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Some models of crack growth in brittle materials

Ph.D. Thesis

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Contents

Introduction

This work is devoted to the study of models of fractures growth in brittle elastic materials; it collects the results obtained during my Ph.D., that are contained in [**77**, **76**, **78**]. We consider quasi-static rate-independent models, as well as rate-dependent ones and the case in which the first ones are limits of the second ones when certain physical parameters vanish. The term *quasi-static* means that, at each instant, the system is assumed to be in equilibrium with respect to its time-dependent data; this setting is typical of systems whose internal time scale is much smaller than that of the loadings. By *rate-independent* system we mean that, if the time-dependent data are rescaled by a strictly monotone increasing function, then the system reacts by rescaling the solutions in the same manner.

We mainly settle our discussion in the framework of the celebrated Griffith's theory [**50**], developed almost a century ago. The fundamental idea is that the crack growth is the result of the competition between the elastic energy released in the process of crack opening and the energy dissipated to create new portions of crack. The involved energy functional can be written as

$$\mathcal{W}(u,\mathit{\Gamma}) + \mathcal{K}(u,\mathit{\Gamma})\,, \tag{1}$$

where $\mathcal{W}(u,\mathit{\Gamma})$ and $\mathcal{K}(u,\mathit{\Gamma})$ represent the bulk elastic energy and the dissipated surface energy, respectively, associated to an elastic deformation u and a crack $\mathit{\Gamma}$. In an isotropic homogeneous material, the latter energy is usually proportional to the fracture surface, thus accounting for the number of broken atomic bonds along the crack, and the proportionality constant is the so-called fracture toughness, which depends on the material.

At the core of Griffith's criterion is the notion of *energy release rate*, corresponding to the derivative of the bulk energy with respect to the variation in length of the fracture. More precisely, let $\Omega\subset\mathbb{R}^2$ be a bounded open domain, corresponding to an uncracked elastic body, and let $\gamma:[0,L]\to\overline{\Omega}$ be a curve parametrized by arc-length. Consider the family $\mathit{\Gamma}(l)$ of cracks of the form $\mathit{\Gamma}(l)=\gamma([0,l])$. Given a boundary loading w, at equilibrium the elastic bulk energy of the unfractured part $\Omega\setminus\mathit{\Gamma}(l)$ of the material is

$$\mathcal{E}^{el}(l) = \min\left\{\mathcal{W}(u,\mathit{\Gamma}(l)):\ u=w \text{ on } \partial\Omega\right\}.$$

Then, for $l\in(0,L)$, the energy release rate is (formally) defined as

$$\mathcal{G}(l) := -\frac{\mathrm{d}\mathcal{E}^{el}(l)}{\mathrm{d}l}. \tag{2}$$

Assume the crack energy $\mathcal{K}(u,\mathit{\Gamma}(l))$ to be proportional to the length of the fracture, *i.e.* $\mathcal{K}(u,\mathit{\Gamma}(l))=\kappa\, l$, and the boundary datum w to be time-dependent. Consequently the bulk energy and the energy release rate are dependent both on l and t, that is, $\mathcal{E}^{el}(l,t)$ and $\mathcal{G}(l,t)$. The aim is to predict the law of the fracture growth, that in this framework corresponds to describe the time evolution of the crack length $l(t)$. Then in Griffith's theory the following conditions need to be satisfied by $l(t)$:

(g1) *irreversibility:* $\dot{l}(t)\geq 0$;
(g2) *stability:* $\mathcal{G}(l(t),t)\leq\kappa$;
(g3) *activation law:* $\big(-\mathcal{G}(l(t),t)+\kappa\big)\dot{l}(t)=0$,

where the dot denotes the derivative with respect to the time variable.

A more recent approach has been developed by Francfort & Marigo [48], who introduced a new formulation of the quasi-static problem, based on the notion of *quasi-static variational evolution* and on two principal assumptions (in addition to the irreversibility of the fracture process):

(GS) *global stability*: at each instant, the system minimizes the energy (1) among all other admissible configurations at that instant;

(E) *energy-dissipation balance*: the increment of internal energy plus the dissipated energy equals the work of the external forces acting on the body.

The main difference between the two formulations is the following: while in Griffith's approach the system satisfies a *first order* minimality condition, in Francfort & Marigo theory it is requested to fulfill a *global* minimality condition. As already noticed in the seminal paper [48], (GS) is not justified by any known thermodynamical argument and induces unnatural time discontinuities in the crack growth process. However, on the other hand, the point of view of Francfort & Marigo allows to overcome some restrictions of Griffith's theory, mainly of being a two-dimensional model and of requiring the crack path to be known in advance. The current research is trying to conjugate the positive aspects of both approaches; in the discussion below and in following chapters, as long as the models are treated more in detail, we will provide a deeper description and a list of references (which is not meant to be complete, it is mainly a selection based on the author personal interests and knowledge).

The model presented in Chapter 2 fits in the Griffith's framework, the one in Chapter 3 in the variational formulation of Francfort & Marigo, while the one in Chapter 4 can be seen as "in between" the two formulations.

In general, we consider the two-dimensional antiplane shear case in the framework of linear elasticity (in Chapters 3 and 4 we extend the results to other settings, like the two-dimensional linearized or nonlinear planar elasticity). The elastic body is represented by a cylinder, whose cross section is a bounded connected open set $\Omega \subset \mathbb{R}^2$. It undergoes a deformation of the form

$$(x,z) \mapsto (x, z + u(x)) \qquad x \in \Omega,\ z \in \mathbb{R}\,,$$

where $u : \Omega \to \mathbb{R}$ is the out-of-plane displacement. The crack section on Ω corresponds to a one-dimensional set $\varGamma \subset \overline{\Omega}$. For a brittle isotropic solid the linear elastic energy and the dissipated energy are of the form

$$\mathcal{W}(u,\varGamma) = \frac{1}{2}\int_{\Omega\setminus\varGamma} |\nabla u|^2 \mathrm{d}x \qquad \text{and} \qquad \mathcal{K}(u,\varGamma) = \int_{\varGamma} \kappa(x)\,\mathrm{d}\mathcal{H}^1(x)\,, \tag{3}$$

respectively, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure and $\kappa > 0$.

The system is driven by a time-dependent boundary displacement $t \mapsto w(t)$, with $w(t) : \partial_D\Omega \to \mathbb{R}$ defined on a subset $\partial_D\Omega$ of the boundary $\partial\Omega$. That is, we look for an evolution $(u(t), \varGamma(t))$ of the displacement and of the crack such that, in addition to some energetic conditions, at each instant it satisfies the constraint $u(t) = w(t)$ on $\partial_D\Omega\setminus\varGamma(t)$. We do not address the issue about crack initiation, which is a critical aspect of Griffith's theory (see, for example, the discussions in [48, 28]), and we always suppose $\varGamma(0) \neq \emptyset$. We also assume that neither body nor surface forces act on the elastic body. Then the quasi-static stability condition corresponds to the fact that the following problem is satisfied at any instant:

$$\begin{cases} \Delta u(t) = 0 & \text{in } \Omega\setminus\varGamma(t) \\ u(t) = w(t) & \text{on } \partial_D\Omega\setminus\varGamma(t) \\ \frac{\partial u(t)}{\partial \nu} = 0 & \text{on } \varGamma(t)\cup\partial\Omega\setminus\partial_D\Omega \end{cases} \tag{4}$$

where ν is the normal vector to $\partial\Omega$ and $\Gamma(t)$.

Each model is characterized by the aforementioned energetic conditions that the evolution has to fulfill. Hence we need to analyze one by one the cases described in this thesis, since they present very different features. However they share the strategy for proving the existence of an evolution $t \mapsto (u(t), \Gamma(t))$, based on a *time discretization* approach: the continuous-time evolution is approximated by discrete-time evolutions obtained by solving incremental minimum problems (see the general strategy described in Section 1.6).

The first part of this work is related to the perplexities that the assumption (GS) in the variational formulation raises from the physical point of view, as already observed by the authors in [**48**]. Without concerns about this issue, the first result about existence of a quasi-static variational evolution in the spirit of Francfort & Marigo has been obtained by Dal Maso & Toader [**38**]; in a second paper [**37**] the same authors suggest to replace (GS) with a sort of *local* stability condition, physically more appropriate. The mathematical idea in order to obtain it is to introduce a penalizing term at the level of the discrete-time incremental problems, in order to penalize large variations of the elastic and/or fracture energy.

In [**22**, **37**] a penalizing term on the elastic energy, related to *viscosity*, is added. In [**58**, **82**, **63**] a *dissipation* on the crack tip is considered. In both approaches first the authors prove the existence of a *rate-dependent evolution*, depending also on the viscosity or dissipation. Then a *quasi-static variational evolution* is obtained as limit of rate-dependent evolutions when the effect of the viscosity or of the dissipation vanishes. The limit evolution is shown to be different by the one obtained in [**38**] with the (GS) assumption, and to satisfy an energy balance and a sort of local stability condition, defined in terms of the energy release rate. Thus Griffith's criterion appears again.

In Chapter 2, we investigate the interaction between the dissipation in the fracture energy and the viscoelastic term in the elastic energy, when they coexist in the rate-dependent evolution. We assume the crack path to be prescribed *a priori*, with an injective arc-length parametrization $\gamma : [0, L] \to \overline{\Omega}$ of class $C^{1,1}$ (the regularity of γ is related to the existence of the energy release rate $\mathcal{G}$, as explained in Section 1.5); the cracks are of the form $\Gamma(l) := \gamma([0, l])$. In accordance with (3), for each $l \in [0, L]$ the elastic energy is given by

$$\mathcal{W}(u, \Gamma(l)) = \frac{\mathfrak{a}}{2} \int_{\Omega\setminus\Gamma(l)} |\nabla u|^2 \,\mathrm{d}x, \tag{5}$$

where $\mathfrak{a} > 0$ is the Young modulus, while the fracture energy is assumed to be proportional to the crack length

$$\mathcal{K}(u, \Gamma(l)) = \kappa\mathcal{H}^1(\Gamma(l)) = \kappa\, l\,.$$

Let us fix the coefficient of viscosity, $\mathfrak{b} > 0$, and the dissipation constant at the crack tip, $\mathfrak{d} > 0$. As in the previously cited papers, given a boundary loading $t \mapsto w(t)$ by means of a time-discretization approach we first prove the existence of a continuous-time rate-dependent evolution $t \mapsto (l^{\mathfrak{b},\mathfrak{d}}(t), u^{\mathfrak{b},\mathfrak{d}}(t))$ satisfying (in a weak sense) the problem

$$\begin{cases} \mathfrak{a}\Delta u^{\mathfrak{b},\mathfrak{d}}(t) + \mathfrak{b}\Delta \dot{u}^{\mathfrak{b},\mathfrak{d}}(t) = 0 & \text{in } \Omega \setminus \Gamma(l^{\mathfrak{b},\mathfrak{d}}(t)) \\ \dfrac{\mathfrak{a}\partial u^{\mathfrak{b},\mathfrak{d}}(t)}{\partial\nu} + \dfrac{\mathfrak{b}\partial \dot{u}^{\mathfrak{b},\mathfrak{d}}(t)}{\partial\nu} = 0 & \text{on } \Gamma(l^{\mathfrak{b},\mathfrak{d}}(t)) \cup \partial\Omega \setminus \partial_D\Omega \\ u^{\mathfrak{b},\mathfrak{d}}(t) = w(t) & \text{on } \partial_D\Omega \setminus \Gamma(l^{\mathfrak{b},\mathfrak{d}}(t)) \end{cases}$$

under proper initial conditions $l^{\mathfrak{b},\mathfrak{d}}(0) = l_0$ and $u^{\mathfrak{b},\mathfrak{d}}(0) = u_0$, and the Griffith-type stability condition, expressed in terms of the energy release rate $\mathcal{G}$,

$$\begin{aligned}
&\dot{l}^{\mathfrak{b},\mathfrak{d}}(t) \geq 0 \\
&-\mathcal{G}\big(l^{\mathfrak{b},\mathfrak{d}}(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)\big) + \kappa l^{\mathfrak{b},\mathfrak{d}}(t) + \mathfrak{d}\dot{l}^{\mathfrak{b},\mathfrak{d}}(t) \geq 0 \\
&\Big[-\mathcal{G}\big(l^{\mathfrak{b},\mathfrak{d}}(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)\big) + \kappa l^{\mathfrak{b},\mathfrak{d}}(t) + \mathfrak{d}\dot{l}^{\mathfrak{b},\mathfrak{d}}(t)\Big]\, \dot{l}^{\mathfrak{b},\mathfrak{d}}(t) = 0\,.
\end{aligned}$$

Here $\mathcal{G}(l, g)$ is the energy release rate relative to the crack $\Gamma(l)$ and to a boundary datum g.

The next step consists in describing the limit behaviour of the sequence of evolutions $(l^{\mathfrak{b},\mathfrak{d}}, u^{\mathfrak{b},\mathfrak{d}})$ when $\mathfrak{b}$ and $\mathfrak{d}$ vanish. It converges (in proper functional spaces) to a rate-independent evolution (l, u), called *vanishing viscosity evolution*, which satisfies the stability condition (4) and the Griffith's criterion (g1)-(g2)-(g3). We remark that an evolution fulfilling these conditions is not necessarily unique.

At this point our first interest lies in understanding the effect of the mutual interaction between the parameters $\mathfrak{b}$ and $\mathfrak{d}$ during the limiting process. We are able to give a complete answer assuming sufficient regularity for the energy release rate $\mathcal{G}$. It turns out unexpectedly that the leading actor in the selection of the evolution (l, u) is the dissipation at the crack tip; the viscosity coefficient $\mathfrak{b}$ does not influence the limit. Indeed the function $l^{\mathfrak{b},\mathfrak{d}}$ satisfies a Cauchy problem

$$\begin{cases} \dot{l}^{\mathfrak{b},\mathfrak{d}}(t) = V_{\mathcal{G}}(\mathfrak{b}, \mathfrak{d}, t, l^{\mathfrak{b},\mathfrak{d}}(t)) \\ l^{\mathfrak{b},\mathfrak{d}}(0) = l_0\,, \end{cases}$$

with the field $V_{\mathcal{G}}$ defined in terms of $\mathcal{G}$. For regular $\mathcal{G}$, the perturbation induced by $\mathfrak{b}$ does not affect too much neither the field $V_{\mathcal{G}}$ nor the solution.

The second interesting fact is shown by means of an example. Under the same smoothness assumptions on $\mathcal{G}$ as above, exploiting a discussion in [82] we prove that the vanishing viscosity evolution $(l(t), u(t))$ does not satisfy the global stability condition (GS), but it really evolves according to a local stability principle.

In the effort of widening the range of application of the energetic theory suggested by Griffith for fracture evolutions, and then renewed by Francfort & Marigo with the definition of variational evolutions, in Chapter 3 we describe a crack growth process taking place in brittle materials with extremely fragile parts, which allow the fracture to grow along highly irregular paths. As it happens in many situations, the reason for investigating this particular setting comes from experimental observations. Different materials, like ceramics, present highly irregular crack surfaces, as reported in several experimental papers, see, e.g., [**13**, **75**]; the fracture shows roughness characteristics suggesting that the appropriate model for it might be given by a fractional Hausdorff dimensional set, rather than by a "smooth" surface. Furthermore in the analysis of real cracks different scales seem to play a role, and patterns of various dimension emerge [**15**]. Theoretical aspects of fracture mechanics in this framework have been developed, for example, by [**12**, **14**, **23**, **84**], among many others.

In order to understand our contribution, we briefly recall some existing results in the context of variational evolutions. As already said, the first complete mathematical analysis of a continuous-time formulation of a variational model in the case of two-dimensional antiplane linear elasticity was given in [**38**] under the assumption that the cracks are compact connected one-dimensional sets of finite length, then extended to plane elasticity by Chambolle [**24**]. The general case of variational evolutions in $\mathbb{R}^n$ is treated by means of the theory of SBV functions, first in [**47**], then in [**36**, **34**] for the case of finite elasticity; in the last cited paper, a suitable notion of convergence of sets, called σ^p-convergence, is introduced. An important feature of all these models is that the path followed by the crack during its evolution is not prescribed, it is instead the result of the energy minimization.

We emphasize that the key tools for [38, 24] are Blaschke Compactness Theorem 1.7.1 and Gołąb Lower Semicontinuity Theorem 1.7.2. For the other cited papers, the main tools are the SBV compactness and lower semicontinuity theorems by Ambrosio [3, 4] and variants of them (see [34] for the σ^p-convergence). However these results are not available in the setting we consider in Chapter 3, as we will see in few lines, so that we are led to introduce some constraints on the admissible cracks.

We study a model in the framework of quasi-static evolutions, so that we can neglect any inertial and viscous effect, and of Griffith's theory, in the sense that the crack advance is controlled by the competition between the elastic energy released due to the crack opening and the energy dissipated by the new crack. The novelty consists in the fact that, instead of a surface dissipation as the one in (3), we consider

$$\widetilde{\mathcal{K}}(u,\Gamma) = \int_{\Gamma} \tilde{\kappa}(x)\, \mathrm{d}\mathcal{H}^d(x)\,, \tag{6}$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure, with $d>1$, and $\tilde{\kappa}>0$. We assume that the set

$$\Gamma^* := \{x \in \overline{\Omega}:\ \tilde{\kappa}(x) < +\infty\}$$

is a pre-assigned curve with $0<\mathcal{H}^d(\Gamma^*)<+\infty$ (and some further property). For instance, Γ^* might be the well-known von Koch curve (see Section 3.7), for which $d=\log 4/\log 3$.

The admissible cracks are compact sets $\Gamma\subset\Gamma^*$ with an *a priori* bounded number of connected components.

It is worth to notice that, in agreement with Griffith's principle, the dissipated energy $\widetilde{\mathcal{K}}(u,\Gamma)$ is still proportional to the number of molecular bonds which are broken to get the fracture.

Under these assumptions, in the energetic framework for rate-independent processes introduced by Mielke (see, e.g., [66]) the main result of the chapter (Theorem 3.2.3) states the existence of a quasi-static evolution in this class of fractures with fractional dimension; more precisely, given a time-dependent boundary loading $t\mapsto w(t)$, we show that there exists an irreversible crack evolution satisfying proper initial data, the global stability condition (GS): at each instant t

$$\mathcal{W}(u(t),\Gamma(t)) + \widetilde{\mathcal{K}}(u(t),\Gamma(t)) \le \mathcal{W}(v,\Gamma) + \widetilde{\mathcal{K}}(v,\Gamma)$$

for every $\Gamma\supset\Gamma(t), \Gamma\subset\Gamma^*$, $v=w(t)$ on $\partial\Omega\setminus\Gamma$, and the energy balance condition (E): for every $0\le s<t\le T$

$$\mathcal{W}(u(t),\Gamma(t)) + \widetilde{\mathcal{K}}(u(t),\Gamma(t)) = \mathcal{W}(u(s),\Gamma(s)) + \widetilde{\mathcal{K}}(u(s),\Gamma(s)) + \int_s^t \langle \partial_\xi W(\nabla u(\tau)), \nabla \dot{w}(\tau)\rangle \mathrm{d}\tau\,,$$

where the last summand represents the power of the force exerted on the boundary to obtain the displacement $w(t)$ on $\partial\Omega\setminus\Gamma(t)$ (see Remark 1.4.2), W being the energy density for $\mathcal{W}$.

Actually we prove the result in a more general setting. We consider a finite number of curves $\Gamma_1^*,\dots,\Gamma_M^*$ which do not need to have the same Hausdorff dimension, and may also intersect each other, provided that the dimension of the intersection is strictly smaller than the dimension of any of the involved curves. The admissible cracks are compact sets $\Gamma\subset\Gamma_1^*\cup\ldots\cup\Gamma_M^*$ with an *a priori* bounded number of connected components (see Subsection 3.1.1 for the precise definition of the class $\mathcal{C}$ of admissible cracks). We are then able to recover a lower semicontinuity result for the crack energy $\widetilde{\mathcal{K}}$ with respect to the Hausdorff convergence of sets in the class $\mathcal{C}$, and to extend to our case, after a careful topological study of the admissible cracks of the class $\mathcal{C}$, a continuity result (Theorem 3.4.1) for the convergence of gradients of solutions to elliptic problems in varying domains.

We remark that the appearance of d-dimensional Hausdorff measures with $d > 1$ has two main consequences. From the point of view of the mathematical setting of the problem, it suggests that the SBV approach is not omnicomprehensive, since jump sets of SBV or $GSBV$ functions defined in domains of $\mathbb{R}^2$ are always 1-rectifiable. The second issue is related to the lower semicontinuity of the Hausdorff measures $\mathcal{H}^d$: as is known, Gołąb Lower Semicontinuity Theorem is not valid when $d > 1$; furthermore, as just said, we also cannot use any lower semicontinuity result related to SBV functions.

With an euristic discussion, let us explain the ideas in the background of the above model. So far Griffith's theory has mainly been studied assuming the material toughness κ in (3) to be bounded both from above and from below:

$$0 < \beta_1 \leq \kappa(x) \leq \beta_2 < +\infty \tag{7}$$

at every point x of the body. By (7), $\mathcal{K}(u, \Gamma)$ in (3) amounts to consider as admissible cracks only sets of finite one-dimensional Hausdorff measure. In [**38**] the admissible cracks are compact sets having an *a priori* bounded number of connected components and finite length, and the displacements are Sobolev functions out of the crack, while in [**47, 34, 36**] the displacements belong to suitable spaces of SBV-type and the cracks are rectifiable sets related to the jump sets of the displacements.

In order to validate Griffith's model in a wider range of possibilities, one should be able to treat cases in which (7) is violated. In the context of homogenization, the extremal case when the material toughness is infinite in some parts of the material was investigated, e.g., in [**39, 10**]. These authors consider materials with a periodic structure, with purely brittle parts separated by unbreakable fibers (where, ideally, $\kappa(x) = +\infty$), and they show that the homogenized material exhibits different behaviours (elastic, cohesive or brittle), depending on the ratio between the width of the brittle parts and of the fibers. In our work, instead, we are interested in the case when the material has extremely fragile parts, represented by the fact that the bound from below in (7) is not guaranteed anymore. Ideally, the crack tends to develop in the most fragile zone, where κ is "small", since it is energetically convenient. The low toughness coefficient allows the crack to grow quite a lot in length, without paying so much in terms of dissipated energy; the consequence is a very irregular crack, concentrated in the fragile zone.

By means of a Γ-convergence approach, we rigorously justify our model with crack energy (6) instead of (3) as a natural extension of the Griffith's setting, in case of a single preassigned crack path Γ^* as above. Indeed our energy functional can be obtained as Γ-limit of energies involving small toughness coefficients and the $\mathcal{H}^1$-measure restricted to polygonal approximations Γ_ε of the cracks $\Gamma \subset \Gamma^*$ with fractional Hausdorff dimension:

$$\mathcal{K}_\varepsilon(u, \Gamma_\varepsilon) = \int_{\Gamma_\varepsilon} \kappa_\varepsilon(x)\, \mathrm{d}\mathcal{H}^1(x)\,,$$

with $\kappa_\varepsilon(x) \to 0$ for x belonging to Γ_ε, as ε vanishes.

The model described above, that for simplicity is discussed in the two-dimensional antiplane shear case in the framework of linear elasticity, is valid even in the planar linearized and nonlinear elasticity cases, while the finite elasticity setting is still difficult to tackle. To our knowledge, the present work is the first attempt to extend the variational approach to fracture evolution in order to encompass fractional dimensional cracks.

The final chapter of the thesis addresses a critical aspect of the mathematical modelling of fractures: branching and kinking of cracks. In case of plane elasticity, as analyzed in [**26, 27**] and the references therein, the debate concerns two different criteria: the principle of local symmetry and the principle of maximal energy release rate. However, already in the simpler situation of antiplane shears, Griffith's theory finds a mathematical obstacle in absence of sufficient regularity of the cracks. Indeed, so far the existence of the energy release rate (2) has been proved in case of $C^{1,1}$ curves by Toader & Lazzaroni [62], and for piecewise-$C^{1,1}$ curves by Negri [71]. The last result is already in the direction of considering models of crack evolutions with kinking phenomena. By a Γ-convergence approach, the energy release rate is showed to depend on the kinking angle in an *implicit* way. Despite the importance of the result, the implicit description is difficult to handle even in simple situations.

In Chapter 4 we address the issue about branching and kinking from a different perspective. The idea is to introduce some general assumptions on the structure of the admissible cracks which allow us to extend Griffith's theory to the case where *branching* and *kinking* are admitted and, at the same time, to define the *front* of the fracture and its *velocity*. For the definition of the last two concepts we have been inspired by [61], where they are introduced by means of a distributional approach and appear in an energy dissipation term.

The class $\mathcal{S}$ of admissible cracks (Definition 4.1.6) contains sets which are finite unions of $C^{1,1}$ curves, with some topological restrictions in order to satisfy good compactness and length continuity properties, and to control the phenomena of branching and kinking during the evolution of the system. In particular we require that branching and kinking points do not accumulate, and that at most a (*a priori* bounded) finite number of branches springs out of each branching point.

The model we study is rate-dependent and contains a dissipative term penalizing the velocity of crack growth at its front. The fracture path is not preassigned, but cracks are only required to belong to the class $\mathcal{S}$ of admissible cracks; we can view this idea as "in between" the standard Griffith's theory, where the crack path is completely pre-assigned, and the variational formulation of Francfort & Marigo, where it is free.

As a standard procedure, given a time-dependent boundary loading $w(t)$ the existence of an evolution $(u(t), \Gamma(t))$, $t \in [0, T]$, is achieved by a time-discretization approach. The evolution satisfies an energy inequality of the form

$$\begin{aligned} &\mathcal{W}(u(b), \Gamma(b)) + \mathcal{H}^1(\Gamma(b)) + \int_a^b \sum_{x \in F(t)} v(x,t)^2 \,\mathrm{d}t \\ \leq\ &\mathcal{W}(u(a), \Gamma(a)) + \mathcal{H}^1(\Gamma(a)) + \int_a^b \int_{\Omega \setminus \Gamma(t)} \nabla u(t) \cdot \nabla \dot{w}(t) \,\mathrm{d}x \,\mathrm{d}t, \end{aligned} \tag{8}$$

for any two instants $0 \leq a < b \leq T$, where $F(t)$ and $v(x,t)$ represent the front of the fracture at time t and its velocity at $x \in F(t)$, respectively.

Inequality (8) is not sufficient to fully characterize the evolution $(u(t), \Gamma(t))$. Furthermore, the presence of the dissipative term

$$\int_a^b \sum_{x \in F(t)} v(x,t)^2 \,\mathrm{d}t$$

prevents the possibility of achieving an energy balance (E) and even a unilateral minimality condition (GS) in the spirit of Francfort & Marigo. We then look for a first order stability condition in the framework of Griffith's theory, in terms of the energy release rate.

So far, the discussion of the model is valid in any elastic regime: linear, linearized, or nonlinear two-dimensional elasticity. However, in order to state the mentioned stability condition we need an explicit formula for the energy release rate, which is available only in the antiplane shear case of linear elasticity (see the very brief survey in Section 1.5); therefore we restrict the discussion to a bulk energy of the form (3).

Differently than in the models discussed earlier, now the cracks might have several front points. Hence the energy release rate is a functional $\mathcal{G}(w, \Gamma, x)$ dependent on the boundary datum w, on the crack Γ and on the point x of the front of Γ. The Griffith stability criterion for the continuous-time evolution $(u(t), \Gamma(t))$ is achieved, locally in space and time, as long as the crack grows at one point of its front. More precisely, if $v(x_0, t_0) > 0$ at an instant $t_0 \in (0, T)$ and for a tip $x_0 \in F(t_0)$, then there exist $\varepsilon > 0$ and $r > 0$ such that for every $t \in (t_0 - \varepsilon, t_0]$ it is $F(t) \cap B_r(x_0) = \{x(t)\}$, and for a.e. $t \in (t_0 - \varepsilon, t_0)$ the following conditions hold

$$
\begin{aligned}
&v(x(t), t) \geq 0 \\
&\mathcal{G}(w(t), \Gamma(t), x(t)) \leq 1 + v(x(t), t) \\
&[-\mathcal{G}(w(t), \Gamma(t), x(t)) + 1 + v(x(t), t)]\, v(x(t), t) = 0\,.
\end{aligned}
$$

Unfortunately we are not able to characterize the evolution in the entire time interval $[0, T]$ and in correspondence of static front tips. The main sources of difficulty are the presence of several branches of the fracture and the approximation procedure by the discrete-time evolutions. Indeed, a careful control of the behaviour of each point of the front is necessary in order to avoid interactions among them that are not physically justified. It is not a simple task to introduce further restrictions on our class of admissible cracks, without considering geometrical settings already discussed in the literature; on the other hand, some work is still necessary in order to remove some of our mathematical constraints not due to mechanics. Anyway, to our knowledge this represents a first attempt to describe a model encompassing branching and kinking, without assuming the crack path to be known *a priori*.

CHAPTER 1

Preliminaries

In this chapter we discuss the mechanical and physical assumptions for the models that we study in this thesis. We briefly review the fundamental contributions to the mathematical theory of fracture in elastic materials: Griffith's theory [50], introduced in the '20s of the past century, and the variational formulation of the problem, proposed by Francfort & Marigo [48] in the '90s. They represent two cornerstones for the subsequent mathematical results on this topics.

The remaining of the chapter deals with other important, and more technical, issues: the existence, in different physical settings, of the functional called *energy release rate*, which plays an important role in the description of fracture growth processes in the framework of Griffith's theory; the general technique to prove the existence of evolutions of cracks in brittle elastic materials, as it will be applied in the following chapters.

Finally we prove some technical results and introduce the main notation used in the thesis.

1.1. Mechanical and physical assumptions

In the thesis, we consider models of growth of fractures in elastic materials. A body is said *elastic* if it deforms when subject to an external loading, and it goes back to the original configuration when unloaded. If $\overline{\Omega}$ is the space domain occupied by the body at rest (unloaded), usually called the *reference configuration*, the new static equilibrium in the loaded state is described by a map $\varphi : \overline{\Omega} \to \mathbb{R}^3$, the *deformation*, and the *deformed configuration* is given by $\varphi(\overline{\Omega})$. In general, the map φ is supposed to satisfy some regular and physical assumptions, in order to properly define the mathematical setting and to describe a physically acceptable model. For example, φ has to be orientation preserving and, since interpenetration of matter should be avoided, injective.

We briefly recall few basics from the classical theory of elasticity, adopting the notation used in [30]. Assume that the body in the deformed configuration occupies the domain $\overline{\varphi(\Omega)}$ and is subject to a boundary loading and to a body force with densities $g^\varphi : \mathit{\Gamma}^\varphi \to \mathbb{R}^3$ and $f^\varphi : \varphi(\Omega) \to \mathbb{R}^3$, respectively, where $\mathit{\Gamma}^\varphi \subset \varphi(\partial\Omega)$. The key axiom of continuum mechanics is the following, as stated in [30].

Axiom 1.1.1 (Cauchy axiom)**.** *There exists a vector field*

$$s^\varphi : \varphi(\Omega) \times S^2 \to \mathbb{R}^3 \,,$$

where $S^2 = \{\nu \in \mathbb{R}^3 : \, |\nu| = 1\}$*, such that*

(1) for any subdomain $A \subset \overline{\varphi(\Omega)}$ *and any* $x \in \mathit{\Gamma}^\varphi \cap \partial A$ *where the unit inner normal* ν *to* $\mathit{\Gamma}^\varphi \cap \partial A$ *exists, it holds*

$$s^\varphi(x, \nu) = g^\varphi(x) \,,$$

(2) axiom of force balance: for any subdomain $A \subset \overline{\varphi(\Omega)}$

$$\int_A f^\varphi(x)\, dx + \int_{\partial A} s^\varphi(x, \nu)\, d\sigma = 0 \,,$$

(3) axiom of momentum balance: for any subdomain $A \subset \overline{\varphi(\Omega)}$

$$\int_A 0x \wedge f^\varphi(x)\, dx + \int_{\partial A} 0x \wedge s^\varphi(x,\nu)\, d\sigma = 0$$

where $0x$ *is the vector connecting the origin* 0 *to the point* $x \in A$.

The vector field s^φ is called *Cauchy stress vector* and, for any subdomain $A \subset \varphi(\Omega)$, it respresents the force per unit deformed area caused by $\varphi(\Omega) \setminus A$ acting on A. Indeed Cauchy axiom expresses the idea that along the boundaries of any subsdomain $A \subset \varphi(\Omega)$ there exist surface forces $s^\varphi(x,\nu)\, d\sigma$, such that at static equilibrium they counterbalance the effects of the given boundary force $g^\varphi(x)\, d\sigma$ and body force $f^\varphi(x)\, dx$.

From Cauchy axiom we deduce the equations that the (unknown) deformation φ must satisfy when the body is at static equilibrium with respect to the applied forces f^φ and g^φ. It is more convenient to describe the equations with respect to the reference configuration Ω, rather than to the unknown deformed one. For this, we introduce the *first Piola-Kirchhoff stress tensor* $T : \Omega \to M^{3\times 3}$, where $M^{3\times 3}$ is the set of 3×3 real matrices. The tensor field T measures the force per unit undeformed area and is related to the Cauchy stress vector s^φ by

$$T(x)\nu = (\det \nabla\varphi(x))\, s^\varphi\big(\varphi(x), (\nabla\varphi(x))^{-t}\nu\big) \qquad \text{for every } \nu \in S^2, x \in \Omega\,.$$

Then, under proper regularity assumptions, the equilibrium equations over Ω are

$$\begin{cases} -\mathrm{div}\, T(x) = f(x) & \text{for } x \in \Omega \\ \nabla\varphi(x) T(x)^t = T(x) \nabla\varphi(x)^t & \text{for } x \in \Omega \\ T(x)\nu = g(x) & \text{for } x \in \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\nu \in S^2$ is the unit outer normal to $\partial\Omega$, $g : \partial\Omega \to \mathbb{R}^3$ is the density of the surface force per unit area, while $f : \Omega \to \mathbb{R}^3$ is the density of body forces per unit volume, both related to g^φ and f^φ through a change of variables.

Notice that the system (1.1) is undetermined since it contains nine unknown functions (three components of the deformatin φ and the components of the first Piola-Kirchhoff stress tensor, which are six due to symmetry reasons, by the second equation in (1.1)), but only three equations. In order to reduce the indeterminacy it is necessary to introduce some *constitutive* relations, dependent on the material under consideration (gas, solid, liquid, elastic, plastic...). In case of *elastic* materials, they are given by

$$T(x) = \widehat{T}(x, \nabla\varphi(x))$$

for some tensor field $\widehat{T} : \Omega \times M^{3\times 3}_+ \to M^{3\times 3}$ ($M^{3\times 3}_+$ is the set of 3×3 matrices with positive determinant). This provides the six missing equations in order to be able to solve the system (1.1). The boundary value problem for the 3-dimensional elasticity is then the following: given a domain $\Omega \subset \mathbb{R}^3$, with $\partial\Omega = \Gamma_0 \cup \Gamma_1$, a tensor field $\widehat{T} : \Omega \times M^{3\times 3}_+ \to M^{3\times 3}$, the loads $f : \Omega \to \mathbb{R}^3$ and $g : \Gamma_1 \to \mathbb{R}^3$, and a boundary deformation $\varphi_0 : \Gamma_0 \to \mathbb{R}^3$, find a deformation $\varphi : \Omega \to \mathbb{R}^3$ orientation preserving and injective, such that it solves the problem

$$\begin{cases} -\mathrm{div}\, \widehat{T}(x, \nabla\varphi) = f & \text{in } \Omega \\ \widehat{T}(x, \nabla\varphi)\, \nu_{\Gamma_1} = g & \text{on } \Gamma_1 \\ \varphi = \varphi_0 & \text{on } \Gamma_0\,, \end{cases} \tag{1.2}$$

where ν_{Γ_1} is the outer normal to Γ_1.

The problem can be formulated in a variational setting. Here we describe the so-called *hyperelastic* materials, *i.e.* materials for which there exists an *elastic energy density* $W : \Omega \times M^{3\times 3}_+ \to \mathbb{R}$ such that the first Piola-Kirchhoff stress tensor is given by

$$\widehat{T}(x, \xi) = \partial_\xi W(x, \xi) \quad \text{for every } x \in \Omega, \xi \in M^{3\times 3}_+\,.$$

In general W is requested to satisfy assumptions that describe physical properties, as frame-indifference, material properties, as homogeneity or isotropy (see [30, 53, 9, 64]), and also proper regularity and boundedness conditions.

For hyperelastic materials the system (1.2) takes the form

$$\begin{cases} -\mathrm{div}\,\partial_\xi W(x,\nabla\varphi) = f & \text{in } \Omega \\ \partial_\xi W(x,\nabla\varphi)\,\nu_{\Gamma_1} = g & \text{on } \Gamma_1 \\ \varphi = \varphi_0 & \text{on } \Gamma_0 \end{cases} \tag{1.3}$$

and it formally corresponds to the Euler-Lagrange equations for the following minimum problem

$$\min_{\varphi=\varphi_0 \text{ on } \Gamma_0} \left\{ \int_\Omega W(x,\nabla\varphi)\,dx - \int_\Omega f\cdot\varphi\,dx - \int_{\Gamma_1} g\cdot\varphi\,d\sigma \right\},$$

with φ also orientation preserving and injective.

In this thesis, we will consider two particular settings of the 3-dimensional elasticity theory described above:

- *Antiplane elasticity*: the body is ideally assumed to be an infinite 3-dimensional cylinder $\Omega\times\mathbb{R}$, where the cross section Ω is a bounded connected open subset of $\mathbb{R}^2$. In this case we consider deformations $\varphi:\Omega\times\mathbb{R}\to\mathbb{R}^3$ of the form
$$\varphi(x_1,x_2,x_3) = (x_1,x_2,x_3+u(x_1,x_2))$$
for $(x_1,x_2,x_3)\in\Omega\times\mathbb{R}$. The problem of finding the deformation φ when external loads are applied is reduced to determine the out-of-plane displacement $u:\Omega\to\mathbb{R}$. Therefore we are led to consider a 2-dimensional scalar problem.
- *Plane elasticity*, both in the nonlinear and linearized case. In this situation, the elastic body is represented by a 2-dimensional bounded connected open set $\Omega\subset\mathbb{R}^2$, and it undergoes a deformation $\varphi:\Omega\to\mathbb{R}^2$.

Remark 1.1.2. In the antiplane case the incompenetration of matter is automatically satisfied, being the deformations orthogonal to the domain Ω.

In order to introduce the fracture problem, we assume that the possible defects of the elastic body are concentrated in a *crack* $\Gamma\subset\overline{\Omega}$ (notice that, in the antiplane case, the actual crack is $\Gamma\times\mathbb{R}$). The set $\overline{\Omega}\setminus\Gamma$ represents the unfractured part of the body, in the reference configuration, which still behaves elastically. The crack Γ can also be seen as the discontinuity set of the deformation φ.

Mathematical models for fracture associate a *dissipation energy* $E^d(\Gamma,\varphi)$ to the set Γ. It represents the energy spent to break the atomic bonds of the elastic body $\overline{\Omega}$ in order to create the crack Γ. The expression for E^d also depends on the material properties of the body; a quite general classification divides materials in ductile ones and brittle ones:

- *ductile* materials undergo large strains and yielding before failure. Hence they can absorb quite a lot of energy and they exhibit extensive plastic deformations ahead of the crack. Examples of ductile materials are steel and aluminium;
- *brittle* materials fail at lower strains than ductile ones. They absorb little strain energy and show almost no plastic deformation prior to fracture by the catastrophic propagation of a crack. Example are given by glass, ceramics, cast iron.

Note that the physical response of materials can change if surrounding conditions do: for example, steel, usually behaving as a ductile material, may have a brittle response at low temperatures.

Moreover, it is important to point out that not all materials can be easily classified in the above categories.

We will only consider models for brittle materials, and the expression for E^d is described more in detail in the following sections and chapters.

1.2. Quasi-static evolutions

Consider a body which in the reference configuration occupies a domain $\overline{\Omega}$ in $\mathbb{R}^3$. We assume that the material defects correspond to a crack set $\Gamma \subset \overline{\Omega}$, while the complementary part $\overline{\Omega} \setminus \Gamma$ behaves as an elastic body.

Let $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$ and suppose that the body is subject to a boundary *loading process*, that is, a continuous map $t \mapsto \varphi_0(t)$ from a time interval $[0, T]$ into a functional space X containing $\varphi_0(t)$, with $\varphi_0(t) : \partial_D\Omega \to \mathbb{R}^3$ for every t. At any instant the uncracked body $\Omega \setminus \Gamma$ undergoes a boundary displacement $\varphi_0(t) : \partial_D\Omega \to \mathbb{R}^3$. Usually, we will consider $X = H^1(\Omega; \mathbb{R}^3)$ and the map $t \mapsto \varphi_0(t)$ will belong to $H^1(0, T; H^1(\Omega; \mathbb{R}^3))$, in particular it will be continuous as a function of time. We will also assume that neither body nor surface forces act on the material, *i.e.* $f + 0$ and $g = 0$ in (1.2) and (1.3).

Under these conditions, a *deformation process* or *evolution* is a map $t \mapsto (\varphi(t), \Gamma(t))$, where $\Gamma(t) \subset \overline{\Omega}$ represents the crack set in the reference configuration, while $\varphi(t) : \Omega \setminus \Gamma(t) \to \mathbb{R}^3$ is an elastic deformation consistent with the imposed boundary loading process at time t, *i.e.* $\varphi(t) = \varphi_0(t)$ on $\partial_D\Omega \setminus \Gamma(t)$.

Notice that in the above definition we did not take into account any equilibrium or stability condition. Any map $t \mapsto (\varphi(t), \Gamma(t))$ satisfying the above requests is addressed to as a deformation process. The concept of *equilibrium* and *stability* are crucial assumptions for the models of crack growth we discuss in the sequel.

Assumption 1.2.1. We will always suppose the loading process $t \mapsto \varphi_0(t)$ to be sufficiently slow so that rate effects, such as inertia and viscous dissipations, can be neglected.

Under Assumption 1.2.1, when at each instant t the configuration $(\varphi(t), \Gamma(t))$ is at static (stable) elastic equilibrium with respect to the load $\varphi_0(t)$, *i.e.* it solves the system (1.3) in $\Omega \setminus \Gamma(t)$ instead of Ω,

$$\begin{cases} -\operatorname{div} \partial_\xi W(x, \nabla\varphi(t)) = 0 & \text{in } \Omega \setminus \Gamma(t) \\ \partial_\xi W(x, \nabla\varphi(t))\, \nu_{\Gamma_1} = 0 & \text{on } \partial_N\Omega \cup \Gamma(t) \\ \varphi(t) = \varphi_0(t) & \text{on } \partial_D\Omega \setminus \Gamma(t)\,, \end{cases}$$

we speak of equilibrium process or, as it is usually called, *quasi-static evolution*. This can often be seen as a minimality condition of the total energy of the system with respect to all other kinematically admissible configurations for the load $\varphi_0(t)$ (see Condition 1.4.1.(ii) in Section 1.4).

Finally, a system is said *rate independent* if it has not an intrinsic time scale: if the time-dependent data are reparametrized by a strictly monotone function, then the system reacts by reparametrizing its solutions in exactly the same way (see [66, Definition 1.1] for a precise definition).

We conclude with some remarks.

Remark 1.2.2. The scope of the study of fracture mechanics is to predict the evolution of the crack $\Gamma(t)$ and of the deformation $\varphi(t)$ of an elastic body Ω, when it is subject to a given time-dependent boundary loading process $t \mapsto \varphi_0(t)$.

Remark 1.2.3. The parameter t need not be identified with the physical time: it is a real ordered positive parameter. Nevertheless, we will always refer to it as *time*.

Remark 1.2.4. Often we will describe the evolution in terms of the *displacement* $u(t) : \Omega \to \mathbb{R}^3$, defined by

$$\varphi(t, x) = x + u(t, x)$$

for every $x \in \Omega$.

Remark 1.2.5. The models we present in Chapter 2 and Chapter 3 describe quasi-static evolutions, while in the one discussed in Chapter 4 a viscous dissipation is present. In any case, inertial effects are always neglected; an investigation of the dynamical problem in fracture mechanics is carried out in [**35**].

1.3. Griffith's theory

Failure of materials is of great importance for material science and for engineering reasons, as for example the problems suffered by the Liberty Ships in the 1940s show. The theories at the beginning of the 20^{th} century were not capable to give a satisfactory solution of the intricacies of materials failure. The light came back with the celebrated work by Griffith [**50**], which completely changed the point of view about materials failure, introducing a criterion based on the energies exchanged during a crack growth process.

It is fair to say immediately that Griffith's theory is not omnicomprehensive and is plagued with some important issues, as we will discuss later. Anyway, it provides a first interesting description of quasi-static evolutions of fractures in brittle materials with a pre-existing crack. Moreover, taking off from this new theory, subsequent important contributions have improved the understanding of the topic.

We describe Griffith's model in a 2-dimensional setting. The elastic body in the reference configuration is given by a bounded connected open set $\Omega \subset \mathbb{R}^2$, whose boundary $\partial\Omega$ is divided in the Dirichlet part $\partial_D\Omega$ and in the Neumann part $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. A loading process $t \mapsto \varphi_0(t)$ is imposed on $\partial_D\Omega$.

We assume the crack path to be known *a priori*, consisting of a pre-assigned 1-dimensional simple curve $\Gamma \subset \overline{\Omega}$, with arc-length parametrization $\gamma : [0, L] \to \overline{\Omega} \subset \mathbb{R}^2$. In addition, we suppose that the crack is a closed connected subset of Γ containing $\gamma(0)$, *i.e.* it is of the form $\Gamma(\ell) := \gamma([0, \ell])$ for some $\ell \in (0, L]$. Notice that under these assumptions the fracture is completely determined by the arc-length value ℓ of its tip.

Quoting Griffith [**50**], the energy dissipated by the crack creation represents the "work [that] must be done against the cohesive forces of the molecules on either side of the crack". It is then an increasing function of the fracture length; in particular it is supposed to have a linear dependence of the form

$$E^d(\Gamma(\ell)) = \kappa\,\mathcal{H}^1(\Gamma(\ell)) = \kappa\,\ell\,,$$

where κ, called *material toughness*, is a constant dependent on the material. Note that the width of the deformation discontinuity between the two lips of the crack is not considered: in Griffith's theory the cohesive forces across the crack, which could be expressed as a function of the jump discontinuity of the deformation field φ across $\Gamma(\ell)$, are neglected; only the measure of the discontinuity set of the deformation is relevant. This assumption, which is certainly not omnicomprehensive, is reasonable for certain conditions and materials, like the brittle ones. Subsequent works by Barenblatt [**11**], Dugdale [**42**], and many other authors, introduce a cohesive type of surface energy on the crack, dependent on the deformation discontinuity between the lips of $\Gamma(\ell)$.

The elastic strain energy of a deformation $\varphi : \Omega \setminus \Gamma(\ell) \to \mathbb{R}^2$ of the unfractured part $\Omega \setminus \Gamma(\ell)$ of the material is given by

$$E(\ell, \varphi) := \int_{\Omega \setminus \Gamma(\ell)} W(x, \nabla\varphi)\, dx\,,$$

where $W : \Omega \times \mathbb{R}^{2\times 2} \to \mathbb{R}$ is the stored-energy density. Given a boundary loading process $t \mapsto \varphi_0(t)$, $\varphi_0(t) : \partial_D\Omega \to \mathbb{R}^2$, the bulk energy of the body $\Omega \setminus \Gamma(\ell)$ at equilibrium is given by

$$E^{el}(t, \ell) := \min\left\{E(\ell, \varphi) :\ \varphi : \Omega \setminus \Gamma(\ell) \to \mathbb{R}^2,\ \varphi = \varphi_0(t) \ \text{ on } \ \partial_D\Omega \setminus \Gamma(\ell)\right\}\,. \tag{1.4}$$

Under proper assumptions on the domain $\Omega \setminus \Gamma$ and on the bulk energy density W, the minimum problem (1.4) admits a solution $\varphi(t, \ell)$. In accord to the assumption of quasi-static growth, at any instant t the deformation $\varphi(t, \ell)$ is in static equilibrium with respect to the load $\varphi_0(t)$. Note that the minimizer $\varphi(t, \ell)$ of $E^{el}(t, \ell)$ is a solution (in some weak sense) of the system

$$\begin{cases} -\mathrm{div}\, \partial_\xi W(x, \nabla\varphi(t, \ell)) = 0 & \text{in } \Omega \setminus \Gamma(\ell) \\ \varphi(t, \ell) = \varphi_0(t) & \text{on } \partial_D\Omega \setminus \Gamma(\ell) \\ \partial_\xi W(x, \nabla\varphi(t, \ell))\nu = 0 & \text{on } \overline{\Omega} \cap \Gamma(\ell) \text{ and } \partial_N\Omega \end{cases} \tag{1.5}$$

where ν is the normal vector to $\partial_N\Omega$ or Γ. Since $\varphi(t, \ell)$ may be discontinuous along the crack, in the second equation it makes no sense to prescribe the boundary deformation also on $\partial_D\Omega \cap \Gamma(\ell)$. The third equation states that the faces of the crack $\Gamma(\ell)$ are traction free, being the material brittle.

The total energy of the system is given by

$$\mathcal{E}(t, \ell) := E^{el}(t, \ell) + E^d(\Gamma(\ell))\,. \tag{1.6}$$

It is not difficult to see that

- for fixed t, the elastic energy E^{el} is monotonic decreasing in ℓ;
- the dissipative term E^d is monotonic increasing in ℓ.

Hence the two summands in (1.6) have an opposite behaviour when we minimize $\mathcal{E}$ with respect to ℓ: while E^{el} favors crack growth, the dissipative term E^d penalizes it.

In order to give meaning to the predictive model, we need to determine the actual crack length $\ell(t)$. Griffith's principle steps in as a selection rule among all possible evolution maps $t \mapsto \tilde{\ell}(t)$. It consists of three conditions:

(G1) the irreversibility of the fracture process;
(G2) a stability condition;
(G3) an activation principle for the crack growth.

The *irreversibility* reflects the physical idea that a fractured brittle material will never heal. Mathematically, this is a requirement about *monotonicity* of the function $t \mapsto \ell(t)$ describing the time evolution of the crack tip.

We highlight an important implicit consequence: roughly speaking, at each instant the system keeps "memory" of the *whole* preceding process. We mean the following: let $\varphi(t, \ell(t))^+(x)$ and $\varphi(t, \ell(t))^-(x)$ be the traces of the elastic deformation $\varphi(t, \ell(t))$ on the two sides of the crack $\Gamma(\ell(t))$ at $x \in \Gamma(\ell(t))$. At a fixed instant t, it might happen that, even though the material is fractured along $\Gamma(l(t))$, the deformation $\varphi(t, \ell(t))$ is not necessarily discontinuous along all of it, that is, $\varphi(t, \ell(t))^+(x) = \varphi(t, \ell(t))^-(x)$ for some $x \in \Gamma(\ell(t))$. Under the irreversibility assumption, the crack should always be seen as

$$\Gamma(\ell(t)) = \bigcup_{\tau \le t} \left\{x \in \Gamma : \varphi(\tau, \ell(\tau))^+(x) \neq \varphi(\tau, \ell(\tau))^-(x)\right\}\,,$$

i.e. it is the union of all past discontinuity sets of the elastic deformation $\varphi(\cdot, \ell(\cdot))$.

The *stability criterion* can be introduced quoting Griffith again [50, p. 166]: "a general theorem which may be stated thus: In an elastic solid body deformed by specified forces applied at its surface, the sum of the potential energy of the applied forces and the strain energy of the body is diminished or unaltered by the introduction of a crack whose surfaces are traction-free. [...] the total decrease in potential energy due to the formation of a crack is equal to the increase in strain energy less the increase in surface energy. The theorem proved above shows that the former quantity must be positive."

To formally express this sentence, the key ingredient is the *energy release rate*. It represents the amount of elastic energy (called strain energy in Griffith's words) dissipated by an infinitesimal crack increase. The energy release rate for the crack $\Gamma(\ell)$ and the boundary loading $\varphi_0(t)$ is formally defined as

$$\mathcal{G}(t, \ell) := - \lim_{h \to 0+} \frac{E^{el}(t, \ell + h) - E^{el}(t, \ell)}{h} \, . \tag{1.7}$$

Of course the definition does not garantee that this functional is well defined, or even exists. Some regularity assumptions on the *a priori* given crack path Γ have to be introduced in order to give mathematical meaning to the formal limit (1.7). In Section 1.5 we summarize the known results on the existence and properties of the energy release rate; they will be useful for the analysis of the models considered in the following chapters. Note that, being $E^{el}(t, \ell + h) \leq E^{el}(t, \ell)$ for $h > 0$, then $\mathcal{G}(t, \ell) \geq 0$.

The quoted Griffith's sentence then translates as

$$\mathcal{G}(t, \ell) - \kappa \leq 0, \tag{1.8}$$

being $\kappa = \frac{dE^d(\Gamma(\ell))}{d\ell}$ the surface energy rate. This inequality states the stationarity of the crack, introducing an upper bound for the release of elastic energy during a crack growth process.

Finally, the *activation principle* couples with (1.8) in order to describe the conditions that allow the fracture to grow. Assuming sufficient regularity of the function $t \mapsto \ell(t)$, it reads as

$$\dot{\ell}(t) > 0 \quad \Rightarrow \quad \mathcal{G}(t, \ell(t)) - \kappa \geq 0 \, .$$

That is, the crack growth is possible only if the released elastic energy is larger than the surface energy dissipated due to the new crack.

Griffith's principle is therefore summarized by the following Kuhn-Tucker conditions:

$$\begin{aligned} &\dot{\ell}(t) \geq 0 \\ &\mathcal{G}(t, \ell(t)) \leq \kappa \\ &\big(\mathcal{G}(t, \ell(t)) - \kappa\big)\, \dot{\ell}(t) = 0 \, , \end{aligned} \tag{1.9}$$

where the dot denotes the derivative with respect to the time variable.

At this point, a number of comments of different kind is due.

Remark 1.3.1. (i) The system (1.9) provides a *first order* optimality condition for the couple $\big(\varphi(t, \ell(t)), \ell(t)\big)$ and the energy $\mathcal{E}(t, \cdot)$, in some proper topology. In Section 1.4 we describe a *global* minimization approach to the fracture problem, introduced by Francfort & Marigo [48], and some *local* variants, in order to tackle some difficulties and "defects" in Griffith's theory.

(ii) In (1.9) it is implicitly assumed that ℓ is differentiable everywhere and that the three conditions hold at any instant t. Unfortunately this is not always the case, and in the literature some weaker forms of (1.9) are present. We report here, and will use in Chapter 2, the definition of solution introduced in [72]:

Definition 1.3.2. A non-decreasing function $\ell : [0, T] \to [\ell_0, L]$ is a *weak solution* of (1.9) if $\ell(0) = \ell_0$, and if it satisfies

$$\mathcal{G}(t, \ell(t)) \leq \kappa$$

and the following *weak activation criterion*:

$$\ell(t+\tau) - \ell(t-\tau) > 0 \text{ for all small } \tau > 0 \quad \Rightarrow \quad \mathcal{G}(t, l) \geq \kappa \text{ for all } l \in [\ell(t-), \ell(t+)] \setminus \{L\}\,.$$

The weak activation criterion states that, if the crack performs an instantaneous elongation, then the system must pass through *unstable* states, *i.e.* $\mathcal{G}(t, l) \geq \kappa$.

Note that if a weak solution $t \mapsto \ell(t)$ is regular enough then it satisfies (1.9).

(iii) Uniqueness of the evolution $\big(\varphi(t, \ell(t)), \ell(t)\big)$ satisfying (1.9) is not guaranteed. Indeed, as we will see, the strategy of proof of existence of an evolution makes use of compactness arguments; generally it is not even possible to recover uniqueness *a posteriori*.

(iv) Griffith's theory is affected by some defects. First of all, the crack path has to be pre-assigned: otherwise we would have too many degrees of freedom for a possible crack path and only one equation to determine it.

A second problematic issue is that, in some conditions, crack initiation is impossible, that is, according to Griffith's principle an unfractured material would never crack, whatever the boundary loading (see [46]). In this thesis we do not address the issue about crack initiation (an investigation is carried out in [28]); we always assume the body to be cracked at the beginning, *i.e.* $\ell(0) = \ell_0$ for some $\ell_0 \in (0, L)$.

In the next section we describe a new formulation of the fracture problem, proposed by Francfort & Marigo [48], which removes some of the defects just mentioned.

1.4. Variational evolutions

The *variational* model of quasi-static crack growth is relatively recent, proposed by Francfort & Marigo [48] in 1998. It represents a variant to Griffith's model, more than an equivalent formulation. Making use of modern mathematical theories, it allows to solve some issues in Griffith's model, like nucleation of cracks and *a priori* knowledge of the fracture path, and it can be generalized to any dimension (as we will do in this brief description). The price to pay for this is to introduce a selection criterion for the evolution based on *global* minimization, rather than on *local* minimization (remember that, as pointed out in Remark 1.3.1.(i), Griffith's model provides a first order optimality condition).

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$; $\overline{\Omega}$ represents an unfractured elastic body. The crack can be any $(\mathcal{H}^{N-1}, N-1)$-rectifiable set Γ contained in $\overline{\Omega}$, with $\mathcal{H}^{N-1}(\Gamma) < +\infty$.

Since the crack path is not pre-assigned, the deformation u is defined almost everywhere in Ω with values in $\mathbb{R}^N$, and might be discontinuous along an $(N-1)$-dimensional set. In the literature, the deformations u belong to suitable spaces X of SBV-type, for which good notions of jump (or discontinuity) set $J(u)$ and of gradient ∇u are defined.

Given a boundary loading $w : \partial_D\Omega \to \mathbb{R}^N$, the set $\mathcal{AC}(w)$ of admissible configurations is

$$\mathcal{AC}(w) := \{(u, \Gamma) : \, \Gamma \text{ rectifiable}, u \in X \text{ with } u = w \text{ on } \partial_D\Omega \setminus \Gamma \text{ and } J(u) \subset \Gamma\}\,. \tag{1.10}$$

The total energy associated to an admissible configuration (u, Γ) is given by the sum of the elastic bulk energy and of the dissipated energy

$$\mathcal{E}(u,\Gamma) = E^{el}(u,\Gamma) + E^{d}(\Gamma)\,. \tag{1.11}$$

The first summand is of the form

$$E^{el}(u,\Gamma) = \int_{\Omega\setminus\Gamma} W(x,\nabla u)\,dx \tag{1.12}$$

with the energy density $W(x,\xi):\Omega\times\mathbb{R}^{N\times N}\to\mathbb{R}$ satisfying suitable regularity assumptions and growth conditions, depending on the model. Instead, the surface crack energy is the functional

$$E^{d}(\Gamma) = \int_{\Gamma\setminus\partial_N\Omega} \kappa(x,\nu(x))\,d\mathcal{H}^{N-1}(x)\,,$$

where, similarly to the Griffith case, κ is the fracture toughness and ν is the unit normal to Γ.

For a boundary loading process $t\mapsto w(t)$, the selection criterion for a variational evolution is based on two postulates, as indicated in [48]: *irreversibility* and *global energy minimization*.

Definition 1.4.1. Given a boundary loading process $t\mapsto w(t)$, a couple deformation-crack $(u(t),\Gamma(t))$ is an *irreversible variational evolution* if $(u(t),\Gamma(t))\in\mathcal{AC}(w(t))$ and the following conditions are satisfied:

(i) irreversibility: $\Gamma(t_1)\subset\Gamma(t_2)$ for every $0\le t_1<t_2$;

(ii) one-sided minimality: at any instant t and for every $(v,\Gamma)\in\mathcal{AC}(w(t))$ such that $\Gamma\supset\Gamma(t)$, it is
$$\mathcal{E}(u(t),\Gamma(t))\le\mathcal{E}(v,\Gamma)\,;$$

(iii) non-dissipativity: the function $t\mapsto\mathcal{E}(u(t),\Gamma(t))$ is absolutely continuous and satisfies
$$\frac{d}{dt}\mathcal{E}(u(t),\Gamma(t)) = \int_{\Omega\setminus\Gamma(t)} \partial_\xi W(x,\nabla u(t)):\nabla\dot w(t)\,dx\,.$$

Remark 1.4.2. In Condition 1.4.1.(iii) the right-hand side represents the power of the force exerted on the boundary to obtain the displacement $w(t)$ on $\partial_D\Omega\setminus\Gamma(t)$. This fact can be seen by integrating by parts the right-hand side in Condition 1.4.1.(iii), when $\partial_D\Omega$ is sufficiently smooth.

Moreover Condition 1.4.1.(iii) is equivalent to the following integral form: for every $0\le s<t$

$$\mathcal{E}(u(t),\Gamma(t)) = \mathcal{E}(u(s),\Gamma(s)) + \int_s^t \Big(\int_{\Omega\setminus\Gamma(\tau)} \partial_\xi W(x,\nabla u(\tau)):\nabla\dot w(\tau)\,dx\Big)\,d\tau\,. \tag{1.13}$$

Remark 1.4.3. Condition 1.4.1.(iii) states the continuity of the map $t\mapsto\mathcal{E}(u(t),\Gamma(t))$; however the function $t\mapsto E^{el}(u(t),\Gamma(t))$ and $t\mapsto E^{d}(u(t),\Gamma(t))$ may be discontinuous. It seems that numerical simulations confirm this fact (see, for example, [16] and references therein).

As announced, the variational formulation solves some of the criticalities in Griffith's model. First of all, the crack path has not to be pre-assigned. Moreover crack initiation is automatically triggered in finite time, thanks to the global minimality postulate 1.4.1.(ii) in Definition 1.4.1; an example is discussed in [48] in case of monotonically increasing boundary loadings, and see also [16, Proposition 4.1].

The first existence result for a variational evolution was proved by Dal Maso & Toader in [38] for the 2-dimensional antiplane case, assuming the crack to have an *a priori* bounded number of connected components. Chambolle [24] then extended it to the plane case, while Francfort & Larsen [47] obtained the result in the generalized antiplane setting, without restricting the number of connected components of the crack, by proving the important result known as *jump* *transfer theorem* [**47**, Theorem 2.1]. Concerning the vectorial case, the first existence proof is given in [**34**], considering a quasi-convex energy density W. The list is not complete at all, since many other contributions then followed, also taking into account the non-interpenetration condition (e.g. [**36**, **45**]).

The drawback of the variational formulation is related to mechanics: due to global minimization, jump discontinuities in the functions $t \mapsto E^{el}(u(t), \Gamma(t))$ and $t \mapsto E^{d}(u(t), \Gamma(t))$ (see Remark 1.4.2) tend to happen earlier than expected. The euristic idea is the following. Consider a non-convex energy with several minimum wells, that can be distinguished in global and local minimum wells. Let (u_0, Γ_0) be a starting configuration dwelling in a global energy well at time $t = 0$. As the time increases, the well deformes; assume that at an instant $t_1 > 0$ it becomes a local minimum well for the energy. Then two possibilities appear:

- according to the global minimality Condition 1.4.1.(ii), the configuration $(u(t), \Gamma(t))$ is forced to jump in a configuration lying on a global minimum well of the energy, without taking into account the potential barriers between the wells;
- according to local minimization, the configuration $(u(t), \Gamma(t))$ stays in the deformed local energy well as long as, at an instant $t_2 > t_1$, the well might disappear. Only at instant t_2 the solution has to jump instantaneously into a new well and it shows a discontinuity, *i.e.* $(u(t_2-), \Gamma(t_2-)) \neq (u(t_2+), \Gamma(t_2+))$. Note that in this case there are no potential barriers to be overcome.

To recover local minimization in the variational approach, it has been proposed to introduce penalizing or regularizing terms in the energy, and to obtain the quasi-static evolution as limit of evolutions globally minimizing these perturbed energies. That is, for every $\varepsilon > 0$ consider an energy $\mathcal{E}_\varepsilon(u, \Gamma)$, which is a perturbation of the original energy $\mathcal{E}$, and construct globally minimizing evolutions $(u_\varepsilon(t), \Gamma_\varepsilon(t))$ for $\mathcal{E}_\varepsilon$. By using compactness arguments, the goal is to obtain an evolution $(u(t), \Gamma(t))$ as limit of $(u_\varepsilon(t), \Gamma_\varepsilon(t))$ when $\varepsilon \to 0$, with $(u(t), \Gamma(t))$ evolving along local minimum wells of the original energy $\mathcal{E}$.

This method is often referred to as *viscosity* or *vanishing viscosity* approach, the parameter ε being the so-called viscosity. It has been adopted in different settings, as discussed in [**67**]; concerning fracture mechanics, we recall for example [**37**, **58**, **82**, **22**, **77**]. This idea will be clarified in Section 1.6, where we discuss the strategy for proving existence of quasi-static evolutions; there, we will show some examples of penalization. In addition, the entire Chapter 2 is dedicated to a model of crack growth with penalizations both on the crack dissipation energy and on the bulk energy.

The local approach presents some difficulties as well. First of all, the notion of locality requires the definition of a distance, which is not really clear in this context. Furthermore, often one only recovers an energy inequality instead of the non-dissipativity equality (1.13), thus it is necessary to complete the description of the process by means of further conditions.

1.4.1. Energetic formulation. Variational rate-independent evolutions of brittle fractures can be described in the language of the *energetic formulation*. The last consists in an abstract approach to the description of rate-independent systems, providing a derivative-free form of the evolution problem which can be adapted to a wide range of models in continuum mechanics. For a complete discussion, we refer to the survey [**66**] and to the references therein.

Evolution problem. Let $\mathcal{Y}$ be the state space, $\mathcal{F} : [0, T] \times \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ the energy and $\Phi : \mathcal{Y} \to [0, +\infty]$ the dissipation potential. Given $y_0 \in \mathcal{Y}$, find $y : [0, T] \to \mathcal{Y}$ with $y(0) = y_0$ and such that for every $t \in [0, T]$ the following conditions hold:

(S) stability: for every $z \in \mathcal{Y}$

$$\mathcal{F}(t, y(t)) \leq \mathcal{F}(t, z) + \Phi(z - y(t));$$

(E) energy balance:

$$\mathcal{F}(t,y(t)) + \mathrm{Diss}_\Phi(y;0,t) = \mathcal{F}(0,y_0) + \int_0^t \partial_t \mathcal{F}(\tau, y(\tau))\,d\tau\,,$$

where $\mathrm{Diss}_\Phi(y;s,t) := \int_s^t \Phi(\dot y(\tau))\,d\tau$.

In the case of brittle fractures, the state space $\mathcal{Y}$ is given by the couples deformation-crack $y = (u, \mathit{\Gamma})$, while the dissipation potential Φ can be expressed in terms of a non-symmetric distance function: $\Phi(\mathit{\Gamma}_1 \setminus \mathit{\Gamma}_0) = \mathcal{D}(\mathit{\Gamma}_0, \mathit{\Gamma}_1)$, where

$$\mathcal{D}(\mathit{\Gamma}_0,\mathit{\Gamma}_1) := \begin{cases} \kappa\,\mathcal{H}^{N-1}(\mathit{\Gamma}_1\setminus\mathit{\Gamma}_0) & \text{if } \mathit{\Gamma}_0 \subset \mathit{\Gamma}_1 \\ +\infty & \text{otherwise.}\end{cases}$$

Note that by setting

$$\mathcal{F}(t,(u,\mathit{\Gamma})) = E^{el}(t,\mathit{\Gamma}) \qquad \mathcal{D}(\mathit{\Gamma}_0,\mathit{\Gamma}_1) = E^d(\mathit{\Gamma}_1\setminus\mathit{\Gamma}_0)$$

with $\kappa(x,\nu)\equiv\kappa$, then (S) and (E) correspond to Conditions 1.4.1.(ii) and 1.4.1.(iii), respectively; the irreversibility Condition 1.4.1.(i) is automatically satisfied thanks to the energy balance equation and to the non-symmetric distance $\mathcal{D}$.

