proof. We can apply Theorem 7.1 to the non-commutative integral φ. Indeed by Lemma 7.4 we have that $|D|[|D|^s, y]_{\sigma_L^s}|D|^{-s}$ is bounded for every $s \in \mathbb{R}$, while by Proposition 7.2 the twist in the commutator $\sigma = \sigma_L$ acts diagonally on $\mathcal{A}$. Therefore we obtain $\varphi(xy) = \varphi(\theta(y)x)$, with $\theta = \sigma^\phi \circ \sigma_L^n$. In the case under consideration we have $\sigma^\phi = \sigma_L^{-a} \circ \sigma_R^b$, so we get

$$\theta = \sigma^\phi \circ \sigma_L^n = \sigma_L^{n-a} \circ \sigma_R^b = \sigma_L^{|b|} \circ \sigma_R^b ,$$

where we have used the fact that the spectral dimension is given by $n = a + |b|$.

For $b < 0$ we have $\theta = \sigma_L^{|b|} \circ \sigma_R^{-|b|}$ and it is clear that there is no solution. On the other hand for $b > 0$ we have $\theta = \sigma_L^b \circ \sigma_R^b$, so the equality $\theta = \vartheta$ holds for $b = 1$. □

This result shows that we can partially fix the arbitrariness in the choice of the weight $\phi^{(a,b)}(\cdot) = \mathrm{Tr}(\Delta_L^{-a}\Delta_R^b \cdot)$. On the other hand, the dependence on the parameter a cancels in the combination $\theta = \sigma^\phi \circ \sigma_L^n$, as seen in the proof above. It is worth pointing out that a similar phenomenon happens also in the spectral triple considered in the previous chapter, where similar techniques were employed. This is expected to happen, quite generically, when the twist in the commutator also appears in the modular group of the weight.

A reasonable criterion to fix this ambiguity is to recover the classical dimension $n = 3$, which fixes $a = 2$. In the last part of this chapter we will consider another condition, more spectral in nature, which also fixes uniquely $a = 2$. It is also of some interest to remark that, if one requires n to be an integer, then the smallest n which is allowed by the previous analysis is indeed $n = 3$. Finally, for examples coming from quantum groups, this ambiguity in the choice of the weight could be related to a similar one that arises in twisted Hochschild homology: indeed it is known that a twist is necessary to avoid the dimension drop, but it happens that one finds a family of such twists, see for example [HaKr05].

Let us also mention what happens for the Podleś sphere. In this case, since the Hilbert space is given by $\mathcal{H}_1 \oplus \mathcal{H}_{-1}$, the modular operator Δ_L^a gives a constant matrix, which can be absorbed in the normalization. Therefore it does not affect the spectral dimension and the modular group of the non-commutative integral. As we mentioned before, the twist in the commutator disappears, since the Podleś sphere is the fixed point algebra of $\mathcal{A}$ under the left action. Then it is easy to repeat the previous analysis, with the result that we must fix the value $b = 1$ if we want to recover the modular group of the Haar state. Moreover it follows from the results of [KrWa10] that the corresponding spectral dimension is $n = 2$. Therefore our results for $SU_q(2)$ restrict in a natural way to the case of the Podleś sphere.

7.6 The Haar state

So far we have only shown that the non-commutative integral φ has the same modular group of the Haar state h, which leaves open the question of whether they are equal. In principle this could happen for some values of a, or maybe for none at all. In this section we show that the non-commutative integral coincides with the Haar state for all allowed values of a. Of course we must normalize φ, since the Haar state satisfies $h(1) = 1$, while in general we do not have $\varphi(1) = 1$. This normalization is achieved by computing $\varphi(1)$, that is the residue of $\zeta^{(a,1)}(z)$ at the spectral dimension $n = a + 1$. The result of this computation is

$$\varphi(1) = \operatorname*{Res}_{z=a+1} \zeta^{(a,1)}(z) = \frac{(q^{-1} - q)^a (q^{a+1} + 1)}{(q^a - q)\log(q)} .$$

We denote the normalized non-commutative integral as $\tilde{\varphi}(x) := \varphi(x)/\varphi(1)$. Notice that the normalization $\varphi(1)$ depends on a. On the other hand we will now show that $\tilde{\varphi}(x)$ is independent of a and recovers the Haar state.

7.6.1 Approximating the GNS representation

To proceed with the computation of the non-commutative integral we use a different representation of $SU_q(2)$. This representation, which we denote by ρ, approximates the GNS representation, as we shall see in the next lemma. It is defined on the generators as

$$\begin{aligned}
\rho(a)\xi^l_{i,j} &:= \sqrt{1 - q^{2(l+i)}}\xi^{l-1/2}_{i-1/2,j-1/2} , \\
\rho(b)\xi^l_{i,j} &:= -q^{l+i+1}\xi^{l+1/2}_{i-1/2,j+1/2} , \\
\rho(c)\xi^l_{i,j} &:= q^{l+i}\xi^{l-1/2}_{i+1/2,j-1/2} , \\
\rho(d)\xi^l_{i,j} &:= \sqrt{1 - q^{2(l+i+1)}}\xi^{l+1/2}_{i+1/2,j+1/2} .
\end{aligned}$$

Here we use the convention that $\xi^l_{i,j}$ is zero whenever the indices are out of bounds (recall that we should have $-l \leq i, j \leq l$). That ρ is a representation of $SU_q(2)$ can be checked by direct computation, for more details see [Kaa11, Proposition 9.4] and references therein.

Lemma 7.7. *For any $x \in \mathcal{A}$ we have the equality*

$$\varphi(x) = \operatorname*{Res}_{z=n} \operatorname{Tr}(\Delta_L^{-a}\Delta_R \rho(x)|D_q|^{-z}) .$$

Proof. We need to show that $\Delta_L^{-a}\Delta_R(x - \rho(x))|D_q|^{-z}$ is trace-class for $z = n$, that is

$$\operatorname{Tr}\left(|\Delta_L^{-a}\Delta_R(x - \rho(x))|D_q|^{-n}|\right) < \infty .$$

Using the fact that $xy - \rho(xy) = (x - \rho(x))\rho(y) + x(y - \rho(y))$, we can restrict our attention to the generators. Moreover, since ρ is a $*$-representation of $SU_q(2)$, it suffices to consider the cases $x = a$ and $x = c$. Recall that for the GNS representation we have the formulae

$$a\xi^l_{i,j} = \alpha^{l+}_{i,j}\xi^{l+1/2}_{i-1/2,j-1/2} + \alpha^{l-}_{i,j}\xi^{l-1/2}_{i-1/2,j-1/2} ,$$
$$c\xi^l_{i,j} = \gamma^{l+}_{i,j}\xi^{l+1/2}_{i+1/2,j-1/2} + \gamma^{l-}_{i,j}\xi^{l-1/2}_{i+1/2,j-1/2} .$$

To prove that the trace is finite, in the case $x = a$, we need to estimate the quantities $|\alpha^{l+}_{i,j}|$ and $\left|\alpha^{l-}_{i,j} - \sqrt{1 - q^{2(l+i)}}\right|$. Recall that we already used the fact that $|\alpha^{l+}_{i,j}| \leq C_+ q^{l+j}$. Similarly one proves that $\left|\alpha^{l-}_{i,j} - \sqrt{1 - q^{2(l+i)}}\right| \leq C_- q^{l+j}$, see [Kaa11, Proposition 9.4]. Therefore we only need to repeat the computation of the sum $\xi^{(a,1)}(z)$ with the factor q^{l+j} inserted. We can easily perform this sum as we did in the computation of the spectral dimension. The result is finite for $z = n$, so that the operator $\Delta_L^{-a}\Delta_R(x - \rho(x))|D_q|^{-n}$ is trace-class.

The computation is completely identical for the case $x = c$, since we have the estimates $|\gamma^{l+}_{i,j}| \leq C'_+ q^{l+j}$ and $\left|\gamma^{l-}_{i,j} - q^{l+i}\right| \leq C'_- q^{l+j}$. Therefore the equality is proven. □

7.6.2 The computation

Using the representation ρ, we can now easily compute $\tilde{\varphi}(x)$ for any $x \in \mathcal{A}$. We show that this non-commutative integral coincides with $h(x)$, where h is the Haar state, independently of the value of the parameter a which appears in the definition of $\tilde{\varphi}$.

