Classification problems for Hamiltonian evolutionary equations and their discretizations

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Contents

CHAPTER 1

Introduction

1.1. Systems of hydrodynamic type and their Hamiltonian structures

The study of dispersive evolutionary partial differential equations (PDEs) systems can be addressed following the promising Dubrovin-Zhang perturbative approach. In the present Thesis, we consider some selected aspects of this general theory, dealing mainly with the subclass of Hamiltonian evolutionary PDEs and their Poisson brackets (PBs). The necessary definitions are provided in the following. However, we refer the reader to the paper **[20]** for a more detailed introduction to the subject (see also **[15]**, **[16]** and references therein).
Chapter 2 is based on **[45]** and concerned with the construction of local PBs on a lattice. Furthermore, in Chapter 3, we explain a new criterium about the existence of tri-Hamiltonian structures for nonlinear wave systems.

Let us start considering the class of first-order quasi-linear system of evolutionary PDEs

$$u^i_t = A^i_p(\mathbf{u})u^p_x, \tag{1.1.1}$$ [1]

where $u^i = u^i(x,t)$, $i = 1, \ldots, N$, are functions of the independent variables x and t. The equations (1.1.1) are also called $1+1$ evolutionary systems, referring to the physical meaning of the independent variables: x is the spatial coordinate and t the temporal one. They appear indeed in many physical applications (e.g. fluid mechanics, gas dynamics, Whitham averaging procedure) but they are also largely applied in differential geometry and topological field theory, starting from a remarkable observation due probably to Riemann. More precisely, PDEs of type (1.1.1) are invariant under local change of the dependent coordinates

$$u^i \longmapsto v^i(\mathbf{u}), \qquad i = 1, \ldots, N, \tag{1.1.2}$$

and the matrix $A^i_p(\mathbf{u})$ transforms as a $(1,1)$-tensor under local transformations of type (1.1.2). According to this fact, the dependent variables $u^1, ..., u^N$ can be seen as coordinates of a point on an N-dimensional manifold $\mathcal{M}^N$ and the equations (1.1.1)

[1] The standard Einstein summation convention will be always adopted, unless otherwise stated.

describe a flow on the loop space

$$\mathcal{L}(\mathcal{M}) \doteq \left\{\mathbb{S}^1 \to \mathcal{M}^N\right\}.$$

Notice that we might forget about boundary conditions since we are interested in techniques coming from the formal calculus of variations.
The Hamiltonian formulation of equations (1.1.1)

$$u^i_t = \left\{u^i, H[\mathbf{u}]\right\} \tag{1.1.3}$$

spells out further interesting geometric properties. However one has to define properly the objects (i.e. Hamiltonian function and Poisson bracket) appearing into the formula (1.1.3). Historically, the successful procedure to define the Hamiltonian formulation of systems (1.1.1) was obtained studying Whitham averaging equations, which describe the evolution of slowly modulated multiphase solutions of PDEs [**19**].

In particular, following the celebrated paper of B. Dubrovin and S. Novikov [**18**], in order to give a precise meaning to the formula (1.1.3) one has first to introduce the so-called *local Poisson structures of hydrodynamic type* (HPBs), which are a suitable infinite-dimensional version of the usual PBs appearing in Classical Mechanics.
Using a physical notation, the Poisson brackets of hydrodynamic type are first-order homogeneous PB, described by

$$\left\{u^i(x), u^j(y)\right\} = g^{ij}\left(\mathbf{u}(x)\right)\delta_x(x-y) + b^{ij}_k\left(\mathbf{u}(x)\right)u^k_x(x)\delta(x-y), \tag{1.1.4}$$

where $i, j = 1\dots, N$, $x, y \in \mathbb{S}^1$ and $\mathbf{u} = (u^1, \dots, u^N)$ are the local coordinates on a smooth N-dimensional manifold $\mathcal{M}^N$. As before, one can notice that such a definition is invariant under local changes (1.1.2): the matrix $g^{ij}(\mathbf{u})$ transforms as a $(0,2)$-tensor and $b^{ij}_k(\mathbf{u})$ as the Christoffel symbols of contra-variant connection.

On the other hand, we define a Hamiltonian functional of hydrodynamic type by

$$H[\mathbf{u}] = \int h(\mathbf{u})dx. \tag{1.1.5}$$

These functionals form a linear space, which can be endowed with a Lie algebra structure, using the PB (1.1.4). Indeed, the bracket between two functionals of hydrodynamic type is still a local functional of the form

$$\{H, K\} \doteq \int\int \frac{\delta H}{\delta u^i(x)}\left(g^{ij}(\mathbf{u}(x))\delta_x(x-y) + b^{ij}_k(\mathbf{u}(x))u^k_x(x)\delta(x-y)\right)\frac{\delta K}{\delta u^j(y)}dx\,dy,$$

where

$$\frac{\delta}{\delta u^i} \doteq \sum_{m\geq 0}(-)^m\partial^m_x\frac{\partial}{\partial u^{i,m}} \tag{1.1.6}$$

is the standard variational derivative.

Severe restrictions must be imposed on the smooth coefficients $g^{ij}(\mathbf{u})$ and $b^{ij}_k(\mathbf{u})$ in order to have a PB. In particular, the first order operator defined by formula (1.1.4) is skew-symmetric (i.e. $\{H, K\} = -\{K, H\}$) if and only if

$$g^{ij} = g^{ji}$$
$$\frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k.$$

Therefore $g^{ij}(\mathbf{u})$ defines a contra-variant (speudo-riemannian) metric on $\mathcal{M}^N$ and $b^{ij}_k(\mathbf{u})$ are the components of a connection compatible with the metric $g^{ij}(\mathbf{u})$. In the non-degenerate case (i.e. $\det g^{ij}(\mathbf{u}) \neq 0$), the Jacobi identity

$$\{\{H, K\}, W\} + \{\{K, W\}, H\} + \{\{W, H\}, K\} \equiv 0 \tag{1.1.7}$$

is satisfied if and only if the connection is torsionless and has zero-curvature (i.e. it is the Levi-Civita connection generated by the contra-variant metric $g^{ij}(\mathbf{u})$). We do not collect here the details about the proof of these statements. However, in the Section 2.4, we will develop analogous arguments dealing with a discrete version of the HPBs.

As a first consequence, for any local non-degenerate Poisson structure of hydrodynamic type (1.1.4), there always exists a set of local coordinates $v^1, \ldots, v^N$ in which the metric g^{ij} is flat, then the PB reads

$$\left\{v^i(x), v^j(y)\right\} = \eta^{ij}\delta_x(x-y), \tag{1.1.8}$$

where η^{ij} is a non-degenerate symmetric constant matrix.

This important result has been the base-point for many developments. The first one was in the direction of increasing the order of HPBs (introducing also inhomogeneity), the so called differential-geometric PBs (DGPBs). In the following we will consider in some detail such DGPBs, which were defined in **[19]** by formulæ of type (1.2.2). More precisely, we deal with a discretization of such formulæ.
Other generalizations were given by increasing the dimension (allowing $x \in \mathbb{R}^N$) or permitting an infinite number of field variables u^i, as in **[10]**.
Moreover, N.I. Grinberg **[24]** obtained, by direct calculations, some necessary and sufficient conditions for the degenerate case, i.e. $\det g^{ij}(\mathbf{u}) = 0$. Her results have been improved by the more recent paper of O.I. Bogoyavlenskij **[5]**.

As a consequence of the formula (1.1.4) the Hamiltonian formulation (1.1.3) is invariant with respect to local change of variable (1.1.2). However, there are certain changes of the independent variables x and t, called *reciprocal transformations*, preserving the class of systems (1.1.1) and destroying the local Hamiltonian formalism. They are defined generalizing the passage from Eulerian to Lagrangian coordinates in fluid dynamics. Let

$$\begin{aligned}&\beta(\mathbf{u})dx + \alpha(\mathbf{u})dt\\ &\nu(\mathbf{u})dx + \mu(\mathbf{u})dt\end{aligned}$$

be two conservation laws of system (1.1.1), written as closed one-forms, then new independent variables $\tilde{t}$ and $\tilde{x}$ are given by

$$\begin{aligned}\tilde{x} &= \beta(\mathbf{u})dx + \alpha(\mathbf{u})dt\\ \tilde{t} &= \nu(\mathbf{u})dx + \mu(\mathbf{u})dt.\end{aligned} \tag{1.1.9}$$

In these new independent variables, equations (1.1.1) take the form

$$u^i_{\tilde{t}} = \tilde{A}^i_p(\mathbf{u})u^p_{\tilde{x}},$$

where

$$\tilde{A} = \frac{\beta A - \alpha\mathbb{I}}{\mu\mathbb{I} - \nu A}$$

and $\mathbb{I}$ is the identity matrix.

In order to obtain an Hamiltonian formalism invariant also with respect to change of variables (1.1.9) O. Mokhov and E. V. Ferapontov proposed a nonlocal version of HPBs (see **[40]**). All these progresses were discussed extensively in a survey of Mokhov **[37]**, which also contains the necessary references.

It is well-known that the study of finite dimensional Poisson manifolds can be naturally embedded into the more general theory about Jacobi manifolds (see the papers by A.A. Kirillov **[27]** and A. Lichnerowicz **[30]**). A Jacobi structure on a finite-dimensional manifold $\mathcal{M}$ is a skew-symmetric bilinear operator on differentiable functions $\{\,\cdot\,,\,\cdot\,\}$, satisfying the Jacobi identity and the following locality condition

$$\operatorname{supp}\{f,g\} \subset \operatorname{supp} f \cap \operatorname{supp} g, \qquad f,g \in C^\infty(\mathcal{M})$$

in replacement of the Leibniz rule, required by the definition of PB.

In **[33]** S.-Q. Liu and Y. Zhang proposed a way to introduce a consistent definition for a infinite-dimensional Jacobi structure of hydrodynamic type. Their general theory applies to evolutionary systems of PDEs of type (1.1.1) and also to the dispersive perturbations (see (1.4.1) below).

Let us conclude the present Section specifying the notion of integrability for systems of type (1.1.1). The following Remarks will be useful in the subsequent Chapters.

Remark 1.1.1 (Integrability for systems of hydrodynamic type)**.** *Given a hyperbolic system of type* (1.1.1) *(i.e. the eigenvalues* $\lambda_1(\mathbf{u}), \dots, \lambda_N(\mathbf{u})$ *of the matrix* $A^i_p(\mathbf{u})$ *are real and pairwise distinct), we can consider the special class of those possessing the Riemann invariants. These systems can be diagonalized by a change of variables of type* (1.1.2)

$$r^i_t = \lambda_i(\mathbf{r})r^i_x,\text{[2]} \qquad r^i = r^i(x,t), \qquad i = 1, \dots, N, \tag{1.1.10}$$

where $r^i = r^i(\mathbf{u})$ *are called Riemann invariants and* $\lambda_i(\mathbf{r})$ *are the characteristic velocities. According to a well-known result of S. Tsarev* [**50**]*, any hyperbolic diagonal system* (1.1.10) *such that the characteristic velocities satisfy the semi-Hamiltonian condition*

$$\partial_{r^k} \frac{\partial_{r^j}\lambda_i}{\lambda_i - \lambda_j} = \partial_{r^j} \frac{\partial_{r^k}\lambda_i}{\lambda_i - \lambda_k}, \qquad i,\ j,\ k \ \text{ distinct} \tag{1.1.11}$$

possesses infinitely many commuting flow, parametrized by N *functions of one variable and the solutions to system* (1.1.10) *can be locally represented by the generalized hodograph method.*
Moreover, one can easily check that any diagonal Hamiltonian system is semi-Hamiltonian and the commuting flows are Hamiltonian, generated by infinitely many linearly independent and involutive first integrals of hydrodynamic type (1.1.5)*.*

An invariant integrability criterion for the equations of hydrodynamic type has been found in [**46**] *and it is based on the vanishing of certain tensors. In particular given a hyperbolic system of type* (1.1.1)*, the vanishing of the Haantjes tensor of the matrix* $A^i_p(\mathbf{u})$ *guarantees its diagonalizability.*

Remark 1.1.2 (Bi-Hamiltonian pencil of hydrodynamic type)**.** *An other perspective in the study of Hamiltonian systems* (1.1.1) *and their integrability has been proposed by F. Magri in* [**34**]*. The notion of compatible Poisson structures was introduced and related to the integration of nonlinear systems of PDEs. The bi-Hamiltonian approach has been extensively developed by I. Gel'fand and I. Dorfman* [**11**] *(see also* [**10**]*), providing a fundamental device for the integrability of Hamiltonian equations. Therefore the description of compatible Poisson structures of certain type and their effective construction have become crucial in the theory of integrable systems and studied by many authors (see* [**7**]*,* [**8**]*,* [**36**] *and references therein).*

A system of type (1.1.1) *is called multi-Hamiltonian if it can be written in the Hamiltonian form*

$$u^i_t = A^i_p(\mathbf{u})u^p_x = \left\{u^i, H_{[k]}[\mathbf{u}]\right\}_{[k]}, \qquad k = 1, 2, \dots, M,$$

[2]Notice that in the diagonal systems there is not summation over the repeated index.

with respect to a finite number of compatible PBs, that is any their linear combination with arbitrary constant coefficients provides a PB of the same type. In the bi-Hamiltonian case a sequence of (independent and in involution) first integrals $K^{[\alpha]}$, is provided by the Lenard-Magri sequence

$$\left\{u^i, K^{[\alpha+1]}[\mathbf{u}]\right\}_{[1]} = \left\{u^i, K^{[\alpha]}[\mathbf{u}]\right\}_{[2]}, \qquad \alpha \geq 0,$$

where $\left\{u^i, K^{[0]}[\mathbf{u}]\right\}_{[1]} \equiv 0$.

1.2. Differential-geometric PBs and their discretizations

In the Chapter 2 we deal with a discrete variant of the differential-geometric PBs (DGPBs), that, although containing many fundamental examples (e.g. Volterra lattices, Toda lattices, Bogoyavlensky lattices (see Yu.B. Suris, [**48**] for a exhaustive list of integrable examples)), has remained quite undeveloped and neglected in the literature. We address the problem of classifying discrete differential-geometric PBs (dDGPBs) of any fixed order on target space of dimension 1 and describing their compatible pencils.

We will explain how these discrete PBs are in one-to-one correspondence with the common points of certain projective hypersurfaces. In addition, they can be reduced to cubic PB of standard Volterra lattice by discrete Miura-type transformations. Moreover, by improving a consolidation lattice procedure, we obtain new families of non-degenerate, vector-valued and first order dDGPBs, which can be considered in the framework of admissible Lie-Poisson group theory.

Let us consider the following class of local PBs

$$\begin{aligned} \left\{u^i_n, u^j_{n+k}\right\}_M &= g^{ij}_k(\mathbf{u}_n, \dots, \mathbf{u}_{n+k}), \quad 0 \leq k \leq M, \\ \left\{u^i_n, u^j_{n+k}\right\}_M &\equiv 0, \qquad\qquad\qquad\qquad\quad k > M, \end{aligned} \tag{1.2.1}$$

defined on the phase space of infinite sequences

$$\begin{aligned} \mathbf{u}: \ \ & \mathbb{Z} \longrightarrow \mathcal{M}^N \\ & n \longmapsto \mathbf{u}_n \doteq \left(u^i_n\right)_{i=1,\dots,N} \end{aligned}$$

with values in the target manifold $\mathcal{M}^N$ of dimension N. The integer M and the dimension N are related by a lattice consolidation procedure of the dependent variables u_n (see formula (1.2.5) below). Fixing the target manifold $\mathcal{M}^N$, the integer number M, called the order of the PB, can be seen as *the locality radius*, i.e. the radius of the maximum local interaction between neighboring lattice variables.

The PBs (1.2.1) have been introduced by B. Dubrovin in **[12]** (see also A. Ya. Mal'tsev, **[35]**), as a discretization of the differential geometric PBs (DGPBs), defined on the loop space $\mathcal{L}(\mathcal{M})$ by the formula

$$\left\{u^i(x), u^j(y)\right\}_M = \sum_{k=0}^{M+1} g_k^{ij}(\mathbf{u}(x), \mathbf{u}_x(x), \ldots, \mathbf{u}^{(k)}(x))\delta^{(M+1-k)}(x-y), \tag{1.2.2}$$

where $i, j = 1, \ldots, N$ and the functions g_k^{ij} are graded-homogeneous differential polynomials, that is $\deg g_k^{ij} = k$ and the gradation on differential polynomials is defined by

$$\deg u_x^i = 1, \ldots, \deg u^{i,(k)} = k, \qquad \deg f(x; \mathbf{u}) = 0.$$

Further generalizations of DGPBs will be described in the Section 1.4.

PBs of type (1.2.1) are associated with lattice Hamiltonian equations of the following form

$$\dot{u}_n^i = \left\{u_n^i, H[\mathbf{u}]\right\}_M = \sum_{m\in\mathbb{Z}} \sum_{p=1}^{N} \left\{u_n^i, u_m^p\right\}_M \frac{\delta H[\mathbf{u}]}{\delta u_m^p}, \tag{1.2.3}$$

where $H[\mathbf{u}] = \sum_{m\in\mathbb{Z}} h(\mathbf{u}_m, \ldots, \mathbf{u}_{m+K})$ for some integer $K \geq 0$ and the function h is defined on a finite interval of the lattice. In addition, we define a discrete formal variational derivative (see formula (1.1.6)) as

$$\frac{\delta H[\mathbf{u}]}{\delta u_n^p} \doteq \frac{\partial}{\partial u_n^p}\left(1 + T^{-1} + \ldots + T^{-K}\right) h(\mathbf{u}_n, \ldots, \mathbf{u}_{n+K}),$$

where T is the standard shift operator, satisfying

$$T^r h(\mathbf{u}_n, \ldots, \mathbf{u}_{n+K}) = h(\mathbf{u}_{n+r}, \ldots, \mathbf{u}_{n+r+K}), \tag{1.2.4}$$

for any integer r. The local Poisson structures of many fundamental integrable systems belong to the class (1.2.1). However, the theory of such discrete PBs is much less developed than the corresponding of DGPBs (1.2.2).

We first observe that, defining new variables of a larger target manifold, according to the formulæ

$$v_n^{i+p} \doteq u_{nM+p}^i, \qquad p = 0, \ldots, M-1, \tag{1.2.5}$$

any discrete PB (1.2.1) can be reduced to the following bracket

$$\begin{aligned} \left\{v_n^i, v_{n+1}^j\right\}_1 &= g_1^{ij}(\mathbf{v}_n, \mathbf{v}_{n+1}) \quad i, j = 1, \ldots, N+M-1, \\ \left\{v_n^i, v_n^j\right\}_1 &= g_0^{ij}(\mathbf{v}_n). \end{aligned} \tag{1.2.6}$$

A classification of such PBs (1.2.6) has been provided in **[12]**, whereas it seems that the higher order PBs have not been studied yet. More precisely, B. Dubrovin proved that if the matrix g_1^{ij} is non-singular, i.e.

$$\det g_1^{ij}(\mathbf{u}_n, \mathbf{u}_n) \neq 0, \tag{1.2.7}$$

the PBs (1.2.6) are induced by admissible Lie-Poisson group structures on the target manifold $\mathcal{M}^N$ (see Definition 2.4.3 and Theorem 2.4.4 below). However, the procedure (1.2.5) leaves some unsolved questions:

(i) what are the relations between PBs (1.2.1) *of order $M > 1$ and admissible Lie-Poisson groups associated to their consolidations?*
(ii) how to produce examples of such admissible Lie-Poisson groups?

In the Chapter 2, in order to give some partial answers, we classify scalar-valued ($N = 1$) PBs (1.2.1) of any positive order M,

$$\{u_n, u_{n+k}\}_M = g_k(u_n, \dots, u_{n+k}), \qquad 1 \leq k \leq M. \tag{1.2.8}$$

We first notice that PBs of type (1.2.1) are invariant under local change of variable

$$u_n^i \longmapsto v_n^i = v^i(\mathbf{u}_n), \qquad i = 1, \dots, N, \tag{1.2.9}$$

where the coefficients g_k^{ij} transform according to the formula

$$g_k^{ij}(\mathbf{u}_n, \dots, \mathbf{u}_{n+k}) \longmapsto \sum_{p,q=1}^{N} \frac{\partial v^i}{\partial u_n^p}(\mathbf{u}_n) g_k^{pq}(\mathbf{u}_n, \dots, \mathbf{u}_{n+k}) \frac{\partial v^j}{\partial u_{n+k}^q}(\mathbf{u}_{n+k}).$$

Two local PBs will be therefore considered equivalent if they can be related by a change of coordinates of type (1.2.9).
Let us be more precise about the classification of scalar-valued PBs (1.2.8). The coefficients $g_k(u_n, \dots, u_{n+k})$ satisfy the system of M^2 bi-linear PDEs imposed by the Jacobi identity (see Section 2.1). It turns out that any PB (1.2.8) is characterized by his leading order function $g_M(u_n, \dots, u_{n+M})$, according to the following results.

Lemma 1.2.1. *For any PB of the form* (1.4.8)*, there exist a set of coordinates* (canonical coordinates) *and an integer $\alpha > 0$, such that the leading order reduces to the form*

$$g_M(u_n, \dots, u_{n+M}) = f^\xi(u_{n+\alpha}, \dots, u_{n+\alpha+\xi}), \qquad \xi \doteq M - 2\alpha,$$

where the function f^ξ is either constant or given by the formula

$$f_n^\xi \doteq f^\xi(u_n, \dots, u_{n+\xi}) = \exp(z_n).$$

Here,

$$z_n \doteq \sum_{i=0}^{\xi} \tau_i \, u_{n+i}, \tag{1.2.10}$$

and the parameters τ_i, $i = 0, \dots, \xi$ satisfy a system of homogeneous polynomial equations (see Theorem 2.1.8 below).

In the case when the leading coefficient $g_M(u_n, \dots, u_{n+M})$ is constant, it is not difficult to prove that all the other coefficients $g_k|_{k=1,\dots,M-1}$ are also constant. The general, non-constant case is described by the following

Theorem 1.2.2. *The coefficients g_k of a non-constant PB (1.2.8) are given, in the canonical coordinates, by suitable linear combinations of the shifted generating function f^ξ, according to the following formulæ*

$$g_{\alpha+\xi+p}(u_n, \dots, u_{n+\alpha+\xi+p}) = \left(\sum_{s=\max(0,p)}^{\min(\alpha+p,\alpha)} \lambda_p^s T^s \right) f^\xi(u_n, \dots, u_{n+\xi}),$$

where $p = -\alpha, \dots, \alpha$, and the constants λ's can be expressed explicitly in terms of the parameters τ's (see equation (2.1.14),below).

This Theorem provides a complete classification of PBs of type (1.4.8). Note that the functional form of the coefficients g_k is fixed by the choice of a finite number of parameters τ's.

According to Theorem 1.2.2 and by performing a lattice consolidation procedure (see formula (1.2.5) above), we will be able to present explicitly some new examples of non-degenerate PB of the form (1.2.6) and describe their associated Lie bi-algebra constants in terms of the parameters τ's (see Section 2.4 below).

In addition to the above results, we also prove a Darboux-type theorem for PBs (1.4.8). This is done by considering the change of variable (1.2.10), that is a generalization of the local one (1.2.9).
Notice that the formula (1.2.10) can be thought as a discrete analogue of Miura transformations, studied in the Hamiltonian PDEs theory (see Section 1.4 and Remark 1.4.1).
Splitting all the variables into $\alpha + \xi$ families according to

$$v_n^{(p)} \doteq z_{(\alpha+\xi)(n-1)+p}, \qquad p = 1, \dots, \alpha + \xi,$$

by direct computation, we obtain that any non-constant PB (1.4.8) in the z-coordinates can be reduced to the following simple form

$$\begin{aligned} \left\{ v_n^{(p)}, v_{n+2}^{(p)} \right\} &= \tau_0 \, \tau_\xi \exp(v_{n+1}^{(p)}) \\ \left\{ v_n^{(p)}, v_{n+1}^{(p)} \right\} &= \tau_0 \, \tau_\xi \left[\exp(v_n^{(p)}) + \exp(v_{n+1}^{(p)}) \right], \end{aligned}$$

that are $\alpha + \xi$ copies of cubic Volterra PB (see Theorem 2.2.2 below).

1.3. Multi-Hamiltonian structure for nonlinear wave systems

Let us now go back to Hamiltonian systems of hydrodynamics type (1.1.1), restricting our attention to the two-components case: $N = 2$.

For a given smooth function $h = h(u, v)$ of the dependent variables $u = u(x, t)$ and $v = v(x, t)$, satisfying the non-degeneracy condition

$$h_{uu}(u, v)h_{vv}(u, v) \neq 0, \tag{1.3.1}$$

we study the following Hamiltonian system

$$\begin{aligned} u_t &= \partial_x \frac{\delta H}{\delta v} \\ v_t &= \partial_x \frac{\delta H}{\delta u} \end{aligned} \tag{1.3.2}$$

Here the PB of hydrodynamic type (1.1.4) is reduced to the Darboux form

$$\{u(x), v(y)\} = \delta_x(x - y), \tag{1.3.3}$$

and $H[u, v] = \int h(u, v)dx$. Notice that u, v turn out to be the flat coordinates for the metric g^{ij} in formula (1.1.4). The first integrals of system (1.3.2) are characterized by the following

Lemma 1.3.1. *Any conservation law $f = f(u, v)$ of the Hamiltonian system (1.3.2) is determined by the solutions to the following PDE*

$$h_{vv}f_{uu} = h_{uu}f_{vv}. \tag{1.3.4}$$

The associated Hamiltonians

$$H_f = \int f(u, v)dx \tag{1.3.5}$$

commute pairwise with respect to the PB (1.3.3).

An important class of Hamiltonian systems (1.3.2) is described by the systems for which the linear PDE (1.3.4) admits a separation variables. Following P.J. Olver and Y. Nutku [**44**], we focus on the *separable Hamiltonian densities*, satisfying

$$\frac{h_{uu}}{h_{vv}} = \frac{Q(u)}{R(v)},$$

where Q and R are arbitrary non-vanishing functions. In particular, we mainly deal with the specific subclass defined by $R(v) = const$. In this case, the equation (1.3.4) reduces to the following PDE

$$f_{uu}(u, v) = Q(u)f_{vv}(u, v), \tag{1.3.6}$$

for some function $Q = Q(u)$, satisfying the genericity condition $Q'(u) \neq 0$. This means that also $h = h(u,v)$ is a solution to (1.3.6).