1.5. Energy release rate

The discussion in Section 1.3 shed light on a main character of Griffith's theory: the energy release rate. In this section we recall the principal results concerning it, without any ambition of completeness, focusing in particular on those that will be applied in the following chapters. The analysis can be subdivided into two main streams: the proofs of the existence of the energy release rate for different geometries of the crack sets and for different elasticity settings, and the investigation about its continuity properties and its dependence on the fracture growth direction.

From now on we restrict to the 2-dimensional setting, both in the antiplane and the in-plane case. In Section 1.3 we formally defined the energy release rate through formula (1.7) for a pre-assigned crack path $\gamma : [0,L] \to \overline{\Omega}$ and a boundary loading process $t \mapsto \varphi_0(t)$. Anyway, for the following discussion we restrict to the static case, eventually letting the crack $\mathit{\Gamma}$ to vary freely (*i.e.* the crack path will not be given *a priori*) in a certain class of sets. Adopting a notation similar to Section 1.3, for a crack $\mathit{\Gamma} \subset \overline{\Omega}$ with $0 < \mathcal{H}^1(\mathit{\Gamma}) < +\infty$, and for a deformation or displacement $u : \Omega\setminus\mathit{\Gamma} \to \mathbb{R}^2$ in the plane case, or $u : \Omega\setminus\mathit{\Gamma} \to \mathbb{R}$ in the antiplane case, the elastic energy is

$$E^{el}(u,\mathit{\Gamma}) = \int_{\Omega\setminus\mathit{\Gamma}} W(x,u(x))\,dx\,.$$

Given a boundary loading $w : \partial_D\Omega \to \mathbb{R}^2$, or $w : \partial_D\Omega \to \mathbb{R}$, the energy of the body at equilibrium is given by

$$\mathcal{E}^{el}(w,\mathit{\Gamma}) := \min\left\{E^{el}(u,\mathit{\Gamma}) :\ u = w \text{ on } \partial_D\Omega\setminus\mathit{\Gamma}\right\}.$$

Analogously to (1.7), we formally define the energy release rate as

$$\mathcal{G}(w,\mathit{\Gamma}) := -\lim_{\substack{\widetilde{\mathit{\Gamma}}\to\mathit{\Gamma}\\ \widetilde{\mathit{\Gamma}}\supset\mathit{\Gamma}}} \frac{\mathcal{E}^{el}(w,\widetilde{\mathit{\Gamma}}) - \mathcal{E}^{el}(w,\mathit{\Gamma})}{\mathcal{H}^1(\widetilde{\mathit{\Gamma}}\setminus\mathit{\Gamma})}\,. \tag{1.14}$$

We now discuss rigorous results concerning the existence of $\mathcal{G}$ and its properties, with a precise description of the geometrical and functional settings.

1.5.1. Two-dimensional antiplane linear elasticity. The general strategy to prove that $\mathcal{G}$ is well defined (*i.e.* the limit (1.14) exists) relies in the *domain differentiation method*, as explained below.

Let Ω be a bounded open simply connected set in $\mathbb{R}^2$ with Dirichlet boundary $\partial\Omega$. Let $\partial_D\Omega$ be a relatively open subset of $\partial\Omega$ with $\mathcal{H}^1(\partial_D\Omega) > 0$, and $\partial_N\Omega := \partial\Omega \setminus \overline{\partial_D\Omega}$.

For the time being, the crack path is pre-assigned: let Γ be a compact non-degenerate simple curve with arc-length parametrization $\gamma : [0, L] \to \overline{\Omega}$, where $L = \mathcal{H}^1(\Gamma)$. Assume that

(i) $\Gamma \cap \partial\Omega = \{\gamma(0), \gamma(L)\}$;
(ii) $\Omega \setminus \Gamma$ has two connected components, Ω^1 and Ω^2, both with Lipschitz boundary;
(iii) $\partial_D\Omega \cap \partial\Omega^1 \neq \emptyset \neq \partial_D\Omega \cap \partial\Omega^2$.

For $\ell \in [0, L]$, we set

$$\Gamma(\ell) := \gamma([0, \ell]) \quad \text{and} \quad \Omega_\ell := \Omega \setminus \Gamma(\ell)\,. \tag{1.15}$$

Let $u : \Omega \to \mathbb{R}$ be the out-of-plane displacement related to a deformation $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}^3$:

$$\varphi(x, z) = (x, z + u(x)) \qquad x \in \Omega, z \in \mathbb{R}\,.$$

We assume the body Ω_ℓ to have a linear elastic response when subject to a given boundary displacement $w \in H^{1/2}(\partial_D\Omega)$, *i.e.* the elastic energy is given by

$$\mathcal{E}^{el}(w, \ell) := \inf\left\{\frac{1}{2}\int_{\Omega_\ell} |\nabla u|^2\, dx : \ u \in H^1(\Omega_\ell),\ u = w \text{ on } \partial_D\Omega\right\}, \tag{1.16}$$

where the equality $u = w$ on $\partial_D\Omega$ is in the sense of traces. By the Direct Method of the Calculus of Variations, there exists a solution u_ℓ to (1.16), and it satisfies in a weak sense the elliptic problem

$$\begin{cases} \Delta u_\ell = 0 & \text{in } \Omega_\ell \\ u_\ell = w & \text{on } \partial_D\Omega \\ \frac{\partial u_\ell}{\partial \nu} = 0 & \text{on } \partial_N\Omega \cup \Gamma(\ell)\,, \end{cases} \tag{1.17}$$

where ν is the normal vector to Γ and $\partial_N\Omega$. That is,

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v\, dx = 0 \quad \text{for all } v \in H^1(\Omega_\ell),\ v = 0 \text{ on } \partial_D\Omega\,.$$

Fixed $\bar\ell \in (0, L)$, the limit (1.14) defining $\mathcal{G}$ corresponds to

$$-\lim_{\ell \to \bar\ell} \frac{\frac{1}{2}\|\nabla u_\ell\|^2_{L^2} - \frac{1}{2}\|\nabla u_{\bar\ell}\|^2_{L^2}}{|\ell - \bar\ell|}\,. \tag{1.18}$$

Note that for $\ell \neq \bar\ell$ the functions u_ℓ and $u_{\bar\ell}$ are defined on different domains: Ω_ℓ and $\Omega_{\bar\ell}$, respectively. The *domain differentiation methods* consists in defining a flow of diffeomorphisms $\Psi_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ such that

- Ψ_ℓ coincides with the identity in a neighbourhood of $\partial\Omega$,
- Ψ_ℓ maps Ω_ℓ into $\Omega_{\bar\ell}$ for $|\ell - \bar\ell|$ small.

Then the function $u_\ell \circ \Psi_\ell^{-1} \in H^1(\Omega_{\bar\ell})$. In order to keep the uniform ellipticity of the problem (1.17), the vector fields Ψ_ℓ need to have sufficient regularity and to be chosen appropriately, depending on the geometric properties of the curve Γ.

Before investigating (1.18) for different regularity assumptions about Γ, we make a couple of remarks.

- Without loss of generality, we assume w to correspond to the trace of a H^1 function defined in Ω. Hence in the following we consider the boundary datum $w \in H^1(\Omega)$, and the equality $u = w$ on $\partial_D\Omega$ is understood in the sense of traces.

- By the regularity theory for elliptic problems, the solution u_ℓ to (1.17) is more regular in the interior of the uncracked region. More precisely, given any two open sets U, V such that $\gamma(\ell) \in U \subset V \subset\subset \Omega$, the solution u_ℓ belongs to $H^2\big(V \setminus (U \cup \Gamma(\ell))\big)$.
  Moreover, the singularity of u_ℓ in a neighbourhood of $\gamma(\ell)$ can be described fairly well, as explained below (see formula (1.19) and (1.21)).

Of course the easiest situation is represented by a straight crack, *i.e.* up to a change of variables we assume $\Gamma = [0, L] \times \{0\}$, still satisfying (i), (ii), (iii) described at the beginning of this subsection. In this case, fixed $\bar\ell \in (0, L)$, the family $(\Psi_\ell)_\ell$ of diffeomorphisms between Ω_ℓ and $\Omega_{\bar\ell}$ can be chosen as

$$\Psi_\ell(x_1, x_2) = (x_1 - (\ell - \bar\ell)\, \psi(x_1, x_2), x_2) \qquad (x_1, x_2) \in \mathbb{R}^2$$

with $\psi : \Omega \to \mathbb{R}$ Lipschitz continuous, $\operatorname{supp}\psi \subset \Omega$ and $\psi \equiv 1$ in a neighbourhood of $\gamma(\bar\ell)$. Notice that Ψ_ℓ is a perturbation of the identity map by a vector field $V(x) = (\psi(x), 0)$ with compact support in Ω and tangent to Γ in a neighbourhood of $\gamma(\bar\ell)$: $\Psi_\ell = id + (\ell - \bar\ell)V$.

The key result for proving the existence of (1.18) is based on the analysis carried out by Destuynder & Djaoua [41] and Grisvard [51, 52] about the singularities of solutions to elliptic problems in polygonal domains. With the current hypotheses on Γ, the solution $u_{\bar\ell}$ can be written in the form

$$u_{\bar\ell} = \mathcal{K}\, r^{1/2} \sin\frac{\theta}{2} + u_R \tag{1.19}$$

where $u_R \in H^2(\omega \setminus \Gamma(\bar\ell))$ for every $\omega \subset\subset \Omega$, $\mathcal{K} > 0$, and (r, θ) are polar coordinate around $\gamma(\bar\ell) = (\bar\ell, 0)$. More precisely, given $x \in \Omega \setminus \gamma([0, \bar\ell]) = \Omega \setminus \big([0, \bar\ell] \times \{0\}\big)$ it is $r = |x - \gamma(\bar\ell)|$, and θ is the angle between $x - \gamma(\bar\ell)$ and the x_1-axis. Hence the solution $u_{\bar\ell}$ splits in a regular part $u_R \in H^2(\omega \setminus \Gamma(\bar\ell))$ and a singular part $\mathcal{K}\, r^{1/2} \sin\frac{\theta}{2} \in H^1(\omega \setminus \Gamma(\bar\ell)) \setminus H^2(\omega \setminus \Gamma(\bar\ell))$. The coefficient $\mathcal{K}$ is called *stress intensity factor*; it is a constant dependent on the material, which describes the relation between stress and strain (in case of linear elastic deformations, stress and strain are proportional to each other).

By means of Grisvard theory [52], in case of a straight crack the energy release rate is proved to exist. Moreover it can be expressed in terms of the stress intensity factor $\mathcal{K}$, through the Irwin formula

$$\mathcal{G}(w, \bar\ell) = \frac{\pi}{4}\mathcal{K}^2\,, \tag{1.20}$$

as observed by Irwin [56] and in [52, Theorem 6.4.1].

The approach for straight cracks can be adopted in a straightforward way for any preassigned crack path Γ of class C^2. Indeed, if $\Gamma = \gamma([0, L])$ with arc-length parametrization $\gamma \in C^2([0, L]; \overline{\Omega})$, fixed $\bar\ell \in (0, L)$ it is possible to write a flow of vector fields $\Psi_\ell : \Omega \to \Omega$ such that

- Ψ_ℓ is the identity close to the boundary $\partial\Omega$,
- $\Psi_\ell(\Omega_\ell) = \Omega_{\bar\ell}$ for $|\ell - \bar\ell|$ small,
- Ψ_ℓ is tangent to Γ in a neighbourhood of $\gamma(\bar\ell)$: $\Psi_\ell(\gamma(\ell)) = \dot\gamma(\ell)$ for $|\ell - \bar\ell|$ small.

An explicit form for Ψ_ℓ can be found in [60, 58]. We stress that, in order to prove the existence of the energy release rate by directly applying the domain differentiation method, the existence of the *second* derivative of γ is essential.

If the crack set is less regular, the best result up to date has been proved by Lazzaroni & Toader [62]. They give proof of the existence of the energy release rate for $C^{1,1}$ cracks. The analysis is much more subtle then in the regular case (cracks of class C^2) and the application of the domain differentiation strategy requires a careful construction of the flow of vector fields that map the varying domains into a fixed one.

The main idea in their proof is to perform a nonlinear change of variables such that, in the modified geometry, the crack is straight in a neighbourhood of the tip and problem (1.17) at $\ell = \bar{\ell}$ transforms in a new one which is still uniformly elliptic. More precisely, assumed the notation in (1.15) with γ of class $C^{1,1}$, exploiting the $C^{1,1}$ regularity of the preassigned crack path Γ and the elliptic nature of the problem (1.17) Lazzaroni & Toader construct a $C^{1,1}$ diffeomorphism $\Psi : \Omega \to \Omega$ (with further properties) such that

- the set $\widetilde{\Gamma}(\bar{\ell}) = \Psi^{-1}(\Gamma(\bar{\ell}))$ is straight in a neighbourhood of the tip $\Psi^{-1}(\gamma(\bar{\ell}))$,
- set $g = w \circ \Psi$ and $v_{\bar{\ell}} = u_{\bar{\ell}} \circ \Psi$, the problem (1.17) becomes

$$\begin{cases} \mathcal{B} v_{\bar{\ell}} = 0 & \text{in } \Omega \setminus \widetilde{\Gamma}(\bar{\ell}) \\ v_{\bar{\ell}} = g & \text{on } \partial_D \Omega \\ \frac{\partial v_{\bar{\ell}}}{\partial \nu} = 0 & \text{on } \partial_N \Omega \cup \widetilde{\Gamma}(\bar{\ell})\,, \end{cases}$$

  with $\mathcal{B}$ still a uniformly elliptic operator.

Then they can apply the results by Grisvard [52] for rectilinear fractures and elliptic operators (before we only described the case of the Laplacian operator, but the analysis for C^2 cracks holds true for more general elliptic operators), obtaining a result similar to (1.19) for $v_{\bar{\ell}}$; in terms of the original problem (1.17), the singularity of the solution $u_{\bar{\ell}}$ is described by

$$u_{\bar{\ell}} - \mathcal{K}\, r^{1/2} \sin\frac{\theta}{2} \in H^2(\omega \setminus \Gamma(\bar{\ell})) \tag{1.21}$$

for every $\omega \subset\subset \Omega$. Here, for $x \in \Omega \setminus \Gamma(\bar{\ell})$, it is $r = |x - \gamma(\bar{\ell})|$, while θ is the angle between the vector $x - \gamma(\bar{\ell})$ and the tangent vector $\dot{\gamma}(\bar{\ell})$. Finally, Lazzaroni & Toader [62] prove that a relation like (1.20) holds even in this situation; the computation needs some care, due to the less regularity assumption.

Remark 1.5.1. We have to highlight some facts contained in [62], that will also be useful in the following chapters.

(i) The results in [62] are valid for uniformly elliptic operators, and not just for the Laplacian problem (1.17).

(ii) In the previous discussion we supposed the crack path Γ to be given *a priori* and we computed the energy release rate $\mathcal{G}(w, \bar{\ell})$ for the set $\Gamma(\bar{\ell})$ strictly contained in Γ. As observed in [62, Remark 2.3], the value $\mathcal{G}(w, \bar{\ell})$ depends only on $\Gamma(\bar{\ell})$ and not on the $C^{1,1}$ extension of $\Gamma(\bar{\ell})$ used to compute it. This will appear evident by the integral formula reported below.

Hence we are no longer restricted to assume the crack path to be known in advance, as long as the fracture set belongs to a suitable class of admissible cracks. Indeed, given any curve Γ_0 of class $C^{1,1}$, in order to compute the energy release rate $\mathcal{G}(w, \Gamma_0)$ at its tip we are free to choose *any* $C^{1,1}$ extension Γ of Γ_0.

(iii) The energy release rate $\mathcal{G}(w, \bar{\ell})$ for the energy (1.16) can be expressed as a volume integral dependent on the displacement gradient $\nabla u_{\bar{\ell}}$ of the minimizer $u_{\bar{\ell}}$. By [62, Proposition 2.4 and Remark 2.5], we have that

$$\begin{aligned} \mathcal{G}(w, \bar{\ell}) &= \int_{\Omega_{\bar{\ell}}} \Big[\frac{(D_1 u_{\bar{\ell}})^2 - (D_2 u_{\bar{\ell}})^2}{2} (D_1 V^1 - D_2 V^2) + D_1 u_{\bar{\ell}} D_2 u_{\bar{\ell}} (D_2 V^1 + D_1 V^2) \Big]\, dx \\ &= -\frac{1}{2} \int_{\Omega_{\bar{\ell}}} |\nabla u_{\bar{\ell}}|^2 \operatorname{div} V \, dx + \int_{\Omega_{\bar{\ell}}} \nabla V \, \nabla u_{\bar{\ell}} \cdot \nabla u_{\bar{\ell}} \, dx\,, \end{aligned}$$

where $V = (V^1, V^2) : \Omega \to \mathbb{R}^2$ is any vector field of class $C^{0,1}$ with compact support in Ω, such that $V(\gamma(\ell)) = \zeta(\gamma(\ell))\,\dot\gamma(\ell)$ for some cut-off function ζ equal to one in a neighbourhood of $\gamma(\bar\ell)$.

In our short review of existing results about the energy release rate, we keep weakening the regularity assumptions on the crack Γ by dropping the C^1 regularity of the crack. At this stage we do not discuss the physical, mechanical or modeling issues related to kinking of cracks and that appear in the settings considered in few lines, and we postpone any comment to Chapter 4, where a model for crack growth allowing for branching and kinking to occur is studied. In Subsection 1.5.2 we briefly comment on the known results for the plane shear elasticity with this same geometry of cracks.

In [29], Chambolle & Lemenant study the case of crack sets Γ which are merely closed and connected, and asymptotic to a half-line at small scales. This property is described by a blow-up condition, requesting that the density of Γ at a point x_0 is $\frac{1}{2}$:

$$\limsup_{r\to 0} \frac{\mathcal{H}^1(\Gamma \cap B_r(x_0))}{2r} = \frac{1}{2}\,.$$

Under the above assumption, Chambolle & Lemenant prove that, up to suitable rescalings and rotations, the minimizer of the elastic energy $E^{el}(w,\Gamma)$ still has an asymptotic expansion of the form (1.21) around x_0. However, it has not yet been proved that an equality of the form (1.20) holds true in such a general setting.

Keeping away from regularity, Negri [71] proves the existence of the energy release rate in the case of kinking. The crack set Γ is a piecewise $C^{1,1}$ curve; more precisely, we can assume to parametrize it by means of a function $f : [-1,1] \to \mathbb{R}$ such that $f|_{[-1,0]}, f|_{[0,1]} \in C^{1,1}$ and $f'_-(0) \neq f'_+(0)$, where f'_- and f'_+ are the left and right derivative of f at 0, respectively. Set $\Gamma(s) = \{(s', f(s')) : \, -1 \le s' \le s\}$, Negri [71] studies the limit

$$\mathcal{G}(w, \Gamma(0)) = \lim_{h\to 0+} \frac{\frac{1}{2}\|\nabla u_0\|^2_{L^2} - \frac{1}{2}\|\nabla u_h\|^2_{L^2}}{\mathcal{H}^1(\Gamma(h)\setminus\Gamma(0))}\,,$$

where for $0 \le h \le 1$ the function $u_h \in H^1(\Omega \setminus \Gamma(h))$ minimizes

$$\frac{1}{2}\int_{\Omega\setminus\Gamma(h)} |\nabla u|^2\,dx$$

among all $u \in H^1(\Omega\setminus\Gamma(h))$ with $u = w$ on $\partial_D\Omega$. We remark that, as already proved in [62, 52] and said previously, the minimizer u_0 splits in regular and singular part like (1.21).

By means of a Γ-convergence approach, in [71, Theorem 8.4 and Theorem 9.1] the energy release rate $\mathcal{G}(w,\Gamma(0))$ is shown to exist, to have an integral representation, and to depend on u_0 only through the stress intensity factor $\mathcal{K}$ of the singular part in (1.21). Furthermore, as one might expect, $\mathcal{G}(w,\Gamma(0))$ also depends on the kinking angle, *i.e.* on the values $f'_-(0)$ and $f'_+(0)$; unfortunately from the point of view of applications of this interesting result, the last dependence is not explicit, hence difficult to handle.

1.5.2. Other elasticity settings. For completeness, we briefly recall a couple of results concerning the cases of nonlinear elastic energies and of plane-shear linearized elasticity.

As it emerges from the previous subsection, the understanding of the energy release rate in the linear antiplane elastic models has already produced interesting results. The literature about the nonlinear case is less flourishing, due to the intricacies of this setting, mainly caused by the absence of unique minimizers of the stored elastic energy. Without a detailed description of the model, here we report the main results and remarks pointed out in [57] in the case of polyconvex energy densities W.

Knees & Mielke [57] consider a two-dimensional bounded open set $\Omega \subset \mathbb{R}^2$, with an *a priori* given straight crack path of the form $[a,b] \times \{0\} \subset \overline{\Omega}$, with $a < 0 < b$ and $(a,b] \times \{0\} \subset \Omega$. Let $w : \Omega \to \mathbb{R}^2$ be a fixed boundary deformation. For $\ell \in (a,b)$, set $\Omega_\ell := \Omega \setminus ([a,l] \times \{0\})$ and consider the energy

$$E^{el}(\ell, u) = \int_{\Omega_\ell} W(\nabla u)\, dx\,, \quad u : \Omega_\ell \to \mathbb{R}^2\,,$$

where the energy density $W : M^{2\times 2} \to [0,+\infty]$ is such that:

- W is *polyconvex*: there exists a convex continuous function $g : \mathbb{R}^5 \to [0,+\infty]$ such that $W(A) = g(A, \det A)$ for every matrix $A \in M^{2\times 2}$;
- W verifies some regularity and p-growth assumptions in order to guarantee the existence of solutions to the minimum problem

$$\mathcal{E}^{el}_{nl}(\ell) := \min\left\{E^{el}(\ell,u) :\ u \in W^{1,p}(\Omega_\ell),\ u = w \text{ on } \partial_D\Omega\right\}.$$

Here nl stands for "nonlinear". Furthermore, since w is fixed, we do not explicit the dependence of the functional on it.

Knees & Mielke investigate the existence of the limit

$$\mathcal{G}_{nl}(0) := -\lim_{\ell \to 0+} \frac{\mathcal{E}^{el}_{nl}(\ell) - \mathcal{E}^{el}_{nl}(0)}{\ell}\,. \tag{1.22}$$

Their main result is the following theorem.

Theorem 1.5.2. [57, Theorem 3.3] *Under proper regularity and growth conditions for W, if $\mathcal{E}^{el}_{nl}(0) < \infty$, then the limit (1.22) exists, is finite, and is given by the formula*

$$\mathcal{G}_{nl}(0) = \max\left\{G_{nl}(\bar u, 0) :\ \bar u \textit{ is a minimizer of } \mathcal{E}^{el}_{nl}(0)\right\},$$

where

$$G_{nl}(\bar u, 0) = -\lim_{\ell \to 0+} \frac{E^{el}(\ell, \bar u) - E^{el}(0, \bar u)}{\ell}\,. \tag{1.23}$$

Some comments are due:

- $\mathcal{G}_{nl}(0)$ and $G_{nl}(\bar u, 0)$ are a "global" and a "local" version of the energy release rate, respectively. As noticed in [57], it is not known if $G(\bar u_1, 0) = G(\bar u_2, 0)$ for different minimizers $\bar u_1$ and $\bar u_2$ of $\mathcal{E}^{el}_{nl}(0)$.
- The proof of the existence of the limits (1.22) and (1.23) is achieved by a domain differentiation method, as in the antiplane case previously described.

For further interesting remarks, we refer the reader to the original paper [57] and also to [59].

Motivated by a debate on kinking criteria, Chambolle, Francfort & Marigo revisited the results on the energy release rate in plane shear elasticity in a couple of papers [26, 27]. Their interest in concentrated on non-smooth (because of kinking) elongations of a straight crack. Without introducing any mathematical object, that would require a quite long description, we underline that in the non-smooth setting considered in [27] the authors introduce a *generalized* notion of energy release rate, which is proved to depend on the growth direction of the crack, *i.e.* on the kinking angle. The existence of this generalized energy release rate is obtained rigorously and is valid for both isotropic and anisotropic models of linearized elasticity.

Furthermore, the proof strategy is very different from the previous one: it does not rely on a domain differentiation method, due to the absence of regularity of the cracks; instead, by means of a blow up argument the authors transfer the problem in the initial bounded domain into a new one defined on an infinite domain.

With this discussion, we conclude the brief drift away from the antiplane shear model.

1.5.3. Continuity of the ERR in the antiplane case. In the framework of antiplane linear elasticity, it is possible to prove the continuity of the energy release rate with respect to the convergence of sets and of boundary loadings; the result is true up to requiring sufficient regularity for the cracks, which, in order to assure the existence of the energy release rate, have to be at least $C^{1,1}$ curves, as seen in Subsection 1.5.1. The continuity property is useful when dealing with approximation procedures in problems of fracture mechanics, and we will use it in Chapters 2 and 4. The next results are mainly proved in [62].

We consider the setting introduced in Subsection 1.5.1, with more general assumptions on the cracks, that, mainly thanks to Remark 1.5.1.(ii), need not be prescribed *a priori*. Let Ω be a bounded open simply connected set in $\mathbb{R}^2$ with Dirichlet boundary $\partial\Omega$. Let $\partial_D\Omega$ be a relatively open subset of $\partial\Omega$ with $\mathcal{H}^1(\partial_D\Omega) > 0$, and $\partial_N\Omega := \partial\Omega \setminus \overline{\partial_D\Omega}$. For any boundary loading $w \in H^1(\Omega)$ and any $C^{1,1}$ compact curve $\Gamma \subset \overline{\Omega}$, in analogy to (1.16) let

$$\mathcal{E}^{el}(w,\Gamma) := \min\left\{\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2\,dx :\ u \in H^1(\Omega\setminus\Gamma),\ u = w \text{ on } \partial_D\Omega\right\}.$$

The above minimum problem is equivalent to the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\setminus\Gamma \\ u = w & \text{on } \partial_D\Omega\setminus\Gamma \\ \frac{\partial u}{\partial\nu} = 0 & \text{on } \partial_N\Omega\cup\Gamma\,, \end{cases} \tag{1.24}$$

which has to be understood in the following variational way:

$$\begin{cases} u \in H^1(\Omega\setminus\Gamma),\ u = w \text{ on } \partial_D\Omega\setminus\Gamma \\ \displaystyle\int_{\Omega\setminus\Gamma} \nabla u\cdot\nabla v\,dx = 0 \quad \text{for all } v \in H^1(\Omega\setminus\Gamma),\ v = 0 \text{ on } \partial_D\Omega\setminus\Gamma\,. \end{cases}$$

Existence and uniqueness of a solution u are assured by the Lax-Milgram lemma in those components of $\Omega\setminus\Gamma$ whose boundaries intersect $\partial_D\Omega\setminus\Gamma$. Note that the solution is not unique if there exists a connected component whose boundary does not intersect $\partial_D\Omega\setminus\Gamma$; however ∇u is always unique.

In [62] the following class of sets is introduced.

Definition 1.5.3. Let $\Gamma_0 \subset \overline{\Omega}$ be a compact nondegenerate curve of class $C^{1,1}$. For $\eta > 0$, $\mathcal{R}_\eta$ is the class of compact curves Γ of class $C^{1,1}$ contained in $\overline{\Omega}$ such that

(i) $\Gamma_0 \subset \Gamma$ and $\Gamma\setminus\Gamma_0 \subset\subset \Omega$;

(ii) for every $x \in \Gamma$ there exists two open balls $B_1, B_2 \subset \Omega$ of radius η such that

$$(B_1\cup B_2)\cap(\Gamma\cup\partial\Omega) = \emptyset \quad\text{and}\quad \overline{B}_1\cap\overline{B}_2 = \{x\}\,.$$

The class $\mathcal{R}_\eta$ turns out to be sequentially compact with respect to convergence of sets in the Hausdorff metric, whose definition is recalled in Section 1.7. Indeed, one can apply compactness arguments using the fact that Condition 1.5.3.(ii) provides a uniform $W^{2,\infty}$ bound for the arc-length parametrization γ of any curve $\Gamma \in \mathcal{R}_\eta$ (see [62, Proposition 2.9]).

The continuity of the energy release rate is proved in [62, Theorem 2.12]; here we rewrite the theorem with a slight generalization.

Theorem 1.5.4. *Let Γ_n be a sequence in $\mathcal{R}_\eta$ converging to $\Gamma \in \mathcal{R}_\eta$ in the Hausdorff metric and let w_n be a sequence in $H^1(\Omega)$ converging to w in $H^1(\Omega)$. Then $\mathcal{G}(w_n,\Gamma_n) \to \mathcal{G}(w,\Gamma)$.*

For the proof we refer to [62] and to Lemma 4.4.7 in Chapter 4, where it is reported with minor changes, adapted to the setting therein. The main ingredients are

- the $W^{2,\infty}$ regularity of the sets $\Gamma_n, \Gamma \in \mathcal{R}_\eta$;
- the integral formula in Remark 1.5.1.(iii) for the energy release rate (with $\Omega \setminus \Gamma_n$ and $\Omega \setminus \Gamma$ instead of $\Omega_{\bar{\ell}}$) in terms of the solutions u_n and u to the elliptic boundary value problems (1.24) with data Γ_n, w_n and Γ, w, respectively;
- a convergence result proven in [38, Theorem 5.1] (and reported below as Theorem 1.7.6) stating that, under the hypotheses in Theorem 1.5.4, the sequence ∇u_n converges to ∇u strongly in $L^2(\Omega;\mathbb{R}^2)$, where u_n and u are as above.

With this we conclude our brief review on the energy release rate, containing the results of major interest for this thesis.

1.6. Existence of quasi-static evolutions by time-discretization

We describe the general strategy adopted to prove the existence of a quasi-static evolution for a given boundary loading $t \mapsto w(t)$ and a given initial datum (u_0, Γ_0). The existence is usually achieved by a *time discretization* approach, using a general scheme introduced by De Giorgi under the name of *minimizing movements method* (see [5]).

Let the total energy of the system be

$$\mathcal{E}(u,\Gamma) = E^{el}(u,\Gamma) + E^d(\Gamma)\,,$$

and assume that both E^{el} and E^d are sequentially lower semicontinuous: if $u_n \to u$ and $\Gamma_n \to \Gamma$, then

$$E^{el}(u,\Gamma) \le \liminf_{n\to+\infty} E^{el}(u_n,\Gamma_n) \quad \text{and} \quad E^d(\Gamma) \le \liminf_{n\to+\infty} E^d(\Gamma_n)\,. \tag{1.25}$$

Remark 1.6.1. The notions of convergence $u_n \to u$ and $\Gamma_n \to \Gamma$ depend on the setting of the model. The choice is done in order to guarantee the sequential lower semicontinuity of the functionals E^{el} and E^d, and the sequential compactness of the minimizing sequences. In particular for $\Gamma_n \to \Gamma$, one has to prove a result that generalizes the Helly's Theorem 1.7.9 for monotone functions.

In Chapters 2 and 4, the cracks Γ are compact sets with $\mathcal{H}^1(\Gamma) < \infty$ and finitely many connected components, and $E^d(\Gamma) = \kappa\mathcal{H}^1(\Gamma)$. By $\Gamma_n \to \Gamma$ we mean that the sequence $(\Gamma_n)_n$ converges to Γ in the Hausdorff metric (whose definition is recalled in Subsection 1.7.1): indeed, by Gołąb's Theorem 1.7.2 the $\mathcal{H}^1$-measure is lower semicontinuous with respect to it, if the sets Γ_n have an *a priori* bounded number of connected components; moreover, Blaschke Theorem 1.7.1 assures the sequential compactness of the minimizing sequences. Concerning the displacements, since we consider the bulk energy

$$E^{el}(u,\Gamma) := \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2\,dx \qquad u \in H^1(\Omega\setminus\Gamma)\,,$$

by $u_n \to u$ we mean that $u_n \to u$ in $L^2(\Omega)$ and $\nabla u_n \to \nabla u$ in $L^2(\Omega;\mathbb{R}^2)$, so that E^{el} is lower semicontinuous; the compactness properties for the displacements are consequence of *a priori* extimates on the sequence $(u_n)_n$. Notice that $u_n \in H^1(\Omega\setminus\Gamma_n)$ while $u \in H^1(\Omega\setminus\Gamma)$; however, since $\mathcal{L}^2(\Gamma_n) = \mathcal{L}^2(\Gamma) = 0$, we can see u_n, u and $\nabla u_n, \nabla u$ as functions in $L^2(\Omega)$ and $L^2(\Omega;\mathbb{R}^2)$, respectively.

In more general settings, different functional spaces and notions of convergence have been introduced. For example, if the displacements belong to a set of SBV or $GSBV$ type, the convergence $u_n \to u$ is given by

$$\begin{aligned}
&u_n(x) \to u(x) \quad \text{for a.e. } x \in \Omega \\
&\nabla u_n \to \nabla u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^{N\times N}) \\
&\limsup_n \mathcal{H}^{N-1}(J(u_n)) < +\infty\,,
\end{aligned}$$

with proper assumptions on the energy density W in (1.12) for E^{el} to be lower semicontinuous. Concerning the cracks, a notion of convergence related to the jump set of SBV functions, called σ^p-convergence, has been introduced. See [**34**, **49**, **47**].

The strategy for proving the existence of quasi-static evolutions contains few main steps. We first describe the case of globally minimizing variational evolutions (see Definition 1.4.1); then we consider a variant in the procedure, in order to obtain a crack growth along trajectories of local minimizers of the total energy, as we discussed in Section 1.4.

Step 1: incremental problem. For any $n \in \mathbb{N}$ let (t_n^i) be an increasing sequence such that

$$0 = t_n^0 < t_n^1 < \ldots < t_n^i < \ldots < t_n^n = T < +\infty \quad \text{or} \quad 0 = t_n^0 < t_n^1 < \ldots < t_n^i \xrightarrow{i\to+\infty} +\infty$$

and

$$\lim_{n\to+\infty} \sup_i \,(t_n^i - t_n^{i-1}) \to 0\,.$$

We define

- $u_n^0 := u_0\,,\ \Gamma_n^0 := \Gamma_0$
- recursively, for $i \geq 1$ let (u_n^i, Γ_n^i) be a minimizer of the incremental minimum problem
$$\min\left\{\mathcal{E}(u,\Gamma) : (u,\Gamma) \in \mathcal{AC}(w(t_n^i)), \Gamma \supset \Gamma_n^{i-1}\right\}\,, \tag{1.26}$$
where $\mathcal{AC}(w)$ is defined in (1.10).

Suppose that, under proper growth and regularity assumptions on the functionals, the minimum problem (1.26) admits a solution for every i. Note that if the set $\mathcal{AC}(w)$ is sequentially compact with respect to the chosen notions of convergence, then the lower semicontinuity of E^{el} and E^d assures the existence of a minimizer of (1.26).

We define the piecewise-constant interpolations in the interval $[0,T]$ (or $[0,+\infty)$) as

$$u_n(t) := u_n^i, \quad \Gamma_n(t) := \Gamma_n^i \qquad \text{for } t_n^i \leq t < t_n^{i+1}\,.$$

Step 2: compactness argument. Using the above iterative scheme, we obtain some estimates on the functions $u_n(t)$ and $\Gamma_n(t)$, which allow to apply compactness arguments. Thus we extract a subsequence of $(u_n(\cdot), \Gamma_n(\cdot))$, independent of t, not relabelled, that converges at any instant t:

$$u_n(t) \to u(t) \quad \text{and} \quad \Gamma_n(t) \to \Gamma(t)\,.$$

It remains to prove that the limit evolution $t \mapsto (u(t), \Gamma(t))$ is a variational evolution according to Definition 1.4.1.

Step 3: irreversibility, minimality and non-dissipativity. The irreversibility Condition 1.4.1.(i) is a consequence of the fact that, by construction, $\Gamma_n(t_1) \subset \Gamma_n(t_2)$ for $0 \leq t_1 \leq t_2$, and of the convergence $\Gamma_n(t) \to \Gamma(t)$, which preserves the inclusions.

In order to check the one-sided minimality Condition 1.4.1.(ii), it is useful to have at your disposal a result like the so-called *jump transfer theorem*, proved by Francfort & Larsen [**47**], then adapted to other settings. The idea is described in the following "qualitative" lemma.

Lemma 1.6.2. *Let $w_n \to w$. Let $(u,\Gamma), (\hat{u}, \widehat{\Gamma}) \in \mathcal{AC}(w)$, with $\widehat{\Gamma} \supset \Gamma$. Let $\Gamma_n \to \Gamma$. Then there exists a sequence $(\hat{u}_n, \widehat{\Gamma}_n) \in \mathcal{AC}(w_n)$ such that $\widehat{\Gamma}_n \supset \Gamma_n$ for every n, $E^d(\widehat{\Gamma}_n) \to E^d(\widehat{\Gamma})$, and $\limsup_n E^{el}(\hat{u}_n, \widehat{\Gamma}_n) \leq E^{el}(\hat{u}, \widehat{\Gamma})$.*

Combining this lemma and the lower semicontinuity (1.25) of the functionals, we recover the minimality condition in Definition 1.4.1 by the minimality of the discrete-time approximating evolutions. Indeed, fix $t \in [0,T]$ and $(v,\Gamma) \in \mathcal{AC}(w(t))$. For every n let $i := i(n,t)$ be such that $t_n^i \leq t < t_n^{i+1}$; by Lemma 1.6.2 there exists a sequence $(v_n,\Gamma_n) \in \mathcal{AC}(w(t_n^i))$ such that $\Gamma_n \supset \Gamma_n(t)$ for every n, $E^d(\Gamma_n) \to E^d(\Gamma)$ and $\limsup_n E^{el}(v_n,\Gamma_n) \leq E^{el}(v,\Gamma)$. Then

$$\begin{aligned}\mathcal{E}(u(t),\Gamma(t)) =& E^{el}(u(t),\Gamma(t)) + E^d(\Gamma(t)) \leq \liminf_{n\to+\infty} E^{el}(u_n(t),\Gamma_n(t)) + E^d(\Gamma_n(t)) \\ &\leq \liminf_{n\to+\infty} E^{el}(v_n,\Gamma_n) + E^d(\Gamma_n) \leq E^{el}(v,\Gamma) + E^d(\Gamma) = \mathcal{E}(v,\Gamma)\,,\end{aligned}$$

where the first inequality is due to (1.25) and the second one to the minimality of $(u_n(t),\Gamma_n(t))$ in (1.26).

Finally, the non-dissipativity Condition 1.4.1.(iii) is consequence of the one-sided minimality and of energy inequalities proved for the piecewise-constant interpolations.

As pointed out in Section 1.4, it would be more desirable to obtain quasi-static variational evolutions of *local* minimizer. A strategy is to penalize, at the level of the discrete-time approximations, the distance between approximating solutions at two consecutive times t_n^i and t_n^{i+1}. Instead of (1.26), in the current literature the iterative problem has been replaced, for example, by the following ones: in [**37**, **22**]

$$\min\left\{\mathcal{E}(u,\Gamma) + \frac{\varepsilon}{\tau}\|u - u_n^{i-1}\|^2_{L^2} : (u,\Gamma) \in \mathcal{AC}(w(t_n^i)), \Gamma \supset \Gamma_n^{i-1}\right\}\,, \tag{1.27}$$

while in [**58**]

$$\min\left\{\mathcal{E}(u,\Gamma) + \frac{\varepsilon}{\tau}E^d(\Gamma\setminus\Gamma_n^{i-1})^2 : (u,\Gamma) \in \mathcal{AC}(w(t_n^i)), \Gamma \supset \Gamma_n^{i-1}\right\}\,. \tag{1.28}$$

The positive parameter ε is called *viscosity* or *friction* constant. The idea is that, at fixed $\varepsilon > 0$, for τ small the ratio ε/τ is large, thus local minimizers (close to u_n^{i-1} and Γ_n^{i-1}) are preferred to global ones.

Applying the time discretization approach at fixed ε, one obtains a variational evolution $(u_\varepsilon(t),\Gamma_\varepsilon(t))$ with an extra-term $\mathcal{P}_\varepsilon$ in the energy balance, accounting for the penalization: for $0 \leq s < t \leq T$

$$\begin{aligned}\mathcal{E}(u_\varepsilon(t),\Gamma_\varepsilon(t)) = \;& \mathcal{E}(u_\varepsilon(s),\Gamma_\varepsilon(s)) + \int_s^t \Big(\int_{\Omega\setminus\Gamma_\varepsilon(\tau)} D_\xi W(x,\nabla u_\varepsilon(\tau)) : \nabla\dot{g}(\tau)\,dx\Big)\,d\tau \\ &+ \mathcal{P}_\varepsilon(u_\varepsilon,\Gamma_\varepsilon;s,t)\,,\end{aligned} \tag{1.29}$$

where $\mathcal{P}_\varepsilon(u_\varepsilon,\Gamma_\varepsilon;s,t) \to 0$ as $\varepsilon \to 0$.

Exploiting again compactness arguments, this time for the evolutions $(u_\varepsilon(\cdot),\Gamma_\varepsilon(\cdot))$, the variational evolution $(u(\cdot),\Gamma(\cdot))$ is obtained as limit when ϵ vanishes. In some particular settings [**58**, **82**, **77**], this evolution has been shown to perform a jump in the configuration space *strictly later* than the evolution of global minimizers previously constructed, as discussed in Section 1.4. Hence the penalization does really play a role in keeping the configuration in a *local* energy well as long as possible.

We remark that the evolutions $(u_\varepsilon(\cdot),\Gamma_\varepsilon(\cdot))$ might be rate-dependent, even if the limit evolution $(u(\cdot),\Gamma(\cdot))$ is not. In this case, usually the rate-dependence is present in the extra-term $\mathcal{P}_\varepsilon$ in the energy balance (1.29).

In Chapter 2 we discuss a model with a penalization both for the displacements and the cracks.

1.7. Technical results

1.7.1. Hausdorff measures and Hausdorff convergence. Both notions can be defined in any metric space X, and the results reported below are valid under further assumptions on X; as a reference, see for example [8]. Since in the following chapters we consider the 2-dimensional real space, we assume here $x = \mathbb{R}^N$ for $N \geq 2$.

For every $d > 0$, the *d-dimensional Hausdorff (outer) measure* $\mathcal{H}^d$ of a set $A \subset \mathbb{R}^N$ is defined as

$$\mathcal{H}^d(A) = m(d) \sup_{\delta>0} \inf \Big\{ \sum_{i\in I} (\operatorname{diam} A_i)^d : A_i \text{ are measurable sets, } A \subset \cup_i A_i, \operatorname{diam} A_i \leq \delta \Big\},$$

where $m(d) = 2^{-d}\Gamma(\frac{1}{2})^d/\Gamma(\frac{d}{2}+1)$, with Γ denoting here the Euler function. For $d \in \mathbb{N}$ the value $m(d)$ corresponds to the volume of the d-dimensional unit sphere.

Let Ω be a bounded open subset of $\mathbb{R}^N$. Given any two compact subsets $K_1, K_2 \subset \overline{\Omega}$, the *Hausdorff distance* between them is defined as

$$\operatorname{dist}_H(K_1, K_2) := \max \left\{ \sup_{x\in K_1} \operatorname{dist}(x, K_2), \sup_{y\in K_2} \operatorname{dist}(y, K_1) \right\},$$

with the convention that $\operatorname{dist}(x, \emptyset) = \operatorname{diam}\Omega$ and $\sup \emptyset = 0$.

Given a sequence K_n of compact sets, we say that it converges to a set K in the Hausdorff metric if $\operatorname{dist}_H(K_n, K) \to 0$ as $n \to +\infty$.

The following compactness result holds:

Theorem 1.7.1 (Blaschke Theorem)**.** *Let K_n be a sequence of compact sets contained in $\overline{\Omega}$. Then there exists a subsequence K_{n_k} and a compact set $K \subset \overline{\Omega}$ such that K_{n_k} converges to K in the Hausdorff metric, as $k \to +\infty$.*

Concerning the lower semicontinuity of the measure $\mathcal{H}^d$ with respect to the Hausdorff convergence of compact connected sets, for $d > 1$ it is usually not true, unless further strong hypotheses are considered; this will be the case in the model of Chapter 3. The situation is different for $d = 1$ in case of connected sets:

Theorem 1.7.2 (Gołąb Theorem)**.** *Let K_n be a sequence of compact connected sets converging to K in the Hausdorff metric. Then K is connected and*

$$\mathcal{H}^1(K) \leq \liminf_{n\to+\infty} \mathcal{H}^1(K_n).$$

1.7.2. Deny-Lions spaces. They were introduced in [40]. For any open set $A \subset \mathbb{R}^N$ the Deny-Lions space $L^{1,2}(A)$ is defined as

$$L^{1,2}(A) := \{u \in L^2_{loc}(A) : \nabla u \in L^2(A; \mathbb{R}^N)\},$$

where ∇u is the distributional gradient of u.

The following facts are proved in [65, Section 1.1.13] and [65, Corollary 1.1.11], respectively:

Proposition 1.7.3. *The set*

$$\{\nabla u : u \in L^{1,2}(A)\}$$

is closed in $L^2(A; \mathbb{R}^N)$.

If A is an open set with Lipschitz boundary, then $L^{1,2}(A) = H^1(A)$.

To give a precise mathematical meaning to the fact that the boundary values of the displacement are imposed, we need to use fine properties of functions in the Deny-Lions space related to the notion of capacity, for which we refer to [43, 54, 86]. Let us only recall that if A is a bounded open set in $\mathbb{R}^N$, the *capacity* of an arbitrary subset E of A is defined as

$$\operatorname{cap}(E,A) := \inf_{u\in\mathcal{U}_E^A} \int_A |\nabla u|^2\,dx\,,$$

where $\mathcal{U}_E^A$ is the set of all functions $u \in H_0^1(A)$ such that $u \geq 1$ a.e. in a neighbourhood of E.

In the sequel we shall use the expression *quasi-everywhere on* E, abbreviated as *q.e. on* E, to indicate that a property holds on a set E except a subset of capacity zero, while we shall use the abbreviation *a.e. on* E when referring to the Lebesgue measure.

Definition 1.7.4. A function $u : A \to \mathbb{R}$ is called *quasi-continuous* if for every $\varepsilon > 0$ there exists a set $E_\varepsilon \subset A$ such that $\operatorname{cap}(E_\varepsilon, A) < \varepsilon$ and $u|_{A\setminus E_\varepsilon}$ is continuous in $A \setminus E_\varepsilon$.

Any function $u \in L^{1,2}(A)$ admits a quasi-continuous representative $\tilde{u}$ (see, e.g., [43, 54, 86]) that can be extended up to the Lipschitz part $\partial_L A$ of the boundary of A; it is characterized by the fact that

$$\lim_{\rho\to 0+} \frac{1}{|B_\rho(x)\cap A|}\int_{B_\rho(x)\cap A} |u(y)-u(x)|\,dy = 0 \qquad \text{for q.e. } x \in A\cup\partial_L A\,.$$

Moreover, if $u_n \to u$ strongly in $H^1(A)$, then a subsequence of $(\tilde{u}_n)$ converges to $\tilde{u}$ q.e. in $A \cup \partial_L A$. We shall always identify each function $u \in L^{1,2}(A)$ with its quasi-continuous representative $\tilde{u}$.

Remark 1.7.5. Let Ω be an open set in $\mathbb{R}^N$. Throughout the thesis, given a function $u \in L^{1,2}(\Omega \setminus K)$ for some K of null $\mathcal{L}^N$ measure, we always extend ∇u to Ω by setting $\nabla u = 0$ a.e. on K. We stress that, however, ∇u is the distributional gradient of u only in $\Omega \setminus K$ and, in general, it does not coincide in Ω with the gradient of an extension of u.

1.7.3. Convergence of minimizers of elliptic problems. In the next chapters we are going to solve elliptic problems in varying 2-dimensional domains. We will need stability results assuring the strong convergence of gradients of minimizers of these problems. The following one has been proved in [38, Theorem 5.1]:

Theorem 1.7.6. *Let Ω be a bounded open subsets in $\mathbb{R}^2$ with Lipschitz boundary; let $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, with $\partial_D\Omega$ relatively open and $\mathcal{H}^1(\partial_D\Omega) > 0$. Let w_n be a sequence in $H^1(\Omega)$ converging to w in $H^1(\Omega)$. Let Γ_n be a sequence of compact sets contained in $\overline{\Omega}$, converging to Γ in the Hausdorff metric, and such that*

- *they have a uniformly bounded number of connected components,*
- $\sup_n \mathcal{H}^1(\Gamma_n) < +\infty$.

Let $u_n \in L^{1,2}(\Omega \setminus \Gamma_n)$ and $u \in L^{1,2}(\Omega\setminus\Gamma)$ be solutions to the minimum problems

$$\min\Big\{\int_{\Omega\setminus\Gamma_n} |\nabla v|^2\,dx :\ v \in L^{1,2}(\Omega\setminus\Gamma_n),\ v = w_n \ \text{ q.e. on } \partial_D\Omega\setminus\Gamma_n\Big\} \tag{1.30}$$

and

$$\min\Big\{\int_{\Omega\setminus\Gamma} |\nabla v|^2\,dx :\ v \in L^{1,2}(\Omega\setminus\Gamma),\ v = w \ \text{ q.e. on } \partial_D\Omega\setminus\Gamma\Big\}, \tag{1.31}$$

respectively. Then $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$.

A minimizer of (1.30) (or, similarly, of (1.31)) solves the elliptic problem with mixed boundary conditions

$$\begin{cases} \Delta u_n = 0 & \text{in } \Omega \setminus \varGamma_n \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{on } \varGamma_n \cup \partial_N \Omega \\ u_n = w_n & \text{on } \partial_D \Omega \setminus \varGamma_n\,, \end{cases}$$

that is,

$$\int_{\Omega \setminus \varGamma_n} \nabla u_n \cdot \nabla v \, dx = 0 \quad \text{for all } v \in L^{1,2}(\Omega \setminus \varGamma_n), v = 0 \text{ on } \partial_D \Omega \setminus \varGamma_n\,.$$

In Chapter 3 we adapt Theorem 1.7.6 to a particular class of compact sets with Hausdorff dimension between one and two (see Section 3.1.1 for the class of sets, and Theorem 3.4.1).

Remark 1.7.7. Results similar to Theorem 1.7.6 have been proved in different settings: Šverak [80] considered Dirichlet problems, while Bucur & Varchon treated the Neumann case in a series of papers [20, 19, 21]. Some of them are based on the so-called Mosco convergence of spaces (see [18, Definition 2.1]):

Definition 1.7.8. Let H be a Hilbert space, and $\{G_n\}_n$, G subsets of H. The sequence G_n converges to G in the sense of Mosco if the following conditions are satisfied:

(M_1) for every $u \in G$ there exists a sequence u_n such that $u_n \in G_n$ and $u_n \xrightarrow{H} u$;

(M_2) if $u_k \in G_{n_k}$ is such that $u_k \xrightarrow{H} u$, then $u \in G$.

We will use a Mosco convergence result in Chapter 3 in order to justify the model described therein.

1.7.4. Monotone functions. We recall a classical result, whose generalizations will be used in the following chapters.

Theorem 1.7.9 (Helly's Theorem). *Let I be a (finite or infinite) interval in $\mathbb{R}$. Let $f_n : I \to [0,1]$ be a sequence of monotone non-decreasing functions. Then there exist a subsequence f_{n_k} and a function $f : I \to [0,1]$ such that $f_{n_k}(t) \to f(t)$ for every $t \in I$. Furthermore, if f is continuous then $f_{n_k} \to f$ uniformly on compact sets contained in I.*

We will also need the following fact:

Lemma 1.7.10. *Let $f, f_n : [0,T] \to \mathbb{R}$ be monotone non-decreasing functions such that $f_n(t) \to f(t)$ for every $t \in [0,T]$. Let f be continuous at $\bar{t} \in [0,T]$. Then for every $t_n \to \bar{t}$ it is $f_n(t_n) \to f(\bar{t})$.*

Proof. Fix $\alpha > 0$. By continuity, there exists $\theta > 0$ such that $|f(t) - f(\bar{t})| < \alpha$ for every $|t - \bar{t}| < 2\theta$, $t \in [0,T]$.

Being $t_n \to \bar{t}$, there exists n_0 such that $|t_n - \bar{t}| < \theta$ for every $n > n_0$, so that

$$|f(t_n) - f(\bar{t})| < \alpha$$

for every $n > n_0$. By monotonicity, $f(\bar{t} - \theta) \le f(t_n) \le f(\bar{t} + \theta)$ for every $n > n_0$.

Pointwise convergence implies that there exists $n_1 \ge n_0$ such that

$$|f_n(\bar{t} - \theta) - f(\bar{t} - \theta)| < \alpha \quad \text{and} \quad |f_n(\bar{t} + \theta) - f(\bar{t} + \theta)| < \alpha$$

for every $n > n_1$.

By continuity of f and the choice of θ, $|f(\bar{t}) - f(\bar{t} \pm \theta)| < \alpha$. Then by monotonicity and by the above inequalities we obtain

$$f(\bar{t}) - 2\alpha < f(\bar{t} - \theta) - \alpha < f_n(\bar{t} - \theta) \le f_n(t_n) \le f_n(\bar{t} + \theta) < f(\bar{t} + \theta) + \alpha < f(\bar{t}) + 2\alpha$$

for every $n > n_1$. Being α arbitrary, the thesis follows. □

1.8. Notation

- By the notation $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ we understand the scalar product and the norm in the Hilbert spaces $L^2(\Omega)$ or $L^2(\Omega;\mathbb{R}^2)$, or $L^2(\Omega\setminus\Gamma)$ or $L^2(\Omega\setminus\Gamma;\mathbb{R}^2)$ when $\mathcal{L}^2(\Gamma)=0$. In any other case or for the sake of clarity, we will specify the space the norm refers to, for example $\|\cdot\|_{H^1(\Omega\setminus\Gamma)}$.
- Given a sequence of compact set Γ_k that converges to a set Γ in the Hausdorff topology as $k\to+\infty$, we will write
$$\Gamma=\mathcal{H}\text{-}\lim_{k\to+\infty}\Gamma_k \quad \text{or} \quad \Gamma_k\xrightarrow{\mathcal{H}}\Gamma\,.$$
- Let A be a subset of $\mathbb{R}^N$. We denote by $\dim_{\mathcal{H}}(A)$ the Hausdorff measure of A, which is defined by
$$\dim_{\mathcal{H}}(A):=\inf\{d\geq 0:\ \mathcal{H}^d(A)=0\}\,.$$
- Given an open interval $I\subset\mathbb{R}$ and a function $f:I\to\mathbb{R}$, for every $t\in I$ we denote
$$f(t-):=\limsup_{\tau\to t-}f(\tau) \qquad f(t+):=\liminf_{\tau\to t+}f(\tau).$$
- Given a set X and a real-valued function $g:X\to\mathbb{R}$, for every $x\in X$ we denote $g(x)^+:=\sup\{g(x),0\}$.
- We denote any constant by C. The constant C may vary also within the same proof and is independent of all the parameters, unless we explicitly write the dependence. It might happen that C is a dimensional constant.

CHAPTER 2

A viscosity-driven crack evolution

In this chapter we present a model of crack growth in brittle materials which couples dissipative effects on the crack tip and viscous effects. We consider the two-dimensional antiplane case in a bounded open domain $\Omega \subset \mathbb{R}^2$. The crack path is assigned *a priori*, with an injective arc-length parametrization $\gamma : [0, L] \to \overline{\Omega}$ of class $C^{1,1}$, and the cracks are of the form $\Gamma(\sigma) = \gamma([0, \sigma])$.

Fixed the Young modulus $\mathfrak{a} > 0$, the coefficient of viscosity $\mathfrak{b} > 0$, the material toughness $\mathfrak{c} > 0$, and the dissipation constant $\mathfrak{d} > 0$, by means of the time-discretization approach discussed in Section 1.6 we first prove the existence of a rate-dependent evolution $(s^{\mathfrak{b},\mathfrak{d}}(t), u^{\mathfrak{b},\mathfrak{d}}(t))$, with $t \in [0, T]$, where $s^{\mathfrak{b},\mathfrak{d}}(t)$ and $u^{\mathfrak{b},\mathfrak{d}}(t)$ are the crack tip position and the out-of-plane elastic displacement, respectively, driven by a time-dependent boundary loading $w(t)$ (*i.e.* we impose $u^{\mathfrak{b},\mathfrak{d}}(t) = w(t)$ on a subset $\partial_D\Omega$ of the boundary $\partial\Omega$). At every instant $t \in [0, T]$ the evolution satisfies the problem

$$\begin{cases} \mathfrak{a}\Delta u^{\mathfrak{b},\mathfrak{d}}(t) + \mathfrak{b}\Delta \dot{u}^{\mathfrak{b},\mathfrak{d}}(t) = 0 & \text{in } \Omega \setminus \Gamma(s^{\mathfrak{b},\mathfrak{d}}(t)) \\ \dfrac{\mathfrak{a}\partial u^{\mathfrak{b},\mathfrak{d}}(t)}{\partial \mathbf{n}} + \dfrac{\mathfrak{b}\partial \dot{u}^{\mathfrak{b},\mathfrak{d}}(t)}{\partial \mathbf{n}} = 0 & \text{on } \Gamma(s^{\mathfrak{b},\mathfrak{d}}(t)) \cup \partial\Omega \setminus \partial_D\Omega \\ u^{\mathfrak{b},\mathfrak{d}}(t) = w(t) & \text{on } \partial_D\Omega \end{cases} \tag{2.1}$$

and the Griffith's conditions

$$\begin{aligned} &\dot{s}^{\mathfrak{b},\mathfrak{d}}(t) \geq 0 \\ &-\mathcal{G}(s^{\mathfrak{b},\mathfrak{d}}(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) + \mathfrak{c}s^{\mathfrak{b},\mathfrak{d}}(t) + \mathfrak{d}\dot{s}^{\mathfrak{b},\mathfrak{d}}(t) \geq 0 \\ &\left[-\mathcal{G}(s^{\mathfrak{b},\mathfrak{d}}(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) + \mathfrak{c}s^{\mathfrak{b},\mathfrak{d}}(t) + \mathfrak{d}\dot{s}^{\mathfrak{b},\mathfrak{d}}(t)\right] \dot{s}^{\mathfrak{b},\mathfrak{d}}(t) = 0\,, \end{aligned}$$

under proper initial data $s^{\mathfrak{b},\mathfrak{d}}(0) = s_0$ and $u^{\mathfrak{b},\mathfrak{d}}(0) = u_0$, where $\mathcal{G}(\sigma, \psi)$ is the energy release rate at the tip $\gamma(\sigma)$ of the crack $\Gamma(\sigma)$, for a boundary loading ψ.

We are then interested in describing the rate-independent evolution obtained as limit of the rate-dependent ones when the dissipative and viscous effects vanish, *i.e.* we let $\mathfrak{b}, \mathfrak{d} \to 0$. This rate-independent evolution, called *vanishing viscosity evolution*, is characterized by the stability condition

$$\begin{cases} \mathfrak{a}\Delta u(t) = 0 & \text{in } \Omega \setminus \Gamma(s(t)) \\ \dfrac{\partial u(t)}{\partial \mathbf{n}} = 0 & \text{on } \Gamma(s(t)) \cup \partial\Omega \setminus \partial_D\Omega \\ u(t) = w(t) & \text{on } \partial_D\Omega \end{cases}$$

and by a Griffith's criterion described in terms of a *weak activation criterion* as in Definition 1.3.2; moreover, in general, the vanishing viscosity evolution does not fulfill any global minimality condition like the one in Definition 1.4.1.(ii). We remark that, while the fracture in the rate-dependent evolutions grows continuously with respect to time, *i.e.* the functions $s^{\mathfrak{b},\mathfrak{d}}(\cdot)$ are continuous, in the rate-independent limit it may exhibit jump discontinuities. Furthermore, in case of monotone increasing boundary loadings $w(t) = tw_0$, the vanishing viscosity evolution corresponds to the one found in [**58**, **82**].

In Section 2.7, under suitable regularity assumptions for the energy release rate $\mathcal{G}$, we discuss the role of the penalizing terms (those containing $\mathfrak{b}$ and $\mathfrak{d}$) in the selection of the vanishing viscosity evolution: we discover that the dissipation on the crack tip is the real responsible of the process, while the viscous effect plays a non-influential role. Under the same conditions, the crack growth of the vanishing viscosity evolution is described by solving a finite-dimensional problem with an "algorithmic" procedure.

Finally, by means of an example we explicitly show that, under some regularity assumptions, the crack growth process of the vanishing viscosity evolution does not follow the trajectory of the globally minimizing variational evolutions, which instead satisfy the one-sided minimality condition in Definition 1.4.1.(ii).

The results of this chapter have been published in [77].

2.1. The geometrical setting

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with Lipschitz boundary $\partial\Omega$. Let Γ be a $C^{1,1}$ simple curve and $\gamma : [0, L] \to \overline{\Omega}$ be its parametrization by arc length, where $L := \mathcal{H}^1(\Gamma)$. We assume the following geometrical landscape (see Figure 1):

- $\Gamma \cap \partial\Omega = \{\gamma(0), \gamma(L)\}$;
- $\Omega \setminus \Gamma = \Omega^1 \cup \Omega^2$, where Ω^1 and Ω^2 are non-empty connected open sets with Lipschitz boundary, and $\Omega^1 \cap \Omega^2 = \emptyset$;
- $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, where $\partial_D\Omega \cap \partial_N\Omega = \emptyset$, $\partial_D\Omega$ is relatively open in $\partial\Omega$ with $\mathcal{H}^1(\partial_D\Omega) > 0$, and $\mathcal{H}^1(\partial_D\Omega \cap \partial\Omega^1) \neq 0 \neq \mathcal{H}^1(\partial_D\Omega \cap \partial\Omega^2)$.

In other words, we assume Γ to split the domain in two connected subdomains, with the Dirichlet boundary laid on the boundary of both subdomains.

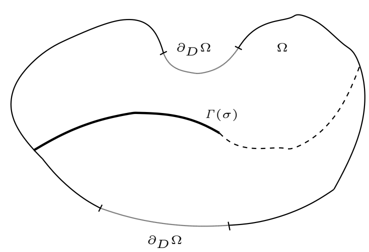


FIGURE 1. The domain Ω and the pre-assigned crack path Γ.

For every $\sigma \in (0, L]$, we set

$$\Gamma(\sigma) := \gamma([0, \sigma]) \quad \text{and} \quad \Omega_\sigma := \Omega \setminus \Gamma(\sigma) .$$

By the regularity hypotheses on Ω, Ω^1 and Ω^2, the trace operators $tr : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ and $tr_i : H^1(\Omega^i) \to H^{1/2}(\partial\Omega^i)$, $i = 1, 2$, are well defined. In particular, for every $v \in H^1(\Omega \setminus \Gamma)$ we define its jump function across Γ, $[v] \in H^{1/2}(\Gamma)$, as

$$[v] = tr_1(v)|_\Gamma - tr_2(v)|_\Gamma .$$

Then the functional space $H^1(\Omega_\sigma)$ corresponds to the set

$$\big\{u \in H^1(\Omega \setminus \Gamma) : [u] = 0 \text{ on } \Gamma \setminus \Gamma(\sigma)\big\} .$$

This fact allows us to work in the fixed Hilbert space $H^1(\Omega \setminus \Gamma)$, and to check the condition on the jump $[u]$ to establish if $u \in H^1(\Omega \setminus \Gamma)$ belongs to one of the smaller spaces $H^1(\Omega_\sigma) \subset H^1(\Omega \setminus \Gamma)$.

We will write u instead of $tr(u)$ whenever from the context it is clear that we are referring to the trace of the function u.

Fix $s_0 \in (0, L)$. For any boundary loading $\psi \in H^1(\Omega_{s_0})$ and any crack $\Gamma(\sigma)$, $\sigma \in [0, L]$, the set of *admissible displacements* is given by

$$\mathcal{AD}(\psi, \sigma) := \left\{ v \in H^1(\Omega_\sigma) \, : \, v = \psi \text{ on } \partial_D \Omega \right\} ,$$

where the last equality is in the sense of traces.

We study the evolution process in a fixed time interval $[0, T]$. When dealing with an element $u \in H^1(0, T; H^1(\Omega \setminus \Gamma))$, we always assume u to be the *continuous* representative (with respect to the time variable) of its class. Therefore it makes sense to consider the pointwise value $u(t)$ for every $t \in [0, T]$. On the Dirichlet part of the boundary, $\partial_D \Omega$, we prescribe a time-dependent boundary displacement which, at each instant $t \in [0, T]$, is given by the value of (the trace of) a function $w \in C^2([0, T]; H^1(\Omega_{s_0}))$ at t.

The initial configuration is the couple (u_0, s_0), where the initial out-of-plane displacement $u_0 \in H^1(\Omega_{s_0})$ is the (weak) solution to

$$\begin{cases} \mathfrak{a}\Delta u_0 = 0 & \text{in } \Omega_{s_0} \\ \frac{\partial u_0}{\partial \mathbf{n}} = 0 & \text{on } \Gamma(s_0) \cup \partial_N \Omega \\ u_0 = w(0) & \text{on } \partial_D \Omega. \end{cases} \tag{2.2}$$

In our computations we will need to define homeomorphisms between the domains $\Omega_{\sigma+\theta}$ and Ω_σ, for $|\theta|$ small, such that $\partial\Omega$ is kept fixed; roughly speaking, we have to slightly "move" the crack tip forward or backward along Γ. This can be done thanks to the regularity assumptions on Γ: fixed $\sigma \in (0, L)$, it is possible to construct a neighbourhood $\omega \subset\subset \Omega$ of $\gamma(\sigma)$ and a $C^{1,1}$ vector field $\eta_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, with $\theta \in \mathbb{R}$, such that η_θ is the identity map in $\mathbb{R}^2 \setminus \omega$, $\eta_\theta(\Gamma) \subset \Gamma$ and

$$\eta_\theta(\Gamma(\sigma)) = \Gamma(\sigma + \theta) \tag{2.3}$$

for $|\theta|$ sufficiently small. Even though η_θ depends on σ, there is no need to make it explicit since it will be clear from the context which fixed σ it refers to.

2.2. The incremental problem

This section is devoted to the study of the incremental problems for the rate-dependent evolutions. We are going to minimize a functional that is a combination of (1.27) and (1.28), containing penalizing terms for both the elastic bulk energy and the fracture energy. We also establish *a priori* estimates in order to obtain solutions to the problem (2.1) by means of compactness arguments.

The positive dimensional parameters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ are fixed throughout the chapter; as already said, they correspond to the Young modulus, the coefficient of viscosity, the material toughness, and the dissipation constant, respectively.

