SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Astrophysics Sector

Astroparticle Physics Curriculum

Gravity beyond General Relativity: theory and phenomenology

Thesis submitted for the degree of

Doctor Philosophiæ

SUPERVISOR:
Prof. Stefano Liberati

CANDIDATE:
Vincenzo Vitagliano

September 2011

Za moju Bubamara

Preface

Despite the notorious achievements of General Relativity, Einstein's theory is under scrutiny due to the lack of a suitable scheme to quantize gravity as well as for the puzzling features it shows both at strong (early universe, black holes) and weak (Dark Energy problem) regime.

The proposal to extend the classical theory of gravity harbours the intriguing goals to cure some of these inconsistencies.

A large class of modifications of General Relativity (GR) has been widely explored in the past; in principle, the main motivation for such early efforts was to solve the problem of non-renormalizability by providing a new framework in which (thanks to higher order corrections in the gravitational action) gravity could be quantized. The analysis of the cosmological implications of such models also showed a number of peculiar features that justified further developments. The ultraviolet modifications that naturally arise at high energy in the context of quantum gravity have been taken into account for their impact on the phenomenology of the very early universe. Furthermore, it was recently argued that alternative infrared extensions of the Einstein-Hilbert (EH) action could be invoked to presumably alleviate the Dark Sector problem.

Bootstrapped by these considerations on the nature of a quantum gravity approach, the research for phenomenological imprints unveiling deviations from GR (plus a cosmological constant term), has recently been the object of much interest but has also triggered an intense debate. Signatures of these departures from GR have been searched both in the late and early universe phenomenology, as well as in the observations of compact objects in astrophysics (black holes and neutron stars).

The huge amount of data already collected from ongoing experiments have stimulated a wealth of new interesting speculations. Promising suggestions to modify the gravitational action span from the generalization of EH action (by including further non-trivial terms obtained from purely geometrical quantities) to the addition of some extra-fields (coupled more or less exotically to the geometry).

If we really are going through the dawn of precision gravity, in few years (world financial crisis permitting) we should be effectively able to restrict the selection of such viable alternative candidates to GR.

To reach this goal, once the theoretical aspects of such models have been developed, we have the pressing need to test them against the experimental results.

The efforts made in the field of experimental gravity cover a wide gamut of possibilities, exploiting a huge amount of different observations at different scale, from the cosmological realm (CMB, BAO, SNe, GRB, galaxy clusters) to smaller galactic scales (rotation curves of galaxies), from solar system experiments to laboratory based research about the unusual matter content of the universe.

To definitely link the theoretical descriptions with the data, a general framework, as much universal (namely model-independent) as possible, is needed; such parametrization should be able to give an unbiased interpretation to the collected data in terms of a set of parameters, whose analytic form depends, for each case, on the specific structure of the model in analysis. The parametrized post-Newtonian formalism and cosmography, for example, provide two different ways (respectively at small and cosmological scale) to implement this approach.

This thesis is devoted to investigate particular classes of alternative theories of gravitation, enlightening their viability from a theoretical perspective and addressing the specific observational imprints which might be able to support or falsify such models.

Contents

Chapter 1

Introduction

The sphinx of contemporary cosmology undoubtedly sets up the most hard riddle for General Relativity. The latest cosmological data sets, and the increasing number of planned satellite missions dedicated to cosmology, provide the starting point for a New Deal where GR cannot settle in without being in an uncomfortable and inadequate position. In fact, the observational evidences of the last two decades (the cosmic acceleration detected by supernova surveys, the study of the dynamics of large scale structure formation, the refined measurements of CMB radiation), raised a radically new theoretical scenario: our universe is undergoing a phase of accelerated expansion after a relatively recent and fast transition from an era of deceleration. This seems to suggest the existence of some unknown exotic energy/matter content in the universe of which, in order to explain the observed accelerated expansion of the universe, at least a dominant part must be violating the Strong Energy Condition. This means that if we think about this presently dominant component in terms of perfect fluid with canonical Equation of State $p = w\rho$, the accelerated phase could be reached only if this behaves as a so called "dark energy" fluid with $w < -1/3$.

Here it is, actually, the fertile soil for the development of several possible theoretical candidates aimed at explaining this wealth of observational data. The easiest and most conservative way to produce a dark energy component is to introduce a cosmological constant Λ acting in the gravitational equations as a perfect fluid with EoS $p_\Lambda = -\rho_\Lambda$. The root of the success of the *concordance model* (also known as Λ-Cold Dark Matter model) is then its surprisingly simplicity with respect to the rich bunch of phenomenologies it is able to account for. Nonetheless, despite such simplicity, this proposal still harbours some very critical problems concerning the value of Λ and its dynamical features.

Observational signatures of the cosmological constant are reproduced if one fixes the typical value of Λ to be of the order of H_0^2, where H_0 denotes the Hubble parameter today. Namely, the corresponding energy density is set to be $\rho_\Lambda \sim 10^{-47}\text{GeV}^4$. On the other hand, relating the cosmological constant to the vacuum energy of matter fields, it is easy to bump into a conceptual hump that apparently cannot be ridden over. In fact, computing the order of magnitude of the vacuum energy density in a quantum field theory approach where the Planck scale identifies the ultraviolet cutoff, ρ_{vacuum} turns out to be of the order $\sim 10^{74}\text{GeV}^4$, that is almost 120 orders of magnitude over the expected ρ_Λ value.

Secondly, ΛCDM, though providing a model for the late time speed-up, must be supported in any case with an inflaton field minimally coupled to gravity. Inflation is a finite period of accelerated expansion at the very early stage of the universe that is believed to occur before the radiation domination epoch. The inflationary era is needed to solve the flatness and horizon problems plaguing big-bang cosmological scenarios, and moreover must be responsible for the observed almost flat spectrum of anisotropies of CMB. Since this accelerated expansion must end somewhen reconnecting itself to the radiation era, a pure cosmological constant cannot account, with no other ingredient, for inflation. A novel component is needed in order to stop the inflationary epoch and to generate the mechanism that sows the seeds of structure formation, namely the primordial inhomogeneities. In this sense, at least a scalar field ϕ with a slowly varying potential appears to be a necessary extra ingredient for the ΛCDM model.

The last doubt about the reliability of the ΛCDM model comes from the so called coincidence problem: in the cosmic history there is just a short window of time during which the energy density of the cosmological constant has the same magnitude of energy density of matter. Assuming the existence of a cosmological constant does not naturally explain by itself the curious circumstance that such era is taking place exactly in this moment, even though equiprobability arguments are still perfectly appliable[1].

[1]For the anthropic principle it is almost a necessity that we are handling this problem in the same moment in which the order of magnitude of the baryon energy density is comparable with

So, why so clumsily small, and why exactly now? Are we just sweeping the dirt under the carpet or are we really piercing the veil of Maya?

Over the last decade, there have been many attempts to build models of effective fluids playing the role of dark energy: the taxonomy of possible explanations, going further with respect to the resurrection of Einstein's cosmological constant ([52] and reference therein), introduces scalar fields as cosmological matter fields. As we have already mentioned, a scalar field, the inflaton, driving the universe in and out from the inflationary era (and possibly responsible for the reheating mechanism after that) is already required to accompany the concordance model. A single field able to act at both early and late time to drive inflation and contemporary acceleration would be a more elegant way to solve two problems in one fell swoop. It is then the next natural step to promote the scalar field, now generally referred to as a quintessence field, so for it to act as a late time dark energy source. The main argument to reject this solution is that, one more time, the orders of magnitude of the physical quantities involved in the model (and in particular the effective mass of the scalar field m_ϕ) are too much small to be naturally justified within the Standard Model of particle physics.

Quintessence and/or inflaton models correspond essentially to modifications of General Relativity at the level of the stress-energy tensor in the Einstein field equations. However, at a more fundamental level, it is possible to think to modify directly *ab initio* the gravitational theory. Analytical mechanics condensates the full information on the dynamics of a gravitational system in the elegant form of the Einstein-Hilbert action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_M\,, \tag{1.1}$$

where S_M denotes the action for matter fields; applying a variational principle (depending on the identification of the dynamical variables, as we will see), Einstein equations arise naturally from (1.1). The invariance of the previous action under diffeomorphisms implies the conservation of the stress-energy tensor and the related field equations.

---

the energy density of a cosmological constant. Before, it was too early for the Earth to exist, later on stars like ours will be dying.

In order to generalise such results it is natural to include some extra-fields (non)-minimally coupled to the geometry (such as Brans-Dicke-like scalars, k-scalars or more complicated vector fields); or to add more complex terms in the action (such as functions of Ricci scalar, Ricci-Ricci or Riemann-Riemann as well as suitable contractions of Cartan torsion tensor).

The choice of generalising the EH action leads to a number of new mathematical and physical issues to be dealt with. Of course, the field equations are no more guaranteed to be second order differential equations like in the General Relativity case; the caravanserai of possible solutions can in principle bring as a gift some new interesting phenomenology concerning both the gravitational domains at low and high energy. However, it is also true that we should operatively work with equations more and more complex, affected by the breakdown of a well-posed formulation of the initial value problem. Many modified gravities share the common feature to contain ghosts (*i.e.*, physical excitations with negative energy eigenvalues); other models can generically exhibit a violation of the Equivalence Principle (leading to a non-conservation of the energy and hence to the introduction of an extra "fifth-force" that is responsible for a modification of the Newtonian dynamics); moreover, all the selected proposals must be checked to have the correct weak field limit and to be stable at the classical and semiclassical level (one must take care to avoid matter instabilities, gravitational instabilities for de Sitter space, and semiclassical instabilities with respect to black hole nucleation). It is usually possible, anyway, to refine the choice of functions and parameters in order to select those models that are viable according to theoretical criteria and experimental constraints.

It is worth noting that the interest with respect this class of modification goes well beyond the already crucial aspect of providing an alternative explanation for the cosmological and astrophysical enigmas. It was already pointed out in the early Sixties that one of the most huge fault of General Relativity, the lack of a straightforward renormalization scheme, could be partially circumvented if higher order curvature corrections were added to the Ricci scalar in the Einstein-Hilbert action, making the theory renormalizable at least at one loop. Moreover, some recent results show that both scalar fields and higher order terms in the curvature invariants emerge in the context of the low energy limit of string models or in dimensionally reduced effective theories obtained from higher dimensional theories, such as Kaluza-Klein.

Up to now we have just focused our attention to modification of the gravitational action to source the observed cosmological phenomena. A comment at this point is due. Even if the assumptions of spatial homogeneity and isotropy of the matter distribution inspired by the Cosmological Principle appear to give an adequate, although approximate, description of the universe on large scales, a very concrete problem with these classical depictions is that the real universe is far from homogeneity and isotropy on small scales and at late epochs. Indeed, the lumpiness of structures and the existence of huge voids are well-known observable properties under some scales. For this reason, although homogeneous and isotropic models with ordinary matter and gravity show good agreement with observations of early times, a hard clash arises in the late universe when deviations from homogeneity and isotropy become significant.

Notwithstanding this, it is still possible to define a scale large enough to recover, at least statistically, the properties of homogeneity and isotropy. The difference between exact and only statistical homogeneity and isotropy is rather subtle: the FLRW models are exactly homogeneous and isotropic, that is the space they describe has a local symmetry, all points and all directions are equivalent. Statistical homogeneity and isotropy, instead, implies that in any taken ensemble of lumpy structures anywhere in the universe, the mean quantities do not depend on its location, orientation or size, provided that it is larger than the homogeneity scale. Now, while the early universe is nearly exactly homogeneous and isotropic (in the meaning of smallness of the amplitude of the perturbations and statistical homogeneity and isotropy of the distribution of the perturbations), at late times (and with non-linear density perturbations) the universe is no longer locally like that. However, the distribution of the nonlinear regions remains statistically homogeneous and isotropic on large scales.

Due to the statistical symmetry, the average expansion rate evaluated inside each patch of universe is always the same (up to statistical fluctuations), but this does not mean that it would be equal to the expansion of a completely smooth spacetime. This is a consequence of the fact that time evolution and averaging do not commute: if we smooth a clumpy distribution and calculate the time evolution of the smooth quantities with the Einstein equation, the result is not the same as if we evolved the full clumpy distribution and took the average at the end.

The fitting problem, *i.e.*, the problem of matching a coarse-grained matter distribution with a spacetime metric obtained with an independent smoothing operator (hence, taking into account the underlined effect of clumpiness) has been firstly discussed in a systematic way in Refs. [63, 64].

Suppose to focus the attention to a certain portion of the universe. The description of such domain will be related to the total amount of details that has been retained. For example, studying the same patch of spacetime on different scales will reveal a picture similar to Fig. 1.1. Here, three different scales have been set (of course, this process is just an approximation, since it is possible to add further intermediate levels of description): Scale 1 shows the details of the lowest level, says density peaks corresponding to the localization of stars; Scale 3 is a middle step that can be referred to as the galaxy scale, where the previous point-like density distribution starts to be smoothed in a continuous profile; Scale 5 is the typical large scale picture of the universe, where most of the usual cosmological models live.

Every level is fully described by the manifold M_i equipped with the metric $g_{ab}^{(i)}$ and by the stress-energy tensor $T_{ab}^{(i)}$; these quantities contain the properties of the specific portion of universe as seen at the $i-$th scale. Since the physical system in analysis is the same, it is possible to define a smoothing operator S'_{ji} that maps the properties of the matter distribution encoded in $T_{ab}^{(i)}$ into $T_{ab}^{(j)}$.

