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Periodic solutions to planar Hamiltonian systems: high multiplicity and chaotic dynamics

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Introduction

This work deals with the problem of the existence and multiplicity of *periodic solutions* to nonautonomous *planar Hamiltonian systems*, collecting some results obtained during my Ph.D. studies [25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. More precisely, we consider the planar differential system

$$Jz' = \nabla_z H(t,z), \qquad z=(x,y)\in\mathbb{R}^2, \tag{1}$$

with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the standard symplectic matrix and $H:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ a regular function, T-periodic in the first variable (with $T>0$ fixed), and we look for periodic solutions to (1) of fixed period, both harmonic (i.e., T-periodic) and subharmonic (i.e., kT-periodic, with $k\geq 2$ an integer, and kT the minimal period, in a suitable sense).

The following brief summary is meant just to give an account of the main problems considered in this work and of the main methods used to analyze them. Each chapter is indeed equipped with a further introduction, containing more detailed descriptions and motivations for our research, as well as a rich bibliography on the corresponding subject.

Let us first recall that (1) is the case $N=1$ of the system of $2N$ ordinary differential equations ($i=1,\ldots,N$)

$$\begin{cases} x_i' = \dfrac{\partial H}{\partial y_i}(t,x_1,\ldots,x_N,y_1,\ldots,y_N) \\[2ex] y_i' = -\dfrac{\partial H}{\partial x_i}(t,x_1,\ldots,x_N,y_1,\ldots,y_N), \end{cases} \qquad z=(x_1,\ldots,x_N,y_1,\ldots,y_N)\in\mathbb{R}^{2N}, \tag{2}$$

nowadays known as Hamilton's equations[1] and describing the evolution of a mechanical system with N degrees of freedom. To be more precise, $(x_i(t),y_i(t))$ ($(q_i(t),p_i(t))$, following tradition) play the role of "generalized coordinates" and "generalized conjugate momenta", while the function $H(t,x,y)$ - the so-called Hamiltonian - represents the total energy of the system (notice that, in the autonomous case, $H(t,x,y)=H(x,y)$ is indeed a first integral for (2)). Of particular physical relevance is the case when the Hamiltonian writes as the

[1] Quoting [128], "the Hamilton's equations appear for the first time in a paper of Lagrange (1809) on perturbation theory, but it was Cauchy (1831) who first gave the true significance of those equations. In 1835, Hamilton put those equations at the basis of his analytical mechanics [...]". With respect to the Lagrangian formulation of classical mechanics, the Hamiltonian formalism provides a better insight on the geometrical properties of the phase space, highlighting its symplectic structure; quoting V.I. Arnold [11], *"Hamiltonian mechanics is geometry in phase space"*.

sum of a kinetic energy $K(t,y) = \frac{1}{2}\sum_{i=1}^{N} y_i^2$ and of a potential energy $V(t,x)$, so that (2) gives, setting[2] $u(t) = x(t)$,

$$u'' + \nabla_u V(t,u) = 0, \qquad u \in \mathbb{R}^N, \tag{3}$$

that is, in the case $N = 1$,

$$u'' + v(t,u) = 0, \qquad u \in \mathbb{R}, \tag{4}$$

for $V(t,x) = \int_0^x v(t,\xi)\, d\xi$. This subclass of Hamiltonian systems has been enormously investigated from the mathematical point of view in the last decades and it has a special role also in this thesis.

Our choice to limit the attention to a planar setting can look unnecessarily restrictive, since nowadays modern functional-analytic techniques (like Topological Degree Theory or, since (2) has a variational structure, Critical Point Theory) permit to successfully tackle the problem in arbitrary dimension. On the other hand, the two-dimensional case deserves a particular attention, since "elementary" - but powerful - tools from Planar Topology can lead to very precise results for (1), which often do not have analogues in higher dimension. This is the main theme of this thesis: in particular, we are interested in the situations in which (1) exhibits *high multiplicity* of periodic solutions and, possibly, *chaotic dynamics*. More in detail, we always assume that (1) admits an equilibrium point[3] (which, without loss of generality, is supposed to be the origin), namely

$$\nabla_z H(t,0) \equiv 0 \tag{5}$$

(that is, for (4), $v(t,0) \equiv 0$), and we look for multiple nontrivial periodic solutions, performing our analysis in two different directions[4].

Part I - Periodic solutions which wind around the origin

In the first part of the thesis (Chapters 2, 3 and 4), we look for *periodic solutions which wind around the origin* (which correspond, in the case of the scalar equation (4), to solutions which change their sign). This is, of course, a very natural choice, since simple autonomous examples - just require, for $H(t,z) = H(z)$,

$$\langle \nabla H(z) | z \rangle > 0, \quad \text{for } z \neq 0, \quad \text{and} \quad \lim_{|z| \to +\infty} H(z) = +\infty,$$

[2]Our choice of using the letter u for the unknown of second order ODEs like (3) and (4) is motivated by the fact that this notation is typically employed in the literature dealing with elliptic PDEs, which represent a natural generalization of (3) and (4).

[3]Equations of this type are called unforced by some authors. Notice that, when a T-periodic solution to (1) is known "a priori" to exist, we can always enter in such a setting via a (time-dependent) linear change of variables, which does not modify the asymptotic properties of the Hamiltonian (see Chapter 4 for some examples in which such a strategy is realized).

[4]We will distinguish between periodic solutions which wind around the origin and periodic solutions which do not wind around the origin. By this, as usual, we mean that their homotopy class in the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{0\})$ is, respectively, nontrival or trivial; see the introductory section "Notation and Terminology" for more details.

(that is, for (4) with $v(t,x) = v(x)$, $v(x)x > 0$ for $x \neq 0$ and $\int_0^{+\infty} v(\xi)\, d\xi = +\infty$) - show that the origin can be a global center for (1): namely, all the nontrivial solutions are periodic, winding around the origin in the same angular direction. We use here the Poincaré-Birkhoff fixed point theorem, in a suitable generalized version, to perform our analysis for the general nonautonomous case, in presence of a dynamics of center type. Indeed, by proving that a "gap" occurs between the number of turns around the origin of the solutions departing from the inner and the outer boundary of a topological annulus, it is possible to ensure the existence of periodic solutions making a precise number of revolutions around the origin: such an additional information is the key point to distinguish among periodic solutions, thus proving sharp multiplicity results. In the same spirit, the information about the number of windings is a crucial fact in establishing the minimality of the period, when trying to detect subharmonic solutions. More in detail, the first part of the thesis is organized as follows.

In Chapter 2 we study the existence of periodic solutions to (1), by comparing the nonlinear system, both at zero and at infinity, with autonomous Hamiltonian systems with Hamiltonian positively homogeneous of degree 2. The main property of such comparison systems is that the origin is an isochronous center (namely, all the nontrivial solutions are periodic, winding around the origin, and have the same minimal period); as a consequence, one can estimate in a very precise manner the number of revolutions of "small" and "large" solutions to (1). In particular, a rotational analysis of a generalized Landesman-Lazer condition for planar systems is performed. Various results are presented, essentially treating the case when (1) is semilinear, sublinear and superlinear at infinity.

In Chapter 3 we describe a general strategy to detect, via the Poincaré-Birkhoff twist theorem, periodic solutions to second order scalar differential equations like (4) in presence of lower/upper T-periodic solutions. Applications are given to pendulum type equations and to Ambrosetti-Prodi type problems. In both cases, a preliminary step in the proof relies on the use of the coincidence degree, which allows to prove the existence and localization of a first T-periodic solution. Via a change of coordinates, such a T-periodic solution is transformed into the origin (in order to have (5) satisfied) and the Poincaré-Birkhoff fixed point theorem eventually provides other periodic solutions, winding around this first one.

In Chapter 4 we consider the second order scalar differential equation

$$u'' + \lambda f(t,u) = 0, \qquad u \in \mathbb{R}, \tag{6}$$

with $\lambda > 0$ playing the role of a parameter. The function $f(t,x)$ is assumed to be supersublinear, i.e.,

$$\lim_{|x|\to 0} \frac{f(t,x)}{x} = \lim_{|x|\to +\infty} \frac{f(t,x)}{x} = 0, \qquad \text{uniformly in } t \in [0,T],$$

so that a gap between the number of revolutions of "small" and "large" solutions is not naturally available. However, we prove that, under a weak sign condition on $f(t,x)$, "intermediate" solutions perform, when $\lambda \to +\infty$, an arbitrarily large number of turns around the origin. Accordingly, it is possible to apply twice (precisely, on a small and on a large annular region) the Poincaré-Birkhoff fixed point theorem, giving pairs of sign-changing periodic solutions to (6).

Part II - Periodic solutions which do not wind around the origin

In the second part of the thesis (Chapters 5 and 6), we look for *periodic solutions which do not wind around the origin*: in such an analysis, we are mainly motivated by the problem of the existence of one-signed periodic solutions to the scalar equation (4). This case is, in principle, more difficult. Indeed, one has to observe, on one hand, that a global dynamics of center type, as in the first part of the thesis, prevents the existence of periodic solutions which do not turn around the origin. Accordingly, to derive results in this direction, one has to impose suitable sign changes for $\langle \nabla_z H(t,z) | z \rangle$ (that is, in the case of (4), for $v(t,x)x$) in order to combine, for system (1), different dynamical scenarios, usually of center and saddle type. On the other hand, it can be hard to provide multiplicity results, since the natural "tag" of the number of windings around the origin here cannot be used to distinguish among solutions and more precise properties of localizations in the plane have to be looked for. Accordingly, different abstract tools have to be employed to perform the analysis. The plan of this second part of the thesis is the following.

In Chapter 5 we consider the general case when, for the planar Hamiltonian system (1), an equilibrium point of saddle type is matched with an asymptotic dynamics of center type. Precisely, we show that such a twist dynamics leads to the existence of periodic solutions which do not wind around the origin, accompanied by a great number of large periodic (subharmonic) solutions around the origin. However, the classical version of the Poincaré-Birkhoff theorem here does not suffice and a recent modified version has to be used.

In Chapter 6 we turn our attention to the existence of positive periodic solutions to second order scalar differential equations of the type

$$u'' + q(t)g(u) = 0, \qquad u \in \mathbb{R}, \tag{7}$$

being $q(t)$ a T-periodic function which changes its sign and $g(x)$ a function satisfying the sign condition $g(x)x > 0$ for $x \neq 0$. Problems of the type (7) are called indefinite (referring to the fact that $q(t)$ assumes both positive and negative values and therefore does not have a definite sign) and they have been widely investigated in the past few decades, as a simple model exhibiting high multiplicity of sign-changing solutions (with different boundary conditions) and, eventually, chaotic dynamics. Such a complex dynamical behavior essentially comes from the alternation, in the time variable, of a saddle type dynamics (when $q(t) \leq 0$) with a center type dynamics (when $q(t) \geq 0$). Here we show that this class of problems gives a natural setting in which existence and multiplicity of positive periodic solutions can be detected, as well, provided that (7) is studied in dependence of real parameters, acting on the weight function $q(t)$. Precisely, in Section 6.1 we are concerned with the case

$$\lim_{x \to 0^+} \frac{g(x)}{x} = \lim_{x \to +\infty} \frac{g(x)}{x} = 0,$$

showing the existence of multiple positive T-periodic solutions, positive subharmonics of any order as well as a complex dynamics (in the coin-tossing sense), entirely generated by positive solutions to (7); in our analysis, we use here a recent approach linked to the theory of topological horseshoes. Finally, in Section 6.2 we prove the existence of multiple positive

T-periodic solutions when

$$\lim_{x\to 0^+}\frac{g(x)}{x}=0,\qquad \lim_{x\to +\infty}\frac{g(x)}{x}=+\infty,$$

and the weight function $q(t)$ satisfies a symmetry condition: such periodic solutions - whose existence and multiplicity is strictly related to the number and the height of the positive humps of $q(t)$ - are provided via solutions of an auxiliary Neumann problem, which is analyzed by a careful shooting technique in the phase plane.

Further remarks and work in progress

This thesis mainly investigates the existence and multiplicity of periodic solutions to planar Hamiltonian systems via a dynamical systems approach. As already anticipated at the beginning of this introduction, various other tools to tackle this very classical problem are available and a complete overview on the subject is highly beyond the purposes of this work. Nevertheless, it seems to be worth briefly mentioning some issues which could be related to the topic of the present thesis, proposing some selected references. Needless to say, this viewpoint is very personal, being strongly influenced by the knowledge and the interests of the author.

The use of Topological Methods in the study of boundary value problems associated with differential systems is, of course, very classical, starting in its modern form with the pioneering paper by Leray and Schauder [109] where the so-called continuation technique was introduced (see [123] for a nice survey on the topic and earlier contributions). For the periodic boundary value problem, a convenient approach in the framework of coincidence degree has been proposed by Mawhin [120]. Such a technique is very general, being useful, in principle, for systems in arbitrary dimension and having an arbitrary (i.e., possibly non Hamiltonian) structure; on the other hand, usually, only existence or low multiplicity of periodic solutions is obtained. For superlinear differential systems (a situation in which the usual a priori bound for periodic solutions is not available and the method of Leray-Schauder can not be directly used), a powerful technique in the framework of Topological Methods has been developed by Capietto, Mawhin and Zanolin [43], matching the global continuation approach with the introduction of an invariant which is strictly related to the rotation number (see the introductory section “Notation and Terminology”) considered in our work; such a technique is particularly well suited for planar problems. The use of polar type coordinates also plays an important role when resonant systems are considered, leading to very precise results of Landesman-Lazer type in the plane [67, 72] (compare with Remark 2.2.5). We finally observe that Rabinowitz’s global bifurcation theory [146] (which could be successfully used to prove high multiplicity results for planar differential systems with separated boundary conditions) cannot be directly applied for the periodic problem since the eigenvalues of $Jz' = \lambda z$ (or of related spectral problems) have even multiplicity. However, bifurcation from infinity has been employed in [115] to prove “multiplicity near resonance” results (on the lines of [125]) for periodic solutions to (4) in presence of symmetry conditions; a more recent contribution, using a degree theory for $SO(2)$-equivariant gradient maps, can be found in [84].

On the other hand, Calculus of Variations and Critical Point Theory provide a natural, powerful tool to treat Hamiltonian systems in arbitrary dimension; their range of applicability, moreover, is not restricted to periodic solutions but it also covers the investigation of homoclinic and heteroclinic solutions and, possibly, complex dynamics. As far as the second order system (3) is considered, well developed minimization arguments, linking theorems, Morse and Lusternik-Schnirelmann theories apply providing existence results for a very broad class of nonlinearities, see [128]. On the other hand, the variational formulation of the periodic problem associated with the general Hamiltonian system (2) leads to the functional

$$\mathscr{F}(z) = \int_0^T \sum_{i=1}^N x_i'(t) y_i(t)\, dt - \int_0^T H(t, x_1(t), \ldots, x_N(t), y_1(t), \ldots, y_N(t))\, dt,$$

which is strongly indefinite (i.e., its leading part $\int_0^T \sum_{i=1}^N x_i'(t) y_i(t)\, dt$ is unbounded both from below and from above), so that the above mentioned tools can not be used directly: investigations in this direction started with the pioneering work by Rabinowitz [149] and some abstract critical point theorems for strongly indefinite functionals were later developed by Benci and Rabinowitz [17] (see [2] for a nice overview on the topic, as well as for a wide bibliography). As for superlinear Hamiltonian systems, the existence of infinitely many T-periodic solutions was first proved by Bahri and Berestycki [13] assuming the Ambrosetti-Rabinowitz superlinearity condition (such a condition, however, is stronger than the planar superlinearity condition suggested by topological tools and used in [43]; see the discussion after Theorem 2.3.8). Hamiltonian systems with an equilibrium point at the origin and asymptotically linear at infinity were studied by Ahmann-Zehnder [9] and Conley-Zehnder [50] via Maslov type index theories. This last investigation naturally suggests a comparison between Variational Methods and the Poincaré-Birkhoff fixed point theorem, showing that always, as far as the planar case is considered, the multiplicity results obtained via the twist theorem are considerably stronger (we refer to the papers [111, 118] and our Remark 1.1.2 for more comments; in the higher dimensional case, multiplicity can be guaranteed when some additional conditions - like convexity, symmetry in the space variable, or for autonomous systems - are required). Multidimensional generalizations of the Poincaré-Birkhoff fixed point theorem, in the framework of variational tools, lead to the very deep questions raised by Arnold's conjectures [11].

Another worth mentioning aspect of our work is the relationship between periodic boundary conditions and other (separated) Sturm-Liouville type boundary conditions which could be considered in association with the planar Hamiltonian system (1). In this regard, we remark that our approach mainly relies on the study of the trajectories of the solutions to (1) in the plane: such an analysis provides information on the Poincaré map associated with (1) and periodic solutions are then found via fixed point theorems. Most of our preliminary estimates, accordingly, could be combined with classical shooting type arguments in order to derive multiplicity results for Sturm-Liouville boundary value problems associated with (1). This is the case, for instance, for most of the technical estimates for the rotation numbers developed in Chapters 2 and 5 (Landesman-Lazer conditions for Sturm-Liouville problems would be, however, a delicate issue), which could lead to corresponding multiplicity results. The precise statements, of course, would have to be written case by case and cannot be summarized here; notice, however, that in our results we actually consider an infinite family of periodic problems (namely, the kT-periodic ones - being k an integer number) thus showing the full strength of the method proposed. Counterparts, for both the Dirichlet and the

Neumann problem, of the results of Chapter 4 are particularly natural and they have been considered in the corresponding paper [34]. Finally, the analogies between the periodic and the Neumann problem when dealing with the existence of positive solutions to the second order scalar equations of Chapter 6 are emphasized in the corresponding introduction.

To conclude, we briefly touch on possible future developments of our research, along some of the directions presented in the present work.
Chapters 2 and 5 provide a quite exhaustive description of the dynamics of a planar Hamiltonian system having an equilibrium point. However, a possibly interesting investigation could be performed about the general requirement of the global continuability for the solutions; to this aim, techniques based on the construction of some spiral-like curves in the plane used to bound the solutions (as in the proof of Lemma 4.2.2) could be useful. For instance, it was proved by Hartman [97] that the global continuability for the solutions is not needed in the case of the second order equation (4), when $v(t,0) \equiv 0$ and $v(t,x)/x \to +\infty$ for $|x| \to +\infty$.
Chapter 3 suggests the following general question: when subharmonic solutions can be obtained between a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ such that $\alpha(t) < \beta(t)$? We do not know the answer; of course simple examples show that additional conditions are needed, but one can imagine many different situations (in the easiest case, when $\alpha(t), \beta(t)$ are solutions) in which subharmonics actually exist. This could be related to the investigation of subharmonic solutions to planar Hamiltonian systems having two, or more, equilibrium points, in connection with some results by Abbondandolo and Franks [1, 81]. Chaotic dynamics for some of the examples considered in Chapters 3 and 4 should be detected as well.
Finally, Chapter 5 can give rise to investigations in different directions. First of all, the existence of subharmonic solutions is, to the best of our knowledge, still to be studied for most of the problems considered. Second, we suspect that, on the lines of [24, 89, 90, 91], various results obtained in the scalar ODEs setting could be extended to the Neumann problem for elliptic partial differential equations (on balls and looking for radial solutions at first, and via variational tools for arbitrary domains as well) and, maybe, for second order systems like (3). At last, it could be nice to study a supersublinear problem (with Dirichlet, Neumann and periodic boundary conditions) when the weight function changes its sign more and more times. Combining the estimates of Section 6.1 with the ones of Section 6.2, one could likely prove an high multiplicity result which seems to be completely new.

Notation and terminology

The plane $\mathbb{R}^2$ is endowed with the Euclidean scalar product $\langle z_1 | z_2 \rangle = x_1 y_1 + x_2 y_2$, for $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{R}^2$, and the corresponding Euclidean norm is denoted by $|z_1| = \sqrt{x_1^2 + y_1^2}$. A symplectic structure is defined as well, via the (standard) symplectic matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The origin $0_{\mathbb{R}^2} = (0,0)$ is simply denoted by 0 and by $\mathbb{R}^2_*$ we mean the punctured plane $\mathbb{R}^2 \setminus \{0\}$. In a similar way, for numeric sets $\mathbb{K} = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, the symbol $\mathbb{K}_*$ denotes $\mathbb{K} \setminus \{0_{\mathbb{K}}\}$. Finally, $\mathbb{K}^+ = \mathbb{K} \cap [0, +\infty[$ (resp., $\mathbb{K}^- = \mathbb{K} \cap]-\infty, 0]$) and $\mathbb{K}^+_* = \mathbb{K} \cap]0, +\infty[$ (resp., $\mathbb{K}^-_* = \mathbb{K} \cap]-\infty, 0[$).

We often use polar coordinates in the plane; that is to say, for $z \in \mathbb{R}^2_*$, we write $z = (\rho \cos\theta, \rho \sin\theta)$, where $\rho > 0$ and $\theta \in \mathbb{R}$. Sometimes, we also identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ (i.e., $z = (x,y) \in \mathbb{R}^2 \sim z = x + iy \in \mathbb{C}$), thus writing $z = \rho e^{i\theta}$. A crucial tool in this framework is the *rotation number* of a planar path, which is defined, for $z = (x,y) : [t_1, t_2] \to \mathbb{R}^2$ absolutely continuous and such that $z(t) \neq 0$ for every $t \in [t_1, t_2]$, as

$$\mathrm{Rot}\,(z(t); [t_1, t_2]) = \frac{1}{2\pi} \int_{t_1}^{t_2} \frac{\langle Jz'(t) | z(t) \rangle}{|z(t)|^2} dt = \frac{1}{2\pi} \int_{t_1}^{t_2} \frac{y(t)x'(t) - x(t)y'(t)}{x(t)^2 + y(t)^2} dt. \tag{8}$$

The rotation number is related to the polar coordinates in the sense that it gives an algebraic count of the (clockwise) angular displacement of the planar path $z(t)$ around the origin, in the time interval $[t_1, t_2]$. Precisely, if $z(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, with $\rho(t), \theta(t)$ absolutely continuous functions and $\rho(t) > 0$ (by the theory of covering spaces [94], such functions always exist), then it can be seen that

$$\mathrm{Rot}\,(z(t); [t_1, t_2]) = \frac{\theta(t_1) - \theta(t_2)}{2\pi}.$$

When $z(t_1) = z(t_2)$ (that is, $z(t)$ is a closed path), then $\mathrm{Rot}\,(z(t); [t_1, t_2])$ is an integer number, which characterizes the homotopy class of the loop $t \mapsto z(t)$ in the fundamental group $\pi_1(\mathbb{R}^2_*) \simeq \mathbb{Z}$ and is often called "winding number". In particular, we say that a loop $z(t)$ winds around the origin (resp., does not wind around the origin) if $\mathrm{Rot}\,(z(t); [t_1, t_2]) \neq 0$ (resp., $\mathrm{Rot}\,(z(t); [t_1, t_2]) = 0$).

We use quite standard notation for function spaces. For instance, C^k (with $k \in \mathbb{N}$) is the space of continuous functions with continuous derivatives up to the k-th order, L^p (with $1 \leq p \leq \infty$) is the Lebesgue space and $W^{k,p}$ (with $k \in \mathbb{N}$ and $1 \leq p \leq \infty$) is the Sobolev space (the domain and codomain of the functions being clear from the context, or explicitly indicated). When we write $C^k_T, L^p_T, W^{k,p}_T$ we mean that the functions are defined on $\mathbb{R}$, T-periodic and (locally) $C^k, L^p, W^{k,p}$. Moreover, we say that a function $\Xi : \mathbb{R} \times \mathcal{O} \to \mathbb{R}^N$ ($N \in \mathbb{N}_*$ and $\mathcal{O} \subset \mathbb{R}^N$ an open subset), T-periodic in the first variable, is L^p-Carathéodory ($1 \le p \le \infty$) if $\Xi(\cdot, \xi)$ is measurable for every $z \in \mathcal{O}$, $\Xi(t, \cdot)$ is continuous for almost every $t \in [0, T]$ and, for every $\mathcal{K} \subset \mathcal{O}$ compact, there exists $h_{\mathcal{K}} \in L^p_T$ such that $|f(t, \xi)| \le h_{\mathcal{K}}(t)$ for almost every $t \in [0, T]$ and every $\xi \in \mathcal{K}$.

We consider first order planar differential systems like

$$Jz' = Z(t, z), \qquad z = (x, y) \in \mathbb{R}^2, \tag{9}$$

with $Z : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ a function which is T-periodic in the first variable and (at least) L^1-Carathéodory. Accordingly, we mean the solutions in the generalized (Carathéodory) sense, i.e., locally absolutely continuous functions (on an interval) solving the differential equation almost everywhere. Of course, when $Z(t, z)$ is continuous on both the variables, every generalized solution is of class C^1 and solves the differential equation for every t (that is, it is a classical solution). We say that system (9) is Hamiltonian if there exists a function $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, T-periodic in the first variable and differentiable in the second one with $\nabla_z H(t, z)$ (at least) L^1-Carathéodory, such that $Z(t, z) = \nabla_z H(t, z)$. When we refer to the second order scalar differential equation

$$u'' + v(t, u) = 0, \qquad u \in \mathbb{R}, \tag{10}$$

with $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a function which is T-periodic in the first variable and (at least) L^1-Carathéodory, we implicitly mean that it is written as the first order planar differential system

$$x' = y, \qquad y' = -v(t, x), \tag{11}$$

which is Hamiltonian, with $H(t, x, y) = \frac{1}{2}y^2 + \int_0^x v(t, \xi)\, d\xi$. Generalized solutions $z(t) = (x(t), y(t))$ to the first order system (11) correspond to generalized solutions $u(t) = x(t)$ to the second order equation (10), i.e., functions of class C^1 with a locally absolutely continuous second derivative, solving the equation almost everywhere.

We are mainly interested in the search for solutions to (9) having period which is an integer multiple of T. Precisely, we say that a solution $z(t)$ is *harmonic* if is T periodic and we say that it is a *subharmonic of order* k (with $k \ge 2$ an integer) if it is kT-periodic, but not lT-periodic for any integer $l = 1, \dots, k-1$ (namely, kT is the minimal period in the class of the integer multiples of T). We recall that, in some cases, it is possible to show that a subharmonic solution of order k has kT has minimal period; for instance, this is the case if the following condition, first introduced in [130], is satisfied:

If $z(t)$ is a periodic function with minimal period qT, for q rational, and $Z(t, z(t))$ is a periodic function with minimal period qT, then q is necessarily an integer.

Such a condition can be seen as an essential time dependence for the nonlinearity $Z(t, z)$, and it is satisfied when it has some particular structures (for instance, when $Z(t, x, y) = (q(t)g(x), y)$ - corresponding to $u'' + q(t)g(u) = 0$ - with $q(t) > 0$ having minimal period T).

Finally, some notation for integer part type functions is useful. Precisely, for $a > 0$, with the symbol $\lfloor a \rfloor$ we mean the greatest integer less than or equal to a, while by $\lceil a \rceil$ we denote the least integer greater than or equal to a. Moreover, we set

$$\mathcal{E}^-(a) = \begin{cases} \lfloor a \rfloor & \text{if } a \notin \mathbb{N} \\ a-1 & \text{if } a \in \mathbb{N}, \end{cases} \qquad \mathcal{E}^+(a) = \begin{cases} \lceil a \rceil & \text{if } a \notin \mathbb{N} \\ a+1 & \text{if } a \in \mathbb{N}, \end{cases}$$

so that $\mathcal{E}^-(a) \leq \lfloor a \rfloor \leq a \leq \lceil a \rceil \leq \mathcal{E}^+(a)$.

Contents

Chapter 1

Preliminaries

This is a preliminary chapter, collecting the main abstract topological results which are employed in this thesis to prove the existence of periodic solutions to planar Hamiltonian systems.

Section 1.1 is devoted to the Poincaré-Birkhoff fixed point theorem, which is used in Chapters 2, 3 and 4. Section 1.2 deals with a recent modified version of the Poincaré-Birkhoff fixed point theorem, employed in Chapter 5. Finally, Section 1.3 collects some results about chaotic dynamics and describes a topological method - the "stretching along the paths" method, SAP for brevity - leading to periodic points and complex dynamics for a planar map: this is the main tool in Chapter 6. The material of the chapter is standard, with the partial exception of Section 1.2, where a detailed exposition, coming from [28], of the application of the modified Poincaré-Birkhoff theorem to Hamiltonian systems which are asymptotically linear at zero is given.

1.1 The "classical" Poincaré-Birkhoff Theorem

The Poincaré-Birkhoff fixed point theorem is a classical result of Planar Topology, ensuring the existence of two fixed points for an area-preserving "twist" homeomorphism of a closed annulus. It was conjectured, and proved in some special cases, by Poincaré [144] in 1912, motivated by the analysis of the restricted three body problem. Later, in 1913, Birkhoff provided a complete proof of the existence of one fixed point [22] and finally, in 1926, of the second one [23]. A detailed and convincing checking of Birkhoff's original arguments - based on an ingenious application of the index of a vector field along a curve - was given in the expository paper by Brown and Neumann [35].

Here is a modern statement of the classical version of the theorem. For $0 < r_i < r_o$, we denote by $\mathcal{A}[r_i, r_o] = \{z \in \mathbb{R}^2 \mid r_i \leq |z| \leq r_o\}$ the closed annulus of radii r_i, r_o ("i" means "inner" and "o" means "outer"). We also define the covering projection

$$\Pi : \mathbb{R}^+_* \times \mathbb{R} \ni (\rho, \theta) \mapsto \sqrt{2\rho}(\cos\theta, \sin\theta) \in \mathbb{R}^2_*, \tag{1.1}$$

making $\mathbb{R}^+_* \times \mathbb{R}$ a (universal) covering space of $\mathbb{R}^2_*$, and we recall that, given a map Ψ :

$\mathcal{D} \subset \mathbb{R}^2_* \to \mathbb{R}^2_*$, the map $\widetilde{\Psi} : \Pi^{-1}(\mathcal{D}) \subset \mathbb{R}^+_* \times \mathbb{R} \to \mathbb{R}^+_* \times \mathbb{R}$ is said to be a lifting of Ψ if $\Pi \circ \widetilde{\Psi} = \Psi \circ \Pi$.

Theorem 1.1.1. *Let $\Psi : \mathcal{A}[r_i, r_o] \to \mathcal{A}[r_i, r_o]$ be an area-preserving homeomorphism. Suppose that Ψ has a lifting $\widetilde{\Psi} : [r_i^2/2, r_o^2/2] \times \mathbb{R} \to [r_i^2/2, r_o^2/2] \times \mathbb{R}$ of the form*

$$\widetilde{\Psi}(\rho, \theta) = (R(\rho, \theta), \theta + \gamma(\rho, \theta)), \tag{1.2}$$

being $R(\rho, \theta), \gamma(\rho, \theta)$ continuous functions, 2π-periodic in the second variable. Finally, assume that:

(i) $R(r_i^2/2, \theta) = r_i^2/2$ and $R(r_o^2/2, \theta) = r_o^2/2$ for every $\theta \in \mathbb{R}$;

(ii) $\gamma(r_i^2/2, \theta)\gamma(r_o^2/2, \theta) < 0$ for every $\theta \in \mathbb{R}$.

Then $\widetilde{\Psi}$ has at least two fixed points $(\rho^{(1)}, \theta^{(1)}), (\rho^{(2)}, \theta^{(2)}) \in]r_i^2/2, r_o^2/2[\times \mathbb{R}$ such that

$$(\rho^{(2)}, \theta^{(2)}) - (\rho^{(1)}, \theta^{(1)}) \neq (0, 2k\pi), \quad \text{for every } k \in \mathbb{Z}. \tag{1.3}$$

As it is clear, condition (1.3) ensures that $z_i = \Pi(\rho^{(1)}, \theta^{(1)})$, $z_2 = \Pi(\rho^{(2)}, \theta^{(2)})$ are distinct fixed points of Ψ, belonging to the open annulus of radii r_i, r_o. Some comments about the statement of the theorem are now in order; see also Figure 1.1.

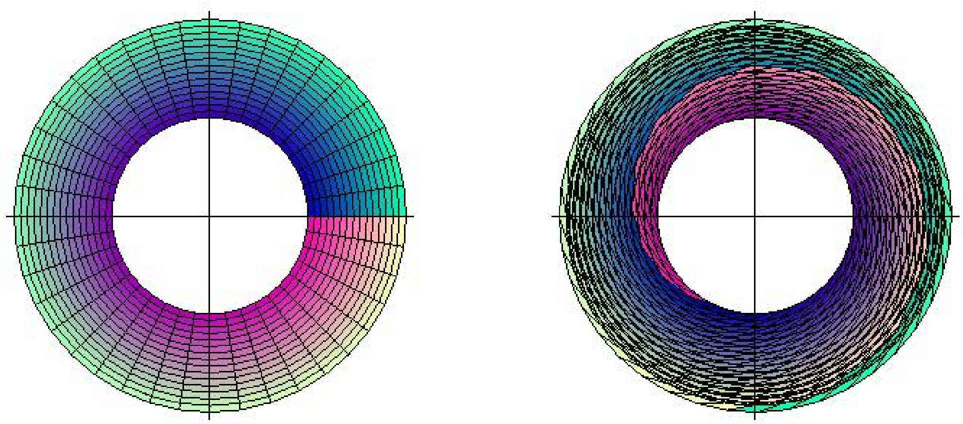


Figure 1.1: A numerical simulation describing the classical version of the Poincaré-Birkhoff fixed point theorem, Theorem 1.1.1. In the figure, different colors are used to highlight the "twist" effect of the area-preserving homeomorphism $\Psi(x, y) = (x\cos(2(x^2+y^2)) + y\sin(2(x^2+y^2)), -x\sin(2(x^2+y^2)) + y\cos(2(x^2+y^2)))$ on the annulus $\mathcal{A}[1, 2]$. Such a map admits the lifting $\widetilde{\Psi}(\rho, \theta) = (\rho, \theta + 2\pi - 4\rho)$, which satisfies all the assumptions of the Poincaré-Birkhoff theorem. In our example, in particular, each point of the annulus is moved by Ψ to a point on the same circumference, the angular displacement being constant on each circumference and monotonically depending on the distance from the origin. It is worth noticing that the map Ψ can be realized as the Poincaré operator, at time $T = 1/2$, of the autonomous planar Hamiltonian system $x' = 4y(x^2+y^2), y' = -4x(x^2+y^2)$ (that is, $Jz' = \nabla H(z)$ for $H(x, y) = (x^2+y^2)^2$).

- The property of area-preservation is meant in the sense that $\mu(\mathcal{B}) = \mu(\Psi(\mathcal{B}))$ for every Borel set $\mathcal{B} \subset \mathcal{A}[r_i, r_o]$, being μ the standard Lebesgue measure in the plane. Notice that, when Ψ is a diffeomorphism, Ψ is area-preserving if and only if $|\det \Psi'| = 1$.

- Since $[r_i^2/2, r_o^2/2] \times \mathbb{R}$ is simply connected, the theory of covering spaces [94] ensures that a lifting $\widetilde{\Psi}$ of Ψ always exists. Moreover, $\widetilde{\Psi}$ satisfies
$$\widetilde{\Psi}(\rho, \theta + 2\pi) = \widetilde{\Psi}(\rho, \theta) + (0, 2k_\Psi \pi),$$
being k_Ψ an integer which depends only on the homotopy class of Ψ. For general (continuous) maps on an annulus, k_Ψ could take any value; here, since Ψ is an homeomorphism, k_Ψ can be only ± 1. Assumption (1.2) requires eventually that $k_\Psi = 1$: that is, Ψ is homotopic to the identity.

- Conditions (i) and (ii) concern the behavior of Ψ at the boundary of the annulus. Precisely, (i) says that both the inner and the outer boundary are invariant for Ψ, while (ii) (the so called *twist condition*) gives a rigorous meaning to the informal expression "the boundaries are rotated in opposite angular directions". Notice that it is really necessary to work with a lifting of Ψ, since the twist condition is meaningless for maps on the annulus; on the other hand, we obtain not only fixed points of Ψ, but fixed points of any lifting satisfying the twist condition (this will be a crucial fact in the applications, see the discussion after Theorem 1.1.3). We remark that, in slightly different contexts (often related to the theories of Kolmogorov-Arnold-Moser and Aubry-Mather, see [92]), it is named "twist condition" the assumption
$$\frac{\partial}{\partial \rho}\gamma(\rho, \theta) > 0, \tag{1.4}$$
i.e., the monotonicity of the twist (which is well-defined also without invoking the lifting to the covering space). To avoid misunderstandings, it can be useful to call "boundary twist condition" the requirement (ii) of Theorem 1.1.1. Notice that the boundary twist condition (ii) and (1.4) are mutually independent. However, when both of them are assumed, the proof of Theorem 1.1.1 is very simple (see, for instance, [51, Proposition 1]).

- A compact way to express the assumptions of Theorem 1.1.1 is obtained by requiring that Ψ is a twist homeomorphism isotopic (i.e., homotopic via homeomorphisms) to the identity. In such a case, indeed, Ψ is homotopic to the identity and preserves each boundary component. Another property of Ψ which could be taken into account is the preservation of the orientation. Indeed, whenever Ψ is homotopic to the identity and preserves the boundaries, then it is orientation-preserving; conversely, an orientation-preserving homeomorphism which leaves each boundary component invariant satisfies (1.2). Recall in particular that, if Ψ is a diffeomorphism, Ψ is orientation-preserving if and only if $\det \Psi' > 0$. Hence, when Ψ is smooth and satisfies the assumptions of Theorem 1.1.1, then $\det \Psi' = 1$. Maps fulfilling this latter property are called symplectic.

- The map $\widetilde{\Psi}_P : [r_i, r_o] \times \mathbb{R} \to [r_i, r_o] \times \mathbb{R}$ defined by $\widetilde{\Psi}_P(\rho, \theta) = (\sqrt{2R(\rho^2/2, \theta)}, \theta + \gamma(\rho^2/2, \theta))$ is a lifting of Ψ, relatively to the covering projection $P(\rho, \theta) = \rho(\cos\theta, \sin\theta)$.

Of course, $\widetilde{\Psi}_P$ has two fixed points, as well, so that Theorem 1.1.1 holds true also if the (more standard) covering projection P is considered. However, it can be useful to express the theorem using Π because, being $\det \Pi' = 1$, the lifting $\widetilde{\Psi}$ is area-preserving too.

Great efforts have been made to generalize Theorem 1.1.1, both in the direction of replacing the area-preservation assumption by a more topological condition (see, among others, [23, 46, 80, 82, 95, 164]) and in the direction of relaxing the requirement of the invariance of the annulus [23, 61, 62, 80, 82, 99, 100, 150]. The history of such generalizations and developments is very interesting and some "delicate" versions of the twist theorem have been settled only very recently; we refer to [51, 76] for a more complete discussion.

As for the application of the theorem to (nonautonomous) planar Hamiltonian systems, the invariance of the annulus is the major drawback. We state here the generalized version which will be used in this thesis; it is due to Rebelo [150, Corollary 2]. In the statement, we will denote by $\mathcal{D}(\Gamma)$ the open bounded region delimited by a Jordan curve $\Gamma \subset \mathbb{R}^2$ (according to the Jordan theorem). Moreover, we will say that Γ is *strictly star-shaped* around the origin if $0 \in \mathcal{D}(\Gamma)$ and every ray emanating from the origin intersects Γ exactly once[1].

Theorem 1.1.2. *Let $\Gamma_i, \Gamma_o \subset \mathbb{R}^2$ be strictly star-shaped Jordan curves around the origin, with $\overline{\mathcal{D}(\Gamma_i)} \subset \mathcal{D}(\Gamma_o)$, and let $\Psi : \overline{\mathcal{D}(\Gamma_o)} \to \Psi(\overline{\mathcal{D}(\Gamma_o)})$ be an area-preserving homeomorphism with $\Psi(0) = 0$. Suppose that $\Psi|_{\overline{\mathcal{D}(\Gamma_o)}\setminus\{0\}}$ has a lifting $\widetilde{\Psi} : \Pi^{-1}(\overline{\mathcal{D}(\Gamma_o)}\setminus\{0\}) \to \Pi^{-1}(\Psi(\overline{\mathcal{D}(\Gamma_o)})\setminus \{0\})$ of the form (1.2), being $R(\rho,\theta), \gamma(\rho,\theta)$ continuous functions, 2π-periodic in the second variable. Finally, assume that*

$$\gamma(\rho,\theta) > 0, \qquad \textit{for every } (\rho,\theta) \in \Pi^{-1}(\Gamma_i),$$

and

$$\gamma(\rho,\theta) < 0, \qquad \textit{for every } (\rho,\theta) \in \Pi^{-1}(\Gamma_o),$$

(or conversely). Then $\widetilde{\Psi}$ has at least two fixed points $(\rho^{(1)},\theta^{(1)}), (\rho^{(2)},\theta^{(2)}) \in \Pi^{-1}(\mathcal{D}(\Gamma_o) \setminus \overline{\mathcal{D}(\Gamma_i)})$, such that (1.3) holds true.

Remark 1.1.1. According to [150, Remark 2] and [62, Lemma 1], the condition $\Psi(0) = 0$ can be replaced by $0 \in \Psi(\mathcal{D}(\Gamma_i))$. We have chosen to deal with this simpler version because, when applying Theorem 1.1.2 to the existence of periodic solutions to Hamiltonian systems, the assumption $\Psi(0) = 0$ just requires the "a priori" existence of a periodic solution, a fact which will be always assumed in the thesis (compare with (5) of the Introduction).

Theorem 1.1.2 is reminiscent of the version of the Poincaré-Birkhoff theorem given by W.-Y. Ding [62], which, however, does not require the strictly star-shapedness of the outer boundary: this is a very delicate point. Indeed, at first it was shown in [119] that the assumption of a strictly star-shaped inner boundary is not eliminable. More recently, [108] provided an example showing that the outer boundary has to be star-shaped too, so that the result stated in [62] does not seem to be correct. We point out that such a mistake probably goes back to the result by Jacobowitz [99] (dealing with an annulus whose inner boundary

[1]We warn the reader that some authors name "star-shaped" the curves satisfying such a property, with the adjective "strictly" referred to the smooth case, when the intersection is, in addition, non-tranverse.

degenerates into a point), which is invoked in the proof by W.-Y. Ding. On the other hand, the proof of Theorem 1.1.2 given in [150] is based on a reduction to the classical version of the twist theorem. Theorem 1.1.2, in the case when Γ_i, Γ_o are circumferences of center the origin, is also contained in [61].

We finally state the semi-abstract result, based on Theorem 1.1.2 together with the concept of rotation number of a planar path (as defined in (8) of the introductory section "Notation and Terminology"), which is our main tool to prove the existence of periodic solutions (winding around the origin) to unforced planar Hamiltonian systems

$$Jz' = \nabla_z H(t,z), \tag{1.5}$$

being $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ a function which is T-periodic in the first variable and differentiable in the second one, with $\nabla_z H(t,z)$ an L^1-Carathéodory function such that $\nabla_z H(t,0) \equiv 0$. Dealing with (1.5), we also assume that *the uniqueness and the global continuability for the solutions to the associated Cauchy problems are ensured* and denote by $z(\cdot;\bar{z})$ the unique solution to (1.5) such that $z(0;\bar{z}) = \bar{z}$.

Theorem 1.1.3. *Let $k \in \mathbb{N}_*$ and $j \in \mathbb{Z}$. Suppose that there exist $\Gamma_i, \Gamma_o \subset \mathbb{R}^2$, strictly star-shaped Jordan curves around the origin, with $\overline{\mathcal{D}(\Gamma_i)} \subset \mathcal{D}(\Gamma_o)$, such that*

$$\mathrm{Rot}\,(z(t;\bar{z});[0,kT]) < j, \qquad \textit{for every } \bar{z} \in \Gamma_i,$$

and

$$\mathrm{Rot}\,(z(t;\bar{z});[0,kT]) > j, \qquad \textit{for every } \bar{z} \in \Gamma_o,$$

(or conversely). Then there exist two kT-periodic solutions $z^{(1)}_{k,j}(t), z^{(2)}_{k,j}(t)$ to (1.5)*, with*

$$z^{(1)}_{k,j}(0), z^{(2)}_{k,j}(0) \in \mathcal{D}(\Gamma_o) \setminus \overline{\mathcal{D}(\Gamma_i)},$$

such that

$$\mathrm{Rot}\,(z^{(1)}_{k,j}(t);[0,kT]) = \mathrm{Rot}\,(z^{(2)}_{k,j}(t);[0,kT]) = j. \tag{1.6}$$

Observe that the condition $\nabla_z H(t,0) \equiv 0$ implies that, if $\bar{z} \neq 0$, then $z(t;\bar{z}) \neq 0$ for every $t \in \mathbb{R}$, so the rotation numbers appearing in the statement of the theorem are well defined. Relation (1.6), of course, has to be meant as an information about the nodal properties of the periodic solutions found (in particular, we recall that, when (1.5) comes from a second order scalar equation $u'' + v(t,u) = 0$, kT-periodic solutions to (1.5) satisfying (1.6) correspond to kT-periodic solution of the scalar equation having exactly $2j$ zeros in $[0,kT[$; see, for instance, [34, Lemma 3.8])) and it is the key point to get multiplicity of solutions. Indeed, when - in the setting of Theorem 1.1.3 - there exist $j_{\min}, j_{\max} \in \mathbb{Z}$, with $j_{\min} < j_{\max}$, such that

$$\mathrm{Rot}\,(z(t;\bar{z});[0,kT]) < j_{\min}, \qquad \text{for every } \bar{z} \in \Gamma_i,$$

and

$$\mathrm{Rot}\,(z(t;\bar{z});[0,kT]) > j_{\max}, \qquad \text{for every } \bar{z} \in \Gamma_o,$$

(or conversely), then one can apply more and more times Theorem 1.1.3 to find $j_{\max} - j_{\min} + 1$ pairs of kT-periodic solutions, with rotation number equal to $j_{\min}, j_{\min}+1, \ldots, j_{\max}-1, j_{\max}$, respectively. Therefore:

the larger the "gap" between the rotation numbers of the solutions departing from the inner and the outer boundary, the greater the number of the periodic solutions obtained.

Initial values, at time $t = 0$, of such kT-periodic solutions will be provided, via Theorem 1.1.2, as fixed points of the k-th iterate of the Poincaré map Ψ associated with (1.5) (see the sketch of the proof below for some details). For $k = 1$, Theorem 1.1.3 gives the existence of T-periodic solutions (i.e., *harmonic solutions*) to the planar Hamiltonian system (1.5). On the other hand, when $k > 1$, it is easy to see that (see, for instance, [59, pp. 523-524]), whenever k, j are relatively prime integers (namely, their greatest common divisor is 1), then the kT-periodic solutions $z^{(1)}_{k,j}(t)$, $z^{(2)}_{k,j}(t)$ are not lT-periodic for any integer $l = 1, \dots, k-1$, so that they are *subharmonic solutions of order* k to (1.5). Notice that subharmonic solutions of order k correspond to fixed points of Ψ^k which are not fixed points of Ψ^l for any $l = 1, \dots, k-1$. We finally remark that, as pointed out in the proof of [152, Theorem 5], it is possible to show that the subharmonic solutions $z^{(1)}_{k,j}(t)$, $z^{(2)}_{k,j}(t)$ do not belong to the same periodicity class, i.e., $z^{(1)}_{k,j}(\cdot) \not\equiv z^{(2)}_{k,j}(\cdot + lT)$ for every integer $l = 1, \dots, k-1$. This means that the orbits of $z_1 = z^{(1)}_{k,j}(0)$ and $z_2 = z^{(2)}_{k,j}(0)$, namely $\mathcal{O}_1 = \{z_1, \Psi(z_1), \dots, \Psi^{k-1}(z_2)\}$ and $\mathcal{O}_2 = \{z_2, \Psi(z_2), \dots, \Psi^{k-1}(z_2)\}$, are disjoint.

Sketch of the proof. Let us consider the Poincaré map $\Psi : \mathbb{R}^2 \ni \bar{z} \mapsto z(T; \bar{z})$. The standard theory of ODEs ensures that Ψ is a global homeomorphism of the plane onto itself; moreover, since $\nabla_z H(t, 0) \equiv 0$, it turns out that $\Psi(0) = 0$. We also have that Ψ is area-preserving: indeed, when $H(t, z)$ is of class C^2 in the second variable, this just follows from the classical Liouville's theorem, while the general case can be treated by approximation (see, as a guide, [132, Remark 2.35]). Notice that the same properties hold for the k-th iterate of Ψ, as well. Define $\widetilde{\Psi}_{k,j} : \mathbb{R}^+_* \times \mathbb{R} \ni (\rho, \theta) \mapsto (R_k(\rho, \theta), \theta + \gamma_{k,j}(\rho, \theta))$, where

$$R_k(\rho, \theta) = |\Psi^k(\Pi(\rho, \theta))|^2/2, \qquad \gamma_{k,j}(\rho, \theta) = 2\pi(j - \mathrm{Rot}(z(t; \Pi(\rho, \theta)); [0, kT])).$$

It is easy to see that $\widetilde{\Psi}_{k,j}$ is a lifting of $\Psi^k|_{\mathbb{R}^2_*} : \mathbb{R}^2_* \to \mathbb{R}^2_*$; moreover, if (ρ^*, θ^*) is a fixed point of $\widetilde{\Psi}_{k,j}$, then $z(t; \Pi(\rho^*, \theta^*))$ is a kT-periodic solution to (1.5) with rotation number equal to j. Since

$$\gamma_{k,j}(\rho, \theta) > 0, \quad \text{for every } (\rho, \theta) \in \Pi^{-1}(\Gamma_i),$$

and

$$\gamma_{k,j}(\rho, \theta) < 0, \quad \text{for every } (\rho, \theta) \in \Pi^{-1}(\Gamma_o),$$

the thesis follows plainly from Theorem 1.1.2 □

Remark 1.1.2. As it is clear, Theorem 1.1.3 is based on the concept of rotation number of a curve around the origin, which deeply relies on the topology of the punctured plane $\mathbb{R}^2_*$. A partial extension of such a tool to the higher dimensional case is represented by the celebrated Conley-Zehnder index [50] (also named Maslov index by some authors; see, for instance, the books [2, 113] and the references therein). Based on the Maslov index, together with Morse theory, partial generalizations of Theorem 1.1.3 to higher dimension have been obtained (see, among others, [50, 111, 112]), despite the 2-dimensional nature of the Poincaré-Birkhoff theorem. However, the main difference is that such results do not provide, in general, a greater number of periodic solutions with a larger "gap" between the Maslov indexes at zero and at infinity. For further comments, we refer to the introductions of [111, 118].

1.2 A “modified” Poincaré-Birkhoff Theorem

In this section we present a “modified” version of the Poincaré-Birkhoff theorem, proved by Margheri, Rebelo and Zanolin [118] (see also [41]). The main difference compared to the “classical” version lies in the fact that the twist condition is considerably weakened (at the inner boundary); as a consequence, only one fixed point is provided.

Here is the statement of the result, dealing with a map defined on a vertical strip contained in $\mathcal{H}^+ = \{(\rho,\theta) \in \mathbb{R}^2 \mid \rho \geq 0\}$.

Theorem 1.2.1. *Let* $\widetilde{\Psi} : [0,R] \times \mathbb{R} \subset \mathcal{H}^+ \to \widetilde{\Psi}([0,R] \times \mathbb{R}) \subset \mathcal{H}^+$ *be an area-preserving homeomorphism of the form* (1.2)*, being* $R(\rho,\theta), \gamma(\rho,\theta)$ *continuous functions,* 2π*-periodic in the second variable, and such that* $R(0,\theta) = 0$ *for every* $\theta \in \mathbb{R}$*. Finally, assume that there exists* $\theta^* \in \mathbb{R}$ *such that*

$$\gamma(0,\theta^*) > 0$$

and

$$\gamma(R,\theta) < 0, \qquad \textit{for every } \theta \in \mathbb{R}.$$

Then $\widetilde{\Psi}$ *has at least one fixed point in* $]0,R[\,\times\mathbb{R}$.

We are now going to derive, on the lines of Theorem 1.1.3, a corollary of Theorem 1.2.1 dealing with periodic solutions to the planar Hamiltonian system (1.5). Notice that, in the proof of Theorem 1.1.3, we lifted - with respect to the covering projection (1.1) - the restriction $\Psi^k|_{\mathbb{R}^2_*} : \mathbb{R}^2_* \to \mathbb{R}^2_*$ (being Ψ the Poincaré map associated with (1.5)) to an homeomorphism defined on the open half-space $\{(\rho,\theta) \in \mathbb{R}^2 \mid \rho > 0\}$. On the other hand, in order to apply Theorem 1.2.1, we need a map defined on (a strip of) the closed half-space $\mathcal{H}^+$, which can not be obtained directly in this manner. From a geometrical point of view, this corresponds to working in a planar annulus whose inner boundary degenerates into a single point, a situation which has been the source of some inaccuracies in previous versions of the Poincaré-Birkhoff theorem (see the brief discussion after Theorem 1.1.2). Here, we propose to show, with some details, how to enter in the setting of Theorem 1.2.1 when the following conditions on the Hamiltonian are fulfilled:

- $(C1)$ $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function,* T*-periodic in the first variable, such that* $\nabla_z H(t,z)$ *exists and is continuous on* $\mathbb{R} \times \mathbb{R}^2$*. Without loss of generality, moreover, we require* $H(t,0) \equiv 0$*;*

- $(C2)$ $\nabla_z H(t,0) \equiv 0$ *and there exists a continuous and* T*-periodic function* $B : \mathbb{R} \to \mathcal{L}_s(\mathbb{R}^2)$ *(we denote here by* $\mathcal{L}_s(\mathbb{R}^2)$ *the vector space of real* 2×2 *symmetric matrices) such that*

$$\lim_{z\to 0} \frac{\nabla_z H(t,z) - B(t)z}{|z|} = 0, \qquad \textit{uniformly in } t \in [0,T]. \tag{1.7}$$

As usual, we also assume that *the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (1.5) *are ensured* and denote by $z(\cdot;\bar{z})$ the unique solution to (1.5) such that $z(0;\bar{z}) = \bar{z}$.

Let us define the Hamiltonian function

$$\widetilde{H}(t,\rho,\theta) = \operatorname{sgn}(\rho) H(t, \sqrt{2|\rho|}e^{i\theta}),$$

for $(t,\theta,\rho) \in \mathbb{R}\times\mathbb{R}^2$. Here, and in the following, we identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, so that we write $e^{i\theta} \in \mathbb{C}$ to denote the vector $(\cos\theta, \sin\theta) \in \mathbb{R}^2$.

Lemma 1.2.1. *The function $\widetilde{H}(t,\rho,\theta)$ is continuous, T-periodic in the first variable, and the partial derivatives $\widetilde{H}_\rho(t,\rho,\theta), \widetilde{H}_\theta(t,\rho,\theta)$ exist and are continuous on $\mathbb{R}\times\mathbb{R}^2$.*

Proof. The continuity, and T-periodicity in the first variable, of $\widetilde{H}$, as well as the existence and continuity of $\widetilde{H}_\theta$, just follow from the assumptions on H (recall that we have assumed $H(t,0)\equiv 0$). For the same reason, $\widetilde{H}_\rho(t,\rho,\theta)$ exists and is continuous for $\rho \neq 0$. From (1.7), moreover, we deduce that

$$\widetilde{H}_\rho(t,\rho,\theta) = \frac{1}{\sqrt{2|\rho|}}\langle \nabla_z H(t,\sqrt{2|\rho|}e^{i\theta}) \mid e^{i\theta}\rangle \to \langle B(t)e^{i\theta} \mid e^{i\theta}\rangle$$

for $\rho \to 0$, uniformly in $(t,\theta)\in\mathbb{R}^2$. At this point, l'Hopital theorem implies that $\widetilde{H}_\rho(t,0,\theta)$ exists (and $\widetilde{H}_\rho(t,\cdot,\theta)$ is continuous by construction). The continuity of $\widetilde{H}_\rho$ with respect to the full variable (t,ρ,θ) is easily seen to be satisfied, too. □

We can thus consider the associated Hamiltonian system

$$\begin{cases} \rho' = \widetilde{H}_\theta(t,\rho,\theta) = \operatorname{sgn}(\rho)\sqrt{2|\rho|}\langle \nabla_z H(t,\sqrt{2|\rho|}e^{i\theta}) \mid Je^{i\theta}\rangle \\ \theta' = -\widetilde{H}_\rho(t,\rho,\theta) = \begin{cases} -\frac{1}{\sqrt{2|\rho|}}\langle \nabla_z H(t,\sqrt{2|\rho|}e^{i\theta}) \mid e^{i\theta}\rangle & \text{if } \rho\neq 0 \\ -\langle B(t)e^{i\theta} \mid e^{i\theta}\rangle & \text{if } \rho = 0, \end{cases} \end{cases} \tag{1.8}$$

for $(t,\rho,\theta)\in\mathbb{R}\times\mathbb{R}^2$.

Lemma 1.2.2. *The uniqueness and the global continuability for the solutions to the initial value problems associated with* (1.8) *are guaranteed.*

Proof. Fix $t_0 \in \mathbb{R}$ and consider the Cauchy problem $(\rho(t_0),\theta(t_0)) = (\rho,\theta)$.
If $\rho = 0$, a standard application of Gronwall's lemma, using (1.7), implies that $\rho(t)\equiv 0$ (indeed, from (1.7) we deduce that for every $m>0$ there exists $C_m>0$ such that $|\nabla_z H(t,z)| \leq C_m|z|$ for $|z|\leq m$). Since $\theta'(t) = -\langle B(t)e^{i\theta(t)} \mid e^{i\theta(t)}\rangle$, the global Lipschitz continuity of the right-hand side implies that $\theta(t)$ is uniquely globally defined, too.
If $\rho\neq 0$, simple calculations show that the function $z(t) = \sqrt{2|\rho(t)|}e^{i\theta(t)}$ is a local solution to (1.5), with $z(t_0) = \sqrt{2|\rho(t_0)|}e^{i\theta(t_0)} \in \mathbb{R}^2_*$. Indeed, from (1.8) we obtain

$$Jz'(t) = \langle \nabla_z H(t,z(t))|Je^{i\theta(t)}\rangle Je^{i\theta(t)} + \langle \nabla_z H(t,z(t))|e^{i\theta(t)}\rangle e^{i\theta(t)}$$

and using the fact that, for every t, $\{e^{i\theta(t)}, Je^{i\theta(t)}\}$ is an orthonormal bases of $\mathbb{R}^2$, we conclude. Since the initial value problems associated with (1.8) have a unique solution and the map $(\rho,\theta)\in\mathbb{R}_*\times\mathbb{R}\mapsto\sqrt{2|\rho|}e^{i\theta}\in\mathbb{R}^2_*$ is a local homeomorphism, we deduce that $\theta(t),\rho(t)$ are locally uniquely defined. Moreover, since $z(t)$ can be globally extended, never reaching the origin, we deduce that $\theta(t),\rho(t)$ can be globally extended too. □

Let us now denote by $(r(\cdot;\rho,\theta),\varphi(\cdot;\rho,\theta))$ the unique solution to (1.8) such that

$$(r(0;\rho,\theta),\varphi(0;\rho,\theta)) = (\rho,\theta).$$

Observe that, along the proof of Lemma 1.2.2, we have showed that

$$z(t;\sqrt{2|\rho|}e^{i\theta}) = \sqrt{2|r(t;\rho,\theta)|}e^{i\varphi(t;\rho,\theta)}, \quad \text{for every } \rho \neq 0. \tag{1.9}$$

Moreover, with similar arguments it is possible to see that, denoting by $z_B(\cdot;\bar{z})$ the unique solution to the linear Hamiltonian system $Jz' = B(t)z$ with $z_B(0;\bar{z}) = \bar{z}$, it holds that

$$z_B(t;e^{i\theta}) = \exp\left(\int_0^t \langle B(s)e^{i\varphi(s;0,\theta)} | Je^{i\varphi(s;0,\theta)}\rangle\, ds\right) e^{i\varphi(t;0,\theta)} \tag{1.10}$$

We are now almost in a position to conclude. Indeed, let us define the Poincaré operator associated with (1.8), that is, $\widetilde{\Psi} : \mathbb{R}^2 \ni (\rho,\theta) \mapsto (r(T;\rho,\theta),\varphi(T;\rho,\theta))$. The following properties hold true.

- $\widetilde{\Psi}$ is an area-preserving (by Liouville's theorem) homeomorphism of the plane; moreover, in view of the 2π-periodicity of $\widetilde{H}(t,\rho,\cdot)$, it has the form (1.2);
- in view of Lemma 1.2.2, $r(t;0,\theta) \equiv 0$ and $r(t;\rho,\theta) > 0$ for every t whenever $\rho > 0$ (so that $\widetilde{\Psi}(\mathcal{H}^+) \subset \mathcal{H}^+$).