Fixed a time-step $\tau \in (0, T)$, for any $u, v \in H^1(\Omega \setminus \Gamma)$ we define the functionals (dependent on τ)

$$E(u, v) := \frac{1}{2}\mathfrak{a}\|\nabla u\|^2 + \frac{\mathfrak{b}}{2\tau}\|\nabla u - \nabla v\|^2$$

and

$$\mathcal{E}(u, v) := \frac{1}{2\mathfrak{a}}\|\mathfrak{a}\nabla u + \frac{\mathfrak{b}}{\tau}(\nabla u - \nabla v)\|^2.$$

By a simple computation it can be seen that, for any fixed $v \in H^1(\Omega \setminus \Gamma)$, the functionals $E(\cdot, v), \mathcal{E}(\cdot, v) : H^1(\Omega \setminus \Gamma) \to \mathbb{R}$ have the same Fréchet differential up to a multiplicative constant. Actually it is

$$\mathcal{E}(u,v) = \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right) E(u,v) - \frac{\mathfrak{b}}{2\tau}\|\nabla v\|^2. \tag{2.4}$$

Consequently, for any fixed $\sigma \in [s_0, L)$, $v \in H^1(\Omega_\sigma)$, $\psi \in H^1(\Omega_{s_0})$ and $\tau \in (0,T)$, the following relation holds true

$$U = \operatorname{argmin}\{\mathcal{E}(u,v) :\ u \in \mathcal{AD}(\psi,\sigma)\} \Leftrightarrow U = \operatorname{argmin}\{E(u,v) :\ u \in \mathcal{AD}(\psi,\sigma)\}. \tag{2.5}$$

The functional E represents a discretized version of the stored elastic energy plus a viscoplastic friction term, energy which should have the form

$$\mathfrak{a}\|\nabla u(t)\|^2 + \mathfrak{b}\int_0^t \|\nabla \dot{u}(\xi)\|^2\, d\xi$$

for an evolution $u \in H^1(0,T; H^1(\Omega \setminus \Gamma))$ of the displacement field. Fixed σ and v, when we minimize $E(\cdot, v)$ (or, equivalently, $\mathcal{E}(\cdot, v)$) we penalize the L^2 distance of the gradients of the two functions u and v, *i.e.* in the discrete-time evolution below we penalize large variations of the displacement gradient with respect to time.

By the algebraic equivalence (2.4), the functional $\mathcal{E}$ provides an equivalent way to select minima, even though it does not have a proper interpretation as energy. Nevertheless, it plays an important part in finding estimates.

According to Griffith's model, the energy dissipated by the crack creation is proportional to the crack length; in our model, we add one more term taking into account the rate of crack increase. As for the viscoelastic part, in the incremental problem below the fracture energy shows two dimensional positive constants $\mathfrak{c}$ and $\mathfrak{d}$ and, for any fixed $\bar{\sigma} \in [s_0, L)$ and every $\sigma \in [\bar{\sigma}, L]$, it has the form

$$\mathfrak{a}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - \bar{\sigma})^2\right).$$

In order to avoid a trivial solution, we really have to consider the adimensional quantity, dependent on the time-step,

$$\mathfrak{a}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1} = \mathfrak{a}\,\frac{\tau}{\mathfrak{a}\tau + \mathfrak{b}}\,.$$

The value

$$\tau_{\mathfrak{b}} := \frac{\tau}{\mathfrak{a}\tau + \mathfrak{b}}$$

can be interpreted as a characteristic time of the viscoelastic material; in Section 2.5 we describe the consequences of neglecting the parameter $\mathfrak{a}\tau_{\mathfrak{b}}$ in the crack energy. Let us observe that, if it were $\mathfrak{b} = 0$, then $\mathfrak{a}\tau_0 = 1$.

We define the *incremental problem with time-step* $\tau \in (0,T)$ in the following way: let $N_\tau \in \mathbb{N}$ be such that $T - \tau < \tau N_\tau \leq T$. Set

- $u_0^\tau := u_0$, $s_0^\tau := s_0$;
- for any $1 \leq i \leq N_\tau$ and $\sigma \geq s_0$, let $u_i^{\tau,\sigma}$ be the unique solution to
$$\min\left\{E(u, u_{i-1}^\tau) :\ u \in \mathcal{AD}(w(i\tau), \sigma)\right\} \tag{2.6}$$
and
$$s_i^\tau \in \operatorname{argmin}\left\{E(u_i^{\tau,\sigma}, u_{i-1}^\tau) + \mathfrak{a}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)^2\right) :\ s_{i-1}^\tau \leq \sigma \leq L\right\}; \tag{2.7}$$

we set $u_i^\tau := u_i^{\tau, s_i^\tau}$.

Remark 2.2.1. Existence and uniqueness of the solution to (2.6) is assured by the direct method of the calculus of variations. In order to prove the existence of a solution to (2.7) it is enough to exploit the compactness of the interval $[s_{i-1}^\tau, L]$ and the convergence result stated in Theorem 1.7.6.

We introduce the piecewise-constant and piecewise-affine interpolants for both the u_i^τ and s_i^τ:

- $u^\tau, \tilde{u}^\tau : [0,T] \to H^1(\Omega \setminus \Gamma)$ as
$$u^\tau(t) := u_i^\tau$$
$$\tilde{u}^\tau(t) := u_i^\tau + \frac{t - i\tau}{\tau}(u_{i+1}^\tau - u_i^\tau)$$
for $i\tau \le t < (i+1)\tau, i = 0, \ldots, N_\tau - 1$, and $u^\tau(t) = \tilde{u}^\tau(t) := u_{N_\tau}^\tau$ for $\tau N_\tau \le t \le T$;
- $s^\tau, \tilde{s}^\tau : [0,T] \to [s_0, L]$ as
$$s^\tau(t) := s_i^\tau$$
$$\tilde{s}^\tau(t) := s_i^\tau + \frac{t - i\tau}{\tau}(s_{i+1}^\tau - s_i^\tau)$$
for $i\tau \le t < (i+1)\tau,\ i = 0, \ldots, N_\tau - 1$, and $s^\tau(t) = \tilde{s}^\tau(t) := s_{N_\tau}^\tau$ for $\tau N_\tau \le t \le T$.

By definition through (2.6), u^τ and $\tilde{u}^\tau$ satisfy the variational equation
$$\mathfrak{a}\langle \nabla u^\tau(t), \nabla\varphi\rangle + \mathfrak{b}\langle \nabla\dot{\tilde{u}}^\tau(t-\tau), \nabla\varphi\rangle = 0 \tag{2.8}$$
for every $\varphi \in H^1\left(\Omega_{s^\tau(t)}\right)$ with $\varphi = 0$ on $\partial_D\Omega$, and $t \in [\tau, N_\tau \tau]$.

Remark 2.2.2. By the equivalence (2.5), the minimum problems (2.6)-(2.7) are equivalent to the following ones:

- $u_0^\tau := u_0$, $s_0^\tau := s_0$;
- for any $1 \le i \le N_\tau$ and $\sigma \ge s_0$, let $u_i^{\tau,\sigma}$ be the unique solution to
$$\min\left\{\mathcal{E}(u, u_{i-1}^\tau) :\ u \in \mathcal{AD}(w_i^\tau, \sigma)\right\}$$
and
$$s_i^\tau \in \operatorname{argmin}\left\{\mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^\tau) + \mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)^2 :\ s_{i-1}^\tau \le \sigma \le L\right\};$$
then set $u_i^\tau := u_i^{\tau, s_i^\tau}$.

For convenience in the discussions below, consider the discretized version of the boundary loading: for every $\tau \in (0,T)$ and $i = 0, \ldots, N_\tau$, set $w_i^\tau := w(i\tau)$ and let w^τ be the piecewise-constant interpolant of the w_i^τ. Then, being $w \in C^2([0,T]; H^1(\Omega_{s_0}))$, it is
$$w_{i+1}^\tau - w_i^\tau = \int_{i\tau}^{(i+1)\tau} \dot{w}(\xi)\, d\xi$$
$$\nabla w_{i+1}^\tau - \nabla w_i^\tau = \int_{i\tau}^{(i+1)\tau} \nabla\dot{w}(\xi)\, d\xi,$$
where the integrals are Bochner integrals (see [2]).

2.2.1. *A priori* estimates. The following lemmas establish some estimates for the families of the displacements $\{u^\tau\}$ and of the crack tips $\{s^\tau\}$. These results will be useful in order to apply compactness arguments.

Lemma 2.2.3. *There exists a non-negative function $\rho(\tau) \to 0$ as $\tau \to 0^+$ such that for every $0 \le i < j \le N_\tau$*

$$\begin{aligned}\frac{1}{2}\mathfrak{a}\|\nabla u_j^\tau\|^2 + \frac{\mathfrak{b}}{2\tau}\sum_{h=i}^{j-1}\|\nabla u_{h+1}^\tau - \nabla u_h^\tau\|^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}s_j^\tau + \frac{\mathfrak{d}}{2\tau}\sum_{h=i}^{j-1}|s_{h+1}^\tau - s_h^\tau|^2\right)&\\ \le \frac{1}{2}\mathfrak{a}\|\nabla u_i^\tau\|^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s_i^\tau + \mathfrak{a}\int_{i\tau}^{j\tau}\langle\nabla u^\tau(\xi), \nabla\dot{w}(\xi)\rangle\,d\xi&\\ + \frac{\mathfrak{b}}{2}\int_{i\tau}^{j\tau}\|\nabla\dot{w}(\xi)\|^2\,d\xi + \rho(\tau).&\end{aligned}$$

PROOF. Taking $\varphi = u_h^\tau + w_{h+1}^\tau - w_h^\tau \in \mathcal{AD}(w_{h+1}^\tau, s_h^\tau)$ as test function in (2.7) (with $i = h+1$), we have

$$\begin{aligned}\frac{1}{2}\mathfrak{a}\|\nabla u_{h+1}^\tau\|^2 + \frac{\mathfrak{b}}{2\tau}\|\nabla u_{h+1}^\tau - \nabla u_h^\tau\|^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}s_{h+1}^\tau + \frac{\mathfrak{d}}{2\tau}|s_{h+1}^\tau - s_h^\tau|^2\right)&\\ \le \frac{1}{2}\mathfrak{a}\|\nabla u_h^\tau + \nabla w_{h+1}^\tau - \nabla w_h^\tau\|^2 + \frac{\mathfrak{b}}{2\tau}\|\nabla w_{h+1}^\tau - \nabla w_h^\tau\|^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s_h^\tau&\\ \le \frac{1}{2}\mathfrak{a}\|\nabla u_h^\tau\|^2 + \mathfrak{a}\int_{h\tau}^{(h+1)\tau}\langle\nabla u^\tau(\xi), \nabla\dot{w}(\xi)\rangle\,d\xi + \frac{1}{2}\mathfrak{a}\|\nabla w_{h+1}^\tau - \nabla w_h^\tau\|^2&\\ + \frac{\mathfrak{b}}{2\tau}\left(\int_{h\tau}^{(h+1)\tau}\|\nabla\dot{w}(\xi)\|\,d\xi\right)^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s_h^\tau&\\ \le \frac{1}{2}\mathfrak{a}\|\nabla u_h^\tau\|^2 + \mathfrak{a}\int_{h\tau}^{(h+1)\tau}\langle\nabla u^\tau(\xi), \nabla\dot{w}(\xi)\rangle\,d\xi&\\ + \frac{1}{2}\mathfrak{a}\left(\max_{0\le k<N_\tau}\int_{k\tau}^{(k+1)\tau}\|\nabla\dot{w}(\xi)\|\,d\xi\right)\int_{h\tau}^{(h+1)\tau}\|\nabla\dot{w}(\xi)\|\,d\xi&\\ + \frac{\mathfrak{b}}{2}\int_{h\tau}^{(h+1)\tau}\|\nabla\dot{w}(\xi)\|^2\,d\xi + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s_h^\tau.&\end{aligned}$$

Iterating over $h = i, \dots, j-1$ and defining

$$\rho(\tau) := \frac{1}{2}\mathfrak{a}T\tau\left(\max_{0\le\xi\le T}\|\nabla\dot{w}(\xi)\|\right)^2$$

the proof is complete. □

Lemma 2.2.4. *There exists a constant $C > 0$, independent of $\mathfrak{b}, \mathfrak{d}, \tau, t$, such that the following estimates hold true for every $\tau \in (0,T)$, $t \in [0,T]$, $j = 1, \dots, N_\tau$:*

$$\|u^\tau(t)\|_{H^1(\Omega\setminus\Gamma)} \le C \tag{2.9}$$

$$\|\tilde{u}^\tau(t)\|_{H^1(\Omega\setminus\Gamma)} \le C \tag{2.10}$$

$$\mathfrak{b}\int_0^{j\tau} \|\nabla\dot{\tilde{u}}^\tau(\xi)\|^2\, d\xi = \frac{\mathfrak{b}}{\tau}\sum_{h=0}^{j-1}\|\nabla u_{h-1}^\tau - \nabla u_h^\tau\|^2 \le C \tag{2.11}$$

$$\|\tilde{u}^\tau\|_{L^2(0,T;H^1(\Omega\setminus\Gamma))} \le CT^{1/2} \tag{2.12}$$

$$\mathfrak{b}\|\dot{\tilde{u}}^\tau\|^2_{L^2(0,T;H^1(\Omega\setminus\Gamma))} \le C. \tag{2.13}$$

PROOF. Fix $t \in [0,T]$ and let $j := j(t) \in 0, \dots, N_\tau - 1$ be such that it satisfies $j\tau \le t < (j+1)\tau$. By the inequality in Lemma 2.2.3 for $i = 0$ we obtain

$$\begin{aligned}&\frac{1}{2}\mathfrak{a}\|\nabla u_j^\tau\|^2 + \frac{\mathfrak{b}}{2}\int_0^{j\tau}\|\nabla\dot{\tilde{u}}^\tau(\xi)\|^2\, d\xi \\ \le&\frac{1}{2}\mathfrak{a}\|\nabla u_0\|^2 + \mathfrak{a}\int_0^{j\tau}\langle\nabla u^\tau(\xi), \nabla\dot{w}(\xi)\rangle\, d\xi + \frac{\mathfrak{b}}{2}\int_0^{j\tau}\|\nabla\dot{w}(\xi)\|^2\, d\xi + \rho(\tau).\end{aligned} \tag{2.14}$$

Hölder's inequality and (2.14) imply

$$\mathfrak{a}\|\nabla u^\tau(t)\|^2 \le C + 2\mathfrak{b}\left(\int_0^t \|\nabla u^\tau(\xi)\|^2\, d\xi\right)^{1/2}\left(\int_0^t \|\nabla\dot{w}(\xi)\|^2\, d\xi\right)^{1/2},$$

where $C > 0$ is independent of $\mathfrak{b}, \mathfrak{d}, \tau, t$. By a refined version of the Gronwall lemma (see [7, Lemma 4.1.8]), we deduce that for every $t \in [0,T]$

$$\left(\int_0^t \|\nabla u^\tau(\xi)\|^2\, d\xi\right)^{1/2} \le (TC)^{1/2} + 2T\|\nabla\dot{w}\|_{L^2(0,T;L^2(\Omega_{s_0};\mathbb{R}^2))}.$$

The last two inequalities imply that $\nabla u^\tau(t)$ is bounded in $L^2(\Omega\setminus\Gamma;\mathbb{R}^2)$ uniformly with respect to $\mathfrak{b}, \mathfrak{d}, \tau, t$. Using the Poincaré inequality we obtain (2.9) and (2.10). Then, considering (2.14), the estimates (2.11) and (2.12) follow. Finally, using the Poincaré inequality for $\dot{\tilde{u}}$

$$\|\dot{\tilde{u}}\|_{L^2(0,T;L^2(\Omega\setminus\Gamma))} \le C\left(\|\nabla\dot{\tilde{u}}\|_{L^2(0,T;L^2(\Omega\setminus\Gamma;\mathbb{R}^2))} + \|w\|_{C^2([0,T];H^1(\Omega_{s_0}))}\right)$$

and (2.11), we obtain (2.13). □

Set

$$\begin{aligned} z_0^\tau &:= \mathfrak{a}w(0) + \mathfrak{b}\dot{w}(0) \\ z_i^\tau &:= \mathfrak{a}w_i^\tau + \frac{\mathfrak{b}}{\tau}\left(w_i^\tau - w_{i-1}^\tau\right) \quad \text{for } 1 \le i \le N_\tau \end{aligned}$$

and call $\tilde{u}_0$ the solution to

$$\begin{cases} \Delta\tilde{u}_0 = 0 & \text{in } \Omega_{s_0} \\ \tilde{u}_0 = \dot{w}(0) & \text{on } \partial_D\Omega \\ \frac{\partial\tilde{u}_0}{\partial\mathbf{n}} = 0 & \text{on } \Gamma(s_0)\cup\partial_N\Omega\,. \end{cases}$$

For $\tau \in (0,T)$ define the incremental problem

- $v_0^\tau := \mathfrak{a}u_0 + \mathfrak{b}\tilde{u}_0$, $\sigma_0^\tau := s_0$;

- for any $1 \le i \le N_\tau$ and $\sigma \ge s_0$, let $v_i^{\tau,\sigma}$ be the unique solution to
$$\min\left\{\|\nabla v\|^2 : v \in \mathcal{AD}(z_i^\tau, \sigma)\right\} \tag{2.15}$$
and
$$\sigma_i^\tau \in \operatorname{argmin}\left\{\frac{1}{\mathfrak{a}}\|\nabla v\|^2 + \mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)^2 : \sigma_{i-1}^\tau \le \sigma \le L\right\}, \tag{2.16}$$
and set $v_i^\tau := v_i^{\tau,\sigma_i^\tau}$.

It is easy to check that for $i = 1$ it is
$$v_1^{\tau,\sigma} := \mathfrak{a}u_1^{\tau,\sigma} + \frac{\mathfrak{b}}{\tau}(u_1^{\tau,\sigma} - u_0^\tau)$$
and
$$\mathcal{E}(u_1^{\tau,\sigma}, u_0^\tau) = \frac{1}{2\mathfrak{a}}\|\nabla v_1^{\tau,\sigma}\|^2$$
for every $\sigma \in [s_0, L]$, so that we can assume $\sigma_1^\tau = s_1^\tau$. Iterating this argument, we suppose $\sigma_i^\tau = s_i^\tau$ for $1 \le i \le N_\tau$ and, consequently,
$$\begin{aligned} v_0^\tau &= \mathfrak{a}u_0 + \mathfrak{b}\tilde{u}_0 \\ v_i^\tau &= \mathfrak{a}u_i^\tau + \tfrac{\mathfrak{b}}{\tau}\left(u_i^\tau - u_{i-1}^\tau\right) \quad \text{for } 1 \le i \le N_\tau \end{aligned} \tag{2.17}$$
so that (2.6)-(2.7) and (2.15)-(2.16) provide the same evolution (up to the relation (2.17) between u_i^τ and v_i^τ), and in addition
$$\mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^\tau) = \frac{1}{2\mathfrak{a}}\|\nabla v_i^{\tau,\sigma}\|^2 \tag{2.18}$$
holds for every $1 \le i \le N_\tau$ and $\sigma \ge s_{i-1}^\tau$.

By the minimality of v_i^τ, we obtain estimates for the crack tip evolution $s^\tau(\cdot)$ as well:

Lemma 2.2.5. *There exists a non-negative function $\tilde{\rho}(\tau) \to 0$ as $\tau \to 0^+$ such that for every $0 \le i < j \le N_\tau$*
$$\begin{aligned} &\frac{1}{2\mathfrak{a}}\|v_j^\tau\|^2 + \mathfrak{c}s_j^\tau + \frac{\mathfrak{d}}{2}\tau\sum_{h=i}^{j-1}\left(\frac{s_{h+1}^\tau - s_h^\tau}{\tau}\right)^2 \\ \le\ &\frac{1}{2\mathfrak{a}}\|v_i^\tau\|^2 + \mathfrak{c}s_i^\tau + \frac{1}{\mathfrak{a}}\sum_{h=i}^{j-1}\langle\nabla v_h^\tau, \nabla z_{h+1}^\tau - \nabla z_h^\tau\rangle + \tilde{\rho}(\tau). \end{aligned} \tag{2.19}$$

Proof. Taking $\varphi = v_{h-1}^\tau + z_h^\tau - z_{h-1}^\tau \in \mathcal{AD}(z_h^\tau, s_{h-1}^\tau)$ as test function in (2.16), it is
$$\begin{aligned} \frac{1}{2\mathfrak{a}}\|\nabla v_h^\tau\|^2 + \mathfrak{c}s_h^\tau + \frac{\mathfrak{d}}{2\tau}(s_h^\tau - s_{h-1}^\tau) &\le \frac{1}{2\mathfrak{a}}\|\nabla v_{h-1}^\tau + \nabla z_h^\tau - \nabla z_{h-1}^\tau\|^2 + \mathfrak{c}s_{h-1}^\tau \\ &\le \frac{1}{2\mathfrak{a}}\|\nabla v_{h-1}^\tau\|^2 + \frac{1}{\mathfrak{a}}\langle\nabla v_{h-1}^\tau, \nabla z_h^\tau - \nabla z_{h-1}^\tau\rangle + \frac{1}{2\mathfrak{a}}\|\nabla z_h^\tau - \nabla z_{h-1}^\tau\|^2 + \mathfrak{c}s_{h-1}^\tau. \end{aligned}$$
Using the assumption $w \in C^2([0,T]; H^1(\Omega_{s_0}))$ and arguing as in the proof of Lemma 2.2.3, it is
$$\sum_{h=1}^{N_\tau}\|\nabla z_h^\tau - \nabla z_{h-1}^\tau\|^2 \le \tilde{\rho}(\tau) \to 0 \quad \text{as } \tau \to 0$$
with $\tilde{\rho}$ dependent only on $\mathfrak{a}$, τ and $\|w\|_{C^2([0,T];H^1(\Omega_{s_0}))}$.

Iterating the inequality above for $i \le h \le j-1$, we obtain the thesis. $\square$

Note that the term

$$\sum_{h=i}^{j} \langle \nabla v^\tau_{h-1}, \nabla z^\tau_h - \nabla z^\tau_{h-1} \rangle$$

in (2.19) has the explicit form

$$\mathfrak{a} \int_{i\tau}^{j\tau} \langle \nabla v^\tau(\xi), \nabla \dot{w}(\xi) \rangle \, d\xi + \mathfrak{b} \int_{i\tau}^{j\tau} \langle \nabla v^\tau(\xi), \frac{\nabla w^\tau(\xi) - 2\nabla w^\tau(\xi - \tau) + \nabla w^\tau(\xi - 2\tau)}{\tau^2} \rangle \, d\xi \,,$$

where we call v^τ the piecewise-constant interpolant of the v^τ_i and, with abuse of notations, $w^\tau(0) - w^\tau(-\tau) := \dot{w}(0)$ in case $i = 0$. As $w \in C^2([0,T]; H^1(\Omega_{s_0}))$, for every $t \in (0,T)$ the difference quotient

$$\frac{\nabla w^\tau(t) - 2\nabla w^\tau(t - \tau) + \nabla w^\tau(t - 2\tau)}{\tau^2}$$

converges strongly in $L^2(\Omega_{s_0}; \mathbb{R}^2)$ to $\nabla \ddot{w}(t)$ as $\tau \to 0$, uniformly with respect to t.

Lemma 2.2.6. *There exists a constant $C > 0$, independent of $\mathfrak{b}, \mathfrak{d}, \tau, t$, such that for every $\tau \in (0,T)$ and $t \in [0,T]$ the following estimates are satisfied:*

$$\|v^\tau(t)\|_{H^1(\Omega \setminus \Gamma)} \leq C \tag{2.20}$$

$$\mathfrak{d} \|\dot{s}^\tau\|^2_{L^2(0,T)} \leq C. \tag{2.21}$$

Proof. Taking $i = 0$ in Lemma 2.2.5 and using Hölder's inequality, it is

$$\begin{aligned} &\frac{1}{2\mathfrak{a}} \|\nabla v^\tau(t)\|^2 + \mathfrak{c} s^\tau(t) + \frac{1}{2} \mathfrak{d} \int_0^t |\dot{s}^\tau(\xi)|^2 \, d\xi \\ &\leq \frac{1}{2\mathfrak{a}} \|\nabla v_0\|^2 + \mathfrak{c} s_0 + \left(\|\dot{w}\|_{L^\infty} + \frac{\mathfrak{b}}{\mathfrak{a}} \|\ddot{w}\|_{L^\infty} \right) \left(\int_0^t \|\nabla v^\tau(\xi)\|^2 \, d\xi \right)^{1/2} + \tilde{\rho}(\tau). \end{aligned} \tag{2.22}$$

Arguing as in Lemma 2.2.4, we obtain

$$\|\nabla v^\tau(t)\| \leq C$$

for every t. Using the Poincaré inequality, since $v^\tau_i = z^\tau_i$ on $\partial_D \Omega$, estimate (2.20) follows. Then (2.21) is consequence of (2.22) and (2.20). □

2.3. Griffith's conditions for (s^τ, u^τ)

In order to achieve a complete description of the evolution of the system, we look for a differential characterization for the evolution $t \mapsto s(t)$ of the crack tip, that will be obtained in Sections 2.4 and 2.6 by means of the energy release rate, in the spirit of Griffith's theory.

First of all we introduce a functional playing the role of the energy release rate at the level of the discrete-time solutions (s^τ, u^τ) defined in Section 2.2, in order to establish a sort of discrete-time version of the Griffith's criterion (1.9).

For every $\sigma \in [s_0, L]$ and $g \in H^{1/2}(\partial_D \Omega)$, let $v(\sigma, g) \in H^1(\Omega_\sigma)$ be the solution to the problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_\sigma \\ v = g & \text{on } \partial_D \Omega \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \Gamma(\sigma) \cup \partial_N \Omega \,, \end{cases} \tag{2.23}$$

and set

$$\mathcal{F}(\sigma, g) := \frac{1}{2} \|\nabla v(\sigma, g)\|^2. \tag{2.24}$$

As already seen in Section 1.5.1, in particular through formula (1.18), the energy release rate $\mathcal{G}$ is defined as the derivative of $\mathcal{F}$ with respect to its "geometric" variable σ (see (2.25) below). The following result states the regularity properties of the functionals $\mathcal{F}$ and $\mathcal{G}$.

Proposition 2.3.1. *The functional $\mathcal{F}$ is continuous from $[s_0, L] \times H^{1/2}(\partial_D\Omega)$ to $\mathbb{R}$. For any fixed $g \in H^{1/2}(\partial_D\Omega)$, the map $\sigma \mapsto \mathcal{F}(\sigma, g)$ is differentiable at every $\sigma \in [s_0, L)$. The energy release rate*

$$\mathcal{G}(\sigma, g) := -\frac{\partial \mathcal{F}(\sigma, g)}{\partial \sigma} \tag{2.25}$$

is continuous in $[s_0, L) \times H^{1/2}(\partial_D\Omega)$.

We do not prove the above proposition since, as discussed in Section 1.5, it is a well known result. The key tools for its proof are Theorem 1.7.6, which moves the issue of the convergence of $\nabla v(\sigma, g)$ to check the convergence of the boundary datum and of the crack variable, and the integral formula for $\mathcal{G}$ in Remark 1.5.1.(iii), which in the current notation reads as

$$\mathcal{G}(\sigma, g) = -\frac{1}{2}\langle \nabla v(\sigma, g), \operatorname{div}(\lambda^\sigma)\, \nabla v(\sigma, g)\rangle + \langle \nabla v(\sigma, g), \nabla \lambda^\sigma\, \nabla v(\sigma, g)\rangle\,, \tag{2.26}$$

where $v(\sigma, g)$ is defined through (2.23), and λ^σ is a Lipschitz continuous vector field such that $\operatorname{supp}(\lambda^\sigma) \subset \Omega$, $\lambda^\sigma(\gamma(\bar\sigma)) = \zeta^\sigma(\gamma(\bar\sigma))\dot\gamma(\bar\sigma)$ for every $\bar\sigma \in [0, L]$, with ζ^σ a cut-off function, equal to one in a neighbourhood of $\gamma(\sigma)$.

Remark 2.3.2. If we fix $s_1 \in (s_0, L)$, then we can assume λ^σ to be the same for any $\sigma \in [s_0, s_1]$: it is $\lambda^\sigma = \lambda$ for every $\sigma \in [s_0, s_1]$, where λ is a Lipschitz continuous vector field such that $\operatorname{supp}(\lambda) \subset \Omega$ and $\lambda(\gamma(\bar\sigma)) = \zeta(\gamma(\bar\sigma))\dot\gamma(\bar\sigma)$ for every $\bar\sigma \in [0, L]$, with ζ a cut-off function, equal to one in a neighbourhood of $\gamma([s_0, s_1])$.

In the following, exploiting the continuity of the trace operator we consider the space $H^1(\Omega_{s_0})$ instead of $H^{1/2}(\partial_D\Omega)$: we assume $\mathcal{F}$ to be defined on $[s_0, L] \times H^1(\Omega_{s_0})$ and, with abuse of notation, we identify every $g \in H^1(\Omega_{s_0})$ with its trace on $\partial_D\Omega$, so that Proposition 2.3.3 still holds true for the functional $\mathcal{F} : [s_0, L] \times H^1(\Omega_{s_0}) \to \mathbb{R}$.

Proposition 2.3.3. *Let $s_1 \in (s_0, L)$ be fixed. Then $\mathcal{G}(\sigma, \cdot) : H^1(\Omega_{s_0}) \to \mathbb{R}$ is Lipschitz continuous, uniformly in $\sigma \in [s_0, s_1]$.*

Proof. Fix $\sigma \in [s_0, s_1]$. For $j = 1, 2$, let $g_j \in H^1(\Omega_{s_0})$ and let $v(\sigma, g_j) \in H^1(\Omega_\sigma)$ be the solution to (2.23) with $g = tr(g_j)$, and write

$$v(\sigma, g_j) = \tilde v(\sigma, g_j) + g_j.$$

Then, for every $\varphi \in H^1(\Omega_\sigma)$ with $\varphi = 0$ on $\partial_D\Omega$, it is

$$0 = \langle \nabla v(\sigma, g_j), \nabla\varphi\rangle = \langle \nabla \tilde v(\sigma, g_j), \nabla\varphi\rangle + \langle \nabla g_j, \nabla\varphi\rangle,$$

i.e.

$$\langle \nabla \tilde v(\sigma, g_j), \nabla\varphi\rangle = -\langle \nabla g_j, \nabla\varphi\rangle \tag{2.27}$$

for any φ as before.

In particular $\tilde v(\sigma, g_j) \in H^1(\Omega_\sigma)$ with $\tilde v(\sigma, g_j) = 0$ on $\partial_D\Omega$. Thus, considering (2.27) and applying Hölder's inequality, it is

$$\begin{aligned}
\|\nabla \tilde v(\sigma, g_1) - \nabla \tilde v(\sigma, g_2)\|^2 =& \langle \nabla \tilde v(\sigma, g_1) - \nabla \tilde v(\sigma, g_2), \nabla \tilde v(\sigma, g_1) - \nabla \tilde v(\sigma, g_2)\rangle \\
=& -\langle \nabla g_1 - \nabla g_2, \nabla \tilde v(\sigma, g_1) - \nabla \tilde v(\sigma, g_2)\rangle \\
\le& \|\nabla g_1 - \nabla g_2\|\, \|\nabla \tilde v(\sigma, g_1) - \nabla \tilde v(\sigma, g_2)\|\,,
\end{aligned}$$

so that $\|\nabla\tilde{v}(\sigma,g_1)-\nabla\tilde{v}(\sigma,g_2)\| \le \|\nabla g_1 - \nabla g_2\|$. Therefore

$$\|\nabla v(\sigma,g_1)-\nabla v(\sigma,g_2)\| \le \|\nabla\tilde{v}(\sigma,g_1)-\nabla\tilde{v}(\sigma,g_2)\| + \|\nabla g_1 - \nabla g_2\| \le 2\,\|\nabla g_1 - \nabla g_2\|\,.$$

By Remark 2.3.2, in the expression (2.26) for $\mathcal{G}$ we can assume $\lambda^\sigma = \lambda$ for every $\sigma \in [s_0, s_1]$. Then, by (2.26) and the above inequality, we obtain

$$|\mathcal{G}(\sigma,g_1)-\mathcal{G}(\sigma,g_1)| \le C\,\|\nabla g_1 - \nabla g_2\| \le C\,\|g_1-g_2\|_{H^1(\Omega_{s_0})}$$

with $C \ge \mathrm{Lip}(\lambda)$, where $\mathrm{Lip}(\lambda)$ is the Lipschitz constant of λ. □

We want to characterize the discrete-time evolution $t \mapsto (s^\tau(t), u^\tau(t))$ in terms of Griffith's theory, with the goal of obtaining a law like (1.9) for the continuous-time evolution by taking the limit $\tau \to 0$ in the conditions for (s^τ, u^τ).

By construction, the maps s^τ and $\tilde{s}^\tau$ are non-decreasing; in particular

$$\dot{\tilde{s}}^\tau(t) \ge 0 \tag{2.28}$$

for every $t \in [0,T]$.

In order to obtain conditions (G2) and (G3) described in Section 1.3, we argue in the following way. By definition (2.15), for every $i \in \{0,\dots,N_\tau\}$ the function $v_i^{\tau,\sigma}$ satisfies the problem

$$\begin{cases} \Delta v_i^{\tau,\sigma} = 0 & \text{in } \Omega_\sigma \\ v_i^{\tau,\sigma} = z_i^\tau & \text{on } \partial_D\Omega \\ \frac{\partial v_i^{\tau,\sigma}}{\partial \mathbf{n}} = 0 & \text{on } \Gamma(\sigma)\partial_N\Omega\,. \end{cases}$$

Having in mind the equality (2.18) and applying Proposition 2.3.1 with $g = z_i^\tau$, the function

$$\sigma \in [s_{i-1}^\tau, L] \mapsto \mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^\tau) = \mathcal{F}(\sigma, z_i^\tau)$$

is differentiable at every $\sigma \in [s_{i-1}^\tau, L)$. For every $\tau \in (0,T)$ and $t \in [0,T]$ such that $s^\tau(t) < L$ we define

$$G(\tau,t) := \mathcal{G}(s^\tau(t), z^\tau(t)) = -\left[\frac{d}{d\sigma}\mathcal{E}(u_{i_\tau}^{\tau,\sigma}, u^\tau(t-\tau))\right]_{\sigma=s^\tau(t)}, \tag{2.29}$$

with $i_\tau := i_\tau(t)$ such that $i_\tau\tau \le t < (i_\tau+1)\tau$.

At this point we use the minimality properties of (s_i^τ, u_i^τ): it is

$$\mathcal{E}(u_i^\tau, u_{i-1}^\tau) + \mathfrak{c}s_i^\tau + \frac{\mathfrak{d}}{2\tau}(s_i^\tau - s_{i-1}^\tau) \le \mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^{\tau,\sigma}) + \mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)$$

for every $\sigma \in [s_{i-1}^\tau, L]$. If $s_i^\tau < L$, then for every $\sigma \in (s_i^\tau, L]$ we have

$$-\frac{\mathcal{E}(u_i^\tau, u_{i-1}^\tau) - \mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^{\tau,\sigma})}{\sigma - s_i^\tau} \le \mathfrak{c} + \frac{\mathfrak{d}}{2\tau}(\sigma + s_i^\tau - 2s_{i-1}^\tau);$$

if in addition $s_i^\tau > s_{i-1}^\tau$, then for every $\sigma \in [s_{i-1}^\tau, s_i^\tau)$ we also have

$$-\frac{\mathcal{E}(u_i^\tau, u_{i-1}^\tau) - \mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^{\tau,\sigma})}{\sigma - s_i^\tau} \ge \mathfrak{c} + \frac{\mathfrak{d}}{2\tau}(\sigma + s_i^\tau - 2s_{i-1}^\tau).$$

By the above inequalities and by the definition of G through the derivative (2.29), we obtain the following two conditions: for every $\tau \in (0,T)$ and every $1 \le i \le N_\tau$ such that $s_i^\tau < L$ it holds:

$$G(\tau, i\tau) \le \mathfrak{c} + \mathfrak{d}\left(\frac{s_i^\tau - s_{i-1}^\tau}{\tau}\right)$$
$$\left[-G(\tau,i\tau) + \mathfrak{c} + \mathfrak{d}\left(\frac{s_i^\tau - s_{i-1}^\tau}{\tau}\right)\right](s_i^\tau - s_{i-1}^\tau) = 0.$$

Before collecting (2.28) and the above result, we introduce the concept of *failure time*, important from now on.

Definition 2.3.4. Let $s : [0,T] \to [s_0, L]$ be a non-decreasing function with $s(0) = s_0$. The instant

$$T_f(s) := \sup\{t \in [0,T] :\ s(t) < L\}$$

is called *failure time* for s.

Thus in the previous analysis we have proved the following fact:

Proposition 2.3.5 (Discrete-time Griffith's criterion)**.** *For every $\tau \in (0,T)$ and every $t \in [0, T_f(\tilde{s}^\tau))$ the following conditions hold true:*

$$\dot{\tilde{s}}^\tau(t) \geq 0 \tag{2.30}$$

$$G(\tau,t) \leq \mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^\tau(t) \tag{2.31}$$

$$\left[-G(\tau,t) + \mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^\tau(t)\right]\dot{\tilde{s}}^\tau(t) = 0. \tag{2.32}$$

2.4. The irreversible viscoelastic evolution

The goal of the section is to describe the rate-dependent fracture problem with continuous-time variable. We investigate the behaviour of the sequence of discrete-time solutions (s^τ, u^τ) as the time-step τ decreases to 0.

Definition 2.4.1. For any $s_0 \in (0,L)$, $w \in C^2([0,T]; H^1(\Omega_{s_0}))$ and u_0 satisfying (2.2), an *irreversible viscoelastic evolution* is a couple

$$(s,u) : [0,T] \to [s_0, L] \times H^1(\Omega \setminus \Gamma)$$

such that $(s(0), u(0)) = (s_0, u_0)$, $s \in H^1(0,T)$ is non-decreasing and

(i) $u \in H^1(0,T; H^1(\Omega\setminus\Gamma))$ and $u(t) \in H^1(\Omega_{s(t)})$ for every $t \in [0,T]$;

(ii) $u(t) = w(t)$ on $\partial_D\Omega$ for every $t \in [0,T]$;

(iii) for a.e. $t \in (0,T)$, for every $\varphi \in H^1(\Omega_{s(t)})$ with $\varphi = 0$ on $\partial_D\Omega$,

$$\mathfrak{a}\langle\nabla u(t), \nabla\varphi\rangle + \mathfrak{b}\langle\nabla\dot{u}(t), \nabla\varphi\rangle = 0;$$

(iv) *Griffith's criterion*: for every $t \in [0, T_f(s))$ the following conditions hold true:

$$\dot{s}(t) \geq 0 \tag{2.33}$$

$$\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) \leq \mathfrak{c} + \mathfrak{d}\dot{s}(t) \tag{2.34}$$

$$[-\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) + \mathfrak{c} + \mathfrak{d}\dot{s}(t)]\,\dot{s}(t) = 0. \tag{2.35}$$

The requirements in the definition above can be rephrased in physical terms. The monotonicity of s means that the crack does not heal, while Condition 2.4.1.(i) affirmes that the jump set of the displacement $u(t)$ is contained in $\Gamma(s(t))$. Conditions 2.4.1.(i)-2.4.1.(iii) tell us that u is a weak solution to the problem

$$\begin{cases} \mathfrak{a}\Delta u(t) + \mathfrak{b}\Delta\dot{u}(t) = 0 & \text{in } \Omega_{s(t)} \\ \mathfrak{a}\frac{\partial u(t)}{\partial \mathbf{n}} + \mathfrak{b}\frac{\partial \dot{u}(t)}{\partial \mathbf{n}} = 0 & \text{on } \Gamma(s(t)) \cup \partial_N\Omega \\ u(t) = w(t) & \text{on } \partial_D\Omega \\ u(0) = u_0 & \\ s(0) = s_0. & \end{cases}$$

Roughly speaking, the uncracked domain $\Omega_{s(t)}$ behaves almost as an elastic body (except for the viscous term $\mathfrak{b}\Delta\dot{u}$). Finally, Condition 2.4.1.(iv) expresses a further relation between u and s, and provides an energetic criterion for the crack growth.

The main result of the section is the following existence theorem, which will be proven combining several lemmas.

Theorem 2.4.2. *For any $s_0 \in (0, L)$, $w \in C^2([0,T]; H^1(\Omega_{s_0}))$ and u_0 satisfying (2.2), there exists an irreversible viscoelastic evolution.*

Consider the discrete-time evolutions (s^τ, u^τ), for $\tau \in (0,T)$, obtained in Section 2.2. The estimates (2.12) and (2.13) assure the existence of a map $u \in H^1(0,T; H^1(\Omega \setminus \Gamma))$ such that

$$\tilde{u}^\tau \rightharpoonup u \tag{2.36}$$

weakly in $H^1(0,T,H^1(\Omega \setminus \Gamma))$ as $\tau \to 0^+$ along a suitable subsequence.

Remark 2.4.3. When we write $\tau \to 0$ we refer to the subsequence selected in (2.36), or to a further subsequence of it.

Concerning the crack tip evolution, by monotonicity of s^τ and Helly's Theorem 1.7.9, we find a further subsequence of $(s^\tau)_{\tau \in (0,T)}$ and a function $s : [0,T] \to [s_0, L]$ such that

$$s^\tau(t) \to s(t) \tag{2.37}$$

for every $t \in [0,T]$, as $\tau \to 0^+$. The function s is non-decreasing, since by pointwise convergence it inherites the monotonicity property of the functions s^τ.

Below we investigate how u and s are mutually related, since so far we do not have any information about the jump set of u. Furthermore, we obtain a regularity estimate for s, since by (2.21) and the fact that $\|s^\tau\|_\infty < L$ we expect it to belong to $H^1(0,T)$ as well.

Lemma 2.4.4. *There exists a subsequence of u^τ, not relabelled, such that $u^\tau(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega \setminus \Gamma)$ for every $t \in [0,T]$.*

Proof. The set

$$B_C := \left\{ v \in H^1(\Omega \setminus \Gamma) \,:\, \|v\|_{H^1(\Omega\setminus\Gamma)} \le C \right\}$$

is a compact subset of $L^2(\Omega)$. The estimate (2.10) implies that $\tilde{u}^\tau(t) \in B_C$ for every $t \in [0,t]$, while by (2.13) it is

$$\|\tilde{u}^\tau(t_1) - \tilde{u}^\tau(t_2)\| \le C(\mathfrak{b})\sqrt{|t_1 - t_2|}$$

for every $t_1, t_2 \in [0,T]$, where $C(\mathfrak{b})$ only depends on $\mathfrak{b}$.

By a refined version of the Ascoli-Arzelà theorem (see [7, Proposition 3.3.1]), there exists $\hat{u} : [0,T] \to B_C$ continuous such that, up to subsequences, for every $t \in [0,T]$

$$\tilde{u}^\tau(t) \to \hat{u}(t) \tag{2.38}$$

strongly in $L^2(\Omega \setminus \Gamma)$ when $\tau \to 0^+$; since (2.36) holds, $\hat{u}(t) = u(t)$ for a.e. t. In particular the equality is true for every $t \in [0,T]$, since we are considering the continuous representative of u in $H^1(0,T; H^1(\Omega \setminus \Gamma))$, and also $\hat{u}$ is continuous.

Fix $t \in [0,T]$. For every τ, set $0 \le i \le N_\tau$ such that $i\tau \le t < (i+1)\tau$. We have

$$\|\tilde{u}^\tau(t) - u^\tau(t)\|_{H^1(\Omega\setminus\Gamma)} = \left(\frac{t - i\tau}{\tau}\right) \|u^\tau_{i+1} - u^\tau_i\|_{H^1(\Omega\setminus\Gamma)} \le \|u^\tau_{i+1} - u^\tau_i\|_{H^1(\Omega\setminus\Gamma)}. \tag{2.39}$$

By the properties of the trace operator and the regularity of w, we obtain

$$\begin{aligned}\|tr(w^\tau_{i+1}-w^\tau_i)\|_{L^2(\partial_D\Omega)} \le& C\|w^\tau_{i+1}-w^\tau_i\|_{H^1(\Omega_{s_0})} = C\left\|\int_{i\tau}^{(i+1)\tau} \dot w(\xi)\,d\xi\right\|_{H^1(\Omega_{s_0})}\\ \le& \int_{i\tau}^{(i+1)\tau}\|\dot w(\xi)\|_{H^1(\Omega_{s_0})}\,d\xi \le CM\tau\end{aligned}$$

with $M := \max_{\xi\in[0,T]}\|\dot w(\xi)\|_{H^1(\Omega_{s_0})}$. This estimate, together with the Poincaré inequality and (2.11), implies

$$\begin{aligned}\|u^\tau_{i+1}-u^\tau_i\| \le& C\left(\|\nabla u^\tau_{i+1}-\nabla u^\tau_i\|+\|tr(u^\tau_{i+1}-u^\tau_i)\|\right)\\ =& C\left(\|\nabla u^\tau_{i+1}-\nabla u^\tau_i\|+\|tr(w^\tau_{i+1}-w^\tau_i)\|\right)\le C\tau,\end{aligned}$$

so that by (2.39) and (2.11) we deduce

$$\|\tilde u^\tau(t)-u^\tau(t)\|_{H^1(\Omega\setminus\Gamma)} \le \|u^\tau_{i+1}-u^\tau_i\|_{H^1(\Omega\setminus\Gamma)}\le C\tau$$

for the fixed t, with C dependent on $\mathfrak{b}$ but not on t. Therefore

$$\sup_{t\in[0,T]}\|\tilde u^\tau(t)-u^\tau(t)\|_{H^1(\Omega\setminus\Gamma)}\to 0 \tag{2.40}$$

as $\tau\to 0$. Since $u^\tau(t)\in B_C$, we conclude by means of (2.38) and (2.40). □

Lemma 2.4.5. *It results $u(t)\in H^1(\Omega_{s(t)})$ for every $t\in[0,T]$.*

Proof. Fix $t\in[0,T]$. If $s(t)=L$, then the claim is automatically satisfied since $u(t)\in H^1(\Omega\setminus\Gamma)$ for every t.

Let assume $s(t)<L$ and let $\alpha\in(0,L-s(t))$. By definition of s through (2.37) and continuity of γ, it is $\Gamma(s^\tau(t))\subset\Gamma(s(t)+\alpha)$ for τ sufficiently small. Since $u^\tau(t)\in H^1(\Omega_{s^\tau(t)})$ for every t, we have $[u^\tau(t)]=0$ on $\Gamma\setminus\Gamma(s(t)+\alpha)$ for τ small enough. By Lemma 2.4.4 and the compactness of the trace operator, up to a subsequence $u^\tau(t)\to u(t)$ $\mathcal{H}^1$-a.e. on Γ, so that $[u(t)]=0$ on $\Gamma\setminus\Gamma(s(t)+\alpha)$. Being α arbitrary, $[u(t)]=0$ on $\Gamma\setminus\Gamma(s(t))$, *i.e.* the thesis holds true. □

Lemma 2.4.6. *The sequence $(\tilde s^\tau)$ converges to s weakly in $H^1(0,T)$ and pointwise for every $t\in[0,T]$. Moreover, $\mathfrak{d}\|\dot s\|^2_{L^2(0,T)}\le C$.*

Proof. By the estimate (2.21), it is $\sup_{\tau\in(0,T)}\|s^\tau\|_{H^1(0,T)}<C(\mathfrak{d})$ for some constant $C(\mathfrak{d})$ dependent only on $\mathfrak{d}$. We deduce the existence of $\hat s\in H^1(0,T)$ such that (up to subsequences) $\tilde s^\tau\rightharpoonup\hat s$ weakly in $H^1(0,T)$. Let us show that $\hat s=s$.

Fix t and for every τ set $0\le i\le N_\tau$ such that $i\tau\le t<(i+1)\tau$. Then

$$\begin{aligned}0\le\tilde s^\tau(t)-s^\tau(t) =& \frac{t-i\tau}{\tau}\left(s^\tau_{i+1}-s^\tau_i\right)\le\tau\dot{\tilde s}^\tau(t)\\ =& \int_{i\tau}^{(i+1)\tau}\dot{\tilde s}^\tau(\xi)\,d\xi\le\tau^{1/2}\left(\int_{i\tau}^{(i+1)\tau}\left(\dot{\tilde s}^\tau(\xi)\right)^2\,d\xi\right)^{1/2}\le\tau^{1/2}C(\mathfrak{d}),\end{aligned}$$

where the last inequality is due to (2.21). Then, considering (2.37), $\tilde s^\tau(t)\to s(t)$ as $\tau\to 0$ for every t and necessarily $\hat s=s$, so that $s\in H^1(0,T)$.

Finally, the estimate for $\dot s$ is a consequence of the weak convergence $\tilde s^\tau\rightharpoonup s$ in $H^1(0,T)$ and (2.21). □

Note that by (2.37) we only deduce that s is monotone, while Lemma 2.4.6 provides the additional information that it is continuous as well: the crack really grows continuously, without the non-physical behaviour of jumps of the fracture set. The responsible of the regular growth is the dissipation at the crack tip.

At this point we would like to define a Griffith's criterion for the couple (s,u), exploiting the one for the discrete-time solutions obtained in Proposition 2.3.5.

Lemma 2.4.7. *It results $s \in C^1((0,T_f(s)) \cup (T_f(s),T))$ and (2.33)-(2.35) hold true for every $t \in [0,T_f(s))$.*

Proof. First of all, we show that $G(\tau,t)$ converges to $\mathcal{G}(s(t),\mathfrak{a}w(t)+\mathfrak{b}\dot{w}(t))$ for every $t \in [0,T_f(s))$ when τ vanishes. Fix $t \in [0,T_f(s))$; for τ small enough it is $s^\tau(t) < L$, so that it is meaningful to consider $G(\tau,t)$. Since $s^\tau(t) \to s(t)$ by (2.37) and $z^\tau(t) \to \mathfrak{a}w(t)+\mathfrak{b}\dot{w}(t)$ in $H^1(\Omega_{s_0})$, the continuity of $\mathcal{G}$ stated in Proposition 2.3.1 implies

$$G(\tau,t) = \mathcal{G}(s^\tau(t),z^\tau(t)) \to \mathcal{G}(s(t),\mathfrak{a}w(t)+\mathfrak{b}\dot{w}(t))$$

as $\tau \to 0$.

Next we show that (2.33)-(2.35) hold for a.e. $t \in [0,T_f(s))$. It is sufficient to prove that (at least for a subsequence of $\tilde{s}^\tau$) $\dot{\tilde{s}}^\tau(t) \to \dot{s}(t)$ for a.e. t and then to pass to the limit in (2.30)-(2.32).

Let $t \in [0,T_f(s))$ be such that $\dot{s}(t)$ exists and consider the sequence $\tilde{s}^\tau$ approximating s. By (2.30), only two situations are possible:

(i) $\dot{\tilde{s}}^\tau(t) > 0$ for any element of the sequence;
(ii) for a subsequence $\tilde{s}^{\tau_j}$ it is $\dot{\tilde{s}}^{\tau_j}(t) = 0$ for every j.

If (i) is the case, then (2.32) forces the equality $\mathfrak{d}\dot{\tilde{s}}^\tau(t) = G(\tau,t) - \mathfrak{c}$ to be satisfied. Since the right-hand side converges (by what proved at the beginning), we obtain that $\dot{\tilde{s}}^\tau(t) \to \vartheta(t) = \frac{1}{\mathfrak{d}}[\mathcal{G}(s(t),\mathfrak{a}w(t)+\mathfrak{b}\dot{w}(t)) - \mathfrak{c}]$.

If we assume (ii), then by (2.31) it is $G(\tau,t) \leq \mathfrak{c}$ and, as τ vanishes, we get

$$\mathcal{G}(s(t),\mathfrak{a}w(t)+\mathfrak{b}\dot{w}(t)) \leq \mathfrak{c}. \tag{2.41}$$

Call $\tilde{s}^{\tau_k}$ the elements in the (at most countable) set $\{\tilde{s}^\tau\} \setminus \{\tilde{s}^{\tau_j}\}$. If there are finitely many $\tilde{s}^{\tau_k}$, then $\lim \dot{\tilde{s}}^\tau(t) = \lim \dot{\tilde{s}}^{\tau_j} = 0$. If there are infinitely many $\tilde{s}^{\tau_k}$, let us show that $\dot{\tilde{s}}^{\tau_k} \to 0$. Repeating the same argument as for (i), $\dot{\tilde{s}}^{\tau_k} \to \vartheta(t) \geq 0$. Then

$$0 \leq \mathfrak{d}\vartheta(t) = \mathfrak{d} \lim_{\tau_k \to 0} \dot{\tilde{s}}^{\tau_k} = \lim_{\tau_k \to 0} G(\tau_k,t) - \mathfrak{c} = \mathcal{G}(s(t),\mathfrak{a}w(t)+\mathfrak{b}\dot{w}(t)) - \mathfrak{c} \leq 0\,,$$

where the last inequality is due to (2.41). Therefore, if (ii) is the case, then $\dot{\tilde{s}}^\tau(t) \to 0$.

The previous analysis shows that a function $\vartheta : [0,T] \to \mathbb{R}$ is defined such that $\dot{\tilde{s}}^\tau(t)$ converges to $\vartheta(t)$ as $\tau \to 0$, for every $t \in [0,T]$. Furthermore ϑ satisfies the following two relations at every $t \in [0,T]$:

$$\begin{aligned} &\mathcal{G}(s(t),\mathfrak{a}w(t)+\mathfrak{b}\dot{w}(t)) \leq \mathfrak{c} + \mathfrak{d}\vartheta(t) \\ &[-\mathcal{G}(s(t),\mathfrak{a}w(t)+\mathfrak{b}\dot{w}(t)) + \mathfrak{c} + \mathfrak{d}\vartheta(t)]\,\vartheta(t) = 0. \end{aligned}$$

In order to prove (2.34) and (2.35) a.e. in $[0,T]$, it is enough to consider the above relations and to observe that, since $s^\tau \rightharpoonup s$ weakly in $H^1(0,T)$, necessarily it has to be $\dot{s}(t) = \vartheta(t)$ for a.e. $t \in [0,T]$. Instead (2.33) is true a.e. in $[0,T]$ by monotonicity of s.

In order to conclude the proof, observe that (2.34) and (2.35) imply that s solves a.e. in $[0,T_f(s))$ the differential relation

$$\mathfrak{d}\dot{s}(t) = [\mathcal{G}(s(t),\mathfrak{a}w(t)+\mathfrak{b}\dot{w}(t)) - \mathfrak{c}]^+\,. \tag{2.42}$$

Since s is continuous in $[0,T]$ and $w \in C^2([0,T];H^1(\Omega_{s_0}))$, the right-hand side in (2.42) is continuous. Then, being s absolutely continuous, it is

$$s(t) = s_0 + \int_0^t \dot{s}(\xi)d\xi = s_0 + \int_0^t \frac{1}{\mathfrak{d}} \left[\mathcal{G}(s(\xi), \mathfrak{a}w(\xi) + \mathfrak{b}\dot{w}(\xi)) - \mathfrak{c}\right]^+ d\xi$$

for every $t \in [0, T_f(s))$. The right-hand side a C^1 function, thus we conclude that $s \in C^1([0, T_f(s)))$ as well; hence (2.33)-(2.35) are satisfied everywhere in $[0, T_f(s))$. Finally, if $T_f(s) < T$, then we have $s \equiv L$ in $[T_f(s), T]$, so that $s \in C^1(T_f(s), L)$. □

Collecting together the above lemmas, we deduce the main result of the section.

Proof of Theorem 2.4.2. Consider u and s obtained by means of the compactness arguments in (2.36) and (2.37), respectively, as limits of u^τ and s^τ as τ decreases to 0.

By construction, we have $u^\tau(0) = u_0$ and $s^\tau(0) = s_0$. Lemma 2.4.4 implies that $u^\tau(0) \rightharpoonup u(0)$ in $H^1(\Omega \setminus \Gamma)$, so that $u(0) = u_0$; concerning s, we obtain $s(0) = s_0$ by the pointwise convergence (2.37). Hence the initial conditions are satisfied.

Lemma 2.4.6 assures the regularity for s, which by construction through Helly's Theorem is non-decreasing since the functions s^τ are.

Condition 2.4.1.(i) is satisfied considering (2.36) and Lemma 2.4.5.

Fix $t \in [0,T]$. It is $u^\tau(t) = w_i^\tau$ on $\partial_D\Omega$, where $i\tau \le t < (i+1)\tau$ for every τ. Combining together Lemma 2.4.4, the compactness of the trace operator and the fact that $w_i^\tau \to w(t)$ strongly in $H^1(\Omega_{s_0})$, we obtain that $u(t) = w(t)$ on $\partial_D\Omega$, *i.e.* Condition 2.4.1.(ii) is verified.

Griffith's criterion 2.4.1.(iv) is established in Lemma 2.4.7.

We are left to prove Condition 2.4.1.(iii). Let $t \in (0,T)$ be a Lebesgue point for $\dot{u}$ and $\varphi \in H^1(\Omega_{s(t)})$ with $\varphi = 0$ on $\partial_D\Omega$. If $s(t) < L$, consider the flow η_θ described in (2.3), with $\theta > 0$, and define $\varphi_\theta(\cdot) := \varphi(\eta_\theta(\cdot))$. If $s(t) = L$, we assume $\varphi_\theta \equiv \varphi$. By the properties of η_θ, $\varphi_\theta \in H^1(\Omega_{s(t)-\theta})$ and $\varphi_\theta = 0$ on $\partial_D\Omega$. By the pointwise convergence in (2.37), for τ sufficiently small it is $s^\tau(t) > s(t) - \theta$; therefore $\varphi_\theta \in H^1(\Omega_{s^\tau(t)})$ and, since s^τ are monotone, we get $\varphi_\theta \in H^1(\Omega_{s^\tau(\xi)})$ for every $\xi \ge t$.

Fix $\delta \in (0, T-t)$. For any $\xi \in [t,T]$ the equality (2.8) holds with φ_θ, and integrating it over $[t, t+\delta]$ we get

$$\int_t^{t+\delta} \left(\mathfrak{a}\langle \nabla u^\tau(\xi), \nabla\varphi_\theta\rangle + \mathfrak{b}\langle \nabla \dot{\tilde{u}}^\tau(\xi-\tau), \nabla\varphi_\theta\rangle\right) d\xi = 0. \tag{2.43}$$

Lemma 2.4.4 assures that

$$\langle \nabla u^\tau(\xi), \nabla\varphi_\theta\rangle \to \langle \nabla u(\xi), \nabla\varphi_\theta\rangle$$

for every $\xi \in [t, t+\delta]$, while considering (2.9) we deduce the estimate

$$|\langle \nabla u^\tau(\xi), \nabla\varphi_\theta\rangle| \le \|\nabla u^\tau(\xi)\| \|\varphi_\theta\| \le C\|\varphi_\theta\|.$$

Then, by the Dominated Convergence Theorem,

$$\int_t^{t+\delta} \langle \nabla u^\tau(\xi), \nabla\varphi_\theta\rangle \, d\xi \to \int_t^{t+\delta} \langle \nabla u(\xi), \nabla\varphi_\theta\rangle \, d\xi.$$

Concerning the other term in (2.43), by (2.36)

$$\int_t^{t+\delta} \langle \nabla \dot{\tilde{u}}^\tau(\xi-\tau), \nabla\varphi_\theta\rangle \, d\xi \to \int_t^{t+\delta} \langle \nabla \dot{u}(\xi), \nabla\varphi_\theta\rangle \, d\xi.$$

Collecting together the two limits above and (2.43), it is

$$\frac{1}{\delta}\int_t^{t+\delta} \left(\mathfrak{a}\langle \nabla u(\xi), \nabla\varphi_\theta\rangle + \mathfrak{b}\langle \nabla \dot{u}(\xi), \nabla\varphi_\theta\rangle\right) d\xi = 0.$$

Since $\varphi_\theta \rightharpoonup \varphi$ in $H^1(\Omega \setminus \Gamma)$ and t is a Lebesgue point for $\dot{u}$, we obtain Condition 2.4.1.(iii) by considering the limits as $\theta \to 0^+$ and $\delta \to 0^+$, in this order. □

Remark 2.4.8. Consider the inequality in Lemma 2.2.3 with $i = 0$:

$$\begin{aligned}\frac{1}{2}\mathfrak{a}\|\nabla u^\tau(t)\|^2 + \frac{\mathfrak{b}}{2}\int_0^t \|\nabla \dot{\tilde{u}}^\tau(\xi)\|^2\, d\xi + \mathfrak{a}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}s^\tau(t) + \frac{\mathfrak{d}}{2}\int_0^t \|\nabla \dot{\tilde{s}}^\tau(\xi)\|^2\, d\xi\right)\\ \le \frac{1}{2}\mathfrak{a}\|\nabla u_0\|^2 + \mathfrak{a}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s^\tau(t_1) + \mathfrak{a}\int_0^t \langle \nabla u^\tau(\xi), \nabla \dot{w}(\xi)\rangle\, d\xi\\ + \frac{\mathfrak{b}}{2}\int_0^t \|\nabla \dot{w}(\xi)\|^2\, d\xi + \rho.\end{aligned}$$

As τ vanishes, the value $\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1}$ vanishes as well, while all the other terms converge, so that we find the inequality:

$$\frac{1}{2}\mathfrak{a}\|\nabla u(t)\|^2 + \frac{\mathfrak{b}}{2}\int_0^t \|\nabla \dot{u}(\xi)\|^2\, d\xi \le \frac{1}{2}\mathfrak{a}\|\nabla u_0\|^2 + \mathfrak{a}\int_0^t \langle \nabla u(\xi), \nabla \dot{w}(\xi)\rangle\, d\xi + \frac{\mathfrak{b}}{2}\int_0^t \|\nabla \dot{w}(\xi)\|^2\, d\xi.$$

In the energy balance above there is no longer trace of the crack energy. Without giving an interpretation at this stage, we underline the analogy of this fact with what proved in [35] in the damped case. We only point out that the absence of the fracture term is probably related to the presence of the viscoelastic term, as the analysis in Section 2.5 seems to suggest.

We conclude the section with some estimates on the irreversible viscoelastic evolution.

Lemma 2.4.9. *Let (s, u) be given by Theorem 2.4.2. Then there exists a constant $C > 0$, independent of $\mathfrak{b}, \mathfrak{d} > 0$ (fixed at the beginning) and t, such that for every $t \in [0, T]$ the following estimates hold:*

$$\|u(t)\|_{H^1(\Omega\setminus\Gamma)} \le C \tag{2.44}$$

$$\|u\|_{L^2(0,T;H^1(\Omega\setminus\Gamma))} \le C \tag{2.45}$$

$$\mathfrak{b}\|\nabla \dot{u}\|^2_{L^2(0,T;L^2(\Omega\setminus\Gamma;\mathbb{R}^2))} \le C \tag{2.46}$$

$$\mathfrak{d}\|\dot{s}\|^2_{L^2(0,T)} \le C\,. \tag{2.47}$$

The proof is a straightforward consequence of Lemma 2.2.3 and (2.36) for what concerns u, and of Lemmas 2.2.5 and 2.4.6 for s.

2.5. A comment on the role of $\tau_{\mathfrak{b}}$

We make clear the role of the parameter

$$\mathfrak{a}\tau_{\mathfrak{b}} = \mathfrak{a}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1}$$

introduced in Section 2.2, in order to justify its presence in front of the fracture energy. We do not prove again every statement, since generally the proofs are similar to those in Section 2.2.

Consider the following discrete-time evolution: for every $\tau \in (0, T)$, let u_0^τ, s_0^τ and $u_i^{\tau,\sigma}$ be defined as in (2.6); for the crack tip s_i^τ, instead of (2.7) we choose

$$s_i^\tau \in \operatorname{argmin}\left\{E(u_i^{\tau,\sigma}, u_{i-1}^\tau) + \mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)^2 :\ s_{i-1}^\tau \le \sigma \le L\right\}. \tag{2.48}$$

As before, set $u_i^\tau := u_i^{\tau, s_i^\tau}$ and define the interpolant functions $u^\tau, \tilde{u}^\tau, s^\tau, \tilde{s}^\tau$.

Arguing as in Lemma 2.2.3, for every $0 \le i < j \le N_\tau$

$$\begin{aligned}
&\frac{1}{2}\mathfrak{a}\|\nabla u_j^\tau\|^2 + \frac{\mathfrak{b}}{2\tau}\sum_{h=i}^{j-1}\|\nabla u_{h+1}^\tau - \nabla u_h^\tau\|^2 + \mathfrak{c}s_j^\tau + \frac{\mathfrak{d}}{2\tau}\sum_{h=i}^{j-1}|s_{h+1}^\tau - s_h^\tau|^2 \\
&\le \frac{1}{2}\mathfrak{a}\|\nabla u_i^\tau\|^2 + \mathfrak{c}s_i^\tau + \mathfrak{a}\int_{i\tau}^{j\tau}\langle \nabla u^\tau(\xi), \nabla \dot{w}(\xi)\rangle\, d\xi \\
&+ \frac{\mathfrak{b}}{2}\int_{i\tau}^{j\tau}\|\nabla \dot{w}(\xi)\|^2\, d\xi + \rho.
\end{aligned}$$

Then Lemma 2.2.4 holds true, since in its proof we do not take into account the fracture term. The main difference is that in the current situation the inequality above provides an L^2 estimate for $\dot{\tilde{s}}^\tau$ too:

$$\mathfrak{d}\|\dot{\tilde{s}}^\tau\|_{L^2(0,T)} \le C$$

for any $\tau \in (0,T)$.

Following the steps of Sections 2.3 and 2.4, by the Helly's Theorem 1.7.9 there exists $s : [0,T] \to [s_0, L]$ pointwise limit of a subsequence of the family $\{s^\tau\}_{\tau\in(0,T)}$, and it satisfies

$$\tilde{s}^\tau \rightharpoonup s$$

weakly in $H^1(0,T)$ as $\tau \to 0^+$, as in Lemma 2.4.6.

In the current framework, the Griffith's criterion equivalent to (2.30)-(2.32) is

$$\begin{aligned}
&\dot{\tilde{s}}^\tau(t) \ge 0 \\
&G(\tau,t) \le \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)\left(\mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^\tau(t)\right) \\
&\left[-G(\tau,t) + \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)\left(\mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^\tau(t)\right)\right]\dot{\tilde{s}}^\tau(t) = 0.
\end{aligned} \tag{2.49}$$

Fix any $t \in [0, T_f(s))$. Since $s^\tau(t) \to s(t) < L$, we can assume $t \in [0, T_f(s^\tau))$ for τ sufficiently small, so that it makes sense to speak of $G(\tau,t)$ for those τ.

Assume that s is not constant in $[0,T]$. Since $s \in H^1(0,T)$, there exists $t \in (0,T)$ such that $\dot{s}(t)$ exists and $\dot{s}(t) > 0$. Hence we can find two sequences $t_j^1 < t < t_j^2$ converging to t with $s(t_j^1) < s(t_j^2)$. By construction of s, there exists τ_j converging to 0 with $s^{\tau_j}(t_j^1) < s^{\tau_j}(t_j^2)$ for every j, so that $\dot{\tilde{s}}^{\tau_j}(t_j) > 0$ for some $t_j \in (t_j^1, t_j^2)$. By construction, $t_j \to t$, while Lemma 1.7.10 implies $s^{\tau_j}(t_j) \to s(t)$. Therefore, by continuity of $\mathcal{G}$, it is

$$G(\tau_j, t_j) \to \mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)).$$

Being $\dot{\tilde{s}}^{\tau_j}(t_j) > 0$, equality (2.49) gives

$$G(\tau_j, t_j) = \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau_j}\right)\left(\mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^{\tau_j}(t_j)\right) > \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau_j}\right)\mathfrak{c}.$$

As $\tau_j \to 0$ the two relations above imply

$$\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) = \lim_{j\to+\infty} G(\tau_j, t_j) = +\infty,$$

which is impossible. We have to conclude that s is necessarily constant. In particular $s \equiv s_0$ and, being continuous in $[0,T]$, $T_f(s) = T$.

