SISSA, International School for Advanced Studies

PhD course in Geometry

Academic Year 2011/2012

# Topics on Multiphase Solutions of the Focusing Nonlinear Schrödinger Equation

Thesis submitted for the degree of

*Doctor Philosophiae*

**Advisor:**
**Prof. Boris Dubrovin**

**Candidate:**
**Pietro Giavedoni**

*November 16th, 2012*

To My Family and to Sanja.

# Contents

# Chapter 1

# Introduction

The nonlinear Schroedinger Equation of focusing type (often abbreviated fNLS) reads as follows:

$$i\psi_t + \psi_{xx} + 2\left|\psi\right|^2 \psi = 0, \qquad x, t \in \mathbb{R}, \quad \psi \in \mathbb{C}. \tag{1.1}$$

The physical applications of this equation constitute by themselves a full justification for its study. The usage of this equation in the theory of ocean waves dates back to 1968 ([19]). Its employment in nonlinear optics started even earlier ([4]).
Beyod its importance in modelling natural phenomena, fNLS equation exhibits the noticeable feature to allow an explicit analytical treatment: In 1971 Zakharov and Shabat found a Lax pair for this equation, which turns out to be solvable by means of Inverse Scattering Method. Later on fNLS was discovered to admit quasi-periodic solutions with $g$ independent phases, for any integer $g$, based on the Riemann theta-function of $g$ variables (see [11]). These can be obtained by means of the algebro-geometric integration procedure for nonlinear PDEs.

### The multi-phase solutions

Let us recall here the construction of multi-phase solutions. Let $g > 0$ be any positive integer. Consider the hyperelliptic Riemann surface associated to the algebraic curve

$$\Gamma : \quad \mu^2 = \prod_{j=1}^{g+1} \left(\lambda - E_j\right)\left(\lambda - \overline{E}_j\right), \tag{1.2}$$

where

$$\operatorname{Im}\left(E_j\right) > 0; \qquad E_j \neq E_k \quad \text{if} \quad j \neq k. \tag{1.3}$$

($\overline{E}_j$ is understood to indicate the complex conjugate of $E_j$). Such a Riemann surface is naturally endowed with an anti-holomorphic involution

$$r : (\mu, \lambda) \to \left(\overline{\mu}, \overline{\lambda}\right), \qquad (\mu, \lambda) \in \Gamma. \tag{1.4}$$

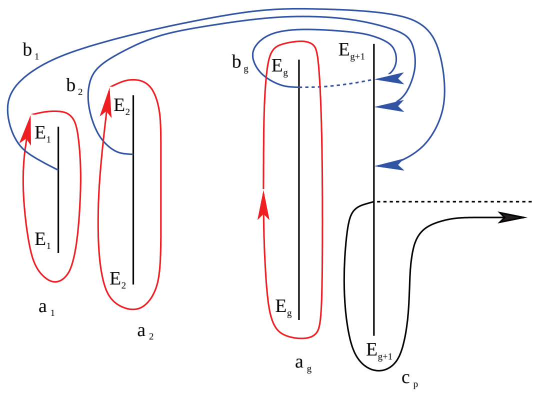



Figure 1.1: *A basis in the homology of* $\Gamma$

In view of this, $\Gamma$ is said to be a real Riemann surface. A standard projection is defined on $\Gamma$:

$$\pi : \Gamma \longrightarrow \mathbb{CP}^1 \tag{1.5}$$

$$(\mu, \lambda) \longrightarrow \lambda \tag{1.6}$$

This defines $\Gamma$ as a two-sheeted covering of $\mathbb{CP}^1$. There are exactly two points $\infty^{\pm} \in \Gamma$ such that

$$\pi\left(\infty^{\pm}\right) = \infty \in \mathbb{CP}^1.$$

As a consequence, one can represent $\Gamma$ as two copies of the compactified complex plane with a certain system of cuts. (One of them is indicated in black in figure 1.1). The coordinate $\mu$ is continuous on each copy of the complex plane with the cuts removed. By our convention, the first sheet is the one on which

$$\mu \sim \lambda^{g+1} \quad \text{as} \quad \lambda \to \infty. \tag{1.7}$$

In order to give an explicit expression to the multi-phase solutions it is necessary to fix a basis in the homology of the surface $\Gamma$. We remark here that this choice doesn't affect the solutions, but only their theta-functional expression.
Following [2] we fix the basis indicated in figure 1.1. It is clear that this one satisfies the defining relations

$$a_i \circ b_j = \delta_{ij}, \qquad a_j \circ a_k = b_j \circ b_k = 0. \tag{1.8}$$

The symbol $\circ$ indicates here the intersection pairing defined in the first homology group of $\Gamma$.
With this specification one can consider the normalized holomorphic differentials

$$\omega_1, \omega_2, \cdots, \omega_g.$$

These are uniquely determined by the conditions

$$\oint_{a_k} \omega_j = 2\pi i \delta_{k,j}, \qquad k, j = 1, 2, \cdots, g. \tag{1.9}$$

The period matrix $\boldsymbol{\tau}$ of $\Gamma$ is a symmetric , $g$-dimensional, complex matrix defined as follows:

$$\tau_{j,k} = \oint_{b_k} \omega_j, \qquad j,k = 1,2,\cdots g. \tag{1.10}$$

This object allows one to construct the corresponding theta function:

$$\theta_{\boldsymbol{\tau}}(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp\left\{\frac{1}{2}\mathbf{m}^T\cdot\boldsymbol{\tau}\cdot\mathbf{m} + \mathbf{m}^T\cdot\mathbf{z}\right\} \tag{1.11}$$
$$\mathbf{m}^T\cdot\mathbf{z} := m_1 z_1 + m_2 z_2 + \cdots + m_g z_g.$$

This function is entire on $\mathbb{C}^g$. Let us fix a local parameter $z$ in the neighborhood of $\infty^+$ and $\infty^-$:

$$z := \frac{1}{\lambda} \qquad P \text{ close to } \infty^\pm. \tag{1.12}$$

This allows one to define the abelian differentials $\mathrm{d}\Omega_1, \mathrm{d}\Omega_2$ and $\mathrm{d}\Omega_3$ imposing the following conditions

- $$\oint_{a_j} \mathrm{d}\Omega_k = 0 \qquad j = 1,2;\ k = 1,2,3. \tag{1.13}$$

- $$\mathrm{d}\Omega_1(P) = \mp\left[\frac{1}{z^2} + \mathcal{O}(1)\right]\mathrm{d}z, \quad P \to \infty^\pm \tag{1.14a}$$
$$\mathrm{d}\Omega_2(P) = \mp\left[\frac{4}{z^3} + \mathcal{O}(1)\right]\mathrm{d}z, \quad P \to \infty^\pm \tag{1.14b}$$
$$\mathrm{d}\Omega_3(P) = \mp\left[\frac{1}{z} + \mathcal{O}(1)\right]\mathrm{d}z, \quad P \to \infty^\pm \tag{1.14c}$$

  where $z = \frac{1}{\lambda}$.

- $\mathrm{d}\Omega_k(P)$ is holomorphic on $\Gamma\backslash\{\infty^+,\infty^-\}$, for $k = 1,2,3$.

Notice that in order to normalize $\mathrm{d}\Omega_3$ one needs to consider a basis in the homology of $\Gamma\backslash\{\infty^+,\infty^-\}$. Let us introduce the vectors

$$V_j = \oint_{b_j}\mathrm{d}\Omega_1, \qquad W_j = \oint_{b_j}\mathrm{d}\Omega_2, \qquad r_j = \int_{C_p}\omega_j, \tag{1.15}$$

$j = 1,2$. Moreover, let the quantities $E, N$ and $\chi$ be defined by means of the asymptotic relations

$$\int_{\overline{E}_3}^P \mathrm{d}\Omega_1 = \pm\left[\lambda - \frac{E}{2} + o(1)\right], \quad P \to \infty^\pm \tag{1.16a}$$
$$\int_{\overline{E}_3}^P \mathrm{d}\Omega_2 = \pm\left[2\lambda^2 + \frac{N}{2} + o(1)\right], \quad P \to \infty^\pm \tag{1.16b}$$
$$\int_{\overline{E}_3}^P \mathrm{d}\Omega_3 = \pm\left[\log(\lambda) - \frac{1}{2}\log(\chi) + o(1)\right], \quad P \to \infty^\pm \tag{1.16c}$$

The function

$$\psi(x,t) = 2\sqrt{-\chi}\frac{\theta_{\boldsymbol{\tau}}(i\mathbf{V}x + i\mathbf{W}t - \mathbf{D} + \mathbf{r})}{\theta_{\boldsymbol{\tau}}(i\mathbf{V}x + i\mathbf{W}t - \mathbf{D})}\exp(-iEx + iNt) \tag{1.17}$$

is a solution to (1.1) provided that the vector $\mathbf{D}$ has the form

$$\mathbf{D} = \begin{pmatrix} i\Delta_1 \\ \vdots \\ i\Delta_g \end{pmatrix}, \quad \Delta_1, \Delta_2, \cdots, \Delta_g \in \mathbb{R} \tag{1.18}$$

## Critical Aspects

In view of the physical importance of fNLS, "explicit" solutions of the type (1.17) seem to be very promising for applications. Nevertheless, their implementation in applied sciences is still far from being easy. Two of the main difficulties in this sense are the following:

**i** These solutions are parametrized by the space of Riemann surfaces endowed with a local parameter in the vicinity of $\infty$. These are quite abstract geometrical objects, not so comfortable to deal with, especially for applied scientists with no background in algebraic geometry. Moreover, this parametrization is redundant in the sense that two different elements of this space can generate the same family of solutions. It is hard to decide when this is the case.

**ii** The numerical computation of (1.17) contains the difficulty of evaluating a multi-dimensional theta-function. Such a difficulty might seem surprising , since the Fourrier coefficients in the theta-series (1.11) eventually decrease faster than exponentially. Nevertheless this problem was put in evidence by Dubrovin, Flickinger and Segur in [7], by means of a numerical example. Its origin lies in the necessity to evaluate a quadratic form over the whole $\mathbb{Z}^g$ in the series (1.11). Selecting the $g$-tuples which give the main contribution turns out to be a delicate issue.

## The Novikov Effectivization of the Formulas

In order to improve point **i**. above Novikov proposed what he called "Effectivization of the formulas for the multi-phase solutions" (see [6]). We will give a precise formulation of his strategy in the case when the genus of $\Gamma$ is two. It is always possible to fix a basis

$$\mathcal{B} = \{a_1, a_2; b_1, b_2\} \tag{1.19}$$

in the homology of $\Gamma$ such that

$$r_\star(a_1) = -a_2, \qquad r_\star(b_1) = b_2. \tag{1.20}$$

The two-phase solutions constructed using this kind of bases have the form[1]

$$\psi(x,t) = A\frac{\theta_\tau(i\mathbf{V}x + i\mathbf{W}t + \boldsymbol{\tau}\cdot\mathbf{p} - 2\pi i\mathbf{q} + \mathbf{r})}{\theta_\tau(i\mathbf{V}x + i\mathbf{W}t + \boldsymbol{\tau}\cdot\mathbf{p} - 2\pi i\mathbf{q})}\exp(-iEx + iNt)\,. \tag{1.21}$$

[1] A detailed justification of this formula can be found in chapter 2, section 2.

In this case, the two-dimensional, complex vectors $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{r}$ have the form

$$\mathbf{V} = \begin{pmatrix} \overline{V} \\ V \end{pmatrix}, \qquad \mathbf{W} = \begin{pmatrix} \overline{W} \\ W \end{pmatrix}, \qquad \mathbf{r} = \begin{pmatrix} \overline{r} \\ r \end{pmatrix}. \tag{1.22}$$

The quantity $E$ and $N$ are real numbers, $A$ is positive

$$E,\ N\ \in\ \mathbb{R}, \qquad A\ >\ 0. \tag{1.23}$$

The period matrix $\boldsymbol{\tau}$ has the form

$$\boldsymbol{\tau} = \begin{bmatrix} \delta + i\gamma & \beta \\ \beta & \delta - i\gamma \end{bmatrix} \qquad \gamma, \delta, \beta \in \mathbb{R}. \tag{1.24}$$

Finally the vectors $\mathbf{p}$ and $\mathbf{q}$ have the form

$$\mathbf{p} = \begin{pmatrix} p \\ -p \end{pmatrix} \qquad \mathbf{q} = \begin{pmatrix} q \\ q \end{pmatrix} \qquad p,\ q\ \in \mathbb{R}. \tag{1.25}$$

(the value of $p$ and $q$ can be arbitrarily chosen). Novikov's idea is the following: Let $\boldsymbol{\tau}$ be the period matrix of some Riemann surface $\Gamma$ as above. Consider a function of $x$ and $t$ of the form (1.21). Forget about the algebro-geometric origin of the parameters appearing in it, the symmetries (1.22) and (1.23) still holding true. Imposing that this is actually a solution for fNLS yields a set of constraints for the quantities

$$V,\ W,\ r,\ A,\ E,\ N. \tag{1.26}$$

One can then select out of these parameters some independent ones which determine all the others. In this way one obtains a parametrization of the two-phase solutions directly in terms of the period matrix $\boldsymbol{\tau}$ together with a finite set of numbers. Algebraic geometry is no more involved in it. In this sense, the new parametrization is, so to say, more effective.

The material contained in this thesis rose from our efforts to put in act this project for the two-phase solutions of the fNLS equation.

## Non-Redundancy Issue and Fundamental Domains

The procedure formulated above exhibits some delicate issues. One of these concerns the non-redundancy of the new parametrization. Indeed, in general there exists more than one basis $\mathcal{B}$ in the homology of $\Gamma$ satisfying relations (1.20). On the other side, the solution (1.21) does not depend on the specific basis in the homology used to construct it. Or, more precisely, only the form of its theta functional representation does. While trying to make the new parametrization non-redundant one is led to the following problem: for every real Riemann surface $\Gamma$ of genus two as above, select a canonical period matrix out of all the ones obtained from bases in the homology of the form (1.19-1.20).

Let us formulate this last issue in some more technical terms. In order to simplify our notations let us introduce

$$\mathbf{w} := \frac{1}{2\pi i}\boldsymbol{\tau}. \tag{1.27}$$

This is actually the period matrix computed with respect to a different normalization of the holomorphic differentials[2]. Let us introduce the group

$$\mathbb{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1 \tag{1.28}$$

where

$$\mathcal{G}_0 := \left\{ \begin{bmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ e & 0 & g & 0 \\ 0 & -e & 0 & g \end{bmatrix}, \; ag - ec = 1; \;\; a, c, e, g \in \mathbb{Z} \right\} \tag{1.29}$$

and

$$\mathcal{G}_1 := \left\{ \begin{bmatrix} 0 & b & 0 & d \\ b & 0 & -d & 0 \\ 0 & f & 0 & h \\ -f & 0 & h & 0 \end{bmatrix}, \; bh - fd = 1; \;\; b, d, f, h \in \mathbb{Z} \right\} \tag{1.30}$$

Notice that $\mathbb{G}$ is a $\mathbb{Z}_2$-graded group. In particular, only $\mathcal{G}_0$ is a (normal) subgroup of $\mathbb{G}$. Consider two bases $\mathcal{B}$ and $\tilde{\mathcal{B}}$ in the homology of $\Gamma$. Let $\mathcal{B}$ satisfy the constraints (1.20). Suppose, moreover, that there exists a matrix

$$G = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \mathbb{G} \tag{1.31}$$

such that

$$\begin{pmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{b}} \end{pmatrix} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}. \tag{1.32}$$

Then $\tilde{\mathcal{B}}$ also satisfies (1.20). Viceversa, let us assume that that both $\mathcal{B}$ and $\tilde{\mathcal{B}}$ satisfy (1.20). Then (1.32) holds for some $G$ belonging to $\mathbb{G}$. Let us introduce the space

$$\mathcal{W} = \left\{ \mathbf{w} = \begin{bmatrix} \tilde{\gamma} + i\tilde{\delta} & i\tilde{\beta} \\ i\tilde{\beta} & -\tilde{\gamma} + i\tilde{\delta} \end{bmatrix}, \mathrm{Im}\,(\mathbf{w}) \text{ positive definite} \right\}. \tag{1.33}$$

All the period matrices $\mathbf{w}$ computed w.r.t. basis in the homology of the form (1.19-1.20) belong to this set. The group $\mathbb{G}$ acts on the space $\mathcal{W}$ in the following way:

$$\mathfrak{M} : \mathbb{G} \times \mathcal{W} \to \mathcal{W} \tag{1.34}$$

$$\mathfrak{M}\,(G, \mathbf{w}) = (P\mathbf{w} + Q)\,(R\mathbf{w} + S)^{-1} \tag{1.35}$$

The matrix $G$ is understood to be of the form (1.31). If two bases satisfy (1.32) then the corresponding period matrices $\mathbf{w}$ and $\tilde{\mathbf{w}}$ are related to each other as follows

$$\tilde{\mathbf{w}} = \mathfrak{M}\,(G, \mathbf{w}). \tag{1.36}$$

[2]In view of this fact, both $\boldsymbol{\tau}$ and $\mathbf{w}$ will be called "period matrix". Every ambiguity will be easily cleared up by the context

In other words, the set of period matrices coming from bases of the form (1.19-1.20) coincides with an orbit of the group $\mathbb{G}$ in the space $\mathcal{W}$. The issue of non-redundancy for the effective parametrization proposed by Novikov reduces to the problem of selecting a canonical representative for each of these orbits. This can be formulated as the quest of a fundamental domain for the modular action of the group $\mathbb{G}$ on the space $\mathcal{W}$. Its definition is recalled here below:

**Definition.** *We say that two points* $\mathbf{w}$ *and* $\mathbf{w}'$ *of* $\mathcal{W}$ *are equivalent if there exists* $G \in \mathbb{G}$ *such that*

$$\mathbf{w}' = \mathfrak{M}\left(G, \mathbf{w}\right). \tag{1.37}$$

*A fundamental domain for* $\mathfrak{M}$ *is a closed subset* $\mathcal{D}$ *of* $\mathcal{W}$ *whose interior part is connected, which satisfies the following three properties:*

**i**_ *For every* $\mathbf{w} \in \mathcal{W}$ *there exists an equivalent point* $\mathbf{w}'$ *belonging to* $\mathcal{D}$.

**ii**_ *If two distinct points,* $\mathbf{w}$ *and* $\mathbf{w}'$*, belonging to* $\mathcal{D}$*, are equivalent, then they both belong to the boundary of* $\mathcal{D}$.

**iii**_ *Every set of equivalent points contained in* $\mathcal{D}$ *has a finite number of elements.*

We solve this problem in chapter 1.

**Theorem.** *A fundamental domain for the action of* $\mathbb{G}$ *on* $\mathcal{W}$ *is given by*

$$\mathcal{D} : \begin{cases} \tilde{\delta} \geq \sqrt{\tilde{\beta}^2 + 1 - \tilde{\gamma}^2} \\ 0 \leq \tilde{\gamma} \leq \frac{1}{2} \end{cases} \tag{1.38}$$

*(The square root in the first inequality is understood to be positive.)*

Moreover, a deeper understanding of the mathematical structure is possible in this case:

**Theorem.** *Introduce, on* $\mathcal{W}$*, the new system of coordinates*

$$\mathcal{I}\left(\mathbf{w}\right) = \frac{\tilde{\delta} - \tilde{\beta}}{\tilde{\delta} + \tilde{\beta}}, \qquad \chi\left(\mathbf{w}\right) = \tilde{\gamma} + \imath\sqrt{\tilde{\delta}^2 - \tilde{\beta}^2}, \qquad \mathbf{w} \in \mathcal{W}. \tag{1.39}$$

*Now consider a modular transformation*

$$\tilde{\mathbf{w}} = \mathfrak{M}\left(G, \mathbf{w}\right) \qquad \mathbf{w} \in \mathcal{W},\ G \in \mathbb{G}. \tag{1.40}$$

*In the new coordinates it acts as follows:*

$$\mathcal{I}\left(\tilde{\mathbf{w}}\right) = \mathcal{I}\left(\mathbf{w}\right) \tag{1.41}$$

*and*

$$\chi\left(\tilde{\mathbf{w}}\right) = \frac{a\chi\left(\mathbf{w}\right) + c}{e\chi\left(\mathbf{w}\right) + g} \tag{1.42}$$

*if* $G$ *belongs to* $\mathbb{G}_+$*, or*

$$\chi\left(\tilde{\mathbf{w}}\right) = \frac{b\overline{\chi\left(\mathbf{w}\right)} - d}{f\overline{\chi\left(\mathbf{w}\right)} - h} \tag{1.43}$$

*if* $G$ *belongs to* $\mathbb{G}_-$*, instead.*

Notice that $\mathcal{I}$ is not affected by the modular action of $\mathbb{G}$, which concentrates only on the second coordinate $\chi$. This last one just undergoes a Moebius transformation, in some cases composed with a complex conjugation.
We also consider the same problem for generic real Riemann surfaces of separated type with just one oval. Their genus is necessarily an even[3] number $2g_0$. We formulate a solution procedure for the general case and work out the calculations in full detail when $2g_0 = 4$. Moreover, we exhibit the analogue of the invariant coordinate $\mathcal{I}$ for all genera. The geometrical meaning of this quantity has not yet been completely understood and it is meant to be part of a work in progress.

This material eventually resulted into a paper which was submitted for a pubblication to IMRN (see [10]). We point out that a different approach to this same problem can be found in [18]. In view of the specific symmetries (1.20) our result turns out to be more convenient for the effectivization of the two-phase solutions.

In [7] Dubrovin, Flickinger and Segur shew that the determination of a fundamental domain for the period matrices can be applied to solve the difficulties rising from the numerical evaluation of the multi-dimensional theta-function (see point **ii** above). This also constituted part of our motivation to tackle this problem. Such an application in the case of fNLS, though, is part of a work in progress and it doesn't appear in this thesis.

### Degenerate Solutions and Local Effectivization

There is another delicate issue concerning the effectivization of the two-phase solutions. After Novikov prescription one should impose that (1.21) is a solution to fNLS equation and so obtain some constraints for the quantities (1.26). Unfortunately, though, these constraints have the form of transcendental equations which are rather difficult to solve. In view of this critical aspect, in chapter 2 and 3 we put in act a different, someway less ambitious approach to the effectivization of the two-phase solutions. Let us formulate it here in some detail.

Our idea consists in not considering all the period matrices, but only those who lie in a certain zone of the fundamental domain. To perform, so to say, a local effectivization. More precisely, we will consider the period matrices of the form (1.24) for which $\delta$ is (negative and) sufficiently large in module:

$$\delta << 0. \tag{1.44}$$

With this purpose, as a first step we consider a configuration of the branch points which yields period matrices satisfying the additional assumption (1.44). A possible one is the following:
Fix $E_1$ and $E_3$ such that

$$\mathrm{Im}\,(E_1) > 0, \qquad \mathrm{Im}\,(E_3) > 0, \qquad \mathrm{Re}\,(E_1) < \mathrm{Re}\,(E_3)$$

Let $\epsilon > 0$ be a small parameter and put

$$E_2 = E_1 + \epsilon, \qquad \overline{E}_2 = \overline{E}_1 + \epsilon. \tag{1.45}$$

[3] see [5], theorem 1.5.3

Consider the family of $\epsilon$-dependent Riemann surfaces

$$\Gamma_\epsilon \quad : \quad \mu^2 = \prod_{j=1}^{3} \left(\lambda - E_j\right)\left(\lambda - \overline{E}_j\right). \tag{1.46}$$

As we recalled above, to this family of curves there corresponds a family of solutions to fNLS of the form

$$\begin{aligned} \psi\left(x,t;p,q;\epsilon\right) =& A\left(\epsilon\right)\frac{\theta_{\boldsymbol{\tau}(\epsilon)}\left(i\mathbf{V}\left(\epsilon\right)x + i\mathbf{W}\left(\epsilon\right)t + \boldsymbol{\tau}\left(\epsilon\right)\cdot\mathbf{p} - 2\pi i\mathbf{q} + \mathbf{r}\left(\epsilon\right)\right)}{\theta_{\boldsymbol{\tau}(\epsilon)}\left(i\mathbf{V}\left(\epsilon\right)x + i\mathbf{W}\left(\epsilon\right)t + \boldsymbol{\tau}\left(\epsilon\right)\cdot\mathbf{p} - 2\pi i\mathbf{q}\right)} \\ &\times \exp\left(-iE\left(\epsilon\right)x + iN\left(\epsilon\right)t\right). \end{aligned} \tag{1.47}$$

Notice that all the algebro-geometric parameters depend now on $\epsilon$. In chapter 2 we study in detail the limit of (1.47) as $\epsilon$ tends to zero. It turns out that $\psi$ exhibits two different qualitative behaviours. When $p$ belongs to the open interval $\left(-\frac{1}{2};\frac{1}{2}\right)$ the solution $\psi$ tends to a plane wave. If instead

$$p = \frac{1}{2}$$

then

$$\lim_{\epsilon\to 0^+}\psi\left(x,t;\frac{1}{2},q;\epsilon\right) = A_0\frac{\cosh\left(\eta x + \phi t - i\sigma\right) + B\cos\left(\xi x + \theta t - 2\pi q - i\rho\right)}{\cosh\left(\eta x + \phi t\right) + B\cos\left(\xi x + \theta t - 2\pi q\right)} \tag{1.48}$$

$$\times \exp\left(-iE_0 x + iN_0 t\right). \tag{1.49}$$

All the parameters in this formula are expressed in terms of $E_1$ and $E_3$ by means of elementary functions. In particular, from explicit calculations $B$ turns out to be a real number between 0 and 1. This result was obtained by studying the behaviour of the algebro-geometric quantities appearing in (1.47). From this analysis one obtains that the period matrix behaves as follows:

$$\boldsymbol{\tau}\left(\epsilon\right) = \begin{bmatrix} 2\log\left(\epsilon\right) & 0 \\ 0 & 2\log\left(\epsilon\right) \end{bmatrix} + \mathcal{O}\left(1\right) \tag{1.50}$$

Al the other parameters exhibit a regular behaviour in the limit, instead. This fact suggests to introduce

$$\varepsilon := \exp\left(\frac{\delta}{2}\right) \tag{1.51}$$

Notice that this new parameter is effective: it does not depend on Riemann surfaces anymore. Instead, it is defined directly in terms of the period matrix $\boldsymbol{\tau}$. Part of the local effectivization of the formulas lies exactly in this step. After (1.51) it is possible to recast (1.21) as follows [4]:

$$\psi\left(x,t\right) = \frac{\mathcal{N}_0\left(x,t\right) + \mathcal{N}_2\left(x,t\right)\varepsilon^2 + \cdots}{\mathcal{D}_0\left(x,t\right) + \mathcal{D}_2\left(x,t\right)\varepsilon^2 + \cdots}\exp\left(iNt\right) \tag{1.52}$$

[4] One can always reduce to the case when $A$ equals one and $q$ and $E$ equal zero using the symmetries of fNLS

Moreover, the algebro-geometric parameters exhibit a regular behaviour with respect to $\varepsilon$ when this approaches zero. One can then proceed as follows: putting $\varepsilon$ equal to zero and substituting into fNLS one selects some appropriate free parameters. The remaining quantities can be expressed as power series of $\varepsilon$ whose coefficients depend on these ones. Considering higher order corrections in (1.52) one obtains a better approximation both for the solution and for the effective parametrization at the same time. The equations involved in this procedure are linear except for the leading term, which yields some algebraic relation. We obtain the following

**Theorem.** *The two-phase solutions to fNLS admit an expansion of the form*

$$\psi(x,t) = \frac{\mathcal{N}(\Phi(x,t), \Psi(x,t))}{\mathcal{D}(\Phi(x,t), \Psi(x,t))} \exp\left[i\frac{v}{2}x + i\left(NA^2 - \frac{v^2}{4}\right)t\right] \tag{1.53}$$

*where*

$$\begin{aligned}\mathcal{D}(\Phi,\Psi) = & \frac{1}{\sqrt{B}}\cosh\Phi + \sqrt{B}\cos\Psi \\ & + 2\varepsilon^2\left\{\cos\gamma\left[B\sqrt{B}\cosh\Phi\cos 2\Psi + \frac{1}{B\sqrt{B}}\cosh 2\Phi\cos\Psi\right]\right. \\ & \left. - \sin\gamma\left[B\sqrt{B}\sinh\Phi\sin 2\Psi + \frac{1}{B\sqrt{B}}\sinh 2\Phi\sin\Psi\right]\right\} + \mathcal{O}\left(\varepsilon^4\right), \\ \mathcal{N}(\Phi,\Psi) = & \mathcal{D}\left(\Phi - i\sigma, \Psi - i\rho\right),\end{aligned} \tag{1.54}$$

*and*

$$\Phi(x,t) = \eta Ax + \left(\phi A^2 - \eta Av\right)t, \qquad \Psi(x,t) = \xi Ax + \left(\theta A^2 - \xi Av\right)t. \tag{1.55}$$

*If the real part $\delta$ of the diagonal entry of the period matrix $\boldsymbol{\tau}$ is (negative and) sufficiently large, the whole family of solutions is parametrized by the vector wave numbers*

$$\xi, \ \eta \quad \in \quad \mathbb{R}, \tag{1.56}$$

*the small parameter*

$$\varepsilon = \exp\left(\frac{\delta}{2}\right) > 0 \tag{1.57}$$

*and the additional quantity*

$$\gamma \ \in \ \mathbb{R}. \tag{1.58}$$

*This is the imaginary part of the first diagonal entry of the period matrix. The quantities*

$$A > 0, \qquad v \in \mathbb{R}, \tag{1.59}$$

*can be fixed arbitrarily: they express the Galilean and scaling invariance for fNLS. For technical convenience let us introduce the auxiliary parameters*

$$\lambda \in (0,\pi), \qquad \mu \in \mathbb{R} \tag{1.60}$$

*such that*

$$\xi = 2\cosh\mu\cos\lambda \qquad \eta = -2\sinh\mu\sin\lambda \tag{1.61}$$

*All the other quantities appearing in the solution can be expressed as follows:*

$$\begin{aligned}\theta = 2\sinh 2\mu\cos 2\lambda + &\frac{2\varepsilon^2}{\cos 2\lambda - \cosh 2\mu}\{\sin\gamma\,[8\sin 2\lambda + (\cosh 2\mu + \cosh 6\mu)\sin 4\lambda - \\ &(\sin 2\lambda + \sin 6\lambda)\cosh 4\mu] + \cos\gamma\,[8\sinh 2\mu + (\cos 2\lambda + \cos 6\lambda) \\ &\sinh 4\mu - (\sinh 6\mu + \sinh 2\mu)\cos 4\lambda]\} + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{1.62}$$

$$\begin{aligned}\phi = -2\cosh 2\mu\sin 2\lambda - &\frac{2\varepsilon^2}{\cos 2\lambda - \cosh 2\mu}\{\cos\gamma\,[-8\sin 2\lambda - (\cosh 2\mu + \cosh 6\mu) \\ &\sin 4\lambda + (\sin 2\lambda + \sin 6\lambda)\cosh 4\mu] + \sin\gamma\,[8\sinh 2\mu - \\ &(\sinh 6\mu + \sinh 2\mu)\cos 4\lambda + (\cos 2\lambda + \cos 6\lambda)\sinh 4\mu]\} + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{1.63}$$

$$\begin{aligned}N = 2 + 8\,\varepsilon^2\,\exp(-4\mu)\,&[6\exp 4\mu\cos\gamma + \cos(\gamma - 4\lambda) + 4\exp 2\mu\cos(\gamma - 2\lambda) + \\ &4\exp 6\mu\cos(\gamma + 2\lambda) + \exp 8\mu\cos(\gamma + 4\lambda)] + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{1.64}$$

$$\begin{aligned}B = \frac{\sin\lambda}{\cosh\mu} + &\frac{\varepsilon^2}{4(\cos 2\lambda - \cosh 2\mu)}\Big\{-\cos\gamma\,\mathrm{sech}^3\mu\Big[2(1 + \cosh 4\mu - \cosh 6\mu) \\ &\sin\lambda + 5\sin 3\lambda + \sin 5\lambda + 2\cosh 2\mu\left(8\cos^2\lambda\sin\lambda + \sin 5\lambda\right)\Big] - \\ &16\cos\gamma\cosh\mu\csc\lambda\sinh^2\mu + 16\cos\lambda\sin\gamma\,(-\sinh\mu + \sinh 3\mu) + \\ &16(\cos\lambda + \cos 3\lambda)\,\mathrm{sech}\mu\sin\gamma\sin^2\lambda\tanh\mu\Big\} + \mathcal{O}\left(\varepsilon^4\right).\end{aligned} \tag{1.65}$$

*Moreover let us put*

$$R = \exp(\rho + i\sigma). \tag{1.66}$$

*One has*

$$\begin{aligned}R = -\exp(-2\mu)(\cos 2\lambda + i\sin 2\lambda) + &\frac{\varepsilon^2\exp(-6\mu)}{\cos 2\lambda - \cosh 2\mu}\{[\exp 8\mu\cos\gamma - \\ &\cos(\gamma - 4\lambda) + \exp 2\mu\,(\cos(\gamma - 6\lambda) + 3(-1 + \exp 4\mu) \\ &\cos(\gamma - 2\lambda) - \exp(4\mu)\cos(\gamma + 2\lambda) + 6\exp 2\mu\sin(\gamma - 2\lambda) \\ &\sin 2\lambda)] + i\,[-\exp 8\mu\sin\gamma + \sin(\gamma - 4\lambda) + \exp 2\mu\,(-\sin(\gamma - 6\lambda) - \\ &3(-1 + \exp 4\mu)\sin(\gamma - 2\lambda) + 6\exp 2\mu\cos(\gamma - 2\lambda)\sin 2\lambda + \\ &\exp 4\mu\sin(\gamma + 2\lambda))]\} + \mathcal{O}\left(\varepsilon^4\right).\end{aligned} \tag{1.67}$$

Considering higher corrections, formulas above grow immediately more complicated: we do not report them here. In principle, though, there is no obstruction to go further with these calculations. In this way one gets an arbitrarily good approximation for the genus two solutions of fNLS, provided that $|\delta|$ is sufficiently large.