The same structural conditions, moreover, are satisfied by the maps

$$\widetilde{\Psi}_{k,j} : \mathbb{R}^2 \ni (\rho,\theta) \mapsto (r(kT;\rho,\theta),\varphi(kT;\rho,\theta) + 2\pi j),$$

for k, j integer numbers. In view of (1.9) and (1.10), and recalling Definition 8, we have, for $\rho > 0$,

$$\widetilde{\Psi}_{k,j}(\rho,\theta) = (r(kT;\rho,\theta), \theta + 2\pi(j - \mathrm{Rot}\,(z(t;\sqrt{2\rho}e^{i\theta});[0,kT]))),$$

and

$$\widetilde{\Psi}_{k,j}(0,\theta) = (0, \theta + 2\pi(j - \mathrm{Rot}\,(z_B(t;e^{i\theta});[0,kT]))).$$

It is clear that, if $(\rho,\theta) \in]0,R[\times\mathbb{R}$ is a fixed point of $\widetilde{\Psi}_{k,j}$, then $z(t;\sqrt{2\rho}e^{i\theta})$ is a kT-periodic solution to (1.5); moreover, we further know that

$$\mathrm{Rot}(z(t;\sqrt{2\rho}e^{i\theta});[0,kT]) = j.$$

We can thus state the following result.

Theorem 1.2.2. *Assume $(C_1),(C_2)$ to be satisfied and let $z(\cdot;\bar{z}), z_B(\cdot;\bar{z})$ be defined as in the previous discussion; moreover, let $k \in \mathbb{N}_*$ and $j \in \mathbb{Z}$. Suppose that there exists $z^* \in \mathbb{R}^2$, with $|z^*| = 1$, such that*

$$\mathrm{Rot}\,(z_B(t;z^*);[0,kT]) < j$$

and that, for a suitable $\widetilde{R} > 0$,

$$\mathrm{Rot}\,(z(t;\bar{z});[0,kT]) > j, \quad \text{for every } \bar{z} \in \mathbb{R}^2 \text{ with } |\bar{z}| = \widetilde{R}.$$

Then there exists a kT-periodic solution $z_{k,j}(t)$ to (1.5), such that

$$\mathrm{Rot}\,(z_{k,j}(t);[0,kT]) = j.$$

We also point out that, combining the previous discussion with Theorem 1.1.2 (see also [108, 118] for further details), it is possible to state the following result, dealing with Hamiltonian systems satisfying condition (C_2), obtained via the classical version of the Poincaré-Birkhoff theorem.

Theorem 1.2.3. *Assume $(C_1),(C_2)$ to be satisfied and let $z(\cdot;\bar{z}), z_B(\cdot;\bar{z})$ be defined as in the previous discussion; moreover, let $k \in \mathbb{N}_*$ and $j \in \mathbb{Z}$. Suppose that*

$$\mathrm{Rot}\,(z_B(t;\bar{z});[0,kT]) < j, \qquad \textit{for every } \bar{z} \in \mathbb{R}^2 \textit{ with } |\bar{z}| = 1,$$

and that, for a suitable $\widetilde{R} > 0$,

$$\mathrm{Rot}\,(z(t;\bar{z});[0,kT]) > j, \qquad \textit{for every } \bar{z} \in \mathbb{R}^2 \textit{ with } |\bar{z}| = \widetilde{R}.$$

Then there exist two kT-periodic solutions $z^{(1)}_{k,j}(t), z^{(2)}_{k,j}(t)$ to (1.5) such that

$$\mathrm{Rot}\,(z^{(1)}_{k,j}(t);[0,kT]) = \mathrm{Rot}\,(z^{(2)}_{k,j}(t);[0,kT]) = j.$$

1.3 Topological horsehoes: the SAP Method

In this section, we present a result about the existence and multiplicity of fixed points and periodic points for planar maps introduced by Papini and Zanolin in [138] and developed in some subsequent papers [139, 140]. Within the same framework one also gets complex dynamics (in a sense that will be made precise in Definition 1.3.1 below). The approach is linked to the theory of topological horseshoes [36, 101], a topic of dynamical systems which has been widely investigated in the past two decades in an attempt to weaken the hyperbolicity assumptions involved in the classical theory of Smale's horseshoes.

The abstract theory we are going to present can be developed in the frame of continuous maps in metric spaces. However, in view of the applications to planar Hamiltonian systems, we restrict ourselves to the case of continuous maps of the plane. For more details and full proofs of the results in their broader generality, we refer the interested reader to [129, 141].

Definition 1.3.1. Let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map. We say that Ψ induces chaotic dynamics on two symbols if there exist two nonempty disjoint compact subsets $\mathcal{K}_0, \mathcal{K}_1 \subset \mathbb{R}^2$ such that:

(i) for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, there exists a sequence $(z_i)_{i\in\mathbb{Z}} \in (\mathbb{R}^2)^{\mathbb{Z}}$ such that, for every $i \in \mathbb{Z}$,

$$z_i \in \mathcal{K}_{s_i} \quad \text{and} \quad z_{i+1} = \Psi(z_i); \tag{1.11}$$

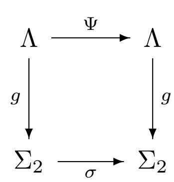
(ii) for every $k \in \mathbb{N}_*$ and for every k-periodic sequence $(s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, there exists a k-periodic sequence $(z_i)_{i\in\mathbb{Z}} \in (\mathbb{R}^2)^{\mathbb{Z}}$ satisfying (1.11).

When we want to emphasize the role of the sets $\mathcal{K}_j$, we also say that Ψ induces chaotic dynamics on two symbols relatively to $\mathcal{K}_0$ and $\mathcal{K}_1$.

Point (i) of our definition of chaotic dynamics corresponds to the concept of *chaos in the coin-tossing sense* [103]; namely, for each sequence $(s_i)_i$ of two symbols ("0" and "1") there exists a trajectory $(z_i)_i$ of Ψ (i.e. a sequence $(z_i)_i$ such that $z_{i+1} = \Psi(z_i)$) which follows the preassigned sequence of symbols, that is, z_i is in $\mathcal{K}_0$ when $s_i = 0$ and z_i is in $\mathcal{K}_1$ if $s_i = 1$. As pointed out by Smale [154], the possibility of reproducing all the possible outcomes of a coin-flipping experiment is one of the paradigms of chaos.
Besides the coin-tossing feature, our definition is enhanced, at point (ii), with respect to the existence of *periodic trajectories* for Ψ. Indeed, given an arbitrary periodic sequence of symbols $(s_i)_i$ (including the constant ones), we have guaranteed not only the existence of a trajectory $(z_i)_i$ wandering from $\mathcal{K}_0$ to $\mathcal{K}_1$ in a periodic fashion, but also the fact that, among such trajectories, at least one is made by periodic points. Hence, for every $k \in \mathbb{N}_*$, Ψ has at least 2^k k-periodic points (for $k = 1$, this in particular means that Ψ as at least two fixed points); moreover, the minimal period of such periodic points coincides with the minimal period of the k-periodic sequence of symbols $(s_i)_i$. We stress that such a property about periodic orbits is not an intrinsic aspect of the coin-tossing dynamics (see the examples [45, p.369] and [101, Example 10], where coin-tossing dynamics without periodic points are produced), although it has a strong relevance from the point of view of the applications. In this direction, some topological tools based on Conley index, Lefschetz number, fixed point index or topological degree have been developed by various authors (see, for instance, [131, 155, 171, 172] and the quotations in [129, 140, 141]).
As a further consequence of Definition 1.3.1 it follows that if Ψ is one-to-one on $\mathcal{K}_0 \cup \mathcal{K}_1$ (a situation which always occurs for the Poincaré map), there exists a nonempty compact set $\Lambda \subset \mathcal{K}_0 \cup \mathcal{K}_1$ which is invariant under Ψ (i.e., $\Psi(\Lambda) = \Lambda$) and such that $\Psi|_\Lambda$ is semiconjugate to the two-sided Bernoulli shift σ on two symbols

$$\sigma : \Sigma_2 \ni (s_i)_{i\in\mathbb{Z}} \mapsto (s_{i+1})_{i\in\mathbb{Z}} \in \Sigma_2,$$

according to the commutative diagram[2]



being $g : \Lambda \to \Sigma_2$ a continuous and surjective function. Moreover, the subset of Λ consisting of the periodic points of Ψ is dense in Λ and the preimage (by g) of any periodic sequence in Σ_2 contains a periodic point of Ψ having the same fundamental period (we refer to [129, 141] for more details). Note that the semiconjugacy to the Bernoulli shift is one of the typical requirements for chaotic dynamics as it implies a positive topological entropy for the map $\Psi|_\Lambda$ [161]. This also shows that Ψ is chaotic in the sense of Block and Coppel (see again [129] and the references therein). Such kind of chaotic dynamics was already considered in [131, 155, 171, 172], quoted above.

[2]Recall that $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ (the set of two-sided sequence of two symbols) is a compact metric space with the distance $d(\mathbf{s}', \mathbf{s}'') = \sum_{i\in\mathbb{Z}} \frac{|s_i' - s_i''|}{2^{|i|}}$ for $\mathbf{s}' = (s_i')_i$ and $\mathbf{s}'' = (s_i'')_i$.

A little bit of terminology is now needed. By a path γ in $\mathbb{R}^2$ we mean a continuous mapping $\gamma : [t_0, t_1] \to \mathbb{R}^2$, while by a sub-path σ of γ we just mean the restriction of γ to a compact subinterval of $[t_0, t_1]$. As a domain for a path, we will often take (without loss of generality) $[t_0, t_1] = [0, 1]$. By an *oriented rectangle* we mean a pair

$$\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-),$$

being $\mathcal{R} \subset \mathbb{R}^2$ homeomorphic to $[0,1]^2$ (namely, a *topological rectangle*) and

$$\mathcal{R}^- = \mathcal{R}_1^- \cup \mathcal{R}_2^-$$

the disjoint union of two compact arcs (by definition, a compact arc is a homeomorphic image of $[0,1]$) $\mathcal{R}_1^-, \mathcal{R}_2^- \subset \partial\mathcal{R}$. The boundary $\partial\mathcal{R}$ is a Jordan curve and therefore $\partial\mathcal{R} \setminus (\mathcal{R}_1^- \cup \mathcal{R}_2^-)$ is the disjoint union of two open arcs. We denote the closure of such open arcs by $\mathcal{R}_1^+$ and $\mathcal{R}_2^+$. According to the Jordan-Schoenflies theorem, we can always label the compact arcs $\mathcal{R}_i^\pm$ $(i = 1, 2)$ and take a homeomorphism h of the plane onto itself, such that $h([0,1]^2) = \mathcal{R}$ and the left and right sides of $[0,1]^2$ are mapped to $\mathcal{R}_1^-$ and $\mathcal{R}_2^-$ while the lower and upper sides of $[0,1]^2$ are mapped to $\mathcal{R}_1^+$ and $\mathcal{R}_2^+$, respectively.

The core of the *stretching along the paths* method is given by the following definition.

Definition 1.3.2. Let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map, $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ oriented rectangles and $\mathcal{H} \subset \mathcal{A}$ a compact subset. We say that $(\mathcal{H}, \Psi)$ stretches $\widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{B}}$ along the paths and write

$$(\mathcal{H}, \Psi) : \widetilde{\mathcal{A}} \Longleftrightarrow\!\!\!\!\!\!\!\!\longrightarrow \widetilde{\mathcal{B}}$$

if for every path $\gamma : [0,1] \to \mathcal{A}$ such that $\gamma(0) \in \mathcal{A}_1^-$ and $\gamma(1) \in \mathcal{A}_2^-$ (or $\gamma(0) \in \mathcal{A}_2^-$ and $\gamma(1) \in \mathcal{A}_1^-$), there exists a subinterval $[t', t''] \subset [0,1]$ such that

$$\gamma(t) \in \mathcal{H} \quad \text{and} \quad \Psi(\gamma(t)) \in \mathcal{B}, \quad \text{for every } t \in [t', t''],$$

and, moreover, $\Psi(\gamma(t'))$ and $\Psi(\gamma(t''))$ belong to different components of $\mathcal{B}^-$.

Based on this definition, we have the following theorem, proved by Pascoletti and Zanolin [143]. It basically relies on a Crossing Lemma of Planar Topology, which, in turns, is equivalent to the planar case of the Poincaré-Miranda theorem.

Theorem 1.3.1. *Let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map and let $\widetilde{\mathcal{P}} = (\mathcal{P}, \mathcal{P}^-)$ be an oriented rectangle. Suppose that there exist two compact disjoint subsets $\mathcal{K}_0, \mathcal{K}_1 \subset \mathcal{P}$ such that*

$$(\mathcal{K}_i, \Psi) : \widetilde{\mathcal{P}} \Longleftrightarrow\!\!\!\!\!\!\!\!\longrightarrow \widetilde{\mathcal{P}}, \quad \text{for } i = 0, 1. \tag{1.12}$$

Then Ψ induces chaotic dynamics on two symbols relatively to $\mathcal{K}_0$ and $\mathcal{K}_1$. Moreover, for each sequence of two symbols $\boldsymbol{s} = (s_i)_{i \in \mathbb{N}_*} \in \{0,1\}^{\mathbb{N}_*}$, there exists a compact connected set $\mathcal{C}_{\boldsymbol{s}} \subset \mathcal{K}_{s_0}$, with

$$\mathcal{C}_{\boldsymbol{s}} \cap \mathcal{P}_1^+ \neq \emptyset \quad \text{and} \quad \mathcal{C}_{\boldsymbol{s}} \cap \mathcal{P}_2^+ \neq \emptyset,$$

such that, for every $w \in \mathcal{C}_{\boldsymbol{s}}$, there exists a sequence $(y_i)_{i \in \mathbb{N}_*}$ with $y_0 = w$ and, for every $i \in \mathbb{N}_*$,

$$y_i \in \mathcal{K}_{s_i} \quad \text{and} \quad \Psi(y_i) = y_{i+1}.$$

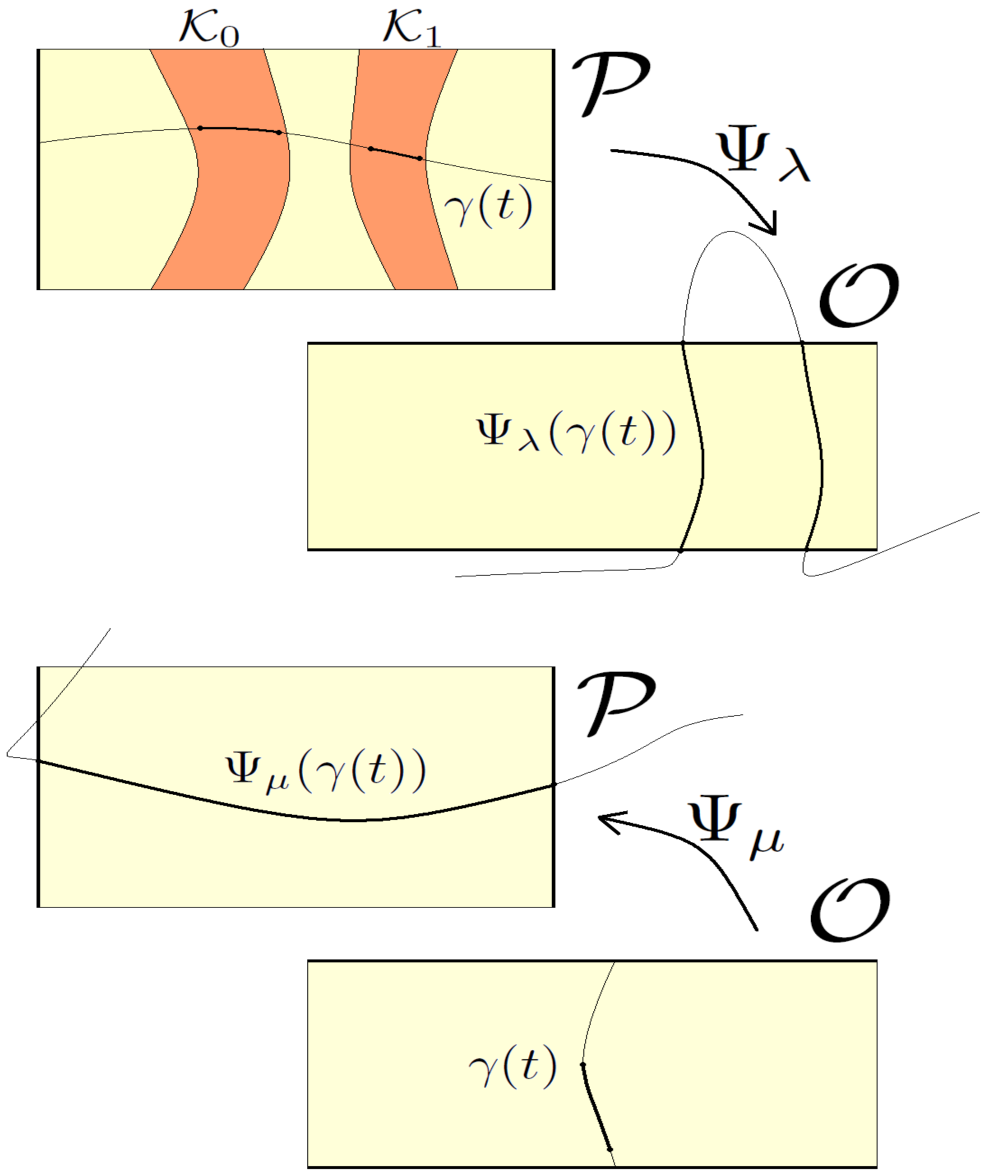


Figure 1.2: A graphical description of Theorem 1.3.2. Two rectangles $\mathcal{P}$ and $\mathcal{O}$ are oriented by selecting two opposite sides. In both the figures, the $[\cdot]^-$-components of the boundaries are indicated with a bold line. Any path $\gamma(t)$ connecting in $\mathcal{P}$ the opposite sides of $\mathcal{P}^-$ contains two sub-paths, with support respectively in $\mathcal{K}_0$ and $\mathcal{K}_1$, which are expanded by Ψ_λ across $\mathcal{O}$ into two sub-paths of $\Psi_\lambda(\gamma(t))$, which join the two opposite sides of $\mathcal{O}^-$ (the upper figure). In the figure below, the action of Ψ_μ is depicted as well. Here, any path $\gamma(t)$ connecting in $\mathcal{O}$ the opposite sides of $\mathcal{O}^-$ contains a sub-path whose image is a sub-path of $\Psi_\mu(\gamma(t))$ in $\mathcal{P}$, joining the opposite sides of $\mathcal{P}^-$.

In this thesis, we will use the following special case of Theorem 1.3.1, which occurs when Ψ splits as the composition of two maps: the first one stretches an oriented rectangle $\widetilde{\mathcal{P}}$ across another one $\widetilde{\mathcal{O}}$, while the second one comes back from $\widetilde{\mathcal{O}}$ to $\widetilde{\mathcal{P}}$. Precisely (see again [143]), we have:

Theorem 1.3.2. *Let* $\Psi_\lambda, \Psi_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ *be continuous maps and let* $\widetilde{\mathcal{P}} = (\mathcal{P}, \mathcal{P}^-)$, $\widetilde{\mathcal{O}} = (\mathcal{O}, \mathcal{O}^-)$ *be oriented rectangles. Suppose that:*

(H_λ) *there exist compact disjoint subsets* $\mathcal{K}_0, \mathcal{K}_1 \subset \mathbb{R}^2$ *such that, for* $i = 0, 1$,

$$(\mathcal{K}_i, \Psi_\lambda) : \widetilde{\mathcal{P}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathcal{O}};$$

(H_μ) $(\mathcal{O}, \Psi_\mu) : \widetilde{\mathcal{O}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathcal{P}}$.

Then (1.12) *is satisfied for the map* $\Psi = \Psi_\mu \circ \Psi_\lambda$ *and therefore the conclusion of Theorem 1.3.1 holds.*

A graphical explanation of Theorem 1.3.2 can be found in Figure 1.2.

Part I

Solutions which wind around the origin

Chapter 2

Planar Hamiltonian systems with a center type dynamics

This chapter, which is based on [26, 27, 29, 30], deals with the planar Hamiltonian system

$$Jz' = \nabla_z H(t,z), \tag{2.1}$$

where $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a given (regular enough) function, T-periodic in the first variable. The existence of an equilibrium point - for simplicity the origin, i.e., $\nabla_z H(t,0) \equiv 0$ - is always assumed and we look for periodic solutions (harmonic and subharmonic) winding around it.

Just to motivate the forthcoming results, we can consider the autonomous Hamiltonian system

$$Jz' = \nabla H(z). \tag{2.2}$$

It is easily seen that, if $H(z)$ is a C^1-function with

$$\langle \nabla H(z) | z \rangle > 0, \quad \text{for } z \neq 0, \qquad \text{and} \qquad \lim_{|z| \to +\infty} H(z) = +\infty,$$

then the phase-portrait associated with (2.2) is that of a global center around the origin: namely, all the nontrivial solutions to (2.2) are periodic; moreover, their orbits are strictly star-shaped Jordan curves around the origin, covered in the clockwise sense. In this situation, an elementary approach to the problem of the existence and multiplicity of periodic solutions of fixed period is available. Indeed, the map which associates to a value $c > \inf_{\mathbb{R}^2} H(z)$ the minimal period of the orbit lying on the energy line $H^{-1}(c)$ (the so called time-map) is continuous [98]; accordingly, by simple continuity-connectedness arguments, kT-periodic solutions ($k \in \mathbb{N}_*$) appear whenever the minimal period of "small" solutions is different enough from the minimal period of "large" solutions[1]. Notice, in particular, that the greater is the gap between the minimal periods, the larger is the number of periodic solutions obtained.

[1] Such a technique has been widely employed especially in the case of second order scalar ODEs, where a simple integral formula for the time-map is available (we will use explicitly this tool in Section 4.1; see also [52] for a general survey on the subject).

The goal of the chapter is to study the nonautonomous Hamiltonian system (2.1), via a suitable version of the Poincaré-Birkhoff fixed point theorem, Theorem 1.1.3. The underlying general technique used to estimate the number of revolutions around the origin of "small" and "large" solutions to (2.1) is that of comparing the nonlinear Hamiltonian system (2.1), both at zero and at infinity, with Hamiltonian systems of the kind

$$Jz' = \nabla V(z), \tag{2.3}$$

being $V : \mathbb{R}^2 \to \mathbb{R}$ a C^1-function satisfying

$$0 < V(\lambda z) = \lambda^2 V(z), \qquad \text{for every } \lambda > 0,\ z \in \mathbb{R}^2_*. \tag{2.4}$$

Such systems have the property that their (nontrivial) solutions are periodic solutions around the origin, with the same minimal period (i.e., the origin is an isochronous center for (2.3)). On the lines of [170], we define a "modified rotation number" associated with a function V satisfying (2.4) - relating it to the classical one - and we use it to count the turns around the origin of the solutions to (2.1).

Our multiplicity results concern three different situations:

(i) a problem *semilinear at infinity*, i.e., when large solutions to (2.1) perform, in a fixed time interval, a finite number of revolutions around the origin. Our main results give the existence of a finite number of T-periodic solutions (Theorem 2.3.1 and Theorem 2.3.2), according to the wideness of the "gap" between the dynamics at zero and at infinity, and subharmonics of large period (Theorem 2.3.5).

(ii) a problem *sublinear at infinity*, i.e., when large solutions to (2.1) do not complete, in a fixed time interval, a full revolution around the origin. Here, we obtain the existence of a finite number of subharmonic solutions of order k (making few turns around the origin) provided that k is large enough (Theorem 2.3.6).

(iii) a problem *superlinear at infinity*, i.e., when large solutions to (2.1) perform an arbitrarily large number of turns around the origin. We obtain, for every integer k, the existence of infinitely many subharmonic solutions of order k, growing in norms towards infinity and making a great number of revolutions around the origin (Theorem 2.3.8).

More comments on the relationships between our results and the existing literature are found along the chapter. It has to be noted that, when dealing with (planar) first order differential systems, there are no standard definitions of sublinearity and superlinearity[2]. Our conditions, which are based on the comparison of the nonlinear system (2.1) with positively homogeneous Hamiltonian systems like (2.3), are quite general, generalizing the standard assumptions of sublinearity and superlinearity for the second order scalar differential equation $u'' + f(t, u) = 0$ (namely, $f(t, x)/x \to 0$ and $f(t, x)/x \to +\infty$ for $|x| \to +\infty$, respectively).

The plan of this chapter is as follows. In Section 2.1, we define our modified rotation number and we prove some crucial properties about it. In Section 2.2, we prove the technical estimates concerning the rotation numbers of small and large solutions to (2.1). Finally, in Section 2.3 we state and prove our multiplicity results.

[2]Incidentally, we observe that, when dealing with Hamiltonian systems, the terms subquadraticity and superquadraticity are often employed, referring to the hypotheses on the Hamiltonian.

2.1 The V-rotation numbers

From now on, we denote by $\mathcal{P}$ the class of the C^1-functions $V : \mathbb{R}^2 \to \mathbb{R}$ which are positively homogeneous of degree 2 and strictly positive, i.e., for every $\lambda > 0$ and for every $z \in \mathbb{R}^2_*$,

$$0 < V(\lambda z) = \lambda^2 V(z).$$

It is easy to see that, if $V \in \mathcal{P}$, then $\lim_{|z|\to+\infty} V(z) = +\infty$ and Euler's formula

$$\langle \nabla V(z) | z \rangle = 2V(z), \quad \text{for every } z \in \mathbb{R}^2,$$

holds true. These properties imply that, for every $c > 0$, the open set $\{V < c\} = \{z \in \mathbb{R}^2 \mid V(z) < c\}$ is a bounded neighborhood of the origin, with boundary

$$\partial\{V < c\} = V^{-1}(c) \subset \mathbb{R}^2_*$$

which turns out to be a strictly star-shaped Jordan curve around the origin.
Henceforth, let $V \in \mathcal{P}$. First of all, recall that, as a consequence of [151, Corollary 1], there is uniqueness for the Cauchy problems associated with

$$Jz' = \nabla V(z). \tag{2.5}$$

Secondly, the Hamiltonian structure implies that every nontrivial solution to (2.5) lies on a level curve $V^{-1}(c)$: thus, it is globally defined and periodic; moreover, it is easy to see that each orbit is covered in the clockwise sense. Denoting by $\varphi_V(t)$ the solution to (2.5) such that $\varphi_V(0) \in \mathbb{R}^+_* \times \{0\}$ and

$$V(\varphi_V(t)) = \frac{1}{2}, \quad \text{for every } t \in \mathbb{R}, \tag{2.6}$$

and by τ_V its minimal period, we can finally define the map $\Pi_V : \mathbb{R} \to V^{-1}(1/2)$, by setting

$$\Pi_V(\theta) = \varphi_V\left(-\frac{\tau_V}{2\pi}\theta\right).$$

It is easy to see that Π_V is a covering projection. Therefore, the standard theory of covering spaces [94] ensures that, for every absolutely continuous path $z : [t_1, t_2] \to \mathbb{R}^2$ such that $z(t) \neq 0$ for every $t \in [t_1, t_2]$, the path

$$[t_1, t_2] \ni t \mapsto \frac{z(t)}{\sqrt{2V(z(t))}} \in V^{-1}(1/2)$$

can be lifted to the covering space $(\mathbb{R}, \Pi_V)$ of $V^{-1}(1/2)$, namely, there exists an absolutely continuous path $\theta_V : [t_1, t_2] \to \mathbb{R}$ such that

$$\frac{z(t)}{\sqrt{2V(z(t))}} = \Pi_V(\theta_V(t)), \quad \text{for every } t \in [t_1, t_2].$$

Moreover, standard calculations show that, for almost every t,

$$\theta'_V(t) = -\frac{2\pi}{\tau_V} \frac{\langle Jz'(t) | z(t) \rangle}{2V(z(t))}.$$

In conclusion, we are led to give the following definition (compare with [170]).

Definition 2.1.1. The (clockwise) V-rotation number (around the origin) of an absolutely continuous path $z = (x, y) : [t_1, t_2] \to \mathbb{R}^2$, such that $z(t) \neq 0$ for every $t \in [t_1, t_2]$, is the number

$$\mathrm{Rot}_V(z(t); [t_1, t_2]) = \frac{\theta_V(t_1) - \theta_V(t_2)}{2\pi},$$

that is, equivalently,

$$\mathrm{Rot}_V(z(t); [t_1, t_2]) = \frac{1}{\tau_V} \int_{t_1}^{t_2} \frac{\langle Jz'(t)|z(t)\rangle}{2V(z(t))} dt = \frac{1}{\tau_V} \int_{t_1}^{t_2} \frac{y(t)x'(t) - x(t)y'(t)}{2V(x(t), y(t))} dt.$$

Note that, as usual, the definition does not depend on the choice of the lifting θ_V.

For the sequel, it is also useful to use Gauss-Green formula in order to compute τ_V as follows. Set

$$A_V = \int_{\{V \leq 1\}} dx\, dy;$$

then we have, using (2.6),

$$\begin{aligned} A_V &= 2\int_{\{V \leq 1/2\}} dx\, dy = \int_{V^{-1}(1/2)^+} (x\, dy - y\, dx) = \int_0^{\tau_V} \langle J\varphi_V'(t))|\varphi_V(t)\rangle\, dt \\ &= \int_0^{\tau_V} \langle \nabla V(\varphi_V(t))|\varphi_V(t)\rangle\, dt = 2\int_0^{\tau_V} V(\varphi_V(t))\, dt = \tau_V. \end{aligned}$$

Incidentally, we notice that this straightly implies that, for every $\lambda > 0$,

$$\mathrm{Rot}_{\lambda V}(z(t); [t_1, t_2]) = \mathrm{Rot}_V(z(t); [t_1, t_2]).$$

We now investigate the relation of a V-rotation number as defined before with the standard one (compare with (8) of the introductory section "Notation and Terminology"), that is, the V-rotation number for $V(x, y) = \frac{1}{2}(x^2 + y^2)$ (in such a case, indeed, $\tau_V = 2\pi$; we will continue to denote this rotation number simply by $\mathrm{Rot}(z(t); [t_1, t_2])$). To begin with, we have the following lemma.

Lemma 2.1.1. *The map $\Lambda_V : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Lambda_V(\theta) = \frac{\pi}{\tau_V} \int_0^{\theta} \frac{d\omega}{V(\cos\omega, \sin\omega)}$$

is an increasing C^1-homeomorphism of $\mathbb{R}$, such that, for every $\theta \in \mathbb{R}$ and for every $k \in \mathbb{Z}$,

$$\Lambda_V(\theta + 2k\pi) = \Lambda_V(\theta) + 2k\pi. \tag{2.7}$$

In particular, for every $k \in \mathbb{Z}$,

$$\Lambda_V(2k\pi) = 2k\pi. \tag{2.8}$$

Proof. Since

$$\Lambda_V'(\theta) = \frac{\pi}{\tau_V} \frac{1}{V(\cos\theta, \sin\theta)} > 0,$$

we have that Λ_V is strictly increasing. Moreover, the 2π-periodicity of the integrand implies that, for every $\theta \in \mathbb{R}$ and for every $k \in \mathbb{Z}$,

$$\Lambda_V(\theta + 2k\pi) = \Lambda_V(\theta) + k\Lambda_V(2\pi);$$

so that the computation in polar coordinates

$$A_V = \int_0^{2\pi} \left(\int_0^{1/\sqrt{V(\cos\omega,\sin\omega)}} \rho \, d\rho \right) d\omega = \frac{1}{2} \int_0^{2\pi} \frac{d\omega}{V(\cos\omega, \sin\omega)} = \frac{\tau_V}{2\pi} \Lambda_V(2\pi)$$

implies (2.7). On the other hand, passing to the limit in (2.7) we conclude that $\Lambda_V(\theta) \to \pm\infty$ for $\theta \to \pm\infty$ and so Λ_V is a homeomorphism of $\mathbb{R}$. Finally, (2.8) follows from (2.7) and the fact that $\Lambda_V(0) = 0$. □

Using the change of variables Λ_V, we can compute a V-rotation number starting from the knowledge of the standard one. Precisely, we have the following proposition.

Proposition 2.1.1. *Let $z : [t_1, t_2] \to \mathbb{R}^2$ be an absolutely continuous path, such that $z(t) \neq 0$ for every $t \in [t_1, t_2]$, and let $\theta : [t_1, t_2] \to \mathbb{R}$ be an absolutely continuous function such that*

$$z(t) = |z(t)|(\cos\theta(t), \sin\theta(t)), \qquad \textit{for every } t \in [t_1, t_2].$$

Then

$$\mathrm{Rot}_V(z(t); [t_1, t_2]) = \frac{\Lambda_V(\theta(t_1)) - \Lambda_V(\theta(t_2))}{2\pi}.$$

Proof. Let $\Theta_V : [t_1, t_2] \to \mathbb{R}$ be the path defined by

$$\Theta_V(t) = \Lambda_V(\theta(t)).$$

Since $\theta(t)$ is absolutely continuous and Λ_V is of class C^1, standard properties of absolutely continuous functions imply that $\Theta_V(t)$ is absolutely continuous too; moreover, for a.e. $t \in [t_1, t_2]$,

$$\Theta_V'(t) = \Lambda_V'(\theta(t))\theta'(t).$$

By a simple computation

$$\begin{aligned} \Theta_V'(t) &= \Lambda_V'(\theta(t))\theta'(t) \\ &= -\frac{\pi}{\tau_V} \frac{1}{V(\cos\theta(t), \sin\theta(t))} \frac{\langle Jz'(t)|z(t)\rangle}{|z(t)|^2} = -\frac{2\pi}{\tau_V} \frac{\langle Jz'(t)|z(t)\rangle}{2V(z(t))}, \end{aligned}$$

whence the conclusion. □

As a consequence, we can deduce the following fundamental properties. Roughly speaking, we can say that Rot_V counts the same number of *complete* clockwise turns around the origin as Rot. The easy proof is omitted (see [26, Proposition 2.1]).

Proposition 2.1.2. *Let $z : [t_1, t_2] \to \mathbb{R}^2$ be an absolutely continuous path, such that $z(t) \neq 0$ for every $t \in [s_1, s_2]$, and $j \in \mathbb{Z}$. Then:*

$$\mathrm{Rot}_V(z(t); [t_1, t_2]) < j \iff \mathrm{Rot}(z; [s_1, s_2]) < j; \tag{2.9}$$

$$\mathrm{Rot}_V(z(t); [t_1, t_2]) > j \iff \mathrm{Rot}(z; [s_1, s_2]) > j. \tag{2.10}$$

Remark 2.1.1. A very useful choice of V is given by the quadratic form

$$V(x,y) = \frac{x^2}{c} + \frac{y^2}{d} \tag{2.11}$$

for some $c, d > 0$; in this case, clearly, $A_V = (\sqrt{cd})\pi$. The asymmetric case

$$V(x,y) = \left(\frac{x^+}{c_1} - \frac{x^-}{c_2}\right)^2 + \left(\frac{y^+}{d_1} - \frac{y^-}{d_2}\right)^2$$

(being $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$) for some $c_1, c_2, d_1, d_2 > 0$ can be considered as well. In particular, in the case of a diagonal quadratic form (2.11), the symmetries of V imply that the homeomorphism Λ_V satisfies the extra property

$$\Lambda_V\left(k\frac{\pi}{2}\right) = k\frac{\pi}{2}, \qquad \text{for every } k \in \mathbb{Z}.$$

From this fact we can easily deduce that

$$\Big|\mathrm{Rot}_V(z(t); [t_1, t_2]) - \mathrm{Rot}(z(t); [t_1, t_2])\Big| < \frac{1}{4}. \tag{2.12}$$

Rotation numbers of this kind have been often considered in literature (at least implicitly, as in [65]) and a systematic treatment is given in [152], where relations (2.9), (2.10) and (2.12) are proved with different arguments. Some other remarks concerning the V-rotation numbers can be found in [26].

We conclude this section by describing the dynamics of Hamiltonian systems of the form

$$Jz' = \gamma(t)\nabla V(z), \tag{2.13}$$

with $V \in \mathcal{P}$ and $\gamma(t) > 0$ a scalar function. As before, we denote by $\varphi_V(t)$ the solution to (2.5), such that $\varphi_V(0)$ is on the positive x semi-axis and $V(\varphi_V(t)) \equiv 1/2$, and by τ_V its minimal period.

Lemma 2.1.2. *Assume that $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function, with $\gamma(t) > 0$ for every $t \in [0, T]$ and*

$$\frac{1}{T}\int_0^T \gamma(t)\, dt = 1. \tag{2.14}$$

Set $\Gamma(t) = \int_0^t \gamma(s)\, ds$; then:

- *there is uniqueness and global continuability for the solutions to the Cauchy problems associated with* (2.13)*;*
- *every solution to* (2.13) *lies on a level curve of V and can be written as*

$$z(t) = C\varphi_V(\Gamma(t) + \theta), \tag{2.15}$$

for suitable constants $C \geq 0$, $\theta \in [0, \tau_V[$.

Notice that assumption (2.14) is not restrictive, up to relabeling V.

Proof. We first prove that (even if (2.13) is not an autonomous Hamiltonian system) V is preserved along the solutions; indeed

$$\frac{d}{dt}V(z(t)) = \langle \nabla V(z(t))|z'(t)\rangle = -\gamma(t)\langle \nabla V(z(t))|J\nabla V(z(t))\rangle = 0.$$

As a consequence, every solution lies on a level curve of V; this ensures the global continuability, since the level lines of V are compact.
We now prove the uniqueness. Let us suppose, by contradiction, that there exist two solutions $z_1(t), z_2(t)$ of (2.13) defined in a neighborhood of $t_0 \in \mathbb{R}$ and such that $z_1(t_0) = z_2(t_0) = z_0 \in \mathbb{R}^2_*$; by the conservation of V and Euler's formula, we get

$$\langle Jz_1'(t)|z_1(t)\rangle = \langle Jz_2'(t)|z_2(t)\rangle = 2\gamma(t)V(z_0) \neq 0. \tag{2.16}$$

Let us define, for (r, s) in a neighborhood of $(1, t_0)$, the C^1 function $P(r, s) = rz_2(s) \in \mathbb{R}^2$. We have that $P(1, t_0) = z_0$, while (2.16) implies that the Jacobian matrix $P'(1, t_0)$ is invertible. Then, the local inversion theorem gives the existence of C^1 maps $r(t), s(t)$, defined in a neighborhood of t_0 and with values in a neighborhood of 1 and t_0 respectively, such that, locally, $z_1(t) = P(r(t), s(t)) = r(t)z_2(s(t))$. The conservation of the energy and the homogeneity of V imply that $r(t) = 1$; hence $z_1(t) = z_2(s(t))$. On the other hand, differentiating this last equality and using relation (2.16), we obtain, since $s(t_0) = t_0$, $s(t) = t$. In conclusion, $z_1(t) = z_2(t)$. The observation that every positive energy level set does not contain the origin implies the uniqueness for $z_0 = 0$ too.
Finally, every $z(t)$ of the form (2.15) is a solution (as it is easily verified). Conversely, since, the map $[0, \tau_V[\ \ni t \mapsto C\varphi_V(t)$ describes a strictly star-shaped curve around the origin in the plane, fixed $t_0 \in \mathbb{R}$ and $\bar{z} \in \mathbb{R}^2$ there exists a solution of the form (2.15) starting from $\bar{z}$ at the time $t = t_0$. □

In the autonomous case (namely, $\gamma(t) \equiv 1$), Lemma 2.1.2 implies that every nontrivial solution to (2.13) is periodic and has minimal period τ_V as well, so that the origin is an isochronous center (see Figure 2.1). In the nonautonomous case, we restrict our attention to the existence of kT-periodic solutions, with k an integer number. Then, it is easy to see that equation (2.13) has a nontrivial kT-periodic solution if and only if $kT/\tau_V \in \mathbb{N}$. In this case, all the nontrivial solutions of (2.13) are kT-periodic and make exactly kT/τ_V turns around the origin in the time $[0, kT]$. In such a situation, we say that (2.13) is at *resonance*. It is well-known that the study of perturbations of resonant equations is usually more difficult, since in general no kT-periodic solutions exist (see, for instance, [67, 71, 74] for perturbations of positively homogeneous Hamiltonian systems, the case we are dealing with). Notice that this concept of resonance includes the classical one for linear scalar second order equations

$$u'' + \lambda u = 0, \qquad \lambda > 0. \tag{2.17}$$

Indeed, (2.17) is written as the first order planar system $Jz' = \nabla V_\lambda(z)$, for $V_\lambda(z) = \frac{1}{2}y^2 + \frac{\lambda}{2}x^2$. By evaluating the minimal period of V_λ, one can see that (2.17) has nontrivial kT-periodic solutions if and only if $\lambda = \left(\frac{2j\pi}{kT}\right)^2$, for a suitable positive integer j. Throughout this chapter, in particular, we use the notation

$$\lambda_j = \left(\frac{2j\pi}{T}\right)^2,$$

i.e., λ_j is the j-th eigenvalue of the differential operator $u \mapsto -u''$ with T-periodic boundary conditions.

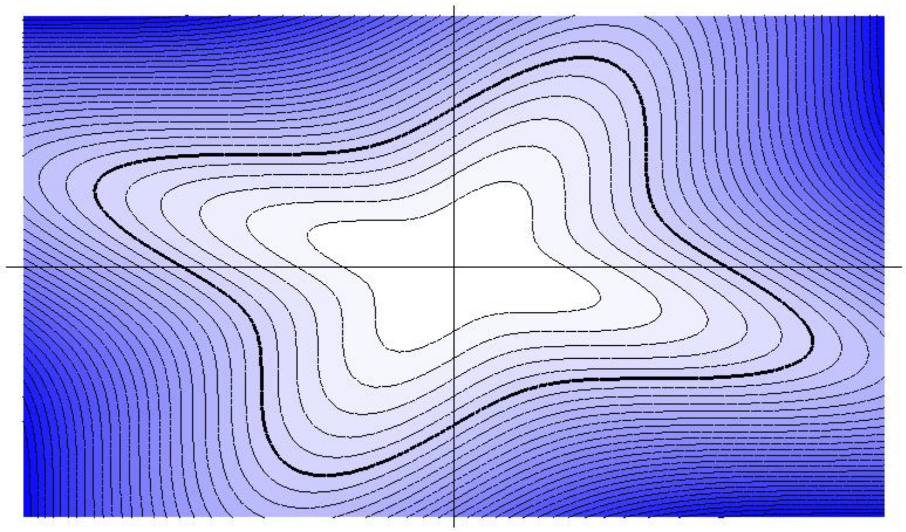


Figure 2.1: The phase portrait of the autonomous Hamiltonian system $Jz' = \nabla V(z)$, with $V(x, y) = x^2 + y^2 + \frac{2xy(x^2-y^2)}{x^2+y^2} \in \mathcal{P}$. Notice that the homogeneity of V implies that all the orbits can be obtained from a fixed one (like, for instance, the one painted in a darker color) via a dilatation. For our dynamical analysis, the essential feature (depending on the exact degree - two - of homogeneity) is that the period is the same for each orbit.

2.2 Some technical estimates

In this section, we perform some estimates for the rotation numbers of both small and large solutions to a planar system. Since the Hamiltonian structure does not play a role at this step, we consider a general first order system like

$$Jz' = Z(t, z), \tag{2.18}$$

being $Z : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ an L^1-Carathéodory function, T-periodic in the first variable. However, we write our system in the form (2.18), namely with the symplectic matrix J at the left-hand side, in order to better compare with the Hamiltonian problem (namely, $Z(t, z) = \nabla_z H(t, z)$), treated in Section 2.3. Corollaries are derived for the second order scalar differential equation

$$u'' + f(t, u) = 0, \tag{2.19}$$

with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ an L^1-Carathéodory function, T-periodic in the first variable.

2.2.1 The nonresonant case

The estimated developed in this subsection essentially refer to a nonresonant situation. We start with a very simple observation; throughout this subsection, $k \in \mathbb{N}_*$.

Lemma 2.2.1. *Let $V \in \mathcal{P}$, $0 \leq r < R \leq +\infty$ and $\beta : \mathbb{R} \times \{z \in \mathbb{R}^2 \mid r < |z| < R\} \to \mathbb{R}$ an L^1-Carathéodory function, T-periodic in the first variable. The following statements hold true.*

- *Assume:*
  - *for almost every $t \in [0,T]$ and for every $r < |z| < R$,*
  $$\frac{\langle Z(t,z)|z\rangle}{2V(z)} \geq \beta(t,z); \tag{2.20}$$
  - *for every $z : [0,kT] \to \mathbb{R}^2$ solving (2.18), with $r < |z(t)| < R$ for every $t \in [0,kT]$, it holds*
  $$\frac{1}{kT}\int_0^{kT} \beta(t,z(t))\,dt > 1. \tag{2.21}$$

  Then, for every $z : [0,kT] \to \mathbb{R}^2$ solving (2.18), with $r < |z(t)| < R$ for every $t \in [0,kT]$, it holds
  $$\mathrm{Rot}\,(z(t);[0,kT]) > \left\lfloor \frac{kT}{\tau_V} \right\rfloor.$$
- *Assume:*
  - *for almost every $t \in [0,T]$ and for every $r < |z| < R$,*
  $$\frac{\langle Z(t,z)|z\rangle}{2V(z)} \leq \beta(t,z); \tag{2.22}$$
  - *for every $z : [0,kT] \to \mathbb{R}^2$ solving (2.18), with $r < |z(t)| < R$ for every $t \in [0,kT]$, it holds*
  $$\frac{1}{kT}\int_0^{kT} \beta(t,z(t))\,dt < 1. \tag{2.23}$$

  Then, for every $z : [0,kT] \to \mathbb{R}^2$ solving (2.18), with $r < |z(t)| < R$ for every $t \in [0,kT]$, it holds
  $$\mathrm{Rot}\,(z(t);[0,kT]) < \left\lceil \frac{kT}{\tau_V} \right\rceil.$$

Proof. We prove the first statement. For $z : [0,kT] \to \mathbb{R}^2$ solving (2.18), with $r < |z(t)| < R$ for every $t \in [0,kT]$, we have, in view of (2.20) and (2.21),

$$\begin{aligned}\mathrm{Rot}_V\,(z(t);[0,kT]) &= \frac{1}{\tau_V}\int_0^{kT} \frac{\langle Jz'(t)|z(t)\rangle}{2V(z(t))}\,dt = \frac{1}{\tau_V}\int_0^{kT} \frac{\langle Z(t,z(t))|z(t)\rangle}{2V(z(t))}\,dt \\ &\geq \frac{1}{\tau_V}\int_0^{kT} \beta(t,z(t))\,dt > \frac{kT}{\tau_V} \geq \left\lfloor \frac{kT}{\tau_V} \right\rfloor.\end{aligned}$$

Hence, the conclusion follows in view of Proposition 2.1.2. The proof of the second statement is analogous, using (2.22) and (2.23). □

As a first consequence, we give a corollary which can be applied when system (2.18) can be compared with a system of the form $Jz' = \gamma(t)\nabla V(z)$, with $V \in \mathcal{P}$, either at 0 or at $+\infty$, in a suitable weak sense. In order to give a unifying statement, we will denote by the symbol $\odot$ either 0 or $+\infty$.

Proposition 2.2.1. *Let $V \in \mathcal{P}$ and $\gamma \in L^1_T$, with*

$$\frac{1}{T}\int_0^T \gamma(t)\,dt = 1. \tag{2.24}$$

Moreover, if $\odot = 0$, assume that $Z(t,0) \equiv 0$. The following statements hold true.

- *If*

$$\liminf_{|z|\to\odot} \frac{\langle Z(t,z)|z\rangle}{2V(z)} \geq \gamma(t), \quad \textit{uniformly for a.e. } t \in [0,T], \tag{2.25}$$

then there exists $\rho > 0$ such that, for every $z : [0,kT] \to \mathbb{R}^2$ solving (2.18) and satisfying, for every $t \in [0,kT]$, $0 < |z(t)| < \rho$ if $\odot = 0$, or $|z(t)| > \rho$ if $\odot = +\infty$, it holds

$$\mathrm{Rot}\,(z(t);[0,kT]) > \mathcal{E}^-\left(\frac{kT}{\tau_V}\right).$$

- *If*

$$\limsup_{|z|\to\odot} \frac{\langle Z(t,z)|z\rangle}{2V(z)} \leq \gamma(t), \quad \textit{uniformly for a.e. } t \in [0,T], \tag{2.26}$$

then there exists $\rho > 0$ such that, for every $z : [0,kT] \to \mathbb{R}^2$ solving (2.18) and satisfying, for every $t \in [0,kT]$, $0 < |z(t)| < \rho$ if $\odot = 0$, or $|z(t)| > \rho$ if $\odot = +\infty$, it holds

$$\mathrm{Rot}\,(z(t);[0,kT]) < \mathcal{E}^+\left(\frac{kT}{\tau_V}\right).$$

Proof. We will only prove the first statement for $\odot = 0$, the other cases being analogous. Define, for $\delta > 0$ small, the function $V_\delta \in \mathcal{P}$ as $V_\delta(z) = (1-\delta)V(z)$ Clearly, uniformly for almost every $t \in [0,T]$,

$$\liminf_{|z|\to 0} \frac{\langle Z(t,z)|z\rangle}{2V_\delta(z)} \geq \frac{1}{1-\delta}\,\gamma(t). \tag{2.27}$$

Since $\frac{1}{T}\int_0^T \frac{1}{1-\delta}\,\gamma(t)\,dt > 1$, we can fix $\epsilon > 0$ such that $\frac{1}{T}\int_0^T \frac{1}{1-\delta}\,(\gamma(t)-\epsilon)\,dt > 1$. In view of (2.27), there exists $\rho > 0$ such that, for almost every t and every $0 < |z| < \rho$,

$$\frac{\langle Z(t,z)|z\rangle}{2V_\delta(z)} \geq \frac{1}{1-\delta}\Big(\gamma(t)-\epsilon\Big),$$

so that, from Lemma (2.2.1), we infer that, for every solution $z(t)$ satisfying $0 < |z(t)| < \rho$,

$$\mathrm{Rot}\,(z(t);[0,kT]) > \left\lfloor \frac{kT}{\tau_{V_\delta}} \right\rfloor.$$

Since

$$\tau_{V_\delta} = \frac{1}{1-\delta}\tau_V \searrow \tau_V$$

for $\delta \to 0$, the conclusion follows. ☐

Remark 2.2.1. Notice that the equality involving $\gamma(t)$ assumed in (2.24) is just a matter of normalization, provided that $\int_0^T \gamma(t)\, dt > 0$ (however, $\gamma(t)$ is allowed to change sign), since (2.25) and (2.26) are invariant under the dilatation

$$\gamma(t) \mapsto \lambda\gamma(t), \qquad V(z) \mapsto \frac{1}{\lambda}V(z), \qquad \text{for } \lambda > 0.$$

Remark 2.2.2. Observe that hypotheses (2.25) and (2.26) can be weakened to hold in an L^1-sense. Precisely, focusing for instance on (2.25) with $\odot = 0$, we can require the following:

- *for every $\epsilon > 0$ there exist $r_\epsilon > 0$ and $\eta_\epsilon \in L^1_T$, with $\int_0^T |\eta_\epsilon(t)|\, dt \leq \epsilon$, such that, for almost every $t \in [0,T]$ and every $0 < |z| < r_\epsilon$,*

$$\frac{\langle Z(t,z)|z\rangle}{2V(z)} \geq \gamma(t) - \eta_\epsilon(t).$$

We now turn our attention to possible corollaries of Lemma 2.2.1 for the scalar second order equation (2.19). The first one concerns a nonresonant case. Again, $\odot$ will denote either 0 or $+\infty$. We postpone our comments after the statement.

Proposition 2.2.2. *If $\odot = 0$, assume $f(t,0) \equiv 0$. The following statements hold true.*

- *Assume that there exists $p \in L^1_T$ such that*

$$\liminf_{|x|\to\odot} \frac{f(t,x)}{x} \geq p(t), \quad \textit{uniformly for a.e. } t \in [0,T]. \tag{2.28}$$

  If there exists $j \in \mathbb{N}_*$ such that

$$\frac{\sqrt{\lambda_j}}{k} < \sup_{\xi>0} \frac{\frac{1}{T}\int_0^T \min\{p(t),\xi\}\, dt}{\sqrt{\xi}}, \tag{2.29}$$

  then there exists $\rho > 0$ such that, for every $u : [0,kT] \to \mathbb{R}$ solving (2.19) and satisfying, for every $t \in [0,kT]$, $0 < u(t)^2 + u'(t)^2 < \rho^2$ if $\odot = 0$, or $u(t)^2 + u'(t)^2 > \rho^2$ if $\odot = +\infty$, it holds

$$\mathrm{Rot}\,((u(t),u'(t));[0,kT]) > j;$$

- *Assume that there exists $q \in L^1_T$ such that*

$$\limsup_{|x|\to\odot} \frac{f(t,x)}{x} \leq q(t), \quad \textit{uniformly for a.e. } t \in [0,T]. \tag{2.30}$$

  If there exists $j \in \mathbb{N}_*$ such that

$$\inf_{\zeta>0} \frac{\frac{1}{T}\int_0^T \max\{q(t),\zeta\}\, dt}{\sqrt{\zeta}} < \frac{\sqrt{\lambda_j}}{k}, \tag{2.31}$$

  then there exists $\rho > 0$ such that, for every $u : [0,kT] \to \mathbb{R}$ solving (2.19) and satisfying, for every $t \in [0,kT]$, $0 < u(t)^2 + u'(t)^2 < \rho^2$ if $\odot = 0$, or $u(t)^2 + u'(t)^2 > \rho^2$ if $\odot = +\infty$, it holds

$$\mathrm{Rot}\,((u(t),u'(t));[0,kT]) < j.$$

Proof. We will only prove the first statement, the second being similar. Anyway, the cases $\odot = 0$ and $\odot = +\infty$ are slightly different, so we will prove both of them. Set $z = (x, y)$, $Z(t, z) = (f(t, x), y)$ and choose $\xi, \epsilon > 0$ such that, in view of (2.29),

$$\sqrt{\lambda_j} < \frac{\frac{1}{T}\int_0^T k \min\{p(t) - \epsilon, \xi\}\, dt}{\sqrt{\xi}}. \tag{2.32}$$

Set $V(x, y) = \frac{1}{2k}\sqrt{\lambda_j \xi}\Big(x^2 + \frac{1}{\xi}y^2\Big)$; computing τ_V, it is easy to see that $\frac{kT}{\tau_V} = j$.

Assume $\odot = 0$. Setting

$$\beta(t) = k\frac{\min\{p(t) - \epsilon, \xi\}}{\sqrt{\lambda_j \xi}},$$

we have, in view of (2.32), that $\frac{1}{kT}\int_0^{kT} \beta(t)\, dt > 1$. On the other hand, by (2.28), there exists $\rho > 0$ such that, for almost every $t \in [0, T]$ and every $|x| < \rho$,

$$f(t, x)x \geq (p(t) - \epsilon)x^2.$$

For $0 < |z| < \rho$, it follows that

$$\begin{aligned} \frac{\langle Z(t, z)|z\rangle}{2V(z)} &= k\frac{f(t, x)x + y^2}{\sqrt{\lambda_j \xi}\Big(x^2 + \frac{1}{\xi}y^2\Big)} \\ &\geq k\frac{\min\{p(t) - \epsilon, \xi\}x^2 + \frac{1}{\xi}\min\{p(t) - \epsilon, \xi\}y^2}{\sqrt{\lambda_j \xi}\Big(x^2 + \frac{1}{\xi}y^2\Big)} \\ &\geq k\frac{\min\{p(t) - \epsilon, \xi\}}{\sqrt{\lambda_j \xi}} = \beta(t), \end{aligned}$$

whence the conclusion by Lemma 2.2.1.

Assume now $\odot = +\infty$. Using (2.28) and the fact that $f(t, x)$ is L^1-Carathéodory, there exists $r \in L^1_T$ such that, for almost every $t \in [0, T]$ and every $x \in \mathbb{R}$,

$$f(t, x)x \geq (p(t) - \epsilon)x^2 - r(t).$$

Setting, for $t \in \mathbb{R}$ and $z \neq 0$,

$$\beta(t, x, y) = k\left(\frac{\min\{p(t) - \epsilon, \xi\}}{\sqrt{\lambda_j \xi}} - \frac{r(t)}{\sqrt{\lambda_j \xi}\Big(x^2 + \frac{1}{\xi}y^2\Big)}\right),$$

we see, in view of (2.32), that it is possible to choose $\rho > 0$ sufficiently large such that

$$\frac{1}{kT}\int_0^{kT} \beta(t, u(t), u'(t))\, dt > 1,$$

for every $u : [0, kT] \to \mathbb{R}$ solving (2.19) such that $u(t)^2 + u'(t)^2 > \rho^2$. On the other hand, with computations similar as before, we have, for almost every $t \in [0, T]$ and every $z \neq 0$, that

$$\frac{\langle Z(t, z)|z\rangle}{2V(z)} \geq \beta(t, z),$$

whence the conclusion by Lemma 2.2.1. □

Remark 2.2.3. Conditions (2.29) and (2.31), this last with λ_j replaced by λ_{j+1}, were introduced by Fabry [65], for a nonlinearity satisfying

$$p(t) \le \liminf_{|x|\to+\infty} \frac{f(t,x)}{x} \le \limsup_{|x|\to+\infty} \frac{g(t,x)}{x} \le q(t), \tag{2.33}$$

in order to ensure the solvability of the kT-periodic problem associated with (2.19). Here, we have seen a separate interpretation of each of the inequalities in (2.33) in terms of the rotation number, and we have shown that the estimates can be carried out in an analogous way both at 0 and at $+\infty$. We highlight the two following special cases of conditions (2.29) and (2.31), referring for simplicity to the case $k = 1$:

1) if

$$\operatorname{ess\,inf}_{[0,T]} p(t) > 0 \quad \text{and} \quad \sqrt{\lambda_j} < \sqrt{\operatorname{ess\,inf}_{[0,T]} p(t)}, \tag{2.34}$$

then (2.29) is satisfied. Indeed, we have

$$\sqrt{\operatorname{ess\,inf}_{[0,T]} p(t)} \le \sup_{\xi>0} \frac{\frac{1}{T}\int_0^T \min\{p(t),\xi\}\,dt}{\sqrt{\xi}},$$

as it can be seen taking $\xi = \operatorname{ess\,inf}_{[0,T]} p(t)$ in the right-hand side. Analogously, if

$$\operatorname{ess\,sup}_{[0,T]} q(t) > 0 \quad \text{and} \quad \sqrt{\operatorname{ess\,sup}_{[0,T]} q(t)} < \sqrt{\lambda_j}, \tag{2.35}$$

then (2.31) is satisfied. Conditions (2.34) and (2.35) are standard nonresonance conditions with respect to λ_j;

2) if

$$\operatorname{ess\,sup}_{[0,T]} p(t) > 0 \quad \text{and} \quad \sqrt{\lambda_j} < \frac{\frac{1}{T}\int_0^T p(t)\,dt}{\sqrt{\operatorname{ess\,sup}_{[0,T]} p(t)}},$$

then (2.29) is satisfied. Indeed, we have

$$\frac{\frac{1}{T}\int_0^T p(t)\,dt}{\sqrt{\operatorname{ess\,sup}_{[0,T]} p(t)}} \le \sup_{\xi>0} \frac{\frac{1}{T}\int_0^T \min\{p(t),\xi\}\,dt}{\sqrt{\xi}},$$

as it can be seen taking $\xi = \operatorname{ess\,sup}_{[0,T]} p(t)$ in the right-hand side. Analogously, if

$$\operatorname{ess\,inf}_{[0,T]} q(t) > 0 \quad \text{and} \quad \frac{\frac{1}{T}\int_0^T q(t)\,dt}{\sqrt{\operatorname{ess\,inf}_{[0,T]} q(t)}} < \sqrt{\lambda_j},$$

then (2.31) is satisfied. This shows, in particular, that conditions (2.29) and (2.31) allow $f(t,x)/x$ to cross an arbitrary number of eigenvalues.

As a second corollary of Lemma 2.2.1, we give another sufficient condition to achieve the same conclusion, which is independent of the previous one. Again, we postpone our comments after the statement; for simplicity, moreover, we deal only with the case $k = 1$.

Proposition 2.2.3. *If $\odot = 0$, assume $f(t,0) \equiv 0$. Assume that there exist $p, q \in L^1_T$ such that*

$$p(t) \le \liminf_{|x|\to\odot} \frac{f(t,x)}{x} \le \limsup_{|x|\to\odot} \frac{f(t,x)}{x} \le q(t), \quad \textit{uniformly for a.e. } t \in [0,T]. \tag{2.36}$$

The following statements hold true.