The above argument shows that, if we consider (2.48) instead of (2.7), then a real crack evolution does not take place since the crack tip stays still, independently of the boundary loading.

2.6. The rate-independent evolution

In the previous sections we never made explicit the dependence of the discrete-time evolutions and of the irreversible viscoelastic evolutions on the parameters $\mathfrak{b}$ and $\mathfrak{d}$. Now we replace $\mathfrak{b}$ and $\mathfrak{d}$ in the previous analysis by $\varepsilon\mathfrak{b}$ and $\nu\mathfrak{d}$, for positive adimensional parameters ε and ν. As discussed in general at the end of Section 1.6, we are interested in investigating the behaviour of the fracture term and of the viscoelastic term as the viscosity coefficient ε and the dissipation coefficient ν vanish.

Unlike for the irreversible viscoelastic evolution, where the fracture has a continuous growth, when ν "disappears" the crack might perform instantaneous increments, even though the boundary loading varies smoothly in time. Despite this fact, we can recover a weaker Griffith's criterion describing the process. We interpret the sudden changes of the fracture as a limit behaviour of fast moving "dissipated" cracks.

From now on, for any $\varepsilon \geq 0$ and $\nu > 0$ we use the notation $(s^{\varepsilon,\nu,\tau}, u^{\varepsilon,\nu,\tau})$ for the discrete-time evolutions defined in Section 2.2, and $(s^{\varepsilon,\nu}, u^{\varepsilon,\nu})$ for the irreversible viscoelastic evolutions obtained in Section 2.4 as limits of $(s^{\varepsilon,\nu,\tau}, u^{\varepsilon,\nu,\tau})$ when $\tau \to 0$.

The main result of the section is Theorem 2.6.5, which states the existence of a particular class of rate-independent evolutions, defined below.

Definition 2.6.1. Let $\sigma : [0,T] \to [s_0, L]$. We say that $t \in [0,T]$ is a *non-constancy instant* for σ if for every neighbourhood U of t there exist $t_1, t_2 \in [0,T] \cap U$ such that $\sigma(t_1) \neq \sigma(t_2)$. We say that $t \in [0,T]$ is a *jump instant* for σ if $\sigma(t-) \neq \sigma(t+)$.

Definition 2.6.2. Let $s_0 \in (0,L)$, $w \in C^2([0,T]; H^1(\Omega_{s_0}))$ and u_0 satisfy (2.2). We call *rate-independent evolution* with initial condition (s_0, u_0) and boundary loading w, a map

$$(s,u) : [0,T] \to [s_0, L] \times H^1(\Omega \setminus \Gamma)$$

such that $(s(0), u(0)) = (s_0, u_0)$, s is left-continuous and the following conditions hold true:

(i) $u \in L^2(0,T; H^1(\Omega \setminus \Gamma))$ and $u(t) \in H^1(\Omega_{s(t)})$ for a.e. $t \in [0,T]$;
(ii) $u(t) = w(t)$ on $\partial_D\Omega$ for a.e. $t \in [0,T]$;
(iii) for a.e. $t \in (0,T)$, for every $\varphi \in H^1(\Omega_{s(t)})$ with $\varphi = 0$ on $\partial_D\Omega$,

$$\mathfrak{a}\langle \nabla u(t), \nabla\varphi \rangle = 0;$$

(iv) *Griffith's criterion*:
- s is non-decreasing;
- for every $t \in [0, T_f(s))$

$$\mathcal{G}(s(t), \mathfrak{a}w(t)) \leq \mathfrak{c}; \tag{2.50}$$

- *weak activation criterion*: if $t \in [0, T_f(s))$ is a non-constancy instant for s, then

$$\mathcal{G}(s(t\pm), \mathfrak{a}w(t)) = \mathfrak{c}; \tag{2.51}$$

  if $t \in [0, T_f(s))$ is a jump instant for s, then

$$\mathcal{G}(\sigma, \mathfrak{a}w(t)) \geq \mathfrak{c} \tag{2.52}$$

  for every $\sigma \in [s(t-), s(t+)]$;
- if $t \in [0, T_f(s))$ and $\mathcal{G}(s(t), \mathfrak{a}w(t)) < \mathfrak{c}$, then s is differentiable at t and $\dot{s}(t) = 0$;

(v) the function $t \mapsto \mathcal{G}(s(t), \mathfrak{a}w(t))$ is continuous in $[0, T_f(s)]$.

As discussed in Remark 1.3.1.(ii) (see Definition 1.3.2), the weak activation criterion has been suggested in [72] in order to relax the differential formulation (1.9) of Griffith's criterion. We stress the fact that it is important to have a criterion valid at every instant in $[0,T]$. Indeed a differential criterion valid only on $[0,T]\backslash\mathcal{N}$, with $\mathcal{L}^1(\mathcal{N})=0$, might make it totally meaningless, since the jump points of s are at most countable and so they might concentrate on $\mathcal{N}$.

Theorem 2.6.3. *For any $s_0\in(0,L)$, $w\in C^2([0,T];H^1(\Omega_{s_0}))$ and u_0 satisfying (2.2), there exists a rate-independent evolution (s,u).*

Theorem 2.6.3 is consequence of the result that we will state and prove below. We first introduce another class of evolutions that turn out to be rate-independent evolutions.

Definition 2.6.4. Let $s_0\in(0,L)$, $w\in C^2([0,T];H^1(\Omega_{s_0}))$ and u_0 satisfy (2.2). We call *vanishing viscosity evolution* with initial condition (s_0,u_0) and boundary loading w, a map

$$(s,u):[0,T]\to[s_0,L]\times H^1(\Omega\setminus\Gamma)$$

for which there exists a sequence $(s^{\varepsilon,\nu},u^{\varepsilon,\nu})_{\varepsilon,\nu}$ of irreversible viscoelastic evolutions satisfying the same initial and boundary data, and such that

$$u^{\varepsilon,\nu}\rightharpoonup u$$

weakly in $L^2(0,T;H^1(\Omega\setminus\Gamma))$ and

$$s^{\varepsilon,\nu}(t)\to s(t)$$

for every $t\in[0,T]$ as $\varepsilon\to0$ and $\nu\to0$.

Theorem 2.6.5. *For any $s_0\in(0,L)$, $w\in C^2([0,T];H^1(\Omega_{s_0}))$ and u_0 satisfying (2.2), there exists a vanishing viscosity evolution (s,u).*

Furthermore, any vanishing viscosity evolution is a rate-independent evolution.

Remark 2.6.6. It is clear that Theorem 2.6.3 is proved as soon as Theorem 2.6.5 is. The last is achieved combining together a number of lemmas.

We will always write $\varepsilon\to0$ even in case $\varepsilon=0$. In this situation, it is understood that we are considering the constant null sequence.

We start identifying a couple (s,u) candidate to satisfy Definition 2.6.4; similarly to Section 2.4, we use a compactness argument. For every $\nu>0$ and $\varepsilon\geq0$, consider an irreversible viscoelastic evolution whose existence is assured by Theorem 2.4.2. By the estimates in Lemma 2.4.9, the sequence $(u^{\varepsilon,\nu})_{\varepsilon\geq0,\nu>0}$ is uniformly bounded in $L^2(0,T;H^1(\Omega\setminus\Gamma))$. Therefore there exists $u\in L^2(0,T;H^1(\Omega\setminus\Gamma))$ such that

$$u^{\varepsilon,\nu}\rightharpoonup u \tag{2.53}$$

weakly in $L^2(0,T;H^1(\Omega\setminus\Gamma))$ as $\varepsilon\to0$ and $\nu\to0$ along suitable sequences.

Concerning the crack tip, Theorem 2.4.2 assures that the functions $s^{\varepsilon,\nu}$ are monotone non-decreasing. Applying Helly's Theorem 1.7.9, there exist a further subsequence of the indices ε,ν found in (2.53), and a function $s\in BV([0,T])$, such that

$$s^{\varepsilon,\nu}(t)\to s(t) \tag{2.54}$$

for every $t\in[0,T]$. The function s is non-decreasing since the functions $s^{\varepsilon,\nu}$ are, and by pointwise convergence $s(t)\in[s_0,L]$ for every $t\in[0,T]$. We can describe more in detail the convergence:

Lemma 2.6.7. *The sequence $(s^{\varepsilon,\nu})$ is monotonically non-increasing with respect to ν,* i.e. *$s^{\varepsilon,\nu_1}(t)\geq s^{\varepsilon,\nu_2}(t)$ for every $t\in[0,T]$ if $0<\nu_1<\nu_2$.*

As a consequence, s is left-continuous.

PROOF. Being $s \in C^1((0,T_f(s) \cup (T_f(s),T))$, equality (2.42) holds true for every $t \in [0,T]$ and not only a.e.; thereby $s^{\varepsilon,\nu}$ solves the Cauchy problem

$$\begin{cases} \dot{s}^{\varepsilon,\nu}(t) = \dfrac{1}{\nu\mathfrak{d}}\,[\mathcal{G}(s^{\varepsilon,\nu}(t),\mathfrak{a}w(t)+\varepsilon\mathfrak{b}\dot{w}(t)) - \mathfrak{c}]^+ \\ s^{\varepsilon,\nu}(0) = s_0. \end{cases}$$

If $\nu_1 < \nu_2$, then s^{ε,ν_1} verifies the differential inequality

$$\dot{s}^{\varepsilon,\nu_1}(t) = \frac{1}{\nu_1\mathfrak{d}}\,[\mathcal{G}(s^{\varepsilon,\nu_1}(t),\mathfrak{a}w(t)+\varepsilon\mathfrak{b}\dot{w}(t)) - \mathfrak{c}]^+ \geq \frac{1}{\nu_2\mathfrak{d}}\,[\mathcal{G}(s^{\varepsilon,\nu_1}(t),\mathfrak{a}w(t)+\varepsilon\mathfrak{b}\dot{w}(t)) - \mathfrak{c}]^+ .$$

By comparison results for differential equations (see [83, Theorem X.8]), we obtain $s^{\varepsilon,\nu_1}(t) \geq s^{\varepsilon,\nu_2}(t)$.

The first part implies that $s(t) \geq s^{\varepsilon,\nu}(t)$ for every $t \in [0,T]$ and every $\varepsilon \geq 0, \nu > 0$. Assume $s(t) - s(t-) > \alpha$ for some $t \in (0,T]$ and $\alpha > 0$; then $s(t) - s(\tau) > \alpha$ for $\tau < t$. For any ε and ν sufficiently small, $s(t) - s^{\varepsilon,\nu}(t) < \frac{\alpha}{2}$, so that $s^{\varepsilon,\nu}(t) - s^{\varepsilon,\nu}(\tau) \geq \frac{\alpha}{2}$ for any $\tau < t$, in contradiction to the continuity of $s^{\varepsilon,\nu}$. □

Lemma 2.6.8. *Up to subsequences, for a.e. $t \in (0,T)$ it is $\nu\dot{s}^{\varepsilon,\nu}(t) \to 0$ as $\nu \to 0$.*

PROOF. Lemma 2.4.6 for the functions $s^{\varepsilon,\nu}$ reads as $\nu\mathfrak{d}\|\dot{s}^{\varepsilon,\nu}\|^2_{L^2(0,T)} \leq C$ with C independent of ν, so that $\nu\dot{s}^{\varepsilon,\nu} \to 0$ strongly in $L^2(0,T)$ as $\nu \to 0$. Then, up to a subsequence, $\nu\dot{s}^{\varepsilon,\nu}(t) \to 0$ for a.e. $t \in (0,T)$. □

Lemma 2.6.9. *Let $t \in (0,T)$ be a jump instant for s. Then there exist subsequences (not relabelled) $\varepsilon,\nu \to 0$ and $t^{\varepsilon,\nu} \in (0,T)$ such that*

(1) $t^{\varepsilon,\nu} \to t$;
(2) $s^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to s(t-)$;
(3) $\mathcal{G}(s^{\varepsilon,\nu}(t^{\varepsilon,\nu}),\mathfrak{a}w(t^{\varepsilon,\nu})+\varepsilon\mathfrak{b}\dot{w}(t^{\varepsilon,\nu})) = \mathfrak{c} + \nu\mathfrak{d}\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu})$.

Similarly, there exists $\hat{t}^{\varepsilon,\nu} \in (0,T)$ such that (1),(3) *and*

(2') $s^{\varepsilon,\nu}(\hat{t}^{\varepsilon,\nu}) \to s(t+)$

are satisfied.

PROOF. Let us discuss only the case $s(t-)$; for the other, $s(t+)$, it is sufficient to argue analogously.

We initially consider the case $\varepsilon > 0$.

Claim: for every $m \in \mathbb{N}$ there exists $\varepsilon(m), \nu(m) > 0$ such that for every $0 < \varepsilon \leq \varepsilon(m), 0 < \nu \leq \nu(m)$ there exists $t_m^{\varepsilon,\nu}$ satisfying

(i) $|t_m^{\varepsilon,\nu} - t| < \frac{1}{m}$;
(ii) $|s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) - s(t-)| < \frac{1}{m}$;
(iii) $\mathcal{G}(s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}),\mathfrak{a}w(t_m^{\varepsilon,\nu})+\varepsilon\mathfrak{b}\dot{w}(t_m^{\varepsilon,\nu})) = \mathfrak{c} + \nu\mathfrak{d}\dot{s}^{\varepsilon,\nu}(t_m^{\varepsilon,\nu})$.

If the claim holds true, then the lemma is proved. Indeed, without loss of generality we can assume $\varepsilon(m+1) < \varepsilon(m), \nu(m+1) < \nu(m)$. If we set

$$t^{\varepsilon,\nu} := t_m^{\varepsilon,\nu} \iff \varepsilon(m+1) < \varepsilon \leq \varepsilon(m), \nu(m+1) < \nu \leq \nu(m)$$

then (1), (2), (3) are consequence of (i),(ii),(iii), respectively.

Proof of the claim. Fix $m \in \mathbb{N}$ such that $\frac{1}{m} < T - t$. There exists $\alpha \in (0, \frac{1}{m})$ such that $|s(t-) - s(\tau)| < \frac{1}{3m}$ for every $t - \alpha < \tau < t$. Fixed $\hat{t} \in (t - \frac{\alpha}{2}, t)$, there exist strictly positive constants $\varepsilon_0(m), \nu_0(m)$ such that

$$|s^{\varepsilon,\nu}(\hat{t}) - s(\hat{t})| < \frac{1}{3m}$$

for every $\varepsilon \le \varepsilon_0(m), \nu \le \nu_0(m)$. Define

$$\hat{t}_m^{\varepsilon,\nu} := \sup\{\xi \ge \hat{t}:\ s^{\varepsilon,\nu}(\xi) = s^{\varepsilon,\nu}(\hat{t})\}.$$

It is $\hat{t}_m^{\varepsilon,\nu} \ge \hat{t} > \hat{t} - \frac{\alpha}{2} > t - \frac{1}{m}$.

By contradiction, assume that there exists a subsequence $(t_j)_j$ of $(\hat{t}_m^{\varepsilon,\nu})$ such that $t_j \ge t + \frac{1}{m}$. Then $s^{\varepsilon_j,\nu_j}(\xi) = s^{\varepsilon_j,\nu_j}(\hat{t})$ for every $\xi \in [\hat{t}, t+\frac{1}{m})$; in particular, $s^{\varepsilon_j,\nu_j}(t+\frac{1}{2m}) = s^{\varepsilon_j,\nu_j}(\hat{t})$. Taking the limit as $\varepsilon_j, \nu_j \to 0$, we obtain

$$s(t+) \le s\left(t + \frac{1}{2m}\right) = s(\hat{t}) \le s(t-) < s(t+),$$

which is a contradiction. Hence there exists $0 < \varepsilon(m) \le \varepsilon_0(m), 0 < \nu(m) \le \nu_0(m)$ such that

$$t - \frac{1}{m} < \hat{t}_m^{\varepsilon,\nu} < t + \frac{1}{m} \tag{2.55}$$

for every $\varepsilon \le \varepsilon(m), \nu \le \nu(m)$.

By definition of $\hat{t}_m^{\varepsilon,\nu}$, (2.55) and continuity of $s^{\varepsilon,\nu}$, for every $\varepsilon \le \varepsilon(m), \nu \le \nu(m)$ there exists $\beta_{\varepsilon,\nu} > 0$ such that

$$\hat{t}_m^{\varepsilon,\nu} + \beta_{\varepsilon,\nu} < t + \frac{1}{m} \quad \text{and} \quad 0 < s^{\varepsilon,\nu}(\hat{t}_m^{\varepsilon,\nu} + \beta_{\varepsilon,\nu}) - s^{\varepsilon,\nu}(\hat{t}_m^{\varepsilon,\nu}) < \frac{1}{3m}.$$

Being $s^{\varepsilon,\nu} \in C^1((0, T_f(s^{\varepsilon,\nu})) \cup (T_f(s^{\varepsilon,\nu}), T))$, necessarily it holds $\dot{s}^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) > 0$ for some $t_m^{\varepsilon,\nu} \in (\hat{t}_m^{\varepsilon,\nu}, \hat{t}_m^{\varepsilon,\nu} + \beta_{\varepsilon,\nu})$.

By choice of $t_m^{\varepsilon,\nu}$, (i) is satisfied.

By continuity of $s^{\varepsilon,\nu}$, it is $s^{\varepsilon,\nu}(\hat{t}_m^{\varepsilon,\nu}) = s^{\varepsilon,\nu}(\hat{t})$ and we have the chain of inequalities

$$\begin{aligned}|s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) - s(t-)| \le &|s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) - s(\hat{t})| + |s(\hat{t}) - s(t-)| \\ \le &s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) - s^{\varepsilon,\nu}(\hat{t}_m^{\varepsilon,\nu}) + |s^{\varepsilon,\nu}(\hat{t}) - s(\hat{t})| + \frac{1}{3m} \\ \le &\frac{1}{3m} + \frac{1}{3m} + \frac{1}{3m} = \frac{1}{m}\end{aligned}$$

and (ii) is achieved.

Finally, since $\dot{s}^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) > 0$, (iii) is a consequence of (2.35).

In case $\varepsilon = 0$, the previous proof holds true by setting $\varepsilon(m) = 0$ for every m and $t^{0,\nu} := t_m^{0,\nu}$ if and only if $\nu(m+1) < \nu \le \nu(m)$. □

Lemma 2.6.10. *For every $t \in [0, T_f(s))$ it is $\mathcal{G}(s(t), \mathfrak{a}w(t)) \le \mathfrak{c}$. If $t \in (0, T_f(s))$ is a non-constancy instant for s, then $\mathcal{G}(s(t\pm), \mathfrak{a}w(t)) = \mathfrak{c}$.*

Proof. Without loss of generality, when $t \in [0, T_f(s))$ is fixed we can assume that $t \in [0, T_f(s^{\varepsilon,\nu}))$ for ε, ν small enough, since $s^{\varepsilon,\nu}(t) \to s(t) < L$.

As already noticed in the proof of Lemma 2.6.7, conditions (2.33)-(2.35) imply that the irreversible viscoelastic evolutions are solutions to the ordinary differential equation

$$\nu\mathfrak{d}\dot{s}^{\varepsilon,\nu}(t) = [\mathcal{G}(s^{\varepsilon,\nu}(t), \mathfrak{a}w(t) + \varepsilon\mathfrak{b}\dot{w}(t)) - \mathfrak{c}]^+ \tag{2.56}$$

for $t \in [0, T_f(s^{\varepsilon,\nu}))$. Fix $t \in [0, T_f(s))$ such that $\nu\dot{s}^{\varepsilon,\nu}(t) \to 0$ when ε and ν vanish. Considering the pointwise convergence (2.54) and the continuity properties of w (by assumption) and $\mathcal{G}$ (see Proposition 2.3.1), from (2.56) we obtain

$$0 = [\mathcal{G}(s(t), \mathfrak{a}w(t)) - \mathfrak{c}]^+ . \tag{2.57}$$

Lemma 2.6.8 implies that (2.57) holds for a.e. $t \in [0, T_f(s))$; by left continuity of s, the equality is verified everywhere in $[0, T_f(s))$. Finally, (2.57) is equivalent to (2.50), and the first part of the statement is proved.

We point out that the inequality (2.50) and the continuity of $\mathcal{G}$ imply that

$$\mathcal{G}(s(t+), \mathfrak{a}w(t)) \le \mathfrak{c}$$

for every $t \in [0, T_f(s))$.

Let now $t \in (0, T_f(s))$ be a non-constancy instant for s. Assume first that t is a jump instant for s and consider the sequence $t^{\varepsilon,\nu}$ defined in Lemma 2.6.9. Let us prove that

$$\nu \dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to 0. \tag{2.58}$$

By contradiction, assume that $\nu \dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to \alpha > 0$. By regularity of w and continuity of $\mathcal{G}$ (Proposition 2.3.1),

$$0 = -\mathcal{G}(s^{\varepsilon,\nu}(t^{\varepsilon,\nu}), \mathfrak{a}w(t^{\varepsilon,\nu}) + \varepsilon\mathfrak{b}\dot{w}(t^{\varepsilon,\nu})) + \mathfrak{c} + \nu\mathfrak{d}\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to -\mathcal{G}(s(t-), \mathfrak{a}w(t)) + \mathfrak{c} + \mathfrak{d}\alpha,$$

so that $\mathcal{G}(s(t-), \mathfrak{a}w(t)) > \mathfrak{c}$, in contradiction to (2.50) proved above. The regularity of w, the continuity of $\mathcal{G}$ (Proposition 2.3.1) and (2.58) allow to conclude

$$0 = -\mathcal{G}(s^{\varepsilon,\nu}(t^{\varepsilon,\nu}), \mathfrak{a}w(t^{\varepsilon,\nu}) + \varepsilon\mathfrak{b}\dot{w}(t^{\varepsilon,\nu})) + \mathfrak{c} + \nu\mathfrak{d}\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to -\mathcal{G}(s(t), \mathfrak{a}w(t)) + \mathfrak{c},$$

i.e. we obtain the thesis, since $s(t) = s(t-)$ by left continuity of s (Lemma 2.6.7).

Similarly for $s(t+)$, we have

$$\nu \dot{s}^{\varepsilon,\nu}(\hat{t}^{\varepsilon,\nu}) \to 0$$

and we deduce that

$$0 = -\mathcal{G}(s^{\varepsilon,\nu}(\hat{t}^{\varepsilon,\nu}), \mathfrak{a}w(\hat{t}^{\varepsilon,\nu}) + \varepsilon\mathfrak{b}\dot{w}(\hat{t}^{\varepsilon,\nu})) + \mathfrak{c} + \nu\mathfrak{d}\dot{s}^{\varepsilon,\nu}(\hat{t}^{\varepsilon,\nu}) \to -\mathcal{G}(s(t+), \mathfrak{a}w(t)) + \mathfrak{c}.$$

Assume now that s is continuous at t and fix a neighbourhood U of t. Since s is not constant in U, for ε, ν small enough $s^{\varepsilon,\nu}$ is not constant in U as well, so that $\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) > 0$ for some $t^{\varepsilon,\nu} \in U \cap (0, T_f(s^{\varepsilon,\nu}))$. Considering a decreasing sequence of neighbourhoods of t converging to t, we find a sequence $t^{\varepsilon,\nu} \to t$ with $\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) > 0$. Since s is continuous at t, Lemma 1.7.10 implies that $s^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to s(t)$. Arguing as in the previous case, we deduce that $\nu\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to 0$. Thanks to the regularity assumption on w, the continuity of $\mathcal{G}$ (Proposition 2.3.1) and Lemma 2.6.8, as before we conclude that

$$0 = -\mathcal{G}(s^{\varepsilon,\nu}(t^{\varepsilon,\nu}), \mathfrak{a}w(t^{\varepsilon,\nu}) + \varepsilon\mathfrak{b}\dot{w}(t^{\varepsilon,\nu})) + \mathfrak{c} + \nu\mathfrak{d}\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to -\mathcal{G}(s(t), \mathfrak{a}w(t)) + \mathfrak{c}\,,$$

and the thesis is proved since $s(t\pm) = s(t)$. □

Lemma 2.6.11. *Let $t \in [0, T_f(s))$ be such that*

$$\mathcal{G}(s(t), \mathfrak{a}w(t)) < \mathfrak{c}. \tag{2.59}$$

Then s is differentiable at t and $\dot{s}(t) = 0$.

Proof. By continuity of $\mathcal{G}$ and w, there exist $\eta, \delta_0 > 0$ such that $\mathcal{G}(\sigma, \mathfrak{a}w(\tau)) < \mathfrak{c}$ for $\sigma \in [s(t) - 2\eta, s(t) + 2\eta]$ and $\tau \in [t - \delta_0, t + \delta_0] \cap [0, T]$. Lemma 2.6.10 and (2.59) imply that s is continuous at t, so that $s(\tau) \in [s(t) - \eta, s(t) + \eta]$ for $\tau \in [t - \delta_1, t + \delta_1] \cap [0, T]$, for some $0 < \delta_1 \le \delta_0$.

By (2.54), it is $s^{\varepsilon,\nu}(t - \delta_1) \ge s(t - \delta_1) - \eta$ and $s^{\varepsilon,\nu}(t + \delta_1) \le s(t + \delta_1) + \eta$ for every ε and ν sufficiently small, say $0 \le \varepsilon < \varepsilon_0$ and $0 < \nu < \nu_0$. Thus we have the chain of inequalities

$$s(t) - 2\eta \le s(t - \delta_1) - \eta \le s^{\varepsilon,\nu}(t - \delta_1) \le s^{\varepsilon,\nu}(\tau) \le s^{\varepsilon,\nu}(t + \delta_1) \le s(t + \delta_1) + \eta \le s(t) + 2\eta$$

for every $\tau \in [t - \delta_1, t + \delta_1] \cap [0, T]$. Consequently

$$\mathcal{G}(s^{\varepsilon,\nu}(\tau), \mathfrak{a}w(\tau)) < \mathfrak{c}$$

for every $\tau \in [t-\delta_1, t+\delta_1] \cap [0,T]$, $0 \leq \varepsilon < \varepsilon_0$ and $0 < \nu < \nu_0$. By the regularity of w and $\mathcal{G}$, we further obtain that, for some $0 < \varepsilon_1 \leq \varepsilon_0$,

$$\mathcal{G}(s^{\varepsilon,\nu}(\tau), \mathfrak{a}w(\tau) + \varepsilon \mathfrak{b}\dot{w}(\tau)) < \mathfrak{c}$$

for every $\tau \in [t-\delta_1, t+\delta_1] \cap [0,T]$, $0 \leq \varepsilon < \varepsilon_1$ and $0 < \nu < \nu_0$. Then (2.35) implies that

$$s^{\varepsilon,\nu}(\tau) = c^{\varepsilon,\nu} \in [s_0, L]$$

for every $\tau \in [t-\delta_1, t+\delta_1] \cap [0,T]$. We deduce that the limit s is constant on $[t-\delta_1, t+\delta_1] \cap [0,T]$, so that it is differentiable at t and $\dot{s}(t) = 0$. □

Lemma 2.6.12. *It results $u^{\varepsilon,\nu}(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega \setminus \Gamma)$ for a.e. $t \in [0,T]$. In addition, $u(t) \in H^1(\Omega_{s(t)})$ for a.e. $t \in [0,T]$.*

Proof. By the estimate (2.44), for every $t \in [0,T]$ there exists $\hat{u}(t) \in H^1(\Omega \setminus \Gamma)$ such that $u^{\varepsilon,\nu}(t) \rightharpoonup \hat{u}(t)$ as ε and ν vanish. Then, since (2.53) holds true, it has to be $\hat{u}(t) = u(t)$ for a.e. $t \in [0,T]$.

The second part of the statement can be proved arguing as in Lemma 2.4.5 at any t for which the weak convergence $u^{\varepsilon,\nu}(t) \rightharpoonup u(t)$ is satisfied. □

Proof of Theorem 2.6.5. Consider the limit (s,u) defined by (2.54) and (2.53), respectively. The couple (s,u) is then a vanishing viscosity evolution with initial condition (s_0, u_0) and boundary loading w. The existence part of the theorem is proved.

Consider now any vanishing viscosity evolution with initial condition (s_0, u_0) and boundary loading w, and let $(s^{\varepsilon,\nu}, u^{\varepsilon,\nu})$ be the correspondent approximating sequence of irreversible viscoelastic evolutions. By Theorem 2.4.2, it is $(s^{\varepsilon,\nu}(0), u^{\varepsilon,\nu}(0)) = (s_0, u_0)$ for every ε, ν, so that the pointwise convergence (2.54) implies $s(0) = s_0$. Lemma 2.6.7 assures the left-continuity for s.

The function u satisfies the boundary condition at a.e. instant since the functions $u^{\varepsilon,\nu}$ do and Lemma 2.6.12 holds true. Therefore Condition 2.6.2.(ii) is proved.

In order to obtain Condition 2.6.2.(iii), argue as in Theorem 2.4.2 and use the fact that $\varepsilon \nabla \dot{u}^{\varepsilon,\nu}$ converges strongly to 0 in $L^2(0,T; L^2(\Omega \setminus \Gamma; \mathbb{R}^2))$, because of (2.46).

Let now prove the weak version 2.6.2.(iv) of Griffith's criterion. By construction, the function s is non-decreasing since the $s^{\varepsilon,\nu}$ are. Inequality (2.50) and the weak activation criterion (2.51) are proved in Lemma 2.6.10. The condition (2.52) on the jump instant can be proved as in [58, Theorem 5.1]. The last requirement has been obtained in Lemma 2.6.11.

Finally, to show Condition 2.6.2.(v) observe that if s is continuous at t, then $\mathcal{G}(s(\cdot), \mathfrak{a}w(\cdot))$ is continuous at t too. If t is a jump instant for s, then

$$\lim_{\substack{\tau \to t \\ \tau < t}} \mathcal{G}(s(\tau), \mathfrak{a}w(\tau)) = \mathcal{G}(s(t-), \mathfrak{a}w(t)) = \mathfrak{c} = \mathcal{G}(s(t+), \mathfrak{a}w(t)) = \lim_{\substack{\tau \to t \\ \tau > t}} \mathcal{G}(s(\tau), \mathfrak{a}w(\tau))$$

where the equalities in the middle are due to (2.51). Therefore $\mathcal{G}(s(\cdot), \mathfrak{a}w(\cdot))$ is continuous at the jump instant of s as well. □

2.7. One-dimensional analysis

Inspired by the analyses proposed in [58, 70], we describe the evolution of the crack tip of the vanishing viscosity evolutions (Definition 2.6.4), highlighting the different behaviour between them and a general rate-independent evolution (Definition 2.6.2). Indeed, the fact of being approximated by irreversible viscoelastic evolutions provides interesting properties. We obtain a one-dimensional analysis of a problem that in principle is infinite dimensional, in the sense that it is initially set in infinite dimensional Sobolev spaces.

In Subsection 2.7.1 an example shows the different behaviour of the globally stable irreversible evolutions introduced in [48] and the vanishing viscosity ones.

First of all observe that, as suggested by Condition 2.6.2.(iv) in Definition 2.6.2, the $\mathfrak{c}$-level set of the energy release rate $\mathcal{G}$ plays an important role. For convenience, we introduce the function $\mathcal{V} : [s_0, L) \times [0, T] \to \mathbb{R}$ defined as

$$\mathcal{V}(\sigma, t) := \mathcal{G}(\sigma, \mathfrak{a}w(t)) - \mathfrak{c} \tag{2.60}$$

and the sets

$$\begin{aligned}
\mathcal{A}^0 &:= \{(t,\sigma) \in [0,T] \times [s_0, L) \ : \ \mathcal{V}(\sigma,t) = 0\} \\
&= \{(t,\sigma) \in [0,T] \times [s_0, L) \ : \ \mathcal{G}(\sigma, \mathfrak{a}w(t)) - \mathfrak{c} = 0\} \\
\mathcal{A}^+ &:= \{(t,\sigma) \in [0,T] \times [s_0, L) \ : \ \mathcal{V}(\sigma,t) > 0\} \\
&= \{(t,\sigma) \in [0,T] \times [s_0, L) \ : \ \mathcal{G}(\sigma, \mathfrak{a}w(t)) - \mathfrak{c} > 0\} \\
\mathcal{A}^- &:= \{(t,\sigma) \in [0,T] \times [s_0, L) \ : \ \mathcal{V}(\sigma,t) < 0\} \\
&= \{(t,\sigma) \in [0,T] \times [s_0, L) \ : \ \mathcal{G}(\sigma, \mathfrak{a}w(t)) - \mathfrak{c} < 0\}\,.
\end{aligned}$$

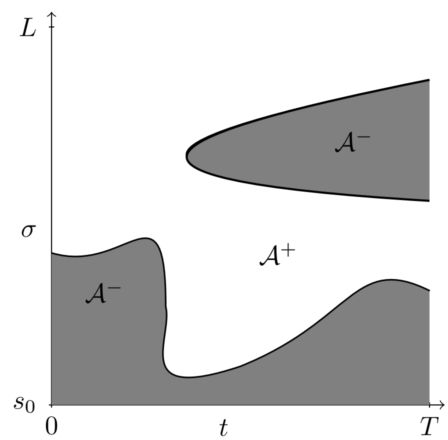


FIGURE 2. The sets $\mathcal{A}^-$, $\mathcal{A}^+$ and $\mathcal{A}^0$ for a sufficiently smooth energy release rate $\mathcal{G}$. $\mathcal{A}^0$ corresponds to the black line separating the gray region $\mathcal{A}^-$ and the white region $\mathcal{A}^+$.

By Definition 2.6.2, the properties of the crack tip function $t \mapsto s(t)$ of a rate-independent evolution, translated in terms of the sets above, are:

- s is non-decreasing and left continuous, with $s(0) = s_0$;
- $(t, s(t)) \in \mathcal{A}^- \cup \mathcal{A}^0$ for every $t \in [0,T]$;
- if t is a non-constancy instant for s, then $(t, s(t\pm)) \in \mathcal{A}^0$;
- if t is a jump instant for s, then $(t, \sigma) \in \mathcal{A}^0 \cup \mathcal{A}^+$ for every $\sigma \in [s(t), s(t+)]$;
- the function $t \mapsto \mathcal{G}(s(t), \mathfrak{a}w(t))$ is continuous.

A priori a function with this behaviour is not unique, thus we really need to characterize the class of vanishing viscosity evolutions.

It remains open the question whether $s^{\varepsilon,\nu}$ and $s^{0,\nu}$ always converge to the same limit for any reasonable $\mathcal{G}$, when ε and ν vanish; the issue arises already at the level of the incremental problems. However, if we assume sufficient regularity for the energy release rate $\mathcal{G}$, we are able to give an answer. Let $\mathcal{G}$ be Lipschitz continuous with respect to both its variables. We already obtained a partial result in Proposition 2.3.3; in order to prove the Lipschitz continuity with respect to the fracture variable σ, we should assume more regularity for the boundary data and for the pre-assigned crack path Γ. Proper assumptions can be deduced by comparison with the result in [72, Appendix A.3]. In the following we will consider only the evolutions $s^{0,\nu}$, and not the $s^{\varepsilon,\nu}$ with $\varepsilon > 0$, since for a Lipschitz continuous $\mathcal{G}$ they both converge to the same limit when $\varepsilon, \nu \to 0$. Indeed, first of all we recall that $s^{\varepsilon,\nu}$ and $s^{0,\nu}$ are solutions to the Cauchy problems

$$\begin{cases} \dot{s}^{\varepsilon,\nu}(t) = \dfrac{1}{\nu\mathfrak{d}}\left[\mathcal{G}(s^{\varepsilon,\nu}(t), \mathfrak{a}w(t) + \varepsilon\mathfrak{b}\dot{w}(t)) - \mathfrak{c}\right]^+ \\ s^{\varepsilon,\nu}(0) = s_0 \end{cases}$$

and

$$\begin{cases} \dot{s}^{0,\nu}(t) = \dfrac{1}{\nu\mathfrak{d}}\mathcal{V}(s^{0,\nu}(t), t)^+ \\ s^{0,\nu}(0) = s_0 \end{cases} \tag{2.61}$$

respectively, where in (2.61) we used the definition (2.60). Being the function $\sigma \in \mathbb{R} \mapsto \sigma^+ = \sup\{\sigma, 0\}$ Lipschitz continuous with constant 1 and denoting by K the Lipschitz constant of $\mathcal{G}$, we have

$$\begin{aligned} |s^{\varepsilon,\nu}(t) - s^{0,\nu}(t)| \leq & \int_0^t \left| [\mathcal{G}(s^{\varepsilon,\nu}(\tau), \mathfrak{a}w(\tau) + \varepsilon\mathfrak{b}\dot{w}(\tau)) - \mathfrak{c}]^+ - \left[\mathcal{G}(s^{0,\nu}(\tau), \mathfrak{a}w(\tau)) - \mathfrak{c}\right]^+ \right| d\tau \\ \leq & \int_0^t \left|\mathcal{G}(s^{\varepsilon,\nu}(\tau), \mathfrak{a}w(\tau) + \varepsilon\mathfrak{b}\dot{w}(\tau)) - \mathcal{G}(s^{0,\nu}(\tau), \mathfrak{a}w(\tau))\right| d\tau \\ \leq & \int_0^t |\mathcal{G}(s^{\varepsilon,\nu}(\tau), \mathfrak{a}w(\tau) + \varepsilon\mathfrak{b}\dot{w}(\tau)) - \mathcal{G}(s^{\varepsilon,\nu}(\tau), \mathfrak{a}w(\tau))| \, d\tau \\ & + \int_0^t \left|\mathcal{G}(s^{\varepsilon,\nu}(\tau), \mathfrak{a}w(\tau)) - \mathcal{G}(s^{0,\nu}(\tau), \mathfrak{a}w(\tau))\right| d\tau \\ \leq & \varepsilon\mathfrak{b}K \int_0^t |\dot{w}(\tau)| \, d\tau + K \int_0^t \left|s^{\varepsilon,\nu}(\tau) - s^{0,\nu}(\tau)\right| d\tau \\ \leq & \varepsilon\mathfrak{b}CTK + K \int_0^t \left|s^{\varepsilon,\nu}(\tau) - s^{0,\nu}(\tau)\right| d\tau. \end{aligned}$$

Gronwall Lemma provides the inequality

$$|s^{\varepsilon,\nu}(t) - s^{0,\nu}(t)| \leq \varepsilon\mathfrak{b}CKTe^{Kt} \leq \varepsilon\mathfrak{b}CKTe^{KT}$$

uniformly in t and ν, so that the claim is proved.

We now describe all the possible behaviours of the crack tip function s of a vanishing viscosity evolution at the initial instant. The propositions below can be seen as different steps in an algorithmic procedure. Since we need a bit of regularity, we assume $\mathcal{A}^0$ to be a C^1 manifold of dimension 1.

Proposition 2.7.1. *If there exists $t \in (0, T]$ such that $[0, t) \times \{s_0\} \subset \mathcal{A}^- \cup \mathcal{A}^0$, then $s(t) = s_0$ for $t \in [0, t_0]$, where $t_0 := \sup\left\{t \in (0, T] : \ [0, t) \times \{s_0\} \subset \mathcal{A}^- \cup \mathcal{A}^0\right\}$.*

Proof. By the regularity assumptions on $\mathcal{G}$, the solution to (2.61) is unique. Since the constant function $\bar{s} \equiv s_0$ solves (2.61) in $[0, t_0)$ for every ν, then it results $s^{0,\nu}(t) = s_0$ for $t \in [0, t_0)$. Being s pointwise limit of the $s^{0,\nu}$ and left-continuous, we have the thesis. $\square$

Proposition 2.7.2. *Assume there exist* $\zeta > 0$ *and a continuous function* $\bar\sigma : [0, t_{\bar\sigma}] \to [s_0, L]$ *such that* $\bar\sigma$ *is increasing,* $\bar\sigma(0) = s_0$ *and*

$$\{(t, \bar\sigma(t)) \ : 0 < t < t_{\bar\sigma}\} \subset \mathcal{A}^0 \tag{2.62}$$

$$\{(t, \sigma) \ : 0 < t < t_{\bar\sigma}, \bar\sigma(t) - \zeta < \sigma < \bar\sigma(t)\} \subset \mathcal{A}^+ \tag{2.63}$$

$$\{(t, \sigma) \ : 0 < t < t_{\bar\sigma}, \bar\sigma(t) < \sigma < \bar\sigma(t) + \zeta\} \subset \mathcal{A}^-. \tag{2.64}$$

Then $s(t) = \bar\sigma(t)$ *for every* $t \in [0, t_{\bar\sigma}]$.

Remark 2.7.3. If $\mathcal{G}$ is regular enough, then (2.62)-(2.64) imply that

$$\partial_\sigma \mathcal{G}(\bar\sigma(t), \mathfrak{a}w(t)) < 0$$

for every $t \in (0, t_{\bar\sigma})$. Therefore the elastic bulk energy $\mathcal{F}$ (defined in Section 2.3 and corresponding also to $\mathcal{E}^{el}$ in (1.16)) is *convex* along $\bar\sigma$, and the proposition states that the crack of the vanishing viscosity evolution grows *continuously* where $\mathcal{F}$ is convex with respect to σ.

Proof. First we prove that $s^{0,\nu}(t) \le \bar\sigma(t)$ for every $\nu > 0$ and $t \in [0, t_{\bar\sigma})$, so that, by pointwise convergence, $s(t) \le \bar\sigma(t)$ for every $t \in [0, t_{\bar\sigma}]$.

By contradiction, assume that for some ν_0 there exists $t \in (0, t_{\bar\sigma})$ with $s^{0,\nu_0}(t) > \bar\sigma(t)$, and define $t_0 := \inf\left\{t \in (0, t_{\bar\sigma}) : \, s^{0,\nu_0}(t) > \bar\sigma(t)\right\}$. There exists $t_1 \ge t_0$ and $\delta > 0$ such that $s^{0,\nu_0}(t_1) = \bar\sigma(t_1)$, $s^{0,\nu_0}(t) > \bar\sigma(t)$ and $(t, s^{0,\nu_0}(t)) \in \mathcal{A}^-$ for every $t \in (t_1, t_1 + \delta)$ (here we used (2.64))). Then, for $t \in (t_1, t_1 + \delta)$, we have

$$0 < \bar\sigma(t) - \bar\sigma(t_1) < s^{0,\nu_0}(t) - s^{0,\nu_0}(t_1) = \frac{1}{\nu\mathfrak{d}} \int_{t_1}^{t} \mathcal{V}\left(s^{0,\nu_0}(\tau), \tau\right)^+ \, d\tau = 0,$$

which is a contradiction.

So far, we have obtained that $s(t) \le \bar\sigma(t)$ for every $t \in [0, t_{\bar\sigma}]$. Defined

$$\bar t := \sup\left\{t \in [0, t_{\bar\sigma}] : \, s(\tau) = \bar\sigma(\tau) \text{ for every } 0 \le \tau \le t\right\},$$

the proof is complete if we show that $\bar t = t_{\bar\sigma}$.

By contradiction, assume that $\bar t < t_{\bar\sigma}$. The definition of $\bar t$ implies the existence of $\tilde t \in (\bar t, \bar t + \delta)$ such that $\bar\sigma(\tilde t) - \zeta < s(\tilde t) < \bar\sigma(\tilde t)$. Being s left continuous, for some $\tilde\delta > 0$ so that $\tilde t - \tilde\delta > \bar t$ and some $0 < \eta < \zeta$, the set

$$\mathcal{D} := \left\{(t, \sigma) \in [0, t_{\bar\sigma}] \times [s_0, L) : \, \tilde t - \tilde\delta \le t \le \tilde t, s(\tilde t - \tilde\delta) - \eta \le \sigma \le s(t)\right\}$$

satisfies $\mathcal{D} \subset\subset \mathcal{A}^+$. Therefore there exists $C > 0$ such that $\mathcal{V} \ge C$ on $\mathcal{D}$.

By pointwise convergence, there exists $\nu_0 > 0$ with $s^{0,\nu_0}(\tilde t - \tilde\delta) > s(\tilde t - \tilde\delta) - \eta$. Since the convergence of the $s^{0,\nu}$ to s is monotone with respect to ν, the chain of inequalities

$$s(\tilde t - \tilde\delta) - \eta < s^{0,\nu_0}(\tilde t - \tilde\delta) \le s^{0,\nu}(\tilde t - \tilde\delta) \le s^{0,\nu}(t) \le s(t)$$

shows that $(t, s^{0,\nu}(t)) \in \mathcal{D}$ for every $t \in [\tilde t - \tilde\delta, \tilde t]$ and $0 < \nu < \nu_0$. Then

$$s(\tilde t) - s(\tilde t - \tilde\delta) + \eta > s^{0,\nu}(\tilde t) - s^{0,\nu}(\tilde t - \tilde\delta) = \frac{1}{\nu\mathfrak{d}} \int_{\tilde t - \tilde\delta}^{\tilde t} \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau \ge \frac{1}{\nu\mathfrak{d}} \tilde\delta C \to +\infty$$

as $\nu \to 0$, which is impossible.

Since the contradiction is due to the assumption $\bar t < t_{\bar\sigma}$, it must be $\bar t = t_{\bar\sigma}$, *i.e.* $s(t) = \bar\sigma(t)$ for every $t \in [0, t_{\bar\sigma}]$. □

In the next proposition, we set $\min \emptyset = +\infty$.

Proposition 2.7.4. *Assume there exists $t \in (0,T]$ such that $(0,t) \times \{s_0\} \subset \mathcal{A}^+$ and define*

$$\bar{s} := \min\big\{L, \min\big\{\sigma \in [s_0, L] : \ (0,\sigma) \in \mathcal{A}^0\big\}\big\}.$$

Then

(1) if $\bar{s} = s_0$, then there exists a continuous increasing function

$$\bar{\sigma} : [0, t_{\bar{\sigma}}] \subset [0,T] \to [s_0, L]$$

with $\bar{\sigma}(0) = s_0$, such that $s = \bar{\sigma}$ for $t \in [0, t_{\bar{\sigma}}]$;

(2) if $s_0 < \bar{s} < L$, let

$$\bar{\sigma} : [0, t_{\bar{\sigma}}] \subset [0,T] \to [s_0, L]$$

be a monotone continuous function such that $\bar{\sigma}(0) = \bar{s}$ and $(t, \bar{\sigma}(t)) \in \mathcal{A}^0$ for every $t \in [0, t_{\bar{\sigma}}]$.

- *If $\bar{\sigma}$ is increasing, then $s(t) = \bar{\sigma}(t)$ for $t \in (0, t_{\bar{\sigma}})$ and $s(0+) = \bar{\sigma}(0)$.*
- *If $\bar{\sigma}$ is strictly decreasing, then $s(t) = \bar{\sigma}(0)$ for every $t \in (0, t_0)$, where*

$$t_0 := \sup\big\{t \in (0,T) : \ (0,t) \times \{\bar{\sigma}(0)\} \subset \mathcal{A}^- \cup \mathcal{A}^0\big\},$$

and $s(0+) = \bar{\sigma}(0) = \bar{s}$;

(3) if $\bar{s} = L$, then $s(t) = L$ for every $t \in (0,T]$ and, consequently, $s(0+) = L$.

Proof. **Case (1)**: being $\bar{s} = s_0$ and $(0,t) \times \{s_0\} \subset \mathcal{A}^+$, the regularity of $\mathcal{A}^0$ implies the existence of a branch $\bar{\sigma} : [0, t_{\bar{\sigma}}] \subset [0,T] \to [s_0, L]$ of $\mathcal{A}^0$ such that $\bar{\sigma}(0) = s_0$ and $\bar{\sigma}$ is increasing. Then the proof is the same as for Proposition 2.7.2, since the geometry around $\bar{\sigma}$ is described by (2.62)-(2.63)-(2.64).

Case (2): first of all observe that, around $\bar{\sigma}$, conditions (2.62)-(2.63)-(2.64) hold true for some $\zeta > 0$ and there exists $\hat{t} \le t_{\bar{\sigma}}$ such that

$$\mathcal{B} := \big\{(t,\sigma) : \ 0 \le t \le \hat{t}, s_0 \le \sigma < \bar{\sigma}(t)\big\} \subset \mathcal{A}^+.$$

Assume first that $\bar{\sigma}$ is strictly increasing. Arguing similarly to the first part of Proposition 2.7.2, we obtain that $s^{0,\nu}(t) \le \bar{\sigma}(t)$ for every $t \in [0, t_{\bar{\sigma}}]$, so that also $s \le \bar{\sigma}$ in the same interval. We now want to prove that the equality holds true.

By contradiction, assume there exists $\tilde{t} \in (0, t_{\bar{\sigma}})$ with $s(\tilde{t}) < \bar{\sigma}(\tilde{t})$. Suppose first that $\tilde{t} < \hat{t}$. By left continuity of s (see Lemma 2.6.7) and $\bar{\sigma}$, there exists a small $\delta > 0$ such that the set

$$\mathcal{D} := \big\{(t,\sigma) : \ t \in [\tilde{t} - \delta, \tilde{t}], s_0 \le \sigma \le s(t)\big\} \subset\subset \mathcal{A}^+,$$

and consequently $\mathcal{V} \ge C$ on $\mathcal{D}$ for some constant $C > 0$. Since $(t, s^{0,\nu}(t)) \in \mathcal{D}$ for every $t \in [\tilde{t} - \delta, \tilde{t}]$ and $\nu > 0$, it is

$$s(\tilde{t}) - s_0 > s^{0,\nu}(\tilde{t}) - s^{0,\nu}(\tilde{t} - \delta) = \frac{1}{\nu \mathfrak{d}} \int_{\tilde{t}-\delta}^{\tilde{t}} \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau \ge \frac{1}{\nu \mathfrak{d}} C\delta \to +\infty$$

as $\nu \to 0$. This is a contradiction; therefore $s(t) = \bar{\sigma}(t)$ for $t \in (0, \hat{t})$ and $\tilde{t} \in [\hat{t}, t_{\bar{\sigma}})$. Arguing as in the second part of Proposition 2.7.2, we obtain again a contradiction. Hence we conclude that it is $s(t) = \bar{\sigma}(t)$ for every $t \in (0, t_{\bar{\sigma}})$ and $s(0+) = \bar{\sigma}(0)$.

If $\bar{\sigma}$ is strictly decreasing, first of all we show the following facts:

(2.i) there exists ν_0 such that for every $0 < \nu < \nu_0$ there exists $t_\nu \in (0, \hat{t})$ with $s^{0,\nu}(t_\nu) = \bar{\sigma}(t_\nu)$;

(2.ii) the sequence t_ν is monotonically converging to 0 as $\nu \searrow 0$.

By contradiction, assume that for every $\nu > 0$ there exists a smaller index $0 < \tilde{\nu} < \nu$ such that for every $t \in (0, \hat{t})$ it is $s^{0,\tilde{\nu}}(t) < \bar{\sigma}(t)$. (Observe that $s^{0,\tilde{\nu}}$ cannot be larger than $\bar{\sigma}$ in the interval $(0, \hat{t})$, otherwise by continuity they would coincide at some instant since $s^{0,\tilde{\nu}}(0) = s_0 < \bar{\sigma}(0)$). Therefore we obtain $s(t) \le \bar{\sigma}(t)$ for $t \in (0, \hat{t})$. Being s non-decreasing and $\bar{\sigma}$ strictly decreasing, it is $s(t) < \bar{\sigma}(t)$ for $t \in [0, \hat{t} - \delta]$ for some small $\delta > 0$. Consequently, for every $t \in [0, \hat{t} - \delta]$ and $0 < \nu < \nu_0$ it is

$$(t, s^{0,\nu}(t)) \in R := \big\{(t, \sigma) : \ 0 \le t \le \hat{t} - \delta, s_0 \le \sigma \le s(\hat{t} - \delta)\big\} \subset\subset \mathcal{A}^+.$$

Since $\mathcal{V} \ge C > 0$ on R for some constant C, we obtain the contradiction

$$s(\hat{t} - \delta) \ge s^{0,\nu}(\hat{t} - \delta) = s_0 + \frac{1}{\nu\mathfrak{d}} \int_0^{\hat{t}-\delta} \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau \ge s_0 + \frac{1}{\nu\mathfrak{d}}(\hat{t} - \delta)C \to +\infty$$

as $\nu \to 0$. Hence (2.i) is proved.

Concerning (2.ii), firstly we show that, if $\nu_1 < \nu_2$, then $t_{\nu_1} \le t_{\nu_2}$. Indeed, if it were $t_{\nu_1} > t_{\nu_2}$, then we would have

$$\bar{\sigma}(t_{\nu_1}) = s^{0,\nu_1}(t_{\nu_1}) \ge s^{0,\nu_1}(t_{\nu_2}) \ge s^{0,\nu_2}(t_{\nu_2}) = \bar{\sigma}(t_{\nu_2}) > \bar{\sigma}(t_{\nu_1}),$$

where the first inequality is due to the monotonicity of the $s^{0,\nu}$ and the second one to the fact that $s^{0,\nu_1} \ge s^{0,\nu_2}$.

Now we prove that $t_\nu \searrow 0$ as $\nu \searrow 0$. By contradiction, assume that $t_\nu \searrow \tilde{t} > 0$. For every $0 < \nu < \nu_0$ (ν_0 selected at step (2.i)) it is

$$s^{0,\nu}(\tilde{t}) \le s^{0,\nu}(t_\nu) = \bar{\sigma}(t_\nu)$$

and, taking the limit as $\nu \to 0$, we get $s(\tilde{t}) \le \bar{\sigma}(\tilde{t})$. Then, by monotonicity of both s and $\bar{\sigma}$, $s(\tilde{t}/2) < \bar{\sigma}(\tilde{t}/2)$. Being $0 < \tilde{t} \le \hat{t}$, for some $C > 0$ it is $\mathcal{V}(\sigma, t) \ge C$ for every $t \in [0, \tilde{t}/2]$ and $\sigma \in [s_0, s(\tilde{t}/2)]$. Repeating the same argument as before,

$$s(\tilde{t}/2) \ge s^{0,\nu}(\tilde{t}/2) = s_0 + \frac{1}{\nu\mathfrak{d}} \int_0^{\tilde{t}/2} \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau \ge s_0 + \frac{1}{\nu\mathfrak{d}} \frac{\tilde{t}}{2} C \to +\infty$$

as $\nu \to 0$, which is a contradiction. Therefore $t_\nu \searrow 0$ as $\nu \searrow 0$, so that (2.ii) is proved as well.

To prove the claim in case (2), observe that the geometry of $\mathcal{A}^-$ in a neighbourhood of $(0, \bar{\sigma}(0))$ is the following: there exists $\tilde{\tau} > 0$ such that

$$B := \{(t, \sigma) : \ 0 < t < \tilde{\tau}, \bar{\sigma}(t) < \sigma < \bar{\sigma}(0)\} \subset \mathcal{A}^-.$$

For ν sufficiently small, $t_\nu < \tilde{\tau}$ and $s^{0,\nu}$ has the form

$$s^{0,\nu}(t) = \begin{cases} s_0 + \dfrac{1}{\nu\mathfrak{d}} \displaystyle\int_0^t \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau & \text{for } t \in [0, t_\nu) \\ s^{0,\nu}(t_\nu) & \text{for } t \in [t_\nu, t_0]. \end{cases}$$

Indeed, with an argument similar to that in Proposition 2.7.1, the unique solution to the Cauchy problem

$$\begin{cases} \dot{\varphi}(t) = \dfrac{1}{\nu\mathfrak{d}} \mathcal{V}(\varphi(t), t)^+ \\ \varphi(t_\nu) = s^{0,\nu}(t_\nu) \end{cases}$$

is $\varphi \equiv s^{0,\nu}(t_\nu)$ in $[t_\nu, t_0]$. Consider $t < t_0$ and, by contradiction, let $s(t) < \bar{\sigma}(0)$, so that also $s^{0,\nu}(t) < \bar{\sigma}(0)$ for every ν. By (2.ii), $t_\nu \to 0$ and $s^{0,\nu}(t_\nu) = \bar{\sigma}(t_\nu) \to \bar{\sigma}(0)$. Hence for ν sufficiently small it is

$$t_\nu < t \quad \text{and} \quad s^{0,\nu}(t_\nu) > s^{0,\nu}(t),$$

which contradicts the monotonicity of $s^{0,\nu}$.

Hence $s(t) = \bar{\sigma}(t)$ for any $0 < t < t_0$ and, by consequence, $s(0+) = \bar{\sigma}(0)$.

Case (3): assuming that $\bar{s} = L$, there are two possibilities:

(3.i) $\mathcal{A}^0 \cap (\{0\} \times [s_0, L]) = \emptyset$.

(3.ii) $\mathcal{A}^0 \cap (\{0\} \times [s_0, L]) = \{(0, L)\}$.

If (3.i) is the case, the set $\mathcal{A}^0 \cap ([0,T] \times (s_0, L))$ is far away from the σ-axis, so that there exists $\tilde{t} > 0$ such that for any $0 < \delta < \tilde{t}$ the set $[\delta, \tilde{t}] \times [s_0, L) \subset \mathcal{A}^+$. Fix $t \in (0, \tilde{t})$. By contradiction, assume that $s(t) < L$. Since $[t/2, t] \times [s_0, s(t)] \subset\subset \mathcal{A}^+$, by continuity of $\mathcal{V}$ there exists $C := C(t) > 0$ such that $\mathcal{V}(\sigma, \tau) \geq C$ for every $\tau \in [t/2, t]$ and $\sigma \in [s_0, s(t)]$. Then, since $s_0 \leq s^{0,\nu}(t) \leq s(t)$ for every $\nu > 0$, we obtain

$$L - s_0 > s(t) - s_0 \geq s^{0,\nu}(t) - s^{0,\nu}(t/2) = \frac{1}{\nu\mathfrak{d}} \int_{t/2}^{t} \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau \geq \frac{1}{\nu\mathfrak{d}} C \frac{t}{2} \to +\infty$$

as $\nu \to 0$, which is a contradiction. We proved that $s(t) = L$ for every $t \in (0, \tilde{t})$. Then $s(t) = L$ for every $t \in (0, T]$ and $s(0+) = L$.

In case (3.ii), there exists a monotone function $\bar{\sigma} : [0,T] \to [s_0, L]$ with $\bar{\sigma}(0) = L$. If $\bar{\sigma} \equiv L$, then the proof is the same as for (2) in case of an increasing function; if $\bar{\sigma}$ is strictly decreasing, argue as in (2) in case of a decreasing function. $\square$

Remark 2.7.5. The above propositions provide a description of the evolution of the crack tip s up to a time $\tilde{t} \in (0, T]$. If $\tilde{t} = T$, then the function s is completely determined over $[0, T]$, otherwise we have to proceed with the analysis. We are not going to prove any further result since, up to modifying slightly the statements and the proofs of Propositions 2.7.1, 2.7.2, 2.7.4, the behaviour of the solution has similar characterizations taking $(\tilde{t}, s(\tilde{t}))$ as starting point instead of $(0, s_0)$.

Remark 2.7.6. The approximation of the vanishing viscosity evolutions by irreversible viscoelastic evolutions plays a key role in the proofs of all the above propositions. In general, the previous characterization is not achievable for a generic rate-independent evolution.

Let us assume more regularity for the $\mathfrak{c}$-level set of $\mathcal{G}$, $\mathcal{A}^0$. In addition to being a C^1 manifold of dimension 1, we require that

- $\nabla \mathcal{V}(\sigma, t) \neq 0$ for every $(t, \sigma) \in \mathcal{A}^0$;
- the singular set

  $$\mathcal{S} := \{(t, \sigma) \in [0,T] \times [s_0, L) \ : \ \partial_\sigma \mathcal{V}(\sigma, t) = 0 \text{ or } \partial_t \mathcal{V}(\sigma, t) = 0\} \cap \mathcal{A}^0$$

  is finite.

Applying the Implicit Function Theorem, there exist finitely many curves

$$\sigma_i : (t_{i,1}, t_{i,2}) \subset (0, T) \to [0, L], \quad i = 1, \ldots, k,$$

such that

- for every i, σ_i is continuous and strictly monotone;
- for every i, the limits

  $$\sigma_i(t_{i,1}) := \lim_{t \to (t_{i,1})^+} \sigma_i(t) \quad \text{and} \quad \sigma_i(t_{i,2}) := \lim_{t \to (t_{i,2})^-} \sigma_i(t)$$

  exist and are finite;
- set

  $$\mathcal{A}_i^0 := \{(t, \sigma_i(t)) : t \in (t_{i,1}, t_{i,2})\}$$

  the graph of σ_i, it is $\mathcal{A}^0 = \cup_{i=1}^k \overline{\mathcal{A}_i^0}$, or, equivalently, $\mathcal{A}^0 \setminus \mathcal{S} = \cup_{i=1}^k \mathcal{A}_i^0$.

In addition, every σ_i verifies one of the following inequalities for every $t \in (t_{i,1}, t_{i,2})$:

$$\partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) < 0 \quad \text{or} \quad \partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) > 0.$$

We are describing a geometry similar to the one in Figure 3.

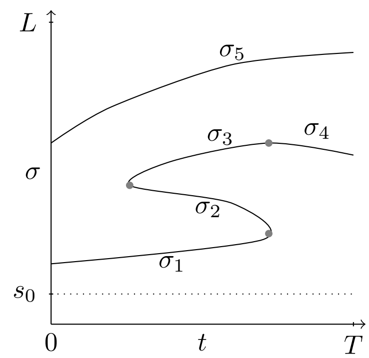


FIGURE 3. Plot of the $\mathfrak{c}$-level set of $\mathcal{G}$, $\mathcal{A}^0$, assuming that it is a C^1 manifold of dimension 1 with finite singular set $\mathcal{S}$.

As said, Propositions 2.7.1, 2.7.2 and 2.7.4 provide an "algorithmic" procedure. They can be quickly adapted to the geometry described above, providing a description of the evolution of $t \mapsto s(t)$ of the crack tip of a vanishing viscosity evolution up to an instant $\tilde{t} \in (0, T]$. While Proposition 2.7.1 is still valid, the new statements for Propositions 2.7.2 and 2.7.4 are the following ones.

Proposition 2.7.7. *Assume there exists $i \in \{1, \dots, k\}$ such that σ_i is strictly increasing, $t_{i,1} = 0$, $\sigma_i(0) = s_0$ and $\partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) < 0$ for every t. Then $s(t) = \sigma_i(t)$ for every $t \in [0, t_{i,2}]$.*

Proposition 2.7.8. *Assume there exists $t \in (0, T]$ such that $(0, t) \times \{s_0\} \subset \mathcal{A}^+$ and define*

$$\bar{s} := \min\{L, \min\{\sigma_i(0) \,:\, 1 \le i \le k \text{ such that } t_{i,1} = 0, \sigma_i(0) \ge s_0, \partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) < 0\}\}.$$

Then

(1) if $\bar{s} = s_0$, then $s(t) = \sigma_i(t)$ for every $t \in (0, t_{i,2})$, where $i \in \{1 \dots, k\}$ is such that $\sigma_i(0) = s_0$ and $\partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) < 0$;

(2) if $s_0 < \bar{s} < L$, set

$$i_0 := \min\{1 \le i \le k :\ t_{i,1} = 0 \text{ and } \sigma_i(0) > s_0\}.$$

If σ_{i_0} is strictly increasing, then $s(t) = \sigma_{i_0}(t)$ for $t \in (0, t_{i_0,2})$ and $s(0+) = \sigma_{i_0}(0)$. If σ_{i_0} is strictly decreasing, then $s(t) = \sigma_{i_0}(0)$ for every $t \in (0, t_0)$, where

$$t_0 := \sup\left\{t \in (0, T) :\ (0, t) \times \{\sigma_{i_0}(0)\} \subset \mathcal{A}^- \cup \mathcal{A}^0\right\},$$

and $s(0+) = \sigma_{i_0}(0)$;

(3) if $\bar{s} = L$, then $s(t) = L$ for every $t \in (0, T]$ and, consequently, $s(0+) = L$.

2.7.1. An example. We present a geometrical setting in which the fracture evolution selected by means of the vanishing viscosity construction jumps later than the globally stable evolution obtained in [48]. We recall and use the example in [82, Section 7]. Toader & Zanini deal with the antiplane 2-dimensional case with pre-assigned crack path $\Gamma = \gamma([-L, L])$ and a monotone increasing loading $w(t,x) = t\psi(x)$ defined on the boundary $\partial\Omega$ of a bounded connected open set $\Omega \subset \mathbb{R}^2$. When considering the case of linearized elasticity and monotone increasing loadings $w(t,x) = t\psi(x)$, the bulk energy $\mathcal{F}(\sigma, w(t))$ in (2.24) has the special form

$$\mathcal{F}(\sigma, w(t)) = t^2 E(\sigma)\,, \tag{2.65}$$

where

$$E(\sigma) := \min\left\{\|\nabla u\|^2 \ : u \in H^1(\Omega \setminus \Gamma(\sigma)), u = \psi \text{ on } \partial_D\Omega\right\}$$

is the energy associated to the boundary loading $w(1,x) = \psi(x)$ and the crack $\Gamma(\sigma) = \gamma([-L, \sigma])$. The quadratic dependence of $\mathcal{F}$ on t is due to the linear nature of the problem. The total energy is then given by

$$t^2 E(\sigma) + \sigma\,, \tag{2.66}$$

where $E^d(\sigma) = \sigma$ is the crack energy (for convenience of exposition, we set the material toughness $\mathfrak{c}$ equal to 1).

In [82] Toader & Zanini construct a boundary loading ψ and a domain Ω in such a way that the elastic energy functional

$$E : [s_0, L] \subset [-L, L] \to \mathbb{R}$$

is *concave* on some subinterval of $[s_0, L]$. In particular, for any $\eta > 0$ they consider the domain

$$\Omega^\eta = B_{-2} \cup T^\eta \cup B_2,$$

where B_{-2} and B_2 are the balls of radius 1 and center in $(-2, 0)$ and $(2, 0)$ respectively, $T^\eta = (-2 + \cos\eta, 2 - \cos\eta) \times (-\sin\eta, \sin\eta)$, and a proper boundary loading ψ_η on $\partial\Omega^\eta$. The crack path is $\Gamma = [-3, 3] \times \{0\}$. The body is assumed to be fractured at time $t = 0$, with initial crack $[-3, -2] \times \{0\}$, and for $\sigma \in [-2, 3)$ set

$$E_\eta(\sigma) := \min\left\{\|\nabla u\|^2 \ : u \in H^1\left(\Omega^\eta \setminus ([-3, \sigma] \times 0)\right), u = \psi_\eta \text{ on } \partial\Omega^\eta\right\}.$$

In this setting (see the discussion for (2.65)-(2.66)), the total energy at time $t > 0$ for the crack $[-3, \sigma] \times \{0\}$ is

$$t^2 E_\eta(\sigma) + \sigma$$

and the function

$$\sigma \in [-2, 3) \mapsto E_\eta(\sigma) \in \mathbb{R}$$

is C^2. The sets $\mathcal{A}^0, \mathcal{A}^-, \mathcal{A}^+$, defined at the beginning of the section, now take the form

$$\begin{aligned}
\mathcal{A}^0_\eta &= \left\{(t, \sigma) \in [0, T] \times [-2, 3] \ : -t^2 E'_\eta(\sigma) = 1\right\} \\
\mathcal{A}^-_\eta &= \left\{(t, \sigma) \in [0, T] \times [-2, 3] \ : -t^2 E'_\eta(\sigma) < 1\right\} \\
\mathcal{A}^+_\eta &= \left\{(t, \sigma) \in [0, T] \times [-2, 3] \ : -t^2 E'_\eta(\sigma) > 1\right\}.
\end{aligned}$$

In [82] the result on the concavity of E_η is achieved by showing the following three facts:

(i) $\limsup_{\eta\to 0^+} E_\eta(2)$ is finite;
(ii) $\liminf_{\eta\to 0^+} E_\eta(-2) = \infty$;
(iii) $\limsup_{\eta\to 0^+} E'_\eta(-2)$ is finite.