General Relativity has been precisely tested on Scale 1, where Einstein equations hold. The goal of physical cosmology should be, at this stage, the research of a geometric prescription describing how to "jump" from the field equations defined on a certain scale to the correspondent equations on another level. Fig. 1.2 gives a schematic view of the underlying coarse-graining between the equations of Scale 1 and those of Scale 3. The maps S, S', S'', S^* determine, respectively, the smoothing procedure linking the two metrics $g_{ab}^{(1)}$ and $g_{ab}^{(3)}$, the new smoothed stress-energy tensor on the Scale 3, the new Einstein tensor, and the correspondence between points of the two different manifolds.

A comment is necessary here. Einstein equations are highly non-linear; this will provide, in general, a lack of commutativity of the smoothing operations. In other words, the products of derivatives involved in the field equations will be responsible

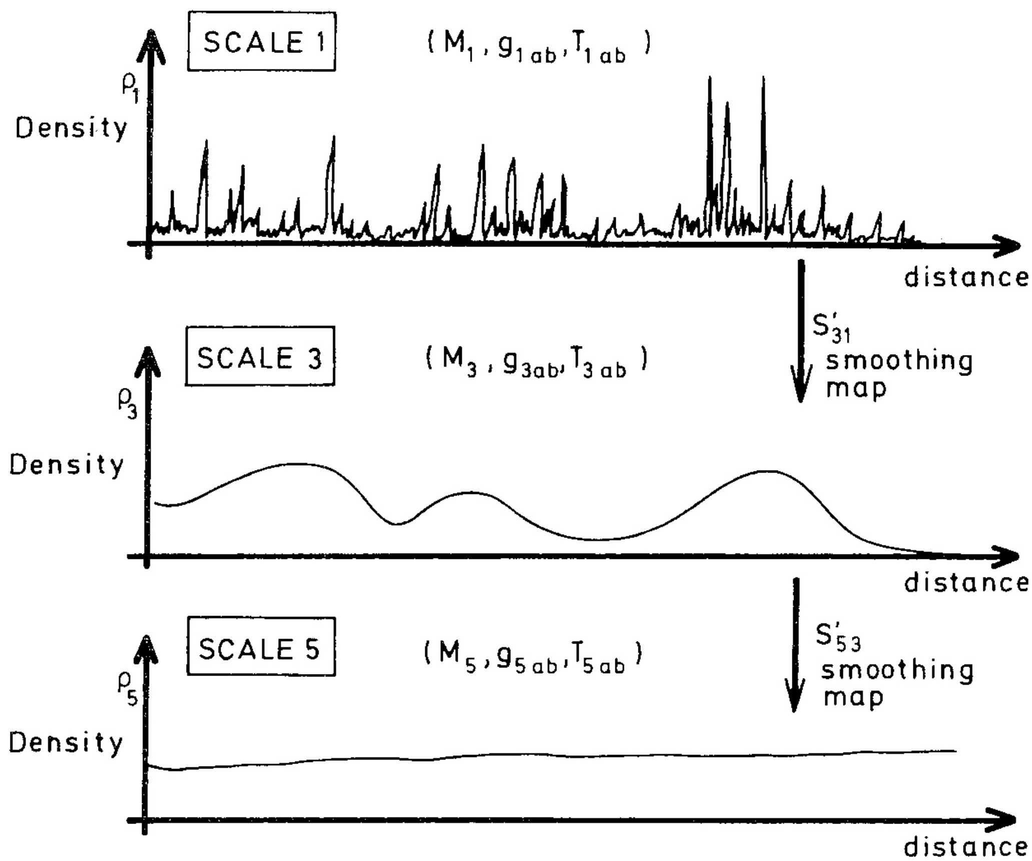

Figure 1.1: Comparison of different density profile scales corresponding to different knowledge of details. The spacetime patch is always the same. Fig. from [63].

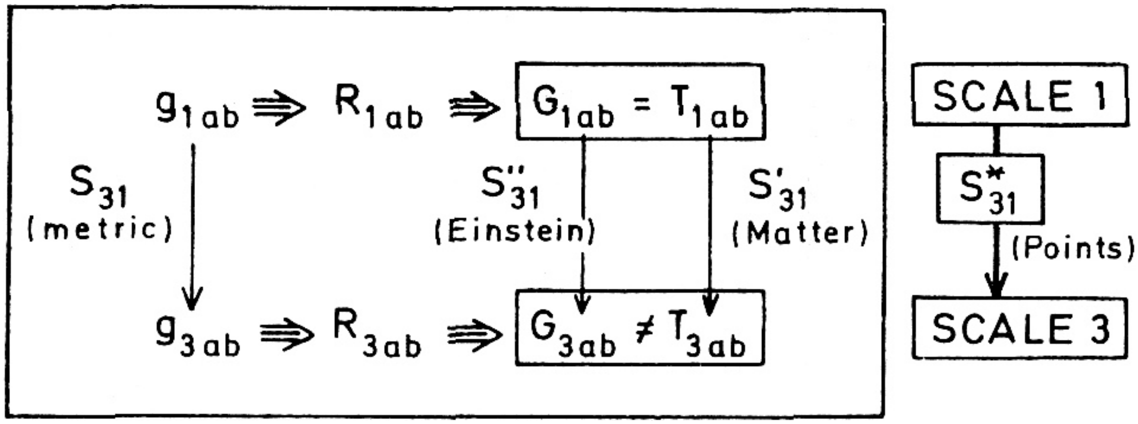

Figure 1.2: Scheme of the different coarse-graining processes applied between geometrical quantities and stress-energy tensors defined on two different levels of description of the same physical system. Due to the high non-linearity of Einstein equations, the smoothing procedures will be affected by a lack of commutativity among them. Fig. from [63].

of having $S' \neq S''$. The shape of the equations at the cosmological level is quite simple to understand: as stated before, General Relativity holds on small scales, but its non-linearity implies

$$\langle G_{\mu\nu}(g_{\mu\nu})\rangle \neq G_{\mu\nu}(\langle g_{\mu\nu}\rangle) \tag{1.2}$$

Usual cosmologies are based on

$$G_{\mu\nu}(\langle g_{\mu\nu}\rangle) = 8\pi G\langle T_{\mu\nu}\rangle + \Lambda\langle g_{\mu\nu}\rangle \tag{1.3}$$

while the correct equations should be

$$\langle G_{\mu\nu}(g_{\mu\nu})\rangle = 8\pi G\langle T_{\mu\nu}\rangle + \Lambda\langle g_{\mu\nu}\rangle \tag{1.4}$$

for some averaging procedure $\langle\cdot\rangle$ defined on a spatial domain $\mathcal{D}$. The picture can be made precise introducing physically infinitesimal volumes or coarse-graining cells, regions large enough to contain a very large sample of structures but much smaller than the scale of the cosmological fluid as a whole. Say $\mathcal{L}$ the typical length of the spatial region $\mathcal{D}$, it must lie somewhere in a range $\mathcal{L}_{hom} \leq \mathcal{L} \ll \mathcal{L}_{syst}$ with $\mathcal{L}_{hom}$, the homogeneity scale, set to be not less than $100h^{-1}$ Mpc (but the precise estimate is disputed) and $\mathcal{L}_{syst}$ being approximately the Hubble scale ($\sim$ 4Gpc). Encompassing the difference between the two different field averaging of the Einstein tensor in a new tensor, says $T^g_{\mu\nu} \equiv G_{\mu\nu}(\langle g_{\mu\nu}\rangle) - \langle G_{\mu\nu}(g_{\mu\nu})\rangle$, the outcome will be a new set of modified Einstein equations:

$$G_{\mu\nu}(\langle g_{\mu\nu}\rangle) = 8\pi G\langle T_{\mu\nu}\rangle + 8\pi G T^g_{\mu\nu} + \Lambda\langle g_{\mu\nu}\rangle \tag{1.5}$$

The implicit assumption in the usual Standard Model approach is the vanishing of $T^g_{\mu\nu}$ term at the cosmological scales while it is not necessarily true. The debate on the order of magnitude of these contributions is still quite heated. It could be even possible that a correction coming from a backreaction term should be taken into account in the usual General Relativity scheme and, *a fortiori*, in its possible modifications, to achieve the goal of a refined fitting of the observational data.

It goes without saying that, at the same time of the development of such models (as well as other schemes not pursued here), we should be able to relate them within an observational framework as much independent as possible from theoretical assumptions. Each theory of gravity previously described, in fact, should have the goal to be a theory with an enriched peculiar phenomenology, in which Einstein gravity is obviously embedded, aimed at explaining more naturally than GR current cosmological observations. Here, another comment should be made. The development of observational cosmology/astrophysics and the identification of some new (even if still controversial) standard candles (or sirens or rulers...) are needed to meaningfully isolate the signature of possible departure from GR on certain scales. In order to establish a universal frame useful in any context, it is possible to implement a model independent approach in the environment of high redshift data, taking care to expand properly the observable distances of objects far and far away than usual SNeIa. The last twenty years have seen the identification of some high redshift objects that can be used for this purpose.

- Supernovae Type Ia (SNeIa)

  A Supernova Type Ia is the result of the violent explosion of a white dwarf star. A white dwarf is a star that has ceased nuclear fusion. In the most common variety of white dwarves, the carbon-oxygen ones, the stars are still able to release huge amount of energy with further fusion reactions. In some peculiar cases, white dwarves are embedded in binary systems allowing them to gradually accrete mass by stealing matter from the binary partners. The mass of a white dwarf is anyway limited to be below the Chandrasekhar limit of about 1.38 solar masses (the maximum mass that can be supported by electron degeneracy pressure). Beyond this limit the white dwarf begins to be unstable: its core reaches the ignition temperature for the conversion of carbon and oxygen in ^{56}Ni as it approaches the limit. Such process sparks off a thermonuclear explosion with a given absolute luminosity: since the mass of the collapsing star is always close to the Chandrasekhar limit, the absolute luminosity of the Supernova is a known parameter. The stability of this value allows these explosions to be used as standard candles to measure the distance to their host galaxies because the visual magnitude of the supernovae depends primarily on the distance.

- Gamma Ray Bursts (GRBs)

  Gamma Ray Bursts are very transient, sudden flashes released as narrow beams of intense radiation at gamma-ray frequencies and lasting typically few seconds, though the afterglows of these explosions can sometimes be detected at longer wavelengths on longer scaletimes (from minutes in X-rays to weeks at radio-wavelengths). These events occur with a rate of about 0.8 burst per day, at unpredictable times and from randomly, isotropically distributed directions in the sky. Most of the observed GRBs are believed to result from a supernova event, as a rapidly rotating, high-mass star collapses to form a neutron star or most probably a black hole. The short bursts (less than 2 seconds) constitute a subclass of GRBs which seems to be originated from a different process, as the merging of the two neutron stars (or a neutron star with a black hole) of a binary system. The discovery of the afterglow emission and of the first optical counterparts, in combination with the localization of the host galaxies, led ultimately to the determination (through optical spectroscopy) of the GRBs cosmological distance scale. Since then, the redshift was estimated for many GRBs up to $z \sim 8$. This high redshift values, combined with the very high fluxes (up to more than 10^5 erg cm^{-2} s^{-1}), make GRBs the most luminous sources in the universe, with isotropic–equivalent radiated energies typically ranging from $\sim 10^{50}$ to more than $\sim 10^{54}$ erg. Even though the GRBs energetics implied by the fluences and redshifts span at least four orders of magnitudes, there are some correlations among observed quantities allowing to know the total energy or the peak luminosity emitted by a specific burst with a great accuracy. Through these relations, GRBs can be promoted to the role of "standard candles".

- Baryon Acoustic Oscillations (BAOs)

  Before recombination and decoupling the Universe is a highly ionized and overdense hot plasma of photons and baryons tightly coupled via Thomson scattering. The opposing action of the radiation pressure and the gravitational collapse results to set up acoustic oscillations in the photon fluid. Taken a single spherical density perturbation in the coupled baryon-photon plasma, it propagates outwards as an acoustic wave. Baryons decouple from radiation at recombination, giving a snapshot of the fluid at the last scattering, the baryon wave stalls and photons are left free to stream away. The destiny of photons is to become the almost completely uniform background radiation we observe nowadays, while baryons remain overdense in a shell of a typical scale s. As time goes by, the gravitational potential well at the origin starts to draw back material. The typical size of the shell formed when the baryon wave stalled is imprinted on the late time matter power spectrum as a density excess. Since baryons interact gravitationally with DM, DM also lumps mainly on the same scale. The result of that wave is reflected in the higher probability that a galaxy has to form in the high density region of the stalled baryon wave. The bump in the 2-point correlation function set at the distance s, namely the radius of the spherical wave, is the consequence of the high probability of finding two galaxies at a distance s one from the other. The acoustic scale s is set by the sound horizon at last scattering, that is the comoving distance a sound wave propagating in a photon-baryon sea covered by the decoupling epoch to the recombination. The sound horizon (and hence the related matter and baryon densities at decoupling) is extremely well constrained by the structure of the acoustic peaks in the CMB. For this reason, this scale provides a potentially excellent standard rules, assuming that the baryon energy density could be known with sufficient precision.