- *If there exists $j \in \mathbb{N}_*$ such that*

$$\lambda_j \le p(t), \quad \textit{with} \quad \lambda_j < \frac{1}{T}\int_0^T p(t)\,dt, \tag{2.37}$$

then there exists $\rho > 0$ such that, for every $u : [0,T] \to \mathbb{R}$ solving (2.19) and satisfying, for every $t \in [0,T]$, $0 < u(t)^2 + u'(t)^2 < \rho^2$ if $\odot = 0$, or $u(t)^2 + u'(t)^2 > \rho^2$ if $\odot = +\infty$, it holds

$$\mathrm{Rot}\,((u(t), u'(t)); [0,T]) > j.$$

- *If there exists $j \in \mathbb{N}_*$ such that*

$$q(t) \le \lambda_j, \quad \textit{with} \quad \frac{1}{T}\int_0^T q(t)\,dt < \lambda_j, \tag{2.38}$$

then there exists $\rho > 0$ such that, for every $u : [0,T] \to \mathbb{R}$ solving (2.19) and satisfying, for every $t \in [0,T]$, $0 < u(t)^2 + u'(t)^2 < \rho^2$ if $\odot = 0$, or $u(t)^2 + u'(t)^2 > \rho^2$ if $\odot = +\infty$, it holds

$$\mathrm{Rot}\,((u(t), u'(t)); [0,T]) < j.$$

Proof. We will only prove the first statement, the second being similar. Anyway, the cases $\odot = 0$ and $\odot = +\infty$ are slightly different, so we will prove both of them. Set $z = (x,y)$, $Z(t,z) = (f(t,x), y)$ and $V(x,y) = \frac{1}{2}(\frac{\lambda_j}{k}x^2 + y^2)$; it is easy to see that $\frac{kT}{\tau_V} = j$. Moreover, write $p(t) = \lambda_j + \eta(t)$, so that $\eta(t) \ge 0$ and $\frac{1}{T}\int_0^T \eta(t)\,dt > 0$.

Assume first $\odot = 0$.

<u>Claim.</u> *There exist $\sigma, r > 0$ such that, for every $u : [0,T] \to \mathbb{R}$ solving (2.19) with $0 < u(t)^2 + u'(t)^2 < r^2$ for every $t \in [0,T]$, it holds*

$$\int_0^T \frac{\eta(t)u(t)^2}{\lambda_j u(t)^2 + u'(t)^2}\,dt \ge \sigma.$$

By contradiction, assume that there exists a sequence of functions $u_n(t)$ solving (2.19), with $0 < u_n(t)^2 + u_n'(t)^2 < 1/n$ for every $t \in [0,T]$, such that

$$\mathcal{I}(u_n) = \int_0^T \frac{\eta(t)u_n(t)^2}{\lambda_j u_n(t)^2 + u_n'(t)^2}\,dt \to 0,$$

for $n \to +\infty$. Set

$$v_n(t) = \tfrac{u_n(t)}{\|u_n\|_{C^1}},$$

and observe that, for every n,

$$v_n''(t) + h(t, u_n(t))v_n(t) = 0, \tag{2.39}$$

with

$$h(t,x) = \begin{cases} \dfrac{f(t,x)}{x} & \text{if } x \neq 0 \\ \\ 0 & \text{if } x = 0. \end{cases}$$

Since $u_n(t)$ has at most a finite number of zeros (otherwise it would have a double zero, which is impossible since $u_n(t)^2 + u_n'(t)^2 > 0$ for every $t \in [0,T]$), it is easy to see that $h(t, u_n(t))$ is measurable, for every n. Moreover, hypothesis (2.36) ensures that there exists $\alpha \in L^1(0,T)$ such that, for almost every $t \in [0,T]$ and every $|x| < 1$,

$$|h(t,x)| \leq \alpha(t). \tag{2.40}$$

As $|u_n(t)| < 1$, then, by the Dunford-Pettis theorem (see [63]) there exists $a \in L^1(0,T)$ such that, up to subsequences, $h(t, u_n(t)) \rightharpoonup a(t)$ in $L^1(0,T)$. On the other hand, a standard argument based on (2.40), on the Dunford-Pettis theorem and on Ascoli's theorem ensures the existence of a nonzero $v \in W^{2,1}(0,T)$ such that, up to subsequences, $v_n \to v$ strongly in $C^1([0,T])$ and weakly in $W^{2,1}(0,T)$ (see [30, Proposition 3.3] for details). Passing to the limit in (2.39), it follows that $v(t)$ satisfies the linear equation

$$v'' + a(t)v = 0,$$

so that, by the uniqueness of the associated Cauchy problem, $v(t)$ has a finite number of zeros, all of them simple. Consequently, $\mathcal{I}(v)$ is well defined, and, since $\eta(t) \geq 0$ and $\frac{1}{T}\int_0^T \eta(t)\,dt > 0$, we deduce that

$$\mathcal{I}(v) = \int_0^T \frac{\eta(t)v(t)^2}{v(t)^2 + v'(t)^2}\,dt > 0.$$

On the other hand, since $v_n \to v$ in $C^1([0,T])$,

$$\mathcal{I}(u_n) = \mathcal{I}(v_n) \to \mathcal{I}(v) = 0,$$

a contradiction which proves the claim. □

We now show that Lemma 2.2.1 implies the conclusion. Fix $0 < \epsilon < \lambda_j \sigma$: by assumption (2.36), there exists $0 < \rho < r$ such that, for almost every $t \in [0,T]$ and for every $|x| < \rho$,

$$f(t,x)x \geq (\lambda_j + \eta(t) - \epsilon)x^2.$$

Hence, defining, for $0 < |z| < \rho$,

$$\beta(t,x,y) = 1 + \frac{(\eta(t) - \epsilon)x^2}{\lambda_j x^2 + y^2},$$

the hypotheses of Lemma 2.2.1 are satisfied, whence the conclusion in the case $\odot = 0$.

Assume now $\odot = +\infty$.

Claim. *There exist $\sigma, R > 0$ such that, for every $u : [0,T] \to \mathbb{R}$ solving (2.19) with $u(t)^2 + u'(t)^2 > R^2$ for every $t \in [0,T]$, it holds*

$$\int_0^T \frac{\eta(t)u(t)^2}{u(t)^2 + u'(t)^2}\, dt \geq \sigma.$$

By contradiction, assume that there exists a sequence of functions $u_n(t)$ solving (2.19), with $u_n(t)^2 + u_n'(t)^2 > n$ for every $t \in [0,T]$, such that

$$\mathcal{I}(u_n) = \int_0^T \frac{\eta(t)u_n(t)^2}{\lambda_j u_n(t)^2 + u_n'(t)^2}\, dt \to 0,$$

for $n \to +\infty$. Set

$$v_n(t) = \tfrac{u_n(t)}{\|u_n\|_{C^1}},$$

and observe that, for every n,

$$v_n''(t) + h(t, u_n(t))v_n(t) + \frac{r(t, u_n(t))}{\|u_n\|_{C^1}} = 0,$$

with

$$h(t,x) = \begin{cases} \dfrac{f(t,x)}{x} & \text{if } |x| \geq 1 \\[2ex] \dfrac{(x+1)}{2} f(t,1) - \dfrac{(x-1)}{2} f(t,-1) & \text{if } |x| \leq 1, \end{cases}$$

and $r(t,x) = f(t,x) - h(t,x)x$. Since $h(t,x)$ and $r(t,x)$ are L^1-Carathéodory, $h(t,u_n(t))$ and $r(t,u_n(t))$ are measurable for every n. Moreover, by (2.36), there exists $\alpha \in L^1(0,T)$ such that, for almost every $t \in [0,T]$ and every $x \in \mathbb{R}$,

$$|h(t,x)| \leq \alpha(t), \qquad |r(t,x)| \leq \alpha(t).$$

Similarly as in the case $\odot = 0$, we conclude the validity of the claim. □

Fix now $0 < \epsilon < \frac{\lambda_j \sigma}{2}$: by assumption (2.36), and the fact that $g(t,x)$ is L^1-Carathéodory, there exists $r \in L^1_T$ such that, for almost every $t \in [0,T]$ and every $x \in \mathbb{R}$,

$$f(t,x)x \geq (\lambda_j + \eta(t) - \epsilon)x^2 - r(t).$$

Now, choose $\rho \geq R$ such that, if $x^2 + y^2 > \rho^2$,

$$\frac{1}{\lambda_j x^2 + y^2} \int_0^T r(t)\, dt < \frac{\sigma}{2}.$$

Defining, for $|z| > \rho$,

$$\beta(t,x,y) = 1 + \frac{(\eta(t) - \epsilon)x^2 - z(t)}{\lambda_j x^2 + y^2},$$

the hypotheses of Lemma 2.2.1 are satisfied, so that the conclusion follows. □

Remark 2.2.4. Conditions (2.37) and (2.38), this last with λ_j replaced by λ_{j+1}, were first introduced by Mawhin and Ward [126], for a nonlinearity satisfying

$$p(t) \leq \liminf_{|x|\to+\infty} \frac{f(t,x)}{x} \leq \limsup_{|x|\to+\infty} \frac{f(t,x)}{x} \leq q(t),$$

ensuring the solvability of (2.19). This situation is usually referred to as *nonuniform nonresonance*. Again, here we have seen an interpretation of each of such inequalities in terms of the rotation number, and we have shown that the estimates can be carried out in an analogous way both at 0 and at $+\infty$.

2.2.2 The resonant case: the Landesman-Lazer conditions

In this section, we consider again the system (2.18), focusing on the role played by a Landesman-Lazer type condition [106, 107] in the estimate of the rotation number of large solutions, when (2.18) is asymptotic, at infinity, to a resonant system of the type $Jz' = \gamma(t)\nabla V(z)$, with $V \in \mathcal{P}$. Notice that the choice of an L^2-framework (namely, with $Z(t,z)$ an L^2-Carathéodory function) would be more typical when dealing with Landesman-Lazer conditions (see Remark 2.2.5), but we emphasize that, for our purposes, it is possible to work in an L^1-setting, as well.

Proposition 2.2.4. *Let $V \in \mathcal{P}$, with $T/\tau_V \in \mathbb{N}$. Moreover, assume that there exist a continuous and T-periodic function $\gamma : \mathbb{R} \to \mathbb{R}$, with $\gamma(t) > 0$ for every $t \in [0,T]$ and*

$$\frac{1}{T}\int_0^T \gamma(t)\, dt = 1, \tag{2.41}$$

and an L^1-Carathéodory function $R : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, T-periodic in the first variable, with

$$\lim_{|z|\to+\infty} \frac{R(t,z)}{|z|} = 0, \quad \textit{uniformly for a.e. } t \in [0,T], \tag{2.42}$$

such that

$$Z(t,z) = \gamma(t)\nabla V(z) + R(t,z). \tag{2.43}$$

Set $\Gamma(t) = \int_0^t \gamma(s)\, ds$. The following statements hold true.

- *Assume:*
  - *for almost every $t \in [0,T]$, for every $z \in \mathbb{R}^2$ with $|z| \leq 1$ and for every $\lambda > 1$,*

$$\langle R(t,\lambda z) | z\rangle \geq \eta(t); \tag{2.44}$$

    for a suitable $\eta \in L^1(0,T)$,
  - *for every $\theta \in [0,T]$,*

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(\Gamma(t)+\omega)) | \varphi_V(\Gamma(t)+\omega)\rangle > 0. \tag{2.45}$$

Then there exists $\rho > 0$ such that, for every $z : [0,T] \to \mathbb{R}^2$ solving (2.18), with $|z(t)| > \rho$ for every $t \in [0,T]$, it holds

$$\mathrm{Rot}\,(z(t);[0,T]) > \frac{T}{\tau_V}.$$

- *Assume:*
  - *for almost every $t \in [0,T]$, for every $z \in \mathbb{R}^2$ with $|z| \leq 1$ and for every $\lambda > 1$,*

$$\langle R(t,\lambda z)|z\rangle \leq \eta(t), \tag{2.46}$$

  for a suitable $\eta \in L^1(0,T)$;
  - *for every $\theta \in [0,T]$,*

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(\Gamma(t)+\omega))|\varphi_V(\Gamma(t)+\omega)\rangle < 0. \tag{2.47}$$

Then there exists $\rho > 0$ such that, for every $z : [0,T] \to \mathbb{R}^2$ solving (2.18), with $|z(t)| > \rho$ for every $t \in [0,T]$, it holds

$$\mathrm{Rot}\,(z(t);[0,T]) < \frac{T}{\tau_V}.$$

Proof. We prove the first statement. For $z : [0,T] \to \mathbb{R}^2$ solving (2.18) and such that $z(t) \neq 0$ for every $t \in [0,T]$, we have, in view of Euler's formula and (2.41),

$$\begin{aligned}
\mathrm{Rot}_V\,(u(t)) &= \frac{1}{\tau_V}\int_0^T \frac{\langle Jz'(t)|z(t)\rangle}{2V(z(t))}\,dt = \frac{1}{\tau_V}\int_0^T \frac{\langle Z(t,z(t))|z(t)\rangle}{2V(z(t))}\,dt \\
&= \frac{1}{\tau_V}\int_0^T \gamma(t)\,\frac{\langle \nabla V(z(t))|z(t)\rangle}{2V(z(t))}\,dt + \frac{1}{\tau_V}\int_0^T \frac{\langle R(t,z(t))|z(t)\rangle}{2V(z(t))}\,dt \\
&= \frac{T}{\tau_V} + \frac{1}{\tau_V}\int_0^T \frac{\langle R(t,z(t))|z(t)\rangle}{2V(z(t))}\,dt.
\end{aligned}$$

In view of Proposition 2.1.2, to conclude the proof it is sufficient to show that there exists $\rho > 0$ such that

$$\int_0^T \frac{\langle R(t,z(t))|z(t)\rangle}{2V(z(t))}\,dt > 0,$$

for every $z : [0,T] \to \mathbb{R}^2$ solving (2.18) and such that $|z(t)| > \rho$ for every $t \in [0,T]$. Let us suppose by contradiction that there exists a sequence of functions $z_n(t)$ solving (2.18) such that $|z_n(t)| \to +\infty$ uniformly, with

$$\int_0^T \frac{\langle R(t,z_n(t))|z_n(t)\rangle}{2V(z_n(t))}\,dt \leq 0. \tag{2.48}$$

We set $v_n(t) = \frac{z_n(t)}{\|z_n\|_{L^\infty}}$; since

$$Jv_n' = \gamma(t)\nabla V(v_n) + \frac{R(t, z_n(t))}{\|z_n\|_{L^\infty}}, \tag{2.49}$$

it follows from (2.42) that there exists $\alpha \in L^1(0,T)$ such that, for almost every $t \in [0,T]$,

$$|v_n'(t)| \leq \alpha(t). \tag{2.50}$$

This implies the equicontinuity of the family $\{v_n(t)\}_n$, so that, by Ascoli's theorem, there exists a nonzero $v \in C([0,T];\mathbb{R}^2)$ such that, up to subsequences, $v_n(t) \to v(t)$ uniformly. On the other hand, in view of (2.50), the Dunford-Pettis theorem (see [63]) applies, yielding the existence of $w \in L^1(]0,T[\,;\mathbb{R}^2)$ such that (up to subsequences) $v_n' \rightharpoonup w$ in $L^1(]0,T[\,;\mathbb{R}^2)$. It is now easy to see that $v \in W^{1,1}(]0,T[\,;\mathbb{R}^2)$, with $v' = w$. Passing to the limit in (2.49), then, we get, in view of (2.42), $Jv' = \gamma(t)\nabla V(v)$, implying, by Lemma 2.1.2, that

$$v(t) = C\varphi_V(\Gamma(t) + \theta),$$

for suitable constants $C > 0$ and $\theta \in [0, \tau_V[$. Performing the change of variables

$$z_n(t) = r_n(t)\varphi_V(\Gamma(t) + \theta_n(t)),$$

with $\theta_n(0) \in [0, \tau_V[$ for every n, it follows that

$$\frac{r_n(t)}{\|z_n\|_{L^\infty}} \to C, \quad \text{uniformly in } t \in [0,T], \tag{2.51}$$

and it can be seen (see [72]) that

$$\theta_n(t) \to \theta, \quad \text{uniformly in } t \in [0,T].$$

Multiplying (2.48) by $\|z_n\|_{L^\infty}$, we get

$$\int_0^T \frac{\langle R(t, r_n(t)\varphi_V(\Gamma(t) + \theta_n(t))) | \varphi_V(\Gamma(t) + \theta_n(t))\rangle}{2\frac{r_n(t)}{\|z_n\|_{L^\infty}} V(\varphi_V(\Gamma(t) + \theta_n(t)))}\,dt \leq 0.$$

Using Fatou's lemma, thanks to (2.44), and noticing that $V(\varphi_V(\Gamma(t) + \theta_n(t))) \equiv \frac{1}{2}$, we get

$$\int_0^T \liminf_{n\to+\infty} \frac{\langle R(t, r_n(t)\varphi_V(\Gamma(t) + \theta_n(t))) | \varphi_V(\Gamma(t) + \theta_n(t))\rangle}{\frac{r_n(t)}{\|z_n\|_{L^\infty}}}\,dt \leq 0.$$

In view of (2.51) and using standard properties of the inferior limit, we infer that

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_V(\Gamma(t) + \omega)) | \varphi_V(\Gamma(t) + \omega)\rangle\,dt \leq 0,$$

a contradiction. □

Remark 2.2.5. Condition (2.44)-(2.45) (resp., (2.46)-(2.47)) was recently introduced by Fonda and Garrione [72] - in an L^2-framework - as a planar version of the scalar Landesman-Lazer condition (2.54)-(2.55) (resp., (2.56)-(2.57), see Corollary 2.2.5) by Fabry [66]. In [72] (see also [78]) it has been used to ensure, for instance, the solvability of the T-periodic problem associated with (2.18) with $Z(t, z)$ of the form (2.43). The main point here, of course, is the integral condition (2.45), while (2.44) is a technical assumption, needed in order to apply Fatou's lemma, and it is satisfied, in particular, when $R(t, z)$ is bounded. Here we have analyzed the relationships between such Landesman-Lazer condition and the rotation number of large-norm solutions of (2.18).

Remark 2.2.6. Notice that, similarly as in Remark 2.2.2, hypothesis (2.42) can be weakened into the following L^1-condition:

- *for every $\epsilon > 0$, there exists $b_\epsilon \in L^1(0, T)$ such that, for almost every $t \in [0, T]$ and every $z \in \mathbb{R}^2$,*
$$|R(t, z)| \leq \epsilon |z| + b_\epsilon(t).$$

Indeed, this is enough to carry out the same proof, in particular to pass to the limit in (2.49).

Let us now focus on the second order case (2.19). As a corollary of Proposition 2.2.4, we have the following result.

Proposition 2.2.5. *Assume that there exist $j \in \mathbb{N}_*$ and $\mu, \nu > 0$, with*

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{j}, \tag{2.52}$$

such that, uniformly for almost every $t \in [0, T]$,

$$\lim_{x \to +\infty} \frac{f(t, x)}{x} = \mu, \quad \lim_{x \to -\infty} \frac{f(t, x)}{x} = \nu. \tag{2.53}$$

The following statements hold true.

- *Assume:*
  - *for almost every $t \in [0, T]$, and every $x \in \mathbb{R}$,*
  $$\operatorname{sgn}(x)(f(t, x) - \mu x^+ + \nu x^-) \geq \eta(t), \tag{2.54}$$
  for a suitable $\eta \in L^1(0, T)$;
  - *for every $\phi(t)$ nontrivial solution (defined on $[0, T]$) to $u'' + \mu u^+ - \nu u^- = 0$,*
  $$\int_{\{\phi > 0\}} \liminf_{x \to +\infty} (f(t, x) - \mu x)\phi(t)\, dt + \int_{\{\phi < 0\}} \limsup_{x \to -\infty} (f(t, x) - \nu x)\phi(t)\, dt > 0. \tag{2.55}$$

  Then there exists $\rho > 0$ such that for every $u : [0, T] \to \mathbb{R}$ solving (2.19), with $u(t)^2 + u'(t)^2 > \rho^2$ for every $t \in [0, T]$, it holds
  $$\operatorname{Rot}((u(t), u'(t)); [0, T]) > j.$$

- *Assume:*
  - *for almost every $t \in [0,T]$, and every $x \in \mathbb{R}$,*
$$\operatorname{sgn}(x)(f(t,x) - \mu x^+ + \nu x^-) \leq \eta(t), \tag{2.56}$$
    for a suitable $\eta \in L^1(0,T)$;
  - *for every $\phi(t)$ nontrivial solution (defined on $[0,T]$) to $u'' + \mu u^+ - \nu u^- = 0$,*
$$\int_{\{\phi>0\}} \limsup_{x\to+\infty}(f(t,x) - \mu x)\phi(t)\,dt + \int_{\{\phi<0\}} \liminf_{x\to-\infty}(f(t,x) - \nu x)\phi(t)\,dt < 0. \tag{2.57}$$

  Then there exists $\rho > 0$ such that for every $u : [0,T] \to \mathbb{R}$ solving (2.19), with $u(t)^2 + u'(t)^2 > \rho^2$ for every $t \in [0,T]$, it holds
$$\operatorname{Rot}((u(t), u'(t)); [0,T]) < j.$$

Proof. Setting $z = (x,y)$, $Z(t,z) = (f(t,x), y)$, $V(x,y) = \frac{1}{2}(y^2 + \mu(x^+)^2 + \nu(x^-)^2)$ and $\gamma(t) \equiv 1$, we briefly show that the hypotheses of Proposition 2.2.4 (actually, in the weaker version of Remark 2.2.6) are satisfied. Using (2.52), it is easy to see that $\frac{T}{\tau_V} = j$. Moreover, (2.53) and the Carathéodory assumption imply that, fixed $\epsilon > 0$, there exists $b_\epsilon \in L^1(0,T)$ such that
$$|f(t,x) - \mu x^+ + \nu x^-| \leq \epsilon|x| + b_\epsilon(t),$$
from which the condition of Remark 2.2.6 follows. Finally, (2.54) implies (2.44), and (2.56) implies (2.46), while for the proof that (2.55) and (2.57) imply (2.45) and (2.47), respectively, we refer to [72, Propositions 3.1-3.2]. □

Remark 2.2.7. Observe that, in this case, it is considered the more general situation of resonance with respect to the Dancer-Fučik spectrum (see [83]), i.e., the spectrum of the asymmetric equation
$$u'' + \mu u^+ - \nu u^- = 0, \qquad \mu, \nu \geq 0, \tag{2.58}$$
where $u^+ = \max\{u, 0\}$, and $u^- = \max\{-u, 0\}$. Indeed, when (2.52) holds, all the nontrivial solutions to (2.58) are T-periodic, and have $2j$ zeros in the interval $[0, T[$. When $\mu = \nu$, (2.52) implies $\mu = \lambda_j$, so that we recover the well known linear theory.

Remark 2.2.8. We do not know if a condition of Landesman-Lazer type can be formulated at zero, to control the behavior of small solutions (clearly, the natural one obtained by the mere replacement of ∞ with 0 is senseless). To this aim, to the best of our knowledge, rougher sign conditions are usually employed.

2.3 Multiplicity results

In this final section, we apply the estimates developed in Section 2.2, together with the Poincaré-Birkhoff fixed point theorem, Theorem 1.1.3, in order to give some multiplicity results for periodic solutions (winding around the origin) to the planar Hamiltonian system
$$Jz' = \nabla_z H(t,z). \tag{2.59}$$

Throughout the section, $H : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is a function which is T-periodic in the first variable and differentiable in the second one, with $\nabla_z H(t,z)$ an L^1-Carathéodory function such that $\nabla_z H(t,0) \equiv 0$. Notice that, even if the results of Section 2.2 are valid in the general context of planar systems, now we need the Hamiltonian structure in order to apply the Poincaré-Birkhoff fixed point theorem. In particular, this is the case for the second order undamped equation (2.19), with $H(t,x,y) = \frac{1}{2}y^2 + \int_0^x f(t,\xi)\,d\xi$.

All our statements exploit both an assumption at zero and an assumption at infinity among the ones previously introduced (so that, for related comments, we refer the reader back to the corresponding sections), in order to show that a twist condition is satisfied. We remark that we limit ourselves to a few combinations of them (including in particular semilinear, sublinear, superlinear problems), but it will be clear that several other statements could be obtained, in the same way. We postpone a sketch of the (standard) proofs in the final part of the section.

2.3.1 Semilinear problem

Our first result deals with a nonresonance situation.

Theorem 2.3.1. *Assume the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (2.59). *Moreover, assume:*

(H_0) $\nabla_z H(t,0) \equiv 0$ *and there exist* $V_0 \in \mathcal{P}$ *and* $\gamma_0 \in L^1_T$, *with* $\frac{1}{T}\int_0^T \gamma_0(t)\,dt = 1$, *such that*

$$\liminf_{|z|\to 0} \frac{\langle \nabla_z H(t,z)|z\rangle}{2V_0(z)} \geq \gamma_0(t), \quad \text{uniformly for a.e. } t \in [0,T];$$

(H_∞) there exist $V_\infty \in \mathcal{P}$ *and* $\gamma_\infty \in L^1_T$, *with* $\frac{1}{T}\int_0^T \gamma_\infty(t)\,dt = 1$, *such that*

$$\limsup_{|z|\to \infty} \frac{\langle \nabla_z H(t,z)|z\rangle}{2V_\infty(z)} \leq \gamma_\infty(t), \quad \text{uniformly for a.e. } t \in [0,T].$$

Finally, assume that there exist $j_0, j_\infty \in \mathbb{N}_*$ *such that*

$$\frac{T}{\tau_{V_\infty}} < j_\infty \leq j_0 < \frac{T}{\tau_{V_0}}. \tag{2.60}$$

Then, for every integer $j \in [j_\infty, j_0]$, *there exist two* T*-periodic solutions* $z_j^{(1)}(t), z_j^{(2)}(t)$ *to* (2.59) *such that*

$$\mathrm{Rot}\,(z_j^{(1)}(t);[0,T]) = \mathrm{Rot}\,(z_j^{(2)}(t);[0,T]) = j.$$

Notice that a symmetric statement can be obtained when we assume a lim sup-inequality in (H_0) and a lim inf-inequality in (H_∞), giving the existence of solutions for any integer $j \in [j_0, j_\infty]$ (when $j_0 \leq j_\infty$). As a consequence, when

$$\lim_{|z|\to 0} \frac{\langle \nabla_z H(t,z)|u\rangle}{2V_0(z)} = \gamma_0(t), \qquad \lim_{|z|\to \infty} \frac{\langle \nabla_z H(t,z)|z\rangle}{2V_\infty(z)} = \gamma_\infty(t),$$

and

$$\frac{T}{\tau_{V_0}} \notin \mathbb{N}, \qquad \frac{T}{\tau_{V_\infty}} \notin \mathbb{N},$$

the statement provides the existence of

$$2\left|\left\lfloor \frac{T}{\tau_{V_0}} \right\rfloor - \left\lfloor \frac{T}{\tau_{V_\infty}} \right\rfloor\right|$$

T-periodic solutions. Notice that this, in particular, holds when

$$\nabla_z H(t,z) = \gamma_0(t)\nabla V_0(z) + o(|z|), \quad \text{for } |z| \to 0,$$

$$\nabla_z H(t,z) = \gamma_\infty(t)\nabla V_\infty(z) + o(|z|), \quad \text{for } |z| \to \infty.$$

The spirit of this kind of results is similar to the ones of [71, Theorem 6], [73, Theorem 1.1]. When (2.59) is asymptotically linear at zero and at infinity, moreover, multiplicity results have been given by Margheri, Rebelo and Zanolin [118], in terms of the Conley-Zehnder indexes of the limit systems (see Remark 1.1.2).

Now we pass to consider a resonant (at infinity) situation, i.e., when, with the previous notation,

$$\frac{T}{\tau_{V_\infty}} = j_\infty \in \mathbb{N}.$$

Of course, in this case, Theorem 2.3.1 can still be applied, giving T-periodic solutions with rotation number equal to j for every $j \in [j_\infty + 1, j_0]$ (if any). However, the existence of solutions making exactly j_∞ revolutions is no longer guaranteed. To recover it, we add a Landesman-Lazer condition.

Theorem 2.3.2. *Assume the uniqueness for the solutions to the Cauchy problems associated with* (2.59)*. Moreover, assume:*

(H_0) $\nabla_z H(t,0) \equiv 0$ *and there exist* $V_0 \in \mathcal{P}$*,* $j_0 \in \mathbb{N}_*$ *and* $\gamma_0 \in L^1_T$*, with*

$$\frac{T}{\tau_{V_0}} > j_0, \qquad \frac{1}{T}\int_0^T \gamma_0(t)\,dt = 1,$$

such that

$$\liminf_{|z|\to 0} \frac{\langle \nabla_z H(t,z) | z\rangle}{2V_0(z)} \geq \gamma_0(t), \quad \textit{uniformly for a.e. } t \in [0,T];$$

(H_∞) there exist $V_\infty \in \mathcal{P}$*,* $j_\infty \in \mathbb{N}_*$*,* $\gamma_\infty \in C_T$ *with* $\gamma_\infty(t) > 0$ *for every* $t \in [0,T]$*, such that*

$$\frac{T}{\tau_{V_\infty}} = j_\infty, \qquad \frac{1}{T}\int_0^T \gamma_\infty(t)\,dt = 1,$$

and an L^1*-Carathéodory function* $R_\infty : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$*,* T*-periodic in the first variable, with*

$$\lim_{|z|\to+\infty} \frac{R_\infty(t,z)}{|z|} = 0, \quad \textit{uniformly for a.e. } t \in [0,T],$$

such that

$$\nabla_z H(t,z) = \gamma_\infty(t)\nabla V_\infty(z) + R_\infty(t,z).$$

Moreover, setting $\Gamma_\infty(t) = \int_0^t \gamma_\infty(s)\,ds$*, suppose that:*

- *for almost every* $t \in [0,T]$*, for every* $z \in \mathbb{R}^2$ *with* $|z| \leq 1$ *and for every* $\lambda > 1$*,*

$$\langle R_\infty(t,\lambda z)|z\rangle \leq \eta(t),$$

for a suitable $\eta \in L^1(0,T)$*;*

- *for every* $\theta \in [0,T]$*,*

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R_\infty(\Gamma_\infty(t), \lambda\varphi_{V_\infty}(\Gamma_\infty(t)+\omega))|\varphi_{V_\infty}(\Gamma_\infty(t)+\omega)\rangle < 0.$$

If $j_\infty \leq j_0$*, then, for every integer* $j \in [j_\infty, j_0]$*, there exist two* T*-periodic solutions* $z_j^{(1)}(t)$ *and* $z_j^{(2)}(t)$ *to* (2.59) *such that*

$$\mathrm{Rot}\,(z_j^{(1)}(t);[0,T]) = \mathrm{Rot}\,(z_j^{(2)}(t);[0,T]) = j.$$

Remark 2.3.1. We remark that, in Theorems 2.3.1 and 2.3.2, $\gamma_0(t)$ and $\gamma_\infty(t)$ are positive functions, at least in mean value. It would be possible to consider negative functions, as well, possibly giving the existence of T-periodic solutions rotating counterclockwise. For the sake of brevity, however, and since this is not possible for the second order scalar equation when written in the standard phase plane setting, we have chosen to present our results from this simpler point of view.

For the second order equation (2.19), we have the following results, which are the counterparts of Theorems 2.3.1 and 2.3.2. Theorem 2.3.3 below can be seen as a more applicable version of the results by Zanini [167].

Theorem 2.3.3. *Assume the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (2.19)*. Moreover, assume:*

(f_0) $f(t,0) \equiv 0$ *and there exist* $j_0 \in \mathbb{N}_*$ *and* $q_0 \in L^1_T$ *with*

$$\sup_{\xi>0} \frac{\frac{1}{T}\int_0^T \{\min q_0(t),\xi\}\,dt}{\sqrt{\xi}} > \sqrt{\lambda_{j_0}},$$

such that

$$\liminf_{x\to 0} \frac{f(t,x)}{x} \geq q_0(t), \quad \textit{uniformly for a.e. } t \in [0,T];$$

(f_∞) there exist $j_\infty \in \mathbb{N}_*$ *and* $q_\infty \in L^1_T$ *with*

$$\inf_{\zeta>0} \frac{\frac{1}{T}\int_0^T \max\{q_\infty(t),\zeta\}\,dt}{\sqrt{\zeta}} < \sqrt{\lambda_{j_\infty}},$$

such that

$$\limsup_{|x|\to+\infty} \frac{f(t,x)}{x} \leq q_\infty(t), \quad \textit{uniformly for a.e. } t \in [0,T].$$

If $j_\infty \leq j_0$, then, for every integer $j \in [j_\infty, j_0]$, there exist two T-periodic solutions $u_j^{(1)}(t)$ and $u_j^{(2)}(t)$ to (2.19) *having exactly $2j$ zeroes in $[0,T[$.*

Finally, Theorem 2.3.4 deals with the situation when the nonlinearity interacts, at infinity, with the Dancer-Fučik spectrum. Even if assumption (f_0) of Theorem 2.3.3 is still suitable, this time we propose a nonuniform nonresonance condition.

Theorem 2.3.4. *Assume the uniqueness for the solutions to the Cauchy problems associated with* (2.19). *Moreover, assume:*

(f_0) $f(t,0) \equiv 0$ and there exist $j_0 \in \mathbb{N}_*$ and $q_0 \in L^1_T$, with

$$q_0(t) \geq \lambda_{j_0} \quad \text{and} \quad \frac{1}{T}\int_0^T q_0(t)\,dt > \lambda_{j_0},$$

such that

$$\lim_{x\to 0}\frac{f(t,x)}{x} = q_0(t), \quad \text{uniformly for a.e. } t \in [0,T];$$

(f_∞) there exist $j_\infty \in \mathbb{N}_*$ and $\mu, \nu > 0$, with

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{j_\infty},$$

such that:

- *uniformly for almost every $t \in [0,T]$,*

$$\lim_{x\to+\infty}\frac{f(t,x)}{x} = \mu, \quad \lim_{x\to-\infty}\frac{f(t,x)}{x} = \nu,$$

- *for almost every $t \in [0,T]$, and every $x \in \mathbb{R}$,*

$$\mathrm{sgn}\,(x)(f(t,x) - \mu x^+ + \nu x^-) \geq \eta(t),$$

  for a suitable $\eta \in L^1(0,T)$,
- *for every $\phi(t)$ nontrivial solution (defined on $[0,T]$) to $u'' + \mu u^+ - \nu u^- = 0$,*

$$\int_{\{\phi>0\}} \limsup_{x\to+\infty}(f(t,x) - \mu x)\phi(t)\,dt + \int_{\{\phi<0\}} \liminf_{x\to-\infty}(f(t,x) - \nu x)\phi(t)\,dt < 0.$$

If $j_\infty \leq j_0$, then, for every integer $j \in [j_\infty, j_0]$, there exist two T-periodic solutions $u_j^{(1)}(t)$ and $u_j^{(2)}(t)$ to (2.19) *having exactly $2j$ zeroes in $[0,T[$.*

We finally show that, whenever any kind of gap between 0 and ∞ is assumed, it is still possible, exploiting a number theory argument developed in [58], to prove the existence of subharmonics of order k, for every k large enough.

Theorem 2.3.5. *Assume the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (2.59). *Moreover, suppose:*

(H_0) $\nabla_z H(t,0) \equiv 0$ *and there exist* $V_0 \in \mathcal{P}$ *and* $\gamma_0 \in L^1_T$, *with* $\frac{1}{T}\int_0^T \gamma_0(t)\,dt = 1$, *such that*

$$\liminf_{|z|\to 0} \frac{\langle \nabla_z H(t,z)|z\rangle}{2V_0(z)} \geq \gamma_0(t), \quad \textit{uniformly for a.e. } t \in [0,T];$$

(H_∞) *there exist* $V_\infty \in \mathcal{P}$ *and* $\gamma_\infty \in L^1_T$, *with* $\frac{1}{T}\int_0^T \gamma_\infty(t)\,dt = 1$, *such that*

$$\limsup_{|z|\to \infty} \frac{\langle \nabla_z H(t,z)|z\rangle}{2V_\infty(z)} \leq \gamma_\infty(t), \quad \textit{uniformly for a.e. } t \in [0,T].$$

Finally, assume that

$$\frac{T}{\tau_{V_\infty}} < \frac{T}{\tau_{V_0}}. \tag{2.61}$$

Then, for every $r \in \mathbb{N}_*$, *there exists an integer* $k^*(r)$ *such that, for every* $k \geq m^*(r)$, *there exist* $2r$ *subharmonics of order* k, *two by two non belonging to the same periodicity class, solving* (2.59).

We remark that condition (2.61) is not sufficient to obtain the existence of a T-periodic solution, since there could be no integers in the interval $\left]\frac{T}{\tau_{V_\infty}}, \frac{T}{\tau_{V_0}}\right[$ (compare with (2.60)). Moreover, observe that, fixed $k \in \mathbb{N}_*$, we are not showing the existence of infinitely many subharmonics of order k; nevertheless, we have subharmonics for any k large enough, and such a number increases with k. Of course, Theorem 2.3.5 just describes a model situation, and the more complicated cases previously considered could also be taken into account.

2.3.2 Sublinear problem

Our main result is the following.

Theorem 2.3.6. *Assume the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (2.59). *Moreover, assume:*

(H_0) $\nabla_z H(t,0) \equiv 0$ *and there exist* $V_0 \in \mathcal{P}$ *and* $\gamma_0 \in L^1_T$, *with* $\frac{1}{T}\int_0^T \gamma_0(t)\,dt = 1$, *such that*

$$\liminf_{|z|\to 0} \frac{\langle \nabla_z H(t,z)|z\rangle}{2V_0(z)} \geq \gamma_0(t), \quad \textit{uniformly for a.e. } t \in [0,T];$$

(H_∞) *there exist sequences* $(V_\infty^n)_n \subset \mathcal{P}$, $(\gamma_\infty^n)_n \subset L^1_T$, *with* $\frac{1}{T}\int_0^T \gamma_\infty^n(t)\,dt = 1$, *such that*

$$\lim_{n\to+\infty} \frac{1}{\tau_{V_\infty^n}} = 0$$

and, for every $n \in \mathbb{N}_*$,

$$\limsup_{|z|\to +\infty} \frac{\langle \nabla_z H(t,z)|z\rangle}{2V_n(z)} \leq \gamma_\infty^n(t), \quad \textit{uniformly for a.e. } t \in [0,T].$$

Then there exists $k^* \in \mathbb{N}_*$ such that, for every integer $k \geq k^*$, there exists an integer $j_*(k)$ such that, for every integer j relatively prime with k and such that $1 \leq j \leq j_*(k)$, system (2.59) *has at least two subharmonic solutions $z^1_{k,j}(t), z^2_{k,j}(t)$ of order k (not belonging to the same periodicity class) with*

$$\mathrm{Rot}(z^{(1)}_{k,j}(t); [0, kT]) = \mathrm{Rot}(z^{(2)}_{k,j}(t); [0, kT]) = j.$$

Moreover, we have the estimate

$$j_*(k) \geq \mathcal{E}^- \left(\frac{kT}{\tau_{V_0}} \right). \tag{2.62}$$

It can be worth noticing some special features of the subharmonic solutions provided by Theorem 2.3.6. First of all, they are accompanied with a sharp nodal characterization. Observe in particular that $j_*(k) \to +\infty$ for $k \to +\infty$ and that, taking $j = 1$, Theorem 2.3.6 ensures the existence of subharmonic solutions of order k for every k large enough: in this case kT is the "true" minimal period (namely, not only the minimal one in the class of integer multiples of T). As pointed out in [40, p. 428], this is a sharper conclusion with respect to those of papers using variational techniques, when typically only a sequence of solutions with minimal periods tending to infinity is provided. Secondly, notice that from (2.62) we can estimate the order of the subharmonics found. In particular, $k^* = 1$ is not excluded (this being the case when $j_*(1) \geq 1$), yielding the existence of $j_*(1)$ pairs of T-periodic solutions.

Assumption (H_∞) of Theorem 2.3.6 is our (weak) sublinearity condition for $\nabla_z H(t, z)$. Indeed, it is verified in the case

$$\limsup_{|z| \to +\infty} \frac{\langle \nabla_z H(t, z) | z \rangle}{|z|^2} \leq 0, \qquad \text{uniformly for a.e. } t \in [0, T], \tag{2.63}$$

by taking $V^n_\infty(z) = |z|^2/n$ and $\gamma^n_\infty(t) \equiv 1$, for every n. On the other hand, it also covers some cases when (2.63) is not fulfilled. It is worth noticing, for instance, that (2.63) always fails in the case of (2.19) (since $H(t, x, y)$ grows quadratically in the y-variable), while (H_∞) is satisfied whenever $f(t, x)/x \to 0$ for $|x| \to +\infty$ (see [26, Corollary 3.1] and [27, Corollary 1.1], taking into account Remark 2.2.2). Accordingly, Theorem 2.3.6 has the following counterpart for the second order scalar equation (2.19).

Theorem 2.3.7. *Assume the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (2.19)*. Moreover, assume:*

(f_0) there exists $q_0 \in L^\infty_T$ with $\int_0^T q_0(t)\, dt > 0$ such that

$$\liminf_{x \to 0} \frac{f(t, x)}{x} \geq q_0(t), \qquad \textit{uniformly in } t \in [0, T];$$

(f_∞) it holds

$$\limsup_{|x| \to +\infty} \frac{f(t, x)}{x} \leq 0, \qquad \textit{uniformly in } t \in [0, T].$$

Then there exists $k^* \in \mathbb{N}_*$ such that, for every integer $k \geq k^*$, there exists an integer $j_*(k)$ such that, for every integer j relatively prime with k and such that $1 \leq j \leq j_*(k)$, equation (2.19) *has at least two subharmonic solutions $u^{(1)}_{k,j}(t), u^{(2)}_{k,j}(t)$ of order k (not belonging to the same periodicity class) with exactly $2j$ zeros in the interval $[0, kT[$. Moreover, we have the estimate*

$$j_*(k) \geq \mathcal{E}^- \left(\frac{k}{2\pi} \sup_{\xi > 0} \frac{\int_0^T \min\{q_0(t), \xi\}\, dt}{\sqrt{\xi}} \right).$$

Notice that, when $q_0 \in L^\infty_T$ with $\int_0^T q_0(t)\, dt > 0$, then

$$\sup_{\xi>0} \frac{\int_0^T \min\{q_0(t), \xi\}\, dt}{\sqrt{\xi}} \geq \frac{\int_0^T q_0(t)\, dt}{\sqrt{\operatorname{ess\,sup}_{[0,T]} q_0(t)}} > 0,$$

so that $j_*(k) \to +\infty$ for $k \to +\infty$. Results about subharmonic solutions of sublinear second order equations like (2.19) were given (pairing the sublinearity assumption with Landesman-Lazer type conditions) by Ding and Zanolin [59], Fonda, Schneider and Zanolin [77], Fonda and Ramos [75]. However, these subharmonics grow in norm as k approaches to infinity: on the contrary, this is not the case for Theorem 2.3.7, whose subharmonics can be quite small (see Chapter 4 for various examples in this direction).

Example 2.3.1. Theorem 2.3.7 straightly applies to the Sitnikov equation. For a detailed description of this classical problem in Celestial Mechanics, as well as of some classical and recent results, we refer to [136]. Here we just recall that such a problem leads to the following equation

$$u'' + \frac{u}{(u^2 + r(t,e)^2)^{3/2}} = 0, \tag{2.64}$$

being $e \in [0, 1[$ the eccentricity of the orbits described by the primaries and $r(\cdot, e)$ implicitly defined by

$$r(t,e) = \frac{1}{2}(1 - e\cos v(t)), \qquad v(t) - e \sin v(t) = t.$$

The possibility of achieving multiplicity results for (2.64) via the Poincaré-Birkhoff theorem is suggested in [167, Example 1], but only a semi-abstract result in term of the weighted eigenvalues of $u'' + \lambda \frac{u}{r(t,e)^3} = 0$ is given there. Here, we can get the following.

Corollary 2.3.1. *For every $k \in \mathbb{N}_*$ and for every integer j relatively prime with k and such that*

$$1 \leq j \leq k \left(\frac{2}{1+e} \right)^{\frac{3}{2}},$$

equation (2.64) *has at least two $2k\pi$-periodic solutions (not $2l\pi$-periodic for $l = 1, \cdots, k-1$) with exactly $2j$ zeros in $[0, 2k\pi[$.*

We point out that the conclusion is optimal for the circular Sitnikov problem (i.e., the autonomous case $e = 0$), while subharmonics with larger number of zeros seem to be lost for greater values of e. This fact seems to be strictly related to the question raised in [136, p. 731].

Remark 2.3.2. A more general version of Theorem 2.3.6 is given in [26, Theorem 4.1], by considering Hamiltonian functions satisfying condition (H_∞) only in an angular region of the plane (that is, for $z = \rho e^{i\theta}$ with $\theta \in [\theta_1, \theta_2]$, being $0 < \theta_2 - \theta_1 < 2\pi$). As a consequence, one can prove that Theorem 2.3.7 still holds true if the sublinearity condition (f_∞) is replaced by the one-side assumption

$$\limsup_{x \to +\infty} \frac{f(t,x)}{x} \leq 0, \qquad \text{uniformly in } t \in [0,T]$$

(see also Lemma 4.2.6 for a proof of this fact). Such an improved version of Theorem 2.3.6 also applies to Lotka-Volterra type planar systems, as described in [26, Section 5].

2.3.3 Superlinear problem

Our main result is the following.

Theorem 2.3.8. *Assume the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (2.59)*. Moreover, assume:*

(H_0) $\nabla_z H(t,0) \equiv 0$ *and there exist* $V_0 \in \mathcal{P}$ *and* $\gamma_0 \in L^1_T$, *with* $\frac{1}{T}\int_0^T \gamma_0(t)\,dt = 1$, *such that*

$$\liminf_{|z| \to 0} \frac{\langle \nabla_z H(t,z) | z \rangle}{2V_0(z)} \leq \gamma_0(t), \quad \textit{uniformly for a.e. } t \in [0,T];$$

(H_∞) there exist sequences $(V^n_\infty)_n \subset \mathcal{P}$, $(\gamma^n_\infty)_n \subset L^1_T$, *with* $\frac{1}{T}\int_0^T \gamma^n_\infty(t)\,dt = 1$, *such that*

$$\lim_{n \to +\infty} \frac{1}{\tau_{V^n_\infty}} = +\infty$$

and, for every $n \in \mathbb{N}_*$,

$$\limsup_{|z| \to +\infty} \frac{\langle \nabla_z H(t,z) | z \rangle}{2V_n(z)} \geq \gamma^n_\infty(t), \qquad \textit{uniformly for a.e. } t \in [0,T].$$

Then, for every $k \in \mathbb{N}_*$, *system* (2.59) *has infinitely many* kT*-periodic solutions. More precisely, for every* $k \in \mathbb{N}_*$, *there exists an integer* $j^*(k)$ *such that, for every integer* j *relatively prime with* k *and such that* $j \geq j^*(k)$, *system* (2.59) *has at least two subharmonic solutions* $z^1_{k,j}(t), z^2_{k,j}(t)$ *of order* k *(not belonging to the same periodicity class) with*

$$\mathrm{Rot}(z^{(1)}_{k,j}(t);[0,kT]) = \mathrm{Rot}(z^{(2)}_{k,j}(t);[0,kT]) = j.$$

Moreover, for every $k \geq 1$ *and* $i = 1,2$,

$$\lim_{j \to +\infty} |z^{(i)}_{k,j}(t)| = +\infty, \qquad \textit{uniformly in } t \in [0,kT],$$

and we have the estimate

$$j^*(k) \leq \mathcal{E}^+\left(\frac{kT}{\tau_{V_0}}\right). \tag{2.65}$$

Notice that condition (H_0) of Theorem 2.3.8 is very mild, since it is satisfied, for instance, when $H(t,z)$ is twice differentiable in the second variable, with $D_z^2 H(t,z)$ an L^1-Caratheodory function. Using the more general version of the Poincaré-Birkhoff fixed point theorem recalled in Remark 1.1.1, one could show that the assumption at zero can be removed at all (up to loosing, of course, estimate (2.65)). Therefore, Theorem 2.3.8 is valid for planar Hamiltonian systems satisfying assumption (H_∞), without requiring $\nabla_z H(t,0) \equiv 0$: we refer to [29, Theorem 2.3] for the precise statement of such a more general result. It can be seen as the Hamiltonian version of a result by Capietto, Mawhin and Zanolin [43, Theorem 4], which proves the existence of a T-periodic solution for a superlinear (in general, non-Hamiltonian) planar system (as remarked in [43, p. 389], in the general case no more than one T-periodic solution can be expected).

Assumption (H_∞) of Theorem 2.3.8 is our (weak) superlinearity condition for $\nabla_z H(t,z)$ and it is more general than the Ambrosetti-Rabinowitz condition ($0 < k < 1/2$, $R \geq 0$)

$$0 < H(t,z) \leq k \langle \nabla_z H(t,z) | z \rangle, \qquad \text{for every } t \in [0,T],\ |z| \geq R, \tag{2.66}$$

which is often employed when system (2.59) is treated with variational techniques (see for instance [2, 13]). Indeed, from (2.66) we get that $H(t,z) \geq a|z|^{1/k}$ for every $t \in [0,T]$, $|z| \geq R$ (with $a > 0$), so that (H_∞) is satisfied with $V_\infty^n(z) = n|z|^2$ and $\gamma_\infty^n(t) \equiv 1$.
Notice also that (H_∞) can be considered as the dual version of the sublinearity condition of Theorem 2.3.6 and, similarly as before, is satisfied in case of (2.19), whenever $f(t,x)/x \to +\infty$ for $|x| \to +\infty$ (see [29, Remark 2.4], taking into account Remark 2.2.2). We do not write here explicitly the counterpart of Theorem 2.3.8 for the second order scalar equation (2.19), since it is a well known result, going back to Hartman and Jacobowitz [97, 99].

Remark 2.3.3. It could look very restrictive to assume the global continuability for the solutions in a superlinear context. Actually, this is not the case since the asymptotic dynamics of (2.59) is of center type. For instance, it is possible to see that the following mild assumption suffices:

- *the function $H(t,z)$ is of class C^1, with*

$$\lim_{|z|\to+\infty} H(t,z) = +\infty, \qquad \textit{uniformly in } t \in [0,T],$$

and, for suitable constants $c, M > 0$,

$$\left| \frac{\partial}{\partial t} H(t,z) \right| \leq c H(t,z), \qquad \textit{for every } t \in [0,T],\ |z| \geq M.$$

For a proof of this fact, as well as for other conditions ensuring the global existence of solutions, we refer to [25, Remark 2.5].

2.3.4 Proofs

All the proofs are based on Theorem 1.1.3, using the estimates developed in Section 2.2.

Indeed, the uniqueness for the initial value problems is always assumed; on the other hand, the global continuability is required in Theorems 2.3.1, 2.3.3, 2.3.5, 2.3.6, 2.3.7 and 2.3.8, while it follows from (H_∞) in Theorem 2.3.2 and (f_∞) in Theorem 2.3.4.

As for the estimates of the rotation numbers of small and large solutions to (2.59), we use Proposition 2.2.1, at zero and at infinity, for Theorems 2.3.1, 2.3.6 and 2.3.8, Propositions 2.2.1 and 2.2.4 for Theorem 2.3.2, Proposition 2.2.2 for Theorems 2.3.3 and 2.3.7, Proposition 2.2.3 and Proposition 2.2.5 for Theorem 2.3.4. The possibility of transferring estimates for solutions which are, respectively, uniformly (i.e., for every $t \in [0, kT]$) small and large to estimates for solutions departing, respectively, from a small and a large circumference with center the origin is ensured by the following facts, usually referred to as "elastic property":

Lemma 2.3.1. *The following hold true:*

(E_0) *for every* $k \in \mathbb{N}_*$ *and for every* $r > 0$*, there exists* $0 < \rho_i < r$ *such that, if* $|\bar{z}| = \rho_i$*, then* $0 < |z(t; \bar{z})| < r$ *for every* $t \in [0, kT]$*;*

(E_∞) *for every* $k \in \mathbb{N}_*$ *and for every* $R > 0$*, there exists* $\rho_o > R$ *such that, if* $|\bar{z}| = \rho_o$*, then* $|z(t; \bar{z})| > R$ *for every* $t \in [0, kT]$*.*

Property (E_0) is a simple consequence of the fact that $\nabla_z H(t, 0) \equiv 0$, together with the uniqueness for the Cauchy problems, while (E_∞) follows from the global continuability, via a compactness argument (see, for instance, [152, Lemma 10]). Even more refined versions of Lemma 2.3.1 will be given in Section 4.2 (see Lemma 4.2.3) and in Section 6.1 (see Lemma 6.1.2).

Finally, the proof of Theorem 2.3.5 requires some more work; we sketch here the argument. It has to be shown that, for every $r \in \mathbb{N}_*$, there exists $k^*(r)$ such that, for every $k \geq k^*(r)$, there exist r integers $j_1^{(k)}, \dots, j_r^{(k)}$, each one relatively prime with k, and such that, for every $i = 1, \dots, r$,

$$\frac{T}{\tau_{V_0}} < \frac{j_i^{(k)}}{k} < \frac{T}{\tau_{V_\infty}}.$$

In view of Theorem 2.3.1, this gives, for every $i = 1, \dots, r$, the existence of two kT-periodic solutions making $j_i^{(k)}$ turns around the origin, whence the conclusion. Dividing the interval $\left] \frac{T}{\tau_{V_\infty}}, \frac{T}{\tau_{V_0}} \right[$ into r subintervals, without loss of generality it is possible to assume $r = 1$. In this case, the argument to achieve the conclusion is based on some properties of prime numbers, and can be found in the proof of [58, Theorem 2.3].

For complete details, we refer to the corresponding proofs in [26, 27, 29, 30].

Chapter 3

Combining Poincaré-Birkhoff Theorem, coincidence degree and lower/upper solutions techniques

This chapter, whose material is based on [33], proposes a general strategy to find, via the Poincaré-Birkhoff theorem, periodic solutions to second order scalar differential equations of the type

$$u'' + f(t,u) = 0, \tag{3.1}$$

being $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function, T-periodic in the first variable, in presence of lower/upper T-periodic solutions.

To motivate our discussion, we can consider the autonomous equation

$$u'' + a \sin u = 0, \qquad a > 0. \tag{3.2}$$

Phase plane analysis (see Figure 3 below) shows that the origin is a local center for (3.2): "small" solutions are periodic, winding around the origin in the clockwise sense. These periodic solutions persist until the two heteroclinic orbits joining the (unstable) equilibria $(-\pi, 0)$ and $(\pi, 0)$ are reached. Using the fact that the minimal period of the periodic solutions approaches infinity as the energy goes to the critical value corresponding to the heteroclinic trajectories, time-map arguments (compare with the introduction of Chapter 2) give the existence of periodic solutions, confined in the strip $\{-\pi < x < \pi\}$ of the phase plane, of every period large enough (the lower bound for their number depending on the parameter $a > 0$). Such a dynamics is reminiscent of that of a sublinear problem (i.e., when large solutions do not complete, in a fixed time interval, a full turn around the origin); however, in this situation the conclusion is completely independent of the asymptotic properties of the nonlinear term $f(x) = a \sin x$, simply relying on the existence of the equilibrium points $(\pm\pi, 0)$.

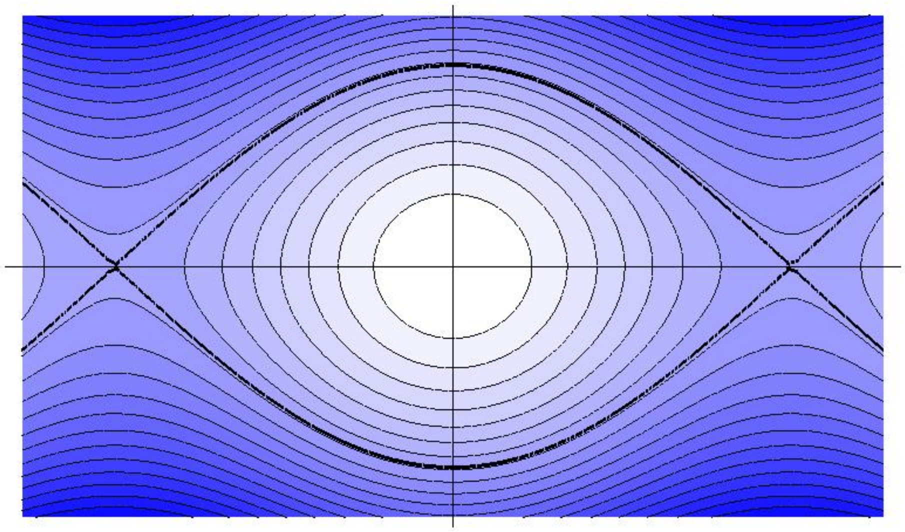


Figure 3.1: The phase portrait of the autonomous equation $u'' + a \sin u = 0$, with $a = 1$. Periodic solutions (around the origin) of arbitrarily large minimal period are found in the region bounded by the two heteroclinic trajectories (painted with a darker color) joining the equilibria $(-\pi, 0)$ and $(\pi, 0)$.

The aim of the present chapter is to show that, in the nonautonomous case (3.1), the same dynamical scenario appears when (keeping an assumption at the origin which gives a local dynamics of center type) the given equation possesses a negative T-periodic lower solution $\alpha(t)$ or a positive T-periodic upper solution $\beta(t)$. Indeed, standard truncations lead to an equivalent one-sided sublinear problem at infinity; according to Theorem 2.3.7 and Remark 2.3.2, the Poincaré-Birkhoff twist theorem now yields the existence of subharmonic solutions of any order, making few turns around the origin. We stress that the result is essentially local in nature, since we do not have conditions on the behavior of the nonlinear term at infinity, and the subharmonic solutions provided lie in a suitable vertical strip of the phase plane.

This technique, although very simple, seems to apply to a variety of different situations and the second part of the chapter provides some examples in this direction. The applications proposed concern parameter dependent pendulum type equations and two different Ambrosetti-Prodi problems, before considered in [18, 48, 163] and [69, 152], respectively.
The analysis is performed in two main steps. First, we prove the existence of a T-periodic solution to (3.1), say $u^*(t)$, via Mawhin's coincidence degree [120] (see also [122] for a modern presentation of the theory): such a solution becomes the "equilibrium" in order to enter in the setting of unforced equations previously introduced in Chapters 1 and 2. It is clear that a suitable localization of $u^*(t)$ has to be provided as well, since we need to show that the local dynamics around it is of center type (providing the inner boundary of the annulus to which the Poincaré-Birkhoff theorem applies). Second, we prove the existence of lower and upper T-periodic solutions, thus having the possibility to conclude (via a truncation argument) as described above.

The plan of this chapter is as follows. In Section 3.1 we present our results, Theorems 3.1.1 and 3.1.2, dealing with subharmonic solutions to (3.1) in presence of lower/upper T-periodic solutions. Section 3.2 and Section 3.3 are devoted to the applications: in the first one we consider pendulum type equations, in the second one we give some Ambrosetti-Prodi type results.

3.1 Subharmonic solutions between lower and upper solutions

In this section, we are concerned with the second order scalar equation (3.1), proving the existence of subharmonic solutions in presence of lower/upper T-periodic solutions. For simplicity, we always assume that $f(t,x)$ *is continuous with continuous partial derivative* $\frac{\partial f}{\partial x}(t,x)$. Notice, on the other hand, that we are not assuming the condition $f(t,0) \equiv 0$.

Here is our main result.