Consequently, along a suitable sequence $\eta_k \to 0^+$, it is

$$E_{\eta_k}(-2) + E'_{\eta_k}(-2)4 > E_{\eta_k}(2),$$

proving that E_{η_k} is necessarily concave in some subinterval of $[-2, 2]$.

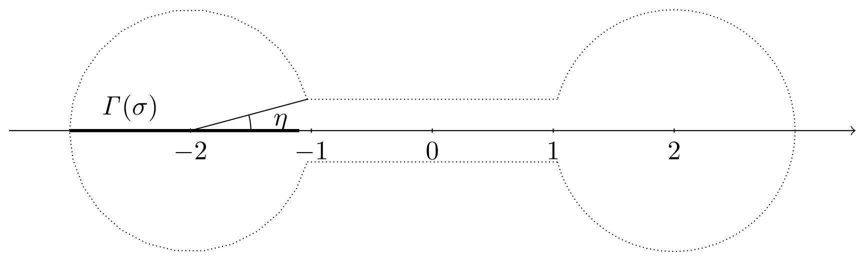


FIGURE 4. The domain Ω^η.

Let us call $E_0(\sigma)$ the elastic energy related to the case where $\Omega = B_{-2}$, the crack set is $[-3, \sigma] \times \{0\}$ for $\sigma \in (-3, -1]$ and the boundary loading is $\sin(\theta/2)$, θ being the angular coordinate between the x-axis and the center $(-2, 0)$ of B_{-2}. In [82], using firstly Irwin's formula (1.20) relating the energy release rate and the stress intensity factor, and then an integral characterization for the last, it is showed that for $\sigma \in [-5/2, -3/2]$ we have

$$\limsup_{\eta \to 0^+} E'_\eta(\sigma) = E'_0(\sigma). \tag{2.67}$$

In order to make clear that $[-3, -2] \times \{0\}$ is the initial crack, below we write $s_0 = -2$. Considering (i),(ii) and (2.67), take $\eta_0 > 0$ such that for any $0 < \eta < \eta_0$ (belonging to a proper subsequence)

$$E_\eta(s_0) + (E'_0(s_0) - 1)(2 - s_0) > E_\eta(2) \tag{2.68}$$

$$\left|E'_\eta(s_0) - E'_0(s_0)\right| < \frac{1}{2}. \tag{2.69}$$

By (2.69) and continuity of E'_η and E'_0, for any η there exists $s_\eta > s_0$ such that

$$\left|E'_\eta(\sigma) - E'_0(\sigma)\right| < \frac{1}{2} \tag{2.70}$$

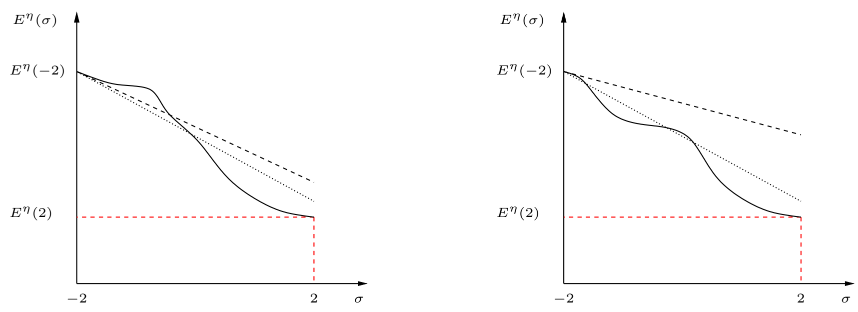
for $\sigma \in [s_0, s_\eta]$.

As proved in [81], E_0 is convex in an interval $[s_0, s_1] \subset [s_0, L]$. Without loss of generality, we can assume $s_\eta \leq s_1$. From (2.70) and convexity of E_0, we deduce

$$E'_\eta(\sigma) > E'_0(\sigma) - \frac{1}{2} \geq E'_0(s_0) - \frac{1}{2}$$

for $\sigma \in [s_0, s_\eta]$, so that Lagrange Theorem implies

$$E_\eta(\sigma) - E_\eta(s_0) = E'_\eta(\xi)(\sigma - s_0) \geq \left(E'_0(s_0) - \frac{1}{2}\right)(\sigma - s_0),$$



(A) (B)

FIGURE 5. Plot of two different cases of the function E_η discussed in the example: in the figure (A), E_η is convex in a neighbouhood of -2, while in (B) it is concave. In principle, we do not know which is the situation, nevertheless for η small enough the request (2.68) is satisfied. The dotted line corresponds to the slope $E_0'(-2) - \frac{1}{2}$, while the dashed one is the tangent to E_η at -2, whose slope is larger than $E_0'(-2) - \frac{1}{2}$, according to (2.69).

where the last inequality is due to the fact that $\xi \in [s_0, \sigma]$. Considering (2.68) too, we obtain

$$\begin{aligned} E_\eta(\sigma) + \left(\frac{1}{2} - E_0'(s_0)\right)\sigma \geq & E_\eta(s_0) + \left(\frac{1}{2} - E_0'(s_0)\right) s_0 \\ > & E_\eta(2) + (1 - E_0'(s_0))(2 - s_0) + \left(\frac{1}{2} - E_0'(s_0)\right) s_0 \\ = & E_\eta(2) + \left(\frac{1}{2} - E_0'(s_0)\right) 2 + \frac{1}{2}(2 - s_0). \end{aligned}$$

Defined $t_0 > 0$ by

$$\frac{1}{t_0^2} = \frac{1}{2} - E_0'(s_0),$$

the above inequality becomes

$$E_\eta(\sigma) + \frac{\sigma}{t_0^2} > E_\eta(2) + \frac{2}{t_0^2} + \frac{1}{2}(2 - s_0) = E_\eta(2) + \frac{2}{t_0^2} + 2 \tag{2.71}$$

for $\sigma \in [s_0, s_\eta]$. The map

$$(t, \sigma) \mapsto E_\eta(\sigma) + \frac{\sigma}{t^2} - E_\eta(2) - \frac{2}{t^2}$$

is continuous in a neighbourhood of $\{t_0\} \times [s_0, L)$, thus by (2.71) we obtain

$$E_\eta(\sigma) + \frac{\sigma}{t^2} > E_\eta(2) + \frac{2}{t^2}$$

for every $t \in [t_\eta, t_0]$ and $\sigma \in [s_0, s_\eta]$, for some $t_\eta < t_0$.

Let $s_G : [0, T] \to [s_0, 3]$ be the globally stable quasi-static evolution. Since at each instant it has to satisfy the global minimality condition

$$t^2 E_\eta(s_G(t)) + s_G(t) \leq t^2 E_\eta(\sigma) + \sigma$$

for every $\sigma \geq \sup_{0 \leq t' < t} s_G(t')$, the discussion above shows that $s_G(t) > s_\eta$ for $t \in [t_\eta, T]$.

Consider now $\eta < \eta_0$ such that

$$E'_\eta(s_0) > E'_0(s_0) - \frac{1}{4}.$$

By choice of η_0, what we achieved above still holds true, in particular the result about the globally stable quasi-static evolution s_G. By continuity of E'_η, there exists $s_0 < \bar{s}_\eta \leq s_\eta$ for which

$$E'_\eta(\sigma) > E'_0(s_0) - \frac{1}{2}$$

for $\sigma \in [s_0, \bar{s}_\eta]$. Then, when σ belongs to this interval and $t \in (0, t_0]$, it is

$$-E'_\eta(\sigma) < \frac{1}{2} - E'_0(s_0) = \frac{1}{t_0^2} \leq \frac{1}{t^2}.$$

In the formalism previously introduced, it is

$$[0, t_0] \times [s_0, \bar{s}_\eta] \subset \mathcal{A}_\eta^-.$$

Denoted by s_V the vanishing viscosity evolution, the analysis at the beginning of the section (Proposition 2.7.1) implies that $s_V(t) = s_0$ for $t \in [0, t_0]$.

Summarizing, we have shown the existence of a domain Ω^η and a boundary loading ψ_η for which the globally stable quasi-static evolution performs a jump in the crack length *strictly* before the vanishing viscosity evolution. Indeed, given the initial crack, $[-3,-2] \times \{0\} \subset \overline{\Omega^\eta}$, there exist $s_\eta \in (-2, L)$, and $0 < t_\eta < t_0 < T$ such that

- the globally stable quasi-static evolution s_G belongs to $[-2, s_\eta]$ for $t \in [0, t_\eta)$ and jumps over s_η at t_η, *i.e.* $s_G(t) \in [-2, s_\eta]$ for $t \in [0, t_\eta)$ and $s_G(t) > s_\eta$ for $t \in [t_\eta, L]$
- any vanishing viscosity evolution s_V is constant on $[0, t_0]$, with $s_V(t) = -2$.

Hence the results in this chapter are a contribution in the search for models of growth of fractures in elastic bodies based on *local* minimization criteria, as discussed in Section 1.4.

CHAPTER 3

A variational model for the quasi-static growth of fractional dimensional brittle fractures

The goal of this chapter is to prove the existence of variational evolutions of fractures with fractional Hausdorff dimension, in the framework of two-dimensional brittle elasticity. The idea is to model the growth of fractures in brittle materials that contain extremely fragile parts, which allow the crack to develop along highly irregular paths.

The interest for this study lies mainly in two reasons. From the point of view of the mathematical setting, the irregularity of cracks of fractional Hausdorff dimension suggests that the approach to fracture mechanics by means of the theory of SBV functions is not omnicomprehensive. Indeed, if the displacements belong to suitable spaces of SBV-type and the fractures are related to the jump sets of the displacements (see, e.g., [**47**, **34**, **36**]), then the cracks are 1-rectifiable, hence they cannot be too irregular. From the point of view of the modeling assumptions, we aim at widening the range of validity of Griffith energetic theory. Indeed, so far the fracture energy of a crack K has been taken of the form

$$\int_K \kappa(x)\, d\mathcal{H}^1(x)\,, \tag{3.1}$$

with the material toughness κ bounded both from above and from below:

$$0 < \beta_1 \leq \kappa(x) \leq \beta_2 < +\infty\,. \tag{3.2}$$

The idea is to violate the lower bound in (3.2).

In this chapter we assume the cracks K to be subsets of *a priori* given curves $\mathcal{K}_1, \ldots, \mathcal{K}_M$, with Hausdorff dimension

$$\dim_{\mathcal{H}}(\mathcal{K}_i) = d_i > 1\,,$$

and, for a set $K \subset \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_M$, we consider a fracture dissipation energy of the form

$$\mathcal{L}(K) = \mathcal{H}^{d_1}(K \cap \mathcal{K}_1) + \ldots + \mathcal{H}^{d_M}(K \cap \mathcal{K}_M)\,.$$

By a time-discretization approach we prove (Theorem 3.2.3) the existence of a variational evolution in the spirit of Francfort & Marigo. Then, by means of Γ-convergence, in Section 3.5 we show, in case of a single curve $\mathcal{K}$, how the crack energy $\mathcal{L}$ can be seen as limit of energies of the form (3.1) when the lower bound in (3.2) is violated.

The results in this chapter are strictly related to the model studied in [**38**]. They have been obtained in collaboration with Rodica Toader, and are contained in [**78**].

3.1. Admissible cracks and displacements

In this section we introduce the class of admissible fractional dimensional cracks and the precise functional setting for the displacements.

Let Ω be a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz boundary; it represents the reference configuration of a brittle elastic body in the antiplane shear case. We fix a relatively open (nonempty) subset $\partial_D\Omega$ of $\partial\Omega$, on which we will impose a Dirichlet boundary condition. We set $\partial_N\Omega = \partial\Omega \setminus \overline{\partial_D\Omega}$; on it a homogeneous Neumann boundary condition will be assumed (in a weak sense).

3.1.1. Admissible cracks. We consider as admissible cracks compact subsets of curves of non-integer Hausdorff dimension having an *a priori* bounded number of connected components.

The curves we have in mind are of the following type: given $d \in (1,2)$, let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous curve such that for some constants $c, L > 0$ it holds

$$\frac{1}{c}|a-b|^{1/d} \leq |\gamma(a)-\gamma(b)| \leq c|a-b|^{1/d} \tag{3.3}$$

and

$$\mathcal{H}^d(\gamma(a,b)) = L(b-a) \tag{3.4}$$

for any $0 \leq a < b \leq 1$.

If $\mathcal{K} := \gamma([0,1])$, then $0 < \mathcal{H}^d(\mathcal{K}) < +\infty$.

As an explicit example of a set $\mathcal{K}$ of the above form, in Section 3.7 we construct a natural parametrization for the von Koch curve, for which $d = \log 4/\log 3$.

Remark 3.1.1. By (3.3), the function $\gamma : [0,1] \to \mathcal{K}$ is invertible with continuous inverse. Hence, if K is a compact connected subset of $\mathcal{K}$ there exist $a, b \in [0,1]$ such that $K = \gamma([a,b])$.

We fix a finite number of sets $\mathcal{K}_1, \ldots, \mathcal{K}_M$ contained in Ω with the property that, for each $m \in \{1,\ldots,M\}$, there exists $d_m \in [1,2[$ such that $\mathcal{K}_m$ is parametrized by a continuous function $\gamma_m : [0,1] \to \mathcal{K}_m$ satisfying (3.3) with $d = d_m$ and some positive constants c_m, L_m, and

$$\mathcal{H}^{d_m}(\gamma_m([a,b])) = L_m(b-a) \qquad \forall a,b \in [0,1],\ a \leq b\,. \tag{3.5}$$

Moreover we assume that

$$\dim_{\mathcal{H}}(\mathcal{K}_{m_1} \cap \mathcal{K}_{m_2}) < \min\{\dim_{\mathcal{H}}(\mathcal{K}_{m_1}), \dim_{\mathcal{H}}(\mathcal{K}_{m_2})\} \qquad \forall m_1 \neq m_2\,. \tag{3.6}$$

The class $\mathcal{C}_p$ of *admissible cracks* is

$$\mathcal{C}_p := \Big\{K \subset \bigcup_{m=1}^{M} \mathcal{K}_m :\ K \text{ nonempty compact set with at most } p \text{ connected components}\Big\}.$$

Note that each connected component of an admissible crack K may contain “pieces" of different Hausdorff dimension.

On this class we will consider the convergence with respect to the Hausdorff distance, recalled in Subsection 1.7.1.

We define the set function

$$\mathcal{L}(K) := \mathcal{H}^{d_1}(K \cap \mathcal{K}_1) + \ldots + \mathcal{H}^{d_M}(K \cap \mathcal{K}_M)\,, \tag{3.7}$$

that will correspond to the fracture dissipation energy. Notice that, by (3.6),

$$\mathcal{H}^{d_m}(K \cap \mathcal{K}_m) = \mathcal{H}^{d_m}\Big(K \setminus \bigcup_{n \neq m} \mathcal{K}_n\Big)$$

for any subset K and $m = 1, \ldots, M$.

3.1.2. Admissible displacements. In the antiplane shear case the body undergoes a deformation of the form

$$(x,z) \in \Omega \times \mathbb{R} \mapsto (x, z+u(x))$$

so that we are led to consider only the out-of-plane component of the displacement, the scalar function $u : \Omega \to \mathbb{R}$. In this situation, if on $\partial_D\Omega$ we impose a bounded displacement $g \in H^1(\Omega) \cap L^\infty(\Omega)$, by a truncation argument we may deduce that the minimizers of the elastic energy $\mathcal{W}(u,K) = \int_{\Omega\setminus K} |\nabla u|^2 dx$ belong to the Sobolev space $H^1(\Omega \setminus K)$. However, in our setting the cracks $K \in \mathcal{C}_p$ are so irregular that, even if they do not disconnect the domain, the H^1 regularity of the boundary datum is not necessarily inherited by the admissible displacements. Therefore we will consider $g \in H^1(\Omega)$ (not necessarily bounded) and we will use for the displacements the *Deny-Lions space* introduced in [40], defined, for any open set $A \subset \mathbb{R}^2$, by

$$L^{1,2}(A) := \{u \in L^2_{loc}(A) : \ \nabla u \in L^2(A;\mathbb{R}^2)\}\,.$$

The main properties of the space $L^{1,2}(A)$ are recalled in Subsection 1.7.2.

See Subsection 1.7.2 also for the notion of *capacity* and the notation *q.e.*, abbreviation of *quasi everywhere*.

3.2. Irreversible quasi-static evolution

For every compact set $K \in \mathcal{C}_p$ and every $g \in H^1(\Omega)$ we consider the minimum elastic energy of the unfractured part of the body, given by

$$E(g,K) := \min_{v \in \mathcal{V}(g,K)} \int_{\Omega\setminus K} |\nabla v|^2 dx\,, \tag{3.8}$$

where

$$\mathcal{V}(g,K) := \{v \in L^{1,2}(\Omega \setminus K) : v = g \quad \text{q.e. on } \partial_D\Omega\ \}\,. \tag{3.9}$$

According to Griffith's theory, the dissipation energy is proportional to the "length" of the crack, *i.e.* to the number of broken atomic bonds; in our setting it is given by the functional $\mathcal{L}$ defined in (3.7). Consequently, the total energy of the system is

$$\mathcal{E}(g,K) := E(g,K) + \mathcal{L}(K)\,. \tag{3.10}$$

Remark 3.2.1. The minimum problem (3.8) admits a solution $u \in \mathcal{V}(g,K)$. Indeed, by standard arguments on the minimization of quadratic forms it is easy to see that u is a solution of (3.8) if and only if it solves the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus K \\ u = g & \text{on } \partial_D\Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } K \cup \partial_N\Omega\,. \end{cases} \tag{3.11}$$

Due to the irregularity of K it is clear that the Neumann boundary condition cannot be satisfied in the classical sense. By a solution of (3.11) we mean a function u which satisfies the following conditions:

$$\begin{cases} u \in L^{1,2}(\Omega\setminus K)\,, \quad u = g \quad \text{q.e. on } \partial_D\Omega\,, \\ \langle \nabla u, \nabla z\rangle = 0 \quad \forall z \in L^{1,2}(\Omega \setminus K)\,,\ z = 0 \quad \text{q.e. on } \partial_D\Omega\,. \end{cases} \tag{3.12}$$

The existence of a solution is assured by the Lax-Milgram lemma. We underline that uniqueness is guaranteed only in the connected components of $\Omega\setminus K$ whose boundary intersects $\partial_D\Omega$; in the connected components for which this is not the case, the solution can be any arbitrary constant, therefore uniqueness is lost. However, ∇u is always unique.

Moreover, the map $g \mapsto \nabla u$ is linear from $H^1(\Omega)$ into $L^2(\Omega \setminus K; \mathbb{R}^2)$ and satisfies the estimate

$$\int_{\Omega \setminus K} |\nabla u|^2 \, dx \leq \int_{\Omega} |\nabla g|^2 \, dx \, .$$

Given a time-dependent boundary displacement $t \mapsto g(t)$, we consider quasi-static evolutions of global minimizers, similar to those introduced in Definition 1.4.1, for which an irreversibility condition and an energy balance condition hold.

Definition 3.2.2. Given $T > 0$ and $g \in AC([0,T]; H^1(\Omega))$, we say that a map $K : [0,T] \to \mathcal{C}_p$ is an *irreversible quasi-static evolution* on $[0,T]$ with imposed boundary condition g if it satisfies the following conditions:

(I) irreversibility: $K(s) \subseteq K(t)$ for $0 \leq s \leq t \leq T$,

(GS) global stability: for every $t \in [0,T]$

$$\mathcal{E}(g(t), K(t)) \leq \mathcal{E}(g(t), K)$$

for every $K \in \mathcal{C}_p$, $K \supseteq K(t)$,

(EB) energy balance: for every s, t with $0 \leq s < t \leq T$

$$\mathcal{E}(g(t), K(t)) = \mathcal{E}(g(s), K(s)) + 2 \int_s^t \langle \nabla u(\tau), \nabla \dot{g}(\tau) \rangle \, d\tau \, ,$$

where $u(\tau)$ is a solution of the minimum problem (3.8) which defines $E(g(\tau), K(\tau))$.

This derivative-free form of the problem is an energetic formulation in the sense of Mielke [66], in which the irreversibility condition can be enclosed in the description of the process by means of the so-called dissipation distance (see Subsection 1.4.1).

We now state the main result of the paper.

Theorem 3.2.3. *Let $T > 0$ and $g \in AC([0,T]; H^1(\Omega))$. Let $p \geq 1$ and $K_0 \in \mathcal{C}_p$. Then there exists an irreversible quasi-static evolution $K \colon [0,T] \to \mathcal{C}_p$ such that $K_0 \subseteq K(0)$ and*

$$\mathcal{E}(g(0), K(0)) \leq \mathcal{E}(g(0), K) \tag{3.13}$$

for every $K \in \mathcal{C}_p$ with $K \supseteq K_0$.

3.3. Properties of sets in $\mathcal{C}_p$ and lower semicontinuity of $\mathcal{L}$

In this section we prove some geometrical, topological and metric properties for the class $\mathcal{C}_p$, the lower semicontinuity of the functional $\mathcal{L}$, and an approximation result for sets in $\mathcal{C}_p$ that will play an important role in the proof of the global minimality conditions (GS) in Definition 3.2.2 and of (3.13). We begin by showing the (sequential) compactness of the class $\mathcal{C}_p$.

Proposition 3.3.1. *If (K_n) is a sequence in $\mathcal{C}_p$, then there exists a subsequence which converges to a set $K \in \mathcal{C}_p$ in the Hausdorff distance.*

Proof. By Blaschke Selection Theorem 1.7.1, there exists a subsequence (not relabelled) converging to a nonempty compact set K. As all K_n are contained in the union $\mathcal{K}_1 \cup \ldots \cup \mathcal{K}_M$, also the limit K is. A simple contradiction argument is enough to prove that the number of connected components of K is at most p. □

We now establish some results on the lower semicontinuity of the Hausdorff measures $\mathcal{H}^d$ (and of the functional $\mathcal{L}$) with respect to the Hausdorff convergence in $\mathcal{C}_p$.

Proposition 3.3.2. *Let (K_n) be a sequence of closed connected nonempty subsets of $\mathcal{K}_1$ converging to K in the Hausdorff metric. Then for every open set $U \subset \Omega$ it holds*

$$\mathcal{H}^{d_1}(K \cap U) \leq \liminf_{n \to +\infty} \mathcal{H}^{d_1}(K_n \cap U). \tag{3.14}$$

PROOF. The set $\mathcal{K}_1 \cap U$ is made of at most countable many connected components $\hat{\mathcal{K}}^i$, of the form

$$\hat{\mathcal{K}}^i = \gamma_1(I^i)$$

with $I^i \subset [0,1]$ an interval, and $I^i \cap I^j = \emptyset$ for $i \neq j$.

Let $K_n = \gamma_1([a_n, b_n])$ and $K = \gamma_1([a,b])$. Then $K_n \cap \hat{\mathcal{K}}^i = \gamma_1([a_n, b_n] \cap I^i)$ and $K \cap \hat{\mathcal{K}}^i = \gamma_1([a,b] \cap I^i)$. By the Hausdorff convergence and (3.3) (with $d = d_1$) we have $a_n \to a$ and $b_n \to b$.

For every $i \in \mathbb{N}$, by (3.5) it holds

$$\mathcal{H}^{d_1}(K \cap \hat{\mathcal{K}}^i) = \lim_{n \to +\infty} \mathcal{H}^{d_1}(K_n \cap \hat{\mathcal{K}}^i).$$

Therefore for every $N \in \mathbb{N}$ we have

$$\begin{aligned}
\mathcal{H}^{d_1}(K \cap \bigcup_{i=1}^N \hat{\mathcal{K}}^i) &= \sum_{i=1}^N \mathcal{H}^{d_1}(K \cap \hat{\mathcal{K}}^i) = \sum_{i=1}^N \lim_{n \to +\infty} \mathcal{H}^{d_1}(K_n \cap \hat{\mathcal{K}}^i) \\
&= \lim_{n \to +\infty} \mathcal{H}^{d_1}(K_n \cap \bigcup_{i=1}^N \hat{\mathcal{K}}^i) \\
&\leq \liminf_{n \to +\infty} \mathcal{H}^{d_1}(K_n \cap U).
\end{aligned}$$

As $N \to \infty$, we obtain (3.14). □

Proposition 3.3.3. *Let (K_n) be a sequence in $\mathcal{C}_1$ converging to K in the Hausdorff metric. Then*

$$\mathcal{L}(K) \leq \liminf_{n \to +\infty} \mathcal{L}(K_n)\,.$$

PROOF. For simplicity, we consider the case $M = 2$. We have to prove that

$$\mathcal{H}^{d_1}(K \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(K \cap \mathcal{K}_2) \leq \liminf_{n \to +\infty} \left(\mathcal{H}^{d_1}(K_n \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(K_n \cap \mathcal{K}_2) \right)\,.$$

If either $K_n \subset \mathcal{K}_1$ for all n large enough, or $K_n \subset \mathcal{K}_2$, the result follows by Proposition 3.3.2 with $U = \Omega$.

Assume now that $K_n \setminus \mathcal{K}_1 \neq \emptyset \neq K_n \setminus \mathcal{K}_2$ for all n large. We first prove that

$$\mathcal{H}^{d_1}(K \setminus \mathcal{K}_2) \leq \liminf_{n \to +\infty} \mathcal{H}^{d_1}(K_n \setminus \mathcal{K}_2)\,. \tag{3.15}$$

For every $\varepsilon > 0$, consider the open set

$$U_\varepsilon := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \mathcal{K}_2) < \varepsilon\}\,.$$

Let V be an open set with $V \subset\subset \mathbb{R}^2 \setminus \overline{U}_\varepsilon$, and define $\delta := \operatorname{dist}(V, \partial U_\varepsilon)$. We claim that the number of connected components C of $K_n \setminus U_\varepsilon$ that intersect V is uniformly bounded with respect to n. Indeed, if $C \cap \partial U_\varepsilon \neq \emptyset$, then by (3.3) and the fact that $C \subset \mathcal{K}_1$ it is

$$\frac{L_1}{c_1} \delta^{d_1} \leq \mathcal{H}^{d_1}(C) \leq L_1.$$

Hence the number of these connected components is at most c_1/δ^{d_1}. If $C \cap \partial U_\varepsilon = \emptyset$, then $C \subset \mathcal{K}_1 \setminus \mathcal{K}_2$ and it is a connected component of K_n, so that actually $C = K_n$.

Let $F_n^1, \dots, F_n^{N_n}$ be the connected components of $K_n \setminus U_\varepsilon$ which intersect V. Up to subsequences, we can assume that $N_n = N \le 1 + c_1/\delta^{d_1}$ for every n and $F_n^i \to F^i$ in the Hausdorff metric, for $i = 1, \dots, N$. Notice that

$$K \cap V \subset F^1 \cup \ldots \cup F^N.$$

Indeed, if $x \in K \cap V$ there exists $x_n \in K_n$ converging to x. For n large enough, $x_n \in V$, so that $x_n \in F_n^{i_n}$ for some $i \in \{1, \dots, N\}$. Therefore, there exists i such that $i_n = i$ for infinitely many n, hence $x \in F^i$.

By the fact that F_n^i and F^i verify the hypotheses of Proposition 3.3.2, and for any fixed n the curves F_n^i are pairwise disjoint, we have

$$\begin{aligned}\mathcal{H}^{d_1}(K \cap V) &\le \sum_{i=1}^N \mathcal{H}^{d_1}(F^i) \le \sum_{i=1}^N \liminf_{n\to+\infty} \mathcal{H}^{d_1}(F_n^i) \\ &\le \liminf_{n\to+\infty} \mathcal{H}^{d_1}(F_n^1 \cup \ldots \cup F_n^N) \le \liminf_{n\to+\infty} \mathcal{H}^{d_1}(K_n \setminus U_\varepsilon) \\ &\le \liminf_{n\to+\infty} \mathcal{H}^{d_1}(K_n \setminus \mathcal{K}_2)\,.\end{aligned}$$

As $V \nearrow \mathbb{R}^2 \setminus \mathcal{K}_2$, we obtain (3.15).

Of course, in an analogous way we can prove that

$$\mathcal{H}^{d_2}(K \setminus \mathcal{K}_1) \le \liminf_{n\to+\infty} \mathcal{H}^{d_2}(K_n \setminus \mathcal{K}_1)\,.$$

Being $\mathcal{H}^{d_j}(\mathcal{K}_1 \cap \mathcal{K}_2) = 0$ for $j = 1, 2$ by (3.6), we can conclude that

$$\begin{aligned}\mathcal{H}^{d_1}(K \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(K \cap \mathcal{K}_2) &= \mathcal{H}^{d_1}(K \setminus \mathcal{K}_2) + \mathcal{H}^{d_2}(K \setminus \mathcal{K}_1) \\ &\le \liminf_{n\to+\infty} \mathcal{H}^{d_1}(K_n \setminus \mathcal{K}_2) + \liminf_{n\to+\infty} \mathcal{H}^{d_2}(K_n \setminus \mathcal{K}_1) \\ &\le \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}(K_n \setminus \mathcal{K}_2) + \mathcal{H}^{d_2}(K_n \setminus \mathcal{K}_1)\right) \\ &= \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}(K_n \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(K_n \cap \mathcal{K}_2)\right)\,.\end{aligned}$$

The general case can be proved similarly. □

Corollary 3.3.4. *Let (K_n) be a sequence in $\mathcal{C}_p$ that converges in the Hausdorff metric to a set K, and let $U \subset \Omega$ be an open set. Then*

$$\mathcal{L}(K \cap U) \le \liminf_{n\to+\infty} \mathcal{L}(K_n \cap U)\,. \tag{3.16}$$

Proof. For simplicity we consider the case when $M = 2$ and the sets K_n are connected. We have to show that for every open set $U \subset \Omega$ it holds

$$\mathcal{H}^{d_1}(K \cap \mathcal{K}_1 \cap U) + \mathcal{H}^{d_2}(K \cap \mathcal{K}_2 \cap U) \le \liminf_{n\to+\infty}(\mathcal{H}^{d_1}(K_n \cap \mathcal{K}_1 \cap U) + \mathcal{H}^{d_2}(K_n \cap \mathcal{K}_2 \cap U))\,. \tag{3.17}$$

Consider V_1 and V_2 open sets, such that $V_1 \subset\subset V_2 \subset\subset U$. Arguing as in the proof of Proposition 3.3.3, the number of connected components $F_n^1, \dots, F_n^{N_n}$ of $K_n \cap \overline{V_2}$ which intersect V_1 is uniformly bounded. As before, we can assume that $N_n = N$ and

$$K \cap V_1 \subset F^1 \cup \ldots \cup F^N,,$$

where F^i is the limit of F^i_n in the Hausdorff metric as $n \to +\infty$, for $i = 1, \dots, N$. Observe that the sequences (F^i_n) satisfy the hypotheses of Proposition 3.3.3. Then we have

$$\begin{aligned}\mathcal{H}^{d_1}(K \cap \mathcal{K}_1 \cap V_1) + \mathcal{H}^{d_2}(K \cap \mathcal{K}_2 \cap V_1) &\le \sum_{i=1}^{N} \mathcal{H}^{d_1}(F^i \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(F^i \cap \mathcal{K}_2) \\ &\le \sum_{i=1}^{N} \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}(F^i_n \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(F^i_n \cap \mathcal{K}_2)\right) \\ &\le \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}\Big(\bigcup_{i=1}^{N} F^i_n \cap \mathcal{K}_1\Big) + \mathcal{H}^{d_2}\Big(\bigcup_{i=1}^{N} F^i_n \cap \mathcal{K}_2\Big)\right) \\ &\le \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}(K_n \cap \mathcal{K}_1 \cap U) + \mathcal{H}^{d_2}(K_n \cap \mathcal{K}_2 \cap U)\right) .\end{aligned}$$

As $V_1 \nearrow U$, we obtain (3.17). □

Corollary 3.3.5. *Let (K_n) be a sequence in $\mathcal{C}_p$ converging to K in the Hausdorff metric. Let (H_n) be a sequence of compact sets converging to H in the Hausdorff metric. Then*

$$\mathcal{L}(K \setminus H) \le \liminf_{n\to+\infty} \mathcal{L}(K_n \setminus H_n) .$$

Proof. For every $\varepsilon > 0$, let $U_\varepsilon := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, H) < \varepsilon\}$. Since, for n large, $H_n \subset U_\varepsilon$, it is $K_n \setminus \overline{U}_\varepsilon \subset K_n \setminus H_n$. By Corollary 3.3.4 with $U = \mathbb{R}^2 \setminus \overline{U}_\varepsilon$, we have

$$\mathcal{L}(K \setminus \overline{U}_\varepsilon) \le \liminf_{n\to+\infty} \mathcal{L}(K_n \setminus \overline{U}_\varepsilon) \le \liminf_{n\to+\infty} \mathcal{L}(K_n \setminus H_n).$$

The thesis follows letting $\varepsilon \to 0$. □

We need to establish a connection between some topological and measure properties of elements in $\mathcal{C}_p$, which will be useful in the proof of Theorem 3.4.1 on the continuity of minimizers of (3.8) as K varies in $\mathcal{C}_p$.

Lemma 3.3.6. *Let $K \in \mathcal{C}_1$ with $\mathcal{L}(K) = 0$. Then $K = \{x\}$.*

Proof. For simplicity, assume $M = 2$. If $K \subset \mathcal{K}_1$ or $K \subset \mathcal{K}_2$ the conclusion follows from Remark 3.1.1 and (3.5).

Assume now that

$$K \setminus \mathcal{K}_1 \neq \emptyset \neq K \setminus \mathcal{K}_2 \tag{3.18}$$

and let

$$U_\varepsilon := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \mathcal{K}_2) < \varepsilon\} .$$

Notice that there exists $\overline{\varepsilon} > 0$ such that $K \setminus U_{\overline{\varepsilon}} \neq \emptyset$. Indeed, otherwise $K \subset \bigcap_{\varepsilon>0} U_\varepsilon = \mathcal{K}_2$ which contradicts $K \setminus \mathcal{K}_2 \neq \emptyset$. As $K \cap \mathcal{K}_2 \neq \emptyset$ we have also $K \cap U_{\overline{\varepsilon}/2} \neq \emptyset$. Since K is connected we deduce that there exists a connected subset C of K which intersects both $\partial U_{\overline{\varepsilon}}$ and $\partial U_{\overline{\varepsilon}/2}$ (otherwise K would have at least two connected components). Then $C \subset \mathcal{K}_1$ and $\operatorname{diam} C > \overline{\varepsilon}/2$. By (3.5) we have $\mathcal{H}^{d_1}(C) > 0$, in contradiction with $\mathcal{L}(K) = 0$. This shows that (3.18) cannot happen, therefore either $K \setminus \mathcal{K}_1 = \emptyset$ or $K \setminus \mathcal{K}_2 = \emptyset$, which is the situation considered at the beginning of the proof. □

Lemma 3.3.7. *For every $l > 0$ there exists a constant $C_l > 0$ such that, if $K \in \mathcal{C}_1$ with $\operatorname{diam} K > l$, then $\mathcal{L}(K) > C_l$.*

Proof. By contradiction, assume that there exists $l > 0$ such that, for every $n \in \mathbb{N}$, there exists $K_n \in \mathcal{C}_1$, with $\operatorname{diam} K_n > l$ and $\mathcal{L}(K_n) \le 1/n$.

Up to subsequences, by Proposition 3.3.1 we can assume that (K_n) converges to a set $K \in \mathcal{C}_1$ in the Hausdorff metric. By the lower semicontinuity of $\mathcal{L}$ (Proposition 3.3.3), we have

$$\mathcal{L}(K) \le \liminf_{n \to +\infty} \mathcal{L}(K_n) = 0\,.$$

Then Lemma 3.3.6 implies that K is a singleton: $K = \{z\}$.

On the other hand, since $\operatorname{diam} K_n > l$ there exists $x_n, y_n \in K_n$ with

$$|x_n - y_n| > l\,. \tag{3.19}$$

By Hausdorff convergence it is $x_n, y_n \to z$, which is clearly a contradiction to (3.19). □

In [73, §2.2] the following definition is given.

Definition 3.3.8. A closed set $A \subset \mathbb{R}^2$ is locally connected if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any two points $x, y \in A$ with $|x - y| < \delta$ we can find a continuum (*i.e.* compact connected set) B with $x, y \in B \subset A$, $\operatorname{diam} B < \varepsilon$.

Lemma 3.3.9. *If $K \in \mathcal{C}_p$ then K is locally connected.*

Proof. We follow the proof of [25, Lemma 1]. It is enough to prove the result for a single connected component of K, since we can choose δ in Definition 3.3.8 smaller than the distance between two connected components. Assume by contradiction that K is not locally connected; hence there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exist $x_n, y_n \in K$ with $|x_n - y_n| < \frac{1}{n}$ with the property that any continuum $B \subset K$ connecting x_n to y_n must have $\operatorname{diam} B > \varepsilon$. Note that such an ε is necessarily less than $\operatorname{diam} K$. Up to subsequences, we may assume that $\lim_n x_n = \lim_n y_n = z \in K$, $x_n \in \mathcal{K}_{m_1}$, and $y_n \in \mathcal{K}_{m_2}$. Then $z \in \mathcal{K}_{m_1} \cap \mathcal{K}_{m_2}$.

For n large enough $x_n, y_n \in B(z, \frac{\varepsilon}{2})$. Let $\widetilde{X}_n$ be the connected component of $K \cap \overline{B(z, \frac{\varepsilon}{2})}$ that contains x_n and $\widetilde{Y}_n$ the one containing y_n. Then $\widetilde{X}_n \cap \widetilde{Y}_n = \emptyset$ (otherwise $\widetilde{X}_n \cup \widetilde{Y}_n$ would be a continuum connecting x_n and y_n of diameter less than ε), therefore either $z \notin \widetilde{X}_n$ or $z \notin \widetilde{Y}_n$. Assume $z \notin \widetilde{X}_n$ for infinitely many indices n. As K is connected and $\operatorname{diam} K > \varepsilon$, $\widetilde{X}_n \cap \partial B(z, \frac{\varepsilon}{2}) \neq \emptyset$. Since $x_n \to z$, for n large enough $\widetilde{X}_n \cap B(z, \frac{\varepsilon}{4}) \neq \emptyset$. Thus $\operatorname{diam} \widetilde{X}_n > \varepsilon/4$ and by Lemma 3.3.7, we have $\mathcal{L}(\widetilde{X}_n) > C_\varepsilon > 0$ for every n. Since, except for a finite number, the sets $\widetilde{X}_n$ are pairwise disjoint, we deduce that $\mathcal{L}(K) = +\infty$, which is impossible since $K \in \mathcal{C}_p$. □

The following approximation results for sets in $\mathcal{C}_p$ are in the spirit of [38, Lemmas 3.5-3.8]. In case their proof is only slightly different, we remark the differences and refer to [38] for the core of it.

Lemma 3.3.10. *Let $p, q \ge 1$. Let (H_n) be a sequence in $\mathcal{C}_p$ converging to H in the Hausdorff metric, and let $K \in \mathcal{C}_q$ be such that $H \subset K$. Then there exists a sequence (K_n) in $\mathcal{C}_q$ such that it converges to K in the Hausdorff metric, $H_n \subset K_n$ and $\mathcal{L}(K_n \setminus H_n) \to \mathcal{L}(K \setminus H)$.*

Its proof is a direct consequence of Lemma 3.3.14 below, for which we need some preliminaries.

Lemma 3.3.11. *Let (H_n) be a sequence in $\mathcal{C}_p$ converging to H in the Hausdorff metric, with $H \in \mathcal{C}_1$. Then there exist a sequence $(\widehat{H}_n)$ in $\mathcal{C}_1$ such that $H_n \subset \widehat{H}_n$, $\widehat{H}_n \to H$ in the Hausdorff metric and $\mathcal{L}(\widehat{H}_n \setminus H_n) \to 0$.*

PROOF. Without loss of generality, we may assume that all the sets H_n have exactly $q \le p$ connected components $H_n^1, \dots H_n^q$ with H_n^i converging to $\widetilde{H}^i$ in the Hausdorff metric, for $i = 1, \dots, q$, with $\widetilde{H}^i \in \mathcal{C}_1$; of course, $H = \widetilde{H}^1 \cup \dots \cup \widetilde{H}^q$.

Being H connected, there exists a finite set of indices $(\sigma_i)_{1 \le i \le l}$ such that $\{\sigma_1, \dots, \sigma_l\} = \{1, \dots, q\}$ and $\widetilde{H}^{\sigma_i} \cap \widetilde{H}^{\sigma_{i+1}} \neq \emptyset$ for every $i = 1, \dots, l-1$. Fixed a point $x^i \in \widetilde{H}^{\sigma_i} \cap \widetilde{H}^{\sigma_{i+1}}$ for every $i = 1, \dots, l-1$, consider $x_n^i \in H_n^{\sigma_i}$ and $y_n^i \in H_n^{\sigma_{i+1}}$ with $x_n^i, y_n^i \to x^i$ as $n \to +\infty$.

Fix $i \in \{1, \dots, l\}$. For every $m = 1, \dots, M$, let

$$I_m := \{n \in \mathbb{N} : \, x_n^i \in \mathcal{K}_m\}.$$

For m with I_m infinite, it is $x^i \in \mathcal{K}_m$. For such indices m and for every $n \in I_m$ consider the arc $X_n^i \subset \mathcal{K}_m$ connecting x_n^i and x^i. Then, by (3.4) and (3.3), we have that $\mathcal{H}^{d_m}(X_n^i) \le c_m |x_n^i - x^i|^{d_m}$, with c_m independent of i and n. Hence $\mathcal{H}^{d_m}(X_n^i) \to 0$ as $n \to +\infty$.

Similarly, defined J_m for the points y_n^i, we choose the sets Y_n^i. Finally we set

$$\widehat{H}_n := H_n \cup \bigcup_{i=1}^{l-1} X_n^i \cup \bigcup_{i=1}^{l-1} Y_n^i .$$

By Lemma 3.3.7 we obtain that X_n^i and Y_n^i converge to $\{x^i\}$ in the Hausdorff metric, so that $\widehat{H}_n \to H$; in addition $\mathcal{L}(\widehat{H}_n \setminus H_n) \to 0$. Finally, being

$$\widehat{H}_n = H_n^{\sigma_1} \cup X_n^1 \cup Y_n^1 \cup H_n^{\sigma_2} \cup \dots \cup H_n^{\sigma_{l-1}} \cup X_n^{l-1} \cup Y_n^{l-1} \cup H_n^{\sigma_l} ,$$

the sets $\widehat{H}_n$ are connected and contained in $\bigcup_{m=1,\dots,M} \mathcal{K}_m$, *i.e* $\widehat{H}_n \in \mathcal{C}_1$. □

Lemma 3.3.12. *If C is a connected subset of $\mathcal{K}_1 \cup \dots \cup \mathcal{K}_M$, then $\mathcal{L}(\overline{C}) = \mathcal{L}(C)$.*

PROOF. For simplicity, we assume $M = 2$. If $C \subset \mathcal{K}_1$, then by Remark 3.1.1 $C = \gamma_1(I)$, where $I \subset [0,1]$ is an interval of the form (a,b), $[a,b)$, $(a,b]$ or $[a,b]$. By (3.4), the thesis follows. The case $C \subset \mathcal{K}_2$ is analogous.

For every $\varepsilon > 0$ let $U_\varepsilon := \{x \in \Omega : \, \mathrm{dist}(x, \mathcal{K}_2) < \varepsilon\}$. Arguing as in the proof of Proposition 3.3.3, the number of connected components F of $C \setminus \mathcal{K}_2$ such that $\overline{F} \cap \mathcal{K}_2 \neq \emptyset \neq F \setminus U_\varepsilon$ is finite, say N_ε. Note that $\overline{C} \setminus U_\varepsilon \subset \bigcup_{i=1}^{N_\varepsilon} \overline{F}_i$. In addition, by construction $F_i \subset \mathcal{K}_1$, $F_i \cap F_j = \emptyset$ for $i \neq j$, and $\mathcal{H}^{d_1}(\overline{F}_i) = \mathcal{H}^{d_1}(F_i)$ by the previous part. Then we have

$$\mathcal{H}^{d_1}(\overline{C} \setminus U_\varepsilon) \le \sum_{i=1}^{N_\varepsilon} \mathcal{H}^{d_1}(\overline{F}_i) = \sum_{i=1}^{N_\varepsilon} \mathcal{H}^{d_1}(F_i) \le \mathcal{H}^{d_1}(C \setminus \mathcal{K}_2) .$$

As $\varepsilon \to 0$, we obtain $\mathcal{H}^{d_1}(\overline{C} \setminus \mathcal{K}_2) \le \mathcal{H}^{d_1}(C \setminus \mathcal{K}_2)$; hence the equality holds.

Similarly, we have $\mathcal{H}^{d_2}(\overline{C} \setminus \mathcal{K}_1) \le \mathcal{H}^{d_2}(C \setminus \mathcal{K}_1)$. Recalling the definition (3.7) of $\mathcal{L}$, and (3.6), the thesis follows. □

Lemma 3.3.13. *Let $K \in \mathcal{C}_1$ and $H \subset K$ be a compact set with $p \ge 2$ connected components $H^1, \dots, H^p$. Then there exists a family of indices $(\sigma_j)_{0 \le j \le l}$, with $\{\sigma_0, \dots, \sigma_l\} = \{1, \dots, p\}$, and a family $(\Gamma_j)_{0 \le j \le l}$ of connected components of $K \setminus H$, such that $\overline{\Gamma}_j$ connects $H^{\sigma_{j-1}}$ with H^{σ_j} for $1 \le j \le l$.*

PROOF. It is enough to argue as in [38, Lemma 3.7], noticing that: by Lemma 3.3.9 the set K is locally connected; by Lemma 3.3.12 it is $\mathcal{L}(\overline{C}_n) = \mathcal{L}(C_n)$, where C_n are defined in the cited result. □

Lemma 3.3.14. *Let (H_n) be a sequence in $\mathcal{C}_p$ converging to H in the Hausdorff metric, and let $K \in \mathcal{C}_1$ be such that $H \subset K$. Then there exists a sequence (K_n) in $\mathcal{C}_1$ such that (K_n) converges to K in the Hausdorff metric, $H_n \subset K_n$ and $\mathcal{L}(K_n \setminus H_n) \to \mathcal{L}(K \setminus H)$.*

PROOF. Following the strategy of [38, Lemma 3.8], apply Lemma 3.3.13, Lemma 3.3.11, Lemma 3.3.12 and Corollary 3.3.5 instead of Lemma 3.7, Lemma 3.6, Proposition 2.5 and Corollary 3.4 in [38], respectively. In the construction of the sets corresponding to X_n^j, Y_n^j and Z_n^i in [38], it is enough to argue as in Lemma 3.3.11. ☐

3.4. Proof of the main result

In this section we prove the existence of a quasi-static evolution for cracks in $\mathcal{C}_p$, satisfying the global minimality condition and the energy balance (Theorem 3.2.3), by the usual time discretization procedure described in Section 1.6. We follow the steps of [38].

We shall need the following result on the convergence of the minimum points of problems (3.8) corresponding to converging sequences in $\mathcal{C}_p$. The statement is completely analogous to that of Theorem 1.7.6, but we cannot directly apply the cited result since it is valid in case of sets of Hausdorff dimension 1; however, considering the properties of $\mathcal{C}_p$ proved in Section 3.3, the proof follows the same steps as for Theorem 1.7.6 (or [38, Theorem 5.1]), as explained below.

Theorem 3.4.1. *Let (K_n) be a sequence in $\mathcal{C}_p$ converging to K in the Hausdorff distance, and let (g_n) be a sequence in $H^1(\Omega)$ which converges to g strongly in $H^1(\Omega)$. Let u_n be a solution of the minimum problem*

$$E(g_n, K_n) = \min_{v \in \mathcal{V}(g_n, K_n)} \int_{\Omega \setminus K_n} |\nabla v|^2 \, dx \,,$$

and let u be a solution of the minimum problem (3.8)

$$E(g, K) = \min_{v \in \mathcal{V}(g, K)} \int_{\Omega \setminus K} |\nabla v|^2 \, dx \,,$$

where $\mathcal{V}(g_n, K_n)$ and $\mathcal{V}(g, K)$ are defined by (3.9). Then $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^2)$.

PROOF. The proof can be done in the same manner as for [38, Theorem 5.1], as long as we check that the key facts therein are satisfied. The first one lies in the application of [38, Theorem 4.3], for which the set K needs to be locally connected; in our case this is assured by Lemma 3.3.9.

The second important step is the following: given any $x \in \overline{\Omega}$, an open rectangle V containing x and an open set $U \subset\subset V$, we need to bound uniformly the number of connected components of $\overline{V} \cap K_n$ which meet U. We can argue in the following way. Let $l = \mathrm{dist}(U, \partial V)$ and let C be a connected component of $\overline{V} \cap K_n$ which meets U. If $C \cap \partial V \neq \emptyset$, then $\mathrm{diam}\, C \geq l$ and by Lemma 3.3.7 there exists a constant C_l such that $\mathcal{L}(C) \geq C_l$. Being $\mathcal{L}(K_n) \leq \lambda$, the number of those connected components is smaller than λ / C_l. If $C \cap \partial V = \emptyset$, then C is a connected component of K_n, and there are at most p of them.

Having established these two key issues, the proof carries on as in the cited result, based on the construction of a harmonic conjugate for u. ☐

Given $\tau > 0$, we denote by N_τ the largest integer such that $\tau N_\tau \leq T$; for $0 \leq i \leq N_\tau$, let $t_i^\tau := i\tau$ and $g_i^\tau := g(t_i^\tau)$. The sets K_i^τ are defined inductively as a solution to the following minimization problem

$$\min_K \left\{ \mathcal{E}(g_i^\tau, K) : K \in \mathcal{C}_p \,, \ K \supseteq K_{i-1}^\tau \right\} , \tag{3.20}$$

where we set $K_{-1}^\tau := K_0$.

Lemma 3.4.2. *There exists a solution of the minimum problem (3.20).*

PROOF. Assume by induction that $K^\tau_{i-1} \in \mathcal{C}_p$. Consider a minimizing sequence (K_n) of problem (3.20). By Proposition 3.3.1, we may assume that (up to a subsequence) (K_n) converges in the Hausdorff distance to some compact set $K \in \mathcal{C}_p$ which contains K^τ_{i-1}. For every n let u_n be a solution of the minimum problem (3.8) which defines $E(g^\tau_i, K_n)$.

By Theorem 3.4.1 the sequence (∇u_n) converges strongly in $L^2(\Omega;\mathbb{R}^2)$ to ∇u, where u is a solution of the minimum problem (3.8) which defines $E(g^\tau_i, K)$. As by Corollary 3.3.4

$$\mathcal{L}(K) \leq \liminf_n \mathcal{L}(K_n)\,,$$

we conclude that $\mathcal{E}(g^\tau_i, K) \leq \liminf_n \mathcal{E}(g^\tau_i, K_n)$. Since (K_n) is a minimizing sequence, this proves that K is a solution of the minimum problem (3.20). □

We define now the piecewise constant functions g_τ, K_τ, and u_τ on $[0,T]$ by setting $g_\tau(t) := g^\tau_i = g(t^\tau_i)$, $K_\tau(t) := K^\tau_i$, and $u_\tau(t) := u^\tau_i$ for $t^\tau_i \leq t < t^\tau_{i+1}$, where u^τ_i is a solution of the minimum problem (3.8) which defines $E(g^\tau_i, K^\tau_i)$.

Lemma 3.4.3. *There exists a positive function $\rho(\tau)$, converging to zero as $\tau \to 0$, such that*

$$\|\nabla u^\tau_j\|^2 + \mathcal{L}(K^\tau_j) \leq \|\nabla u^\tau_i\|^2 + \mathcal{L}(K^\tau_i) + 2\int_{t^\tau_i}^{t^\tau_j} \langle \nabla u_\tau(t), \nabla \dot g(t)\rangle\, dt + \rho(\tau) \tag{3.21}$$

for $0 \leq i < j \leq N_\tau$.

PROOF. Let us fix an integer r with $i \leq r < j$. From the absolute continuity of g we have

$$g^\tau_{r+1} - g^\tau_r = \int_{t^\tau_r}^{t^\tau_{r+1}} \dot g(t)\, dt\,,$$

where the integral is a Bochner integral for functions with values in $H^1(\Omega)$. This implies that

$$\nabla g^\tau_{r+1} - \nabla g^\tau_r = \int_{t^\tau_r}^{t^\tau_{r+1}} \nabla \dot g(t)\, dt\,, \tag{3.22}$$

where the integral is a Bochner integral for functions with values in $L^2(\Omega;\mathbb{R}^2)$.

As $u^\tau_r + g^\tau_{r+1} - g^\tau_r \in L^{1,2}(\Omega\backslash K^\tau_r)$ and $u^\tau_r + g^\tau_{r+1} - g^\tau_r = g^\tau_{r+1}$ q.e. on $\partial_D\Omega\backslash K^\tau_r$, we have

$$\mathcal{E}(g^\tau_{r+1}, K^\tau_r) \leq \|\nabla u^\tau_r + \nabla g^\tau_{r+1} - \nabla g^\tau_r\|^2 + \mathcal{L}(K^\tau_r)\,. \tag{3.23}$$

By the minimality of u^τ_{r+1} and by (3.20) it is

$$\|\nabla u^\tau_{r+1}\|^2 + \mathcal{L}(K^\tau_{r+1}) = \mathcal{E}(g^\tau_{r+1}, K^\tau_{r+1}) \leq \mathcal{E}(g^\tau_{r+1}, K^\tau_r)\,. \tag{3.24}$$

From (3.22), (3.23), and (3.24) we obtain

$$\begin{aligned}
\|\nabla u^\tau_{r+1}\|^2 + \mathcal{L}(K^\tau_{r+1}) &\leq \|\nabla u^\tau_r + \nabla g^\tau_{r+1} - \nabla g^\tau_r\|^2 + \mathcal{L}(K^\tau_r) \\
&\leq \|\nabla u^\tau_r\|^2 + \mathcal{L}(K^\tau_r) + 2\int_{t^\tau_r}^{t^\tau_{r+1}} \langle \nabla u^\tau_r, \nabla \dot g(t)\rangle\, dt + \Big(\int_{t^\tau_r}^{t^\tau_{r+1}} \|\nabla \dot g(t)\|\, dt\Big)^2 \\
&\leq \|\nabla u^\tau_r\|^2 + \mathcal{L}(K^\tau_r) + 2\int_{t^\tau_r}^{t^\tau_{r+1}} \langle \nabla u_\tau(t), \nabla \dot g(t)\rangle\, dt + \sigma(\tau)\int_{t^\tau_r}^{t^\tau_{r+1}} \|\nabla \dot g(t)\|\, dt\,,
\end{aligned}$$

where

$$\sigma(\tau) := \max_{0\leq r<N_\tau} \int_{t^\tau_r}^{t^\tau_{r+1}} \|\nabla \dot g(t)\|\, dt \;\longrightarrow\; 0$$

by the absolute continuity of the integral. Iterating this inequality for $i \leq r < j$ we get (3.21) with $\rho(\tau) := \sigma(\tau)\int_0^T \|\nabla \dot g(t)\|\, dt$. □

Lemma 3.4.4. *There exists a constant λ, depending only on g and K_0, such that*

$$\|\nabla u_i^\tau\| \leq \lambda \quad \textit{and} \quad \sum_{m=1}^{M} \mathcal{H}^{d_m}(K_i^\tau \cap \mathcal{K}_m) \leq \lambda \tag{3.25}$$

for every $\tau > 0$ and for every $0 \leq i \leq N_\tau$.

PROOF. As g_i^τ is admissible for the problem (3.8) which defines $E(g_i^\tau, K_i^\tau)$, by the minimality of u_i^τ we have $\|\nabla u_i^\tau\| \leq \|\nabla g_i^\tau\|$, hence $\|\nabla u_\tau(t)\| \leq \|\nabla g_\tau(t)\|$ for every $t \in [0,T]$. As $t \mapsto g(t)$ is absolutely continuous with values in $H^1(\Omega)$ the function $t \mapsto \|\nabla \dot{g}(t)\|$ is integrable on $[0,T]$ and there exists a constant $C > 0$ such that $\|\nabla g(t)\| \leq C$ for every $t \in [0,T]$. This implies the first bound in (3.25).

The latter inequality follows now from Lemma 3.4.3 and from the inequality $\|\nabla u_0^\tau\|^2 + \mathcal{L}(K_0^\tau) \leq \|\nabla g(0)\|^2 + \mathcal{L}(K(0))$, which is an obvious consequence of (3.20) for $i = 0$. □

At this point we have all the elements to obtain a continuous-time evolution as limit of discrete-time ones when the time step τ vanishes.

By a generalization of Helly's Theorem (see, e.g., [38]), there exists a subsequence of K_τ, not relabelled, and an increasing function $K : [0,T] \to \mathcal{C}_p$ such that

$$K_\tau(t) \to K(t)$$

in the Hausdorff metric for every $t \in [0,T]$.

In the rest of this section, when we write $\tau \to 0$, we always refer to the sequence given above by Helly's Theorem.

For every $t \in [0,T]$ let $u(t)$ be a solution of the minimum problem (3.8) which defines $E(g(t), K(t))$. Then

$$\mathcal{E}(g(t), K(t)) = \|\nabla u(t)\|^2 + \mathcal{L}(K(t)).$$

Lemma 3.4.5. *For every $t \in [0,T]$ we have $\nabla u_\tau(t) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$.*

PROOF. As $u_\tau(t)$ is a solution of the minimum problem (3.8) which defines $E(g_\tau(t), K_\tau(t))$, and $g_\tau(t) \to g(t)$ strongly in $H^1(\Omega)$, the conclusion follows from Theorem 3.4.1. □

Lemma 3.4.6. *For every $t \in [0,T]$ we have*

$$\mathcal{E}(g(t), K(t)) \leq \mathcal{E}(g(t), K) \quad \forall\, K \in \mathcal{C}_p\,,\ K \supset K(t)\,. \tag{3.26}$$

Moreover

$$\mathcal{E}(g(0), K(0)) \leq \mathcal{E}(g(0), K) \quad \forall\, K \in \mathcal{C}_p\,,\ K \supset K_0\,. \tag{3.27}$$

PROOF. Fix $t \in [0,T]$. By construction, $K(t)$ is the limit of the sequence $(K_\tau(t))$ in the Hausdorff metric as τ vanishes. Fix $K \in \mathcal{C}_p$ with $K \supset K(t)$. Applying Lemma 3.3.10 we find a sequence (K_τ) in $\mathcal{C}_p$ with $K_\tau \supset K_\tau(t)$, such that $K_\tau \to K$ in the Hausdorff metric and $\mathcal{L}(K_\tau \setminus K_\tau(t)) \to \mathcal{L}(K \setminus K(t))$.

Consider the minimizers v_τ and v of the elastic energies corresponding to $E(g_\tau(t), K_\tau)$ and $E(g(t), K)$, respectively. By Theorem 3.4.1 we have that $\nabla v_\tau \to \nabla v$ strongly in $L^2(\Omega; \mathbb{R}^2)$. By the choice of $K_\tau(t)$ as minimizers of (3.20), it is $\mathcal{E}(g_\tau(t), K_\tau(t)) \leq \mathcal{E}(g_\tau(t), K_\tau)$, which implies $\|\nabla u_\tau(t)\|^2 \leq \|\nabla v_\tau\|^2 + \mathcal{L}(K_\tau \setminus K_\tau(t))$. By Lemma 3.4.5 and the properties of the sequence (K_τ), we obtain $\|\nabla u(t)\|^2 \leq \|\nabla v\|^2 + \mathcal{L}(K \setminus K(t))$. To get (3.26) it is now enough to add $\mathcal{L}(K(t))$ to both sides of the last inequality.

The proof for (3.27) is similar, exploiting the minimality of $K_\tau(0)$ in (3.20) with respect to all sets $K \in \mathcal{C}_p$ containing K_0, and applying Corollary 3.3.5 for the functional $\mathcal{L}$. □

The previous lemma proves the global minimality conditions (GS) in Definition 3.2.2 and also (3.13).

Finally, after a technical result, we will deal with the energy balance (EB), the only missing property in Theorem 3.2.3.

Lemma 3.4.7. *For every $K \in \mathcal{C}_p$ the function $\mathcal{E}(\cdot, K) : H^1(\Omega) \to \mathbb{R}$ is of class C^1, and for every $g, h \in H^1(\Omega)$ it is*

$$\partial_g \mathcal{E}(g, K)[h] = 2\langle \nabla u(g, K), \nabla h \rangle \,, \tag{3.28}$$

where $u(g, K)$ is the solution to the minimum problem (3.8).

Proof. Being K fixed, for simplicity of notation we write $u_g := u(g, K)$. By linearity, for every $\eta \in \mathbb{R}$ it is $u_{g+\eta h} = u_g + \eta u_h$ a.e. in Ω. Then

$$\begin{aligned} \mathcal{E}(g + \eta h, K) - \mathcal{E}(g, K) &= \|\nabla u_{g+\eta h}\|^2 - \|\nabla u_g\|^2 \\ &= 2\eta \, \langle \nabla u_g, \nabla u_h \rangle + \eta^2 \|\nabla u_h\|^2 = 2\eta \, \langle \nabla u_g, \nabla h \rangle + \eta^2 \|\nabla u_h\|^2, \end{aligned}$$

where the last equality is obtained by (3.12) with $z = u_h - h$, since $u_h - h \in L^{1,2}(\Omega \setminus K)$ and $u_h - h = 0$ q.e. on $\partial_D \Omega$. Dividing by $\eta \neq 0$ and letting η vanish, we get (3.28). Finally, the C^1-regularity is consequence of the continuity of the map $g \mapsto \nabla u(g, K)$ (see Theorem 3.4.1). $\square$

Lemma 3.4.8. *For every s, t with $0 \le s < t \le T$*

$$\mathcal{E}(g(t), K(t)) = \mathcal{E}(g(s), K(s)) + 2 \int_s^t \langle \nabla u(r), \nabla \dot g(r) \rangle \, dr \,. \tag{3.29}$$

Proof. The strategy is to show that the map $t \mapsto \mathcal{E}(g(t), K(t))$ is absolutely continuous on $[0, T]$, with pointwise derivative $2\langle \nabla u(t), \nabla \dot g(t) \rangle$ for a.e. $t \in [0, T]$.

Let us fix s, t with $0 \le s < t \le T$, and $\tau > 0$. Applying Lemma 3.4.3 we obtain

$$\|\nabla u_\tau(t)\|^2 + \mathcal{L}(K_\tau(t) \backslash K_\tau(s)) \le \|\nabla u_\tau(s)\|^2 + 2 \int_{s_\tau}^{t_\tau} \langle \nabla u_\tau(r), \nabla \dot g(r) \rangle \, dr + \rho(\tau) \,, \tag{3.30}$$

with $\rho(\tau)$ converging to zero as $\tau \to 0$, where s_τ, t_τ are the discrete times such that $s_\tau \le s < s_\tau + \tau$, $t_\tau \le t < t_\tau + \tau$. For every $r \in [0, T]$ we have, by Lemma 3.4.5, that $\nabla u_\tau(r) \to \nabla u(r)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ as $\tau \to 0$, and, by Lemma 3.4.4, that $\|\nabla u_\tau(r)\| \le \lambda$. Moreover, by Corollary 3.3.5 we get

$$\mathcal{L}(K(t) \backslash K(s)) \le \liminf_{\tau \to 0} \, \mathcal{L}(K_\tau(t) \backslash K_\tau(s)) \,,$$

so that, passing to the limit in (3.30) as $\tau \to 0$, we obtain

$$\mathcal{E}(g(t), K(t)) \le \mathcal{E}(g(s), K(s)) + 2 \int_s^t \langle \nabla u(r), \nabla \dot g(r) \rangle \, dr \,. \tag{3.31}$$

To prove the opposite inequality note that, by the global stability (GS) of Definition 3.2.2 we have $\mathcal{E}(g(s), K(s)) \le \mathcal{E}(g(s), K(t))$, and by Lemma 3.4.7

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(t)) = 2 \int_s^t \langle \nabla u(r, t), \nabla \dot g(r) \rangle \, dr \,,$$

where $u(r, t)$ is a solution of the minimum problem (3.8) which defines $E(g(r), K(t))$. Therefore

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s)) \ge 2 \int_s^t \langle \nabla u(r, t), \nabla \dot g(r) \rangle \, dr \,. \tag{3.32}$$

Since for $s \le r \le t$ the uniform bounds $\|\nabla u(r)\| \le \|\nabla g(r)\| \le C$ and $\|\nabla u(r,t)\| \le \|\nabla g(r)\| \le C$ hold, from (3.31) and (3.32) we obtain

$$\big|\mathcal{E}(g(t),K(t)) - \mathcal{E}(g(s),K(s))\big| \le 2\,C \int_s^t \|\nabla \dot g(r)\|\,dr\,,$$

which proves the absolute continuity of the map $t \mapsto \mathcal{E}(g(t),K(t))$.

Observe that by Theorem 3.4.1 $\nabla u(r,t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ as $r \to t$. Dividing now both (3.31) and (3.32) by $t-s$ and letting $s \to t-$, we get

$$\lim_{s\to t-} \frac{\mathcal{E}(g(t),K(t)) - \mathcal{E}(g(s),K(s))}{t-s} = 2\,\langle \nabla u(t), \nabla \dot g(t)\rangle$$

for a.e. $t \in [0,T]$, and thus the proof is concluded. □

3.5. Fractional dimensional crack evolution as limit of one-dimensional ones

In this section we show that the energy functional considered in the previous sections arises as a natural extension of the Griffith's setting; indeed, it can be obtained as Γ-limit of energies involving small toughness coefficients and the $\mathcal{H}^1$-measure restricted to polygonal approximations of the curves with fractional Hausdorff dimension. We illustrate this idea in the case of a single curve $\mathcal{K}$.

Let $\mathcal{K}$ be a curve of the form $\mathcal{K} = \gamma([0,1])$ with γ satisfying (3.3) and (3.4), and $d \in (1,2)$. For $n \in \mathbb{N}$ we construct a sequence of polygonal approximations $\mathcal{K}^n$ in the following way: define $\gamma_n : [0,1] \to \mathbb{R}^2$ as

$$\gamma_n(s) := \gamma(i/n) + (ns - i)(\gamma((i+1)/n) - \gamma(i/n))$$

for $i/n \le s < (i+1)/n$ and $i = 0,\dots,n-1$, and set $\mathcal{K}^n := \gamma_n([0,1])$. By (3.3), it is

$$\mathcal{K}^n \to \mathcal{K} \tag{3.33}$$

in the Hausdorff metric, as $n \to +\infty$.