# Chapter 2

# Period matrices of Real Riemann Surfaces and Fundamental Domains

## 2.1 Formulation of the Problem

The *Siegel upper half plane of degree g*, usually denoted $\mathbb{H}_g$, is defined as the space of $g \times g$ complex, symmetric matrices whose imaginary part is positive definite. The *modular group of dimension* $2g$ is the group of $2g \times 2g$ symplectic matrices whose entries are integer numbers. We will denote it by the symbol $\mathbf{Sp}(2g, \mathbb{Z})$. There exists a well-known action of the modular group on the Siegel upper half plane, the so called *modular action*:

$$\mathfrak{M} : \mathbf{Sp}(2g, \mathbb{Z}) \times \mathbb{H}_g \longrightarrow \mathbb{H}_g. \tag{2.1}$$

To every matrix

$$G = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathbf{Sp}(2g, \mathbb{Z}) \tag{2.2}$$

and every point $\mathbf{w}$ of $\mathbb{H}_g$ it associates

$$\mathfrak{M}(G, \mathbf{w}) = (P\mathbf{w} + Q)(R\mathbf{w} + S)^{-1}. \tag{2.3}$$

Now, let $g$ and $n$ be positive integers such that

$$g + 1 = 2p + n \tag{2.4}$$

for some non-negative integer $p$. Let us define the space

$$\mathcal{W}_{g,n} = \left\{ \mathbf{w} \in \mathbb{H}_g \text{ such that } V\mathbf{w}V = -\overline{\mathbf{w}} \right\} \tag{2.5}$$

where the matrix $V$ is given by

$$V = \begin{pmatrix} 0 & Id_p & 0 \\ Id_p & 0 & 0 \\ 0 & 0 & Id_{n-1} \end{pmatrix}. \tag{2.6}$$

Let us also introduce the group

$$\mathbb{G}_{g,n} = \{G \in \mathbf{Sp}\,(2g, \mathbb{Z}) \text{ such that } GT = TG\} \tag{2.7}$$

where $T$ denotes the $2g \times 2g$ matrix

$$T = \begin{pmatrix} V & 0 \\ 0 & -V \end{pmatrix}. \tag{2.8}$$

**Proposition 2.1.1.** *The modular action can be restricted to an action of $\mathbb{G}_{g,n}$ on the space $\mathcal{W}_{g,n}$: for every matrix*

$$G \in \mathbb{G}_{g,n} \tag{2.9}$$

*and every point*

$$\mathbf{w} \in \mathcal{W}_{g,n} \tag{2.10}$$

*one has that*

$$\mathfrak{M}\,(G, \mathbf{w}) \in \mathcal{W}_{g,n}. \tag{2.11}$$

*Proof.* From the definition of $\mathbb{G}_{g,n}$ one can immediately deduce the following relations:

$$P = VPV, \quad Q = -VQV, \quad R = -VRV, \quad S = VSV. \tag{2.12}$$

Now let us put

$$\mathbf{w}' = (P\mathbf{w} + Q)\,(R\mathbf{w} + S)^{-1}. \tag{2.13}$$

Applying (2.12), one obtains that

$$V\mathbf{w}'V = [(VPV)\,(V\mathbf{w}V) + (VQV)]\,[(VRV)\,(V\mathbf{w}V) + (VSV)]^{-1} \tag{2.14}$$

$$= -\,[P\overline{\mathbf{w}} + Q]\,[R\overline{\mathbf{w}} + S]^{-1} = -\overline{\mathbf{w}'}. \tag{2.15}$$

By definition, this implies that $\mathbf{w}'$ belongs to $\mathcal{W}_{g,n}$. □

The study of real Riemann surfaces leads to address the problem of determining a fundamental domain for the modular action of the group $\mathbb{G}_{g,n} \subset \mathbf{Sp}\,(2g, \mathbb{Z})$ on the space $\mathcal{W}_{g,n}$. Its definition is recalled here below:

**Definition 2.1.2.** *We say that two points $\mathbf{w}$ and $\mathbf{w}'$ of $\mathcal{W}_{g,n}$ are equivalent if there exists $G \in \mathbb{G}_{g,n}$ such that*

$$\mathbf{w}' = \mathfrak{M}\,(G, \mathbf{w}). \tag{2.16}$$

*A fundamental domain for $\mathfrak{M}$ is a closed subset $\mathcal{D}$ of $\mathcal{W}_{g,n}$ whose interior part is connected, which satisfies the following three properties:*

**i_** *For every $\mathbf{w} \in \mathcal{W}_{g,n}$ there exists an equivalent point $\mathbf{w}'$ belonging to $\mathcal{D}$.*

**ii_** *If two distinct points, $\mathbf{w}$ and $\mathbf{w}'$, belonging to $\mathcal{D}$, are equivalent, then they both belong to the boundary of $\mathcal{D}$.*

**iii_** *Every set of equivalent points contained in $\mathcal{D}$ has a finite number of elements.*

Let us recall that a compact Riemann surface $\Gamma$ of genus $g$ is said to be *real* or *symmetric* when it is endowed with an anti-holomorphic involution $r$ (see [5] and [8] for a complete account of this subject).
One can consider the locus of points which are invariant with respect to $r$. Its connected components are at most $g+1$ and they are usually called the *ovals* of the surface.
When $\Gamma$ with this locus removed has two connected components, it is said to be *separated*. Let this be the case, and assume[1] that the number of ovals is $n$. Then it is possible to fix a basis in the homology of the form

$$\begin{array}{c} a'_1, a'_2, \ldots, a'_p, a''_1, a''_2, \ldots, a''_p, a_1, a_2, \ldots, a_{n-1} \\ b'_1, b'_2, \ldots, b'_p, b''_1, b''_2, \ldots, b''_p, b_1, b_2, \ldots, b_{n-1} \end{array} \tag{2.19}$$

satisfying the following conditions:

- $$a'_k \circ b'_k = a''_k \circ b''_k = a_j \circ b_j = 1 \tag{2.20}$$

  for every $k = 1, 2, \ldots, p$ and every $j = 1, 2, \ldots, n-1$, the intersection number of any two other elements of the basis being zero.

- $$\begin{array}{ll} r_\star\left(a'_k\right) = -a''_k, & k = 1, 2, \ldots, p; \\ r_\star\left(b'_k\right) = b''_k, & k = 1, 2, \ldots, p; \\ r_\star\left(a_j\right) = -a_j, & j = 1, 2, \ldots, n-1; \\ r_\star\left(b_j\right) = b_j, & j = 1, 2, \ldots, n-1. \end{array} \tag{2.21}$$

  where $r_\star$ denotes the morphism of the homology into itself induced by $r$.

Given any Riemann surface (not necessrily a real one) with a fixed basis in the homology $\mathcal{B}$, one can compute the corresponding *period matrix*[2]. This is a point in the Siegel upper half plane (see [16] and [8] for more details).
Any other basis in the homology $\tilde{\mathcal{B}}$ on the same surface can be expressed in terms of the cycles of $\mathcal{B}$ by means of a modular matrix.
If this is blockwise written as

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathbf{Sp}\,(2g, \mathbb{Z}) \tag{2.24}$$

[1] In the case of a separated surface one has that

$$1 \le n \le g+1 \tag{2.17}$$

and

$$n \equiv g+1 \qquad \text{mod } 2. \tag{2.18}$$

For more details see [5], theorem 1.5.3

[2] In this procedure, the normalized holomorphic differentials

$$\omega'_1, \omega'_2 \ldots, \omega'_p, \omega''_1, \omega''_2 \ldots, \omega''_p, \omega_1, \omega_2 \ldots, \omega_{n-1} \tag{2.22}$$

are involved. Throughout this chapter we will use the convention

$$\oint_{a'_k} \omega'_k = \oint_{a''_k} \omega''_k = \oint_{a_j} \omega_j = 1 \tag{2.23}$$

for $k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n-1$.

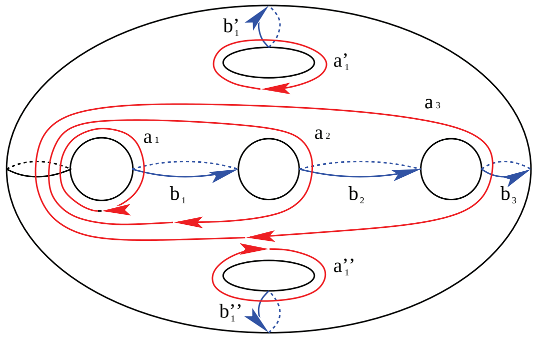



Figure 2.1: *A basis of cycles of the form (2.19-2.21) on a real, separable Riemann surface of genus five with four ovals.*

then the corresponding period matrices $\mathbf{t}$ and $\mathbf{t}'$ are related by the modular transformation

$$\mathbf{t}' = (P\mathbf{t} + Q)(R\mathbf{t} + S)^{-1}. \tag{2.25}$$

Now, in the case of a real Riemann surface with a basis $\mathcal{B}$ of the form (2.19-2.21), the period matrix belongs to the space $\mathcal{W}_{g,n}$. Moreover, the changes of basis in the homology which preserve the form (2.19-2.21) are exactly the ones given by matrices of $\mathbb{G}_{g,n}$.

In this chapter we formulate a general procedure to determine a fundamental domain for the modular action of the group $\mathbb{G}_{2g_0,1}$ on the space $\mathcal{W}_{2g_0,1}$. This corresponds to considering those real Riemann surfaces which are involved in the construction of algebro-geometric solutions to fNLS equation. In this case one has the following simpler characterizations:

$$\mathcal{W}_{2g_0,1} = \left\{ \mathbf{w} \in \mathbb{H}_g \text{ such that } \mathbf{w} = \begin{bmatrix} \mathbf{z} & \mathbf{x} \\ -\overline{\mathbf{x}} & -\overline{\mathbf{z}} \end{bmatrix}; \mathbf{z}, \mathbf{x} \in Mat\,(g_0 \times g_0, \mathbb{C}) \right\}, \tag{2.26}$$

$$\mathbb{G}_{2g_0,1} = \left\{ G \in \mathbf{Sp}\,(4g_0, \mathbb{Z}) \text{ such that } G = \begin{bmatrix} A & B & C & D \\ B & A & -D & -C \\ E & F & G & H \\ -F & -E & H & G \end{bmatrix} \right\}. \tag{2.27}$$

It is understood that the matrices $A, B, C, D, E, F, G$ and $H$ belong to $Mat\,(g_0 \times g_0, \mathbb{Z})$.

We briefly sketch our methods and results here below:
Our first step consists in a reformulation of the problem (Section 2.2).
Let us denote with $\mathbf{GL}\,(g, \mathbb{Z})$ the group of all $g$-dimensional, unimodular matrices with integer entries. Let us also introduce the space $Sym_{>0}\,(g, \mathbb{R})$ of all real symmetric and positive definite matrices of dimension $g$.
The *congruence action*

$$\mathfrak{C} : Sym_{>0}\,(g, \mathbb{R}) \times \mathbf{GL}\,(g, \mathbb{Z}) \longrightarrow Sym_{>0}\,(g, \mathbb{R}) \tag{2.28}$$

is defined as follows

$$\mathfrak{C}(\Sigma, G) = G^T \Sigma G. \tag{2.29}$$

The modular and the congruence actions are closely related by a result of Siegel (Theorem 2.2.1): There exists an injective and smooth map

$$\mathbf{\Sigma} : \mathbb{H}_g \longrightarrow Sym_{>0}(g, \mathbb{R}) \tag{2.30}$$

such that the following diagram is commutative for every $G$ belonging to $\mathbf{Sp}(2g, \mathbb{Z})$

$$\begin{array}{ccc} \mathbb{H}_g & \xrightarrow{\mathfrak{M}(G,\_)} & \mathbb{H}_g \\ \downarrow \mathbf{\Sigma} & & \downarrow \mathbf{\Sigma} \\ Sym_{>0}(2g, \mathbb{R}) & \xrightarrow{\mathfrak{C}(\_,G^{-1})} & Sym_{>0}(2g, \mathbb{R}) \end{array} \tag{2.31}$$

Let us define the space

$$\mathcal{S}_{2g_0,1} = \mathbf{\Sigma}\left(\mathcal{W}_{2g_0,1}\right). \tag{2.32}$$

The congruence action can be restricted to an action of the group $\mathbb{G}_{2g_0,1}$ on $\mathcal{S}_{2g_0,1}$. Now, let $\mathcal{D}'$ be a fundamental domain for this last one, and put

$$\mathcal{D} = \mathbf{\Sigma}^{-1}(\mathcal{D}'). \tag{2.33}$$

Using Siegel's result one can show that the set $\mathcal{D}$ so defined is a fundamental domain for the modular action restricted to $\mathbb{G}_{2g_0,1}$ and $\mathcal{W}_{2g_0,1}$.
The original issue can then be reformulated as the quest of such a $\mathcal{D}'$.

The main technical tool to tackle this new problem is the map $\mathbf{P}$, introduced in Section 2.3. Due to its several remarkable properties this might turn out to be of general interest by itself, beyond its role in this specific context. To the best of our knowledge, this map was never considered in the literature before.
The domain of $\mathbf{P}$ is the set of all $4g_0$-dimensional, real and symplectic matrices of the form

$$\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & -\delta & -\gamma \\ \pi & \rho & \xi & \eta \\ -\rho & -\pi & \eta & \xi \end{bmatrix} \in \mathbf{Sp}(4g_0, \mathbb{R}). \tag{2.34}$$

Both $\mathcal{S}_{2g_0,1}$ and $\mathbb{G}_{2g_0,1}$ are contained in it.
Its explicit definition is very simple:

$$\mathbf{P}\left(\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & -\delta & -\gamma \\ \pi & \rho & \xi & \eta \\ -\rho & -\pi & \eta & \xi \end{bmatrix}\right) = \begin{bmatrix} \alpha+\beta & \gamma-\delta \\ \pi+\rho & \xi-\eta \end{bmatrix} \tag{2.35}$$

After this, $\mathbf{P}$ turns out to be a bijection onto $\mathbf{GL}(2g_0, \mathbb{R})$, the space of all real and invertible matrices of dimension $2g_0$. It maps $\mathcal{S}_{2g_0,1}$ onto $Sym_{>0}(2g_0, \mathbb{R})$.

Moreover **P** respects both the matrix product and transposition. As a consequence the following diagram is commutative for every $G$ belonging to $\mathbb{G}_{2g_0,1}$:

$$\begin{array}{ccc} \mathcal{S}_{2g_0,1} & \xrightarrow{\mathfrak{C}(\_,G)} & \mathcal{S}_{2g_0,1} \\ \downarrow \mathbf{P} & & \downarrow \mathbf{P} \\ Sym_{>0}\left(2g_0,\mathbb{R}\right) & \xrightarrow{\mathfrak{C}(\_,\mathbf{P}(G))} & Sym_{>0}\left(2g_0,\mathbb{R}\right) \end{array} \qquad G \in \mathbb{G}_{2g_0,1} \tag{2.36}$$

This construction allows a further reformulation of the problem, in the same spirit as above:
Let us consider the group $\mathbb{K}_{2g_0,1}$ defined as

$$\mathbb{K}_{2g_0,1} = \mathbf{P}\left(\mathbb{G}_{2g_0,1}\right). \tag{2.37}$$

This is a finite-index subgroup of $\mathbf{GL}\left(2g_0,\mathbb{Z}\right)$. Obviously, the congruence action can be restricted to an action of $\mathbb{K}_{2g_0,1}$ on $Sym_{>0}\left(2g_0,\mathbb{R}\right)$; let $\mathcal{D}''$ be a fundamental domain for it.
Due to the properties of **P** the set $\mathcal{D}'$ defined as

$$\mathcal{D}' = \mathbf{P}^{-1}\left(\mathcal{D}''\right) \tag{2.38}$$

turns out to be a fundamental domain for the congruence action restricted to $\mathbb{G}_{2g_0,1}$ and $\mathcal{S}_{2g_0,1}$. The original problem is then equivalent to the quest of such a $\mathcal{D}''$.
This second reformulation appears in Section 2.4. It is given the name of Reduction of the Problem because it halves, so to say, the dimension of the matrices involved in the issue.

The advantage of these subsequent reformulations is that we finally get to an explicitly solvable problem.
Indeed, the Minkowski Reduction Theory provides us with a fundamental domain for the congruence action of the whole group $\mathbf{GL}\left(2g_0,\mathbb{Z}\right)$ on $Sym_{>0}\left(2g_0,\mathbb{R}\right)$.
On the other side, the index of $\mathbb{K}_{2g_0,1}$ in $\mathbf{GL}\left(2g_0,\mathbb{Z}\right)$ is finite.
In view of these two facts, a fundamental domain $\mathcal{D}''$ for the congruence action of $\mathbb{K}_{2g_0,1}$ on $Sym_{>0}\left(2g_0,\mathbb{R}\right)$ can be computed by means of standard "gluing" techniques.

The case in which $2g_0$ equals 2 is particularly interesting (Section 2.5). It exhibits the remarkable peculiarity that the group $\mathbb{K}_{2,1}$ coincides with the whole $\mathbf{GL}\left(2,\mathbb{Z}\right)$. As a consequence, the two reformulations above reconduct the original problem to a classical one already solved: the reduction of positive definite binary quadratic forms. No usage of "gluing" techniques is required in this case.
The formulas expressing the result are very simple: Let us denote by

$$\mathbf{w} = \begin{pmatrix} \tilde{\gamma} + \imath\tilde{\delta} & \imath\tilde{\beta} \\ \imath\tilde{\beta} & -\tilde{\gamma} + \imath\tilde{\delta} \end{pmatrix} \qquad \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \in \mathbb{R} \tag{2.39}$$

the generic element of $\mathcal{W}_{2,1}$; a fundamental domain $\mathcal{D}$ for the modular action $\mathfrak{M}$ of the group $\mathbb{G}_{2,1} \subset \mathbf{Sp}\left(4,\mathbb{Z}\right)$ on this last one is given by the following system of inequalities:

$$\mathcal{D}: \begin{cases} \tilde{\delta} \geq \sqrt{\tilde{\beta}^2 + 1 - \tilde{\gamma}^2} \\ 0 \leq \tilde{\gamma} \leq \frac{1}{2} \end{cases} \tag{2.40}$$

(The square root in the first inequality is understood to be positive.)
Moreover, a deeper understanding of the mathematical structure is possible in this case:
Let $G$ belong to $\mathbb{G}_{2,1}$. One has that[3] either

$$G = \begin{bmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ e & 0 & g & 0 \\ 0 & -e & 0 & g \end{bmatrix}, \qquad ag - ec = 1 \tag{2.42}$$

or

$$G = \begin{bmatrix} 0 & b & 0 & d \\ b & 0 & -d & 0 \\ 0 & f & 0 & h \\ -f & 0 & h & 0 \end{bmatrix}, \qquad bh - fd = 1. \tag{2.43}$$

Introduce, on $\mathcal{W}_{2,1}$, the new system of coordinates

$$\mathcal{I}(\mathbf{w}) = \frac{\tilde{\delta} - \tilde{\beta}}{\tilde{\delta} + \tilde{\beta}}, \qquad \chi(\mathbf{w}) = \tilde{\gamma} + \imath\sqrt{\tilde{\delta}^2 - \tilde{\beta}^2}, \qquad \mathbf{w} \in \mathcal{W}_{2,1}. \tag{2.44}$$

Now consider a modular transformation

$$\mathbf{w}' = \mathfrak{M}(G, \mathbf{w}) \qquad \mathbf{w} \in \mathcal{W}_{2,1},\ g \in \mathbb{G}_{2,1}. \tag{2.45}$$

In the new coordinates it acts as follows:

$$\mathcal{I}(\mathbf{w}') = \mathcal{I}(\mathbf{w}) \tag{2.46}$$

and

$$\chi(\mathbf{w}') = \frac{a\chi(\mathbf{w}) + c}{e\chi(\mathbf{w}) + g} \tag{2.47}$$

if $G$ has the form (2.42), or

$$\chi(\mathbf{w}') = \frac{b\overline{\chi(\mathbf{w})} - d}{f\overline{\chi(\mathbf{w})} - h} \tag{2.48}$$

if (2.43) holds, instead.
In other words, we find out that $\mathcal{I}$ is an invariant quantity: it is not affected by the modular action of $\mathbb{G}_{2,1}$, which concentrates only on the second coordinate

[3] More precisely, $\mathbb{G}_{2,1}$ is a $\mathbb{Z}_2$-group of the form

$$\mathbb{G}_{2,1} = \mathcal{G}_0 \sqcup \mathcal{G}_1. \tag{2.41}$$

$\mathcal{G}_0$ and $\mathcal{G}_1$ contain the matrices of the form (2.42) and (2.43) respectively. Notice that only $\mathcal{G}_0$ is a (normal) subgroup of $\mathbb{G}_{2,1}$.

$\chi$. This last one just undergoes a Moebius transformation, in some cases composed with a complex conjugation.
The theory developed in Sections 2.2 and 2.3 allowed us to individuate the higher dimensional analogue of the invariant quantity $\mathcal{I}$ and to prove that also in this case it is not affected by the modular action of $\mathbb{G}_{2g_0,1}$ (Section 2.4). A full understanding of the geometrical meaning of this quantity has not yet been achieved though, and it is meant to be part of a work in progress.
Finally, the explicit calculation of a fundamental domain has been worked out in the case when $2g_0$ equals 4. The result is not so simple to write down; a rather detailed account of the formulas there involved can be found in Section 2.6.

Before detailing our calculations let us point out that a different approach to the problem of a canonical form for period matrices of real Riemann surfaces can be found in [18].

## 2.2 Reformulation of the Problem

In this section we show how the original problem can be refomulated in terms of a restriction of the congruence action $\mathfrak{C}$, introduced in (2.28-2.29).

For every matrix

$$\mathbf{w} = \lambda + \imath\mu \;\in\; \mathbb{H}_g \tag{2.49}$$

belonging to the Siegel upper half plane of degree $g$, let us define

$$\mathbf{\Sigma}(\mathbf{w}) = \begin{bmatrix} \mu^{-1} & -\mu^{-1}\lambda \\ -\lambda\mu^{-1} & \mu + \lambda\mu^{-1}\lambda \end{bmatrix} \qquad \mathbf{w} \in \mathbb{H}_g. \tag{2.50}$$

This map already appeared in Siegel's investigations on the modular group, from which we drew some inspiration.
It relates the modular and the congruence actions:

**Theorem 2.2.1.** *i_ $\mathbf{\Sigma}$ is a bijection onto the space of all real, symmetric and positive definite matrices of dimension $2g$ which are also symplectic.*

*ii_ For every G belonging to the modular group, the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{H}_g & \xrightarrow{\mathfrak{M}(G,\_)} & \mathbb{H}_g \\ \downarrow{\scriptstyle\mathbf{\Sigma}} & & \downarrow{\scriptstyle\mathbf{\Sigma}} \\ Sym_{>0}(2g,\mathbb{R}) & \xrightarrow{\mathfrak{C}(\_,G^{-1})} & Sym_{>0}(2g,\mathbb{R}) \end{array} \qquad G \in \mathbf{Sp}(2g,\mathbb{Z}) \tag{2.51}$$

For a proof see [17], pag. 148, theorem 1.

Now, since the original problem is concerned with the proper subset $\mathcal{W}_{2g_0,1} \subset \mathbb{H}_{2g_0}$, let us introduce the space

$$\mathcal{S}_{2g_0,1} := \mathbf{\Sigma}\left(\mathcal{W}_{2g_0,1}\right). \tag{2.52}$$

In view of point **ii** of theorem 2.2.1, it is easy to realize that $\mathfrak{C}$ can be restricted to an action of the group $\mathbb{G}_{2g_0,1}$ on $\mathcal{S}_{2g_0,1}$. Moreover the following diagram is commutative, for every $G$ belonging to $\mathbb{G}_{2g_0,1}$:

$$\begin{array}{ccc} \mathcal{W}_{2g_0,1} & \xrightarrow{\mathfrak{M}(G,\_)} & \mathcal{W}_{2g_0,1} \\ \downarrow \boldsymbol{\Sigma} & & \downarrow \boldsymbol{\Sigma} \\ \mathcal{S}_{2g_0,1} & \xrightarrow{\mathfrak{C}(\_,G^{-1})} & \mathcal{S}_{2g_0,1} \end{array} \qquad G \in \mathbb{G}_{2g_0,1} \tag{2.53}$$

This immediately gives the following

**Corollary 2.2.2.** *Let $\mathcal{D}'$ be a fundamental domain for the congruence action of the group $\mathbb{G}_{2g_0,1} \subset \mathbf{Sp}(4g_0, \mathbb{Z})$ on $\mathcal{S}_{2g_0,1}$. Then the set*

$$\mathcal{D} := \boldsymbol{\Sigma}^{-1}(\mathcal{D}') \tag{2.54}$$

*is a fundamental domain for the modular action of the same group $\mathbb{G}_{2g_0,1}$ on $\mathcal{W}_{2g_0,1}$.*

In view of this, the original problem is equivalent to the quest of such a $\mathcal{D}'$. To this purpose, a more explicit characterization of $\mathcal{S}_{2g_0,1}$ will be useful:

**Proposition 2.2.3.** *The set $\mathcal{S}_{2g_0,1}$ consists of all real, symplectic, symmetric and positive definite matrices $\Sigma$ of dimension $4g_0$, which have the following form*

$$\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & -\delta & -\gamma \\ \gamma^T & -\delta^T & \xi & \eta \\ \delta^T & -\gamma^T & \eta & \xi \end{bmatrix} \in \mathbf{Sp}(4g_0, \mathbb{R}) \cap Sym_{>0}(4g_0, \mathbb{R}) \tag{2.55}$$

*for some $\alpha, \beta, \gamma, \delta, \xi$ and $\eta$ belonging to $Mat(g_0 \times g_0, \mathbb{R})$.*

*Proof.* In view of theorem 2.2.1, point **i**, it will be sufficient to prove that a matrix $\mathbf{w}$ of the Siegel upper half plane belongs to $\mathcal{W}_{2g_0,1}$ if and only if its image $\boldsymbol{\Sigma}(\mathbf{w})$ has the form (2.55). This last condition is equivalent to the relation

$$T[\boldsymbol{\Sigma}(\mathbf{w})]T = \boldsymbol{\Sigma}(\mathbf{w}) \tag{2.56}$$

where

$$T = \begin{bmatrix} V & 0 \\ 0 & -V \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & Id \\ Id & 0 \end{bmatrix}. \tag{2.57}$$

Using the explicit definition (2.50), one can show that (2.56) holds if and only if the following relations do:

$$V\lambda V = -\lambda, \qquad V\mu V = \mu. \tag{2.58}$$

But this happens if and only if $\mathbf{w}$ belongs to $\mathcal{W}_{2g_0,1}$, so the proof is complete. □

Next sections will be dedicated to determine a fundamental domain $\mathcal{D}'$ for the congruence action of $\mathbb{G}_{2g_0,1}$ on $\mathcal{S}_{2g_0,1}$.

## 2.3 Reduction Toolkit

In this section we introduce some objects which will be, so to say, useful tools in our task.