Let us consider the Lie algebra of local Hamiltonians of type (1.3.5), endowed with the PB (1.3.3). Our interest in the abelian Lie subalgebra, parametrized by the solutions $f = f(u,v)$ to equation (1.3.6), is motivated by the following physical and geometric reasons.

Remark 1.3.2 (Motivations from applications)**.** *The Hamiltonian systems* (1.3.2) *with Hamiltonian densities satisfying* (1.3.6) *appear in a number of physically important models of gas dynamics, e.g.*

$$\begin{aligned} u_t &= (uv)_x \\ v_t &= \left(\frac{1}{2}v^2 + P'(u)\right)_x \end{aligned} \tag{1.3.7}$$

where $u = u(x,t)$ represents the density, $v = v(x,t)$ the velocity and the function $P(u)$ can be related to the pressure. System (1.3.7) *is Hamiltonian with PB of type* (1.3.3) *and Hamiltonian density*

$$h(u,v) = \frac{1}{2}uv^2 + P(u).$$

The paradigmatic example of class (1.3.7) *is given by the equations of polytropic gas dynamics (see* (3.1.23) *below), which exhaust also the generic two-dimensional algebraic Frobenius potential (see* (3.1.21) *below).*

Other examples come from the one-dimensional nonlinear elastic waves (see Whitham' book **[53]**, *Example 12, paq. 123), where the Hamiltonian density is given by*

$$h(u,v) = \frac{1}{2}v^2 + P(u).$$

Particular choices of the function $P = P(u)$ describe different models of phase transitions, for examples the case

$$P(u) = -\frac{(1+u)^{-\gamma}}{\gamma} \tag{1.3.8}$$

corresponds to the Euler equation for nonlinear acoustics (see **[25]***).*

Remark 1.3.3 (Motivations from WDVV equations)**.** *The theory of homogeneous Frobenius manifold provides a procedure to associate a hierarchy of first-order commuting quasi-linear PDEs of type* (1.1.1) *to any solution $F = F(\mathbf{u})$ of the WDVV associativity equations (see* **[13]***)*

$$\frac{\partial^3 F}{\partial u^\alpha \partial u^\beta \partial u^p}\eta^{pq}\frac{\partial^3 F}{\partial u^q \partial u^\tau \partial u^\lambda} = \frac{\partial^3 F}{\partial u^\lambda \partial u^\beta \partial u^p}\eta^{pq}\frac{\partial^3 F}{\partial u^q \partial u^\tau \partial u^\alpha}, \quad \forall\ \alpha,\beta,\lambda,\tau, \tag{1.3.9}$$

where

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial u^1 \partial u^\alpha \partial u^\beta} \tag{1.3.10}$$

is a constant symmetric non-degenerate matrix and $\eta^{\alpha\lambda}\eta_{\lambda\beta} = \delta^\alpha_\beta$ *is the Kronecker symbol. Notice that the variable* u^1 *is marked due to* (1.3.10)*. For any fixed* $\mathbf{u}$*, one can define the structure constants of an associative algebra, by setting*

$$c^\gamma_{\alpha\beta}(\mathbf{u}) \doteq \eta^{\gamma\lambda} \frac{\partial^3 F}{\partial u^\lambda\, \partial u^\alpha\, \partial u^\beta}(\mathbf{u}),$$

where $c^\gamma_{1\beta} = \delta^\gamma_\beta$*. Let now* $f = f(\mathbf{u})$ *be a solution to the following system of linear differential equations*

$$\frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} = c^\gamma_{\alpha\beta}(\mathbf{u}) \frac{\partial^2 f}{\partial u^1\, \partial u^\gamma}, \tag{1.3.11}$$

and consider the Hamiltonian systems of hydrodynamic type

$$\mathbf{u}_t = \{\mathbf{u}, H_f[\mathbf{u}]\} \tag{1.3.12}$$

with local Hamiltonian

$$H_f = \int f(\mathbf{u}) dx, \tag{1.3.13}$$

and PB in the constant form (1.1.8)*. Then, the Hamiltonians* (1.3.13) *commute pairwise and satisfy a maximality conditions (i.e. the abelian Lie subalgebra of Hamiltonians* (1.3.13) *depends on* N *arbitrary functions of one variable, see Section 1.4). Therefore the system* (1.3.11) *can be considered as a completely integrable Hamiltonian systems of PDEs.*

Starting from a Hamiltonian system, one can naturally wonder if a multi-Hamiltonian presentation is also possible (see Remark 1.1.2 above).

From the general theory (see [**14**]), we know that any systems of first-order quasilinear PDEs, associated to a Frobenius manifold, possesses a quasihomogeneous bi-Hamiltonian pencil of PBs of hydrodynamic type, i.e. any Frobenius manifold can be related to a certain bi-Hamiltonian structure on its loop space. However, the viceversa holds true only under a certain regularity condition on the bi-Hamiltonian structure.

This suggests us to investigate, at least in some particular cases, the notion of bi-Hamiltonian systems of hydrodynamic type, generalizing those coming from the Frobenius world. In other word, we consider any solution $F = F(\mathbf{u})$ to WDVV equations (1.3.9), such that the linear space of solutions to (1.3.11) is an abelian Lie sub-algebra, with respect to a bi-Hamiltonian pencil of HPB. We are therefore relaxing the quasihomogeneity condition, forgetting the role played by the Euler vector field.

Actually, when we are dealing with the Frobenius case the Lie subalgebra of commuting Hamiltonian densities (1.3.13) is tri-Hamiltonian (see [**47**]). At least in the two-components case, this would be always the case.

Let us focus on the two-components case, where the system (1.3.11) reduces to (1.3.6), with $u^1 = v$ and $u^2 = u$ and the WDVV associativity equations (1.3.9) are empty.

In the following, we would like to describe necessary and sufficient conditions on function $Q = Q(u)$ such that the Hamiltonians (1.3.5) commute with respect to a pencil of compatible PBs of hydrodynamic type (1.1.4), without any further homogeneity conditions, extending therefore Frobenius examples.
We are looking for a potential of the form

$$F = \frac{1}{2}uv^2 + \varphi(u),$$

where $\varphi(u)$ is an arbitrary function. Let $Q = Q(u)$ be defined by the third derivative of the function φ,

$$Q \doteq \varphi'''(u),$$

then the main result of Chapter 3 can be stated as a criterium for the existence of local multi-Hamiltonian structures.

Theorem 1.3.4. *The Hamiltonians H_f (1.3.5), with $f = f(u,v)$ solution to (1.3.6) commute pairwise with respect to a pencil of three compatible PBs of hydrodynamic type if and only if the function $Q = Q(u)$ is a solution to the following third order autonomous ODE*

$$\left(\frac{Q}{Q'}\right)'' = k^2 Q \cdot \left(\frac{Q}{Q'}\right) \tag{1.3.14}$$

for some arbitrary constant k.

Remark 1.3.5. *As immediate consequence of Remark 1.1.1 we have that any Hamiltonian two-components system of type (1.3.2) is integrable and usually, the local Hamiltonians (1.3.5) commute with respect to pencil of PBs. However, this Poisson pencil is often generated by the local PB (1.3.3) and a non-local one. The above result gives us a complete description of Hamiltonian systems of type (1.3.2), which are bi-Hamiltonian in the class of local PBs (1.1.4).*

Given an Hamiltonian density $h = h(u, v)$, we compute the ratio

$$Q = \frac{h_{uu}}{h_{vv}}.$$

If the resulting function Q depends on one variable and solves equation (1.3.14) we can reconstruct the local tri-Hamiltonan structures of system (1.3.2). Notice that as in the Frobenius case, any two-components bi-Hamiltonian system (1.3.2) is always tri-Hamiltonian and there are not further local compatible Poisson structures of hydrodynamic type. Moreover, we will see that solutions to equation (1.3.14) can be expressed in terms of hyperbolic functions.

1.4. (Formal) integrable Hamiltonian perturbations of wave systems

In the present Section we do one more step, analyzing the integrability of two-components evolutionary systems of PDEs, in which we are allowing corrections in the higher derivatives. Let us be more rigorous, collecting the necessary definitions in the general N-components setting.

A large class of perturbations of the hyperbolic system (1.1.1) is formally described by the following systems of evolutionary PDEs

$$u^i_t = A^i_p(\mathbf{u})u^p_x + \sum_{k\geq 1} \epsilon^k B^i_{[k]}(\mathbf{u}; \mathbf{u}_x, \dots, \mathbf{u}^{(k)}), \quad i = 1, \dots, N, \tag{1.4.1}$$

where ϵ is a small dispersive parameter and $B^i_{[k]}(\mathbf{u}; \mathbf{u}_x, \dots, \mathbf{u}^{(k)})$ are graded homogeneous polynomials in the jet variables $\mathbf{u}_x$, $\mathbf{u}_{xx}$, ...
For any integers $i = 1, \dots, N$ and $m > 0$, one can introduce the gradation $\deg u^{i,(m)} = m$. Therefore the coefficients $B^i_{[m]}(\mathbf{u}; \mathbf{u}_x, \dots, \mathbf{u}^{(m)})$ are required to be homogeneous of degree m, i.e. $\deg B^i_{[m]} = m$, (see also Remark 1.4.3 below).

The system (1.1.1) is called the *hydrodynamic limit* or *dispersionless limit* of system (1.4.1). Our study will be formal in the sense that we do not care about the convergence of the series appearing in the perturbative formulæ.
A perturbed system of type (1.4.1) is called Hamiltonian if it can be presented in the form

$$u^i_t = \left\{u^i(x), H[\mathbf{u}]\right\}, \tag{1.4.2}$$

where the PB is given by

$$\left\{u^i(x), u^j(y)\right\} = \sum_{m\geq 0} \epsilon^m \left\{u^i(x), u^j(y)\right\}^{[m]} \tag{1.4.3}$$

and for any $m \geq 0$,

$$\left\{u^i(x), u^j(y)\right\}^{[m]} = \sum_{k=0}^{m+1} a^{ij}_{m,k}(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^{(k)})\delta^{(m+1-k)}(x-y),$$

is a DGPB (1.2.2) of order m. Here the coefficients $a^{ij}_{m,k}$ are homogeneous polynomials in the derivatives of degrees

$$\deg a^{ij}_{m,k} = k, \quad k = 0, 1, \ldots, m+1.$$

Moreover, the corresponding Hamiltonian functionals are

$$H[\mathbf{u}] = \sum_{k\geq 0} \epsilon^k H^{[k]}[\mathbf{u}] = \int \left[\sum_{k\geq 0} \epsilon^k h^{[k]}(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^k)\right] dx \tag{1.4.4}$$

and

$$\deg h^{[k]}(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^{(k)}) = k.$$

Remark 1.4.1. *(Miura-type transformation). Let us introduce the generalization of the changes of variables* (1.1.2)*, the so-called Miura-type transformations. They are defined by*

$$v^i = F^i_{[0]}(\mathbf{u}) + \sum_{k\geq 1} \epsilon^k F^i_{[k]}(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^{(k)}), \quad i = 1, \ldots, n, \tag{1.4.5}$$

$$\deg F^i_{[k]}(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^{(k)}) = k, \quad \det\left(\frac{\partial F^i_{[0]}}{\partial u^j}\right)_{ij} \neq 0,$$

where, as usual, the coefficients $F^i_{[k]}(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^{(k)})$ are assumed to depend polynomially on the derivatives. The Miura-type transformations (1.4.5) *form a group (i.e. they are invertible at least formally).*

Moreover the classes of evolutionary PDEs (1.4.1)*, local PBs* (1.4.3) *and local Hamiltonians* (1.4.4) *are invariant with respect to the action of the group of Miura-type transformations* (1.4.5)*. Two invariant elements of the perturbative theory are therefore equivalent if they can be related by a generalized Miura-type transformation. This is the starting point of the Dubrovin-Zhang classification program of Hamiltonian dispersive PDEs* (1.4.2) *(see* **[20]***).*

The freedom in choosing the right set of coordinates up to Miura-type transformations (1.4.5) can be used in order to reduce the PBs (1.4.3) to a normal form. This will be very useful in the following description of integrable perturbations.

Remark 1.4.2 (Poisson cohomology triviality)**.** *The local PBs of type* (1.4.3) *can be seen as a perturbation of PBs of hydrodynamic type*

$$\left\{u^i(x), u^j(y)\right\}^{[0]} = a^{ij}_{0,0}(\mathbf{u})\delta_x(x-y) + a^{ij}_{0,1}(\mathbf{u}; \mathbf{u}_x)\delta(x-y), \tag{1.4.6}$$

In Section 1.1 we have already noticed that if the non-degeneracy condition

$$\det a^{ij}_{0,0} \neq 0 \tag{1.4.7}$$

is satisfied there exists a change of variable reducing (1.4.6) to the constant form (1.1.8). Then, we may assume that PB (1.4.3) has the form

$$\left\{u^i(x), u^j(y)\right\} = \eta^{ij}\delta_x(x-y) + \mathcal{O}(\epsilon). \tag{1.4.8}$$

Let us notice that the first correction is a 2-cocycle in the Poisson cohomology of the PB (1.1.8), which can be eliminated by a Miura-type transformation using the triviality of the Poisson cohomology in positive degrees in ϵ proven in [**23**] *(see also* [**9**] *and* [**20**]*).*

As an immediate consequence of Remark 1.4.2, we have that any system of Hamiltonian PDEs (1.4.2) satisfying the nondegeneracy assumption can be reduced to the following normal form

$$u^i_t = \eta^{ip}\partial_x \frac{\delta H[\mathbf{u}]}{\delta\, u^p(x)}, \tag{1.4.9}$$

where we have fixed the PB in the normal form (1.1.8) and $H[\mathbf{u}]$ of the perturbed type (1.4.4). Notice that the class of systems of type (1.4.9) is invariant with respect to the subclass of Miura type-transformations (1.4.5) given by canonical transformation of the PB (1.1.8), i.e.

$$u^i \longrightarrow u^i + \epsilon\left\{u^i(x), F\right\} + \epsilon^2\left\{\left\{u^i(x), F\right\}, F\right\} + \mathcal{O}(\epsilon^3), \tag{1.4.10}$$

where $F[\mathbf{u}]$ is a Hamiltonian function of the form (1.4.4) and the PB is given by (1.1.8).

Let us add one more comment on Remark 1.4.2, underlining the importance of the polynomiality assumption in formulæ (1.4.1), (1.4.3), (1.4.4).

Remark 1.4.3 (Polynomiality and quasi-triviality)**.** *The polynomiality assumption can be relaxed by allowing more general class of transformations, the so called quasi-Miura transformations*

$$v^i = G^i_{[0]}(\mathbf{u}) + \sum_{k\geq 1}\epsilon^k G^i_{[k]}(\mathbf{u}; \mathbf{u}_x, \dots, \mathbf{u}^{(m_k)}), \quad m_k = \left[\frac{3k}{2}\right], \quad i = 1, \dots, N, \tag{1.4.11}$$

where, $G^i_{[k]}$, $k \geq 0$, are now rational functions of the jets variables $\mathbf{u}_x, \mathbf{u}_{xx}, \dots$.
Then any semisimple bi-Hamiltonian structure of the form (1.4.1) is quasi-trivial (see [**17**]*). This means that applying a quasi-Miura transformation the equations (1.4.1) and their bi-Hamiltonian structure can be reduced to the leading order term. A quasi-triviality result is available also for a more general class of scalar evolutionary perturbations (1.4.1) (see* [**32**]*). The proof presented in* [**32**] *has been adapted to the scalar Hamiltonian evolutionary equation of type (1.4.1) (see* [**31**]*), however the general Hamiltonian case is still missing.*

In the Remark 1.1.1 we have described the notion of integrability for hydrodynamic type systems (1.1.1). Now, we should generalize this property to the dispersive perturbations (1.4.1). Following **[15]**, one can say that a Hamiltonian perturbation of the form (1.4.2) is integrable if any first integral of the unperturbed system (1.4.9) can be extended to a first integral of the deformed system (1.4.2).

More precisely, let us recall that the dispersionless limit of system (1.4.9) is integrable if and only if the Lie algebra of first integrals $H_f^{[0]} = \int f(\mathbf{u})dx$, defined by solution $f = f(\mathbf{u})$ to

$$\left\{H^{[0]}, F^{[0]}\right\} = 0, \tag{1.4.12}$$

or equivalently

$$\frac{\partial^2 f}{\partial u^i \partial u^\alpha}\eta^{ij}\frac{\partial^2 h}{\partial u^j \partial u^\beta} - \frac{\partial^2 f}{\partial u^i \partial u^\beta}\eta^{ij}\frac{\partial^2 h}{\partial u^j \partial u^\alpha} = 0 \quad \alpha, \beta = 1, \dots, N, \tag{1.4.13}$$

depends on the maximal number N of arbitrary functions of one variable. Then, the perturbed system (1.4.2) is called $M-$integrable if there exists a linear differential operator

$$\begin{aligned} \text{(1.4.14)}\quad D^M &= D_{[0]} + \epsilon D_{[1]} + \epsilon^2 D_{[1]} + \ldots + \epsilon^M D_{[M]} \\ D_{[0]} &= \mathrm{id}, \quad D_{[k]} = \sum_{i_1,\ldots,i_{m(k)}} d^{[k]}_{i_1,\ldots,i_{m(k)}}(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^{(k)}) \frac{\partial^{m(k)}}{\partial u^{i_1} \ldots \partial u^{i_{m(k)}}}, \\ \deg d^{[k]}_{i_1,\ldots,i_{m(k)}}&(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^{(k)}) = k, \quad m(k) = \left[\frac{3k}{2}\right], \qquad k \geq 1 \end{aligned}$$

such that, for any two solutions $f = f(\mathbf{u})$, $g = g(\mathbf{u})$ to the equation (1.4.13) the Hamiltonians defined (up to x-total derivatives) by

$$H_f^{[M]}[\mathbf{u}] = \int D^M f(\mathbf{u})dx, \quad H_g^{[M]}[\mathbf{u}] = \int D^M g(\mathbf{u})dx \tag{1.4.15}$$

commute within the $M+1$ approximation in ϵ,

$$\left\{H_f^{[M]}, H_g^{[M]}\right\} = \mathcal{O}(\epsilon^{M+1}). \tag{1.4.16}$$

The perturbed Hamiltonian system (1.4.2) is called integrable if it is M-integrable for any $M \geq 0$.

Remark 1.4.4 (About uniqueness)**.** *The uniqueness of a D-operator can be guaranteed by the following regularity condition.*
The Lie subalgebra of commuting Hamiltonians $H_f^{[0]}$, where $f = f(\mathbf{u})$ is a solution to (1.4.13) is said regular if given any functional

$$G = \int g(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots, \mathbf{u}^{(k)})dx, \;\; k \geq 1,$$

with $g = g(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots, \mathbf{u}^{(k)})$ *a differential polynomial, such that*

$$\left\{G, H_f^{[0]}\right\} \equiv 0,$$

then $g = \partial_x \tilde{g}$, *where* $\tilde{g} = \tilde{g}(\mathbf{u}; \mathbf{u}_x, \dots, \mathbf{u}^{(k-1)})$. *For instance, this conditions is always fulfilled in the scalar case* ($N = 1$). *When* $N = 2$, *one can prove that the Lie subalgebra defined by equation* (1.3.6) *is regular.*

On the other hand, proving the existence of such a D-operator is really challenging, even for the scalar case (see [**15**]). In the Chapter 3 (Section 3.2) we will consider the class of two-components systems and we will address the problem of extending up to the third order the classification of integrable Hamiltonian perturbation of the following nonlinear wave system

$$\begin{aligned} u_t &= v_x \\ v_t &= P''(u)u_x, \end{aligned} \tag{1.4.17}$$

where $P(u)$ is a smooth function of the independent variable $u = u(x, t)$. In particular, we will prove that any integrable Hamiltonian perturbation of the third order in ϵ is trivial (see Theorem 3.2.8 below).

Notice that systems of type (1.4.17) belong to the class of those considered above in Section 1.3 (see also Chapter 3, below). Moreover the dispersionless limits of many higher order PDEs (e.g. generalized Fermi-Pasta-Ulam systems (3.1.30), Boussinesq equation (3.1.24), nonlinear Schrödinger equation (3.1.26), Ablovitz-Ladik equation (3.1.34)), can be written in the nonlinear wave systems form (1.4.17).

1.5. Future perspectives

We conclude the Chapter adding a natural question which can be now addressed. Let us recall that in the scalar case (i.e. $N = 1$), systems of type (1.1.1) reduce to

$$u_t = a(u)u_x, \tag{1.5.1}$$

for some smooth function $a = a(u)$. Any equation (1.5.1) is actually bi-Hamiltonian with respect to the pencil of local PBs, generated by

$$\{u(x), u(y)\}_1 = \delta_x(x - y) \tag{1.5.2}$$

and

$$\{u(x), u(y)\}_2 = q(u)\delta_x(x - y) + \frac{1}{2}q_x(u)u_x\delta(x - y), \tag{1.5.3}$$

for any function $q = q(u)$. In this case, the D-operator (with respect to PB (1.5.2)) is defined up to the choice of two arbitrary functions of one variables $c = c(u)$ and

$p = p(u)$:

$$D\,f = f(u) + \epsilon^2 c\, f'''(u) + \epsilon^4 \left[\left(p\, f'''(u) + \frac{6}{5} c^2 f^{(IV)} \right) u_{xx}^2 - \frac{c}{2} \left(c'' f^{(IV)}(u) + c' h^{(V)}(u) + \right.\right.$$
$$\left.\left. + \frac{c}{3} f^{(VI)} - \frac{1}{6} \left(p' f^{(IV)} + p\, f^{(V)} \right) \right) u_x^4 \right] + \mathcal{O}(\epsilon^6) dx$$

See **[15]**, **[16]** for the details.
Therefore, we obtain by definition that the class of Hamiltonian integrable perturbations of equation (1.5.1) with respect to PB (1.5.2), is parametrized by such arbitrary functions, appearing also in the theory of central invariants (see **[17]**).

In addition, if (and only if) the function $c = c(u)$ is non-vanishing, there exists a choice of the function $q(u)$, such that the dispersionless PB (1.5.3) admits a M-order deformation and the Hamiltonians

$$H_f^{[M]}[u] = \int D^M f(\mathbf{u}) dx, \qquad H_g^{[M]}[u] = \int D^M g(\mathbf{u}) dx,$$

commute with respect to such deformation

$$\left\{ H_f^{[M]}, H_g^{[M]} \right\}_2 \equiv \mathcal{O}(\epsilon^{M+1}).$$

We have therefore constructed a bi-Hamiltonian family of commuting, perturbed first integrals.

The same procedure can be in principle applied to the two-components systems. Indeed, combining Theorem 1.2.2 and the above results about integrable two-components perturbations, we aim to study the bi-Hamiltonian integrable perturbations of systems (1.4.17) at least within the ϵ^2 approximations.

CHAPTER 2

Discrete scalar-valued PBs

2.1. Classification of (α, ξ)-brackets

This Chapter is devoted to the classification of the scalar-valued PBs given by the formulæ

$$\{u_n, u_{n+k}\}_M = g_k(u_n, \ldots, u_{n+k}), \qquad 1 \le k \le M. \tag{2.1.1}$$

The coefficients $g_k(u_n, \ldots, u_{n+k})$ are *locally analytic functions* (see R. Yamilov [**54**] for the detailed definition), satisfying for all values of the independent variables u_n, $n \in \mathbb{Z}$, the following bi-linear PDEs given by Jacobi identity

$$\{\{u_n, u_{n+p}\}, u_{n+p+q}\} = \{u_n, \{u_{n+p}, u_{n+p+q}\}\} + \{\{u_n, u_{n+p+q}\}, u_{n+p}\}, \tag{p, q}$$

explicitly,

$$\begin{array}{c}
\sum_{i=0}^{p} g_p(u_n, \ldots, u_{n+p})_{,u_{n+p-i}} g_{q+i}(u_{n+p-i}, \ldots, u_{n+p+q}) + \\
-\sum_{i=0}^{q} g_q(u_{n+p}, \ldots, u_{n+p+q})_{,u_{n+p+i}} g_{p+i}(u_n, \ldots, u_{n+p+i}) \\
\shortparallel \\
\sum_{i=0}^{p} g_{p+q}(u_n, \ldots, u_{n+p+q})_{,u_{n+i}} g_{p-i}(u_{n+i}, \ldots, u_{n+p}) + \\
-\sum_{i=0}^{q} g_{p+q}(u_n, \ldots, u_{n+p+q})_{,u_{n+p+i}} g_i(u_{n+p}, \ldots, u_{n+p+i})
\end{array}$$

where $p, q = 1, \ldots, M$ and $g_0(\cdot) \equiv 0$, $g_k(\cdot) \equiv 0$, $k > M$.

Before going into the details of the classification results, we introduce a couple of well-known integrable lattice systems which will provide our foundamental examples.