Theorem 3.1.1. *Let us suppose that:*

(A_1) *there exists a T-periodic solution $u^*(t)$ to* (3.1) *satisfying*

$$\int_0^T \frac{\partial f}{\partial x}(t, u^*(t))\, dt > 0; \tag{3.3}$$

(A_2) *there exists a T-periodic function $\alpha(t)$, of class C^2, such that*

$$\alpha''(t) + f(t,\alpha(t)) \geq 0, \qquad \text{for every } t \in [0,T]; \tag{3.4}$$

(A_3) *there exists a T-periodic function $\beta(t)$, of class C^2, such that*

$$\beta''(t) + f(t,\beta(t)) \leq 0, \qquad \text{for every } t \in [0,T]. \tag{3.5}$$

Assume, moreover, that

$$\alpha(t) < u^*(t) < \beta(t), \qquad \text{for every } t \in [0,T]. \tag{3.6}$$

Then there exists $k^* \in \mathbb{N}_*$ such that, for every integer $k \geq k^*$, there exists an integer $j_*(k)$ such that, for every integer j relatively prime with k and such that $1 \leq j \leq j_*(k)$, equation (3.1) *has at least two subharmonic solutions $u^{(1)}_{k,j}(t), u^{(2)}_{k,j}(t)$ of order k (not belonging to the same periodicity class) such that, for $i = 1,2$, $u^{(i)}_{k,j}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$ and*

$$\alpha(t) \leq u^{(i)}_{k,j}(t) \leq \beta(t), \qquad \text{for every } t \in \mathbb{R}. \tag{3.7}$$

Proof. Define the function $\widetilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by setting

$$\widetilde{f}(t,x) = \begin{cases} f(t,\alpha(t)) & \text{if } x \leq \alpha(t) \\ f(t,x) & \text{if } \alpha(t) < x \leq \beta(t) \\ f(t,\beta(t)) & \text{if } \beta(t) < x \end{cases}$$

Being $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, we have that $\widetilde{f}(t,x)$ is well defined, T-periodic in the first variable, continuous and locally Lipschitz continuous in x and globally bounded. Moreover, by the regularity assumption on $f(t,x)$, the partial derivative $\frac{\partial \widetilde{f}}{\partial x}(t,x) = \frac{\partial f}{\partial x}(t,x)$ exists in a neighborhood of $(t, u^*(t))$. We further claim the following:

<u>Claim:</u> *if $u(t)$ is a kT-periodic solution to*

$$u'' + \widetilde{f}(t,u) = 0 \tag{3.8}$$

such that $u(t) - u^*(t)$ has at least one zero in $\mathbb{R}$, then $\alpha(t) \leq u(t) \leq \beta(t)$ for every $t \in \mathbb{R}$, so that $u(t)$ solves (3.1) *as well.*

We prove that $u(t) \leq \beta(t)$, the other inequality being analogous. Let us suppose by contradiction that, for some $t^* \in \mathbb{R}$, $\beta(t^*) < u(t^*)$ and define $[t^-, t^+]$ to be the largest interval containing t^* and such that $\beta(t) < u(t)$ for every $t \in]t^-, t^+[$. Since $u^*(t) < \beta(t)$ and, by kT-periodicity, $u(t) - u^*(t)$ has at least one zero in every interval of length kT, we deduce that $-\infty < t^- < t^+ < +\infty$. We clearly have

$$(u - \beta)(t^-) = (u - \beta)(t^+) = 0 \tag{3.9}$$

and

$$(u - \beta)'(t^-) \geq 0 \geq (u - \beta)'(t^+); \tag{3.10}$$

moreover, for every $t \in]t^-, t^+[$,

$$\begin{aligned}(u-\beta)''(t) &= u''(t) - \beta''(t) \geq -\tilde{f}(t,u(t)) + f(t,\beta(t)) \\ &= -f(t,\beta(t)) + f(t,\beta(t)) = 0.\end{aligned}$$

Together with (3.9) and (3.10), this implies that $(u - \beta)(t) = 0$ for every $t \in]t^-, t^+[$, a contradiction. □

According to the above discussion, to conclude it is enough to prove the existence of subharmonic solutions $u(t)$ to (3.8) such that $u(t) - u^*(t)$ changes sign. We proceed as follows: set, for $(t,x) \in \mathbb{R} \times \mathbb{R}$,

$$g(t,x) = \widetilde{f}(t, x + u^*(t)) - \widetilde{f}(t, u^*(t))$$

and consider the equation

$$v'' + g(t,v) = 0. \tag{3.11}$$

Since $g(t,0) \equiv 0$, this is an unforced equation (like those in Chapter 2) and, as it is clear, $v(t)$ is a solution to (3.11) if and only if $u(t) = u^*(t) + v(t)$ is a solution to (3.8). As a consequence, the global continuability for the solutions to (3.11) holds. Moreover, it is easy to see that $v(t)$ is a sign-changing subharmonic of order k to (3.11) if and only if $u(t)$ is a subharmonic of order k to (3.1) such that $u(t) - u^*(t)$ changes sign. To conclude, we are going to use Theorem 2.3.7. Indeed, the regularity of $\widetilde{f}(t,x)$ ensures that the uniqueness for the solutions to the Cauchy problems holds; moreover, since $\widetilde{f}(t,x)$ is bounded, condition (f_∞) of Theorem 2.3.7 holds true. Finally, we check that (f_0) holds true with the choice $q(t) = \frac{\partial f}{\partial x}(t, u^*(t))$. Indeed, since there exists $\frac{\partial \widetilde{f}}{\partial x}(t,x) = \frac{\partial f}{\partial x}(t,x)$ in a neighborhood of $(t, u^*(t))$, Langrange theorem implies

$$\frac{g(t,x)}{x} = \frac{\widetilde{f}(t, x + u^*(t)) - \widetilde{f}(t, u^*(t))}{x} = \frac{\partial f}{\partial x}(t, \xi(t,x)),$$

for a suitable $\xi(t,x)$ such that $|\xi(t,x)-u^*(t)| \leq |x|$. Letting $x \to 0$, in view of the uniform continuity of $\frac{\partial f}{\partial x}(t,x)$ on compact subsets, we get

$$\frac{\partial f}{\partial x}(t,\xi(t,x)) \to \frac{\partial f}{\partial x}(t,u^*(t)) = q(t), \quad \text{uniformly in } t \in [0,T],$$

and hence $g(t,x)/x \to q(t)$ for $x \to 0$, as desired. □

Remark 3.1.1. A T-periodic function satisfying the differential inequality (3.4) (resp., (3.5)) is usually referred to as a (classical) *lower solution* (resp., *upper solution*) to the T-periodic problem associated with equation (3.1) (weaker concepts of lower/upper solutions could be introduced as well). Notice that in Theorem 3.1.1 we assume the existence of a T-periodic solution $u^*(t)$ satisfying (3.6) and it is well known that between a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ satisfying $\alpha(t) \leq \beta(t)$ a T-periodic solution always exists (see, for instance, [55]). Nevertheless, to our purposes the explicit knowledge of $u^*(t)$ is essential in view of the assumption (3.3), while, on the other hand, the existence of $\alpha(t)$ and $\beta(t)$ only plays the role of an additional information which allows to perform a suitable truncation. In this regard, it can be noticed that this fact also allows us to provide a localization of the subharmonic solutions produced, which indeed lie in a vertical strip of the phase plane determined by $\alpha(t)$ and $\beta(t)$ (see (3.7)). Observe finally, that, if $\alpha(t)$ is a solution, then the uniqueness for the solutions to the Cauchy problems further implies that $\alpha(t) < u^{(i)}_{k,j}(t)$ for every $t \in \mathbb{R}$. Analogously, $u^{(i)}_{k,j}(t) < \beta(t)$, whenever $\beta(t)$ is a solution.

Remark 3.1.2. Another fact that it may be worth mentioning here is related to the instability property of the periodic solutions obtained via the lower/upper solutions technique. Indeed, according to [160] (see also [54] for previous results in this direction), under the assumptions of Theorem 3.1.1 and when at least one between $\alpha(t)$ and $\beta(t)$ is not a solution to (3.1), there exists a T-periodic solution to (3.1), satisfying (3.6), which is unstable both in the past and in the future. On the other hand, according to [134, 135], our condition (3.3) is not incompatible with the stability of the solution $u^*(t)$ itself: this is not a contradiction. In fact, in our setting $\alpha(t)$ and/or $\beta(t)$ may well be solutions to (3.1) (indeed, this will occur in some of the applications of Sections 3.2 and 3.3); moreover, we require the existence of a T-periodic solution $u^*(t)$ satisfying (3.3), but this does not prevent the existence of other (possibly unstable) T-periodic solutions.

We now state a variant of Theorem 3.1.1 in which we assume solely the existence of an upper solution $\beta(t)$; clearly, a dual result, in presence of a lower solution $\alpha(t)$, could be obtained as well. For our theorem we suppose that *the global continuability for the solutions to* (3.1) *is ensured* (see Remark 3.1.3 below).

Theorem 3.1.2. *Assume* (A_1) *and* (A_3) *of Theorem 3.1.1 and suppose that*

$$u^*(t) < \beta(t), \qquad \textit{for every } t \in [0,T].$$

Then the same conclusion of Theorem 3.1.1 holds, with (3.7) *replaced by*

$$u^{(i)}_{k,j}(t) \leq \beta(t), \qquad \textit{for every } t \in \mathbb{R}.$$

Proof. The proof is similar to that of Theorem 3.1.1, using the truncation

$$\widetilde{f}(t,x) = \begin{cases} f(t,x) & \text{if } x \leq \beta(t) \\ f(t,\beta(t)) & \text{if } \beta(t) < x \end{cases}$$

and the variant of Theorem 2.3.7 described in Remark 2.3.2. □

Remark 3.1.3. The hypothesis of global continuability for the solutions to (5.2) is needed only to ensure that such a property holds for the solutions to the truncated equation

$$u'' + \tilde{f}(t,u) = 0. \tag{3.12}$$

Notice that this request was not made in Theorem 3.1.1 because in the corresponding proof we have performed a two-sided truncation, obtaining $\tilde{f}(t,x)$ bounded and thus having guaranteed the continuability of the solutions to (3.12). In this regard, we could replace the global existence hypothesis of Theorem 3.1.2 by other conditions ensuring the global continuability for the solutions to (3.12). For instance, a simple possible alternative condition could be the following:

- *there exist three constants* $A, B, d > 0$ *such that*

$$|f(t,x)| \leq A|x| + B, \quad \textit{for every } x \leq -d,\, t \in [0,T].$$

It is rather easy to produce examples of differential equations satisfying the above one-sided growth assumption and possessing, at the same time, solutions which are not globally defined on $\mathbb{R}$. Other, more refined conditions could be given as well (see, among others, [49, 68, 78, 97]).

We conclude this section with a few examples in which our results immediately apply. It is worth mentioning that both the equations (3.13) and (3.16) considered below deal with nonlinearities $f(t,x)$ which are odd functions in the x-variable. Accordingly, solutions always occur in pairs $(u(t), -u(t))$ and we cannot exclude, in the situation, for instance, of Theorem 3.1.1, that $u^{(1)}_{k,j}(t) \equiv -u^{(2)}_{k,j}(t)$. This fact, however, is purely accidental and does not affect the conclusion of Poincaré-Birkhoff twist theorem asserting, in general, the existence of pairs of fixed points. Indeed, after Example 3.1.1 and Example 3.1.2 we will propose some straightforward generalizations to second order equations with no symmetry conditions.

Example 3.1.1. Consider a frictionless unforced simple pendulum with a periodically moving support (which can be equivalently described as a pendulum with a stationary support in a space with a periodically varying constant of gravity [57]). Mechanical models of this type lead to differential equations of the form

$$u'' + a(t)\sin u = 0, \tag{3.13}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function. In this case we can apply Theorem 3.1.1, by taking $u^*(t) \equiv 0$, $\alpha(t) \equiv -\pi$ and $\beta(t) \equiv \pi$. With this choice of $u^*(t)$, condition (3.3) is satisfied if and only if $\bar{a} = T^{-1} \int_0^T a(t)\,dt > 0$. Therefore, the following result holds.

Corollary 3.1.1. *Suppose that $\bar{a} > 0$. Then there exists $k^* \in \mathbb{N}_*$ such that, for every integer $k \geq k^*$, there exists an integer $j_*(k)$ such that, for every integer j relatively prime with k and such that $1 \leq j \leq j_*(k)$, equation (3.13) has at least two subharmonic solutions $u^{(1)}_{k,j}(t), u^{(2)}_{k,j}(t)$ of order k (not belonging to the same periodicity class) such that, for $i = 1, 2$, $u^{(i)}_{k,j}(t)$ has exactly $2j$ zeros in the interval $[0, kT[$ and*

$$-\pi < u^{(i)}_{k,j}(t) < \pi, \qquad \text{for every } t \in \mathbb{R}.$$

The case in which $\bar{a} < 0$ follows again from Theorem 3.1.1, by starting from $u^*(t) \equiv \pi$, $\alpha(t) \equiv 0$ and $\beta(t) \equiv 2\pi$. In this situation, the solutions found rotate around $(\pi, 0)$ in the phase plane, with the dynamics of an inverted pendulum. Notice also that, in view of the estimate for $j_*(k)$ provided by Theorem 2.3.7, here the following lower bound can be given:

$$j_*(k) \geq \mathcal{E}^- \left(\frac{kT}{2\pi} \frac{\bar{a}}{\sqrt{\max_{[0,T]} a(t)}} \right).$$

We point out that, even if Corollary 3.1.1 applies to the unforced pendulum type equation (3.13), we do not need many special features of the sine function (in particular, its oddness and the fact that it has zero mean value in a period). Indeed, with the same argument, we can derive an application of Theorem 3.1.1 to an equation of the form

$$u'' + a(t)g(u) = 0, \tag{3.14}$$

with $g : \mathbb{R} \to \mathbb{R}$ a $2L$-periodic function of class C^1 and such that

$$g(0) = 0 \quad \text{and} \quad g'(0) > 0. \tag{3.15}$$

Indeed, from the periodicity of $g(x)$ and (3.15), we obtain that $g(x)$ vanishes at some points in $]-2L, 0[$ as well as at some points in $]0, 2L[$. Hence the constants $L_- = \max\{x < 0 \mid g(x) = 0\}$, $L_+ = \min\{x > 0 \mid g(x) = 0\}$ are well defined and, by construction, $g(x) < 0$ for $x \in]L_-, 0[$ and $g(x) > 0$ for $x \in]0, L_+[$. Note that, with this approach, the periodicity of the potential $G(x) = \int_0^x g(\xi)\, d\xi$ is not required. For other results concerning (3.14) with $g(x)$ having a periodic potential, see [124, 153].

Example 3.1.2. As a second example, we consider a model studied by Belmonte-Beitia and Torres [16] arising from the search for some special solutions to a nonlinear Schrödinger equation. The equation under consideration takes the form

$$u'' + \mu u - p(t)u^3 = 0, \tag{3.16}$$

where $\mu > 0$ and $p : \mathbb{R} \to \mathbb{R}$ is a strictly positive, continuous and T-periodic function (comparing with the notation in [16], we have $\mu = -2\lambda$ and $p(t) = 2g(x)$, for $t = x$). Following [16] we introduce the constants

$$\xi_1 = \sqrt{\frac{\mu}{\min_{[0,T]} p(t)}}, \qquad \xi_2 = \sqrt{\frac{\mu}{\max_{[0,T]} p(t)}}.$$

A simple computation shows that ξ_2 is a constant lower solution and ξ_1 is a constant upper solution to equation (3.16). By symmetry, also the constant functions $-\xi_1$ and $-\xi_2$ are, respectively, a lower and an upper solution to the same equation. A direct application of Theorem 3.1.1 can then be given with the following choices: $u^*(t) \equiv 0$, $\alpha(t) \equiv -\xi_1$ and $\beta(t) \equiv \xi_1$. Observe that the average condition (3.3) is automatically satisfied, since $\mu > 0$. Therefore, the following result holds.

Corollary 3.1.2. *With the above positions, there exists $k^* \in \mathbb{N}_*$ such that, for every integer $k \geq k^*$, there exists an integer $j_*(k)$ such that, for every integer j relatively prime with k and such that $1 \leq j \leq j_*(k)$, equation* (3.16) *has at least two subharmonic solutions $u^{(1)}_{k,j}(t), u^{(2)}_{k,j}(t)$ of order k (not belonging to the same periodicity class) such that, for $i = 1, 2$, $u^{(i)}_{k,j}(t)$ has exactly $2j$ zeros in the interval $[0, kT[$ and*

$$-\xi_1 \leq u^{(i)}_{k,j}(t) \leq \xi_1, \quad \textit{for every } t \in \mathbb{R}. \tag{3.17}$$

If we like to improve estimate (3.17), we can take advantage of the fact, as proved in [16], there exist two T-periodic solutions to (3.16), say $\rho_-(t)$ and $\rho_+(t)$, such that, for every $t \in [0, T]$,

$$-\xi_1 \leq \rho_-(t) \leq -\xi_2, \qquad \xi_2 \leq \rho_+(t) \leq \xi_1$$

With this further information, we can apply Theorem 3.1.1 with $\alpha(t) = \rho_-(t)$ and $\beta(t) = \rho_+(t)$ and obtain the sharper estimate

$$\rho_-(t) < u^{(i)}_{k,j}(t) < \rho_+(t), \quad \text{for every } t \in \mathbb{R}.$$

Moreover, as in Example 3.1.1, we can provide a lower bound for $j_*(k)$ as follows:

$$j_*(k) \geq \mathcal{E}^-\left(\frac{kT\sqrt{\mu}}{2\pi}\right).$$

Again, we have not used the oddness of the nonlinearity and our result easily extends to the more general equation

$$u'' + \mu u - p(t)g(u) = 0, \tag{3.18}$$

with $g : \mathbb{R} \to \mathbb{R}$ a C^1-function such that $g(x)x > 0$ for $x \neq 0$, $g'(0) = 0$ and $\lim_{|x|\to\infty} g(x)/x = +\infty$. Arguing like in [16], we can prove the existence of a maximal negative T-periodic solution $\rho_-(t)$ and a minimal positive T-periodic solution $\rho_+(t)$ (this follows using a lower/upper solutions technique). Now we can apply Theorem 3.1.1 with the positions $u^*(t) \equiv 0$, $\alpha(t) = \rho_-(t)$ and $\beta(t) = \rho_+(t)$. As before, the average condition (3.3) is automatically satisfied since $\mu > 0$. In this manner, we can extend Corollary 3.1.2 to the nonsymmetric case of equation (3.18).

3.2 Second order ODEs of pendulum type

In this section, we propose to develop a consequence of Theorem 3.1.1 which applies to forced pendulum type equations. As ideal model for our investigation, we consider the equation

$$u'' + \mu \sin u = e(t), \tag{3.19}$$

where $\mu > 0$ and $e : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function. For what follows it is also convenient to split $e(t)$ as

$$e(t) = \bar{e} + \tilde{e}(t), \qquad \text{with} \quad \bar{e} = \frac{1}{T}\int_0^T e(t)\,dt.$$

Integrating both sides of (3.19), we see that a necessary condition (see [124, 127]) for the existence of kT-periodic solutions, $k \in \mathbb{N}_*$, to (3.19) is $\bar{e} \in [-\mu, \mu]$. On the other hand, it is well known that, if

$$\bar{e} \in \,]-\mu, \mu[\,,$$

then, for every sufficiently small $\tilde{e}(t)$, equation (3.19) has at least two T-periodic solutions (geometrically distinct, i.e., non differing by a multiple of 2π). Precisely, considering the two solutions of the equation

$$\mu \sin x = \bar{e}, \qquad \text{with} \quad x \in \,]\pi, \pi],$$

namely

$$x_0 = \arcsin(\bar{e}/\mu) \quad \text{and} \quad x_1 = \begin{cases} \pi - x_0 & \text{if } x_0 \geq 0 \\ -\pi - x_0 & \text{if } x_0 < 0, \end{cases}$$

one can prove (via degree theory) that, whenever $\tilde{e}(t)$ is small, T-periodic solutions to (3.19) exist near x_0 and x_1, respectively (see, for instance, [44, 85, 122]). Usually, the smallness of $\tilde{e}(t)$ is expressed in terms of its L^1-norm on $[0, T]$ or of the oscillation of some of its primitives (and, clearly, such a bound depends on μ: the smaller is μ, the larger upper bound for $\tilde{e}(t)$ is available).

In such a framework, our aim is to prove a result about subharmonic solutions. More in general, starting from an autonomous equation of the form

$$u'' + \mu h(t, u) = 0,$$

for which we assume the existence of three constant solutions $N^-, 0, N^+$, with $N^- < 0 < N^+$ (which mimic the three consecutive constant solutions $-\pi, 0, \pi$ of $u'' + \mu \sin u = 0$), we are going to show the existence of infinitely many subharmonic solutions to the perturbed equation

$$u'' + \mu h(t, u) = e(t), \tag{3.20}$$

provided that $e(t)$ is sufficiently small. We will make the convenient assumption that

$$\int_0^T e(t)\,dt = 0 \tag{3.21}$$

and, accordingly, we will denote by $\mathcal{E}(t)$ the unique T-periodic function such that

$$\mathcal{E}''(t) = e(t) \quad \text{and} \quad \|\mathcal{E}\|_\infty = \frac{1}{2}\text{Osc}(\mathcal{E}),$$

being $\text{Osc}(\mathcal{E}) = \max_{[0,T]} \mathcal{E}(t) - \min_{[0,T]} \mathcal{E}(t)$.

With these preliminaries, we can state the following result.

Theorem 3.2.1. *Let $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function, T-periodic in the first variable and with continuous partial derivative $\frac{\partial h}{\partial x}(t,x)$, and let $e : \mathbb{R} \to \mathbb{R}$ be a continuous and T-periodic function satisfying* (3.21). *Suppose that, for suitable real numbers N^-, A, B, N^+ and a constant $\delta > 0$ satisfying*

$$N^- + \delta \le A < 0 < B \le N^+ - \delta,$$

the following conditions hold true:

- *for every $t \in [0,T]$,*
$$h(t,N^-) = h(t,0) = h(t,N^+) = 0; \tag{3.22}$$
- *for every $t \in [0,T]$ and $x \in [N^\pm - \delta, N^\pm + \delta]$,*
$$h(t,x)(x - N^\pm) \le 0; \tag{3.23}$$
- *for every $t \in [0,T]$ and $x \in]A,0[\cup]0,B[$,*
$$h(t,x)x > 0; \tag{3.24}$$
- *for every $t \in [0,T]$ and $x \in]A,B[$,*
$$\frac{\partial h}{\partial x}(t,x) > 0. \tag{3.25}$$

Finally, assume that

$$\mathrm{Osc}(\mathcal{E}) < \min\{\delta, -A, B\}. \tag{3.26}$$

Then there exists $\mu_* > 0$ such that, for every $\mu \in]0,\mu_*]$:

i) there exists a T-periodic solution $u^*(t)$ to (3.20) *such that*

$$A < u^*(t) < B, \quad \textit{for every } t \in \mathbb{R};$$

ii) there exist two T-periodic solutions $u^-(t), u^+(t)$ to (3.20) *such that*

$$N^\pm - \delta < u^\pm(t) < N^\pm + \delta, \quad \textit{for every } t \in \mathbb{R};$$

iii) there exists $k_\mu^* \in \mathbb{N}_*$ such that, for every integer $k \ge k_\mu^*$ there exists an integer $m_{k,\mu}$ such that, for every integer j relatively prime with k and such that $1 \le j \le m_{k,\mu}$, equation (3.20) *has at least two subharmonic solutions $u_{k,j}^{(1)}(t), u_{k,j}^{(2)}(t)$ of order k (not belonging to the same periodicity class) such that $u_{k,j}^{(i)}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0,kT[$ and*

$$u^-(t) < u_{k,j}^{(i)}(t) < u^+(t), \quad \textit{for every } t \in \mathbb{R}.$$

Proof. First, define
$$M = \max_{(t,x)\in[0,T]\times[A,B]} |h(t,x)|$$
and choose $\mu_*, R > 0$ so that
$$\frac{\mu_* M T^2}{12} < R < \min\left\{-\frac{A}{2} - \|\mathcal{E}\|_\infty, \frac{B}{2} - \|\mathcal{E}\|_\infty\right\}. \tag{3.27}$$
Such a choice is possible, in view of (3.26). Now, fix $\mu \in]0, \mu_*]$.

We first show that the conclusion at point $i)$ holds true, by proving that there exists a T-periodic solution $x^*(t)$ to
$$x'' + \mu h(t, x + \mathcal{E}(t)) = 0, \tag{3.28}$$
such that
$$A + \|\mathcal{E}\|_\infty < x^*(t) < B - \|\mathcal{E}\|_\infty, \quad \text{for every } t \in \mathbb{R}.$$
Setting $u^*(t) = x^*(t) + \mathcal{E}(t)$, this implies the conclusion.
Our argument is closely related to the one in [169]. Let C_T be the Banach space of the continuous and T-periodic functions $x : \mathbb{R} \to \mathbb{R}$, with the norm $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$; moreover, for $x \in C_T$, set $\bar{x} = \frac{1}{T}\int_0^T x(t)\,dt$ and $\tilde{x}(t) = x(t) - \bar{x}$. Define the open set
$$\Omega = \left\{x \in C_T \mid \bar{x} \in \left]\frac{A}{2}, \frac{B}{2}\right[, \|\tilde{x}\|_\infty < R\right\}$$
and observe first of all that, if $x \in \overline{\Omega}$, then relation (3.27) implies that
$$A + \|\mathcal{E}\|_\infty < x(t) < B - \|\mathcal{E}\|_\infty, \quad \text{for every } t \in [0,T]. \tag{3.29}$$
We are going to show, via coincidence degree's theory, that equation (3.28) has a solution $x^* \in \overline{\Omega}$. In view of [122, Theorem 2.4], this is true if:

a) for every $\alpha \in]0,1[$, the equation
$$x'' + \alpha\mu h(t, x + \mathcal{E}(t)) = 0, \tag{3.30}$$
has no T-periodic solutions $x \in \partial\Omega$;

b) it holds that
$$\int_0^T \mu h\left(t, \frac{A}{2} + \mathcal{E}(t)\right) dt \neq 0 \neq \int_0^T \mu h\left(t, \frac{B}{2} + \mathcal{E}(t)\right) dt;$$

c) it holds that
$$\deg_{\mathrm{B}}\left(x \mapsto \int_0^T \mu h(t, x + \mathcal{E}(t))\,dt, \left]\frac{A}{2}, \frac{B}{2}\right[, 0\right) \neq 0,$$
where "$\deg_{\mathrm{B}}$" denotes the (one-dimensional) Brouwer degree.

Conditions b) and c) follow from (3.24). Indeed, suppose that $s = B/2$. In this case we have $0 < s+\mathcal{E}(t) < B$, so that $h(t, s+\mathcal{E}(t)) > 0$ for every $t \in [0,T]$. Therefore, $\int_0^T \mu h(t, s+\mathcal{E}(t))\,dt > 0$. Similarly, one can check that $\int_0^T \mu h(t, s+\mathcal{E}(t))\,dt < 0$ for $s = A/2$. As consequence, b) holds and c) is satisfied with degree equal to one.
For what concerns condition a), observe first of all that whenever $x(t)$ is a T-periodic solution to (3.30) ($0 < \alpha < 1$) such that (3.29) holds true, in view of the Sobolev inequality (see [128]) and since $x(t) + \mathcal{E}(t) \in [A, B]$ for every $t \in [0,T]$, we have

$$\begin{aligned} \|\tilde{x}\|_\infty^2 &\leq \frac{T}{12}\int_0^T x'(t)^2\,dt = -\frac{T}{12}\int_0^T x''(t)\tilde{x}(t)\,dt = \\ &= \frac{T}{12}\int_0^T \alpha\mu h(t, x+\mathcal{E}(t))\tilde{x}(t)\,dt \leq \frac{\mu_* M T^2}{12}\|\tilde{x}\|_\infty < R\|\tilde{x}\|_\infty. \end{aligned}$$

From such a priori bound we conclude that in order to show condition a) it is sufficient to prove that (3.30) has no T-periodic solutions $x(t)$ such that $\bar{x} = \frac{A}{2}$ or $\bar{x} = \frac{B}{2}$ with $\|\tilde{x}\|_\infty < R$. Assume to the contrary that this is the case and, just to fix the ideas, that $\bar{x} = \frac{B}{2}$. Then

$$\|\mathcal{E}\|_\infty < \frac{B}{2} - R < x(t) = \bar{x} + \tilde{x}(t) < \frac{B}{2} + R < B - \|\mathcal{E}\|_\infty,$$

so that $0 < x(t) + \mathcal{E}(t) < B$ for every $t \in [0,T]$, a contradiction in view of (3.24) (just integrate (3.30) and divide by $\alpha > 0$ to get $0 = \int_0^T h(t, x(t) + \mathcal{E}(t))\,dt$).

The conclusion at point *ii*) follows in a direct way from the lower/upper solutions technique. Indeed, in view of (3.26) and (3.23), it is easy to verify that the functions

$$\alpha^-(t) = N^- - \frac{\delta}{2} + \mathcal{E}(t), \quad \beta^-(t) = N^- + \frac{\delta}{2} + \mathcal{E}(t)$$

are, respectively, a lower and an upper solution to (3.20) with $a^-(t) \leq \beta^-(t)$. Hence, the existence of a T-periodic solution $u^-(t)$ to (3.20) satisfying, for $t \in [0,T]$,

$$N^- - \delta < N^- - \frac{\delta}{2} + \mathcal{E}(t) \leq u^-(t) \leq N^- + \frac{\delta}{2} + \mathcal{E}(t) < N^- + \delta,$$

follows immediately. A symmetric argument shows the existence of $u^+(t)$.

Finally, the conclusion at point *iii*) follows from Theorem 3.1.1, with the choice $\alpha(t) = u^-(t)$ and $\beta(t) = u^+(t)$. Indeed, (3.3) follows from (3.25), since $A < u^*(t) < B$ for every $t \in [0,T]$. □

When applied to the forced pendulum equation (3.19), Theorem 3.2.1 immediately gives the following:

Corollary 3.2.1. *Assume that $e : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function satisfying* (3.21) *and*

$$\mathrm{Osc}(\mathcal{E}) < \frac{\pi}{2}\,. \tag{3.31}$$

Then there exists $\mu_* > 0$ such that, for every $\mu \in]0, \mu_*]$:

i) there exists a T-periodic solution $u^*(t)$ to (3.19) *such that*

$$-\frac{\pi}{2} < u^*(t) < \frac{\pi}{2}, \quad \textit{for every } t \in \mathbb{R};$$

ii) there exist a T-periodic solutions $u^-(t)$ to (3.19) *such that*

$$-\frac{3}{2}\pi < u^-(t) < -\frac{\pi}{2}, \quad \textit{for every } t \in \mathbb{R};$$

iii) there exists $k^*_\mu \in \mathbb{N}_*$ such that, for every integer $k \geq k^*_\mu$ there exists an integer $m_{k,\mu}$ such that, for every integer j relatively prime with k and such that $1 \leq j \leq m_{k,\mu}$, equation (3.19) *has at least two subharmonic solutions $u^{(1)}_{k,j}(t), u^{(2)}_{k,j}(t)$ of order k (not belonging to the same periodicity class) such that $u^{(i)}_{k,j}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$ and*

$$u^-(t) < u^{(i)}_{k,j}(t) < u^-(t) + 2\pi, \quad \textit{for every } t \in \mathbb{R}.$$

Proof. It is enough to apply Theorem 3.2.1 with the positions $N^- = -\pi, A = -\frac{\pi}{2}, B = \frac{\pi}{2}, N^+ = \pi$ and $\delta = \frac{\pi}{2}$. Notice that here, in view of the 2π-periodicity of $x \mapsto \sin x$, the solution lying near π is just $u^+(t) = u^-(t) + 2\pi$. □

Existence results for pendulum type equations based on upper bounds on $\mathrm{Osc}(\mathcal{E})$ have been previously obtained. For instance, using topological degree methods, the condition

$$\mathrm{Osc}(\mathcal{E}) \leq \pi, \tag{3.32}$$

paired with other assumptions, has been used to obtain existence results [53, 121]. Our condition (3.31) is clearly more restrictive than (3.32); however, it allows to obtain a multiplicity result for subharmonic solutions as well. Note also that, according to (3.27), an estimate for μ_* can be given, in the sense that our result is true for

$$\mu < \frac{6}{T^2}\left(\frac{\pi}{2} - \mathrm{Osc}(\mathcal{E})\right).$$

We stress that, in spite of an enormous amount of literature dealing with the existence of T-periodic solutions to the forced pendulum equation (3.19) (see for instance the survey [124]), far fewer results are available for subharmonic solutions [79, 153, 166]. In particular, [79, 153] provide generic type results under suitable nondegeneracy conditions on the associated energy functionals, while [166] is closer in spirit to Corollary 3.2.1, showing the existence of infinitely many subharmonics for $e(t)$ small in the L^2-norm. Even if our assumption (3.31) is in general not comparable with the ones in [166], Corollary 3.2.1, as usual when trying to make a comparison between variational methods and the Poincaré-Birkhoff theorem, provides a sharper conclusion for what concerns the multiplicity, minimal period and localization of the subharmonics found.

3.3 Some Ambrosetti-Prodi type results

In this final section, we propose to study, via Theorem 3.1.2, the existence of harmonic and subharmonic solutions to second order scalar parameter dependent ordinary differential equations of the form

$$u'' + g(u) = \lambda + e(t), \tag{3.33}$$

being $g : \mathbb{R} \to \mathbb{R}$ a C^1-function, $e : \mathbb{R} \to \mathbb{R}$ a continuous and T-periodic function with

$$\int_0^T e(t)\,dt = 0 \tag{3.34}$$

and $\lambda \in \mathbb{R}$ a parameter. Observe that, in this setting, (3.34) is not restrictive, up to relabelling $e(t)$ and λ. Throughout the section, given a continuous T-periodic function $u : \mathbb{R} \to \mathbb{R}$, we set (as before) $\bar{u} = \frac{1}{T}\int_0^T u(t)\,dt$, $\tilde{u}(t) = u(t) - \bar{u}$ and $\|u\|_{L^p_T} = \left(\int_0^T |u(t)|^p\,dt\right)^{1/p}$ (for $p = 1, 2$).

Our goal is to find, for every λ in a suitable interval, a T-periodic solution $u_\lambda(t)$ to (3.33) which plays the role of $u^*(t)$ in Theorem 3.1.2. More precisely, we will look for some $u_\lambda(t)$ such that the following property is satisfied:

($\mathscr{P}$) *There exists $k^*_\lambda \in \mathbb{N}_*$ such that, for every integer $k \geq k^*_\lambda$ there exists an integer $j_*(k,\lambda)$ such that, for every integer j relatively prime with k and such that $1 \leq j \leq j_*(k,\lambda)$, equation (3.33) has at least two subharmonic solutions $u^{(1)}_{k,j,\lambda}(t), u^{(2)}_{k,j,\lambda}(t)$ of order k (not belonging to the same periodicity class) such that, for $i = 1, 2$, $u^{(i)}_{k,j,\lambda}(t) - u_\lambda(t)$ has exactly $2j$ zeros in the interval $[0, kT[$.*

As in Section 3.2, coincidence degree will be the main tool to find and localize (in dependence of λ) such a solution $u_\lambda(t)$. To this aim, we present a preliminary result, Lemma 3.3.1 below, which deals with the case when $g(x)$ is strictly increasing on a half-line. Our result, although technical, permits a unifying treatment of the forthcoming applications.

Lemma 3.3.1. *Assume that there exists $d \in \mathbb{R}$ such that $g(x)$ is strictly increasing on $[d, +\infty[$ (respectively, on $]-\infty, -d]$). Moreover, suppose that there exists $K > 0$ such that:*

- *for every $\lambda \in [g(d), g(+\infty)[$ (resp., $\lambda \in]g(-\infty), g(-d)]$), for every $\alpha \in]0,1[$ and for every T-periodic solution $u(t)$ to*

$$u'' + \alpha g(u) = \alpha(\lambda + e(t)) \tag{3.35}$$

satisfying $u(t) > d$ (resp., $u(t) < -d$) for every $t \in [0,T]$, it holds that

$$\|u'\|_{L^1_T} < K.$$

Then there exists $\lambda^* \in [g(d), g(+\infty)[$ (resp., $\lambda^* \in]g(-\infty), g(-d)]$) such that, for every $\lambda \in [\lambda^*, g(+\infty)[$ (resp., $\lambda \in]g(-\infty), \lambda^*]$), there exists a T-periodic solution $u^*_\lambda(t)$ to (3.33) such that $u^*_\lambda(t) > d$ (resp., $u^*_\lambda(t) < -d$) for every $t \in \mathbb{R}$. Moreover, for $\lambda \to g(+\infty)$, it holds that

$$u^*_\lambda(t) \to +\infty, \qquad \textit{uniformly in } t \in [0,T], \tag{3.36}$$

(resp., $u^*_\lambda(t) \to -\infty$, uniformly in t, for $\lambda \to g(-\infty)$).

Proof. For every $\lambda \in [g(d), g(+\infty)[$, let us denote by x_λ the unique real number in $[d, +\infty[$ such that $g(x_\lambda) = \lambda$; clearly, $x_\lambda \to +\infty$ for $\lambda \to g(+\infty)$. Let us choose now $\lambda^* \geq g(d)$ so large that $x_{\lambda^*} - K > d$ and fix $\lambda \in [\lambda^*, g(+\infty)[$.
Let C_T be the Banach space of the continuous and T-periodic functions $u : \mathbb{R} \to \mathbb{R}$, with the norm $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$, and, for $a, b \in \mathbb{R}$ with $a < b$, consider the open set

$$\Omega(a, b) = \{u \in C_T \mid a < u(t) < b, \text{ for every } t \in \mathbb{R}\}.$$

We are going to show that equation (3.33) has a solution $u^*_\lambda \in \overline{\Omega}$ for

$$\Omega = \Omega(a, b) \quad \text{with} \quad a = x_\lambda - K,\ b = x_\lambda + K.$$

By [122, Theorem 2.4], this is true if:

a) for every $\alpha \in]0, 1[$, the equation (3.35) has no T-periodic solutions $u(t)$ such that $a \leq u(t) \leq b$ for every $t \in \mathbb{R}$ and $u(\tilde{t}) \in \{a, b\}$ for some $\tilde{t} \in [0, T]$;

b) it holds that

$$\int_0^T (g(a) - \lambda - e(t))\, dt \neq 0 \neq \int_0^T (g(b) - \lambda - e(t))\, dt;$$

c) it holds that

$$\deg_{\mathrm{B}} \left(x \mapsto \int_0^T (g(x) - \lambda - e(t))\, dt,\ ]a, b[\,, 0 \right) \neq 0,$$

where "$\deg_{\mathrm{B}}$" denotes the (one-dimensional) Brouwer degree.

Indeed, conditions b) and c) follow easily from the fact that $g(x)$ is strictly increasing on $[d, +\infty[$ and the choice of λ. We prove condition a). Let $u(t)$ be a T-periodic solution of (3.35) such that $u(t) \geq x_\lambda - K > d$; then, for every $s, t \in [0, T]$,

$$|u(s) - u(t)| = \left| \int_t^s u'(\tau)\, d\tau \right| \leq \int_0^T |u'(t)|\, dt < K. \tag{3.37}$$

On the other hand, integrating equation (3.35) and dividing by $\alpha > 0$, we get

$$\frac{1}{T} \int_0^T g(u(s))\, ds = \lambda,$$

which implies that, for some $t^* \in [0, T]$, $g(u(t^*)) = \lambda$. Since $u(t^*) > d$, we get $u(t^*) = x_\lambda$. On the other hand, by assumption we know that $u(\tilde{t}) \in \{x_\lambda - K, x_\lambda + K\}$ for some $\tilde{t} \in [0, T]$. Hence

$$|u(t^*) - u(\tilde{t})| = K,$$

in contradiction with (3.37). Finally, (3.36) follows from the fact that $x_\lambda - K \leq u^*_\lambda(t) \leq x_\lambda + K$, since $x_\lambda \to +\infty$. The proof of the symmetric case follows a similar argument. □

Our first application deals with a problem previously considered by Cid-Sanchez, Ward and Bereanu-Mawhin [18, 48, 163]. The general framework is that of a bounded nonlinearity $g(x)$ satisfying the basic assumption:

$$g(x) > 0, \quad \textit{for every } x \in \mathbb{R}, \qquad \textit{and} \qquad \lim_{|x| \to +\infty} g(x) = 0. \tag{3.38}$$

In such a situation, the results in [18, 48, 163] guarantee the existence of two T-periodic solutions for small positive values of the parameter λ. Our goal is to show that, by adding an asymptotic condition on the derivative $g'(x)$, such harmonic solutions can be localized in a precise manner, and they are accompanied by the existence of infinitely many subharmonic solutions with prescribed nodal properties.

Theorem 3.3.1. *Assume* (3.38) *and suppose that there exists $d > 0$ such that*

$$g'(x) > 0, \quad \textit{for every } x < -d, \qquad \textit{and} \qquad g'(x) < 0, \quad \textit{for every } x > d. \tag{3.39}$$

Set $M = \max_{\mathbb{R}} g(x)$. Then there exists $\lambda^* \in]0, M]$ such that, for every $\lambda \in]0, \lambda^*]$:

i) there exists a unique T-periodic solution $u_\lambda^+(t)$ to (3.33) *such that $u_\lambda^+(t) > d$ for every $t \in \mathbb{R}$;*

ii) there exists a T-periodic solution $u_\lambda^-(t)$ to (3.33) *such that $u_\lambda^-(t) < -d$ for every $t \in \mathbb{R}$;*

iii) property ($\mathscr{P}$) holds with respect to $u_\lambda^-(t)$, with

$$u_{k,j,\lambda}^{(i)}(t) < u_\lambda^+(t), \qquad \textit{for every } t \in \mathbb{R}.$$

Moreover, for $\lambda \to 0^+$,

$$u_\lambda^+(t) \to +\infty \qquad \textit{and} \qquad u_\lambda^-(t) \to -\infty,$$

uniformly in $t \in \mathbb{R}$.

Proof. We split our arguments into three steps.

First of all, we prove that, for $\lambda > 0$ sufficiently small, there exists a unique T-periodic solution $u_\lambda^+(t)$ of (3.33) such that $u_\lambda^+(t) > d$ for every $t \in \mathbb{R}$. To this aim, let us denote by x_λ the unique real number in $[d, +\infty[$ such that $g(x_\lambda) = \lambda$; moreover, let $\mathcal{E}(t)$ be the unique T-periodic function such that $\mathcal{E}''(t) = e(t)$ and $\int_0^T \mathcal{E}(t)\, dt = 0$. For $\lambda > 0$ small enough, we have $x_\lambda > d + 2\|\mathcal{E}\|_\infty$; accordingly, choose $m, M > 0$ such that

$$d < m - \|\mathcal{E}\|_\infty \leq m + \|\mathcal{E}\|_\infty \leq x_\lambda \leq M - \|\mathcal{E}\|_\infty.$$

An appropriate choice can be $m = x_\lambda - \|\mathcal{E}\|_\infty$ and $M = x_\lambda + \|\mathcal{E}\|_\infty$. It is easy to see that the T-periodic functions

$$\alpha^+(t) = m + \mathcal{E}(t), \qquad \beta^+(t) = M + \mathcal{E}(t)$$

are such that $d < \alpha^+(t) < \beta^+(t)$ and are, respectively, a lower and an upper solution to equation (3.33). The existence of a T-periodic solution $u_\lambda^+(t) > d$ follows then from the lower/upper solution method. Moreover, by the same estimates it follows that $u_\lambda^+(t) \to +\infty$ uniformly in $t \in [0, T]$. Finally, the uniqueness of $u_\lambda^+(t)$ follows from a direct argument using $g'(x) < 0$ for $x > d$.

Secondly, we show that the condition at point *ii*) holds true. To this aim, we are going to use Lemma 3.3.1. Indeed, $g(x)$ is strictly increasing on $]-\infty,-d]$. Moreover, for $\lambda \in]0,M]$, $\alpha \in]0,1[$ and every T-periodic solution $u(t)$ to (3.35) satisfying $u(t) < -d$ for every $t \in \mathbb{R}$, it holds that

$$\|u''\|_{L^1_T} \leq 2M + \|e\|_{L^1_T}.$$

Letting $t^* \in [0,T]$ an instant such that $u'(t^*) = 0$, the previous relation implies that, for every $t \in [0,T]$,

$$|u'(t)| = \left| \int_{t^*}^{t} u''(s)\, ds \right| \leq 2M + \|e\|_{L^1_T},$$

so that the assumption of Lemma 3.3.1 is satisfied by choosing $K > (2M + \|e\|_{L^1_T})T$.

Finally, the conclusion at point *iii*) follows from Theorem 3.1.2, with the choice $u^*(t) = u_\lambda^-(t)$ and $\beta(t) = u_\lambda^+(t)$. Indeed, (3.3) follows from (3.39) since $u_\lambda^-(t) < -d$ for every $t \in \mathbb{R}$. □

Notice that the uniqueness of $u_\lambda^-(t)$ is not guaranteed, in general. However, it can be achieved by adding some conditions on $g(x)$ (for instance we could suppose that $0 < g'(x) < \left(\frac{2\pi}{T}\right)^2$ for every $x < -d$).

Our second application deals with a classical situation first considered by Fabry, Mawhin and Nkashama [69]. Basically, we have in mind to consider the case when

$$\lim_{|x|\to+\infty} g(x) = +\infty \tag{3.40}$$

In such a situation, the results in [69] ensure the existence of two T-periodic solutions to (3.33) for large values of λ. Here, again, by adding conditions on $g'(x)$ we provide further information about the localization of such T-periodic solutions, as well as the existence of subharmonic solutions with prescribed nodal properties. It is worth noticing that results providing the existence of subharmonic solutions for (3.33) in case when (3.40) is satisfied can be deduced also from [152, Theorem 10]. We postpone more comments about this point after the statement of our result.

Theorem 3.3.2. *Assume* (3.40) *and suppose that there exist* $d > 0$ *and* $0 < l < \left(\frac{2\pi}{T}\right)^2$ *such that*

$$g'(x) < 0, \ \ \textit{for every } x < -d, \qquad \textit{and} \qquad 0 < g'(x) \leq l, \ \ \textit{for every } x > d. \tag{3.41}$$

Then there exists $\lambda^* > 0$ *such that, for every* $\lambda \geq \lambda^*$*:*

- *i) there exists a unique* T*-periodic solution* $u_\lambda^-(t)$ *to* (3.33) *such that* $u_\lambda^-(t) < -d$ *for every* $t \in \mathbb{R}$*;*
- *ii) there exists a unique* T*-periodic solution* $u_\lambda^+(t)$ *to* (3.33) *such that* $u_\lambda^+(t) > d$ *for every* $t \in \mathbb{R}$*;*
- *iii) property* ($\mathscr{P}$) *holds with respect to* $u_\lambda^+(t)$*, with*

$$u_\lambda^-(t) < u_{k,j,\lambda}^{(i)}(t), \qquad \textit{for every } t \in \mathbb{R}.$$

Moreover, for $\lambda \to +\infty$,

$$u^+_\lambda(t) \to +\infty, \qquad \text{and} \qquad u^-_\lambda(t) \to -\infty,$$

uniformly in $t \in \mathbb{R}$.

Proof. Similarly as in the proof of Theorem 3.3.1, we split our arguments into three steps.

The existence and uniqueness of a T-periodic solution $u^-_\lambda(t)$ of (3.33) such that $u^-_\lambda(t) < -d$ for every $t \in \mathbb{R}$ follows in a similar way as in the proof of point $i)$ of Theorem 3.3.1 (using upper and lower solutions techniques like in [69] and the fact that $g'(x) < 0$ for $x < -d$).

To show the conclusion at point $ii)$, we use again Lemma 3.3.1; indeed, $g(x)$ is strictly increasing on $[d, +\infty[$ with $g(+\infty) = +\infty$. For simplicity of notations, we set $g_\lambda(x) = g(x) - \lambda$. Assume now that $\lambda \geq g(d)$, $\alpha \in]0,1[$ and $u(t)$ is a T-periodic solution $u(t)$ of (3.35) such that $u(t) > d$ for every $t \in \mathbb{R}$. Multiplying equation (3.35) by $\tilde{u}(t)$ and integrating, we get

$$\begin{aligned}\int_0^T u'(t)^2\,dt &= \alpha \int_0^T g_\lambda(u(t))\tilde{u}(t)\,dt - \alpha \int_0^T e(t)\tilde{u}(t)\,dt \\ &= \alpha \int_0^T (g_\lambda(u(t)) - g_\lambda(\bar{u}))\tilde{u}(t)\,dt - \alpha \int_0^T e(t)\tilde{u}(t)\,dt.\end{aligned}$$

On the other hand, since $u(t), \bar{u} > d$ and $g(x)$ is strictly increasing on $[d, +\infty[$, Lagrange theorem implies that

$$g_\lambda(u(t)) - g_\lambda(\bar{u}) = g'_\lambda(\xi(t))(u(t) - \bar{u}) = g'_\lambda(\xi(t))\tilde{u}(t)$$

for a suitable $\xi(t) > d$. Hence we get

$$\|u'\|^2_{L^2_T} \leq l\|\tilde{u}\|^2_{L^2_T} + \|e\|_{L^1_T}\|\tilde{u}\|_\infty,$$

which implies, in view of the Sobolev and Wirtinger inequalities (see [128]), that

$$\left(1 - l\left(\frac{T}{2\pi}\right)^2\right)\|u'\|^2_{L^2_T} \leq \left(\frac{T}{12}\right)^{1/2}\|e\|_{L^1_T}\|u'\|_{L^2_T}.$$

In conclusion, $\|u'\|_{L^2_T}$ is bounded so that $\|u'\|_{L^1_T}$ is bounded too and the assumption of Lemma 3.3.1 are satisfied. Finally, the uniqueness of $u^+_\lambda(t)$ comes from the fact that $0 < g'(x) < \left(\frac{2\pi}{T}\right)^2$ for $x > d$, by a direct argument.

Finally, the conclusion at point $iii)$ follows from the dual version of Theorem 3.1.2, with the choice $u^*(t) = u^+_\lambda(t)$ and $\alpha(t) = u^-_\lambda(t)$. Indeed, according to Remark 3.1.3, the global continuability for the solutions to the modified equation

$$u'' + \tilde{f}(t,u) = 0,$$

where $f(t,x) = g(x) - \lambda - e(t)$ and

$$\tilde{f}(t,x) = \begin{cases} f(t,x) & \text{if } x \geq u^-_\lambda(t) \\ f(t, u^-_\lambda(t)) & \text{if } x < u^-_\lambda(t), \end{cases}$$

is guaranteed since $|g'(x)|$ is uniformly bounded for $x > d$. Moreover (3.3) follows from (3.41) since $u^+_\lambda(t) > d$ for every $t \in \mathbb{R}$. □

As already anticipated, the problem of the existence of subharmonic solutions for (3.33) in case when (3.40) is satisfied has been analyzed in [152]. In particular, the following result can be deduced from [152, Theorem 10].

Proposition 3.3.1 (From [152]). *Assume* (3.40)*; moreover, suppose that:*

$$0 < \liminf_{x \to +\infty} g'(x) \leq \limsup_{x \to +\infty} g'(x) < +\infty.$$

Then for every r there exists k_r^* such that, for every $k \geq k_r^*$ there exists $\lambda_{r,k}^*$ such that, for $\lambda > \lambda_{r,k}^*$, equation (3.33) *has at least $2r$ subharmonics of order k.*

Trying to make a comparison between such a result and our Theorem 3.3.2, the following facts may be emphasized.

- The assumption on the derivative $g'(x)$ considered in Theorem 3.3.2 does not prevent the possibility that $\liminf_{x \to +\infty} g'(x) = 0$, which is excluded in [152, Theorem 10].
- For a fixed (large) value of λ, the subharmonics found in Theorem 3.3.2 have a sharper nodal characterization with respect to the ones in [152, Theorem 10]. Indeed, we find subharmonic solutions rotating around the fixed harmonic solution $u_\lambda^+(t)$, whose existence is not considered in [152].

Remark 3.3.1. Combining our argument with [168, Theorem 2.1], the same conclusion of Theorem 3.3.2 can be proved assuming, instead of (3.41), the following condition:

- *there exist $d, k > 0$ such that $g'(x) < 0$ for $x < -d$ and*

$$\lim_{x \to +\infty} g'(x) = k \neq \left(\frac{2\pi m}{T} \right)^2, \qquad \textit{for every } m \in \mathbb{N}_*.$$

We point out that a nonresonance condition is really needed to prove that $u_\lambda^+(t) \to +\infty$ for $\lambda \to +\infty$. To see this, consider the equation

$$u'' + k|u| = \lambda + \sin(\sqrt{k}t), \tag{3.42}$$

with $k = \left(\frac{2\pi m}{T}\right)^2$ for some $m \in \mathbb{N}_*$. We claim that, for every $\lambda > 0$, equation (3.42) has no positive T-periodic solutions. Indeed, suppose that a positive T-periodic solution $u(t)$ exists. Then, multiplying (3.42) by $\sin(\sqrt{k}t)$ and integrating on $[0, T]$, we get

$$\int_0^T u''(t) \sin(\sqrt{k}t)\, dt + k \int_0^T u(t) \sin(\sqrt{k}t)\, dt = \int_0^T \sin^2(\sqrt{k}t)\, dt;$$

integrating twice by parts the first term on the left-hand side, we find

$$0 = \int_0^T \sin^2(\sqrt{k}t)\, dt,$$

a contradiction.

Chapter 4

Pairs of sign-changing solutions to parameter dependent second order ODEs

This chapter, coming from [34], deals with sign-changing T-periodic solutions to the second order scalar differential equation

$$u'' + \lambda f(t,u) = 0, \tag{4.1}$$

being $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function T-periodic in the first variable, with $f(t,0) \equiv 0$, and $\lambda > 0$ a real parameter. We are interested in the situation in which the nonlinear term $f(t,x)$ satisfies growth conditions, at zero and at infinity, of supersublinear type[1], namely

$$\lim_{|x|\to 0} \frac{f(t,x)}{x} = \lim_{|x|\to +\infty} \frac{f(t,x)}{x} = 0, \qquad \text{uniformly in } t \in [0,T]. \tag{4.2}$$

Boundary value problems associated with (parameter dependent) supersublinear differential equations have become very popular after the pioneering work by Rabinowitz [147], proving the existence of a pair of positive solutions to the Dirichlet problem associated with (4.1) when $\lambda > 0$ is large enough (actually, the Dirichlet problem for the PDE $-\Delta u = \lambda f(x,u)$ is considered in [147]; on the other hand, more restrictive conditions on the nonlinear term are required). As a matter of fact, the existence of two positive T-periodic solutions cannot be guaranteed for (4.1) under the same conditions: we refer to Section 6.1 for a complete discussion and some results in this direction [31, 32].

As for sign-changing solutions in the ODEs setting (4.1), the only result we know is the one by Rabinowitz [148], dealing again with the Dirichlet problem and showing the existence

[1] The choice of this terminology, which is not completely standard, is due to the fact that a possible function $f(t,x)$ satisfying (4.2) behaves, in the x-variable, like x^α with $\alpha > 1$ near zero and like x^β with $0 \le \beta < 1$ near infinity. It is also consistent with the analogous situation for the Dirichlet problem, where the term supersublinear is referred to a function having slope less than the first eigenvalue at zero and at infinity [55, p. 361].

of unbounded connected components of (nodal) solution $\mathcal{C}_k^{\pm} \subset \mathbb{R} \times C^1([0,T])$. More in detail, for every positive integer k, there exists $\Lambda_k > 0$ such that for every $\lambda > \Lambda_k$ there exist at least two distinct points $(\lambda, u), (\lambda, v) \in \mathcal{C}_k^{\pm}$ with $u(t), v(t)$ solutions to (4.1) vanishing at $t = 0$ and $t = T$ and having precisely k zeros in $]0, T[$. Moreover $u'(0), v'(0) > 0$ for $(\lambda, u), (\lambda, v) \in \mathcal{C}_k^{+}$ and $u'(0), v'(0) < 0$ for $(\lambda, u), (\lambda, v) \in \mathcal{C}_k^{-}$. The assumptions on $f(t,x)$ in [148] are essentially the same as those considered in [147] for the case of positive solutions (and they now regard two-sided conditions on $f(t,x)$, with $x \in \mathbb{R}$); precisely, a superlinear condition at zero

$$\lim_{x\to 0} \frac{f(t,x)}{x} = 0, \qquad \text{uniformly in } t \in [0,T], \tag{4.3}$$

a sign condition,

$$f(t,x)x > 0, \qquad \text{for } \ t \in [0,T], \ 0 < |x| < r_0, \tag{4.4}$$

and a strong sublinearity condition at infinity given by

$$f(t,x)x < 0, \qquad \text{for } \ t \in [0,T], \ |x| > r_1. \tag{4.5}$$

The aim of this chapter is to provide a similar result for the periodic problem. Actually, we are able to prove our theorem for a wider class of nonlinearities, by relaxing the above conditions (4.3) and (4.5) into one-sided growth restrictions on $f(t,x)$ at zero and infinity (that is, only for $x > 0$ or for $x < 0$, and for x near zero or for x near infinity, in all the possible combinations), and by replacing the sign condition (4.4), which is uniform with respect to $t \in [0,T]$, with a weaker one. This, in particular, permits us to obtain some applications to equations with sign-indefinite weight (i.e., when $f(t,x) = q(t)g(x)$ and $q(t)$ is of nonconstant sign), thus providing a connection with the results in Chapter 6.

Our proof relies again on the Poincaré-Birkhoff fixed point theorem, Theorem 1.1.3. Compared to the results in Chapter 2, where periodic solutions are provided by exploiting a gap between the number of revolutions around the origin of “small” and “large” solutions to (4.1) (coming from different growth conditions for the nonlinear term at zero and at infinity), the situation here is rather different. Indeed, the supersublinear assumption (4.2) (or one of its more general one-sided versions given along the chapter) implies that small solutions behave like large solutions, and they both do not complete a full turn around the origin in a fixed time interval. On the other hand, our sign-condition allows us to prove that “intermediate” solutions perform, when $\lambda \to +\infty$, an arbitrarily large number of turns around the origin. Accordingly, it is possible to apply twice, on a small and on a large annular region, the Poincaré-Birkhoff fixed point theorem, giving pairs of sign-changing periodic solutions to (6); see also Figure 4 below.

The plan of the chapter is the following. In Section 4.1 we analyze the second order conservative equation $u'' + \lambda g(u) = 0$ by a standard phase plane approach based on time-map techniques. Section 4.2 is split into three parts. In the first one we state (with a few comments) our main result. In the second part we develop some technical estimates for the rotation numbers; one of these results requires a modified version of a classical theorem on flow-invariant sets [7] which is presented with all the details in Section 4.4. The last part of Section 4.2 is devoted to the proof of the main result. Finally, in Section 4.2 we show some applications and propose a few variants which can be easily derived from the main result.

We remark that, on the lines of [148], Sturm-Liouville boundary value problems associated with (4.1) are worth to be investigated, as well. Even if, in this dissertation, we have preferred to focus on the case of periodic solutions only, analogous results can be provided for separated boundary conditions (including Dirichlet and Neumann ones) by combining the estimates of Section 4.2 with a shooting argument (we refer to [34] for the precise statements of the results, as well as for complete proofs). We also emphasize that, on the lines of Section 6.2 and using the estimates developed therein, we could provide a result about chaotic-like dynamics (including the existence of infinitely many subharmonics) for sign-changing solutions to (4.1) for $\lambda > 0$ large enough. This investigation, however, is postponed to a future work.

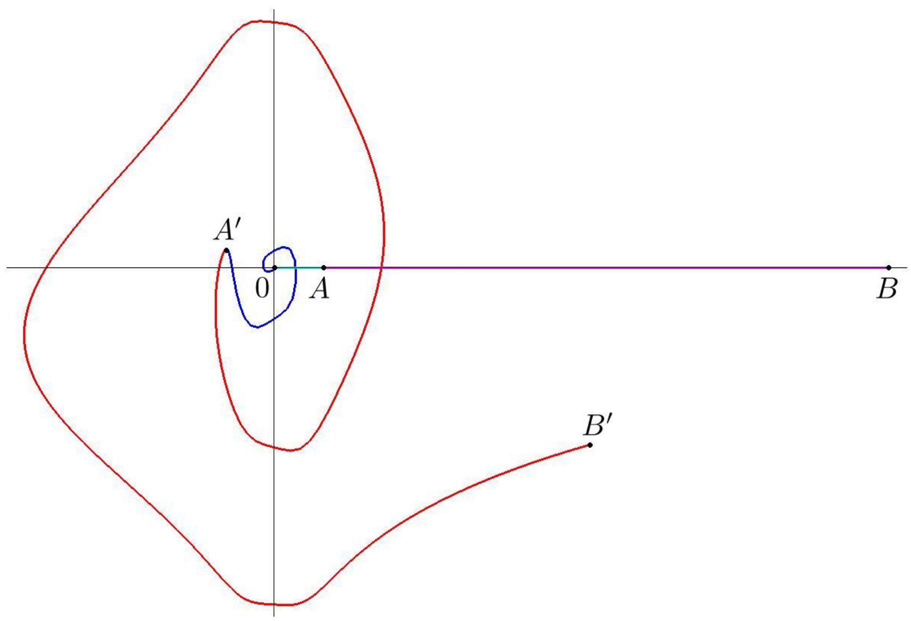


Figure 4.1: A numerical simulation describing the crucial geometrical features of supersublinear problems like (4.1). In particular, in the figure it is shown the manner in which an half-line from the origin is deformed by the Poincaré map Ψ associated with the planar Hamiltonian system $x' = y, y' = -\lambda f(x)$, for $f(x) = \frac{4x^3}{1+x^4}$ and $T = 1$. For the value $\lambda = 50$, the positive x-axis is transformed into a double spiral winding around the origin. More precisely, the image of the segment $\overline{0A}$ through Ψ is the small spiral-like arc connecting 0 to A', while the image of the segment $\overline{AB}$ is the large spiral-like arc connecting A' to B'. Notice that the six nontrivial intersections of $\Psi(\overline{0B})$ with the x-axis correspond to six nodal solutions to the Neumann problem associated with $u'' + \lambda f(u) = 0$, starting with a positive initial value $x(0)$ with $(x(0), 0) \in \overline{0B}$. Indeed, we have two solutions with respectively: one zero, two zeros, three zeros in $]0, T[$ (exactly). For each pair of such solutions, one is with a "small" initial value (namely with $(x(0), 0) \in \overline{0A}$) and the other is with a "large" initial value (namely with $(x(0), 0) \in \overline{AB}$). Other six solutions can be obtained from initial points on the negative x-axis. The spiral will make more winds around the origin as λ grows, thus producing more pairs of solutions to the Neumann problem. Along the chapter, we will prove that such kind of geometry provides multiple pairs of T-periodic solutions, as well.

4.1 The autonomous case: a time-map approach

In this section, we focus our attention on the autonomous differential equation

$$u'' + \lambda g(u) = 0, \tag{4.6}$$

being $\lambda > 0$ a parameter and $g : \mathbb{R} \to \mathbb{R}$ a continuous function satisfying the sign condition

$$g(x)x > 0, \qquad \textit{for every } x \neq 0. \tag{4.7}$$

Our analysis of the solutions to (4.6) will be performed by means of the study of the planar Hamiltonian system

$$\begin{cases} x' = y \\ y' = -g(x), \end{cases} \tag{4.8}$$

using the simple observation that a solution $(x(t), y(t))$ to (4.8) corresponds to a solution $u(t) = x(\sqrt{\lambda}t)$ to (4.6).