Let us start with an auxiliary map. Consider the vector space

$$\mathcal{V}_{2g_0,1} := \left\{ \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & -\delta & -\gamma \\ \pi & \rho & \xi & \eta \\ -\rho & -\pi & \eta & \xi \end{bmatrix} \in Mat\,(4g_0 \times 4g_0, \mathbb{R}) \right\}. \tag{2.59}$$

It is understood that the matrices $\alpha, \beta, \gamma, \delta, \pi, \rho, \xi$ and $\eta$ belong to $Mat\,(g_0 \times g_0, \mathbb{R})$. We define the map

$$\mathbf{G} : \mathcal{V}_{2g_0,1} \longrightarrow Mat\,(2g_0 \times 2g_0, \mathbb{R}) \times Mat\,(2g_0 \times 2g_0, \mathbb{R}) \tag{2.60}$$

as follows:

$$\mathbf{G}\,(\Sigma) = \left( \begin{bmatrix} \alpha + \beta & \gamma - \delta \\ \pi + \rho & \xi - \eta \end{bmatrix}, \begin{bmatrix} \alpha - \beta & \gamma + \delta \\ \pi - \rho & \xi + \eta \end{bmatrix} \right), \qquad \forall \Sigma \in \mathcal{V}_{2g_0,1}. \tag{2.61}$$

Let us also introduce the notation

$$J = \begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix} \tag{2.62}$$

for the *standard symplectic matrix*. Its dimension will be clear from the context.

**Lemma 2.3.1.** *Let $\Sigma$ and $\Sigma'$ belong to $\mathcal{V}_{2g_0,1}$ and put*

$$\mathbf{G}\,(\Sigma) = (\sigma, \tau)\,, \quad \mathbf{G}\,(\Sigma') = (\sigma', \tau')\,. \tag{2.63}$$

*One has the following*[4]*:*

**i_** **G** *is an isomorphism of vector spaces from $\mathcal{V}_{2g_0,1}$ to $Mat\,(2g_0 \times 2g_0, \mathbb{R}) \times Mat\,(2g_0 \times 2g_0, \mathbb{R})$.*

**ii_** $\mathbf{G}\,(\Sigma\Sigma') = (\sigma\sigma', \tau\tau')$

**iii_** $\mathbf{G}\left(\Sigma^T\right) = \left(\sigma^T, \tau^T\right)$

**iv_** *$\Sigma$ is symmetric and positive definite if and only if both $\sigma$ and $\tau$ are.*

**v_** *$\Sigma$ is symplectic if and only if*

$$\left(-J\tau^T J\right)\sigma = Id. \tag{2.65}$$

*Proof.* Points **i**, **ii** and **iii** can be verified by means of a straightforward calculation.

[4]The space $\mathcal{V}_{2g_0,1}$ is closed with respect to matrix product. Indeed, a $4g_0$-dimensional, real matrix belongs to $\mathcal{V}_{2g_0,1}$ if and only if it satisfies the relation

$$T\Sigma T = \Sigma. \tag{2.64}$$

where $T$ is the idempotent matrix defined in (2.57).

**iv_** Since **G** is bijective (see point **i**,) one has that

$$\Sigma = \Sigma^T \Longleftrightarrow \mathbf{G}(\Sigma) = \mathbf{G}\left(\Sigma^T\right) \tag{2.66}$$

but by point **iii**,

$$\mathbf{G}\left(\Sigma^T\right) = \left(\sigma^T, \tau^T\right) \tag{2.67}$$

so

$$\Sigma = \Sigma^T \Longleftrightarrow (\sigma, \tau) = \left(\sigma^T, \tau^T\right) \tag{2.68}$$

and $\Sigma$ is symmetric if and only if both $\sigma$ and $\tau$ are; let us suppose that this is the case. One can verify that the two identities

$$\begin{pmatrix} \underline{\alpha}^T & \underline{\beta}^T \end{pmatrix} \sigma \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \underline{\alpha}^T & \underline{\alpha}^T & \underline{\beta}^T & -\underline{\beta}^T \end{pmatrix} \Sigma \begin{pmatrix} \underline{\alpha} \\ \underline{\alpha} \\ \underline{\beta} \\ -\underline{\beta} \end{pmatrix} \tag{2.69}$$

$$\begin{pmatrix} \underline{\alpha}^T & \underline{\beta}^T \end{pmatrix} \tau \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \underline{\alpha}^T & -\underline{\alpha}^T & \underline{\beta}^T & \underline{\beta}^T \end{pmatrix} \Sigma \begin{pmatrix} \underline{\alpha} \\ -\underline{\alpha} \\ \underline{\beta} \\ \underline{\beta} \end{pmatrix} \tag{2.70}$$

hold true for every column vector $\underline{\alpha}, \underline{\beta}$ belonging to $\mathbb{R}^{g_0}$. This implies that if $\Sigma$ is positive definite, then also $\sigma$ and $\tau$ are.
Viceversa, the quadratic form given by $\Sigma$ can be expressed in terms of the quadratic forms corresponding to $\sigma$ and $\tau$ in the following way:

$$\begin{aligned} \begin{pmatrix} \underline{\alpha}^T & \underline{\beta}^T & \underline{\gamma}^T & \underline{\delta}^T \end{pmatrix} \Sigma \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \\ \underline{\gamma} \\ \underline{\delta} \end{pmatrix} &= \tfrac{1}{2} \begin{pmatrix} \left(\underline{\alpha} + \underline{\beta}\right)^T & \left(\underline{\gamma} - \underline{\delta}\right)^T \end{pmatrix} \sigma \begin{pmatrix} \left(\underline{\alpha} + \underline{\beta}\right) \\ \left(\underline{\gamma} - \underline{\delta}\right) \end{pmatrix} \\ &+ \tfrac{1}{2} \begin{pmatrix} \left(\underline{\alpha} - \underline{\beta}\right)^T & \left(\underline{\gamma} + \underline{\delta}\right)^T \end{pmatrix} \tau \begin{pmatrix} \left(\underline{\alpha} - \underline{\beta}\right) \\ \left(\underline{\gamma} + \underline{\delta}\right) \end{pmatrix}. \end{aligned} \tag{2.71}$$

So, if both $\sigma$ and $\tau$ are positive definite, then also $\Sigma$ is.

**v_** By definition, $\Sigma$ is symplectic if and only if

$$\Sigma^T J \Sigma = J \tag{2.72}$$

which, by carrying out the products in the left-hand side, reduces to the following system of matrix-equations:

$$\begin{cases} -\pi^T\alpha + \rho^T\beta + \alpha^T\pi - \beta^T\rho = 0 & (2.73) \\ -\pi^T\beta + \rho^T\alpha + \alpha^T\rho - \beta^T\pi = 0 & (2.74) \\ -\xi^T\gamma + \eta^T\delta + \gamma^T\xi - \delta^T\eta = 0 & (2.75) \\ -\xi^T\delta + \eta^T\gamma + \gamma^T\eta - \delta^T\xi = 0 & (2.76) \\ -\pi^T\gamma - \rho^T\delta + \alpha^T\xi + \beta^T\eta = Id & (2.77) \\ -\pi^T\delta - \rho^T\gamma + \alpha^T\eta + \beta^T\xi = Id & (2.78) \end{cases}$$

Adding and subtracting (2.73) and (2.74) one gets the equations

$$\begin{cases} -(\pi-\rho)^T(\alpha+\beta)+(\alpha-\beta)^T(\pi+\rho)=0 & (2.79)\\ -(\pi+\rho)^T(\alpha-\beta)+(\alpha+\beta)^T(\pi-\rho)=0 & (2.80)\end{cases}$$

which are equivalent one to the other. So (2.73) and (2.74) together are equivalent to the unique equation

$$-(\pi-\rho)^T(\alpha+\beta)+(\alpha-\beta)^T(\pi+\rho)=0. \tag{2.81}$$

Analogously, (2.75) and (2.76) together are equivalent to

$$-(\xi-\eta)^T(\gamma+\delta)+(\gamma-\delta)^T(\xi+\eta)=0. \tag{2.82}$$

Equations (2.77) and (2.78) are indeed equivalent to the system

$$\begin{cases} -(\pi+\rho)^T(\gamma+\delta)+(\alpha+\beta)^T(\xi+\eta)=Id & (2.83)\\ -(\pi-\rho)^T(\gamma-\delta)+(\alpha-\beta)^T(\xi-\eta)=Id & (2.84)\end{cases}$$

So $\Sigma$ is symplectic if and only if it satisfies the following system of equations:

$$\begin{cases} -(\pi-\rho)^T(\alpha+\beta)+(\alpha-\beta)^T(\pi+\rho)=0 & (2.85)\\ -(\xi-\eta)^T(\gamma+\delta)+(\gamma-\delta)^T(\xi+\eta)=0 & (2.86)\\ -(\pi+\rho)^T(\gamma+\delta)+(\alpha+\beta)^T(\xi+\eta)=Id & (2.87)\\ -(\pi-\rho)^T(\gamma-\delta)+(\alpha-\beta)^T(\xi-\eta)=Id & (2.88)\end{cases}$$

but after transposition of (2.86) and (2.87), this turns out to be equivalent to the condition

$$\begin{pmatrix} (\xi+\eta)^T & -(\gamma+\delta)^T \\ -(\pi-\rho)^T & (\alpha-\beta)^T \end{pmatrix}\begin{pmatrix} (\alpha+\beta) & (\gamma-\delta) \\ (\pi+\rho) & (\xi-\eta) \end{pmatrix}=Id \tag{2.89}$$

which is simply

$$\left(-J\tau^T J\right)\sigma = Id. \tag{2.90}$$

□

Let us introduce the group[5]

$$\tilde{\mathcal{S}}_{2g_0,1}=\left\{\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & -\delta & -\gamma \\ \pi & \rho & \xi & \eta \\ -\rho & -\pi & \eta & \xi \end{bmatrix}\in \mathbf{Sp}(4g_0,\mathbb{R})\right\}. \tag{2.91}$$

It is understood that the matrices $\alpha,\beta,\gamma,\delta,\pi,\rho,\xi$ and $\eta$ belong to $Mat(g_0\times g_0,\mathbb{R})$. Notice that both $\mathbb{G}_{2g_0,1}$ and $\mathcal{S}_{2g_0,1}$ are contained in it.
Let us also define

$$\mathbf{P}:\tilde{\mathcal{S}}_{2g_0,1}\longrightarrow Mat(2g_0\times 2g_0,\mathbb{R}) \tag{2.92}$$

[5]One can easily verify that this is a group using the argument quoted in footnote 4

as follows:

$$\mathbf{P}(\Sigma) = \begin{bmatrix} \alpha+\beta & \gamma-\delta \\ \pi+\rho & \xi-\eta \end{bmatrix}, \qquad \forall \Sigma \in \tilde{\mathcal{S}}_{2g_0,1} \tag{2.93}$$

Many properties of this map can be deduced from lemma 2.3.1:

**Corollary 2.3.2.** *The map* $\mathbf{P}$ *is a homeomorphism from* $\tilde{\mathcal{S}}_{2g_0,1}$ *to* $\mathbf{GL}(2g_0, \mathbb{R})$*, this last one consisting of all real invertible matrices of dimension* $2g_0$*.*
*Moreover it respects both matrix product*

$$\mathbf{P}(\Sigma\Sigma') = \mathbf{P}(\Sigma)\,\mathbf{P}(\Sigma') \qquad \forall \Sigma, \Sigma' \in \tilde{\mathcal{S}}_{2g_0,1} \tag{2.94}$$

*and transposition*

$$\mathbf{P}\left(\Sigma^T\right) = [\mathbf{P}(\Sigma)]^T \qquad \forall \Sigma \in \tilde{\mathcal{S}}_{2g_0,1} \tag{2.95}$$

*Proof.* The inverse of $\mathbf{P}$ is the map

$$\mathbf{Q} : \mathbf{GL}(2g_0, \mathbb{R}) \longrightarrow \mathcal{S}_{2g_0,1} \tag{2.96}$$

defined as follows:

$$\mathbf{Q}(\sigma) = \mathbf{G}^{-1}\left(\sigma, -J\left(\sigma^{-1}\right)^T J\right) \qquad \forall \sigma \in \mathbf{GL}(2g_0, \mathbb{R}) \tag{2.97}$$

Notice that the image of $\mathbf{Q}$ is contained in $\tilde{\mathcal{S}}_{2g_0,1}$ because of point **v** of the lemma. Since both $\mathbf{Q}$ and $\mathbf{P}$ are continuous, this last one is a homeomorphism. Moreover $\mathbf{P}$ respects both matrix product and transposition due to point **ii** and **iii** of the lemma. □

**Corollary 2.3.3.** *The map* $\mathbf{P}$ *restricts to a homeomorphism from* $\mathcal{S}_{2g_0,1}$ *to* $Sym_{>0}(2g_0, \mathbb{R})$*.*

*Proof.* From point **iv** of lemma 2.3.1, one immediately has that

$$\mathbf{P}\left(\mathcal{S}_{2g_0,1}\right) \subset Sym_{>0}(2g_0, \mathbb{R}). \tag{2.98}$$

Viceversa, let $\sigma$ belong to $Sym_{>0}(2g_0, \mathbb{R})$. The matrix

$$-J\left(\sigma^{-1}\right)^T J = J^T\left(\sigma^{-1}\right)^T J \tag{2.99}$$

is also positive definite.
From the proof of the previous corollary, we know that

$$\mathbf{P}^{-1}(\sigma) = \mathbf{G}^{-1}\left(\sigma, -J\left(\sigma^{-1}\right)^T J\right) \in \tilde{\mathcal{S}}_{2g_0,1}. \tag{2.100}$$

In view of point **iv** of the lemma, this last matrix is also positive definite and so it belongs to $\mathcal{S}_{2g_0,1}$. It follows that

$$\mathbf{P}\left(\mathcal{S}_{2g_0,1}\right) = Sym_{>0}(2g_0, \mathbb{R}) \tag{2.101}$$

and the proof is complete. □

Finally let us introduce the group

$$\mathbb{K}_{2g_0,1} = \left\{ g \in \mathbf{GL}\left(2g_0, \mathbb{Z}\right) \text{ s.t. } g^T J g = J + 2M, \text{ for some } M \in Mat\left(2g_0 \times 2g_0, \mathbb{Z}\right) \right\}. \quad (2.102)$$

One has the following

**Corollary 2.3.4.** *The map* $\mathbf{P}$ *restricts to an isomorphism of groups from* $\mathbb{G}_{2g_0,1}$ *to* $\mathbb{K}_{2g_0,1}$.

*Proof.* Let $G$ belong to $\mathbb{G}_{2g_0,1}$ and put

$$\mathbf{G}\left(G\right) = \left(\sigma, \tau\right). \quad (2.103)$$

From point **v** of the lemma, one has that

$$\left(-J\tau^T J\right)\sigma = Id \quad (2.104)$$

Since both $\sigma$ and $\tau$ have integer coefficients, this implies that

$$\sigma \in \mathbf{GL}\left(2g_0, \mathbb{Z}\right). \quad (2.105)$$

Equation (2.104) can be rewritten as follows:

$$\tau^T J\sigma = J. \quad (2.106)$$

On the other side, from definition (2.61) one has that

$$\tau = \sigma - 2\tilde{M} \quad (2.107)$$

for some $\tilde{M}$ belonging to $Mat\left(2g_0 \times 2g_0, \mathbb{Z}\right)$, in this case. Plugging (2.107) into (2.106) one obtains that

$$\sigma^T J\sigma = J + 2M \quad (2.108)$$

for some $M$ belonging to $Mat\left(2g_0 \times 2g_0, \mathbb{Z}\right)$.
Viceversa, let $g$ belong to $\mathbb{K}_{2g_0,1}$. Let us put

$$\mathbf{P}^{-1}\left(g\right) = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & -\delta & -\gamma \\ \pi & \rho & \xi & \eta \\ -\rho & -\pi & \eta & \xi \end{bmatrix} \qquad g \in \mathbb{K}_{2g_0,1} \quad (2.109)$$

From the proof of corollary 2.3.2 we know that

$$\mathbf{P}^{-1}\left(g\right) = \mathbf{G}^{-1}\left(g, -J\left(g^{-1}\right)^T J\right) \qquad g \in \mathbb{K}_{2g_0,1}. \quad (2.110)$$

Recalling the definition of $\mathbf{G}$ one can easily write

$$\begin{bmatrix} \alpha & \gamma \\ \pi & \xi \end{bmatrix} = \frac{1}{2}\left[g - J\left(g^{-1}\right)^T J\right], \quad \begin{bmatrix} \beta & -\delta \\ \rho & -\eta \end{bmatrix} = \frac{1}{2}\left[g + J\left(g^{-1}\right)^T J\right] \quad (2.111)$$

Now, since

$$g^T Jg = J + 2M \quad (2.112)$$

for some $M$ belonging to $Mat\left(2g_0 \times 2g_0, \mathbb{Z}\right)$, one also has that

$$g + J\left(g^{-1}\right)^T J = 2\tilde{M} \tag{2.113}$$

for some $\tilde{M}$ belonging to $Mat\left(2g_0 \times 2g_0, \mathbb{Z}\right)$. Since $g$ has an inverse with integer entries, it follows immediately that

$$g - J\left(g^{-1}\right)^T J = 2\tilde{N} \tag{2.114}$$

for some $\tilde{N}$ belonging to $Mat\left(2g_0 \times 2g_0, \mathbb{Z}\right)$.
Equations (2.111) together with (2.113) and (2.114) imply that $\mathbf{P}^{-1}(g)$ has integer coefficients. So it belongs to $\mathbb{G}_{2g_0,1}$. □

**Remark 2.3.5.** *The whole modular group* $\mathbf{Sp}\left(2g_0, \mathbb{Z}\right)$ *is contained in* $\mathbb{K}_{2g_0,1}$. *For every matrix*

$$\begin{bmatrix} A & C \\ E & G \end{bmatrix} \in \mathbf{Sp}\left(2g_0, \mathbb{Z}\right) \tag{2.115}$$

*one has that*

$$\mathbf{P}\left(\begin{bmatrix} A & 0 & C & 0 \\ 0 & A & 0 & -C \\ E & 0 & G & 0 \\ 0 & -E & 0 & G \end{bmatrix}\right) = \begin{bmatrix} A & C \\ E & G \end{bmatrix} \tag{2.116}$$

*It is easy to show that the argument of the function* $\mathbf{P}$ *in the left-hand side belongs to* $\mathbb{G}_{2g_0,1}$.
*Since*

$$\begin{bmatrix} Id & 0 \\ 0 & -Id \end{bmatrix} \in \mathbb{K}_{2g_0,1} \tag{2.117}$$

*the following inclusion also holds:*

$$\mathbf{Sp}\left(2g_0, \mathbb{Z}\right) \cdot \begin{bmatrix} Id & 0 \\ 0 & -Id \end{bmatrix} \subset \mathbb{K}_{2g_0,1}. \tag{2.118}$$

*For every*

$$\begin{bmatrix} B & -D \\ F & -H \end{bmatrix} \in \mathbf{Sp}\left(2g_0, \mathbb{Z}\right) \cdot \begin{bmatrix} Id & 0 \\ 0 & -Id \end{bmatrix} \tag{2.119}$$

*one has that*

$$\mathbf{P}\left(\begin{bmatrix} 0 & B & 0 & D \\ B & 0 & -D & 0 \\ 0 & F & 0 & H \\ -F & 0 & H & 0 \end{bmatrix}\right) = \begin{bmatrix} B & -D \\ F & -H \end{bmatrix} \tag{2.120}$$

*and the argument of* $\mathbf{P}$ *again belongs to* $\mathbb{G}_{2g_0,1}$.

**Remark 2.3.6.** *In general, the group $\mathbb{G}_{2g_0,1}$ is not exhausted by the block-sparse matrices appearing as argument of* $\mathbf{P}$ *in (2.116) and (2.120). A counterexample is given by*

$$G = \begin{bmatrix} 1 & 3 & 0 & 3 & 0 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -3 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & -3 & 0 & -3 & 1 \end{bmatrix} \in \mathbb{G}_{4,1}. \tag{2.121}$$

*One has*

$$\mathbf{P}(G) = \begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.122}$$

## 2.4 Reduction of the Problem

We can now reformulate further the original problem, using the instruments developed in the previous section. This process bears the name of reduction because it halves the dimension of the matrices involved in the issue.

**Proposition 2.4.1.** *The following diagram*

$$\begin{array}{ccc} \mathcal{S}_{2g_0,1} & \xrightarrow{\mathfrak{C}(\_,G)} & \mathcal{S}_{2g_0,1} \\ \downarrow \mathbf{P} & & \downarrow \mathbf{P} \\ Sym_{>0}(2g_0,\mathbb{R}) & \xrightarrow{\mathfrak{C}(\_,\mathbf{P}(G))} & Sym_{>0}(2g_0,\mathbb{R}) \end{array} \qquad G \in \mathbb{G}_{2g_0,1} \tag{2.123}$$

*is commutative for every $G$ belonging to $\mathbb{G}_{2g_0,1}$.*

*Proof.* Let $\Sigma$ and $G$ belong to $\mathcal{S}_{2g_0,1}$ and $\mathbb{G}_{2g_0,1}$ respectively. Due to corollary 2.3.2 one has

$$\mathbf{P}[\mathfrak{C}(\Sigma,G)] = \mathbf{P}\left[G^T\Sigma G\right] = [\mathbf{P}(G)]^T\,\mathbf{P}(\Sigma)\,\mathbf{P}(G), \qquad \Sigma \in \mathcal{S}_{2g_0,1},\ G \in \mathbb{G}_{2g_0,1}. \tag{2.124}$$

On the other side, simply by definition,

$$\mathfrak{C}[\mathbf{P}(\Sigma),\mathbf{P}(G)] = [\mathbf{P}(G)]^T\,\mathbf{P}(\Sigma)\,\mathbf{P}(G), \qquad \Sigma \in \mathcal{S}_{2g_0,1},\ G \in \mathbb{G}_{2g_0,1}. \tag{2.125}$$

□

As a consequence, one has the following

**Theorem 2.4.2.** *Let $\mathcal{D}''$ be a fundamental domain for the congruence action of $\mathbb{K}_{2g_0,1}$ on $Sym_{>0}(2g_0,\mathbb{R})$. Then the set*

$$\mathcal{D}' := \mathbf{P}^{-1}(\mathcal{D}'') \tag{2.126}$$

*is a fundamental domain for the congruence action of $\mathbb{G}_{2g_0,1}$ on $\mathcal{S}_{2g_0,1}$.*

*Proof.* The proof is a standard argument based on proposition 2.4.1 and corollaries 2.3.3 and 2.3.4. □

The original problem is thus equivalent to the quest of such a $\mathcal{D}''$. The remaining sections will be dedicated to this issue. This will also emphasize the full advantage of our approach.

Another consequence of proposition 2.4.1 is worth pointing out:
For every $\mathbf{w}$ belonging to $\mathcal{W}_{2g_0,1}$, let us define

$$\mathcal{I}(\mathbf{w}) = \det[(\mathbf{P} \circ \mathbf{\Sigma})(\mathbf{w})] \qquad \mathbf{w} \in \mathcal{W}_{2g_0,1}. \tag{2.127}$$

**Corollary 2.4.3.** *Let* $\mathbf{w}$ *and* $G$ *belong to* $\mathcal{W}_{2g_0,1}$ *and* $\mathbb{G}_{2g_0,1}$ *respectively. Consider the modular transformation*

$$\mathbf{w}' = \mathfrak{M}(G, \mathbf{w}). \tag{2.128}$$

*One has*

$$\mathcal{I}(\mathbf{w}') = \mathcal{I}(\mathbf{w}). \tag{2.129}$$

*Proof.* In view of commutative diagram (2.53),

$$\mathbf{\Sigma}[\mathfrak{M}(G, \mathbf{w})] = \mathfrak{C}\left[\mathbf{\Sigma}(\mathbf{w}), G^{-1}\right] \qquad \mathbf{w} \in \mathcal{W}_{2g_0,1},\ G \in \mathbb{G}_{2g_0,1} \tag{2.130}$$

Taking the image via $\mathbf{P}$ of both sides, and applying proposition 2.4.1

$$(\mathbf{P} \circ \mathbf{\Sigma})[\mathfrak{M}(G, \mathbf{w})] = \mathfrak{C}\left[(\mathbf{P} \circ \mathbf{\Sigma})(\mathbf{w}), \mathbf{P}\left(G^{-1}\right)\right]. \tag{2.131}$$

Considering the determinants, and recalling the explicit definition (2.29),

$$\det\{(\mathbf{P} \circ \mathbf{\Sigma})[\mathfrak{M}(G, \mathbf{w})]\} = \det\left[\mathbf{P}\left(G^{-1}\right)\right]^2 \det[(\mathbf{P} \circ \mathbf{\Sigma})(\mathbf{w})] \tag{2.132}$$

The thesis follows from this last equation, in view of the fact that $\mathbb{K}_{2g_0,1}$ is contained in $\mathbf{GL}(2g_0, \mathbb{Z})$. □

From corollary 2.3.3, one has

$$(\mathbf{P} \circ \mathbf{\Sigma})\left(\mathcal{W}_{2g_0,1}\right) = Sym_{>0}(2g_0, \mathbb{R}). \tag{2.133}$$

As a consequence, the quantity $\mathcal{I}$ is not constant over the whole $\mathcal{W}_{2g_0,1}$. In other words, it is not a trivial invariant with respect to the modular action $\mathfrak{M}$ of the group $\mathbb{G}_{2g_0,1} \subset \mathbf{Sp}(4g_0, \mathbb{Z})$.
To the best of our understanding, no geometrical interpretation of this quantity is available so far.

## 2.5 The case $2g_0 = 2$

We start here to put in concrete action the previously developed abstract theory. The simplest case when $2g_0$ equals 2 allows an elegant solution, together with a deeper understanding of the structure of the problem.

**Lemma 2.5.1.** *One has*

$$\mathbb{K}_{2,1} = \mathbf{GL}(2,\mathbb{Z}). \tag{2.134}$$

*Moreover, if $G$ belongs to $\mathbb{G}_{2,1}$, there are only two possible cases: either*

$$G = \begin{bmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ e & 0 & g & 0 \\ 0 & -e & 0 & g \end{bmatrix}, \qquad ag - ec = 1 \tag{2.135}$$

*or*

$$G = \begin{bmatrix} 0 & b & 0 & d \\ b & 0 & -d & 0 \\ 0 & f & 0 & h \\ -f & 0 & h & 0 \end{bmatrix}, \qquad bh - fd = 1. \tag{2.136}$$

*Proof.* As pointed out in remark 2.3.5, one has the following inclusions:

$$\mathbf{Sp}(2,\mathbb{Z}) \subset \mathbb{K}_{2,1}, \qquad \mathbf{Sp}(2,\mathbb{Z}) \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \subset \mathbb{K}_{2,1}. \tag{2.137}$$

On the other side,

$$\mathbf{GL}(2,\mathbb{Z}) = \mathbf{Sp}(2,\mathbb{Z}) \cup \mathbf{Sp}(2,\mathbb{Z}) \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.138}$$

This gives (2.134).
Now, let $G$ belong to $\mathbb{G}_{2,1}$. Then, either

$$\mathbf{P}(G) \in \mathbf{Sp}(2,\mathbb{Z}) \tag{2.139}$$

or

$$\mathbf{P}(G) \in \mathbf{Sp}(2,\mathbb{Z}) \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.140}$$

More explicitly, one has

$$\mathbf{P}(G) = \begin{bmatrix} a & c \\ e & g \end{bmatrix} \qquad ag - ec = 1 \tag{2.141}$$

in the former case, and

$$\mathbf{P}(G) = \begin{bmatrix} b & -d \\ f & -h \end{bmatrix} \qquad bh - fd = 1 \tag{2.142}$$

in the latter one.
Always in view of remark 2.3.5, (2.141) and (2.142) yield (2.135) and (2.136) respectively. □

**Remark 2.5.2.** *The explicit characterization of the group $\mathbb{G}_{2,1}$ given in the previous lemma can also be obtained by a direct calculation, starting from (2.27)*

By means of (2.134) we are reconducted to a classical problem whose solution is well-known:
Let us indicate with

$$\left[\begin{array}{cc} \phi & \chi \\ \chi & \psi \end{array}\right] \in Sym_{>0}\,(2,\mathbb{R}) \tag{2.143}$$

the generic element of $Sym_{>0}\,(2,\mathbb{R})$. A fundamental domain $\mathcal{D}''$ for the congruence action of the group $\mathbb{K}_{2,1} = \mathbf{GL}\,(2,\mathbb{Z})$ on this space is described by the following system of inequalities

$$\mathcal{D}'' : \left\{ \begin{array}{c} \phi \leq \psi \\ -\phi \leq 2\chi \leq 0 \end{array}\right. \tag{2.144}$$

This result was already known to Lagrange and Hermite.

Let us denote with

$$\mathbf{w} = \left( \begin{array}{cc} \tilde{\gamma} + \imath\tilde{\delta} & \imath\tilde{\beta} \\ \imath\tilde{\beta} & -\tilde{\gamma} + \imath\tilde{\delta} \end{array}\right) \in \mathcal{W}_{2,1} \qquad \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \in \mathbb{R} \tag{2.145}$$

the generic element of $\mathcal{W}_{2,1}$.
The imaginary part of $\mathbf{w}$ is positive definite if and only if

$$\tilde{\delta} > |\tilde{\beta}|. \tag{2.146}$$

For the map $\mathbf{\Sigma}$ defined in (2.50) one has the following explicit expression:

$$\mathbf{\Sigma}\,(\mathbf{w}) = \frac{1}{\tilde{\delta}^2 - \tilde{\beta}^2} \left( \begin{array}{cccc} \tilde{\delta} & -\tilde{\beta} & -\tilde{\gamma}\tilde{\delta} & -\tilde{\gamma}\tilde{\beta} \\ -\tilde{\beta} & \tilde{\delta} & \tilde{\gamma}\tilde{\beta} & \tilde{\gamma}\tilde{\delta} \\ -\tilde{\gamma}\tilde{\delta} & \tilde{\gamma}\tilde{\beta} & \tilde{\delta}\left(\tilde{\gamma}^2 + \tilde{\delta}^2 - \tilde{\beta}^2\right) & \tilde{\beta}\left(\tilde{\gamma}^2 + \tilde{\delta}^2 - \tilde{\beta}^2\right) \\ -\tilde{\gamma}\tilde{\beta} & \tilde{\gamma}\tilde{\delta} & \tilde{\beta}\left(\tilde{\gamma}^2 + \tilde{\delta}^2 - \tilde{\beta}^2\right) & \tilde{\delta}\left(\tilde{\gamma}^2 + \tilde{\delta}^2 - \tilde{\beta}^2\right) \end{array}\right). \tag{2.147}$$

Applying $\mathbf{P}$ to both sides of this equality, one gets

$$\left(\mathbf{P} \circ \mathbf{\Sigma}\right)(\mathbf{w}) = \frac{1}{\tilde{\delta} + \tilde{\beta}} \left( \begin{array}{cc} 1 & -\tilde{\gamma} \\ -\tilde{\gamma} & \tilde{\gamma}^2 + \tilde{\delta}^2 - \tilde{\beta}^2 \end{array}\right). \tag{2.148}$$

These simple facts, together with the results of the previous sections, suffice to prove the following

**Theorem 2.5.3.** *A fundamental domain $\mathcal{D}$ for the modular action $\mathfrak{M}$ of the group $\mathbb{G}_{2,1} \subset \mathbf{Sp}\,(4,\mathbb{Z})$ over $\mathcal{W}_{2,1}$ is described by the following system of inequalities:*

$$\mathcal{D} : \left\{ \begin{array}{l} \tilde{\delta} \geq \sqrt{\tilde{\beta}^2 + 1 - \tilde{\gamma}^2} \\ 0 \leq \tilde{\gamma} \leq \frac{1}{2} \end{array}\right. \tag{2.149}$$

*(The square root in the first inequality is understood to be positive.)*

*Proof.* Due to corollaries 2.4.3 and 2.2.2, such a fundamental domain $\mathcal{D}$ can be found by simply considering

$$\mathcal{D} = \left(\mathbf{P} \circ \mathbf{\Sigma}\right)^{-1}\left(\mathcal{D}''\right). \tag{2.150}$$

In view of (2.148), one gets an explicit description of this set operating the substitution

$$\phi = \frac{1}{\tilde{\delta}+\tilde{\beta}}, \quad \chi = -\frac{\tilde{\gamma}}{\tilde{\delta}+\tilde{\beta}}, \quad \psi = \frac{\tilde{\gamma}^2+\tilde{\delta}^2-\tilde{\beta}^2}{\tilde{\delta}+\tilde{\beta}} \tag{2.151}$$

into the system (2.144). This leads to (2.149). □

With this result, the original problem can be considered completely solved when $2g_0$ equals 2.
The theory developed up to now, though, allows a deeper insight into the structure of the modular action of $\mathbb{G}_{2,1}$ on $\mathcal{W}_{2,1}$.