We define the “toughness coefficients”

$$\kappa_n^i = \frac{L}{n\,|\gamma((i+1)/n) - \gamma(i/n)|}\,,$$

for $i = 0,\dots,n-1$, where $L = \mathcal{H}^d(\mathcal{K})$, and set $\kappa_n(x) = \kappa_n^i$ if $x \in \gamma_n([i/n,(i+1)/n))$. Finally, we introduce the set-function

$$\mathcal{L}_n(K) := \int_{K\cap\mathcal{K}^n} \kappa_n(x)\,d\mathcal{H}^1(x)\,.$$

Lemma 3.5.1. *Let (K_n) be a sequence of compact connected sets such that $K_n \subset \mathcal{K}^n$ for every n. Assume that (K_n) converges to K in the Hausdorff metric. Then K is a compact connected set, contained in $\mathcal{K}$, and*

$$\mathcal{L}_n(K_n) \to \mathcal{H}^d(K)\,.$$

Proof. The set K is compact, connected and contained in $\mathcal{K}$ by the properties of the Hausdorff convergence (and (3.33)). For every n, it is $K_n = \gamma_n([a_n,b_n])$ for some $a_n, b_n \in [0,1]$, and $K = \gamma([a,b])$ for $a,b \in [0,1]$. It is not difficult to verify that $a_n \to a$ and $b_n \to b$. Set $i_n, j_n \in \{0,\dots,1/n\}$ such that $i_n/n \le a_n < (i_n+1)/n$ and $j_n/n \le b_n < (j_n+1)/n$, we have

$$\mathcal{L}_n(K_n) = L(j_n - (i_n+1)) + \kappa_n^{i_n}\big|\gamma((i_n+1)/n) - \gamma(a_n)\big| + \kappa_n^{j_n}\big|\gamma(b_n) - \gamma(j_n/n)\big|\,,$$

which converges to $L(b-a)$ as $n \to +\infty$. Being $\mathcal{H}^d(K) = L(b-a)$ by (3.4), the lemma is proved. □

On the other hand, given a compact connected set $K \subset \mathcal{K}$, there exists a sequence K_n of compact connected sets such that $K_n \subset \mathcal{K}^n$, $K_n \to K$ in the Hausdorff distance and

$$\mathcal{L}_n(K_n) \to \mathcal{H}^d(K)\,. \tag{3.34}$$

Indeed, being $K = \gamma([a,b])$, it is enough to take

$$K_n := \gamma_n([a,b])\,. \tag{3.35}$$

Then Lemma 3.5.1 provides (3.34).

Remark 3.5.2. The length of the approximating polygonals K_n in the previous lemma tends to infinity:

$$\begin{aligned}\mathcal{H}^1(K_n) &\geq \sum_{h=i_n+1}^{j_n} |\gamma(h/n) - \gamma((h+1)/n)| \\ &\geq c^{-1} \sum_{h=i_n+1}^{j_n} (1/n)^{1/d} = c^{-1}\frac{L}{b-a} n^{1-1/d} + o(1) \to +\infty\,.\end{aligned}$$

Conversely, the toughness coefficients κ_n vanish, so that the lower bound in (3.2) (or (7) in the Introduction) is violated: indeed

$$\sup_i \kappa_n^i = \sup_i \frac{L}{n\,|\gamma((i+1)/n) - \gamma(i/n)|} \leq c\, n^{-(1-1/d)} \to 0$$

as $n \to +\infty$, being $d > 1$.

We consider the functionals

$$F(u,g,K) := \begin{cases} \displaystyle\int_{\Omega\setminus K} |\nabla u|^2\,dx + \mathcal{H}^d(K) & \text{if } K \subset \mathcal{K}, g \in H^1(\Omega) \text{ and } u \in \mathcal{V}(g,K) \\ +\infty & \text{otherwise} \end{cases}$$

and

$$F_n(u,g,K) := \begin{cases} \displaystyle\int_{\Omega\setminus K} |\nabla u|^2\,dx + \mathcal{L}_n(K) & \text{if } K \subset \mathcal{K}^n, g \in H^1(\Omega) \text{ and } u \in \mathcal{V}(g,K) \\ +\infty & \text{otherwise}\,, \end{cases}$$

where $\mathcal{V}(g,K)$ is defined in (3.9) for $K \subset \mathcal{K}$ and similarly when $K \subset \mathcal{K}^n$. The two functionals are related in the following way.

Theorem 3.5.3. *Let (K_n) be a sequence of compact sets with at most p connected components and $K_n \subset \mathcal{K}^n$, and assume it converges to K in the Hausdorff metric. Let (g_n) be a sequence converging to g in $H^1(\Omega)$. Then $F_n(\cdot, g_n, K_n)$ Γ-converges to $F(\cdot, g, K)$ with respect to the weak convergence in L^2 of the gradients.*

The proof of the above theorem will be a consequence of the result below, proved in [33, Theorem 6.3], and that we rewrite for the ease of the reader. Similar results, concerning Dirichlet and Neumann boundary data, were proved, e.g., in [80] and [20, 19, 25], respectively.

Theorem 3.5.4. *Let (g_n) be a sequence in $H^1(\Omega)$ converging to g in $H^1(\Omega)$, and let (K_n) be a sequence of compact subsets of $\overline{\Omega}$ converging to K in the Hausdorff metric. Assume that $|K_n|$ converges to $|K|$ and that K_n have a uniformly bounded number of connected components. Then the space*

$$H_n := \left\{\nabla u\, 1_{\Omega\setminus K_n} :\ u \in L^{1,2}(\Omega \setminus K_n),\ u = g_n \ \text{ on } \ \partial_D\Omega\right\}$$

converges to

$$H := \left\{ \nabla u \, 1_{\Omega\setminus K} : \; u \in L^{1,2}(\Omega \setminus K), \; u = g \;\; on \;\; \partial_D \Omega \right\}$$

in the sense of Mosco [68], i.e. *the following two conditions hold:*

(M_1) *for every* $u \in L^{1,2}(\Omega \setminus K)$ *with* $u = g$ *on* $\partial_D\Omega$ *there exists a sequence* $u_n \in L^{1,2}(\Omega \setminus K_n)$ *with* $u = g_n$ *on* $\partial_D\Omega$*, such that* $\nabla u_n \, 1_{\Omega\setminus K_n}$ *converges strongly to* $\nabla u \, 1_{\Omega\setminus K}$ *in* $L^2(\Omega;\mathbb{R}^2)$*;*

(M_2) *if* (h_n) *is a sequence of indices that tends to* $+\infty$*, and* (u_n) *is a sequence such that* $u_n \in L^{1,2}(\Omega \setminus K_{h_n})$ *with* $u_n = g_{h_n}$ *on* $\partial_D\Omega$ *for every* n *and* $\nabla u_n \, 1_{\Omega\setminus K_{h_n}}$ *converges weakly in* $L^2(\Omega;\mathbb{R}^2)$ *to* ψ*, then there exists a function* $u \in L^{1,2}(\Omega \setminus K)$ *with* $u = g$ *on* $\partial_D\Omega$ *and* $\psi = \nabla u \, 1_{\Omega\setminus K}$*.*

Proof of Theorem 3.5.3. Let us observe immediately that the hypotheses on the K_n and K in Theorem 3.5.4 are satisfied: indeed the K_n have at most p connected components and, since $\mathcal{L}_n(K_n) < \infty$ and $\mathcal{H}^d(K) < \infty$, it is $|K_n| = |K| = 0$. Below we apply Theorem 3.5.4 with $H_n = \{\nabla u : \; u \in \mathcal{V}(g_n, K_n)\}$ and $H = \{\nabla u : \; u \in \mathcal{V}(g, K)\}$.

$\Gamma - \liminf$ *inequality.* Let $u \in \mathcal{V}(g, K)$ and let (u_n) be a sequence such that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^2)$. We may assume that

$$F_n(u_n, g_n, K_n) \leq C \tag{3.36}$$

for some $C > 0$ for every n (otherwise the $\Gamma - \liminf$ inequality is trivially satisfied); hence $u_n \in \mathcal{V}(g_n, K_n)$ for every n. By Lemma 3.5.1 it is

$$\mathcal{H}^d(K) = \lim_{n\to+\infty} \mathcal{L}_n(K_n)\,.$$

Since

$$\int_{\Omega\setminus K} |\nabla u|^2 \, dx \leq \liminf_{n\to+\infty} \int_{\Omega\setminus K_n} |\nabla u_n|^2 \, dx\,,$$

we get

$$F(u, g, K) \leq \liminf_{n\to+\infty} F_n(u_n, g_n, K_n)\,.$$

$\Gamma - \limsup$ *inequality.* Consider a function $u \in \mathcal{V}(g, K)$ and the sequence $u_n \in \mathcal{V}(g_n, K_n)$ provided by (M_1) in Theorem 3.5.4. Then ∇u_n converges to ∇u strongly in $L^2(\Omega;\mathbb{R}^2)$ and

$$\begin{aligned} F(u, g, K) &= \int_{\Omega\setminus K} |\nabla u|^2 \, dx + \mathcal{H}^d(K) \\ &= \lim_{n\to+\infty} \int_{\Omega\setminus K_n} |\nabla u_n|^2 \, dx + \mathcal{L}_n(K_n) = \lim_{n\to+\infty} F_n(u_n, g_n, K_n)\,. \end{aligned}$$ □

At this point we want to prove that the evolutions described in Theorem 3.2.3 are indeed limits of irreversible quasi-static crack evolutions $t \mapsto (u_n(t), K_n(t))$ (of global minimizers) whose crack set $K_n(t)$ is 1-dimensional and contained in $\mathcal{K}^n$, with fracture dissipation energy given by

$$\mathcal{L}_n(K_n(t)) = \int_{K_n(t)} \kappa_n(x) \, d\mathcal{H}^1(x)\,.$$

In analogy to Sections 3.1 and 3.2, we define the set

$$\mathcal{C}_p^n := \Big\{ K \subset \mathcal{K}^n : \; K \text{ nonempty compact set with at most } p \text{ connected components} \Big\}$$

and the energy functional

$$\mathcal{E}_n(g, K) := \min_{u\in\mathcal{V}(g,K)} \int_{\Omega\setminus K} |\nabla u|^2 dx + \mathcal{L}_n(K)\,.$$

The results in [38] (in particular [38, Theorem 7.1]) guarantee the existence of irreversible quasi-static crack evolutions $t \mapsto (u_n(t), K_n(t))$ for the total energy $\mathcal{E}_n$,with the constraint $K_n(t) \subset \mathcal{K}^n$, $K_n(t)$ having at most p connected components (with p prescribed *a priori*), and satisfying conditions analogous to those in Theorem 3.2.3. More precisely, for every n, given $K_n^0 \subset \mathcal{K}^n$ and $g \in AC([0,T]; H^1(\Omega))$, there exists an evolution $t \in [0,T] \mapsto K_n(t) \subset \mathcal{K}^n$ fulfilling the following conditions:

(I_n) $\quad K_n^0 \subseteq K_n(\tau) \subseteq K_n(t)$ for $0 \le \tau \le t \le T$;

(GS_n) $\quad \mathcal{E}_n(g(0), K_n(0)) \le \mathcal{E}_n(g(0), K) \quad \forall\, K \in \mathcal{C}_p^n\,,\ K \supseteq K_n^0$, and for $0 \le t \le T$

$$\mathcal{E}_n(g(t), K_n(t)) \le \mathcal{E}_n(g(t), K) \quad \forall\, K \in \mathcal{C}_p^n\,,\ K \supseteq K_n(t)\,;$$

(EB_n) $\quad$ for every s, t with $0 \le s < t \le T$

$$\mathcal{E}_n(g(t), K_n(t)) = \mathcal{E}_n(g(s), K_n(s)) + 2\int_s^t \langle \nabla u_n(\tau), \nabla \dot{g}(\tau)\rangle \, d\tau\,,$$

where $u_n(t)$ is the unique solution of the minimum problem defining $\mathcal{E}_n(g(t), K_n(t))$.

Theorem 3.5.5. *For every $n \in \mathbb{N}$, let $t \to K_n(t)$ be an irreversible quasi-static evolution satisfying* (I_n) - (GS_n) - (EB_n) *and such that $K_n(t) \subset \mathcal{K}^n$ for every $t \in [0,T]$. Then there exist a subsequence, not relabelled, and an evolution $t \mapsto K(t)$, such that it satisfies the conditions in Theorem 3.2.3 and $K_n(t)$ converges to $K(t)$ in the Hausdorff metric for every $t \in [0,T]$.*

Proof. Monotonicity of the maps $t \mapsto K_n(t)$ due to (I_n), and Helly's Theorem [38, Theorem 6.3], guarantee the existence of a subsequence (not relabelled) and of an increasing set-function $t \mapsto K(t)$ such that, for every $t \in [0,T]$, $K_n(t)$ converges to $K(t)$ in the Hausdorff metric. Since $(\mathcal{K}^n)$ converges to $\mathcal{K}$ in the Hausdorff metric and $K_n(t) \subset \mathcal{K}^n$, it is $K(t) \subset \mathcal{K}$. Moreover $K(t)$ has at most p connected components, so that $K(t) \in \mathcal{C}_p$ for every t. Hence condition (I) in Theorem 3.2.3 is satisfied.

We have to check the global unilateral minimality conditions (3.13) and (GS) at any instant t. Fix $t \in [0,T]$ and $K \in \mathcal{C}_p$ with $K \supset K(t)$ for $t > 0$, and with $K \supset K_0$ if $t = 0$.

We claim that there exists a sequence (K_n) converging to K in the Hausdorff metric and such that, for every n, K_n has at most p connected components and $K_n(t) \subset K_n \subset \mathcal{K}^n$.

By the minimality of $K_n(t)$, corresponding to (GS_n), we have

$$\mathcal{E}_n(g(t), K_n(t)) \le \mathcal{E}_n(g(t), K_n)\,,$$

where K_n is the sequence provided by the claim above. By Theorem 3.5.3 and the properties of Γ-convergence (the functionals $F_n(\cdot, g(t), K_n)$ and $F_n(\cdot, g(t), K_n(t))$ are asymptotically sequentially coercive; see [31, Chapter 7]) we get the convergence of the minima:

$$\mathcal{E}_n(g(t), K_n) = \min_{u \in \mathcal{V}(g(t), K_n)} F_n(u, g(t), K_n) \to \mathcal{E}(g(t), K) = \min_{u \in \mathcal{V}(g(t), K)} F(u, g(t), K)$$

and, analogously,

$$\mathcal{E}_n(g(t), K_n(t)) \to \mathcal{E}(g(t), K(t))\,. \tag{3.37}$$

The three relations above prove conditions (GS) and (3.13).

The conservation of the energy (EB) follows by (EB_n) and (3.37).

Proof of the claim.

We now illustrate how to construct the sets K_n; the main issue is to fulfil the condition on the maximum number of connected components. Let $t \in [0,T]$ be fixed. Assume that

$$K(t) = \gamma([a_1, b_1]) \cup \ldots \cup \gamma([a_q, b_q])$$

for some $q \leq p$, with $b_i < a_{i+1}$ for $i = 1, \ldots, q-1$. Without loss of generality, we can assume that the sets $K_n(t)$ have r connected components for every n, more precisely they are of the form

$$K_n(t) = \gamma_n([a_1^n, b_1^n]) \cup \ldots \cup \gamma_n([a_r^n, b_r^n])$$

with $b_j^n < a_{j+1}^n$ for $j = 1, \ldots, r-1$.

In general, $r \geq q$. If $r > q$ we want to substitute the set $K_n(t)$ with a set $\widetilde{K}_n(t)$ having exactly q connected components, containing $K_n(t)$ and still converging to $K(t)$ in the Hausdorff metric. The construction can be done in the following way. We firstly observe that

$$\gamma_n([a_{i_n}^n, b_{i_n}^n] \cup \ldots \cup [a_{h_n}^n, b_{h_n}^n]) \to \gamma([a_i, b_i])$$

in the Hausdorff metric if and only if

$$a_{i_n}^n \to a_i \qquad b_{h_n}^n \to b_i \qquad a_l^n - b_{l-1}^n \to 0$$

for $l = i_n + 1, \ldots, h_n$.

Let $\eta > 0$ be such that $a_{i+1} - b_i > 3\eta$ for all $i = 1, \ldots, q-1$. Set $\alpha_1^n := a_1^n$ and $\beta_1^n := b_j^n$ with the index j satisfying

$$b_j^n < a_2 - \eta \leq a_{j+1}^n\,,$$

and $\beta_q^n = b_r^n$. For $i = 2, \ldots, q-1$ we define the intervals $[\alpha_i^n, \beta_i^n] = [a_j^n, b_h^n]$, where the indices j, h are such that

$$b_{j-1}^n < a_i - \eta \leq a_j^n < b_h^n \leq b_i + \eta < a_{h+1}^n\,.$$

Set

$$\widetilde{K}_n(t) := \gamma_n([\alpha_1^n, \beta_1^n]) \cup \ldots \cup \gamma_n([\alpha_q^n, \beta_q^n])\,.$$

By construction, $K_n(t) \subset \widetilde{K}_n(t) \subset \mathcal{K}^n$ and $\widetilde{K}_n(t)$ has q connected components; by the previous observation, $\widetilde{K}_n(t)$ converges to $K(t)$ in the Hausdorff metric.

Let $K \in \mathcal{C}_p$ with $K \supset K(t)$. It is of the form

$$K = \gamma([c_1, d_1]) \cup \ldots \cup \gamma([c_s, d_s])$$

for some $s \leq p$. Notice that, by inclusion, every interval $[a_i, b_i]$ is contained in an interval $[c_j, d_j]$. It is not difficult to verify that the set

$$K_n := \gamma_n([c_1, d_1]) \cup \ldots \cup \gamma_n([c_s, d_s]) \cup \widetilde{K}_n(t)$$

fulfils the requests of the claim: it has the same number of connected components as K (hence less then p), contains $K_n(t)$, is a subset of $\mathcal{K}^n$, and converges to K in the Hausdorff metric. □

The result above is consistent with the justification of the model, as discussed in the Introduction, when the lower bound in (3.2) (or (7) in the Introduction) is violated (see Remark 3.5.2). Indeed, where the material becomes more and more fragile the $\mathcal{H}^1$ measure of the crack is no longer appropriate for the dissipative term, and it is necessary to introduce fractional Hausdorff measures in order to take into account the increased roughness of the fracture in the fragile area.

3.6. The linearized and nonlinear cases

The results of the previous sections, which for simplicity have been proved in the antiplane linear setting, can be extended to more general frameworks, in particular to the vectorial 2-dimensional setting, corresponding to the mode I and mode II fracture models, both in the nonlinear and linearized case.

3.6.1. Nonlinear elasticity. Our setting can be extended to the case of hyperelastic materials, under suitable assumptions on the nonlinear energy density that guarantee the existence of global minimizers. We consider both the antiplane case and the plane case. We briefly discuss the main steps.

The bulk energy for a deformation v of the unfractured part of the body $\Omega \setminus K$ is given by the functional

$$\int_{\Omega\setminus K} W(x, \nabla v(x))\, dx\,,$$

where $W : \Omega \times \mathbb{R}^{N\times 2} \to \mathbb{R}$ is a given energy density, dependent on the material. Here $N = 1$ in the antiplane case, with v describing an out-of-plane vertical deformation; $N = 2$ if v describes an in-plane deformation.

We assume W to satisfy the following properties:

- W is a Carathéodory function;
- for every $x \in \Omega$ the function $\xi \mapsto W(x, \xi)$ is C^1 and *quasiconvex*, *i.e.* for every $\xi \in \mathbb{R}^{N\times 2}$ and for every $\phi \in C^1_c(\Omega; \mathbb{R}^N)$
$$\frac{1}{|\Omega|}\int_{\Omega} W(x, \xi + \nabla\phi(y))\, dy \geq W(\xi)\,;$$
- for some constants $a_0, a_1 > 0$ and a non-negative function $b \in L^1(\Omega)$ it is
$$a_0|\xi|^2 \leq W(x, \xi) \leq a_1|\xi|^2 + b(x) \tag{3.38}$$
for every $x \in \Omega$ and $\xi \in \mathbb{R}^{N\times 2}$.

Note that for $N = 1$ quasiconvexity and convexity coincide.

Similarly to (3.9), for every $g \in H^1(\Omega; \mathbb{R}^N)$ and $K \in \mathcal{C}_p$ we define the set

$$\mathcal{V}_N(g, K) := \{w \in L^{1,2}(\Omega\setminus K; \mathbb{R}^N) : w = g \quad \text{q.e. on } \partial_D\Omega\}$$

and we consider the functional

$$\mathcal{W}(g, K, v) := \begin{cases} \displaystyle\int_{\Omega\setminus K} W(x, \nabla v(x))\, dx & \text{if } v \in \mathcal{V}_N(g, K) \\ +\infty & \text{otherwise}\,. \end{cases}$$

Proposition 3.6.1. *Let (g_n) be a sequence converging to g in $H^1(\Omega; \mathbb{R}^N)$. Let (K_n) be a sequence in $\mathcal{C}_p$ converging to K in the Hausdorff metric. Let $v_n \in \mathcal{V}(g_n, K_n)$ be such that (∇v_n) converges to ψ weakly in $L^2(\Omega; \mathbb{R}^{N\times 2})$. Then $\psi = \nabla v$ for some $v \in \mathcal{V}(g, K)$, and*

$$\mathcal{W}(g, K, v) \leq \liminf_{n\to+\infty} \mathcal{W}(g_n, K_n, v_n)\,. \tag{3.39}$$

Proof. The existence of $v \in \mathcal{V}(g, K)$ with $\psi = \nabla v$ is consequence of Theorem 3.5.4 (when $N = 2$, hence $v_n(x) = (v^1_n(x), v^2_n(x))$, it is enough to apply it to each component v^1_n, v^2_n).

Consider a subsequence (v_{n_m}) of (v_n) such that

$$\liminf_{n\to+\infty} \mathcal{W}(g_n, K_n, v_n) = \lim_{m\to+\infty} \mathcal{W}(g_{n_m}, K_{n_m}, v_{n_m}).$$

Consider a Lipschitz open set $\omega \subset\subset (\Omega \setminus K) \cup \partial_D\Omega$ with $\mathcal{H}^1(\partial\omega \cap \partial_D\Omega) > 0$. By Hausdorff convergence, $K_n \cap \omega = \emptyset$ for n sufficiently large. As ω has a Lipschitz boundary, $v_n \in H^1(\omega; \mathbb{R}^N)$ for every n. By Rellich Theorem and the convergence in $H^1(\Omega; \mathbb{R}^N)$ of (g_n) to g, there exists a subsequence (not relabelled) of (v_{n_m}) that converges to v strongly in $L^2(\omega; \mathbb{R}^N)$.

Therefore (v_{n_m}) converges to v weakly in $H^1(\omega;\mathbb{R}^N)$ and we can apply the semicontinuity result [1, Theorem II.4] to obtain

$$\begin{aligned}\int_\omega W(x,\nabla v(x))\,dx &\le \liminf_{m\to+\infty}\int_\omega W(x,\nabla v_{n_m}(x))\,dx\\ &\le \liminf_{m\to+\infty}\int_{\Omega\setminus K_{n_m}} W(x,\nabla v_{n_m}(x))\,dx\\ &= \lim_{m\to+\infty}\mathcal{W}(g_{n_m},K_{n_m},v_{n_m}) = \liminf_{n\to+\infty}\mathcal{W}(g_n,K_n,v_n)\,,\end{aligned}$$

where the last inequality is due to the fact that $W\ge 0$ and $\omega\subset\Omega\setminus K_{n_m}$ for m large. As $\omega\nearrow\Omega\setminus K$ we obtain

$$\mathcal{W}(g,K,v)\le\liminf_{n\to+\infty}\mathcal{W}(g_n,K_n,v_n)\,. \qquad\square$$

Corollary 3.6.2. *For every g,K, the minimum problem*

$$\min_{w\in\mathcal{V}_N(g,K)}\mathcal{W}(g,K,w) \tag{3.40}$$

has a solution.

The following result is the counterpart of Theorem 3.4.1 in the nonlinear setting.

Proposition 3.6.3. *Let (K_n) be a sequence in $\mathcal{C}_p$ converging to K in the Hausdorff metric, and let (g_n) be a sequence converging to g in $H^1(\Omega;\mathbb{R}^N)$. For every n let $v_n\in\mathcal{V}_N(g_n,K_n)$ be a minimizer of $\mathcal{W}(g_n,K_n,\cdot)$, and assume that*

$$\sup_n\mathcal{W}(g_n,K_n,v_n)<+\infty\,. \tag{3.41}$$

Then, up to subsequences, ∇v_n converges to ∇v weakly in $L^2(\Omega;\mathbb{R}^{N\times 2})$, with $v\in\mathcal{V}_N(g,K)$ which minimizes $\mathcal{W}(g,K,\cdot)$.

Proof. By (3.41) and (3.38), it results that $\sup_n\|\nabla v_n\|<+\infty$. Hence, up to subsequences, (∇v_n) converges to a function ψ weakly in $L^2(\Omega;\mathbb{R}^{N\times 2})$. Theorem 3.5.4 guarantees the existence of a function $v\in\mathcal{V}_N(g,K)$ with $\nabla v=\psi$ (as before, when $N=2$, *i.e.* $v_n(x)=(v_n^1(x),v_n^2(x))$, it is enough to apply it to each component v_n^1, v_n^2).

It remains to show that v minimizes $\mathcal{W}(g,K,\cdot)$ in $\mathcal{V}_N(g,K)$. Let $w\in\mathcal{V}_N(g,K)$; by (M_1) in Theorem 3.5.4, there exists a sequence (w_n) with $w_n\in\mathcal{V}_N(g_n,K_n)$ and ∇w_n converging to ∇w strongly in $L^2(\Omega;\mathbb{R}^{N\times 2})$. Up to subsequences, we can assume that $\nabla w_n(x)\to\nabla w(x)$ for a.e. $x\in\Omega$, so that $W(x,\nabla w_n(x))\to W(x,\nabla w(x))$ for a.e. $x\in\Omega$; by the growth assumption (3.38) and the Generalized Dominated Convergence Theorem, we obtain

$$\int_\Omega W(x,\nabla w_n(x))\,dx\to\int_\Omega W(x,\nabla w(x))\,dx\,.$$

Finally, by the lower semicontinuity result in Proposition 3.6.1 and by the minimality of the v_n it follows

$$\mathcal{W}(g,K,v)\le\liminf_{n\to+\infty}\mathcal{W}(g_n,K_n,v_n)\le\liminf_{n\to+\infty}\mathcal{W}(g_n,K_n,w_n)=\mathcal{W}(g,K,w)\,,$$

which proves that v is a minimizer of $\mathcal{W}(g,K,\cdot)$ in $\mathcal{V}_N(g,K)$. $\square$

For $g\in H^1(\Omega;\mathbb{R}^N)$ and $K\in\mathcal{C}_p$ we define

$$\mathcal{E}_{nl}(g,K):=\inf_{w\in\mathcal{V}_N(g,K)}\mathcal{W}(g,K,w)+\mathcal{L}(K)\,.$$

At this point, considering Proposition 3.6.1, Proposition 3.6.3 and the lower semicontinuity of the functional $\mathcal{L}$ (see Corollary 3.3.4), in order to show the existence of a quasi-static crack evolution in the context of nonlinear elasticity it is sufficient to argue as for Theorem 3.2.3. In other words, we can prove the following result:

Theorem 3.6.4. *Let $T > 0$ and $g \in AC([0,T]; H^1(\Omega;\mathbb{R}^N))$, with $N = 1, 2$. Let $p \geq 1$ and $K_0 \in \mathcal{C}_p$. Then there exists a function $K\colon [0,T] \to \mathcal{C}_p$ such that*

(I_{nl}) $K_0 \subseteq K(s) \subseteq K(t)$ *for* $0 \leq s \leq t \leq T$,

(GS_{nl}) for every $0 \leq t \leq T$

$$\mathcal{E}_{nl}(g(t), K(t)) \leq \mathcal{E}_{nl}(g(t), K)$$

for all $K \in \mathcal{C}_p$ *with* $K \supseteq K(t)$.
Moreover, $\mathcal{E}_{nl}(g(0), K(0)) \leq \mathcal{E}_{nl}(g(0), K)$ *for all* $K \in \mathcal{C}_p$ *with* $K \supseteq K_0$,

(EB_{nl}) for every s, t *with* $0 \leq s < t \leq T$

$$\mathcal{E}_{nl}(g(t), K(t)) = \mathcal{E}_{nl}(g(s), K(s)) + \int_s^t \langle D_\xi W(x, \nabla v(\tau)), \nabla \dot{g}(\tau)\rangle \, d\tau \,,$$

where $v(\tau)$ *is a solution of the minimum problem* (3.40) *with* $g(\tau)$ *and* $K(\tau)$.

3.6.2. Linearized elasticity. The extension of our model of crack growth to the linearized case cannot be done in a straightforward way by means of Korn's inequality: indeed, due to the irregularity of the crack sets, it cannot be applied. Instead, the key role is played by the approximation result proved by Chambolle [24, Theorem 1] (see also [18]), which can be used similarly to Theorem 3.4.1 in the proof of existence of minimizers for the energy $\mathcal{E}_{sym}$ introduced below. Roughly speaking, [24, Theorem 1] states that if $\mathbb{R}^2 \setminus \Omega$ has a finite number of connected components then $H^1(\Omega;\mathbb{R}^2)$ is dense in $\{u \in L^2_{loc}(\Omega;\mathbb{R}^2) : e(u) \in L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})\}$. Here

$$e(u) := \frac{\nabla u + (\nabla u)^T}{2}$$

is the symmetrized gradient of u, and $\mathbb{R}^{2\times 2}_{sym}$ is the space of 2×2 symmetric matrices.

Let A be a positive definite quadratic form on the space of symmetric matrices, *i.e.*, $A\xi : \xi \geq C|\xi|^2$ for every $\xi \in \mathbb{R}^{2\times 2}_{sym}$, where ":" denotes the scalar product between matrices, and $C > 0$. Combining together [38, Theorem 7.1], [24, Theorem 1] and Theorem 3.2.3, we can state that Theorem 3.2.3 holds true for the energy

$$\mathcal{E}_{sym}(g, K) := \min_{v \in \mathcal{V}_{sym}(g,K)} \int_{\Omega\setminus K} Ae(v) : e(v)\, dx + \sum_{m=1}^{M} \mathcal{H}^{d_m}(K \cap \mathcal{K}_m)\,, \tag{3.42}$$

where

$$\mathcal{V}_{sym}(g, K) := \{v \in L^2_{loc}(\Omega \setminus K;\mathbb{R}^2) : e(v) \in L^2(\Omega \setminus K;\mathbb{R}^{2\times 2}_{sym}),\ v = g \quad \text{q.e. on } \partial_D\Omega\}\,.$$

Indeed, the approximation theorem [24, Theorem 1], together with the metric and topological properties shown in Section 3.3 and used to extend the results in [38], can be applied in order to prove the lower semicontinuity of $\mathcal{E}_{sym}(\cdot,\cdot)$ with respect to the convergence of functions g_n to g in $H^1(\Omega)$ and of sets $K_n \in \mathcal{C}_p$ to K in the Hausdorff metric, and to construct appropriate recovery sequences in order to obtain (GS) and (3.13) in Theorem 3.2.3 with $\mathcal{E}_{sym}$ instead of $\mathcal{E}$, and $e(u), e(g)$ instead of $\nabla u, \nabla g$ in the condition (EB).

3.7. The von Koch curve

The von Koch curve, denoted in this subsection by $\mathcal{K}$, represents a significative example for the class of admissible fractal cracks considered in this paper. Therefore, let us describe now the constructive iterative process that defines this self-similar fractal starting from the segment $[0,1]\times\{0\}\subset\mathbb{R}^2$, and provides a parametrization which satisfies (3.3) and (3.4).

With reference to Figure 1, for $i=1,\dots,4$ let $S_i:\mathbb{R}^2\to\mathbb{R}^2$ be the unique similitude that maps the segment $[0,1]\times\{0\}\subset\mathbb{R}^2$ into the segment l_1^i (with length $1/3$) and has positive determinant. It results (see for example [55]) that the von Koch curve is the unique compact set $\mathcal{K}$ such that

$$\mathcal{K}=\bigcup_{i=1}^{4}S_i(\mathcal{K})\,.$$

We now construct iteratively a parametrization for the von Koch curve.

Let $\gamma_0:[0,1]\to\mathbb{R}^2$ be such that $\gamma_0([0,s])=[0,s]\times\{0\}$.

Let $\gamma_1:[0,1]\to\mathbb{R}^2$ be a continuous parametrization of the set $\tilde{\mathcal{K}}_1$ as in Figure 1, such that $\gamma_1(0)=0\in\mathbb{R}^2$ and $\mathcal{H}^1(\gamma_1([0,s]))=\frac{4}{3}s$. It results that $\gamma_1([(i-1)/4,i/4])=l_1^i$ for $i=1,\dots,4$.

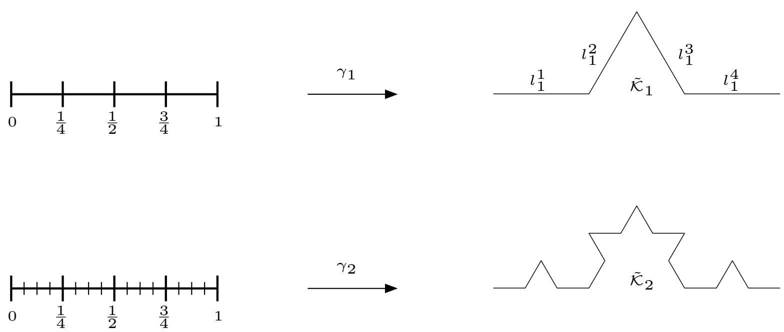


FIGURE 1. The first and second iterations in the construction of the natural parametrization γ of the von Koch curve.

Iteratively construct the set $\tilde{\mathcal{K}}_2=\bigcup_{i=1,\dots,4}S_i(\tilde{\mathcal{K}}_1)$ and its continuous parametrization $\gamma_2:[0,1]\to\mathbb{R}^2$ such that $\gamma_2(0)=0\in\mathbb{R}^2$, $\mathcal{H}^1(\gamma_2([0,s]))=\left(\frac{4}{3}\right)^2s$ and $\gamma_2([(i-1)/4^2,i/4^2])=l_2^i$ for $i=1,\dots,4^2$.

It results that for any $n\in\mathbb{N}$ it is

$$\|\gamma_n-\gamma_{n+1}\|_\infty=\frac{1}{3^{n+1}}\frac{\sqrt{3}}{2}$$

and, as consequence, for any $n,j\in\mathbb{N}$ we have

$$\|\gamma_n-\gamma_{n+j}\|_\infty\le\frac{1}{3^n}\frac{3\sqrt{3}}{4}.$$

Therefore the sequence γ_n is a Cauchy sequence in $(C([0,1];\mathbb{R}^2),\|\cdot\|_\infty)$, and there exists a continuous function $\gamma:[0,1]\to\mathbb{R}^2$ such that

$$\gamma_n\to\gamma \tag{3.43}$$

uniformly on $[0,1]$.

The sequence of compact sets $\tilde{\mathcal{K}}_n$ converges in the Hausdorff metric to the von Koch curve $\mathcal{K}$. This fact, together with the uniform convergence (3.43), implies that $\gamma([0,1]) = \mathcal{K}$.

It can be proved that $\mathcal{K}$ has Hausdorff dimension

$$d := \frac{\log 4}{\log 3}$$

and $0 < \mathcal{H}^d(\mathcal{K}) < +\infty$.

The map γ we just obtained corresponds to the one that in [74] is called *natural parametrization*. The following result shows that γ fulfils (3.3) and (3.4).

Proposition 3.7.1. *There exists a constant $c > 0$ such that for any $a, b \in [0,1]$ the natural parametrization γ satisfies*

$$\frac{1}{c}|a-b|^{1/d} \le |\gamma(a) - \gamma(b)| \le c|a-b|^{1/d} \tag{3.44}$$

and, for $a < b$,

$$\mathcal{H}^d(\gamma(a,b)) = (b-a)\mathcal{H}^d(\mathcal{K})\,.$$

Proof. The first statement is proved in [74, Theorem 1].

Concerning the second fact, firstly note that, by construction, the von Koch curve $\mathcal{K}$ and the parametrization γ have the following self-similarity property: for every $n \in \mathbb{N}$ and $j = 1, \ldots, 4^n - 1$ there exists an affine isometry $\Phi_n^j : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\Phi_n^j\left(\gamma\Big(\frac{j}{4^n}, \frac{j+1}{4^n}\Big)\right) = \gamma\Big(0, \frac{1}{4^n}\Big)\,.$$

For any $s, h \in [0,1]$ let $i_n^s, i_n^h \in \{1, \ldots, 4^n\}$ be such that

$$\frac{i_n^s}{4^n} \le s < \frac{i_n^s + 1}{4^n} \quad \text{and} \quad \frac{i_n^h}{4^n} \le h < \frac{i_n^h + 1}{4^n}\,.$$

For n sufficiently large (so that $i_n^h \ge 2$) it is

$$(s, s+h) = (s, (i_n^s + 1)/4^n) \cup [(i_n^s + 1)/4^n, (i_n^s + i_n^h)/4^n] \cup ((i_n^s + i_n^h)/4^n, s+h)\,.$$

Then, being the Φ_n^j Lipschitz continuous maps with Lipschitz constant equal to 1, we have

$$\begin{aligned}
\mathcal{H}^d(\gamma(s,s+h)) =& \mathcal{H}^d\Big(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\Big) + \sum_{j=i_n^s+1}^{i_n^s+i_n^h-1} \mathcal{H}^d\Big(\gamma\Big(\frac{j}{4^n}, \frac{j+1}{4^n}\Big)\Big) \\
&+ \mathcal{H}^d\Big(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\Big) \\
=& \mathcal{H}^d\Big(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\Big) + \sum_{j=i_n^s+1}^{i_n^s+i_n^h-1} \mathcal{H}^d\Big(\Phi_n^j\Big(\gamma\Big(\frac{j}{4^n}, \frac{j+1}{4^n}\Big)\Big)\Big) \\
&+ \mathcal{H}^d\Big(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\Big) \\
=& \mathcal{H}^d\Big(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\Big) + \sum_{j=i_n^s+1}^{i_n^s+i_n^h-1} \mathcal{H}^d\Big(\gamma\Big(0, \frac{1}{4^n}\Big)\Big) + \mathcal{H}^d\Big(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\Big) \\
=& \mathcal{H}^d\Big(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\Big) + (i_n^h - 2)\mathcal{H}^d\Big(\gamma\Big(0, \frac{1}{4^n}\Big)\Big) + \mathcal{H}^d\Big(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\Big).
\end{aligned}$$

Since γ is $(1/d)$-Hölder continuous by (3.44), it holds that

$$\mathcal{H}^d\left(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\right) \leq C(d)\left(\frac{i_n^s+1}{4^n} - s\right) \leq C(d)\frac{1}{4^n} \to 0$$

and

$$\mathcal{H}^d\left(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\right) \leq C(d)\left(s+h-\frac{i_{i_n^s}+N_n^h}{4^n}\right) \leq 2C(d)\frac{1}{4^n} \to 0$$

as $n \to +\infty$, with $C(d)$ independent of t and h. Hence we obtain

$$\begin{aligned}\mathcal{H}^d(\gamma(s,s+h)) &= \lim_{n\to+\infty}(i_n^h-2)\mathcal{H}^d\left(\gamma\Big(0,\frac{1}{4^n}\Big)\right)\\ &= \lim_{n\to+\infty}(1_n^h-2)\frac{1}{4^n}\mathcal{H}^d\left(\gamma(0,1)\right) = h\mathcal{H}^d(\mathcal{K}),\end{aligned}$$

where, in the second equality, we used the self-similiarity property of $\mathcal{K}$, that is, $\mathcal{K} = \gamma([0,1])$ contains exactly 4^n distinct copies of $\gamma([0,1/4^n])$.

Consider now $0 \leq a < b \leq 1$. Set $s = a$ and $h = b-a$ in the above argument, the thesis follows. □

CHAPTER 4

A model for crack growth with branching and kinking

In the mathematical description of fracture mechanics, at present the phenomena of branching and kinking of cracks are still tricky matters. In this chapter we propose an evolution model where both phenomena are admitted, and the crack path is not assigned *a priori*. In order to overcome some of the mathematical difficulties, we introduce some geometrical constraints on the admissible cracks, so that accumulation of branching points and kinking points, and micro-cracking around the tips are avoided.

We discuss the problem in the framework of linear elasticity in the two-dimensional antiplane shear case. The results up to Subsection 4.4.1 can be treated in more general settings, like linearized and nonlinear planar elasticity, appealing to the stability results in [**24, 18**] and [**33**] instead of Theorem 1.7.6 on the convergence of minimizers of elliptic problems, similarly to what we have done in Section 3.6. However, in Subsection 4.4.2 we will need an explicit formula for the energy release rate: in the antiplane case it is proved in [**62**] (see Remark 1.5.1.(iii)), while it is lacking in the other regimes for our geometric setting (see [**57, 59**] in case of polyconvex elastic energy densities and a preassigned crack path, and [**27**] for the linearized case).

As a standard procedure, we first consider a discrete-time approximation $t \mapsto (\Gamma_\tau(t), u_\tau(t))$ driven by a boundary datum $w_\tau(t)$, where τ is the incremental step, while $\Gamma_\tau(t)$ and $u_\tau(t)$ are the crack and the displacement, respectively. At each incremental step, the minimum problem that selects the proper approximation takes into account terms related to the discrete velocity of the front (see (4.32)).

As $\tau \to 0$, we recover a continuous-time rate-dependent evolution $(\Gamma(t), u(t))$ with boundary datum $w(t)$ (where $w_\tau(t) \to w(t)$) as limit of the sequence $(\Gamma_\tau(t), u_\tau(t))$, such that

- the crack growth is irreversible, *i.e.* $\Gamma(t') \subset \Gamma(t)$ for any $t' < t$;
- it satisfies an energy inequality containing the term

$$\int_a^b \sum_{p \in F(t)} v(p,t)^2 \, dt \, .$$

  It represents the energy dissipated at the crack front $F(t)$ of the crack $\Gamma(t)$ due to the speed $v(p,t)$ of crack growth at the tip $p \in F(t)$;
- a Griffith's principle holds, as long as the front advances: if at an instant t_0 the velocity $v(p(t_0), t_0)$ of a point $p(t_0)$ of the front $F(t_0)$ is strictly positive, then for a.e. $t < t_0$ and close to t_0 the following conditions are satisfied

$$\begin{aligned} & v(t,p(t)) \geq 0 \\ & \mathcal{G}(w(t), \Gamma(t), p(t)) \leq 1 + v(t,p(t)) \\ & [-\mathcal{G}(w(t), \Gamma(t), p(t)) + 1 + v(t,p(t))]\, v(t,p(t)) = 0, \end{aligned} \tag{4.1}$$

  where $p(t)$ belongs to the front $F(t)$ of $\Gamma(t)$ and $p(t) \to p(t_0)$ as $t \nearrow t_0$. In (4.1), the function $\mathcal{G}(w, \Gamma, p)$ is the energy release rate relative to a time-dependent loading w, a crack Γ and a tip p of Γ.

The notion of velocity of the crack front is described with two different approaches, one of which, obtained by means of a distributional argument, reminds that by Larsen, Ortiz & Richardson in [61].

Unfortunately if a branch of the crack does not grow, we cannot prove a stability condition like (4.1). The main difficulty dwells on the approximation procedure by the discrete-time evolutions. Indeed, the stationarity of a branch of the continuous-time evolution does not garantee a similar behaviour of the approximating discrete-time sequence, which could present several different growth conditions, as explained at the end of the chapter.

To our knowledge, this is a first attempt to describe a model with branching and kinking of fractures without assuming the crack path to be assigned *a priori*; furthermore, also the rate of growth of the cracks at many tips is taken into account. Nevertheless, since the description of the process is not complete, the investigation needs to go on.

The results of this chapter will appear in [76].

4.1. Geometrical setting and admissible cracks

In this section we introduce the geometrical setting of the model. We describe the class of admissible cracks and prove its compactness with respect to the Hausdorff convergence of sets. Throughout the section, we comment on the meaning of some mathematical and geometrical constraints necessary to obtain this result.

Let Ω be a bounded connected open subset of $\mathbb{R}^2$, with Lipschitz boundary $\partial\Omega$. Let $\partial_D\Omega \subset \partial\Omega$ be relatively open, with $\mathcal{H}^1(\partial_D\Omega) > 0$; we refer to $\partial_D\Omega$ as the Dirichlet part of the boundary.

We consider a slight modification of the family of curves $\mathcal{R}_\eta$ introduced in [62, 63] and reported in Definition 1.5.3; we replace Condition 1.5.3.(i) by Condition 4.1.1.(i) below.

Definition 4.1.1. For any $\eta > 0$, $\mathcal{R}_\eta$ denotes the set of compact curves Γ of class $C^{1,1}$ in $\overline{\Omega}$ such that

(i) $\mathcal{H}^1(\Gamma) > 0$ and $\Gamma \subset\subset \Omega$;

(ii) for every point $x \in \Gamma$ there exist two open balls B_1, B_2 of radius η, such that $(B_1 \cup B_2) \cap (\Gamma \cup \partial\Omega) = \emptyset$ and $\overline{B}_1 \cap \overline{B}_2 = \{x\}$.

Proposition 4.1.2. *The following facts hold:*

(i) every curve $\Gamma \in \mathcal{R}_\eta$ is simple.

(ii) Fix $\Gamma_0 \in \mathcal{R}_\eta$. Then the set

$$\{\Gamma \in \mathcal{R}_\eta :\ \Gamma_0 \subset \Gamma\}$$

is compact with respect to the Hausdorff convergence.

(iii) Consider $\Gamma \in \mathcal{R}_\eta$ and its arc-length parametrization γ. Then, set $L_\Gamma = \mathcal{H}^1(\Gamma)$, it is $\gamma \in W^{2,\infty}((0, L_\Gamma); \mathbb{R}^2)$. Furthermore there exists a constant $C > 0$ such that

$$\|\gamma\|_{W^{2,\infty}((0,L_\Gamma);\mathbb{R}^2)} \le C$$

for any $\Gamma \in \mathcal{R}_\eta$.

(iv) Let $(\Gamma_k)_k \subset \mathcal{R}_\eta$ be such that $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$. Then $\mathcal{H}^1(\Gamma_k) \to \mathcal{H}^1(\Gamma)$.

(v) Let $(\Gamma_k)_k \subset \mathcal{R}_\eta$ be such that $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$, with $L = \mathcal{H}^1(\Gamma) > 0$. Let $L_k = \mathcal{H}^1(\Gamma_k)$, and γ_k, γ be the arc-length parametrizations of Γ_k and Γ, respectively. Define $\tilde{\gamma}_k(s) := \gamma_k\big(\frac{L_k}{L}s\big)$. Then $\tilde{\gamma}_k \in W^{2,\infty}((0, L); \mathbb{R}^2)$ and

$$\tilde{\gamma}_k \rightharpoonup \gamma$$

weakly in $W^{2,\infty}((0,L);\mathbb{R}^2)$. *In particular, by the fundamental theorem of calculus, it holds that*

$$\tilde{\gamma}_k \to \gamma \quad \textit{pointwise} \tag{4.2}$$

$$\dot{\tilde{\gamma}}_k \to \dot{\gamma} \quad \textit{pointwise.} \tag{4.3}$$

PROOF. Properties 4.1.2.(i), 4.1.2.(ii), 4.1.2.(iii) and 4.1.2.(v) are already proved in [62].

Concerning Property 4.1.2.(iv), if $\mathcal{H}^1(\Gamma) > 0$, then the conclusion follows by applying Property 4.1.2.(iii) and (4.3) in the formula for the length of a C^1 curve.

If $\mathcal{H}^1(\Gamma) = 0$, then $\Gamma = \{x\}$ since it is connected and nonempty. For any $r > 0$, it is $\Gamma_k \subset B_r(x)$ for k sufficiently large. Then the bound on the curvature for $\Gamma_k \in \mathcal{R}_\eta$ (assured by the $W^{2,\infty}$-bound in 4.1.2.(iii)) implies that $\mathcal{H}^1(\Gamma_k) \le Cr$ for a constant C independent of r, so that $\mathcal{H}^1(\Gamma_k) \to 0$ as $k \to +\infty$. □

In the following, we call *endpoints* of $\Gamma \in \mathcal{R}_\eta$ the points $\gamma(0)$ and $\gamma(L)$, where γ is the arc-length parametrization of Γ and $L = \mathcal{H}^1(\Gamma)$.

The admissible crack sets will be finite unions of elements in $\mathcal{R}_\eta$, satisfying some topological restrictions in order to control the phenomena of branching and kinking. To define a proper class of cracks, for any curve in $\mathcal{R}_\eta$ we introduce two kinds of neighbourhoods, called *1-sided pencil-like neighbourhood* and *2-sided pencil-like neighbourhood*, which depend on two parameters that will not change along the paper:

$$\beta \in (0,\eta) \quad \text{and} \quad \theta \in (0,\pi/2).$$

Fixed $\Gamma \in \mathcal{R}_\eta$, set $L := \mathcal{H}^1(\Gamma)$ and let $\gamma : [0,L] \to \mathbb{R}^2$ be the arc-length parametrization of Γ. Considered the rectangle

$$R = (0,L) \times (-\beta,\beta)$$

and the extended one

$$R_e = (0,L+\beta) \times (-\beta,\beta)\,,$$

we define the maps $\Phi : R \to \mathbb{R}^2$ and $\Phi_e : R_e \to \mathbb{R}^2$ as

$$\Phi(s,z) := \gamma(s) + z\dot{\gamma}(s)^\perp \quad \text{if } (s,z) \in R$$

and

$$\Phi_e(s,z) := \begin{cases} \gamma(s) + z\dot{\gamma}(s)^\perp & \text{if } (s,z) \in R \\ \gamma(L) + (s-L)\dot{\gamma}(L) + z\dot{\gamma}(L)^\perp & \text{if } (s,z) \in [L,L+\beta) \times (-\beta,\beta). \end{cases}$$

By the regularity of γ (see Proposition 4.1.2), the maps Φ and Φ_e are homeomorphisms from R into $\Phi(R)$ and from R_e into $\Phi_e(R_e)$, respectively.

We consider the subset of R

$$P_2 := \{(s,z) \in R :\ 0 < s < L, |z| < \min\{s\tan\theta, \beta, (L-s)\tan\theta\}\}$$

and define the *2-sided pencil-like neighbourhood* of Γ (see Fig. 1) as

$$\mathcal{P}_2(\Gamma) := \Phi(P_2).$$

In coordinates, it is

$$\mathcal{P}_2(\Gamma) = \left\{\gamma(s) + z\dot{\gamma}(s)^\perp : 0 < s < L, |z| < \min\{s\tan\theta, \beta, (L-s)\tan\theta\}\right\}.$$

We consider the subset of R_e

$$\begin{aligned} P_1 := &\{(s,z) \in R :\ 0 < s \le L, |z| < \min\{s\tan\theta, \beta\}\} \\ &\cup \left\{(s,z) \in R_e :\ L \le s < L+\beta, |z| < \min\{s\tan\theta, \sqrt{\beta^2 - (s-L)^2}\}\right\} \end{aligned}$$

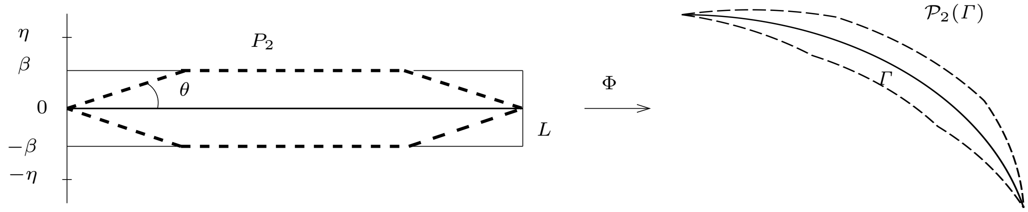


FIGURE 1. The 2-sided pencil-like neighbourhood $\mathcal{P}_2(\Gamma)$ for a curve $\Gamma \in \mathcal{R}_\eta$.

and define the *1-sided pencil-like neighbourhood* of Γ (see Fig. 2) as

$$\mathcal{P}_1(\Gamma, p) := \Phi_e(P_1)$$

with $p = \gamma(0)$. In coordinates, it is

$$\begin{aligned}\mathcal{P}_1(\Gamma, p) = &\big\{\gamma(s) + z\dot\gamma(s)^\perp : 0 < s \le L, |z| < \min\{s\tan\theta, \beta\}\big\}\\ &\cup \Big\{\gamma(L) + (s-L)\dot\gamma(L) + z\dot\gamma(L)^\perp : L \le s < L + \beta\\ &\quad \text{and } |z| < \min\{s\tan\theta, \sqrt{\beta^2 - (s-L)^2}\}\Big\}.\end{aligned}$$

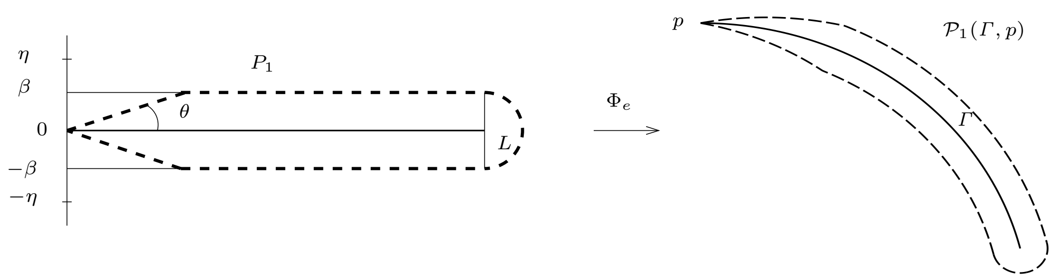


FIGURE 2. The 1-sided pencil-like neighbourhood $\mathcal{P}_1(\Gamma, p)$ for a curve $\Gamma \in \mathcal{R}_\eta$.

Remark 4.1.3. With abuse of terminology, we use the name *neighbourhood* for the open sets $\mathcal{P}_1$ and $\mathcal{P}_2$. They are not neighbourhoods of Γ, but only of $\Gamma \setminus \{p_0, p_1\}$ in case of $\mathcal{P}_2(\Gamma)$ and of $\Gamma \setminus \{p_0\}$ in case of $\mathcal{P}_1(\Gamma, p_0)$, where p_0, p_1 are the endpoints of Γ. In particular, the following holds: if $\mathcal{P}_2(\Gamma_1) \cap \Gamma_2 = \emptyset$ and $\Gamma_1 \cap \Gamma_2 \neq \emptyset$, then they intersect in (one of or both) their endpoints. Similarly, if $\mathcal{P}_1(\Gamma_1, p_0) \cap \Gamma_2 = \emptyset$ and $\Gamma_1 \cap \Gamma_2 \neq \emptyset$, then they intersect in p_0.

We consider finite unions of curves in $\mathcal{R}_\eta$ satisfying the following properties:

$$K = \bigcup_{j=1}^{m} \widetilde{K}_j \tag{4.4}$$

such that

(i) K is connected;
(ii) $\widetilde{K}_j \in \mathcal{R}_\eta$ for any j;

(iii) if $\widetilde{K}_j \cap \widetilde{K}_l \neq \emptyset$ for $j \neq l$, then they intersect in (one of or both) their endpoints;
(iv) if $\widetilde{K}_i \cup \widetilde{K}_j \in \mathcal{R}_\eta$, then there exists $\widetilde{K}_l$, $l \neq i, j$, such that $\widetilde{K}_i \cap \widetilde{K}_j \cap \widetilde{K}_l \neq \emptyset$;
(v) let p_0, p_1 be the endpoints of $\widetilde{K}_j$. Assume that $p_0 \in \widetilde{K}_j \cap \widetilde{K}_{l_0}$ and $p_1 \in \widetilde{K}_j \cap \widetilde{K}_{l_1}$ for some $l_0, l_1 \neq j$. Then

$$\mathcal{P}_2(\widetilde{K}_j) \cap \widetilde{K}_l = \emptyset$$

for every $l \neq j$;
(vi) let p_0, p_1 be the endpoints of $\widetilde{K}_j$. Assume that $p_0 \in \widetilde{K}_j \cap \widetilde{K}_{l_0}$ for some $l_0 \neq j$ and $p_1 \notin \widetilde{K}_l$ for any $l \neq j$. Then

$$\mathcal{P}_1(\widetilde{K}_j, p_0) \cap \widetilde{K}_l = \emptyset$$

for every $l \neq j$.

Each $\widetilde{K}^j$ is called a *branch* of K.

Definition 4.1.4. We divide the points of a set K as in (4.4) in three groups:

- the set T_K of *crack tip points*: $x \in K$ belongs to T_K if there exists $r > 0$ such that $K \cap \overline{B_r(x)}$ is an element of $\mathcal{R}_\eta$ with endpoint x.
- The set S_K of *singular points*: $x \in K$ belongs to S_K if there exist two unit vectors $v_1, v_2 \in \mathbb{R}^2$ tangent to K at x such that $v_1 \cdot v_2 \neq \pm 1$.
- The set R_K of *regular points*: $x \in K$ belongs to R_K if there exists $r > 0$ such that $K \cap \overline{B_r(x)}$ is an element of $\mathcal{R}_\eta$ and x in the relative interior of K.

Remark 4.1.5. Conditions (4.4).(iii) and (4.4).(iv) represent a sort of "maximality" condition of each branch in the class $\mathcal{R}_\eta$ with respect to inclusion.

Conditions (4.4).(v) and (4.4).(vi) are mathematical restrictions, which will be necessary to prove compactness of a suitable class of crack sets. We require that each branch is surrounded by an off-limit zone for the other branches; it is represented by the pencil-like neighbourhoods. Around one of (both) the tips, the 1-sided (2-sided) pencil-like neighbourhoods have the shape of a curvilinear triangle with vertex at the tip and vertex angle of 2θ. The triangular shape is necessary in order to permit the branching phenomenon at the tip; the condition on the angle bounds the number of branches that can develop from each singular point, as proved in Lemma 4.1.9 below.

For any K of this form we can define the sets $I_1(K)$ and $I_2(K)$ such that

$$\begin{aligned} &\widetilde{K}_j \in I_1(K) \quad \text{if and only if } \widetilde{K}_j \text{ satisfies the assumption in (4.4).(vi)} \\ &\widetilde{K}_l \in I_2(K) \quad \text{if and only if } \widetilde{K}_l \text{ satisfies the assumption in (4.4).(v).} \end{aligned} \tag{4.5}$$

It holds that:

- T_Γ is the set of endpoints p_1 of $\widetilde{K}_j$, for some $j \in I_1(K_l)$;
- S_Γ is the set of endpoints p_0, p_1 of $\widetilde{K}_j$, for some $j \in I_2(K_l)$;
- R_Γ is the set $\Gamma \setminus (T_\Gamma \cup S_\Gamma)$.

Definition 4.1.6. Let δ, λ be positive constants, with

$$\delta \geq \beta\Big(\frac{2}{\tan\theta} + 1\Big) \quad \text{and} \quad \lambda \geq \frac{\beta}{\tan\theta}. \tag{4.6}$$

We define the class $\mathcal{S}$ of *admissible cracks* as the set of curves Γ of the form

$$\Gamma = \bigcup_{j=1}^{N} K_j \tag{4.7}$$

such that $N \in \mathbb{N}$ and

(i) each K_j is of the form (4.4) and verifies Conditions (4.4).(i)-(4.4).(vi);

(ii) it is

$$K_j \cap \mathcal{P}_1(\widetilde{K}_l, p_0) = \emptyset \qquad \text{for every } \widetilde{K}_l \in I_1(K_m) \text{ and } m \neq j$$
$$K_j \cap \mathcal{P}_2(\widetilde{K}_l) = \emptyset \qquad \text{for every } \widetilde{K}_l \in I_2(K_m) \text{ and } m \neq j;$$

(iii) $\mathcal{H}^1(K_j) \geq \lambda$ for every j;

(iv) defined $S_\Gamma := \bigcup_{j=1}^N S_{K_j}$, for every $x_1, x_2 \in S_\Gamma$ with $x_1 \neq x_2$ it is

$$|x_1 - x_2| \geq \delta. \tag{4.8}$$

An example of an element $\Gamma \in \mathcal{S}$ is showed in Fig. 3.

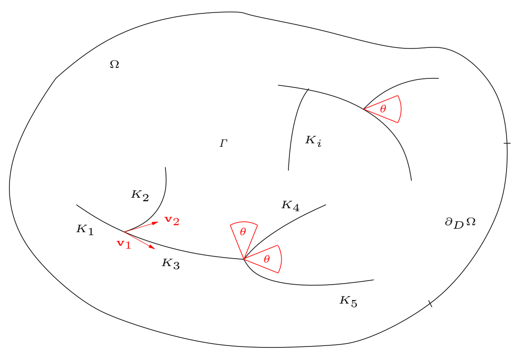


FIGURE 3. A domain $\Omega \subset \mathbb{R}^2$ with a crack $\Gamma \in \mathcal{S}$. The unit vectors $v_1, v_2 \in \mathbb{R}^2$ are tangent to Γ at a singular point; note that $v_1 \cdot v_2 \neq \pm 1$. The red triangles give an idea of the meaning of the one- and two-sided pencil-like neighbourhoods.

Remark 4.1.7. We briefly comment the assumptions (4.6). Notice that, if $\widetilde{K} \in \mathcal{R}_\eta$, γ is its arc-length parametrization and $\mathcal{H}^1(\widetilde{K}) \geq \beta(1 + 1/\tan\theta)$, then for every $s \geq \beta(1 + 1/\tan\theta)$ it is

$$B_\beta(\gamma(s)) \subset \mathcal{P}_1(\widetilde{K}, \gamma(0)).$$

If $\mathcal{H}^1(\widetilde{K}) \geq \delta$, then there exists $x \in \partial\mathcal{P}_2(\widetilde{K})$ with $\operatorname{dist}(x, \widetilde{K}) = \beta$. Indeed, this is true for

$$x = \gamma(s) + z\dot{\gamma}(s)^\perp$$

with $\beta/\tan\theta \leq s \leq \beta/\tan\theta + \beta$, for which $z = \pm\beta$.

The constraint (4.6) on λ and Condition 4.1.6.(iii) assure that, if a connected component K contains only one branch, *i.e.* $K \in \mathcal{R}_\eta$, then at each tip of K the 1-sided pencil-like neighbourhood contains a half-ball of radius β.

The class $\mathcal{S}$ is endowed with the topology induced by the Hausdorff metric defined on sets (see Subsection 1.7.1). The main result of the section is the following theorem, whose proof is achieved after several lemmas which will clarify the geometric meaning of the parameters and of the objects introduced for the definition of the class $\mathcal{S}$.

Theorem 4.1.8. *The class $\mathcal{S}$ is compact with respect to the Hausdorff convergence.*

Lemma 4.1.9. *There exists a constant $M \in \mathbb{N}$ such that every $\Gamma \in \mathcal{S}$ has at most M branches. In addition, also the number of singular points and tip points is uniformly bounded.*

PROOF. By (4.8), in $\overline{\Omega}$ there can be at most

$$\frac{1}{\frac{\sqrt{3}}{4}\delta^2}|\Omega|$$

singular points, where $\sqrt{3}\,\delta^2/4$ is the area of an equilateral triangle of side δ. By Conditions (4.4).(v) and (4.4).(vi) on the 1-sided and 2-sided pencil-like neighbourhoods, from each singular point at most $2\pi/\theta$ branches can develop. Therefore, $\frac{2\pi}{\theta}\frac{\sqrt{3}}{4\delta^2}|\Omega|$ is the maximum number of branches for an element $\Gamma \in \mathcal{S}$.

The final statement is an easy consequence of the previous part: the number of singular points and tip points is bounded by the number of equilateral triangles of side δ and by the number of branches, respectively. □

Lemma 4.1.10. *There exists a positive constant C such that $\mathcal{H}^1(\Gamma) \le C$ for every $\Gamma \in \mathcal{S}$.*

PROOF. Apply Proposition 4.1.2.(iii) and Lemma 4.1.9. □

Lemma 4.1.11. *There exists an increasing function $\rho : (0,+\infty) \to (0,\beta]$ satisfying the following property: if $K \in \mathcal{R}_\eta$ and γ is its arc-length parametrization, then*

$$B_{\rho(s)}(\gamma(s)) \subset \mathcal{P}_1(K,\gamma(0)),$$

for every $s \in (0,\mathcal{H}^1(K)]$.

PROOF. By elementary geometrical arguments, it is not difficult to observe that the bound on the curvature in the class $\mathcal{R}_\eta$ implies that

$$\rho(s) = \frac{\tan\theta}{2}s$$

satisfies the above property. □

Lemma 4.1.12. *Let $(\Gamma_k)_k \subset \mathcal{R}_\eta$ be such that $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$ and $\mathcal{H}^1(\Gamma) > 0$. Then*

$$\partial\mathcal{P}_2(\Gamma_k) \xrightarrow{\mathcal{H}} \partial\mathcal{P}_2(\Gamma) \quad \text{and} \quad \overline{\mathcal{P}_2(\Gamma_k)} \xrightarrow{\mathcal{H}} \overline{\mathcal{P}_2(\Gamma)}.$$

If $p_k^0 \to p^0$, then

$$\partial\mathcal{P}_1(\Gamma_k,p_k^0) \xrightarrow{\mathcal{H}} \partial\mathcal{P}_1(\Gamma,p^0) \quad \text{and} \quad \overline{\mathcal{P}_1(\Gamma_k,p_k^0)} \xrightarrow{\mathcal{H}} \overline{\mathcal{P}_1(\Gamma,p^0)}.$$

PROOF. By the compactness property for $\mathcal{R}_\eta$, Γ belongs to $\mathcal{R}_\eta$. We now adopt the notation of Proposition 4.1.2.

We consider first the case of $\partial\mathcal{P}_2$. In terms of the parametrization $\tilde{\gamma}_k$, it is

$$\mathcal{P}_2(\Gamma_k) = \left\{\tilde{\gamma}_k(s) + \frac{L}{L_k}z\dot{\tilde{\gamma}}_k(s)^\perp : 0 < s < L, |z| < \min\left\{\frac{L_k}{L}s\tan\theta, \beta, \frac{L_k}{L}(L-s)\tan\theta\right\}\right\}.$$

To obtain the claim, we verify the Kuratowski convergence, which in our setting is equivalent to the Hausdorff convergence (see for example [8]).

• First we show that, given a sequence $y_j \in \partial\mathcal{P}_2(\Gamma_{k_j})$ with $y_j \to y$ as $j \to +\infty$, then $y \in \partial\mathcal{P}_2(\Gamma)$. By definition of $\mathcal{P}_2$,

$$y_j = \tilde{\gamma}_{k_j}(s_j) + \frac{L}{L_{k_j}}z_j\dot{\tilde{\gamma}}_{k_j}(s_j).$$

Up to subsequences, $s_j \to s \in [0,L]$. By the bound $\|\tilde{\gamma}_{k_j}\|_{W^{2,\infty}((0,L);\mathbb{R}^2)} \le C$, (4.2), and (4.3), we have

$$\tilde{\gamma}_{k_j}(s_j) \to \gamma(s) \quad \text{and} \quad \dot{\tilde{\gamma}}_{k_j}(s_j) \to \dot{\gamma}(s). \tag{4.9}$$

In order to conclude, we should consider different scenarios:

$$\beta < \min\{s\tan\theta, (L-s)\tan\theta\}$$
$$s\tan\theta \le \beta \le (L-s)\tan\theta$$
$$(L-s)\tan\theta \le \beta \le s\tan\theta.$$

We only discuss the first case, for the others it is enough to argue similarly. Since $L_{k_j}/L \to 1$ and $s_j \to s$, it is $\beta < \min\left\{\frac{L_{k_j}}{L}s_j\tan\theta, \frac{L_{k_j}}{L}(L-s_j)\tan\theta\right\}$ for j sufficiently large, so that $z_j = \beta$. Then, by (4.9), we obtain that $y = \gamma(s) + \beta\,\dot\gamma(s)$, which belongs to $\partial\mathcal{P}_2(\mathit{\Gamma})$.