We first recall that, in view of (4.7), the uniqueness for the Cauchy problems associated with (4.8) is ensured [151]. Set

$$G(x) = \int_0^x g(\xi)\, d\xi;$$

as well known, each solution to (4.8) lies on a level line of the energy function

$$\mathcal{E}(x, y) = \frac{1}{2}y^2 + G(x),$$

so that the global structure of the phase portrait associated with (4.8) is affected by $G(\pm\infty)$. We are now going to define some time-map functions associated with (4.8). For every real number $c > 0$, define $]\omega^-(c), \omega^+(c)[$ as the maximal interval such that

$$G(x) < \frac{c^2}{2}, \qquad \text{for every } x \in]\omega^-(c), \omega^+(c)[\,,$$

and set, noticing that $\omega^-(c) < 0 < \omega^+(c)$,

$$\tau^-(c) = \int_{\omega^-(c)}^0 \frac{d\xi}{\sqrt{c^2 - 2G(\xi)}}, \qquad \tau^+(c) = \int_0^{\omega^+(c)} \frac{d\xi}{\sqrt{c^2 - 2G(\xi)}}.$$

Since the integrand is nonnegative, using Lebesgue integration theory we have that $\tau^-(c)$ and $\tau^+(c)$ are well defined, as functions with values in the set $]0, +\infty]$. We have the following scenario:

- if $G(+\infty) = +\infty$, then $\omega^+(c) \in]0, +\infty[$ and the map $c \mapsto \omega^+(c)$ is continuous; as a consequence (using standard properties of Lebesgue integrals) $\tau^+(c) \in]0, +\infty[$ and the map $c \mapsto \tau^+(c)$ is continuous, as well;
- if $G(+\infty) < +\infty$, we distinguish two cases:
  - if $c \in]0, \sqrt{2G(+\infty)}[\,$, then $\omega^+(c) \in]0, +\infty[$ and the same considerations as before hold true;

- if $c \geq \sqrt{2G(+\infty)}$, then $\omega^+(c) = +\infty$ and $\tau^+(c) = +\infty$.

A symmetric situation holds with respect to $\omega^-(c)$ and $\tau^-(c)$, depending whether $G(-\infty) = +\infty$ or $G(-\infty) < +\infty$.

Set

$$c_\infty^- = \sqrt{2G(-\infty)}, \quad c_\infty^+ = \sqrt{2G(+\infty)}, \quad c_\infty = \min\{c_\infty^-, c_\infty^+\}.$$

If $c < c_\infty^+$ (resp., $c < c_\infty^-$), then $\tau^+(c)$ (resp., $\tau^-(c)$) is the time needed by a solution to (4.8) to cover, in the clockwise sense, the piece of orbit from $(0, c)$ to $(\omega^+(c), 0)$ (resp., from $(\omega^-(c), 0)$ to $(0, c)$). Hence, the origin is a local center for the solution to (4.8). Indeed, every nontrivial small solution is periodic, winding around the origin in the clockwise sense. In detail, for every $c \in]0, c_\infty[$ the energy level line

$$\Gamma^c = \{(x, y) \in \mathbb{R}^2 \mid \mathcal{E}(x, y) = c^2/2\}$$

is a closed orbit surrounding the origin with minimal period

$$\tau(c) = 2(\tau^-(c) + \tau^+(c)).$$

Notice that the center is global if and only if $c_\infty = +\infty$. On the other hand, if $c_\infty < +\infty$, some of the energy level lines are unbounded. More precisely, let us suppose that $c_\infty^+ < +\infty$. In this case the solution of (4.8) departing for the point $(0, y_0)$ with $y_0 > 0$, meets again the y-axis (at the symmetric point $(0, -y_0)$ if and only if $y_0 < c_\infty^+$. Otherwise, if $y_0 \geq c_\infty^+$, the corresponding solution $(x(t), y(t))$ of (4.8) lies on the unbounded line $y = \sqrt{y_0^2 - 2G(x)}$ in the first quadrant and hence $x(t)$ is a positive decreasing function with $x(\infty) \geq 0$ and such that $2G(x(\infty)) = y_0^2$. Similar considerations apply when $c_\infty^- < +\infty$; see Figure 4.1.

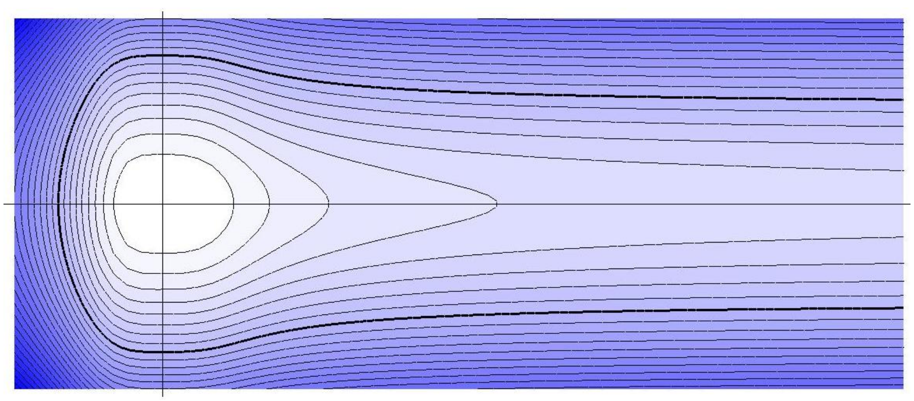


Figure 4.2: Example of energy level lines when $c_\infty^- = +\infty$ and $c_\infty^+ < +\infty$. Here $g(x) = 5x^3/(1 + x^4)$ for $x \leq 0$ and $g(x) = x^2/(1 + x^4)$ for $x \geq 0$.

Our aim now is to use the time-maps $\tau^\pm$ as a tool to provide multiplicity results for periodic solutions to (4.6). To this end, we propose a unified approach which is independent of the finiteness of the values $c_\infty^\pm$. From this point of view, we need to take into account both the cases in which the time-maps are finite or are associated with an unbounded semi-orbit and thus are infinite. Hence we compactify $\mathbb{R}$ to the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ by introducing the distance

$$d_{\overline{\mathbb{R}}}(x, y) = |\arctan(x) - \arctan(y)|,$$

where we agree that $\arctan(\pm\infty) = \pm\frac{\pi}{2}$. Clearly, the relative topology induced by $d_{\overline{\mathbb{R}}}$ on $\mathbb{R} \subset \overline{\mathbb{R}}$ coincides with the standard topology. With these preliminaries, we have the following.

Lemma 4.1.1. *The maps $\tau^-, \tau^+ : \mathbb{R}^+_* \to \overline{\mathbb{R}}$ are continuous.*

Proof. Let us fix $c_0 > 0$; we prove the continuity of τ^+ at c_0, the other case being analogous. In view of the previous discussion, the only situation to analyze is when $c_0 = c^+_\infty = \sqrt{2G(+\infty)}$. In this case, $\tau^+(c) = +\infty$ for $c \geq c_0$, so that the continuity in a right neighborhood of c_0 is ensured. The continuity in a left neighborhood follows from Fatou's lemma; indeed, since $\omega^+(c) \to +\infty$ for $c \to c_0^-$, we have

$$\begin{aligned} +\infty &= \int_0^{+\infty} \frac{1}{\sqrt{c_0^2 - 2G(\xi)}}\, d\xi = \int_0^{+\infty} \left(\lim_{c \to c_0^-} \frac{\chi_{[0,\omega^+(c)]}(\xi)}{\sqrt{c^2 - 2G(\xi)}} \right) d\xi \\ &\leq \liminf_{c \to c_0^-} \int_0^{+\infty} \frac{\chi_{[0,\omega^+(c)]}(\xi)}{\sqrt{c^2 - 2G(\xi)}}\, d\xi = \liminf_{c \to c_0^-} \tau^+(c). \end{aligned}$$

This ends the proof. □

We now introduce suitable conditions for the behavior of $g(x)$ at zero and at infinity among which we are going to select the hypotheses for our theorem. Namely, we set

(g_0^+) $$\lim_{x \to 0^+} \frac{g(x)}{x} = 0,$$

(g_0^-) $$\lim_{x \to 0^-} \frac{g(x)}{x} = 0,$$

(g_∞^+) $$\lim_{x \to +\infty} \frac{g(x)}{x} = 0,$$

(g_∞^-) $$\lim_{x \to -\infty} \frac{g(x)}{x} = 0.$$

In the sequel, we say that $g(x)$ satisfies (g_0) if at least one of the two conditions $(g_0^\pm)$ holds. A similar convention is applied to define (g_∞). The following preliminary results are classical (see, for instance, [133]); we give the proofs for completeness.

Lemma 4.1.2. *Assume (g_0^+) (resp., (g_0^-)). Then*

$$\lim_{c \to 0^+} \tau^+(c) = +\infty, \qquad \textit{(resp.,}\ \lim_{c \to 0^+} \tau^-(c) = +\infty\textit{).}$$

Proof. Let us fix $\epsilon > 0$; there exists $\delta > 0$ such that $g(x) \leq \epsilon x$ for $x \in]0, \delta]$. For c small enough, one has $\omega^+(c) \leq \delta$, so that, for every $\xi \in [0, \omega^+(c)]$,

$$c^2 - 2G(\xi) = 2(G(\omega^+(c)) - G(\xi)) = 2 \int_\xi^{\omega^+(c)} g(x)\, dx \leq \epsilon(\omega^+(c)^2 - \xi^2).$$

Hence, for every c small enough,

$$\tau^+(c) \geq \frac{1}{\sqrt{\epsilon}} \int_0^{\omega^+(c)} \frac{d\xi}{\sqrt{\omega^+(c)^2 - \xi^2}} = \frac{\pi}{2\sqrt{\epsilon}},$$

whence the conclusion for $\epsilon \to 0^+$. □

Lemma 4.1.3. *Assume* (g_∞^+) *(resp.,* (g_∞^-)*). Then*

$$\lim_{c \to (c_\infty^+)^-} \tau^+(c) = +\infty, \qquad \textit{(resp.,} \lim_{c \to (c_\infty^-)^-} \tau^-(c) = +\infty\textit{).}$$

Proof. When $c_\infty^+ < +\infty$ the conclusion follows from Lemma 4.1.1, since $\tau^+(c_\infty^+) = +\infty$. Assume now $c_\infty^+ = +\infty$. Let us fix $\epsilon > 0$; there exists $M > 0$ such that $g(x) \leq \epsilon x$ for $x \geq M$. For c large enough, one has $\omega^+(c) \geq 2M$, so that, for $\xi \in [M, \omega^+(c)]$,

$$c^2 - 2G(\xi) = 2(G(\omega^+(c)) - G(\xi)) = 2 \int_\xi^{\omega^+(c)} g(x)\, dx \leq \epsilon(\omega^+(c)^2 - \xi^2).$$

Hence, for every c large enough,

$$\begin{aligned} \tau^+(c) &\geq \frac{1}{\sqrt{\epsilon}} \int_M^{\omega^+(c)} \frac{d\xi}{\sqrt{\omega^+(c)^2 - \xi^2}} = \frac{1}{\sqrt{\epsilon}} \left(\frac{\pi}{2} - \arcsin\left(\frac{M}{\omega^+(c)} \right) \right) \\ &\geq \frac{1}{\sqrt{\epsilon}} \left(\frac{\pi}{2} - \arcsin\left(\frac{1}{2} \right) \right), \end{aligned}$$

whence the conclusion for $\epsilon \to +\infty$. □

We are now in the position to state and prove our main result for this section.

Theorem 4.1.1. *Assume* (4.7), $(g_0), (g_\infty)$ *and fix* $T > 0$. *Then, for every integer* $m \geq 1$, *there exists* $\Lambda_m^* > 0$ *such that, for every* $\lambda > \Lambda_m^*$, *equation* (4.6) *has at least* $2m$ T*-periodic solutions. Precisely, for every* $j = 1, \ldots, m$, *there are at least two* T*-periodic solutions having exactly* $2j$ *zeros in* $[0, T[$; *for one of such solutions* $|u(t)| + |u'(t)|$ *is "small" and for the other one* $|u(t)| + |u'(t)|$ *is "large".*

Proof. At first, we fix $c^* \in]0, c_\infty[$; for an integer $m \geq 1$, we define

$$\Lambda_m^* = \left(\frac{2m}{T} (\tau^+(c^*) + \tau^-(c^*)) \right)^2. \tag{4.9}$$

Fix now $\lambda > \lambda_m^*$ and $j \in \{1, \dots, m\}$; it is easy to see that a T-periodic solution $u_j(t)$ to (4.6), with exactly $2j$ zeros in $[0, T[$, exists if and only if

$$\frac{2j}{\sqrt{\lambda}} \left(\tau^+(c) + \tau^-(c)\right) = T, \tag{4.10}$$

for a suitable $c > 0$. In view of (4.9), it holds that

$$\frac{2j}{\sqrt{\lambda}} \left(\tau^+(c^*) + \tau^-(c^*)\right) < T;$$

on the other hand, in view of Lemma 4.1.2 and of Lemma 4.1.3, there exist $c_1, c_2 > 0$, with $c_1 < c^* < c_2 < c_\infty$, such that (according to whether (f_0^+) or (f_0^-) is satisfied)

$$T < \frac{2}{\sqrt{\lambda}}\tau^+(c_1) < +\infty \quad \text{or} \quad T < \frac{2}{\sqrt{\lambda}}\tau^-(c_1) < +\infty$$

and (according to whether (f_∞^+) or (f_∞^-) is satisfied)

$$T < \frac{2}{\sqrt{\lambda}}\tau^+(c_2) < +\infty \quad \text{or} \quad T < \frac{2}{\sqrt{\lambda}}\tau^-(c_2) < +\infty.$$

Summing up, we have

$$\frac{2j}{\sqrt{\lambda}} \left(\tau^+(c^*) + \tau^-(c^*)\right) < T < \frac{2}{\sqrt{\lambda}} \left(\tau^+(c_1) + \tau^-(c_1)\right)$$

and

$$\frac{2j}{\sqrt{\lambda}} \left(\tau^+(c^*) + \tau^-(c^*)\right) < T < \frac{2}{\sqrt{\lambda}} \left(\tau^+(c_2) + \tau^-(c_2)\right).$$

By Bolzano theorem, we get the existence of $c_1^\# \in]c_1, c^*[$ and $c_2^\# \in]c^*, c_2[$ satisfying (4.10), giving the two T- periodic solutions to (4.6), with exactly $2j$ zeros in $[0, T[$, as desired. □

Observe that, since (4.6) is an autonomous equation, every T-periodic solution gives rise to a whole family of T-periodic solutions (all its time translations). More precisely, in the nonautonomous case, we will show the existence of four (two small and two large) T-periodic solutions with exactly $2j$ zeros, for every $j = 1, \dots, m$ (see Theorem 4.2.1).

Remark 4.1.1. We remark that conditions (g_0) and (g_∞) are just some natural assumptions which guarantee that the time-maps go to infinity at zero and at infinity, respectively. They are, however, not the optimal conditions and known sharper assumptions are available. For instance, according to Opial [133], the conclusion of Lemma 4.1.3 is still true if we assume, instead of (g_∞), the more general hypothesis

(G_∞)
$$\lim_{x \to +\infty} \frac{G(x)}{x^2} = 0 \quad \text{or} \quad \lim_{x \to -\infty} \frac{G(x)}{x^2} = 0.$$

Remark 4.1.2. The sign condition (4.7) is assumed, in this introductory section, only for the sake of simplicity. It could be replaced by the following local hypothesis:

- *there exists* $\delta > 0$ *such that* $g(x)x > 0$ *for* $0 < |x| \leq \delta$.

In this case, we can take a maximal open interval $]x_*, x^*[$ with $-\infty \leq x_* < -\delta < 0 < \delta < x^* \leq +\infty$ such that $g(x)x > 0$ on $]x_*, x^*[\setminus\{0\}$. If $x^* < +\infty$ (resp., $x_* > -\infty$) we must have $g(x^*) = 0$ (resp., $g(x_*) = 0$). Then we can modify $g(x)$ to a new function $\widetilde{g}(x)$ which coincides with $g(x)$ on $]x_*, x^*[$ and vanishes elsewhere. The phase plane analysis for the truncated equation $u'' + \lambda\widetilde{g}(u) = 0$ can be performed with some minor modifications with respect to the arguments exposed above. An elementary maximum principle reasoning allows to conclude that the solutions found have range in $[x_*, x^*]$ and, therefore, are solutions to our original equation as well (for further details, see Corollary 4.3.2).

4.2 The main result

In this section, we state and prove our main results for the second order scalar differential equation

$$u'' + \lambda f(t, u) = 0, \tag{4.11}$$

being $\lambda > 0$ a parameter and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function, T-periodic in the first variable and such that $f(t, 0) \equiv 0$. Our goal is to extend the result of Section 4.1 to the nonautonomous equation (4.11).

4.2.1 Assumptions and statement

At first we introduce a sign condition which generalizes (4.7). Our hypothesis is of local nature in x and nonuniform in t. More precisely, we assume:

(f_g) *there exist* $\delta > 0$, *two continuous functions* $g_1, g_2 : [-\delta, \delta] \to \mathbb{R}$, *with*

$$0 < g_1(x)x \leq g_2(x)x, \quad \text{for every } 0 < |x| \leq \delta,$$

and a nondegenerate compact interval I_0 *such that*

$$g_1(x)x \leq f(t, x)x \leq g_2(x)x, \quad \text{for every } t \in I_0,\ |x| \leq \delta.$$

Note that in the special case $f(t, x) = g(x)$, the above condition is fulfilled if and only if $g(x)x > 0$ for $0 < |x| \leq \delta$ and we can take $g_1 = g_2 = g$ on $[-\delta, \delta]$. If, moreover, the function $f(t, x)$ takes the form

$$f(t, x) = q(t)g(x),$$

with $q : \mathbb{R} \to \mathbb{R}$ continuous and T-periodic, then the condition (f_g) is satisfied provided that $q(t_0) > 0$ for some $t_0 \in \mathbb{R}$ and $g(x)x > 0$ for all $0 < |x| \leq \delta$. Indeed, in such a situation, we can choose as I_0 a suitable interval (containing t_0) such that $0 < \min_{I_0} q(t)$ and take $g_1(x) = (\min_{I_0} q(t))g(x)$, $g_2(x) = (\max_{I_0} q(t))g(x)$. In the following, without loss of generality (just replace $f(t, x)$ with $f(t - \inf I_0, x)$), we assume that

$$I_0 = [0, \tau] \subset [0, T].$$

Next, we introduce some growth assumptions for $f(t, x)$ at zero and at infinity which represent a natural generalization to the nonautonomous case of the conditions (g_0) and

(g_∞) considered above. Precisely, and similarly as before, by (f_0) we mean that at least one of the two conditions

$$(f_0^+) \qquad \limsup_{x\to 0^+} \frac{f(t,x)}{x} \leq 0, \quad \textit{uniformly in } t \in [0,T],$$

$$(f_0^-) \qquad \limsup_{x\to 0^-} \frac{f(t,x)}{x} \leq 0, \quad \textit{uniformly in } t \in [0,T],$$

holds. In the same way, by (f_∞) we express the fact that at least one of the two conditions

$$(f_\infty^+) \qquad \limsup_{x\to +\infty} \frac{f(t,x)}{x} \leq 0, \quad \textit{uniformly in } t \in [0,T],$$

$$(f_\infty^-) \qquad \limsup_{x\to -\infty} \frac{f(t,x)}{x} \leq 0, \quad \textit{uniformly in } t \in [0,T],$$

holds. Notice that, due to the local and nonuniform nature of the sign condition (f_g), the above growth restrictions look more general than the corresponding hypotheses considered in Section 4.1. It is clear however that, in the special case when $f(t,x) = g(x)$ with $g(x)x > 0$ for every $x \neq 0$, the new growth conditions coincide with the previous ones. We also observe that (f_∞) is fulfilled whenever

$$f(t,x)x \leq 0, \qquad \text{for } t \in [0,T] \text{ and } x \text{ positive (or negative) with } |x| \text{ large},$$

a generalization of condition (4.5) considered in [148, Theorem 3].

To conclude with the list of the hypotheses for our main result, we add a technical condition which is required by the particular approach that we follow. Namely, we suppose that

- *the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (4.11) *are ensured.*

In order to propose a few explicit assumptions which guarantee the above requirement for the solutions of the initial value problems, we suppose that $f(t,x)$ *is locally Lipschitz continuous in* x *and grows at most linearly at infinity.* This latter assumption can be replaced by the knowledge of some specific properties, e.g. the existence of a negative lower solution or a positive upper solution or some information about the sign of $f(t,x)$ for $|x|$ large; see Section 4.3 for a discussion about this topic, accompanied by the presentation of some examples. However, as explained in [104, 170], there are several situations in which an explicit reference to the uniqueness and global continuability can be omitted, since we can enter in the required setting via standard tricks.

In such a framework the following result holds.

Theorem 4.2.1. *Assume* $(f_g), (f_0), (f_\infty)$. *Then, for every integer* $m \geq 1$, *there exists* $\Lambda_m^* > 0$ *such that, for each* $\lambda > \Lambda_m^*$, *equation* (4.11) *has at least* $4m$ T*-periodic solutions. Precisely, for every* $j = 1, \ldots, m$, *there are at least four* T*-periodic solutions having exactly* $2j$ *zeros in* $[0,T[$; *for two of such solutions* $|u(t)| + |u'(t)|$ *is "small" and for the other two* $|u(t)| + |u'(t)|$ *is "large".*

The terms "small" and "large" referred to the solutions in the theorem can be expressed in a form which is precisely described in the corresponding proof. The constants Λ_m^* depend (besides, of course, on m) only on $g_1(x), g_2(x)$ and the length of the interval I_0 .

Dealing with the nonautonomous equation (4.11), we have assumed the continuity of the function $f(t,x)$. The results could be extended in the L^∞-Carathéodory setting (compare with (i) in Remark 6.1.2).

4.2.2 Technical estimates

We prove here some technical estimates for the rotation numbers of the solutions to the second order scalar differential equation

$$x'' + w(t,x) = 0, \tag{4.12}$$

being $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function such that $w(t,0) \equiv 0$. With this, we mean that (4.12) is written as the planar Hamiltonian system

$$\begin{cases} x' = y \\ y' = -w(t,x), \end{cases} \tag{4.13}$$

and we evaluate the rotation number of the planar path $(x(t), x'(t))$, being $x(t)$ a solution to (4.12) (so that $(x(t), x'(t))$ solves (4.13)). For further convenience, we also observe that, passing to standard polar coordinates, system (4.13) is transformed into

$$\begin{cases} \rho' = R(t,\rho,\theta) = (\rho\cos\theta - w(t,\rho\cos\theta))\sin\theta \\ \\ \theta' = \Theta(t,\rho,\theta) = -\sin^2\theta - \dfrac{w(t,\rho\cos\theta)\cos\theta}{\rho}. \end{cases} \tag{4.14}$$

As usual we assume that the uniqueness and the global continuability for the solutions to the Cauchy problems associated with (4.12) are guaranteed. Notice that in (4.12) we have suppressed the parameter $\lambda > 0$; indeed, an equation like (4.12) will be obtained from (4.11) after a rescaling in the time variable (see the beginning of the next subsection for the details).

A preliminary simple fact is expressed in the next lemma. It asserts that the rotation numbers of solutions to equations like (4.12) is bounded from below (the precise lower bound being given by $-1/2$) independently of the time interval on which the solution is considered.

Lemma 4.2.1. *Let $x : \mathbb{R} \to \mathbb{R}$ be a solution to* (4.12) *and $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$. Then*

$$\mathrm{Rot}\,((x(t), x'(t)); [t_0, t_1]) > -\frac{1}{2}.$$

Proof. The result follows by observing that $\theta'(t) < 0$ whenever $\theta(t) = \frac{\pi}{2} + j\pi$ for some $j \in \mathbb{Z}$. This makes the sets $\{(\rho,\theta) \in \mathbb{R}_*^+ \times \mathbb{R} \mid \theta \le \frac{\pi}{2} + j\pi\}$ positively invariant (according to Definition 4.4.1) relatively to the open domain $\mathbb{R}_*^+ \times \mathbb{R}$ and with respect to the differential system (4.14). See [59, Lemma 2.3] or [60, Step 3] for similar considerations. □

Our next result guarantees that, under some weak sign conditions, the rotation number can become arbitrarily large provided that the time interval on which the solutions are considered is broad enough. Precisely, we have the following.

Lemma 4.2.2. *Let $\delta > 0$ and let $g_1, g_2 : [-\delta, \delta] \to \mathbb{R}$ be two continuous functions such that*

$$0 < g_1(x)x \le g_2(x)x, \quad \text{for every } x \neq 0. \tag{4.15}$$

Then, for every integer $j \ge 1$, there exist $\tau_j^* > 0$ and $r_j^* \in]0, \delta[$ such that, for every $\tau > \tau_j^*$ and for every continuous function $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with $w(t, 0) \equiv 0$ and such that

$$g_1(x)x \le w(t, x)x \le g_2(x)x, \quad \text{for every } t \in [0, \tau],\ x \in [-\delta, \delta], \tag{4.16}$$

it holds that, for every $x : \mathbb{R} \to \mathbb{R}$ solution to (4.12) with $x(0)^2 + x'(0)^2 = (r_j^*)^2$,

$$\mathrm{Rot}\left((x(t), x'(t)); [0, \tau]\right) > j. \tag{4.17}$$

Remark 4.2.1. Notice that the constants τ_j^* and r_j^* depend only on j, the comparison functions $g_1(x), g_2(x)$ and the the size τ of the time interval on which (4.16) is satisfied, namely, our estimate is uniform with respect to all the functions $w(t, x)$ which are bounded by the same comparison functions.

Proof. We are going to prove a local result, that is to say, concerning solutions in a neighborhood of the origin. For technical reasons, however, it will be convenient to suppose that $g_1(x), g_2(x)$ are defined on the whole real line, satisfying (4.15) and

$$\lim_{|x|\to+\infty} G_1(x) = \lim_{|x|\to+\infty} G_2(x) = +\infty, \tag{4.18}$$

being $G_1(x) = \int_0^x g_1(\xi)\, d\xi$ and $G_2(x) = \int_0^x g_2(\xi)\, d\xi$. Moreover, we define the energy functions

$$\mathcal{E}_{G_1}(x, y) = \frac{1}{2}y^2 + G_1(x), \qquad \mathcal{E}_{G_2}(x, y) = \frac{1}{2}y^2 + G_2(x),$$

respectively. Finally, for further convenience we set, for $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2_*$,

$$S(t, x, y) = \frac{(x - w(t, x))y}{x^2 + y^2}, \qquad U(t, x, y) = -\frac{y^2 + w(t, x)x}{x^2 + y^2},$$

so that, comparing with (4.14), we have

$$R(t, \rho, \theta) = \rho S(t, \rho\cos\theta, \rho\sin\theta), \qquad \Theta(t, \rho, \theta) = U(t, \rho\cos\theta, \rho\sin\theta). \tag{4.19}$$

The proof follows an argument previously employed in [59, 68, 78, 97]. It consists in constructing some spiral-like curves in the phase plane which bound from above and from below the trajectories of (4.12). With the aid of such curves one can prove that if a solution $z(t) = (x(t), y(t))$ to (4.13) has a certain gap in amplitude, expressed by $\big||z(s_1)| - |z(s_0)|\big|$, then it must have performed a certain number of turns around the origin. Such a fact is justified by the analysis of the energy levels associated with the comparison systems

$$x' = y, \qquad y' = -g_1(x)$$

and

$$x' = y, \qquad y' = -g_2(x),$$

which, according to our preliminary analysis in Section 4.1, are closed curves around the origin. Roughly speaking, this argument goes as follows: let $z(t) = (x(t), y(t))$ be a (nontrivial) solution to (4.13) such that $|x(t)| \leq \delta$ for every $t \in [s_0, s_1] \subset [0, \tau]$. By simple computations, one can see that

$$\frac{d}{dt}\mathcal{E}_{G_1}(z(t)) \leq 0, \text{ for } x(t)y(t) \geq 0, \quad \text{and} \quad \frac{d}{dt}\mathcal{E}_{G_1}(z(t)) \geq 0, \text{ for } x(t)y(t) \leq 0,$$

and, symmetrically,

$$\frac{d}{dt}\mathcal{E}_{G_2}(z(t)) \geq 0, \text{ for } x(t)y(t) \geq 0, \quad \text{and} \quad \frac{d}{dt}\mathcal{E}_{G_2}(z(t)) \leq 0, \text{ for } x(t)y(t) \leq 0.$$

As a consequence, in order to bound $z(t)$ from below, we can use the level lines of $\mathcal{E}_{G_2}$ in the first and the third quadrant and those of $\mathcal{E}_{G_1}$ in the second and the fourth quadrant. Analogously, the level lines of $\mathcal{E}_{G_1}$ in the first and the third quadrant and those of $\mathcal{E}_{G_2}$ in the second and the fourth quadrant can be used to obtain upper bounds for $z(t)$. Taking into account that $z(t)$ winds around the origin in the clockwise sense, we can define two spirals departing from $(x(0), y(0))$ which provide a control for the solution. Such spirals are constructed by gluing level lines of $\mathcal{E}_{G_1}$ and $\mathcal{E}_{G_2}$ in alternate manner (see [68, Figure 2]). Although we believe that the argument exposed above is sufficiently convincing, we prefer to present all the details in a more formal proof, by passing to the polar coordinates and using the theory of positively invariant sets.

We introduce the auxiliary functions $\mathscr{M}_{\pm}(x, y) : \mathbb{R}^2_* \to \mathbb{R}$, defined as

$$\mathscr{M}_+(x, y) = \begin{cases} \dfrac{(g_1(x) - x)y}{g_1(x)x + y^2} & \text{if } xy \leq 0 \\[2ex] \dfrac{(g_2(x) - x)y}{g_2(x)x + y^2} & \text{if } xy \geq 0 \end{cases}$$

and

$$\mathscr{M}_-(x, y) = \begin{cases} \dfrac{(g_1(x) - x)y}{g_1(x)x + y^2} & \text{if } xy \geq 0 \\[2ex] \dfrac{(g_2(x) - x)y}{g_2(x)x + y^2} & \text{if } xy \leq 0 \end{cases}$$

and consider the associated first order differential equations

$$\frac{dr}{d\theta} = r\mathscr{M}_+(r\cos\theta, r\sin\theta), \quad (r, \theta) \in \mathbb{R}^+_* \times \mathbb{R}, \tag{4.20}$$

$$\frac{dr}{d\theta} = r\mathscr{M}_-(r\cos\theta, r\sin\theta), \quad (r, \theta) \in \mathbb{R}^+_* \times \mathbb{R}. \tag{4.21}$$

For every (θ_0, ρ_0) with $\theta_0 \in \mathbb{R}$ and $\rho_0 > 0$, we denote, respectively, by $r_+(\cdot;\theta_0,\rho_0)$ and $r_-(\cdot;\theta_0,\rho_0)$ the solutions to (4.20) or (4.21) satisfying the initial condition $r(\theta_0) = \rho_0$. The geometrical meaning of these solutions is the following: the maps

$$\mathbb{R} \ni \theta \mapsto (r_\pm(\theta;\theta_0,\rho_0)\cos\theta, r_\pm(\theta;\theta_0,\rho_0)\sin\theta)$$

parameterize two spiralling curves in the plane, passing through

$$(x_0, y_0) = (\rho_0\cos\theta_0, \rho_0\sin\theta_0)$$

and obtained by alternating (along the motion from one quadrant to another) level curves of $\mathcal{E}_{G_1}$ and $\mathcal{E}_{G_2}$. Such spirals are precisely the ones needed to bound (from below and above) in the plane the solutions to (4.13). The uniqueness of the solutions $r_\pm(\theta)$ follows at once from the fundamental theory of ODEs if we assume $g_1(x), g_2(x)$ continuously differentiable. One could prove the uniqueness under the sole assumption of continuity thanks to the sign condition (4.15), arguing like in [151]. On the other hand, (4.15) jointly with (4.18) guarantee the global existence of the solutions. The 2π-periodicity of $\mathscr{M}_\pm$ in the θ-variable implies that, for every $\theta, \theta_0 \in \mathbb{R}$ and $\rho_0 > 0$,

$$r_\pm(\theta;\theta_0 + 2\pi,\rho_0) = r_\pm(\theta + 2\pi;\theta_0,\rho_0).$$

Notice that, from the sign condition (4.16), we deduce that

$$\begin{cases} U(t,x,y) < 0 \\ \mathscr{M}_-(x,y) \le \dfrac{S(t,x,y)}{U(t,x,y)} \le \mathscr{M}_+(x,y), \end{cases} \quad \text{for } t \in [0,\tau],\ |x| \le \delta,\ y \in \mathbb{R},\ x^2 + y^2 > 0. \tag{4.22}$$

At this point, we define, for every integer $j \ge 1$ and for every $\rho_0 > 0$, the parameters

$$m_j^*(\rho_0) = \inf_{\substack{\theta_0 \in [0,2\pi[\\ \theta \in [\theta_0 - 2j\pi, \theta_0]}} r_+(\theta;\theta_0,\rho_0), \quad M_j^*(\rho_0) = \sup_{\substack{\theta_0 \in [0,2\pi[\\ \theta \in [\theta_0 - 2j\pi, \theta_0]}} r_-(\theta;\theta_0,\rho_0).$$

The number $M_j^*(\rho_0)$ provides an upper bound for the modulus of a spiral associated with (4.21) and departing from the circumference $\rho = \rho_0$, while performing an angular displacement of $2j\pi$. Similarly, $m_j^*(\rho_0)$ gives a lower bound for the modulus of the spiral associated with (4.20). For $\delta > 0$ as in (4.16) and any fixed $j \ge 1$, we choose $r_j^* \in]0,\delta[$ such that

$$M_{j+1}^*(r_j^*) < \delta.$$

Subsequently, we fix two numbers $\check{r}_j$ and $\hat{r}_j$ such that

$$0 < \check{r}_j < m_{j+1}^*(r_j^*) \le r_j^* \le M_{j+1}^*(r_j^*) < \hat{r}_j < \delta. \tag{4.23}$$

Once we have chosen $\hat{r}_j$ and $\check{r}_j$, we can define

$$\delta_j^* = \inf_{\check{r}_j \le \sqrt{x^2+y^2} \le \hat{r}_j} \frac{g_1(x)x + y^2}{x^2 + y^2} > 0.$$

At last, we set

$$\tau_j^* = \frac{2\pi j}{\delta_j^*}$$

and, finally, we are in the position to verify that the conclusion of the lemma holds true.

Fix $\tau > \tau_j^*$ and $w(t,x)$ as in the statement; moreover, let $x(t)$ be a solution to (4.12) with $x(0)^2 + x'(0)^2 = (r_j^*)^2$. For simplicity of notation, we set $z(t) = (x(t), x'(t))$. Passing to the polar coordinates

$$x(t) = \rho(t)\cos\theta(t), \qquad x'(t) = \rho(t)\sin\theta(t), \tag{4.24}$$

we have $\rho(0) = r_j^*$ and, without loss of generality, $\theta(0) = \theta_0 \in [0, 2\pi[$. With these positions, (4.17) turns out to be equivalent to

$$\theta(\tau) - \theta(0) < -2j\pi. \tag{4.25}$$

Define $\sigma \in]0, \tau]$ as the maximal number such that

$$\check{r}_j \leq \rho(t) = |z(t)| \leq \hat{r}_j\,, \quad \text{for every } t \in [0,\sigma].$$

By the choice of $\hat{r}_j$ we have also that $|x(t)| < \delta$ for all $t \in [0,\sigma]$ and therefore we can take advantage of the assumption (4.16) as long as $t \in [0,\sigma]$.

We now argue differently according to the fact that $\sigma = \tau$ or $\sigma < \tau$.

- Suppose that $\sigma = \tau$. In this case, we directly obtain that

$$\begin{aligned}\mathrm{Rot}\,(z(t);[0,\tau]) &= \frac{1}{2\pi}\int_0^\tau \frac{x'(t)^2 + w(t,x(t))x(t)}{x(t)^2 + x'(t)^2}\,dt \\ &\geq \frac{1}{2\pi}\int_0^\tau \frac{x'(t)^2 + g_1(x(t))x(t)}{x(t)^2 + x'(t)^2}\,dt \geq \delta_j^* \frac{\tau}{2\pi} > \delta_j^* \frac{\tau_j^*}{2\pi} = j\end{aligned}$$

and hence the thesis follows.

- Assume that $\sigma < \tau$. In this case, the maximality of σ implies that

$$\check{r}_j \leq \rho(t) \leq \hat{r}_j\,, \quad \text{for every } t \in [0,\sigma], \quad \text{and} \quad \rho(\sigma) \in \{\check{r}_j\,, \hat{r}_j\}.$$

We consider that case in which $\rho(\sigma) = \hat{r}_j$; the treatment of the other situation is completely symmetric (involving the consideration of r_+ instead of r_-) and thus is omitted.

We want to prove that

$$\theta(\sigma) - \theta(0) < -2(j+1)\pi \tag{4.26}$$

holds. Indeed, from (4.26) one can conclude easily by observing that $\theta(\tau) - \theta(\sigma) < \pi$ (as a consequence of Lemma 4.2.1) and therefore $\theta(\tau) - \theta(0) < -2j\pi - \pi$, yielding (4.25).

If, by contradiction, we suppose that (4.26) is not true, from $\theta'(t) < 0$ for all $t \in [0,\sigma]$, we obtain

$$-2(j+1)\pi < \theta(t) - \theta(0) < 0, \quad \text{for every } t \in]0, \sigma[\,.$$

Let $\gamma : \mathbb{R} \to \mathbb{R}_*^+$ be the solution to (4.21) with $r(\theta_0) = r_j^*$, that is $\gamma(\theta) = r_-(\theta;\theta_0, r_j^*)$. From the definition of $M_j^*(\rho_0)$ and the choice of $\hat{r}_j$ in (4.23), we can find $\varepsilon > 0$ such that

$$\gamma(\theta) < \hat{r}_j\,, \quad \text{for every } \theta \in]\theta_0 - 2(j+1)\pi - \varepsilon, \theta_0 + \varepsilon[\,.$$

Moreover, from (4.19) and (4.22) it follows that

$$R(t,\gamma(\theta),\theta) \leq \gamma'(\theta)\Theta(t,\gamma(\theta),\theta), \quad \text{for every } t \in [0,\tau],\ \theta \in]\theta_0 - 2(j+1)\pi - \varepsilon, \theta_0 + \varepsilon[\,,$$

so that, using Corollary 4.4.1 with the positions $\theta_1 = \theta_0 - \varepsilon$, $\theta_2 = \theta_0 + 2(j+1)\pi + \varepsilon$ and $I = [0, \sigma]$, we easily conclude that

$$\rho(t) \leq \gamma(\theta(t)), \quad \text{for every } t \in [0, \sigma],$$

and hence

$$\rho(t) < \hat{r}_j\,, \quad \text{for every } t \in [0, \sigma],$$

thus contradicting the hypothesis that $\rho(\sigma) = \hat{r}_j$. □

Our final results show that, under conditions like (f_0) (respectively, (f_∞)), for any fixed time interval, small solutions (respectively, large solutions) cannot complete a turn. Both Lemma 4.2.4 and Lemma 4.2.6 below (as well as their symmetric versions) rely on the following well known estimate, the so called "elastic property" as it refers to the fact that solutions departing small (respectively, large) remain small (respectively, large) in a uniform manner. Precisely, (E_0) below follows from the fact that $w(t, 0) \equiv 0$, together with the uniqueness of the solutions to the Cauchy problems, while (E_∞) relies on a compactness argument in view of the global continuability. For more details, see [31, Lemma 3.1].

Lemma 4.2.3. *In the above setting, there exist two continuous functions $\eta, \nu : \mathbb{R}^+_* \to \mathbb{R}$, with $0 < \eta(s) \leq s \leq \nu(s)$ for all $s > 0$, such that for every $x : \mathbb{R} \to \mathbb{R}$ solution to (4.12) we have:*

(E_0) *for every $r > 0$,*

$$\min_{t \in J} \sqrt{x(t)^2 + x'(t)^2} \leq \eta(r) \Longrightarrow \max_{t \in J} \sqrt{x(t)^2 + x'(t)^2} \leq r;$$

(E_∞) *for every $R > 0$,*

$$\max_{t \in J} \sqrt{x(t)^2 + x'(t)^2} \geq \nu(R) \Longrightarrow \min_{t \in J} \sqrt{x(t)^2 + x'(t)^2} \geq R.$$

We consider at first the case of rotation numbers for small solutions.

Lemma 4.2.4. *Let $J \subset \mathbb{R}$ be a compact interval and suppose that*

$$\limsup_{x \to 0^+} \frac{w(t,x)}{x} \leq 0, \quad \textit{uniformly in } t \in J. \tag{4.27}$$

Then, there exists $r_0 > 0$ such that, for every $x : \mathbb{R} \to \mathbb{R}$ nontrivial solution to (4.12) such that

$$\min_{t \in J} \sqrt{x(t)^2 + x'(t)^2} \leq r_0,$$

it holds that

$$\mathrm{Rot}\,((x(t), x'(t)); J) < 1,$$

If, moreover, for some $t' \in J$ it holds that $x(t') \geq 0$ and $x'(t') \leq 0$, then

$$\mathrm{Rot}\,((x(t), x'(t)); J \cap [t', +\infty[) < 1/4.$$

Proof. By (4.27) we have that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$w(t,x)x \le \varepsilon x^2, \quad \text{for every } t \in J,\ x \in [0,\delta_\varepsilon].$$

Let $x(t)$ be a nontrivial solution to (4.12) satisfying

$$|x(t)| \le \delta_\varepsilon, \quad \text{for every } t \in J. \tag{4.28}$$

Passing to polar coordinates as in (4.24) we study the angular variation of the solution in order to show that the rotation number must be small if we take care of choosing ε sufficiently small. To this aim, we prove the following claim.

Claim. *Let us fix $\alpha \in]0,\pi/2[$. Then there exists $\varepsilon^* = \varepsilon^*_\alpha > 0$ such that, for each nontrivial solution satisfying (4.28) for $\varepsilon \le \varepsilon^*$, the set $\theta(J)$ does not contain any interval of the form $[\theta_1,\theta_2]$ for some $\theta_1 = \theta(t_1) \in [-\pi/2 + 2k\pi, 2k\pi]$, $\theta_2 = \theta(t_2) \in [2k\pi, \pi/2 + 2k\pi]$ (with $k \in \mathbb{Z}$), for $t_1 < t_2$ and $\theta_2 - \theta_1 = \alpha$.*

Suppose, by contradiction, that there exist $t', t'' \in J$ $(t' < t'')$ and a pair θ_1, θ_2 as above such that

$$\theta(t') = \theta_1\,,\ \theta(t'') = \theta_2 \quad \text{and} \quad \theta(t) \in]\theta_1,\theta_2[\,, \ \text{ for } t \in]t',t''[\,.$$

By the choice of $[\theta_1,\theta_2]$, we have that $x(t) = \rho(t)\cos\theta(t) \in [0,\delta_\varepsilon]$, for all $t \in [t',t'']$. Therefore, from (4.14), we have, for every $t \in [t',t'']$,

$$\theta'(t) \le \sin^2\theta(t) + \varepsilon\cos^2\theta(t),$$

that is

$$\frac{\theta'(t)}{\sin^2\theta(t) + \varepsilon\cos^2\theta(t)} \le 1. \tag{4.29}$$

Let us introduce the auxiliary function

$$\Psi(\theta) = \int_0^\theta \frac{d\xi}{\sin^2\omega + \varepsilon\cos^2\omega}\,,$$

so that (4.29) writes as

$$\Psi'(\theta(t))\theta'(t) \le 1, \quad \text{for every } t \in [t',t''].$$

An integration over $[t',t'']$ gives

$$\Psi(\theta(t'')) - \Psi(\theta(t')) \le \int_{t'}^{t''} dt \le t'' - t' \le |J|,$$

that is

$$\int_{\theta_1=\theta(t')}^{\theta_2=\theta(t'')} \frac{d\theta}{\sin^2\theta + \varepsilon\cos^2\theta} = \Psi(\theta_2) - \Psi(\theta_1) \le |J|.$$

On the other hand,

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin^2\theta + \varepsilon\cos^2\theta} = \frac{1}{\sqrt{\varepsilon}}\left(\arctan(\varepsilon^{-1/2}\tan\theta_2) - \arctan(\varepsilon^{-1/2}\tan\theta_1)\right) = \phi_{\theta_1,\theta_2}(\varepsilon).$$

At this point, we recall that

$$\tan\theta_1 \le 0 \le \tan\theta_2, \qquad \text{with } \ \theta_2 - \theta_1 = \alpha > 0.$$

Then, an analysis for the search for the minimum of the function

$$[-\alpha/2, \alpha/2] \ni s \mapsto \arctan\Big(\varepsilon^{-1/2}\tan(s + \tfrac{\alpha}{2})\Big) - \arctan\Big(\varepsilon^{-1/2}\tan(s - \tfrac{\alpha}{2})\Big)$$

shows that

$$\phi_{\theta_1,\theta_2}(\varepsilon) \ge \frac{1}{\sqrt{\varepsilon}}\,\min\Big\{2\arctan\Big(\varepsilon^{-1/2}\tan(\tfrac{\alpha}{2})\Big), \arctan\Big(\varepsilon^{-1/2}\tan(\alpha)\Big)\Big\}.$$

The above minimum can be achieved by the first function or the second one, depending whether $0 < \varepsilon < 1$ or $\varepsilon > 1$. In any case,

$$\lim_{\varepsilon\to 0^+} \phi_{\theta_1,\theta_2}(\varepsilon) = +\infty,$$

uniformly on $\theta_1 \le 0$ and $\theta_2 \ge 0$ with $\theta_2 - \theta_1 = \alpha$. In particular, if $0 < \varepsilon < 1$, we find

$$\phi_{\theta_1,\theta_2}(\varepsilon) \ge \frac{\alpha}{\sqrt{\varepsilon}}\,.$$

Thus a contradiction is achieved provided that $\varepsilon \in]0,1[$ is chosen so small that $\varepsilon < (\alpha/|J|)^2$. □

Let us fix now $\alpha = \pi/4$ and take $0 < \varepsilon \le \varepsilon^*$. Suppose, by contradiction, that $x(t)$ is a nontrivial solution satisfying (4.28) and such that $\mathrm{Rot}\,((x(t), x'(t)); J) \ge 1$. Then, from the definition of rotation number, it follows that $\theta(\max J) - \theta(\min J) \ge 2\pi$ and, consequently, the interval $[\theta(\min J), \theta(\max J)]$ (contained in $\theta(J)$) must contain (mod 2π) at least one interval between $[-\pi/4, 0]$ and $[0, \pi/4]$. This clearly contradicts the Claim (thanks to the choice of ε). Finally, we invoke the first part of Lemma 4.2.3, which guarantees that the choice $r_0 = \eta(\delta_\varepsilon)$ is sufficient for the conclusion to hold. The proof of the second part of the lemma is omitted as it follows from the Claim using an analogous argument. □

A symmetric version of our result reads as follows.

Lemma 4.2.5. *Let $J \subset \mathbb{R}$ be a compact interval and suppose that*

$$\limsup_{x\to 0^-} \frac{w(t,x)}{x} \le 0, \qquad \textit{uniformly in } \ t \in J.$$

Then, there exists $r_0 > 0$ such that, for every $x : \mathbb{R} \to \mathbb{R}$ nontrivial solution to (4.12) *such that*

$$\min_{t\in J} \sqrt{x(t)^2 + x'(t)^2} \le r_0,$$

it holds that

$$\mathrm{Rot}\,((x(t), x'(t)); J) < 1,$$

If, moreover, for some $t' \in J$ it holds that $x(t') \le 0$ and $x'(t') \ge 0$, then

$$\mathrm{Rot}\,((x(t), x'(t)); J \cap [t', +\infty[) < 1/4.$$

We consider now the rotation numbers of large solutions.

Lemma 4.2.6. *Let $J \subset \mathbb{R}$ be a compact interval and suppose that*

$$\limsup_{x\to+\infty} \frac{w(t,x)}{x} \le 0, \quad \textit{uniformly in } t \in J. \tag{4.30}$$

Then, there exists $R_0 > 0$ such that, for every $x : \mathbb{R} \to \mathbb{R}$ solution to (4.12) such that

$$\max_{t\in J} \sqrt{x(t)^2 + x'(t)^2} \ge R_0,$$

it holds that

$$\mathrm{Rot}\,((x(t), x'(t)); J) < 1,$$

If, moreover, for some $t' \in J$ it holds that $x(t') \ge 0$ and $x'(t') \le 0$, then

$$\mathrm{Rot}\,((x(t), x'(t)); J \cap [t', +\infty[) < 1/4.$$

Proof. By (4.30) we have that for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$w(t,x)x \le \varepsilon x^2 + M_\varepsilon, \quad \text{for every } t \in J,\ x \ge 0.$$

Let $x(t)$ be a solution to (4.12) such that, for $z(t) = (x(t), x'(t))$,

$$|z(t)|^2 \ge \frac{M_\varepsilon}{\varepsilon}, \quad \text{for every } t \in J. \tag{4.31}$$

Passing again to the polar coordinates as in (4.24) we prove the following claim.

Claim. *Let us fix $\alpha \in]0, \pi/2[$. Then there exists $\varepsilon^* = \varepsilon^*_\alpha > 0$ such that, for each solution satisfying (4.31) for $\varepsilon \le \varepsilon^*$, the set $\theta(J)$ does not contain any interval of the form $[\theta_1, \theta_2]$ for some $\theta_1 = \theta(t_1) \in [-\pi/2 + 2k\pi, 2k\pi]$, $\theta_2 = \theta(t_2) \in [2k\pi, \pi/2 + 2k\pi]$ (with $k \in \mathbb{Z}$), for $t_1 < t_2$ and $\theta_2 - \theta_1 = \alpha$.*

Suppose, by contradiction, that there exist $t', t'' \in J$ ($t' < t''$) and a pair θ_1, θ_2 as above such that

$$\theta(t') = \theta_1\,,\ \theta(t'') = \theta_2 \quad \text{and} \quad \theta(t) \in]\theta_1, \theta_2[\,, \ \text{for } t \in]t', t''[\,.$$

By the choice of $[\theta_1, \theta_2]$, we have that $x(t) = \rho(t)\cos\theta(t) \ge 0$, for all $t \in [t', t'']$. Therefore, from (4.14), it holds that, for every $t \in [t', t'']$,

$$\theta'(t) \le \sin^2\theta(t) + \varepsilon\cos^2\theta(t) + \frac{M_\varepsilon}{\rho(t)^2}.$$

Using (4.31) for $0 < \varepsilon < 1$, so that $M_\varepsilon/|z(t)|^2 \le \sin^2\theta(t) + \varepsilon\cos^2\theta(t)$, we obtain

$$\theta'(t) \le 2(\sin^2\theta(t) + \varepsilon\cos^2\theta(t)).$$

From now on, the proof of the Claim proceeds like that of the analogous Claim in Lemma 4.2.4 and we skip the details. A contradiction is achieved provided that $\varepsilon \in]0, 1[$ is chosen so small that $\varepsilon < (\alpha/2|J|)^2$. □

Let us fix now $\alpha = \pi/4$ and take $0 < \varepsilon \le \varepsilon^*$. Suppose, by contradiction, that $x(t)$ is a solution to (4.12) such that, for $z(t) = (x(t), x'(t))$, (4.31) holds true and $\mathrm{Rot}\,(z(t); J) \geq 1$. Then, from the definition of rotation number, it follows that $\theta(\max J) - \theta(\min J) \geq 2\pi$ and, consequently, the interval $[\theta(\min J), \theta(\max J)]$ (contained in $\theta(J)$) must contain (mod 2π) at least one interval between $[-\pi/4, 0]$ and $[0, \pi/4]$. This clearly contradicts the Claim (thanks to the choice of ε). Finally, we invoke the second part of Lemma 4.2.3, which guarantees that the choice

$$R_0 = \nu\left(\left(\frac{M_\varepsilon}{\varepsilon}\right)^{1/2}\right)$$

is sufficient for the conclusion to hold. The proof of the second part of the lemma is omitted as it follows from the Claim using an analogous argument. □

Symmetrically, we also have the following.

Lemma 4.2.7. *Let $J \subset \mathbb{R}$ be a compact interval and suppose that*

$$\limsup_{x \to -\infty} \frac{w(t,x)}{x} \leq 0, \quad \textit{uniformly in } t \in J.$$

Then, there exists $R_0 > 0$ such that, for every $x : \mathbb{R} \to \mathbb{R}$ solution to (4.12) such that

$$\max_{t \in J} \sqrt{x(t)^2 + x'(t)^2} \geq R_0,$$

it holds that

$$\mathrm{Rot}\,((x(t), x'(t)); J) < 1,$$

If, moreover, for some $t' \in J$ it holds that $x(t') \leq 0$ and $x'(t') \geq 0$, then

$$\mathrm{Rot}\,((x(t), x'(t)); J \cap [t', +\infty[) < 1/4.$$

4.2.3 Proof

First of all, for $\lambda > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}$, we set

$$w(t, x, \lambda) = f\left(\frac{t}{\sqrt{\lambda}}, x\right).$$

The function $w(t, x, \lambda)$ is $\sqrt{\lambda}T$ periodic in the first variable and all the hypotheses on $f(t, x)$ now are transferred to the function $w(t, x, \lambda)$; in particular, we have

$$g_1(x)x \leq w(t, x, \lambda) \leq g_2(x)x, \quad \text{for every } t \in \sqrt{\lambda}I_0 = [0, \sqrt{\lambda}\tau],\ |x| \leq \delta,$$

for the same functions $g_1(x), g_2(x)$ as in (f_g). Observe, moreover, that if $(x(t), y(t))$ is a solution to the planar Hamiltonian system

$$\begin{cases} x' = y \\ y' = -w(t, x, \lambda), \end{cases} \tag{4.32}$$

then $u(t) = x(\sqrt{\lambda}\, t)$ is a solution to (4.11); moreover $x(t)$ is $\sqrt{\lambda}T$ periodic if and only if $u(t)$ is T-periodic. We are going to use Theorem 1.1.3 to prove the existence (when λ is large) of $\sqrt{\lambda}T$-periodic solutions to (4.32): that is, we look for fixed points of the Poincaré map $\Psi_\lambda : \mathbb{R}^2 \ni \bar{z} \mapsto z(\sqrt{\lambda}T; \bar{z})$, being $z(\cdot; \bar{z})$ the solution to (4.32) satisfying $z(0; \bar{z}) = \bar{z}$.

For an integer $m \geq 1$, let us define

$$\Lambda_m^* = \left(\frac{\tau_{m+1}^*}{|I_0|}\right)^2 ,$$

where τ_{m+1}^* is the constant given by Lemma 4.2.2. Now we fix $\lambda > \Lambda_m^*$; we will apply Lemma 4.2.2 with $I = \sqrt{\lambda} I_0$ and the subsequent lemmas with $J = [0, \sqrt{\lambda} T]$.

By the choice of Λ_m^* we have that, for every $\bar{z} \in \mathbb{R}^2$ with $|\bar{z}| = r_{m+1}^*$,

$$\mathrm{Rot}\,(z(t;\bar{z}); I) > m + 1.$$

Using Lemma 4.2.1, we find

$$\mathrm{Rot}\,(z(t;\bar{z}); J) = \mathrm{Rot}\,(z(t;\bar{z}); I) + \mathrm{Rot}\,(z(t;\bar{z}); [\sqrt{\lambda}\tau, \sqrt{\lambda}T]) > (m+1) - \frac{1}{2} > m.$$

Assumption (f_0) permits to apply Lemma 4.2.4 or Lemma 4.2.5. In any case, there exists $r_0 > 0$ such that, for every $\bar{z} \in \mathbb{R}^2$ with $0 < |\bar{z}| \leq r_0$, it holds

$$\mathrm{Rot}\,(z(t;\bar{z}); J) < 1.$$

Note that r_0 depends on λ and, moreover, $r_0 < r_{m+1}^*$. Then, from Theorem 1.1.3, we get, for every $j = 1, \ldots, m$, the existence of at least two $\sqrt{\lambda}T$-periodic solutions $z_j^{(1)}(t)$ and $z_j^{(2)}(t)$ to (4.32) such that $(i = 1, 2)$ $r_0 < |z_j^{(i)}(0)| < r_{m+1}^*$ and

$$\mathrm{Rot}\,(z_j^{(i)}(t); J) = j.$$

As remarked at the beginning of the subsection, this implies the existence, for every $j = 1, \ldots, m$, of at least two T-periodic solution $u_j^{(1)}(t)$ and $u_j^{(1)}(t)$ to (4.11) having precisely $2j$ zeros in $[0, T[$.

On the other hand, assumption (f_∞) permits to apply Lemma 4.2.6 or Lemma 4.2.7. In any case, there exists $R_0 > 0$ such that, for every $\bar{z} \in \mathbb{R}^2$ with $|\bar{z}| \geq R_0$, it holds:

$$\mathrm{Rot}\,(z(t;\bar{z}); J) < 1.$$

Also the constant R_0 depends on λ. Moreover, we have $R_0 > r_{m+1}^*$. In this manner, again from Theorem 1.1.3, we get, for every $j = 1, \ldots, m$, the existence of at least two $\sqrt{\lambda}T$-periodic solutions $\check{z}_j^{(1)}(t)$ and $\check{z}_j^{(2)}(t)$ to (4.32) such that $(i = 1, 2)$ $r_{m+1}^* < |\check{z}_j^{(i)}(0)| < R_0$ and

$$\mathrm{Rot}\,(\check{z}_j^{(i)}(t); J) = j.$$

That is, for every $j = 1, \ldots, m$, we have found two T-periodic solution $\check{u}_j^{(1)}(t)$ and $\check{u}_j^{(2)}(t)$ to (4.11) having precisely $2j$ zeros in $[0, T[$.

Summarizing, we have found $4m$ T-periodic solutions of (4.11); precisely, for each $j = 1, \ldots, m$ there are at least four with exactly $2j$ zeros in $[0, T[$: two small solutions and two large solutions.

4.3 Variants of the main result, remarks and applications

In this section, we propose some applications of Theorem 4.2.1; our model case here is the differential equation with weight

$$u'' + \lambda q(t)g(u) = 0, \tag{4.33}$$

being $q : \mathbb{R} \to \mathbb{R}$ a continuous and T-periodic function and $g : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz continuous function with $g(0) = 0$. Our main interest concerns the assumption of global continuability for the solutions, which will be no longer explicitly required.

At first, we consider the following Landesman-Lazer type condition:

(LL) *Let $d > 0$ and let $\gamma_1, \gamma_2 : [0, T] \to \mathbb{R}$ be continuous functions, with*

$$\int_0^T \gamma_1(t)\, dt \leq 0 \leq \int_0^T \gamma_2(t)\, dt,$$

such that, for every $t \in [0, T]$,

$$f(t, x) \leq \gamma_1(t), \quad \textit{for all } x \leq -d, \quad \textit{and} \quad f(t, x) \geq \gamma_2(t), \quad \textit{for all } x \geq d.$$

As proved in [59, 60], condition (LL), when paired with a one-sided sublinear growth at infinity (in particular, (f_∞) is enough), guarantees the global continuability of the solutions. We can therefore deduce the following Corollary for equation (4.33).

Corollary 4.3.1. *Suppose that $g(x)x > 0$ for $x \neq 0$; moreover,*

$$\lim_{x \to 0^+} \frac{g(x)}{x} = 0 \quad \textit{or} \quad \lim_{x \to 0^-} \frac{g(x)}{x} = 0$$

and

$$\lim_{x \to +\infty} \frac{g(x)}{x} = 0 \quad \textit{or} \quad \lim_{x \to -\infty} \frac{g(x)}{x} = 0.$$

Finally, assume that $q(t) \geq 0$ for every $t \in [0, T]$, with $q(t) \not\equiv 0$. Then the conclusion of Theorem 4.2.1 holds for equation (4.33).

Proof. By the assumptions, it is obvious that (f_0) and (f_∞) hold for $f(t, x) = q(t)g(x)$. Moreover, the global continuability of the solutions is guaranteed because $f(t, x)$ satisfies a (LL)-type condition with $\gamma_1(t) = \gamma_2(t) \equiv 0$. Finally, the sign condition (f_g) is fulfilled as explained at the beginning of Section 4.2. □

The assumption on $g(x)/x$ at $\pm\infty$ could be replaced by

$$\lim_{x \to +\infty} \frac{G(x)}{x^2} = 0 \quad \text{or} \quad \lim_{x \to -\infty} \frac{G(x)}{x^2} = 0,$$

for $G(x) = \int_0^x g(\xi)\, d\xi$ (in this case, instead of using Lemma 4.2.6 and Lemma 4.2.7, the proof that large solutions have small rotation numbers follows from a result in [59]).

Corollary 4.3.1 extends to (4.33) the result obtained in Section 4.1 for the autonomous equation $u'' + \lambda g(u) = 0$. Examples of equations to which Corollary 4.3.1 applies are given, for instance, by

$$u'' + \lambda q(t)|u|^p(\exp u - 1) = 0,$$

for $p > 1$ and $q(t) \geq 0$, not identically zero.