Example 2.1.1. *[Bogoyavlensky lattices]*
The Bogoyavlensky lattice (BL) on an infinite lattice has been introduced in [**3**] *(see also* [**4**], [**41**]*) by the following equations*

$$\dot{u}_n = u_n \left(\sum_{k=1}^{M} u_{n+k} - \sum_{k=1}^{M} u_{n-k} \right), \tag{2.1.2}$$

where M is a fixed positive integer. Performing the local change of variable

$$u_n \longmapsto \log u_n,$$

we obtain equations

$$\dot{u}_n = \sum_{k=1}^{M} \exp(u_{n+k}) - \sum_{k=1}^{M} \exp(u_{n-k}) \tag{2.1.3}$$

which admit a Hamiltonian representation with respect to a constant PB of type (2.1.1)

$$\{u_n, u_{n+k}\}_M = 1, \quad k = 1, \ldots, M, \tag{2.1.4}$$

and Hamiltonian functional $H[\mathbf{u}] = \sum_k \exp(u_k)$*. We refer the reader to the Suris' book* **[48]** *(see Chapter 17) for a complete study of the Bogoyavlensky lattices, including the following related systems*

$$\dot{z}_n = z_n \left(\prod_{k=1}^{M} z_{n+k} - \prod_{k=1}^{M} z_{n-k} \right) \tag{2.1.5}$$

and

$$\dot{z}_n = z_n^2 \left(\prod_{k=1}^{M-1} z_{n+k} - \prod_{j=1}^{M-1} z_{n-k} \right). \tag{2.1.6}$$

Here we notice that for any fixed M*, equations* (2.1.5) *and* (2.1.6) *transform into* (2.1.2) *under the non-invertible changes of variable* $u_n = \prod_{k=0}^{M} z_{n+k}$ *and* $u_n = \prod_{k=0}^{M-1} z_{n+k}$ *respectively. Analogous transformations will be considered in Section 2.2 (see formula* (2.2.1) *below), where we will prove a Darboux-type Theorem for PB* (2.1.1)*.*
A bi-Hamiltonian formulations for the Bogoyavlensky lattices (2.1.2)*,* (2.1.5)*,* (2.1.6) *has been described by many authors (see* **[57]**, **[26]**, **[52]***). However, it turns out that the second compatible PB depends on an infinite number of independent variables* u_n*, therefore it is not local and does not belong to our class* (2.1.1)*.*

Example 2.1.2. *[Volterra lattice or discrete KdV equation]*
Our classification procedure will be illustrated by the particular case of equations (2.1.3)*, corresponding to* $M = 1$ *and known as the Volterra lattice (VL),*

$$\dot{u}_n = \exp(u_{n+1}) - \exp(u_{n-1}). \tag{2.1.7}$$

Equations (2.1.7) *can be presented into a bi-Hamiltonian form (see* **[21]***),*

$$\dot{u}_n = \{u_n, H_1[\mathbf{u}]\}_1 = \{u_n, H_2[\mathbf{u}]\}_2\,,$$

according to the following local PBs and local Hamiltonian functionals

$$\begin{array}{ll} H_1[\mathbf{u}] = \sum_k \exp(u_k) & \{u_n, u_{n+1}\}_1 \;=\; 1, \\ H_2[\mathbf{u}] = \dfrac{1}{2} \sum_k u_k & \begin{array}{l}\{u_n, u_{n+2}\}_2 \;=\; \exp(u_{n+1}) \\ \{u_n, u_{n+1}\}_2 \;=\; \exp(u_n) + \exp(u_{n+1}).\end{array} \end{array} \tag{2.1.8}$$

These PBs belong to the class (2.1.1)*. Moreover, the coefficients of the cubic PB* $\{\cdot, \cdot\}_2$ *are characterized by suitable linear combination of the leading order term, given by the exponential functions* $f_n^0 \doteq f^0(u_n) = \exp(u_n)$*. It will be clear later on that the formulæ* (2.1.8) *reproduce the typical behavior of PBs* (2.1.1)*.*

2.1.1. The leading-order coefficient. By the definition of locality radius, the leading-order term g_M might depend on variables $u_n, \dots, u_{n+M}$, i.e.

$$g_M = g_M(u_n, \dots, u_{n+M})$$

and

$$\frac{\partial g_M}{\partial u_n} \cdot \frac{\partial g_M}{\partial u_{n+M}} \neq 0.$$

Lemma 2.1.3. *There exist* canonical coordinates *and an integer* $\alpha > 0$ *such that the leading term* g_M *reduces to the form*

$$g_M(u_n, \dots, u_{n+M}) = f(u_{n+\alpha}, \dots, u_{n+M-\alpha}),$$

where

$$f_{,u_{n+\alpha}} \cdot f_{,u_{n+M-\alpha}} \neq 0 \tag{2.1.9}$$

if f *is a non-constant function.*

Let us denote

$$\xi \doteq M - 2\alpha,$$

the notation f_n^ξ indicates an arbitrary function f depending on variables from u_n up to $u_{n+\xi}$, that is

$$f_n^\xi \doteq f^\xi(u_n, \dots, u_{n+\xi}).$$

Proof. Let us consider the bi-linear PDEs $[p, q]$, with $p, q = 1, \dots, M$. We first focus our attention on equation

$$[M, M] \qquad \{\{u_n, u_{n+M}\}, u_{n+2M}\} = \{u_n, \{u_{n+M}, u_{n+2M}\}\}$$

which provides us

$$\log g_M(u_n, \dots, u_{n+M})_{,u_{n+M}} = \log g_M(u_{n+M}, \dots, u_{n+2M})_{,u_{n+M}} = \hat{a}(u_{n+M})$$

for some arbitrary function $\hat{a}(u_{n+M})$. Solving this logarithmic equation, we obtain the following factorization

$$g_M(u_n, \dots, u_{n+M}) = a(u_n) f_{n+\alpha}^{M-\alpha-\beta}(u_{n+\alpha}, \dots, u_{n+M-\beta}) a(u_{n+M})$$

where $\log a(u_n)_{,u_n} = \hat{a}(u_n)$ and $f_{n+\alpha}^{M-\alpha-\beta} \doteq f(u_{n+\alpha}, \dots, u_{n+M-\beta})$ is a constant function or such that

$$f_{,u_{n+\alpha}} \cdot f_{,u_{n+M-\beta}} \neq 0,$$

for some non-negative integers $\alpha, \beta \geq 1$.
Performing a local change of the variables $u_n \longmapsto \tilde{u}_n = \varphi(u_n)$, we reduce to

$$g_M(\tilde{u}_n, \dots, \tilde{u}_{n+M}) = f_{n+\alpha}^{M-\alpha-\beta}(\tilde{u}_{n+\alpha}, \dots, \tilde{u}_{n+M-\beta}).$$

Finally, we prove that $\alpha = \beta$. Indeed, from equation

$$[M, M-\beta] \qquad \{\{u_n, u_{n+M}\}, u_{n+2M-\beta}\} = \{u_n, \{u_{n+M}, u_{n+2M-\beta}\}\}$$

after some elementary computations, we obtain

$$\begin{gathered} \log g_M(u_{n+\alpha}, \dots, u_{n+M-\beta})_{,u_{n+M-\beta}} \\ \shortparallel \\ g_{M-\beta}(u_{n+M}, \dots, u_{n+2M-\beta})_{,u_{n+M}} g_M(u_{n+M-\beta+\alpha}, \dots, u_{n+2(M-\beta)})^{-1} \end{gathered}$$

where the variables appearing on the left-hand side do not intersect with those appearing on the right-hand side. Therefore there exists a non-zero constant k, such that

$$g_{M-\beta}(u_{n+M}, \dots, u_{n+2M-\beta})_{,u_{n+M}} = k\, g_M(u_{n+M-\beta+\alpha}, \dots, u_{n+2(M-\beta)})$$

and this PDE makes sense only if $\alpha \geq \beta$.
Applying the same procedure to equation

$$[M-\alpha, M] \qquad \{\{u_n, u_{n+M-\alpha}\}, u_{n+2M-\alpha}\} = \{u_n, \{u_{n+M-\alpha}, u_{n+2M-\alpha}\}\}$$

that explicitly is

$$\begin{gathered} g_{M-\alpha}(u_n, \dots, u_{n+M-\alpha})_{,u_{n+M-\alpha}} g_M(u_{n+\alpha}, \dots, u_{n+M-\beta})^{-1} \\ \shortparallel \\ \log g_M(u_{n+M}, \dots, u_{n+2M-\beta-\alpha})_{,u_{n+M}} \end{gathered}$$

we obtain $\alpha \leq \beta$. Therefore it must be $\alpha = \beta$. □

For any fixed order M, the leading order functions, given by Lemma 2.1.3, define essentially different classes of PBs (2.1.1). This suggests us the following

Definition 2.1.4. *Let (α, ξ) be a pair of non-negative integers with $\alpha \geq 1$, we call (α, ξ)-brackets the class of non-constant PBs (2.1.1) of order $M = 2\alpha + \xi$ expressed in the canonical coordinates, i.e. $g_M(u_n, \dots, u_{n+M}) = f_{n+\alpha}^{\xi}(u_{n+\alpha}, \dots, u_{n+\alpha+\xi})$.*

In the constant case (i.e. $g_M \equiv \text{const}$) one can immediately prove the following

Proposition 2.1.5. *If there exists a set of coordinates reducing the leading order term to the constant form $g_M(u_n, \dots, u_{n+M}) = \sigma_M$, for some non-zero constant σ_M, then all the coefficients g_k, $k = 1, \dots, M$ are constant in such coordinates, i.e.*

$$\{u_n, u_{n+k}\}_M = g_k(u_n, \dots, u_{n+k}) = \sigma_k, \qquad k = 1, \dots, M, \tag{2.1.10}$$

where σ_k are complex constants.

When the constants σ_k are normalized (i.e. $\sigma_k \equiv 1,\ k = 1, \ldots, M$), the PB (2.1.10) becomes the constant bracket (2.1.4) of the Bogoyavlensky lattice.

2.1.2. Classification theorem.

Theorem 2.1.6. *For any PB (2.1.1) of order M, there exist a set of coordinates and an integer $\alpha \geq 1$, such that the coefficients $g_k(u_n, \ldots, u_{n+k})$, $k = 1, \ldots, M$ are given by linear combination of the suitably shifted function f_n^ξ (see Lemma 2.1.3)*

$$f_n^\xi(u_n, \ldots, u_{n+\xi}) = \exp\left(\sum_{i=0}^{\xi} \tau_i \, u_{n+i}\right), \tag{2.1.11}$$

where $\tau_i|_{i=0,\ldots,\xi}$ are complex parameters. Explicitly, we have

$$g_{\alpha+\xi+p}(u_n, \ldots, u_{n+\alpha+\xi+p}) = \left(\sum_{s=\max(0,p)}^{\min(\alpha+p,\alpha)} \lambda_p^s \, T^s\right) f_n^\xi(u_n, \ldots, u_{n+\xi}), \tag{2.1.12}$$

where $\xi \doteq M - 2\alpha$, $p = -\alpha, \ldots, \alpha$, T is the shift operator (1.2.4) and λ_{s-r}^s are scalars such that

(i) they satisfy the multiplication rule

$$\lambda_{s-r}^s = \lambda_{\alpha-r}^\alpha \, (\lambda_0^\alpha)^{-1} \, \lambda_{\alpha-s}^\alpha \qquad s, r = 0, \ldots, \alpha, \tag{2.1.13}$$

(ii) denoting $\theta \doteq \min(\alpha, \xi)$, they can be expressed by explicit formulæ in terms of the $\theta + 1$ parameters $\tau_0, \tau_1, \ldots, \tau_{\theta-1}; \tau_\xi$

$$\begin{array}{lcll} \lambda_{\alpha-r}^\alpha & = & \det \mathcal{A}_r \left(\{(-)^s \tau_0^{-1} \, \tau_s\}_{s\geq 0}\right) & r = 0, \ldots, \alpha - 1 \\ \lambda_0^\alpha & = & \tau_0 \, (\tau_\xi)^{-1} & \end{array} \tag{2.1.14}$$

where $\mathcal{A}_r$ is the band-Toeplitz matrix

$$\mathcal{A}_r(\mathbf{a}) = \begin{pmatrix} a_1 & a_0 & 0 & \ldots & 0 \\ a_2 & a_1 & a_0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{r-1} & \ldots & \ldots & a_1 & a_0 \\ a_r & \ldots & \ldots & a_2 & a_1 \end{pmatrix} \tag{2.1.15}$$

associated to the sequence $\mathbf{a} = (a_0, a_1, \ldots)$.

Here and below we adopt the notation $\tau_p \equiv 0$ if $p < 0$ or $p > \xi$.

Remark 2.1.7. *The scalars appearing in formula (2.1.12) can be easily visualized looking at the rows of the following rhombus*

$$
\begin{array}{lccccccc}
g_{(\alpha+\xi)+\alpha} & \longrightarrow & & & & & \lambda^{\alpha}_{\alpha} \\
g_{(\alpha+\xi)+\alpha-1} & \longrightarrow & & & & \lambda^{\alpha-1}_{\alpha-1} & \lambda^{\alpha}_{\alpha-1} \\
\vdots & & & & \iddots & \vdots & \vdots \\
g_{(\alpha+\xi)+1} & \longrightarrow & & \lambda^{1}_{1} & \dots & \lambda^{\alpha-1}_{1} & \lambda^{\alpha}_{1} \\
g_{(\alpha+\xi)} & \longrightarrow & \lambda^{0}_{0} & \lambda^{1}_{0} & \dots & \lambda^{\alpha-1}_{0} & \lambda^{\alpha}_{0} \\
g_{(\alpha+\xi)-1} & \longrightarrow & \lambda^{0}_{-1} & \lambda^{1}_{-1} & \dots & \lambda^{\alpha-2}_{-1} & \lambda^{\alpha-1}_{-1} \\
\vdots & & \vdots & \vdots & \iddots & & \\
g_{(\alpha+\xi)-\alpha+1} & \longrightarrow & \lambda^{0}_{-\alpha+1} & \lambda^{1}_{-\alpha+1} & & & \\
g_{(\alpha+\xi)-\alpha} & \longrightarrow & \lambda^{0}_{-\alpha} & & & &
\end{array}
\tag{2.1.16}
$$

We complete the description above, specifying the non-trivial constraints that the Jacobi identity imposes on parameters τ's in the following

Theorem 2.1.8. *Let (α, ξ) be a pair of non-negative integers and define the sequence $\sigma_q|_{q\geq 0} = (-)^q \lambda^{\alpha}_{\alpha-q}|_{q\geq 0}$. Then, the (α,ξ)-brackets are in one-to-one correspondence with the intersection points of the θ projective hypersurfaces in $\mathbb{CP}^{\theta}$ defined by the homogeneous polynomial equations*

(i) if $\xi \geq 2\alpha - 1$, for any $p = 0, \dots, \alpha - 1$,

$$(\tau_0)^{p+1} \det \mathcal{A}_{\xi-p}\left(\{\sigma_q\}_{q\geq 0}\right) = (\tau_\xi)^{p+1} \det \mathcal{A}_p\left(\{\sigma_{\alpha-q}\}_{q\geq 0}\right) \tag{2.1.17}$$

(ii) if $\xi < \alpha$, for any $p = 0, \dots, \xi - 1$,

$$(\tau_0)^{\xi-p} \tau_p = (\tau_\xi)^{\xi-p+1} \det \mathcal{A}_{\xi-p}\left(\{\sigma_{\alpha-q}\}_{q\geq 0}\right) \tag{2.1.18}$$

(iii) if $\alpha \leq \xi < 2\alpha - 1$, for any $p = 0, \dots, \xi - \alpha$

$$(\tau_0)^{p+1} \det \mathcal{A}_{\xi-p}\left(\{\sigma_q\}_{q\geq 0}\right) = (\tau_\xi)^{p+1} \det \mathcal{A}_p\left(\{\sigma_{\alpha-q}\}_{q\geq 0}\right) \tag{2.1.19}$$

and, for any $p = \xi - \alpha + 1, \dots, \alpha - 1$

$$(\tau_0)^{\xi-p} \tau_p = (\tau_\xi)^{\xi-p+1} \det \mathcal{A}_{\xi-p}\left(\{\sigma_{\alpha-q}\}_{q\geq 0}\right). \tag{2.1.20}$$

The case $\theta = 0$ is described below in the Example 2.1.4.1.

2.1.3. Proof. We follow a direct approach to the proof of Theorem 2.1.6, analyzing the bi-linear equations [p, q] from Jacobi identity. In our computations the generalized Fibonacci sequences will play an important role. Let us collect some elementary properties about them.

Remark 2.1.9. *For any positive integer $k \geq 2$, the k-generalized Fibonacci sequence $F_n|_{n\geq 0}$ is defined by the order k linear homogeneous recurrence relations*

$$
\begin{array}{lcll}
F_n & = & \sum_{i=1}^{k} (-)^{i+1} a_i \, F_{n-i}, & \text{for } n \geq 1, \\
F_0 & \equiv & 1 &
\end{array}
\tag{2.1.21}
$$

for arbitrary coefficients a_i, $i = 0, \ldots, k$.
The generating function for $F_n|_{n\geq 0}$ is given by

$$\varphi(t) = \sum_{n\geq 0} F_n t^n = \frac{1}{1 - a_1 t + a_2 t^2 - \ldots + (-)^k a_k t^k}$$

and using the theory of lower triangular Toeplitz matrices (see for example **[49]**, **[55]***), one can express the k-generalized Fibonacci numbers in terms of determinants of matrices with entries given by the coefficients a_i of the recurrence equation,*

$$F_n = \det(a_{1-i+j})_{1\leq i,j\leq n} = \det \mathcal{A}_n\left(\mathbf{a}\right), \tag{2.1.22}$$

where $\mathbf{a} = (a_0 \equiv 1, a_1, \ldots, a_k, a_{k+1} \equiv 0, \ldots)$ and $\mathcal{A}$ the band Toeplitz matrix defined by (2.1.15).

Let us consider a PB of type (2.1.1). According to the Lemma 2.1.3 we can choose certain coordinates on the target manifold $\mathcal{M}$ such that the leading term g_M reduces to the following form

$$f^{\xi}_{n+\alpha} = f^{\xi}_{n+\alpha}(u_{n+\alpha}, \ldots, u_{n+\alpha+\xi}).$$

In our opinion, the simplest way to obtain the functional form of the coefficients $g_k\,|_{k=1,\ldots,M}$ comes from considering equations

$$[M-\alpha-p, M] \quad \left\{\left\{u_n, u_{n+\alpha+\xi-p}\right\}, u_{n+2M-\alpha-p}\right\} = \left\{u_n, \left\{u_{n+M-\alpha-p}, u_{n+2M-\alpha-p}\right\}\right\}$$

$$[M, M-\alpha-p] \qquad \left\{\left\{u_n, u_{n+M}\right\}, u_{n+2M-\alpha-p}\right\} = \left\{u_n, \left\{u_{n+M}, u_{n+2M-\alpha-p}\right\}\right\}$$

for any $p = 0, \ldots, \alpha+\xi-1$.

2.1.3.1. *Formula* (2.1.12). Let us describe a detailed analysis for a couple of simpler cases (i.e. $p = 0, 1$), which will suggest us how to address the general computations. For the sake of simplicity, we suppose $\xi > \alpha$.

$p = 0$. Equations $[M-\alpha, M]$ and $[M, M-\alpha]$ yield

$$\left(\log f^{\xi}_n\right)_{,u_n} = \tau_0 \qquad , \left(\log f^{\xi}_n\right)_{,u_{n+\xi}} = \tau_\xi$$

for some non-zero constants τ_0 and τ_ξ (see condition (2.1.9)). Therefore the function $g_{\alpha+\xi}$ has to be of the form

$$g_{\alpha+\xi}(u_n, \ldots, u_{n+\alpha+\xi}) = \lambda^0_0\, f^{\xi}_n + \ldots + \lambda^{\alpha}_0\, f^{\xi}_{n+\alpha},$$

where dots stay for any arbitrary function depending at the most on the variables $u_{n+1}, \ldots, u_{n+\alpha+\xi-1}$ and $\lambda^0_0 \doteq (\tau_0)^{-1}\,\tau_\xi$, $\lambda^{\alpha}_0 \doteq \tau_0\,(\tau_\xi)^{-1}$.

$p = 1$. Analogously, equations $[M-\alpha-1, M]$ and $[M, M-\alpha-1]$ provide us

$$\left(\log f_n^\xi\right)_{,u_{n+1}} = \tau_1, \qquad \left(\log f_n^\xi\right)_{,u_{n+\xi-1}} = \tau_{\xi-1},$$

for some constants τ_1, $\tau_{\xi-1}$ and

$$\begin{gathered} g_{\alpha+\xi-1}(u_n, \dots, u_{n+\alpha+\xi-1})_{,u_{n+\alpha+\xi-1}} \\ \shortparallel \\ \tau_0\, g_{M-1}(u_n, \dots, u_{n+2\alpha+\xi-1}) + \tau_1\, f^\xi(u_{n+\alpha}, \dots, u_{n+\alpha+\xi}) \end{gathered}$$

$$\begin{gathered} g_{\alpha+\xi-1}(u_n, \dots, u_{n+\alpha+\xi-1})_{,u_n} \\ \shortparallel \\ \tau_\xi\, g_{M-1}(u_{n-\alpha}, \dots, u_{n+\alpha+\xi-1}) + \tau_{\xi-1}\, f^\xi(u_{n-1}, \dots, u_{n+\xi-1}). \end{gathered}$$

It necessarily follows that the constants τ_1, $\tau_{\xi-1}$ are non-zero and the coefficient g_{M-1} has to be of the form

$$g_{M-1}(u_{n+\alpha-1}, \dots, u_{n+\alpha+\xi}) = \lambda_{\alpha-1}^{\alpha-1} f_{n+\alpha-1}^\xi + \dots + \lambda_{\alpha-1}^{\alpha} f_{n+\alpha}^\xi$$

where $\lambda_{\alpha-1}^{\alpha-1} \doteq (\tau_\xi)^{-1}\, \tau_{\xi-1}$ and $\lambda_{\alpha-1}^{\alpha} \doteq -(\tau_0)^{-1}\, \tau_1$. In addition the dots stay for any arbitrary function $h = h_{n+\alpha}^{\xi-1}$ depending at the most on the independent variables $u_{n+\alpha}, \dots, u_{n+\alpha+\xi-1}$.

Remark 2.1.10. *Notice that the function $h_{n+\alpha}^{\xi-1}$ can be understood as the leading order function of a $(\alpha, \xi-1)$-bracket:*

$$h_{n+\alpha}^{\xi-1} \doteq f_{n+\alpha}^{\xi-1}(u_{n+\alpha}, \dots, u_{n+M-\alpha-1}).$$

We decide to forget about the contributions provided by the leading order functions $f^{\xi'}$ of lower order PBs (i.e. $M' < M$), postponing to Section 2.3 the problem of classifying the compatible pairs of PBs (2.1.1).

According to Remark 2.1.10, we solve equations $[M-\alpha-1, M]$ and $[M, M-\alpha-1]$, finding

$$g_{\alpha+\xi-1}(u_n, \dots, u_{n+\alpha+\xi-1}) = \lambda_{-1}^0\, f_n^\xi + \dots + \lambda_{-1}^{\alpha-1}\, f_{n+\alpha-1}^\xi,$$

where $\lambda_{-1}^0 \doteq (\lambda_0^\alpha)^{-1}\, \lambda_{\alpha-1}^\alpha$, $\lambda_{-1}^{\alpha-1} \doteq \lambda_0^\alpha\, \lambda_{\alpha-1}^{\alpha-1}$ and dots stay for any arbitrary function depending at the most on variables $u_{n+1}, \dots, u_{n+\alpha+\xi-2}$.

Iterating this procedure, one can show that the coefficients $g_k\,|_{k=1,\dots,M}$ depend on lattice variables according to formulæ

$$\begin{aligned} g_{\alpha+\xi+k} &= g_{\alpha+\xi+k}(u_{n+k}, \dots, u_{n+\alpha+\xi}), \quad k = 0, \dots, \alpha \\ g_{\alpha+\xi-k} &= g_{\alpha+\xi-k}(u_n, \dots, u_{n+\alpha+\xi-k}), \quad k = 1, \dots, \alpha+\xi-1 \end{aligned}$$

and

$$f^\xi(u_n, \dots, u_{n+\xi}) = \lambda_\alpha^\alpha \exp\left(\sum_{i=0}^{\xi} \tau_i\, u_{n+i}\right),$$

where

(i) λ_α^α is a free multiplicative constant, that can be normalized (i.e. $\lambda_\alpha^\alpha \equiv 1$) choosing a suitable rescaling of the coordinates: $u_n \longmapsto k\, u_n$, for some constant k,

(ii) $\tau_i|_{i=0,\dots,\xi}$ are non-zero complex parameters, according to condition (2.1.9). We denote $\tau_i \equiv 0$, if $i<0$ or $i>\xi$.

Moreover, solving the equations $[M-\alpha-p, M]$ and the symmetric ones $[M, M-\alpha-p]$, we obtain the formulæ (2.1.12) for the coefficients $g_k|_{k=1,\dots,M}$.

2.1.3.2. *The multiplication rule for constants λ's.* Let us focus our attention on the set of equations $[M-\alpha-p, M]_{p=0,\dots,\alpha+\xi-1}$ (analogous results come from the symmetric equations $[M, M-\alpha-p]_{p=0,\dots,\alpha+\xi-1}$).
The constraints on constants λ's can be encoded into the following linear system

$$\begin{pmatrix} \tau_\xi\, \lambda_{-p}^{\alpha-p} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_{\alpha-p}^{\alpha-p} & 0 & \dots & \dots & 0 \\ \lambda_{\alpha-p}^{\alpha-p+1} & \lambda_{\alpha-p+1}^{\alpha-p+1} & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ \lambda_{\alpha-p}^{\alpha-1} & \lambda_{\alpha-p+1}^{\alpha-1} & \dots & \lambda_{\alpha-1}^{\alpha-1} & 0 \\ \lambda_{\alpha-p}^{\alpha} & \lambda_{\alpha-p+1}^{\alpha} & \dots & \lambda_{\alpha-1}^{\alpha} & \lambda_{\alpha}^{\alpha} \end{pmatrix} \begin{pmatrix} \tau_0 \\ \tau_1 \\ \vdots \\ \tau_{p-1} \\ \tau_p \end{pmatrix} \tag{2.1.23}$$

where λ_{s-r}^s is non-zero if and only if $s,r=0,\dots,\alpha$ and, recursively on p, we have defined

(i) $\lambda_{-p}^{\alpha-p} \doteq \lambda_0^\alpha\, \lambda_{\alpha-p}^{\alpha-p}$,

(ii) $\lambda_{\alpha-p}^{\alpha-p+s} \doteq -\tau_0^{-1} \left[\sum_{i=1}^s \tau_i\, \lambda_{\alpha-p+i}^{\alpha-p+s}\right]$, $s=1,\dots,p$.

By induction on p, combining (i) and (ii) one can prove the following multiplication rule

$$\lambda_{s-r}^s = \lambda_s^s\, \lambda_{\alpha-r}^\alpha, \qquad s,r=0,\dots,\alpha, \tag{2.1.24}$$

that allows us to describe all constants λ's in terms of the constants $\{\lambda_{\alpha-s}^\alpha, \lambda_{\alpha-s}^{\alpha-s}\}_{s=0,\dots,\alpha}$, appearing on the edges of rhombus (2.1.16).