**Theorem 2.5.4.** *Let us introduce the following system of coordinates on the space* $\mathcal{W}_{2,1}$*:*

$$\mathcal{I}(\mathbf{w}) = \frac{\tilde{\delta}-\tilde{\beta}}{\tilde{\delta}+\tilde{\beta}} \qquad \chi(\mathbf{w}) = \tilde{\gamma} + \imath\sqrt{\tilde{\delta}^2-\tilde{\beta}^2}, \qquad \mathbf{w}\in\mathcal{W}_{2,1}. \tag{2.152}$$

*The square root is chosen to be positive.*
*Consider the modular transformation given by an element of* $\mathbb{G}_{2,1}$*:*

$$\mathbf{w}' = \mathfrak{M}(G,\mathbf{w}) \qquad \mathbf{w}\in\mathcal{W}_{2,1}, G\in\mathbb{G}_{2,1}. \tag{2.153}$$

*In terms of the new coordinates, this acts as follows:*

$$\mathcal{I}(\mathbf{w}') = \mathcal{I}(\mathbf{w}) \tag{2.154}$$

*and*

$$\chi(\mathbf{w}') = \frac{a\chi(\mathbf{w})+c}{e\chi(\mathbf{w})+g} \tag{2.155}$$

*if G has the form (2.135), while*

$$\chi(\mathbf{w}') = \frac{b\overline{\chi(\mathbf{w})}-d}{f\overline{\chi(\mathbf{w})}-h} \tag{2.156}$$

*if (2.136) holds.*

*Proof.* The explicit expression for $\mathcal{I}(\mathbf{w})$ in (2.152) agrees with its general definition given in section 2.4. To see this it is sufficient to plug (2.148) into (2.127).
Formula (2.154) is just the content of corollary 2.4.3.
Gluing together (2.53) and (2.123) one gets the following commutative diagram:

$$\begin{array}{ccc} \mathcal{W}_{2g_0,1} & \xrightarrow{\mathfrak{M}(G,\_)} & \mathcal{W}_{2g_0,1} \\ \downarrow{\scriptstyle \mathbf{P}\circ\mathbf{\Sigma}} & & \downarrow{\scriptstyle \mathbf{P}\circ\mathbf{\Sigma}} \\ Sym_{>0}(2g_0,\mathbb{R}) & \xrightarrow{\mathfrak{C}(\_,\mathbf{P}(G^{-1}))} & Sym_{>0}(2g_0,\mathbb{R}) \end{array} \qquad G\in\mathbb{G}_{2,1}. \tag{2.157}$$

Let us introduce

$$\mathbf{u}(\mathbf{w}) := \frac{1}{\sqrt{\mathcal{I}(\mathbf{w})}}\left[(\mathbf{P}\circ\mathbf{\Sigma})(\mathbf{w})\right], \qquad \mathbf{w}\in\mathcal{W}_{2,1} \tag{2.158}$$

where the square root is again chosen to be positive. Using (2.157) one can write

$$\begin{aligned}
\mathbf{u}(\mathbf{w}') &= \frac{1}{\sqrt{\mathcal{I}(\mathbf{w}')}}\left[(\mathbf{P}\circ\boldsymbol{\Sigma})(\mathbf{w}')\right] \\
&= \frac{1}{\sqrt{\mathcal{I}(\mathbf{w}')}}\left\{\mathfrak{C}\left[(\mathbf{P}\circ\boldsymbol{\Sigma})(\mathbf{w}),\mathbf{P}\left(G^{-1}\right)\right]\right\} \\
&= \left[\mathbf{P}\left(G^{-1}\right)\right]^T\left\{\frac{1}{\sqrt{\mathcal{I}(\mathbf{w})}}\left[(\mathbf{P}\circ\boldsymbol{\Sigma})(\mathbf{w})\right]\right\}\left[\mathbf{P}\left(G^{-1}\right)\right] \\
&= \left[\mathbf{P}\left(G^{-1}\right)\right]^T\left[\mathbf{u}(\mathbf{w})\right]\left[\mathbf{P}\left(G^{-1}\right)\right]. \qquad (2.159)
\end{aligned}$$

In view of definition (2.127), one has that

$$\det(\mathbf{u}) = 1 \qquad \forall\mathbf{w}\in\mathcal{W}_{2,1}. \qquad (2.160)$$

This means that $\mathbf{u}(\mathbf{w})$ is not only symmetric and positive definite but also symplectic for every $\mathbf{w}$ belonging to $\mathcal{W}_{2,1}$. As a consequence of theorem 2.2.1, then, there exists a unique point $\chi(\mathbf{w})$ in the Siegel upper half plane of degree one such that

$$\boldsymbol{\Sigma}(\chi(\mathbf{w})) = \mathbf{u}(\mathbf{w}), \qquad \mathbf{w}\in\mathcal{W}_{2,1} \qquad (2.161)$$

Using (2.50),(2.148) and (2.127) one can verify that this definition of $\chi$ coincides with the more explicit one given in (2.152).
Now, suppose that $G$ has the form (2.135). In this case,

$$\mathbf{P}(G) = \begin{bmatrix} a & c \\ e & g \end{bmatrix} \in \mathbf{Sp}(2,\mathbb{Z}). \qquad (2.162)$$

By point **ii** of theorem 2.2.1, (2.159) gives (2.155).
Let $G$ have the form (2.136), instead. One has that

$$\mathbf{P}(G)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} b & d \\ f & h \end{bmatrix} \in \mathbf{Sp}(2,\mathbb{Z}). \qquad (2.163)$$

Equation (2.159) can be rewritten as follows:

$$\mathbf{u}(\mathbf{w}') = \left[\left(\mathbf{P}(G)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)^{-1}\right]^T\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{u}(\mathbf{w})\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)\left(\mathbf{P}(G)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)^{-1}. \qquad (2.164)$$

By means of (2.50), one can verify that

$$\boldsymbol{\Sigma}\left(-\overline{\chi(\mathbf{w})}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{u}(\mathbf{w})\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (2.165)$$

Again by point **ii** of theorem 2.2.1 equality (2.164) implies (2.156) □

**Proposition 2.5.5.** *Consider the modular transformation associated to an element of* $\mathbb{G}_{2,1}$*, like the one in (2.153). One has*

$$sgn\left(\tilde{\beta}'\right) = sgn\left(\tilde{\beta}\right). \qquad (2.166)$$

*Proof.* The explicit expression for $\mathcal{I}(\mathbf{w})$ given in (2.152) implies that

$$\begin{cases} \mathcal{I}(\mathbf{w}) > 1 & \text{if} \quad -\tilde{\delta} < \tilde{\beta} < 0 \\ \mathcal{I}(\mathbf{w}) = 1 & \text{if} \quad \tilde{\beta} = 0 \\ \mathcal{I}(\mathbf{w}) < 1 & \text{if} \quad 0 < \tilde{\beta} < \tilde{\delta} \end{cases} \tag{2.167}$$

On the other side, in view of (2.154) one has

$$\mathcal{I}(\mathbf{w}) \gtreqless 1 \Leftrightarrow \mathcal{I}(\mathbf{w}') \gtreqless 1. \tag{2.168}$$

It follows that

$$\tilde{\beta} \gtreqless 0 \Leftrightarrow \tilde{\beta}' \gtreqless 0. \tag{2.169}$$

□

## 2.6 The general case

At the beginning of the last century Minkowski studied the congruence action of $\mathbf{GL}(n,\mathbb{Z})$ on $Sym_{>0}(n,\mathbb{R})$: he exhibited, for every $n \geq 2$, a fundamental domain $\mathcal{M}$ which can be described by a finite set of inequalities

$$\mathcal{M} := \{\sigma \in Sym_{>0}(n,\mathbb{R}): \ f_1(\sigma) \geq 0,\ f_2(\sigma) \geq 0, \ldots, f_{m_n}(\sigma) \geq 0.\} \tag{2.170}$$

where $f_1, f_2, \ldots, f_{m_n}$ are certain linear homogeneous expressions of the entries of $\sigma$. For low dimensions these expressions were explicitly determined: see [1] and [15].
After considerations of the previous sections, we now need a fundamental domain for the congruence action, on $Sym_{>0}(2g_0,\mathbb{R})$, of the group $\mathbb{K}_{2g_0,1}$, which is stricly contained in $\mathbf{GL}(2g_0,\mathbb{Z})$ for $2g_0 \geq 4$.
The index of this subgroup is finite. It can be easily computed and it equals[6]

$$\frac{\text{card}(\mathbf{GL}(2g_0,\mathbb{Z}_2))}{\text{card}(\mathbf{Sp}(2g_0,\mathbb{Z}_2))} \tag{2.174}$$

(this is an advantage of the reduction procedure). In similar situations, a standard technique seems to be the following one:

- Select a representantive for each left coset of $\mathbb{K}_{2g_0,1}$ in $\mathbf{GL}(2g_0,\mathbb{Z})$.
- Act with each representantive on the Minkowski fundamental domain $\mathcal{M}$ and consider the union of all the sets obtained this way.

[6]Consider the quotient map

$$\phi : \mathbf{GL}(2g_0,\mathbb{Z}) \longrightarrow \mathbf{GL}(2g_0,\mathbb{Z}_2) \tag{2.171}$$

which associates to every unimodular matrix its class modulo 2 elementwise: it is surjective and

$$\phi\left(\mathbb{K}_{2g_0,1}\right) = \mathbf{Sp}(2g_0,\mathbb{Z}_2), \tag{2.172}$$

(see [13], lemma 4 and [14] theorem 1 respectively); moreover

$$\text{Ker}\left(\phi\right) \subset \mathbb{K}_{2g_0,1}. \tag{2.173}$$

Because of these facts $\phi$ induces a bijection between the left cosets of $\mathbb{K}_{2g_0,1}$ in $\mathbf{GL}(2g_0,\mathbb{Z})$ and the ones of $\mathbf{Sp}(\mathbf{2g_0},\mathbb{Z}_2)$ in $\mathbf{GL}(2g_0,\mathbb{Z}_2)$.

- If its interior part is connected, this union is a fundamental domain for $\mathbb{K}_{2g_0,1}$.

As we are not aware of any reference for the theoretical justification of this procedure, we formulate one suitable for this setting here below

**Notation.** Let $g$ be an element of $\mathbf{GL}(n,\mathbb{Z})$. The symbol $\mathfrak{C}_g$ will denote the map

$$\mathfrak{C}_g : Sym_{>0}(n,\mathbb{R}) \longrightarrow Sym_{>0}(n,\mathbb{R}) \tag{2.175}$$

defined as follows

$$\mathfrak{C}_g(\sigma) = \mathfrak{C}(\sigma, g) = g^T \sigma g, \qquad \sigma \in Sym_{>0}(n,\mathbb{R}). \tag{2.176}$$

**Lemma 2.6.1.** *Let $\sigma$ be an interior point of $\mathcal{M}$, and $g \in \mathbf{GL}(n,\mathbb{Z})$ such that $\mathfrak{C}(\sigma, g) \in \mathcal{M}$. Then $g = \pm Id$.*

*Proof.* By definition of fundamental domain, if $\mathfrak{C}(\sigma, g) \in \mathcal{M}$ then

$$\mathfrak{C}(\sigma, g) = \sigma. \tag{2.177}$$

Now, let $\tilde{\tilde{\mathcal{B}}}_\sigma$ be a neighborhood of $\sigma$ contained in $\mathcal{M}$. Due to the continuity of the map $\mathfrak{C}_g$ there exists a neighborhood of $\sigma$, say $\tilde{B}_\sigma$, such that

$$\mathfrak{C}_g\left(\tilde{\mathcal{B}}_\sigma\right) \subset \tilde{\tilde{\mathcal{B}}}_\sigma. \tag{2.178}$$

Let us put $\mathcal{B}_\sigma = \tilde{\mathcal{B}}_\sigma \cap \operatorname{int}(\mathcal{M})$. The points of $\mathcal{B}_\sigma$ are all interior to $\mathcal{M}$ and remain inside $\mathcal{M}$ when the map $\mathfrak{C}_g$ is applied, so one has

$$\mathfrak{C}_g(\rho) = \rho, \quad \forall \rho \in \mathcal{B}_\sigma. \tag{2.179}$$

But $\mathcal{B}_\sigma$ is an open subset of $Sym_{>0}(n,\mathbb{R})$ and $\mathfrak{C}_g$ is a linear map, so

$$\mathfrak{C}_g = Id_{Sym_{>0}(n,\mathbb{R})}. \tag{2.180}$$

This implies that $g = \pm Id$. □

**Lemma 2.6.2.** *Let $\sigma \in \mathcal{M}$. Every neighborhood of $\sigma$ contains interior points of $\mathcal{M}$.*

*Proof.* This is a direct consequence of the fact that $\mathcal{M}$ is a convex set with non-empty interior. □

**Lemma 2.6.3.** *Let $\sigma$ be an interior point of*

$$\bigcup_{i=1}^{n} \mathfrak{C}(\mathcal{M}, g_i) \tag{2.181}$$

*for some $g_1, g_2, \ldots, g_n \in \mathbf{GL}(n,\mathbb{Z})$ and let $g \in \mathbf{GL}(n,\mathbb{Z})$ satisfy*

$$\mathfrak{C}(\rho, g) = \sigma \tag{2.182}$$

*for some $\rho \in \mathcal{M}$. Then*

$$g = \pm g_i \quad \text{for some } 1 \le i \le n. \tag{2.183}$$

*Proof.* Let $\mathcal{B}_\sigma$ be a neighborhood of $\sigma$ contained in $\bigcup_{i=1}^n \mathfrak{C}(\mathcal{M}, g_i)$, and let $\mathcal{B}_\rho$ be a neighborhood of $\rho$ such that

$$\mathfrak{C}\left(\mathcal{B}_\rho, g\right) \subset \mathcal{B}_\sigma \tag{2.184}$$

(such a neighborhood exists due to continuity of the map $\mathfrak{C}_g$).
Let $\nu$ be an interior point of $\mathcal{M}$ contained in $\mathcal{B}_\rho$; then

$$\mathfrak{C}(\nu, g) \in \bigcup_{i=1}^{n} \mathfrak{C}(\mathcal{M}, g_i), \tag{2.185}$$

that is

$$\mathfrak{C}(\nu, g) \in \mathfrak{C}(\mathcal{M}, g_i) \quad \text{for some } \ 1 \le i \le n \tag{2.186}$$

which is equivalent to

$$\mathfrak{C}\left(\nu, g g_i^{-1}\right) \in \mathcal{M} \quad \text{for some } \ 1 \le i \le n. \tag{2.187}$$

From this, applying lemma 2.6.1 one gets

$$g g_i^{-1} = \pm Id \quad \text{for some } \ 1 \le i \le n. \tag{2.188}$$

□

**Proposition 2.6.4.** *Let us consider the left cosets of* $\mathbb{K}_{2g_0,1}$ *in* $\mathbf{GL}(2g_0, \mathbb{Z})$

$$L_1, L_2, \ldots, L_m, \tag{2.189}$$

*and fix a representative for each of them:*

$$g_1 \in L_1, g_2 \in L_2, \ldots, g_m \in L_m. \tag{2.190}$$

*If*

$$int\left(\bigcup_{i=1}^{m} \mathfrak{C}(\mathcal{M}, g_i)\right) \tag{2.191}$$

*is a connected set then*

$$\bigcup_{i=1}^{m} \mathfrak{C}(\mathcal{M}, g_i) \tag{2.192}$$

*is a fundamental domain for the congruence action of* $\mathbb{K}_{2g_0,1}$ *on* $Sym_{>0}(2g_0, \mathbb{R})$.

*Proof.* The nontrivial part of the proof is the verification of point $\mathbf{ii}_-$ of definition 2.1.2.
Let $\rho$ be an interior point of $\bigcup_{i=1}^m \mathfrak{C}(\mathcal{M}, g_i)$ and $\sigma$ belong to $\mathfrak{C}(\mathcal{M}, g_1)$; also suppose that

$$\rho = \mathfrak{C}(\sigma, g) \tag{2.193}$$

for some $g \in \mathbf{GL}(2g_0, \mathbb{Z})$. This relation can be rewritten as

$$\rho = \mathfrak{C}\left(\mathfrak{C}\left(\sigma, g_1^{-1}\right), (g_1 g)\right), \tag{2.194}$$

where $\mathfrak{C}\left(\sigma, g_1^{-1}\right)$ belongs to $\mathcal{M}$. Applying lemma 2.6.3 one gets

$$g_1 g = \pm g_k \quad \text{for some } 1 \le k \le m, \tag{2.195}$$

that is

$$g = \pm g_1^{-1} g_k \quad \text{for some } 1 \le k \le m. \tag{2.196}$$

Now, if $k = 1$ then $g = \pm Id$ and $\sigma = \rho$.
If $k \neq 1$, instead, $g$ does not belong to $\mathbb{K}_{2g_0,1}$, because by construction $g_1$ and $g_k$ belong to different cosets.
In the same way, one can treat the case in which $\sigma$ belongs to $\mathfrak{C}\left(\mathcal{M}, g_j\right)$ for $j = 2, \ldots, m$ and finally prove that $\rho$ cannot be equivalent (in the $\mathbb{K}_{2g_0,1}$-sense) to any other point of $\bigcup_{i=1}^{m} \mathfrak{C}\left(\mathcal{M}, g_i\right)$. □

At this point, the issue is about selecting a set of representatives such that the set in (2.191) is connected. Next lemma gives a working criterion to individuate some identifications on the border of $\mathcal{M}$ which will serve as "building blocks" in this task.

**Lemma 2.6.5.** *Let $g \in \mathbf{GL}(n, \mathbb{Z})$. Let us assume that there exists a point $\rho_0 \in \mathcal{M}$ with the following two properties:*

- *There exists an integer $k_\rho$ such that*

$$f_{k_\rho}(\rho_0) = 0 \quad \text{and} \quad f_j(\rho_0) > 0 \quad \text{for } j \neq k_\rho. \tag{2.197}$$

- *Let us put $\sigma_0 = \mathfrak{C}(\rho_0, g)$; there exists an integer $k_\sigma$ such that*

$$f_{k_\sigma}(\sigma_0) = 0 \quad \text{and} \quad f_j(\sigma_0) > 0 \quad \text{for } j \neq k_\sigma. \tag{2.198}$$

  *(In particular, $\sigma_0$ belongs to $\mathcal{M}$.)*

*Then the set*

$$int\left(\mathcal{M} \cup \mathfrak{C}(\mathcal{M}, g)\right) \tag{2.199}$$

*is connected.*

*Proof.* There exists an open ball $\mathcal{B}_{\rho_0}$ with center in $\rho_0$ such that every $\rho$ belonging to $\mathcal{B}_{\rho_0}$ satisfies

$$f_j(\rho) > 0, \qquad j \neq k_\rho. \tag{2.200}$$

If the radius of $\mathcal{B}_{\rho_0}$ is sufficiently small, its image through $\mathfrak{C}_g$, say $\mathcal{B}_{\sigma_0}$, has the analogous property: every $\sigma$ belonging to $\mathcal{B}_\sigma$, satisfies

$$f_j(\sigma) > 0, \qquad j \neq k_\sigma. \tag{2.201}$$

Let $\mathcal{B}_{\rho_0}$ satisfy both properties above, and define the following three sets:

$$\mathcal{B}^+_{\rho_0} = \left\{\rho \in \mathcal{B}_{\rho_0} \text{ such that } f_{k_\rho}(\rho) > 0\right\}, \tag{2.202}$$

$$\mathcal{B}^-_{\rho_0} = \left\{\rho \in \mathcal{B}_{\rho_0} \text{ such that } f_{k_\rho}(\rho) < 0\right\}, \tag{2.203}$$

$$\mathcal{B}^0_{\rho_0} = \left\{\rho \in \mathcal{B}_{\rho_0} \text{ such that } f_{k_\rho}(\rho) = 0\right\}. \tag{2.204}$$

Analogously we define $\mathcal{B}^+_{\sigma_0}$, $\mathcal{B}^-_{\sigma_0}$ and $\mathcal{B}^0_{\sigma_0}$.
Consider $\rho \in \mathcal{B}^0_{\rho_0}$. Suppose that

$$f_{k_\sigma}\left(\mathfrak{C}\left(\rho, g\right)\right) > 0. \tag{2.205}$$

Then $\mathfrak{C}(\rho, g)$ belongs to the interior of $\mathcal{M}$ and is equivalent to another point of $\mathcal{M}$, which is absurd. Suppose instead that

$$f_{k_\sigma}\left(\mathfrak{C}\left(\rho, g\right)\right) < 0 \tag{2.206}$$

and put $\rho = \rho_0 + \nu$. Then $\rho_0 - \nu$ still belongs to $\mathcal{B}^0_{\rho_0}$ and one easily proves that

$$f_{k_\sigma}\left(\mathfrak{C}\left(\left(\rho_0 - \nu\right), g\right)\right) > 0, \tag{2.207}$$

which again is absurd. It follows that

$$\mathfrak{C}\left(\mathcal{B}^0_{\rho_0}, g\right) \subset \mathcal{B}^0_{\sigma_0}, \tag{2.208}$$

This implies that there are only two possibilities:

$$\mathfrak{C}\left(\mathcal{B}^+_{\rho_0}, g\right) = \mathcal{B}^+_{\sigma_0} \quad \text{or} \quad \mathfrak{C}\left(\mathcal{B}^+_{\rho_0}, g\right) = \mathcal{B}^-_{\sigma_0} \tag{2.209}$$

Since both $\mathcal{B}^+_{\rho_0}$ and $\mathcal{B}^+_{\sigma_0}$ are open sets contained in the interior of $\mathcal{M}$, the first of (2.209) can occur only in the trivial case $g = \pm Id$. In all other cases, then, $\mathcal{B}^-_{\sigma_0}$ is contained in $\mathfrak{C}(\mathcal{M}, g)$ and

$$\mathcal{B}_{\sigma_0} \subset \left(\mathcal{M} \cup \mathfrak{C}\left(\mathcal{M}, g\right)\right). \tag{2.210}$$

The thesis now follows from (2.210) by means of standard arguments of general topology. □

Suppose now that the matrices

$$g^1, g^2, \ldots, g^l \in \mathbf{GL}\left(n, \mathbb{Z}\right) \tag{2.211}$$

all satisfy the hypotheses of lemma 2.6.5. More generally one can consider finite collections of matrices of the form

$$Id,\ g^{j_1},\ g^{j_2} g^{j_1},\ g^{j_k} g^{j_{k-1}} \ldots g^{j_1}, \tag{2.212}$$

where $1 \leq j_1, j_2, \ldots, j_k \leq l$. One can prove that also in this case the set

$$\operatorname{int}\left(\mathcal{M} \cup \mathfrak{C}\left(\mathcal{M}, g^{j_1}\right) \cup \mathfrak{C}\left(\mathcal{M}, g^{j_2} g^{j_1}\right) \cup \ldots \cup \mathfrak{C}\left(\mathcal{M}, g^{j_k} g^{j_{k-1}} \ldots g^{j_1}\right)\right) \tag{2.213}$$

is connected.
It is possible to use this simple fact to try to determine algorithmically a set of matrices of $\mathbf{GL}(n, \mathbb{Z})$ which satisfies the hypotheses of proposition 2.6.4.
We report an example here below.

### 2.6.1 The case $2g_0 = 4$

Let us denote by

$$\sigma = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} & \sigma_{1,4} \\ \sigma_{1,2} & \sigma_{2,2} & \sigma_{2,3} & \sigma_{2,4} \\ \sigma_{1,3} & \sigma_{2,3} & \sigma_{3,3} & \sigma_{3,4} \\ \sigma_{1,4} & \sigma_{2,4} & \sigma_{3,4} & \sigma_{4,4} \end{bmatrix} \in Sym_{>0}(4, \mathbb{R}) \tag{2.214}$$

the generic real, symmetric and positive definite matrix of dimension 4. A fundamental domain $\mathcal{M}$ for the congruence action of $\mathbf{GL}(4, \mathbb{Z})$ on $Sym_{>0}(4, \mathbb{R})$ can be obtained by imposing the following conditions:

**i_**

$$\sigma_{1,1} \leq \sigma_{2,2} \leq \sigma_{3,3} \leq \sigma_{4,4}. \tag{2.215}$$

**ii_**

$$\sigma_{1,2} \geq 0, \quad \sigma_{2,3} \geq 0, \quad \sigma_{3,4} \geq 0. \tag{2.216}$$

**iii_**

$$\mathbf{m} \cdot \sigma \cdot \mathbf{m}^T \geq \sigma_{i,i} \tag{2.217}$$

for every

$$\mathbf{m} \in \left\{ \begin{array}{l} (1,0,1,0),\ (1,0,0,1),\ (0,1,0,1),\ (-1,1,0,0),\ (-1,0,1,0), \\ (-1,0,0,1),\ (0,-1,1,0),\ (0,-1,0,1),\ (0,0,-1,1),\ (0,1,-1,1), \\ (1,-1,0,1),\ (-1,1,0,1),\ (1,0,-1,1),\ (-1,0,-1,1),\ (1,-1,1,0), \\ (1,-1,1,1),\ (1,1,-1,1),\ (-1,-1,1,1),\ (-1,1,-1,1),\ (1,-1,-1,1) \end{array} \right\}, \tag{2.218}$$

the index $i$ in the right-hand side of (2.217) depending on $\mathbf{m}$ as follows:

$$i := \max \left\{ j \text{ such that } m_j \neq 0, \mathbf{m} = \left(m_j\right)_{j=1}^{4} \right\}. \tag{2.219}$$

This explicit result is due to E.S. Barnes and M.J. Cohn (see [1]). We used it to determine a fundamental domain $\mathcal{D}''$ for the congruence action of $\mathbb{K}_{4,1}$ on $Sym_{>0}(4, \mathbb{R})$. Our calculations are summarized here below:

The following elements of $\mathbf{GL}(4, \mathbb{Z})$ satisfy the hypotheses of lemma 2.6.5:

$$
\begin{aligned}
g^1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, &
g^2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
g^3 &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, &
g^4 &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
g^5 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, &
g^6 &= \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
g^7 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, &
g^8 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\
g^9 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, &
g^{10} &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
g^{11} &= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, &
g^{12} &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.
\end{aligned}
\tag{2.220}
$$

This can be verified considering for each matrix above the corresponding point of $\mathcal{M}$ from the list below:

$$
\begin{aligned}
&\rho_0^1 = \begin{bmatrix} 10 & 1 & 1 & 1 \\ 1 & 11 & 1 & 1 \\ 1 & 1 & 11 & 1 \\ 1 & 1 & 1 & 12 \end{bmatrix}, \qquad
\rho_0^2 = \begin{bmatrix} 10 & 1 & 1 & 1 \\ 1 & 10 & 1 & 1 \\ 1 & 1 & 11 & 1 \\ 1 & 1 & 1 & 12 \end{bmatrix}, \\
&\rho_0^3 = \begin{bmatrix} 10 & 1 & -5 & 1 \\ 1 & 11 & 1 & 1 \\ -5 & 1 & 12 & 1 \\ 1 & 1 & 1 & 13 \end{bmatrix}, \qquad
\rho_0^4 = \begin{bmatrix} 10 & 1 & 1 & -5 \\ 1 & 11 & 1 & 1 \\ 1 & 1 & 12 & 1 \\ -5 & 1 & 1 & 13 \end{bmatrix}, \\
&\rho_0^5 = \begin{bmatrix} 11 & 1 & 1 & 1 \\ 1 & 12 & 1 & -6 \\ 1 & 1 & 13 & 1 \\ 1 & -6 & 1 & 14 \end{bmatrix}, \qquad
\rho_0^6 = \begin{bmatrix} 10 & 5 & -1 & 1 \\ 5 & 11 & 1 & 1 \\ -1 & 1 & 12 & 1 \\ 1 & 1 & 1 & 13 \end{bmatrix}, \\
&\rho_0^7 = \begin{bmatrix} 11 & 1 & 1 & 1 \\ 1 & 12 & 6 & -1 \\ 1 & 6 & 13 & 1 \\ 1 & -1 & 1 & 14 \end{bmatrix}, \qquad
\rho_0^8 = \begin{bmatrix} 10 & 1 & 1 & 1 \\ 1 & 11 & 1 & 1 \\ 1 & 1 & 12 & 6 \\ 1 & 1 & 6 & 13 \end{bmatrix}, \\
&\rho_0^9 = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 5 & 1 & -2 \\ 1 & 1 & 7 & 3 \\ 1 & -2 & 3 & 8 \end{bmatrix}, \qquad
\rho_0^{10} = \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 5 & 1 & 2 \\ 1 & 1 & 6 & 1 \\ -1 & 2 & 1 & 7 \end{bmatrix}, \\
&\rho_0^{11} = \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 5 & 2 \\ -1 & 1 & 2 & 6 \end{bmatrix}, \qquad
\rho_0^{12} = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 5 & 2 & 1 \\ -1 & 2 & 6 & 1 \\ 1 & 1 & 1 & 7 \end{bmatrix}
\end{aligned}
\tag{2.221}
$$

A set of elements of $\mathbf{GL}(4,\mathbb{Z})$ satisfying the hypotheses of proposition 2.6.4 is the following:

$$
\begin{array}{lll}
g_1 = Id, & g_{11} = g^7 g^5 g^1, & g_{21} = g^9 g^2, \\
g_2 = g^1, & g_{12} = g^{12} g^5 g^1, & g_{22} = g^4, \\
g_3 = g^3 g^1, & g_{13} = g^7 g^1, & g_{23} = g^6 g^4, \\
g_4 = g^4 g^3 g^1, & g_{14} = g^{10} g^7 g^1, & g_{24} = g^6, \\
g_5 = g^5 g^3 g^1, & g_{15} = g^{10} g^1, & g_{25} = g^2 g^6, \\
g_6 = g^9 g^3 g^1, & g_{16} = g^{12} g^{10} g^1, & g_{26} = g^9 g^6, \\
g_7 = g^4 g^1, & g_{17} = g^{12} g^1, & g_{27} = g^8, \\
g_8 = g^7 g^4 g^1, & g_{18} = g^2, & g_{28} = g^{11}. \\
g_9 = g^{12} g^4 g^1, & g_{19} = g^3 g^2, & \\
g_{10} = g^5 g^1, & g_{20} = g^6 g^2, &
\end{array}
\tag{2.222}
$$

In fact, by means of a computer it is easy to verify that no two such matrices belong to the same left coset and, for what we observed above, the index of $\mathbb{K}_{4,1}$ in $\mathbf{GL}(4,\mathbb{Z})$ is 28; so this list is composed of exactly one representantive for each left coset. Moreover, (2.222) is the union of families of the type (2.212); thus it gives place to a set whose interior is connected.
We emphasize our final result in the following

**Theorem 2.6.6.** *A fundamental domain $\mathcal{D}''$ for the congruence action of $\mathbb{K}_{4,1}$ on*

*$Sym_{>0}(4, \mathbb{R})$ is given by*

$$\mathcal{D}'' = \bigcup_{j=1}^{28} \mathfrak{C}\left(\mathcal{M}, g_j\right) \tag{2.223}$$

*where the matrices $g_j$ are listed in (2.222).*

### 2.6.2 Discussion

The method presented here works, in principle, when $2g_0$ is an arbitrary positive and even integer but it requires the explicit description of the Minkowski fundamental domain $\mathcal{M}$.
Such a description, though not completely non-redundant, is available for $2g_0 = 6$ (see [15]); it would be interesting to try to work out the calculations in this case, with a more systematic use of a computer.
To the best of our knowledge, no explicit description of $\mathcal{M}$ is yet available when $2g_0$ is equal or larger than 8.