A different possibility (of purely local nature) to satisfy both the assumption of global continuability and (f_∞) is based on some truncation arguments using lower/upper solutions techniques. Such a procedure has been already developed in Chapter 3 with all the details; here we limit ourselves to emphasize the following corollary, dealing with equation (4.33).

Corollary 4.3.2. *Suppose that* $g(x)x > 0$ *for* $0 < |x| \leq \delta$ *and*

$$\lim_{x\to 0^+} \frac{g(x)}{x} = 0 \quad \text{or} \quad \lim_{x\to 0^-} \frac{g(x)}{x} = 0.$$

Moreover, let exist $x_* < -\delta$ *and* $x^* > \delta$ *such that*

$$g(x_*) = g(x^*) = 0.$$

Finally, assume that $q(t_0) > 0$ *for some* $t_0 \in [0, T]$*. Then the conclusion of Theorem 4.2.1 holds for equation* (4.33) *and all the corresponding solutions take value in the open interval* $]x_*, x^*[$.

Proof. Fefine $\widetilde{g}(x) = g(\min\{x^*, \max\{x_*, x\}\})$, so that $\widetilde{g}(x) = g(x)$ on $[x_*, x^*]$ and $\tilde{g}(x)$ vanishes elsewhere; we look for solutions to the truncated equation

$$u'' + \lambda q(t)\tilde{g}(u) = 0. \tag{4.34}$$

The uniqueness and global continuability are obviously guaranteed. Since $\widetilde{g}(x) = g(x)$ for $|x| \leq \delta$ and $\max_{t\in[0,T]} q(t) > 0$, we have (f_g) and (f_0) satisfied, as well. Finally, (f_∞) trivially holds since $\widetilde{g}(x) = 0$ for $|x|$ large. Then Theorem applies to (4.34) and an elementary direct argument (see the proof of Theorem 3.1.1) ensures that the solutions produced take values in $]x_*, x^*[$ and hence they solve (4.33) as well. □

Examples of equations to which Corollary 4.3.2 applies are given, for instance, by

$$u'' + \lambda q(t)|u|^p \sin u = 0,$$

for $p > 1$ and $\max_{[0,T]} q(t) > 0$. Notice that the weight function $q(t)$ can here change its sign, thus providing a link between the results of this chapter and those in Chapter 6.

4.4 Appendix: a result on flow-invariant sets

The proof of one of our key lemmas for the rotations numbers (Lemma 4.2.2) is based on a comparison argument which is justified by a result on flow-invariant set for nonautonomous differential system. In order to make our work self-contained we present here some specific details on the subject; similar results can be also found in [7, 70].

Throughout the section, $\Omega \subset \mathbb{R}^N$ will denote an open set and $\Xi : J_0 \times \Omega \to \mathbb{R}^N$ a continuous function with $J_0 \subset \mathbb{R}$ an interval (that will be not restrictive to assume open). We are interested in conditions ensuring the (positive) invariance of a set $\mathcal{M} \subset \Omega$, for the solutions to the differential system

$$\xi' = \Xi(t, \xi), \tag{4.35}$$

according to the following definition.

Definition 4.4.1. *Let $\mathcal{M} \subset \Omega$. We say that $\mathcal{M}$ is positively invariant for* (4.35) *if, for every $\xi : dom(\xi) \to \Omega$ solution to* (4.35) *(being $dom(\xi) \subset J_0$ an interval) and $t_0 \in dom(\xi)$,*

$$\xi(t_0) \in \mathcal{M} \implies \xi(t) \in \mathcal{M}, \qquad \text{for every } t \in]t_0, +\infty[\cap\, dom(\xi).$$

In the above definition, we have denoted by $\mathrm{dom}(\xi)$ any interval on which a solution $\xi(t)$ is defined (not necessarily a maximal one). We remark that, in literature, various concepts of (positive) invariance have been considered by different authors. In some classical results (like, for instance, the Nagumo theorem) a weaker form of invariance is considered, namely, that for any initial point belonging to $\mathcal{M}$ at least one local solution exists belonging to $\mathcal{M}$ in the forward time. Our definition refers to all the possible solutions. Clearly, such different points of view coincide when the (forward) uniqueness for the solutions to the initial value problems is guaranteed.

In the following, the topological notions of interior, boundary and closure of $\mathcal{M}$ will always be meant in the subspace topology; moreover, for $p \in \mathbb{R}^N$ and $\epsilon > 0$, we will denote by $\mathcal{B}(p, \epsilon)$ the open ball of center p and radius ϵ.

Lemma 4.4.1. *Let $\mathcal{M} \subset \Omega$ be closed. Assume that, for every point $p \in \partial\mathcal{M}$, there exist $\epsilon_p > 0$ with $\mathcal{B}(p, \epsilon_p) \subset \Omega$ and a C^1-function $V_p : \mathcal{B}(p, \epsilon_p) \to \mathbb{R}$ such that:*

- *(i) $\mathcal{M} \cap \mathcal{B}(p, \epsilon_p) = \{q \in \mathcal{B}(p, \epsilon_p) \mid V_p(q) \le 0\}$,*
- *(ii) for every $t \in J_0$, there exists a right neighborhood $\mathcal{I}_t$ of t such that*

$$\langle \nabla V_p(q) \mid \Xi(s, q)\rangle \le 0, \qquad \text{for every } s \in \mathcal{I}_t,\ q \in \mathcal{B}(p, \epsilon_p) \setminus \mathcal{M}.$$

Then $\mathcal{M}$ is positively invariant for (4.35).

Proof. Assume, by contradiction, that there exists $\xi : \mathrm{dom}(\xi) \to \Omega$ solution to (4.35) and $t_0, t_1 \in \mathrm{dom}(\xi)$, with $t_0 < t_1$, such that $\xi(t_0) \in \mathcal{M}$, $\xi(t_1) \notin \mathcal{M}$. The set $J = \{s \in [t_0, t_1] \mid \xi(s) \in \mathcal{M}\}$ is compact and nonempty; set $t^* = \max J < t_1$ and $p^* = \xi(t^*)$. Since $p^* \in \partial\mathcal{M}$ and $\xi(t) \notin \mathcal{M}$ for $t \in]t^*, t_1]$, assumption (ii) implies that

$$\langle \nabla V_{p^*}(\xi(t)) \mid \Xi(t, \xi(t))\rangle \le 0, \qquad \text{for every } t \in]t^*, t^* + \delta],$$

being $\delta > 0$ so small that $[t^*, t^* + \delta] \subset \mathcal{I}_{t^*}$ and $\xi(t) \in \mathcal{B}(p^*, \epsilon_{p^*})$ for every $t \in]t^*, t^* + \delta]$. Hence,

$$V_{p^*}(\xi(t^* + \delta)) - V_{p^*}(\xi(t^*)) = \int_{t^*}^{t^*+\delta} \langle \nabla V_{p^*}(\xi(t)) \mid \Xi(t, \xi(t))\, dt \le 0. \tag{4.36}$$

On the other hand, (i) implies that $V_{p^*}(\xi(t^*)) \le 0 < V_{p^*}(\xi(t^* + \delta))$, contradicting (4.36). $\square$

We now turn our attention to the case in which the set $\mathcal{M}$ is a sublevel set of a scalar function V. This situation occurs in various applications where V is a Lyapunov tpe function. Accordingly, let us suppose that

$$\mathcal{M} = \{p \in \Omega \mid V(p) \leq 0\}, \tag{4.37}$$

for a function $V : \Omega \to \mathbb{R}$ of class C^1. Notice that, in view of the continuity of V, $\mathcal{M} \subset \Omega$ is a closed set (relatively to Ω).

Proposition 4.4.1. *Suppose that the uniqueness for the Cauchy problems associated with* (4.35) *is ensured and let* $\mathcal{M} \subset \Omega$ *be defined as in* (4.37)*. Moreover, assume that:*

- $\nabla V(p) \neq 0$*, for every* $p \in \partial\mathcal{M}$*;*
- *for every* $t \in J_0$ *and for every* $p \in \partial\mathcal{M}$*,*

$$\langle \nabla V(p) \mid \Xi(t,p)\rangle \leq 0. \tag{4.38}$$

Then $\mathcal{M}$ *is positively invariant for* (4.35)*.*

Proof. When (4.38) is satisfied with the strict inequality, the thesis follows directly from Lemma 4.4.1 (even without assuming the uniqueness for the Cauchy problems). Indeed, taking, for every $p \in \partial\mathcal{M}$, $\epsilon_p > 0$ small enough and $V_p = V|_{B(p,\epsilon_p)}$, one has that (i) is clearly satisfied, while (ii) follows from (4.38), using the continuity of V.
We now show the conclusion in the general case. Assume, by contradiction, that $\xi : \mathrm{dom}(\xi) \to \Omega$ is a solution to (4.35), $t_0, t_1 \in \mathrm{dom}(\xi)$ with $t_0 < t_1$ and $\xi(t_0) \in \mathcal{M}$, while $\xi(t_1) \notin \mathcal{M}$. Consider the differential system

$$\xi' = \Xi_n(t,\xi) = \Xi(t,\xi) - \frac{1}{n}\nabla V(\xi). \tag{4.39}$$

Since $\Xi_n(t,\xi) \to \Xi(t,\xi)$ uniformly on compact subsets of $J_0 \times \Omega$, the general theory of ODEs implies that, for every n large enough, (4.39) has a (not necessarily unique) solution $\xi_n(t)$ defined on $[t_0,t_1]$ and such that $\xi_n(t_0) = \xi(t_0)$; moreover, since we are assuming that the Cauchy problems associated with (4.35) have a unique solution, $\xi_n(t) \to \xi(t)$ uniformly on $[t_0,t_1]$. For every $t \in J_0$ and for every $p \in \partial\mathcal{M}$, it holds

$$\langle \nabla V(p) \mid \Xi_n(t,p)\rangle = \langle \nabla V(p) \mid \Xi(t,p)\rangle - \frac{1}{n}|\nabla V(p)|^2 < 0.$$

Hence, in view of the first part of the proof, $\xi_n(t_1) \in \mathcal{M}$. Passing to the limit, we get - since $\mathcal{M}$ is closed in Ω - $\xi(t_1) \in \mathcal{M}$, a contradiction. □

Remark 4.4.1. We recall that, when $\mathcal{M} \subset \Omega$ is defined as in (4.37), one has $\partial\mathcal{M} \subset V^{-1}(0)$, the equality being satisfied whenever 0 is a regular value of V, i.e., $\nabla V(p) \neq 0$ for every $p \in V^{-1}(0)$. Hence, the conclusion of Proposition 4.4.1 holds true in the particular case when 0 is a regular value of V and (4.38) is fulfilled for every $t \in J_0$ and $p \in V^{-1}(0)$. See [7] for similar results in the autonomous case.

We finally give a corollary of Proposition 4.4.1 which will be directly applied in Section 4.2. Let us consider the differential system

$$\begin{cases} \rho' = R(t, \rho, \theta) \\ \theta' = \Theta(t, \rho, \theta), \end{cases} \tag{4.40}$$

being $R, \Theta : [0, \tau] \times \mathbb{R}^+_* \times]\theta_1, \theta_2[\to \mathbb{R}$ continuous functions (with $-\infty \leq \theta_1 < \theta_2 \leq +\infty$). In this situation, we have the following.

Corollary 4.4.1. *Suppose that the uniqueness for the Cauchy problems associated with* (4.40) *is ensured and let* $\gamma :]\theta_1, \theta_2[\to \mathbb{R}$ *be a function of class* C^1*, with* $\gamma(\theta) > 0$ *for every* $\theta \in \mathbb{R}$*. Assume*

$$R(t, \gamma(\theta), \theta) \leq \gamma'(\theta)\Theta(t, \gamma(\theta), \theta), \qquad \text{for every } t \in [0, \tau],\ \theta \in]\theta_1, \theta_2[\,. \tag{4.41}$$

Then, for every $(\rho, \theta) : I \to \mathbb{R}^+_* \times]\theta_1, \theta_2[$ *solution to* (4.40) *(being* $I \subset [0, \tau]$ *an interval) and* $t_0 \in I$*,*

$$\rho(t_0) \leq \gamma(\theta(t_0)) \implies \rho(t) \leq \gamma(\theta(t)), \qquad \text{for every } t \in]t_0, +\infty[\cap I.$$

Proof. Set $\Omega = \mathbb{R}^+_* \times]\theta_1, \theta_2[$ and $V(\rho, \theta) = \rho - \gamma(\theta)$. Since $\nabla V(\rho, \theta) = (1, -\gamma'(\theta))$, one has that 0 is a regular value of V; moreover (4.41) implies (4.38). Hence, the thesis follows from Proposition 4.4.1, taking into account Remark 4.4.1. □

Part II

Solutions which do not wind around the origin

Chapter 5

Planar Hamiltonian systems with an equilibrium point of saddle type

In this chapter, coming from [28], we deal again with the planar Hamiltonian system

$$Jz' = \nabla_z H(t, z), \tag{5.1}$$

being $H : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ a function T-periodic in the first variable and such that $\nabla_z H(t, 0) \equiv 0$. The general idea is that a dynamics of saddle type (with a slight abuse in terminology, see Section 5.2) at the origin, matched with an asymptotic dynamics of center type, gives rise to T-periodic solutions which do not wind around the origin.

Our results are mainly motivated by the problem of the existence of one-signed (i.e., positive or negative) T-periodic solutions to the second order scalar ordinary differential equation

$$u'' + f(t, u) = 0, \tag{5.2}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, T-periodic in the first variable, such that $f(t, 0) \equiv 0$. Probably, the most classical approach in the search for one-signed solutions to boundary value problems is based on the use of some forms of the Krasnoselskii fixed point theorem on compression and expansion of cones. As explained in [159, Introduction], such a technique, which requires the conversion of the differential equation (5.2) into an integral equation via the Green's function, is particularly well suited for separated boundary conditions, but presents some difficulties when dealing with the periodic problem. Indeed, the differential operator $u \mapsto -u''$ is not invertible with T-periodic boundary conditions, so that one is usually led to study equations of the form

$$u'' + a(t)u + g(t, u) = 0,$$

with $u \mapsto -u'' - a(t)u$ invertible, by imposing suitable conditions both on the sign of the Green's function and on the nonlinear term $g(t, x)$ or its potential $G(t, x) = \int_0^x g(t, \xi)\, d\xi$, near zero and infinity (see, for instance, [12, 116, 159] and the references therein). Such conditions, however, are often not so nicely recognized, when referred to (5.2).

On the other hand, considering the autonomous case $f(t,x) = f(x)$, one sees that there is a natural way to ensure the existence of one-signed T-periodic solutions. Indeed, if we assume that $f(x)$ is of class C^1 in a neighborhood of $x = 0$ and satisfies

$$f'(0) < 0 \quad \text{and} \quad \liminf_{|x|\to+\infty} f(x)\mathrm{sgn}\,(x) > 0, \tag{5.3}$$

then $f(x)$ has both a positive and a negative zero on the real line and such zeros are, of course, constant one-signed T-periodic solutions. In particular, by looking at the phase plane portrait (see Figure 5 below) of the autonomous equation $u'' + f(u) = 0$, with $f(x)$ satisfying (5.3), we see that the origin is a (local) saddle equilibrium point, while “large” solutions wind around the origin in the clockwise sense.

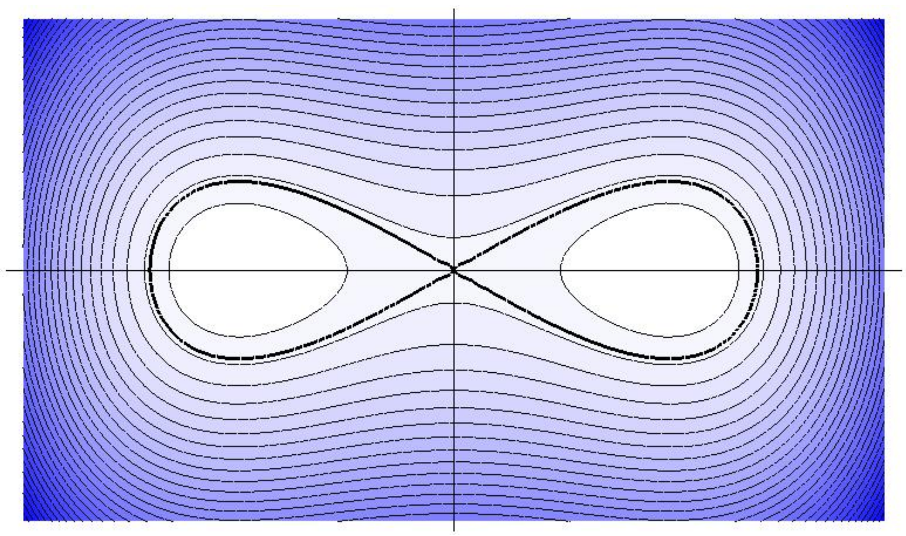


Figure 5.1: The phase portrait of the autonomous equation $u'' + f(u) = 0$, with $f(x) = -x + x^3$. The origin is a local saddle, with two homoclinic orbits (painted with a darker color). “Small” periodic solutions, which do not wind around the origin, are found in the regions bounded by the two homoclinic orbits, both in the left and in the right halfplane of the phase-plane. Moreover, “large” periodic solutions around the origin appear, as well.

In this chapter, we deal with the nonautonomous case (5.2) and we use a recent modified version of the Poincaré-Birkhoff fixed point theorem, Theorem 1.2.2, to detect such kind of “twist dynamics” between zero and infinity[1]. Our main results, Theorem 5.1.1 and Theorem

[1] It is worth noticing that the classical version of the theorem can not succeed, since - in order to have satisfied the boundary twist condition leading to one-signed T-periodic solutions - one should have to show that small solutions to (5.2) move in the counterclockwise sense in the phase-plane. But, also in the simple situation in which (5.2) has, near the origin, the linearized equation $u'' + q_0 u = 0$ (with $q_0 < 0$), one is led to consider - switching to polar coordinates - the equation $-\theta' = \sin^2\theta + q_0\cos^2\theta$, which always has solutions with $\theta' > 0$ as well as solutions with $\theta' < 0$.

5.1.2, are based on two mutually independent generalizations of (5.3) to the nonautonomous case. They both exploit an assumption near $x = 0$ (see hypothesis (f_0) of Section 5.1), which generalizes in a natural way the condition $f'(0) < 0$. Such a condition provides information on the behavior of the solutions to the linearization (at the origin) of equation (5.2) and, accordingly, implies the inner boundary twist condition of the (modified) Poincaré-Birkhoff theorem. On the other hand, the behavior of large solutions to (5.2) (that is, the outer boundary twist condition) is ruled by a classical Landesman-Lazer condition (with respect to the principal eigenvalue $\lambda_0 = 0$ of the periodic problem, see condition (f^1_∞)) in Theorem 5.1.1 and by a comparison with a linear (possibly, sign-indefinite) problem in Theorem 5.1.2 (see condition (f^2_∞)). We stress that, besides one-signed T-periodic solutions, we also get the existence of infinitely many sign-changing subharmonic solutions (with a precise nodal characterization), as the qualitative analysis of the autonomous case suggests. It has to be noticed that the proof of the existence of subharmonic solutions is more standard and it simply relies on more classical versions of the Poincaré-Birkhoff fixed point theorem, (precisely, Theorem 1.2.3).

The plan of the chapter is the following. In Section 5.1, we state and prove our main results, Theorem 5.1.1 and Theorem 5.1.2. In Section 5.2, we propose (partial) extensions of the results to the planar Hamiltonian system (5.1), when the existence of an equilibrium point of saddle type is assumed (see condition (H_0)). In particular, Theorem 5.2.2 is on the lines of Theorem 5.1.1, using a recent generalization of the Landesman-Lazer condition to planar systems [87] (see (H^1_∞)), while Theorem 5.2.1 is the counterpart of Theorem 5.1.2 and it is based on the comparison of the nonlinear system (5.1) with positively homogeneous Hamiltonian systems (see (H^1_∞)), similarly as in Chapter 2.

5.1 One-signed solutions to second order ODEs

In this section, we state and prove our main results (Theorem 5.1.1 and Theorem 5.1.2) dealing with the existence of one-signed harmonic (i.e., T-periodic) solutions and sign-changing subharmonic solutions to the second order scalar ordinary differential equation

$$u'' + f(t, u) = 0. \tag{5.4}$$

We first give the statement of both the results, together with some brief comments on the assumptions and on the relationship with the existing literature; the proofs are postponed at the end of the section.

Henceforth, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, which is T-periodic in the first variable. Moreover, we assume the following condition:

(f_0) $f(t, 0) \equiv 0$ *and there exists a continuous and* T*-periodic function* $q_0 : \mathbb{R} \to \mathbb{R}$*, with*

$$q_0(t) \leq 0, \quad \text{for every } t \in [0, T], \quad \text{and} \quad \int_0^T q_0(t)\, dt < 0,$$

such that

$$\lim_{x \to 0} \frac{f(t, x)}{x} = q_0(t), \qquad \text{uniformly in } t \in [0, T].$$

Writing equation (5.4) in Hamiltonian form, we immediately see that assumption (C_2) of Section 1.2 is fulfilled with $B(t) = \begin{pmatrix} q_0(t) & 0 \\ 0 & 1 \end{pmatrix}$.

We state the first main result.

Theorem 5.1.1. *Suppose that the uniqueness for the solutions to the Cauchy problems associated with* (5.4) *is guaranteed. Moreover, assume condition* (f_0) *and*

(f^1_∞) *it holds that*

$$\lim_{|x|\to+\infty} \frac{f(t,x)}{x} = 0, \qquad \textit{uniformly in } t \in [0,T], \tag{5.5}$$

and

$$\int_0^T \limsup_{x\to-\infty} f(t,x)\,dt < 0 < \int_0^T \liminf_{x\to+\infty} f(t,x)\,dt. \tag{5.6}$$

Then the following conclusions hold true:

i) there exist a positive T-periodic solution $u_p(t)$ and a negative T-periodic solution $u_n(t)$ to (5.4)*;*

ii) there exists $k^* \in \mathbb{N}_*$ such that, for every integer $k \geq k^*$, equation (5.4) *has at least two subharmonic solutions $u_k^{(1)}(t), u_k^{(2)}(t)$ of order k (not belonging to the same periodicity class) with exactly two zeros in the interval $[0, kT[$.*

Assumption (5.5) is a sublinearity condition, in the x-variable, for the nonlinear term $f(t,x)$ (and we recall that it implies the global continuability for the solutions to (5.4)), while (5.6) is the well known Landesman-Lazer condition (with respect to the principal eigenvalue $\lambda_0 = 0$ of the T-periodic problem, compare with Section 2.2). Notice that, in order for the integrals in (5.6) to make sense, it is implicitly assumed that there exists $\eta \in L^1(0,T)$ such that

$$f(t,x)\operatorname{sgn}(x) \geq \eta(t), \qquad \text{for every } t \in [0,T],\ x \in \mathbb{R}.$$

Under similar assumptions to those of Theorem 5.1.1, the existence of two nontrivial T-periodic solutions to (5.4) is proved by Su and Zhao [156, Theorem 1] via Morse theory techniques, but no information concerning their sign is given.

We now state the second main result.

Theorem 5.1.2. *Suppose that the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (5.4) *are guaranteed. Moreover, assume condition* (f_0) *and*

(f^2_∞) *there exist $q_\infty \in L^\infty_T$, with $\int_0^T q_\infty(t)\,dt > 0$, such that*

$$\liminf_{|x|\to+\infty} \frac{f(t,x)}{x} \geq q_\infty(t), \quad \textit{uniformly in } t \in [0,T]. \tag{5.7}$$

Then the following conclusions hold true:

i) *there exist a positive T-periodic solution $u_p(t)$ and a negative T-periodic solution $u_n(t)$ to* (5.4)*;*

ii) *there exists $k^* \in \mathbb{N}_*$ such that, for every integer $k \geq k^*$, there exists an integer $j_*(k)$ such that, for every integer j relatively prime with k and such that $1 \leq j \leq j_*(k)$, equation* (5.4) *has at least two subharmonic solutions $u^{(1)}_{k,j}(t), u^{(2)}_{k,j}(t)$ of order k (not belonging to the same periodicity class) with exactly $2j$ zeros in the interval $[0, kT[$; moreover, we have the estimate*

$$j_*(k) \geq n_k = \mathcal{E}^- \left(\frac{k}{2\pi} \sup_{\xi > 0} \frac{\int_0^T \min\{q_\infty(t), \xi\}\, dt}{\sqrt{\xi}} \right). \tag{5.8}$$

Assumption (5.7) has to be compared with condition (f_0) of Theorem 2.3.7 (see also Remark 2.2.3); notice that only a condition on the mean value of the weight function $q_\infty(t)$ is required. In the definite-sign case $q_\infty(t) \geq 0$ (together with an assumption near the origin related to (f_0)), the existence of one-signed T-periodic solutions to (5.4) is proved by Gasiński and Papageorgiou [88, Proposition 3.3]. Moreover, in [88, Theorem 4.1], some other T-periodic solutions are provided. Theorem 5.1.2 gives more information in this direction. Indeed, from (5.8) it is possible to estimate in a sharp way the order of the subarmonics produced, so that multiple sign-changing T-periodic solutions can appear depending on the "size" of the weight function $q_\infty(t)$. In particular, we can state the following corollary.

Corollary 5.1.1. *Suppose that the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (5.4) *are guaranteed. Moreover, assume condition* (f_0) *and, for a suitable integer $m \geq 1$ and $q_\infty(t) \equiv q_\infty > 0$,*

$$\liminf_{|x| \to +\infty} \frac{f(t,x)}{x} \geq q_\infty > \lambda_m, \quad \textit{uniformly in } t \in [0,T],$$

being $\lambda_m = \left(\frac{2m\pi}{T}\right)^2$ the m-th eigenvalue of the differential operator $u \mapsto -u''$ with T-periodic boundary conditions. Then, equation (5.4) *has a positive T-periodic solution, a negative T-periodic solution and, for every integer j with $1 \leq j \leq m$, two sign-changing T-periodic solutions with exactly $2j$ zeros in the interval $[0, T[$.*

Remark 5.1.1. In [88], the emphasis is on the case when double resonance at infinity occurs, namely when

$$\lambda_h \leq \liminf_{|x| \to +\infty} \frac{f(t,x)}{x} \leq \limsup_{|x| \to +\infty} \frac{f(t,x)}{x} \leq \lambda_{h+1},$$

for some integer $h \geq 1$. When the existence and multiplicity of periodic solutions to equations like (5.4) is studied with functional analytic techniques, this situation presents peculiar difficulties due to a lack of compactness, so that some additional conditions (like the Landesman-Lazer one) have to be added. We stress that, with the Poincaré-Birkhoff fixed point theorem approach, the situation is very different since the estimates of the rotation numbers can be performed in any case, and no nonresonance assumptions are needed. However, as shown in Section 2.2, the Landesman-Lazer condition can be useful to obtain sharper multiplicity results. For instance, according to Proposition 2.2.5, the conclusion of Corollary 5.1.1 still holds true whenever $\lim_{|x| \to +\infty} \frac{f(t,x)}{x} = \lambda_m$ and a Landesman-Lazer condition, with respect to the m-th eigenvalue, is fulfilled.

Remark 5.1.2. As for the existence of one-signed T-periodic solutions, a common interpretation of both Theorem 5.1.1 and Theorem 5.1.2 is provided by the concept of nonresonance with respect to the principal eigenvalue $\lambda_0 = 0$ of the T-periodic problem. From this point of view, (f_0) is a nonuniform nonresonance assumption at the origin (see [126] and Remark 2.2.4), while $(f^1_\infty), (f^2_\infty)$ both require $f(t,x)$ to be nonresonant with respect to λ_0 at infinity. In particular, (f^2_∞) directly imposes that $f(t,x)/x$ is "far away" from zero for $|x|$ large (even if in a quite mild sense), while in (f^1_∞) a Landesman-Lazer condition is added in the sublinear case $f(t,x)/x \to 0$. Results providing the existence of positive solutions to second order differential equations in terms of nonresonance conditions with respect to the principal eigenvalue are very common in literature, dealing with the Dirichlet problem [56, 117].

We now give the proofs of the results.

Proof of Theorem 5.1.1 and Theorem 5.1.2. We first state, in three separate claims, some consequences of (f_0), (f^1_∞) and (f^2_∞) from the point of view of the rotation numbers of the solutions. The technical proofs are postponed at the end of the section. To this aim, let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function, T-periodic in the first variable. For simplicity of exposition, we will always assume that $g(t,0) \equiv 0$ and that the uniqueness and the global continuability for the solutions to the Cauchy problems associated with

$$u'' + g(t,u) = 0 \tag{5.9}$$

are guaranteed.

Claim 1. *For every integer $k \geq 1$ and for every $u : \mathbb{R} \to \mathbb{R}$ solving $u'' + q_0(t)u = 0$, it holds that*

$$\mathrm{Rot}((u(t),u'(t));[0,kT]) < 1; \tag{5.10}$$

moreover, if $|u(0)| = 1$ and $u'(0) = 0$, then

$$\mathrm{Rot}((u(t),u'(t));[0,T]) < 0. \tag{5.11}$$

Claim 2. *Let $g(t,x)$ satisfy assumption (f^1_∞). Then there exist an integer $k^* \in \mathbb{N}_*$ and $\widetilde{R} > 0$ such that, for every integer $k \geq k^*$ and for every $u : \mathbb{R} \to \mathbb{R}$ solving (5.9) with $u(0)^2 + u'(0)^2 = \widetilde{R}^2$, it holds that*

$$\mathrm{Rot}((u(t),u'(t));[0,kT]) > 1; \tag{5.12}$$

moreover, there exists $R > 0$ such that, for every $u : \mathbb{R} \to \mathbb{R}$ solving (5.9) with $u(0)^2 + u'(0)^2 = R^2$, it holds that

$$\mathrm{Rot}((u(t),u'(t));[0,T]) > 0. \tag{5.13}$$

Claim 3. *Let $g(t,x)$ satisfy assumption (f^2_∞). Then there exist an integer $k^* \in \mathbb{N}_*$ such that, for every integer $k \geq k^*$, there exists $\widetilde{R}_k > 0$ such that for every $u : \mathbb{R} \to \mathbb{R}$ solving (5.9) with $u(0)^2 + u'(0)^2 = \widetilde{R}_k^2$, it holds that*

$$\mathrm{Rot}((u(t),u'(t));[0,kT]) > n_k \geq 1; \tag{5.14}$$

moreover, there exists $\hat{R} > 0$ *such that, for every* $u : \mathbb{R} \to \mathbb{R}$ *solving* (5.9) *with* $u(0)^2 + u'(0)^2 = \hat{R}^2$*, it holds that*

$$\mathrm{Rot}((u(t), u'(t)); [0, T]) > 0. \tag{5.15}$$

We are now ready to prove Theorem 5.1.1 and Theorem 5.1.2.

We first deal with conclusion i), both in case of Theorem 5.1.1 and Theorem 5.1.2, proving the existence of a positive T-periodic solution $u_p(t)$; the existence of a negative T-periodic solution $u_n(t)$ follows using a similar argument. Define the function

$$\widetilde{f}(t, x) = \mathrm{sgn}\,(x) f(t, |x|) = \begin{cases} f(t, x) & \text{if } x \geq 0 \\ -f(t, -x) & \text{if } x < 0 \end{cases}$$

and consider the auxiliary equation

$$u'' + \widetilde{f}(t, u) = 0. \tag{5.16}$$

Notice that $\widetilde{f}(t, x)$ satisfies assumption (f_0) - with the same $q_0(t)$ - as well. In view of (5.11) and (5.13) for Theorem 5.1.1, and of (5.11) and (5.15) for Theorem 5.1.2, the existence of a T-periodic solution, say $u_p(t)$, to (5.16) satisfying

$$\mathrm{Rot}((u_p(t), u_p'(t)); [0, T]) = 0 \tag{5.17}$$

follows from Theorem 1.2.2, for $k = 1$ and $j = 0$. Moreover, (5.17) implies that $u_p(t)$ has constant sign and, since $\widetilde{f}(t, x)$ is odd in the x-variable, we can assume that it is positive. Hence $u_p(t)$ solves (5.4), thus concluding the proof.

We now pass to conclusion ii). In the case of Theorem 5.1.1, the thesis follows directly by applying Theorem 1.2.3, with $k \geq k^*$ and $j = 1$, to equation (5.4), in view of (5.10) and (5.12). In the case of Theorem 5.1.2, the thesis follows again from Theorem 1.2.3, with $k \geq k^*$ and $1 \leq j \leq n_k$, to equation (5.4), in view of (5.10) and (5.14). □

We end the section with the proofs of our technical claims.

Proof of Claim 1. The proof of (5.10) is quite classical (following from the fact that $q_0(t) \leq 0$) and it will be omitted; see also Proposition 2.2.2. Concerning (5.11) (and focusing, for instance, on the case $u(0) = 1$), we preliminarily observe that $u(t)$ has to be convex on $[0, T]$. Indeed, setting $J = \{s \in [0, T] \mid u''(t) \geq 0, t \in [0, s]\}$, it is clear that J is a nonempty (since $0 \in J$) closed interval. Suppose by contradiction that $t^* = \sup J < T$; then $u(t^*) \geq 1$, so that $u(t) > 0$ in a right neighborhood of t^*. Accordingly, $u''(t) = -q(t)u(t) \geq 0$ in a right neighborhood of t^*, a contradiction. As a consequence, $u(t) \geq 1$ and $u'(t) \geq 0$ for every $t \in [0, T]$, so that

$$u'(T) = -\int_0^T q_0(s) u(s)\, ds > 0,$$

from which the claim follows. □

Proof of Claim 2. For the proof of (5.12), we refer to [77, Lemma 4.3] (see also [60]). For the proof of (5.13), we first recall that from (5.6) we can deduce, using the definition of inferior limit and the monotone convergence theorem, the existence of a constant $M > 0$ and functions $h_+, h_- \in L^1(0,T)$, with

$$\int_0^T h_-(t)\,dt < 0 < \int_0^T h_+(t)\,dt, \tag{5.18}$$

such that

$$g(t,x)x \geq h_+(t)x^+ - h_-(t)x^-, \qquad \text{for every } |x| \geq M \tag{5.19}$$

(being, as usual, $x^+(t) = \max\{x(t),0\}$ and $x^-(t) = \max\{-x(t),0\}$). In view of (E_∞) of Lemma 2.3.1, it is enough to prove the existence of $R^* > 0$ such that (5.13) holds true whenever $u(t)^2 + u'(t)^2 \geq (R^*)^2$ for every $t \in [0,T]$. To see this, assume by contradiction that there exists a sequence of functions $u_n(t)$ solving (5.9), with $u_n(t)^2 + u_n'(t)^2 \geq n^2$ for every $t \in [0,T]$, such that, for n large enough,

$$2\pi \mathrm{Rot}((u_n(t),u_n'(t));[0,T]) = \int_0^T \frac{g(t,u_n(t))u_n(t) + u_n'(t)^2}{u_n(t)^2 + u_n'(t)^2}\,dt \leq 0. \tag{5.20}$$

Setting $x_n(t) = \frac{u_n(t)}{\|u_n\|_{C^1}}$, we have that $x_n(t)$ solves

$$x_n''(t) + \frac{g(t,u_n(t))}{\|u_n\|_{C^1}} = 0.$$

By standard arguments, in view of (5.5), it is seen that there exists a nonzero $x \in H^2(0,T)$ such that, up to subsequences, $x_n \to x$ strongly in $C^1([0,T])$, with $x(t)$ solving the equation $x''(t) = 0$. We deduce that it has to be $x(t) = at + b$, for suitable $a, b \in \mathbb{R}$ with $a^2 + b^2 > 0$. We now distinguish two cases. If $a \neq 0$, then we have that

$$\int_0^T \frac{u_n'(t)^2}{u_n(t)^2 + u_n'(t)^2}\,dt = \int_0^T \frac{x_n'(t)^2}{x_n(t)^2 + x_n'(t)^2}\,dt \to \int_0^T \frac{x'(t)^2}{x(t)^2 + x'(t)^2}\,dt > 0$$

and, using (5.5) again,

$$\int_0^T \frac{g(t,u_n(t))u_n(t)}{u_n(t)^2 + u_n'(t)^2}\,dt = \int_0^T \frac{g(t,u_n(t))x_n(t)}{\|u_n\|_{C^1}(x_n(t)^2 + x_n'(t)^2)}\,dt \to 0.$$

Recalling (5.20), we get a contradiction.
If $a = 0$ (and assuming for instance that $b > 0$), we have that $u_n(t) \to +\infty$ uniformly so that, using (5.19), we get, for n large,

$$\begin{aligned} 0 &\geq 2\pi \mathrm{Rot}((u_n(t),u_n'(t));[0,T]) \geq \int_0^T \frac{g(t,u_n(t))u_n(t)}{u_n(t)^2 + u_n'(t)^2}\,dt \\ &\geq \int_0^T \frac{h_+(t)u_n(t)}{u_n(t)^2 + u_n'(t)^2}\,dt. \end{aligned}$$

Multiplying by $\|u_n\|_{C^1}$ and passing to the limit, we obtain

$$0 \geq \liminf_{n\to+\infty} \int_0^T \frac{h_+(t)x_n(t)}{x_n(t)^2 + x_n'(t)^2}\,dt = \frac{1}{b}\int_0^T h_+(t)\,dt,$$

which is a contradiction in view of (5.18). □

Proof of Claim 3. It follows from Proposition 2.2.2, taking into account (E_∞) of Lemma 2.3.1 again. □

5.2 The planar Hamiltonian system

In this section, we give partial generalizations of the results of the previous Section 5.1 to a planar Hamiltonian system

$$Jz' = \nabla_z H(t,z), \tag{5.21}$$

being $H : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ a function continuous and T-periodic in the first variable, and such that $\nabla_z H(t,z)$ exists and is continuous on $\mathbb{R}\times\mathbb{R}^2$. Moreover, we assume the following condition:

(H_0) $\nabla_z H(t,0) \equiv 0$ *and there exist a continuous and* T*-periodic function* $B : \mathbb{R} \to \mathcal{L}_s(\mathbb{R}^2)$ *(we denote here by* $\mathcal{L}_s(\mathbb{R}^2)$ *the vector space of real* 2×2 *symmetric matrices) and* $z^* \in \mathbb{R}^2$*, with* $|z^*| = 1$*, such that*

$$\langle B(t)z^*|z^*\rangle < 0, \quad \textit{for every } t\in[0,T], \tag{5.22}$$

and

$$\lim_{z\to 0} \frac{\nabla_z H(t,z) - B(t)z}{|z|} = 0, \qquad \textit{uniformly in } t\in[0,T]. \tag{5.23}$$

Some remarks about this hypothesis are in order. First of all, condition (5.23) just means that the Hamiltonian system (5.21) can be linearized at zero (that is, condition (C_2) of Section 1.2). On the other hand, (5.22) specifies the nature of $z = 0$ as an equilibrium point of (5.21). In particular, notice that when $B(t) \equiv B \in \mathcal{L}_s(\mathbb{R}^2)$, then (5.22) is satisfied if and only if at least one of the (real) eigenvalues of the symmetric matrix B is negative. In this case, one can have two possibilities:

- both the eigenvalues are nonpositive, so that the solutions to the linear autonomous Hamiltonian system $Jz' = Bz$ are equilibrium points or rotate around the origin counterclockwise;
- one eigenvalue is negative and the other one is positive, so that the origin is a saddle equilibrium point for the system $Jz' = Bz$.

From our point of view, the common feature of this two different dynamical scenarios is that not all the solutions to $Jz' = Bz$ move in the clockwise sense; anyway, the second possibility seems to be the most interesting so that, with slight abuse in terminology, we will say that systems satisfying (H_0) have an equilibrium point of *saddle type*.

Notice that, when (5.21) comes from the second order equation (5.4), then condition (H_0) is a strengthening of condition $(f_0)$of Section 5.1: indeed, it is satisfied for $B(t) = \begin{pmatrix} q_0(t) & 0 \\ 0 & 1 \end{pmatrix}$ and $z^* = (1,0)$, provided that $q_0(t) < 0$ for every $t\in[0,T]$.

Remark 5.2.1. It can worth recalling that to every linear Hamiltonian system $Jz' = B(t)z$ (even in dimension greater than two) can be assigned an integer number $i(-B)$, the Conley-Zehnder index [50] (see Remark 1.1.2; we write here $i(-B)$ to be consistent with the definition of the Conley-Zehnder index as given in [2], where the Hamiltonian system is written with the symplectic matrix on the right-hand side). In the particular case of planar Hamiltonian systems and in the T-nonresonant case (i.e., when the only T-periodic solution to $Jz' = B(t)z$ is the trivial one) a detailed analysis of the relationship between the Conley-Zehnder index and the rotation numbers of the solutions in the plane is available [118]. In particular, according to [118, Lemma 4], we can deduce that (5.22)-(5.23) imply that $i(-B) \leq 0$ (indeed, from $i(-B) \geq 1$ follows that every solution to $Jz' = B(t)z$ has a negative rotation number, and we will show - see Claim 1 at the end of the section - that this is not the case).

We now state our first main result. As in Section 2.1, let us denote by $\mathcal{P}$ the class of the C^1-functions $V : \mathbb{R}^2 \to \mathbb{R}$ which are positively homogeneous of degree 2 and strictly positive, i.e., for every $\lambda > 0$ and for every $z \in \mathbb{R}^2_*$,

$$0 < V(\lambda z) = \lambda^2 V(z);$$

moreover, recall that the origin is a global isochronous center for the autonomous Hamiltonian system

$$Jz' = \nabla V(z), \qquad V \in \mathcal{P}, \tag{5.24}$$

that is to say, all the nontrivial solutions to (5.24) are periodic - around the origin - with the same minimal period τ_V (and rotate in the clockwise sense, see Section 2.1 again for more details). Theorem 5.2.1 below combines, for the nonlinear planar Hamiltonian systems (5.21), an equilibrium point of saddle type with an asymptotic dynamics like that of (5.24) and it can be seen as the counterpart of Theorem 5.1.2 for a planar Hamiltonian system. Notice, however, that here, at point *i*), only one T-periodic solution is obtained.

Theorem 5.2.1. *Suppose that the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (5.21) *are guaranteed. Moreover, assume condition* (H_0) *and*

(H^2_∞) *there exist* $V_\infty \in \mathcal{P}$ *and* $\gamma_\infty \in L^1_T$, *with* $\int_0^T \gamma_\infty(t)\,dt = 1$, *such that*

$$\liminf_{|z|\to\infty} \frac{\langle \nabla_z H(t,z) | z\rangle}{2V(z)} \geq \gamma_\infty(t), \quad \text{uniformly in } t \in [0,T].$$

Then the following conclusions hold true:

i) there exists a T*-periodic solution* $z_0(t)$ *to* (5.21) *such that*

$$\mathrm{Rot}(z_0(t); [0,T]) = 0;$$

ii) there exists $k^* \in \mathbb{N}_*$ *such that, for every integer* $k \geq k^*$, *there exists an integer* $j_*(k)$ *such that, for every integer* j *relatively prime with* k *and such that* $1 \leq j \leq j_*(k)$, *system* (5.21) *has at least two subharmonic solutions* $z^{(1)}_{k,j}(t), z^{(2)}_{k,j}(t)$ *of order* k *(not belonging to the same periodicity class) with*

$$\mathrm{Rot}(z^{(1)}_{k,j}(t); [0,kT]) = \mathrm{Rot}(z^{(2)}_{k,j}(t); [0,kT]) = j.$$

Moreover, we have the estimate

$$j_*(k) \geq n_k = \mathcal{E}^- \left(\frac{kT}{\tau_{V_\infty}} \right). \tag{5.25}$$

We now state our second main result. To this aim, we denote by $\mathcal{P}^*$ the class of the C^1-functions $V : \mathbb{R}^2 \to \mathbb{R}$ which are positively homogeneous of degree 2 and nonnegative, i.e., for every $\lambda > 0$ and for every $z \in \mathbb{R}^2_*$,

$$0 \leq V(\lambda z) = \lambda^2 V(z).$$

Of course, $\mathcal{P} \subset \mathcal{P}^*$. Systems of the type

$$Jz' = \nabla V(z), \qquad V \in \mathcal{P}^* \setminus \mathcal{P}, \tag{5.26}$$

have been recently considered by Garrione [87] as a possible generalization of the scalar second order equation $u'' = 0$ which, indeed, turns out to be equivalent to system (5.26) with $V(x, y) = \frac{1}{2}y^2 \in \mathcal{P}^* \setminus \mathcal{P}$. In [87, Theorem 3.2], a Landesman-Lazer type condition is introduced to ensure the solvability of the T-periodic problem associated with a sublinear perturbation of (5.26). Here we show - in the spirit of Proposition 2.2.4 - that such a condition provides enough information about the rotation numbers of large solutions to nonlinear Hamiltonian systems which are asymptotic at infinity to systems of the type (5.26). Accordingly, combining again with assumption (H_0), we get the existence of a nontrivial T-periodic solution with zero rotation number.

Theorem 5.2.2. *Suppose that the uniqueness for the solutions to the Cauchy problems associated with* (5.21) *is guaranteed. Moreover, assume condition* (H_0) *and*

(H^1_∞) *there exists* $V \in \mathcal{P}^*$, *with* $\nabla V(z)$ *locally Lipschitz continuous, such that*

$$\lim_{|z| \to +\infty} \frac{\nabla_z H(t, z) - \nabla V(z)}{|z|} = 0, \qquad \text{uniformly in } t \in [0, T]. \tag{5.27}$$

Moreover, setting $R(t, z) = \nabla_z H(t, z) - \nabla V(z)$, *suppose that:*

- *for every* $t \in [0, T]$, *for every* $z \in \mathbb{R}^2$ *with* $|z| \leq 1$ *and for every* $\lambda > 1$,

$$\langle R(t, \lambda z) | z \rangle \geq \eta(t), \tag{5.28}$$

for a suitable $\eta \in L^1(0, T)$,

- *for every* $\xi \in \mathbb{R}^2 \setminus \{0\}$ *satisfying* $\nabla V(\xi) = 0$,

$$\int_0^T \liminf_{(\lambda, \eta) \to (+\infty, \xi)} \langle R(t, \lambda \eta) | \eta \rangle > 0. \tag{5.29}$$

Then there exists a T*-periodic solution* $z_0(t)$ *to* (5.21) *such that*

$$\mathrm{Rot}(z_0(t); [0, T]) = 0.$$

Theorem 5.2.2 is a partial generalization of Theorem 5.1.1. Indeed, (5.29) is a Landesman-Lazer type condition; in particular, according to [87, Remark 4.1], assumption (H^1_∞) implies assumption (f^1_∞). However, only one T-periodic solution is obtained and no conclusions about subharmonic solutions are provided, at all.

We now sketch the proofs of our results.

Proof of Theorem 5.2.1 and Theorem 5.2.2. Similarly as in the proofs of Theorems 5.1.1 and 5.1.2, the conclusions now follow from the following claims. The main difference is that here we always work directly on (5.21) (and, as a consequence, only one T-periodic solution is obtained).

Claim 1. *For every integer $k \geq 1$ and for every $z : \mathbb{R} \to \mathbb{R}$ solving $Jz' = B(t)z$, it holds that*

$$\mathrm{Rot}(z(t)); [0, kT]) < \frac{1}{2}; \tag{5.30}$$

moreover, if $z(0) = z^*$, then

$$\mathrm{Rot}(z(t); [0, T]) < 0. \tag{5.31}$$

To see this, we just write the solution $z(t)$ as $z(t) = \sqrt{2\rho(t)}e^{i\theta(t)}$, with $\rho(t), \theta(t)$ of class C^1 and $\rho(t) > 0$ (compare with (8) of the introductory section "Notation and Terminology"), and set $z^* = e^{i\theta^*}$, with $\theta^* \in [0, 2\pi[$. In view of (1.8), assumption (H_0) implies that $-\theta'(t) < 0$ whenever $z(t) = \lambda z^*$ for $\lambda \neq 0$, namely $\theta(t) \equiv \theta^* \bmod \pi$. As a consequence, $\theta(t_2) < \theta(t_1) + \pi$ for every $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, so that (5.30) follows. With the same arguments, we also deduce the validity of (5.31). □

Claim 2. *Let $H(t, z)$ satisfy assumption (H^1_∞). Then there exists $R > 0$ such that, for every $z : \mathbb{R} \to \mathbb{R}$ solving (5.21) with $|z(0)| = R$, it holds that*

$$\mathrm{Rot}(z(t); [0, T]) > 0. \tag{5.32}$$

Observe first that, in view of (E_∞) of Lemma 2.3.1 (the global continuability for the solutions to (5.21) follows from (5.27), since $\nabla V(z)$ grows at most linearly), it is enough to show that there exists $R^* > 0$ such that (5.32) holds true whenever $|z(t)| \geq R^*$ for every $t \in [0, T]$. To see this, assume by contradiction that there exists a sequence of functions $z_n(t)$ solving (5.21), with $|z_n(t)| \geq n$ for every $t \in [0, T]$, such that, for n large enough,

$$2\pi\mathrm{Rot}(z_n(t); [0, T]) = \int_0^T \frac{\langle \nabla V(z_n(t)) + R(t, z_n(t)) | z_n(t) \rangle}{|z_n(t)|^2}\, dt \leq 0. \tag{5.33}$$

For further convenience, recall also that, in view of Euler's formula, we have

$$\begin{aligned} \int_0^T \frac{\langle \nabla V(z_n(t)) + R(t, z_n(t)) | z_n(t) \rangle}{|z_n(t)|^2}\, dt \quad &= \quad \int_0^T \frac{2V(z_n(t))}{|z_n(t)|^2}\, dt \\ &\quad + \int_0^T \frac{\langle R(t, z_n(t)) | z_n(t) \rangle}{|z_n(t)|^2}\, dt. \end{aligned} \tag{5.34}$$

Setting $w_n(t) = \frac{z_n(t)}{\|z_n\|_\infty}$, the function $w_n(t)$ satisfies

$$Jw_n'(t) = \nabla V(w_n(t)) + \frac{R(t, z_n(t))}{\|z_n\|_\infty},$$

and, by standard arguments using (5.27), it is seen that there exists $0 \not\equiv w \in H^1(]0, T[; \mathbb{R}^2)$ such that $w_n \to w$ uniformly, with $w(t)$ solving $Jw'(t) = \nabla V(w(t))$. Notice that, since

$\nabla V(z)$ is locally Lipschitz continuous and $w(t) \not\equiv 0$, it has to be $w(t) \neq 0$ for every $t \in [0,T]$; moreover, $\nabla V(w(t)) = 0$ if and only if $V(w(t)) = 0$.
We now distinguish two cases. If $w(t)$ is nonconstant, then, using (5.27), we get

$$\int_0^T \frac{\langle R(t, z_n(t))|z_n(t)\rangle}{|z_n(t)|^2}\, dt = \int_0^T \frac{\langle R(t, z_n(t))|w_n(t)\rangle}{\|z_n\|_\infty |w_n(t)|^2}\, dt \to 0.$$

On the other hand, since $V(w(t)) \equiv V(w(0)) > 0$ (otherwise $w(0)$ would be an equilibrium point for $Jz' = \nabla V(z)$ and, by uniqueness, $w(t)$ should be constant), we have

$$\int_0^T \frac{2V(z_n(t))}{|z_n(t)|^2}\, dt = \int_0^T \frac{2V(w_n(t))}{|w_n(t)|^2}\, dt \to \int_0^T \frac{2V(w(t))}{|w(t)|^2}\, dt > 0.$$

In view of (5.33) and (5.34), we have a contradiction.
If $w(t)$ is constant, it has to be $w(t) \equiv \xi$, for a suitable $\xi \in \mathbb{R}^2 \setminus \{0\}$ such that $\nabla V(\xi) = 0$. In this case, we have, using (5.34) and the nonnegativeness of V,

$$0 \geq 2\pi \mathrm{Rot}(z_n(t); [0,T]) \geq \int_0^T \frac{\langle R(t, z_n(t))|z_n(t)\rangle}{|z_n(t)|^2}\, dt.$$

Multiplying by $\|z_n\|_\infty$, we get

$$0 \geq \int_0^T \frac{\langle R(t, \|z_n\|_\infty w_n(t))|w_n(t)\rangle}{|w_n(t)|^2}\, dt,$$

so that, since $w_n(t) \to \xi$ uniformly, using (5.28) and Fatou's lemma we obtain

$$0 \geq \int_0^T \liminf_{n\to+\infty} \langle R(t, \|z_n\|_\infty w_n(t))|w_n(t)\rangle\, dt,$$

contradicting (5.29). □

<u>Claim 3.</u> *Let $H(t,z)$ satisfy assumption $(H^2_{\widetilde{\infty}})$. Then there exist an integer $k^* \in \mathbb{N}_*$ such that, for every integer $k \geq k^*$, there exists $\widetilde{R}_k > 0$ such that, for every $z : \mathbb{R} \to \mathbb{R}$ solving (5.21) with $|z(0)| = \widetilde{R}_k$, it holds that*

$$\mathrm{Rot}(z(t); [0, kT]) > n_k \geq 1;$$

moreover, there exists $\hat{R} > 0$ such that, for every $z : \mathbb{R} \to \mathbb{R}$ solving (5.21) with $|z(0)| = \hat{R}$, it holds that

$$\mathrm{Rot}(z(t); [0,T]) > 0.$$

This just follows from Proposition 2.2.1, taking into account (E_∞) of Lemma 2.3.1. □

Chapter 6

Positive solutions to parameter dependent second order ODEs with indefinite weight: multiplicity and complex dynamics

In this chapter, based on [25, 31, 32], we are concerned with positive periodic solutions to second order scalar differential equations of the type

$$u'' + q(t)g(u) = 0, \tag{6.1}$$

where $q : \mathbb{R} \to \mathbb{R}$ is a bounded and T-periodic function which changes its sign and $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function such that $g(0) = 0$ and $g(x)x > 0$ for $x \neq 0$.

Boundary value problems associated with equations like (6.1) are called - referring to the fact that $q(t)$ assumes both positive and negative values and therefore does not have a definite sign - "indefinite". They have been widely investigated in the past fifty years, starting with the works by Kiguradze and Waltman [102, 162], devoted to the study of the oscillatory behavior of the solutions to the superlinear equation $u'' + q(t)u^{2n-1} = 0$ (see also [165] for extensions to more general nonlinearities and a rather exhaustive list of references on the subject). In particular, the periodic problem for (6.1) was considered by Butler [38, 39] for $g(x)$ having superlinear growth at infinity or sublinear growth at zero, respectively. Further results in this direction have been obtained in [137, 158]; in both papers solutions with a large number of zeros in the intervals of positivity of the weight were produced. This, in turn, led to the study of chaotic dynamics [42, 158] for (6.1) as well as for

$$u'' + cu' + q(t)g(u) = 0;$$

precisely, the authors obtained solutions to (6.1) globally defined on $\mathbb{R}$ and, again, with a complex oscillatory behavior expressed in terms of their number of zeros.

On the other hand, starting with the nineties, various authors have investigated the existence of positive solutions to boundary value problems (typically, the Dirichlet or the Neumann one) associated with the nonlinear elliptic partial differential equation

$$\Delta u - ku + q(x)g(u) = 0, \qquad x \in \Omega \subset \mathbb{R}^N, \tag{6.2}$$

with a sign changing $q(x)$ (see, among others, [4, 8, 14, 15, 20]; a large list of references on the subject is contained in [140]); in particular, multiplicity results for positive solutions to equation (6.2), with Dirichlet boundary conditions on a bounded domain Ω, have been obtained in [24, 47, 89, 90, 91, 93, 114] in the superlinear case. When $k = 0$, however, the Neumann problem for (6.2) as well as the periodic problem for (6.1) exhibit some peculiar features which prevent, for certain classes of nonlinearities $g(x)$, the existence of positive solutions. In particular, the following two facts are worth to be emphasized here:

- if $g'(0) = 0$, then the linearization at zero of (6.1) is resonant with respect to the principal eigenvalue $k_0 = 0$ of both the Neumann and the periodic problem, and no positive solutions in general exist (see [31, Proposition 2.2]);
- if $g(x)$ is of class C^1 on $\mathbb{R}^+_*$ and $g'(x) > 0$ for $x > 0$, then positive solutions (for both the Neumann and the periodic problem) can exist only if

$$\int_0^T q(t)\, dt < 0. \tag{6.3}$$

This is easily seen (see [15] and [31, Proposition 2.1]), since - using the fact that $g(u(t)) > 0$ and integrating by parts, in view of the boundary conditions -

$$-\int_0^t q(t)\, dt = \int_0^T \frac{u''(t)}{g(u(t))}\, dt = \int_0^T g'(u(t)) \left(\frac{u'(t)}{g(u(t))}\right)^2 dt > 0.$$

The aim of this chapter is to show that, due to the complex dynamics of (6.1), multiple positive periodic solutions can appear in some cases. In our analysis, which is performed via phase plane techniques, the weight function $q(t)$ plays a special role. In particular, according to the previously mentioned nonexistence results, we assume that $q(t)$ takes the form

$$q(t) = a_{\lambda,\mu}(t) = \lambda a^+(t) - \mu a^-(t), \tag{6.4}$$

being $\lambda, \mu > 0$ real parameters and $a^+(t), a^-(t)$ the positive and the negative part of a (sign-indefinite) weight function $a(t)$, and we obtain multiplicity results for λ, μ in suitable ranges of values.

More precisely, the plan of the chapter is the following. In Section 6.1, we analyze a supersublinear problem (namely, when $g(x)$ satifies (6.7); compare with the introduction of Chapter 4), obtaining the existence of two positive T-periodic solutions, positive subharmonics of any order as well as a complex dynamics of coin-tossing type entirely generated by positive solutions to (6.1), when - in (6.4) - λ, μ are large. Our main tool here is a recent result (coming from the theory of topological horseshoes) about fixed points, periodic points and chaotic dynamics for a planar map, see Section 1.3. In Section 6.1, we study a superlinear problem (namely, for $g(x)$ fulfilling (6.32)), obtaining the existence of multiple positive T-periodic solutions when - in (6.4) - μ is large ($\lambda > 0$ is fixed) and $a_{\lambda,\mu}(t)$ satisfies a symmetry condition. Solutions are found here by solving, via an elementary shooting technique in the phase-plane, an auxiliary Neumann problem. More detailed descriptions and motivations for both problems can be found in the corresponding sections.

We emphasize that the study of (6.1) with respect to the existence and multiplicity of positive periodic solutions appears rather new from the point of view of the existing literature and various interesting issues seem to be worth to be investigated.

6.1 Supersublinear problem: chaotic dynamics via topological horseshoes

Our research, in this section, is partially inspired by a classical result by Rabinowitz [147], dealing with multiple positive solutions to the Dirichlet problem associated with

$$\Delta u + \lambda f(x,u) = 0, \qquad x \in \Omega \subset \mathbb{R}^N. \tag{6.5}$$

The basic hypotheses on (6.5) require $f(x,z)/z \to 0$ for $z \to 0^+$ uniformly in $x \in \overline{\Omega}$ and, moreover, assumptions like $f(x,z) \geq 0$ and $\not\equiv 0$ on $\overline{\Omega} \times [0,\bar{z}]$, as well as $f(x,z) \leq 0$ on $\overline{\Omega} \times [\bar{z},\infty)$. Pairs of positive solutions appear for $\lambda > \hat{\lambda} > 0$ (see [147, Corollary 1.34] and also [5, 6, 56, 64, 110] for related results). The hypothesis $f(x,z) \leq 0$ on large values of z can be replaced by other conditions ensuring the existence of a priori bounds for the solutions [147, Remark 1.14], implying (roughly speaking) that $f(x,z)$ has sublinear growth at infinity in the z variable.

Such considerations lead us to the search for positive periodic solutions to

$$u'' + a_{\lambda,\mu}(t)g(u) = 0, \tag{6.6}$$

being:

- $a_{\lambda,\mu}(t) = \lambda a^+(t) - \mu a^-(t)$, with $a^+(t), a^-(t)$ the positive and the negative part of a (sign-indefinite) continuous and T-periodic function $a : \mathbb{R} \to \mathbb{R}$,
- $g : \mathbb{R}^+ \to \mathbb{R}$ a locally Lipschitz continuous function, with $g(0) = 0$ and $g(x) > 0$ for $x > 0$, satisfying
$$\lim_{x \to 0^+} \frac{g(x)}{x} = \lim_{x \to +\infty} \frac{g(x)}{x} = 0. \tag{6.7}$$

The results in [147] for (6.5), jointly with the necessary condition (6.3) when $g'(x) > 0$, suggest that multiple positive periodic solutions can appear for λ, μ both large enough.

To perform our analysis we adopt a dynamical systems approach by looking for fixed points and periodic points of the Poincaré map $\Psi_{\lambda,\mu}$ associated with the planar Hamiltonian system equivalent to (6.6). The different qualitative behavior of the solutions to (6.6) when $a_{\lambda,\mu}(t) \geq 0$ (center type dynamics) and when $a_{\lambda,\mu}(t) \leq 0$ (saddle type dynamics) produces a rich dynamics for $\Psi_{\lambda,\mu}$, so that we can apply Theorem 1.3.2, which ensures not only the existence of at least two fixed points for $\Psi_{\lambda,\mu}$ in the right half-plane (which correspond to initial points of two positive T-periodic solutions) but also the presence of a full symbolic dynamics on two symbols.