- Hubble (HST)

  The usual common weakness of the measurements described before, is that they are largely based on integrated cosmological parameters to determine the cosmic expansion history itself. It is possible to bypass such limitation measuring directly the rate dz/dt via the differential-age technique. This method allows to date galaxies with respect to a fiducial model rather than computing absolute ages. Measuring the age difference δt between two galaxies that formed at the same time but separated by a small redshift interval δz, one can recover the derivative dz/dt from the ratio $\delta z/\delta t$. All selected galaxies need to have similar metallicities and low star formation rates, so that the average age of their stars would far exceed the age difference δt between the two galaxies. The whole technique relies on the possibility to find a proper sample of these galaxies with the properties previously sketched. There is a strong empirical evidence for a population of galaxies harboured in clusters, whose star-formation activity ceased at high redshift, $z \sim 2-3$. Since that time, the stellar population has been passively evolving without further episodes of star formation; less than 1% of the present population has been formed at $z < 1$. Such circumstances make these galaxies good candidates to provide a sort of "standard clock" to accurately determine the rate of change of the universe as a function of redshift.

- Cosmic Microwave Background (CMB)

  Until recombination, the rapid collisions of photons with free electrons kept the radiation in thermal equilibrium with the hot matter: the radiation field then had a Planck spectrum. Matter became cooler and less dense as time passed, up to when recombination and last scattering epoch are reached. Eventually radiation began a free expansion; the photons emerging the last scattering surface undergo negligible additional scattering and absorption until they arrive to us. Notwithstanding this, the spectrum of radiation keeps the same black-body form even after the photons went out of equilibrium with matter.

> The temperature of this background radiation is extremely uniform across the universe; the only deviations from uniformity are regions of different angular sizes, with temperature varying from the mean with typical fluctuations of the order $\delta T/T_{CMB} \sim 10^{-5}$. Temperature primary anisotropies in the CMB originate as a consequence of small-amplitude inhomogeneities in the almost uniform cosmic mass distribution at the end of the inflationary era, at the very early universe. The gravitational mass potential of these mass density inhomogeneities attracts the photon-baryon plasma while the pressure of the fluid is working in the opposite direction. Under these competing forces, different regions of the universe starts to periodically fluctuate around the mean density state, alternating expansions to contractions; in the fluid, consequently, sound waves are formed at different wavelengths. When the universe emerge from the inflationary era, the acoustic oscillations are stationary and hence everywhere in phase. When decoupling of matter and photons has been achieved, the imprint of the under/overdense regions is encapsulated in the radiation field as spots on the CMB sky, with slightly different temperatures and with particular sizes; in other words, the power spectrum of CMB fluctuations must have a discrete shape with peaks and throats corresponding to those preferred scales. It turns out that measurements of the angular scales at the positions of the acoustic peaks (and their relative heights) can determine most of the parameters describing cosmological models.

The outline of this thesis is as follows. Chapter 2 gives a brief review on the possible modifications of General Relativity. The field equations for scalar-tensor theories and modified gravities are derived. In particular the metric and Palatini approaches are introduced. We also consider generalized Palatini theories of gravity, i.e., theories with a connection which is independent of the metric and an action allowed to contain higher order curvature invariants than the Ricci scalar of this connection. We show that, unlike Palatini f(R) theories, where the connection can be algebraically eliminated in favour of the metric and the matter fields, the connection of generalized Palatini theories in principle does carry dynamics and cannot be eliminated.

Chapter 3 is devoted to the study of metric-affine theories of gravity. In such an approach, the metric and the affine (not necessarily symmetric) connection are independent quantities. Furthermore, the action should include covariant derivatives of the matter fields, with the covariant derivative naturally defined using the independent connection. As a result, in metric-affine theories a direct coupling involving matter and connection is also present. The role and the dynamics of the connection in such theories is explored.

In chapter 4 the backreaction of inhomogeneities on the cosmic dynamics is studied in the context of scalar-tensor gravity. Due to terms of indefinite sign in the non-canonical effective energy tensor of the Brans-Dicke-like scalar field, extra contributions to the cosmic acceleration can arise. Brans-Dicke and metric f(R) gravity are presented as specific examples. Certain representation problems of the formalism peculiar to these theories are pointed out.

In chapter 5 we perform a cosmographic analysis using several cosmological observables such as the Hubble parameter, the luminosity distance moduli and the volume distance. These quantities are determined using the data sets already sketched: the Hubble parameter as measured from surveys of galaxies, the luminosity distance from Supernovae and Gamma Ray Bursts data, the Baryonic Acoustic Oscillations as seen in the power spectra of the distribution of galaxies, the ratio between the angular diameter distance to the last scattering surface and the sound horizon at last scattering as measured from the cosmic microwave background power spectrum. This data set allows to put constraints on the cosmographic expansion up to fifth order.

Conclusions and future perspectives are discussed in the last chapter 6.

Chapter 2

Ouverture: alternative actions, variational principles and field equations

2.1 Scalar-tensor theories of gravity

The very first proposals of alternative to Einstein gravity were strongly motivated by the attempt to incorporate Mach's principle, which is not explicitly embodied in General Relativity. Mach's principle states that the local inertial frame is determined by the action of distant objects in the universe. As a consequence, the gravitational coupling at a spacetime point is not absolute but is determined by surrounding matter and, therefore, becomes a function of the spacetime location. Brans-Dicke theory was the first alternative to Einstein GR, and the prototype of alternative theories of gravity. The variable gravitational "constant" corresponding to a scalar field coupled non-minimally to the geometry constitutes a more satisfactory implementation of Mach's principle than GR and allows the cosmological distribution of matter to affect local gravitational experiments.

To take into account the scalar field in the mediation of the gravitational interaction, the Brans–Dicke theory must rely on the action

$$S_{BD} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega_0}{\phi}(\nabla_\mu\phi\nabla^\mu\phi) - V(\phi)\right] + S_M(g_{\mu\nu},\psi), \qquad (2.1)$$

where ϕ is the scalar field, ω_0 is the dimensionless Brans–Dicke parameter and

$S_M = \int d^4x\sqrt{-g}\mathcal{L}_M$ is the action describing any form of ordinary matter but the scalar field. The fact that the matter action does not depend on the Brans–Dicke field ϕ means that the scalar field is not coupled to the matter, while the term ϕR assures the non-minimally coupling to gravity. For such reason, Brans–Dicke theory can be classified as a metric theory of gravity: matter responds only to the metric whilst the scalar field shares with the metric the only role to generate the spacetime curvature [69].

The potential $V(\phi)$ appearing in (2.1) provides a generalization of the cosmological constant thanks to which the scalar field can eventually play the role of quintessence.

It is now clear how Mach's principle is encountered in this theory: the effective gravitational constant is now related to the ratio G/ϕ that depends on the gravitational dynamics itself. The field ϕ is usually chosen to be positive in order to get a positive-defined gravitational constant.

Brans–Dicke theory has only the ω_0 parameter as new free parameter with respect to GR. The available tests of gravitational theories in the weak field limit seem to suggest a big value for the Brans–Dicke parameter, $|\omega_0| > 40\,000$, even though the theoretical perspective of the low-energy limit of string theories suggests $\omega_0 \sim \mathcal{O}(1)$ as the most natural choice. However, the very large value of ω_0 on the one hand tends to make the theory indistinguishable from General Relativity [1], whilst on the other hand implies a rather unattractive fine tuning. Consequently, Brans–Dicke theory has been quickly discarded as a viable alternative to General Relativity; nonetheless, it lays the foundations for a whole class of models involving a scalar field generalization to GR, the scalar-tensor theories of gravity. A general form for the action of such theories is

$$S_{ST} = \frac{1}{16\pi\, G}\int d^4x\sqrt{-g}\left[\phi\, R - \frac{\omega(\phi)}{\phi}(\nabla_\mu\phi\nabla^\mu\phi) - V(\phi)\right] + S_M(g_{\mu\nu},\psi), \qquad (2.2)$$

[1] The statement that $\omega_0 \to \infty$ makes the corresponding limiting Brans–Dicke theory indistinguishable from GR is true only when applied at the fundamental level of the action. Some exact solutions associated with systems with traceless stress-energy tensor, in fact, do not yield to the correct General Relativity correspondent solutions for $\omega_0 \to \infty$. This issue is related to the conformal invariance of the full Brans–Dicke action when it includes conformal matter. For further details [2]

where now $\omega(\phi)$ is some function of the scalar-field [2] ϕ. In order to derive the field equations, we must perform an independent variation with respect to the metric and the scalar field of (2.2); after few manipulations, the resulting equations will be

$$G_{\mu\nu} = \frac{8\pi\, G}{\phi} T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2}\Big(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi\Big) + \\ + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) - \frac{V}{2\phi}g_{\mu\nu}\,, \tag{2.3}$$

$$\Box\phi = \frac{1}{2\omega(\phi)+3}\left(8\pi G T - \frac{d\omega(\phi)}{d\phi}\nabla^\lambda\phi\nabla_\lambda\phi + \phi\frac{dV}{d\phi} - 2V\right), \tag{2.4}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$ is the stress-energy tensor, ∇ denotes covariant differentiation and $\Box \equiv \nabla^\mu\nabla_\mu$. Ordinary matter, moving by definitions on the geodesics of the gravitational metric, is covariantly conserved.

By setting $\omega(\phi) = \omega_0$ we can get the simpler field equations for Brans–Dicke theory with a potential $V(\phi)$. From the cosmological point of view the dependence of the parameter ω on the scalar field ϕ, and hence on its possible variation in time and space, allows for some interesting phenomenology. Having an ω parameter depending on time, for example, gives the possibility to have a small ω in the early stage of the universe, while its value can become large at late times. In this way significant deviations from General Relativity are still permitted in the early universe, whilst the present constraints on ω can be fulfilled.

In scalar-tensor theories, performing a conformal transformation $g_{\mu\nu} \to \Omega\,\tilde{g}_{\mu\nu}$ with the choice of the conformal factor $\Omega = \sqrt{G\phi}$, brings the gravitational action (2.2) into what is called the Einstein frame form (to be counterposed to the Jordan frame expressed by (2.2)). Let us write the matter field action as

$$S_M = \int d^4x\sqrt{-g}\alpha_M\mathcal{L}_M\,, \tag{2.5}$$

where α_M is the coupling constant of ordinary matter and $\mathcal{L}_M$ is the Lagrangian density of the matter fields. Defining the Einstein frame scalar field by the differential relation

$$d\tilde{\phi} = \sqrt{\frac{2\omega(\phi)+3}{16\pi G}}\frac{d\phi}{\phi}\,, \tag{2.6}$$

[2] More complex theories involving multiple scalar fields have been investigated.

and the scalar field potential as

$$U(\tilde{\phi}) = \frac{V(\phi(\tilde{\phi}))}{(G\phi)^2}\,, \tag{2.7}$$

the scalar-field action can be rewritten as

$$S_{EST} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} - U(\tilde{\phi}) + \tilde{\alpha}_M(\tilde{\phi})\mathcal{L}_M\right] \tag{2.8}$$

where $\tilde{\nabla}_\mu$ is the covariant derivative with respect to the transformed metric tensor $\tilde{g}_{\mu\nu}$; note also that now the coupling is described by $\tilde{\alpha}_M(\tilde{\phi}) = \alpha_M/(G\phi)^2$. This form of the action can be seen as the action of General Relativity with a canonical scalar field having positive-definite kinetic energy density. The most important difference is in the fact that the matter coupling "constant" can vary in space and time. Because of this coupling, the modified matter stress-energy tensor obeys to a modified conservation equation implying changes to the geodesics equation deviation and the violation of the Equivalence Principle in the Einstein frame.

2.2 The dynamics of modified actions

Actions obtained by including functions of other possible linear and quadratic contractions of the Riemann tensor $(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$ constitute a second class of Extended Theories of Gravitation; in particular one of the simplest modifications one can propose is the $f(R)$ gravity in which the Lagrangian density is chosen to be an arbitrary function of the Ricci scalar R

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,f(R) + S_M. \tag{2.9}$$

Einstein field equations and the corresponding dynamics of the system described by the action (2.9) can be derived following mainly three different approaches. The first one is the standard metric formalism in which the field equations are derived by the variation of the action with respect to the metric tensor $g_{\mu\nu}$.

The second is the Palatini formalism in which the metric tensor and the affine connection $\Gamma^\lambda{}_{\mu\nu}$ are two independent variables when we vary the action. Note that, while these two approaches give rise to different field equations for a generic non-linear $f(R)$ Lagrangian density, both variational principles lead to the same set of equations for the Einstein–Hilbert Lagrangian.

In both the metric and Palatini approach, the matter piece of the action (2.9) does not couple to the connection, namely $S_M = S_M(g_{\mu\nu}, \psi)$, with ψ encapsulating all the ordinary matter fields. In the last possible approach, the metric-affine one, we will leave aside this assumption, and the generic matter action will be rewritten in the most generic way, $S_M = S_M(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi)$.

2.2.1 Metric approach

Variation with respect to the metric of the action (2.9) yields to the following field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right] f'(R) = \kappa\, T_{\mu\nu}, \tag{2.10}$$

where ∇_μ is the covariant derivative associated with the *Levi-Civita connection* of the metric.