# Chapter 3

# A Degeneration of Two-Phase solutions

## 3.1 Formulation of the problem

Let $E_1$ and $E_3$ be two complex numbers such that

$$\mathrm{Re}\left(E_1\right) < \mathrm{Re}\left(E_3\right), \qquad \mathrm{Im}\left(E_1\right) > 0, \qquad \mathrm{Im}\left(E_3\right) > 0. \tag{3.1}$$

Let $\epsilon$ be real and positive. Put

$$E_2 = E_1 + \epsilon, \qquad \epsilon > 0. \tag{3.2}$$

Consider the $\epsilon$-dependent hyperelliptic curve of genus two

$$\begin{aligned} \Gamma_\epsilon \quad : \quad \mu^2 =& \prod_{j=1}^{3}\left(\lambda - E_j\right)\left(\lambda - \overline{E}_j\right) \\ =& \lambda^6 - S_1\left(\epsilon\right)\lambda^5 + S_2\left(\epsilon\right)\lambda^4 + \cdots \end{aligned} \tag{3.3}$$

Following [2] let us recall that the two-phase solutions associated to this family of curves read as follows:

$$\tilde{\psi}\left(x,t;\Delta_1,\Delta_2;\epsilon\right) = 2\sqrt{-\tilde{\chi}\left(\epsilon\right)}\frac{\theta_{\tilde{\boldsymbol{\tau}}(\epsilon)}\left[i\tilde{\mathbf{V}}\left(\epsilon\right)x + i\tilde{\mathbf{W}}\left(\epsilon\right)t - \tilde{\mathbf{D}} + \tilde{\mathbf{r}}\left(\epsilon\right)\right]}{\theta_{\tilde{\boldsymbol{\tau}}(\epsilon)}\left[i\tilde{\mathbf{V}}\left(\epsilon\right)x + i\tilde{\mathbf{W}}\left(\epsilon\right)t - \tilde{\mathbf{D}}\right]} \tag{3.4}$$

$$\times \exp\left[-i\tilde{E}\left(\epsilon\right)x + i\tilde{N}\left(\epsilon\right)t\right]. \tag{3.5}$$

where

$$\tilde{\mathbf{D}} = \begin{pmatrix} i\Delta_1 \\ i\Delta_2 \end{pmatrix}, \quad \Delta_1, \Delta_2 \in \mathbb{R}. \tag{3.6}$$

Considering the basis in the homology of $\Gamma_\epsilon$ Fig 3.1, the parameters appearing in (3.4) are defined as follows

$$\tau_{i,j}\left(\epsilon\right) = \oint_{b_j}\tilde{\omega}_i \qquad i,j = 1,2. \tag{3.7}$$

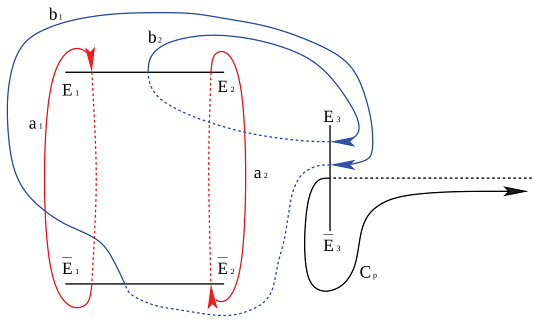



Figure 3.1: *The basis $\tilde{B}$ in the homology of* $\Gamma_\epsilon$.

$$\tilde{V}_j(\epsilon) = \oint_{\tilde{b}_j} d\tilde{\Omega}_1, \qquad \tilde{W}_j(\epsilon) = \oint_{\tilde{b}_j} d\tilde{\Omega}_2, \qquad \tilde{r}_j(\epsilon) = -\int_{\tilde{b}_j} d\tilde{\Omega}_3 \qquad j = 1, 2. \tag{3.8}$$

Here $\tilde{\omega}_1, \tilde{\omega}_2$ indicate the normalized holomorphic differentials, while $d\tilde{\Omega}_1, d\tilde{\Omega}_2$ and $d\tilde{\Omega}_3$ are the normalized abelian differentials with poles only in $\infty^\pm$ such that

$$\int_{\overline{E}_3}^{P} d\tilde{\Omega}_1 = \pm\left(\lambda - \frac{\tilde{E}(\epsilon)}{2} + o(1)\right), \quad P \to \infty^\pm \tag{3.9}$$

$$\int_{\overline{E}_3}^{P} d\tilde{\Omega}_2 = \pm\left(2\lambda^2 + \frac{\tilde{N}(\epsilon)}{2} + o(1)\right), \quad P \to \infty^\pm \tag{3.10}$$

$$\int_{\overline{E}_3}^{P} d\tilde{\Omega}_3 = \pm\left(\log(\lambda) - \left(\frac{1}{2}\right)\log(\tilde{\chi}(\epsilon)) + o(1)\right), \quad P \to \infty^\pm \tag{3.11}$$

These relations also define the quantities $\tilde{E}$, $\tilde{N}$ and $\tilde{\chi}$. Notice that all the algebro-geometric parameters now depend on $\epsilon$.

In this chapter we study the asymptotic behaviour of the solution

$$\tilde{\psi}(x, t; \Delta_1, \Delta_2; \epsilon), \qquad \epsilon \to 0^+ \tag{3.12}$$

in the limit when $\epsilon$ tends to zero from the right. The parameters $\Delta_1$ and $\Delta_2$ are assumed to be fixed constants independent of $\epsilon$.
In order to perform this investigation one needs to study the limiting behaviour of all the algebro-geometric quantities. This analysis is significantly simplified if the two-phase solutions are constructed using an appropriate basis in the homology of $\Gamma_\epsilon$. A possible one is the basis $\mathcal{B}$ indicated in Fig 3.3. Notice that it respects the symmetries

$$r_\star(a_1) = -a_2, \qquad r_\star(b_1) = b_2 \tag{3.13}$$

where $r$ is the anti-holomorphic involution

$$r \quad : \quad (\mu, \lambda) \quad \to \quad (\overline{\mu}, \overline{\lambda}). \tag{3.14}$$

The convenience of such basis in this framework is actually our motivation to consider specifically symmetries (3.13) in our attempt to put in act the effectivization of the two-phase solutions to fNLS equation.

The main point of our strategy is then to recast (3.4) in terms of the new basis in the homology. This calculation is worked out in section 2. The behaviour of the algebro-geometric parameters in the limit when $\epsilon$ tends to zero is studied in section 3. It turns out that all of them admit a finite limit except for the period matrix. The latter has the asymptotic behavior

$$\boldsymbol{\tau}(\epsilon) = \begin{bmatrix} 2\log\epsilon & 0 \\ 0 & 2\log\epsilon \end{bmatrix} + \mathcal{O}(1) \qquad \epsilon \to 0^+. \tag{3.15}$$

In section 4 we study the degeneration of the corresponding theta-function and finally determine the limit of the solution. For generic values of $\Delta_1$ and $\Delta_2$ this is a plane wave. If instead

$$\Delta_1 = \Delta_2 \tag{3.16}$$

one obtains

$$\begin{aligned} \lim_{\epsilon\to 0^+} \psi\left(x, t; \frac{1}{2}, q; \epsilon\right) = -2\sqrt{-\chi_0}\frac{\cosh(\Phi - i\sigma) + B\cos(\Psi - 2\pi q - i\rho)}{\cosh\Phi + B\cos(\Psi - 2\pi q)} \\ \times \exp(-iE_0 x + iN_0 t). \end{aligned} \tag{3.17}$$

Here we have put

$$\Phi = \eta x + \phi t, \qquad \Psi = \xi x + \theta t. \tag{3.18}$$

All the parameters appearing in (3.17-3.18) can be expressed in terms of $E_1$ and $E_3$ by means of elementary functions (see theorem 3.4.3). This explicit solution to fNLS was first discovered by Zakharov and Gelash in 2011, who obtained it by means of a local $\overline{\partial}$-problem (see [9]).

**Observation 3.1.1.** *Consider two arbitrary points $E_1$ and $E_2$ in the complex plane keeping valid the conditions*

$$Im(E_1) > 0, \qquad Im(E_2) > 0, \qquad Re(E_1) > Re(E_1). \tag{3.19}$$

*It is always possible to determine a real Moebius transformation of the form*

$$w(z) = \frac{Az + B}{Cz + D}, \qquad z \in \mathbb{C},\ A, B, C, D \in \mathbb{R} \tag{3.20}$$

*such that the images of these points lie on a horizontal line:*

$$Im(w(E_1)) = Im(w(E_2)). \tag{3.21}$$

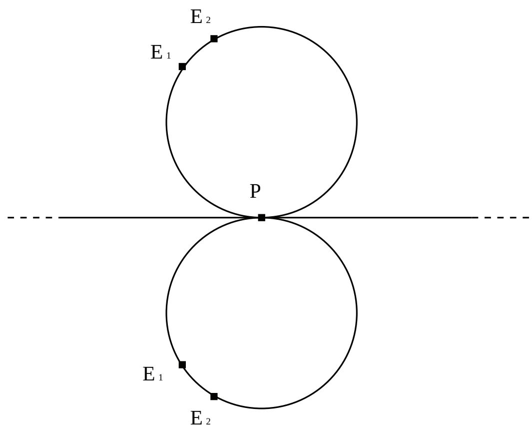
*Since the coefficients of (3.20) are real, one will also have*

$$Im\left(w\left(\overline{E}_1\right)\right) = Im\left(w\left(\overline{E}_2\right)\right). \tag{3.22}$$

*Such a transformation can be determined in the following way (see Fig 3.2): Consider the circle $\mathcal{C}$ passing through the ponits $E_1$, $E_2$ and tangent to the real line in some point $P$. The Schwarz reflected of $\mathcal{C}$ is a circle passing through $\overline{E}_1$, $\overline{E}_2$ and tangent to the real line in the same point $P$. There exists a real Moebius transformation of the form (3.20) such that*

$$w(P) = \infty. \tag{3.23}$$

Figure 3.2: *How to reduce to the case $\epsilon > 0$.*




*The images $w(\mathcal{C})$ and $w(\overline{\mathcal{C}})$ via this one are two lines. Still they are the Schwarz reflected of each other. Moreover, each of them has no intersection with the other or with the real line if not at $\infty$. As a consequence, this transformation satisfies both (3.21) and (3.22). In view of this fact, the requirement for the small parameter $\epsilon$ to be real does not imply any loss of generality while studying the limit of the solution (3.4) and its first correction.*

## 3.2 Changing the basis in the homology

It seems to be a standard in the literature that algebro-geometric solutions to fNLS equation are constructed starting from the basis in the homology indicated in Fig 3.1. But this is not the only possibility and, most important, it is far from being the most convenient to perform our calculations. The basis

$$\mathcal{B} = \{a_1, a_2; b_1, b_2\} \tag{3.24}$$

indicated in Fig 3.3 is much more suitable to our purposes, indeed. In this section we will reformulate the original problem in terms of this basis.

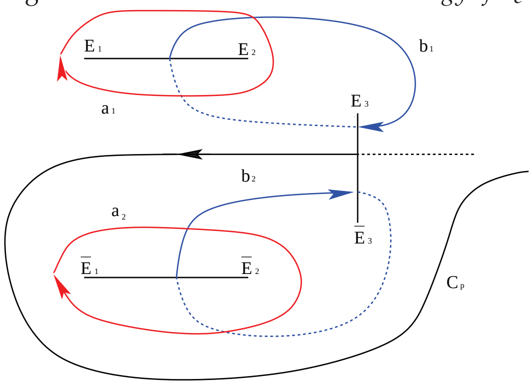
Let us introduce the normalized holomorphic differentials $\omega_1$ and $\omega_2$ individuated by the condition

$$\oint_{a_j} \omega_k = (2\pi i)\,\delta_{jk}. \qquad j,k = 1,2. \tag{3.25}$$

Let us also consider the meromorphic differentials $d\Omega_1$, $d\Omega_2$ and $d\Omega_3$ determined by the following conditions:

- $$\oint_{a_j} d\Omega_k = 0 \qquad j = 1,2;\ k = 1,2,3. \tag{3.26}$$

Figure 3.3: *The new basis $\mathcal{B}$ in the homology of $\Gamma_\epsilon$.*




(Notice that the normalization is now taken w.r.t. the new basis in the homology)

- 

$$d\Omega_1(P) = \mp\left[\frac{1}{z^2} + \mathcal{O}(1)\right] dz, \quad P \to \infty^\pm \tag{3.27a}$$

$$d\Omega_2(P) = \mp\left[\frac{4}{z^3} + \mathcal{O}(1)\right] dz, \quad P \to \infty^\pm \tag{3.27b}$$

$$d\Omega_3(P) = \mp\left[\frac{1}{z} + \mathcal{O}(1)\right] dz, \quad P \to \infty \tag{3.27c}$$

where $z = \frac{1}{\lambda}$.

- $d\Omega_k(P)$ is holomorphic on $\Gamma_\epsilon \backslash \{\infty^+, \infty^-\}$.

Let us consider the function

$$\begin{aligned}\psi(x,t;p,q;\epsilon) =& 2\sqrt{-\chi(\epsilon)}\exp\left(\mathbf{r}^T(\epsilon)\cdot\mathbf{p}\right)\\ &\frac{\theta_{\boldsymbol{\tau}(\epsilon)}\left(i\mathbf{V}(\epsilon)x + i\mathbf{W}(\epsilon)t + \boldsymbol{\tau}(\epsilon)\cdot\mathbf{p} - 2\pi i\mathbf{q} + \mathbf{r}(\epsilon)\right)}{\theta_{\boldsymbol{\tau}(\epsilon)}\left(i\mathbf{V}(\epsilon)x + i\mathbf{W}(\epsilon)t + \boldsymbol{\tau}(\epsilon)\cdot\mathbf{p} - 2\pi i\mathbf{q}\right)}\\ &\exp\left(-iE(\epsilon)x + iN(\epsilon)t\right)\end{aligned} \tag{3.28}$$

Here $\mathbf{p}$ and $\mathbf{q}$ are arbitrary vectors of the form

$$\mathbf{p} = \begin{pmatrix} p \\ -p \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} q \\ q \end{pmatrix}, \qquad p, q \in \mathbb{R}. \tag{3.29}$$

All the remaining parameters appearing in (3.28) are geometrical quantities to be computed w.r.t. the basis in the homology of $\Gamma_\epsilon$ indicated in Fig 3.3:

The value of $E, N$ and $\chi$ is deduced by the asymptotic relations

$$\int_{\overline{E}_3}^{P} \mathrm{d}\Omega_1 = \pm \left[\lambda - \frac{E}{2} + o(1)\right], \quad P \to \infty^{\pm} \tag{3.30a}$$

$$\int_{\overline{E}_3}^{P} \mathrm{d}\Omega_2 = \pm \left[2\lambda^2 + \frac{N}{2} + o(1)\right], \quad P \to \infty^{\pm} \tag{3.30b}$$

$$\int_{\overline{E}_3}^{P} \mathrm{d}\Omega_3 = \pm \left[\log(\lambda) - \frac{1}{2}\log(\chi) + o(1)\right], \quad P \to \infty^{\pm} \tag{3.30c}$$

The vectors $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{r}$ are defined as[1]

$$V_j = \oint_{b_j} \mathrm{d}\Omega_1, \qquad W_j = \oint_{b_j} \mathrm{d}\Omega_2, \qquad r_j = \int_{C_p} \omega_j, \tag{3.31}$$

for $j = 1, 2$. One has the following

**Proposition 3.2.1.** *Let*

$$p = \frac{1}{2}\left(\frac{\Delta_1 - \Delta_2}{\pi} - 1\right), \qquad q = \frac{\Delta_2}{2\pi}. \tag{3.32}$$

*Then*

$$\tilde{\psi}(x, t; \Delta_1, \Delta_2; \epsilon) = \psi(x, t; p, q; \epsilon) \qquad \forall\, x, t \in \mathbb{R};\ \forall \epsilon > 0. \tag{3.33}$$

As an obvious consequence of this result, the original problem of studying the behaviour of $\tilde{\psi}$ is equivalent to the analogous problem for $\psi$.

$$\lim_{\epsilon \to 0^+} \tilde{\psi}(x, t; \Delta_1, \Delta_2; \epsilon) = \lim_{\epsilon \to 0^+} \psi(x, t; p, q; \epsilon) \tag{3.34}$$

On the other side, calculations for $\psi$ are much faster and simpler because the basis in the homology is more appropriate for them. In view of this fact in the next sections we will compute the right hand side of (3.34). The remaining part of this one will be dedicated to the proof of proposition 3.2.1 instead.

Let us first observe that the two basis in the homology introduced above are related to each other by the following linear transformation:

$$\begin{pmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \tag{3.35}$$

where

$$D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.36}$$

[1]Throughout this chapter we used the path indicated in Fig 3.1 to compute $\mathbf{r}$, while its genuine definition would prescribe to use the one indicated in fig 3.3. This small discrepancy, though, does not affect our calculations, which remain rigorous.

As prescribed by the theory of homolgy

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}\,(4, \mathbb{Z})\,. \tag{3.37}$$

Let us define the matrix

$$M := 2\pi i D + C \cdot \boldsymbol{\tau}. \tag{3.38}$$

**Lemma 3.2.2.** *The abelian differentials defined above are related to each other as follows:*

**i_**

$$\tilde{\boldsymbol{\omega}} = 2\pi i \left(M^T\right)^{-1} \cdot \boldsymbol{\omega}. \tag{3.39}$$

**ii_**

$$d\tilde{\Omega}_1 = d\Omega_1 - \mathbf{V}^T \cdot C^T \cdot \left(M^T\right)^{-1} \cdot \boldsymbol{\omega}. \tag{3.40}$$

**iii_**

$$d\tilde{\Omega}_2 = d\Omega_2 - \mathbf{W}^T \cdot C^T \cdot \left(M^T\right)^{-1} \cdot \boldsymbol{\omega}. \tag{3.41}$$

**iv_**

$$d\tilde{\Omega}_3 = d\Omega_3 + \mathbf{r}^T \cdot C^T \cdot \left(M^T\right)^{-1} \cdot \boldsymbol{\omega}. \tag{3.42}$$

*Proof.* **i_** One has

$$\tilde{\boldsymbol{\omega}} = X \cdot \boldsymbol{\omega} \tag{3.43}$$

for some two dimensional, complex, constant matrix $X$ to be determined. From the normalization condition one obtains

$$\begin{aligned} 2\pi i \delta_{jk} &= \oint_{\tilde{b}_k} \tilde{\omega}_j \\ &= \oint_{D_{kh} a_h + C_{kh} b_h} X_{jm} \omega_m \\ &= X_{jm} \left[ D_{kh} \oint_{a_h} \omega_m + C_{kh} \oint_{b_h} \omega_m \right] \\ &= \left(X^T\right)_{mj} \left[ (2\pi i)\, D_{km} + C_{kh} \tau_{hm} \right]. \end{aligned} \tag{3.44}$$

It is understood that Einstein summation notation has been used. In vectorial notation (3.44) reads

$$X^T \cdot M = 2\pi i \mathrm{Id}. \tag{3.45}$$

Equation (3.39) follows immediately.

**ii − iii_** Recall that

$$\tilde{a}_j = (D \cdot \mathbf{a} + C \cdot \mathbf{b})_j = D_{jh} a_h + C_{jh} b_h, \qquad j = 1, 2. \tag{3.46}$$

Plugging this into the normalization condition for $\mathrm{d}\Omega_1$ one gets

$$0 = \oint_{\tilde{a}_j} \mathrm{d}\tilde{\Omega}_1 = D_{jh} \oint_{a_h} \mathrm{d}\tilde{\Omega}_1 + C_{jh} \oint_{b_h} \mathrm{d}\tilde{\Omega}_1. \tag{3.47}$$

Now, since $\mathrm{d}\tilde{\Omega}_1$ and $\mathrm{d}\Omega_1$ have poles in the same points, with the same principal parts, one has

$$\mathrm{d}\tilde{\Omega}_1 - \mathrm{d}\Omega_1 = Y_s \omega_s \tag{3.48}$$

for some complex constants $Y_1$ and $Y_2$. Using (3.48) one can rewrite (3.47) as follows:

$$0 = D_{jh} \oint_{a_h} (\mathrm{d}\Omega_1 + Y_s \omega_s) + C_{jh} \oint_{b_h} (\mathrm{d}\Omega_1 + Y_s \omega_s) \tag{3.49}$$

$$= D_{jh} \left( \oint_{a_h} \omega_s \right) Y_s + C_{jh} \oint_{b_h} \mathrm{d}\Omega_1 + C_{jh} \left( \oint_{b_h} \omega_s \right) Y_s \tag{3.50}$$

$$= D_{jh} (2\pi i) \, \delta_{hs} Y_s + C_{jh} V_h + C_{jh} \tau_{hs} Y_s \tag{3.51}$$

$$= M_{js} Y_s + C_{jh} V_h \qquad j = 1, 2. \tag{3.52}$$

In vectorial form, this last one reads

$$M \cdot \mathbf{Y} + C \cdot \mathbf{V} = 0 \tag{3.53}$$

or equivalently

$$\mathbf{Y} = -M^{-1} \cdot C \cdot \mathbf{V} \tag{3.54}$$

Plugging (3.54) into (3.48) yields (3.40). The proof of (3.41) and (3.42) is analogous.

□

**Lemma 3.2.3.** *One has*

**i_**

$$\tilde{\boldsymbol{\tau}} = 2\pi i \, (A \cdot \boldsymbol{\tau} + 2\pi i B) \cdot (C \cdot \boldsymbol{\tau} + 2\pi i D)^{-1} \tag{3.55}$$

**ii_**

$$\tilde{\mathbf{r}} = 2\pi i \left( M^T \right)^{-1} \cdot \mathbf{r} \tag{3.56}$$

**iii_**

$$\tilde{\mathbf{V}} = 2\pi i \left( M^T \right)^{-1} \cdot \mathbf{V} \tag{3.57}$$

**iv_**

$$\tilde{\mathbf{W}} = 2\pi i \left( M^T \right)^{-1} \cdot \mathbf{W} \tag{3.58}$$

*Proof.* **i_** see [8].

**ii**_ This is an obvious consequence of (3.39).

**iii** − **iv**_ For the ease of notation let us put

$$\mathbf{Y} := -M^{-1} \cdot C \cdot \mathbf{V}. \tag{3.59}$$

In this way,

$$\mathrm{d}\tilde{\Omega}_1 = \mathrm{d}\Omega_1 + \mathbf{Y}^T \cdot \boldsymbol{\omega}. \tag{3.60}$$

Simply applying the different definitions one has

$$\tilde{V}_j = \oint_{\tilde{b}_j} \mathrm{d}\tilde{\Omega}_1 \tag{3.61}$$

$$= \oint_{B_{jh}a_h + B_{jh}a_h} \mathrm{d}\Omega_1 + \oint_{B_{jh}a_h + A_{jh}b_h} Y_1\omega_1 + Y_2\omega_2 \tag{3.62}$$

$$= A_{jh}V_h + \left[ B_{jh} \oint_{a_h} \omega_s + A_{jh} \oint_{b_h} \omega_s \right] Y_s \tag{3.63}$$

$$= A_{jh}V_h + \left[ 2\pi i B_{js} + A_{jh}\tau_{hs} \right] Y_s \qquad j = 1, 2. \tag{3.64}$$

In vectorial notation this reads

$$\tilde{\mathbf{V}} = A \cdot \mathbf{V} + (2\pi i B + A \cdot \tau) \cdot \mathbf{Y}. \tag{3.65}$$

Using (3.59) and recalling (3.38) this last equation can be rewritten as follows:

$$\begin{aligned}
\tilde{\mathbf{V}} &= A \cdot \mathbf{V} - (2\pi i B + A \cdot \boldsymbol{\tau}) \cdot (2\pi i D + C \cdot \boldsymbol{\tau})^{-1} \cdot C \cdot \mathbf{V} \\
&= \frac{1}{2\pi i} \left[ 2\pi i A - 2\pi i \,(A \cdot \boldsymbol{\tau} + 2\pi i B)\,(C \cdot \boldsymbol{\tau} + 2\pi i D)^{-1}\, C \right] \mathbf{V} \\
&= \frac{1}{2\pi i} \,(2\pi i A - \tilde{\boldsymbol{\tau}} \cdot C) \cdot \mathbf{V} \\
&= \left[ \frac{1}{2\pi i} \left( 2\pi i A^T - C^T \cdot \tilde{\boldsymbol{\tau}} \right) \right]^T \cdot \mathbf{V}.
\end{aligned} \tag{3.66}$$

Here above we have used the transformation law for the period matrix (3.55).
By means of the same property one easily obtains:

$$\frac{1}{2\pi i} \left( 2\pi i A^T - C^T \cdot \tilde{\boldsymbol{\tau}} \right) \cdot \left( \frac{1}{2\pi i} M \right) = \tag{3.67}$$

$$\frac{1}{2\pi i} \left[ 2\pi i A^T - 2\pi i C^T \,(2\pi i B + A \cdot \boldsymbol{\tau})\,(2\pi i D + C \cdot \boldsymbol{\tau})^{-1} \right] \cdot \left[ \frac{1}{2\pi i} \,(2\pi i D + C \cdot \boldsymbol{\tau}) \right] = \tag{3.68}$$

$$\frac{1}{2\pi i} \left[ A^T \cdot (2\pi i D + C \cdot \boldsymbol{\tau}) - C^T \cdot (2\pi i B + A \cdot \boldsymbol{\tau}) \right] = \tag{3.69}$$

$$\frac{1}{2\pi i} \left[ 2\pi i \left( A^T \cdot D - C^T \cdot B \right) + \left( A^T \cdot C - C^T \cdot A \right) \cdot \boldsymbol{\tau} \right] = Id. \tag{3.70}$$

The last equality holds because the matrix $G$ belongs to $\mathbf{Sp}(4,\mathbb{Z})$. So

$$\frac{1}{2\pi i}\left(2\pi i A^T - C^T\tilde{\boldsymbol{\tau}}\right) = \left(\frac{1}{2\pi i}M\right)^{-1} \tag{3.71}$$

Plugging (3.71) into (3.66) one gets (3.57). The proof of (3.58) is completely analogous.

□

**Lemma 3.2.4.** *The quantities $E, N$ and $\chi$ are related to the quantities $\tilde{E}, \tilde{N}$ and $\tilde{\chi}$ as follows:*

**i_**

$$\tilde{E} = E + \mathbf{V}^T \cdot C^T \cdot \left(M^{-1}\right)^T \cdot \mathbf{r} \tag{3.72}$$

**ii_**

$$\tilde{N} = N - \mathbf{W}^T \cdot C^T \cdot \left(M^{-1}\right)^T \cdot \mathbf{r} \tag{3.73}$$

**iii_**

$$\tilde{\chi} = \chi \exp\left[-\mathbf{r}^T \cdot C^T \cdot \left(M^{-1}\right)^T \cdot \mathbf{r}\right] \tag{3.74}$$

*Proof.* From the definition of $\tilde{E}$, one immediately deduces the following equivalent one:

$$\tilde{E} = 2\lim_{\lambda\to\infty}\left[\lambda - \int_{\overline{E}_3}^{\lambda} d\tilde{\Omega}_1\right]. \tag{3.75}$$

Making use of (3.40) one obtains

$$\begin{aligned}\tilde{E} &= 2\lim_{\lambda\to\infty}\left\{\lambda - \int_{\overline{E}_3}^{\lambda}\left[d\Omega_1 - \mathbf{V}^T \cdot C^T \cdot \left(M^{-1}\right)^T \cdot \boldsymbol{\omega}\right]\right\} \\ &= 2\lim_{\lambda\to\infty}\left[\lambda - \int_{\overline{E}_3}^{\lambda} d\Omega_1\right] + \mathbf{V}^T \cdot C^T \cdot \left(M^{-1}\right)^T \cdot \mathbf{r} \\ &= E + \mathbf{V}^T \cdot C^T \cdot \left(M^{-1}\right)^T \cdot \mathbf{r}\end{aligned} \tag{3.76}$$

This is (3.72). The proof of (3.73) and (3.74) is completely analogous.