Theorem 7.8. *For any $x \in \mathcal{A}$ we have $\tilde{\varphi}(x) = h(x)$, where h is the Haar state.*

Proof. Recall that the Haar state h takes the following values on the generators

$$h(a^i b^j c^k) = h(d^i b^j c^k) = \delta_{i,0}\delta_{j,k}(-1)^k[k+1]_q^{-1} .$$

Using the previous lemma and the explicit formulae for the approximate representation, it is not difficult to see that the non-commutative integral must have the following form

$$\tilde{\varphi}(a^i b^j c^k) = \tilde{\varphi}(d^i b^j c^k) = \delta_{i,0}\delta_{j,k}\tilde{\varphi}(b^j c^k) .$$

Therefore, to prove that $\tilde{\varphi}$ is the Haar state, it only remains to show that $\tilde{\varphi}(b^n c^n)$ coincides with $h(b^n c^n)$. Using the representation ρ we can immediately compute

$$\rho(b^n c^n)\xi^l_{i,j} = (-1)^n q^{n(l+i+1)} q^{n(l+i)} \xi^l_{i,j} = (-1)^n q^{2nl} q^{2ni} q^n \xi^l_{i,j} .$$

Now we only need to repeat the computation of Proposition 7.5 by inserting the factor $(-1)^n q^{2nl} q^{2ni} q^n$. We omit this computation. The result is that there is a simple pole at $z = a+1$, that is the spectral dimension, whose residue is non-zero. Explicitely we obtain

$$\tilde{\varphi}(b^n c^n) = (-1)^n q^n \frac{q^2 - 1}{q^{2(n+1)} - 1} = (-1)^n [n+1]_q^{-1} .$$

This coincides with $h(b^n c^n)$, so the proof is complete. □

7.7 An additional requirement

In the previous section we have shown that the non-commutative integral coincides with the Haar state, regardless of the value of the parameter a. Of course, as we mentioned before, since the spectral dimension is given by $n = a + 1$, we have the natural choice $a = 2$ which gives the classical dimension. In this section we propose a different criterion to fix this free parameter, which is based on the value of a certain residue of the zeta function. It will turn out that this criterion is satisfied only for $a = 2$.

To formulate this criterion we note that, by requiring the non-commutative integral to be equal to the Haar state, we have imposed a condition on the residue at $z = n$ of the zeta function. But the zeta function contains much more information than this residue: indeed in the classical case we know, for example from the heat kernel expansion, that also the other residues contain geometrical information, like the scalar curvature and various contractions of the Riemann tensor. We can therefore look at these other residues to impose an additional requirement. In particular we can look at the next non-trivial coefficient of the expansion.

For this reason we briefly recall how the heat kernel expansion works, and how we can use it for our needs. Let M be a compact Riemannian manifold of dimension n with a fixed metric g. Consider a second order operator of Laplace-type, which locally can be written as

$$P = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E) \ . \tag{7.1}$$

For any smooth function f on M we can consider the operator $f\exp(-tP)$, for $t > 0$. Then there is an asymptotic expansion of $\mathrm{Tr}(f\exp(-tP))$, for $t \downarrow 0$, which is given by

$$\mathrm{Tr}(fe^{-tP}) \sim \sum_{k=0}^{\infty} t^{(k-n)/2} a_k(f, P) \ ,$$

where the coefficients $a_k(f, P)$ can be expressed as integrals of local invariants of M. For a manifold without boundary only the even coefficients are non-zero. In the following we will consider only the first two non-zero coefficients, which read as follows

$$a_0(f, P) = (4\pi)^{-n/2} \int_M f\sqrt{g} d^n x \ , \qquad a_2(f, P) = (4\pi)^{-n/2} 6^{-1} \int_M f(6E + R)\sqrt{g} d^n x \ .$$

Here R is the scalar curvature associated to the metric g.

The coefficients of the heat kernel expansion are closely related to the residues of the zeta function. Indeed, in the case of a positive P, consider the zeta function defined as

$\zeta(z, f, P) = \mathrm{Tr}(fP^{-z})$. Then the heat kernel coefficients $a_k(f, P)$ are given by

$$a_k(f, P) = \operatorname*{Res}_{z=(n-k)/2} \Gamma(z)\zeta(z, f, P).$$

We remark that, from this relation, we obtain yet another justification for our definition of the non-commutative integral. Indeed, for the zeta function of a first order operator, like the Dirac operator D, the residue at the spectral dimension n is proportional to the coefficient $a_0(f, P)$, which as recalled above is proportional to the integral of f.

We now want to look at the next non-trivial coefficient of the expansion, which is given by $a_2(f, P)$. This corresponds, for the zeta function defined in terms of the Dirac operator, to the residue at $z = n - 2$. We first compute this coefficient for the classical limit of our Dirac operator D_q. In the following we set $f = 1$ and we drop the dependence on f in the notation.

7.7.1 The commutative limit of D_q

The manifold corresponding to the group $SU(2)$ is the 3-sphere. On this space we consider the Laplace-Beltrami operator $\Delta = -g^{\mu\nu}\nabla_\mu\nabla_\nu$, which has eigenvalues $k(k+2)$ with multiplicity $(k+1)^2$, where $k \in \mathbb{N}_0$. It is not hard to show that $a_2(\Delta) = \sqrt{\pi}/4$. Since for the operator Δ we have from (7.1) that $E = 0$, it follows that the scalar curvature is $R = 6$.

In the non-commutative case the operator D_q satisfies $D_q^2 = \chi^{-1}\Delta_L^{-1}C_q$, where χ is a constant matrix, Δ_L acts as $\Delta_L\xi^l_{i,j} = q^{2j}\xi^l_{i,j}$ and C_q is the Casimir of $SU_q(2)$. In particular C_q has the eigenvalues $[l + 1/2]_q^2$ with multiplicity $(2l + 1)^2$, where $l = \frac{1}{2}\mathbb{N}_0$. Now in the commutative limit $q \to 1$ the matrix χ reduces to the identity, Δ_L reduces to the identity operator and the eigenvalues of C_q become $(l + 1/2)^2$. Therefore we see that, upon writing $k = l/2$, we reduce to the classical situation of an operator C with eigenvalues $\frac{1}{4}(k+1)^2$ and multiplicity $(k + 1)^2$, where $k \in \mathbb{N}_0$. It is clear that C is related to the Laplace-Beltrami operator Δ by a rescaling and the addition of a costant, that is $C = \frac{1}{4}\Delta + \frac{1}{4}$.

Therefore we can compare C_q with its classical limit C. To this end we look at the heat kernel coefficients of the operator C, specifically at $a_2(C)$. We can easily obtain it from the knowledge of $a_2(\Delta)$ in the following way: the rescaling $\Delta \to \frac{1}{4}\Delta$ has the effect a conformal transformation $g \to 4g$ of the metric, which in turn changes the scalar curvature by $R \to \frac{1}{4}R$. Then the addition of the constant $\frac{1}{4}$ simply sets $E = -\frac{1}{4}$. Therefore we obtain

$$6E + R \to -\frac{6}{4} + \frac{R}{4} = 0 \ ,$$

where we have used the fact that for the 3-sphere the scalar curvature is $R = 6$. In other words we have that $a_2(C) = 0$. Another way to check that this is the case is directly via the zeta function. Indeed, after removing the zero eigenvalue, it is simple to compute

$$\zeta(z, C) = \sum_{k=1}^{\infty}(k+1)^2 4^{-z}(k+1)^{-2z} = 4^{-z}(\zeta(2z-2)-1) \ .$$

This function is regular at $z = (3-2)/2 = 1/2$, so that we have $a_2(C) = 0$.

7.7.2 The requirement on the residue

Let us now get back to our original problem. We have seen that the non-commutative integral recovers the Haar state, independently of the value of a. But now, from the previous discussion, we have a natural requirement that could possibly fix this ambiguity: since, as we have seen, in the non-commutative case the role of C is played by D_q^2, we can try to impose the analogue of the condition $a_2(C) = 0$. This means that we can require

$$\operatorname*{Res}_{z=n-2} \Gamma(z)\zeta^{(a,1)}(z) = 0.$$

Recall that the spectral dimension n depends on a, since $n = a+1$. The next proposition shows that this fixes the natural value $a = 2$, corresponding to the classical dimension.

Proposition 7.9. *The residue of $\zeta^{(a,1)}(z)$ at $z = n-2$ is zero if and only if $a = 2$.*

Proof. From the proof of Proposition 7.5 we have that

$$\zeta^{(a,1)}(z) = \frac{q^{-z/2} + q^{z/2}}{(q^{-1}-q)^{-z}} \sum_{k=0}^{\infty} \begin{pmatrix} z+k-1 \\ k \end{pmatrix} S_k^{(a,1)}(z) \ ,$$

where $S_k^{(a,1)}(z)$ is given by

$$S_k^{(a,1)}(z) = \frac{q^{k+z/2}\left(1-q^{2k+z}\right)}{\left(1-q^{z-(a+1)+k}\right)\left(1-q^{z-(a-1)+k}\right)\left(1-q^{a+1+k}\right)\left(1-q^{a-1+k}\right)} \ .$$

Since the spectral dimension is given by $n = a+1$, we should take the residue at $z = n-2 = a-1$. Notice that for this value of z the zeta function $\zeta^{(a,1)}(z)$ has two poles, coming respectively from the terms $k=0$ and $k=2$. Omitting a common prefactor, these are given by

$$\frac{q^{z/2}\left(1-q^{z}\right)}{\left(1-q^{z-a-1}\right)\left(1-q^{z-a+1}\right)\left(1-q^{a+1}\right)\left(1-q^{a-1}\right)} + \frac{1}{2}z(z+1)\frac{q^{2+z/2}\left(1-q^{4+z}\right)}{\left(1-q^{z-a+1}\right)\left(1-q^{z-a+3}\right)\left(1-q^{a+3}\right)\left(1-q^{a+1}\right)} \ .$$

Taking now the limit $z \to a-1$ in the regular terms we get

$$\frac{q^{(a-1)/2}\left(1-q^{a-1}\right)}{\left(1-q^{-2}\right)\left(1-q^{z-a+1}\right)\left(1-q^{a+1}\right)\left(1-q^{a-1}\right)} + \frac{1}{2}a(a-1)\frac{q^{2+(a-1)/2}\left(1-q^{a+3}\right)}{\left(1-q^{z-a+1}\right)\left(1-q^{2}\right)\left(1-q^{a+3}\right)\left(1-q^{a+1}\right)} \ .$$