2.1.3.3. *Constants λ's as function of parameters τ's.* We start looking at the last row of the matrix appearing in formula (2.1.23). If $0\le p<\alpha$, we have the following order p recursive relations

$$\lambda_{\alpha-p}^\alpha = -(\tau_0)^{-1} \sum_{i=1}^p \tau_i\, \lambda_{\alpha-p+i}^\alpha. \tag{2.1.25}$$

Denoting $F_p \doteq \lambda_{\alpha-p}^\alpha$, $a_p \doteq (-)^p \tau_0^{-1}\, \tau_p$, we recognize the p-generalized Fibonacci sequence (see equation (2.1.21) above). Then, according to Remark 2.1.9, we can express

$$\lambda^{\alpha}_{\alpha-p}\,(\tau_0,\dots,\tau_p) = \det \mathcal{A}_p\left(\{(-)^s \tau_0^{-1}\,\tau_s\}_{s\geq 0}\right) \qquad p=0,\dots,\alpha-1.$$

Moreover, in order to express the constants $\lambda^{\alpha-s}_{\alpha-s}\,|_{s=0,\dots,\alpha}$ as functions of the constants $\lambda^{\alpha}_{\alpha-s}\,|_{s=0,\dots,\alpha}$, we look at equations

$$[M-q, M-\alpha+q] \qquad \{\{u_n, u_{n+M-q}\}, u_{n+2M-\alpha}\} = \{u_n, \{u_{n+M-q}, u_{n+2M-\alpha}\}\},$$

for any $q=0,\dots,\alpha$. They split into

$$\begin{array}{ll} (a) & g_{(\alpha+\xi)+\alpha-q}(u_{n+\alpha-q},\dots,u_{n+\alpha+\xi})_{,u_{n+\alpha+\xi}} = \lambda^{\alpha}_{\alpha-q}\, f^{\xi}(u_{n+\alpha},\dots,u_{n+\alpha+\xi})_{,u_{n+\alpha+\xi}} \\ (b) & g_{(\alpha+\xi)+q} \quad (u_{n+q},\dots,u_{n+\alpha+\xi}))_{,u_{n+q}} = \lambda^{q}_{q}\, f^{\xi}(u_{n+q},\dots,u_{n+q+\xi}))_{,u_{n+q}} \end{array}$$

that are compatible only if

$$\lambda^q_q = (\lambda^{\alpha}_0)^{-1}\,\lambda^{\alpha}_{\alpha-q}. \tag{2.1.26}$$

Substituting (2.1.26) into the formula (2.1.24), we achive the multiplication rule (2.1.13).

2.1.3.4. *Constraints for parameters τ's.* Supposing $\xi > \alpha$, we analyze the equations where $\alpha \leq p \leq \xi$. Analogously to (2.1.25), we have the following linear homogeneous recurrence relation of order α

$$\tau_p = -\sum_{q=1}^{\alpha} \lambda^{\alpha}_{\alpha-q}\tau_{p-q}, \qquad \text{for any } p\geq\alpha. \tag{2.1.27}$$

For any $q=1,\dots,\alpha$, denoting $a_q \doteq (-)^q\lambda^{\alpha}_{\alpha-q}$, we arrive at $\tau_p = \sum_{q=1}^{\alpha}(-)^{q+1}a_q\,\tau_{p-q}$, that is

$$\tau_p\,(\tau_0,\dots,\tau_{\alpha-1};\tau_\xi) = \tau_0\,\det\mathcal{A}_p\left(\{(-)^s\lambda^{\alpha}_{\alpha-s}\}_{s\geq 0}\right), \qquad p=\alpha,\dots,\xi. \tag{2.1.28}$$

Finally, when $\xi+1\leq p<\alpha+\xi$, we obtain analogue relations with respect to the case $1\leq p<\alpha$: the equations $[M-\alpha-p, M]$ give us

$$\sum_{i=0}^{q}\lambda^{\alpha}_{q-i}\tau_{\xi-i} = 0, \qquad q=1,\dots,\alpha-1.$$

Following again Remark 2.1.9, for any $q=1,\dots,\alpha-1$, we obtain

$$\tau_{\xi-q}\,(\tau_0,\dots,\tau_{\alpha-1};\tau_\xi) = \tau_\xi\,\det\mathcal{A}_q\left(\{(-)^s(\lambda^{\alpha}_0)^{-1}\,\lambda^{\alpha}_s\}_{s\geq 0}\right) \tag{2.1.29}$$

Observe that formulæ (2.1.28) and (2.1.29) have to be consistent: this provides us the constraints on parameters τ's. According to the values of α and ξ, we immediately find out the homogeneous polynomial equations given by Theorem 2.1.8.

2.1.3.5. *Remaining equations.* Due to the relation (2.1.26), the symmetric set of equations $[M, M-\alpha-p]_{p=0,\dots,\alpha+\xi-1}$ give us the same contraints.

Moreover, replacing expression (2.1.12) on equations $[p,q]_{p,q=1,\dots,M}$, with straightforward computations that generalize the previous ones, we can obtain that the relations for the parameters $\tau_0, \dots, \tau_{\theta-1}; \tau_\xi$ describe above are necessary and sufficient.

2.1.4. Examples. The previous computations appear quite cumbersome. Let us look at a couple of significative examples, in which the above description simplifies. We consider in detail the cases in which the leading order function depend on the minimal and the maximal number of independent variables u_n.

2.1.4.1. $(\alpha, 0)$-*brackets.* The leading order coefficient can be choosen in the form

$$g_{2\alpha} = f^0(u_{n+\alpha}) = \exp(\tau_0\, u_{n+\alpha})$$

and substituting into the formulæ for the constants λ's, we obtain

$$\begin{aligned}\{u_n, u_{n+2\alpha}\}_M &= \exp\left(\tau_0\, u_{n+\alpha}\right)\\ \{u_n, u_{n+\alpha}\}_M &= \exp\left(\tau_0\, u_n\right) + \exp\left(\tau_0\, u_{n+\alpha}\right)\end{aligned}$$

where τ_0 is a free complex constant. Notice that the cubic PB of Volterra lattice (2.1.8) belongs on the class of $(1,0)$-brackets.

2.1.4.2. $(1, \xi)$-*brackets.* These PBs are given in the canonical coordinates by the formulæ

$$\begin{aligned}\{u_n, u_{n+M}\} &= \exp(z_{n+1})\\ \{u_n, u_{n+M-1}\} &= \tau_0^{-1}\,\tau_\xi \exp(z_n) + \tau_0\,\tau_\xi^{-1}\exp(z_{n+1})\\ \{u_n, u_{n+M-2}\} &= \exp(z_n)\end{aligned}$$

where $z_n \doteq \sum_{i=0}^{\xi}(-\tau_0)^{i+1}(\tau_\xi^{-1})^i u_{n+i}$ and the pair $(\tau_0,\, \tau_\xi)$ belongs to the set of points of $\mathbb{CP}^1$, described by the following equation (see equation (2.1.17) above)

$$\tau_0^{1+\xi} + (-)^{1+\xi}\tau_\xi^{1+\xi} = 0 \qquad \text{where} \quad [\tau_0 : \tau_\xi] \in \mathbb{CP}^1.$$

Analogously, the $(\alpha, 1)$-brackets are parametrized by the points $[\tau_0 : \tau_1] \in \mathbb{CP}^1$, satisfying

$$\tau_0^{\alpha+1} + (-)^{\alpha+1}\tau_1^{\alpha+1} = 0.$$

The present list of particular examples can be easily extended following the recipe of Theorems 2.1.6 and 2.1.8. For the sake of brevity, in the next example we provide the details only for the first non trivial case in which the parameters τ's are described by the intersection points of certain hypersufaces.

2.1.4.3. *A multi-parameters example:* $(2,2)$*-brackets.* It turns out that the $(2,2)$-brackets are characterized by the leading function

$$g_6(u_{n+2}, u_{n+3}, u_{n+4}) = \exp(\sum_{i=0}^{2} \tau_i u_{n+2+i}),$$

where the three parameters τ_0, τ_1, τ_2 are constrained to satisfy the following system of homogeneous polynomial equations

$$\begin{array}{rcl} (\tau_0)^2 & = & (\tau_2)^2 \\ (\tau_1)^2 & = & 2\tau_0\,\tau_2. \end{array}$$

2.2. A Darboux-type theorem

Theorem 2.1.6 says that, up to local point-wise change of coordinates, for any PB (2.1.1), there exists a pair of non-negative integers (α, ξ), such that the leading order function g_M can be considered of the following form

$$g_M = f^{\xi}_{n+\alpha} = \exp\left(\sum_{i=0}^{\xi} \tau_i\, u_{n+\alpha+i}\right),$$

where $\tau_i|_{i=0,\dots,\xi}$ are some non-zero constants.

In the present Section, we deal with the problem of finding out canonical forms of PB (2.1.1) with respect to a suitable extension of the class of admissible variables transformations. Let us be more precise with the following

Definition 2.2.1. *Let (s_1, s_2) be a pair of non negative integers, we call a discrete Miura-type transformation any map of the form $z_n = \varphi(u_{n-s_1}, \dots, u_{n+s_2})$ that is a canonical transformation, i.e. a change of coordinates preserving the class of PBs.*

Notice that these discrete Miura-type transformations, as the continuous ones, are *differential substitutions* (i.e. they depend on $u_{n+1}, u_{n+2}, \dots$) and therefore only formally invertible.

Given a PB (2.1.1) in the canonical variables u_n, we define new coordinates according to the formula

$$z_n = \sum_{i=0}^{\xi} \tau_i\, u_{n+i}. \tag{2.2.1}$$

By direct computation we prove the following

Theorem 2.2.2. *Any (α, ξ)-bracket is mapped by the Miura-type transformation (2.2.1) into the following $(\alpha + \xi, 0)$-bracket,*

$$\begin{aligned} \{z_n, z_{n+2(\alpha+\xi)}\} &= \tau_0\,\tau_\xi \exp(z_{n+\alpha+\xi}) \\ \{z_n, z_{n+\alpha+\xi}\} &= \tau_0\,\tau_\xi\,[\exp(z_n) + \exp(z_{n+\alpha+\xi})]\,. \end{aligned} \tag{2.2.2}$$

Corollary 2.2.3. *Subdividing all the particles-variables into $\alpha+\xi$ families, according to the formula $v_n^{(p)} \doteq z_{(\alpha+\xi)(n-1)+p}$, $p = 1, \dots, \alpha+\xi$, the PB (2.2.2) splits into $\alpha+\xi$ copies of the cubic Volterra PB (see (2.1.8) above)*

$$\left\{v_n^{(p)}, v_{n+2}^{(p)}\right\} = \tau_0\, \tau_\xi \exp(v_{n+1}^{(p)})$$
$$\left\{v_n^{(p)}, v_{n+1}^{(p)}\right\} = \tau_0\, \tau_\xi \left[\exp(v_n^{(p)}) + \exp(v_{n+1}^{(p)})\right].$$

In addition, any constant PB of order M, $\{u_n, u_{n+k}\} = \sigma_k$, with $k = 1, \dots, M$, can be reduced, up to a normalization of the constants σ_k, to the quadratic PB of Bogoyavlensky lattice (BL) (see [**48**]*, Chapter 17), performing the lattice splitting $v_n^{(p)} \doteq u_{M(n-1)+p}$,*

$$\begin{array}{rcl} \left\{v_n^{(p)}, v_{n+1}^{(q)}\right\} & = & \sigma_{M+q-p} \\ \left\{v_n^{(p)}, v_n^{(q)}\right\} & = & \sigma_{q-p}, \end{array} \tag{2.2.3}$$

where $\sigma_0 \equiv 0$ and $p, q = 1, \dots, M$.

Proof. We are interested in the expression of (α, ξ)-brackets in z-coordinates. Let p be a positive integer, we compute the following brackets

$$\begin{array}{rl} \{z_n, z_{n+p}\} & = \{\sum_{i=0}^{\xi} \tau_i\, u_{n+i}, \sum_{j=0}^{\xi} \tau_j\, u_{n+p+j}\} = \\ & = \sum_{i=0}^{\xi} \tau_i \left[\sum_{j\in J_a^+(i,p)} \tau_j \{u_{n+i}, u_{n+p+j}\} + \sum_{j\in J_a^-(i,p)} \tau_j \{u_{n+p+j}, u_{n+i}\}\right] \end{array}$$

where the sets of admissible j's $J_a^{\pm}(i,p)$ are defined by

$$\begin{array}{l} J_a^+(i,p) \doteq \{0, \dots \xi\} \cap \{i+\xi-p, \dots, i+2\alpha+\xi-p\} \\ J_a^-(i,p) \doteq \{0, \dots \xi\} \cap \{i-2\alpha-\xi-p, \dots, i-\xi-p\}. \end{array}$$

As in proof of the previous Section, we suppose $\xi > \alpha$. We first notice that when $p > 2(\alpha+\xi)$ the PB $\{z_n, z_{n+p}\}$ vanishes.
Let us start considering the set $J_a^-(i,p)$. It is non-empty only if $i = \xi$ and $p = 0$. When $p = 0$, $J_a^+(i,p)$ is non-empty only if $i = 0$. In this case the summation vanishes, indeed

$$\begin{array}{rcl} \{z_n, z_n\} & = & \{\sum_{i=0}^{\xi} \tau_i\, u_{n+i}, \sum_{j=0}^{\xi} \tau_j\, u_{n+j}\} \\ & = & \sum_{j>i} \tau_i\, \tau_j \{u_{n+i}, u_{n+j}\} - \sum_{i>j} \tau_i\, \tau_j \{u_{n+j}, u_{n+i}\} = 0. \end{array}$$

If $p \geq 1$, we can reduce to evaluate

$$\{z_n, z_{n+p}\} = \sum_{i=0}^{\xi} \tau_i \left[\sum_{j\in J_a^+(i,p)} \tau_j \{u_{n+i}, u_{n+p+j}\}\right]. \tag{2.2.4}$$

In the following steps, we give some details about the complete computations. Enforcing a recursive procedure on p, for any i that runs from 0 to ξ, we describe the set of admissible j: $J_a^+(i,p)$.

Step 1: $p = 2(\alpha+\xi)$. Then $J_a^+(i,p) \neq \varnothing$ if and only if $i = \xi$ and

$$\left\{z_n, z_{n+2(\alpha+\xi)}\right\} = \tau_0\,\tau_\xi\,\left\{u_{n+\xi}, u_{n+2(\alpha+\xi)}\right\} = \tau_0\,\tau_\xi \exp(z_{n+\alpha+\xi}).$$

Step 2: $p = 2(\alpha+\xi) - q$, $q = 1, \ldots, \xi$. In the following table: fixed p, we describe the non-empty sets $J_a^+(i,p)$, as the index i changes.

p	$i : J_a^+(i,p) \neq \varnothing$	$J_a^+(i,p)$					
$2(\alpha+\xi)-q$	$i=\xi$	$\{q-2\alpha$	$\ldots$	$\ldots$	$\ldots$	$q-1$	$q\}$
	$\vdots$	$\{\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots\}$
	$i=\xi-q+2\alpha$	$\{\ \ 0$	1	$\ldots$	$\ldots$	$2\alpha-1$	$2\alpha\}$
	$\vdots$	$\{$	$\ddots$	$\ldots$	$\ldots$	$\ldots$	$\ldots\}$
	$i=\xi-q+\alpha$	$\{$		0	$\ldots$	$\alpha-1$	$\alpha\}$
	$\vdots$	$\{$			$\ddots$	$\ldots$	$\ldots\}$
	$i=\xi-q+1$	$\{$				0	$1\}$
	$i=\xi-q$	$\{$					$0\}$

Looking at sets $J_a^+(i,p)$, we distinguish three cases depending on the number of elements $\#J_a^+(i,p)$:

$$\begin{array}{ccccccc} (i) & & & \#J_a^+(i,p) & = & 2\alpha+1 \\ (ii) & 1 & \leq & \#J_a^+(i,p) & \leq & \alpha+1 \\ (iii) & \alpha+1 & < & \#J_a^+(i,p) & < & 2\alpha+1 \end{array}$$

(i) *When* $\#J_a^+(i,p) = 2\alpha+1$, we have

$$\sum_{i=0}^{\xi} \tau_i \sum_{j\in\{i+\xi-p,\ldots,i+2\alpha+\xi-p\}} \tau_j\,\{u_{n+i}, u_{n+p+j}\} = \sum_{i=0}^{\xi} \tau_i \sum_{t=0}^{2\alpha} \tau_{r-t}\{u_{n+i}, u_{n+i+M-t}\}$$

where $r \doteq i + M - p$. Now,

$$\begin{aligned} \textstyle\sum_{t=0}^{2\alpha} \tau_{r-t}\{u_{n+i}, u_{n+i+M-t}\} &= \textstyle\sum_{s=0}^{\alpha} \sum_{t=s}^{s+\alpha} \tau_{r-t}\lambda_{\alpha-t}^{\alpha-s} f^{\xi}(u_{n+\alpha-s}, \ldots, u_{n+\alpha+\xi-s}) \\ &= \textstyle\sum_{s=0}^{\alpha} \lambda_{\alpha-s}^{\alpha-s} f^{\xi}(u_{n+\alpha-s}, \ldots, u_{n+\alpha+\xi-s}) \sum_{t=s}^{s+\alpha} \tau_{r-t}\lambda_{\alpha-t+s}^{\alpha} \end{aligned}$$

and $\sum_{t=s}^{s+\alpha} \tau_{r-t}\lambda_{\alpha-t+s}^{\alpha} = \sum_{q=0}^{\alpha} \tau_{r-s-q}\lambda_{\alpha-q}^{\alpha} \equiv 0$, according to the recurrence relations (2.1.27) for the parameters τ's.

(ii) *When* $1 \leq \#J_a^+(i,p) \leq \alpha+1$, the set $J_a^+(i,p)$ is given by $J_a^+(i,p) = \{0, \ldots, r\}$, for some $0 \leq r \leq \alpha$. The summation (2.2.4) can be written in the following way

$$\sum_{i=\xi-q}^{\xi-q+\alpha} \tau_i \left[\sum_{j\in\{0,\ldots,r\}} \tau_j\,\{u_{n+i}, u_{n+p+j}\}\right] \qquad r \doteq i - \xi + q.$$

Noticing that $\sum_{j\in\{0,\dots,r\}} \tau_j \left\{u_{n+i}, u_{n+p+j}\right\} = \sum_{s=0}^{r} \tau_{r-s}\{u_{n+i}, u_{n+i+M-s}\} = \tau_0\, \lambda^{\alpha-r}_{\alpha-r} f^{\xi}_{n+i+\alpha-r}$ we achieve

$$\sum_{i=\xi-q}^{\xi-q+\alpha} \tau_i \left[\sum_{j\in\{0,\dots,r\}} \tau_j \left\{u_{n+i}, u_{n+p+j}\right\}\right] = \left[\sum_{i=\xi-q}^{\xi-q+\alpha} \tau_i\, \lambda^{\alpha}_{i-\xi+q}\right] f^{\xi}_{n+\alpha+\xi-q}$$

and $\sum_{i=\xi-q}^{\xi-q+\alpha} \tau_i\, \lambda^{\alpha}_{i-\xi+q} = \sum_{t=0}^{\alpha} \tau_{\xi-q+t}\, \lambda^{\alpha}_{t} \equiv 0$, see also the recurrence relation (2.1.27).
(iii) *When* $\alpha + 1 < \#J^+_a(i,p) < 2\alpha + 1$, the set $J^+_a(i,p)$ is $J^+_a(i,p) = \{0,\dots,r\}$, for some $\alpha < r \le 2\alpha$. With analogous calculations, one can directly prove that the summation $\sum_{j\in\{0,\dots,r\}} \tau_j \left\{u_{n+i}, u_{n+p+j}\right\}$ vanishes.

Step 3: $p = 2\alpha + \xi - q$, $q = 1,\dots,\alpha+1$. The sets $J^+_a(i,p)$ are given by

p	$i : J^+_a(i,p) \neq \varnothing$	$J^+_a(i,p)$					
$2\alpha+\xi-q$	$i=\xi$	$\{\xi-2\alpha+q$	$\dots$	ξ			$\}$
	$\vdots$	$\{\ \dots$	$\dots$	$\dots$	$\ddots$		$\}$
	$\vdots$	$\{\ \dots$	$\dots$	$\dots$	$\dots$	$\ddots$	$\}$
	$i=\xi-q$	$\{\ \xi-2\alpha$	$\dots$	$\dots$	$\dots$	$\xi-1$	$\xi\}$
	$\vdots$	$\{\ \dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots\}$
	$i=2\alpha-q$	$\{\ 0$	$\dots$	$\dots$	$\dots$	$\dots$	$2\alpha\}$
	$\vdots$	$\{$	$\ddots$	$\dots$	$\dots$	$\dots$	$\dots\}$
	$\vdots$	$\{$		$\ddots$	$\dots$	$\dots$	$\dots\}$
	$i=0$	$\{$			0	$\dots$	$q\}$

We find out again the two situations

$$\begin{array}{lcccl} (i) & & \#J^+_a(i,p) & = & 2\alpha+1 \quad \checkmark \\ (ii) & \alpha+1 & < \#J^+_a(i,p) & < & 2\alpha+1 \quad \checkmark \end{array}$$

which we have been already studied in the previous Step. Then the summation (2.2.4) vanishes.

Step 4: $p = \alpha + \xi$. In this case, looking at the following table

p	$J^+_{i,p}$	$i : J^+_a \neq \varnothing$	$J^+_a(i)$
$\alpha+\xi$	$\{i-\alpha,\dots,i+\alpha\}$	$i=\xi$	$\{\xi-\alpha \quad \dots \quad \dots \quad \xi \qquad\qquad \}$
		$\vdots$	$\{ \quad \dots \quad \dots \quad \dots \quad \dots \quad \ddots \qquad \}$
		$\vdots$	$\{ \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \ddots \quad \}$
		$i=\xi-\alpha$	$\{\xi-2\alpha \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \xi\}$
		$\vdots$	$\{ \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots\}$
		$i=\alpha$	$\{ \quad 0 \quad \dots \quad \dots \quad \dots \quad \dots \qquad 2\alpha\}$
		$\vdots$	$\{ \qquad \ddots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots\}$
		$\vdots$	$\{ \qquad\qquad \ddots \quad \dots \quad \dots \quad \dots \quad \dots\}$
		$i=0$	$\{ \qquad\qquad\qquad 0 \quad \dots \quad \dots \quad \alpha\}$

we have three different cases

$$\begin{array}{lcccccl}
(i) & & & \#J^+_a(i,p) & = & 2\alpha+1 & \checkmark \\
(ii) & \alpha+1 & < & \#J^+_a(i,p) & < & 2\alpha+1 & \checkmark \\
(iii) & & & \#J^+_a(i,p) & = & \alpha+1 & ?
\end{array}$$

Only the last one provides some contributions. Indeed,

$$\textstyle\sum_{j\in\{0,\dots,\alpha\}} \tau_j \{u_{n+i}, u_{n+p+j}\} \quad = \quad \tau_0\, \lambda_0^0 f^\xi(u_{n+i},\dots,u_{n+i+\xi})$$

and

$$\textstyle\sum_{j\in\{\xi-\alpha,\dots,\xi\}} \tau_j \{u_{n+i}, u_{n+p+j}\} \quad = \quad \tau_\xi\, \lambda_0^\alpha f^\xi(u_{n+\alpha+i},\dots,u_{n+i+\alpha+\xi}).$$

Our summation (2.2.4) becomes

$$\{z_n, z_{n+\alpha+\xi}\} = \tau_0\,\tau_\xi\, f^\xi(z_n) + \tau_0\,\tau_\xi\, f^\xi(z_{n+\alpha+\xi}) = \tau_0\,\tau_\xi \left[\exp(z_n) + \exp(z_{n+\alpha+\xi})\right].$$

Finally, **Step 5**: $p = 0,\dots,\xi+\alpha-1$ can be analyzed similarly to Step 2 and Step 3. $\square$

2.3. Compatible pairs

We are now in a position to complete the Remark 2.1.10 about compatible pairs (P, P') of PBs (2.1.1). We provide some necessary conditions that we also expect to be sufficient. This might be the starting point for a future classification of the still little-understood bi-Hamiltonian higher order scalar-valued difference equations.

Let P and P' be two PBs of type (2.1.1). In principle, their leading order functions are given respectively by

$$g_M(u_n,\dots,u_{n+M}) = a_M(u_n)\, f^\xi(u_{n+\alpha},\dots,u_{n+\alpha+\xi})\, a_M(u_{n+M}) \tag{2.3.1}$$

and

$$g'_{M'}(u_n,\dots,u_{n+M'}) = a'_{M'}(u_n)\, f'^{\xi'}(u_{n+\alpha'},\dots,u_{n+\alpha'+\xi'})\, a'_{M'}(u_{n+M'}) \tag{2.3.2}$$

where $f^{\xi}_{n+\alpha} = \sigma_M \exp\left(\sum_{p=0}^{\xi}\tau_p\, u_{n+\alpha+p}\right)$ and $f'^{\xi'}_{n+\alpha'} = \sigma_{M'} \exp\left(\sum_{p=0}^{\xi'}\tau'_p\, u_{n+\alpha'+p}\right)$, for some non-zero constant σ_M and $\sigma_{M'}$. It is not restrictive to suppose $M \geq M'$. Moreover the following Lemma imposes severe restriction on g_M and $g'_{M'}$

Lemma 2.3.1. *A pair of non-constant PBs (2.1.1) (P, P') forms a pencil of PBs only if there exists a local change of variables, reducing the leading coefficients (2.3.1) and (2.3.2) to the formulæ*

$$g_M(u_n,\dots,u_{n+M}) = f^{\xi}(u_{n+\alpha},\dots,u_{n+\alpha+\xi}) \tag{2.3.3}$$

and

$$g'_{M'}(u_n,\dots,u_{n+M'}) = f'^{\xi'}(u_{n+\alpha},\dots,u_{n+\alpha+\xi'}) \tag{2.3.4}$$

for some functions f^{ξ} and $f'^{\xi'}$. Notice that $\alpha = \alpha'$.