□

The theta function of genus two with characteristics

$$\begin{bmatrix}\mathbf{g}\\ \mathbf{h}\end{bmatrix} \in \mathbb{R}^4 \tag{3.77}$$

can be defined as follows:

$$\theta_{\boldsymbol{\tau}}\begin{bmatrix}\mathbf{g}\\ \mathbf{h}\end{bmatrix}(\mathbf{z}) = \exp\left\{\frac{1}{2}\mathbf{g}^T \cdot \boldsymbol{\tau} \cdot \mathbf{g} + (\mathbf{z} + 2\pi i\mathbf{h})^T \cdot \mathbf{g}\right\}\theta_{\boldsymbol{\tau}}(\mathbf{z}+\mathbf{e}), \quad \mathbf{z} \in \mathbb{C}^2. \tag{3.78}$$

Here

$$\mathbf{e} = 2\pi i\mathbf{h} + \boldsymbol{\tau} \cdot \mathbf{g}. \tag{3.79}$$

Notice that for $\mathbf{g} = \mathbf{h} = \mathbf{0}$ one gets the standard theta function . From the classical theory one has the following

**Proposition 3.2.5.** *Theta functions with characteristics computed w.r.t. the basis $\tilde{\mathcal{B}}$ and $\mathcal{B}$ (Fig 3.1 and 3.3 respectively) are related by the following formula:*

$$\begin{aligned}&\theta_{\tilde{\tau}}\begin{bmatrix}\tilde{\mathbf{g}}\\ \tilde{\mathbf{h}}\end{bmatrix}\left[2\pi i\left(M^T\right)^{-1}\cdot\mathbf{z}\right]\\ &=\kappa\,(\det M)^{\frac{1}{2}}\exp\left\{\frac{1}{2}\sum_{i\leq j\leq 2}z_iz_j\frac{\partial\log\det M}{\partial\tau_{ij}}\right\}\theta_{\tau}\begin{bmatrix}\mathbf{g}\\ \mathbf{h}\end{bmatrix}(\mathbf{z}).\end{aligned}\tag{3.80}$$

*The characteristics are meant to satisfy the following equation*

$$\begin{bmatrix}\tilde{\mathbf{g}}\\ \tilde{\mathbf{h}}\end{bmatrix}=\begin{pmatrix}D & -C\\ -B & A\end{pmatrix}\cdot\begin{bmatrix}\mathbf{g}\\ \mathbf{h}\end{bmatrix}+\frac{1}{2}\begin{bmatrix}diag\left(C\cdot D^T\right)\\ diag\left(A\cdot B^T\right)\end{bmatrix}\tag{3.81}$$

*$\kappa$ is a complex constant.*

For a proof of this proposition in its full generality see [8] and references therein. The value of $\kappa$ and the determination of the quare root of the determinant of $M$ will be irrelevant to our purposes. In order to simplify formula (3.80) in our specific case we prove use the following

**Lemma 3.2.6.** *One has*

$$\sum_{i\leq j\leq 2}z_iz_j\frac{\partial\log\det M}{\partial\tau_{ij}}=\mathbf{z}^T\cdot C^T\cdot\left(M^{-1}\right)^T\cdot\mathbf{z},\quad\mathbf{z}=\begin{pmatrix}z_1\\ z_2\end{pmatrix}\in\mathbb{C}^2.\tag{3.82}$$

*Moreover*

$$M^{-1}\cdot C=C^T\cdot\left(M^{-1}\right)^T.\tag{3.83}$$

*Proof.* The thesis follows from a straightforward though long calculation. □

We are finally ready to prove proposition 3.2.1:

*Proof of proposition 3.2.1.* Using lemma (3.2.3) and (3.2.4) one can rewrite

$$\begin{aligned}&\tilde{\psi}(x,t;\Delta_1,\Delta_2;\epsilon)\\ &=2\sqrt{-\chi}\exp\left[-\left(i\mathbf{V}^Tx+i\mathbf{W}^Tt+\frac{1}{2}\mathbf{r}^T\right)\cdot C^T\cdot\left(M^{-1}\right)^T\right.\\ &\quad\left.\cdot\,\mathbf{r}\right]\frac{\theta_{\tilde{\tau}}\left\{\left(\frac{1}{2\pi i}M^{-1}\right)^T\cdot\left[i\mathbf{V}x+i\mathbf{W}t-\left(\frac{1}{2\pi i}M^T\right)\cdot\tilde{\mathbf{D}}+\mathbf{r}\right]\right\}}{\theta_{\tilde{\tau}}\left\{\left(\frac{1}{2\pi i}M^{-1}\right)^T\cdot\left[i\mathbf{V}x+i\mathbf{W}t-\left(\frac{1}{2\pi i}M^T\right)\cdot\tilde{\mathbf{D}}\right]\right\}}\exp\left(-iEx+iNt\right).\end{aligned}\tag{3.84}$$

Now, for the ease of notation let us put

$$\mathbf{z}=i\mathbf{V}x+i\mathbf{W}t-\left(\frac{1}{2\pi i}M^T\right)\cdot\tilde{\mathbf{D}}.\tag{3.85}$$

Using the transformation law (3.80) one can simplify the ratio appearing in (3.84) as follows:

$$
\begin{aligned}
&\frac{\theta_{\tilde{\tau}}\left[\left(\frac{1}{2\pi i}M^T\right)^{-1}\cdot(\mathbf{z}+\mathbf{r})\right]}{\theta_{\tilde{\tau}}\left[\left(\frac{1}{2\pi i}M^T\right)^{-1}\cdot\mathbf{z}\right]}\\
&=\frac{\kappa\,(\det M)^{\frac{1}{2}}\exp\left[\frac{1}{2}(\mathbf{z}+\mathbf{r})^T\cdot C^T\cdot\left(M^{-1}\right)^T\cdot(\mathbf{z}+\mathbf{r})\right]\theta_\tau\begin{bmatrix}\mathbf{g}\\\mathbf{h}\end{bmatrix}(\mathbf{z}+\mathbf{r})}{\kappa\,(\det M)^{\frac{1}{2}}\exp\left[\frac{1}{2}\mathbf{z}^T\cdot C^T\cdot\left(M^{-1}\right)^T\cdot\mathbf{z}\right]\theta_\tau\begin{bmatrix}\mathbf{g}\\\mathbf{h}\end{bmatrix}(\mathbf{z})}\\
&=\exp\left[\mathbf{z}^T\cdot C^T\cdot\left(M^{-1}\right)^T\cdot\mathbf{r}+\frac{1}{2}\mathbf{r}^T\cdot C^T\cdot\left(M^{-1}\right)^T\cdot\mathbf{r}\right]\cdot\frac{\theta_\tau\begin{bmatrix}\mathbf{g}\\\mathbf{h}\end{bmatrix}(\mathbf{z}+\mathbf{r})}{\theta_\tau\begin{bmatrix}\mathbf{g}\\\mathbf{h}\end{bmatrix}(\mathbf{z})}.
\end{aligned}
\tag{3.86}
$$

Notice that in the last equality we used (3.83). Resubstituting **z** by expression (3.85), plugging (3.86) in (3.84) and simplifying one gets

$$
\begin{aligned}
&\tilde{\psi}(x,t;\Delta_1,\Delta_2;\epsilon)\\
&\quad=2\sqrt{-\chi}\exp\left[-\frac{1}{2\pi i}\tilde{\mathbf{D}}^T\cdot C\right.\\
&\qquad\left.\cdot\,\mathbf{r}\right]\frac{\theta_\tau\begin{bmatrix}\mathbf{g}\\\mathbf{h}\end{bmatrix}\left[i\mathbf{V}x+i\mathbf{W}t-\left(\frac{1}{2\pi i}M^T\right)\cdot\tilde{\mathbf{D}}+\mathbf{r}\right]}{\theta_\tau\begin{bmatrix}\mathbf{g}\\\mathbf{h}\end{bmatrix}\left[i\mathbf{V}x+i\mathbf{W}t-\left(\frac{1}{2\pi i}M^T\right)\cdot\tilde{\mathbf{D}}\right]}\exp(-iEx+iNt)
\end{aligned}
\tag{3.87}
$$

Now, in our specific case

$$
\begin{bmatrix}\tilde{\mathbf{g}}\\\tilde{\mathbf{h}}\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\\mathbf{0}\end{bmatrix}. \tag{3.88}
$$

Plugging (3.36) and (3.88) into equation (3.81) yields

$$
\mathbf{g}=\begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\qquad\mathbf{h}=\begin{bmatrix}0\\0\end{bmatrix}. \tag{3.89}
$$

Let us put

$$
\mathbf{e}=\boldsymbol{\tau}\cdot\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}. \tag{3.90}
$$

Using property (3.78) one can rewrite $\tilde{\psi}$ as follows:

$$
\begin{aligned}
\tilde{\psi}(x,t;\Delta_1,\Delta_2;\epsilon)=2\sqrt{-\chi}\exp&\left[-\frac{1}{2\pi i}\tilde{\mathbf{D}}^T\cdot C\cdot\mathbf{r}+\mathbf{g}^T\right.\\
&\left.\cdot\,\mathbf{r}\right]\frac{\theta_\tau\left[i\mathbf{V}x+i\mathbf{W}t-\left(\frac{1}{2\pi i}M^T\right)\cdot\tilde{\mathbf{D}}+\mathbf{e}+\mathbf{r}\right]}{\theta_\tau\left[i\mathbf{V}x+i\mathbf{W}t-\left(\frac{1}{2\pi i}M^T\right)\cdot\tilde{\mathbf{D}}+\mathbf{e}\right]}\exp(-iEx+iNt)
\end{aligned}
\tag{3.91}
$$

Some simple algebraic manipulations show that

$$\mathbf{e} - \frac{1}{2\pi i} M^T \cdot \tilde{\mathbf{D}} = -2\pi i \mathbf{q} + \boldsymbol{\tau} \cdot \mathbf{p} \tag{3.92a}$$

and that

$$-\frac{1}{2\pi i} \tilde{\mathbf{D}}^T \cdot C \cdot \mathbf{r} + \mathbf{g}^T \cdot \mathbf{r} = \mathbf{p}^T \cdot \mathbf{r}. \tag{3.92b}$$

Plugging (3.92a) and (3.92b) into (3.91) gives the thesis. □

## 3.3 Degeneration of the geometrical quantities

In this section we consider the behaviour of the geometrical quantities $\mathbf{V}$, $\mathbf{W}$, $\mathbf{r}$, $E$, $N$ and $\chi$ appearing in (3.28), in the limit when $\epsilon$ tends to zero from the right.

**Proposition 3.3.1.** *For small values of $\epsilon$, the period matrix $\boldsymbol{\tau}(\epsilon)$ behaves as follows:*

$$\boldsymbol{\tau}(\epsilon) = \begin{bmatrix} 2log(\epsilon) & 0 \\ 0 & 2log(\epsilon) \end{bmatrix} + \begin{bmatrix} \alpha_0 & \beta_0 \\ \beta_0 & \alpha_0 \end{bmatrix} + o(1), \quad \epsilon \to 0^+. \tag{3.93}$$

*Here*[2]

$$\begin{aligned} \alpha_0 &= -4\log(2) + 2log\left[\frac{E_1 - Re(E_3)}{\sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)}} - 1\right] \\ &\quad - 2\log\left[E_1 - E_3 - \sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)}\right] \\ &\quad + 2\log\left\{-i\left[\sqrt{(\lambda - E_3)\left(\lambda - \overline{E}_3\right)} - \lambda + \sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)} + E_1\right]\right\}\Bigg|_{\lambda=E1}^{\lambda=E_3} \end{aligned} \tag{3.96}$$

*and*

$$\begin{aligned} \beta_0 = 2\log\Bigg\{ & \frac{1}{\left(E_1 - \overline{E}_1\right)\left(E_3 - \overline{E}_3\right)} \Bigg[2\sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)} \\ & \cdot \sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} - (E_1 - E_3)\left(\overline{E}_1 - \overline{E}_3\right) - \left(E_1 - \overline{E}_3\right)\left(\overline{E}_1 - E_3\right)\Bigg]\Bigg\} \end{aligned} \tag{3.97}$$

[2] Throughout this chapter, functions of the form

$$f(x) = \sqrt{(x - E_3)\left(x - \overline{E}_3\right)}, \qquad g(x) = \sqrt{(x - E_1)(x - E_2)}, \qquad \text{etc.} \tag{3.94}$$

are meant to be evaluated on the branch on which their real part is positive for large and positive $x$. Expressions of the form

$$\sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)}, \qquad \sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} \tag{3.95}$$

are meant to be understood as $f(E_1)$, $f\left(\overline{E}_1\right)$

Notice that the argument of the logarithm in (3.97) is real and positive. In order to prove proposition 3.3.1 let us recall that

$$\nu_1 = \frac{\mathrm{d}\lambda}{\mu}, \qquad \nu_2 = \frac{\lambda \mathrm{d}\lambda}{\mu} \tag{3.98}$$

are holomorphic differentials on $\Gamma_\epsilon$. Let us introduce the matrices

$$\mathcal{A} = \begin{bmatrix} \oint_{a_1} \nu_1 & \oint_{a_2} \nu_1 \\ \oint_{a_1} \nu_2 & \oint_{a_2} \nu_2 \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} \oint_{b_1} \nu_1 & \oint_{b_2} \nu_1 \\ \oint_{b_1} \nu_2 & \oint_{b_2} \nu_2 \end{bmatrix}. \tag{3.99}$$

It is easy to verify that

$$\boldsymbol{\tau} = (2\pi i)\, \mathcal{A}^{-1} \cdot \mathcal{B}. \tag{3.100}$$

**Lemma 3.3.2.** *The matrix $\mathcal{A}$ has the form*

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & -\overline{\mathcal{A}_{11}} \\ \mathcal{A}_{21} & -\overline{\mathcal{A}_{21}} \end{bmatrix}. \tag{3.101}$$

*As $\epsilon$ tends to zero from the right the following asymptotic expansions hold true:*

$$\mathcal{A}_{11} = \frac{-\pi}{Im\,(E_1)\, \sqrt{(E_1 - E_3)\,(E_1 - \overline{E}_3)}} + \mathcal{O}\,(\epsilon)\,, \quad \epsilon \to 0 \tag{3.102}$$

$$\mathcal{A}_{12} = \frac{-\pi E_1}{Im\,(E_1)\, \sqrt{(E_1 - E_3)\,(E_1 - \overline{E}_3)}} + \mathcal{O}\,(\epsilon)\,, \quad \epsilon \to 0 \tag{3.103}$$

*Proof.* Using (3.13) one gets

$$\mathcal{A}_{12} = \oint_{a_2} \nu_1 = -\oint_{r_\star(a_1)} \nu_1 = -\oint_{a_1} r^\star\,(\nu_1) = \oint_{a_1} \overline{\nu_1} = -\overline{\mathcal{A}_{11}}. \tag{3.104}$$

In the same way one obtains that $\mathcal{A}_{22} = -\overline{\mathcal{A}_{21}}$. So (3.101) is proved.
Due to Dominated Convergence theorem

$$\begin{aligned} \lim_{\epsilon \to 0^+} \mathcal{A}_{11}\,(\epsilon) &= \oint_{a_1} \frac{\mathrm{d}\lambda}{(\lambda - E_1)\,(\lambda - \overline{E}_1)\, \sqrt{(\lambda - E_3)\,(\lambda - \overline{E}_3)}} \\ &= \frac{-\pi}{\mathrm{Im}\,(E_1)\, \sqrt{(E_1 - E_3)\,(E_1 - \overline{E}_3)}}. \end{aligned} \tag{3.105}$$

Moreover, the derivative

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\epsilon} \frac{1}{\sqrt{\prod_{j=1}^{3} (\lambda - E_j)\,(\lambda - \overline{E}_j)}} = \frac{1}{2} \Bigg[ & \frac{1}{\lambda - (E_1 + \epsilon)} \\ & + \frac{1}{\lambda - (\overline{E}_1 + \epsilon)} \Bigg] \frac{1}{\sqrt{\prod_{j=1}^{3} (\lambda - E_j)\,(\lambda - \overline{E}_j)}} \end{aligned} \tag{3.106}$$

is a unformly bounded function if $\epsilon$ is small enough. By standard theorems of calculus it follows that the first correction of $\mathcal{A}_{11}$ is linear and (3.102) is proved. Now let us consider

$$\mathcal{A}_{21} = \oint_{a_1} \frac{(\lambda - E_1)\,\mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}(\lambda - E_j)(\lambda - \overline{E}_j)}} + E_1 \oint_{a_1} \frac{\mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}(\lambda - E_j)(\lambda - \overline{E}_j)}} \tag{3.107}$$

Again due to Dominated Convergnce theorem

$$\lim_{\epsilon\to 0^+} \oint_{a_1} \frac{(\lambda - E_1)\,\mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}(\lambda - E_j)(\lambda - \overline{E}_j)}} = \oint_{a_1} \frac{\mathrm{d}\lambda}{(\lambda - \overline{E}_1)\sqrt{(\lambda - E_3)(\lambda - \overline{E}_3)}} = 0. \tag{3.108}$$

With the same procedure as above, one can prove that the first correction of $\mathcal{A}_{21}$ is linear in $\epsilon$, as this last one tends to zero. Expansion (3.103) follows from this and the lemma is proved.

□

**Lemma 3.3.3.** *The matrix $\mathcal{B}$ has the form*

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{11} & \overline{\mathcal{B}_{11}} \\ \mathcal{B}_{21} & \overline{\mathcal{B}_{21}} \end{bmatrix}. \tag{3.109}$$

*As $\epsilon$ tends to zero, the following asymptotic expansions hold true:*

$$\mathcal{B}_{11} = \frac{-2}{(E_1 - \overline{E}_1)\sqrt{(E_1 - E_3)(E_1 - \overline{E}_3)}}\log(\epsilon) + \mathcal{B}^0_{11} + o(1), \quad \epsilon \to 0^+ \tag{3.110}$$

$$\mathcal{B}_{21} = \frac{-2E_1}{(E_1 - \overline{E}_1)\sqrt{(E_1 - E_3)(E_1 - \overline{E}_3)}}\log(\epsilon) + \mathcal{B}^0_{21} + o(1), \quad \epsilon \to 0^+. \tag{3.111}$$

*The coefficients $\mathcal{B}^0_{11}$ and $\mathcal{B}^0_{21}$ are given by*

$$\begin{aligned}
\mathcal{B}^0_{11} = {} & \frac{2}{(E_1 - \overline{E}_1)\sqrt{(E_1 - E_3)(E_1 - \overline{E}_3)}} \Bigg( 2log(2) \\
& - log\left[\frac{E_1 - Re(E_3)}{\sqrt{(E_1 - E_3)(E_1 - \overline{E}_3)}} - 1\right] \\
& + log\left[u(\lambda(E_3)) + E_1 - \sqrt{(E_1 - E_3)(E_1 - \overline{E}_3)}\right] \\
& - log\left\{-i\left[\sqrt{(E_1 - E_3)(E_1 - \overline{E}_3)} + u(\lambda) + E_1\right]\right\}\Bigg|_{\lambda=\lambda(E_1)}^{\lambda=\lambda(E_3)}\Bigg) \\
& - \frac{2}{(E_1 - \overline{E}_1)\sqrt{(\overline{E}_1 - E_3)(\overline{E}_1 - \overline{E}_3)}} \Bigg( log\left\{-i\left[\sqrt{(\overline{E}_1 - E_3)(\overline{E}_1 - \overline{E}_3)} - u(\lambda) - \overline{E}_1\right]\right\} \\
& - log\left\{i\left[\sqrt{(\overline{E}_1 - E_3)(\overline{E}_1 - \overline{E}_3)} + u(\lambda) + \overline{E}_1\right]\right\}\Bigg)\Bigg|_{\lambda=\lambda(E_1)}^{\lambda=\lambda(E_3)}
\end{aligned} \tag{3.112}$$

*and*

$$\mathcal{B}^0_{21} = E_1\mathcal{B}^0_{11} + \frac{2}{\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\left(\log\left\{i\left[u(\lambda)+\overline{E}_1 - \sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}\right]\right\} - \log\left\{i\left[u(\lambda)+\overline{E}_1 + \sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}\right]\right\}\right)\Big|_{\lambda=E_1}^{\lambda=E_3} \tag{3.113}$$

*Proof.* Let us start proving (3.112). One can easily reduce to integrate a function along a path contained in the complex plane (the first sheet of $\Gamma$):

$$\oint_{b_1}\nu_1 = 2\int_{E_1+\epsilon}^{E_3}\nu_1 = 2\int_{E_1+\epsilon}^{E_3}\left[\frac{i}{\sqrt{\lambda - E_1}\sqrt{\lambda-(E_1+\epsilon)}}g_\epsilon(\lambda)\right]\mathrm{d}\lambda \tag{3.114}$$

We have put

$$g_\epsilon(\lambda) = \frac{1}{\sqrt{\lambda-\overline{E}_1}\sqrt{\lambda-\left(\overline{E}_1+\epsilon\right)}\sqrt{i(\lambda-E_3)}\sqrt{i\left(\lambda-\overline{E}_3\right)}}, \qquad \epsilon\in\mathbb{R} \tag{3.115}$$

Since the integrand has singularities both in $E_1+\epsilon$ and in $E_3$, it is convenient to split this integral as follows:

$$\int_{E_1+\epsilon}^{E_3}\nu_1 = \int_{E1+\epsilon}^{F}\nu_1 + \int_F^{E_3}\nu_1 \tag{3.116}$$

F just needs to be an interior point of the integration path.
By an elementary calculation,

$$\frac{1}{\sqrt{\lambda-E_1}\sqrt{\lambda-(E_1+\epsilon)}} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\log\left[(\lambda-E_1)+\sqrt{\lambda-E_1}\sqrt{\lambda-(E_1+\epsilon)}-\frac{\epsilon}{2}\right] \tag{3.117}$$

for $\mathrm{Re}(\lambda) > \mathrm{Re}(E_1)+\epsilon$. Integration by parts yields

$$\int_{E_1+\epsilon}^{F}\left[\frac{i}{\sqrt{\lambda-E_1}\sqrt{\lambda-(E_1+\epsilon)}}g_\epsilon(\lambda)\right]\mathrm{d}\lambda = \tag{3.118}$$

$$i\log\left[(\lambda-E_1)+\sqrt{\lambda-E_1}\sqrt{\lambda-(E_1+\epsilon)}-\frac{\epsilon}{2}\right]g_\epsilon(\lambda)\,|_{E_1+\epsilon}^{F}$$

$$-i\int_{E_1}^{F}\log\left[(\lambda-E_1)+\sqrt{\lambda-E_1}\sqrt{\lambda-(E_1+\epsilon)}-\frac{\epsilon}{2}\right]\frac{\mathrm{d}}{\mathrm{d}\lambda}g_\epsilon(\lambda)\,\mathrm{d}\lambda.$$

One can now apply the theorem of Dominated Convergence, and obtain that

$$\lim_{P\to E_1}\int_{E_1}^{F}\log\left[(\lambda-E_1)+\sqrt{\lambda-E_1}\sqrt{\lambda-(E_1+\epsilon)}-\frac{\epsilon}{2}\right]\frac{\mathrm{d}}{\mathrm{d}\lambda}g_\epsilon(\lambda)\,\mathrm{d}\lambda = \int_{E_1}^{F}\log\left[2(\lambda-E_1)\right]\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}g_0(\lambda)\right]\mathrm{d}\lambda.$$

Integrating by parts again, one obtains

$$\int_{E_1}^{F} \log\left[2\left(\lambda - E_1\right)\right]\left[\frac{\mathrm{d}}{\mathrm{d}\lambda} g_0\left(\lambda\right)\right]\mathrm{d}\lambda =$$
$$\lim_{P\to E_1}\left\{\log\left[2\left(\lambda - E_1\right)\right] g_0\left(\lambda\right)|_P^F - \int_P^F \frac{1}{\lambda - E_1} g_0\left(\lambda\right)\mathrm{d}\lambda\right\} \tag{3.119}$$

One has to consider the limit, in the r.h.s. because two reciprocally erasing divergences appear after integration by parts. Plugging (3.119) into (3.118)

$$\int_{E_1+\epsilon}^{F} \nu_1 = -i\log\left(\epsilon\right) g_0\left(E_1\right) + i\log\left(2\right) g_0\left(E_1\right) \tag{3.120}$$
$$+i\lim_{P\to E_1}\left[\log\left(2\left(P - E_1\right)\right) g_0\left(P\right) + \int_P^F \frac{1}{\lambda - E_1} g_0\left(\lambda\right)\mathrm{d}\lambda\right]$$

By means of the substitution

$$u\left(\lambda\right) = \sqrt{\left(\lambda - E_3\right)\left(\lambda - \overline{E}_3\right)} - \lambda \tag{3.121}$$

one obtains the primitive

$$\int \frac{1}{\lambda - E_1} g_0\left(\lambda\right)\mathrm{d}\lambda = \frac{2}{\left(E_1 - \overline{E}_1\right)\sqrt{\left(E_1 - E_3\right)\left(E_1 - \overline{E}_3\right)}} \cdot \operatorname{atan}\left[\frac{i\left(u + E_1\right)}{\sqrt{\left(E_1 - E_3\right)\left(E_1 - \overline{E}_3\right)}}\right]$$
$$- \frac{2}{\left(E_1 - \overline{E}_1\right)\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}} \cdot \operatorname{atan}\left[\frac{i\left(u + \overline{E}_1\right)}{\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\right]$$

Using this formula, one finds

$$\int_{E_1+\epsilon}^{F} \nu_1 = i g_0\left(E_1\right)\Bigg(-\log\left(\epsilon\right) + 2\log\left(2\right) \tag{3.122}$$
$$+ \log\left[u\left(\lambda\left(F\right)\right) + E_1 - \sqrt{\left(E_1 - E_3\right)\left(E_1 - \overline{E}_3\right)}\right]$$
$$- \log\left[\frac{E_1 - \operatorname{Re}\left(E_3\right)}{\sqrt{\left(E_1 - E_3\right)\left(E_1 - \overline{E}_3\right)}} - 1\right]$$
$$- \log\left\{-i\left[\sqrt{\left(E_1 - E_3\right)\left(E_1 - \overline{E}_3\right)} + u\left(\lambda\right) + E_1\right]\right\}\Bigg|_{\lambda=\lambda(P)}^{\lambda=\lambda(F)}\Bigg)$$
$$- \frac{1}{\left(E_1 - \overline{E}_1\right)\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\Bigg(\log\left\{-i\left[\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} - u\left(\lambda\right) - \overline{E}_1\right]\right\}\Bigg|_{\lambda=\lambda(P)}^{\lambda=\lambda(F)}$$
$$- \log\left\{i\left[\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} + u\left(\lambda\right) + \overline{E}_1\right]\right\}\Bigg|_{\lambda=\lambda(P)}^{\lambda=\lambda(F)}\Bigg).$$

The logarithm in the formula above can be chosen to have a branch cut on the half line $(-\infty, 0]$. On the remaining part of the integration path one can directly apply the theorem of Dominated Convergence and obtain

$$\begin{aligned}
\lim_{\epsilon\to 0^+}\int_F^{E_3}\nu_1 \quad = \quad & \frac{1}{\left(E_1-\overline{E}_1\right)\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}}\left(\log\left[u\left(\lambda\right)+E_1-\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}\right]\right. \\
& \left.\log\left\{-i\left[\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}+u\left(\lambda\right)+E_1\right]\right\}\right)\Bigg|_{\lambda=\lambda(F)}^{\lambda=\lambda(E_3)} \\
& -\frac{1}{\left(E_1-\overline{E}_1\right)\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}}\left(\log\left\{-i\left[\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}\right.\right.\right. \\
& \left.\left.\left.-u\left(\lambda\right)-\overline{E}_1\right]\right\}-\log\left\{i\left[\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}+u\left(\lambda\right)+\overline{E}_1\right]\right\}\right)\Bigg|_{\lambda=\lambda(F)}^{\lambda=\lambda(E_3)}.
\end{aligned}$$

Putting together the two parts one gets

$$\begin{aligned}
\int_{E_1+\epsilon}^{E_3}\nu_1 \quad = \quad & \frac{-1}{\left(E_1-\overline{E}_1\right)\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}}\log\left(\epsilon\right) \\
& +\frac{1}{\left(E_1-\overline{E}_1\right)\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}}\left(2\log\left(2\right)-\log\left[\frac{E_1-\mathrm{Re}\left(E_3\right)}{\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}}-1\right]\right. \\
& +\log\left[u\left(\lambda\left(E_3\right)\right)+E_1-\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}\right] \\
& \left.-\log\left\{-i\left[\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}+u\left(\lambda\right)+E_1\right]\right\}\Bigg|_{\lambda=\lambda(E_1)}^{\lambda=\lambda(E_3)}\right) \qquad (3.123) \\
& -\frac{1}{\left(E_1-\overline{E}_1\right)\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}}\left(\log\left\{-i\left[\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}\right.\right.\right. \\
& \left.\left.\left.-u\left(\lambda\right)-\overline{E}_1\right]\right\}-\log\left\{i\left[\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}+u\left(\lambda\right)+\overline{E}_1\right]\right\}\right)\Bigg|_{\lambda=\lambda(E_1)}^{\lambda=\lambda(E_3)}
\end{aligned}$$

This proves (3.112). Let us now consider (3.113) We easily reduce to an integration along a path in the complex plane:

$$\oint_{b_1}\nu_2 = 2\int_{E_1+\epsilon}^{E_3}\frac{\lambda \mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}\left(\lambda-E_j\right)\left(\lambda-\overline{E}_j\right)}} \qquad (3.124)$$

It is convenient to consider the obvious identity

$$\begin{aligned}
\int_{E_1+\epsilon}^{E_3}\frac{\lambda \mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}\left(\lambda-E_j\right)\left(\lambda-\overline{E}_j\right)}} = & \int_{E_1+\epsilon}^{E_3}\frac{\left[\lambda-\left(E_1+\epsilon\right)\right]\mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}\left(\lambda-E_j\right)\left(\lambda-\overline{E}_j\right)}} \\
& +\int_{E_1+\epsilon}^{E_3}\frac{\left(E_1+\epsilon\right)\mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}\left(\lambda-E_j\right)\left(\lambda-\overline{E}_j\right)}}. \qquad (3.125)
\end{aligned}$$

Using formula (3.123)

$$\int_{E_1+\epsilon}^{E_3} \frac{(E_1+\epsilon)\,\mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}\left(\lambda-E_j\right)\left(\lambda-\overline{E}_j\right)}} = E_1\int_{E_1+\epsilon}^{E_3} \frac{\mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}\left(\lambda-E_j\right)\left(\lambda-\overline{E}_j\right)}} + \mathcal{O}\left(\epsilon\log\left(\epsilon\right)\right). \tag{3.126}$$

On the other side,

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\int_{E_1+\epsilon}^{E_3} \frac{\left[\lambda-(E_1+\epsilon)\right]\mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}\left(\lambda-E_j\right)\left(\lambda-\overline{E}_j\right)}} \\
&= \int_{E_1}^{E_3}\frac{\mathrm{d}\lambda}{\left(\lambda-\overline{E}_1\right)\sqrt{\left(\lambda-E_3\right)\left(\lambda-\overline{E}_3\right)}} \\
&= \frac{1}{\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}}\left(\log\left\{i\left[u\left(\lambda\right)+\overline{E}_1-\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}\right]\right\}\right. \\
&\qquad \left.-\log\left\{i\left[u\left(\lambda\right)+\overline{E}_1+\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}\right]\right\}\right)\Bigg|_{\lambda=E_1}^{\lambda=E_3}
\end{aligned} \tag{3.127}$$

□

*proof of proposition 3.3.1.* From lemma 3.3.2 and 3.3.3 one has

$$\begin{aligned}
\mathcal{A}^{-1} &= \frac{1}{2\pi i}\begin{bmatrix}\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)} & 0 \\ 0 & -\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}\end{bmatrix} \\
&\quad\cdot\begin{bmatrix}\overline{E}_1 & -1 \\ -E_1 & 1\end{bmatrix}+\mathcal{O}\left(\epsilon\right),
\end{aligned} \tag{3.128}$$

and

$$\begin{aligned}
\mathcal{B} &= i\begin{bmatrix}1 & 1 \\ E_1 & \overline{E}_1\end{bmatrix} \\
&\quad\cdot\begin{bmatrix}\frac{1}{\mathrm{Im}(E_1)\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}} & 0 \\ 0 & \frac{-1}{\mathrm{Im}(E_1)\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}}\end{bmatrix} \\
&\quad\cdot\log\left(\epsilon\right)+\begin{bmatrix}\mathcal{B}_{11}^0 & \overline{\mathcal{B}_{11}^0} \\ \mathcal{B}_{21}^0 & \overline{\mathcal{B}_{21}^0}\end{bmatrix}+o\left(1\right), \quad \epsilon\to 0
\end{aligned} \tag{3.129}$$

Plugging these two expressions into (3.100) yields (3.93). □

**Proposition 3.3.4.** *As $\epsilon$ tends to zero from the right, one has*

$$\mathbf{V}\left(\epsilon\right)=\begin{pmatrix}-2\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)} \\ -2\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}\end{pmatrix}+\mathcal{O}\left(\epsilon\right), \quad \epsilon\to 0^+ \tag{3.130}$$

*and*

$$\mathbf{W}(\epsilon) = \begin{pmatrix} -4\left[E_1 + Re\left(E_3\right)\right] \sqrt{\left(E_1 - E_3\right)\left(E_1 - \overline{E}_3\right)} \\ -4\left[\overline{E}_1 + Re\left(E_3\right)\right] \sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} \end{pmatrix} + \mathcal{O}(\epsilon), \quad \epsilon \to 0^+ \tag{3.131}$$

*Proof.* Put $\mathcal{C} = 2\pi i \mathcal{A}^{-1}$. With this notation

$$\omega_i = \mathcal{C}_{ij} \nu_j, \qquad i = 1, 2. \tag{3.132}$$

where $\omega_{1,2}$ are the normalized holomorphic differentials. It is possible to prove (see [3], pages 155-156) that

$$V_i = 2\mathcal{C}_{i2}, \quad W_i = 4\left(\mathcal{C}_{i1} + \frac{\mathcal{C}_{i2} S_1}{2}\right) \qquad i = 1, 2. \tag{3.133}$$

Here

$$S_1 = 2\sum_{i=1}^{3} \mathrm{Re}\left(E_i\right).$$

In view of this fact, the proof reduces to a straightforward calculation. □

**Proposition 3.3.5.** *As $\epsilon$ tends to zero from the right one has*

$$E(\epsilon) = 2Re\left(E_3\right) + \mathcal{O}\left(\epsilon^2\right), \quad \epsilon \to 0^+ \tag{3.134a}$$

$$N(\epsilon) = -2\left[2Re\left(E_3\right)^2 - Im\left(E_3\right)^2\right] + \mathcal{O}\left(\epsilon^2\right), \quad \epsilon \to 0^+. \tag{3.134b}$$

*Notice that the first correction of both E and N is of order two in $\epsilon$.*

In order to prove this proposition let us introduce the notation

$$\mathrm{d}\Omega_3 = \frac{\lambda^2 + d_2(\epsilon)\lambda + d_1(\epsilon)}{\mu} \tag{3.135}$$

for the normalized abelian differential $\mathrm{d}\Omega_3$ defined in section 3.2.