Here, some surface terms have been discarded. It is interesting to note that in the EH case the procedure one should follow in order to cancel the surface term is rather straightforward, since such terms are already related to the variation of a total divergence. For what concerns modified actions, instead, this circumstance is not satisfied. It is anyway possible to circumvent the problem by noting that, due to the presence of higher order derivatives of the metric, there will be further degrees of the freedom to fix on the boundary. However, the field equations (2.10) would be unaffected by the fixing chosen: from a purely classical perspective the field equations are well posed.

The field equations (2.10) are fourth order partial differential equations in the metric. Considering the Einstein-Hilbert Lagrangian density, one has $f(R) = R$ and $f'(R) = 1$; hence the Einstein equations reduce to the General Relativity ones. Taking into account the trace of the field equations yields

$$3\Box f' + f'(R)R - 2f(R) = \kappa\, T. \tag{2.11}$$

It is easy to realize that in the General Relativity limit $R = -\kappa T$, that is the Ricci scalar is completely determined by the matter content. In modified gravity, instead, the not-vanishing of the Dalembertian term gives rise to a new propagating degree of freedom, the scalaron $\Phi = f'(R)$, whose dynamics is described by equation (2.11).

A simple calculation shows that the left hand side of (2.10) is solenoidal, namely its covariant divergence vanishes; this result reveals the generalization of the Bianchi identity and implies that the stress-energy tensor $T_{\mu\nu}$ is covariantly conserved. This can also be recognized *a priori* as a consequence of the minimal coupling between matter and metric and of the usual arguments based on the invariance of the action under diffeomorphism.

A last remark is the following. All the curvature terms modulo the Einstein tensor may be moved on the right hand side of field equations to appear as a further effective stress-energy tensor of a "curvature fluid". Einstein equations (2.10) assume then the form

$$G_{\mu\nu} = \frac{1}{f'(R)}\left[\kappa T_{\mu\nu} + g_{\mu\nu}\frac{[f(R) - Rf'(R)]}{2} + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\right] = \\ = \frac{\kappa}{f'(R)}\left(T_{\mu\nu} + T^{(\text{curv})}_{\mu\nu}\right). \tag{2.12}$$

Note that the effective energy density of this effective energy-momentum tensor is not defined to be positive, and that energy conditions are not satisfied *a priori*.

2.3 The dynamics of generalized Palatini Theories of Gravity

Einstein's equations can be derived by varying the Einstein–Hilbert action with respect to the metric. They can also be derived by what is formally the same action, by assuming that the connection is independent of the metric and performing independent variations with respect to the metric and the connection. This is called a Palatini variation and it can be found in some textbooks, see for example Ref. [1]. Note that in the Palatini variation the independent connection is assumed to not enter the matter action.

Even though both standard metric and Palatini variations of (what is formally) the Einstein–Hilbert action lead to equivalent systems of field equations, this is not the case for more general actions. A typical example of actions that have been widely studied with both variational principles are $f(R)$ actions, see Refs. [3, 4, 5, 6, 7] for reviews. Indeed there is by now a long literature on $f(R)$ theories with an independent, symmetric connection which does not couple to the matter, dubbed Palatini $f(R)$ theories of gravity [33, 8, 9].

Even though these theories are not equivalent to the theory corresponding to the same action obtained with simple metric variation, they are nevertheless still metric theories according to the Thorne–Will definition [3] [10]. In fact, the independent connection in Palatini $f(R)$ gravity does not actually carry any dynamics. It is really an auxiliary field that can be eliminated in favour of the metric and the matter fields [11, 12, 13]. This result has recently been generalized to $f(R)$ theories with non-symmetric connections, *i.e.* theories that allow for torsion [14].

The fact that in Palatini $f(R)$ gravity the independent connection results to be non-dynamical can be viewed as a blessing at first: no extra degrees of freedom are introduced with respect to General Relativity, so one needs not worry about pathologies usually associated with such degrees of freedom (ghost modes, instabilities etc.) or conflicts with current experimental bounds on their existence. However, one soon realizes that having a theory with second order dynamics and still different from General Relativity actually requires a drastic departure from the latter. Indeed, a number of viability issues plague generic models of Palatini $f(R)$ gravity, and all of these shortcomings have their origin at the peculiar differential structure of the theory [18].

Palatini $f(R)$ gravity models with infrared corrections with respect to General Relativity have been shown to be in conflict with the standard model of particle physics [15, 19] and to violate solar system tests as their post-Newtonian metric has an algebraic dependence on the matter fields [16, 20]. Singularities have been shown to arise on the surface of well known spherically symmetric matter configuration [18], showing the theory at best incomplete and providing a very strong viability criterion. This criterion is almost independent of the functional form of the Lagrangian, the only exception being Lagrangians with corrections which become important only in

[3] Quoting Thorne and Will [10], a metric theory is a theory that satisfies the metric postulates, *i.e.* a theory for which "(i) gravity is associated, at least in part, to a symmetric tensor, the *metric* and (ii) the response of matter and fields to gravity is described by $\nabla_\mu T^{\mu\nu} = 0$, where ∇_μ is the divergence with respect to the metric and $T^{\mu\nu}$ is the stress-energy tensor for all matter and nongravitational fields."

the far ultraviolet (as in this case the singularities manifest at scales where non-classical effects take over) [21] .

On the other hand, $f(R)$ actions are a special class and there is no reason for one to restrict to those. In fact from an effective field theory point of view such a restriction can be considered a severe fine tuning. It is, therefore, interesting to consider more general actions. Then a question naturally arises: will these more general actions share the property of Palatini $f(R)$ of actually having a non-dynamical connection? Or will at least some of the degrees of freedom hiding in the connection be excited? This is what we would like to address here. The answer is ultimately related to whether such more general theories would suffer by the same shortcomings as Palatini $f(R)$ gravity, which, as mentioned, can be traced back to the presence of the non-dynamical connection.

Generalized Palatini theories of gravity have been considered to some extent in the literature. In Ref. [22] the cosmology of Lagrangians of the form $f(R^{(\mu\nu)}R_{(\mu\nu)})$ was studied (here and thereafter round parentheses stand for symmetrization while curved brackets indicate antisymmetrization). In Ref. [23] the focus was on theories of the form $R+f(R^{(\mu\nu)}R_{(\mu\nu)})$. Finally, in Refs. [24, 25, 26] Lagrangians of the more general form $f(R, R^{\mu\nu}R_{\mu\nu})$ were studied. In fact, in Ref. [24] the very question that we are posing here was considered and it was claimed that the connection can indeed be eliminated. We argue that this claim in not correct, at least unless one imposes extra *a priori* restrictions on the connection or the action.

In next section we illustrate briefly how the connection can be algebraically eliminated in the case of $f(R)$ theories. This will serve as a brief review of the results in the literature. In section 2.4 we move on to consider more general actions and we argue that the connection cannot be eliminated for generic actions. We discuss some special cases that constitute exceptions and we show that they do not include the action considered in Refs. [24, 25, 26], contrary to what was claimed there. We also give an easy but characteristic example of a generalized Palatini theory with extra degrees of freedom with respect to General Relativity.

Before going further it is worth emphasizing that throughout this section we are considering theories in which the independent connection does not enter the matter action, *i.e.* it does not couple to the matter fields. One can clearly question if this is the most sensible choice, and in fact it would be very reasonable to allow for the independent connection to define the covariant derivative and, therefore, couple to (at least) some matter fields. $f(R)$ theories of this type, dubbed metric-affine $f(R)$ theories of gravity, have been introduced in [27]. We will consider them and their generalizations as the subject of the next chapter.

2.3.1 Palatini $f(R)$ actions as an example

We start by briefly reviewing how the independent connection can be eliminated in Palatini $f(R)$ gravity. For simplicity we restrict ourselves to a symmetric connection, even though the results can be generalized to a non symmetric one. We refer the reader to Ref. [14] for details.

Consider the action

$$S = \frac{1}{16\pi l_p^2} \int dx^4 \sqrt{-g} f(\mathcal{R}) + S_M(\psi, g_{\mu\nu}), \tag{2.13}$$

where g is the determinant of the metric $g_{\mu\nu}$, $\mathcal{R}_{\mu\nu}$ is the Ricci tensor of the independent connection, $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$, S_M is the matter action, ψ collectively denotes the matter fields (note that the connection does not enter the matter action) and l_p has dimensions of a length. Varying the action independently with respect to the metric and the connection gives the following set of field equations, after some manipulations:

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.14}$$

$$\nabla_\lambda \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\right) = 0, \tag{2.15}$$

where ∇_μ is the covariant derivative defined with the independent connection, a prime denotes differentiation with respect to the argument,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{2.16}$$

and $\kappa = 8\pi\, l_p^2$. The right-hand side of eq. (2.15) vanishes thanks to our assumption that the matter action is independent of the connection. Details of the variation can be found in section 4.1 of Ref. [27].

Eq. (2.15) can be solved for the connection to give

$$\Gamma^\lambda{}_{\mu\nu} \;=\; \left\{{}^{\lambda}{}_{\mu\nu}\right\} + \frac{1}{2f'}\Big[2\partial_{(\mu} f' \delta^\lambda_{\nu)} - g^{\lambda\sigma} g_{\mu\nu} \partial_\sigma f'\Big]. \tag{2.17}$$

The trace of eq. (2.14) is

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T, \tag{2.18}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. This is actually an algebraic equation in $\mathcal{R}$ which can generically be solved to give $\mathcal{R}$ as a function of T. $f \propto \mathcal{R}^2$ is an exception, which leads to a conformally invariant theory [9, 17]. This exception, as well as choices of f for which eq. (2.18) has no root will not be considered further (in this case there are also no solutions of the full field equations [9]). Expressing $\mathcal{R}$ as a function of T via eq. (2.18) and using the result to eliminate the $\mathcal{R}$ dependence in the right-hand side of eq. (2.17) expresses the independent connection algebraically in terms of the metric and the matter fields. One can then proceed and eliminate the connection from the field equations. See, for example, Ref. [14] for more details and the final form of the field equations.

This establishes that the connection does not carry any dynamics for $f(\mathcal{R})$ action.

2.3.2 Equivalence between $f(R)$ and Brans–Dicke theory

Metric and Palatini formulations of $f(R)$ gravity are dynamically equivalent to specific scalar-tensor theories, where the derivative of the $f(R)$ function assumes the role of an effective scalar field degree of freedom. From the point of view of classical mechanics, two theories can be considered dynamically equivalent if it is possible to recover the field equations of one of the two from the other by a suitable redefinition of gravitational and matter fields; the same statement can be made at the level of the actions (for an extended discussion see [3]). Here we want to explicitly show that metric and Palatini $f(R)$ are different representations of Brans–Dicke theory with respectively $\omega_0 = 0$ and $-3/2$ and specific potentials.

Let us start with the $f(R)$ action (2.9) analysed in the metric version. Introducing the scalar field $\phi \equiv R$, then it is pretty trivial to realize that the action (2.9) can be easily rewritten in the following form

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big[\chi(\phi)R - V(\phi)\Big] + S_M(g_{\mu\nu}, \psi)\,, \tag{2.19}$$

where $\chi = f'(\phi)$ and $V(\phi) = \phi f'(\phi) - f(\phi)$; we implicitly suppose to have $f''(R) \neq 0$. On the other side, varying the action (2.19) with respect to the scalar field ϕ yields

$$R\frac{d\chi}{d\phi} - \frac{dV}{d\phi} = (R - \phi)f''(R) = 0\,, \tag{2.20}$$

implying $\phi = R$ when $f''(R) \neq 0$. The action introduced (2.19) has the form of a Brans–Dicke theory

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big[\chi R - \frac{\omega_0}{\chi}\nabla_\mu\chi\nabla^\mu\chi - U(\chi)\Big] + S_M(g_{\mu\nu},\psi)\,, \tag{2.21}$$

with Brans–Dicke field χ, Brans–Dicke parameter $\omega_0 = 0$ and potential $U(\chi) = V(\phi(\chi))$. A Brans–Dicke theory having $\omega_0 = 0$ was initially proposed by O'Hanlon to generate a Yukawa correction to the Newtonian potential in the weak field limit.

The field equations obtained by varying the action (2.19) with respect to the metric and to the scalar field are

$$G_{\mu\nu} = \frac{\kappa}{\chi}T_{\mu\nu} - \frac{1}{2\chi}g_{\mu\nu}U(\chi) + \frac{1}{\chi}\left(\nabla_\mu\nabla_\nu\chi - g_{\mu\nu}\Box\chi\right), \tag{2.22}$$

$$R = \frac{dU(\chi)}{d\chi}. \tag{2.23}$$

Such equations can in principle be derived even directly from the Einstein equations of metric approach, using the same field redefinition. The Ricci scalar in (2.23) can be replaced by taking the trace of (2.22), getting the equation of motion of the scalar field ϕ for a given matter distribution

$$3\Box\chi + 2U(\chi) - \chi\frac{dU}{d\chi} = \kappa T\,. \tag{2.24}$$

Palatini $f(R)$ gravity can also be recast into a special Brans–Dicke theory with a scalar field potential. The Palatini action is equivalent to

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big[f(\chi) + f'(\chi)(\mathcal{R} - \chi)\Big] + S_M(g_{\mu\nu},\psi)\,. \tag{2.25}$$

Varying this action with respect to χ, it is straightforward to verify that $\chi = \mathcal{R}$. If now we use a scalar field $\phi \equiv f'(\chi)$ and the fact that the Ricci scalar of the Palatini connection can be seen as the scalar curvature of a new metric $h_{\mu\nu}$ conformally related to $g_{\mu\nu}$ by $h_{\mu\nu} = f(\mathcal{R})g_{\mu\nu}$, then the action (2.24), discarding a boundary term, can be rewritten as a Brans–Dicke theory with $\omega_0 = -3/2$ and a potential given by $V(\phi) = \phi\chi(\phi) - f(\chi(\phi))$

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big[\phi R + \frac{3}{2\phi}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\Big] + S_M(g_{\mu\nu},\psi)\,. \tag{2.26}$$

where R and ∇_μ are now respectively the Ricci scalar of the metric $g_{\mu\nu}$ and the covariant derivative related to its Christoffel symbols.