This expression can be rearranged as

$$\frac{q^{(a-1)/2}}{1-q^{a+1}}\left(\frac{1}{1-q^{-2}}+\frac{1}{2}a(a-1)\frac{q^2}{1-q^2}\right)\frac{1}{1-q^{z-a+1}} = \frac{q^{(a-1)/2}}{1-q^{a+1}}\frac{q^2}{1-q^2}\left(-1+\frac{1}{2}a(a-1)\right)\frac{1}{1-q^{z-a+1}} \ .$$

The residue of this term is then non-zero unless we have $\frac{1}{2}a(a-1)=1$, whose solutions are $a=-1$ and $a=2$. But we known from Proposition 7.5 that we have to impose the conditions $a \pm 1 > 0$ for the spectral dimension to exists. This excludes the case $a=-1$. $\square$

Chapter 8

Quantum dimension and quantum projective spaces

In this chapter we show that the family of spectral triples for quantum projective spaces introduced by D'Andrea and Dąbrowski in [D'ADą10], which have spectral dimension equal to zero, can be reconsidered as modular spectral triples by taking into account the action of the element $K_{2\rho}$ or its inverse. The spectral dimension computed in this sense coincides with the dimension of the classical projective spaces. The connection with the well known notion of quantum dimension of quantum group theory is pointed out.

8.1 Introduction

Quantum homogeneous spaces provide an excellent testing ground to study how quantum groups fit into the framework of non-commutative geometry developed by Connes. An important result in this respect is given in [Krä04], where a Dirac operator D is defined on quantized irreducible generalized flag manifolds, which yields a Hilbert space realization of the covariant first-order differential calculus constructed in [HeKo04]. This means, in particular, that the commutator of D with an element of the coordinate algebra is a bounded operator, which is one of the defining properties of a spectral triple. The other essential property, that of the compactness of the resolvent of D, has not been proven, even though it is expected to hold. In particular it can be checked for the simplest case to which this construction applies, that is the Podleś sphere. In this case the Dirac operator D coincides with the Dirac operator introduced in [DąSi03], which has compact resolvent.

Among the class of q-deformed irreducible flag manifolds are the quantum projective spaces $\mathbb{C}P_q^\ell$, the simplest example of which is again the Podleś sphere, which is obtained for $\ell = 1$. The case of $\mathbb{C}P_q^2$ has been studied in [DDL08] and then generalized in [D'ADą10] to $\mathbb{C}P_q^\ell$ with $\ell \geq 2$. The starting point is the introduction of the q-analogue of the module of antiholomorphic differential k-forms Ω^k. More generally the modules Ω^k_N are considered, with

$N \in \mathbb{Z}$, corresponding essentially to $\Omega^k = \Omega^k_0$ twisted by certain line bundles. The Hilbert space completion of these is denoted by H_N. For each of these an unbounded self-adjoint operator D_N is introduced, which has bounded commutators with the coordinate algebra $\mathcal{A}(\mathbb{C}P^\ell_q)$. The main result is that $(\mathcal{A}(\mathbb{C}P^\ell_q), H_N, D_N)$ is a family of equivariant spectral triples.

It turns out that these spectral triples are 0^+-summable, in the sense that the operator $(D_N^2 + 1)^{-\epsilon/2}$ is trace-class for every $\epsilon > 0$. The detailed computation of the spectrum clearly reveals why this is the case: the eigenvalues of this operator grow like a q-number, so exponentially, while their multiplicities grow like a polynomial. We recall that in the classical case it is the balance between the growth of the eigenvalues and their multiplicities that allows to recover the dimension of the manifold in consideration. In this case the eigenvalues grow much faster than their multiplicities, which explains the 0^+-summability.

In this chapter we consider a simple modification of the above construction, which fits into the framework of modular spectral triples. The idea is to consider the action of the element $K_{2\rho}$, which implements the modular group of the Haar state of $\mathcal{A}(\mathbb{C}P^\ell_q)$. In particular we compute the spectral dimension associated to D with respect to the weight $\mathrm{Tr}(K_{2\rho}\cdot)$, with the result that it coincides with the classical dimension. This computation is linked with an important concept in the theory of quantum groups, which is the notion of quantum dimension. We also point out that, as a consequence of a property of the quantum dimension, the same result for the spectral dimension is obtained by considering $K_{2\rho}^{-1}$. This in turn is connected with some results from twisted Hochschild (co)homology.

As in the previous chapters, the motivation comes from the notion of integration which, from the point of view of spectral triples, is defined in terms of D. As we mentioned above, in the case of quantum projective spaces this procedure gives a spectral dimension equal to zero. But, more importantly, this procedure does not allow to recover the natural notion of integration that is available on these spaces, which is given by the Haar state. As in the previous chapter, requiring the non-commutative integral to have the same modular properties of the Haar state immediately brings us into the realm of modular spectral triples.

8.2 Quantum projective spaces

In this section we provide some background on quantum projective spaces, which we denote by $\mathbb{C}P^\ell_q$ for $\ell \geq 2$ and $0 < q < 1$. These are q-deformations of complex projective spaces of real dimension 2ℓ. The case $\ell = 1$ of this construction coincides with the standard Podleś sphere and is well known in the literature. We take our definitions and notations from [D'ADą10].

To define quantum projective spaces we first define the Hopf $*$-algebra $U_q(\mathfrak{su}(\ell+1))$ and its dual $\mathcal{A}(SU_q(\ell+1))$, which can be considered as the algebra of representative functions on the quantum $SU(\ell+1)$ group. The coordinate algebra $\mathcal{A}(\mathbb{C}P^\ell_q)$ of the quantum projective space $\mathbb{C}P^\ell_q$ is then defined as the fixed point subalgebra of $\mathcal{A}(SU_q(\ell+1))$ for the action of a suitable Hopf subalgebra of $U_q(\mathfrak{su}(\ell+1))$. We now review these notions.

For $0 < q < 1$ we denote by $U_q(\mathfrak{su}(\ell+1))$ the $*$-algebra generated by $K_i = K_i^*$, K_i^{-1}, E_i and $F_i = E_i^*$, with $i = 1, \cdots, \ell$, and with relations

$$\begin{gathered}
[K_i, K_j] = 0, \quad K_i E_i K_i^{-1} = q E_i, \\
K_i E_j K_i^{-1} = q^{-1/2} E_j \quad \text{if } |i-j| = 1, \\
K_i E_j K_i^{-1} = E_j \quad \text{if } |i-j| > 1, \\
[E_i, F_j] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}}, \\
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } |i-j| = 1, \\
[E_i, E_j] = 0 \quad \text{if } |i-j| > 1.
\end{gathered}$$

We call $U_q(\mathfrak{su}(\ell))$ the Hopf $*$-subalgebra generated by the elements $K_i = K_i^*$, K_i^{-1}, E_i and $F_i = E_i^*$ with $i = 1, \cdots, \ell-1$. Its commutant is the Hopf $*$-subalgebra $U_q(\mathfrak{u}(1))$ generated by the element $K_1 K_2^2 \cdots K_\ell^\ell$ and its inverse. This is a positive operator in all the representations we consider, so we can define its root of order $2/(\ell+1)$ by

$$\hat{K} = (K_1 K_2^2 \cdots K_\ell^\ell)^{2/(\ell+1)}.$$

The following element will play a central role in this chapter:

$$K_{2\rho} = (K_1^\ell K_2^{2(\ell-1)} \cdots K_j^{j(\ell-j+1)} \cdots K_\ell^\ell)^2.$$

Here the symbol ρ denotes the Weyl vector of the Lie algebra $\mathfrak{su}(\ell+1)$, see for example [KlSc] for its role in q-deformations of semisimple Lie algebras. One important property of this element is that it implements the square of the antipode, in the sense that $S^2(h) = K_{2\rho} h K_{2\rho}^{-1}$ for any $h \in U_q(\mathfrak{su}(\ell+1))$. More importantly for us, it also implements the modular group of the Haar state of $\mathcal{A}(SU_q(\ell+1))$, as we will see in a moment.

We are interested in representations in which the K_j's are represented by positive operators. Such irreducible finite-dimensional $*$-representations of $U_q(\mathfrak{su}(\ell+1))$ are labeled by ℓ non-negative integers. Writing $n = (n_1, \cdots, n_\ell)$, we denote by V_n the vector space carrying the representation ρ_n with weight n. These are highest weight representations, so there exists a vector v which is annihilated by all the E_j's and satisfies $\rho_n(K_i) v = q^{n_i/2} v$.

We now introduce the coordinate algebra $\mathcal{A}(SU_q(\ell+1))$. It is the Hopf $*$-algebra generated by the elements u^i_j, with $i, j = 1, \cdots, \ell+1$), and with relations

$$\begin{gathered}
u^i_k u^j_k = q u^j_k u^i_k, \quad u^k_i u^k_j = q u^k_j u^k_i, \quad \text{for } i < j, \\
[u^i_l, u^j_k] = 0, \quad [u^i_k, u^j_l] = (q - q^{-1}) u^i_l u^j_k, \quad \text{for } i < j,\ k < l.
\end{gathered}$$

and with the determinant relation

$$\sum_{p \in S_{\ell+1}} (-q)^{\|p\|} u^1_{p(1)} \cdots u^{\ell+1}_{p(\ell+1)} = 1.$$

The $*$-structure is defined as in [D'ADą10].