Proof. Let P and P' be PBs defined respectively by leading function (2.3.1) and (2.3.2). From equation

$$\{\{u_n, u_{n+M'}\}, u_{n+M+M'}\} = \{u_n, \{u_{n+M'}, u_{n+M+M'}\}\} \tag{$[M, M']$}$$

we find

$$(\log a_M(u_n))_{,u_n} = \left(\log a'_{M'}(u_n)\right)_{,u_n},$$

that implies $a_M(u_n) = k\, a'_{M'}(u_n)$ for some constant k. A suitable change of coordinates leads us to (2.3.3) and (2.3.4), where the constant k has been absorbed in the multiplicative constant $\sigma_{M'}$. Now, look at the equation

$$\{\{u_n, u_{n+M}\}, u_{n+M+M'-\alpha}\} = \{u_n, \{u_{n+M}, u_{n+M+M'-\alpha}\}\} \tag{$[M, M'-\alpha]$}$$

which makes sense if and only if $M' - \alpha \leq M' - \alpha'$, i.e. $\alpha \geq \alpha'$. In such case we have

$$f^{\xi}_{n+\alpha, u_{n+\alpha+\xi}}\, f^{\xi'}_{n+\alpha+\xi+\alpha'} = g'_{\alpha+\xi'}(u_{n+M},\dots,u_{n+\alpha+\xi+M'})_{,u_{n+M}}\, f^{\xi}_{n+\alpha}$$

that gives $g'_{\alpha+\xi'}(u_{n+M},\dots,u_{n+M+M'-\alpha})_{,u_{n+M}} = \tau_\xi\, f^{\xi'}(u_{n+\alpha+\xi+\alpha'},\dots,u_{n+\alpha+\alpha'+\xi+\xi'})$. The last equation makes sense if $\alpha \leq \alpha'$, therefore it follows that $\alpha = \alpha'$. Furthermore, it is not restrictive to suppose $\xi > \xi'$. □

Theorem 2.3.2. *A pair of non-constant PBs (2.1.1) (P, P'), defined by leading functions (2.3.3) and (2.3.4) for certain parameters τ and τ' satisfying algebraic constraints of Theorem 2.1.8, provides a pencil of PBs only if*

$$\tau_p = \tau'_p = \tau_{\xi-\xi'+p}, \tag{2.3.5}$$

for any $p = 0, \dots, \xi'$.

Proof. Let P and P' be a pair of PBs (2.1.1) respectively of order $M = 2\alpha + \xi$ and $M' = 2\alpha + \xi'$ (i.e. $M > M'$), leading functions (2.3.3) and (2.3.4) and suppose that any linear combination $\mu\, P + \nu\, P'$, for μ, ν arbitrary constants, is a PB of order M (i.e. it satisfies the bi-linear PDEs, coming from Jacobi identity).
Analogously to the procedure followed in the proof of Theorem 2.1.6, we find necessary conditions looking recursively at equations

$$[M', M-\alpha-p]\,, \quad [M-\alpha-p, M'] \quad p = 0, \dots, \alpha+\xi'$$

When $p = 0$, equation

$$[M', M-\alpha] \qquad \{\{u_n, u_{n+M'}\}, u_{n+M+M'-\alpha}\} = \{u_n, \{u_{n+M'}, u_{n+M+M'-\alpha}\}\}$$

gives us

$$\log f^{\xi'}(u_{n+\alpha}, \dots, u_{n+\alpha+\xi'})_{,u_{n+\alpha+\xi'}} = \lambda_0^0 \log f^{\xi}(u_{n+M'}, \dots, u_{n+M'+\xi})_{,u_{n+M'}}$$

that enables us to identify $\tau'_{\xi'} \equiv \tau_\xi$. Analogously, equation $[M-\alpha, M']$ provides $\tau'_0 \equiv \tau_0$. Iterating this procedure, when $1 \le p \le \alpha$, equations $[M-\alpha-p, M']$ restricted to the function $f^{\xi'}$ originate some constraints that can be organized in the following matrix form

$$\begin{pmatrix} \tau_\xi\, \lambda'^{\alpha-p}_{-p} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda'^{\alpha-p}_{\alpha-p} & 0 & \dots & \dots & 0 \\ \lambda'^{\alpha-p+1}_{\alpha-p} & \lambda'^{\alpha-p+1}_{\alpha-p+1} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \lambda'^{\alpha-1}_{\alpha-p} & \lambda'^{\alpha-1}_{\alpha-p+1} & \dots & \lambda'^{\alpha-1}_{\alpha-1} & 0 \\ \lambda'^{\alpha}_{\alpha-p} & \lambda'^{\alpha}_{\alpha-p+1} & \dots & \dots & \lambda'^{\alpha}_{\alpha} \end{pmatrix} \begin{pmatrix} \tau_0 \\ \tau_1 \\ \vdots \\ \tau_{p-1} \\ \tau_p \end{pmatrix}$$

and recursively on p, we obtain that $\lambda'^{\alpha}_{\alpha-p} \equiv \lambda^{\alpha}_{\alpha-p}$, for any $p = 0, \dots, \alpha$. Adding the contribution coming from equations $[M', M-\alpha-p]$, the following constraints on the parameters τ's hold

$$\begin{array}{rcl} \tau_p & = & \tau'_p \\ \tau_{\xi-p} & = & \tau'_{\xi'-p} \end{array} \qquad p = 0, \dots, \alpha-1.$$

The formula (2.3.5) immediately follows. □

2.3.1. Compatibility with constant brackets. We devote this sub-section to study the compatibility of pairs (P, P') of PBs (2.1.1), where P' can be reduced by local change of coordinates to the constant form.

Lemma 2.3.3. *Any non-constant PB (2.1.1) of order $M = 2\alpha + \xi$, with $\xi > 0$ is compatible with a constant bracket of order M' only if $M' \le \alpha$. Moreover, the constant coefficients $\sigma_i|_{i=1,\dots,M'}$ have to satisfy the following recurrences*

$$\begin{array}{rl} \sigma_{\alpha-k} & = \left(\sum_{i=0}^{k} \lambda^{\alpha-i}_{\alpha-k}\right) \sigma_\alpha, \quad k = 0, \dots, \alpha, \\ \sigma_0 & \equiv 0. \end{array} \tag{2.3.6}$$

PROOF. Let us fix a pair of PBs (2.1.1) (P, P') such that their leading functions are of the form

$$\begin{aligned} g_M(u_n,\dots,u_{n+M}) &= a_M(u_n)\, f^{\xi}(u_{n+\alpha},\dots,u_{n+\alpha+\xi})\, a_M(u_{n+M}) \\ g'_{M'}(u_n,\dots,u_{n+M'}) &= a_{M'}(u_n)\, \sigma_{M'}\, a_{M'}(u_{n+M'}),\ \ \sigma_{M'} \neq 0. \end{aligned}$$

As in the proof of Lemma 2.3.3, from equation $[M, M']$, we obtain

$$\left(\log a_M(u_n)\right)_{,u_n} = \left(\log a'_{M'}(u_n)\right)_{,u_n},$$

that is $a_M(u_n) = k\, a_{M'}(u_n)$, for some constant k.
Moreover, after a change of variables, we can reduce to consider

$$\begin{aligned} g_M(u_n,\dots,u_{n+M}) &= f^{\xi}(u_{n+\alpha},\dots,u_{n+\alpha+\xi}) \\ g'_{M'}(u_n,\dots,u_{n+M'}) &= \sigma_{M'}. \end{aligned}$$

Let us first consider $\alpha < M' \leq M$. From the equation

$$[M, M'-\alpha] \qquad \left\{\left\{u_n, u_{n+M}\right\}, u_{n+M+M'-\alpha}\right\} = \left\{u_n, \left\{u_{n+M}, u_{n+M+M'-\alpha}\right\}\right\}$$

we obtain $f^{\xi}(u_{n+\alpha},\dots,u_{n+\alpha+\xi})_{,u_{n+\alpha+\xi}}\sigma_{M'} = 0$ that implies $\sigma_{M'} \equiv 0$.
When $\xi > 0$, for any $k = 1,\dots,\alpha$ we have $M - k > \alpha$. Focusing our attention on equations

$$[k, M-k] \quad \left\{\left\{u_n, u_{n+k}\right\}, u_{n+M}\right\} = \left\{u_n, \left\{u_{n+k}, u_{n+M}\right\}\right\} + \left\{\left\{u_n, u_{n+M}\right\}, u_{n+k}\right\},$$

we find the constraints system $\lambda^{\alpha-k}_{\alpha-k}\sigma_\alpha = \tau_0^{-1}\sum_{i=0}^{k}\sigma_{\alpha+i-k}\,\tau_i$ that can be written in the form (2.3.6).

□

Remark 2.3.4. *When $\xi = 0$, we immediately obtain that any $(\alpha, 0)$-bracket is compatible with a constant PB of order α, given by the formulæ*

$$\begin{aligned} \{u_n, u_{n+\alpha}\} &= \sigma_\alpha \\ \{u_n, u_{n+\alpha-s}\} &\equiv 0, \qquad s = 1,\dots,\alpha-1. \end{aligned}$$

2.3.2. Examples. We complete this Section, adding the details for the relevant family of $(1, \xi)$-brackets, where we are able to prove that our necessary conditions are also sufficient. Let P be a $(1, \xi)$-bracket,

- if $\tau_0 \neq \tau_\xi$, according to Theorem 2.3.2, there are not $(1, \xi')$-brackets P', such that the pair (P, P') forms a pencil of PBs. Looking for constant brackets, we have that P is compatible only with first order constant brackets if and only if $\xi = 0, 1$.

- if $\tau_0 = \tau_\xi$, then necessarily $M = 2K$ for some positive integer K. Any pair of $(1, 2q)$-brackets, with $q = 1,\dots,K$, defined by building functions of the form

$$f^{2q}(u_{n+1},\dots,u_{n+2q-1})=\exp(z^{(2q)}_{n+1}),\qquad \text{where}\ \ z^{(2q)}_n\doteq\sum_{i=0}^{2(q-1)}(-\tau_0)^i u_{n+i},$$

defines a pencil.

2.4. On non-degenerate, vector-valued PBs

In order to provide some examples of vector-valued PBs, we define new lattice variables, according to the following

Proposition 2.4.1. *Let us consider a pair of positive integers (M,K), with $K\leq M$. Any scalar-valued PB (2.1.1) of order M, according to the formula*

$$v^{1+p}_n\doteq u^1_{nK+p},\quad p=0,\dots,K-1,$$

is transformed into a non-degenerate PB (i.e. the leading order is given by a non-singular matrix) of order A and target space of dim. K iff $M=AK$, for some positive integer A.

We are interested in vector-valued PBs of first order (i.e. $A=1$), therefore we apply Proposition (2.4.1), with $K=M$. Let us first recall some preliminary definitions. A Lie group G, with a PB $\{\cdot,\cdot\}_G$ is a *Lie-Poisson group* if the multiplication $\mu:G\times G\to G$ is a mapping of Poisson manifolds, where on $G\times G$ is defined the bracket $\{\varphi,\psi\}_{G\times G}(g,h)=\{\varphi(\,,h),\psi(\,,h)\}_G(g)+\{\varphi(g,\,),\psi(g,\,)\}_G(h)$, with $h,g\in G$. Let c^k_{ij} be the structure constants of a Lie algebra $\mathfrak{g}$. The couple $(\mathfrak{g},\gamma)$ is a *Lie bi-algebra* if and only if

(i) γ is a 1-cocycle on $\mathfrak{g}$ with values on $\mathfrak{g}\otimes\mathfrak{g}$, where $\mathfrak{g}$ acts on $\mathfrak{g}\otimes\mathfrak{g}$ by the adjoint representation $\mathrm{ad}^{(2)}_\xi=\mathrm{ad}_\xi\otimes\mathbf{1}+\mathbf{1}\otimes\mathrm{ad}_\xi$, that means: $\delta_{\mathrm{ad}}\gamma=0$, i.e.

$$\mathrm{ad}^{(2)}_\xi(\gamma(\eta))-\mathrm{ad}^{(2)}_\eta(\gamma(\xi))-\gamma([\xi,\eta])=0,$$

or, fixed a basis of $\mathfrak{g}$, $c^\epsilon_{rs}\gamma^{pq}_\epsilon=c^p_{\epsilon s}\gamma^{\epsilon q}_r+c^q_{\epsilon s}\gamma^{p\epsilon}_r-c^p_{\epsilon r}\gamma^{\epsilon q}_s-c^q_{\epsilon r}\gamma^{p\epsilon}_s$.

(ii) ${}^t\gamma:\mathfrak{g}^*\otimes\mathfrak{g}^*\to\mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}^*$: $[\xi,\eta]_{g^*}={}^t\gamma(\xi\otimes\eta)$.

The correspondence between the Lie-Poisson groups and the Lie bi-algebras is clarified by the following

Theorem 2.4.2. *Let G be a Lie group, with tangent Lie algebra $\mathfrak{g}$. Then locally a Lie-Poisson structure on G is uniquely (up to isomorphism) determined by a Lie algebra structure on the dual space $\mathfrak{g}^*$, then $\mathfrak{g}$ is a Lie bialgebra $(\mathfrak{g},\gamma)$.*

Proof. All the details of the proof can be found in **[28]**. Here we recall some ideas about the direction from Lie bi-algebra to Lie-Poisson group, which will be useful in the following. Fixed a basis of $(\mathfrak{g},\gamma)$, the constants γ_k^{pq} define a Lie bracket on $\mathfrak{g}^*$. Moreover, solving the differential equation

$$\gamma_k^{pq}=\partial_k\pi_G^{pq}|_e,$$

we can define $\{\varphi,\psi\}_G\doteq\partial_p\varphi\pi_G^{pq}\partial_q\psi$, that is a PBs because the compatibility condition of the system

$$\left\{\begin{array}{rcl}\partial_k\pi_G^{pq}&=&c_{sk}^p\pi_G^{sq}+c_{sk}^q\pi_G^{ps}+\gamma_k^{pq}\\ \pi_G^{pq}|_e&=&0\end{array}\right. \tag{2.4.1}$$

is guaranteed by (i), i.e. γ is a $1-$ cocycle on $\mathfrak{g}$. □

We focus on the particular class of Lie-Poison groups, given by the following

Definition 2.4.3. *A Lie-Poisson group* $(G,\{\,\cdot\,,\,\cdot\,\}_G)$ *is called* **admissible** *if there exist:*

(i) a skew-symmetric matrix $\mathbf{k}\in\Lambda^2\mathfrak{g}$*, such that the cohomologous 1-cocycle* $\tilde\gamma$*, defined by* $\tilde\gamma\doteq\delta_{\mathrm{ad}}\mathbf{k}+\gamma$*, i.e.*

$$\tilde\gamma_t^{pq}\doteq\gamma_t^{pq}+c_{st}^pk^{sq}+k^{ps}c_{st}^q,$$

provides a Lie algebra structure on $\mathfrak{g}^*$*. Notice that* $\mathbf{k}$ *has to satisfy the Yang-Baxter equation*

$$k^{sq}c_{st}^pk^{tr}+k^{ps}c_{st}^qk^{tr}+k^{ps}c_{sq}^rk^{tq}=\gamma_s^{pq}k^{sr}+\gamma_s^{rp}k^{sq}+\gamma_s^{qr}k^{sp},$$

(ii) a Lie algebra homomorphism

$$\mathbf{r}:(\mathfrak{g}^*,\gamma_s^{pq})\longrightarrow(\mathfrak{g},c_{pq}^s)$$

such that $\mathbf{r}_*:r_*^{pq}=r^{qp}$ *defines a Lie algebra homomorphism*

$$\mathbf{r}_*:(\mathfrak{g}^*,\tilde\gamma_s^{pq})\rightarrow\left(\mathfrak{g},c_{pq}^s\right).$$

We are now in a position to formulate the Dubrovin's theorem [**12**].

Theorem 2.4.4. *An admissible Lie-Poisson group* $(G,\{\;,\;\}_G)$ *together with corresponding matrices* $\mathbf{r}$*,* $\mathbf{k}$ *defines a PB of the form*

$$\begin{array}{rcl}\left\{u_n^i,u_n^j\right\}_1&=&h^{ij}\left(\mathbf{u}_n\right)\\ \left\{u_n^i,u_{n+1}^j\right\}_1&=&g^{ij}\left(\mathbf{u}_n,\mathbf{u}_{n+1}\right)\end{array} \tag{2.4.2}$$

where $\mathbf{u}_n\in G$ *for all* n*, according to the following formulæ*

$$\{\varphi\left(\mathbf{u}_n\right),\psi\left(\mathbf{u}_{n+1}\right)\}_1\doteq\partial_\alpha\varphi\left(\mathbf{u}_n\right)r^{\alpha\beta}\partial'_\beta\psi\left(\mathbf{u}_{n+1}\right)$$

where ∂_α *and* ∂'_β *are left- and right-invariant vector fields on* G*,*

$$\{\varphi\left(\mathbf{u}_n\right),\psi\left(\mathbf{u}_n\right)\}_1\doteq h^{\alpha\beta}\left(\mathbf{u}_n\right)\partial_\alpha\varphi\left(\mathbf{u}_n\right)\partial_\beta\psi\left(\mathbf{u}_n\right)$$

where $h^{\alpha\beta}(\mathbf{u}_n) \doteq \pi_G^{\alpha\beta}(\mathbf{u}_n) + \mathrm{Ad}^{(2)}_{\mathbf{u}^{-1}} k^{\alpha\beta}$ *and* $\pi_G^{\alpha\beta}(\mathbf{u}_n)$ *is determined by the system* (2.4.1)*. Here* φ, ψ *are arbitrary smooth functions on* G*. Viceversa, all brackets of such form are obtained in this way under the non-degeneracy condition* (1.2.7)

$$\det g^{ij}(\mathbf{u}_n, \mathbf{u}_n) \neq 0.$$

Remark 2.4.5. *This Theorem does not seem to have simple applications in the lattice systems, appearing in the literature. For example, one can notice that the fundamental Toda lattice has three well-known local compatible PBs, but only the quadratic one is non-degenerate and can be represent using the previous Theorem on the algebra of* $\mathrm{Aff}^0\mathbb{R}^1$*, group of affine transformations of the straight line.*

Proof. For convenience of the reader we derive here a direct proof of Theorem 2.4.4. Let us start considering a PB of type (2.4.2). The Jacobi identity can be recast into the following cases, corresponding to differential equations for the coefficients h^{ij} and g^{ij}

$$\left\{\left\{u_n^i, u_{n+1}^j\right\}, u_{n+2}^k\right\} = \left\{u_n^i, \left\{u_{n+1}^j, u_{n+2}^k\right\}\right\} \tag{2.4.3}$$

$$\left\{\left\{u_n^i, u_n^j\right\}, u_{n+1}^k\right\} + \left\{\left\{u_n^j, u_{n+1}^k\right\}, u_n^i\right\} + \left\{\left\{u_{n+1}^k, u_n^i\right\}, u_n^j\right\} = 0 \tag{2.4.4}$$

$$\left\{\left\{u_n^i, u_n^j\right\}, u_{n-1}^k\right\} + \left\{\left\{u_n^j, u_{n-1}^k\right\}, u_n^i\right\} + \left\{\left\{u_{n-1}^k, u_n^i\right\}, u_n^j\right\} = 0 \tag{2.4.5}$$

$$\left\{\left\{u_n^i, u_n^j\right\}, u_n^k\right\} + \left\{\left\{u_n^j, u_n^k\right\}, u_n^i\right\} + \left\{\left\{u_n^k, u_n^i\right\}, u_n^j\right\} = 0. \tag{2.4.6}$$

In the following, for the sake of simplicity we denote $\mathbf{u} \doteq \mathbf{u}_n$, $\mathbf{v} \doteq \mathbf{u}_{n+1}$ and $\mathbf{w} \doteq \mathbf{u}_{n+2}$. Let us first notice that the identity (2.4.6) simply means that $h^{ij}(\mathbf{u})$ defines a Poisson structure on the target manifold M^N. The identity (2.4.3) is equivalent to equation

$$\frac{\partial g^{ij}(\mathbf{u}, \mathbf{v})}{\partial v^s} g^{sk}(\mathbf{v}, \mathbf{w}) = \frac{\partial g^{is}(\mathbf{u}, \mathbf{v})}{\partial v^s} g^{jk}(\mathbf{v}, \mathbf{w}) \tag{2.4.7}$$

from which, according to the nondegeneracy condition (1.2.7), one obtain the following factorization for the matrix $g^{ij}(\mathbf{u}, \mathbf{v})$

$$g^{ij}(\mathbf{u}, \mathbf{v}) = L^i_\alpha(\mathbf{u}) r^{\alpha\,\beta} R^j_\beta(\mathbf{v}),$$

where $L = L(\mathbf{u})$, $R = R(\mathbf{v})$ and r are invertible matrices. Notice that the matrix r is constant. Now, denoting $\dfrac{\partial}{\partial v^s} \doteq \partial_s$, equation (2.4.7) provides the following commutativity condition

$$\partial_s R^{\alpha k}(\mathbf{v})\, L^s_\beta(\mathbf{v}) = R^{\alpha s}(\mathbf{v})\, \partial_s L^k_\beta(\mathbf{v}),$$

that is

$$[L_\beta, R_\alpha]^k = 0,$$

where $R^{\alpha j} = r^{\alpha\beta} R^j_\beta$. The vectors $\{L_\alpha\}_{\alpha=1,\dots,N}$ and $\{R_\beta\}_{\beta=1,\dots,N}$ are therefore our candidates for the left- and right-invariant vector fields of a Lie algebra.

Let us now consider the identity (2.4.5), that is

$$\begin{aligned}\partial_{u^s} h^{ij}(\mathbf{u})\, g^{sk}(\mathbf{u},\mathbf{v}) = \quad & \partial_{u^s} g^{ik}(\mathbf{u},\mathbf{v})\, h^{sj}(\mathbf{u}) - \partial_{v^s} g^{ik}(\mathbf{u},\mathbf{v})\, g^{js}(\mathbf{u},\mathbf{v}) + \\ & -\partial_{u^s} g^{jk}(\mathbf{u},\mathbf{v})\, h^{si}(\mathbf{u}) + \partial_{v^s} g^{jk}(\mathbf{u},\mathbf{v})\, g^{is}(\mathbf{u},\mathbf{v})\end{aligned} \tag{2.4.8}$$

or equivalently

$$\left[\partial_{u^s} h^{ij}(\mathbf{u})\, L^s_\lambda(\mathbf{u}) - L^i_{\lambda,s}(\mathbf{u})\, h^{sj}(\mathbf{u}) + L^j_{\lambda,s}(\mathbf{u})\, h^{si}(\mathbf{u})\right] R^{\lambda k}$$
$$\shortparallel$$
$$L^i_\alpha(\mathbf{u})\, L^j_\beta(\mathbf{u}) \left[R^{\beta k}_{,s}(\mathbf{v})\, R^{\alpha s}(\mathbf{v}) - R^{\alpha k}_{,s}(\mathbf{v})\, R^{\beta s}(\mathbf{v})\right].$$

Using the non-degeneracy condition (1.2.7), we are able to separate the terms depending on variable $\mathbf{u}$ from that ones depending on $\mathbf{v}$. In particular, moving on left-hand side all the terms depending on $\mathbf{u}$, we obtain

$$\left[R^\alpha, R^\beta\right]^k(\mathbf{v}) = \Gamma^{\alpha\beta}_\lambda R^{\lambda k}(\mathbf{v}), \tag{2.4.9}$$

where $\Gamma^{\alpha\beta}_\lambda$ are therefore the structure's constants of a Lie algebra. Notice that the relations (2.4.9) are equivalent to

$$r^{\alpha\mu} r^{\beta\nu} \left[R_\mu, R_\nu\right]^k(\mathbf{v}) = r^{\lambda\tau}\Gamma^{\alpha\beta}_\lambda R^k_\tau(\mathbf{v}).$$

Denoting $\left[R_\mu, R_\nu\right]^k(\mathbf{v}) = \gamma^\epsilon_{\mu\nu} R^k_\epsilon(\mathbf{v})$, the matrix $\mathbf{r}$ defines a Lie algebra homomorphism

$$\mathbf{r} : \left(\mathfrak{g}^*, \Gamma^{\alpha\beta}_\lambda\right) \longrightarrow \left(\mathfrak{g}, \gamma^\epsilon_{\mu\nu}\right),$$

where $\mathfrak{g}^* \doteq \operatorname{span}\{R^\alpha\}_{\alpha=1,\dots,N}$ and $\mathfrak{g} \doteq \operatorname{span}\{R_\mu\}_{\mu=1,\dots,N}$.