- Given $y = \gamma(s) + z\dot\gamma(s) \in \partial\mathcal{P}_2(\mathit{\Gamma})$, we have to construct a sequence $y_k \in \partial\mathcal{P}_2(\mathit{\Gamma}_k)$ converging to y. Define

$$y_k := \tilde\gamma_k(s) + \frac{L}{L_k} z_k \dot{\tilde\gamma}_k(s) \in \partial\mathcal{P}_2(\mathit{\Gamma}_k)$$

where, for k large, we set

  * if $\beta < \min\left\{\frac{L_k}{L}s\tan\theta, \frac{L_k}{L}(L-s)\tan\theta\right\}$, then $z_k = \beta$;
  * if $\frac{L_k}{L}s\tan\theta \le \beta \le \frac{L_k}{L}(L-s)\tan\theta$, then $z_k = \min\left\{\beta, \frac{L_k}{L}s\tan\theta\right\}$;
  * if $\frac{L_k}{L}(L-s)\tan\theta \le \beta \le \frac{L_k}{L}s\tan\theta$, then $z_k = \min\left\{\beta, \frac{L_k}{L}(L-s)\tan\theta\right\}$.

Using the pointwise convergence of $\tilde\gamma_k$ and $\dot{\tilde\gamma}_k$, it is easy to verify that $y_k \to y$.

For the case of $\partial\mathcal{P}_1$, in terms of $\tilde\gamma_k$, with $\tilde\gamma_k(0) = p_k^0$, the set $\mathcal{P}_1(\mathit{\Gamma}_k, p_k^0)$ is

$$\begin{aligned}\mathcal{P}_1(\mathit{\Gamma}_k, p_k) = &\left\{\tilde\gamma_k(s) + \frac{L}{L_k} z\dot{\tilde\gamma}_k(s)^\perp : 0 < s < L, |z| < \min\{\frac{L_k}{L}s\tan\theta, \beta\}\right\} \\ &\cup\left\{\tilde\gamma_k(L) + (s-L)\dot{\tilde\gamma}_k(L) + \frac{L}{L_k} z\dot{\tilde\gamma}_k(L)^\perp : L < s < L + \frac{L}{L_k}\beta\right. \\ &\left.\text{and } |z| < \min\left\{\frac{L_k}{L}s\tan\theta, \sqrt{\beta^2 - \frac{L_k^2}{L^2}(s-L)^2}\right\}\right\}.\end{aligned}$$

The proof follows the steps of the previous one, with the following differences.

- Let $y_j \in \partial\mathcal{P}_1(\mathit{\Gamma}, p_{k_j}^0)$ be such that $y_j \to y$. It is $y_j = \gamma_{k_j}(s_j) + x_j$ and, up to subsequences, $s_j \to s$, where
  * if $0 \le s < L$, then for j large
$$x_j = \frac{L}{L_{k_j}} z_j \dot{\tilde\gamma}_{k_j}(s_j)^\perp; \tag{4.10}$$
  * if $L < s \le L + \beta$, then for j large
$$x_j = (s_j - L)\dot{\tilde\gamma}_{k_j}(L) + \frac{L}{L_{k_j}} z_j \dot{\tilde\gamma}_{k_j}(L)^\perp; \tag{4.11}$$
  * if $s = L$, then there exists a further subsequence such that either (4.10) or (4.11) hold for every term of the subsequence.
- Given $y \in \partial\mathcal{P}_1(\mathit{\Gamma}, p^0)$, we write $y = \gamma(s) + x$, with

$$\begin{aligned} x &= z\dot\gamma(s)^\perp && \text{if } 0 \le s \le L \\ x &= (s-L)\dot\gamma(L) + z\dot\gamma(L)^\perp && \text{if } L \le s \le L + \beta. \end{aligned}$$

Then one defines $y_k \in \partial\mathcal{P}_1(\mathit{\Gamma}_k, p_k^0)$ by considering $\tilde\gamma_k(s)$, $\dot{\tilde\gamma}_k(s)$ and arguing similarly as for $\mathcal{P}_2$ in order to choose z_k appropriately.

Similar arguments hold for the closed sets $\overline{\mathcal{P}_2(\mathit{\Gamma}_k)}$ and $\overline{\mathcal{P}_1(\mathit{\Gamma}_k, p_k^0)}$. $\square$

Lemma 4.1.13. *Let $(\Gamma_k^1)_k, (\Gamma_k^2)_k \subset \mathcal{R}_\eta$ be two sequences such that*

$$\Gamma_k^1 \xrightarrow{\mathcal{H}} \Gamma^1 \quad \textit{and} \quad \Gamma_k^2 \xrightarrow{\mathcal{H}} \Gamma^2 , \tag{4.12}$$

and assume that $\mathcal{H}^1(\Gamma^1) > 0$. The following properties hold:

- *if for every k*

$$\begin{aligned} &\mathcal{P}_1(\Gamma_k^1, p_k) \cap \Gamma_k^2 = \emptyset \\ &p_k \to p \quad \textit{with} \quad p_k \in T_{\Gamma_k^1}, \end{aligned} \tag{4.13}$$

  then $p \in T_{\Gamma^1}$ and $\mathcal{P}_1(\Gamma^1, p) \cap \Gamma^2 = \emptyset$;
- *if $\mathcal{P}_2(\Gamma_k^1) \cap \Gamma_k^2 = \emptyset$ for every k, then $\mathcal{P}_2(\Gamma^1) \cap \Gamma^2 = \emptyset$.*

Proof. Being $p_k \in T_{\Gamma_k^1}$, it is $p_k = \tilde{\gamma}_k(0)$ for the arc-length parametrization γ_k. Since $\tilde{\gamma}_k(0) \to \gamma(0)$ by Proposition 4.1.2, it is $p \in T_{\Gamma^1}$.

By contradiction, assume that there exists $x \in \mathcal{P}_1(\Gamma^1, p^1) \cap \Gamma^2$. Let $r > 0$ be such that $\mathrm{dist}(x, \partial\mathcal{P}_1(\Gamma^1, p^1)) = 4r$. By (4.12), there exists a sequence $x_k \in \Gamma_k^2$ converging to x; in particular $x_k \in B_r(x)$ for k large. By Lemma 4.1.12, for k large enough we have $B_r(x) \subset \mathcal{P}_1(\Gamma_k^1, p_k^1)$, so that

$$x_k \in \Gamma_k^2 \cap B_r(x) \subset \Gamma_k^2 \cap \mathcal{P}_1(\Gamma_k^1, p_k^1),$$

in contradiction to (4.13).

The second property can be proved in a similar manner. □

Lemma 4.1.14. *Let $\Gamma_k \in \mathcal{S}$ be a sequence converging to Γ in the Hausdorff metric. Then $\mathcal{H}^1(\Gamma_k) \to \mathcal{H}^1(\Gamma)$.*

Proof. Without loss of generality, we can assume that

$$\Gamma_k = \bigcup_{i=1}^{N} \widetilde{K}_k^i \tag{4.14}$$

with $\widetilde{K}_k^i$ branches of Γ_k and

$$\widetilde{K}_k^i \xrightarrow{\mathcal{H}} \hat{K}^i, \tag{4.15}$$

for some $N \le M$ (M given in Lemma 4.1.9). Indeed, if this is not the case, for every $N \le M$ we consider the subsequence $\Gamma_{N,k}$ of elements having N branches and, up to relabelling, we can assume that (4.15) holds.

First notice that, having Γ_k a uniformly bounded number of connected components, by Gołąb's Theorem 1.7.2 we obtain immediately

$$\mathcal{H}^1(\Gamma) \le \liminf_{k \to +\infty} \mathcal{H}^1(\Gamma_k). \tag{4.16}$$

If $\mathcal{H}^1(\hat{K}^i) = 0$, then $\hat{K}^i = \{x^i\}$. We argue as in the proof of 4.1.2.(iv) of Proposition 4.1.2: the bound on the curvature for $\widetilde{K}_k^i \in \mathcal{R}_\eta$ and (4.15) imply that $\mathcal{H}^1(\widetilde{K}_k^i) \to 0$.

If for $i \neq l$ it is $\mathcal{H}^1(\hat{K}^i), \mathcal{H}^1(\hat{K}^l) > 0$, then Lemma 4.1.13 and Remark 4.1.3 imply that

$$\hat{K}^i \cap \hat{K}^l \subset \{p_0^i, p_1^i\}$$

with p_0^i, p_1^i endpoints of $\hat{K}^i$. Hence $\mathcal{H}^1(\hat{K}^i \cup \hat{K}^l) = \mathcal{H}^1(\hat{K}^i) + \mathcal{H}^1(\hat{K}^l)$. Applying Proposition 4.1.2.(iv),

$$\mathcal{H}^1(\hat{K}^i) = \lim_{k \to +\infty} \mathcal{H}^1(\widetilde{K}_k^i), \tag{4.17}$$

for every i. Considering (4.16), we have

$$\mathcal{H}^1(\Gamma) \leq \liminf_{k\to+\infty} \mathcal{H}^1(\Gamma_k) = \liminf_{k\to+\infty} \sum_{\substack{i=1 \\ \mathcal{H}^1(\hat{K}^i)=0}}^{N} \mathcal{H}^1(\widetilde{K}_k^i) = \sum_{i=1}^{N} \mathcal{H}^1(\hat{K}^i) = \mathcal{H}^1(\Gamma).$$

Since (4.17) holds, in the above relation all $\liminf$ are actually limits, so that we obtain the thesis. □

Proof of Theorem 4.1.8. Let $(\Gamma_k)_k$ be a sequence in $\mathcal{S}$.

We assume that for each k the set Γ_k is connected; the general case can be obtained with similar arguments. Hence Γ_k has the form (4.4):

$$\Gamma_k = \bigcup_{j=1}^{N_k} \widetilde{K}_k^j.$$

Being $N_k \leq M$ for every k (see Lemma 4.1.9), without loss of generality we can assume that $N_k = N'$ for all k. By Blaschke compactness Theorem 1.7.1, up to subsequences $\Gamma_k \xrightarrow{\mathcal{H}} \Gamma$ for a compact connected set Γ. It remains to prove that $\Gamma \in \mathcal{S}$.

Lemma 4.1.14 and Condition 4.1.6.(iii) for Γ_k imply that $\mathcal{H}^1(\Gamma_k) \to \mathcal{H}^1(\Gamma)$; therefore $\lambda \leq \mathcal{H}^1(\Gamma) < +\infty$, *i.e.* Condition 4.1.6.(iii) for Γ.

Applying again Blaschke's Theorem 1.7.1, up to relabelling the $\widetilde{K}_k^j$, we can assume $\widetilde{K}_k^j \xrightarrow{\mathcal{H}} \hat{K}^j$ for some compact set $\hat{K}^j$, for $j = 1,\dots,N'$; of course, $\Gamma = \hat{K}^1 \cup \ldots \cup \hat{K}^{N'}$. By Proposition 4.1.2.(iv)

$$\mathcal{H}^1(\hat{K}^j) = \lim_{k\to+\infty} \mathcal{H}^1(\widetilde{K}_k^j)$$

and we relabel the sets $\hat{K}^j$ so that $\mathcal{H}^1(\hat{K}^j) > 0$ for $j = 1,\dots,N''$ for some $N'' \leq N'$ and $\mathcal{H}^1(\hat{K}^j) = 0$ for $j = N''+1,\dots,N'$ (in this case $\hat{K}^j = \{x_j\}$). Proposition 4.1.2 implies also that

$$\hat{K}^j \in \mathcal{R}_\eta \tag{4.18}$$

for $j = 1,\dots,N''$. Being Γ and $\hat{K}^j$ connected,

$$\Gamma = \bigcup_{j=1}^{N''} \hat{K}^j\,. \tag{4.19}$$

Thanks to (4.18) and (4.19), Γ can be described as a finite union of $C^{1,1}$ curves in $\mathcal{R}_\eta$. We write

$$\Gamma = \bigcup_{i=1}^{N} \widetilde{K}^i$$

is such a way that Conditions (4.4).(i)-(4.4).(iv) in (4.4) are satisfied. We are left to check Conditions (4.4).(v) and (4.4).(vi). Firstly we remark that, by Lemma 4.1.13 and Remark 4.1.3, if $\hat{K}^j \cap \hat{K}^l \neq \emptyset$, then they intersect at most in their endpoints. Hence for every $i = 1,\dots,N$ it is

$$\widetilde{K}^i = \bigcup_{j\in I_i} \hat{K}^j$$

for a set of indices $I_i \subset \{1,\dots,N'\}$.

Assume that $\widetilde{K}^1$ satisfies the assumptions in (4.4).(v). If $\widetilde{K}^1 = \hat{K}^j$ for some j, then Lemma 4.1.13 implies that $\mathcal{P}_2(\widetilde{K}^1) \cap \hat{K}^l = \mathcal{P}_2(\hat{K}^j) \cap \hat{K}^l = \emptyset$ for every $l \neq j$. Therefore $\mathcal{P}_2(\widetilde{K}^1) \cap \widetilde{K}^h = \emptyset$ for all $h = 2,\dots N$, which is exactly condition (4.4).(v).

Assume now that, up to relabel the $\hat{K}^j$, it is

$$\widetilde{K}^1 = \hat{K}^1 \cup \ldots \cup \hat{K}^l$$

with $\hat{K}^j \cap \hat{K}^{j+1} = \{y^j\}$, and that for the remaining $\hat{K}^j$ it is

$$\bigcup_{j=l+1}^{N''} \hat{K}^j = \bigcup_{i=2}^{N} \widetilde{K}^i.$$

Necessarily, for every k large enough the set $\widetilde{K}^1_k \cup \ldots \cup \widetilde{K}^l_k$ is connected and $\widetilde{K}^j_k \cap \widetilde{K}^{j+1}_k = \{y^j_k\}$, with

$$\widetilde{K}^1_k, \ldots, \widetilde{K}^l_k \in I_2(\Gamma_k), \tag{4.20}$$

and $I_2(\Gamma_k)$ defined in (4.5). This claim is consequence of Lemma 4.1.11 and the fact that $\mathcal{H}^1(\widetilde{K}^j_k) \geq \frac{1}{2}\mathcal{H}^1(\hat{K}^j) > 0$ for k large: by contradiction, if one of the $\widetilde{K}^j_k$ belongs to $I_1(\Gamma_k)$, then Lemma 4.1.11 implies that a tip remains at a positive distance (independent of k) from all the other branches, so that the same holds for its Hausdorff limit; as a consequence, $\widetilde{K}^1$ cannot belong to $I_2(\Gamma)$.

Call y^0 the endpoint of $\hat{K}^1$ not belonging to $\hat{K}^2$, and y^l the endpoint of $\hat{K}^l$ not belonging to $\hat{K}^{l-1}$. We have $\mathcal{H}^1(\hat{K}^j) \geq |y^{j-1} - y^j| \geq \delta$ for $j = 1, \ldots, l$, since $y^i_k \to y^i$ and $|y^i_k - y^h_k| \geq \delta$ for $0 \leq i < h \leq l$. This remark on the lengths, the assumption (4.6) on δ and the comments in Remark 4.1.7 imply

$$\mathcal{P}_2(\widetilde{K}^1) = \mathcal{P}_1(\hat{K}^1, y^0) \cup (\hat{K}^2)_\beta \cup \ldots (\hat{K}^{l-1})_\beta \cup \mathcal{P}_1(\hat{K}^l, y^l)\,, \tag{4.21}$$

where $(\hat{K}^j)_\beta = \{x \in \Omega :\ \mathrm{dist}(x, \hat{K}^j) < \beta\}$. In particular $\mathcal{P}_2(\widetilde{K}^1) \supset \mathcal{P}_2(\hat{K}^1) \cup \ldots \cup \mathcal{P}_2(\hat{K}^l)$.

To prove Condition (4.4).(v) we argue by contradiction. Assume that $\mathcal{P}_2(\widetilde{K}^1) \cap \widetilde{K}^i \neq \emptyset$ for some $i \in \{2, \ldots, N\}$; since by Lemma 4.1.13 and (4.20) it is $\mathcal{P}_2(\hat{K}^h) \cap \hat{K}^m = \emptyset$ for all $h = 1, \ldots l$ and $m \neq l$, there exists

$$x \in \widetilde{K}^i \cap \mathcal{P}_2(\widetilde{K}^1) \setminus \Big(\mathcal{P}_2(\hat{K}^1) \cup \ldots \cup \mathcal{P}_2(\hat{K}^l)\Big).$$

More precisely,

$$x \in (\hat{K}^j)_\beta \setminus \mathcal{P}_2(\hat{K}^j) \tag{4.22}$$

for some $j \in \{1 \ldots, l\}$ and $x \in \hat{K}^m$ for some $m > l$, with $\hat{K}^m$ having endpoints y^{m-1}, y^m. Let γ^j be the arc-length parametrization of $\hat{K}^j$; then

$$x = \gamma^j(s) + z\dot{\gamma}^j(s)^\perp \tag{4.23}$$

for some $|z| < \beta$ and, because of (4.22),

$$\text{either} \quad s \in [0, \beta/\tan\theta) \quad \text{or} \quad s \in (\mathcal{H}^1(\hat{K}^j) - \beta/\tan\theta, \mathcal{H}^1(\hat{K}^j)]. \tag{4.24}$$

Assume $(4.24)_1$ and remember that $y^{m-1} = \gamma^j(0)$ and $y^j = \gamma^j(\mathcal{H}^1(\hat{\Gamma}^j))$. Then

$$\begin{aligned}
\min\big\{|x - y^{m-1}|, |x - y^m|\big\} \geq& \min\big\{|y^{m-1} - y^{j-1}|, |y^m - y^{j-1}|\big\} - |y^{j-1} - x| \\
=& \min\big\{|y^{m-1} - y^{j-1}|, |y^m - y^{j-1}|\big\} \\
&- |\gamma^j(0) - \gamma^j(s) + z\dot{\gamma}^j(s)^\perp| \\
\geq& \delta - (|z| + s) \geq \delta - \beta - \frac{\beta}{\tan\theta} \\
\geq& \beta\Big(\frac{2}{\tan\theta} - 1\Big) - \beta - \frac{\beta}{\tan\theta} = \frac{\beta}{\tan\theta}.
\end{aligned}$$

If $(4.24)_2$ holds, then by substituting y^{j-1} with y^j in the above chain of inequalities we get

$$\min\left\{|x-y^{m-1}|,|x-y^m|\right\} \geq \frac{\beta}{\tan\theta}.$$

Hence $x=\gamma^m(s)$ with $s\in[\beta/\tan\theta, \mathcal{H}^1(\hat{\Gamma}^m)-\beta/\tan\theta]$; then, by choice of δ (see Remark 4.1.7), this condition implies that $\operatorname{dist}(x,\hat{K}^i)\geq\beta$ for every $i\neq m$, in contradiction to (4.23), since $|z|<\beta$.

Condition (4.4).(v) is now proved.

In order to check Condition (4.4).(vi) one argues similarly. Instead of (4.21), one has to observe that

$$\mathcal{P}_1(\widetilde{K}^1,y^0)=\mathcal{P}_1(\hat{K}^1,y^0)\cup(\hat{K}^2)_\beta\cup\ldots\cup(\hat{K}^l)_\beta.$$

This concludes the proof when the sets Γ_k are connected. □

In conclusion, we prove some further geometrical results which will be useful in the following sections.

Lemma 4.1.15. *Let $\Gamma_k,\Gamma\in\mathcal{S}$ and $\Gamma_k\xrightarrow{\mathcal{H}}\Gamma$. Then for every $p\in T_\Gamma$ there exists a sequence $(p_k)_k$, with $p_k\in T_{\Gamma_k}$, such that $p_k\to p$.*

Proof. By contradiction, assume that there exist $r>0$ and a subsequence of Γ_k, not relabelled, such that

$$\operatorname{dist}(p,T_{\Gamma_k})\geq r \tag{4.25}$$

for every k. Without loss of generality, we can assume that $r\in(0,\eta)$ and r satisfies the properties defining a crack tip (see Definition 4.1.4). Define $K:=\Gamma\cap\overline{B_r(p)}\in\mathcal{R}_\eta$ and notice that $K\cap\partial B_r(p)=\{y\}$.

Two cases are possible: either

$$\operatorname{dist}(p,S_{\Gamma_k})\geq r \tag{4.26}$$

for every k (possibly by replacing the previous r with a smaller one), or there exists a sequence $x_k\in S_{\Gamma_k}$ such that

$$x_k\to p. \tag{4.27}$$

If (4.26) holds, by Hausdorff convergence there exists $x_k\to p$ with $x_k\in\Gamma_k\setminus(S_{\Gamma_k}\cup T_{\Gamma_k})$. Let K_k be the branch of Γ_k containing x_k; $K_k\in\mathcal{R}_\eta$. Set $\widetilde{K}_k:=K_k\cap\overline{B_r(p)}$, then $\widetilde{K}_k\in\mathcal{R}_\eta$. Fix $\varepsilon\in(0,\eta/2)$; by Hausdorff convergence, for k sufficiently large $\widetilde{K}_k:=K_k\cap\overline{B_r(p)}\subset(K)_\varepsilon:=\{x\in\Omega:\operatorname{dist}(x,K)<\varepsilon\}$, hence Condition 4.1.1.(ii) in Definition 4.1.1 is not satisfied, in contradiction to the fact that $\widetilde{K}_k\in\mathcal{R}_\eta$.

If (4.27) is the case, then there exist at least two branches K_k^1 and K_k^2 of Γ_k containing x_k and such that $K_k^i\setminus\overline{B_r(p)}\neq\emptyset$ (because we are assuming (4.25)); let $y_k^i\in K_k^i\cap\partial B_r(p)$. For k sufficiently large we can assume that $|x_k-p|<r/2$, so that

$$\mathcal{H}^1\left(K_k^i\cap\partial B_r(p)\right)\geq|x_k-y_k^i|\geq r/2.$$

By Lemma 4.1.11, it must be

$$|y_k^1-y_k^2|\geq\rho(r/2). \tag{4.28}$$

Taken $\varepsilon<\frac{1}{2}\min\{\rho(r/2),r/2\}$, by Hausdorff convergence we have

$$K_k^i\cap\overline{B_r(p)}\subset(K)_\varepsilon \quad\text{and}\quad |y_k^i-y|<\varepsilon,$$

which imply that $|y_k^1-y_k^2|<2\varepsilon<\rho(r/2)$, in contradiction to (4.28). □

Lemma 4.1.16. *Let $\Gamma,\Gamma_k\in\mathcal{S}$ and $\Gamma_k\xrightarrow{\mathcal{H}}\Gamma$. Assume that for a tip $p\in T_\Gamma$ there exist $p_k^1,p_k^2\in T_{\Gamma_k}$, $p_k^1\neq p_k^2$, converging to p. Then there exists $y_k\in S_{\Gamma_k}$ converging to p.*

PROOF. We argue by contradiction. Assume that

$$\operatorname{dist}(p, S_{\Gamma_k}) \geq r$$

for some $r > 0$. As in Lemma 4.1.15, without loss of generality, we can assume that $r \in (0, \eta)$ and r satisfies the properties defining a crack tip. Define $K := \Gamma \cap \overline{B_r(p)} \in \mathcal{R}_\eta$.

For k sufficiently large, $|p_k^i - p| < r/2$, so that the connected components K_k^i containing p_k^i, for $i = 1, 2$, satisfy the bound from below

$$\mathcal{H}^1(K_k^i) \geq r/2.$$

Then, by Lemma 4.1.11, it has to be $|p_k^1 - p_k^2| \geq \rho(r/2)$, in contradiction to the fact that p_k^1 and p_k^2 converge both to p. □

4.2. The incremental problem

This section is devoted to the study of the discrete-time approximation of the continuous-time evolution. At each incremental step, the fracture is permitted to grow simultaneously at many tips and to develop new branches. In order to avoid non-physical interactions between them, we need to perform a sort of localization argument to treat each tip separately. This is obtained by keeping trace of the fracture increments at each step in the discrete-time approach. At the end of the section, we establish some *a priori* estimates and properties of the discrete-time evolutions.

We study the evolution problem in the fixed time interval $[0, 1]$. On $\partial_D\Omega$ we prescribe a time-dependent boundary displacement which, at each instant $t \in [0, 1]$, is given by the value $w(t)$ of (the trace of) a function

$$w \in H^1(0, 1; H^1(\Omega))$$

at t.

The initial configuration is the couple (u_0, Γ_0) where $u_0 \in H^1(\Omega \setminus \Gamma_0)$ is solution to

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega \setminus \Gamma_0 \\ \frac{\partial u_0}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_0 \cup \partial\Omega \setminus \partial_D\Omega \\ u_0 = w(0) & \text{on } \partial_D\Omega \end{cases} \tag{4.29}$$

and Γ_0 belongs to the class $\mathcal{S}$ of admissible cracks. By solution to (4.29) we mean that

$$\int_{\Omega\setminus\Gamma_0} \nabla u_0 \cdot \nabla v \, dx = 0 \qquad \text{for every } v \in H^1(\Omega \setminus \Gamma_0),\ v = 0 \text{ on } \partial_D\Omega\,.$$

Given $\Gamma_1, \Gamma_2 \in \mathcal{S}$ with $\Gamma_1 \subset \Gamma_2$, let

$$\mathcal{C}(\Gamma_1, \Gamma_2)$$

be the set of connected components of $\Gamma_2 \setminus \Gamma_1$. Notice that every element $\mathfrak{c} \in \mathcal{C}(\Gamma_1, \Gamma_2)$ is a finite union of $C^{1,1}$ curves. In particular, $\bar{\mathfrak{c}}$ satisfies Conditions (4.4).(i)-(4.4).(vi) in (4.4) and

$$\mathfrak{c} = \bar{\mathfrak{c}} \setminus \Gamma_1. \tag{4.30}$$

In addition, if $\overline{\mathfrak{c}'} \cap \overline{\mathfrak{c}''} \neq \emptyset$ for two distinct components $\mathfrak{c}', \mathfrak{c}'' \in \mathcal{C}(\Gamma_1, \Gamma_2)$, then

$$\overline{\mathfrak{c}'} \cap \overline{\mathfrak{c}''} \subset S_{\Gamma_2} \cap \Gamma_1. \tag{4.31}$$

We now construct the discrete-time evolution with incremental time step $\tau > 0$ and initial datum (u_0, Γ_0) satisfying (4.29). Define

- $u_\tau^0 := u_0$ and $\Gamma_\tau^0 := \Gamma_0$;

- recursively u_τ^i and Γ_τ^i as minimizers of

$$\|\nabla u\|^2 + \mathcal{H}^1(\Gamma) + \frac{1}{\tau} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma_\tau^{i-1}, \Gamma)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \tag{4.32}$$

under the constraints $\Gamma \in \mathcal{S}$, $\Gamma_\tau^{i-1} \subset \Gamma$, $u \in H^1(\Omega \setminus \Gamma)$ and $u = w(i\tau)$ on $\partial_D \Omega$.

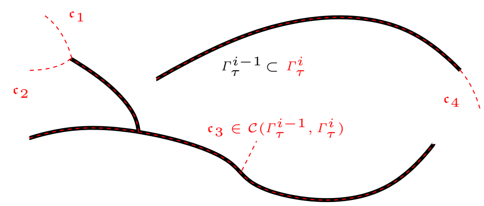


FIGURE 4. At each incremental step the fracture is permitted to grow simultaneously at many tips and to develop new branches.

Proposition 4.2.1. *The minimum problem* (4.32) *has a solution.*

For its proof we need the following lower semicontinuity result, together with Theorem 1.7.6 about the convergence of gradients of solutions to elliptic problems in varying domains.

Lemma 4.2.2. *Consider $\hat{\Gamma}, \Gamma \in \mathcal{S}$ and a sequence $(\Gamma_k)_k \subset \mathcal{S}$ such that*

$$\hat{\Gamma} \subset \Gamma_k \quad \textit{for any } k \quad \textit{and} \quad \Gamma_k \xrightarrow{\mathcal{H}} \Gamma\,. \tag{4.33}$$

Then $\hat{\Gamma} \subset \Gamma$ and

$$\sum_{\mathfrak{c} \in \mathcal{C}(\hat{\Gamma}, \Gamma)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \le \liminf_{k \to +\infty} \sum_{\mathfrak{c} \in \mathcal{C}(\hat{\Gamma}, \Gamma_k)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2.$$

PROOF. The fact that $\hat{\Gamma} \subset \Gamma$ is a direct consequence of (4.33).

We claim that for every $\mathfrak{c} \in \mathcal{C}(\hat{\Gamma}, \Gamma)$ there exists a sequence $\mathfrak{c}_k \in \mathcal{C}(\hat{\Gamma}, \Gamma_k)$ such that $\mathfrak{c}_k \xrightarrow{\mathcal{H}} \mathfrak{c}$. By contradiction, assume that the claim does not hold for some $\mathfrak{c} \in \mathcal{C}(\hat{\Gamma}, \Gamma)$. By (4.33), there exist $\mathfrak{c}_k^1, \mathfrak{c}_k^2 \in \mathcal{C}(\hat{\Gamma}, \Gamma_k)$ such that

$$\mathfrak{c}_k^1 \xrightarrow{\mathcal{H}} \mathfrak{c}^1 \qquad \mathfrak{c}_k^2 \xrightarrow{\mathcal{H}} \mathfrak{c}^2$$

with $\mathcal{H}^1(\mathfrak{c}^1) > 0$, $\mathcal{H}^1(\mathfrak{c}^2) > 0$, $\mathfrak{c}^1 \cup \mathfrak{c}^2 \subset \mathfrak{c}$ and $\mathfrak{c}^1 \cup \mathfrak{c}^2$ connected. The set $\bar{\mathfrak{c}}^1 \cap \bar{\mathfrak{c}}^2$ contains finitely many points. Notice that, being $\mathfrak{c}^1 \cup \mathfrak{c}^2$ connected and $\left(\mathfrak{c}^1 \cup \mathfrak{c}^2\right) \cap \hat{\Gamma} = \emptyset$, there exists $x \in \left(\bar{\mathfrak{c}}^1 \cap \bar{\mathfrak{c}}^2\right) \setminus \hat{\Gamma}$. In particular, by (4.30) it is $x \in \mathfrak{c}^1 \cap \mathfrak{c}^2$.

By Hausdorff convergence, there exist $x_k^1 \in \mathfrak{c}_k^1$, $x_k^2 \in \mathfrak{c}_k^2$ with

$$x_k^1, x_k^2 \to x. \tag{4.34}$$

By assumption (4.8) for the singular points of sets in $\mathcal{S}$, up to subsequences we can assume that either

$$x_k^1 \in \Gamma_k \setminus S_{\Gamma_k} \tag{4.35}$$

for every k and there exists $C > 0$ such that

$$\mathrm{dist}(x_k^1, S_{\Gamma_k} \cap \bar{\mathfrak{c}}_k^1) \ge C, \tag{4.36}$$

or the analogous properties hold true for x_k^2 and $\mathfrak{c}_k^2$. Indeed,

- if both $x_k^1, x_k^2 \in S_{\Gamma_k}$, then (4.8) is contradicted, because $x_k^1 \neq x_k^2$ and $|x_k^1 - x_k^2| \to 0$.
- If both

$$\operatorname{dist}(x_k^1, S_{\Gamma_k} \cap \bar{\mathfrak{c}}_k^1), \operatorname{dist}(x_k^2, S_{\Gamma_k} \cap \bar{\mathfrak{c}}_k^2) \to 0,$$

let $y_k^1 \in S_{\Gamma_k} \cap \bar{\mathfrak{c}}_k^1$ with $|x_k^1 - y_k^1| = \operatorname{dist}(x_k^1, S_{\Gamma_k} \cap \bar{\mathfrak{c}}_k^1)$, and the same for y_k^2. Notice that $y_k^1, y_k^2 \to x$.

If $y_k^1 = y_k^2 =: y_k$, then $y_k \in \bar{\mathfrak{c}}_k^1 \cap \bar{\mathfrak{c}}_k^2$ and, by (4.31) with $\Gamma_1 = \hat{\Gamma}$, it is $y_k \in \hat{\Gamma}$. Hence $x \in \hat{\Gamma}$, in contradiction to the choice of x.

If $y_k^1 \neq y_k^2$, then (4.8) is contradicted by the fact that $|y_k^1 - y_k^2| \to 0$.

Assume (4.35) and (4.36). Let $\widetilde{K}_k$ be the branch in Γ_k containing x_k^1 and γ_k its arc-length parametrization. Then $x_k^1 = \gamma_k(s_k)$ for some $s_k \in (0, \mathcal{H}^1(\widetilde{K}_k))$; by (4.36), necessarily $s_k \geq C$. Hence Lemma 4.1.11 implies that

$$\operatorname{dist}(x_k^1, \widetilde{K}'_k) \geq \rho(C)$$

for all branches $\widetilde{K}'_k$ different than $\widetilde{K}_k$. In particular,

$$|x_k^1 - x_k^2| \geq \rho(C)$$

for every k, in contradiction to (4.34).

To conclude, let $\mathcal{C}(\hat{\Gamma}, \Gamma) = \{\mathfrak{c}^1, \dots, \mathfrak{c}^m\}$ and consider $\mathfrak{c}_k^i \in \mathcal{C}(\hat{\Gamma}, \Gamma_k)$ such that $\mathfrak{c}_k^i \xrightarrow{\mathcal{H}} \mathfrak{c}^i$, which exist for what we just proved. Then, by the fact that $\mathcal{H}^1(\bar{\mathfrak{c}}^i) = \mathcal{H}^1(\mathfrak{c}^i)$, $\mathcal{H}^1(\bar{\mathfrak{c}}_k^i) = \mathcal{H}^1(\mathfrak{c}_k^i)$ and all components $\bar{\mathfrak{c}}^i, \bar{\mathfrak{c}}_k^i$ satisfy Conditions (4.4).(i)-(4.4).(vi), we can apply Lemma 4.1.14 to get $\mathcal{H}^1(\mathfrak{c}_k^i) \to \mathcal{H}^1(\mathfrak{c}^i)$ for $i = 1, \dots, m$. Finally

$$\begin{aligned}
\sum_{\mathfrak{c} \in \mathcal{C}(\hat{\Gamma}, \Gamma)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 &= \sum_{i=1}^{m} \left(\mathcal{H}^1(\mathfrak{c}^i)\right)^2 = \lim_{k \to +\infty} \sum_{i=1}^{m} \left(\mathcal{H}^1(\mathfrak{c}_k^i)\right)^2 \\
&\leq \liminf_{k \to +\infty} \sum_{\mathfrak{c} \in \mathcal{C}(\hat{\Gamma}, \Gamma_k)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 .
\end{aligned}$$

□

Proof of Proposition 4.2.1. Fix $i \in \{1, \dots, N_\tau\}$. For any $\Gamma \in \mathcal{S}$ let u_Γ be the minimizer of

$$\min\left\{\|\nabla v\|^2 : \, v \in H^1(\Omega \setminus \Gamma) \text{ with } v = w(i\tau) \text{ on } \partial_D \Omega\right\} .$$

Consider a minimizing sequence $\Gamma_k \in \mathcal{S}$ for (4.32). By compactness of the class $\mathcal{S}$, there exist a subsequence, not relabelled, and an element $\widetilde{\Gamma} \in \mathcal{S}$ such that $\Gamma_k \xrightarrow{\mathcal{H}} \widetilde{\Gamma}$. Lemma 4.1.14 implies that

$$\mathcal{H}^1(\Gamma_k) \to \mathcal{H}^1(\widetilde{\Gamma}) . \tag{4.37}$$

Since $\Gamma_\tau^{i-1} \subset \Gamma_k$ for every k and Lemma 4.2.2 holds, it follows that $\Gamma_\tau^{i-1} \subset \widetilde{\Gamma}$ and

$$\sum_{\mathfrak{c} \in \mathcal{C}(\Gamma_\tau^{i-1}, \widetilde{\Gamma})} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \leq \liminf_{k \to +\infty} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma_\tau^{i-1}, \Gamma_k)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 . \tag{4.38}$$

Applying Theorem 1.7.6 we obtain that $\nabla u_{\Gamma_k} \to \nabla u_{\widetilde{\Gamma}}$ strongly in $L^2(\Omega; \mathbb{R}^2)$.

Collecting together (4.37), (4.38) and the last fact, we obtain

$$\begin{aligned}
&\|\nabla u_{\widetilde{\Gamma}}\|^2 + \mathcal{H}^1(\widetilde{\Gamma}) + \frac{1}{\tau} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma_\tau^{i-1}, \widetilde{\Gamma})} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \\
&\leq \lim_{k \to +\infty} \left(\|\nabla u_{\Gamma_k}\|^2 + \mathcal{H}^1(\Gamma_k)\right) + \frac{1}{\tau} \liminf_{k \to +\infty} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma_\tau^{i-1}, \Gamma_k)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \\
&\leq \inf \left\{ \|\nabla u\|^2 + \mathcal{H}^1(\Gamma) + \frac{1}{\tau} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma_\tau^{i-1}, \Gamma)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \right\},
\end{aligned}$$

i.e. the couple $(u_{\widetilde{\Gamma}}, \widetilde{\Gamma})$ minimizes (4.32) (notice that, by definition of $u_{\widetilde{\Gamma}}$, it is $u_{\widetilde{\Gamma}} = w(i\tau)$ on $\partial_D \Omega$). □

We introduce the following functions:

- the piecewise-constant interpolant for the displacement $u_\tau : [0,1] \to L^2(\Omega)$ as
$$u_\tau(t) := u_\tau^i$$
for $i\tau \leq t < (i+1)\tau, i = 0, \ldots, N_\tau - 1$, and $u_\tau(t) := u_\tau^{N_\tau}$ for $\tau N_\tau \leq t \leq 1$;
- the piecewise-constant interpolant for the crack set $\Gamma_\tau : [0,1] \to \mathcal{S}$ as
$$\Gamma_\tau(t) := \Gamma_\tau^i$$
for $i\tau \leq t < (i+1)\tau, i = 0, \ldots, N_\tau$, and $\Gamma_\tau(t) := \Gamma_\tau^{N_\tau}$ for $\tau N_\tau \leq t \leq 1$;
- the piecewise-constant and piecewise-affine interpolants of the fracture length $\ell_\tau, \tilde{\ell}_\tau$: $[0,1] \to \mathbb{R}$ as
$$\ell_\tau(t) := \mathcal{H}^1(\Gamma_\tau^i) \quad \text{and} \quad \tilde{\ell}_\tau(t) := \mathcal{H}^1(\Gamma_\tau^i) + \frac{t - i\tau}{\tau} \mathcal{H}^1(\Gamma_\tau^{i+1} \setminus \Gamma_\tau^i)$$
for $i\tau \leq t < (i+1)\tau, i = 0, \ldots, N_\tau$, and $\ell_\tau(t) = \tilde{\ell}_\tau(t) := \mathcal{H}^1(\Gamma_\tau^{N_\tau})$ for $\tau N_\tau \leq t \leq 1$.

Notice that $u_\tau(t)$ solves the problem

$$\begin{cases} \Delta u_\tau(t) = 0 & \text{in } \Omega \setminus \Gamma_\tau(t) \\ \frac{\partial u_\tau(t)}{\partial \nu} = 0 & \text{on } \Gamma_\tau(t) \cup \partial\Omega \setminus \partial_D \Omega \\ u_\tau(t) = w_\tau(t) & \text{on } \partial_D \Omega \end{cases} \tag{4.39}$$

with $w_\tau(t) := w(i\tau)$ for $i\tau \leq t < (i+1)\tau$, and also that

$$\tilde{\ell}_\tau(t) := \mathcal{H}^1(\Gamma_\tau^i) + \frac{t - i\tau}{\tau} \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma_\tau^i, \Gamma_\tau^{i+1})} \mathcal{H}^1(\mathfrak{c}).$$

Remark 4.2.3. To be precise, by construction $u_\tau(t) \in H^1(\Omega \setminus \Gamma_\tau(t))$. Since $\mathcal{L}^2(\Gamma_\tau(t)) = 0$, sometimes we will consider u_τ as a map taking values in $L^2(\Omega)$. Similarly, we will write $\nabla u_\tau(t) \in L^2(\Omega; \mathbb{R}^2)$; notice that $\nabla u_\tau(t)$ is the distributional gradient of $u_\tau(t)$ only in $\Omega \setminus \Gamma_\tau(t)$ but, in general, it does not coincide in Ω with the gradient of an extension of $u_\tau(t)$.

Since $w \in H^1(0,1; H^1(\Omega))$, for any $0 \leq a < b \leq 1$ it is

$$w(b) - w(a) = \int_a^b \dot{w}(t)\, dt \quad \text{and} \quad \nabla w(b) - \nabla w(a) = \int_a^b \nabla \dot{w}(t)\, dt,$$

where the integrals are Bochner integrals (see [2]). It is also true that

$$\| \int_a^b \dot{w}(t)\, dt \| \leq \int_a^b \|\dot{w}(t)\|\, dt \quad \text{and} \quad \| \int_a^b \nabla \dot{w}(t)\, dt \| \leq \int_a^b \|\nabla \dot{w}(t)\|\, dt,$$

which will be used below.

Proposition 4.2.4. *There exists a bounded non-negative function* $\varpi : (0,1) \to [0,+\infty)$ *such that* $\varpi(\tau) \to 0$ *as* $\tau \to 0$ *and for any* $0 \le i < j \le N_\tau$ *the following inequality holds:*

$$\begin{aligned} &\|\nabla u_\tau^j\|^2 + \mathcal{H}^1(\Gamma_\tau^j) + \frac{1}{\tau}\sum_{h=i}^{j-1} \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma_\tau^h,\Gamma_\tau^{h+1})} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \\ &\le \|\nabla u_\tau^i\|^2 + \mathcal{H}^1(\Gamma_\tau^i) + 2\int_{i\tau}^{j\tau} \langle \nabla u_\tau(t), \nabla \dot w(t)\rangle\, dt + \varpi(\tau). \end{aligned} \tag{4.40}$$

Proof. Consider the function $u = u_\tau^h + w((h+1)\tau) - w(h\tau)$. Since $u \in H^1(\Omega \setminus \Gamma_\tau^h)$ and $u = w((h+1)\tau)$ on $\partial_D\Omega$, the couple (u, Γ_τ^h) can be used as competitor in (4.32) at the $h+1$ step. Then

$$\begin{aligned} &\|\nabla u_\tau^{h+1}\|^2 + \mathcal{H}^1(\Gamma_\tau^{h+1}) + \frac{1}{\tau}\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma_\tau^h,\Gamma_\tau^{h+1})} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \\ \le& \|\nabla u_\tau^h + \nabla w((h+1)\tau) - \nabla w(h\tau)\|^2 + \mathcal{H}^1(\Gamma_\tau^h) \\ =& \|\nabla u_\tau^h\|^2 + 2\langle \nabla u_\tau(t), \nabla w((h+1)\tau) - \nabla w(h\tau)\rangle \\ &+ \|\nabla w((h+1)\tau) - \nabla w(h\tau)\|^2 + \mathcal{H}^1(\Gamma_\tau^h) \\ \le& \|\nabla u_\tau^h\|^2 + \mathcal{H}^1(\Gamma_\tau^h) + 2\int_{h\tau}^{(h+1)\tau} \langle \nabla u_\tau(t), \nabla \dot w(t)\rangle\, dt \\ &+ \left(\max_{0\le n<N_\tau} \int_{n\tau}^{(n+1)\tau} \|\nabla \dot w(t)\|\, dt\right) \int_{h\tau}^{(h+1)\tau} \|\nabla \dot w(t)\|\, dt. \end{aligned}$$

Iterating over $h = i, \dots, j-1$ and defining

$$\varpi(\tau) := \left(\max_{0\le n<N_\tau} \int_{n\tau}^{(n+1)\tau} \|\nabla \dot w(t)\|\, dt\right) \int_0^1 \|\nabla \dot w(t)\|\, dt,$$

we obtain the thesis. □

Lemma 4.2.5. *There exists a constant* $C > 0$*, independent of* τ *and* t*, such that the following estimates hold true for every* $\tau \in (0,1)$ *and* $t \in [0,1]$*:*

$$\|\nabla u_\tau(t)\| \le C \tag{4.41}$$

$$\frac{1}{\tau}\sum_{i=0}^{N_\tau-1} \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma_\tau^i,\Gamma_\tau^{i+1})} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \le C \tag{4.42}$$

$$\mathcal{H}^1(\Gamma_\tau(t)) \le C. \tag{4.43}$$

Proof. Fix $t \in [0,1]$ and let $j = j(t) \in 0, \dots, N_\tau - 1$ be such that it satisfies $j\tau \le t < (j+1)\tau$. By the inequality in Proposition 4.2.4 for $i = 0$, we obtain

$$\|\nabla u_\tau^j\|^2 + \frac{1}{\tau}\sum_{i=0}^{j-1} \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma_\tau^i,\Gamma_\tau^{i+1})} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \le \|\nabla u_0\|^2 + \int_0^{j\tau} \langle \nabla u_\tau(\xi), \nabla \dot w(\xi)\rangle\, d\xi + \varpi(\tau). \tag{4.44}$$

Hölder's inequality and (4.44) imply

$$\frac{1}{\tau}\sum_{i=0}^{j} \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma_\tau^i,\Gamma_\tau^{i+1})} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 \le C + \left(\int_0^t \|\nabla u_\tau(\xi)\|^2\, d\xi\right)^{1/2} \left(\int_0^t \|\nabla \dot w(\xi)\|^2\, d\xi\right)^{1/2} \tag{4.45}$$

and

$$\|\nabla u_\tau(t)\|^2 \le C + \left(\int_0^t \|\nabla u_\tau(\xi)\|^2 \, d\xi\right)^{1/2} \left(\int_0^t \|\nabla \dot w(\xi)\|^2 \, d\xi\right)^{1/2},$$

where $C>0$ is independent of τ and t.

By a refined version of the Gronwall lemma (see [7, Lemma 4.1.8]), we deduce that for every $t\in[0,1]$

$$\left(\int_0^t \|\nabla u_\tau(\xi)\|^2 \, d\xi\right)^{1/2} \le C\big(1+\|\nabla \dot w\|_{L^2(0,1;L^2(\Omega;\mathbb{R}^2))}\big)$$

The last two inequalities imply that $\nabla u_\tau(t)$ is bounded in $L^2(\Omega;\mathbb{R}^2)$ uniformly with respect to τ, t, *i.e.* (4.41). Then, considering (4.45) and (4.41), the estimate (4.42) follows.

Finally, Lemma 4.1.10 implies (4.43). □

Lemma 4.2.6. *It is $\tilde\ell_\tau \in H^1(0,1)$ and*

$$\|\tilde\ell_\tau\|_{H^1(0,1)} \le C$$

for every τ, with C independent of τ.

Proof. By Lemma 4.1.10, $\mathcal{H}^1(\Gamma)\le C$ for any $\Gamma\in\mathcal{S}$. Then

$$0\le \tilde\ell_\tau(t) \le \mathcal{H}^1(\Gamma^i_\tau) + \mathcal{H}^1(\Gamma^{i+1}_\tau) \le 2C$$

for any t and τ.

Observe that Lemma 4.1.9 implies that at each step the set $\mathcal{C}(\Gamma^i_\tau,\Gamma^{i+1}_\tau)$ contains at most M elements. Then

$$\begin{aligned}\int_0^1 |\dot{\tilde\ell}_\tau(t)|^2\,dt &= \sum_{i=0}^{N_\tau-1}\int_{i\tau}^{(i+1)\tau} \Big|\frac{1}{\tau}\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma^i_\tau,\Gamma^{i+1}_\tau)} \mathcal{H}^1(\mathfrak{c})\Big|^2\,dt\\ &\le 2^M \sum_{i=0}^{N_\tau-1}\frac{1}{\tau}\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma^i_\tau,\Gamma^{i+1}_\tau)} \big(\mathcal{H}^1(\mathfrak{c})\big)^2 \le 2^M C,\end{aligned}$$

where the last inequality is consequence of (4.42). □

4.3. The continuous-time evolution

In this section we select a continuous-time evolution $t\mapsto(u(t),\Gamma(t))$ as limit of discrete-time ones, esploiting the *a priori* estimates of the previous section and compactness results. Among all evolutions $t\mapsto(\widetilde u(t),\widetilde\Gamma(t))\in L^2(\Omega)\times\mathcal{S}$ with $t\mapsto\widetilde\Gamma(t)$ monotone and $\widetilde u(t)\in H^1(\Omega\setminus\widetilde\Gamma(t))$ in static equilibrium with respect to the boundary datum $w(t)$, the above selection provides the evolution $t\mapsto(u(t),\Gamma(t))$ with additional properties, as explained in Section 4.4.

By construction, the set functions $\Gamma_\tau:[0,1]\to\mathcal{S}$ are monotone increasing (with respect to the inclusion ordering). Considering the version of the Helly's Theorem proved in [38, Theorem 6.3], there exists a subsequence (not relabelled) Γ_τ and a map $\Gamma:[0,1]\to 2^\Omega$ such that

$$\Gamma_\tau(t)\xrightarrow{\mathcal{H}}\Gamma(t) \tag{4.46}$$

for every $t\in[0,1]$. By the compactness result in Theorem 4.1.8, it is $\Gamma:[0,1]\to\mathcal{S}$.

Concerning the displacements u_τ, the following convergence result holds. Let $u(t) \in H^1(\Omega \setminus \Gamma(t))$ be the solution to

$$\begin{cases} \Delta u(t) = 0 & \text{in } \Omega \setminus \Gamma(t) \\ \frac{\partial u(t)}{\partial \nu} = 0 & \text{on } \Gamma(t) \cup \partial\Omega \setminus \partial_D\Omega \\ u(t) = w(t) & \text{on } \partial_D\Omega\,. \end{cases} \tag{4.47}$$

Since $\Gamma_\tau(t) \xrightarrow{\mathcal{H}} \Gamma(t)$, $w_\tau(t) \to w(t)$ strongly in $H^1(\Omega)$, and (4.39) and (4.47) hold, by applying Theorem 1.7.6 we conclude that

$$\nabla u_\tau(t) \to \nabla u(t) \tag{4.48}$$

strongly in $L^2(\Omega;\mathbb{R}^2)$ for every t. Furthermore the bound (4.41) implies

$$\|\nabla u(t)\| \leq C\,, \tag{4.49}$$

with C independent of t.

Now we analyze the approximation process, in order to obtain the growth properties of the evolution $t \mapsto (u(t), \Gamma(t))$ announced at the beginning of the section.

Applying the classical Helly's Theorem 1.7.9, there exists a subsequence (not relabelled) ℓ_τ and a function $\ell : [0,1] \to \mathbb{R}$ such that

$$\ell_\tau(t) \to \ell(t) \tag{4.50}$$

for every $t \in [0,1]$. By Lemma 4.1.14 and (4.46), it is $\ell(t) = \mathcal{H}^1(\Gamma(t))$.

Proposition 4.3.1. *The function ℓ obtained in (4.50) belongs to $H^1(0,1)$. In particular, $\tilde{\ell}_\tau(t) \to \ell(t)$ for $t \in [0,1]$ and $\tilde{\ell}_\tau \rightharpoonup \ell$ weakly in $H^1(0,1)$.*

Proof. By the uniform bound proved in Lemma 4.2.6, up to subsequences it is

$$\tilde{\ell}_\tau \rightharpoonup \tilde{\ell} \tag{4.51}$$

weakly in $H^1(0,1)$ for some $\tilde{\ell} \in H^1(0,1)$, and

$$\|\tilde{\ell}\|_{H^1(0,1)} \leq \liminf_{\tau \to 0} \|\tilde{\ell}_\tau\|_{H^1(0,1)} \leq C.$$

By definition of $\tilde{\ell}_\tau$ and ℓ_τ, we have

$$\begin{aligned} 0 \leq \tilde{\ell}_\tau(t) - \ell_\tau(t) =& \frac{t - i\tau}{\tau}\mathcal{H}^1(\Gamma_\tau^{i+1} \setminus \Gamma_\tau^i) \leq \tau\dot{\tilde{\ell}}_\tau(t) \\ =& \int_{i\tau}^{(i+1)\tau} \dot{\tilde{\ell}}_\tau(\xi)\, d\xi \leq \tau^{1/2} \left(\int_{i\tau}^{(i+1)\tau} |\dot{\tilde{\ell}}_\tau(\xi)|^2\, d\xi \right)^{1/2} \leq \tau^{1/2} C, \end{aligned}$$

where i is such that $i\tau \leq t < (i+1)\tau$, and the last inequality is due to Lemma 4.2.6. Then, considering (4.50), as $\tau \to 0$ we obtain that $\tilde{\ell}_\tau(t) \to \ell(t)$ for $t \in [0,1]$. Finally, by uniqueness of the limit, it is $\tilde{\ell} = \ell$ a.e. in $[0,1]$, so that by (4.51) we conclude. □

Corollary 4.3.2. *The set function $\Gamma : [0,1] \to \mathcal{S}$ is continuous with respect to the Hausdorff convergence.*

Proof. For any $t \in (0,1)$ we define the left- and right-limit of $\Gamma(\cdot)$ at t as

$$\Gamma^-(t) := \overline{\bigcup_{t'<t} \Gamma(t')} \quad \text{and} \quad \Gamma^+(t) := \bigcap_{t'>t} \Gamma(t').$$

By compactness of $\mathcal{S}$, both limits belong to $\mathcal{S}$ and it is easy to check that $\Gamma^-(t) \subset \Gamma^+(t)$. Let $t'_n < t < t''_n$ be sequences converging to t; then, by monotonicity of $\Gamma(\cdot)$, we have

$$0 \leq \mathcal{H}^1(\Gamma^+(t) \setminus \Gamma^-(t)) \leq \mathcal{H}^1(\Gamma(t''_n) \setminus \Gamma(t'_n)) = \ell(t''_n) - \ell(t'_n) \to 0,$$

where the last limit is due to the continuity of ℓ, as consequence of Proposition 4.3.1.

If, by contradiction, it is $\Gamma^-(t) \neq \Gamma^+(t)$, the above discussion implies that

$$\Gamma^+(t) = \Gamma^-(t) \cup A(t)$$

with $\mathcal{H}^1(A(t)) = 0$. This contradicts the fact that each connected component of $\Gamma^+(t)$ has length at least λ (as requested by Definition 4.1.6.(iii)). □

We analyze the approximation process in correspondence of the tips of the crack $\Gamma(t)$; the presence of several branches makes the scenario rich.

For simplicity of notation, set

$$T(t) := T_{\Gamma(t)} \quad \text{and} \quad S(t) := S_{\Gamma(t)}$$

and

$$T_\tau(t) := T_{\Gamma_\tau(t)} \quad \text{and} \quad S_\tau(t) := S_{\Gamma_\tau(t)}.$$

For every $t \in (0,1]$ we define

$$MT(t) := \Gamma(t) \setminus \bigcup_{t'<t} \Gamma(t');$$

by Corollary 4.3.2 and the geometric properties of the class $\mathcal{S}$, it is not difficult to prove that

$$MT(t) = T(t) \setminus \bigcup_{t'<t} T(t'),$$

motivating the notation MT which stands for "moving tips". We call

$$\mathcal{A}_0 := \{t \in (0,1] : \ MT(t) \neq \emptyset\} \tag{4.52}$$

the set of instants when the fracture has really grown, at least at one tip.

We cannot exclude *a priori* that a tip of the continuous-time evolution is the limit point of (finitely) many tips of the approximating discrete-time evolutions. If this happens, we have some difficulties in characterizing the exact behaviour of the continuous-time process (see the comments at the end of Subsection 4.4.2). Hence below we introduce and describe the properties of a subset $\mathcal{A}$ of $\mathcal{A}_0$, containing the instants t such that every moving tip at $t \in \mathcal{A}$ is approximated exactly by one tip of each discrete-time evolution. The set $\mathcal{A}$ will play an important role later, in the description of a stability criterion for the continuous-time evolution.

Lemma 4.3.3. *Let $\mathcal{A}$ be the set of instants $t \in \mathcal{A}_0$ such that for every $p \in MT(t)$ there exist a neighbourhood U of p and a value $\nu(t,p) > 0$ such that for every $\tau \leq \nu(t,p)$ the following two conditions hold:*

- *$T_\tau(t) \cap U$ contains one and only one element, denoted $p_\tau(t)$;*
- *$S_\tau(t) \cap U = \emptyset$.*

Then $\mathcal{A}_0 \setminus \mathcal{A}$ is finite.

Proof. By definition of the class $\mathcal{S}$ and Lemma 4.1.9, the cardinality of $S_\tau(1)$ is uniformly bounded with respect to τ. Up to considering a subsequence, we can assume that

$$S_\tau(1) = \{x_\tau^1, \dots, x_\tau^M\}$$

and $x_\tau^j \to x^j$ as $\tau \to 0$, for $j = 1, \dots, M$. Notice that $|x^j - x^l| \geq \delta$ if $j \neq l$, since the same holds for x_τ^j and x_τ^l (see Condition 4.1.6.(iv)).

By Proposition 4.1.15 and since $T(t)$ contains finitely many points (see again Lemma 4.1.9), for every $t \in [0,1]$ and $r > 0$ there exists $\tilde{\nu}(t,r) > 0$ such that

$$B_r(p) \cap T_\tau(t) \neq \emptyset$$

for every $p \in T(t)$ and $\tau < \tilde{\nu}(t,r)$.

Let $t \in \mathcal{A}_0 \setminus \mathcal{A}$. Then there exists $p \in MT(t)$ such that for every $r > 0$ and every $\nu \in (0, \tilde{\nu}(t,r))$ there exists $\tau_\nu < \nu$ such that $T_{\tau_\nu}(t) \cap B_r(p)$ has at least two elements or $S_{\tau_\nu}(t) \cap B_r(p) \neq \emptyset$.

In the first case, by Lemma 4.1.16 there exists $y_{\tau_\nu} \in S_{\tau_\nu}(t)$ such that $y_{\tau_\nu} \to p$ as $\nu \to 0$ (so $\tau_\nu \to 0$). Being $S_{\tau_\nu}(t) \subset S_{\tau_\nu}(1)$, then $y_{\tau_\nu} = x^{j_\nu}_{\tau_\nu}$ for some $j_\nu \in \{1, \dots, M\}$; it follows that $p = x^j$ for some j.

Similarly, if $S_{\tau_\nu}(t) \cap B_r(p) \neq \emptyset$ then $p = x^j$ for some j.

Since $p(= x^j) \notin MT(t')$ for every $t' \in \mathcal{A}_0 \setminus \{t\}$ (being $MT(t') \neq MT(t'')$ for any distinct $t', t'' \in \mathcal{A}_0$) and the points x^j are a finite number, then also $\mathcal{A}_0 \setminus \mathcal{A}$ is finite. □

By definition of crack tip, for any fixed $\hat{t} \in (0,1]$ there exists $r_1(\hat{t}) \in (0, \eta)$ such that

$$\overline{B_{r_1(\hat{t})}(p)} \cap \Gamma(\hat{t})$$

is a curve in $\mathcal{R}_\eta$ for every $p \in T(\hat{t})$. In addition, $r_1(\hat{t})$ can be chosen so that

$$\begin{aligned} &\partial B_{r_1(\hat{t})}(p) \cap \Gamma(\hat{t}) = \{x(\hat{t}, p)\} \\ &\mathcal{H}^1(B_{r_1(\hat{t})}(p) \cap \Gamma(\hat{t})) \leq \lambda \\ &\overline{B_{r_1(\hat{t})}(p)} \cap S(\hat{t}) = \emptyset. \end{aligned} \tag{4.53}$$

It results that the points p and $x(\hat{t}, p)$ belong to the same branch of $\Gamma(\hat{t})$ and $x(\hat{t}, p) \notin S(\hat{t})$. Since the function $\Gamma : [0,1] \to \mathcal{S}$ is monotone and continuous with respect to the Hausdorff convergence (see Corollary 4.3.2), and (4.53) holds, for instants t in a left neighbourhood of $\hat{t}$ and for each $p \in T(\hat{t})$ the set $B_{r_1(\hat{t})}(p) \cap T(t)$ has exactly one element, labelled $p(t)$, *i.e.*

$$B_{r_1(\hat{t})}(p) \cap T(t) = \{p(t)\}, \tag{4.54}$$

and

$$T(t) = \left\{p(t) : \ p \in T(\hat{t})\right\}. \tag{4.55}$$

We are able to estimate the size of the left neighbourhood of $\hat{t}$ in which the above conditions hold. Indeed, define

$$\alpha_1(\hat{t}, p) := \inf\left\{t \in [0, \hat{t}) : x(\hat{t}, p) \in \Gamma(t)\right\},$$

where $x(\hat{t}, p)$ has been introduced in (4.53). Then we have

$$\begin{aligned} r_1(\hat{t}) \leq \mathcal{H}^1(B_{r_1(\hat{t})}(p) \cap \Gamma(\hat{t})) &\leq \int_{\alpha_1(\hat{t},p)}^{\hat{t}} \dot{\ell}(t)\, dt \\ &\leq \left(\hat{t} - \alpha_1(\hat{t}, p)\right)^{1/2} \left(\int_{\alpha_1(\hat{t},p)}^{\hat{t}} |\dot{\ell}(t)|^2\, dt\right)^{1/2} \\ &\leq C\left(\hat{t} - \alpha_1(\hat{t}, p)\right)^{1/2}, \end{aligned}$$

so that

$$\hat{t} - \alpha_1(\hat{t}, p) \geq C\, r_1(\hat{t})^2$$

with C independent of $\hat{t}$ and p.

Since $T(\hat{t})$ contains finitely many points, if we set

$$\alpha_1(\hat{t}) := \max\{\alpha_1(\hat{t}, p) : \ p \in T(\hat{t})\}, \tag{4.56}$$

then (4.54) and (4.55) hold for every $t \in (\alpha_1(\hat{t}), \hat{t}]$.

Remark 4.3.4. Notice that we cannot infer anything about the local behaviour of $\Gamma(\cdot)$ at the instants after $\hat{t}$, since new branches might spring out at some tip $p \in T(\hat{t})$.

Lemma 4.3.5. *For every $\hat{t} \in (0,1] \setminus \mathcal{A}_0$ there exists $\alpha(\hat{t}) \in [\alpha_1(\hat{t}), \hat{t})$ such that $\Gamma(t) = \Gamma(\hat{t})$ for every $t \in (\alpha(\hat{t}), \hat{t}]$. In particular $T(t) = T(\hat{t})$ for every $t \in (\alpha(\hat{t}), \hat{t}]$.*

PROOF. It is a straightforward consequence of the definition of MT and $\mathcal{A}_0$. □

By definition of $\mathcal{A}$, at instants $t \in \mathcal{A}$ each crack tip is locally approximated by exactly one tip, while singular points of the approximating sequence remain "distant" (see Fig. 5). The following lemma shows that these properties are preserved locally in a left neighbourhood of every instant in $\mathcal{A}$. The importance of this result lies in the fact that this left neighbourhood is not necessarily entirely contained in $\mathcal{A}$.

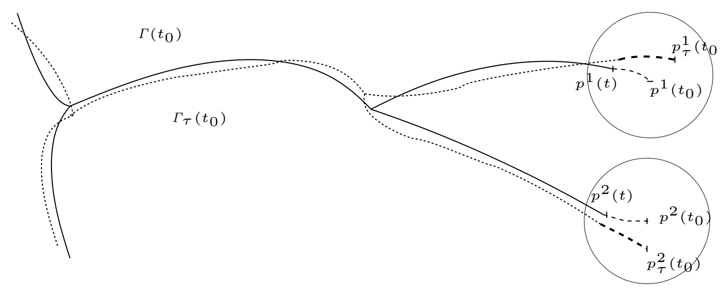


FIGURE 5. The crack set $\Gamma(t_0)$ and, dotted, a discrete-time approximating crack set $\Gamma_\tau(t_0)$ at an instant $t_0 \in \mathcal{A}$, in correspondence of two tips $p^1(t_0), p^2(t_0) \in MT(t_0)$.

Lemma 4.3.6. *Let $\hat{t} \in \mathcal{A}$. Then there exist $\alpha(\hat{t}) \in [\alpha_1(\hat{t}), \hat{t})$, $\nu(\hat{t}) > 0$ and $r(\hat{t}) \in (0, \eta)$ such that the following facts hold for every $t \in (\alpha(\hat{t}), \hat{t}]$:*

(i) if $p \in T(\hat{t}) \setminus MT(\hat{t})$, then $p \in T(t)$;

(ii) if $p \in MT(\hat{t})$, then for every $\tau < \nu(\hat{t})$ the set $B_{r(\hat{t})}(p) \cap T_\tau(t)$ has exactly one element, that we label $p_\tau(t)$.

PROOF. If $p \in T(\hat{t}) \setminus MT(\hat{t})$, then argue as in Lemma 4.3.5 and call $\beta_1(\hat{t})$ what therein is $\alpha(\hat{t})$.

Consider now $p \in MT(\hat{t})$. By definition of $\mathcal{A}$, there exist $r(\hat{t}) > 0$ and $\nu_1(\hat{t}) > 0$ such that

$$T_\tau(\hat{t}) \cap B_{r(\hat{t})}(p) = \{p_\tau(\hat{t})\} \quad \text{and} \quad S_\tau(\hat{t}) \cap B_{r(\hat{t})}(p) = \emptyset \tag{4.57}$$

for every $p \in MT(\hat{t})$. In particular, we can choose $r(\hat{t}) \in (0, r_1(\hat{t})]$, where $r_1(\hat{t})$ was introduced in (4.53), and such that

$$\sup\left\{\mathcal{H}^1(K) : K \in \mathcal{R}_\eta, K \subset \overline{B_{r(\hat{t})}(0)}\right\} < \lambda. \tag{4.58}$$

The above conditions on $r(\hat{t})$ imply that

$$\Gamma_\tau(\hat{t}) \cap \overline{B_{r(\hat{t})}(p)} \in \mathcal{R}_\eta$$

and every connected component of $\Gamma_\tau(\hat{t})$ is not completely contained in $\overline{B_{r(\hat{t})}(p)}$, because of (4.58) and of the constraint given by Condition 4.1.6.(iii).

For simplicity of notation, in the remaining of the proof we write

$$r = r(\hat{t}) \quad \text{and} \quad \nu_1 = \nu_1(\hat{t}).$$

Fix $p \in MT(\hat{t})$. Let $\nu = \nu(\hat{t}) \in (0, \nu_1)$ be such that $p_\tau(\hat{t}) \in B_{r/2}(p)$ for every $\tau < \nu$ (such ν exists since, by Proposition 4.1.15, it is $p_\tau(\hat{t}) \to p$ as $\tau \to 0$); if necessary, later we will replace ν with a smaller one. By (4.57) and (4.58), it follows that

$$\mathcal{H}^1(\Gamma_\tau(\hat{t}) \cap B_r(p)) \geq \frac{r}{2} \tag{4.59}$$

for every $\tau < \nu$.