There is a non-degenerate pairing $\langle \cdot, \cdot \rangle$ between $U_q(\mathfrak{su}(\ell+1))$ and $\mathcal{A}(SU_q(\ell+1))$, which is used to define the canonical left and right actions as $h \triangleright a = a_{(1)} \langle h, a_{(2)} \rangle$ and $a \triangleleft h = \langle h, a_{(1)} \rangle a_{(2)}$, where we use Sweedler's notation for the coproduct. This pairing can be extended to include also the action of the element $\hat{K}$ and its inverse.

There is a faithful state on $\mathcal{A}(SU_q(\ell+1))$, called the Haar state and denoted by φ, which generalizes the properties of the Haar integral in the classical case. However, differently from the classical case, the Haar state is not a trace on $\mathcal{A}(SU_q(\ell+1))$. In particular its modular group is implemented by the element $K_{2\rho}$, in the sense that

$$\varphi(ab) = \varphi(bK_{2\rho} \triangleright a \triangleleft K_{2\rho}). \tag{8.1}$$

Consider now the left action of $U_q(\mathfrak{su}(\ell+1))$ on $\mathcal{A}(SU_q(\ell+1))$ defined by

$$\mathcal{L}_h a = a \triangleleft S^{-1}(h).$$

It can be used to define the coordinate algebra $\mathcal{A}(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ as

$$\mathcal{A}(S_q^{2\ell+1}) = \{a \in \mathcal{A}(SU_q(\ell+1)) : \mathcal{L}_h a = \varepsilon(h) a, \forall h \in U_q(\mathfrak{su}(\ell))\}.$$

Finally, using the generator of $U_q(\mathfrak{u}(1))$, which we denoted by $\hat{K}$, we define the coordinate algebra $\mathcal{A}(\mathbb{C}P_q^\ell)$ of the quantum projective space $\mathbb{C}P_q^\ell$ as

$$\mathcal{A}(\mathbb{C}P_q^\ell) = \{a \in \mathcal{A}(S_q^{2\ell+1}) : \mathcal{L}_{\hat{K}} a = a\}.$$

Having defined the coordinate algebra $\mathcal{A}(\mathbb{C}P_q^\ell)$, the next step in order to build a spectral triple is to introduce a Hilbert space, on which elements of this algebra act as bounded operators. Recall that the projective spaces $\mathbb{C}P^\ell$ are only spin^c manifolds when ℓ is even. Then one possibility is to complete the space of antiholomorphic forms, with the idea of defining a Dolbeault-Dirac operator acting on it. This is the strategy followed in [DDL08] for the case $\ell = 2$, where a q-analogue of the space of antiholomorphic forms is introduced.

This strategy is generalized in [D'ADą10] for all quantum projective spaces. We denote by Ω^k their q-analogue of the space of antiholomorphic k-form. More generally, they also consider the possibility of twisting this module of k-forms by a line bundle Γ_N, with the resulting space being denoted by Ω_N^k, and with the space of forms corresponding to the case $N = 0$. Since the left and right canonical actions of $U_q(\mathfrak{su}(\ell+1))$ are mutually commuting, the space $\bigoplus_{k=0}^{\ell} \Omega_N^k$ carries a left action of $U_q(\mathfrak{su}(\ell+1))$ and can be decomposed into irreducible representations. We record this result in the schematic form

$$\begin{aligned} \Omega_N^0 &= \bigoplus_{m\in\mathbb{N}} V_{(m+c_1,0,\cdots,0,m+c_2)}, \\ \Omega_N^k &= \bigoplus_{m\in\mathbb{N}} V_{(m+c_3,0,\cdots,0,m+c_4)+e_k} \oplus V_{(m+c_5,0,\cdots,0,m+c_6)+e_{k+1}} \quad \text{for } 1\leq k\leq \ell-1, \\ \Omega_N^\ell &= \bigoplus_{m\in\mathbb{N}} V_{(m+c_7,0,\cdots,0,m+c_8)}. \end{aligned} \tag{8.2}$$

Here $c_1,\cdots,c_8$ are integers depending on k and N, but independent of m. The Hilbert spaces obtained as a completions of $\bigoplus_{k=0}^{\ell} \Omega_N^k$ are denoted by H_N.

The q-analogue of the Dolbeault operator, which we denote by $\bar\partial$, maps Ω_N^k into Ω_N^{k+1} and satisfies $\bar\partial^2=0$. Similarly the adjoint $\bar\partial^\dagger$ maps Ω_N^{k+1} into Ω_N^k and satisfies $(\bar\partial^\dagger)^2=0$. A family of Dolbeault-Dirac operators, denoted by D_N for $N\in\mathbb{Z}$, is defined by taking suitable linear combinations of $\bar\partial$ and $\bar\partial^\dagger$ on each Ω_N^k. The operator D_0 is the q-analogue of the Dolbeault-Dirac operator on $\mathbb{C}P^\ell$, while D_N is the twist of D_0 with the Grassmannian connection of a certain line bundle. In particular, if ℓ is odd and $N=(\ell+1)/2$, then D_N is the q-analogue of the Dirac operator for the Fubini-Study metric.

Here we do not need the precise form of D_N, but only an asymptotic form of its eigenvalues. In particular, for our purposes, this turns out to be independent on the value of N. Using the decomposition (8.2), it is possible to compute the eigenvalues of $|D_N|$ when restricted to the space Ω_N^k. The information that we need is that these eigenvalues grow like q^{-m} with $m\in\mathbb{N}$, see the discussion at the end of [D'ADą10].

8.3 Twisted trace property

We now consider the restriction of the Haar state of $\mathcal{A}(SU_q(\ell+1))$ to $\mathcal{A}(\mathbb{C}P_q^\ell)$. It follows, using the definitions given in the previous section, that any element $a\in\mathcal{A}(\mathbb{C}P_q^\ell)$ is invariant under the right action of $K_{2\rho}$, that is $a\triangleleft K_{2\rho}=a$. Therefore the modular property of the Haar state of $\mathcal{A}(SU_q(\ell+1))$, given by (8.1), for $a,b\in\mathcal{A}(\mathbb{C}P_q^\ell)$ becomes

$$\varphi(ab)=\varphi(bK_{2\rho}\triangleright a).$$

As we have remarked in the introduction, the non-commutative integral, defined in the usual sense of spectral triples in terms of D_N, does not coincide with the Haar state, since the former is a trace while the latter is not. This fact provides a motivation to introduce a twist in the definition of the non-commutative integral, as we now explain.

We denote by $K_{2\rho}$ the closure of the unbounded operator on H_N acting via the left action of $K_{2\rho}$ on $\mathcal{A}(SU_q(\ell+1))$. It is a positive and invertible operator. For the moment let us suppose that the operator $K_{2\rho}(D_N^2+1)^{-z/2}$ is trace-class for $\mathrm{Re}(z)>n$, where $n\geq 0$.

Suppose furthermore that the residue at $z = n$ of its trace exists. Then, similarly to the usual non-commutative integral, we can define a linear functional on $\mathcal{A}(\mathbb{C}P^\ell_q)$ by

$$\psi(a) = \operatorname*{Res}_{z=n} \operatorname{Tr}(K_{2\rho} a (D_N^2 + 1)^{-z/2}).$$

Then it can be shown, as done in Appendix D, that we have

$$\begin{aligned}\psi(ab) &= \operatorname*{Res}_{z=n} \operatorname{Tr}(K_{2\rho} ab (D_N^2 + 1)^{-z/2}) \\ &= \operatorname*{Res}_{z=n} \operatorname{Tr}(b K_{2\rho} a (D_N^2 + 1)^{-z/2}) \\ &= \operatorname*{Res}_{z=n} \operatorname{Tr}(K_{2\rho} K_{2\rho}^{-1} b K_{2\rho} a (D_N^2 + 1)^{-z/2}).\end{aligned} \tag{8.3}$$

The non-trivial equality is the first one, which can be shown using similar methods to the usual case. Then, since $K_{2\rho}^{-1} b K_{2\rho} = K_{2\rho}^{-1} \triangleright b$ for any $b \in \mathcal{A}(\mathbb{C}P^\ell_q)$, we have the property

$$\psi(ab) = \psi(K_{2\rho}^{-1} \triangleright ba),$$

which is equivalent to the property of equation (8.1). Therefore in this way we obtain a linear functional on $\mathcal{A}(\mathbb{C}P^\ell_q)$ which has the modular property of the Haar state. Of course we have yet to show that our assumptions are justified, namely those on the trace-class property and on the residue. This will be done in later sections.

This kind of construction fits into the framework of modular spectral triples, whose relevant definitions can be found in Chapter 4.

8.4 Quantum dimension

Motivated by the previous section, we now want to introduce the tools needed to compute the spectral dimension of D_N with respect to the weight defined by $\operatorname{Tr}(K_{2\rho}\cdot)$. This computation is strictly related to the notion of quantum dimension, that we now review.