The fact that Palatini $f(R)$ gravity has been shown to be dynamically equivalent to Brans–Dicke theory with Brans–Dicke parameter $\omega_0 = -3/2$ [15, 16, 17] irrespectively of how general the connection is allowed to be [14], gives also an explanation about why in a generic Palatini $f(R)$ the independent connection plays just the role of an auxiliary field. The Brans–Dicke theory with $\omega_0 = -3/2$, in fact, is a particular theory within the Brans–Dicke class in which the scalar does not carry any dynamics and can be algebraically eliminated in favour of the matter fields.

2.4 More general actions within Palatini approach

We would now like to explore the dynamics of more general Palatini theories of gravity. Our aim is to illustrate that for actions which contain generic higher order curvature invariants the *independent connection* cannot be algebraically eliminated (differently from the restricted $f(\mathcal{R})$ case). However, let us first point out that, as mentioned in section 2.3, in Ref. [24] the following class of actions was considered

$$S = \frac{1}{16\pi l_p^2}\int dx^4\sqrt{-g}f(\mathcal{R}, \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}) + S_M(\psi, g_{\mu\nu}), \tag{2.27}$$

and there it was claimed that the connection can indeed be eliminated in such theories. This claim would obviously contradict our previous statement: even though action (2.27) is restricted, it is still much more general than those in the $f(\mathcal{R})$ class. In what follows we shall show that this contradiction is due to an implicit and unjustified assumption made in Ref. [24] regarding the symmetries of the Ricci tensor in Palatini theories.

We start by recalling that the Ricci tensor is given in term of the connection as

$$\mathcal{R}_{\mu\nu} = \partial_\lambda\Gamma^\lambda{}_{\mu\nu} - \partial_\nu\Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\lambda}. \tag{2.28}$$

We can define the non-metricity of the connection as

$$Q_{\lambda\alpha\beta} = -\nabla_\lambda g_{\alpha\beta}. \tag{2.29}$$

For what comes next we will restrict ourselves to a symmetric connection for simplicity. One could easily generalize our approach to include non-symmetric connections. However, it is obvious that if our claim is true for a symmetric connection it will continue to be true for a non-symmetric one, since the eventual antisymmetric part could just introduce a further dynamical degree of freedom.

A symmetric connection can be written as

$$\Gamma^{\rho}{}_{\alpha\beta} = \left\{{}^{\rho}{}_{\alpha\beta}\right\} + \frac{1}{2} g^{\rho\lambda} \left[Q_{\alpha\beta\lambda} + Q_{\beta\alpha\lambda} - Q_{\lambda\alpha\beta} \right], \tag{2.30}$$

where $\left\{{}^{\rho}{}_{\alpha\beta}\right\}$ denotes the Levi-Civita connection of $g_{\mu\nu}$. The non-metricity vector is defined as

$$Q_\mu = \frac{1}{4} Q_{\mu\nu}{}^{\nu}. \tag{2.31}$$

Then, for a symmetric connection, the antisymmetric part of the Ricci tensor is given by

$$\mathcal{R}_{[\alpha\beta]} = -\partial_{[\beta}\Gamma^{\lambda}{}_{\alpha]\lambda} = -2\nabla_{[\beta} Q_{\alpha]} \tag{2.32}$$

It should then be clear that $\mathcal{R}_{\mu\nu}$ is not necessarily symmetric even for a symmetric connection. Further restrictions on the non-metricity would have to be imposed to achieve that, which would restrict the connection.

Consider now the action

$$S = \frac{1}{16\pi l_p^2} \int dx^4 \sqrt{-g} \left[\mathcal{R} + l_p^2 \mathcal{R}_{\mu\nu} (a\mathcal{R}^{\mu\nu} + b\mathcal{R}^{\nu\mu}) \right] \tag{2.33}$$

Clearly this is not the most general action one could think of, but it is general enough for our purposes and simple enough to make the calculations tractable. Note that as long as $\mathcal{R}_{[\mu\nu]} \neq 0$ the last two terms are not equal. In fact (2.33) can be written as

$$S = \frac{1}{16\pi l_p^2} \int dx^4 \sqrt{-g} \Big[\mathcal{R} + c_1\, l_p^2\, \mathcal{R}_{(\mu\nu)} \mathcal{R}^{(\mu\nu)} + c_2\, l_p^2\, \mathcal{R}_{[\mu\nu]} \mathcal{R}^{[\mu\nu]} \Big], \tag{2.34}$$

where $c_1 = a + b$ and $c_2 = a - b$. Note also that for $b = 0$, or $c_1 = c_2$, action (2.33) reduces to the simplest model within the class given in action (2.27), *i.e.*, to the case where f is linear in both invariants.

We now vary the action independently with respect to the metric and the connection. The variation with respect to the metric yields

$$\begin{aligned} \mathcal{R}_{(\mu\nu)} - \frac{1}{2}\mathcal{R} g_{\mu\nu} + 2c_1\, l_p^2\, \mathcal{R}_{(\alpha\mu)} \mathcal{R}_{(\beta\nu)} g^{\alpha\beta} \\ +2c_2\, l_p^2\, \mathcal{R}_{[\alpha\mu]} \mathcal{R}_{[\beta\nu]} g^{\alpha\beta} - \frac{1}{2} c_1\, l_p^2\, \mathcal{R}_{(\alpha\beta)} \mathcal{R}^{(\alpha\beta)} g_{\mu\nu} \\ -\frac{1}{2} c_2\, l_p^2\, \mathcal{R}_{[\alpha\beta]} \mathcal{R}^{[\alpha\beta]} g_{\mu\nu} = \kappa T_{\mu\nu}. \end{aligned} \tag{2.35}$$

Interestingly, the trace of the previous equation leads to $\mathcal{R} = -\kappa T$. The variation with respect to the connection yields

$$\begin{aligned} -\nabla_\lambda \left[\sqrt{-g}\left(g^{\mu\nu} + 2\,c_1\,l_p^2\mathcal{R}^{(\mu\nu)}\right)\right] + \nabla_\sigma \left(\sqrt{-g}g^{\sigma(\mu}\right)\delta^{\nu)}{}_\lambda \\ +c_1\,l_p^2\,\nabla_\sigma\left[\sqrt{-g}\mathcal{R}^{(\mu\sigma)}\delta^\nu{}_\lambda + \sqrt{-g}\mathcal{R}^{(\nu\sigma)}\delta^\mu{}_\lambda\right] \\ +c_2\,l_p^2\,\nabla_\sigma\left[\sqrt{-g}\mathcal{R}^{[\mu\sigma]}\delta^\nu{}_\lambda + \sqrt{-g}\mathcal{R}^{[\nu\sigma]}\delta^\mu{}_\lambda\right] = 0. \end{aligned} \tag{2.36}$$

Eq. (2.36) can be simplified by taking its trace and using it to replace the terms containing divergences. This leads to

$$\nabla_\lambda \left[\sqrt{-g}\left(g^{\mu\nu} + 2\,c_1\,l_p^2\mathcal{R}^{(\mu\nu)}\right)\right] + \frac{2}{3}c_2\,l_p^2\,\nabla_\sigma\left[\sqrt{-g}\mathcal{R}^{[\mu\sigma]}\delta^\nu{}_\lambda + \sqrt{-g}\mathcal{R}^{[\nu\sigma]}\delta^\mu{}_\lambda\right] = 0. \tag{2.37}$$

Eqs. (2.35) and (2.37) should reduce to eqs. (3) and (4) of Ref. [24] for a linear function f when we set $b = 0$ or $c_1 = c_2 = a$. This is not the case however. The two sets of equations actually differ by terms including $\mathcal{R}_{[\mu\nu]}$. The fact that $\mathcal{R}_{[\mu\nu]}$ does not generically vanish for an independent connection, even a symmetric one as shown above, seems to have been overlooked in Ref. [24] and subsequently in Refs. [25, 26]. Hence, these terms were ignored there.[4]

If one would indeed make the assumption that $\mathcal{R}_{[\mu\nu]} = 0$ then, for any values of a and b the system of equations would reduce to

$$\mathcal{R}_{(\mu\nu)} - \frac{1}{2}\left(\mathcal{R} + c_1\,l_p^2\,\mathcal{R}_{(\alpha\beta)}\mathcal{R}^{(\alpha\beta)}\right)g_{\mu\nu} + 2c_1\,l_p^2\,\mathcal{R}_{(\alpha\mu)}\mathcal{R}_{(\beta\nu)}g^{\alpha\beta} = \kappa T_{\mu\nu}, \tag{2.38}$$

$$\nabla_\lambda \left[\sqrt{-g}\left(g^{\mu\nu} + 2\,c_1\,l_p^2\,\mathcal{R}^{(\mu\nu)}\right)\right] = 0. \tag{2.39}$$

The assumption that $\mathcal{R}_{[\mu\nu]} = 0$ is equivalent to the requirement

$$\nabla_{[\nu}Q_{\mu]} = 0, \tag{2.40}$$

which essentially would mean that Q_μ is the gradient of a scalar. Interestingly, one gets the exact same equations by assuming that $a = b$ or $c_2 = 0$ (which is different than the case considered in Ref. [24, 25, 26]), without imposing any constraints on $\mathcal{R}_{[\mu\nu]}$ and consequently on the non-metricity. This choice of parameters correspond to an action which depends only on $\mathcal{R}_{(\mu\nu)}$.

Let us concentrate on these two cases for the moment, for which one can indeed apply the arguments of Ref. [24]. Notice that eq. (2.38) is actually an algebraic

[4] In Refs. [22, 23], on the other hand, $\mathcal{R}_{\mu\nu}$ was explicitly assumed to be symmetric *a priori*.

equation in $\mathcal{R}_{(\mu\nu)}$. That is to say, one could solve algebraically for the components of $\mathcal{R}_{(\mu\nu)}$, in terms of the components of $T_{\mu\nu}$ and $g_{\mu\nu}$ (even though it might not be possible to express the result in tensorial form). This could also be seen by thinking of eq. (2.38) as a matrix equation. Hence, $\mathcal{R}_{(\mu\nu)}$ in eq. (2.39) can be thought of as depending only on the matter fields and the metric, not on the connection.

Now, eq. (2.39) can be written as

$$\nabla_\lambda \left[\sqrt{-h}h^{\mu\nu}\right] = 0, \tag{2.41}$$

where $h_{\mu\nu}$ is a symmetric metric implicitly defined via the relationship

$$\sqrt{-h}h^{\mu\nu} = \sqrt{-g}\left(g^{\mu\nu} + 2\,c_1\,l_p^2\,\mathcal{R}^{(\mu\nu)}\right). \tag{2.42}$$

Eq. (2.41) implies that the independent connection is the Levi-Civita connection of $h_{\mu\nu}$. Since $h_{\mu\nu}$ can be expressed in terms of the $g_{\mu\nu}$ and $T_{\mu\nu}$ one can then use the steps listed here in order to completely eliminate the independent connection $\Gamma^\lambda{}_{\mu\nu}$.

As mentioned above, what was just described works for the specific choice of parameters $a = b$ or $c_2 = 0$ or if ones imposes *a priori* that $\mathcal{R}_{[\mu\nu]} = 0$, which corresponds to eq. (2.40). In the latter case, one would think that eq. (2.40) might impose an extra condition. However, it is trivially satisfied when eq. (2.39), or better yet eq. (2.41) is satisfied. That is because a *sufficient* condition for a symmetric connection to lead to a symmetric Ricci tensor is for it to be the Levi-Civita connection of *some* metric. This can be easily shown by replacing the Levi-Civita expression for a connection in eq. (2.32).

Even though we derived the results presented above using an action linear in Ricci squared invariants, there is no reason to believe that they are not more general than that. In fact, one should be able to eliminate a symmetric connection, in favour of the matter field and the metric, whenever only invariants constructed with the symmetric part of the Ricci tensor are considered in the action, *e.g.* for Lagrangians of the form $f(\mathcal{R}, \mathcal{R}^{(\mu\nu)}\mathcal{R}_{(\mu\nu)})$. However, this is not the case for actions of the form $f(\mathcal{R}, \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu})$ as claimed in Ref. [24].

Let us see that in more detail. We return to more generic choices of parameters. Since the antisymmetric part of the Ricci enters the field equations now, the situation changes radically. Eq. (2.35) cannot be used to algebraically determine the full Ricci tensor, even at the component level, in term of the matter fields and the metric.

Recall that if the Ricci is not assumed to be symmetric it has 16 independent components and eq. (2.35) corresponds to just 10 component equations. This is enough to argue that the presence of derivatives of $\mathcal{R}_{\mu\nu}$ in eq. (2.36) will make this equation a dynamical one in the independent connection. Therefore, one will not be able to eliminate the connection algebraically anymore.