Moreover, it is not restrictive to suppose

$$h^{ij}(\mathbf{u}) = h^{\alpha\beta}(\mathbf{u}) L^i_\alpha(\mathbf{u}) L^j_\beta(\mathbf{u}),$$

then (2.4.8) becomes

$$\begin{aligned}L^s_\lambda(\mathbf{u}) h^{\alpha\beta}_{,s}(\mathbf{u}) + h^{\epsilon\beta}(\mathbf{u})\left[L^i_{\epsilon,s}(\mathbf{u}) L^s_\lambda(\mathbf{u}) - L^i_{\lambda,s}(\mathbf{u}) L^s_\epsilon(\mathbf{u})\right] L^{-1\,\alpha}_i(\mathbf{u}) + \\ + h^{\alpha\epsilon}(\mathbf{u})\left[L^j_\epsilon(\mathbf{u}) L^s_\lambda(\mathbf{u}) - L^j_{\lambda,s}(\mathbf{u}) L^s_\epsilon(\mathbf{u})\right] L^{-1\,\beta}_j(\mathbf{u}) = \Gamma^{\alpha\beta}_\lambda\end{aligned}$$

that is

$$L^s_\lambda(\mathbf{u}) h^{\alpha\beta}_{,s}(\mathbf{u}) + h^{\epsilon\beta}(\mathbf{u})\left[L_\lambda, L_\epsilon\right]^i L^{-1\,\alpha}_i(\mathbf{u}) + h^{\alpha\epsilon}(\mathbf{u})\left[L_\lambda, L_\epsilon\right]^j L^{-1\,\beta}_j(\mathbf{u}) = \Gamma^{\alpha\beta}_\lambda. \tag{2.4.10}$$

The above procedure can be repeated for the identity (2.4.4). We have

$$\left[\partial_{u^s} h^{ij}(\mathbf{v}) R^{s\alpha}(\mathbf{v}) - R^{i\alpha}_{,s}(\mathbf{v}) h^{sj}(\mathbf{v}) + R^{j\alpha}_{,s}(\mathbf{v}) h^{is}(\mathbf{v})\right] L^k_\alpha(\mathbf{u})$$
$$\shortparallel$$
$$R^{\alpha i}(\mathbf{v}) R^{\beta j}(\mathbf{v}) \left[L^k_{\beta,s}(\mathbf{u}) L^s_\alpha(\mathbf{u}) - L^k_{\alpha,s}(\mathbf{u}) L^s_\beta(\mathbf{u})\right],$$

then

$$[L_\alpha, L_\beta]^k(\mathbf{u}) = c^\lambda_{\alpha\beta} L^k_\lambda(\mathbf{u}) \tag{2.4.11}$$

and analogously

$$r^{\mu\alpha} r^{\nu\beta} c^\tau_{\mu\nu} = C^{\alpha\beta}_\lambda r^{\tau\lambda}.$$

Therefore the matrix $\mathbf{r}^t$ defines a Lie algebra homomorphism

$$\mathbf{r}^* = \mathbf{r}^t : \left(\mathfrak{g}^*, C^{\alpha\beta}_\lambda\right) \longrightarrow \left(\mathfrak{g}, c^\epsilon_{\mu\nu}\right).$$

Up to now, we have built a homogeneous space. Let us choose the identity $\mathbf{e}$, i.e. a point such that

$$L^i_\alpha(\mathbf{e}) = R^i_\alpha(\mathbf{e}).$$

We have $\gamma^\lambda_{\alpha\beta} = c^\lambda_{\alpha\beta}$ and therefore the left- and right-invariant vector fields, defining a Lie algebra $\left(\mathfrak{g}, c^\gamma_{\alpha\beta}\right)$. Moreover, as a consequence of Theorem 2.4.2, any Lie algebra can be uniquely associated (up to isomorphism) to a connected and simply connected Lie group, G, defined by $G = \exp \mathfrak{g}$.
In addition, we can use the matrix $r = r^{\alpha\beta} \in \mathfrak{g}l(n, \mathbb{R})$ to define a two Lie algebra structure on $\mathfrak{g}^*$, such that

$$\mathbf{r} : \left(\mathfrak{g}^*, \Gamma^{\alpha\beta}_\lambda\right) \to \left(\mathfrak{g}, c^\lambda_{\alpha\beta}\right)$$

and

$$\mathbf{r}^* = \mathbf{r}^t : \left(\mathfrak{g}^*, C^{\alpha\beta}_\lambda\right) \to \left(\mathfrak{g}, c^\lambda_{\alpha\beta}\right)$$

are Lie algebra homomorphism.

The equation (2.4.10) can be written in the following form

$$\begin{aligned} &\partial_\lambda h^{\alpha\beta} = c^\alpha_{\lambda\epsilon} h^{\beta\epsilon} + c^\beta_{\lambda\epsilon} h^{\epsilon\alpha} + \Gamma^{\alpha\beta}_\lambda \\ &h^{\alpha\beta}|_e = k^{\alpha\beta} \end{aligned}$$

for some constant matrix $\mathbf{k}$. We look for solutions of the form

$$h^{\alpha\beta}(\mathbf{u}) \doteq \pi^{\alpha\beta}_G(\mathbf{u}) + k^{\alpha\beta}(\mathbf{u})$$

where $\pi_G(\mathbf{u})$ is the Lie-Poisson structure on G,

$$\begin{array}{rcl} \partial_\lambda \pi^{\mu\nu}_G &=& c^\mu_{\epsilon\lambda} \pi^{\epsilon\nu}_G + c^\nu_{\epsilon\lambda} \pi^{\mu\epsilon}_G + \gamma^{\mu\nu}_\lambda \\ \partial_\lambda \pi^{\mu\nu}_G|_e &=& \gamma^{\mu\nu}_\lambda \\ \pi^{\mu\nu}_G|_e &=& 0 \end{array} \tag{2.4.12}$$

and $\mathbf{k} = \mathbf{k}(\mathbf{u})$ satisfies

$$\begin{aligned} \partial_\lambda k^{\alpha\beta}(\mathbf{u}) &= k^{\beta\epsilon}(\mathbf{u}) c^{\alpha}_{\lambda\epsilon} + k^{\epsilon\alpha}(\mathbf{u}) c^{\beta}_{\lambda\epsilon} = \mathrm{ad}^{(2)}_{\lambda} k^{\alpha\beta}(\mathbf{u}) \\ k^{\alpha\beta}|_e &= k^{\alpha\beta} \end{aligned} \tag{2.4.13}$$

that is

$$k^{\alpha\beta}(\mathbf{u}) = \mathrm{Ad}^{(2)}_{u^{-1}} k^{\alpha\beta}.$$

Furthermore, starting from the PB (2.4.2), we have obtained

$$g^{ij}(\mathbf{u}, \mathbf{v}) = L^i_\alpha(\mathbf{u}) r^{\alpha\beta} R^j_\beta(\mathbf{v}) \tag{2.4.14}$$

$$h^{ij}(\mathbf{u}) = L^i_\alpha(\mathbf{u}) \left[\pi^{\alpha\beta}_G(\mathbf{u}) + \mathrm{Ad}^{(2)}_{u^{-1}} k^{\alpha\beta} \right] L^j_\beta(\mathbf{u}). \tag{2.4.15}$$

Finally, let us observe that evaluating (2.4.13) on the group identity $\mathbf{e}$ we obtain

$$\Gamma^{\alpha\beta}_\lambda - C^{\alpha\beta}_\lambda = k^{\epsilon\beta} c^{\alpha}_{\epsilon\lambda} + k^{\alpha\epsilon} c^{\beta}_{\epsilon\lambda}$$

then $\mathbf{k}$ is a skew-symmetric matrix and $\Gamma^{\alpha\beta}_\lambda$, $C^{\alpha\beta}_\lambda$ define two Lie algebra structure on $\mathfrak{g}^*$. As 2-cocycles they are cohomologous, i.e.

$$\Gamma = C + \delta\mathbf{k},$$

and $\mathbf{k}$ (as a r-matrix on $\mathfrak{g}$) has to satisfy

$$[\mathbf{k}, \mathbf{k}]^{\alpha\beta\gamma} = C^{\alpha\beta}_\epsilon k^{\epsilon\gamma} + C^{\lambda\alpha}_\epsilon k^{\epsilon\beta} + C^{\beta\lambda}_\epsilon k^{\epsilon\alpha}.$$

On the other hand one can associated to any admissible Lie-Poisson group $(G, \mathbf{r}, \mathbf{k})$ a PB of type (2.4.2).
We have just to prove that the formulæ (2.4.14) and (2.4.15) define a PB, i.e. the identities (2.4.3), (2.4.4) and (2.4.6) are satisfied.

Remark 2.4.6. *Performing a rescaling procedure, one can consider the continuous limits of differential-difference systems of type* (1.2.3) *and their Hamiltonian realization through the discrete PBs* (2.4.2).

Denoting $u^i_n = u^i(\epsilon n)$ *and replacing in the Hamiltonian structure* (2.4.2) *the Kronecker symbols*

$$\begin{aligned} &\delta_{n,m} \qquad \textit{by} \qquad \frac{1}{\epsilon}\delta(x-y), \\ &\delta_{n,m+1} \qquad \textit{by} \qquad \frac{1}{\epsilon}\delta(x-y-\epsilon), \end{aligned}$$

we obtain that the continuous limit of PB of type (2.4.2) *can be described by*

$$\left\{ u^i(x), u^j(x) \right\} = \frac{1}{\epsilon}\omega^{ij}(\mathbf{u})\delta(x-y) + g^{ij}(\mathbf{u})\delta_x(x-y) + b^{ij}_k(\mathbf{u}) u^k_x \delta(x-y). \tag{2.4.16}$$

This class of PBs is an extension of the PBs of hydrodynamic type, where we have added the ultra-local term $\omega^{ij}(\mathbf{u})$*. An interesting problem suggested in* **[20]** *is the*

study of formal perturbations of PBs (2.4.16). When the ultra-local PB $\omega^{ij}(\mathbf{u})$ is non-degenerate then a triviality Theorem (as in the Remark 1.4.2) can be proven. However the general case seems still to be not understood.
Let us look at a reduction of PBs (2.4.2), considering when $h^{ij} \equiv 0$. From equation (2.4.8) one has that the vector fields L_α and R_α coincide, so

$$g^{ij}\left(\mathbf{u},\mathbf{v}\right) = L^i_\alpha\left(\mathbf{u}\right) C^{\alpha\beta} L^j_\beta\left(\mathbf{v}\right)$$

and we could find a change of coordinates such that $g^{ij}\left(\mathbf{u},\mathbf{v}\right) = C^{\alpha\beta}$. In the continuous case this correspond to the flatness of the metric $g^{ij}\left(\mathbf{u},\mathbf{v}\right)$.

□

A new class of non-degenerate vector-valued ($N > 1$) PBs (2.4.2) is described by the following

Theorem 2.4.7. *Any (α,ξ)-bracket, after the lattice consolidation procedure, becomes a non-degenerate PB of the form (2.4.2), with associated Lie bi-algebra given by*

$$\begin{array}{ll} \mathfrak{g}_{(\alpha,\xi)} = \operatorname{span}\left\{L_s\right\}_{s=1,\dots,M}, & \left[L_p,L_q\right](\mathbf{v}_n) = c^r_{p\,q}\,L_r(\mathbf{v}_n) \\ \mathfrak{g}^*_{(\alpha,\xi)} = \operatorname{span}\left\{R^s\right\}_{s=1,\dots,M}, & \left[R^p,R^q\right](\mathbf{v}_n) = \gamma^{p\,q}_r\,R^r(\mathbf{v}_n), \end{array}$$

where the summation is over the repeated index r and $c^r_{p\,q}$ and $\gamma^{p\,q}_r$ are certain constants depending on the parameters τ's.

Proof. Choosing $K = M$ in the Proposition 2.4.1, we have

$$g^{ij}(\mathbf{v}_n,\mathbf{v}_{n+1}) = \begin{pmatrix} \left\{v^1_n, v^1_{n+1}\right\}_M & & \\ \vdots & \ddots & \\ \left\{v^M_n, v^1_{n+1}\right\}_M & \dots & \left\{v^M_n, v^M_{n+1}\right\}_M \end{pmatrix}$$

and $h^{ij}(\mathbf{v}_n) = g^{ji}(\mathbf{v}_n,\mathbf{v}_n) - g^{ij}(\mathbf{v}_n,\mathbf{v}_n)$. Substituting the explicit formulæ(2.1.12), we find out a decomposition of the leading order $g^{ij}(\mathbf{v}_n,\mathbf{v}_{n+1}) = L^i_\mu(\mathbf{v}_n) R^{\mu\,j}(\mathbf{v}_{n+1})$. The matrix $L(\mathbf{v}_n)$ is given by $L(\mathbf{v}_n) \doteq \Lambda_L \operatorname{diag}(l_1,\dots,l_M)(\mathbf{v}_n)$ where $l_s(v^{\alpha+s}_n,\dots,v^{\alpha+\xi+s}_n) = \exp\left(\sum_{i=0}^{\xi} \tau_i\, v^{\alpha+s+i}_n\right)$ and

$$\Lambda_L \doteq \begin{pmatrix} \lambda^\alpha_\alpha & & & & & \\ \vdots & \lambda^\alpha_\alpha & & & & \\ \lambda^0_0 & \vdots & \ddots & & & \\ & \lambda^0_0 & & \ddots & & \\ & & \ddots & & \ddots & \\ & & & \lambda^0_0 & \dots & \lambda^\alpha_\alpha \end{pmatrix}$$

Analogously the matrix $R(\mathbf{v}_{n+1})$ factorizes into $R(\mathbf{v}_{n+1}) \doteq \operatorname{diag}(r_1, \ldots, r_M)\,\Lambda_R$, where $r_s(v^1_{n+1}, \ldots, v^{s-\alpha}_{n+1}) = \exp\left(\sum_{i=0}^{\alpha-s+1} \tau_{\xi-i} v^{s-\alpha-i}_{n+1}\right)$,

$$\Lambda_R \doteq \begin{pmatrix} \lambda^\alpha_\alpha & & & & & \\ \vdots & \lambda^\alpha_\alpha & & & & \\ \lambda^\alpha_0 & \vdots & \ddots & & & \\ & \lambda^\alpha_0 & & \ddots & & \\ & & \ddots & & \ddots & \\ & & & \lambda^\alpha_0 & \ldots & \lambda^\alpha_\alpha \end{pmatrix}$$

We are using the notation $v^s_n \equiv 0$ if $s < 1$ or $s > M$. Moreover, as in the previous Sections we are supposing $\xi > \alpha$.

Let $L_s|_{s=1,\ldots,M}$ be the s-th column of matrix L and $\mathfrak{g}_{(\alpha,\xi)} \doteq \operatorname{span}\{L_s\}$ as vector space. By direct computation, we find that the following commutators

$$[L_p, L_q]^k(\mathbf{v}_n) = L^s_q(\mathbf{v}_n)\, L^k_{p,s}(\mathbf{v}_n) - L^s_p(\mathbf{v}_n)\, L^k_{q,s}(\mathbf{v}_n) = c^r_{p\,q}\, L_r(\mathbf{v}_n) \tag{2.4.17}$$

equip $\mathfrak{g}_{(\alpha,\xi)}$ of a Lie algebra structure. At first, we observe that if $\alpha + \xi < q \leq M$, $L_q(\mathbf{v}_n)$ are constant vector fields. Then

$$[L_p, L_q]^k(\mathbf{v}_n) = \sigma_{p\,q}\, L_p(\mathbf{v}_n), \qquad p = 1, \ldots, M,$$

for some constants $\sigma_{p\,q}$ that can be expressed in terms of parameters τ.

In particular, observe that $[L_s, L_{\alpha+\xi+t}]^k(\mathbf{v}_n) = \delta_{s,t}\, L_s$, where $s, t = 1, \ldots, \alpha$ and δ is the Kronecker symbol.

Let us now consider $p, q = 1, \ldots, \alpha + \xi$. It is not restrictive to suppose $q > p$. Denoting $t \doteq q - p$, the formulæ(2.4.17) for the commutators reduce to

$$[L_p, L_{p+t}]^k(\mathbf{v}_n) = L^s_{p+t}(\mathbf{v}_n)\, L^k_{p,s}(\mathbf{v}_n).$$

We distinguish two cases:

(i) when $t \leq \alpha$, we have $[L_p, L_{p+t}]^k(\mathbf{v}_n) = \lambda^{\alpha-k+p}_{\alpha-k+p} \sum_{i=0}^t \lambda^{t-i}_{t-i}\tau_i\, l_{p+t}(\mathbf{v}_n) l_p(\mathbf{v}_n)$. Let us focus on the summation $\sum_{i=0}^t \lambda^{t-i}_{t-i}\tau_i$. According to formulæ(2.1.13) and (2.1.27), we have

$$\sum_{i=0}^t \lambda^{t-i}_{t-i}\tau_i = \lambda^0_0 \sum_{i=0}^t \lambda^\alpha_{\alpha-t+i}\tau_i \equiv 0,$$

(ii) when $t > \alpha$, we obtain $[L_p, L_{p+t}]^k(\mathbf{v}_n) = \lambda^{\alpha-k+p}_{\alpha-k+p} \sum_{i=0}^\alpha \lambda^{\alpha-i}_{\alpha-i}\tau_{t-\alpha+i}\, l_{p+t}(\mathbf{v}_n) l_p(\mathbf{v}_n)$, and we have $\sum_{i=0}^\alpha \lambda^{\alpha-i}_{\alpha-i}\tau_{t-\alpha+i} = \lambda^0_0 \sum_{i=0}^\alpha \lambda^\alpha_i \tau_{t-\alpha+i} \equiv 0$, according to (2.1.27).

Finally, starting from the rows of matrix R, we can repeat the procedure above finding the Lie algebra structure on $\mathfrak{g}^*$. □

CHAPTER 3

Multi-Hamiltonian wave systems and integrable perturbations

3.1. Compatible Poisson pencils for systems of hyperbolic conservation laws

In the present Section we prove Theorem 1.3.4, providing the details about the explicit realization of a tri-Hamiltonian pencil of PBs.

Let us first recall what we have observed in Remark 1.3.5. Any two-components hyperbolic system of hydrodynamic type (1.3.2) is integrable. Indeed, one can always reduce the system in the diagonal form and the abelian Lie subalgebra of first integrals

$$H_f = \int f(u,v)dx,$$

where $f = f(u,v)$ is a solution to equation (1.3.4), i.e.

$$f_{uu}h_{vv} = f_{vv}h_{uu},$$

depends on two arbitrary functions of one variable. We focus our attention on the separable case. This means that the Hamiltonian densities $h = h(u,v)$ and $f = f(u,v)$ satisfy the linear PDEs

$$h_{uu} = Q(u)h_{vv} \quad \text{and} \quad f_{uu} = Q(u)f_{vv}, \tag{3.1.1}$$

for some function $Q = Q(u)$. Dealing with this particular class, we are going to obtain constraints on function Q which guarantee the pairwise commutativity of the first integrals H_f with respect to a pencil of three compatible PBs of hydrodynamic type (1.1.4).

Let $a_i = a_i(u,v)$, $i = 1,2,3$ and $b_j = b_j(u,v)$, $j = 1,2$ be smooth functions of the dependent variables u and v and π a skew-symmetric first order differential operator of the following form

$$\pi \doteq \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}\partial_x + \begin{pmatrix} \frac{1}{2}(a_1)_x & \left((a_2)_u - b_1\right)u_x + \left((a_2)_v - b_2\right)v_x \\ b_1u_x + b_2v_x & \frac{1}{2}(a_3)_x \end{pmatrix}. \tag{3.1.2}$$

The differential operator (3.1.2) satisfies the Jacobi identity if and only if the following constraints are fulfilled,

$$a_3 = a_1 Q \tag{3.1.3}$$

$$2b_1 = (a_1)_v Q \tag{3.1.4}$$

$$2b_2 = (a_1)_u \tag{3.1.5}$$

$$2b_1 = 2(a_2)_u - (a_3)_v \tag{3.1.6}$$

$$2Qb_2 = 2Q(a_2)_v - (a_3)_u \tag{3.1.7}$$

$$(b_1)_v = (b_2)_u. \tag{3.1.8}$$

Moreover, if the above constraints (3.1.3)-(3.1.8) holds true, any pair of functions $f = f(u,v)$ and $g = g(u,v)$ solutions to (3.1.1), define commuting first integrals with respect to the PB $\{\,\cdot\,,\,\cdot\,\}_\pi$ associated to the operator π, that is

$$\{H_f, H_g\}_\pi \equiv 0.$$

Let us look carefully at equations (3.1.3)-(3.1.8), in order to extract constrains on $Q = Q(u)$. Starting from equation (3.1.8), we introduce the potential $\lambda = \lambda(u,v)$, such that $\lambda_u = b_1$ and $\lambda_v = b_2$. Substituting into the equations (3.1.4) and (3.1.5) we easily find

$$2\lambda_u = Q\,(a_1)_v \tag{3.1.9}$$

and

$$2\lambda_v = (a_1)_u \tag{3.1.10}$$

therefore λ has to satisfy the following compatibility equation

$$\lambda_{vv} = \left(\frac{\lambda_u}{Q}\right)_u.$$

From equation (3.1.6), written in terms of the function $\lambda(u,v)$, we obtain

$$2\lambda_u = (a_2)_u = (a_3)_v,$$

which can be integrated in order to obtain an expression for the coefficient a_2,

$$a_2(u,v) = 2\lambda(u,v) + \psi(v),$$

for some integration function $\psi(v)$ to be determined. Finally, from equation (3.1.7), we obtain an explicit expression for the function $a_1 = a_1(u,v)$,

$$a_1 = 2\psi' \frac{Q}{Q'},$$

where $\psi = \psi(v)$ and $Q = Q(u)$ and prime indicates ordinary derivative. Let us denote $\xi \doteq \dfrac{Q}{Q'}$, the equations (3.1.9) and (3.1.10) above become

$$\lambda_u = \psi'' Q \xi \tag{3.1.11}$$

$$\lambda_v = \psi' \xi'. \tag{3.1.12}$$

Equation (3.1.12) can be easily integrate, providing an explicit expression for the function λ,

$$\lambda(u,v) = \psi \xi' + \rho(u), \tag{3.1.13}$$

where we have introduced an arbitrary integration function $\rho = \rho(u)$. Formula (3.1.13) can be substituted into equation (3.1.11), obtaining the following compatibility differential equation

$$\psi(v)\xi''(u) + \rho'(u) = \psi''(v)Q(u)\xi(u). \tag{3.1.14}$$

Let us first notice that in the case $\psi \equiv \mathit{const}$ we reconstruct our starting PB (1.1.8) up to multiplication by a non-zero constant. It is therefore not restrictive to consider only the nonconstant case $\psi = \psi(v)$ in the analysis of equation (3.1.14).

Equation (3.1.14) makes sense only in the following case

$$(i) \qquad \begin{cases} \psi''(v) = \alpha_2, \\ \rho'(u) = \alpha_2 Q(u)\xi(u), \\ \xi''(u) \equiv 0. \end{cases}$$

$$(ii) \qquad \begin{cases} \psi''(v) = k^2\psi(v), \\ \rho'(u) \equiv 0, \\ \xi''(u) = k^2 Q(u)\xi(u). \end{cases}$$

for some constant arbitrary constants α_2 and k^2. It turns out that in order to have solution to (3.1.14) the function $Q = Q(u)$ must satisfy equation

$$\left(\frac{Q}{Q'}\right)'' = k^2 Q \left(\frac{Q}{Q'}\right), \tag{3.1.15}$$

for some constant k^2, eventually zero (as in the case (i)).

Examples. Let us now spell out the cases (i) and (ii). We will cover some well-known examples coming from the theory of integrable PDEs, giving a unitary and elementary descriptions of their tri-Hamiltonian structures.

Case (i): The function $\xi = \xi(u)$ is a linear function of u

$$\xi(u) = \beta_1 u + \beta_0,$$

for some constant β_0 and β_1 determined by the function $Q = Q(u)$. Indeed, equation (3.1.15) implies that

$$Q' = \frac{Q}{\beta_1 u + \beta_0} \tag{3.1.16}$$

and therefore the function $Q = Q(u)$ has to be chosen among the functions of following form

$$Q(u) = \beta_2 \left(\beta_1 u + \beta_0\right)^{1/\beta_1}, \tag{3.1.17}$$

where β_2 is an integration constant. Let us now reconstruct the coefficients

$$\begin{aligned} a_i &= a_i(u, v), \quad i = 0, 1, 2 \\ b_j &= b_j(u, v), \quad j = 1, 2. \end{aligned}$$

From equation $\psi'' = \alpha_2$, we have that the function $\psi = \psi(v)$ is polynomial of the second order in v, that is

$$\psi(v) = \frac{\alpha_2}{2} v^2 + \alpha_1 v + \alpha_0,$$

for some arbitrary constants α_i. The family of compatible PBs of hydrodynamics type is therefore described by

$$\begin{aligned} a_1 = &\ \ 2(\alpha_2 v + \alpha_1)(\beta_1 u + \beta_0) \\ a_2 = &\ \ (2\beta_1 + 1)\left(\frac{\alpha_2}{2} v^2 + \alpha_1 v + \alpha_0\right) + 2\alpha_2 \rho(u) \\ a_3 = &\ \ Q(u) a_1 \\ b_1 = &\ \ \alpha_2 Q(u)(\beta_1 u + \beta_0) \\ b_2 = &\ \ \beta_1(\alpha_2 v + \alpha_1). \end{aligned} \tag{3.1.18}$$

where here $\rho = \rho(u)$ is defined (up to a constant) by the quadrature

$$\rho'(u) = Q(u)\xi(u). \tag{3.1.19}$$

Notice that α_i, $i = 1, 2, 3$ are arbitrary constants whilst β_j, $j = 1, 2$ are fixed by the choice of $Q(u)$.

Remark 3.1.1. *We still have the freedom of multiplying the PB* (3.1.2) *by a non-zero constant. This allows us to reduce the number of arbitrary constants. The formulæ* (3.1.18) *can be normalized in the following way*

$$\begin{aligned} a_1 = &\ \ \alpha(2v + \mu)(\beta_1 u + \beta_0) \\ a_2 = &\ \ v^2 + 2\alpha\rho(u) + \mu v + \nu \\ a_3 = &\ \ Q(u) a_1, \\ b_1 = &\ \ \alpha Q(u)(\beta_1 u + \beta_0) \end{aligned} \tag{3.1.20}$$

$$b_2 = \quad \beta_1 \alpha (v + \frac{\mu}{2}).$$

where the constant α *is defined by* β_1 *according to the following formula*

$$\alpha = \frac{2}{2\beta_1 + 1},$$

and μ*,* ν *are defined as the ratio among the arbitrary constants* α_i *in the following way*

$$\mu = \frac{2\,\alpha_1}{\alpha_2}, \quad \nu = \frac{2\,\alpha_0}{\alpha_2}.$$

This means that the above formulæ (3.1.20) describe a pencil of three compatible HPBs.

Let us now specialized the above formulæ, considering the examples possessing a Frobenius manifold description. They will be included in our case (i).
The theory of homogeneous Frobenius manifold is an efficient tool providing integrable hierarchies of evolutionary PDEs. When $N = 2$, the WDVV associativity equations (1.3.9) are empty and the moduli space of 2-dim. Frobenius manifolds is parameterized by the charge d. Recall that any solution $F = F(u, v)$ can be presented in the following form

$$F = \frac{1}{2} u v^2 + \varphi(u),$$

for some function $\varphi = \varphi(u)$, defined by the quasi-homogeneity conditions. Following **[20]**, it is more convenient to introduce parameter κ s.t.

$$d = 1 - \frac{2}{\kappa}.$$

The analytic solutions to the homogeneity condition

$$\mathcal{L}ie_E F = (3 - d)F, \qquad E = u\frac{\partial}{\partial u} + (1 - d)v\frac{\partial}{\partial v}$$

are exhausted by the following classes

$$\varphi(u) = \begin{cases} \dfrac{u^{\kappa+1}}{\kappa^2 - 1} & \kappa \neq -1, 0, 1, \\ u^2 \log u & \kappa = 1, \\ \log u & \kappa = -1, \\ \exp u & \kappa = 0. \end{cases} \tag{3.1.21}$$

Let us observe that for any function $\varphi(u)$ of the above classification the third derivative $Q(u) = \varphi'''(u)$ satisfies equation (3.1.16), therefore it can be written in the form (3.1.17) for some constants β_j and our procedure can be applied.