**Lemma 3.3.6.** *As $\epsilon$ tends to zero from the right one has*

$$d_1(\epsilon) = E_1\overline{E}_1 + Re\left(E_1\right)\epsilon + \mathcal{O}\left(\epsilon^2\right), \quad \epsilon \to 0^+ \tag{3.136a}$$

$$d_2(\epsilon) = -2Re\left(E_1\right) - \epsilon + \mathcal{O}\left(\epsilon^2\right), \quad \epsilon \to 0^+. \tag{3.136b}$$

*Proof.* Let us put

$$d_1(\epsilon) = d_1^0 + d_1^1\epsilon + \mathcal{O}\left(\epsilon^2\right), \quad \epsilon \to 0^+ \tag{3.137}$$

$$d_2(\epsilon) = d_2^0 + d_1^1\epsilon + \mathcal{O}\left(\epsilon^2\right), \quad \epsilon \to 0^+. \tag{3.138}$$

The following expansion with rispect to $\epsilon$ holds uniformly on the $a$-cycles:

$$\frac{1}{\sqrt{\prod_{j=1}^{3}\left(\lambda - E_j\right)\left(\lambda - \overline{E}_j\right)}} = \frac{1}{\left(\lambda - E_1\right)\left(\lambda - \overline{E}_1\right)\sqrt{\left(\lambda - E_3\right)\left(\lambda - \overline{E}_3\right)}}\left[1 + \frac{\lambda - \mathrm{Re}\left(E_1\right)}{\left(\lambda - E_1\right)\left(\lambda - \overline{E}_1\right)}\epsilon + \mathcal{O}\left(\epsilon^2\right)\right] \tag{3.139}$$

Imposing the normalization conditions

$$\oint_{a_j} \frac{\left[\left(\lambda^2 + d_2^0\lambda + d_1^0\right) + \left(d_2\lambda + d_1^1\right)\epsilon\right]}{(\lambda - E_1)\left(\lambda - \overline{E}_1\right)\sqrt{(\lambda - E_3)\left(\lambda - \overline{E}_3\right)}}\left[1 + \frac{\lambda - \mathrm{Re}\,(E_1)}{(\lambda - E_1)\left(\lambda - \overline{E}_1\right)}\epsilon\right] = \mathcal{O}\left(\epsilon^2\right) \tag{3.140}$$

for $j = 1, 2$ yields (3.136). □

*Proof of proposition 3.3.5.* Recalling the definition of $S_1(\epsilon)$ and $S_2(\epsilon)$ given in (3.3) the following formulas for $E(\epsilon)$ and $N(\epsilon)$ hold:

$$E(\epsilon) = S_1(\epsilon) + 2d_2(\epsilon) \tag{3.141a}$$

$$N(\epsilon) = -2\left[2d_1(\epsilon) + \frac{3}{4}(S_1(\epsilon))^2 - S_2(\epsilon) + S_1(\epsilon)\, d_2(\epsilon)\right] \tag{3.141b}$$

(see [3], pages 155-156.) Since $E_2 = E_1 + \epsilon$, one has the following expansions:

$$S_1(\epsilon) = [4\mathrm{Re}\,(E_1) + 2\mathrm{Re}\,(E_3)] + 2\epsilon + \mathcal{O}\left(\epsilon^2\right) \tag{3.142a}$$

$$\begin{aligned} S_2(\epsilon) = &\left[2E_1\overline{E}_1 + 8\mathrm{Re}\,(E_1)\,\mathrm{Re}\,(E_3) + E_3\overline{E}_3 + 4\mathrm{Re}\,(E_1)^2\right] \\ &+ [6\mathrm{Re}\,(E_1) + 4\mathrm{Re}\,(E_3)]\,\epsilon + \mathcal{O}\left(\epsilon^2\right). \end{aligned} \tag{3.142b}$$

Plugging (3.142) and (3.136) into (3.141) one gets (3.134). □

**Proposition 3.3.7.** *As $\epsilon$ tends to zero from the right one has*

$$\chi(\epsilon) = -\frac{Im\,(E_3)^2}{4} + \mathcal{O}\left(\epsilon^2\right), \quad \epsilon \to 0^+ \tag{3.143}$$

*Notice that the first correction is of order two in $\epsilon$.*

*Proof.* Plugging (3.136) into (3.135) yields

$$\mathrm{d}\Omega_3 = \frac{\mathrm{d}\lambda}{\sqrt{(\lambda - E_3)\left(\lambda - \overline{E}_3\right)}} + \mathcal{O}\left(\epsilon^2\right) \tag{3.144}$$

Directly from the the definition of $\chi(\epsilon)$ given in section 3.2 one can deduce that

$$\begin{aligned} \chi(\epsilon) &= \left\{2\lim_{\lambda\to\infty}\left[\log(\lambda) - \int_{\overline{E}_3}^{\lambda} \mathrm{d}\Omega_3\right]\right\} \\ &= \exp\left\{2\lim_{\lambda\to\infty}\left[\log(\lambda) - \int_{\overline{E}_3}^{\lambda}\frac{\mathrm{d}\lambda}{\sqrt{(\lambda - E_3)\left(\lambda - \overline{E}_3\right)}}\right]\right\} + \mathcal{O}\left(\epsilon^2\right) \end{aligned} \tag{3.145}$$

Now, by elementary methods one obtains

$$\int_{\overline{E}_3}^{\lambda}\frac{\mathrm{d}x}{\sqrt{(x - E_3)\left(x - \overline{E}_3\right)}} = \log\left[x - \mathrm{Re}\,(E_3) + \sqrt{(x - E_3)\left(x - \overline{E}_3\right)}\right]\Bigg|_{x=\overline{E}_3}^{x=\lambda} \tag{3.146}$$

$$= \log(\lambda) - \log\left[\frac{\mathrm{Im}\,(E_3)}{2i}\right] + o(1), \quad \lambda \to \infty. \tag{3.147}$$

Plugging this one into (3.145) yields the thesis.

□

**Proposition 3.3.8.** *As $\epsilon$ tends to zero from the right one has*

$$\mathbf{r}(\epsilon) = \begin{pmatrix} \overline{r}_0 \\ r_0 \end{pmatrix} + \mathcal{O}(\epsilon), \quad \epsilon \to 0^+. \tag{3.148}$$

*The quantity $r_0$ is defined as follows*

$$r_0 = -2\log\left\{ \frac{-2\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} + \left[\left(\overline{E}_1 - E_3\right) + \left(\overline{E}_1 - \overline{E}_3\right)\right]}{\left(E_3 - \overline{E}_3\right)} \right\}. \tag{3.149}$$

**Observation 3.3.9.** *The determination of the logarithm in (3.149) will be irrelevant to our purposes.*

*Proof.* An equivalent definition of $\mathbf{r}$ is the following:

$$\begin{aligned} \mathbf{r} &= 2\pi i \cdot \mathcal{A}^{-1} \begin{pmatrix} \int_{Cp} \nu_1 \\ \int_{Cp} \nu_2 \end{pmatrix} \\ &= 2 \cdot (2\pi i) \cdot \mathcal{A}^{-1} \begin{pmatrix} \int_{\overline{E}_3}^{\infty^+} \frac{\mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}\left(\lambda - E_j\right)\left(\lambda - \overline{E}_j\right)}} \\ \int_{\overline{E}_3}^{\infty^+} \frac{\lambda \mathrm{d}\lambda}{\sqrt{\prod_{j=1}^{3}\left(\lambda - E_j\right)\left(\lambda - \overline{E}_j\right)}} \end{pmatrix}. \end{aligned} \tag{3.150}$$

Recalling (3.128), we have already proved that

$$\begin{aligned} (2\pi i) \cdot \mathcal{A}^{-1} = &\begin{bmatrix} \sqrt{\left(E_1 - E_3\right)\left(E_1 - \overline{E}_3\right)} & 0 \\ 0 & -\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} \end{bmatrix} \\ &\cdot \begin{bmatrix} \overline{E}_1 & -1 \\ -E_1 & 1 \end{bmatrix} + \mathcal{O}(\epsilon), \end{aligned} \tag{3.151}$$

By elementary methods one obtains

$$\begin{aligned} &\frac{1}{\left(\lambda - E_1\right)\left(\lambda - \overline{E}_1\right)\sqrt{\left(\lambda - E_3\right)\left(\lambda - \overline{E}_3\right)}} \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{ \frac{2i}{\left(E_1 - \overline{E}_1\right)\sqrt{\left(E_1 - E_3\right)\left(E_1 - \overline{E}_3\right)}} \cdot \arctan\left[\frac{i\left(u + E_1\right)}{\sqrt{\left(E_1 - E_3\right)\left(E_1 - \overline{E}_3\right)}}\right] \right. \\ &\quad \left. - \frac{2i}{\left(E_1 - \overline{E}_1\right)\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}} \cdot \arctan\left[\frac{i\left(u + \overline{E}_1\right)}{\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\right] \right\} \end{aligned} \tag{3.152}$$

and

$$\frac{\lambda}{(\lambda - E_1)\left(\lambda - \overline{E}_1\right)\sqrt{(\lambda - E_3)\left(\lambda - \overline{E}_3\right)}}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\frac{-2i}{\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\arctan\left[\frac{-i\left(u + \overline{E}_1\right)}{\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\right]\right.$$
$$+ \frac{2iE_1}{\left(E_1 - \overline{E}_1\right)\sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)}} \cdot \arctan\left[\frac{i\left(u + E_1\right)}{\sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)}}\right]$$
$$\left. - \frac{2iE_1}{\left(E_1 - \overline{E}_1\right)\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}} \cdot \arctan\left[\frac{i\left(u + \overline{E}_1\right)}{\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\right]\right\}. \tag{3.153}$$

In the formulas above we have put

$$u = \sqrt{(\lambda - E_3)\left(\lambda - \overline{E}_3\right)} - \lambda. \tag{3.154}$$

Plugging (3.151), (3.152) and (3.153) into (3.150) and working out the calculations yields

$$\lim_{\epsilon \to 0^+} r(\epsilon) = \begin{pmatrix} -4i \arctan\left[\frac{i(u+E_1)}{\sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)}}\right]\Bigg|_{\overline{E}_3}^{\infty^+} \\ -4i \arctan\left[\frac{i\left(u+\overline{E}_1\right)}{\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\right]\Bigg|_{\overline{E}_3}^{\infty^+} \end{pmatrix}. \tag{3.155}$$

Since

$$u\left(\overline{E}_3\right) = -\overline{E}_3 \tag{3.156}$$

and

$$\lim_{\lambda \to \infty} u(\lambda) = -\mathrm{Re}\left(E_3\right) \tag{3.157}$$

one can easily perform the evaluation in (3.155) and obtain

$$\lim_{\epsilon \to 0^+} \mathbf{r}(\epsilon) \tag{3.158}$$
$$= \begin{pmatrix} -4i\left\{\arctan\left[\frac{i\left(E_1 - \mathrm{Re}(E_3)\right)}{\sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)}}\right] - \arctan\left[\frac{i\left(E_1 - \overline{E}_3\right)}{\sqrt{(E_1 - E_3)\left(E_1 - \overline{E}_3\right)}}\right]\right\} \\ -4i\left\{\arctan\left[\frac{i\left(\overline{E}_1 - \mathrm{Re}(E_3)\right)}{\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\right] - \arctan\left[\frac{i\left(\overline{E}_1 - \overline{E}_3\right)}{\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)}}\right]\right\} \end{pmatrix}.$$

An elementary though long semplification of this formula gives the leading term of (3.148). Finally, a direct calculation shows that

$$\left| \frac{\mathrm{d}}{\mathrm{d}\epsilon} \frac{1}{\sqrt{\prod_{j=1}^{3} \left(\lambda - E_j\right) \left(\lambda - \overline{E}_j\right)}} \right| \leq M \cdot \frac{1}{\left| \left(\lambda - E_1\right) \left(\lambda - \overline{E}_1\right) \sqrt{\left(\lambda - E_3\right) \left(\lambda - \overline{E}_3\right)} \right|} \tag{3.159}$$

for some positive constant $M$ and all $\lambda$ belonging to $Cp$. The theorem of differentiation under the sign of integral, then, guarantees that the first correction of $\int_{\overline{E}_3}^{\infty^+} \nu_1$ is linear in $\epsilon$. The same holds for $\int_{\overline{E}_3}^{\infty^+} \nu_2$. This fact, together with lemma 3.3.2 implies that the first correction of $\mathbf{r}$ is also linear in $\epsilon$.

□

## 3.4 Degeneration of the Solution

Let us consider the function

$$\theta_{\boldsymbol{\tau}(\epsilon)} \left(\mathbf{z} + \boldsymbol{\tau}(\epsilon) \cdot \mathbf{p}\right), \quad \mathbf{z} \in \mathbb{C}^2. \tag{3.160}$$

where $\boldsymbol{\tau}(\epsilon)$ satisfies the prescription of proposition 3.3.1. That is

$$\boldsymbol{\tau}(\epsilon) = \begin{bmatrix} 2\log(\epsilon) & 0 \\ 0 & 2\log(\epsilon) \end{bmatrix} + \begin{bmatrix} \alpha_0 & \beta_0 \\ \beta_0 & \overline{\alpha_0} \end{bmatrix} + o(1), \quad \epsilon \to 0^+. \tag{3.161}$$

Let us recall here that

$$\mathbf{p} = \begin{pmatrix} p \\ -p \end{pmatrix}, \quad p \in \mathbb{R}. \tag{3.162}$$

In this section we willl study the behaviour of (3.160) in the limit when $\epsilon$ is small. This will allow us to complete the calculation of the limit of the solution $\psi(x, t; p, q; \epsilon)$ as $\epsilon$ tends to zero from the right.

**Proposition 3.4.1.** *If*

$$-\frac{1}{2} < p < \frac{1}{2} \tag{3.163}$$

*then*

$$\theta_{\boldsymbol{\tau}(\epsilon)} \left[\mathbf{z} + \boldsymbol{\tau}(\epsilon) \cdot \mathbf{p}\right] = 1 + S(\mathbf{z}; p)\, \epsilon^{1-2|p|} + o\left(\epsilon^{1-2|p|}\right), \quad \epsilon \to 0^+; \mathbf{z} \in \mathbb{C}^2 \tag{3.164}$$

*The form of the function $S(\mathbf{z}; p)$ depends on $p$ as follows:*

- *If $0 < p < \frac{1}{2}$,*

$$S(\mathbf{z}; p) = \exp\left[\frac{\alpha_0}{2} + p\left(\beta_0 - \alpha_0\right) - z_1\right] + \exp\left[\frac{\overline{\alpha}_0}{2} + p\left(\beta_0 - \overline{\alpha}_0\right) + z_2\right] \tag{3.165}$$

- *If* $-\frac{1}{2} < p < 0$,

$$S(\mathbf{z}; p) = \exp\left[\frac{\alpha_0}{2} + p(\alpha_0 - \beta_0) + z_1\right] + \exp\left[\frac{\overline{\alpha}_0}{2} + p(\overline{\alpha}_0 - \beta_0) - z_2\right] \tag{3.166}$$

- *If* $p = 0$,

$$S(\mathbf{z}; 0) = \exp\left(\frac{\alpha_0}{2} + z_1\right) + \exp\left(\frac{\alpha_0}{2} - z_1\right) + \exp\left(\frac{\overline{\alpha}_0}{2} + z_2\right) + \exp\left(\frac{\overline{\alpha}_0}{2} - z_2\right) \tag{3.167}$$

*If, instead,*

$$p = \frac{1}{2} \tag{3.168}$$

*then*

$$\lim_{\epsilon \to 0^+} \theta_{\boldsymbol{\tau}(\epsilon)}\left[\mathbf{z} + \boldsymbol{\tau}(\epsilon) \cdot \mathbf{p}\right] = 1 + \exp\left(-z_1 + \frac{\beta_0}{2}\right) + \exp\left(z_2 + \frac{\beta_0}{2}\right) + \exp(z_2 - z_1), \quad \mathbf{z} \in \mathbb{C}^2 \tag{3.169}$$

*The estimate of the error in (3.164) and the convergence of (3.169) are understood to be uniform w.r.t.* $\mathbf{z}$ *over compact subsets of* $\mathbb{C}^2$.

Let us introduce the notation

$$\boldsymbol{\tau}(\epsilon) = 2\log(\epsilon)\,\mathrm{Id} + \boldsymbol{\tau}_{conv}(\epsilon) \tag{3.170}$$

so that

$$\boldsymbol{\tau}_{conv}(\epsilon) = \begin{bmatrix} \alpha_0 & \beta_0 \\ \beta_0 & \alpha_0 \end{bmatrix} + o(1), \quad \epsilon \to 0^+ \tag{3.171}$$

In order to prove the proposition above we will need the following

**Lemma 3.4.2.** *Let R and C be some positive constants. There exist* $\epsilon_0 > 0$ *and* $M > 0$ *such that for all* $0 < \epsilon < \epsilon_0$

$$\sum_{\mathbf{m} \in \mathbb{Z}^2} \left| \exp\left\{\frac{1}{2}\mathbf{m}^T \cdot \left[\boldsymbol{\tau}_{conv}(\epsilon) + C \cdot Id \cdot \log(\epsilon)\right] \cdot \mathbf{m} + \mathbf{m}^T \cdot \mathbf{z}\right\}\right| \leq M, \quad |z| \leq R. \tag{3.172}$$

*Proof.* Since

$$|\exp(w)| = \exp[\mathrm{Re}(w)], \qquad w \in \mathbb{C} \tag{3.173}$$

the following equality holds:

$$\begin{aligned} &\sum_{\mathbf{m} \in \mathbb{Z}^2} \left| \exp\left\{\frac{1}{2}\mathbf{m}^T \cdot \left[\boldsymbol{\tau}_{conv}(\epsilon) + C \cdot \mathrm{Id} \cdot \log(\epsilon)\right] \cdot \mathbf{m} + \mathbf{m}^T \cdot \mathbf{z}\right\}\right| \\ &= \sum_{\mathbf{m} \in \mathbb{Z}^2} \exp\left(\frac{1}{2}\mathbf{m}^T \cdot \{\mathrm{Re}[\boldsymbol{\tau}_{conv}(\epsilon)] + C \cdot \mathrm{Id} \cdot \log(\epsilon)\} \cdot \mathbf{m} + \mathbf{m}^T \cdot \mathrm{Re}(\mathbf{z})\right). \end{aligned} \tag{3.174}$$

Now, there exists $\epsilon_0 > 0$ such that for all $\epsilon$ between 0 and $\epsilon_0$ one has

$$\mathbf{m}^T \cdot \{\mathrm{Re}\,[\boldsymbol{\tau}_{conv}(\epsilon)] + C \cdot \mathrm{Id} \cdot \log(\epsilon)\} \cdot \mathbf{m} \leq -\mathbf{m}^T \cdot \mathbf{m}, \quad \mathbf{m} \in \mathbb{Z}^2 \qquad (3.175)$$

This is evident, if one observes that the eigenvalues of the matrix

$$\mathrm{Re}\,[\boldsymbol{\tau}_{conv}(\epsilon)] + [C \cdot \log(\epsilon) + 1] \cdot \mathrm{Id} \qquad (3.176)$$

both tend to $-\infty$ as $\epsilon$ tends to zero from the right. As a consequence,

$$\begin{aligned}
&\sum_{\mathbf{m}\,\in\mathbb{Z}^2} \exp\left(\frac{1}{2}\mathbf{m}^T \cdot \{\mathrm{Re}\,[\boldsymbol{\tau}_{conv}(\epsilon)] + C \cdot \mathrm{Id} \cdot \log(\epsilon)\} \cdot \mathbf{m} + \mathbf{m}^T \cdot \mathrm{Re}\,(\mathbf{z})\right) \\
&\leq \sum_{\mathbf{m}\in\mathbb{Z}^2} \exp\left\{-\frac{1}{2}\mathbf{m}^T \cdot \mathbf{m} + \mathbf{m}^T \cdot \mathrm{Re}\,(\mathbf{z})\right\} \\
&= \theta_{-Id}\,[\mathrm{Re}\,(\mathbf{z})]\,.
\end{aligned} \qquad (3.177)$$

Since the function $\theta_{-Id}$ is entire, its absolute value is bounded on compact subsets of $\mathbb{C}^2$. So there exists a constant $M$ satisiying (3.172) and the lemma is proved □

*Proof of proposition 3.4.1.* In view of (3.170) one can rewrite

$$\begin{aligned}
\theta_{\boldsymbol{\tau}(\epsilon)}\left[\mathbf{z} + \boldsymbol{\tau}(\epsilon) \cdot \mathbf{p}\right] = \sum_{\mathbf{m}\in\mathbb{Z}^2} \exp\Big\{ & F_p(\mathbf{m}) \log(\epsilon) \\
&+ \left[\frac{1}{2}\mathbf{m}^T \cdot \boldsymbol{\tau}_{conv}(\epsilon) \cdot (\mathbf{m} + 2\mathbf{p}) + \mathbf{m}^T \cdot \mathbf{z}\right]\Big\}.
\end{aligned} \qquad (3.178)$$

Here we have defined

$$F_p(\mathbf{m}) = \mathbf{m}^T \cdot \mathbf{m} + 2\mathbf{m}^T \cdot \mathbf{p}, \quad \mathbf{m} \in \mathbb{Z}^2 \qquad (3.179)$$

Let us consider the case

$$0 < p < \frac{1}{2}. \qquad (3.180)$$

One has to prove that

$$\lim_{\epsilon\,\to 0^+} \frac{\theta_{\boldsymbol{\tau}(\epsilon)}\left[\mathbf{z} + \boldsymbol{\tau}(\epsilon) \cdot \mathbf{p}\right] - \left[1 + S\left(\mathbf{z}; p\right)\epsilon^{1-2p}\right]}{\epsilon^{1-2p}} = 0. \qquad (3.181)$$

Let us define the set

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad (3.182)$$

Using (3.178) it is straightforward to verify that

$$\begin{aligned}
1 + S\left(z; p\right)\epsilon^{1-2p} = \sum_{\mathbf{m}\in\mathcal{N}} \exp\Big\{ & F_p(\mathbf{m}) \log(\epsilon) \\
&+ \left[\frac{1}{2}\mathbf{m}^T \cdot \boldsymbol{\tau}_{conv}(\epsilon) \cdot (\mathbf{m} + 2\mathbf{p}) + \mathbf{m}^T \cdot \mathbf{z}\right]\Big\} + o\left(\epsilon^{1-2p}\right).
\end{aligned} \qquad (3.183)$$

In view of this relation, (3.181) is equivalent to

$$\lim_{\epsilon \to 0^+} \sum_{\mathbf{m} \in \mathbb{Z}^2 \backslash \mathcal{N}} \exp \left\{ \left[ F_p (\mathbf{m}) - \left(1 - 2p\right) \right] \log (\epsilon) + \left[ \frac{1}{2} \mathbf{m}^T \cdot \boldsymbol{\tau}_{conv} (\epsilon) \cdot (\mathbf{m} + 2\mathbf{p}) + \mathbf{m} \cdot \mathbf{z} \right] \right\} \tag{3.184}$$

Now, there exists a positive constant $C_p$ such that

$$F_p (\mathbf{m}) - \left(1 - 2p\right) > C_p \left(m^2 + n^2\right), \quad \mathbf{m} \in \mathbb{Z}^2 \backslash \mathcal{N} \tag{3.185}$$

This is a consequence of the following inequality

$$F_p (\mathbf{m}) > 1 - 2p, \quad \mathbf{m} \in \mathbb{Z}^2 \backslash \mathcal{N} \tag{3.186}$$

By means of (3.185) one gets

$$\sum_{\mathbf{m} \in \mathbb{Z}^2 \backslash \mathcal{N}} \left| \exp \left\{ \left[ F_p (\mathbf{m}) - (1 - 2p) \right] \log (\epsilon) + \left[ \frac{1}{2} \mathbf{m}^T \cdot \boldsymbol{\tau}_{conv} (\epsilon) \cdot (\mathbf{m} + 2\mathbf{p}) + \mathbf{m}^T \cdot \mathbf{z} \right] \right\} \right| \leq \tag{3.187}$$

$$\epsilon^{\frac{C_p}{2}} \sum_{\mathbf{m} \in \mathbb{Z}^2 \backslash \mathcal{N}} \left| \exp \left\{ \frac{1}{2} \mathbf{m}^T \cdot \left[ \boldsymbol{\tau}_{conv} (\epsilon) + \frac{C_p}{2} \log (\epsilon) \, Id \right] \cdot \mathbf{m} + \mathbf{m}^T \cdot \left[ \boldsymbol{\tau}_{conv} (\epsilon) \cdot \mathbf{p} + \mathbf{z} \right] \right\} \right| \tag{3.188}$$

In view of this, (3.181) follows from lemma 3.4.2. The remaining points of the thesis can be proved using this same argument, with only minor modifications.

□

We are now ready for the main theorem of this chapter[3].

**Theorem 3.4.3.** *If*

$$-\frac{1}{2} < p < \frac{1}{2} \tag{3.189}$$

*then*

$$\psi \left(x, t; p, q; \epsilon\right) = 2 \sqrt{-\chi^0} \exp \left(-iE^0 x + iN^0 t\right) \left(1 + R \left(x, t; p, q\right) \epsilon^{1-2|p|} + o \left(\epsilon^{1-2|p|}\right)\right) \tag{3.190}$$

*for all $q \in \mathbb{R}$. The form of the coefficient $R \left(x, t; p, q\right)$ depends on $p$:*

- *If $0 < p < \frac{1}{2}$,*

$$R \left(x, t; p, q\right) = \left[\exp \left(r_0\right) - 1\right] \exp \left[ iV_0 x + iW_0 t - 2\pi i q + \frac{\overline{\alpha}_0}{2} + p \left(\beta_0 - \overline{\alpha}_0\right) \right] + \left[\exp \left(-\overline{r}_0\right) - 1\right] \exp \left[ -i\overline{V}_0 x - i\overline{W}_0 t + 2\pi i q + \frac{\alpha_0}{2} + p \left(\beta_0 - \alpha_0\right) \right] \tag{3.191}$$

---

[3]It is well-known that if $\psi$ is a solution to fNLS, also $e^{id}\psi$ is, for every real $d$. In view of this fact, we will relax our precision on this kind of factors.