Given a finite-dimensional irreducible representation T of a Drinfeld-Jimbo algebra $U_q(\mathfrak{g})$, its *quantum dimension* is defined as the number $\operatorname{Tr}(T(K_{2\rho}))$, where the trace is taken over the vector space that carries the representation T, see for example [KlSc]. In the classical case, that is for $q = 1$, the quantum dimension coincides with the dimension of the vector space. In the context of quantum groups the notion of quantum dimension appears, for example, in the q-analogue of the Schur orthogonality relations.

In the classical case, if we consider a finite-dimensional representation of a Lie algebra $\mathfrak{g}$ with highest weight Λ, the dimension of the associated vector space V_Λ can be computed from the *Weyl dimension formula*, which reads as

$$\dim V_\Lambda = \prod_{\alpha > 0} \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where the product is over the positive roots and ρ is the Weyl vector, defined as the half-sum of the positive roots. There is also a q-analogue of this formula, see for example [ChPr]. It allows to compute the quantum dimension of a representation with highest weight Λ as

$$\dim_q V_\Lambda = \prod_{\alpha>0} \frac{[(\Lambda+\rho,\tilde{\alpha})]}{[(\rho,\tilde{\alpha})]},$$

where we use the usual notion of q-number

$$[x] = \frac{q^{-x}-q^{x}}{q^{-1}-q}$$

and $\tilde{\alpha} = 2\alpha/(\theta,\theta)$ where θ is the highest root. Note that an explicit normalization is needed for the positive roots, differently from the classical case.

Our aim is now to compute the quantum dimension for any of the irreducible representations that appear in the decomposition (8.2). More precisely we are only interested in the asymptotics of this value when $m \to \infty$, since this is the only contribution that matters in the computation of the spectral dimension.

We need to review some facts about the root system of $\mathfrak{su}(\ell+1)$, whose elements can be considered as vectors in $\mathbb{R}^{\ell+1}$. The simple roots are given by $\alpha_i = e_i - e_{i+1}$ with $1 \leq i \leq \ell$. The positive roots are given by $\alpha_{ij} = e_i - e_j$, with $1 \leq i < j \leq \ell+1$, and we note that they can be written in terms of the simple roots as $\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k$. Their scalar product is $(\alpha_{ij},\alpha_{ij}) = 2$. In particular $(\theta,\theta) = 2$, so that $\tilde{\alpha} = \alpha$ in the Weyl formula.

We also need the basis of the fundamental weights, which we denote by ω_i. They are connected to the simple roots via the Cartan matrix A as $\alpha_i = \sum_{j=1}^{\ell} A_{ij}\omega_j$. The fundamental weights are dual to the simple roots in the sense that

$$\frac{2(\alpha_i,\omega_j)}{(\alpha_i,\alpha_i)} = \delta_{ij}.$$

Since in our case $(\alpha_i,\alpha_i) = 2$ this relation becomes $(\alpha_i,\omega_j) = \delta_{ij}$. Finally we recall that the Weyl vector ρ, which is usually defined as the half-sum of the positive roots, can be written in the basis of the fundamental weights in the simple form $\rho = \sum_{j=1}^{\ell} \omega_j$.

Proposition 8.1. *Let $\Lambda = n_1\omega_1 + n_a\omega_a + n_\ell\omega_\ell$ be a weight, where $n_1 = m + c_1$, $n_\ell = m + c_2$ with $m \in \mathbb{N}$, $c_1, c_2 \in \mathbb{Z}$ and $n_a = 0, 1$ with $2 \leq a \leq \ell - 1$. Then the quantum dimension of the representation with weight Λ is $\dim_q(V_\Lambda) = O(q^{-2\ell m})$ for $m \to \infty$.*

Proof. Let us introduce the notation

$$S_i = \prod_{j=i+1}^{\ell+1} \frac{[(\Lambda+\rho,\alpha_{ij})]}{[(\rho,\alpha_{ij})]},$$

so that $\dim_q(V_\Lambda)$ is given by the product of the S_i, that is

$$\dim_q(V_\Lambda) = \prod_{i=1}^{\ell} S_i.$$

Let us consider first the case $i = 1$. Using the formulae $(\alpha_i, \omega_j) = \delta_{ij}$ and $\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k$ it is immediate to show that we have

$$(\Lambda, \alpha_{ij}) = \begin{cases} n_1 & j \leq a, \\ n_1 + n_a & a < j < \ell + 1, \\ n_1 + n_a + n_\ell & j = \ell + 1. \end{cases}$$

Then for $m \to \infty$ we obtain

$$S_1 = \prod_{j=2}^{\ell+1} \frac{[(\Lambda + \rho, \alpha_{ij})]}{[(\rho, \alpha_{ij})]} = O([m]^{\ell-1}[2m]).$$

Similarly for $2 \leq i \leq a$ we have

$$(\Lambda, \alpha_{ij}) = \begin{cases} 0 & j \leq a, \\ n_a & a < j < \ell + 1, \\ n_a + n_\ell & j = \ell + 1. \end{cases}$$

and for $m \to \infty$ we obtain

$$S_i = \prod_{j=i+1}^{\ell+1} \frac{[(\Lambda + \rho, \alpha_{ij})]}{[(\rho, \alpha_{ij})]} = O([m]).$$

Finally for $i \geq a + 1$ we have

$$(\Lambda, \alpha_{ij}) = \begin{cases} 0 & j < \ell + 1 \\ n_\ell & j = \ell + 1 \end{cases}$$

and for $m \to \infty$ we obtain

$$S_i = \prod_{j=i+1}^{\ell+1} \frac{[(\Lambda + \rho, \alpha_{ij})]}{[(\rho, \alpha_{ij})]} = O([m]).$$

Putting all together we find

$$\begin{aligned}\dim_q(V_\Lambda) &= S_1\left(\prod_{i=2}^{a} S_i\right)\left(\prod_{i=a+1}^{\ell} S_i\right)\\ &= O([m]^{\ell-1}[2m])O([m])^{a-1}O([m])^{\ell-a}\\ &= O([m]^{2\ell-2}[2m]).\end{aligned}$$

Finally, since for $x \to \infty$ we have $[x] = O(q^{-x})$, we conclude that $\dim_q(V_\Lambda) = O(q^{-2\ell m})$. □

8.5 Spectral dimension

Given the result of the previous section, it is now easy to prove the main result of this chapter.

Theorem 8.2. *We have that* $\mathrm{Tr}(K_{2\rho}(D_N^2+1)^{-z/2}) < \infty$ *for* $\mathrm{Re}(z) > 2\ell$ *and the residue at* $z = 2\ell$ *exists. In particular the associated spectral dimension is* 2ℓ.

Proof. The Hilbert space H_N is the completion of $\bigoplus_{k=0}^{\ell} \Omega_N^k$ and each Ω_N^k can be decomposed into irreducible representations of $U_q(\mathfrak{su}(\ell+1))$ as in (8.2). As shown in [D'ADą10], the operator D_N^2 restricted to the space Ω_N^k can be expressed in terms of the Casimir operator of $U_q(\mathfrak{su}(\ell+1))$. Therefore it acts as a multiple of the identity in each irreducible representation.

The only representations which appear in the decomposition (8.2) are those of weight $(m + c_{1,k,N}, 0, \cdots, 0, m + c_{2,k,N}) + e_k$, where $m \in \mathbb{N}$, $2 \leq k \leq \ell$ and $c_{1,k,N}, c_{2,k,N}$ are some integers depending on k and N. We denote the vector space that carries such a representation by $V_{m,k,N}$ and the corresponding eigenvalue of D_N^2 by $\lambda_{m,k,N}^2$. Finally denoting by $\mathrm{Tr}_{m,k,N}$ the trace on the vector space $V_{m,k,N}$ we have that

$$\mathrm{Tr}_{m,k,N}(K_{2\rho}(D_N^2+1)^{-z/2}) = \dim_q(V_{m,k,N})(\lambda_{m,k,N}^2+1)^{-z/2}.$$

From Proposition 8.1 we know that, for $m \to \infty$, we have $\dim_q(V_{m,k,N}) = O(q^{-2\ell m})$. Since $\lambda_{m,k,N} = O(q^{-m})$ we conclude that $\mathrm{Tr}_{m,k,N}(K_{2\rho}(D_N^2+1)^{-z/2}) = O(q^{(z-2\ell)m})$.

Finally the trace can be written in the form

$$\begin{aligned}\mathrm{Tr}(K_{2\rho}(D_N^2+1)^{-z/2}) &= \sum_{k=0}^{\ell}\sum_{m=1}^{\infty} \mathrm{Tr}_{m,k,N}(K_{2\rho}(D_N^2+1)^{-z/2})\\ &= \sum_{m=1}^{\infty} O(q^{(z-2\ell)m}).\end{aligned}$$

The series $\sum_{m=1}^{\infty} q^{(z-2\ell)m}$ is absolutely convergent for $\mathrm{Re}(z) > 2\ell$ and has a non-zero residue at $z = 2\ell$, from which the statement of the theorem follows. □

We now give a few comments on this result. As we mentioned in the introduction, in the classical case the computation of the spectral dimension hinges on the balance between the growth of the eigenvalues of D and the growth of their multiplicities. On the other hand, in the case of q-deformations the eigenvalues of D grow like q-numbers, therefore exponentially, while their multiplicities only grow polynomially. This has the consequence of giving a spectral dimension equal to zero for the spectral triples $(\mathcal{A}(\mathbb{C}P^\ell_q), H_N, D_N)$. Intuitively, considering the weight $\mathrm{Tr}(K_{2\rho}\cdot)$ has the effect of replacing the multiplicities of the eigenvalues with their q-analogues, therefore restoring the balance in the computation. Indeed it can be argued that in this context the notion of quantum dimension is the most natural one, as seen from its role in the formulation of the quantum orthogonality relations.