The plan of the section is the following. In Subsection 6.1.1 we state our main result (Theorem 6.1.1), while Subsection 6.1.2 is devoted to the proof. Finally, in Subsection 6.1.3 we give a further result for (6.6), obtained via a variational approach.

6.1.1 Statement of the main result

Let $a(t)$, $g(x)$ be as before; moreover, for technical convenience, from now on we will identify $g(x)$ with its null extension[1]

$$g^0(x) = g(x^+) = \begin{cases} g(x) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{6.9}$$

As usual, we write (6.6) as the equivalent planar Hamiltonian system

$$\begin{cases} x' = y \\ y' = -a_{\lambda,\mu}(t)g(x). \end{cases} \tag{6.10}$$

By the assumptions on $g(x)$ and $a(t)$ the uniqueness and the global continuability of the solutions to the Cauchy problems associated with (6.10) are guaranteed. Thus we can define the Poincaré map

$$\Psi_{\lambda,\mu} : \mathbb{R}^2 \ni z \mapsto \zeta_{\lambda,\mu}(T; 0, z),$$

where, for every $s \in [0,T]$ and $z \in \mathbb{R}^2$, $\zeta_{\lambda,\mu}(\cdot; s, z)$ denotes the solution to (6.10) satisfying the initial condition $\zeta_{\lambda,\mu}(s; s, z) = z$. By the fundamental theory of ODEs, $\Psi_{\lambda,\mu}$ turns out to be a global homeomorphism of the plane onto itself and a two-sided trajectory $(z_i)_i$ of $\Psi_{\lambda,\mu}$ (that is, $z_{i+1} = \Psi_{\lambda,\mu}(z_i)$) corresponds to a globally defined solution $\zeta(t)$ to (6.10) such that $\zeta(iT) = z_i$ for every $i \in \mathbb{Z}$. Moreover, by the T-periodicity of $a(t)$, a k-periodic trajectory (with $k \in \mathbb{N}_*$) corresponds to a kT-periodic solution.

Here is the statement of our main result.

Theorem 6.1.1. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a locally Lipschitz continuous function such that $g(0) = 0$ and $g(x) > 0$ for $x > 0$. Suppose that:*

(g_0) $$\lim_{x \to 0^+} \frac{g(x)}{x} = 0;$$

[1] Dealing with such an extension, it is useful to prove the following simple result.

Lemma 6.1.1. *Let $u(t)$ be a solution to*

$$u'' + a_{\lambda,\mu}(t)g^0(u) = 0, \tag{6.8}$$

such that $u(t_j) \geq 0$, for a two-sided sequence $(t_j)_{j\in\mathbb{Z}}$ with $t_j \to \pm\infty$ for $j \to \pm\infty$. Then $u(t) \geq 0$ for every $t \in \mathbb{R}$. In particular, every kT-periodic solution to (6.8) different from a nonpositive constant is a nontrivial nonnegative solution to (6.6).

Proof. Let us suppose by contradiction that $u(\tilde{t}) < 0$ for some $\tilde{t} \in \mathbb{R}$ and, just to fix the ideas, let us assume that $u'(\tilde{t}) \leq 0$ (if $u'(\tilde{t}) \geq 0$ the argument is completely symmetric). Define $t^* = \sup\{t \geq \tilde{t} \mid u(s) < 0 \text{ for every } s \in [\tilde{t}, t[\}$; by continuity of $u(t)$, $\tilde{t} < t^*$. Moreover, $u''(t) = 0$ for a.e. $t \in]\tilde{t}, t^*[$, which implies that

$$u(t) = u'(\tilde{t})(t - \tilde{t}) + u(\tilde{t}), \quad \text{for every } t \in [\tilde{t}, t^*[.$$

By the assumptions, $t^* < +\infty$ since $t^* \leq t_j$ for some j. Then $u(t^*) < 0$; by continuity $u(t) < 0$ in a right neighborhood of t^*, a contradiction. □

$$(g_\infty) \qquad \lim_{x\to+\infty} \frac{g(x)}{x} = 0 \quad \textit{and} \quad \liminf_{x\to+\infty} g(x) > 0.$$

Moreover, assume that, for some $\tau \in]0,T[$,

$$(a_*) \qquad \begin{array}{lll} a(t) \geq 0, & \textit{for every } t \in [0,\tau], & \textit{with } \int_0^\tau a(t)\,dt > 0, \\ a(t) \leq 0, & \textit{for every } t \in [\tau,T], & \textit{with } \int_\tau^T a(t)\,dt < 0. \end{array}$$

Then there exists $\lambda^* > 0$ *such that, for every* $\lambda > \lambda^*$, *there exist two disjoint compact subsets* $\mathcal{K}_0, \mathcal{K}_1$ *of the first quadrant (with* $0 \notin \mathcal{K}_0 \cup \mathcal{K}_1$*) and there exists* $\mu^*(\lambda) > 0$ *such that, for every* $\mu > \mu^*(\lambda)$, *the conclusion of Theorem 1.3.1 holds true for the Poincaré map* $\Psi_{\lambda,\mu}$ *associated with (6.10). In particular,* $\Psi_{\lambda,\mu}$ *induces chaotic dynamics on two symbols relatively to* $\mathcal{K}_0$ *and* $\mathcal{K}_1$.

We now give an interpretation of the result in terms of the dynamical properties of the solutions. On one hand, for each two-sided sequence $(s_i)_i$ with $s_i \in \{0,1\}$, there exists a solution $\zeta(t) = (u(t), u'(t))$ to (6.10) (so that $u(t)$ solves (6.6)) such that

$$\zeta(iT) \in \mathcal{K}_{s_i}, \quad \text{for every } i \in \mathbb{Z}. \tag{6.11}$$

Notice that, in view of Lemma 6.1.1 and the uniqueness for the solutions to the Cauchy problems, such solutions satisfy $u(t) > 0$ for all $t \in \mathbb{R}$ and hence correspond to *positive solutions* to equation (6.6).
On the other hand, given a k-periodic sequence $(s_i)_i$ with $s_i \in \{0,1\}$, there exists a kT-periodic solution $\zeta(t)$ to (6.10) satisfying (6.11) and, again, corresponding to a positive solution to (6.6). In particular, if $k \geq 2$ is the minimal period of $(s_i)_i$, then $\zeta(t)$ is not lT-periodic for any $l = 1,\ldots,k-1$, that is $\zeta(t)$ is a subharmonic solution of order k to system (6.10).
Hence, Theorem 6.1.1 provides the existence of at least *two positive T-periodic solutions* to equation (6.6), as well as the existence of *positive subharmonics of order k for every $k \geq 2$*.
By the sign assumptions on the coefficient $a(t)$, we also know that such solutions $u(t)$ are concave on $[iT, \tau+iT]$, convex on $[\tau+iT,(i+1)T]$; furthermore, $u'(t)$ vanishes at least once in $]iT,\tau+iT[$ and at least once in $]\tau+iT,(i+1)T[$.

For the proof of Theorem 6.1.1, we will apply Theorem 1.3.2 by splitting $\Psi_{\lambda,\mu}$ as

$$\Psi_{\lambda,\mu} = \Psi_\mu \circ \Psi_\lambda\,, \tag{6.12}$$

where Ψ_λ and Ψ_μ are the homeomorphisms defined by

$$\Psi_\lambda : \mathbb{R}^2 \ni z \mapsto \zeta_{\lambda,\mu}(\tau;0,z), \qquad \Psi_\mu : \mathbb{R}^2 \ni z \mapsto \zeta_{\lambda,\mu}(T;\tau,z), \tag{6.13}$$

with $\tau \in]0,T[$ as above. Note that, from the assumptions on the weight $a(t)$, it follows that $a(t) = a^+(t)$ on $[0,\tau]$ and $a(t) = -a^-(t)$ on $[\tau,T]$. Hence (consistently with the notation used) Ψ_λ does not depend on μ and, similarly, Ψ_μ does not depend on λ.

Remark 6.1.1. Without loss of generality, and for convenience in view of the proof, we suppose that $\tau \in]0,T[$ is chosen so that

$$-\int_\tau^{\tau+\eta} a(t)\,dt = \int_\tau^{\tau+\eta} a^-(t)\,dt > 0$$

for every $\eta \in]0, T-\tau]$.

6.1.2 An auxiliary lemma and proof of the main result

Some more specific notation is useful in this section. First of all, for every $z \in \mathbb{R}^2_*$, we routinely pass to standard polar coordinates $z = \rho(\cos\theta, \sin\theta)$; we also often identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, thus writing $z = \rho e^{i\theta}$. Given $\theta \in]-2\pi, 2\pi[$, we denote by $\mathcal{L}(\theta)$ and $\mathcal{L}[\theta]$ respectively the open and closed half lines

$$\mathcal{L}(\theta) = \{z = \rho e^{i\theta} \in \mathbb{R}^2_* \mid \rho > 0\}, \qquad \mathcal{L}[\theta] = \mathcal{L}(\theta) \cup \{0\}.$$

For $\theta_1, \theta_2 \in [-2\pi, 2\pi]$ with $0 < \theta_2 - \theta_1 < 2\pi$, we denote by $\mathcal{S}(\theta_1, \theta_2)$ and $\mathcal{S}[\theta_1, \theta_2]$ respectively the open and closed angular sectors

$$\mathcal{S}(\theta_1, \theta_2) = \{z = \rho e^{i\theta} \in \mathbb{R}^2_* \mid \rho > 0,\ \theta_1 < \theta < \theta_2\},$$

$$\mathcal{S}[\theta_1, \theta_2] = \{z = \rho e^{i\theta} \in \mathbb{R}^2_* \mid \rho > 0,\ \theta_1 \le \theta \le \theta_2\} \cup \{0\}.$$

Moreover, given $0 \le r_1 \le r_2$, we denote by $\mathcal{A}[r_1, r_2]$ the closed annulus centered ad the origin and of radii r_1, r_2, that is

$$\mathcal{A}[r_1, r_2] = \{z \in \mathbb{R}^2 \mid r_1 \le |z| \le r_2\}.$$

To begin with, before passing to the proof of Theorem 6.1.1, we first develop some estimates for the solutions to parameter dependent supersublinear equations like

$$u'' + \lambda f(t, u) = 0. \tag{6.14}$$

These results will be applied to the study of equation (6.6) in the time interval $[0, \tau]$, namely for $f(t, x) = a^+(t)g(x)$; nevertheless, we give them in a slightly more general context than needed, since we think that they may have some independent interest. In particular, our analysis leads us to a two-solutions theorem (Proposition 6.1.1) for a special Sturm-Liouville problem associated with equation (6.14), which will play an important role in the proof of Theorem 6.1.1.

To be more precise, in the following $f : [0, \tau] \times \mathbb{R} \to \mathbb{R}$ will be an L^∞-Carathéodory function, locally Lipschitz continuous in the x-variable, that is:

- for every $x \in \mathbb{R}$, $f(\cdot, x) \in L^\infty([0, \tau])$;
- for every $r > 0$, there exists $\chi_r \in L^\infty([0, \tau], \mathbb{R}^+)$ such that, for a.e. $t \in [0, \tau]$ and for every $x, y \in [-r, r]$,
$$|f(t, x) - f(t, y)| \le \chi_r(t)|x - y|.$$

Moreover, we will suppose that:

(f_0) $\qquad f(t, 0) \equiv 0 \quad$ *and* $\quad \displaystyle\lim_{x \to 0} \frac{f(t, x)}{x} = 0,$ *uniformly for a.e.* $t \in [0, \tau]$;

(f_∞) $\qquad \displaystyle\lim_{|x| \to +\infty} \frac{f(t, x)}{x} = 0,$ *uniformly for a.e.* $t \in [0, \tau]$.

For every $s \in [0,\tau]$ and for every $z = (x,y) \in \mathbb{R}^2$, we denote by $u_\lambda(\cdot; s, z)$ the unique solution to the Cauchy problem

$$\begin{cases} u'' + \lambda f(t,u) = 0 \\ (u(s), u'(s)) = (x,y). \end{cases}$$

By the assumption (f_∞), $u_\lambda(\cdot; s, z)$ is globally defined on $[0,\tau]$. Moreover, we define the continuous map [96]

$$\varphi_\lambda : [0,\tau] \times [0,\tau] \times \mathbb{R}^2 \ni (t,s,z) \mapsto (u_\lambda(t;s,z), u'_\lambda(t;s,z)).$$

We rephrase here the "elastic property" (see Lemma 4.2.3) in a form which will be particularly suited for the rest of the proof.

Lemma 6.1.2. *Assume (f_0) and (f_∞). Then, the following hold true:*

(E_0) *for every $r > 0$, there exists $\eta(r) \in]0, r]$ such that, for every $s, t \in [0,\tau]$,*

$$|z| \leq \eta(r) \Longrightarrow |\varphi_\lambda(t,s,z)| \leq r;$$

(E_∞) *for every $R > 0$, there exists $\nu(R) \in [R, +\infty[$ such that, for every $s, t \in [0,\tau]$,*

$$|z| \geq \nu(R) \Longrightarrow |\varphi_\lambda(t,s,z)| \geq R.$$

For the following, it is useful to express the solutions to (6.14) in polar coordinates. To this aim, we preliminarily observe that the fact that $f(t,0) \equiv 0$ (together with the uniqueness for the Cauchy problems) implies that $\varphi_\lambda(t;s,z) \neq 0$ for every $s,t \in [0,\tau]$, provided that $z \neq 0$. Then, for every $z = re^{i\alpha}$, with $r = |z| > 0$ and $\alpha \in]-\pi,\pi[$, and for every $s \in [0,\tau]$, we denote by $\theta_\lambda(\cdot; s, \alpha, r)$ the unique continuous function with

$$\theta_\lambda(s; s, \alpha, r) = \alpha \in]-\pi, \pi[\tag{6.15}$$

and such that

$$\varphi_\lambda(t;s,z) = |\varphi_\lambda(t;s,z)| e^{i\theta_\lambda(t;s,\alpha,r)}, \quad \text{for every } t \in [0,\tau]. \tag{6.16}$$

We also set,

$$\rho_\lambda(t;s,\alpha,r) = |\varphi_\lambda(t;s,z)|.$$

By the continuity of $\varphi_\lambda(t,s,z)$ and standard results on the theory of paths lifting [94], we know that $\rho_\lambda(t;s,\alpha,r)$ and $\theta_\lambda(t;s,\alpha,r)$ depend continuously on (t,s,α,r), for $s,t \in [0,\tau]$, $r > 0$ and $\alpha \in]-\pi,\pi[$.

The next lemma gives a first result about the angular coordinate $\theta_\lambda(t;s,\alpha,r)$ for $\alpha = 0$ and for small or large values of r. The details of the proof are omitted, since they are very similar to the ones leading to Lemma 4.2.4 and Lemma 4.2.6 (see, however, [32, Lemma 3.2]).

Lemma 6.1.3. *Assume (f_0) and (f_∞). Then, for every $\epsilon > 0$ and for every $\lambda > 0$, there exist r_*, r^* with $0 < r_* < r^*$ such that, for every $r \in]0, r_*[\cup]r^*, +\infty[$ and for every $s \in [0,\tau]$, it holds*

$$\varphi_\lambda(t;s,(r,0)) \in \mathcal{S}(-\arctan\epsilon, \arctan\epsilon)$$

for every $t \in [0,\tau]$.

Combining Lemma 6.1.2 with Lemma 6.1.3, we get the following corollary.

Corollary 6.1.1. *Assume* (f_0), (f_∞). *Then for every* $\lambda > 0$ *and for every* $\alpha \in]-\pi, \pi[$ *with* $\alpha \neq 0$, *there exist* $\underline{r}, \overline{r}$ *with* $0 < \underline{r} < \overline{r}$, *such that for every* $r \in]0, \underline{r}[\cup]\overline{r}, +\infty[$ *and for every* $s \in [0, \tau]$, *it holds that*

$$\varphi_\lambda(t; s, re^{i\alpha}) \notin \mathcal{L}[0]$$

for every $t \in [0, \tau]$.

Proof. Choose $\epsilon > 0$ so small that $\arctan \epsilon < |\alpha|$ and let $r_* > 0$ be given by Lemma 6.1.3. By Lemma 6.1.2, there exists $\underline{r} > 0$ such that

$$r \in]0, \underline{r}] \quad \Longrightarrow \quad |\varphi_\lambda(t; s, re^{i\alpha})| < r_*, \qquad \text{for every } s, t \in [0, \tau].$$

Suppose by contradiction that, for some $s_1, t_1 \in [0, \tau]$, we have $\varphi_\lambda(t_1; s_1, re^{i\alpha}) \in \mathcal{L}[0]$, that is $\varphi_\lambda(t_1; s_1, re^{i\alpha}) = (\rho, 0)$ for some $\rho \geq 0$. The fact that $\varphi_\lambda(t; s, 0) = 0$ implies that $\rho \neq 0$; moreover, $\rho < r_*$. Hence, by Lemma 6.1.3,

$$re^{i\alpha} = \varphi_\lambda(s_1; t_1, \varphi_\lambda(t_1; s_1, re^{i\alpha})) \in \mathcal{S}(-\arctan \epsilon, \arctan \epsilon),$$

a contradiction. The existence of $\overline{r} > 0$ can be proved in an analogous way, using r^*. □

Finally, assuming more information about the sign of $f(t, x)$, we get a two-solutions theorem, which can be compared with the one in [147].

Proposition 6.1.1. *Assume* (f_0) *and* (f_∞). *Moreover, suppose that*

$$f(t, x) \geq 0 \quad \textit{for a.e. } t \in [0, \tau], \ \textit{ for every } x \in \mathbb{R}$$

and that, for some $m > 0$, $0 < d_0 < d_1$ *and* $0 \leq s_0 < s_1 \leq \tau$,

$$(f_+) \qquad f(t, x) \geq m, \quad \textit{for a.e. } t \in [s_0, s_1], \ \textit{ for every } x \in [d_0, d_1].$$

Then there exists $\lambda^* > 0$ *such that, for every* $\lambda > \lambda^*$, *there exists* $\epsilon^*(\lambda) > 0$ *such that, for all* $\theta_1, \theta_2 \in [\frac{\pi}{2}, \frac{\pi}{2} + \epsilon^*(\lambda)[$, *equation* (6.14) *has at least two solutions* $u_1(t), u_2(t)$ *with (for* $i = 1, 2$*)* $\max_{[0,\tau]} u_i(t) > 0$ *and satisfying the boundary conditions*

$$(u_i(0), u_i'(0)) \in \mathcal{L}(\theta_1) \quad \textit{and} \quad (u_i(\tau), u_i'(\tau)) \in \mathcal{L}(-\theta_2).$$

Proof. Let us define

$$s^* = \frac{s_0 + s_1}{2}, \qquad \delta = \frac{s_1 - s_0}{4}$$

and consider the solution $u_\lambda(\cdot) = u_\lambda(\cdot; s^*, (d_1, 0))$ to the Cauchy problem

$$\begin{cases} u'' + \lambda f(t, u) = 0 \\ (u(s^*), u'(s^*)) = (d_1, 0); \end{cases}$$

clearly, $u_\lambda(t)$ is defined on the whole interval $[0, \tau]$. Choose $\lambda^* > 0$ so large that

$$\delta \geq \frac{1}{\sqrt{\lambda^*}} \int_{d_0}^{d_1} \frac{d\xi}{\sqrt{2m(d_1 - \xi)}}, \qquad d_0 - \sqrt{2\lambda^* m(d_1 - d_0)}\, \delta \leq 0 \tag{6.17}$$

and fix $\lambda > \lambda^*$. We split our proof into some steps.

Step 1. First of all, we claim that $u_\lambda(0) < 0$, $u'_\lambda(0) > 0$ and $u_\lambda(\tau) < 0$, $u'_\lambda(\tau) < 0$. To see this, we recall first of all that, by $f \geq 0$ and (f_+),

$$u_\lambda(t) < u_\lambda(s^*) = d_1, \qquad \operatorname{sgn}(t - s^*)u'_\lambda(t) < 0$$

for every $t \in [0, \tau]$, with $t \neq s^*$. Let $[s', s''] \subset [0, \tau]$ be the maximal interval containing s^* such that $d_0 \leq u_\lambda(t)$ for every $t \in [s', s'']$. We observe that

$$s_0 + \delta < s' \quad \text{and} \quad s'' < s_1 - \delta.$$

In fact, if we assume by contradiction that $s' \leq s_0 + \delta$, so that $d_0 \leq u_\lambda(s_0 + \delta) < u_\lambda(t) < u_\lambda(s^*) = d_1$, for every $t \in]s_0 + \delta, s^*[$, we have that, for for a.e. $t \in [s_0 + \delta, s^*]$,

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}u'_\lambda(t)^2 + \lambda m u_\lambda(t)\right) &= u'_\lambda(t)(u''_\lambda(t) + m) \\ &= \lambda u'_\lambda(t)(-f(t, u_\lambda(t)) + m) \leq 0. \end{aligned}$$

Hence, the map $t \mapsto \frac{1}{2}u'_\lambda(t)^2 + \lambda m u_\lambda(t)$ is weakly decreasing on $[s_0 + \delta, s^*]$ and, therefore,

$$\frac{1}{2}u'_\lambda(t)^2 \geq \lambda m(d_1 - u_\lambda(t)), \qquad \text{for every } t \in [s_0 + \delta, s^*],$$

that is

$$\frac{u'_\lambda(t)}{\sqrt{2m\lambda(d_1 - u_\lambda(t))}} \geq 1.$$

An integration on $[s_0 + \delta, s^*]$ implies

$$\frac{1}{\sqrt{\lambda}} \int_{s_0+\delta}^{s^*} \frac{u'_\lambda(t)}{\sqrt{2m(d_1 - u_\lambda(t))}}\, dt \geq s^* - s_0 - \delta = \delta\,.$$

Using this last inequality and recalling the choice of $\lambda > \lambda^*$, with λ^* as in (6.17), we find

$$\begin{aligned} \frac{1}{\sqrt{\lambda^*}} \int_{d_0}^{d_1} \frac{d\xi}{\sqrt{2m(d_1 - \xi)}} &\leq \delta \leq \frac{1}{\sqrt{\lambda}} \int_{s_0+\delta}^{s^*} \frac{u'_\lambda(t)}{\sqrt{2m(d_1 - u_\lambda(t))}}\, dt \\ &\leq \frac{1}{\sqrt{\lambda}} \int_{u_\lambda(s_0+\delta)}^{d_1} \frac{d\xi}{\sqrt{2m(d_1 - \xi)}} \leq \frac{1}{\sqrt{\lambda}} \int_{d_0}^{d_1} \frac{d\xi}{\sqrt{2m(d_1 - \xi)}}\,, \end{aligned}$$

a contradiction. With a completely symmetric argument it is possible to show that $s'' < s_1 - \delta$. In this manner, we have found a maximal interval $[s', s''] \subset]s_0 + \delta, s_1 - \delta[$, with $s' < s^* < s''$, such that $u_\lambda(t) \in [d_0, d_1]$ for every $t \in [s', s'']$ and, moreover,

$$u_\lambda(s') = u_\lambda(s'') = d_0 \quad \text{and} \quad u'_\lambda(s') > 0 > u'_\lambda(s'').$$

By the assumption $f \geq 0$, we also know that $u_\lambda(t) < d_0\,, u'_\lambda(t) \geq u'_\lambda(s')$ for every $t \in [0, s'[$ and $u_\lambda(t) < d_0\,, u'_\lambda(t) \leq u'_\lambda(s'')$ for every $t \in]s'', \tau]$.
Repeating the same computations as above, for $t \in [s', s^*]$, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}u'_\lambda(t)^2 + \lambda m u_\lambda(t)\right) &= u'_\lambda(t)(u''_\lambda(t) + m) \\ &= \lambda u'_\lambda(t)(-f(t, u_\lambda(t)) + m) \leq 0, \end{aligned}$$

which implies that

$$u'_\lambda(s') \geq \sqrt{2\lambda m(d_1 - d_0)}.$$

Hence (from $f \geq 0$) we deduce that

$$u'_\lambda(t) \geq \sqrt{2\lambda m(d_1 - d_0)}\,, \qquad \text{for every } t \in [0, s'].$$

By another integration and using $f \geq 0$ again, we have

$$\begin{aligned} u_\lambda(0) &= u_\lambda(s') - \int_0^{s'} u'_\lambda(t)\,dt \leq d_0 - \sqrt{2\lambda m(d_1 - d_0)}s' \\ &< d_0 - \sqrt{2\lambda m(d_1 - d_0)}(s_0 + \delta) < d_0 - \sqrt{2\lambda^* m(d_1 - d_0)}\,\delta \leq 0. \end{aligned}$$

We have thus proved that $u_\lambda(0) < 0$ and $u'_\lambda(0) > 0$. A completely symmetric argument shows that $u_\lambda(\tau) < 0$ and $u'_\lambda(\tau) < 0$.

Step 2. From now on, $\lambda > \lambda^*$ is fixed and we can define

$$\epsilon^* = \epsilon^*(\lambda) = \min\{\arctan|u_\lambda(0)/u'_\lambda(0)|\,, \arctan|u_\lambda(\tau)/u'_\lambda(\tau)|\} > 0.$$

Then we choose θ_1, θ_2 such that

$$\theta_1\,, \theta_2 \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \epsilon^*(\lambda)\right[$$

and we claim that there are at least two solutions to (6.14) as required in the statement of the theorem. To this aim, we turn to consider the angular coordinate $\theta_\lambda(t; s, \alpha, r)$ as defined in (6.15), (6.16). Here, moreover, we also notice that the fact that $f \geq 0$ implies that

$$-\pi < \theta_\lambda(t; s, \alpha, r) < \pi, \qquad \text{for every } s, t \in [0, \tau].$$

The results of the first step of the proof, together with this remark, imply that

$$\theta_\lambda(0; s^*, 0, d_1) \geq \frac{\pi}{2} + \epsilon^*, \qquad \theta_\lambda(\tau; s^*, 0, d_1) \leq -\frac{\pi}{2} - \epsilon^*.$$

On the other hand, Lemma 6.1.3 implies that there exist $r_*, r^* > 0$ such that, for every $r \in]0, r_*[\cup]r^*, +\infty[$,

$$-\frac{\pi}{2} < \theta_\lambda(0; s^*, 0, r), \theta_\lambda(\tau; s^*, 0, r) < \frac{\pi}{2}.$$

Then, we have that

$$\begin{cases} \theta_\lambda(0; s^*, 0, d_1) > \theta_1, & \theta_\lambda(0; s^*, 0, r) < \theta_1, \\ \theta_\lambda(\tau; s^*, 0, d_1) < -\theta_2, & \theta_\lambda(\tau; s^*, 0, r) > -\theta_2, \end{cases} \qquad \text{for every } r \in]0, r_*[\cup]r^*, +\infty[\,.$$

We produce a first solution (which will be the "smaller" one). By applying the intermediate value theorem to the continuous function

$$[r_*/2, d_1] \ni r \mapsto \min\{\theta_\lambda(0; s^*, 0, r) - \theta_1, -\theta_\lambda(\tau; s^*, 0, r) - \theta_2\},$$

we find a maximal value $\tilde{r}$ with $\tilde{r} \in [r_*, d_1[$ such that

$$\theta_\lambda(0; s^*, 0, \tilde{r}) = \theta_1 \quad \text{or} \quad \theta_\lambda(\tau; s^*, 0, \tilde{r}) = -\theta_2.$$

If both the equations are satisfied, we are done. Just to fix the ideas, assume that we have $\theta_\lambda(0; s^*, 0, \tilde{r}) = \theta_1$, while $\theta_\lambda(\tau; s^*, 0, \tilde{r}) < -\theta_2$. Consider now the function $r \mapsto \theta_\lambda(\tau; 0, \theta_1, r)$. Denoting by

$$\hat{r} = \rho_\lambda(0; s^*, 0, \tilde{r}),$$

we know that

$$\theta_\lambda(\tau; 0, \theta_1, \hat{r}) < -\theta_2.$$

On the other hand, Corollary 6.1.1 and the fact that $\theta_\lambda(\tau; \tau, \theta_1, r) = \theta_1 > 0$ imply that there exists $\underline{r} > 0$ such that

$$\theta_\lambda(\tau; 0, \theta_1, r) > 0 > -\theta_2 \quad \text{for every } r \in]0, \underline{r}[\,.$$

Hence, applying the Bolzano theorem again we find $\check{r} \in [\underline{r}, \hat{r}[$ such that

$$\theta_\lambda(\tau; 0, \theta_1, \check{r}) = -\theta_2.$$

In conclusion, for $u_1(t) = u_\lambda(t; 0, \check{z})$, with $\check{z} = \check{r}e^{i\theta_1}$, we have that

$$(u_1(0), u_1'(0)) \in \mathcal{L}(\theta_1), \qquad (u_1(\tau), u_1'(\tau)) \in \mathcal{L}(-\theta_2).$$

From the proof it follows also that $u_1(t)$ is concave on $[0, \tau]$ and $\max_{[0,\tau]} u_1(t) \leq \tilde{r} < d_1$. The second solution (which will be the "larger" one) is obtained following a symmetric argument and the details are omitted. In particular, for such a solution we have $\max_{[0,\tau]} u_2(t) > d_1$ and hence $u_1(t) \not\equiv u_2(t)$. □

Now, we are ready to pass to the proof of Theorem 6.1.1.

Proof of Theorem 6.1.1. The proof will be divided into three main steps.

Step 1: Study of system (6.10) in the interval $[0, \tau]$

In this first step of the proof, we study the dynamical properties (in the phase plane) of the solutions to system (6.10) in the time interval $[0, \tau]$. The final goal will be the definition of an oriented rectangle $\widetilde{\mathcal{P}}$ (included in the first quadrant of the plane) and of two disjoint compact subsets $\mathcal{K}_0, \mathcal{K}_1 \subset \mathcal{P}$.

To this aim, we will extensively use the previous results of this subsection. Indeed, we preliminarily observe that, setting $f(t, x) = a^+(t)g(x)$ for $(t, x) \in [0, \tau] \times \mathbb{R}$, we have that $f \geq 0$ and the assumptions (f_0), (f_∞) are satisfied. Moreover, thanks to the continuity of $a^+(t)$ and $g(x)$, also (f_+) holds true. So, keeping the previous notation, we still denote, for every $s \in [0,\tau]$ and for every $z = (x,y) \in \mathbb{R}^2$, by $u_\lambda(\cdot;s,z)$ the solution to the Cauchy problem

$$\begin{cases} u'' + \lambda a^+(t) g(u) = 0 \\ (u(s), u'(s)) = (x,y), \end{cases}$$

which is unique and globally defined on $[0,\tau]$; moreover, we set

$$\zeta_\lambda(\cdot;s,z) = (u_\lambda(\cdot;s,z), u'_\lambda(\cdot;s,z)).$$

Clearly, we have

$$\Psi_\lambda(z) = \zeta_\lambda(\tau;0,z), \qquad \text{for every } z \in \mathbb{R}^2,$$

where Ψ_λ is defined in (6.13).

We are particularly interested in the manner in which Ψ_λ rotates points of the first quadrant. Then, we continue to denote, for every $r > 0$ and $\alpha \in \,]-\pi,\pi[\,$, by $\theta_\lambda(t;s,\alpha,r)$ the angular coordinate associated with $\zeta_\lambda(t;s,re^{i\alpha})$, as defined in (6.15), (6.16). We recall that such a function satisfies the property:

$$\theta_\lambda(t^*;s,\alpha,r) \le \frac{\pi}{2} \Longrightarrow \theta_\lambda(t;s,\alpha,r) \le \frac{\pi}{2}\,, \qquad \text{for every } t \ge t^*. \tag{6.18}$$

Moreover, the fact that $f(t,x) = 0$ for every $(t,x) \in \mathbb{R}\times\,]-\infty,0]$ implies that

$$-\pi < \theta_\lambda(t;s,\alpha,r) < \pi, \qquad \text{for every } s,t \in [0,\tau]. \tag{6.19}$$

From this remark, we in particular infer that, for every $z \notin \mathcal{L}[-\pi]$ ($z = re^{i\alpha}$ with $r > 0$ and $\alpha \in \,]-\pi,\pi[$) and $\alpha_1, \alpha_2 \in \,]-\pi,\pi[$ with $\alpha_1 < \alpha_2$,

$$\Psi_\lambda(z) \in \mathcal{S}[\alpha_1,\alpha_2] \Longleftrightarrow \alpha_1 \le \theta_\lambda(\tau;0,\alpha,r) \le \alpha_2. \tag{6.20}$$

The next two results (Lemma 6.1.4 and Lemma 6.1.5) provide a precise description of the rotation properties associated with Ψ_λ. In Lemma 6.1.4 we construct two arcs of the first quadrant $\mathcal{S}[0,\frac{\pi}{2}]$ which are moved by Ψ_λ to arcs of the third quadrant $\mathcal{S}[-\pi,-\frac{\pi}{2}]$.

Lemma 6.1.4. *There exists $\lambda^* > 0$ such that for every $\lambda > \lambda^*$ there exist two arcs $\beta_0, \beta_1 : [0,1] \to \mathbb{R}^2_*$ such that, for $i = 0,1$:*

i) $\beta_i(0) \in \mathcal{L}(0)$, $\beta_i(1) \in \mathcal{L}(\frac{\pi}{2})$;

ii) $\beta_i(\sigma) \in \mathcal{S}[0,\frac{\pi}{2}]$ *for every* $\sigma \in [0,1]$;

iii) $\beta_0([0,1]) \cap \beta_1([0,1]) = \emptyset$;

iv) $\Psi_\lambda(\beta_i(\sigma)) \in \mathcal{S}[-\pi,-\frac{\pi}{2}]$ *for every* $\sigma \in [0,1]$.

Proof. Fix $\lambda^* > 0$ as in Proposition 6.1.1 and take $\lambda > \lambda^*$. Next, select $\tilde{\theta}, \hat{\theta} \in \,]\frac{\pi}{2}, \frac{\pi}{2} + \epsilon^*(\lambda)[\,$, with $\tilde{\theta} \ne \hat{\theta}$, and consider the continuous function given by

$$\mathbb{R}^+_* \ni r \mapsto \theta_\lambda(0;\tau,-\tilde{\theta},r).$$

Corollary 6.1.1 and the fact that $\theta_\lambda(0;0,-\tilde{\theta},r)=-\tilde{\theta}<0$ imply that there exists $\underline{r}>0$ such that

$$\theta_\lambda(0;\tau,-\tilde{\theta},r)<0, \qquad \text{for every } r\in\,]0,\underline{r}[\,.$$

On the other hand, an application of Proposition 6.1.1 with $\theta_1=\frac{\pi}{2}$ and $\theta_2=\tilde{\theta}$ gives the existence of a solution $u(t)$ (actually, two solutions) to $u''+\lambda a^+(t)g(u)=0$, with

$$(u(0),u'(0))\in\mathcal{L}(\tfrac{\pi}{2}), \qquad (u(\tau),u'(\tau))\in\mathcal{L}(-\tilde{\theta}).$$

Hence, for some $\tilde{r}>0$, $\theta_\lambda(0;\tau,-\tilde{\theta},\tilde{r})=\frac{\pi}{2}+2k\pi$ for some $k\in\mathbb{Z}$. By the fact that $-\pi<\theta_\lambda(0;\tau,-\tilde{\theta},\tilde{r})<\pi$, we conclude that $k=0$, namely $\theta_\lambda(0;\tau,-\tilde{\theta},\tilde{r})=\pi/2$. By standard connectivity arguments we deduce the existence of an interval $[r_1,r_2]$ such that

$$\theta_\lambda(0;\tau,-\tilde{\theta},r_1)=0, \qquad \theta_\lambda(\tau;0,-\tilde{\theta},r_2)=\frac{\pi}{2}$$

and

$$0<\theta_\lambda(0;\tau,-\tilde{\theta},r)<\frac{\pi}{2}, \qquad \text{for every } r\in\,]r_1,r_2[\,.$$

Set, for $0\le\sigma\le1$,

$$\beta_0(\sigma)=\varphi_\lambda(0;\tau,(r_1+\sigma(r_2-r_1))e^{-i\tilde{\theta}}).$$

Then *i)* and *ii)* of the statement are clearly satisfied. Moreover, for every $\sigma\in[0,1]$,

$$\Psi_\lambda(\beta_0(\sigma))=\varphi_\lambda(\tau;0,\beta_0(\sigma))=(r_1+\sigma(r_2-r_1))e^{-i\tilde{\theta}},$$

which implies that also *iv)* holds true.
Repeating the previous arguments for $\hat{\theta}$, we get the existence of β_1.
Finally, *iii)* follows from the uniqueness for the solutions to the Cauchy problems. □

From now on (and without loss of generality) we label β_0 and β_1 in such a manner that $\beta_0(0)<\beta_1(0)$, so that β_1 is external with respect to β_0 (and viceversa).

Now we fix $\lambda>\lambda^*$ and $\epsilon>0$ small. The next lemma shows that small and large points of the first quadrant $\mathcal{S}[0,\frac{\pi}{2}]$, on the contrary, rotate little with Ψ_λ.

Lemma 6.1.5. *There exist $r<\min_{[0,1]}|\beta_0|$ and $R>\max_{[0,1]}|\beta_1|$ such that for every $z\in\mathcal{S}[0,\frac{\pi}{2}]$ with $|z|=r$ or $|z|=R$, it holds*

$$\Psi_\lambda(z)\in\mathcal{S}[-\arctan\epsilon,\tfrac{\pi}{2}].$$

Proof. Let $\epsilon>0$ be fixed as above and consider the corresponding $r_*>0$ given by Lemma 6.1.3. Lemma 6.1.2 implies that there exists $r>0$ such that

$$|z|\le r\Longrightarrow|\zeta_\lambda(t;0,z)|<r_*, \qquad \text{for every } t\in[0,\tau]. \tag{6.21}$$

Now, let $z\in\mathcal{S}[0,\frac{\pi}{2}]$ with $|z|=r$ (that is, $z=re^{i\alpha}$ with $0\le\alpha\le\frac{\pi}{2}$) and consider the angular coordinate $\theta_\lambda(t;0,\alpha,r)$. We claim that

$$-\arctan\epsilon\le\theta_\lambda(\tau;0,\alpha,r)\le\frac{\pi}{2},$$

which, in view of (6.20), implies the conclusion. Indeed, being $0 \leq \alpha \leq \frac{\pi}{2}$, from (6.18) we deduce that $\theta_\lambda(\tau; 0, \alpha, r) \leq \frac{\pi}{2}$.
Suppose by contradiction that $\theta_\lambda(\tau; 0, \alpha, r) < -\arctan\epsilon$. Then, it has to be $\theta_\lambda(\tilde{t}; 0, \alpha, r) = 0$ for some $\tilde{t} \in [0, \tau]$, namely $\zeta_\lambda(\tilde{t}; 0, z) = (\tilde{r}, 0)$ for some $\tilde{r} > 0$. Now, from (6.21) we have that $\tilde{r} < r_*$ and so, using Lemma 6.1.3, we get

$$\zeta_\lambda(\tau; 0, z) = \zeta_\lambda(\tau; \tilde{t}, \zeta_\lambda(\tilde{t}; 0, z))) \in \mathcal{S}(-\arctan\epsilon, \arctan\epsilon),$$

a contradiction. With a similar argument, we prove the existence of R. □

We are now ready to define the first oriented rectangle. Set $\widetilde{\mathcal{P}} = (\mathcal{P}, \mathcal{P}^-)$, where

$$\mathcal{P} = \mathcal{A}[r, R] \cap \mathcal{S}[0, \tfrac{\pi}{2}]$$

and $\mathcal{P}^- = \mathcal{P}_1^- \cup \mathcal{P}_2^-$ is defined by

$$\mathcal{P}_1^- = \mathcal{A}[r, r] \cap \mathcal{S}[0, \tfrac{\pi}{2}], \qquad \mathcal{P}_2^- = \mathcal{A}[R, R] \cap \mathcal{S}[0, \tfrac{\pi}{2}].$$

Moreover, we define $\mathcal{K}_0, \mathcal{K}_1 \subset \mathcal{P}$ to be the compact regions of $\mathcal{P}$ internal to $\bar{\beta}_0 = \beta_0([0,1])$ and external to $\bar{\beta}_1 = \beta_1([0,1])$ respectively, as in Figure 6.1. We end this first part of the proof by defining

$$\begin{aligned} r_* &= \inf\left\{ |\zeta_\lambda(t; 0, z)| \,\middle|\, t \in [0, \tau], z \in \mathcal{A}[r, R] \cap \mathcal{S}[0, \tfrac{\pi}{2}] \right\} > 0, \\ R^* &= \sup\left\{ |\zeta_\lambda(t; 0, z)| \,\middle|\, t \in [0, \tau], z \in \mathcal{A}[r, R] \cap \mathcal{S}[0, \tfrac{\pi}{2}] \right\} < +\infty. \end{aligned} \tag{6.22}$$

Step 2: Study of system (6.10) in the interval $[\tau, T]$

In this second step, we pass to study the dynamical properties of the solutions to system (6.10) in the time interval $[\tau, T]$. This will lead us to the definition of an oriented rectangle $\widetilde{\mathcal{O}}$, contained in the fourth quadrant $\mathcal{S}[-\frac{\pi}{2}, 0]$.

For every $s \in [\tau, T]$ and for every $z = (x, y) \in \mathbb{R}^2$, let us denote by $u_\mu(\cdot; s, z)$ the solution to the Cauchy problem

$$\begin{cases} u'' - \mu a^-(t) g(u) = 0 \\ (u(s), u'(s)) = (x, y), \end{cases}$$

which is unique and globally defined on $[\tau, T]$; moreover, we set

$$\zeta_\mu(\cdot; s, z) = (u_\mu(\cdot; s, z), u'_\mu(\cdot; s, z)).$$

Clearly, we have

$$\Psi_\mu(z) = \zeta_\mu(T; \tau, z), \qquad \text{for every } z \in \mathbb{R}^2,$$

where Ψ_μ is defined in (6.13). Finally, let $\lambda, \epsilon > 0$ be fixed as in the previous step and define, for simplicity of notation,

$$\Gamma = \mathcal{L}[-\arctan\epsilon] \cap \mathcal{A}[r_*, R^*], \tag{6.23}$$

being r_*, R^* defined in (6.22).

We will be mainly interested in the expansion properties of the map Ψ_μ. In particular, Lemma 6.1.6 below shows that points of Γ are mapped by Ψ_μ into points of the first quadrant of large norm.

Lemma 6.1.6. *There exists $\mu_1^*(\lambda) > 0$ such that for every $\mu > \mu_1^*(\lambda)$ and for every $z \in \Gamma$, it holds*

$$u_\mu(T;\tau,z) \geq R \quad \text{and} \quad u'_\mu(T;\tau,z) \geq R.$$

Proof. We preliminarily observe that, for every $z \in \Gamma$,

$$r_\epsilon = \frac{r_*}{\sqrt{1+\epsilon^2}} \leq u_\mu(\tau;\tau,z) \leq \frac{R^*}{\sqrt{1+\epsilon^2}} = R_\epsilon,$$

and

$$-\epsilon R_\epsilon \leq u'_\mu(\tau;\tau,z) = -\epsilon u_\mu(\tau;\tau,z) \leq -\epsilon r_\epsilon.$$

Define $r_0 = r_\epsilon/2$ and set

$$g_{r_0} = \min_{x \geq r_0} g(x) > 0.$$

Step 1. We claim that, if μ is sufficiently large, then $u_\mu(t;\tau,z) \geq r_0$ for every $t \in [\tau, T]$.
By contradiction, let $\tilde{t} \in [\tau, T]$ be the first instant such that $u_\mu(\tilde{t};\tau,z) = r_0$. Then, we have, for every $t \in [\tau, \tilde{t}]$,

$$\begin{aligned} u'_\mu(t;\tau,z) &= u'_\mu(\tau;\tau,z) + \mu \int_\tau^t a^-(s) g(u_\mu(s;\tau,z))\, ds \\ &\geq u'_\mu(\tau;\tau,z) \geq -\epsilon R_\epsilon. \end{aligned}$$

Integrating the above inequalities between τ and $\tilde{t}$, we get

$$r_0 - r_\epsilon \geq u_\mu(\tilde{t};\tau,z) - u_\mu(\tau;\tau,z) \geq -\epsilon R_\epsilon(\tilde{t} - \tau),$$

which implies that

$$\frac{r_\epsilon - r_0}{\epsilon R_\epsilon} \leq \tilde{t} - \tau. \tag{6.24}$$

On the other hand we have

$$\begin{aligned} 0 &\geq u'_\mu(\tilde{t};\tau,z) = u'_\mu(\tau;\tau,z) + \mu \int_\tau^{\tilde{t}} a^-(s) g(u_\mu(s;\tau,z))\, ds \\ &\geq -\epsilon R_\epsilon + \mu g_{r_0} \int_\tau^{\tilde{t}} a^-(s)\, ds, \end{aligned}$$

which implies that

$$\mu g_{r_0} \int_\tau^{\tilde{t}} a^-(s)\, ds \leq \epsilon R_\epsilon.$$

Combining the above inequality with (6.24), we get

$$\mu g_{r_0} \int_\tau^{\tau + \frac{r_\epsilon - r_0}{\epsilon R_\epsilon}} a^-(s)\, ds \leq \mu g_{r_0} \int_\tau^{\tilde{t}} a^-(s)\, ds \leq \epsilon R_\epsilon.$$

Since $\int_{\tau}^{\tilde{t}} a^{-}(s)\,ds > 0$ by Remark 6.1.1, we get a contradiction if μ is large enough.

Step 2. Being $u_\mu(t;\tau,z) \geq r_0$ for every $t \in [\tau,T]$, we have, for every $t \in [\tau,T]$,

$$u_\mu''(t;\tau,z) = \mu a^{-}(t) g(u_\mu(t;\tau,z)) \geq \mu g_{r_0} a^{-}(t);$$

setting $A^{-}(t) = \int_{\tau}^{t} a^{-}(s)\,ds$ and integrating we get, for every $t \in [\tau,T]$,

$$u_\mu'(t;\tau,z) \geq -\epsilon R_\epsilon + \mu g_{r_0} A^{-}(t). \tag{6.25}$$

In particular,

$$u_\mu'(T;\tau,z) \geq -\epsilon R_\epsilon + \mu g_{r_0} A^{-}(T) \geq R$$

if μ is sufficiently large. Finally, (6.25) implies that

$$\begin{aligned} u_\mu(T;\tau,z) &= u_\mu(\tau;\tau,z) + \int_{\tau}^{T} u_\mu'(s;\tau,z)\,ds \\ &\geq r_\epsilon - \epsilon R_\epsilon(T-\tau) + \mu g_{r_0} \int_{\tau}^{T} A^{-}(t)\,dt \geq R \end{aligned}$$

for μ large enough. □

In our last technical result we look for points of the fourth quadrant which are moved by Ψ_μ to $\mathcal{L}(0)$. Actually, we show that points of the compact region $\mathcal{A}[0,R^*] \cap \mathcal{S}[-\frac{\pi}{2},0]$ which are moved by Ψ_μ onto $\mathcal{L}(0)$ actually can reach $\mathcal{L}(0)$ only in a small neighborhood of the origin.

Lemma 6.1.7. *There exists $\mu_2^*(\lambda) > 0$ such that for every $\mu > \mu_2^*(\lambda)$ and for every $x \geq r$, it holds*

$$u_\mu(\tau;T,(x,0)) \geq R^* + 1 \quad \text{and} \quad u_\mu'(\tau;T,(x,0)) \leq -(R^*+1).$$

Proof. Defining as above $g_r = \min_{x \geq r} g(x)$, by convexity, we get

$$u_\mu(t;T,(x,0)) \geq x \geq r, \quad \text{for every } t \in [\tau,T].$$

Moreover, defined $B^{-}(t) = \int_{t}^{T} a^{-}(s)\,ds$, we have for every $t \in [\tau,T]$

$$\begin{aligned} u_\mu'(t;T,(x,0)) &= u_\mu'(T;T,(x,0)) - \mu \int_{t}^{T} a^{-}(s) g(u_\mu(s;T,(x,0)))\,ds \\ &\leq -\mu g_r B^{-}(t). \end{aligned}$$

In particular

$$u_\mu'(\tau;T,(x,0)) \leq -\mu g_r B^{-}(\tau) \leq -(R^*+1)$$

if μ is large enough (being $B^{-}(\tau) > 0$). Finally, integrating we get

$$\begin{aligned} u_\mu(\tau;T,(x,0)) &= u_\mu(T;T,(x,0)) - \int_{t}^{T} u_\mu'(s;T,(x,0))\,ds \\ &= x - \int_{t}^{T} u_\mu'(s;T,(x,0))\,ds \\ &\geq r + \mu g_r \int_{\tau}^{T} B^{-}(s)\,ds \geq R^* + 1 \end{aligned}$$

if μ is large enough. □

Set $\mu^*(\lambda) = \max(\mu_1^*(\lambda), \mu_2^*(\lambda))$ and fix $\mu > \mu^*(\lambda)$. We conclude this second step by defining a second oriented rectangle $\widetilde{\mathcal{O}} = (\mathcal{O}, \mathcal{O}^-)$. We set

$$\mathcal{O} = \mathcal{A}[r_*, R^*] \cap \mathcal{S}[-\tfrac{\pi}{2}, -\arctan\epsilon],$$

with $\mathcal{O}^- = \mathcal{O}_1^- \cup \mathcal{O}_2^-$ given by

$$\mathcal{O}_1^- = \Gamma, \qquad \mathcal{O}_2^- = \mathcal{L}[-\tfrac{\pi}{2}] \cap \mathcal{A}[r_*, R^*]$$

(recall the definition of Γ in (6.23); see Figure 6.1).

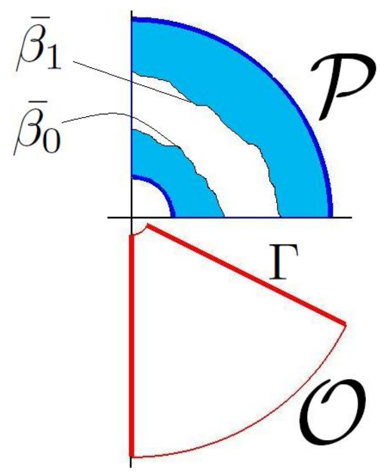


Figure 6.1: A graphical comment to the definition of the sets $\mathcal{P}$ and $\mathcal{O}$. The sets $\mathcal{K}_0$ and $\mathcal{K}_1$ are painted with a darker color. The parts of the boundaries corresponding to $\mathcal{P}^-$ and $\mathcal{O}^-$ are drawn with bold lines.

Step 3: Conclusion

We are in the position to conclude the proof, by applying Theorem 1.3.2 to the Poincaré map $\Psi_{\lambda,\mu}$ of (6.10). As in (6.12), we have

$$\Psi_{\lambda,\mu} = \Psi_\mu \circ \Psi_\lambda.$$

We claim that (H_λ) of Theorem 1.3.2 holds, that is, for $i = 0, 1$,

$$(\mathcal{K}_i, \Psi_\lambda) : \mathcal{P} \Leftrightarrow\!\!\!\!\!\rightarrow \mathcal{O}.$$

We prove our assertion for the compact set $\mathcal{K}_0$; for $\mathcal{K}_1$ the argument is very similar and the details will be omitted. Let $\gamma : [0,1] \to \mathcal{P}$ be a path such that $|\gamma(0)| = r$ and $|\gamma(1)| = R$. Recalling the definition of r_* and R^* in (6.22), we get

$$r_* \le |\Psi_\lambda(\gamma(\sigma))| \le R^*, \qquad \text{for every } \sigma \in [0,1]. \tag{6.26}$$

Now, let $\alpha : [0,1] \to \mathbb{R}$ be the unique continuous functions with $\alpha(\sigma) \in [0, \frac{\pi}{2}]$ for every $\sigma \in [0,1]$ and such that $\gamma(\sigma) = |\gamma(\sigma)|e^{i\alpha(\sigma)}$. Consider the continuous function

$$[0,1] \ni \sigma \mapsto \theta_\lambda(\tau; 0, |\gamma(\sigma)|, \alpha(\sigma)).$$

Let $[0,\sigma_1]$ be the maximal interval such that $\gamma(\sigma) \in \mathcal{K}_0$ for every $\sigma \in [0,\sigma_1]$. From Lemma 6.1.4 and Lemma 6.1.5 we deduce that

$$\Psi_\lambda(\gamma(0)) \in \mathcal{S}[-\arctan\epsilon, \tfrac{\pi}{2}] \quad \text{and} \quad \Psi_\lambda(\gamma(\sigma_1)) \in \mathcal{S}[-\pi, -\tfrac{\pi}{2}].$$

In view of (6.20), this means that

$$-\arctan\epsilon \le \theta_\lambda(\tau; 0, |\gamma(0)|, \alpha(0)) \le \frac{\pi}{2} \quad \text{and} \quad -\pi \le \theta_\lambda(\tau; 0, |\gamma(\sigma_1)|, \alpha(\sigma_1)) \le -\frac{\pi}{2},$$

which in particular imply that

$$\theta_\lambda(\tau; 0, |\gamma(\sigma_1)|, \alpha(\sigma_1)) < -\frac{\pi}{2} < -\arctan\epsilon < \theta_\lambda(\tau; 0, |\gamma(0)|, \alpha(0)).$$

By standard connectivity arguments, there exists a subinterval $[\sigma_2,\sigma_3] \subset [0,\sigma_1]$ such that

$$\theta_\lambda(\tau; 0, |\gamma(\sigma_2)|, \alpha(\sigma_2)) = -\arctan\epsilon, \qquad \theta_\lambda(\tau; 0, |\gamma(\sigma_3)|, \alpha(\sigma_3)) = -\frac{\pi}{2}$$

and

$$-\frac{\pi}{2} < \theta_\lambda(\tau; 0, |\gamma(\sigma)|, \alpha(\sigma)) < -\arctan\epsilon, \quad \text{for every } \sigma \in]\sigma_2, \sigma_3[\,.$$

Being

$$\Psi_\lambda(\gamma(\sigma)) = \varphi_\lambda(\tau; 0, \gamma(\sigma)) = |\varphi_\lambda(\tau; 0, \gamma(\sigma))| e^{i\theta_\lambda(\tau;0,|\gamma(\sigma)|,\alpha(\sigma))}$$

and in view of the radial estimate (6.26), the claim follows.

We claim that (H_μ) of Theorem 1.3.2 holds, that is

$$(\mathcal{O}, \Psi_\mu) : \mathcal{O} \Leftrightarrow\!\!\!\!\!\!\not\;\;\; \mathcal{P}.$$

Indeed, take $\gamma : [0,1] \to \mathcal{O}$ such that $\gamma(0) \in \mathcal{O}_2^-$ and $\gamma(1) \in \mathcal{O}_1^- = \Gamma$. Writing the angular coordinate $\theta_\mu(t; s, \alpha, r)$ associated with $\zeta_\mu(t; s, re^{i\alpha})$, the fact that $g(x) = 0$ for $x \le 0$ implies that relations (6.18), (6.19) hold true for such a coordinate. In particular, we infer that

$$\Psi_\mu(\gamma(\sigma)) \in \mathcal{S}(-\pi, \tfrac{\pi}{2}), \quad \text{for every } \sigma \in [0,1] \tag{6.27}$$

and, moreover,

$$\Psi_\mu(\gamma(0)) \in \mathcal{S}(-\pi, -\tfrac{\pi}{2}).$$

On the other hand, Lemma 6.2.3 implies that

$$\Psi_\mu(\gamma(1)) \in [R, +\infty[\times [R, +\infty[\,.$$

By the intermediate value theorem, and writing $\Psi_\mu = (\Psi_\mu^1, \Psi_\mu^2)$, there exists a subinterval $[\sigma_1, \sigma_2] \subset [0,1]$ such that

$$\Psi_\mu^2(\gamma(\sigma_1)) = 0, \qquad \Psi_\mu^2(\gamma(\sigma_2)) = R$$

and

$$0 < \Psi^2_\mu(\gamma(\sigma)) < R, \quad \text{for every } \sigma \in]\sigma_1, \sigma_2[\,. \tag{6.28}$$

Moreover, we have

$$\Psi^1_\mu(\gamma(\sigma_1)) < r;$$

indeed, if we suppose $r \leq \Psi^1_\mu(\gamma(\sigma_1))$, then Lemma 6.1.7 would imply that

$$|\gamma(\sigma_1)| = |\zeta_\mu(\tau; T, \zeta_\mu(T; \tau, \gamma(\sigma_1)))| = |\zeta_\mu(\tau; T, \Psi_\mu(\gamma(\sigma_1)))| \geq R^* + 1,$$

a contradiction. Finally, (6.27) and (6.28) imply that

$$0 < \Psi^1_\mu(\gamma(\sigma)), \quad \text{for every } \sigma \in [\sigma_1, \sigma_2],$$

and the claim is proved.

The conclusion of the proof of Theorem 6.1.1 now follows from Theorem 1.3.2. □

A visual description of the above proof is contained in Figure 6.2.

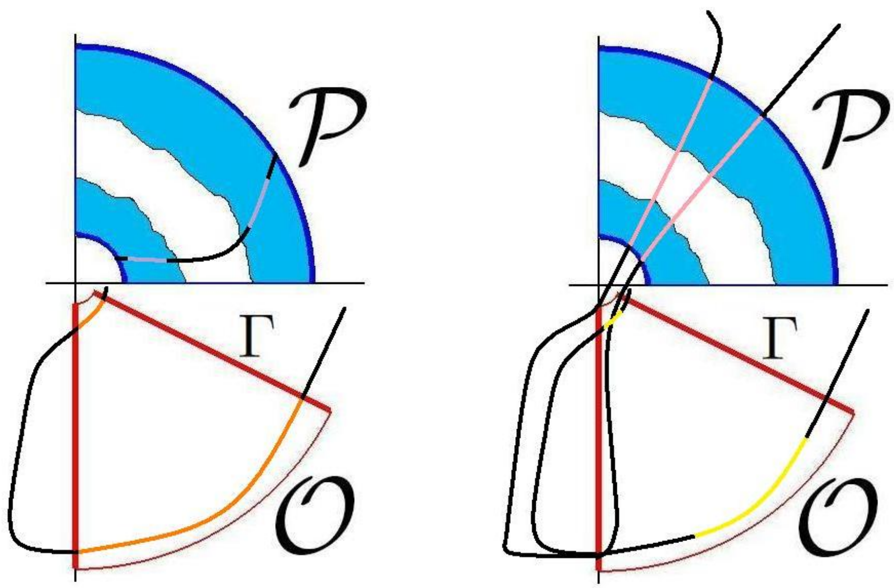


Figure 6.2: A graphical comment to the proof of Theorem 6.1.1. In the left side of the figure, we see a path $\gamma(t)$ joining the opposite sides of $\mathcal{P}^-$ and its image through Ψ_λ, $\sigma(t) = \Psi_\lambda(\gamma(t))$. The two sub-paths of $\gamma(t)$ in purple color are mapped into the orange sub-paths of $\sigma(t)$, so that $(\mathcal{K}_i, \Psi_\lambda) : \mathcal{P} \Leftrightarrow\!\!\!\rightarrow \mathcal{O}$ for $i = 1, 2$. In the right side of the figure we see the image of $\sigma(t)$ through Ψ_μ. The sub-paths in yellow color are mapped into the pink sub-paths, showing that $(\mathcal{O}, \Psi_\mu) : \mathcal{O} \Leftrightarrow\!\!\!\rightarrow \mathcal{P}$. In conclusion, $(\mathcal{K}_i, \Psi_{\lambda,\mu}) : \mathcal{P} \Leftrightarrow\!\!\!\rightarrow \mathcal{P}$ for $i = 1, 2$, obtaining a "horseshoe geometry".

Remark 6.1.2. We notice that the estimates performed along the proof of Theorem 6.1.1 and its preliminary lemmas allow the following extensions, without significant modifications in the arguments.

(*i*) Instead of the continuity of $a(t)$, we can suppose $a : \mathbb{R} \to \mathbb{R}$ measurable, bounded and T-periodic, provided that there exist $\delta > 0$ and $[\sigma, \rho] \subset [0, \tau]$ such that

$$a(t) \geq \delta, \quad \text{for a.e. } t \in [\sigma, \rho],$$

and, of course, with (a_*) of Theorem 6.1.1 satisfied for almost every t. In particular, a piecewise continuous weight $a(t)$ satisfying (a_*) is allowed in Theorem 6.1.1.

(*ii*) Since both the claims of Step 3 above are stable with respect to small perturbations of the Poincaré maps Ψ_λ and Ψ_μ , we have that Theorem 6.1.1 holds true for the perturbed equation

$$u'' + a_{\lambda,\mu}(t)g(u) + \epsilon h(t, u, u') = 0$$

for $|\epsilon|$ smaller than a constant depending on (λ, μ) (one has just to repeat the same arguments as in [142, pp.200-202]). Thus, in particular, one can add a small friction $\epsilon\, cu'$ in (6.6).