As a simple but characteristic example let us consider the specific choice $a = -b$, or $c_1 = 0$, in which case the equations reduce to

$$\begin{aligned} \mathcal{R}_{(\mu\nu)} - \frac{1}{2}\Big(\mathcal{R} &+ c_2\, l_p^2\, \mathcal{R}_{[\alpha\beta]}\mathcal{R}^{[\alpha\beta]}\Big) g_{\mu\nu} \\ &+ 2c_2\, l_p^2\, \mathcal{R}_{[\alpha\mu]}\mathcal{R}_{[\beta\nu]} g^{\alpha\beta} = \kappa T_{\mu\nu}, \end{aligned} \tag{2.43}$$

$$\begin{aligned} \nabla_\lambda \left[\sqrt{-g} g^{\mu\nu}\right] &+ \frac{2}{3} c_2\, l_p^2\, \nabla_\sigma \left[\sqrt{-g}\mathcal{R}^{[\mu\sigma]}\right] \delta^\nu{}_\lambda \\ &+ \frac{2}{3} c_2\, l_p^2\, \nabla_\sigma \left[\sqrt{-g}\mathcal{R}^{[\nu\sigma]}\right] \delta^\mu{}_\lambda = 0. \end{aligned} \tag{2.44}$$

Contracting eq. (2.44) with the metric yields

$$g_{\mu\nu}\nabla_\lambda \left[\sqrt{-g} g^{\mu\nu}\right] = -\frac{4}{3} c_2\, l_p^2\, g_{\lambda\nu}\nabla_\mu \left[\sqrt{-g}\mathcal{R}^{[\nu\mu]}\right]. \tag{2.45}$$

On the other hand, one can straightforwardly show that

$$\nabla_\mu \left[\sqrt{-g}\mathcal{R}^{[\nu\mu]}\right] = \sqrt{-g}\bar{\nabla}_\mu \left[\mathcal{R}^{[\nu\mu]}\right], \tag{2.46}$$

where $\bar{\nabla}_\mu$ denote the covariant derivative defined with the Levi-Civita connection of $g_{\mu\nu}$. Using eq. (2.46) and eqs. (2.29) and (2.31), one can rewrite eq. (2.45) as

$$c_2\, l_p^2\, \bar{\nabla}_\mu \left[\mathcal{R}^{[\nu\mu]}\right] - 3Q^\nu = 0, \tag{2.47}$$

while eq. (2.44) takes the simple form

$$Q_{\lambda\mu\nu} = 2g_{\mu\nu}Q_\lambda - 2g_{\lambda\mu}Q_\nu - 2g_{\lambda\nu}Q_\mu. \tag{2.48}$$

Thus, the non-metricity can now be fully determined in terms of Q_ν. The independent connection is then given by

$$\Gamma^\lambda{}_{\mu\nu} = \left\{{}^{\lambda}_{\mu\nu}\right\} - 3g_{\mu\nu}Q^\lambda + \delta^\lambda{}_\mu Q_\nu + \delta^\lambda{}_\nu Q_\mu, \tag{2.49}$$

and $\mathcal{R}_{(\mu\nu)}$ can be expressed in terms of the Ricci tensor of $g_{\mu\nu}$, $R_{\mu\nu}$ and Q_ν as

$$\mathcal{R}_{(\mu\nu)} = R_{\mu\nu} - 3g_{\mu\nu}\bar{\nabla}_\sigma Q^\sigma - 6Q_\mu Q_\nu. \tag{2.50}$$

Taking a divergence of eq. (2.47) on can show that

$$\bar{\nabla}_\nu Q^\nu = 0 \tag{2.51}$$

Thus, eqs. (2.43) and (2.44) are equivalent to the more familiar system

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = -s\,\kappa\, F_{\alpha\mu}F_{\beta\nu}g^{\alpha\beta} + s\frac{1}{4}\kappa F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}$$
$$+\kappa\, m^2\, A_\mu A_\nu - \frac{1}{2}\kappa\, m^2\, A^\sigma A_\sigma g_{\mu\nu} + \kappa T_{\mu\nu}, \tag{2.52}$$
$$\bar{\nabla}_\mu F^{\mu\nu} + s\, m^2 A^\nu = 0. \tag{2.53}$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, $A_\mu = \sqrt{|c_2|/(4\pi)}\,Q_\mu$ and $m^2 = 3/(|c_2|l_p^2)$ and $s = sign(c_2)$. One can use these redefinitions and eqs. (2.50) and (2.32) to rewrite action (2.34) when $c_1 = 0$ as

$$S = \frac{1}{16\pi l_p^2}\int dx^4\sqrt{-g}R + S_F + S_M(\psi, g_{\mu\nu}), \tag{2.54}$$

where

$$S_F = \frac{1}{2}\int dx^4\sqrt{-g}\Big[s\,\frac{1}{2}\,F_{\mu\nu}F^{\mu\nu} - m^2\,A^\mu A_\mu\Big]. \tag{2.55}$$

One can easily verify that eqs. (2.52) and (2.53) can be straightforwardly derived by varying action (2.54) with respect to $g_{\mu\nu}$ and A_μ respectively. Action (2.54), and consequently also action (2.34) with $c_1 = 0$, correspond to General Relativity with matter and a massive vector field, also know as the Einstein–Proca field. This specific example was actually considered by Buchdahl in Ref. [29], where action (2.33) with $a = -b$ was proposed as a "geometrization" of the Einstein–Proca field.

One should have $s = -1$, *i.e.* c_2 negative, for the vector field to not be a ghost and the theory to be quantum mechanically stable. This choice leads also to classical stability [our signature here is $(-+++)$]. In any case, irrespective of its physical relevance, this theory serves as a simple example of how higher order curvature invariants introduce extra degrees of freedom. It also demonstrates through the restriction in the sign of c_2 how the dynamics of these extra degrees of freedom can potentially lead to pathologies.

As an aside, note that the connection given in eq. (2.49) is a typical example of a symmetric connection for which $\mathcal{R}_{[\mu\nu]} \neq 0$: A_ν satisfies eq. (2.53) which is well known to admit non-constant solutions. Because of the relation between A_ν and Q_ν and the relation between $\mathcal{R}_{[\mu\nu]}$ and Q_ν given in eq. (2.32), one can easily infer that the theory admits solutions with $\mathcal{R}_{[\mu\nu]} \neq 0$.

2.4.1 Summary

We have considered generalized Palatini theories of gravity, *i.e.*, theories with a connection which is independent of the metric and an action allowed to contain higher order curvature invariants than the Ricci scalar of this connection. We have shown that, unlike Palatini $f(R)$ theories, this connection does carry dynamics and cannot be algebraically eliminated. We gave as a simple, known, example the specific choice of action that is dynamically equivalent to the Einstein–Proca system (Einstein gravity plus a massive vector field). We also identified some specific actions which constitute exceptions, and for which the independent connection can indeed be algebraically eliminated.

Our results disagree with those of Refs. [24, 25, 26]. The reason appears to be that in Refs. [24, 25, 26] the fact that the Ricci tensor of a symmetric connection is not necessarily symmetric unless extra constraint are imposed has been overlooked or it has been implicitly assumed that the Ricci tensor is indeed symmetric due to some restriction on the connection.

We have not considered here theories where the independent connection is coupled to the matter as this will be the subject of next chapter.

Chapter 3

The dynamics of metric-affine theories of gravity

3.1 The dynamics of metric-affine gravity

As we have seen in the previous chapter, in General Relativity the spacetime geometry is fully described by the metric. That is to say, the metric does not only define distances, which is its primary role, but also defines parallel transport, as it is used to construct the Levi-Civita connection. However, in principle this does not have to be the case. The metric and the connection can be independent quantities. In this case one would need field equations that would determine the dynamics of both the metric and the connection.

How can one construct such a theory? As stated before, the Palatini variation, an independent variation with respect to the metric and the connection of what is formally the Einstein–Hilbert action, is considered as an alternative way to arrive to Einstein's equations. Indeed the variation with respect to the connection leads to a non-dynamical equation fixing the latter to be equal to the Levi-Civita connection of the metric, and under this condition the field equations for the metric become Einstein's equations. However, there is a very crucial implicit assumption: that the matter action does not depend on the connection. This is equivalent to assume that any covariant derivative eventually contained in the Lagrangian density of matter fields, is defined with the Levi-Civita connection of the metric instead of the independent connection. Then, the independent connection does not really carry the geometrical meaning described previously, see Refs. [12, 31] for a discussion.

If instead one allows the connection to enter the matter action, the resulting theory will not generically be General Relativity [32]. Additionally, one can easily argue that within a metric-affine setting the Einstein–Hilbert form of the action is not necessarily well motivated anyway: under the assumption that the connection is the Christoffel symbol of the metric, the Einstein–Hilbert action is indeed the unique diffeomorphism invariant action which leads to second order field equations (modulo topological terms and total divergences). However, this is not the case if the connection is allowed to be independent and it is not assumed to be symmetric: in this case there are other invariants one should in principle include in the action, even with the same dimensions as the Ricci scalar.

The situation gets more complicated once one decides to consider the role of higher order terms. Again, such actions have been studied mostly under the simplifying (but geometrically unappealing) assumption that the connection does not enter the matter action.

What we would like to understand here is what happens when one jumps from the Palatini approach, to the more general and better motivated metric-affine approach, where the independent connection is allowed to enter the matter action, define the covariant derivative, and, therefore, retain its geometrical significance. In particular, we would like to understand under which circumstances this connection becomes an auxiliary field, which can be algebraically eliminated, and when it actually does carry dynamics. Note that there are well known examples, such as Einstein–Cartan theory [36] (which is a metric-affine theory with the additional constraint that the connection is metric, but not symmetric), where the independent connection can be eliminated algebraically, leading to General Relativity with extra matter interactions. In this specific case, this is a four-fermion interaction. See also Ref. [37] for an example of a more general action with the same property. What happens for more general theories, however, and especially how the dynamics of the connection will be affected by considering higher order terms in the action, has not been systematically understood.

In order to address this issue we follow an approach motivated by effective field theory. We will consider the metric-affine action as an effective action, possibly arising from some more fundamental theory at some appropriate limit. We will then employ power counting in order to construct the most general action order by order. This will allow us to arrive at model independent statements and avoid considering fine-tuned actions, which can lead to misleading results.

3.2 General setup for metric-affine theories

We start by clarifying our notation and conventions. The covariant derivative of the connection $\Gamma^{\rho}{}_{\mu\nu}$ acting on a tensor is defined as

$$\nabla_{\mu} A^{\nu}{}_{\sigma} = \partial_{\mu} A^{\nu}{}_{\sigma} + \Gamma^{\nu}{}_{\alpha\mu} A^{\alpha}{}_{\sigma} - \Gamma^{\alpha}{}_{\sigma\mu} A^{\nu}{}_{\alpha} \,. \tag{3.1}$$

It is important to stress that the position of indices must be taken very carefully into account, since the connection are not assumed to be symmetric. The antisymmetric part of the connection is commonly referred to as the Cartan torsion tensor

$$S_{\mu\nu}{}^{\lambda} \equiv \Gamma^{\lambda}{}_{[\mu\nu]} \,. \tag{3.2}$$

The failure of the connection to covariantly conserve the metric is measured by the non-metricity tensor

$$Q_{\lambda\mu\nu} \equiv -\nabla_{\lambda} g_{\mu\nu} \,. \tag{3.3}$$

Torsion and non-metricity have a pretty clear geometrical meaning. Consider two geodesics ℓ_1 and ℓ_2 (see Fig. 3.1) with unit tangent vector $t_1^{\mu} = \frac{dx^{\mu}}{ds_1}$ and $t_2^{\mu} = \frac{dx^{\mu}}{ds_2}$ respectively. Both the geodesics start at the point A. Let's parallel transport t_1^{μ} along ℓ_2 using the connection $\Gamma^{\lambda}{}_{\mu\nu}$; it will end in a final vector $\tilde{t}_1^{\mu}$ at point C, at the distance $ds_2 = d\ell_2$ from A; the new vector $\tilde{t}_1^{\mu}$ defines the direction of the geodesics $\tilde{\ell}_1$ finally arriving at the point D_2 at a distance $d\tilde{s}_1 = d\ell_1$ from C. Using the same procedure on the other side, we find that in a curved spacetime this four-sided figure is not necessarily a closed loop; the tensor encapsulating the information of the non-closure of the figure is the torsion tensor. The difference between the positions of two points D_1 and D_2 is in fact up to second order corrections

$$x^{\lambda}(D_2) - x^{\lambda}(D_1) = \left[\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}\right] dx^{\mu} dx^{\nu} = S_{\mu\nu}{}^{\lambda} t_1^{\mu} t_2^{\nu} d\ell_1 d\ell_2 \,. \tag{3.4}$$