The same result for the spectral dimension is obtained by considering $K_{2\rho}^{-1}$, as follows from a general property of the quantum dimension.

Corollary 8.3. *The results of Theorem 8.2 remain valid if $K_{2\rho}$ is replaced by $K_{2\rho}^{-1}$.*

Proof. This follows from the identity $\mathrm{Tr}(K_{2\rho}^{-1}) = \mathrm{Tr}(K_{2\rho})$, where the trace is taken on the vector space of an irreducible finite-dimensional representation, which is a general property of the quantum dimension. For a proof see for example [KlSc]. □

This simple corollary is interesting in view of its possible applications to twisted Hochschild (co)homology, as we now proceed to explain. It is known that for quantum groups there is a dimension drop in Hochschild homology: this means that, if G is a semisimple group and we denote by $\mathcal{A}(G_q)$ the associated quantized algebra of functions, then we have $H_n(\mathcal{A}(G_q)) = 0$, where n denotes the classical dimension of G. On the other hand, by using twisted Hochschild homology, that is by twisting appropriately the notion of Hochschild homology, it is possible to avoid this dimension drop. This was observed first in [HaKr05] for $SL_q(2)$ by direct computation and then generalized in [BrZh08] to the general case.

Similar results hold for quantum homogeneous spaces as the Podleś spheres, as shown by the computations in [Had07]. For results on a more general class of quantum homogeneous spaces see [Krä12]. In particular, let us consider the case of the standard Podleś sphere. If we denote by ϑ_P the modular group of the Haar state, then the dimension drop in Hochschild homology is avoided by considering the twist ϑ_P^{-1}. Then the volume form, being a twisted cycle, will pair non-trivially with a twisted cocycle with twist ϑ_P^{-1}.

In view of the results mentioned above, we expect that they continue to hold also for the projective spaces $\mathcal{A}(\mathbb{C}P^\ell_q)$. Therefore, if we denote by ϑ the modular group in this case, we expect to avoid the dimension drop in homology by twisting with ϑ^{-1}. Therefore, in view of our results, we can define a a twisted cocycle which has a chance of pairing non-trivially with the volume form.

Corollary 8.4. *The functional on $\mathcal{A}(\mathbb{C}P^\ell_q)^{\otimes(2\ell+1)}$ defined by*

$$\tilde{\psi}(a_0, \cdots, a_{2\ell}) = \operatorname*{Res}_{z=2\ell} \mathrm{Tr}(K_{2\rho}^{-1} a_0 [D_N, a_1] \cdots [D_N, a_{2\ell}] (D_N^2 + 1)^{-z/2})$$

is a twisted cocycle with twist ϑ^{-1}.

Proof. It follows from Corollary 8.3 that this functional is well-defined. That it is a twisted cocycle with twist ϑ^{-1} follows from the twisted trace property shown in equation (8.3), with $K_{2\rho}$ replaced by $K_{2\rho}^{-1}$, and from standard computations. □

For the case of the Podleś sphere, it is shown in [KrWa10] that such a twisted cocycle is indeed non-trivial, when D_N is taken to be the Dirac operator introduced in [DąSi03].

Appendix A

Non-unital spectral triples

In this appendix we briefly discuss the problem of extending the definition of spectral triple to cover the case of non-compact spaces. That the definition needs to be modified can be seen already by considering the elementary case of $\mathbb{R}^n$: in this case, it is not difficult to check that the Dirac operator does not have compact resolvent, so that at least this condition must be modified. Here only this issue and the related one of summability will be discussed, for a more general discussion we refer to [GGISV04].

A possible argument to arrive at the necessary modification is the following. In the compact case we have seen that, by taking the Dixmier trace of the operator $f(D^2+1)^{-n/2}$, with n being the dimension of the manifold, we obtain the integral of f (up to a constant). Clearly when f is the identity function we obtain the volume of the manifold, which is infinite in the non-compact case. On one hand, this shows that only functions with suitable decay conditions must be considered in the algebra. On the other hand, it shows that the resolvent condition can be modified by taking into account multiplication by such functions.

Therefore the operator $f(D^2+1)^{-1/2}$ is required to be compact for all $f \in \mathcal{A}$, where $\mathcal{A}$ is a suitable algebra of functions. This condition has been considered for the first time in [Con95]. It is clear that, in the compact setting, it is equivalent to the one that we have previously introduced. The definition of spectral triple is modified as follows.

Definition A.1. Let $\mathcal{A}$ be a $*$-subalgebra of $B(H)$, where H is a Hilbert space. We call the triple $(\mathcal{A}, H, D)$ a (non-compact) spectral triple if

1. D is a self-adjoint operator,
2. $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$,
3. $a(D^2+1)^{-1/2}$ is compact for all $a \in \mathcal{A}$.

In the following example we examine in more detail the case of $\mathbb{R}^n$.

Example A.1. *For the flat space $\mathbb{R}^n$ the Dirac operator is given by $D = -i\gamma^\mu\partial_\mu$ and its square is the Laplacian. Consider the following zeta function*

$$\zeta_f(z) = \mathrm{Tr}\left(f(D^2+\mu^2)^{-z/2}\right) \ .$$

Here $\mu > 0$ is needed for invertibility. It is easy to see that, if f is "good enough", for example if we take f to be a Schwartz function, then the trace can be rewritten as

$$\zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n} \int \left(\xi^2+\mu^2\right)^{-z/2} d^n\xi \int f(x) d^n x \ ,$$

where the coefficient $2^{[n/2]}$ comes from the trace over the spinor bundle. The integral over ξ is finite for $\mathrm{Re}(z) > n$ *and we get*

$$\int \left(\xi^2+\mu^2\right)^{-z/2} d^n\xi = \pi^{n/2}\mu^{n-z}\frac{\Gamma\left(\frac{z-n}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} \ . \tag{A.1}$$

We obtain an analytic continuation using well-known properties of the gamma function, and we find that the only singularities of $\xi_f(z)$ are simple poles. Indeed $\Gamma(z)$ has poles on the negative real axis at $z = 0, -1, -2, \cdots$, so that the function $\Gamma\left(\frac{z-n}{2}\right)$ has poles at $z = n - 2m$, where $m \in \mathbb{N}_0$. When n is even the poles at $z = 0, -2, -4, \cdots$ are canceled by the zeroes of $\Gamma\left(\frac{z}{2}\right)$. Then the result is that $\zeta_f(z)$ has simple poles at $z = n, n-2, \cdots, 2$ when n is even, and has simple poles at $z = n, n-2, \cdots, 1, -1, -3, \cdots$ when n is odd.

We can easily compute the residue at $z = n$ of $\zeta_f(z)$, which is given by

$$\operatorname*{Res}_{z=n} \zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n}\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int f(x) d^n x \ .$$

We then recover the integral of f up to a constant, with the parameter μ disappearing in this expression.

Appendix B

Hopf algebras

In this appendix we give some elementary notions of Hopf algebras and quantum groups. The material is standard and can be found for example in [KlSc]. We take our ground field to be $\mathbb{C}$ and the tensor product to be the *algebraic* tensor product.

Let us recall that an associative algebra with unit is a linear space A with a bilinear mapping $(a, b) \mapsto ab$ and a non-zero element $1 \in A$ such that

$$(ab)c = a(bc), \qquad 1a = a = a1.$$

We can rewrite these two conditions in a more "categorical" fashion by considering multiplication as the linear map $m : A \otimes A \to A$ defined by $m(a \otimes b) = ab$ and the unit as the linear map $\eta : \mathbb{C} \to A$ such that $\eta(1) = 1$. In terms of these maps the two conditions become

$$\begin{aligned} m \circ (m \otimes \mathrm{id}) &= m \circ (\mathrm{id} \otimes m), \\ m \circ (\eta \otimes \mathrm{id}) = \mathrm{id} &= m \circ (\mathrm{id} \otimes \eta), \end{aligned}$$

where in the second line we identify $\mathbb{C} \otimes A$ with A. It is immediate to obtain the notion of (associative) coalgebra by dualizing these conditions.

Definition B.1. An associative *coalgebra* with counit (or shortly coalgebra) is a linear space A with linear maps $\Delta : A \to A \otimes A$ (*coproduct*) and $\varepsilon : A \to \mathbb{C}$ (*counit*) such that

$$\begin{aligned} (\Delta \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \Delta) \circ \Delta, \\ (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} &= (\mathrm{id} \otimes \varepsilon) \circ \Delta. \end{aligned}$$

The compatibility of these two notions gives a bialgebra structure.

Definition B.2. A *bialgebra* is a linear space A which has the structure of both an algebra and a coalgebra, and such that $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{C}$ are algebra homomorphisms.

At this stage there is no requirement for the existence of the inverse of an element. A weaker notion is provided by the antipode map, which makes a bialgebra into a Hopf algebra.