- *If* $-\frac{1}{2} < p < 0$,

$$R\left(x,t;p,q\right) = \left[\exp\left(\overline{r}_0\right) - 1\right] \exp\left[i\overline{V}_0 x + i\overline{W}_0 t - 2\pi i q + \frac{\alpha_0}{2} + p\left(\alpha_0 - \beta_0\right)\right]$$
$$+ \left[\exp\left(-r_0\right) - 1\right] \exp\left[-iV_0 x - iW_0 t + 2\pi i q + \frac{\overline{\alpha}_0}{2} + p\left(\overline{\alpha}_0 - \beta_0\right)\right] \tag{3.192}$$

- *If* $p = 0$,

$$\begin{aligned} R\left(x,t;p,q\right) = & \left[\exp\left(\overline{r}_0\right) - 1\right] \exp\left(\frac{\alpha_0}{2} + i\overline{V}_0 x + i\overline{W}_0 t - 2\pi i q\right) \\ & + \left[\exp\left(-\overline{r}_0\right) - 1\right] \exp\left(\frac{\alpha_0}{2} - i\overline{V}_0 x - i\overline{W}_0 t + 2\pi i q\right) \\ & + \left[\exp\left(r_0\right) - 1\right] \exp\left(\frac{\overline{\alpha}_0}{2} + iV_0 x + iW_0 t - 2\pi i q\right) \\ & + \left[\exp\left(-r_0\right) - 1\right] \exp\left(\frac{\overline{\alpha}_0}{2} - iV_0 x - iW_0 t + 2\pi i q\right) \end{aligned} \tag{3.193}$$

*If instead*

$$p = \frac{1}{2}$$

*then*

$$\lim_{\epsilon \to 0^+} \psi\left(x,t;\frac{1}{2},q;\epsilon\right) = \tag{3.194}$$
$$-2\sqrt{-\chi_0}\frac{\cosh\left(\eta x + \phi t - i\sigma\right) + \exp\left(\frac{\beta_0}{2}\right)\cos\left(\xi x + \theta t - 2\pi q - i\rho\right)}{\cosh\left(\eta x + \phi t\right) + \exp\left(\frac{\beta_0}{2}\right)\cos\left(\xi x + \theta t - 2\pi q\right)} \exp\left(-iE_0 x + iN_0 t\right)$$

*The error estimate in (3.190) and (3.194) is to be considered uniform on compact subsets of the $(x,t)$-plane. The values of the parameters appearing in the formulas above are expressed in terms of $E_1$ and $E_3$ as follows:*

$$E^0 = 2Re\left(E_3\right) \tag{3.195a}$$

$$\chi^0 = -\frac{Im\left(E_3\right)^2}{4} \tag{3.195b}$$

$$N^0 = -2\left[2Re\left(E_3\right)^2 - Im\left(E_3\right)^2\right] \tag{3.195c}$$

$$V^0 = -2\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} \tag{3.195d}$$

$$W^0 = -4\left[\overline{E}_1 + Re\left(E_3\right)\right]\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} \tag{3.195e}$$

$$r^0 = -2\log\left\{\frac{-2\sqrt{\left(\overline{E}_1 - E_3\right)\left(\overline{E}_1 - \overline{E}_3\right)} + \left[\left(\overline{E}_1 - E_3\right) + \left(\overline{E}_1 - \overline{E}_3\right)\right]}{\left(E_3 - \overline{E}_3\right)}\right\}. \tag{3.195f}$$

$\beta^0$

$$= 2\log\left\{\frac{1}{\left(E_1-\overline{E}_1\right)\left(E_3-\overline{E}_3\right)}\left[2\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}\sqrt{\left(\overline{E}_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)}\right.\right.$$
$$\left.\left.-\left(E_1-E_3\right)\left(\overline{E}_1-\overline{E}_3\right)-\left(E_1-\overline{E}_3\right)\left(\overline{E}_1-E_3\right)\right]\right\} \tag{3.195g}$$

$\alpha^0$

$$= -4\log(2) + 2log\left[\frac{E_1-Re\left(E_3\right)}{\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}}-1\right]$$
$$-2\log\left[E_1-E_3-\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}\right]$$
$$+2\log\left\{-i\left[\sqrt{\left(\lambda-E_3\right)\left(\lambda-\overline{E}_3\right)}-\lambda+\sqrt{\left(E_1-E_3\right)\left(E_1-\overline{E}_3\right)}+E_1\right]\right\}\Bigg|_{\lambda=E1}^{\lambda=E_3} \tag{3.195h}$$

*Moreover,*

$$\xi = Re\left(V^0\right) \qquad \eta = Re\left(V^0\right) \qquad \theta = Re\left(W^0\right) \tag{3.196}$$
$$\phi = Im\left(W^0\right) \qquad \rho = Re\left(r^0\right) \qquad \sigma = Im\left(r^0\right). \tag{3.197}$$

*Proof.* Both (3.190) and (3.194) are a direct consequence of proposition 3.4.1, together with some standard manipulations of trigonometric and hyperbolic functions. Expressions (3.195) can be easily deduced using propositions 3.3.1, 3.3.4, 3.3.5, 3.3.7, 3.3.8 from previous section. □

**Observation 3.4.4.** *From (3.195g) one obtains*

$$B := \exp\left(\frac{\beta}{2}\right) = \frac{1}{\left|E_1-\overline{E}_1\right|\left|E_3-\overline{E}_3\right|}\left(\left|E_1-E_3\right|-\left|E_1-\overline{E}_3\right|\right)^2. \tag{3.198}$$

*So B is real and positive. Moreover, using the subadditivity of the norm,one gets*

$$\left|\left|E_1-E_3\right|-\left|E_1-\overline{E}_3\right|\right| < \left|E_3-\overline{E}_3\right|, \tag{3.199}$$
$$\left|\left|E_1-E_3\right|-\left|E_1-\overline{E}_3\right|\right| = \left|\left|E_1-E_3\right|-\left|\overline{E}_1-E_3\right|\right| < \left|E_1-\overline{E}_1\right|. \tag{3.200}$$

*It follows that*

$$0 < B < 1. \tag{3.201}$$

*In view of the last inequality, the limit solution (3.194) is well defined for all the values of x and t.*

**Observation 3.4.5.** *Using proposition 1.5.5 it is possible to prove that (3.201) is not peculiar to this specific degenracy situation. Indeed, it holds for all real, separated Riemann surfaces of genus two with one oval only. Formula (3.194) and this specific context are concretely instructive about the role played by the topological type of $\Gamma$ in the smoothness of the two-phase solutions.*

# Chapter 4

# Local Effectivization of the Two-Phase Solutions

## 4.1 Introduction

In this chapter we put in act the effectivization of the two-phase solutions in a certain regime. It is clear from the calculations of the previous chapter that such solutions can be written as follows:

$$\psi(x,t;p,q) = A\frac{\theta_{\boldsymbol{\tau}}(i\mathbf{V}x + i\mathbf{W}t + \boldsymbol{\tau}\cdot\mathbf{p} - 2\pi i\mathbf{q} + \mathbf{r})}{\theta_{\boldsymbol{\tau}}(i\mathbf{V}x + i\mathbf{W}t + \boldsymbol{\tau}\cdot\mathbf{p} - 2\pi i\mathbf{q})}\exp(-iEx + iNt) \tag{4.1}$$

In the formula above $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{r}$ are vectors of $\mathbb{C}^2$ having the form

$$\mathbf{V} = \begin{pmatrix} \overline{V} \\ V \end{pmatrix}, \qquad \mathbf{W} = \begin{pmatrix} \overline{W} \\ W \end{pmatrix}, \qquad \mathbf{r} = \begin{pmatrix} \overline{r} \\ r \end{pmatrix}. \tag{4.2}$$

The parameters $E$ and $N$ are real, $A$ is positive. The period matrix $\boldsymbol{\tau}$ has the form[1]

$$\boldsymbol{\tau} = \begin{bmatrix} \delta + i\gamma & \beta \\ \beta & \delta - i\gamma \end{bmatrix}, \qquad \delta, \gamma, \beta \in \mathbb{R}. \tag{4.4}$$

Finally, $\mathbf{p}$ and $\mathbf{q}$ are two-dimensional, real vectors of the form

$$\mathbf{p} = \begin{pmatrix} p \\ -p \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} q \\ q \end{pmatrix}, \qquad p, q \in \mathbb{R}. \tag{4.5}$$

The parameters $p$ and $q$ are free. Without loss of generality their range can be restricted to

$$p \in \left(-\frac{1}{2}; \frac{1}{2}\right], \qquad q \in \left(-\frac{1}{2}; \frac{1}{2}\right]. \tag{4.6}$$

[1] Throughout this chapter, the holomorphic differentials $\omega_1$ and $\omega_2$ are understood to be normalized as follows

$$\oint_{a_j} \omega_k = 2\pi i\delta_{jk}, \qquad j,k = 1,2. \tag{4.3}$$

In this case, the period matrix has negative definite real part. Referring to (4.4), this implies that $\delta < 0$.

After the effectivization prescription one should plug (4.1) into the focusing NLS equation and impose to get an identity. This way one would obtain a set of equations for the quantities $V$,$W$,$r$,$E$,$N$ and $A$. Choosing in appropriate way some free parameters these equations would yield the effective parametrization of the two-phase solutions. This process, though, leads to some transcendental equations for the quantities involved in (4.1). Since these ones are rather difficult to solve, we will consider a different, less ambitious approach.

## 4.2 Local Effectivization

Before starting let us observe that using an appropriate scaling and the other symmetries of the fNLS equation (4.1) can be recast as follows:

$$\psi\left(x,t;p\right)=\frac{\theta_{\boldsymbol{\tau}}\left(i\mathbf{V}x+i\mathbf{W}t+\boldsymbol{\tau}\cdot\mathbf{p}+\mathbf{r}\right)}{\theta_{\boldsymbol{\tau}}\left(i\mathbf{V}x+i\mathbf{W}t+\boldsymbol{\tau}\cdot\mathbf{p}\right)}\exp\left(iNt\right). \tag{4.7}$$

We will consider the case

$$p=\frac{1}{2}. \tag{4.8}$$

In chapter two, Theorem 2.5.3, we have exhibited a fundamental domain for $\mathbf{w}=(2\pi i)^{-1}\boldsymbol{\tau}$. A fundamental domain for $\boldsymbol{\tau}$, then, can be easily deduced[2]:

$$\mathcal{D}\quad:\quad\begin{cases}\delta\leq-\sqrt{4\pi^2+\beta^2-\gamma^2}\\ 0\leq\gamma<\pi\end{cases} \tag{4.9}$$

(The square root in the first inequality is understood to be positive.) We have already pointed out the necessity to choose a period matrix $\boldsymbol{\tau}$ belonging to $\mathcal{D}$ in order to get a non-redundant parametrization of the two-phase solutions to fNLS. Our starting point here consists in the additional assumption

$$\delta<<0. \tag{4.10}$$

In other words, we restrict to consider a specific zone in the fundamental domain $\mathcal{D}$. This is the reason for the adjective "local". In view of (4.10) let us introduce the parameter

$$\varepsilon:=\exp\left(\frac{\delta}{2}\right). \tag{4.11}$$

Let us also introduce the notation

$$B:=\exp\left(\frac{\beta}{2}\right),\qquad R:=\exp\left(r\right). \tag{4.12}$$

[2]One simply needs to perform the following substitution:

$$\begin{cases}\beta=-2\pi\tilde{\beta}\\ \gamma=2\pi\tilde{\gamma}\\ \delta=-2\pi\tilde{\delta}\end{cases}$$

The solution (4.7) can then be rewritten in the form[3]:

$$\psi\left(x,t;\varepsilon\right)=\frac{\mathcal{N}_0\left(x,t\right)+\varepsilon^2\mathcal{N}_2\left(x,t\right)+\cdots}{\mathcal{D}_0\left(x,t\right)+\varepsilon^2\mathcal{D}_2\left(x,t\right)+\cdots}\exp\left(iNt\right). \tag{4.13}$$

Here

$$\mathcal{N}_M\left(x,t\right)=\sum_{(m,n)\in\mathcal{S}_M}R^n\overline{R}^mB^{n+2mn-m}e^{\frac{i\gamma}{2}[m(m+1)-n(n-1)]}\exp\left(nz-m\overline{z}\right)$$

and

$$\mathcal{D}_M\left(x,t\right)=\sum_{(m,n)\in\mathcal{S}_M}B^{n+2mn-m}e^{\frac{i\gamma}{2}[m(m+1)-n(n-1)]}\exp\left(nz-m\overline{z}\right). \tag{4.14}$$

The set $\mathcal{S}_M$ is defined as follows

$$\mathcal{S}_M:=\{(m,n)\ \text{ s.t. }\ m\left(m+1\right)+n\left(n-1\right)=M;\ m,n\in\mathbb{Z}\} \tag{4.15}$$

Notice that this one has finite cardinality and it is empty for odd $M$. Moreover we have put

$$z:=iVx+iWt. \tag{4.16}$$

In our choice the free parameters will be

$$V\in\mathbb{C},\qquad\gamma,\ \varepsilon\in\mathbb{R}. \tag{4.17}$$

The remaining ones will be expressed as follows [4]:

$$\begin{aligned}W&=W_0+W_2\varepsilon^2+o\left(\varepsilon^2\right),\qquad R=R_0+R_2\varepsilon^2+o\left(\varepsilon^2\right)\\N&=N_0+N_2\varepsilon^2+o\left(\varepsilon^2\right),\qquad B=B_0+B_2\varepsilon^2+o\left(\varepsilon^2\right).\end{aligned} \tag{4.18}$$

After plugging (4.13) into the fNLS equation we impose to obtain a formal identity to all orders in $\varepsilon$. This yields a triangular (infinite) system of equations for the coefficients of the expansions (4.18) which can be recursively determined in terms of the free parameters (4.17).
Let us start with considering only the terms of order zero in $\varepsilon$ in the expansion (4.13). In this case $\psi$ reduces to

$$\psi_0\left(x,t\right)=\frac{\mathcal{N}_0\left(x,t\right)}{\mathcal{D}_0\left(x,t\right)}e^{iNt}. \tag{4.19}$$

This one coincides with the limit of $\psi\left(x,t;\varepsilon\right)$ when $\epsilon$ goes to zero. From (4.2) and (4.14) one obtains

$$\mathcal{N}_0\left(x,t\right)=1+\frac{B}{R}\exp\left(\overline{z}\right)+BR\exp\left(z\right)+\frac{R}{R}\exp\left(z+\overline{z}\right) \tag{4.20}$$

$$\mathcal{D}_0\left(x,t\right)=1+B\exp\left(\overline{z}\right)+B\exp\left(z\right)+\exp\left(z+\overline{z}\right). \tag{4.21}$$

[3]We omit to indicate the dependence of $\psi$ on $p$, since the value of this last one has already been fixed to $\frac{1}{2}$.

[4]A rigorous justification of these power series expansion is suggested below in the section. The corrections of order one in $\varepsilon$ do not appear here because both $\mathcal{N}_1$ and $\mathcal{D}_1$ in (4.13) vanish identically.

Let us plug $\psi_0$ into the focusing NLS equation and impose to get an identity in $x$ and $t$. This turns out to be equivalent to annihilate all the coefficents of a polynomial in the two variables

$$\exp(z), \qquad \exp(\overline{z}). \tag{4.22}$$

So one gets the following algebraic relations

$$V^2 + 2R_0 - W_0 - 2 = 0 \tag{4.23a}$$

$$R_0\left(V^2 + W_0 - 2\right) + 2 = 0 \tag{4.23b}$$

$$\begin{aligned}(R_0 - 1)\left(\overline{R}_0 - 1\right)\left[\left(R_0 - \overline{R}_0\right) + B_0^2\left(1 - 2R_0 + R_0\overline{R_0}\right)\right]\\ - R_0\left[\left(\overline{R}_0 - R_0\right) + B_0^2\left(1 - 2\overline{R}_0 + R_0\overline{R}_0\right)\right]V\overline{V} = 0.\end{aligned} \tag{4.23c}$$

Moreover

$$N_0 = 2. \tag{4.24}$$

The solutions of the system (4.23) can be conveniently parametrized as follows:

$$\begin{aligned}V &= 2\left(\cosh\mu\cos\lambda - i\sinh\mu\sin\lambda\right)\\ W^0 &= 2\left(\sinh 2\mu\cos 2\lambda - i\cosh 2\mu\sin 2\lambda\right)\\ R^0 &= -\exp\left(-2\mu\right)\left(\cos 2\lambda + i\sin 2\lambda\right)\\ B^0 &= \frac{\sin\lambda}{\cosh\mu}\end{aligned} \tag{4.25}$$

Both $\lambda$ and $\mu$ are understood to be real. The function $\psi_0$ is actually an exact solution of the fNLS equation. After some elementary manipulations one realizes that it already appeared in the previous chapter, formula (3.194). Indeed, the last one was obtained studying the two-phase solutions in the limit when the parameter $\epsilon$ tends to zero. Now, in view of (4.11) and proposition 3.3.1 the two parameters $\epsilon$ and $\varepsilon$ are related as follows:

$$\varepsilon = \kappa\epsilon + o(1) \tag{4.26}$$

Here $\kappa$ is some positive constant. Due to this fact, the limit of the solutions performed with respect to either of the small paramters need to give the same result. One can verify that the quantities (3.195) satisfy the equations (4.23). (Because of the assumptions at the beginning of this section, one has to take $E_3 = i$.) Moreover, the change of variables (4.26) can be proved to be analytic. Using this and the fact that the quantities $N$, $W$, $R$ and $B$ are analytic in $\epsilon$ formulas (4.18) can be rigorously justified (see [8] for more details).

**Observation 4.2.1.** *One needs to impose the restriction*

$$0 < \lambda < \pi. \tag{4.27}$$

*Indeed, from observation 3.4.4 and 3.4.5, one must have*

$$0 < B < 1. \tag{4.28}$$

In the next step we consider (4.13) up to its first correction in $\varepsilon$. That is

$$\psi_2\left(x,t;\varepsilon\right)=\frac{\mathcal{N}_0\left(x,t\right)+\varepsilon^2\mathcal{N}_2\left(x,t\right)}{\mathcal{D}_0\left(x,t\right)+\varepsilon^2\mathcal{D}_2\left(x,t\right)}e^{iNt} \tag{4.29}$$

Let us plug (4.29) into fNLS and impose to get an identity up to the order two in $\varepsilon$. Analogously to above, this is equivalent to annihilate all the coefficients of a Laurent polynomial in the variables (4.22). In this way one obtains some linear constraints for the quantities $N_2$, $B_2$, $W_2$ and $R_2$ which are finally expressed in terms of the free parameters $\gamma$ and $V$.We summarize the result of our calculations subject to some elementary semplifications in the following

**Theorem 4.2.2.** *The two-phase solutions to fNLS admit an expansion of the form*

$$\psi\left(x,t\right)=\frac{\mathcal{N}\left(\Phi\left(x,t\right),\Psi\left(x,t\right)\right)}{\mathcal{D}\left(\Phi\left(x,t\right),\Psi\left(x,t\right)\right)}\exp\left[i\frac{v}{2}x+i\left(NA^2-\frac{v^2}{4}\right)t\right] \tag{4.30}$$

*where*

$$\begin{aligned}\mathcal{D}\left(\Phi,\Psi\right)=&\frac{1}{\sqrt{B}}\cosh\Phi+\sqrt{B}\cos\Psi\\&+2\varepsilon^2\left\{\cos\gamma\left[B\sqrt{B}\cosh\Phi\cos2\Psi+\frac{1}{B\sqrt{B}}\cosh2\Phi\cos\Psi\right]\right.\\&\left.-\sin\gamma\left[B\sqrt{B}\sinh\Phi\sin2\Psi+\frac{1}{B\sqrt{B}}\sinh2\Phi\sin\Psi\right]\right\}+\mathcal{O}\left(\varepsilon^4\right),\\\mathcal{N}\left(\Phi,\Psi\right)=&\mathcal{D}\left(\Phi-i\sigma,\Psi-i\rho\right),\end{aligned} \tag{4.31}$$

*and*

$$\Phi\left(x,t\right)=\eta Ax+\left(\phi A^2-\eta Av\right)t,\qquad\Psi\left(x,t\right)=\xi Ax+\left(\theta A^2-\xi Av\right)t. \tag{4.32}$$

*If the real part $\delta$ of the diagonal entry of the period matrix $\boldsymbol{\tau}$ is (negative and) sufficiently large, the whole family of solutions is parametrized by the vector wave numbers*

$$\xi,\ \eta\quad\in\quad\mathbb{R}, \tag{4.33}$$

*the small parameter*

$$\varepsilon=\exp\left(\frac{\delta}{2}\right)>0 \tag{4.34}$$

*and the additional quantity*

$$\gamma\ \in\ \mathbb{R}. \tag{4.35}$$

*This is the imaginary part of the first diagonal entry of the period matrix. The quantities*

$$A>0,\qquad v\in\mathbb{R}, \tag{4.36}$$

*can be fixed arbitrarily: they express the Galilean and scaling invariance for fNLS. For technical convenience let us introduce the auxiliary parameters*

$$\lambda\in\left(0,\pi\right),\qquad\mu\in\mathbb{R} \tag{4.37}$$

*such that*

$$\xi = 2\cosh\mu\cos\lambda \qquad \eta = -2\sinh\mu\sin\lambda \tag{4.38}$$

*All the other quantities appearing in the solution can be expressed as follows:*

$$\begin{aligned}\theta = 2\sinh 2\mu\cos 2\lambda + &\frac{2\varepsilon^2}{\cos 2\lambda - \cosh 2\mu}\left\{\sin\gamma\left[8\sin 2\lambda + (\cosh 2\mu + \cosh 6\mu)\sin 4\lambda - \right.\right.\\ &\left.(\sin 2\lambda + \sin 6\lambda)\cosh 4\mu\right] + \cos\gamma\left[8\sinh 2\mu + (\cos 2\lambda + \cos 6\lambda)\right.\\ &\left.\left.\sinh 4\mu - (\sinh 6\mu + \sinh 2\mu)\cos 4\lambda\right]\right\} + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{4.39}$$

$$\begin{aligned}\phi = -2\cosh 2\mu\sin 2\lambda - &\frac{2\varepsilon^2}{\cos 2\lambda - \cosh 2\mu}\left\{\cos\gamma\left[-8\sin 2\lambda - (\cosh 2\mu + \cosh 6\mu)\right.\right.\\ &\left.\sin 4\lambda + (\sin 2\lambda + \sin 6\lambda)\cosh 4\mu\right] + \sin\gamma\left[8\sinh 2\mu - \right.\\ &\left.\left.(\sinh 6\mu + \sinh 2\mu)\cos 4\lambda + (\cos 2\lambda + \cos 6\lambda)\sinh 4\mu\right]\right\} + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{4.40}$$

$$\begin{aligned}N = 2 + 8\,\varepsilon^2\,\exp(-4\mu)&\left[6\exp 4\mu\cos\gamma + \cos(\gamma - 4\lambda) + 4\exp 2\mu\cos(\gamma - 2\lambda) + \right.\\ &\left.4\exp 6\mu\cos(\gamma + 2\lambda) + \exp 8\mu\cos(\gamma + 4\lambda)\right] + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{4.41}$$

$$\begin{aligned}B = \frac{\sin\lambda}{\cosh\mu} + &\frac{\varepsilon^2}{4(\cos 2\lambda - \cosh 2\mu)}\left\{-\cos\gamma\,\mathrm{sech}^3\mu\left[2(1 + \cosh 4\mu - \cosh 6\mu)\right.\right.\\ &\left.\sin\lambda + 5\sin 3\lambda + \sin 5\lambda + 2\cosh 2\mu\left(8\cos^2\lambda\sin\lambda + \sin 5\lambda\right)\right] - \\ &16\cos\gamma\cosh\mu\csc\lambda\sinh^2\mu + 16\cos\lambda\sin\gamma(-\sinh\mu + \sinh 3\mu) + \\ &\left.16(\cos\lambda + \cos 3\lambda)\,\mathrm{sech}\mu\sin\gamma\sin^2\lambda\tanh\mu\right\} + \mathcal{O}\left(\varepsilon^4\right).\end{aligned} \tag{4.42}$$

*Moreover let us put*

$$R = \exp(\rho + i\sigma). \tag{4.43}$$

*One has*

$$\begin{aligned}R = -\exp(-2\mu)(\cos 2\lambda + i\sin 2\lambda) + &\frac{\varepsilon^2\exp(-6\mu)}{\cos 2\lambda - \cosh 2\mu}\left\{\left[\exp 8\mu\cos\gamma - \right.\right.\\ &\cos(\gamma - 4\lambda) + \exp 2\mu\left(\cos(\gamma - 6\lambda) + 3(-1 + \exp 4\mu)\right.\\ &\cos(\gamma - 2\lambda) - \exp(4\mu)\cos(\gamma + 2\lambda) + 6\exp 2\mu\sin(\gamma - 2\lambda)\\ &\left.\left.\sin 2\lambda\right)\right] + i\left[-\exp 8\mu\sin\gamma + \sin(\gamma - 4\lambda) + \exp 2\mu\left(-\sin(\gamma - 6\lambda) - \right.\right.\\ &3(-1 + \exp 4\mu)\sin(\gamma - 2\lambda) + 6\exp 2\mu\cos(\gamma - 2\lambda)\sin 2\lambda + \\ &\left.\left.\left.\exp 4\mu\sin(\gamma + 2\lambda)\right)\right]\right\} + \mathcal{O}\left(\varepsilon^4\right).\end{aligned} \tag{4.44}$$

Considering higher corrections, formulas above grow immediately more complicated: we do not report them here. In principle, though, there is no obstruction to go further with these calculations. In this way one gets an arbitrarily good approximation for the genus two solutions of fNLS, provided that $|\delta|$ is sufficiently large.

**Observation 4.2.3.** *Since we are working in the limit for $\delta$ (negative and) large in its absolute value, one can assume that the first and the third inequalities in the system (4.9) are satisfied. Imposing the conditions*

$$0 \leq \gamma < \pi, \qquad 0 < \lambda < \pi \tag{4.45}$$

*then, the free parameters*

$$\varepsilon, \quad \gamma, \quad \mu, \quad \lambda \quad \in \quad \mathbb{R} \tag{4.46}$$

*turn out to be non-redundant: to different values of these ones there correspond different solutions to the fNLS equation in the form (4.13).*

# Acknowledgements

I wish to thank Prof. B.A. Dubrovin sincerely. For all the things he taught me. For allowing me to try to learn a little bit of his conception of mathematics. For being definitely strict and tough, but always kind and completely respectful to me.

I wish to thank Prof. T. Grava, for constantly assisting me during this PhD. Her encouragement and help were literally essential to finish these studies.

I wish to thank Prof. M. Bertola, who was exceptionally available to me for discussions and collaboration during the two months I spent in Montreal. I also wish to thank him for his kind and open attitude, and for revising my thesis giving many helpful suggestions.

I wish to thank Prof. K.T. MacLaughlin, Prof. A. Tovbis, Prof. J. Harnad, Prof. G. Biondini, Prof. D. Korotkin and Dr. A. Moro for valuable discussions and useful suggetions.

More personally, I wish to thank my family. It was always there for me, and one time more during these studies. Especially during the hardest and less encouraging periods. It represented my guarantee and it proved to appreciate me in spite of marks and academic achievements. I wish to dedicate this thesis to you.

I wish to thank Sanja, because. This thesis, gathering my efforts during the last four years, is dedicated also to you.

I wish to thank my friends Antonio, Giacomo, Sergio, Yanfang, Simona, Garchia, Alessandra, Francesco M., Francesco B., Ferenc, Ben, Davide, Alejo, Andrea, Devis, Alain, Gianluca, Ali, Sahar, Michal, Shima, Zhaleh, Sasha, Diana and several others which I cannot remember right now. You are the people with whom I shared laugh and tears. Sadness and hope. It was not always easy, but thanks to you, looking back I just can say it was great!

# Bibliography

[1] Barnes E.S., Cohn M.J., On Minkowski Reduction of Positive Quaternary Quadratic Forms, *Mathematika* 23(1976), 156-158.

[2] Belokolos E.D., Bobenko A.I., Enol'ski V.Z., Its A.R., Matveev V.B. *Algebro-Geometric Approach to Nonlinear Integrable Equations* Springer Series in Nonlinear Dynamics, Springer-Verlag, 1994

[3] Bobenko A.I., Klein C. (Eds.), *Computational Approach to Riemann Surfaces*, Lecture Notes in Mathematics, 2013, Springer-Verlag, 2011.

[4] R.Y. Chiao, E. Garmire and C.H. Townes, *Self-Trapping of Optical Beams* Phys. Rev. Lett 13, 479 (1964) (see also the corresponding *Erratum*, Phys. Rev. Lett. 14, 1056 (1965)).

[5] Cirre F.J., Bujalance E., Gromadzki G., Gamboa J.M., *Symmetries of Compact Riemann Surfaces* Lecture Notes in Mathematics, 2007, Springer, 2010.

[6] Dubrovin B. A., Theta functions and non-linear equations, *Russian Math. Surveys 36:2 (1981),11-92*.

[7] Dubrovin B.A., Flickinger R., Segur H., Three-Phase Solutions of the Kadomtsev-Petviashvili Equation, *Studies in Applied Mathematics* 99:137-203 (1997)

[8] Fay J., *Theta Functions on Riemann Surfaces*, Lecture Notes in Mathematics, 352, Springer-Verlag, 1973.

[9] Gelash A. A., Zakharov V. E., Soliton on Unstable Condensate, arXiv: 1109.0620v2, preprint.

[10] Giavedoni P., Period Matrices of Real Riemann Surfaces and Fundamental Domains, arXiv:1202.3560v2, preprint.

[11] Its A.R., Kotlyarov V.P., Explicit expressions for the solutions of nonlinear Schrodinger equation, Dockl. Akad. Nauk. SSSR, S. A, V. 965., N. 11, 1976.

[12] Minkowski H., Diskontinuitätsbereich für arithmetische Aequivalenz, *Ges.Abh.*, 2, 53-103, 1911.

[13] Newman M., Reiner I., Inclusion Theorems for Congruence Subgroups, *Transactions of the American Mathematical Society* Vol. 91, No.3 (Jun.,1959), 369-379.

[14] Newman M., Smart J.R., Symplectic modulary groups, *Acta Arithmetica* IX(1964), 83-89.

[15] Ryshkov S.S., The theory of Hermite-Minkowski Reduction of Positive Definite Quadratic Forms, *J. Soviet Math* 6:6 (1976), 651-671.

[16] Siegel C.L., *Topics in Complex Function Theory, vol. II*, Wiley-Interscience, 1971.

[17] Siegel C.L., *Topics in Complex Function Theory, vol. III*, Wiley-Interscience, 1973.

[18] Silhol R., Normal Forms for Period Matrices of Real Curves of Genus 2 and 3, *Journal of Pure and Applied Algebra* 87 (1993) 79-92

[19] Zakharov V. E., J.Appl.Mech.and Tech Phys. 9(2), 190 (1968).