Define

$$t_\tau := \inf\{t \in [0, \hat{t}) : \ \Gamma_\tau(t') \cap B_r(p) \neq \emptyset \ \text{ for every } \ t' \in (t, \hat{t})\}.$$

If $t_\tau = 0$ for any $\tau < \nu$, set $\beta_2(\hat{t}) := \alpha_1(\hat{t})$, where $\alpha_1(\hat{t})$ was defined in (4.56). If $t_\tau > 0$ for some $\tau < \nu$, we argue in the following way: for any τ let $i_\tau, j_\tau \in \mathbb{N}$ be such that

$$i_\tau \leq t_\tau < (i_\tau + 1)\tau \quad \text{and} \quad j_\tau \leq \hat{t} < (j_\tau + 1)\tau.$$

By (4.59), Lemma 4.2.6 and Hölder inequality, we have

$$\begin{aligned} \frac{r}{2} \leq & \mathcal{H}^1(\Gamma_\tau(\hat{t}) \setminus \Gamma_\tau(t_\tau)) = \sum_{h=i_\tau}^{j_\tau} \sum_{C \in \mathcal{C}(\Gamma_\tau^h, \Gamma_\tau^{h+1})} \mathcal{H}^1(C) = \int_{i_\tau \tau}^{(j_\tau+1)\tau} \dot{\hat{\ell}}_\tau(t)\, dt \\ \leq & \int_{t_\tau - \tau}^{\hat{t}+\tau} \dot{\hat{\ell}}_\tau(t)\, dt \leq (\hat{t} - t_\tau + 2\tau)^{1/2} \left(\int_0^1 |\dot{\hat{\ell}}_\tau(t)|^2\, dt \right)^{1/2} \leq C(\hat{t} - t_\tau + 2\tau)^{1/2} \end{aligned} \tag{4.60}$$

with $C > 0$ independent of τ and $\hat{t}$. Define

$$\beta_2(\hat{t}) := \max\left\{ \alpha_1(\hat{t}),\ \hat{t} + \nu(\hat{t}) - \frac{r^2}{4C^2} \right\}$$

and choose $\nu(\hat{t})$ such that, in addition to being smaller than ν_1, it satifies

$$\nu(\hat{t}) \leq \frac{1}{4} \frac{r^2}{4C^2}.$$

Then $\beta_2(\hat{t}) < \hat{t}$ and, taking into account (4.60), for every $\tau < \nu(\hat{t})$ it is

$$t_\tau \leq \hat{t} + \tau - \frac{r^2}{4C^2} \leq \hat{t} + \nu(\hat{t}) - \frac{r^2}{4C^2} \leq \beta_2(\hat{t}).$$

Summarizing, we have shown that for every $\tau < \nu(\hat{t})$ and $t \in (\beta_2(\hat{t}), \hat{t}]$

$$\Gamma_\tau(t) \cap B_{r(\hat{t})}(p) \neq \emptyset \quad \text{and} \quad \overline{\Gamma_\tau(t) \cap B_{r(\hat{t})}(p)} \in \mathcal{R}_\eta.$$

Then we can conclude that $T_\tau(t) \cap B_{r(\hat{t})}(p)$ has only one element, denoted by $p_\tau(t)$.

Since $MT(\hat{t})$ contains finitely many points, we can choose $r(\hat{t})$, $\beta_2(\hat{t})$ and $\nu(\hat{t})$ such that the above property holds for every $p \in MT(\hat{t})$.

Finally, define $\alpha(\hat{t}) := \max\{\beta_1(\hat{t}), \beta_2(\hat{t})\}$, so that both (i) and (ii) are valid for any instant $t \in (\alpha(\hat{t}), \hat{t}]$. □

4.3.1. Velocity of the crack tips. In this subsection we introduce the notion of velocity of the front $T(t)$ of the fracture. It will play a role in the dissipative term of the energy and for a Griffith-like stability criterion for the crack growth. We provide two equivalent descriptions, both interesting for different reasons. We first introduce the velocity by means of a distributional approach, based on the theory of absolutely continuous maps with values in the space of bounded measures. This point of view gives a picture of the situation in the whole of Ω, and it somehow reminds the approach suggested in [61]. Instead, the second description is local and is based on the parametrization of the branches of the crack.

At the very beginning, we summarize what we know so far about the crack growth $t \mapsto \Gamma(t)$. As observed in Corollary 4.3.2, the set function $\Gamma(\cdot) : [0,1] \to \mathcal{S}$ is continuous with respect to the Hausdorff topology in $\mathcal{S}$. By construction of the class $\mathcal{S}$, it is $\mathrm{card}(S(1)) \leq M$ for some $M \in \mathbb{N}$. Since the map $t \mapsto S(t)$ is monotone increasing with respect to inclusion, there exists a partition of the interval $[0,1]$

$$0 = a_0 < a_1 < \ldots < a_n = 1 \tag{4.61}$$

such that

- $S(t) = S(t')$ for every $t, t' \in (a_i, a_{i+1}]$;
- $\mathrm{card}(S(t)) < \mathrm{card}(S(t'))$ for any $t \leq a_i < t'$.

In the time intervals $(a_i, a_{i+1}]$ new branches of the fracture can appear; being $S(\cdot)$ constant, they necessarily originate at some point in $S(t)$, for $t \in (a_i, a_{i+1}]$. Together with any new branch, also a new tip appears; by monotonicity of $\Gamma(\cdot)$, for any $t, t' \in (a_i, a_{i+1}]$ with $t' < t$, it has to be $\mathrm{card}(T(t')) \leq \mathrm{card}(T(t))$.

We can again establish a sort of stability from the left: as seen in the discussion in Section 4.3, for every $t \in (a_i, a_{i+1}]$ there exists $\alpha_1(t) < t$ (defined in (4.56)) such that (4.55) holds, *i.e.*

$$\mathrm{card}(T(t)) = \mathrm{card}(T(t')) \tag{4.62}$$

for every $t' \in (\alpha_1(t), t]$; by (4.55), notice that $\alpha_1(t) \geq a_i$. Hence we can further subdivide each interval $(a_i, a_{i+1}]$ with a partition $a_i = b_i^0 < b_i^1 < \ldots < b_i^{n_i} = a_{i+1}$ such that

- $\mathrm{card}(T(t)) = \mathrm{card}(T(t'))$ if $t, t' \in (b_i^k, b_i^{k+1}]$;
- $\mathrm{card}(T(t)) < \mathrm{card}(T(t'))$ if $a_i < t \leq b_i^k < t' < a_{i+1}$.

Actually, above we have proved the following fact.

Lemma 4.3.7. *There exists a partition*

$$0 = t_0 < t_1 < \ldots < t_{N+1} = 1$$

of the interval $[0,1]$ *such that one of the following alternatives holds:*

(i) if $t, t' \in (t_i, t_{i+1}]$, *then* $S(t) = S(t')$ *and* $\mathrm{card}(T(t)) = \mathrm{card}(T(t'))$;
(ii) if $t \leq t_i < t'$, *then either* $S(t) = S(t')$ *and* $\mathrm{card}(T(t)) < \mathrm{card}(T(t'))$, *or* $S(t) \neq S(t')$.

Lemma 4.3.8. *Consider a sequence* $(\Gamma_k)_k \subset \mathcal{S}$ *such that* $\Gamma_k \xrightarrow{\mathcal{H}} \widehat{\Gamma}$. *Then for every* $\psi \in C_b(\Omega)$

$$\int_{\Gamma_k} \psi \, d\mathcal{H}^1 \to \int_{\widehat{\Gamma}} \psi \, d\mathcal{H}^1. \tag{4.63}$$

In other words, the sequence of measures $\mu_k := \mathcal{H}^1 \llcorner \Gamma_k$ *converges to* $\widehat{\mu} := \mathcal{H}^1 \llcorner \widehat{\Gamma}$ *weakly* in* $\mathcal{M}_b(\Omega)$.

Proof. It is enough to use the regularity of the curves in $\mathcal{R}_\eta$, in particular the parametrization provided by Proposition 4.1.2.(v). □

Let $\mu : [0,1] \to \mathcal{M}_b(\Omega)$ be the map defined as

$$\mu(t) := \mathcal{H}^1 \llcorner \Gamma(t). \tag{4.64}$$

Proposition 4.3.9. *The map* $\mu : [0,1] \to \mathcal{M}_b(\Omega)$ *belongs to the space* $AC([0,1]; \mathcal{M}_b(\Omega))$.

PROOF. By definition of μ, it results that

$$|\mu(t)|(\Omega) = \mathcal{H}^1(\Gamma(t)) = \ell(t),$$

where ℓ was introduced in (4.50) and $\ell \in H^1(0,1)$ by Proposition 4.3.1.

Let $\psi \in C_0(\Omega)$. Then for every $0 \le a < b \le 1$ it is

$$\begin{aligned}
|\langle \psi, \mu(b) - \mu(a)\rangle| = \left| \int_\Omega \psi \, d(\mu(b) - \mu(a)) \right| &= \left| \int_{\Gamma(b)\setminus\Gamma(a)} \psi \, d\mathcal{H}^1 \right| \\
&\le \|\psi\|_\infty \, \mathcal{H}^1(\Gamma(b) \setminus \Gamma(a)) = \|\psi\|_\infty \, \big(\mathcal{H}^1(\Gamma(b)) - \mathcal{H}^1(\Gamma(a))\big) \\
&= \|\psi\|_\infty \int_a^b \dot{\ell}(\xi) \, d\xi.
\end{aligned}$$

Taking the supremum over all $\psi \in C_0(\Omega)$ with $\|\psi\|_\infty \le 1$, we obtain

$$|\mu(b) - \mu(a)|(\Omega) \le \int_a^b \dot{\ell}(t) \, dt.$$

Since $\ell \in H^1(0,1)$, by the absolute continuity of the integral with respect to the integration domain we obtain the thesis. □

In accordance with the results in [32, Appendix], for a.e. $t \in [0,1]$ there exists

$$\dot{\mu}(t) := w^* - \lim_{s \to t} \frac{\mu(s) - \mu(t)}{s - t},$$

and $\dot{\mu}(t) \in \mathcal{M}_b(\Omega)$. We mean that for a.e. $t \in [0,1]$ there exists a Radon measure $\dot{\mu}(t) \in \mathcal{M}_b(\Omega)$ such that

$$\langle \psi, \dot{\mu}(t) \rangle = \lim_{s \to t} \langle \psi, \frac{\mu(s) - \mu(t)}{s - t} \rangle$$

for every $\psi \in C_0(\Omega)$.

We describe the "structure" of these measures, in order to introduce a *distributional* notion of velocity.

Proposition 4.3.10. *For a.e.* $t \in [0,1]$

$$\operatorname{supp} \dot{\mu}(t) \subset T(t).$$

PROOF. Consider $\hat{t} \in [0,1] \setminus \{t_1, \dots, t_N\}$ for which $\dot{\mu}(\hat{t})$ exists, where $t_1, \dots, t_N$ are given in Lemma 4.3.7. Fix $\psi \in C_0(\Omega)$ such that $\operatorname{supp} \psi \subset \Omega \setminus T(\hat{t})$. Taken $r_1(\hat{t})$ and $\alpha_1(\hat{t})$ as in (4.53) and (4.56) respectively, for every $t \in (\alpha_1(\hat{t}), \hat{t}]$ it is

$$\Gamma(t) \setminus \bigcup_{p \in T(\hat{t})} B_{r_1(\hat{t})}(p) = \Gamma(\hat{t}) \setminus \bigcup_{p \in T(\hat{t})} B_{r_1(\hat{t})}(p).$$

Let $r \in (0, r_1(\hat{t}))$ be such that

$$\operatorname{supp} \psi \subset \Omega \setminus \bigcup_{p \in T(\hat{t})} \overline{B_r(p)}.$$

By continuity of the set function $\Gamma(\cdot)$ with respect to the Hausdorff converge (see Corollary 4.3.2), if we repeat for r the discussion done for $r_1(\hat{t})$ and $\alpha_1(\hat{t})$, we obtain that there exists $t_r < \hat{t}$ such that

$$\Gamma(t) \setminus \bigcup_{p \in T(\hat{t})} B_r(p) = \Gamma(\hat{t}) \setminus \bigcup_{p \in T(\hat{t})} B_r(p)$$

for every $t \in (t_r, \hat{t}]$. Therefore

$$\langle \psi, \frac{\mu(t) - \mu(\hat{t})}{t - \hat{t}} \rangle = \frac{1}{\hat{t} - t} \int_{\Gamma(\hat{t}) \setminus \Gamma(t)} \psi \, dx = 0$$

for every $t \in (t_r, \hat{t})$. Taking the limit as $t \to \hat{t}-$, since $\dot{\mu}(\hat{t})$ exists we get $\langle \psi, \dot{\mu}(\hat{t}) \rangle = 0$.

We have shown that, for every $t \in (0,1)$ for which $\dot{\mu}(t)$ exists, if $\psi \in C_0(\Omega)$ with $\operatorname{supp} \psi \subset \Omega \setminus T(t)$ then $\langle \psi, \dot{\mu}(t) \rangle = 0$. Therefore $\operatorname{supp} \dot{\mu}(t) \subset T(t)$. □

As a consequence of Proposition 4.3.10, for a.e. $t \in (0,1)$

$$\dot{\mu}(t) \ll \mathcal{H}^0 \llcorner T(t).$$

Definition 4.3.11. We call (*distributional*) *velocity* of the crack tip $p \in T(t)$ the value $v(t,p)$, where

$$\dot{\mu}(t) = \sum_{p \in T(t)} v(t,p) \, \delta_p \tag{4.65}$$

and δ_x is the Dirac measure concentrated at $x \in \mathbb{R}^2$.

Now we pass to the second approach for the description of the front velocity, which will lead to an equivalent definition.

Consider $\hat{t} \in (t_i, t_{i+1})$, with t_i introduced in Lemma 4.3.7, and $r_1(\hat{t})$, $\alpha_1(\hat{t})$ as in (4.53) and (4.56). Fixed $p \in T(\hat{t})$, we can describe the curve

$$\Gamma(\hat{t}) \cap \overline{B_{r_1(\hat{t})}(p)} \in \mathcal{R}_\eta$$

by means of an arc-length parametrization $\gamma : [0, L^{\hat{t},p}] \to \mathbb{R}^2$ (here $L^{\hat{t},p} := \mathcal{H}^1(\Gamma(\hat{t}) \cap \overline{B_{r_1(\hat{t})}(p)})$) and an increasing function $\sigma : [\alpha_1(\hat{t}), \hat{t}] \to [0, L^{\hat{t},p}]$ such that for every $t \in (\alpha_1(\hat{t}), \hat{t}]$ it is

$$\sigma(t) = \mathcal{H}^1(\Gamma(t) \cap \overline{B_{r_1(\hat{t})}(p)}) \quad \text{and} \quad \gamma(\sigma(t)) = p(t)\,,$$

where $p(t)$ is the unique element in $T(t) \cap B_{r_1(\hat{t})}(p)$ (see (4.54)). Since the curves in $\mathcal{R}_\eta$ belong to $W^{2,\infty}$ and $\ell(\cdot) = \mathcal{H}^1(\Gamma(\cdot))$ is in $H^1(0,1)$, it results that $\gamma \in W^{2,\infty}$ and $\sigma \in H^1(\alpha_1(\hat{t}), \hat{t})$, hence

$$\gamma(\sigma(\cdot)) \in H^1((\alpha_1(\hat{t}), \hat{t}); \mathbb{R}^2).$$

Then, for a.e. $t \in (\alpha_1(\hat{t}), \hat{t})$, we define the velocity of the crack tip $p(t)$ as

$$\mathbf{v}(t, p(t)) := \dot{\sigma}(t) \, \dot{\gamma}(\sigma(t))$$

and

$$\tilde{v}(t, p(t)) := |\mathbf{v}(t, p(t))| = \dot{\sigma}(t). \tag{4.66}$$

It is not difficult to see that the two notions (4.65) and (4.66) coincide, *i.e.*

$$v(t, p(t)) = \tilde{v}(t, p(t)) \tag{4.67}$$

for a.e. $t \in [0,1]$. Indeed, assume that $\dot\mu(t)$ and $\dot\sigma(t)$ exist for some $t \in [0,1] \setminus \{t_1, \dots, t_N\}$, with t_i as in Lemma 4.3.7. For $s \in (\alpha_1(t), t)$ it is $\Gamma(t) \setminus \Gamma(s) = \gamma(\sigma((s,t]))$. Fixed $p(t) \in T(t)$, for $\psi \in C_b(\Omega)$ with $\operatorname{supp}\psi \subset B_{r_1(t)}(p(t))$ and $\psi(p(t)) = 1$ it is

$$\begin{aligned}\frac{1}{t-s}\int_{\Gamma(t)\setminus\Gamma(s)} \psi \, d\mathcal{H}^1 =& \frac{1}{t-s}\int_s^t \psi(\gamma(\sigma(\xi)))\,|\frac{d}{d\xi}(\gamma(\sigma(\xi)))|\,d\xi \\ =& \frac{1}{t-s}\int_s^t \psi(\gamma(\sigma(\xi)))\,\dot\sigma(\xi)\,d\xi.\end{aligned}$$

As $s \nearrow t$, the left-hand side converges to

$$\langle \psi, \dot\mu(t)\rangle = \langle \psi, \sum_{p \in T(t)} v(t,p)\delta_p\rangle = \psi(p(t))\, v(t,p(t)) = v(t,p(t)), \tag{4.68}$$

while the right-hand side to

$$\psi(\gamma(\sigma(t)))\,\dot\sigma(t) = \psi(p(t))\,\tilde v(t,p(t)) = \tilde v(t,p(t)).$$

Hence (4.67) is proved.

Similarly to the map $\mu : [0,1] \to \mathcal{M}_b(\Omega)$ defined in (4.64), we introduce $\mu_\tau : [0,1] \to \mathcal{M}_b(\Omega)$ as

$$\mu_\tau(t) := \mathcal{H}^1 \llcorner \Gamma_\tau(t)\,.$$

Lemma 4.3.8 and (4.46) imply that

$$\mu_\tau(t) \rightharpoonup \mu(t) \tag{4.69}$$

weakly* in $\mathcal{M}_b(\Omega)$, for every $t \in [0,1]$. Observe that if $r \in (0,\eta)$, then for every $x \in \Omega$

$$\mu(t)(\partial B_r(x)) = 0. \tag{4.70}$$

Indeed, being $r < \eta$, the constraint on the curvature of the curves $K \in \mathcal{R}_\eta$ implies that the set $K \cap \partial B_r(x)$ contains finitely many points, and consequently the same holds for the set $\Gamma \cap \partial B_r(x)$ for every $\Gamma \in \mathcal{S}$. Then, by (4.69) and (4.70), we obtain that

$$\mu_\tau(t)(B_r(x)) \to \mu(t)(B_r(x))$$

for every $r \in (0,\eta)$ and $x \in \Omega$.

Lemma 4.3.12. *Let $\hat t \in \mathcal{A}$, $r(\hat t)$ given by Lemma 4.3.6 and $p \in MT(\hat t)$. For every $t \in (\alpha(\hat t), \hat t)$ and τ such that $t + \tau \in (\alpha(\hat t), \hat t]$, the set*

$$(\Gamma_\tau(t+\tau) \setminus \Gamma_\tau(t)) \cap B_{r(\hat t)}(p) \tag{4.71}$$

is either empty or connected.

Proof. Assume that the set is not empty. By choice of $\hat t$, it is $\Gamma_\tau(\hat t) \cap \overline{B_{r(\hat t)}(p)} \in \mathcal{R}_\eta$ and $T_\tau(\hat t) \cap B_{r(\hat t)}(p) = \{p_\tau(\hat t)\}$. If for some t and τ the set in (4.71) has two or more connected components, then they must be separated by points or arcs of curve contained in $\Gamma_\tau(t)$: there exists $\mathfrak{c}$ connected component of $\Gamma_\tau(t)$ with $\mathfrak{c}$ strictly contained in $B_{r(\hat t)}(p)$ and $\bar{\mathfrak{c}} \in \mathcal{R}_\eta$ (because $\bar{\mathfrak{c}} \subset \Gamma_\tau(\hat t) \cap \overline{B_{r(\hat t)}(p)} \in \mathcal{R}_\eta$). By the fact that (4.58) is verified for our choice of $r(\hat t)$, it is $\mathcal{H}^1(\mathfrak{c}) < \lambda$. However this is impossible, since $\Gamma_\tau(t) \in \mathcal{S}$ and all its connected components must have length at least λ. □

Hence by the previous lemma we conclude that, for τ sufficiently small and $t \in (\alpha(\hat t), \hat t]$, it is

$$(\Gamma_\tau(t+\tau) \setminus \Gamma_\tau(t)) \cap B_{r(\hat t)}(p) = \mathfrak{c}_\tau^{p_\tau(t)}$$

for the connected component $\mathfrak{c}_\tau^{p_\tau(t)} \in \mathcal{C}(\Gamma_\tau(t), \Gamma_\tau(t+\tau))$ with $p_\tau(t) \in \mathfrak{c}_\tau^{p_\tau(t)}$.

We introduce the following notion of *discrete velocity*. For any $p \in T_{\Gamma_\tau(t+\tau)} \setminus T_{\Gamma_\tau(t)}$ we set

$$v_\tau(t,p) := \frac{1}{\tau}\mathcal{H}^1(\mathfrak{c}_\tau^p) \tag{4.72}$$

where, as above, $\mathfrak{c}_\tau^p$ is the connected component in $\mathcal{C}(\Gamma_\tau(t), \Gamma_\tau(t+\tau))$ containing p. If $p \in T_{\Gamma_\tau(t+\tau)} \cap T_{\Gamma_\tau(t)}$, we simply set $v_\tau(t,p) := 0$.

Remark 4.3.13. Let us underline once more that the connected components $\mathfrak{c}_\tau^p$ above might not be $C^{1,1}$ arcs of curve, but they might kink or contain several branches.

In conclusion of the section, we establish a result which relates v_τ and v in small time intervals.

Proposition 4.3.14. *For every $\hat{t} \in [0,1] \setminus (\mathcal{A}_0 \setminus \mathcal{A})$ let $\alpha(\hat{t})$ be as in Lemmas 4.3.5 or 4.3.6. Then for every interval $(a,b) \subset (\alpha(\hat{t}), \hat{t})$ it holds*

$$\int_a^b v(t,p(t))^2\,dt \le \liminf_{\tau\to 0} \int_a^b v_\tau(t,p_\tau(t))^2\,dt,$$

where, if $t \in \mathcal{A}$, then $p(t)$ and $p_\tau(t)$ are as in Lemma 4.3.6.

Proof. Fixed $\hat{t} \in [0,1] \setminus (\mathcal{A}_0 \setminus \mathcal{A})$, consider $p \in T(\hat{t}) \setminus MT(\hat{t})$ (if this set is not empty). Let $\alpha(\hat{t})$ be as in Lemma 4.3.5 or in Lemma 4.3.6:

$$\Gamma(t) \cap \overline{B_{r_1(\hat{t})}(p)} = \Gamma(\hat{t}) \cap \overline{B_{r_1(\hat{t})}(p)}$$

for all $t \in (\alpha(\hat{t}), \hat{t}]$. Using the definition (4.66) of $\tilde{v}$ and the notation introduced therein, it results that $\tilde{v}(t,p(t)) = \tilde{v}(t,p) = 0$. Since (4.67) holds true, it is

$$v(t,p(t)) = \tilde{v}(t,p(t)) = 0$$

for a.e. $t \in (\alpha(\hat{t}), \hat{t}]$. Therefore for any $(a,b) \subset (\alpha(\hat{t}), \hat{t})$ we have

$$0 = \int_a^b v(t,p(t))^2\,dt \le \int_a^b v_\tau(t,p_\tau(t))^2\,dt. \tag{4.73}$$

If $\hat{t} \in \mathcal{A}$ and $p \in MT(\hat{t})$, let $\alpha(\hat{t})$, $\nu(\hat{t})$, $r(\hat{t})$, $p(t)$, $p_\tau(t)$ be as in Lemma 4.3.6. For $t \in (\alpha(\hat{t}), \hat{t})$, by Lemma 4.3.6 and Lemma 4.3.12 it results that it is either $v_\tau(t,p_\tau(t)) = 0$ or $v_\tau(t,p_\tau(t)) = \frac{1}{\tau}\,\mathcal{H}^1(\mathfrak{c}_\tau^{p_\tau(t)})$, where $\mathfrak{c}_\tau^{p_\tau(t)}$ is the connected component in $\mathcal{C}(\Gamma_\tau(t), \Gamma_\tau(t+\tau))$ containing $p_\tau(t)$. Set

$$\begin{aligned}
\tilde{\ell}_\tau^{p,r(\hat{t})}(t) &:= \mathcal{H}^1\big(\Gamma_\tau(t) \cap B_{r(\hat{t})}(p)\big) + \frac{t - i\tau}{\tau}\mathcal{H}^1\big((\Gamma_\tau(t+\tau) \setminus \Gamma_\tau(t)) \cap B_{r(\hat{t})}(p)\big)\\
\ell^{p,r(\hat{t})}(t) &:= \mathcal{H}^1\big(\Gamma(t) \cap B_{r(\hat{t})}(p)\big) = \mu(t)\big(B_{r(\hat{t})}(p)\big)\,.
\end{aligned}$$

Arguing as in Proposition 4.3.1, it results that $\ell^{p,r(\hat{t})} \in H^1(0,1)$ and

$$\tilde{\ell}_\tau^{p,r(\hat{t})}(\cdot) \rightharpoonup \ell^{p,r(\hat{t})}(\cdot) \tag{4.74}$$

weakly in $H^1(0,1)$.

For τ small enough $\mathfrak{c}_\tau^{p_\tau(t)} \subset B_{r(\hat{t})}(p)$, so that

$$\dot{\tilde{\ell}}_\tau^{p,r(\hat{t})}(t) = \frac{1}{\tau}\mathcal{H}^1\big(\mathfrak{c}_\tau^{p_\tau(t)} \cap B_{r(\hat{t})}(p)\big) = \frac{1}{\tau}\mathcal{H}^1\big(\mathfrak{c}_\tau^{p_\tau(t)}\big) = v_\tau(t,p_\tau(t))\,.$$

By definition of $\ell^{p,r(\hat{t})}$, for a.e. $t \in (\alpha(\hat{t}), \hat{t})$ we have

$$\dot{\ell}^{p,r(\hat{t})}(t) = \dot{\mu}(t)(B_{r(\hat{t})}p) = \sum_{q \in T(t)} v(t,q)\, \delta_q\big(B_{r(\hat{t})}(p)\big) = v(t, p(t))\,,$$

where the last equality is due to the fact that $T(t) \cap B_{r(\hat{t})}(p) = \{p(t)\}$ for $t \in (\alpha(t), t)$ (see Lemma 4.3.6). By (4.74), in particular it is

$$v_\tau(\cdot, p_\tau(\cdot)) \rightharpoonup v(\cdot, p(\cdot)) \tag{4.75}$$

weakly in $L^2(\alpha(\hat{t}), \hat{t})$. Hence, by (4.74) for every $(a,b) \subset (\alpha(\hat{t}), \hat{t})$ we have

$$\begin{aligned}\int_a^b v(t,p(t))^2\, dt = \int_a^b |\dot{\ell}^{p,r}(t)|^2\, dt &\le \liminf_{\tau \to 0} \int_a^b |\dot{\tilde{\ell}}^{p,r}_\tau(t)|^2\, dt \\ &= \liminf_{\tau \to 0} \int_a^b v_\tau(t, p_\tau(t))^2\, dt\,,\end{aligned}$$

and this concludes the proof. □

4.4. Properties of the continuous-time evolution

In this section we give a characterization of the evolution $t \mapsto (u(t), \Gamma(t))$ selected in Section 4.3. Indeed, the approximation by means of the discrete-time evolutions obtained in Section 4.2 provides $(u(t), \Gamma(t))$ with further interesting properties.

In the following, $\Gamma(t)$ is the family of sets obtained in (4.46) and $u(t)$ is the solution to the problem (4.47).

4.4.1. Energy inequality. We want to obtain an energy inequality for the continuous-time evolution (see Proposition 4.4.4). The presence of several branches of the fracture requires a careful control of the approximation process by the discrete-time evolutions, in order to obtain the proper dissipation energy due to the crack increase rate.

Rewritten with the notation introduced in Section 4.3, inequality (4.40) has the form

$$\begin{aligned}&\|\nabla u_\tau(b)\|^2 + \mathcal{H}^1(\Gamma_\tau(b)) + \int_{i\tau}^{k\tau} \sum_{p \in T_\tau(t)} v_\tau(t,p)^2\, dt \\ &\le \|\nabla u_\tau(a)\|^2 + \mathcal{H}^1(\Gamma_\tau(a)) + 2\int_{i\tau}^{k\tau} \langle \nabla u_\tau(t), \nabla \dot{w}(t)\rangle\, dt + \varpi(\tau),\end{aligned} \tag{4.76}$$

where $a < b$, $0 \le i\tau \le a < (i+1)\tau$ and $k\tau \le b < (k+1)\tau \le T$ for some $i, k \in \{0, \dots, N_\tau\}$, $i \le k$.

Lemma 4.4.1. *For every $t \in [0,1)$ it is $\Gamma_\tau(t+\tau) \xrightarrow{\mathcal{H}} \Gamma(t)$ as $\tau \to 0$.*

Proof. Fix $t \in [0,1)$ and let $i \in \{0, \dots, N_\tau\}$ be such that $i\tau \le t < (i+1)\tau$. Set

$$\widetilde{\Gamma}(t) := \mathcal{H} - \lim_{\tau \to 0} \Gamma_\tau(t+\tau), \tag{4.77}$$

which exists (up to subsequences) and belongs to the family $\mathcal{S}$ by compactness of this class (see Theorem 4.1.8).

By contradiction, assume $\widetilde{\Gamma}(t) \setminus \Gamma(t) \neq \emptyset$. Being $\Gamma_\tau(t) \subset \Gamma_\tau(t+\tau)$, the limit sets verify the same inclusion, *i.e.* $\Gamma(t) \subset \widetilde{\Gamma}(t)$. By continuity (with respect to the Hausdorff convergence) of the measure $\mathcal{H}^1$ restricted to sets in $\mathcal{S}$ (see Lemma 4.1.14), we have

$$\begin{aligned}
0 &\le \mathcal{H}^1(\widetilde{\Gamma}(t) \setminus \Gamma(t)) = \mathcal{H}^1(\widetilde{\Gamma}(t)) - \mathcal{H}^1(\Gamma(t)) \\
&= \lim_{\tau\to 0} \mathcal{H}^1(\Gamma_\tau(t+\tau)) - \lim_{\tau\to 0} \mathcal{H}^1(\Gamma_\tau(t)) = \lim_{\tau\to 0} \mathcal{H}^1(\Gamma_\tau(t+\tau) \setminus \Gamma_\tau(t)) \\
&= \lim_{\tau\to 0} \mathcal{H}^1\left(\Gamma_\tau^{i+1} \setminus \Gamma_\tau^i\right) = \lim_{\tau\to 0} \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma_\tau^i, \Gamma_\tau^{i+1})} \mathcal{H}^1(\mathfrak{c}) = \lim_{\tau\to 0} \int_{i\tau}^{(i+1)\tau} \dot{\hat{\ell}}_\tau(\xi)\, d\xi \\
&\le \lim_{\tau\to 0} \tau^{1/2} \left(\int_{i\tau}^{(i+1)\tau} |\dot{\hat{\ell}}_\tau(\xi)|^2\, d\xi \right)^{1/2} \le C \lim_{\tau\to 0} \tau^{1/2} = 0,
\end{aligned}$$

where the last inequality is due to Lemma 4.2.6.

Hence the set $\widetilde{\Gamma}(t) \setminus \Gamma(t)$ is composed by isolated points, which contradicts the fact that $\widetilde{\Gamma}(t) \in \mathcal{S}$ (Condition 4.1.6.(iii) is not satisfied). Therefore $\widetilde{\Gamma}(t) = \Gamma(t)$, which, taking into account (4.77), concludes the proof. $\square$

Lemma 4.4.2. *For any $t \in [0,1)$ the functions $\nabla u_\tau(t)$ and $\nabla u_\tau(t+\tau)$ converge to $\nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ as $\tau \to 0$.*

Proof. Fix $t \in [0,1)$ and for every τ let $i \in \{0,\dots,N_\tau\}$ be such that $i\tau \le t < (i+1)\tau$. We already proved in (4.48) that $\nabla u_\tau(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$.

Concerning the other claim, we argue as for (4.48): $u(t+\tau)$ is solution to the problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \setminus \Gamma_\tau^{i+1} \\ v = w((i+1)\tau) & \text{on } \partial_D\Omega \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_\tau^{i+1}. \end{cases}$$

Then, in order to apply again Theorem 1.7.6, we notice that $w((i+1)\tau) \to w(t)$ strongly in $H^1(\Omega)$ and, by Lemma 4.4.1, $\Gamma_\tau^{i+1} = \Gamma_\tau(t+\tau) \xrightarrow{\mathcal{H}} \Gamma(t)$. $\square$

The only remaining term to analyze is the dissipation energy due to the crack growth rate. Then we will have all the tiles to recompose the mosaic. In the following we apply the results at the end of Subsection 4.3.1.

Let t_i be defined as in Lemma 4.3.7. The set

$$F := \{t_0, \cdots, t_N\} \cup (\mathcal{A}_0 \setminus \mathcal{A})$$

is finite (see Lemma 4.3.3). We write $F = \{t'_0, \dots, t'_{N_1}\}$ with $t'_i < t'_{i+1}$ and for $t \in (t'_i, t'_{i+1})$ we define

$$I(t) = (\alpha(t), t] \cap (t'_i, t'_{i+1}),$$

where $\alpha(t)$ is given by Lemma 4.3.5 if $t \notin \mathcal{A}_0$ and by Lemma 4.3.6 if $t \in \mathcal{A}$. The following fact holds:

Lemma 4.4.3. *For every $\tilde{t} \in (t'_i, t'_{i+1})$ there exists a countable set $A(\tilde{t}) \subset (t'_i, t'_{i+1})$ such that*

$$(t'_i, \tilde{t}] = \bigcup_{t\in A(\tilde{t})} I(t)$$

and $I(t) \cap I(t') = \emptyset$ for every $t, t' \in A(\tilde{t})$, $t \neq t'$.

Proof. Fix $\tilde t \in (t'_i, t'_{i+1})$ and define

$$\iota_{\tilde t} := \inf\left\{t \in [t'_i, \tilde t] : (t, \tilde t] \text{ can be covered by countably many disjoint } I(\cdot)\right\}. \tag{4.78}$$

Of course $\iota_{\tilde t} < \tilde t$ since it is enough to consider $I(\tilde t)$ to obtain that $\iota_{\tilde t} \le \inf I(\tilde t)$.

By contradiction, assume that $\iota_{\tilde t} > t'_i$. Then the set $I(\iota_{\tilde t}) \cup (\iota_{\tilde t}, \tilde t]$ is an interval of the form $(a, \tilde t]$, is covered by (at most) countably many disjoint intervals $I(t)$ and

$$\inf\left(I(\iota_{\tilde t}) \cup (\iota_{\tilde t}, \tilde t]\right) = \inf I(\iota_{\tilde t}) < \iota_{\tilde t},$$

in contradiction to the definition (4.78). Therefore $\iota_{\tilde t} = t'_i$. □

We want to establish the following lower semicontinuity result about the dissipation at the crack front: for $(a,b) \subset (0,1)$

$$\int_a^b \sum_{p \in T(t)} v(t,p)^2 \, dt \le \liminf_{\tau \to 0} \int_a^b \sum_{p \in T_\tau(t)} v_\tau(t,p)^2 \, dt\,. \tag{4.79}$$

We first prove it in a time interval $(a,b) \subset I(\hat t)$ for any $\hat t \in (0,1) \setminus F$, then we extend it to the case $(a,b) \subset (t'_i, t'_{i+1})$ and finally to $(a,b) \subset (0,1)$.

If $\hat t \in (0,1) \setminus \mathcal{A}_0$, then Proposition 4.3.14, and in particular (4.73), provides the inequality in $I(\hat t) \cap (a,b)$:

$$0 = \int_{I(\hat t)\cap(a,b)} \sum_{p \in T(t)} v(t,p)^2 \, dt \le \int_{I(\hat t)\cap(a,b)} \sum_{p \in T_\tau(t)} v_\tau(t,p)^2 \, dt.$$

If $\hat t \in \mathcal{A}$, then applying again Proposition 4.3.14 we obtain:

$$\begin{aligned}
\int_{I(\hat t)\cap(a,b)} \sum_{p \in T(t)} v(t,p)^2 \, dt &= \int_{I(\hat t)\cap(a,b)} \sum_{p \in MT(t)} v(t,p)^2 \, dt \\
&\quad + \int_{I(\hat t)\cap(a,b)} \sum_{p \in T(t)\setminus MT(t)} v(t,p)^2 \, dt \\
&= \int_{I(\hat t)\cap(a,b)} \sum_{p \in MT(\hat t)} v(t,p(t))^2 \, dt \\
&\quad + \int_{I(\hat t)\cap(a,b)} \sum_{p \in T(\hat t)\setminus MT(\hat t)} v(t,p(t))^2 \, dt \\
&= \sum_{p \in MT(\hat t)} \int_{I(\hat t)\cap(a,b)} v(t,p(t))^2 \, dt + 0 \\
&\le \sum_{p \in MT(\hat t)} \liminf_{\tau \to 0} \int_{I(\hat t)\cap(a,b)} v_\tau(t,p_\tau(t))^2 \, dt \\
&\le \liminf_{\tau \to 0} \sum_{p \in MT(\hat t)} \int_{I(\hat t)\cap(a,b)} v_\tau(t,p_\tau(t))^2 \, dt \\
&\le \liminf_{\tau \to 0} \int_{I(\hat t)\cap(a,b)} \sum_{p \in T_\tau(t)} v_\tau(t,p)^2 \, dt.
\end{aligned}$$

Assume now that $(a,b) \subset (t'_i, t'_{i+1})$ and consider a sequence $\hat{t}_k \nearrow b$. Using the two inequalities above, the countable additivity of the integral and Lemma 4.4.3, we have

$$\begin{aligned}
\int_a^{\hat{t}_k} \sum_{p \in T(t)} v(t,p)^2\, dt &= \sum_{\hat{t} \in A(\hat{t}_k)} \int_{I(\hat{t}) \cap (a,b)} \sum_{p \in T(t)} v(t,p)^2\, dt \\
&\le \sum_{\hat{t} \in A(\hat{t}_k)} \left(\liminf_{\tau \to 0} \int_{I(\hat{t}) \cap (a,b)} \sum_{p \in T_\tau(t)} v_\tau(t,p)^2\, dt \right) \\
&\le \liminf_{\tau \to 0} \left(\sum_{\hat{t} \in A(\hat{t}_k)} \int_{I(\hat{t}) \cap (a,b)} \sum_{p \in T_\tau(t)} v_\tau(t,p)^2\, dt \right) \\
&= \liminf_{\tau \to 0} \int_a^{\hat{t}_k} \sum_{p \in T_\tau(t)} v_\tau(t,p)^2\, dt \\
&\le \liminf_{\tau \to 0} \int_a^{b} \sum_{p \in T_\tau(t)} v_\tau(t,p)^2\, dt.
\end{aligned}$$

As $k \to +\infty$, we get (4.79) when $(a,b) \subset (t'_i, t'_{i+1})$.

Finally, if $(a,b) \subset (0,1)$, then it is enough to argue as above in $(a,b) \cap (t'_i, t'_{i+1})$ and then sum over i, in order to obtain that (4.79) holds.

Proposition 4.4.4. *For all $0 \le a < b \le 1$, the couple (u, Γ) defined by (4.46) and (4.47) satisfies the following energy inequality:*

$$\begin{aligned}
&\|\nabla u(b)\|^2 + \mathcal{H}^1(\Gamma(b)) + \int_a^b \sum_{p \in T(t)} v(t,p)^2\, dt \\
&\le \|\nabla u(a)\|^2 + \mathcal{H}^1(\Gamma(a)) + 2\int_a^b \langle \nabla u(t), \nabla \dot{w}(t) \rangle\, dt.
\end{aligned}$$

Proof. We choose i and k as in (4.76). In the following series of inequalities, we apply in sequence: Lemma 4.4.2 and Lemma 4.4.1, together with Lemma 4.1.14 and the inequality (4.79); the inequality (4.76) (or, equivalently, (4.40) with $j = k+1$); again Lemma 4.4.2 and Lemma 4.1.14. Hence we have

$$\begin{aligned}
&\|\nabla u(b)\|^2 + \mathcal{H}^1(\Gamma(b)) + \int_a^b \sum_{p \in T(t)} v(t,p)^2\, dt \\
&\le \liminf_{\tau \to 0} \left(\|\nabla u_\tau(b+\tau)\|^2 + \mathcal{H}^1(\Gamma_\tau(b+\tau)) + \int_a^b \sum_{p \in T_\tau(t)} v_\tau(t,p)^2\, dt \right) \\
&\le \liminf_{\tau \to 0} \left(\|\nabla u_\tau(b+\tau)\|^2 + \mathcal{H}^1(\Gamma_\tau(b+\tau)) + \int_{i\tau}^{(k+1)\tau} \sum_{p \in T_\tau(t)} v_\tau(t,p)^2\, dt \right) \\
&\le \liminf_{\tau \to 0} \left(\|\nabla u_\tau(a)\|^2 + \mathcal{H}^1(\Gamma_\tau(a)) + 2\int_{i\tau}^{(k+1)\tau} \langle \nabla u_\tau(t), \nabla \dot{w}(t) \rangle\, dt + \varpi(\tau) \right) \\
&= \|\nabla u(a)\|^2 + \mathcal{H}^1(\Gamma(a)) + 2\int_a^b \langle \nabla u(t), \nabla \dot{w}(t) \rangle\, dt.
\end{aligned}$$

$\square$

Remark 4.4.5. The discussion described so far can be obtained also in the linearized and nonlinear planar cases, applying the results in [24, 18, 33] as done in Section 3.6. However, from now on we have to restrict to the linear antiplane shear case, in order to exploit the integral representation of the energy release rate provided in [62] (see Section 1.5 and Remark 1.5.1.(iii)).

4.4.2. Energy release rate and Griffith's principle. In order to complete the characterization of the evolution process $(u(t), \Gamma(t))$, we aim at obtaining a description in terms of Griffith's theory. In our framework we are able to achieve this goal as long as the crack set does not change direction abruptly, does not bifurcate and does not stay still (see Theorem 4.4.9 below). In those situations it is not even clear what would be the proper choice for predicting the direction in which the fracture is more likely to grow (see the discussion in [27, 26]).

The key functional is the energy release rate, whose definition and properties in case of a single curve are treated in Section 1.5; since in the current situation the crack might have several branches, the discussion needs a local argument.

For any $\Gamma \in \mathcal{S}$ and any function $g \in H^1(\Omega)$, we consider the elastic energy related to the body $\Omega \setminus \Gamma$ and the boundary displacement g, given by

$$E^{el}(g, \Gamma) := \inf \left\{ \|\nabla u\|^2 : \, u \in H^1(\Omega \setminus \Gamma), u = g \text{ on } \partial_D \Omega \right\}. \tag{4.80}$$

For a tip $p \in T_\Gamma$, we say that $\widetilde{\Gamma}$ is an *extension of* Γ *at* p if $\Gamma \subset \widetilde{\Gamma}$, $\widetilde{\Gamma} \setminus \Gamma$ is connected and there exists $r > 0$ as in Definition 4.1.4 of crack tip such that $\widetilde{\Gamma} \setminus \Gamma \subset\subset B_r(p)$ and $\widetilde{\Gamma} \cap \overline{B_r(p)} \in \mathcal{R}_\eta$.

Remark 4.4.6. Notice that any extension $\widetilde{\Gamma}$ belongs to $\mathcal{S}$, at least when $\mathcal{H}^1(\widetilde{\Gamma} \setminus \Gamma)$ is small.

In order to compute the energy release rate at a fixed $p \in T_\Gamma$, fix an extension $\widetilde{\Gamma}^p$ of Γ at p and consider the family $(\widetilde{\Gamma}^p_s)_s$ of extensions of Γ at p such that

$$\widetilde{\Gamma}^p_s \subset \widetilde{\Gamma}^p \quad \text{and} \quad \mathcal{H}^1(\widetilde{\Gamma}^p_s \setminus \Gamma) = s\,.$$

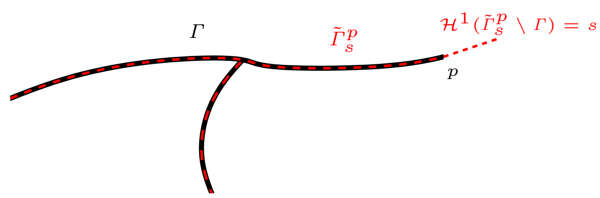


FIGURE 6. The extension $\widetilde{\Gamma}^p_s$ of Γ.

According to the results in Section 1.5, we define the energy release rate at $p \in T_\Gamma$, for a boundary displacement $g \in H^1(\Omega)$, by means of the limit

$$\mathcal{G}(g, \Gamma, p) := - \lim_{s \to 0^+} \frac{E^{el}(g, \widetilde{\Gamma}^p_s) - E^{el}(g, \Gamma)}{s}, \tag{4.81}$$

which exists and, as noticed in Remark 1.5.1.(ii), is independent of the chosen extension $\widetilde{\Gamma}^p$.

Fix $i \in \{1, \dots, N_\tau\}$ and $p \in T_{\Gamma^i_\tau}$. Consider a family of extensions $\widetilde{\Gamma}^p_s$ of Γ^i_τ at p, as above. By the minimality property of Γ^i_τ and u^i_τ, we obtain

$$\begin{aligned} &\|\nabla u^i_\tau\|^2 + \mathcal{H}^1(\Gamma^i_\tau) + \frac{1}{\tau} \sum_{C \in \mathcal{C}(\Gamma^{i-1}_\tau, \Gamma^i_\tau)} \big(\mathcal{H}^1(\mathfrak{c})\big)^2 \\ &\leq \|\nabla u_{\widetilde{\Gamma}^p_s}\|^2 + \mathcal{H}^1(\widetilde{\Gamma}^p_s) + \frac{1}{\tau} \sum_{\mathfrak{c} \in \mathcal{C}(\widetilde{\Gamma}^{i-1}_\tau, \widetilde{\Gamma}^p_s)} \big(\mathcal{H}^1(\mathfrak{c})\big)^2, \end{aligned} \tag{4.82}$$

where $u_{\widetilde{\Gamma}^p_s}$ is the minimizer of the problem (4.80) with $g = w(i\tau)$ and $\Gamma = \widetilde{\Gamma}^p_s$. Set

$$\mathfrak{c}^p_s := \widetilde{\Gamma}^p_s \setminus \Gamma^i_\tau,$$

and note that $\mathcal{H}^1(\mathfrak{c}^p_s) = s$ and, by definition of extension, $p \in \overline{\mathfrak{c}^p_s}$.

If $p \in T_{\Gamma^i_\tau} \cap T_{\Gamma^{i-1}_\tau}$, then $\widetilde{\Gamma}^p_s \setminus \Gamma^{i-1}_\tau = \mathfrak{c}^p_s \cup \big(\Gamma^i_\tau \setminus \Gamma^{i-1}_\tau\big)$ and

$$\sum_{\mathfrak{c} \in \mathcal{C}(\Gamma^{i-1}_\tau, \widetilde{\Gamma}^p_s)} \big(\mathcal{H}^1(\mathfrak{c})\big)^2 = \big(\mathcal{H}^1(\mathfrak{c}^p_s)\big)^2 + \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma^{i-1}_\tau, \Gamma^i_\tau)} \big(\mathcal{H}^1(\mathfrak{c})\big)^2.$$

Since $\mathcal{H}^1(\mathfrak{c}^p_s) = s$, by the above relation and (4.82) we obtain

$$-\frac{\|\nabla u_{\widetilde{\Gamma}^p_s}\|^2 - \|\nabla u^i_\tau\|^2}{s} \leq 1 + \frac{1}{\tau} s.$$

Therefore, recalling the definition (4.81) of $\mathcal{G}$, as $s \to 0^+$ we get

$$\mathcal{G}(w(i\tau), \Gamma^i_\tau, p) \leq 1.$$

Assume now that

$$p \in T_{\Gamma^i_\tau} \setminus T_{\Gamma^{i-1}_\tau}. \tag{4.83}$$

Then $p \in \mathfrak{c}^p$ for (only) one $\mathfrak{c}^p \in \mathcal{C}(\Gamma^{i-1}_\tau, \Gamma^i_\tau)$. It results that $\mathfrak{c}^p \cup \mathfrak{c}^p_s$ is connected,

$$\widetilde{\Gamma}^p_s \setminus \Gamma^{i-1}_\tau = (\mathfrak{c}^p \cup \mathfrak{c}^p_s) \cup \big(\Gamma^i_\tau \setminus (\Gamma^{i-1}_\tau \cup \mathfrak{c}^p)\big)$$

and

$$\sum_{\mathfrak{c} \in \mathcal{C}(\Gamma^{i-1}_\tau, \widetilde{\Gamma}^p_s)} \big(\mathcal{H}^1(\mathfrak{c})\big)^2 = \big(\mathcal{H}^1(\mathfrak{c}^p \cup \mathfrak{c}^p_s)\big)^2 + \sum_{\substack{\mathfrak{c} \in \mathcal{C}(\Gamma^{i-1}_\tau, \Gamma^i_\tau) \\ \mathfrak{c} \neq \mathfrak{c}^p}} \big(\mathcal{H}^1(\mathfrak{c})\big)^2.$$

It follows that

$$\begin{aligned} -\frac{\|\nabla u_{\widetilde{\Gamma}^p_s}\|^2 - \|\nabla u^i_\tau\|^2}{s} &\leq 1 + \frac{1}{\tau} \frac{\big(\mathcal{H}^1(\mathfrak{c}^p \cup \mathfrak{c}^p_s)\big)^2 - \big(\mathcal{H}^1(\mathfrak{c}^p)\big)^2}{s} \\ &= 1 + \frac{1}{\tau} \frac{s^2 + 2s\mathcal{H}^1(\mathfrak{c}^p)}{s}, \end{aligned}$$

and, as $s \to 0^+$, we get

$$\mathcal{G}(w(i\tau), \Gamma^i_\tau, p) \leq 1 + \frac{2}{\tau} \mathcal{H}^1(\mathfrak{c}^p).$$

If (4.83) is the case, then also the following sets can be considered in the minimization problem (4.32): $\widehat{\Gamma}^p_s \in \mathcal{S}$ such that $\Gamma^{i-1}_\tau \subset \widehat{\Gamma}^p_s \subset \Gamma^i_\tau$, the set $\Gamma^i_\tau \setminus \widehat{\Gamma}^p_s$ is connected, $p \in \Gamma^i_\tau \setminus \widehat{\Gamma}^p_s$ and $\mathcal{H}^1(\Gamma^i_\tau \setminus \widehat{\Gamma}^p_s) = s$. In this case we have that

$$\widehat{\Gamma}^p_s \setminus \Gamma^{i-1}_\tau = \Big(\mathfrak{c}^p \cap (\widehat{\Gamma}^p_s \setminus \Gamma^{i-1}_\tau)\Big) \cup \big(\Gamma^i_\tau \setminus (\Gamma^{i-1}_\tau \cup \mathfrak{c}^p)\big)$$

and

$$\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma_\tau^{i-1},\widehat{\Gamma}_s^p)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2 = \left(\mathcal{H}^1\left(\mathfrak{c}^p\cap(\widehat{\Gamma}_s^p\setminus\Gamma_\tau^{i-1})\right)\right)^2 + \sum_{\substack{\mathfrak{c}\in\mathcal{C}(\Gamma_\tau^{i-1},\Gamma_\tau^i)\\ \mathfrak{c}\neq\mathfrak{c}^p}} \left(\mathcal{H}^1(\mathfrak{c})\right)^2.$$

Inequality (4.82) holds even in this case, with $\widehat{\Gamma}_s^p$ instead of $\widetilde{\Gamma}_s^p$, and we obtain

$$\begin{aligned}-\left(\|\nabla u_{\widehat{\Gamma}_s^p}\|^2-\|\nabla u_\tau^i\|^2\right)\leq&\mathcal{H}^1(\widehat{\Gamma}_s^p)-\mathcal{H}^1(\Gamma_\tau^i)\\ &+\frac{1}{\tau}\left(\left(\mathcal{H}^1(\mathfrak{c}^p\cap(\widehat{\Gamma}_s^p\setminus\Gamma_\tau^{i-1}))\right)^2-\left(\mathcal{H}^1(\mathfrak{c}^p)\right)^2\right)\\ =&-s+\frac{1}{\tau}\left(s^2-2s\mathcal{H}^1(\mathfrak{c}^p\cap(\widehat{\Gamma}_s^p\setminus\Gamma_\tau^{i-1}))\right).\end{aligned}$$

Dividing by $-s$ and letting $s\to 0^+$, since

$$\mathcal{H}^1(\mathfrak{c}^p\cap(\widehat{\Gamma}_s^p\setminus\Gamma_\tau^{i-1}))\to\mathcal{H}^1(\mathfrak{c}^p)$$

we obtain the reverse inequality

$$\mathcal{G}(w(i\tau),\Gamma_\tau^i,p)\geq 1+\frac{2}{\tau}\mathcal{H}^1(\mathfrak{c}^p).$$

Using the definition of discrete velocity introduced in (4.72), we can restate the above discussion in terms of a discrete Griffith's principle: for every $t\in(0,1)$ and $p_\tau(t)\in T_\tau(t)$

$$v_\tau(t,p_\tau(t))\geq 0 \tag{4.84}$$
$$\mathcal{G}(w_\tau(t),\Gamma_\tau(t),p_\tau(t))\leq 1+2\,v_\tau(t,p_\tau(t)) \tag{4.85}$$
$$[-\mathcal{G}(w_\tau(t),\Gamma_\tau(t),p_\tau(t))+1+2\,v_\tau(t,p_\tau(t))]\,v_\tau(t,p_\tau(t))=0. \tag{4.86}$$

We now look for a similar stability criterion for the continuous-time evolution. We will see that, in the case of moving tips, this is achievable. On the other hand, when dealing with static tips a number of problematic issues might appear.

As recalled in Remark 1.5.1.(iii), the energy release rate has the following integral representation in terms of the displacement gradient. Let $K\in\mathcal{R}_\eta$ and γ be its arc-length parametrization. Consider $p\in T_K$, $p=\gamma(\mathcal{H}^1(K))$. Then

$$\begin{aligned}\mathcal{G}(g,K,p)=\int_{\Omega\setminus K}\Bigg[&\frac{(D_1u_K)^2-(D_2u_K)^2}{2}\,(D_1V^1-D_2V^2)\\ &+D_1u_KD_2u_K(D_2V^1+D_1V^2)\Bigg]\,dx,\end{aligned} \tag{4.87}$$

where u_K minimizes $E^{el}(g,K)$, $\nabla u_K=(D_1u_K,D_2u_K)$, and $V=(V_1,V_2)$ is any vector field of class $C^{0,1}$ with compact support in Ω such that $V(\gamma(s))=\dot\gamma(s)$ for s in a neighbourhood of $\mathcal{H}^1(K)$ (recall that $p=\gamma(\mathcal{H}^1(K))$). This explicit formula will be useful in the sequel.

The following lemma is a slight variant of [62, Theorem 2.12], also recalled in Subsection 1.5.3 as Theorem 1.5.4. As announced in Subsection 1.5.3, for the sake of completeness and clarity we prove the lemma below, following the proof of [62, Theorem 2.12]. We remind that the set $\mathcal{A}$ was introduced in Lemma 4.3.3.

Lemma 4.4.7. *Fix $\hat t\in\mathcal{A}$ and let $\alpha(\hat t)$, $\nu(\hat t)$, $r(\hat t)$ and $p_\tau(t)$ be as in Lemma 4.3.6, and $p(t)$ as in (4.55). Then, for every $p\in MT(\hat t)$,*

$$\mathcal{G}(w_\tau(t),\Gamma_\tau(t),p_\tau(t))\to\mathcal{G}(w(t),\Gamma(t),p(t)) \tag{4.88}$$

for every $t\in(\alpha(\hat t),\hat t]$.

PROOF. We set

$$K^p_\tau(t) := \Gamma_\tau(t) \cap \overline{B_{r(\hat t)}(p)} \quad \text{and} \quad K^p(t) := \Gamma(t) \cap \overline{B_{r(\hat t)}(p)}$$

for any $t \in (\alpha(\hat t), \hat t]$. As seen in Lemma 4.3.6, $K^p_\tau(t) \in \mathcal{R}_\eta$ and $K^p_\tau(t) \xrightarrow{\mathcal{H}} K^p(t)$.

Consider γ_τ and γ arc-length parametrizations of $K^p_\tau(\hat t)$ and $K^p(\hat t)$, respectively, with $\gamma_\tau(0), \gamma(0) \in \partial B_{r(\hat t)}(p)$. Set $L := \mathcal{H}^1(K^p(\hat t))$, it is $p = \gamma(L)$. Using the notation of Proposition 4.1.2, $\tilde\gamma_\tau$ converges to γ in the weak* topology of $W^{2,\infty}([0,L];\mathbb{R}^2)$. We extend each $K^p_\tau(\hat t)$ adding a segment along the tangent direction to the tip $p_\tau(\hat t) = \tilde\gamma_\tau(L)$ and the same for $K^p(\hat t)$ at $p = \gamma(L)$. Using the Implicit Function Theorem, the bound on the curvature in Definition 4.1.1.(ii) and the choice of $r(\hat t)$, these extended curves are graphs of some $C^{1,1}$ scalar functions φ_τ, φ. We fix two coordinate axes such that the extension of $K^p_\tau(\hat t)$ is described by $(x_1, \varphi_\tau(x_1))$ and the extension of $K^p(\hat t)$ is described by $(x_1, \varphi(x_1))$. Fix a cut-off function ζ supported in $B_{r(\hat t)}(p)$. Given a point $x = (x_1, x_2) \in B_{r(\hat t)}(p)$, define the vector fields $V_\tau(x) := \zeta(x)\,(1, \frac{d}{dx_1}\varphi_\tau(x_1))$; similarly we define a vector field V related to φ. By the weak* convergence of $\tilde\gamma_\tau$ to γ in $W^{2,\infty}([0,L];\mathbb{R}^2)$, we obtain that ∇V_τ converges to ∇V weakly* in $L^\infty(\Omega;\mathbb{R}^4)$.

Observe that, according to the formula (4.87), the vector fields introduced above are suitable for the integral representation of the energy release rate for the curves $K^p_\tau(t)$ and $K^p(t)$ for every $t \in (\alpha(\hat t), \hat t]$ (and not only for $t = \hat t$). That is, we have the following equality:

$$\begin{aligned}\mathcal{G}(w_\tau(t), \Gamma_\tau(t), p_\tau(t)) = \int_{\Omega\setminus\Gamma_\tau(t)} \Big[&\frac{(D_1 u_\tau(t))^2 - (D_2 u_\tau(t))^2}{2}(D_1 V^1_\tau - D_2 V^2_\tau) \\ &+ D_1 u_\tau(t) D_2 u_\tau(t)(D_2 V^1_\tau + D_1 V^2_\tau)\Big]\,dx \end{aligned} \tag{4.89}$$

and similarly for $\mathcal{G}(w(t), \Gamma(t), p(t))$.

Since the sequence ∇V_τ converges to ∇V weakly* in $L^\infty(\Omega;\mathbb{R}^4)$ as $\tau \to 0$ and, as proved in (4.48), $\nabla u_\tau(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ for all $t \in (\alpha(\hat t), \hat t]$, we obtain the claimed pointwise convergence. ∎

Lemma 4.4.8. *Assume the same hypotheses as in Lemma 4.4.7. Then, for every $1 \le q < \infty$,*

$$\mathcal{G}(w(\cdot), \Gamma(\cdot), p(\cdot)) \in L^q(\alpha(\hat t), \hat t)$$

and

$$\mathcal{G}(w_\tau(\cdot), \Gamma_\tau(\cdot), p_\tau(\cdot)) \to \mathcal{G}(w(\cdot), \Gamma(\cdot), p(\cdot))$$

in $L^q(\alpha(\hat t), \hat t)$.

PROOF. By means of the Dunford-Pettis Theorem (see [6]) and (4.75), we obtain that the functions $v_\tau(\cdot, p_\tau(\cdot))$ are equiintegrable in $(\alpha(\hat t), \hat t)$. Being

$$0 \le \mathcal{G}(w_\tau(t), \Gamma_\tau(t), p_\tau(t)) \le 1 + 2\,v_\tau(t, p_\tau(t)),$$

the sequence $\mathcal{G}(w_\tau(\cdot), \Gamma_\tau(\cdot), p_\tau(\cdot))$ is equiintegrable too. Then, considering Lemma 4.4.7, by Vitali's Theorem (see [79, Chapter 6, Exercise 9]) we have that $\mathcal{G}(w(\cdot), \Gamma(\cdot), p(\cdot)) \in L^1(\alpha(\hat t), \hat t)$ and

$$\int_{\alpha(\hat t)}^{\hat t} \mathcal{G}(w_\tau(t), \Gamma_\tau(t), p_\tau(t))\,dt \to \int_{\alpha(\hat t)}^{\hat t} \mathcal{G}(w(t), \Gamma(t), p(t))\,dt\,.$$

Since $\mathcal{G}$ is non-negative, the last limit means that

$$\|\mathcal{G}(w_\tau(\cdot), \Gamma_\tau(\cdot), p_\tau(\cdot))\|_{L^1(\alpha(\hat t),\hat t)} \to \|\mathcal{G}(w(\cdot), \Gamma(\cdot), p(\cdot))\|_{L^1(\alpha(\hat t),\hat t)}.$$

Then, applying [6, Proposition 1.33] and the pointwise convergence (4.88), we obtain that

$$\mathcal{G}(w_\tau(\cdot), \Gamma_\tau(\cdot), p_\tau(\cdot)) \to \mathcal{G}(w(\cdot), \Gamma(\cdot), p(\cdot))$$

in $L^1(\alpha(\hat{t}), \hat{t})$.

Finally, observe that, by the integral formula (4.89) for the energy release rate, it results

$$\mathcal{G}(w_\tau(\cdot), \Gamma_\tau(\cdot), p_\tau(\cdot)),\ \mathcal{G}(w(\cdot), \Gamma(\cdot), p(\cdot)) \in L^\infty(\alpha(\hat{t}), \hat{t}).$$

Indeed, the maps ∇V_τ and ∇V are uniformly bounded in L^∞ by the $W^{2,\infty}$ norms of γ_τ and γ introduced in Lemma 4.4.7; for $\nabla u_\tau(t)$ and $\nabla u(t)$ we use (4.41) and (4.49) to have a uniform bound.

The L^∞ bound uniform in τ and the L^1 convergence proved above are sufficient to conclude the proof. □

The concluding main result of this section is proved in the following theorem.

Theorem 4.4.9. *Fix $\hat{t} \in \mathcal{A}$ and let $\alpha(\hat{t})$, $\nu(\hat{t})$, $r(\hat{t})$ be as in Lemma 4.3.6, and $p(t)$ as in (4.55). Then, for every $p \in MT(\hat{t})$, the following conditions hold for a.e. $t \in (\alpha(\hat{t}), \hat{t})$:*

$$v(t, p(t)) \geq 0 \tag{4.90}$$

$$\mathcal{G}(w(t), \Gamma(t), p(t)) \leq 1 + 2\, v(t, p(t)) \tag{4.91}$$

$$[-\mathcal{G}(w(t), \Gamma(t), p(t)) + 1 + 2\, v(t, p(t))]\, v(t, p(t)) = 0. \tag{4.92}$$

Proof. Fix t such that $\dot{\mu}(t)$ exists. Consider $\psi \in C_b(\Omega)$ with $\operatorname{supp}\psi \subset B_{r_1(t)}(p(t))$, $\psi \geq 0$ and $\psi(p(t)) = 1$. Then, by (4.68), it is

$$v(t, p(t)) = \langle \psi, \dot{\mu}(t) \rangle = \lim_{s \nearrow t} \frac{1}{t-s} \int_{\Gamma(t) \setminus \Gamma(s)} \psi \, d\mathcal{H}^1 \geq 0\,,$$

hence (4.90) holds.

Let $(a, b) \subset (\alpha(\hat{t}), \hat{t})$. By the weak convergence (4.75), Lemma 4.4.8 and (4.85), we obtain

$$\int_a^b \mathcal{G}(w(t), \Gamma(t), p(t))\, dt \leq \int_a^b [1 + 2\, v(t, p(t))]\ dt.$$

If $t' \in (\alpha(\hat{t}), \hat{t})$ is a Lebesgue point of the function $-\mathcal{G}(w(\cdot), \Gamma(\cdot), p(\cdot)) + 1 + 2\, v(\cdot, p(\cdot))$, by the inequality above we obtain

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{t'-\varepsilon}^{t'} [-\mathcal{G}(w(t), \Gamma(t), p(t)) + 1 + 2\, v(t, p(t))]\ dt \\ &= -\, \mathcal{G}(w(t'), \Gamma(t'), p(t')) + 1 + 2\, v(t', p(t')). \end{aligned}$$

Therefore (4.91) holds true a.e. in $(\alpha(\hat{t}), \hat{t})$.

The inequalities (4.90) and (4.91) trivially imply

$$[-\mathcal{G}(w(t), \Gamma(t), p(t)) + 1 + 2\, v(t, p(t))]\, v(t, p(t)) \geq 0 \tag{4.93}$$

for a.e. $t \in (\alpha(\hat{t}), \hat{t})$. Then, considering (4.86), the weak convergence (4.75) and Lemma 4.4.8, we have the following chain of inequalities

$$\begin{aligned}
0 &\leq \int_{\alpha(\hat{t})}^{\hat{t}} [-\mathcal{G}(w(t), \Gamma(t), p(t)) + 1 + 2\, v(t, p(t))]\, v(t, p(t))\, dt \\
&\leq \lim_{\tau \to 0} \int_{\alpha(\hat{t})}^{\hat{t}} [-\mathcal{G}(w_\tau(\cdot), \Gamma_\tau(\cdot), p_\tau(\cdot))\, v_\tau(t, p_\tau(t)) + v_\tau(t, p_\tau(t))]\, dt \\
&\quad + \liminf_{\tau \to 0} \int_{\alpha(\hat{t})}^{\hat{t}} 2\, v_\tau(t, p_\tau(t))^2\, dt \\
&\leq \liminf_{\tau \to 0} \int_{\alpha(\hat{t})}^{\hat{t}} [-\mathcal{G}(w_\tau(t), \Gamma_\tau(t), p_\tau(t)) + 1 + 2\, v_\tau(t, p_\tau(t))]\, v_\tau(t, p_\tau(t)) = 0,
\end{aligned}$$

i.e.

$$\int_{\alpha(\hat{t})}^{\hat{t}} [-\mathcal{G}(w(t), \Gamma(t), p(t)) + 1 + 2\, v(t, p(t))]\, v(t, p(t))\, dt = 0.$$

Together with (4.93), this equality implies (4.92) for a.e. $t \in (\alpha(\hat{t}), \hat{t})$. $\square$

In conclusion, we would like to explain some of the difficulties that arise in the characterization of the behaviour of points in $T(\hat{t}) \setminus MT(\hat{t})$. In general, our method does not provide information about unilateral minimality properties for the continuous-time evolution, therefore any property concerning it needs to be obtained by the limit behaviour of the discrete-time evolutions.

In case of static tips, we are not able to prove a result like Lemma 4.3.6, which plays a key role in the proof of the subsequent results. For example, a static tip might be approximated by a discrete-time sequence of cracks that kink near the tip. The approximation procedure suggests that, in this situation, many direction of growth for the crack tip (of the continuous-time evolution) are possible: which would be the preferred one? How to deal with the energy release rate $\mathcal{G}$, which, as proved by Negri [71], depends on the kinking angle?

Unfortunately, in the mathematical setting we proposed it is not possible to avoid this kind of situations and a complete description of the growth process remains an open problem. Anyway, it is not a simple task to introduce further restrictions on the geometrical properties of the crack sets in the class $\mathcal{S}$, without finding some geometrical setting already discussed in the literature (see for example [63, 58]). On the other hand, our geometrical constraints are necessary in order to avoid some mathematical "pathologies" (like accumulation of singular points) that would arise if branching is admissible and those constraints were absent. To our knowledge, this is a first attempt to consider kinking and branching in the framework of Griffith's theory without assuming the crack path to be known *a priori*, and much work still needs to be done.

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