Definition B.3. A *Hopf algebra* is a bialgebra A together with a linear mapping $S : A \to A$ (*antipode*) such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta.$$

For a linear space V let us also define the *flip* $\sigma : V \otimes V \to V \otimes V$ as the linear map such that $\sigma(v_1 \otimes v_2) = v_2 \otimes v_1$. An algebra A is commutative if $m \circ \sigma = m$, so that we say that a coalgebra is *cocommutative* if $\sigma \circ \Delta = \Delta$.

We now turn to some examples, which show that an Hopf algebra structure can be used to accomodate both (functions on) groups and universal enveloping algebras of Lie algebras.

Example B.1. *Let $\mathfrak{g}$ be a Lie algebra and $U\mathfrak{g}$ its universal enveloping algebra. Then $U\mathfrak{g}$ can be made into a Hopf algebra as follows. We define the maps Δ, ε and S on $\mathfrak{g}$ as*

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = -X.$$

These maps are declared to be unital on $\mathbb{C}$. The maps Δ and ε can be extended as algebra homomorphisms and S as an algebra anti-homomorphism, and it is not difficult to check that all the Hopf algebra axioms are satisfied.

This Hopf algebra is cocommutative and moreover the antipode satisfies $S^2 = \mathrm{id}$.

Example B.2. *Let G be a finite group and $\mathcal{F}(G)$ the $\mathbb{C}$-valued functions on G. For a finite group we have the isomorphism $\mathcal{F}(G \times G) \cong \mathcal{F}(G) \otimes \mathcal{F}(G)$. Using this fact we can define a coproduct by $\Delta(f)(g_1, g_2) = f(g_1 g_2)$. Similarly we define the counit and the antipode as $\varepsilon(f)(g) = f(1)$ and $S(f)(g) = f(g^{-1})$. With these maps $\mathcal{F}(G)$ is a Hopf algebra.*

Notice that $\mathcal{F}(G)$ is always commutative, but is cocommutative only if G is commutative. Also in this case the square of the antipode gives the identity.

This example can be extended to compact groups, but in this case we do not have the isomorphism $\mathcal{F}(G \times G) \cong \mathcal{F}(G) \otimes \mathcal{F}(G)$. One possibility is to work with representative functions, that is matrix elements of finite-dimensional representation of G. Another possibility is to consider a completion of the tensor product.

It is possible to introduce a star structure on a Hopf algebra. This requires some compatibility with the Hopf algebra structure.

Definition B.4. A Hopf $*$-algebra is a Hopf algebra A which, as an algebra, is a $*$-algebra and such that Δ and ε are $*$-homomorphisms.

Proposition B.1. *In a Hopf $*$-algebra A we have $S \circ * \circ S \circ * = \mathrm{id}$.*

Another important notion is that of duality. It can be motivated by considering the case of the dual vector space, denoted by A', of a finite dimensional Hopf algebra A. This space becomes an algebra with multiplication defined as $fg(a) = (f \otimes g)\Delta(a)$ for $f, g \in A'$ and

$a \in A$. We can also define a coproduct, counit and antipode as follows: $\Delta(f)(a \otimes b) = f(ab)$, $\varepsilon(f) = f(1)$ and $S(f)(a) = f(S(a))$. With these maps A' becomes a Hopf algebra, which is obtained by dualizing A. This situation is generalized as follows.

Definition B.5. Let U and A be two bialgebras. A *dual pairing* of U and A is a bilinear map $\langle \cdot, \cdot \rangle : U \times A \to \mathbb{C}$ such that

$$\langle \Delta_U(f), a_1 \otimes a_2 \rangle = \langle f, a_1 a_2 \rangle, \qquad \langle f_1 \otimes f_2, \Delta_A(a) \rangle = \langle f_1 f_2, a \rangle,$$
$$\langle 1_U, a \rangle = \varepsilon_A(a), \qquad \langle f, 1_A \rangle = \varepsilon_U(f),$$

for all $f, f_1, f_2 \in U$ and $a, a_1, a_2 \in A$. The pairing is called *non-degenerate* if $\langle f, a \rangle = 0$ for all $f \in U$ implies $a = 0$ and $\langle f, a \rangle = 0$ for all $a \in A$ implies $f = 0$.

It turns out that, if U and A are Hopf algebras, then we have also $\langle S_U(f), a \rangle = \langle f, S_A(a) \rangle$. It can be shown that there is such a pairing between the algebra of representative functions of a group and its universal enveloping algebra.

A quantum group can be considered as a non-commutative and non-cocommutative Hopf algebra. An important class of non-trivial examples is given by q-deformations of universal enveloping algebras, which we now define.

Definition B.1 (Drinfeld-Jimbo algebras)**.** Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank l, with Cartan matrix (a_{ij}) and $d_i = (\alpha_i, \alpha_i)/2$, where α_i are the simple roots and the inner product is defined in terms of the Killing form. Let q be a non-zero complex number and define $q_i = q^{d_i}$. Then $U_q(\mathfrak{g})$ is defined to be the algebra generated by the $4l$ generators K_i, K_i^{-1}, E_i, F_i with defining relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$
$$K_i E_j K_i^{-1} = q_i^{a_{ij}/2} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}/2} F_j,$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}},$$

plus the quantum Serre relations, see [KlSc].

Theorem B.2. *There is a unique Hopf algebra structure on the algebra $U_q(\mathfrak{g})$ with comultiplication Δ, counit ε and antipode S such that*

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1}$$
$$\Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + K_i^{-1} \otimes F_i,$$
$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$
$$S(K_i) = K_i^{-1}, \quad S(E_i) = -q_i E_i, \quad S(F_i) = -q_i^{-1} F_i.$$

It is clear from the relations that this Hopf algebra is non-commutative and non-cocommutative.

As we have seen in the examples at the beginning of this section, given a Lie group it is possible to describe both its universal enveloping algebra and its space of representative functions in terms of Hopf algebras. In the realm of q-deformations the counterpart of the former is given by Drinfeld-Jimbo algebras, while in the compact case the counterpart of the latter is given by compact quantum groups. We mention the important fact that there is a non-trivial dual pairing between these two classes of Hopf algebras.

Compact quantum groups (or more precisely their coordinate algebras) are particular Hopf $*$-algebras, which can be characterized in terms of several equivalent conditions. Usually they are defined in terms of a condition on the corepresentations. A corepresentation is essentially the dual notion of representation for Lie groups.

Definition B.6. A Hopf $*$-algebra A is called a *compact quantum group* if A is the linear span of all matrix elements of finite-dimensional unitary corepresentations of A.

We do not need the precise definition of corepresentation, since we will use the following equivalent characterization in terms of Haar states.

Theorem B.2. *A Hopf $*$-algebra is a compact quantum group if and only if there exists a linear functional h on A such that $(h \otimes \mathrm{id}) \circ \Delta = h(a)1$ and $h(a^* a) > 0$ for all $a \in A$ with $a \neq 0$. Such a functional is called the* Haar state *and it is unique.*

Appendix C

(Twisted) Hochschild homology

Hochschild homology is a homology theory for associative algebras over rings. For a reference and more connections with non-commutative geometry see [Kha]. In the following A is an associative algebra over $\mathbb{C}$ and M is an A-bimodule.

Definition C.1. Let $C_n(A) = M \otimes A^{\otimes n}$, where $n \in \mathbb{N}$. Define $b : C_n(A) \to C_{n-1}(A)$ as

$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_n) = &\sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &+ (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned} \tag{C.1}$$

It can be proven that $b^2 = 0$, so that b is a boundary operator and $(C_n(A), b)$ is a chain complex. We define the *Hochschild homology* of the algebra A with coefficients in M, denoted by $H_*(A, M)$, as the homology of the complex $(C_n(A), b)$. In the special case when $M = A$ we denote this as $HH_*(A)$ and simply speak of the Hochschild homology of A.

To understand the significance of Hochschild homology in the context of non-commutative geometry we must briefly mention a theorem of Connes [Con85], which generalizes a theorem of Hochschild, Kostant and Rosenberg. In this setting we consider the algebra $C^\infty(M)$ of smooth functions on a compact manifold M and consider its *continuous* Hochschild homology. This is an extension of Hochschild homology which takes into account the topology of $C^\infty(M)$. Without getting into details, the important result is that the continuous Hochschild homology of $C^\infty(M)$ is isomorphic to the space of differential forms on M.

Therefore Hochschild homology provides a definition of the space of differential forms that does not make use of the commutativity of the algebra $C^\infty(M)$. As such, it is a good candidate for defining the analogue of differential forms for non-commutative spaces.

Let us point out that, in the case of a compact manifold M, the dimension of the space of differential forms is given in this context by the *Hochschild dimension*. This is the largest $n \in \mathbb{N}$ such that $HH_n(C^\infty(M))$ is different from zero. This number of course coincides with the dimension of the manifold M.

If we consider now a non-commutative space which is a "deformation" of a compact manifold, it is by no means obvious that the Hochschild dimension coincides with the classical dimension. It turns out to be true for non-commutative spaces like the non-commutative tori, but generically it is not the case for quantum groups. In this case we speak of dimension drop, since the Hochschild dimension turns out to be smaller than the classical one.

In these cases it is possible, at least in simple examples, to provide a modification of Hochschild homology which prevents the dimension drop. This is one of the motivation behind the introduction of twisted Hochschild homology [KMT03]. Here we refer to [BrZh08] and in particular to their notion of twisted Hochschild dimension.