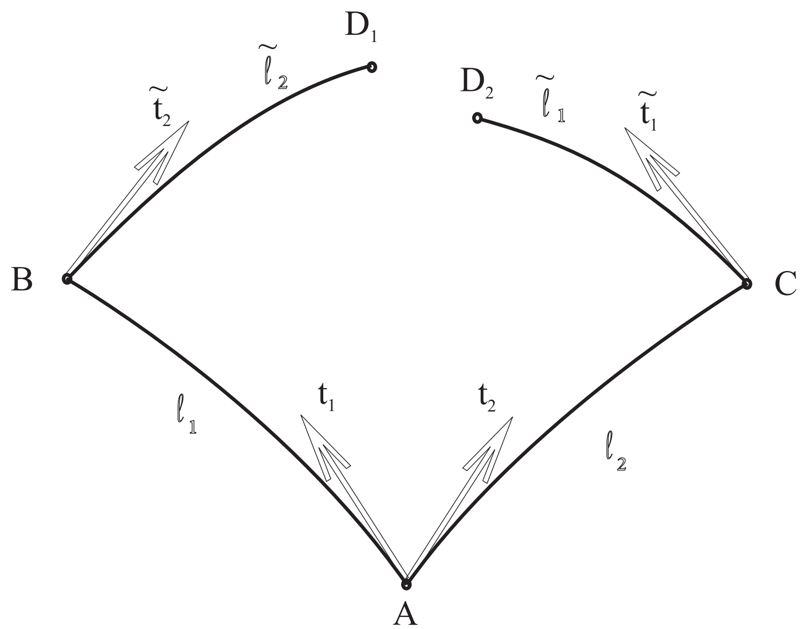

Figure 3.1: Geometrical meaning of torsion tensor. From [41]

A not vanishing non-metricity tensor, instead, is symptomatic of a lack of preservation of inner products (and in particular lengths and standard angles between two vectors) during parallel transport; in fact, if we suppose to have two vectors u^μ and v^ν, a generic vector field w^μ and $Q_{\lambda\mu\nu} \neq 0$, then performing the parallel transport along the curve with tangent vector w^μ we obtain

$$D_{\mathbf{w}}\left(g_{\mu\nu}u^\mu v^\nu\right) = \left(D_{\mathbf{w}}g_{\mu\nu}\right)u^\mu v^\nu = u^\mu v^\nu \nabla_\xi g_{\mu\nu} dw^\xi = -u^\mu v^\nu Q_{\xi\mu\nu} dw^\xi \,. \tag{3.5}$$

Using the connection one can construct the Riemann tensor

$$\mathcal{R}^\mu{}_{\nu\sigma\lambda} = -\partial_\lambda \Gamma^\mu{}_{\nu\sigma} + \partial_\sigma \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\alpha\sigma}\Gamma^\alpha{}_{\nu\lambda} - \Gamma^\mu{}_{\alpha\lambda}\Gamma^\alpha{}_{\nu\sigma} \,. \tag{3.6}$$

which has no dependence on the metric. Notice that the Riemann tensor here has only one obvious symmetry: it is antisymmetric in the last two indices. All other symmetries one might be accustomed to from General Relativity are not present for an arbitrary connection [42]. Without any use of the metric we can also define as $\mathcal{R}_{\mu\nu}$ the Ricci tensor built with the connection $\Gamma^\rho{}_{\mu\nu}$

$$\mathcal{R}_{\mu\nu} \equiv \mathcal{R}^\lambda{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\lambda} \,. \tag{3.7}$$

$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ is the corresponding Ricci scalar.

Note that there is an intrinsic ambiguity in the definition of the Ricci tensor in metric-affine theories as the limited symmetries of the Riemann tensor allow now for an alternative definition as

$$\hat{\mathcal{R}}_{\mu\nu} \equiv \mathcal{R}^{\sigma}{}_{\sigma\mu\nu} = -\partial_\nu \Gamma^{\sigma}{}_{\sigma\mu} + \partial_\mu \Gamma^{\sigma}{}_{\sigma\nu} \,. \tag{3.8}$$

This tensor is called the homothetic curvature. For a symmetric connection it is equal to the antisymmetric part of $\mathcal{R}_{\mu\nu}$ and, therefore, it need not be separately considered. This is not the case for a non-symmetric connection. Note however, that the homothetic curvature is fully antisymmetric and as such it leads to a vanishing scalar when contracted with the metric[1].

As already mentioned at the beginning of this chapter, the key characteristic of metric-affine gravity is that the affine connection $\Gamma^{\rho}{}_{\mu\nu}$ is not assumed to have any *a priori* relation with the metric. On the other hand, it is assumed to define parallel transport and the covariant derivative of matter fields, so it inevitably enters the matter action, see Ref. [12] for a discussion. That is, in metric-affine gravity couplings between the connection and the matter fields are allowed. This is the main difference from (generalized) Palatini theories of gravity, as mentioned earlier. The action will, therefore be of the following general form

$$S = S_G + S_M = \int d^4x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}, \Gamma^{\rho}{}_{\mu\nu}) + \mathcal{L}_M \left(g_{\mu\nu}, \Gamma^{\rho}{}_{\mu\nu}, \psi \right) \right] \,, \tag{3.9}$$

where g is the determinant of the metric $g_{\mu\nu}$, ψ collectively denotes the matter fields, and S_M is the matter action. We have written the dependence of $\mathcal{L}_M$ on the various fields explicitly to avoid confusion here, but we will suppress it from now on and just use S_M instead, in order to lighten the notation. Clearly, specific choices of matter fields will not couple to the connection, such as scalar fields or gauge fields. Scalar fields have no spin and, for any affine space, their covariant derivatives are always reduced to partial derivatives. Therefore, neither of these fields will introduce

[1]See Ref. [31] for a more detailed discussion about the ambiguities in the definition of the Ricci tensor. Note also that, unlike the usual Ricci tensor, the homothetic curvature tensor has a direct physical interpretation: it measures the change of the length of a vector when it is transported along a closed loop. When the homothetic curvature vanishes, the connection is volume preserving, *i.e.* volumes do not change during parallel transport.

torsion or extra non-metricity. Things are a little bit more subtle for gauge fields; we can observe for example that the electromagnetic field tensor can be expressed in terms of the standard exterior derivative that is already a covariant object on any differential manifold and respects gauge invariance. Preserving the gauge invariance property does not allow us to apply the minimal coupling procedure in this case. Since the Maxwell field is not minimally coupled to geometry, photons do not feel the presence of torsion in a metric-affine theory. Gauge fields arising from local invariance with respect to a non-Abelian symmetry group, also share with Maxwell fields the characteristic immunity from the minimal coupling prescription, since they can be minimally coupled to torsion only breaking the gauge symmetry. See Ref. [31, 36] for a detailed discussion on such matters.

One now needs to specify the exact form of the Lagrangian $\mathcal{L}_G$. In Ref. [32] an action linear in $\mathcal{R}$ was consider and in Ref. [31] the most general $f(\mathcal{R})$ family was studied extensively. Instead of an *ad hoc* choice inspired by some similarity with the Einstein–Hilbert action and its generalizations, we would like to follow here an effective field theory approach so to consider the most general action possible at each order. To construct this action, we should carry on a power counting analysis which will reveal the whole set of appropriate terms order by order. We set $c = 1$ and we can choose the engineering dimensions

$$[dx] = [dt] = [l] \tag{3.10}$$

where l is a place holder symbols with dimension of a length. Then we have

$$[g_{\mu\nu}] = [1]\,, \quad [\sqrt{-g}dx^4] = [l^4]\,, \quad [\Gamma^{\lambda}{}_{\mu\nu}] = [l^{-1}]\,, \quad [\mathcal{R}_{\mu\nu}] = [l^{-2}]\,. \tag{3.11}$$

Now consider as a simple example the action

$$S_G = \frac{1}{l_p^2}\int dx^3 dt\sqrt{-g}\mathcal{R}\,. \tag{3.12}$$

Requiring that this action is dimensionless implies that the coupling constant l_p must have dimensions of a length which can then be naturally associated to the Planck length. What we mean by order of the gravitational theory is also clear now: we mean the highest order in l_p^{-1} powers appearing in the Lagrangian (which, since one cannot choose the metric and the connection to be dimensionless at the same time, does not correspond to the order in its derivatives).

3.3 Second order action

Clearly the action written above is not the most general one we could write in metric-affine gravity. It is just an example inspired by the analogy with standard GR and the Einstein–Hilbert action. To begin with, we could include a cosmological constant term, which is of lower order. But such a term would not play any important role in our arguments so we will omit it for simplicity. What other terms can we write at the second order? Under the assumption that the connection is torsionless and metric compatible (Levi-Civita), there exist no other term which respects diffeomorphism invariance, as it is well known. But, in the more general metric-affine setting we are considering here, there is at least two more tensors one could imagine using in order to construct invariants:

- The aforementioned "second Ricci" tensor $\hat{\mathcal{R}}_{\mu\nu}$. However, this tensor has dimensions $[l^{-2}]$ and is antisymmetric, so there is no invariant quantity one can construct out of it at second order;

- the Cartan torsion tensor of eq. (3.2), which has the same dimensions as $\Gamma^{\lambda}{}_{\mu\nu}$. Therefore, terms with one derivative of $S_{\mu\nu}{}^{\lambda}$ or terms quadratic in $S_{\mu\nu}{}^{\lambda}$ will be of the same order as $\mathcal{R}$.

Due to the symmetries of $S_{\mu\nu}{}^{\lambda}$ there is only a single term with a derivative we can write

$$g^{\mu\nu}\nabla_{\mu}S_{\nu\sigma}{}^{\sigma}\,. \tag{3.13}$$

For the same reason, there are just three terms quadratic in $S_{\mu\nu}{}^{\lambda}$ one can write

$$g^{\mu\nu}S_{\mu\lambda}{}^{\lambda}S_{\nu\sigma}{}^{\sigma}\,,\qquad g^{\mu\nu}S_{\mu\lambda}{}^{\sigma}S_{\nu\sigma}{}^{\lambda}\,,\qquad g^{\mu\alpha}g^{\nu\beta}g_{\lambda\gamma}S_{\mu\nu}{}^{\lambda}S_{\alpha\beta}{}^{\gamma}\,. \tag{3.14}$$

Note that the term in eq. (3.13) has been considered by Papapetrou and Stachel in [39].

A subtle point is the following. The term in eq. (3.13) is not a total divergence as ∇_{μ} is not defined with the Levi-Civita connection of the metric. On the other hand, one can think to decompose the connection as

$$\Gamma^{\lambda}{}_{\mu\nu}=\left\{{}^{\lambda}{}_{\mu\nu}\right\}+C^{\lambda}{}_{\mu\nu}\,, \tag{3.15}$$

i.e. in its Levi-Civita part and the rest. Now, using this decomposition we can split the covariant derivative in (3.13) in a metric compatible part, which will lead to a total divergence, and the rest, which will lead to terms consisting of contractions between $C^{\lambda}{}_{\mu\nu}$ and the Cartan torsion tensor. Since the non-metricity is not zero, these terms are different than the ones already considered above in eq. (3.14). Thus the term in eq. (3.13) is non-trivial.

This brings us to another puzzle though: $C^{\lambda}{}_{\mu\nu}$ is a tensor, so why not consider terms constructed with it as well? Actually, $C^{\lambda}{}_{\mu\nu}$ can always be decomposed in terms of torsion $S_{\mu\nu}{}^{\lambda}$ and non-metricity $Q_{\lambda\mu\nu}$, so the question then reduces to whether we should also consider terms constructed with $Q_{\lambda\mu\nu}$ or not. From a power counting/field theory perspective nothing prevents us from doing so, and these would indeed be terms of the same order. However, from this perspective we should also consider, for instance, the Ricci tensor of the metric $R_{\mu\nu}$. In fact, $Q_{\lambda\mu\nu}$ and $R_{\mu\nu}$ share a common characteristic which is crucial for our discussion: They cannot be expressed without using derivatives of the metric (even if instead of (3.3) one tries to define $Q_{\lambda\mu\nu}$ using the connection, then still the Levi-Civita connection would be needed as well). Therefore, the puzzle reduces to whether or not we should be considering invariants constructed with derivatives of the metric.

Clearly, field theoretic considerations cannot give an answer to this question. Such terms should be considered unless we are willing to invoke some principle excluding them, along the line of minimal coupling in General Relativity. Such a principle has been discussed in Ref. [31]. In simple terms it would be the requirement that the metric be used only for raising and lowering indices. We choose to follow this prescription here, as it seems sensible from a geometrical perspective (the purpose of the metric being to measure distances) and it significantly reduces the number of terms one can consider.

Another way to reduce the number of terms without invoking a minimal coupling principle would be to require the connection to be metric compatible. This would force $Q_{\lambda\mu\nu}$ to vanish, without necessarily implying torsion has to vanish as well. We would then remain with exactly the same terms written above. However, in this case the term in eq. (3.13) would indeed differ from the first term in eq. (3.14) only by a total surface term and one would be able to omit it.