For generic $\kappa \neq -1, 0, 1$, we have

$$F(u,v) = \frac{1}{2}\,uv^2 + \frac{u^{\kappa+1}}{\kappa^2-1} \tag{3.1.22}$$

and the associated Hamiltonian flow (after changing of the sign of the time variable t) coincides with the equations of motion of one-dimensional polytropic gas with the equation of state $p = \frac{\kappa}{\kappa+1}u^{\kappa+1}$:

$$\begin{aligned} u_t &= (u\,v)_{,x} \\ v_t &= \left(\frac{v^2}{2} + \frac{u^\kappa}{\kappa-1}\right)_{,x}. \end{aligned} \tag{3.1.23}$$

The function $Q = Q(u)$ is given by $Q(u) = \kappa u^{\kappa-2}$, therefore

$$\beta_0 = 0 \qquad \beta_1 = \frac{1}{\kappa-2} \qquad \text{and} \qquad \alpha = \frac{2(\kappa-2)}{\kappa}.$$

Applying the formulæ (3.1.20), we can easily described the tri-Hamiltonian structure for system (3.1.23)

$$\begin{aligned} \{u(x),u(y)\}_{\mu,\nu} &= \frac{1}{\kappa}\left[2(2v+\mu)u\,\delta'(x-y) + (2u\,v_x + \mu u_x)\,\delta(x-y)\right], \\ \{v(x),u(y)\}_{\mu,\nu} &= (v^2 + 4\frac{u^\kappa}{\kappa} + \mu v + \nu)\,\delta'(x-y) + (2u^{\kappa-1})u_x\delta(x-y) \\ &\quad + \frac{1}{\kappa}(2v+\mu)v_x\delta(x-y), \\ \{v(x),v(y)\}_{\mu,\nu} &= u^{\kappa-1}(4v+2\mu)\,\delta'(x-y) + \left(u^{\kappa-1}(2v+\mu)\right)_x\,\delta(x-y). \end{aligned}$$

The bi-Hamiltonian structure for the polytropic gas equations has been found by P. Olver [**43**]. The particular value $\kappa = 3$ corresponds to the dispersionless limit of Boussinesq equation

$$u_{tt} = (uu_x)_x + u_{xxxx}. \tag{3.1.24}$$

An other example is given by the integrable Euler equation presented in formula (1.3.8) (see [**25**]). Euler's equation

$$\psi_{tt} - (1+\psi_x)^{-(1+\gamma)}\psi_{xx} = 0 \tag{3.1.25}$$

governs the propagation of plane sound waves of finite amplitude. Introducing the potential $\phi = \phi(x,t)$ equation (3.1.25) reads in the following evolutionary form

$$\begin{aligned} \psi_t &= \phi_x \\ \phi_t &= -\frac{(1+\psi_x)^{-\gamma}}{\gamma} \qquad \gamma \neq 0, \end{aligned}$$

and denoting $u \doteq 1 + \psi_x$, $v \doteq \phi_x$,

$$\begin{aligned} u_t &= v_x \\ v_t &= u^{-(1+\gamma)} u_x, \end{aligned}$$

which belongs to the class of equations of hydrodynamic type. The Hamiltonian is

$$h(u, v) = \frac{1}{2} v^2 - \frac{u^{-\gamma}}{\gamma},$$

and therefore

$$\frac{h_{uuu}}{h_{uu}} = \frac{-3-\gamma}{u}.$$

The formulæ (3.1.24) remain valid also for the exceptional values $\kappa = \pm 1$ where the expression for the potential of the Frobenius manifold becomes respectively

$$\begin{aligned} F_1(u, v) = & \quad \frac{1}{2} u v^2 + u^2 \log u \\ F_{-1}(u, v) = & \quad \frac{1}{2} u v^2 + \log u. \end{aligned}$$

The first case corresponds to the dispersionless limit of the focusing/defocusing non-linear Schrödinger equation

$$i\epsilon\psi_t + \frac{1}{2}\epsilon^2 \psi_{xx} \pm |\psi|^2 \psi = 0. \tag{3.1.26}$$

Indeed, introducing the following coordinates

$$u = |\psi|^2, \qquad v = \frac{\epsilon}{2i} \left(\log\psi - \log\bar{\psi} \right)_x$$

the NLS equation can be written in the following Hamiltonian form

$$\begin{aligned} & u_t + (uv)_x = 0 \\ & v_t + v v_x \mp u_x = \frac{1}{4} \epsilon^2 \left(\frac{u_x x}{u} - \frac{1}{2} \frac{u_x^2}{u} \right)_x \end{aligned} \tag{3.1.27}$$

with respect to the PB (1.3.3) and the perturbed Hamiltonian

$$H_{NLS}[u, v] = \int \left[\frac{1}{2} (u^2 \mp u v^2) - \frac{\epsilon^2}{2} \frac{u_x^2}{4u} \right] dx.$$

Let us now move to the exceptional 2-dimensional Frobenius manifold with the charge $d = 1$ (i.e. $\kappa = 0$)

$$F = \frac{1}{2} (v)^2 u + e^u \tag{3.1.28}$$

which corresponds to the so-called long wave limit of the Toda lattice equation

$$u_{tt} - (e^u)_{xx} = 0. \tag{3.1.29}$$

The Toda equations

$$\ddot{u}_n = e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}},$$

belongs to the class of generalized Fermi-Pasta-Ulam (FPU) systems, describing one-dimensional systems of particles q_n with neighboring interaction ruled by some potential $P = P(q_n - q_{n-1})$

$$H = \sum_k \frac{1}{2}p_k^2 + P(q_k - q_{k-1}). \tag{3.1.30}$$

In the long-wave limit, any FPU system reduces to the nonlinear wave system (1.4.17)

$$\begin{aligned} u_t &= v_x, \\ v_t &= P''(u)u_x, \end{aligned}$$

which provides the dispersionless limit of Toda equations, choosing $P(u) = e^u$. Therefore, we have that $Q(u) = P''(u) = e^u$ and $\xi(u) = 1$. This implies $\beta_0 = 1$ and $\beta_1 = 0$. From the above formulæ (3.1.20) we have

$$\begin{aligned} a_1 =& \quad 4v + 2\mu \\ a_2 =& \quad v^2 + 4e^u + \mu v + \nu, \\ a_3 =& \quad e^u a_1. \end{aligned}$$

and the well-known tri-Hamiltonian structure reads

$$\begin{aligned} \{u(x), u(y)\}_{\mu,\nu} &= (4v + 2\mu)\,\delta'(x-y) + 2v_x\delta(x-y) \\ \{u(x), v(y)\}_{\mu,\nu} &= (v^2 + 4e^u + \mu v + \nu)\,\delta'(x-y) + (2e^u v)\,u_x\delta(x-y) + \\ &\quad + (2v + \mu)\,v_x\delta(x-y) \\ \{v(x), v(y)\}_{\mu,\nu} &= e^u\,(4v + 2\mu)\,\delta'(x-y) + \left(e^u(2v + \mu)\right)_x\,\delta(x-y). \end{aligned}$$

Remark 3.1.2. *All the examples coming from the Frobenius manifolds are such that the function $Q = Q(u)$ satisfies (3.1.16), where the constant β_1 and β_2 are not both different from zero. In particular for the Toda equations we have to choose $\beta_0 = 1$ and $\beta_1 = 0$, whilst for the other cases we have $\beta_0 = 0$ and $\beta_1 = const$. Let us notice that in our procedure we are allowing the generic case $\beta_1 \neq 0$ and $\beta_0 \neq 0$. This case can be reduced to the choice $\beta_0 = 0$ and $\beta_1 = const$ by a suitable translation $u \mapsto u + c$, where*

$$c = \frac{\beta_0}{\beta_1}.$$

Case (ii): The function $\xi = \xi(u)$ and $\psi(v)$ satisfy the following second order ODEs

$$\xi'' = k^2 Q\xi \tag{3.1.31}$$

and

$$\psi'' = k^2\psi, \tag{3.1.32}$$

for some non-zero constant k^2 and $\rho \equiv const$. The function ψ is determined up to some constants α_i, $i = 1, 2$ by the formula

$$\psi(v) = \alpha_1 e^{k\,v} - \alpha_2 e^{-k\,v}.$$

The entire pencil can be normalized multiplying by a non-zero constant. Let us introduce

$$\mu = \frac{\alpha_2}{\alpha_1}, \quad \nu = \frac{\alpha_0}{\alpha_1},$$

where α_0 is an other constant, then the coefficients $a_i = a_i(u, v)$ and $b_j = b_j(u, v)$ can be easily reconstruct starting from the solution $\xi = \xi(u)$ of the equation (3.1.31) and $\psi = \psi(v)$. The explicit formulæ are

$$\begin{aligned} a_1 &= \; 2\psi'\xi \\ a_2 &= \; (2\xi' + 1)\psi + \nu \\ b_1 &= \; \psi'' Q\xi \\ b_2 &= \; \psi'\xi', \end{aligned} \tag{3.1.33}$$

where

$$\psi(v) = e^{k\,v} - \mu e^{-k\,v}.$$

Example 3.1.3 (Ablowitz-Ladick)**.** *A non-trivial solution to the equation* (3.1.15) *(with $k^2 = 1$) is given by the function*

$$Q(u) = \frac{e^u}{e^u - 1},$$

which comes from the long-wave limit of the complexified Ablowitz-Ladick (AL) lattice [**1**]*. This Hamiltonian system is given by the pair of equations*

$$i\dot{a}_n = \frac{1}{2}(1 - a_n b_n)(a_{n-1} + a_{n+1})$$

$$i\dot{b}_n = \frac{1}{2}(1 - a_n b_n)(b_{n-1} + b_{n+1})$$

describing the time evolution of two sequences of complex variables a_n and b_n, with $n \in \mathbb{Z}$. Following [**15**]*, we introduce new variables*

$$w_n = \log(1 - a_n b_n)$$

$$v_n = \frac{1}{2}\left(\log\frac{a_n}{a_{n-1}} - \log\frac{b_n}{b_{n-1}}\right).$$

With this choice of the dependent variables and after interpolation and time rescaling

$$w_n = w(\epsilon n, \epsilon t), \qquad v_n = v(\epsilon n, \epsilon t),$$

one obtains in the dispersionless limit a Hamiltonian system of type (1.1.1). The PB can be reduced to the Darboux form (1.3.3) by a new change of variable

$$u = \frac{\epsilon\partial_x}{e^{\epsilon\partial_x}-1}w.$$

As shown in [**6**]*, in the variables u and v the dispersionless Hamiltonian becomes*

$$H_{AL}[u,v] = \int (1-e^u) cosh\, v\, dx.$$

The Ablowits-Ladick lattice can be associated to the following solution to WDVV equations

$$F(u,v) = \frac{1}{2}uv^2 + Li_3(e^u),$$

where the n^{th}-logarithm is obtained by analytic continuation of the function

$$Li_n(x) \doteq \sum_{k\geq 1} \frac{x^k}{k^n}, \qquad \text{for} \quad |x|<1.$$

In addition, we immediately have $\xi(u) = 1-e^u$ and the formulæ (3.1.33) described the tri-Hamiltonian structure for Ablowitz-Ladick

$$\begin{aligned}
\{u(x),u(y)\}_{\mu,\nu} &= 2\left(e^v+\mu e^{-v}\right)(1-e^u)\ \delta'(x-y) + \left(\left(e^v+\mu e^{-v}\right)(1-e^u)\right)_x \delta(x-y),\\
\{v(x),u(y)\}_{\mu,\nu} &= \left((1-2e^u)\left(e^v-\mu e^{-v}\right)+\nu\right)\ \delta'(x-y) - \left(e^v-\mu e^{-v}\right)e^u u_x\,\delta(x-y) + \\
&\quad -\left(e^v+\mu e^{-v}\right)e^u v_x\,\delta(x-y)\\
\{v(x),v(y)\}_{\mu,\nu} &= -2\left(e^v+\mu e^{-v}\right)e^u\,\delta'(x-y) + \left(\left(e^v+\mu e^{-v}\right)e^u\right)_x \delta(x-y).
\end{aligned}$$

3.1.1. Looking for solutions to equation (3.1.15). We devote this subsection to the study of general solutions to equation (3.1.15)

$$\left(\frac{Q}{Q'}\right)'' = k^2 Q\left(\frac{Q}{Q'}\right)$$

Remark 3.1.4. *Let k be a non-vanishing constant and $Q = Q(u)$ be a solution to the equation (3.1.15), then the function $Q = Q(u/k)$ is a solution to the equation*

$$\left(\frac{Q}{Q'}\right)'' = Q\left(\frac{Q}{Q'}\right). \tag{3.1.34}$$

Thanks to Remark 3.1.4 it is not restrictive to focus our attention on equation (3.1.34), that is equivalent to the following autonomous equation

$$Q''' + Q''\left(\frac{Q'}{Q} - 2\frac{Q''}{Q'}\right) + QQ' = 0. \tag{3.1.35}$$

Any autonomous ODE can be reduced to an ODE of lower order by changing the dependent variable from $Q = Q(u)$ to

$$W(Q) = Q'(u), \quad \text{i.e.,} \quad \int \frac{dQ}{W(Q)} = u. \tag{3.1.36}$$

Equation (3.1.35) transforms into

$$W'' - \frac{(W')^2}{W} + \frac{W'}{Q} + \frac{Q}{W} = 0, \tag{3.1.37}$$

where $W = W(Q)$ and $W' = \dfrac{dW}{dQ}$. The second order non-linear equation (3.1.37) belongs to the class of canonical equations (Type II) considered by E. L. Ince [**29**], p. 335. In particular, the further substitution $Q = e^Z$ provides

$$W'' - \frac{(W')^2}{W} + \frac{e^{3\,Z}}{W} = 0, \tag{3.1.38}$$

that can be solved in terms of hyperbolic functions.

3.1.2. Compatible pairs of flat metrics. We complete this Section comparing our procedure to the related problem of describing nonsingular pairs of compatible flat metric, which has been studied by many authors, see O. I. Mokhov [**38**], E. V. Ferapontov [**22**], B. A. Dubrovin [**14**] and references therein. The main difference in our approach is that we are using flat coordinates instead of the usual canonical ones.

Let us consider hyperbolic Hamiltonian systems of hydrodynamic type (1.1.1) admitting Riemann invariants. Therefore, there exist suitable coordinates $\mathbf{r} = (r^1, \dots, r^N)$, such that the system reduces to the diagonal form

$$r^i_t = \lambda^i(\mathbf{r}) r^i_x, \qquad i = 1, \dots, N$$

and the metric g^{ij} appearing in formula for the PB (1.1.4) becomes diagonal. The Riemann invariants are also called canonical coordinates in the Frobenius theory. We have already noticed that the cases $N = 1$ or $N = 2$ are always integrable, i.e. one can always reduce to diagonal systems.

It was proven by Mokhov [**38**] that an arbitrary nonsingular pair of metrics (g_1, g_2) is compatible if and only if there exist local coordinates $\mathbf{r} = (r^1, \dots, r^N)$ such that

$$g_1^{ij}(\mathbf{r}) = h^i(\mathbf{r})^2 \delta_{ij}, \tag{3.1.39}$$

and

$$g_2^{ij}(\mathbf{r}) = \frac{h^i(\mathbf{r})^2}{\rho^i(r^i)} \delta_{ij}, \tag{3.1.40}$$

where $\rho^i(r^i)$, $i = 1, \dots, N$ are non-vanishing arbitrary functions of one-variables.

The problem of describing diagonal flat metric (3.1.39) is a classical issue in differential geometry and it can be addressed by introducing the *rotational coefficients*

$$\beta_{ij}(\mathbf{r}) = \frac{\partial_i h_j}{h_i}. \tag{3.1.41}$$

The flatness of the metric g_1^{ij} is equivalent to finding a curvilinear orthogonal coordinate systems. Locally, such coordinate systems are determined by the Darboux equations, i.e. the following nonlinear integrable system for the rotational coefficients

$$\begin{aligned} &\frac{\partial \beta_{ij}}{\partial r^k} = \beta_{ik}\beta_{kj}, \qquad i \neq j, i \neq k, j \neq k \\ &\sum_{s\neq i,\, s\neq j}^{N} \beta_{si}\beta_{sj} + \frac{\partial \beta_{ij}}{\partial r^i} + \frac{\partial \beta_{ji}}{\partial r^j} = 0, \qquad i \neq j. \end{aligned} \tag{3.1.42}$$

Notice that the curvilinear orthogonal coordinates are defined by $N(N-1)/2$ arbitrary functions of two variables [**2**].

The problem of nonsingular compatible flat metrics has been solved by Mokhov (see [**39**]) using the Zakharov method [**56**]. Here, we are interested in the particular case, given by symmetric rotation coefficients

$$\beta_{ij} = \beta_{ji}.$$

In the language of Frobenius manifold [**13**] this is equivalent to require the flatness of the unit vector. We obtain a reduction of the Darboux-Egorov system

$$\begin{aligned} &\frac{\partial \beta_{ij}}{\partial r^k} = \beta_{ik}\beta_{kj}, \qquad i \neq j, i \neq k, j \neq k \\ &\sum_{k=1}^{N} \frac{\partial \beta_{ij}}{\partial r^k} = 0 \\ &\sum_{k=1}^{N} \rho_k(r_k)\frac{\partial \beta_{ij}}{\partial r_k} + \frac{1}{2}\left(\rho_i'(r_i) + \rho_j'(r_j)\right)\beta_{ij} = 0, \end{aligned} \tag{3.1.43}$$

where the last equation guarantees the flatness of the second metric (3.1.40).

When $N = 2$, the above system (3.1.43) is linear. One obtains that the rotation coefficient β_{12} depends on the difference $r_1 - r_2$ and has to satisfies

$$2(\rho_1 - \rho_2)\beta_{12}' + (\rho_1' + \rho_2')\beta_{12} = 0,$$

which is equivalent to

$$(\partial_{r_1} - \partial_{r_2})\log\beta_{12} + (\partial_{r_1} + \partial_{r_2})\log(\rho_1 - \rho_2) = 0.$$

The last equation can be easily integrated, obtaining

$$\rho_1(r_1) - \rho_2(r_2) = \frac{\psi(r_1 + r_2)}{\beta_{12}(r_1 - r_2)}, \tag{3.1.44}$$

where $\psi = \psi(r_1 + r_2)$ is an arbitrary integration function. Differentiating formula (3.1.44) with respect to r_1 and r_2 we obtain constraints on β_{12} and ψ. Denoting $\tilde{u} \doteq r_1 - r_2$, $v \doteq r_1 + r_2$, then $\beta_{12} = \beta_{12}(\tilde{u})$, $\psi = \psi(v)$ and

$$\beta_{12}\left(\frac{1}{\beta_{12}}\right)'' = k^2 \tag{3.1.45}$$

$$\psi'' = k^2\psi, \tag{3.1.46}$$

where the constant k plays the same role as in equations (i) and (ii) (see also Remark 3.1.5 below).

We obtain the following formulæ

$$(i)' \qquad k = 0 \qquad \begin{cases} \psi(v) = & \psi_1 v + \psi_0 \\ \beta_{12}(\tilde{u}) = & \dfrac{1}{\beta_1\tilde{u} + \beta_0} \\ \rho_1(r_1) = & \psi_1\beta_1 r_1^2 + (\psi_1\beta_0 + \psi_0\beta_1)r_1 + \psi_0\beta_0, \\ \rho_2(r_1) = & \psi_1\beta_1 r_2^2 + (\psi_1\beta_0 - \psi_0\beta_1)r_2 \end{cases}$$

$$(ii)' \qquad k \neq 0 \qquad \begin{cases} \psi(v) = & \psi_+ e^{kv} + \psi_- e^{-kv} \\ \beta_{12}(\tilde{u}) = & \dfrac{1}{\beta_+ e^{k\tilde{u}} + \beta_- e^{-k\tilde{u}}} \\ \rho_1(r_1) = & \psi_+\beta_+ e^{2kr_1} + \psi_-\beta_- e^{-2kr_1} \\ \rho_2(r_2) = & -\psi_+\beta_- e^{2kr_2} - \psi_-\beta_+ e^{-2kr_2} \end{cases}$$

which agree with the previous description (see also Remark 3.1.5) and with formulæ presented by E. Ferapontov in **[22]**.

Remark 3.1.5. *Let us explain the equivalence between equations* (3.1.45) *and* (3.1.15), *considering the nonlinear wave system*

$$\begin{aligned} u_t &= v_x \\ v_t &= P''(u)u_x. \end{aligned}$$

In the domain of hyperbolicity $P''(u) > 0$, *one can introduce the Riemann invariants by setting*

$$r_\pm = v \pm \tau(u), \qquad \textit{where} \qquad \tau'(u) \doteq \sqrt{P''(u)}.$$

Therefore, the system (3.1.47) *reduces to the diagonal form*

$$(r_\pm)_t = \pm\sqrt{P''(u)}\,(r_\pm)_x\,. \tag{3.1.47}$$

Denoting $Q(u) \doteq P''(u)$*, we obtain*

$$\begin{aligned} dr_+ &= dv + \sqrt{Q(u)}du \\ dr_- &= dv - \sqrt{Q(u)}du \end{aligned}$$

and

$$dv = \frac{1}{2}\left(dr_+ + dr_-\right)$$

$$du = \frac{1}{2\sqrt{Q}}\left(dr_+ - dr_-\right)$$

so that the flat metric $ds^2 = 2dudv$ *in the Riemann invariants becomes diagonal*

$$ds^2 = \frac{1}{2\sqrt{Q}}\left(dr_+^2 - dr_-^2\right).$$

Following the notation introduced in formula (3.1.39)*, we have*

$$h_1(u) = \frac{1}{\sqrt{2}Q^{1/4}}, \qquad h_2(u) = \frac{i}{\sqrt{2}Q^{1/4}}.$$

In order to compute the rotation coefficient $\beta_{12}(u)$*, let us first introduce the operators*

$$\frac{\partial}{\partial v} = \frac{\partial}{\partial r_+} + \frac{\partial}{\partial r_-}$$

$$\frac{\partial}{\partial u} = \sqrt{Q}\left(\frac{\partial}{\partial r_+} - \frac{\partial}{\partial r_-}\right)$$

and

$$\frac{\partial}{\partial r_-} = \frac{1}{2}\left(\frac{\partial}{\partial v} - \frac{1}{\sqrt{Q}}\frac{\partial}{\partial u}\right)$$

$$\frac{\partial}{\partial r_+} = \frac{1}{2}\left(\frac{\partial}{\partial v} + \frac{1}{\sqrt{Q}}\frac{\partial}{\partial u}\right)$$

Substituting into the formula (3.1.41)*, we immediately obtain*

$$\beta_{12}(u) = \frac{-i}{8}\frac{Q'}{Q\sqrt{Q}}.$$

Notice that the symmetry of the rotational coefficient β_{12} *is preserved (i.e.* $\beta_{12} = \beta_{21}$*). We are now in a position to compare equations* (3.1.45) *and* (3.1.15)*. Indeed*

$$\frac{1}{\sqrt{Q}}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{Q}}\frac{\partial}{\partial u}\left(\frac{Q\sqrt{Q}}{Q'}\right)\right) - k^2\frac{Q\sqrt{Q}}{Q'} = 0$$

reduces exactly to our equation (3.1.15)*.*

3.2. D-operators within ϵ^3 approximation

We complete the study of nonlinear wave systems (1.4.17) addressing the problem of existence of D-operator (see Section 1.4), extending Dubrovin results (see **[15]**) up to the third order in the small dispersive parameter ϵ. Let us briefly summarize the study of linear and quadratic perturbations.

Theorem 3.2.1. **[15]** *Any 2-integrable perturbation of the nonlinear wave equation* (3.1.47) *is described by a function $\rho = \rho(u, v)$, satisfying the following equation*

$$\rho_{uu} - P''(u)\rho_{vv} = \frac{1}{2}\left(\log P''(u)\right)_u \rho_u. \tag{3.2.1}$$

One can show that equation (3.2.1) is equivalent to the D'Alembert equation in Riemann invariants $r_{1,2} = r_{1,2}(u, v)$ (see (3.1.47) for the definition),

$$\rho_{uu} - P''(u)\rho_{vv} - \frac{P'''(u)}{2P''(u)}\rho_u = -4P''\frac{\partial^2\rho}{\partial r_1 \partial r_2}$$

and $\rho = \rho(u, v) = \rho_1(r_1) + \rho_2(r_2)$.

Corollary 3.2.2. *Any 2-integrable perturbation of the nonlinear wave equation* (1.4.17) *is parameterized by two arbitrary functions of one variable $\rho_{1,2} = \rho_{1,2}(r_{1,2})$.*

Our first step in proving the Theorem is to notice that any 1-integrable perturbation of the nonlinear wave equation (1.4.17) can be eliminated by a canonical transformation of type (1.4.10). Therefore, all the Hamiltonian perturbation (1.4.2) can be reduced (up to x-total derivative) to the form

$$H_{\text{pert}} = \int \left[\frac{1}{2}v^2 + P(u) - \frac{\epsilon^2}{2}\left(a_1(u, v)u_x^2 + 2a_2(u, v)u_x v_x + a_3(u, v)v_x^2\right)\right] dx, \tag{3.2.2}$$

where the coefficients $a_i(u, v)$, $i = 1, 2, 3$ are arbitrary functions of the independent variables u and v. Therefore, given $f = f(u, v)$ a solution of equation (1.3.6),

$$f_{uu} = P''(u) f_{vv} \tag{3.2.3}$$

we are looking for a perturbed Hamiltonian of the form (up to total $x-$derivative)

$$H_f = \int \left[f(u, v) + \frac{\epsilon^2}{2}\left(a_1^{[f]}(u, v)u_x^2 + 2a_2^{[f]}(u, v)u_x v_x + a_3^{[f]} v_x^2\right)\right] dx,$$

such that

$$\{H_f, H_{\text{pert}}\} = \mathcal{O}(\epsilon^3).$$

The computation of such bracket gives us a functional that can be written in power of the small parameter ϵ. Let $\int \mathcal{I} dx$ be the functional we obtain, where the integrand $\mathcal{I}$ is a differential homogeneous polynomials of degree 3 as well the variational derivatives $\delta_u \mathcal{I}$ and $\delta_v \mathcal{I}$.