Definition C.2. Let σ be an automorphism of A and denote by ${}_\sigma A$ the algebra A with left action twisted by σ, that is A with bimodule structure $a \cdot b \cdot c = \sigma(a)bc$. Then the *twisted Hochschild dimension* is defined as the maximum of the Hochschild dimension of $H_*(A, {}_\sigma A)$ over all the automorphisms σ of A.

It turns out that this definition avoids the dimension drop in several examples coming from quantum groups, see [Had07, HaKr05] and also [BrZh08].

Appendix D

Proof of the twisted trace property

In this appendix we want to give a proof of the equality appearing in equation (8.3). The proof holds quite generally, so that we need not make reference to this particular example.

We consider a triple $(\mathcal{A}, H, D)$ with $\mathcal{A} \subset B(H)$ and Δ_ϕ a positive invertible operator acting on H. The assumptions we make are essentially those of a modular spectral triple:

1. $\Delta_\phi a \Delta_\phi^{-1} \in \mathcal{A}$ for any $a \in \mathcal{A}$,
2. Δ_ϕ and $(D^2+1)^{-1/2}$ commute,
3. $[D, a]$ extends to a bounded operator for any $a \in \mathcal{A}$.

We also make the following summability assumptions:

1. $\Delta_\phi (D^2+1)^{-z/2}$ is trace-class for all $\mathrm{Re}(z) > p$, with fixed $p \in \mathbb{R}$,
2. $z \mapsto \mathrm{Tr}(\Delta_\phi (D^2+1)^{-z/2})$ has a meromorphic extension with a simple pole at $z = p$.

We note in passing that these two summability conditions can be related to the semifinite theory, see [CRSS07]. In any case, we can define a linear functional on $\mathcal{A}$ by

$$\psi(a) = \operatorname*{Res}_{z=p} \mathrm{Tr}(\Delta_\phi a (D^2+1)^{-z/2}).$$

It is well defined, since using Hölder's inequality we find

$$|\mathrm{Tr}(\Delta_\phi a (D^2+1)^{-z/2})| \leq \|\Delta_\phi a \Delta_\phi^{-1}\| \mathrm{Tr}(|\Delta_\phi (D^2+1)^{-z/2}|)$$

and by assumption $\Delta_\phi a \Delta_\phi^{-1} \in \mathcal{A}$ for any $a \in \mathcal{A}$.

Proposition D.1. *Given the assumptions above, the linear functional $\psi : \mathcal{A} \to \mathbb{C}$ satisfies the twisted trace property $\psi(ab) = \psi(\Delta_\phi^{-1} b \Delta_\phi a)$ for all $a, b \in \mathcal{A}$.*

Proof. The crucial step of the proof is to show that

$$\psi(ab) = \operatorname*{Res}_{z=p} \mathrm{Tr}(\Delta_\phi ab (D^2+1)^{-z/2}) = \operatorname*{Res}_{z=p} \mathrm{Tr}(\Delta_\phi a (D^2+1)^{-z/2} b),$$

or equivalently that the following residue vanishes

$$\operatorname*{Res}_{z=p} \operatorname{Tr}(\Delta_\phi a[(D^2+1)^{-z/2}, b]) = 0.$$

It is enough to show that the function $g(z) = \operatorname{Tr}(\Delta_\phi[(D^2+1)^{-z/2}, b])$ is holomorphic in a neighbourhood of p, since using Hölder's inequality we have

$$|\operatorname{Tr}(\Delta_\phi a[(D^2+1)^{-z/2}, b])| \leq \|\Delta_\phi a \Delta_\phi^{-1}\| \operatorname{Tr}(|\Delta_\phi[(D^2+1)^{-z/2}, b]|).$$

It is also easy to see that we can restrict our attention to real values of z, so we will write $z = s$ with $s \in (p-\epsilon, p+\epsilon)$ and $\epsilon > 0$ to be fixed later.

We proceed similarly to [GVF, Theorem 10.20], but taking care of the presence of the modular operator Δ_ϕ. First of all we write $p = k\bar{r}$, with fixed $k \in \mathbb{N}$ and $0 < \bar{r} < 1$ (notice that if p is an integer we can set $k = 2p$ and $\bar{r} = 1/2$). With this convention we can write any s, in a sufficiently small neighbourhood of p, as $s = kr$ for some $0 < r < 1$.

Then, using simple commutator identities, we obtain

$$\begin{aligned}
[(D^2+1)^{-s/2}, b] &= \sum_{j=1}^{k} (D^2+1)^{-(j-1)r/2}[(D^2+1)^{-r/2}, b](D^2+1)^{-(k-j)r/2} \\
&= -\sum_{j=1}^{k} (D^2+1)^{-jr/2}[(D^2+1)^{r/2}, b](D^2+1)^{-(k-j+1)r/2}.
\end{aligned}$$

We introduce the notation

$$R_j = (D^2+1)^{-jr/2}[(D^2+1)^{r/2}, b](D^2+1)^{-(k-j+1)r/2}.$$

Let p_j and q_j be numbers such that $p_j^{-1} + q_j^{-1} = 1$. Then we have

$$\begin{aligned}
\Delta_\phi R_j = \Delta_\phi^{p_j^{-1}} \Delta_\phi^{q_j^{-1}} (D^2+1)^{-jr/2} \Delta_\phi^{-q_j^{-1}} \Delta_\phi^{q_j^{-1}} \\
\times [(D^2+1)^{r/2}, b] \Delta_\phi^{-q_j^{-1}} \Delta_\phi^{q_j^{-1}} (D^2+1)^{-(k-j+1)r/2}.
\end{aligned}$$

Since we assumed that D and Δ_ϕ commute, this can be rewritten as

$$\Delta_\phi R_j = \Delta_\phi^{p_j^{-1}} (D^2+1)^{-jr/2}[(D^2+1)^{r/2}, \Delta_\phi^{q_j^{-1}} b \Delta_\phi^{-q_j^{-1}}] \Delta_\phi^{q_j^{-1}} (D^2+1)^{-(k-j+1)r/2}.$$

Now from Hölder's inequality it follows that

$$\operatorname{Tr}(|\Delta_\phi R_j|) \leq C_j \operatorname{Tr}(\Delta_\phi (D^2+1)^{-jp_j r/2})^{p_j^{-1}} \operatorname{Tr}(\Delta_\phi (D^2+1)^{-(k-j+1)q_j r/2})^{q_j^{-1}},$$

where $C_j = \|[(D^2+1)^{r/2}, \Delta_\phi^{q_j^{-1}} b \Delta_\phi^{-q_j^{-1}}]\|$. It follows from general arguments, which use the boundedness of $[D, a]$ for every $a \in \mathcal{A}$, that this quantity is finite, see [GVF, Lemma 10.17].

Now we want to choose p_j and q_j in such a way that the operators $\Delta_\phi(D^2+1)^{-jp_jr/2}$ and $\Delta_\phi(D^2+1)^{-(k-j+1)q_jr/2}$ are trace-class, which in turn would show that $\Delta_\phi R_j$ is trace-class. Since by assumption we have that $\Delta_\phi(D^2+1)^{-z/2}$ is trace-class for all $\mathrm{Re}(z) > p$, this implies the inequalities $jp_jr > p$ and $(k-j+1)q_jr > p$. Let us set

$$p_j = \frac{s}{r(j-1/2)}, \quad q_j = \frac{s}{r(k-j+1/2)},$$

and notice that they satisfy the equality $p_j^{-1} + q_j^{-1} = 1$, as they should. For $s \geq p$ it is immediate to see that the inequalities $jp_jr > p$ and $(k-j+1)q_jr > p$ are satisfied. Then consider the case $s = p-\epsilon$, with $\epsilon > 0$. In this case the first inequality is satisfied for $\epsilon < p/2j$ and the second one for $\epsilon < p/2(k-j+1)$. Then we fix ϵ, once and for all, by requiring it to be the smallest value such that these inequalities are satisfied for all $j \in \{1, \cdots, k\}$.

Therefore we have proven that $\Delta_\phi R_j$ is trace-class and, since

$$\Delta_\phi[(D^2+1)^{-s/2}, b] = -\sum_{j=1}^{k} \Delta_\phi R_j,$$

the same is true for this operator when $s \in (p-\epsilon, p+\epsilon)$. Then we conclude that

$$\psi(ab) = \operatorname*{Res}_{z=p} \mathrm{Tr}(\Delta_\phi ab(D^2+1)^{-z/2}) = \operatorname*{Res}_{z=p} \mathrm{Tr}(\Delta_\phi a(D^2+1)^{-z/2}b).$$

The rest of the proof is now trivial. Using the trace property we get

$$\begin{aligned}
\psi(ab) &= \operatorname*{Res}_{z=p} \mathrm{Tr}(\Delta_\phi a(D^2+1)^{-z/2}b) \\
&= \operatorname*{Res}_{z=p} \mathrm{Tr}(b\Delta_\phi a(D^2+1)^{-z/2}) \\
&= \operatorname*{Res}_{z=p} \mathrm{Tr}(\Delta_\phi\Delta_\phi^{-1} b\Delta_\phi a(D^2+1)^{-z/2}).
\end{aligned}$$

But this shows that $\psi(ab) = \psi(\Delta_\phi^{-1} b\Delta_\phi a)$, which concludes the proof. $\square$

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