(*iii*) We can also deal with differential equations involving nonlinear differential operators, as

$$(\phi(u'))' + a_{\lambda,\mu}(t)g(u) = 0,$$

with $\phi : \mathbb{R} \to \mathbb{R}$ an odd increasing homeomorphism satisfying upper and lower σ-conditions at the origin and at infinity, like in [86]. Hypotheses (g_0) and (g_∞) have to be modified accordingly as, in the new version, they will involve corresponding limits at zero and at infinity for $g(x)/\phi(x)$.

6.1.3 A further result via critical point theory

Theorem 6.1.1 provides a quite rich set of information about the dynamical properties of the solutions to (6.6). The prizes which have to be paid are the choice of a simple nodal configuration for the weight function $a(t)$ (that is, (a_*) of Theorem 6.1.1; we claim, however that more complicate configurations could be assumed at the expense of heavier computations) and the (not sharp) choice of a large $\mu > 0$. A natural question is whether such restrictions can be removed; Theorem 6.1.2 below provides a partial answer in this direction.

Theorem 6.1.2. *Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous and T-periodic function of nonconstant sign and let $g : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function, with $g(0) = 0$ and $g(x) > 0$ for $x > 0$, such that:*

(g_0) *there exists $\alpha > 1$ such that*

$$\lim_{x\to 0^+} \frac{g(x)}{x^\alpha} = l_\alpha > 0;$$

(g_∞) *there exists $0 \leq \beta < 1$ such that*

$$\limsup_{x\to+\infty} \frac{g(x)}{x^\beta} < +\infty \quad \text{and} \quad \lim_{x\to+\infty} \frac{G(x)}{x^{2\beta}} = +\infty.$$

Then there exists $\lambda^* > 0$ *such that, for every* $\lambda > \lambda^*$ *and for*

$$\mu > \lambda \left(\frac{\int_0^T a^+(t)\, dt}{\int_0^T a^-(t)\, dt} \right), \tag{6.29}$$

equation (6.6) *has at least two positive* T*-periodic solutions.*

Remark 6.1.3. Conditions (g_0) and (g_∞) of Theorem 6.1.2 imply that $g(x)$ satisfies (6.7). Notice that (g_∞) may be seen as a generalized Ahmad-Lazer-Paul (nonresonance) condition at infinity [157] for the potential energy $F(t,x) = a(t)G(x)$. Recall that the classical Ahmad-Lazer-Paul condition [3] (see also [128, Theorems 1.5 and 4.8] for the periodic problem), read in our context, requires that $\int_0^T F(t,x)\, dt \to \pm\infty$ for $x \to \infty$, with $f(t,x) = \frac{\partial}{\partial x} F(t,x)$ bounded, i.e., (g_∞) with $\beta = 0$. Observe also that (g_∞) is satisfied when $g(x)$ has a precise (sublinear) power-growth at infinity, namely if, for some $0 \leq \beta < 1$,

$$(g'_\infty) \qquad \lim_{x \to +\infty} \frac{g(x)}{x^\beta} = l_\beta > 0.$$

Indeed, in this case, l'Hopital rule implies that $\frac{G(x)}{x^{\beta+1}} \to \frac{l_\beta}{\beta+1}$ for $x \to +\infty$.

Notice that, since

$$\mu > \lambda \left(\frac{\int_0^T a^+(t)\, dt}{\int_0^T a^-(t)\, dt} \right) \Longleftrightarrow \int_0^T a_{\lambda,\mu}(t)\, dt < 0,$$

the choice for μ in (6.29) is sharp, see (6.3). The existence of positive subharmonic solutions to (6.6) under this sharp assumption is still to be investigated.

The proof of Theorem 6.1.2 uses a variational approach on the Sobolev space $H^1_T = W^{1,2}_T$ of locally absolutely continuous and T-periodic functions with a locally L^2-derivative. Indeed (see for example [128, Corollary 1.1]), T-periodic solutions to (6.6) correspond to critical points of the C^1 functional $\mathscr{I}_{\lambda,\mu} : H^1_T \to \mathbb{R}$ defined by

$$\mathscr{I}_{\lambda,\mu}(u) = \frac{1}{2} \int_0^T u'(t)^2\, dt - \int_0^T a_{\lambda,\mu}(t) G(u(t))\, dt.$$

Roughly speaking, a first solution can be characterized as a global minimum point for $\mathscr{I}_{\lambda,\mu}$ on H^1_T, while a second one can be provided by a classical Mountain Pass procedure [10]. By maximum-principle arguments, both these solutions can be shown to be nontrivial and positive. For the complete details of the proof, we refer to [31].

6.2 Superlinear problem: multiple solutions via a shooting technique

In this section, we address our investigation to the existence of positive periodic solutions to (6.1), when $g'(0) = 0$ and $g(x)$ is superlinear at infinity; as far as we know, this problem has never been considered explicitly in literature. However, as remarked in the introduction of the chapter, analogies can be found with the Neumann problem associated with the elliptic partial differential equation

$$\Delta u + q(x)g(u) = 0, \qquad x \in \Omega \subset \mathbb{R}^N. \tag{6.30}$$

In the model case $g(x) = x^{\gamma+1}$ with $\gamma > 0$, problem (6.30) has been extensively studied, with variational methods strongly depending on the homogeneity of the nonlinear term, by Berestycki, Capuzzo-Dolcetta and Nirenberg [20], who proved the existence of at least one positive solution under the mean value condition $\int_\Omega q(x)\,dx < 0$ (compare with (6.3)). The same conclusion is achieved in [4, 19], for a more general superlinear nonlinearity satisfying suitable extra assumptions at zero and at infinity (the first paper is based on variational techniques again, while the second relies on a priori estimates, degree theory and bifurcation arguments). These results have been subsequently extended to more general equations involving the p-Laplacian operator [145]. However, to the best of our knowledge, no multiplicity results in the general setting are available in literature.

Our aim is to show that multiple positive T-periodic solutions can appear for

$$u'' + a_\mu(t)g(u) = 0, \tag{6.31}$$

being:

- $a_\mu(t) = a^+(t) - \mu a^-(t)$, with $a^+(t), a^-(t)$ the positive and the negative part of a (sign-indefinite) continuous, T-periodic and even-symmetric function $a : \mathbb{R} \to \mathbb{R}$
- $g : \mathbb{R}^+ \to \mathbb{R}$ a locally Lipschitz continuous function, with $g(0) = 0$ and $g(x) > 0$ for $x > 0$, satisfying
$$\lim_{x\to 0^+} \frac{g(x)}{x} = 0, \qquad \lim_{x\to +\infty} \frac{g(x)}{x} = +\infty. \tag{6.32}$$

Similarly as in Section 6.1, solutions are found when μ is large enough[2] .

Our symmetry condition on the weight function $a_\mu(t)$ allows us to find T-periodic solution starting from solutions to an (indefinite superlinear) Neumann problem on $[0, T/2]$. We then use shooting type arguments (inspired from a paper by Gaudenzi, Habets and Zanolin [89], dealing with a Dirichlet problem), combining the oscillatory properties of the solutions in the intervals of positivity of the weight function with blow-up phenomena in the intervals of negativity, to analyze the Neumann problem.

The plan of the section is the following. In Subsection 6.2.1 we state our main result (Theorem 6.2.1), together with some brief comments on the assumptions. Subsection 6.2.2 is devoted to the proof.

6.2.1 Statement of the main result

The main result of the section is the following.

[2] Comparing with (6.4), we have set here $\lambda = 1$, since such a parameter does not play a role.

Theorem 6.2.1. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a locally Lipschitz continuous function such that $g(0) = 0$ and $g(x) > 0$ for $x > 0$. Setting $G(x) = \int_0^x g(\xi)\, d\xi$, let us assume that:*

(g_0)
$$\lim_{x\to 0^+} \frac{g(x)}{x} = 0;$$

(g_∞)
$$\lim_{x\to +\infty} \frac{g(x)}{x} = +\infty \quad \textit{and} \quad \int^{+\infty} \frac{dx}{\sqrt{G(x)}} < +\infty;$$

(g_∞^*)
$$\limsup_{x\to +\infty} \int_x^{+\infty} \frac{d\xi}{\sqrt{G(\xi) - G(x)}} < +\infty.$$

Moreover, let $a : \mathbb{R} \to \mathbb{R}$ be a continuous, even-symmetric and T-periodic function such that, for some σ, τ with $0 < \sigma < \tau < T/2$,

$$\begin{array}{lll} a(t) \geq 0, & \textit{for every } t \in [0,\sigma], & \textit{with } \int_0^\sigma a(t)\, dt > 0,\\ a(t) \leq 0, & \textit{for every } t \in [\sigma,\tau], & \textit{with } \int_\sigma^\tau a(t)\, dt < 0,\\ a(t) \geq 0, & \textit{for every } t \in [\tau,T/2], & \textit{with } \int_\tau^{T/2} a(t)\, dt > 0. \end{array}$$

Then there exists $\mu^* > 0$ such that, for every $\mu > \mu^*$, equation (6.31) *has at least three positive, even-symmetric, T-periodic solutions.*

A few comments about the assumptions are in order. Hypothesis (g_0) and the first condition in (g_∞) correspond to (6.32). The second requirement of (g_∞) is the well known Keller-Osserman condition and, according to [37, Theorem 2], it is a necessary condition for the existence of a solution to (6.31) which blows up in an interval of negativity of $a_\mu(t)$. Finally, hypothesis (g_∞^*) is a time-map assumption for the autonomous equation

$$u'' - g(u^+) = 0. \tag{6.33}$$

In fact, it is immediate to check that (g_∞^*) holds true if and only if

$$\limsup_{c\to -\infty} \sqrt{2} \int_{G^{-1}(-c)}^{+\infty} \frac{d\xi}{\sqrt{G(\xi) + c}} < +\infty \tag{6.34}$$

and the integral in (6.34) is just the time for the orbit of (6.33) passing through the point $(G^{-1}(-c), 0)$. As proved in [137, Appendix], (g_∞^*) is fulfilled whenever (g_∞) holds true and one of the following conditions is satisfied:

- for every x large, $g(x) \geq h(x)$, for some continuous and monotone function $h : \mathbb{R}^+ \to \mathbb{R}$ satisfying (g_∞);
- there exists a constant $k > 1$ such that
$$\liminf_{x\to +\infty} \frac{G(kx)}{G(x)} > 1.$$

It is worth noticing that this last condition is related to the Karamata's theory of slowly varying functions [21]. Needless to say, the model nonlinearity $g(x) = x^{\gamma+1}$ (with $\gamma > 0$) satisfies all such assumptions $(g_0), (g_\infty), (g_\infty^*)$.

6.2.2 Proof of the main result

First of all, we observe that, if $u(t)$ ($t \in [0, T/2]$) is a positive solution to the Neumann problem

$$\begin{cases} u'' + a_\mu(t) g(u) = 0 \\ u'(0) = u'(T/2) = 0, \end{cases} \tag{6.35}$$

then the function defined (with a slight abuse in notation) by $u(t) = u(|t|)$ for $t \in [-T/2, T/2]$ and extended by T-periodicity is a positive and even-symmetric T-periodic solution to (6.31). Hence, we are going to prove the existence of three positive solutions to (6.35).

Let us fix $\rho \in]\sigma, \tau[$ and $\epsilon, \delta > 0$ such that

$$a^-(t) \geq \delta, \qquad \text{for every } t \in [\rho - \epsilon, \rho]; \tag{6.36}$$

as in Section 6.1, moreover, we will identify $g(x)$ with its null extension $g^0(x)$, defined in (6.9). We split our proof into three steps.

Step 1: Forward shooting

For every $x \geq 0$ and $\mu \geq 0$, let $u_\mu(\cdot; x)$ be the solution to the Cauchy problem

$$\begin{cases} u'' + a_\mu(t) g(u) = 0 \\ (u(0), u'(0)) = (x, 0) \end{cases} \tag{6.37}$$

and denote by $[0, t^+_\mu(x)[$ its maximal interval of (forward) continuability in $[0, T/2]$; it is a well known fact in the theory of initial value problems for ODEs that the function $x \mapsto t^+_\mu(x)$ is lower semicontinuous. Set

$$\mathcal{D}^+_\mu = \{x \geq 0 \mid t^+_\mu(x) > \rho\}$$

and define the translation operator

$$\varphi_\mu : \mathcal{D}^+_\mu \ni x \mapsto (u_\mu(\rho; x), u'_\mu(\rho; x)).$$

We recall that, for every $S \subset \mathcal{D}^+_\mu$, S and $\varphi_\mu(S)$ are homeomorphic. The final goal of this first step of the proof is to construct two disjoint intervals contained in $\mathcal{D}^+_\mu$, whose images under φ_μ are contained in the right half-plane and connect the y-axis with $(+\infty, +\infty)$. Precisely, we will prove the following.

Lemma 6.2.1. *There exists $\mu^*_1 > 0$ such that, for every $\mu > \mu^*_1$, there exist ξ_1, ξ_2, ξ_3 with $0 < \xi_1 \leq \xi_2 < \xi_3$ such that:*

- $[0, \xi_1[\cup]\xi_2, \xi_3] \subset \mathcal{D}^+_\mu$*;*
- *for every $x \in]0, \xi_1[\cup]\xi_2, \xi_3[$, $u_\mu(t; x) > 0$ for every $t \in [0, \rho]$;*
- $\varphi_\mu(0) = (0, 0)$ *and* $\varphi_\mu(\xi_3) = (0, R)$ *with* $R < 0$*;*
- $\lim_{x \to \xi_1^-} \varphi_\mu(x) = \lim_{x \to \xi_2^+} \varphi_\mu(x) = (+\infty, +\infty)$.

The proof of such a lemma is based on a careful study of the behavior of the solutions to (6.37) in the time interval $[0,\rho]$. In particular, a special care has to be paid to the blow-up properties of such solutions. Hence, before passing to the proof of Lemma 6.2.1, we establish the following auxiliary results.

Lemma 6.2.2. *There exist $x_*, \mu_1^* > 0$ such that:*

i) $t_0^+(x) = T/2$ *for every* $x \geq 0$ *and* $u_0(t;x) > 0$ *for every* $t \in [0,\rho]$ *and* $x \in]0, x_*]$*;*

ii) for every $\mu \geq 0$*, it holds that* $t_\mu^+(x) > \sigma$ *for every* $x \geq 0$ *and* $u_\mu(t;x) > 0$ *for every* $t \in [0,\rho] \cap [0, t_\mu^+(x)[$ *and* $x \in]0, x_*]$*;*

iii) for every $\mu > \mu_1^*$*, it holds that* $t_\mu^+(x_*) \leq \rho$*.*

Proof. We prove separately the different claims.

The fact that $t_0^+(x) = T/2$ for every $x \geq 0$ follows from the observation that $g(x) = 0$ for every $x \leq 0$ and, by concavity, $u_0(t;x) \leq x$ for every $t \in [0, t_0(x)[$. Next, define x_* small enough such that, for every $x \in]0, x_*]$,

$$\sup_{t \in [0,\sigma], x \in]0, x_*]} a^+(t) \frac{g(x)}{x} < \left(\frac{\pi}{2\rho}\right)^2.$$

Let us suppose by contradiction that, for some $x \in]0, x_*]$, there exists $t^* \in [0,\rho]$ such that $u_0(t^*;x) = 0$. Without loss of generality, we can suppose that $u_0(t;x) > 0$ for every $t \in [0, t^*[$; moreover, as observed above, we have that $u_0(t;x) \leq x$ for every $t \in [0, t^*]$. Setting $\omega = \frac{\pi}{2t^*}$ (and supposing $t^* \in [\sigma, \rho]$, the other case being even simpler), we get

$$\begin{aligned}
0 &= \Big[\omega u_0(t;x) \sin(\omega t) + u_0'(t;x)\cos(\omega t)\Big]_{t=0}^{t^*} \\
&= \int_0^{t^*} \left[\omega^2 - a_0(t)\, \frac{g(u_0(t;x))}{u_0(t;x)}\right] u_0(t;x) \cos(\omega t)\, dt \\
&= \int_0^{\sigma} \left[\omega^2 - a^+(t)\, \frac{g(u_0(t;x))}{u_0(t;x)}\right] u_0(t;x) \cos(\omega t)\, dt \, + \\
&\quad + \int_\sigma^{t^*} \omega^2 u_0(t;x) \cos(\omega t)\, dt \\
&\geq \int_0^{\sigma} \left[\left(\frac{\pi}{2\rho}\right)^2 - a^+(t)\, \frac{g(u_0(t;x))}{u_0(t;x)}\right] u_0(t;x) \cos(\omega t)\, dt > 0,
\end{aligned}$$

which is a contradiction.

Fix $\mu \geq 0$ and $x \geq 0$. Similarly as above, since $u_\mu(t;x) \leq x$ for every $t \in [0,\sigma] \cap [0, t_\mu^+(x)[$ and $g(x) = 0$ for $x \leq 0$, we deduce that $t_\mu^+(x) > \sigma$. Moreover, it is clear that $u_\mu(t;x) = u_0(t;x)$, $u_\mu'(t;x) = u_0'(t;x)$ for every $t \in [0,\sigma]$. Hence, since $u_\mu''(\cdot;x) \geq 0$ on $[\sigma,\rho] \cap [\sigma, t_\mu^+(x)[$, we have that $u_\mu(t;x) \geq u_0(t;x)$ for every $t \in [0,\rho] \cap [0, t_\mu^+(x)[$. The conclusion now follows from point *i*).

Let x_* be as before and suppose by contradiction that there exists $\mu_k \to +\infty$ such that $t^+_{\mu_k}(x_*) > \rho$. As observed in the proof of point ii), and using the fact that $u_0(\cdot; x_*)$ is non increasing and concave, we get that, for every k,

$$u_{\mu_k}(t; x_*) \geq u_0(t; x_*) \geq u_0(\rho; x_*), \quad \text{for every } t \in [\sigma, \rho].$$

and

$$u'_{\mu_k}(t; x_*) \geq u'_0(t; x_*) = u'_0(\sigma; x_*), \quad \text{for every } t \in [\sigma, \rho].$$

Set

$$m = \inf_{x \geq u_0(\rho; x_*)} g(x) > 0;$$

then we have, for every k and for every $t \in [\sigma, \rho - \epsilon]$,

$$\begin{aligned} u'_{\mu_k}(t; x_*) &= u'_{\mu_k}(\sigma; x_*) + \int_\sigma^t u''_{\mu_k}(s; x_*)\, ds \\ &\geq u'_0(\sigma; x_*) + \mu_k \int_\sigma^t a^-(s) g(u_{\mu_k}(s; x_*))\, ds \\ &\geq u'_0(\sigma; x_*) + \mu_k m \int_\sigma^t a^-(s)\, ds. \end{aligned}$$

Setting $A_-(t) = \int_\sigma^t a^-(s)\, ds$ and integrating on $[\sigma, \rho - \epsilon]$, we have moreover, for every k,

$$\begin{aligned} u_{\mu_k}(\rho - \epsilon; x_*) &\geq u_{\mu_k}(\sigma; x_*) + u'_0(\sigma; x_*)(\rho - \epsilon - \sigma) + \mu_k m \int_\sigma^{\rho - \epsilon} A^-(s)\, ds \\ &\geq u_0(\sigma; x_*) + u'_0(\sigma; x_*)(\rho - \epsilon - \sigma) + \mu_k m \int_\sigma^{\rho - \epsilon} A^-(s)\, ds. \end{aligned}$$

Being, in view of (6.36), $a^-(\rho - \epsilon) \geq \delta$, we have that $\int_\sigma^{\rho-\epsilon} A^-(s)\, ds > 0$ and $\int_\sigma^{\rho-\epsilon} a^-(s)\, ds > 0$; then we can conclude that

$$\lim_{k \to +\infty} u_{\mu_k}(\rho - \epsilon; x_*) = +\infty, \tag{6.38}$$
$$\lim_{k \to +\infty} u'_{\mu_k}(\rho - \epsilon; x_*) = +\infty.$$

In particular, for k large enough and for every $t \in]\rho - \epsilon, \rho]$,

$$u'_{\mu_k}(t; x_*) > u'_{\mu_k}(\rho - \epsilon; x_*) \geq 0. \tag{6.39}$$

Hence, we get, for every $t \in]\rho - \epsilon, \rho]$,

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} u'_{\mu_k}(t; x_*)^2 - \mu_k \delta G(u_{\mu_k}(t; x_*)\right) &= u'_{\mu_k}(t; x_*)(u''_{\mu_k}(t; x_*) - \mu_k \delta g(u_{\mu_k}(t; x_*))) \\ &\geq u'_{\mu_k}(t; x_*)(u''_{\mu_k}(t; x_*) - \mu_k a^-(t) g(u_{\mu_k}(t; x_*))) \\ &= 0. \end{aligned}$$

Integrating between $\rho - \epsilon$ and $t \in [\rho - \epsilon, \rho]$, we obtain

$$\frac{1}{2} u'_{\mu_k}(t; x_*)^2 - \mu_k \delta G(u_{\mu_k}(t; x_*)) \geq -\mu_k \delta G(u_{\mu_k}(\rho - \epsilon; x_*)),$$

that is (since, from (6.39), $G(u_{\mu_k}(t;x_*)) > G(u_{\mu_k}(\rho-\epsilon;x_*))$ for $t\in\,]\rho-\epsilon,\rho]$)

$$\frac{u'_{\mu_k}(t;x_*)}{\sqrt{2\mu_k\delta(G(u_{\mu_k}(t;x_*)) - G(u_{\mu_k}(\rho-\epsilon;x_*))}} \geq 1.$$

Then we have

$$\begin{aligned}
\epsilon &= \int_{\rho-\epsilon}^{\rho} dt \leq \int_{\rho-\epsilon}^{\rho} \frac{u'_{\mu_k}(t;x_*)}{\sqrt{2\mu_k\delta(G(u_{\mu_k}(t;x_*)) - G(u_{\mu_k}(\rho-\epsilon;x_*))}}\, dt \\
&= \sqrt{\frac{1}{2\mu_k\delta}} \int_{u_{\mu_k}(\rho-\epsilon;x_*)}^{u_{\mu_k}(\rho;x_*)} \frac{d\xi}{\sqrt{G(\xi) - G(u_{\mu_k}(\rho-\epsilon;x_*))}} \\
&\leq \sqrt{\frac{1}{2\mu_k\delta}} \int_{u_{\mu_k}(\rho-\epsilon;x_*)}^{+\infty} \frac{d\xi}{\sqrt{G(\xi) - G(u_{\mu_k}(\rho-\epsilon;x_*))}}.
\end{aligned}$$

Taking into account assumption (g^*_∞) and (6.38), we get a contradiction for k large enough. □

Lemma 6.2.3. *There exists a positive solution (i.e, $u(t) > 0$ for every $t \in [0,\sigma[$) to the boundary value problem*

$$\begin{cases} u'' + a^+(t)g(u) = 0 \\ u'(0) = 0, \quad u(\sigma) = 0. \end{cases} \tag{6.40}$$

Proof. Being $a^+(t)$ continuous, with $a^+ \geq 0$ and $a^+ \not\equiv 0$ on $[0,\sigma]$, [105, Corollary 3.6] implies that there exists a nontrivial solution to (6.40) such that $u(t) > 0$ for every $t \in]0,\sigma[$. The uniqueness for the solutions to the Cauchy problems implies that $u(0) > 0$, too. □

Proof of Lemma 6.2.1. Fix $\mu > \mu_1^*$. By point *ii*) of Lemma 6.2.2, $t^+_\mu(x) > \sigma$ for every $x \geq 0$ and hence the set

$$\mathcal{E}_\mu = \{x > 0 \mid u_\mu(t;y) > 0, \text{ for every } t \in [0,\sigma],\, y \in]0,x]\}$$

is well defined. Moreover, $\mathcal{E}_\mu$ is nonempty, since $x_* \in \mathcal{E}_\mu$, as a consequence of point *ii*) of Lemma 6.2.2 again. Set $\xi = \sup \mathcal{E}_\mu$; in view of Lemma 6.2.3, $x_* < \xi < +\infty$. Next define the set

$$\mathcal{F}_\mu = \{x \in [0,\xi] \mid t^+_\mu(x) \leq \rho\};$$

again this set is nonempty since $x_* \in \mathcal{F}_\mu$, as consequence of point *iii*) of Lemma 6.2.2. Set $\xi_1 = \inf \mathcal{F}_\mu$ and $\xi_2 = \sup \mathcal{F}_\mu$. Since $t^+_\mu(x)$ is lower semicontinuous, $\mathcal{F}_\mu$ is a closed set; being $0, \xi \notin \mathcal{F}_\mu$, we deduce that $0 < \xi_1 \leq x_* \leq \xi_2 < \xi$. Moreover, by construction it holds that $[0,\xi_1[\cup]\xi_2,\xi] \subset \mathcal{D}^+_\mu$ and, in view of point *ii*) of Lemma 6.2.2, we know that $u_\mu(t;x) > 0$, for every $t \in [0,\rho]$ and $x \in]0,\xi_1[$. We claim that

$$\lim_{x\to\xi_1^-} \varphi_\mu(x) = \lim_{x\to\xi_2^+} \varphi_\mu(x) = (+\infty,+\infty). \tag{6.41}$$

Just to fix the ideas, we prove the relation for $x \to \xi_2^+$. We first verify that $\lim_{x\to\xi_2^+} u_\mu(\rho;x) = +\infty$. Let us suppose by contradiction that, for a sequence $x_k \to \xi_2^+$, $u_\mu(\rho;x_k) \leq M$ and fix $\eta > 0$ small. Since $u_\mu(\cdot;x_k) \to u_\mu(\cdot;\xi_2)$ in $C^1([0,\sigma])$, we have that, for k large enough,

$$|u_\mu(\sigma;x_k) - u_\mu(\sigma;\xi_2)| \leq \eta, \qquad |u'_\mu(\sigma;x_k) - u'_\mu(\sigma;\xi_2)| \leq \eta.$$

By convexity arguments, it is easy to see that, for k large enough and for every $t \in [0,\rho]$,

$$(u'_\mu(\sigma;\xi_2)-\eta)(\rho-\sigma)+u_\mu(\sigma;\xi_2)-\eta \le u_\mu(t;x_k) \le \max(\xi_2+\eta, M); \tag{6.42}$$

hence, the sequence $u_\mu(\cdot;x_k)$ is uniformly bounded on $[0,\rho]$. Moreover, since $u''_\mu(t;x_k) = a_\mu(t)g(u_\mu(t;x_k))$ and $u'_\mu(t;x_k) = \int_0^t u''_\mu(s;x_k)\,ds$, we conclude that $u_\mu(\cdot;x_k)$ is bounded in $C^2([0,\rho])$. Hence, up to subsequences, $u_\mu(\cdot;x_k)$ converges (weakly in $W^{2,2}(0,\rho)$ and strongly in $C^1([0,\rho])$) to a function $u_\mu(t)$ with $u_\mu(0)=\xi_2$, $u'_\mu(0)=0$ and such that

$$u''_\mu(t)+a_\mu(t)g(u_\mu(t))=0, \qquad \text{for every } t \in [0,\rho].$$

Since $\xi_2 \in \mathcal{F}_\mu$ (recall that $\mathcal{F}_\mu$ is a closed set), this is a contradiction. To conclude the proof of (6.41), it is sufficient to observe that, by convexity of $u_\mu(\cdot;x)$ on $[\sigma,\rho]$,

$$u'_\mu(\rho;x) \ge \frac{u_\mu(\rho;x)-u_\mu(\sigma;x)}{\rho-\sigma}$$

and that the right-hand side goes to infinity as $x \to \xi_2^+$, since $0 < u_\mu(\sigma;x) \le x$. Notice that for $x \to \xi_1^-$ the proof is even simpler because, in (6.42), we know that $u_\mu(t;x_k) > 0$ for every $t \in [0,\rho]$ and every $x_k < \xi_1$.
Observe now that relation (6.41) implies that $u_\mu(t;x) > 0$ for every $t \in [0,\rho]$ and for x in a right neighborhood of ξ_2. Indeed, if $u_\mu(\tilde{t};x) \le 0$ for some $\tilde{t} \in [0,\rho]$, then $u_\mu(\rho;x) \le 0$ too. Thus, the set

$$\mathcal{G}_\mu = \{x \in]\xi_2,\xi] \mid \ u_\mu(t;y) > 0, \text{ for every } t \in [0,\rho],\, y \in]\xi_2,x]\}$$

is nonempty and we set $\xi_3 = \sup \mathcal{G}_\mu$. It is easily seen that $\xi_2 < \xi_3 < \xi$ and that $\varphi_\mu(\xi_3) = (0,R)$ with $R<0$. Moreover, by definition, $u_\mu(t;x) > 0$ for every $t \in [0,\rho]$ and for every $x \in]\xi_2,\xi_3[$. Recalling (6.41), the proof is concluded. □

Step 2: Backward shooting

This second step is completely symmetric to the previous one. Define, for $x \ge 0$ and $\mu \ge 0$, $v_\mu(\cdot;x)$ as the unique backward solution to the Cauchy problem

$$\begin{cases} v''+a_\mu(t)g(v)=0 \\ (v(T/2),v'(T/2))=(x,0) \end{cases}$$

and denote by $]t^-_\mu(x),T/2]$ its maximal interval of (backward) continuability in $[0,T/2]$. Then $x \mapsto t^-_\mu(x)$ is upper semicontinuous and we can define the set

$$\mathcal{D}^-_\mu = \{x \ge 0 \mid t^-_\mu(x) < \rho\}$$

and the translation operator

$$\psi_\mu : \mathcal{D}^-_\mu \ni x \mapsto (v_\mu(\rho;x), v'_\mu(\rho;x)).$$

We can prove the following analogous of Lemma 6.2.1.

Lemma 6.2.4. *There exists $\mu_2^* > 0$ such that, for every $\mu > \mu_2^*$, there exist η_1, η_2, η_3 with $0 < \eta_1 \leq \eta_2 < \eta_3$ such that:*

- $[0, \eta_1[\cup]\eta_2, \eta_3] \subset \mathcal{D}_\mu^-$;
- *for every* $x \in]0, \eta_1[\cup]\eta_2, \eta_3[$, $v_\mu(t; x) > 0$ *for every* $t \in [\rho, T]$;
- $\psi_\mu(0) = (0, 0)$ *and* $\psi_\mu(\eta_3) = (0, S)$ *with* $S > 0$;
- $\lim_{x \to \eta_1^-} \psi_\mu(x) = \lim_{x \to \eta_2^+} \psi_\mu(x) = (+\infty, -\infty)$.

Step 3: Conclusion

Define $\mu^* = \max\{\mu_1^*, \mu_2^*\}$ and fix $\mu > \mu^*$. By standard connectivity arguments, the following facts hold true:

- $\varphi_\mu(]0, \xi_1[)$ intersects $\psi_\mu(]\eta_2, \eta_3[)$;
- $\psi_\mu(]0, \eta_1[)$ intersects $\varphi_\mu(]\xi_2, \xi_3[)$;
- $\varphi_\mu(]\xi_2, \xi_3[)$ intersects $\psi_\mu(]\eta_2, \eta_3[)$.

These intersection points are pairwise distinct because of the uniqueness for the solutions to the Cauchy problems and it is clear that each of them corresponds to a value $(u(\rho), u'(\rho))$ of a positive solution to the Neumann problem (6.35).

A numerical simulation is plotted in Figure 6.2.2 below.

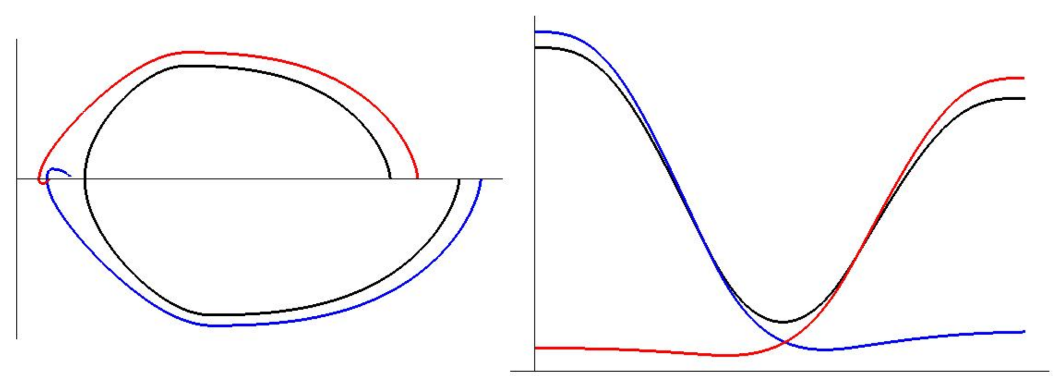


Figure 6.3: A numerical experiment, plotting the three solutions to the Neumann problem on $[0, T/2]$ associated with $u'' + (a^+(t) - \mu a^-(t))g(u) = 0$. Here, $T = 6$, $g(x) = x^2$, $\mu = 15$, $a^+(t) = 0.9\sin(\pi t)$ for $0 \leq t \leq 1$, $= 0$ for $0 \leq t \leq 2$, $= \sin(\pi t)$ for $2 \leq t \leq 3$ and $a^-(t) = 0$ for $0 \leq t \leq 1$, $= -\sin(\pi t)$ for $0 \leq t \leq 2$, $= 0$ for $2 \leq t \leq 3$. Solutions are plotted in the phase-plane $\{(u, u')\}$ (figure on the left) and in the plane $\{(t, u(t))\}$ (figure on the right).

Bibliography

[1] A. Abbondandolo, *Sub-harmonics for two-dimensional Hamiltonian systems*, NoDEA Nonlinear Differential Equations Appl. **6** (1999), 341–355.

[2] A. Abbondandolo, Morse Theory for Hamiltonian systems, Chapman & Hall/CRC Research Notes in Mathematics **425**, Boca Raton, 2001.

[3] S. Ahmad, A.C. Lazer and J.L. Paul, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J. **25** (1976), 933-944.

[4] S. Alama and G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities*, Calc. Var. Partial Differential Equations **1** (1993), 439–475.

[5] H. Amann, *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Functional Analysis **11** (1972), 346–384.

[6] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

[7] H. Amann, Ordinary Differential Equations. An Introduction to Nonlinear Analysis, de Gruyter Studies in Mathematics **13**, Walter de Gruyter & Co., Berlin, 1990.

[8] H. Amann and J. López-Gómez, *A priori bounds and multiple solutions for superlinear indefinite elliptic problems*, J. Differential Equations **146** (1998), 336–374.

[9] H. Amann and E. Zehnder, *Periodic solutions of asymptotically linear Hamiltonian systems*, Manuscripta Math. **32** (1980), 149–189.

[10] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349-381.

[11] V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York-Heidelberg, 1978.

[12] F.M. Atici and G.Sh. Guseinov, *On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions*, J. Comput. Appl. Math. **132** (2001), 341–356

[13] A. Bahri and H. Berestycki, *Forced vibrations of superquadratic Hamiltonian systems*, Acta Math. **152** (1984), 143–197.

[14] C. Bandle, M.A. Pozio and A. Tesei, *The asymptotic behavior of the solutions of degenerate parabolic equations*, Trans. Amer. Math. Soc. **303** (1987), 487–501.

[15] C. Bandle, M.A. Pozio and A. Tesei, *Existence and uniqueness of solutions of nonlinear Neumann problems*, Math. Z. **199** (1988), 257–278.

[16] J. Belmonte-Beitia and P.J. Torres, *Existence of dark soliton solutions of the cubic nonlinear Schrödinger equation with periodic inhomogeneous nonlinearity*, J. Nonlinear Math. Phys. **15** (2008), 65–72.

[17] V. Benci and P.H. Rabinowitz, *Critical points theorem for indefinite functionals*, Invent. Math **52** (1979), 241–273.

[18] C. Bereanu and J. Mawhin, *Multiple periodic solutions of ordinary differential equations with bounded nonlinearities and φ-Laplacian*, NoDEA Nonlinear Differential Equations Appl. **15** (2008), 159–168.

[19] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems*, Topol. Methods Nonlinear Anal. **4** (1994), 59–78.

[20] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems*, NoDEA Nonlinear Differential Equations Appl. **2** (1995), 553–572.

[21] N.H. Bingham, C.M. Goldie and J.L. Teugels, Regular variation, Encyclopedia of Mathematics and its Applications **27**, Cambridge University Press, Cambridge, 1987.

[22] G.D. Birkhoff, *Proof of Poincaré's geometric theorem*, Trans. Amer. Math. Soc. **14** (1913), 14–22.

[23] G.D. Birkhoff, *An extension of Poincaré's last geometric theorem*, Acta Math. **47** (1926), 297–311.

[24] D. Bonheure, J.M. Gomes and P. Habets, *Multiple positive solutions of superlinear elliptic problems with sign-changing weight*, J. Differential Equations **214** (2005), 36–64.

[25] A. Boscaggin, *A note on a superlinear indefinite Neumann problem with multiple positive solutions*, J. Math. Anal. Appl. **377** (2011), 259–268.

[26] A. Boscaggin, *Subharmonic solutions of planar Hamiltonian systems: a rotation number approach*, Adv. Nonlinear Stud. **11** (2011), 77–103.

[27] A. Boscaggin, *Subharmonic solutions of planar Hamiltonian systems via the Poincaré-Birkhoff theorem*, Matematiche (Catania) **66** (2011), 115–122.

[28] A. Boscaggin, *One-signed harmonic solutions and sign-changing subharmonic solutions to scalar second order differential equations*, Adv. Nonlinear Stud. **12** (2012), 445–463.

[29] A. Boscaggin, *Periodic solutions to superlinear planar Hamiltonian systems*, Port. Math. **69** (2012), 127–140

[30] A. Boscaggin and M. Garrione, *Resonance and rotation numbers for planar Hamiltonian systems: multiplicity results via the Poincaré-Birkhoff theorem*, Nonlinear Anal. **74** (2011), 4166–4185.

[31] A. Boscaggin and F. Zanolin, *Pairs of positive periodic solutions of second order nonlinear equations with indefinite weight*, J. Differential Equations **252** (2012), 2900–2921.

[32] A. Boscaggin and F. Zanolin, *Positive periodic solutions of second order nonlinear equations with indefinite weight: multiplicity results and complex dynamics*, J. Differential Equations **252** (2012), 2922–2950.

[33] A. Boscaggin and F. Zanolin, *Subharmonic solutions for nonlinear second order equations in presence of lower and upper solutions*, Discrete Contin. Dyn. Syst. **33** (2013), 89–110.

[34] A. Boscaggin and F. Zanolin, *Pairs of nodal solutions for a class of nonlinear problems with one-sided growth conditions*, to appear on Adv. Nonlinear Stud.

[35] M. Brown and W.D. Neumann, *Proof of the Poincaré-Birkhoff fixed point theorem*, Michigan Math. J. **24** (1977), 21–31.

[36] K. Burns and H. Weiss, *A geometric criterion for positive topological entropy*, Comm. Math. Phys. **172** (1995), 95–118.

[37] T. Burton and R. Grimmer, *On continuability of solutions of second order differential equations*, Proc. Amer. Math. Soc. **29** (1971), 277–283.

[38] G.J. Butler, *Rapid oscillation, nonextendability, and the existence of periodic solutions to second order nonlinear ordinary differential equations*, J. Differential Equations **22** (1976), 467–477.

[39] G.J. Butler, *Periodic solutions of sublinear second order differential equations*, J. Math. Anal. Appl. **62** (1978), 676–690.

[40] N.P. Các and A.C. Lazer, *On second order, periodic, symmetric, differential systems having subharmonics of all sufficiently large orders*, J. Differential Equations **127** (1996), 426–438.

[41] J. Campos, A. Margheri, R. Martins and C. Rebelo, *A note on a modified version of the Poincaré-Birkhoff theorem*, J. Differential Equations **203** (2004), 55–63.

[42] A. Capietto, W. Dambrosio and D. Papini, *Superlinear indefinite equations on the real line and chaotic dynamics*, J. Differential Equations **181** (2002), 419–438.

[43] A. Capietto, J. Mawhin and F. Zanolin, *A continuation approach to superlinear periodic boundary value problems*, J. Differential Equations **88** (1990), 347–395.

[44] A. Capietto, J. Mawhin and F. Zanolin, *Continuation theorems for periodic perturbations of autonomous systems*, Trans. Amer. Math. Soc. **329** (1992), 41-72.

[45] M.C. Carbinatto, J. Kwapisz and K. Mischaikow, *Horseshoes and the Conley index spectrum*, Ergodic Theory Dynam. Systems **20** (2000), 365–377.

[46] P.H. Carter, *An improvement of the Poincaré-Birkhoff fixed point theorem*, Trans. Amer. Math. Soc. **269** (1982), 285–299.

[47] K.-C. Chang and M.-Y. Jiang, *Morse theory for indefinite nonlinear elliptic problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 139–158.

[48] J.Á. Cid and L. Sanchez, *Periodic solutions for second order differential equations with discontinuous restoring forces*, J. Math. Anal. Appl. **288** (2003), 349–364.

[49] C.V. Coffman and D.F. Ullrich, *On the continuation of solutions of a certain non-linear differential equation*, Monatsh. Math. **71** (1967), 385-392.

[50] C. Conley and E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*, Comm. Pure Appl. Math. **37** (1984), 207–253.

[51] F. Dalbono and C. Rebelo, *Poincaré-Birkhoff fixed point theorem and periodic solutions of asymptotically linear planar Hamiltonian systems*, Rend. Sem. Mat. Univ. Politec. Torino **60** (2002), 233–263.

[52] W. Dambrosio, *Time-map techniques for some boundary value problems*, Rocky Mountain J. Math. **28** (1998), 885–926.

[53] E.N. Dancer, *On the use of asymptotics in nonlinear boundary value problems*, Ann. Mat. Pura Appl. (4) **131** (1982), 167-185.

[54] E.N. Dancer and R. Ortega, *The index of Lyapunov stable fixed points in two dimensions*, J. Dynam. Differential Equations **6** (1994), 631–637.

[55] C. De Coster and P. Habets, Two-point Boundary Value Problems: Lower and Upper Solutions, Mathematics in Science and Engineering **205**, Elsevier B. V., Amsterdam, 2006.

[56] D.G. De Figueiredo, *Positive solutions of semilinear elliptic problems*, Differential equations (São Paulo, 1981), pp. 34–87, Lecture Notes in Math. **957**, Springer, Berlin-New York, 1982.

[57] J.P. Den Hartog, Mechanical Vibrations, Dover, New York, 1985.

[58] T. Ding, R. Iannacci and F. Zanolin, *Existence and multiplicity results for periodic solutions of semilinear Duffing equations*, J. Differential Equations **105** (1993), 364–409.

[59] T. Ding and F. Zanolin, *Subharmonic solutions of second order nonlinear equations: a time-map approach*, Nonlinear Anal. **20** (1993), 509–532.

[60] T. Ding and F. Zanolin, *Periodic solutions and subharmonic solutions for a class of planar systems of Lotka-Volterra type*, World Congress of Nonlinear Analysts '92, Vol. I-IV (Tampa, FL, 1992), 395–406, de Gruyter, Berlin, 1996.

[61] W.Y. Ding, *Fixed point of twist mappings and periodic solutions of ordinary differential equations*, Acta Math. Sinica **25** (1982), 227–235 (Chinese).

[62] W.Y. Ding, *A generalization of the Poincaré-Birkhoff theorem*, Proc. Amer. Math. Soc. **88** (1983), 341–346.

[63] N. Dunford and J. T. Schwartz, Linear Operators. I. General Theory, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London 1958.

[64] L.H. Erbe, S. Hu and H. Wang, *Multiple positive solutions of some boundary value problems*, J. Math. Anal. Appl. **184** (1994), 640–648.

[65] C. Fabry, *Periodic solutions of the equation* $x'' + f(t,x) = 0$, Séminaire de Mathématique **117** (1987), Louvain-la-Neuve.

[66] C. Fabry, *Landesman-Lazer conditions for periodic boundary value problems with asymmetric nonlinearities*, J. Differential Equations **116** (1995), 405–418.

[67] C. Fabry and A. Fonda, *Periodic solutions of perturbed isochronous Hamiltonian systems at resonance*, J. Differential Equations **214** (2005), 299–325.

[68] C. Fabry and P. Habets, *Periodic solutions of second order differential equations with superlinear asymmetric nonlinearities*, Arch. Math. (Basel) **60** (1993), 266-276.

[69] C. Fabry, J. Mawhin and M.N. Nkashama, *A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations*, Bull. London Math. Soc. **18** (1986), 173–180.

[70] M.L.C. Fernandes and F. Zanolin, *Repelling conditions for boundary sets using Liapunov-like functions. I. Flow-invariance, terminal value problem and weak persistence*, Rend. Sem. Mat. Univ. Padova **80** (1988), 95-116 (1989).

[71] A. Fonda, *Positively homogeneous Hamiltonian systems in the plane*, J. Differential Equations **200** (2004), 162–184.

[72] A. Fonda and M. Garrione, *Double resonance with Landesman-Lazer conditions for planar systems of ordinary differential equations*, J. Differential Equations **250** (2011), 1052–1082.

[73] A. Fonda and L. Ghirardelli, *Multiple periodic solutions of Hamiltonian systems in the plane*, Topol. Methods Nonlinear Anal. **36** (2010), 27-38.

[74] A. Fonda and J. Mawhin, *Planar differential systems at resonance*, Adv. Differential Equations **11** (2006), 1111–1133.

[75] A. Fonda and M. Ramos, *Large-amplitude subharmonic oscillations for scalar second-order differential equations with asymmetric nonlinearities*, J. Differential Equations, **109** (1994), 354–372.

[76] A. Fonda, M. Sabatini and F. Zanolin, *Periodic solutions of perturbed Hamiltonian systems in the plane by the use of the Poincaré-Birkhoff Theorem*, to appear in Topol. Methods Nonlinear Anal.

[77] A. Fonda, Z. Schneider and F. Zanolin, *Periodic oscillations for a nonlinear suspension bridge model*, J. Comput. Appl. Math. **52** (1994), 113–140.

[78] A. Fonda and A. Sfecci, *A general method for the existence of periodic solutions of differential systems in the plane*, J. Differential Equations **252** (2012), 1369–1391.

[79] A. Fonda and M. Willem, *Subharmonic oscillations of forced pendulum-type equations*, J. Differential Equations **81** (1989), 215–220.

[80] J. Franks, *Generalizations of the Poincaré-Birkhoff theorem*, Ann. of Math. (2) **128** (1988), 139–151.

[81] J. Franks, *Geodesics on S^2 and periodic points of annulus homeomorphisms*, Invent. Math. **108** (1992), 403–418.

[82] J. Franks, *Erratum to: "Generalizations of the Poincaré-Birkhoff theorem"*, Ann. of Math. (2) **164** (2006), 1097–1098.

[83] S. Fučik, Solvability of Nonlinear Equations and Boundary Value Problems, D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1980.

[84] J. Fura and S. Rybicki, *Periodic solutions of second order Hamiltonian systems bifurcating from infinity*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), 471–490.

[85] M. Furi, M.P. Pera and M. Spadini, *Multiplicity of forced oscillations for scalar differential equations*, Electron. J. Differential Equations 2001, **36**, 9 pp. (electronic).

[86] M. García-Huidobro, R. Manásevich and F. Zanolin, *A Fredholm-like result for strongly nonlinear second order ODEs*, J. Differential Equations **114** (1994), 132–167.

[87] M. Garrione, *Resonance at the first eigenvalue for first order systems in the plane: vanishing Hamiltonians and the Landesman-Lazer condition*, Differential Integral Equations **25** (2012), 505–526.

[88] L. Gasiński and N. Papageorgiou, *A multiplicity theorem for double resonant periodic problems*, Adv. Nonlinear Stud. **10** (2010), 819–836.

[89] M. Gaudenzi, P. Habets and F. Zanolin, *An example of a superlinear problem with multiple positive solutions*, Atti Sem. Mat. Fis. Univ. Modena **51** (2003), 259–272.

[90] M. Gaudenzi, P. Habets and F. Zanolin, *A seven-positive-solutions theorem for a superlinear problem*, Adv. Nonlinear Stud. **4** (2004), 149–164.

[91] P.M. Girão and J.M. Gomes, *Multibump nodal solutions for an indefinite superlinear elliptic problem*, J. Differential Equations **247** (2009), 1001–1012.

[92] C. Golé, Symplectic twist maps. Global variational techniques, Advanced Series in Nonlinear Dynamics **18**, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[93] R. Gómez-Reñasco and J. López-Gómez, *The effect of varying coefficients on the dynamics of a class of superlinear indefinite reaction-diffusion equations*, J. Differential Equations **167** (2000), 36–72.

[94] M. Greenberg, Lectures on Algebraic Topology, W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[95] L. Guillou, *Théorème de translation plane de Brouwer et généralisations du théorème de Poincaré-Birkhoff*, Topology **33** (1994), 331–351 (French).

[96] J.K. Hale, Ordinary Differential Equations. Second edition, Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980.

[97] P. Hartman, *On boundary value problems for superlinear second order differential equations*, J. Differential Equations **26** (1977), 37–53.

[98] M. Henrard and F. Zanolin, *Bifurcation from a periodic orbit in perturbed planar Hamiltonian systems*, J. Math. Anal. Appl. **277** (2003), 79–103.

[99] H. Jacobowitz, *Periodic solutions of $x'' + f(t, x) = 0$ via the Poincaré-Birkhoff theorem*, J. Differential Equations **20** (1976), 37–52.

[100] H. Jacobowitz, *Corrigendum: The existence of the second fixed point: a correction to "Periodic solutions of $x'' + f(x, t) = 0$ via the Poincaré-Birkhoff theorem"*, J. Differential Equations **25** (1977), 148–149.

[101] J. Kennedy and J.A. Yorke, *Topological horseshoes*, Trans. Amer. Math. Soc. **353** (2001), 2513–2530.

[102] I.T. Kiguradze, *A note on the oscillation of solutions of the equation $u'' + a(t)|u|^n \operatorname{sgn} u = 0$*, Časopis Pěst. Mat. **92** (1967), 343–350 (Russian).

[103] U. Kirchgraber and D. Stoffer, *On the definition of chaos*, Z. Angew. Math. Mech. **69** (1989), 175–185.

[104] M.A. Krasnosel'skiĭ, The Operator of Translation along the Trajectories of Differential Equations, Translations of Mathematical Monographs, Vol. **19**, American Mathematical Society, Providence, R.I. 1968.

[105] K.Q. Lan and J.R.L. Webb, *Positive solutions of semilinear differential equations with singularities*, J. Differential Equations **148** (1998), 407–421.

[106] E.M. Landesman and A.C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. **19** (1970), 609–623.

[107] A.C. Lazer and D.E. Leach, *Bounded perturbations of forced harmonic oscillators at resonance*, Ann. Mat. Pura Appl. (4) **82** (1969), 49–68.

[108] P. Le Calvez and J. Wang, *Some remarks on the Poincaré-Birkhoff theorem*, Proc. Amer. Math. Soc. **138** (2010), 703–715.

[109] J. Leray et J. Schauder, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup. (3) **51** (1934), 45–78 (French).

[110] P.-L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev. **24** (1982), 441–467.

[111] Z. Liu, J. Su and Z.-Q. Wang, *A twist condition and periodic solutions of Hamiltonian systems*, Adv. Math. **218** (2008), 1895–1913.

[112] Y. Long, *Maslov-type index, degenerate critical points, and asymptotically linear Hamiltonian systems*, Sci. China Ser. A **33** (1990), 1409–1419.

[113] Y. Long, Index Theory for Symplectic Paths with Applications, Progress in Mathematics **207**, Birkhäuser Verlag, Basel, 2002.

[114] J. López-Gómez, *Varying bifurcation diagrams of positive solutions for a class of indefinite superlinear boundary value problems*, Trans. Amer. Math. Soc. **352** (2000), 1825–1858.

[115] R. Ma, *Bifurcation from infinity and multiple solutions for periodic boundary value problems*, Nonlinear Anal. **42** (2000), 27–39.

[116] R. Ma, C. Gao and R. Chen, *Existence of positive solutions of nonlinear second-order periodic boundary value problems*, Bound. Value Probl. **2010**, 18 pp.

[117] R. Manásevich, F.I. Njoku and F. Zanolin, *Positive solutions for the one-dimensional p-Laplacian*, Differential Integral Equations **8** (1995), 213–222.

[118] A. Margheri, C. Rebelo and F. Zanolin, *Maslov index, Poincaré-Birkhoff theorem and periodic solutions of asymptotically linear planar Hamiltonian systems*, J. Differential Equations **183** (2002), 342–367.

[119] R. Martins and A.J. Ureña, *The star-shaped condition on Ding's version of the Poincaré-Birkhoff theorem*, Bull. Lond. Math. Soc. **39** (2007), 803–810.

[120] J. Mawhin, *Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces*, J. Differential Equations **12** (1972), 610–636.

[121] J. Mawhin, *Recent results on periodic solutions of the forced pendulum equation*, Rend. Istit. Mat. Univ. Trieste **19** (1987), 119-129.

[122] J. Mawhin, *Topological degree and boundary value problems for nonlinear differential equations*, Topological methods for ordinary differential equations (Montecatini Terme, 1991), Lecture Notes in Math. **1537**, Springer, Berlin, (1993), 74-142.

[123] J. Mawhin, *Boundary value problems for nonlinear ordinary differential equations: from successive approximations to topology*, Development of mathematics 1900-1950 (Luxembourg, 1992), Birkhäuser, Basel, (1994), 443–477.

[124] J. Mawhin, *Global results for the forced pendulum equation*, Handbook of differential equations, Elsevier/North-Holland, Amsterdam, (2004), 533–589.

[125] J. Mawhin and K. Schmitt, *Nonlinear eigenvalue problems with the parameter near resonance*, Ann. Polon. Math. **51** (1990), 241–248.

[126] J. Mawhin and J.R. Ward, *Nonresonance and existence for nonlinear elliptic boundary value problems*, Nonlinear Anal. **6** (1981), 677–684.

[127] J. Mawhin and M. Willem, *Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations*, J. Differential Equations **52** (1984), 264-287.

[128] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences **74**, Springer-Verlag, New York, 1989.

[129] A. Medio, M. Pireddu and F. Zanolin, *Chaotic dynamics for maps in one and two dimensions: a geometrical method and applications to economics*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **19** (2009), 3283–3309.

[130] R. Michalek and G. Tarantello, *Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems*, J. Differential Equations **72** (1988), 28–55.

[131] K. Mischaikow, M. Mrozek and A. Szymczak, *Chaos in the Lorenz equations: a computer assisted proof. III. Classical parameter values*, J. Differential Equations **169** (2001), 17–56.

[132] J. Moser and E.J. Zehnder, Notes on dynamical systems, Courant Lecture Notes in Mathematics **12**, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2005.

[133] Z. Opial, *Sur les périodes des solutions de l'équation différentielle* $x'' + g(x) = 0$, Ann. Polon. Math **10** (1961), 49–72 (French).

[134] R. Ortega, *The twist coefficient of periodic solutions of a time-dependent Newton's equation*, J. Dynam. Differential Equations **4** (1992), 651-665.

[135] R. Ortega, *Periodic solutions of a Newtonian equation: stability by the third approximation*, J. Differential Equations **128** (1996), 491-518.

[136] R. Ortega and A. Rivera, *Global bifurcation from the center of mass in the Sitnikov problem*, Discrete Contin. Dyn. Syst. Ser. B **14** (2010), 719–732.

[137] D. Papini and F. Zanolin, *A topological approach to superlinear indefinite boundary value problems*, Topol. Methods Nonlinear Anal. **15** (2000), 203–233.

[138] D. Papini and F. Zanolin, *Periodic points and chaotic-like dynamics of planar maps associated to nonlinear Hill's equations with indefinite weight*, Georgian Math. J. **9** (2002), 339–366.

[139] D. Papini and F. Zanolin, *Fixed points, periodic points, and coin-tossing sequences for mappings defined on two-dimensional cells*, Fixed Point Theory Appl. **2004** (2004), 113–134.

[140] D. Papini and F. Zanolin, *On the periodic boundary value problem and chaotic-like dynamics for nonlinear Hill's equations*, Adv. Nonlinear Stud. **4** (2004), 71–91.

[141] A. Pascoletti, M. Pireddu and F. Zanolin, *Multiple periodic solutions and complex dynamics for second order ODEs via linked twist maps*, Proceedings of the 8th Colloquium on the Qualitative Theory of Differential Equations (Szeged, 2007), Electron. J. Qual. Theory Differ. Equ., Szeged, **14** (2008), 1–32.

[142] A. Pascoletti and F. Zanolin, *Chaotic dynamics in periodically forced asymmetric ordinary differential equations*, J. Math. Anal. Appl. **352** (2009), 890–906.

[143] A. Pascoletti and F. Zanolin, *Example of a suspension bridge ODE model exhibiting chaotic dynamics: a topological approach*, J. Math. Anal. Appl. **339** (2008), 1179–1198.

[144] H. Poincaré, *Sur un théorème de géométrie*, Rend. Circ. Mat. Palermo **33** (1912), 375–407.

[145] S.I. Pohozaev and A. Tesei, *Existence and nonexistence of solutions of nonlinear Neumann problems*, SIAM J. Math. Anal. **31** (1999), 119–133.

[146] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Functional Analysis **7** (1971), 487–513.

[147] P.H. Rabinowitz, *Pairs of positive solutions of nonlinear elliptic partial differential equations*, Indiana Univ. Math. J. **23** (1973/74), 173–186.

[148] P.H. Rabinowitz, *A note on pairs of solutions of a nonlinear Sturm-Liouville problem*, Manuscripta Math. **11** (1974), 273–282.

[149] P.H. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. **31** (1978), 157–184.

[150] C. Rebelo, *A note on the Poincaré-Birkhoff fixed point theorem and periodic solutions of planar systems*, Nonlinear Anal. **29** (1997), 291–311.

[151] C. Rebelo, *A note on uniqueness of Cauchy problems associated to planar Hamiltonian systems*, Portugal. Math. **57** (2000), 415–419.

[152] C. Rebelo and F. Zanolin, *Multiplicity results for periodic solutions of second order ODEs with asymmetric nonlinearities*, Trans. Amer. Math. Soc. **348** (1996), 2349–2389.

[153] E. Serra, M. Tarallo and S. Terracini, *Subharmonic solutions to second-order differential equations with periodic nonlinearities*, Nonlinear Anal. **41** (2000), 649–667.

[154] S. Smale, *Finding a horseshoe on the beaches of Rio*, Math. Intelligencer **20** (1998), 39–44.

[155] R. Srzednicki and K. Wójcik, *A geometric method for detecting chaotic dynamics*, J. Differential Equations **135** (1997), 66–82.

[156] J. Su and L. Zhao, *Multiple periodic solutions of ordinary differential equations with double resonance*, Nonlinear Anal. **70** (2009), 1520–1527.

[157] C.-L. Tang, *Periodic solutions for nonautonomous second systems with sublinear nonlinearity*, Proc. Amer. Math. Soc. **126** (1998), 3263–3270.

[158] S. Terracini and G. Verzini, *Oscillating solutions to second-order ODEs with indefinite superlinear nonlinearities*, Nonlinearity **13** (2000), 1501–1514.

[159] P.J. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations **190** (2003), 643–662.

[160] A.J. Ureña, *Dynamics of periodic second-order equations between an ordered pair of lower and upper solutions*, Adv. Nonlinear Stud. **11** (2011), 675–694.

[161] P. Walters, An Introduction to Ergodic Theory, Graduate Texts in Mathematics **79**, Springer-Verlag, New York-Berlin, 1982.

[162] P. Waltman, *An oscillation criterion for a nonlinear second order equation*, J. Math. Anal. Appl. **10** (1965), 439–441.

[163] J.R. Ward, *Periodic solutions of ordinary differential equations with bounded nonlinearities*, Topol. Methods Nonlinear Anal. **19** (2002), 275–282.

[164] H.E. Winkelnkemper, *A generalization of the Poincaré-Birkhoff theorem*, Proc. Amer. Math. Soc. **102** (1988), 1028–1030.

[165] J.S.W. Wong, *Oscillation criteria for second order nonlinear differential equations involving general means*, J. Math. Anal. Appl. **247** (2000), 489–505.

[166] J. Yu, *The minimal period problem for the classical forced pendulum equation*, J. Differential Equations **247** (2009), 672–684.

[167] C. Zanini, *Rotation numbers, eigenvalues, and the Poincaré-Birkhoff theorem*, J. Math. Anal. Appl. **279** (2003), 290–307.

[168] C. Zanini and F. Zanolin, *A multiplicity result of periodic solutions for parameter dependent asymmetric non-autonomous equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **12** (2005), 343–361.

[169] F. Zanolin, *Remarks on multiple periodic solutions for nonlinear ordinary differential systems of Liénard type*, Boll. Un. Mat. Ital. B (6) **1** (1982), 683–698.

[170] F. Zanolin, *Continuation theorems for the periodic problem via the translation operator*, Rend. Sem. Mat. Univ. Politec. Torino **54** (1996), 1–23.

[171] P. Zgliczyński, *Fixed point index for iterations of maps, topological horseshoe and chaos*, Topol. Methods Nonlinear Anal. **8** (1996), 169–177.

[172] P. Zgliczyński and M. Gidea, *Covering relations for multidimensional dynamical systems*, J. Differential Equations **202** (2004), 32–58.

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