Remark 3.2.3. *One can easily check that the equations $\delta_v\mathcal{I} = 0$ and $\delta_u\mathcal{I} = 0$ provide equivalent sets of constraints. We give below an evidence of this fact, however a general understanding is still missing (see also Remark 3.2.9 below). Given*

$$\begin{aligned}\mathcal{I} = \quad & d^1(u)u_{xxx} + d^2(u)v_{xxx} + (d^3(u)u_x + d^4(u)v_x)u_{xx} + (d^5(u)u_x + d^6(u)v_x)v_{xx} + \\ & (d^7(u)u_x + d^8(u)v_x)u_x^2 + (d^9(u)u_x + d^{10}(u)v_x)v_x^2\end{aligned}$$

we find the following set of constraints

$$\delta_u\mathcal{I} = 0 \quad \textit{if and only if} \quad \begin{cases} d^4 - d^5 = d^1_v - d^2_u & (i)_u \\ 2d^7 = d^3_u - d^1_{uu} & (ii)_u \\ 2d^9 = 3d^4_v - d^5_v - 3d^1_{vv} + d^6_u & (iii)_u \\ 2d^8 = d^3_v + 2d^4_u - 3d^1_{uv} & (iv)_u \\ d^9_v = d^4_{vv} - d^1_{vvv} + d^{10}_u & (v)_u \end{cases} \tag{3.2.4}$$

$$\delta_v\mathcal{I} = 0 \quad \textit{if and only if} \quad \begin{cases} d^4 - d^5 = d^1_v - d^2_u & (i)_v \\ 2d^{10} = d^6_v - d^2_{vv} & (ii)_v \\ 2d^8 = 3d^2_{uu} + d^3 - 3d^5_u + d^4_u & (iii)_v \\ 2d^9 = d^6_u + 2d^5_v - 3d^2_{uv} & (iv)_v \\ d^8_u = d^5_{uu} - d^2_{uuu} + d^7_v & (v)_v \end{cases} \tag{3.2.5}$$

By simple computation we can obtain constraints (3.2.5) from those described in (3.2.4).

Thanks to the previous Remark 3.2.3, it is not restrictive to consider only the vanishing of $\delta_v\mathcal{I}$. From the equation $(i)_v$ and $(ii)_v$, we have respectively

$$a_1^{[f]}(u,v) = a_3^{[f]}P''(u) - a_1(u,v)f_{vv}(u,v) + a_3(u,v)f_{uu}(u,v),$$

and

$$(a_2^{[f]}(u,v) + a_2(u,v)f_{vv}(u,v))_v = (a_3^{[f]}(u,v) + a_3(u,v)f_{vv}(u,v))_u + 2a_2(u,v)f_{vvv}(u,v).$$

Let us introduce the following functions

$$\begin{aligned}\alpha(u,v) &\doteq a_2^{[f]}(u,v) + a_2(u,v)f_{vv}(u,v) \\ \beta(u,v) &\doteq a_3^{[f]}(u,v) + a_3(u,v)f_{vv}(u,v),\end{aligned}$$

related by the first order PDE

$$\alpha_v = \beta_u + 2a_2 f_{vvv}. \tag{3.2.6}$$

The remaining equations $(iii)_v$, $(iv)_v$ and $(v)_v$ give respectively

$$\begin{aligned}&\beta(u,v)P'''(u) + 2\left[a_1(u,v) + a_3(u,v)P''(u)\right] f_{uvv}(u,v) + \\ &\qquad 4a_2(u,v)P''(u)f_{vvv}(u,v) = 0,\end{aligned} \tag{3.2.7}$$

$$\begin{aligned}&\beta(u,v)_vP''(u) - 2\left[a_1(u,v) + a_3(u,v)P''(u)\right] f_{vvv}(u,v) + \\ &\qquad -\alpha(u,v)_u - 2a_2(u,v)f_{uvv}(u,v) = 0,\end{aligned} \tag{3.2.8}$$

and

$$\alpha(u,v)P'''(u) + P''(u)^2 a_2(u,v)_v + \\ -P''(u)\left(\alpha(u,v)_u - 2a_3(u,v)f_{uvv}(u,v)\right) = 0. \tag{3.2.9}$$

Now, the equations (3.2.7), (3.2.8) and (3.2.9) allow us to obtain explicit formulæ for the functions $\alpha(u,v)$ and $\beta(u,v)$. Let $s = s(u,v)$ be

$$s(u,v) \doteq a_1(u,v) + P''(u)a_3(u,v),$$

then

$$\alpha = -2\frac{P''}{P'''}\left[s\, f_{vvv} + 2a_2\, f_{uvv}\right]$$

and

$$\beta = -2\frac{1}{P'''}\left[s\, f_{uvv} + 2a_2\, f_{uuv}\right].$$

Therefore, substituting in the equation (3.2.6) we have

$$\left[s_u - (\log P''')'s - 2P''(a_2)_v\right] f_{uvv} = \left[s_v - \left[\log P'' - 2\log P'''\right]' a_2 - 2(a_2)_u\right] f_{vvv}.$$

Due to the arbitrary of the function $f = f(u,v)$ among the solutions of equation (1.3.6), we obtain the system

$$\begin{aligned} s_u &= (\log P''')'s + 2P''(a_2)_v \\ s_v &= \left[\log P'' - 2\log P'''\right]' a_2 + 2(a_2)_u. \end{aligned} \tag{3.2.10}$$

The compatibility condition for the system (3.2.10) implies the equation (3.2.1), where

$$\rho_u = \frac{P''}{P'''}a_2.$$

Moreover, the term containing the function $s = s(u,v)$ can be eliminated by a canonical transformation generated by the Hamiltonian

$$K[u,v] = \epsilon \int k(u,v)v_x dx,$$

with $k = k(u,v)$ given by $k_u(u,v) = s(u,v)$.

Remark 3.2.4. *Any integrable quadratic perturbation of the wave system (1.4.17) of the diagonal form (i.e. $a_2(u,v) \equiv 0$)*

$$H_{\text{pert}} = \int \left[\frac{1}{2}v^2 + P(u) - \frac{\epsilon^2}{2}\left(a_1(u,v)u_x^2 + a_3(u,v)v_x^2\right)\right] dx$$

is trivial (up to Miura-type transformation). This means that the appearing of invariant functions ρ_i is related to the diagonal term $a_2(u,v)$.

Remark 3.2.5. *Let us consider $a_1(u) = P''(u)$ (as in the FPU systems (3.1.30), see* **[15]***), equation (2.1.25) leads to*

$$P''(u) = kP'''(u)$$

and therefore to have an integrable deformation the potential $P(u)$ has to be choosen in the following way

$$P(u) = \kappa_0 e^{ku} + \kappa_1 u + \kappa_2, \tag{3.2.11}$$

where κ_i, $i = 0, 1, 2$ and k are arbitrary constant. The potential (3.2.11) turns out to be that one for the Toda lattice (3.1.29).

Remark 3.2.6. *Chosen the linear solution $\rho(u, v) = c_1 v + c_0$ to (3.2.1) for some constants c_0 and c_1, we explicitly have*

$$\begin{aligned} H_{\text{pert}} &= \int \left[\frac{1}{2} v^2 + P(u) + \frac{\epsilon^2}{4} c_1 P'''(u) u_x^2 \right] dx, \\ H_f &= \int \left[f(u, v) + \frac{\epsilon^2}{2} c_1 \left(\left(\frac{1}{2} P'''(u) f_{vv}(u, v) + P''(u) f_{uvv}(u, v) \right) u_x^2 + \right.\right. \\ &\qquad \left.\left. + 2 f_{uuv}(u, v) u_x v_x + f_{uvv}(u, v) v_x^2 \right) \right] dx, \end{aligned}$$

where $f = f(u, v)$ satisfies equation (3.2.3) and c_1 can be eliminated by a rescaling of the small parameter ϵ. It could be interesting to consider the problem of extending this particular family of 2-integrable Hamiltonian systems to the fourth order in ϵ.

Cubic term. We look for the 3-integrable Hamiltonian perturbation of the equation (1.4.17), beginning with the following

Lemma 3.2.7. *Up to Miura transformations all the third order Hamiltonian perturbation (1.4.2) can be reduced to the form*

$$H^{[3]} = \int \left(a_1(u, v) u_x^3 + a_2(u, v) u_x^2 v_x + a_3(u, v) u_x v_x^2 + a_4(u, v) v_x^4 \right) dx, \tag{3.2.12}$$

where the coefficients $a_i(u, v)$ are arbitrary functions of the independent variables u and v.

Theorem 3.2.8. *Any 3^{rd} order integrable Hamiltonian perturbation of the nonlinear wave equation (1.4.17) is trivial.*

Proof. Given $f = f(u, v)$ a solution of equation (1.3.6), we are looking for a perturbed Hamiltonian of the form (up to total $x-$derivative)

$$H_f = \int \left[f(u, v) + \epsilon^3 \left(a_1^{[f]}(u, v) u_x^3 + a_2^{[f]}(u, v) u_x^2 v_x + a_3^{[f]}(u, v) u_x v_x^2 + a_4^{[f]}(u, v) v_x^4 \right) \right] dx,$$

such that

$$\{H_f, H_{\text{pert}}\} = \mathcal{O}(\epsilon^4).$$

The computation of such bracket gives us a functional that can be written in power of the small parameter ϵ. Let us denote

$$\int \sum_{k \geq 0} \epsilon^k \mathcal{I}^{[k]}(u, v; u_x, v_x, \ldots, u^{(k)}, v^{(k)}) dx \doteq \{H_f, H_{\text{pert}}\},$$

where $\mathcal{I}^{[k]}(u, v; u_x, v_x, \dots, u^{(k)}, v^{(k)})$ are graded homogeneous differential polynomials of the degree k.

Remark 3.2.9. *As in the Remark 3.2.3, by a direct computation one can show that the vanishing of $\delta_u \mathcal{I}^{[3]}$ and $\delta_v \mathcal{I}^{[3]}$ provides two sets of equivalent constraints.*

Then, we could restrict our attention to the constraints coming from the vanishing of $\delta_v \mathcal{I}^{[3]}$. We obtain the following system of seven independent equations

$$3a_1^{[f]}(u,v) = a_3^{[f]}(u,v)P''(u) + 3a_1(u,v)f_{vv}(u,v) - a_3(u,v)f_{uu}(u,v) \tag{3.2.13}$$

$$a_2^{[f]}(u,v) = 3a_4^{[f]}(u,v)P''(u) + a_2(u,v)f_{vv}(u,v) - 3a_4(u,v)f_{uu}(u,v) \tag{3.2.14}$$

$$\begin{aligned} a_4^{[f]}(u,v)P'''(u) = \; & a_1(u,v)f_{vvv}(u,v) + a_2(u,v)f_{uvv}(u,v) + \\ & + a_3(u,v)f_{uuv}(u,v) + a_4(u,v)f_{uuu}(u,v) \end{aligned} \tag{3.2.15}$$

$$\begin{aligned} a_3^{[f]}(u,v)_u = \; & 3a_4^{[f]}(u,v)_v P''(u) + 3a_2(u,v)f_{vvv}(u,v) + \\ & a_3(u,v)_u f_{vv}(u,v) + 4a_3(u,v)f_{uvv}(u,v) + \\ & -4a_4(u,v)_v f_{uu}(u,v) + 3a_4(u,v)f_{uuv}(u,v) \end{aligned} \tag{3.2.16}$$

$$\begin{aligned} 2a_3^{[f]}(u,v)P'''(u) = \; & \left(a_3^{[f]}(u,v)_u - a_2^{[f]}(u,v)_v\right) P''(u) + \\ & 6a_1(u,v)f_{uvv}(u,v) + a_2(u,v)_v f_{uu}(u,v) + \\ & -a_3(u,v)_u f_{uu}(u,v) + 4a_2(u,v)f_{uuv}(u,v) + \\ & 2a_3(u,v)f_{uuu}(u,v) \end{aligned} \tag{3.2.17}$$

$$\begin{aligned} a_3^{[f]}(u,v)_v = \; & 3a_4^{[f]}(u,v)_u + a_3(u,v)_v f_{vv}(u,v) + \\ & -2a_3(u,v)f_{vvv}(u,v) - 3a_4(u,v)_u f_{vv}(u,v) + \\ & -6a_4(u,v)f_{uvv}(u,v) \end{aligned} \tag{3.2.18}$$

$$\begin{aligned} 2a_2^{[f]}(u,v)P'''(u) = \; & \left(a_2^{[f]}(u,v)_u - 3a_1^{[f]}(u,v)_v\right) P''(u) + \\ & 3a_1(u,v)_v f_{uu}(u,v) - a_2(u,v)_u f_{uu}(u,v) + \\ & 6a_1(u,v)f_{uuv}(u,v) + 2a_2(u,v)f_{uuu}(u,v). \end{aligned} \tag{3.2.19}$$

Let us explain how to solve the system of equations (3.2.13) - (3.2.19). We first select the equations that allow us to express the functions $a_i^{[f]}(u,v)$ in terms of the coefficients $a_i(u,v)$, then from the remaining equations we obtain constraints on $a_i(u,v)$. Let us start from equation (3.2.15), which allows us to express explicitly the coefficient $a_4^{[f]}(u,v)$ in terms of the functions $a_i(u,v)$. Substituting into the equation (3.2.14) and (3.2.16), we find $a_2^{[f]}(u,v)$ and $a_3^{[f]}(u,v)_u$ as functions of $a_i(u,v)$. In particular, we have

$$
\begin{aligned}
a_3^{[f]}(u,v)_u = \quad & 3\left[a_1(u,v)f_{vvv}(u,v) + a_2(u,v)f_{uvv}(u,v) + \right. \\
& \left. a_3(u,v)f_{uuv}(u,v) + a_4(u,v)f_{uuu}(u,v)\right]_v \frac{P''(u)}{P'''(u)} + \\
& 3a_2(u,v)f_{vvv}(u,v) + a_3(u,v)_u f_{vv}(u,v) + \\
& 4a_3(u,v)f_{uvv}(u,v) - 4a_4(u,v)_v f_{uu}(u,v) \\
& + 3a_4(u,v)f_{uuv}(u,v).
\end{aligned} \tag{3.2.20}
$$

Substituting (3.2.20) into equation (3.2.17), we obtain $a_3^{[f]}(u,v)$ as functions of $a_i(u,v)$.

Next, considering equation (3.2.18) and using the freedom in the choice of the function $f = f(u,v)$ among the solution of equation (1.3.6), we find out that $a_4(u,v)_{vv} \equiv 0$. Therefore

$$a_4(u,v) = \alpha(u)v + \beta(u),$$

for some arbitrary functions $\alpha = \alpha(u)$ and $\beta = \beta(u)$. Moreover, we have

$$a_2(u,v) = -a_4(u,v)P''(u), \quad a_3(u,v) \equiv 0,$$

and $a_1 = a_1(u)$ is determined up to some constant by the following equation

$$a_1'(u) = a_1(u)\left(\log P'''(u)\right)' + \alpha(u).$$

The compatibility between equation (3.2.16) and (3.2.17) provides us that $a_2(u,v) = 0$, therefore $a_4(u,v) = 0$ and $a_1(u) = cP'''(u)$, for some constant c. Notice that the last equation (3.2.19) do not provide further constraints on the coefficients $a_i(u,v)$. Collecting together all the informations we arrive at the following family of commuting Hamiltonians

$$
\begin{aligned}
H_f = \int [\, f + c\,\epsilon^3 \left(\left(P'''(u)f_{vv} + P''(u)f_{uvv}\right)u_x^3 + 3f_{uuv}u_x^2 v_x + \right. \\
\left. 3f_{uvv}u_x v_x^2 + f_{vvv}v_x^3\right)]\, dx,
\end{aligned} \tag{3.2.21}
$$

where c is an arbitrary constant, which can be eventually eliminated by rescaling the small parameter ϵ. The function $f = f(u,v)$ satisfies equation (1.3.6), therefore

$$f_{uuu} = P'''(u)f_{vv} + P''(u)f_{uvv}$$

and the formula (3.2.21) can be rewritten in the following way

$$
\begin{aligned}
H_f &= \int \left[f + \epsilon^3\left(f_{uuu}u_x^3 + 3f_{uuv}u_x^2 v_x + 3f_{uvv}u_x v_x^2 + f_{vvv}v_x^3\right)\right] dx \\
&= \int \left[f + \epsilon^3\left(\frac{d^3}{dx^3}f(u,v) - \frac{d}{dx}\left(f(u,v)u_{xx}\right) - \frac{d}{dx}\left(f(u,v)v_{xx}\right)\right)\right] dx \equiv 0.
\end{aligned}
$$

The triviality of any 3^{rd} order integrable Hamiltonian perturbation is proven. It would be interesting to prove a general result about the triviality of any odd order integrable perturbation. $\square$

Bibliography

[1] M.J. Ablowitz, J.F. Ladik *Nonlinear differential-difference equations and Fourier analysis.* J. Math. Phys. **17** (1976), 1011-1018.

[2] L. Bianchi, *Opere Vo. 3: Sistemi tripli ortogonali.* Edizioni Cremonese (1955).

[3] O. I. Bogoyavlensky, *Some constructions of integrable dynamical systems*, USSR Math. Izv. **31** (1988), 47-75.

[4] O. I. Bogoyavlensky, *Integrable dynamical systems associated with the KdV equation*, USSR Math. Izv. **31** (1988), 435-454.

[5] O. I. Bogoyavlensky, *Invariant foliations for the PBs of hydrodynamic type*, Physics Letters A **360** (2007) 539-544.

[6] A. Brini, G. Carlet and P. Rossi, *Integrable hierarchies and the mirror model of local* $\mathbb{CP}^1$, Physica D, (2011), http://dx.doi.org/10.1016/j.physd.2011.09.011

[7] D.B. Cooke, *Classification results and Darboux theorem for low-order Hamiltonian operators*, J. Math. Phys. 1991. **32:1**, 109-119.

[8] D.B. Cooke, *Compatibility conditions for Hamiltonian pairs*, J. Math. Phys. 1991. **32:11**, 3071-3076.

[9] L. Degiovanni, F. Magri, V. Sciacca, *On deformation of Poisson manifolds of hydrodynamic type*, Comm. Math. Phys. **253:1** (2005), 1-24.

[10] I. Ya. Dorfman, *Dirac structures and integrability of nonlinear evolution equations,* Nonlinear Sci. Theory Appl. (John Wiley & Sons, 1993) 176 pp.

[11] I. Ya. Dorfman and I. M. Gelfand, *Hamiltonian operators and algebraic structures related to them,* Funct. Anal. Appl., 1979, **13**, 248-262.

[12] B.A. Dubrovin, *Differential-geometric PBs on a lattice*, Func. Anal. and Appl., **23** (1989), 131-133.

[13] B.A. Dubrovin, *Geometry of 2D topological field theories*, Lect. Notes Math. (1996) **1620** p. 120-384.

[14] B.A. Dubrovin, *Flat pencils of metrics and Frobenius manifolds*, Proceedings Taniguchi Symposium "Integrable Systems and Algebraic Geometry", Editors M.-H. Saito, Y.Shimizu and K.Ueno, 47-72. World Scientific, 1998.

[15] B.A. Dubrovin, *On universality of critical behaviour in Hamiltonian PDEs*, Amer. Math. Soc. Transl., **224** (2008) 59-109.

[16] B.A. Dubrovin, T. Grava and C. Klein, *On universality of critical behaviour in the focusing nonlinear Schroedinger equation, elliptic umbilic catastrophe and the tritronquee solution to the Painleve-I equation*, J. Nonlinear Sci. **19:1** (2009) 57-94.

[17] B.A. Dubrovin, S.-Q. Liu and Y. Zhang *On Hamiltonian perturbations of hyperbolic systems of conservation law, I: Quasi-triviality of bi-Hamiltonian perturbations*, Comm. Pure Appl. Math. **59:4** (2006) 559-615.

[18] B.A. Dubrovin and S.P. Novikov, *Hamiltonian formalism of one-dimensional systems of hydordynamic type and the Bogolyubov-Whitham averaging method*, Soviet Math. Dokl. **270** (1983), 665-669.

[19] B.A. Dubrovin and S.P. Novikov, *On PBs of hydrodynamic type*, Soviet Math. Dokl. **279** (1984), 294-297.

[20] B.A. Dubrovin and Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, (2001) http://arxiv.org/abs/math/0108160v1

[21] L. Fadeev and L. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer, (1986), Berlin.

[22] E. V. Ferapontov, *Hamiltonian systems of hydrodynamic type and their realization on hypersurfaces of a pseudo-Euclidean space*, J. Soviet Math. 1991. **55**, 1970-1995.

[23] E. Getzler, *A Darboux theorem for Hamiltonian operators in the formal calculus of variations*, Duke Math. J. 2002. **111**, 535-560.

[24] N.I. Grinberg, *On PBs of hydrodynamic type with a degenerate metric*, Russian Math. Surveys **40:4** (1985), 231-244.

[25] H. Gümral and Y. Nutku, *Multi-Hamiltonian structure of equations of hydrodynamic type*, J. Math. Phys.**31:11** (1990), 2606-2611.

[26] K. Hikami and R. Inoue, *The Hamiltonian Structure of the Bogoyavlensky Lattice*, J. Phys. Society Japan, **68:3** (1999), 776-794.

[27] A.A. Kirillov, *Local Lie algebras*, Russian Math Survey **31:4** (1976), 55-75.

[28] Y. Kosmann-Schwarzbach, *Lie bialgebras, poisson lie groups, and dressing transformations*, Integrability of Nonlinear Systems, 2003.

[29] E. L. Ince, *Ordinary differential equations*, Dover Publications, (1958) ISBN 0486603490.

[30] A. Lichnerowicz, *Les variété de Jacobi et leurs algébres de Lie associées*, J. Math. Pures et Appl. **57** (1978), 453-488.

[31] S.-Q. Liu, C. -Z. Wu and Y. Zhang *On properties of Hamiltonian structures for a class of evolutionary PDEs*, Lett. Math. Phys. **84** (2008) 47-63.

[32] S.-Q. Liu and Y. Zhang *On Quasi-triviality and Integrability of a Class of Scalar Evolutionary PDEs*, J. Geom. Phys. **57** (2006) 101-119.

[33] S.-Q. Liu and Y. Zhang *Jacobi Structures of Evolutionary Partial Differential Equations*, Adv. Math. **227** (2011) 73-130.

[34] F. Magri, *A simple model of integrable Hamiltonian equation*, J. Math. Phys. **19** (1978), 1156-1162.

[35] A. Ya. Mal'tsev, *The averaging of Hamiltonian strutures in the discrete variant of Whitham's method*, Uspekhi Mat. Nauk, **53:1** (319) (1998), 219-220

[36] O. I. Mokhov, *Hamiltonian differential operators and contact geometry*, Functional Anal. Appl. (1987) **21:3**, 217-223.

[37] O. I. Mokhov, *Symplectic and Poisson structures on loop spaces of smooth manifolds, and integrable systems*, Russian Math. Surveys **53:3**, (1998) 515-622

[38] O. I. Mokhov, *Compatible and almost compatible pseudo-Riemannian metrics*, Functional Anal. Appl. (2001) **35:2**, 100-110.

[39] O. I. Mokhov, *Integrability of the equations for nonsingular pairs of compatible flat metrics*, Teoret. Mat. Fiz., **130:2** (2002), 233Ð250.

[40] O. I. Mokhov and E. V. Ferapontov *Nonlocal Hamiltonian operators of hydrodynamic type related to metrics of constant curvature*, Russian Math. Surveys (1990), **45:3** 218-219.

[41] K. Narita, *Soliton solution to extended Volterra equations*, J. Phys. Soc. Jpn. **51** (1982), 1682-1685.

[42] F. Neyzi, *Diagonalization and Hamiltonian structures of hyperbolic systems*, J. Math. Phys. **30:8** (1989), 1695-1698.

[43] P. Olver, *Hamiltonian perturbation theory and water waves.* In: Fluids and plasmas: geometry and dynamics (Boulder,1983), Contemp. Math. **28**, Amer. Math. Soc., Providence, R.I. (1984) 231-249.

[44] P. Olver and Y. Nutku, *Hamiltonian structures for systems of hyperbolic conservation laws*, J. Math. Phys. **29**, (1988) 1610.

[45] E. Parodi, *On classification of discrete, scalar-valued PBs*, J. Geom. Physics **62** (2012), 2059-2076.

[46] M. V. Pavlov, R. A. Sharipov and S. I. Svinolupov, *Invariant integrability criterion for equations of hydrodynamic type*, Func. Anal. App. **30:1** (1996), 18-29

[47] M. V. Pavlov and S. P. Tsarev , *Tri-Hamiltonian structures of the Egorov systems of hydrodynamic types*, Func. Anal. App. **37:1** (2003), 38- 54

[48] Yu. B. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach,* Progress in Mathematics, Vol. 219. BirkhŁuser, (2003), xxi+1070 pp.

[49] W. T. Trench, *Inversion of Toeplitz band matrices*, Math. Comp., **28**, (1974), 1089-1095.

[50] S. P. Tsarev, *The geometry of hamiltonian systems of hydrodynamic type. The generalized hodograph method*, Izv. Akad. Nauk SSSR Ser. Mat., **54:5** (1990), 1048Ð1068.

[51] V. Volterra, *Leçons sur la theorie mathematique de la lutte pour la vie*, (Paris: Gautheir-Villars).

[52] J.-P. Wang, *Recursion operator of the Narita-Itoh-Bogoyavlensky lattice*, http://arxiv.org/abs/1111.6874, (2011).

[53] G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley & Sons (1974).

[54] R. Yamilov, *Symmetries as integrability criteria for differential difference equations*, J. Phys. A: Math. Gen. **39**, (2006) R541- R623.

[55] Sheng-liang Yang, *On the k-generalized Fibonacci numbers and high-order linear recurrence relations*, J. App. Math. and Comp. **196**, (2008) 850- 857.

[56] V.E. Zakharov, *Description of the n-orthogonal curvilinear coordinate systems and Hamiltonian integrable systems of hydrodynamic type, I: Integration of the Lamé equations*, Duke Math. J. **94:1** (1998), 103-139.

[57] H. Zhang, G.-Z. Tu, W. Oevel and B. Fuchssteiner, *Symmetries, conserved quantities, and hierarchies for some lattice systems with soliton structure*, J. Math. Phys. **32** (1991), 1908-1918.