Let us then consider the most general second-order action as we have just constructed it in our setting

$$S=\frac{1}{16\,\pi\,l_p^2}\int dx^4\sqrt{-g}\left(g^{\mu\nu}\mathcal{R}_{\mu\nu}+a_1 g^{\mu\nu}\nabla_\mu S_{\nu\sigma}{}^{\sigma}+a_2 g^{\mu\nu}S_{\mu\lambda}{}^{\lambda}S_{\nu\sigma}{}^{\sigma}\right.$$
$$\left.+a_3 g^{\mu\nu}S_{\mu\lambda}{}^{\sigma}S_{\nu\sigma}{}^{\lambda}+a_4 g^{\mu\alpha}g^{\nu\beta}g_{\lambda\gamma}S_{\mu\nu}{}^{\lambda}S_{\alpha\beta}{}^{\gamma}\right)+S_M\,, \tag{3.16}$$

where the a_i's represent the various coupling constant. Varying independently with respect to metric and connection yields

$$\mathcal{R}_{(\mu\nu)}-\frac{1}{2}g_{\mu\nu}\mathcal{R}+a_1\left\{\nabla_{(\mu}S_{\nu)}-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha S_\beta\right\}$$
$$+a_2\left\{-\frac{1}{2}g_{\mu\nu}S_\alpha S^\alpha+S_\mu S_\nu\right\}+a_3\left\{-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}S_{\alpha\lambda}{}^{\sigma}S_{\beta\sigma}{}^{\lambda}+S_{\mu\lambda}{}^{\sigma}S_{\nu\sigma}{}^{\lambda}\right\}$$
$$+a_4\left\{-\frac{1}{2}g_{\mu\nu}S_{\rho\sigma\lambda}S^{\rho\sigma\lambda}+2S_{\alpha\mu}{}^{\lambda}S^{\alpha}{}_{\nu\lambda}-S_{\rho\sigma\mu}S^{\rho\sigma}{}_{\nu}\right\}=\kappa T_{\mu\nu}\,, \tag{3.17}$$

$$\frac{1}{\sqrt{-g}}\left[-\nabla_\lambda\left(\sqrt{-g}g^{\mu\nu}\right)+\nabla_\sigma\left(\sqrt{-g}g^{\sigma\mu}\right)\delta^{\nu}{}_{\lambda}-a_1\nabla_\alpha\left(\sqrt{-g}g^{\alpha[\mu}\right)\delta^{\nu]}_{\lambda}\right]$$
$$+\left(2-a_1\right)g^{\mu\nu}S_\lambda-2S^{(\mu}\delta^{\nu)}{}_{\lambda}+2(a_1+a_2-1)S^{[\mu}\delta^{\nu]}{}_{\lambda}+2a_3 g^{\alpha[\mu}S_{\alpha\lambda}{}^{\nu]}$$
$$+2a_4 g^{\alpha[\mu}g^{\nu]\beta}g_{\lambda\gamma}S_{\alpha\beta}{}^{\gamma}=\kappa\Delta_\lambda{}^{\mu\nu}\,. \tag{3.18}$$

where $S_\alpha\equiv S_{\alpha\beta}{}^{\beta}$, $\kappa=8\,\pi\,l_p^2$ and

$$T_{\mu\nu}\equiv-\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\,,\qquad\qquad \Delta_\lambda{}^{\mu\nu}\equiv-\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta\Gamma^{\lambda}{}_{\mu\nu}}\,. \tag{3.19}$$

$\Delta_\lambda{}^{\mu\nu}$ is known as the *hypermomentum* and, as already identified in [40], it encapsulates all the information related to the spin angular momentum of matter, the intrinsic part of dilation current and the shear current. $T_{\mu\nu}$ on the other hand is sometimes referred to as the stress-energy tensor, in analogy with General Relativity. However, it should be stressed that this terminology might be misleading within the metric-affine framework as this tensor does not have the properties usually associated with the stress-energy tensor is General Relativity. For instance, it is not necessarily divergence free, it does not reduce to the special relativistic stress energy tensor at a suitable limit and of course it does describe only some properties of matter, given the existence of $\Delta_\lambda{}^{\mu\nu}$ as well. In fact, it is best if it is just considered as nothing more than a short hand notation for the functional derivative of the matter action with respect to the metric.

Our present aim is to check whether it is possible to fully eliminate the connection from the field equations. Let us consider the contraction of λ index in (3.18) respectively with μ and ν

$$\frac{3}{2}\frac{a_1}{\sqrt{-g}}\nabla_\mu\left(\sqrt{-g}g^{\mu\nu}\right) = \kappa\Delta_\mu{}^{\mu\nu} + (4a_1+3a_2+a_3+2a_4)S^\nu\,, \tag{3.20}$$

$$\frac{6-3a_1}{2\sqrt{-g}}\nabla_\nu\left(\sqrt{-g}g^{\mu\nu}\right) = \kappa\Delta_\nu{}^{\mu\nu} - (2a_1+3a_2+a_3+2a_4-6)S^\mu\,. \tag{3.21}$$

Combining these two equations gives S_ν and the trace $\nabla_\mu\left(\sqrt{-g}g^{\mu\nu}\right)$ as functions of the hypermomentum

$$S^\nu = \frac{\kappa}{(1-a_1)a_1+3a_2+a_3+2a_4}\left[(a_1-1)\Delta^\nu - \tilde{\Delta}^\nu\right]\,, \tag{3.22}$$

$$\frac{1}{\sqrt{-g}}\nabla_\mu\left(\sqrt{-g}g^{\mu\nu}\right) \;=\; \frac{2}{3}\left\{\kappa\Delta^\nu + \frac{\kappa(a_1+3)\left[(a_1-1)\Delta^\nu - \tilde{\Delta}^\nu\right]}{(1-a_1)a_1+3a_2+a_3+2a_4}\right\}\,, \tag{3.23}$$

where we defined the two quantities $\Delta^\mu \equiv \Delta_\alpha{}^{(\alpha\mu)}$ and $\tilde{\Delta}^\mu \equiv \Delta_\alpha{}^{[\alpha\mu]}$. Eq. (3.23) can be inserted in (3.18) to eliminate the second and the third term in order to get

$$\begin{aligned}
&\frac{1}{\sqrt{-g}}\left[-\nabla_\lambda\left(\sqrt{-g}g^{\mu\nu}\right)\right] + 2a_3 g^{\alpha[\mu}S_{\alpha\lambda}{}^{\nu]} + 2a_4 g^{\alpha[\mu}g^{\nu]\beta}g_{\lambda\gamma}S_{\alpha\beta}{}^\gamma = \\
&= \kappa\Delta_\lambda{}^{\mu\nu} - \frac{2}{3}\left[\kappa\Delta^\mu + (a_1+3)S^\mu\right]\delta^\nu{}_\lambda + \frac{2}{3}a_1\left\{\kappa\Delta^{[\mu} + (a_1+3)S^{[\mu}\right\}\delta^{\nu]}{}_\lambda - \\
&\quad - (2-a_1)\,g^{\mu\nu}S_\lambda + 2S^{(\mu}\delta^{\nu)}{}_\lambda - 2(a_1+a_2-1)S^{[\mu}\delta^{\nu]}{}_\lambda\,,
\end{aligned} \tag{3.24}$$

while we will refrain from replacing S_ν for compactness.

Using the identities

$$\nabla_\mu\sqrt{-g} = \partial_\mu\sqrt{-g} - \Gamma^\alpha{}_{\alpha\mu}\sqrt{-g} \tag{3.25}$$

and

$$g_{\mu\nu}\partial_\lambda(\sqrt{-g}g^{\mu\nu}) = 2\sqrt{-g}\,\partial_\lambda\ln\sqrt{-g} \tag{3.26}$$

we can write the trace of eq. (3.24) with the metric in the μ and ν indices as

$$\frac{\partial_\lambda\sqrt{-g}}{\sqrt{-g}} = -\frac{1}{2}\kappa g_{\mu\nu}\Delta_\lambda{}^{\mu\nu} + \frac{1}{3}\kappa g_{\mu\lambda}\Delta^\mu + \left(4-\frac{5}{3}a_1\right)S_\lambda + \Gamma^\alpha{}_{\alpha\lambda}\,. \tag{3.27}$$

Eliminating the density related terms and suitably lowering the indices, eq. (3.24) can eventually take the form

$$\begin{aligned}
&\partial_\lambda g_{\sigma\rho} - \Gamma^{\mu}{}_{\rho\lambda} g_{\mu\sigma} - \Gamma^{\nu}{}_{\sigma\lambda} g_{\nu\rho} + 2a_3 S_{[\sigma|\lambda|\rho]} + 2a_4 S_{\sigma\rho\lambda} = \\
&= g_{\sigma\rho}\left(4 - \frac{5}{3}a_1\right) S_\lambda + \frac{1}{3}\kappa g_{\sigma\rho}\Delta_\lambda - \frac{1}{2}\kappa g_{\sigma\rho}\Delta_\lambda{}^{\mu}{}_{\mu} + \\
&\quad +\kappa\Delta_{\lambda\sigma\rho} - (2-a_1)\, g_{\sigma\rho} S_\lambda + 2S_{(\sigma}g_{\rho)\lambda} - 2(a_1+a_2-1)S_{[\sigma}g_{\rho]\lambda} + \\
&\quad +\frac{2}{3}(a_1+3)\left(a_1 S_{[\sigma}g_{\rho]\lambda} - S_\sigma g_{\rho\lambda}\right) + \frac{2}{3}\kappa\left(a_1\Delta_{[\sigma}g_{\rho]\lambda} - \Delta_\sigma g_{\rho\lambda}\right)\,. \qquad (3.28)
\end{aligned}$$

We can now split this last expression in its antisymmetric and symmetric part with respect to the two indices σ and ρ

$$\begin{aligned}
2a_3 S_{[\sigma|\lambda|\rho]} + 2a_4 S_{\sigma\rho\lambda} &= \Theta_{\lambda\sigma\rho}\,, \\
\partial_\lambda g_{\sigma\rho} - \Gamma^{\mu}{}_{\rho\lambda} g_{\mu\sigma} - \Gamma^{\nu}{}_{\sigma\lambda} g_{\nu\rho} &= \kappa\Delta_{\lambda(\sigma\rho)} - \frac{2}{3}\left[\kappa\Delta_{(\sigma} + a_1 S_{(\sigma}\right] g_{\rho)\lambda} \\
-g_{\sigma\rho}\Big[\left(\frac{2}{3}a_1 - 2\right) S_\lambda &+ \frac{1}{2}\kappa\Delta_\lambda{}^{\mu}{}_{\mu} - \frac{1}{3}\kappa\Delta_\lambda\Big]\,, \qquad (3.29)
\end{aligned}$$

where we have introduced the short hand notation

$$\Theta_{\lambda\sigma\rho} \equiv \kappa\Delta_{\lambda[\sigma\rho]} + \frac{2}{3}(a_1-1)\left[\kappa\Delta_{[\sigma} + \left(a_1 - \frac{3a_2}{a_1-1}\right) S_{[\sigma}\right] g_{\rho]\lambda}\,; \qquad (3.30)$$

it is worth noting than, by virtue of (3.22), $\Theta_{\lambda\sigma\rho}$ is just a function of the matter fields encoded in the hypermomenta. Adding suitable permutations of (3.29) and (3.29) we obtain

$$S_{\rho\nu\mu} = \frac{a_3}{2a_3(a_3+a_4) - 4a_4^2}\left[\Theta_{\nu\rho\mu} - \Theta_{\rho\nu\mu} - \left(2\frac{a_4}{a_3} - 1\right)\Theta_{\mu\rho\nu}\right]\,, \qquad (3.31)$$

$$\begin{aligned}
\Gamma^{\xi}{}_{(\sigma\rho)} &= \left\{{}^{\xi}{}_{\sigma\rho}\right\} + 2S_{(\rho}{}^{\xi}{}_{\sigma)} - \frac{1}{2}\kappa g^{\xi\lambda}(-\Delta_{\lambda(\sigma\rho)} + \Delta_{\rho(\sigma\lambda)} + \Delta_{\sigma(\rho\lambda)}) \\
&\quad -\frac{\kappa}{3}g^{\xi\lambda}\left(2\Delta_{(\sigma}g_{\rho)\lambda} - 3\Delta_\lambda g_{\sigma\rho}\right) + g^{\xi\lambda}\left[S_\lambda g_{\sigma\rho} + \left(\frac{2}{3}a_1 - 2\right) S_{(\sigma}g_{\rho)\lambda}\right] \\
&\quad -\frac{\kappa}{4}g^{\xi\lambda}\left[g_{\sigma\rho}\Delta_{\lambda\mu}{}^{\mu} - 2g_{\lambda(\rho}\Delta_{\sigma)\mu}{}^{\mu}\right]\,. \qquad (3.32)
\end{aligned}$$

Eqs. (3.31) and (3.32) give the antisymmetric and symmetric parts of the connection algebraically in terms of the hypermomentum and the metric. Under the condition that the matter action depends at most linearly on the connection, the above statement is equivalent to saying that we have algebraically expressed the connection in terms of the matter fields and the metric. This assumption is indeed satisfied for all common matter actions, such as scalar and gauge field, in which the matter action does not depend on the connection, and fermions, where the matter action is linear in the connection. This condition can be violated for some fields, for example massive vector fields, especially if not trivial couplings between the connection and the matter are introduced. However, as long as the matter action contains only first order derivatives of the matter fields (in order for the matter fields to satisfy second order equations of motion), ${\Delta_\lambda}^{\mu\nu}$ will only depend algebraically on the connection. This implies that, even though some more complicated manipulations will be required, the connection can always be expressed algebraically in terms of the matter field and the metric (at least at the component level).

This establishes that the independent connection in (up to) second order metric-affine actions does not carry any dynamics and it can be algebraically eliminated. Consider now using eqs. (3.32) and (3.31) to completely eliminate the connection in eq. (3.17). One would then get an equations of the form

$$R_{\mu\nu}-\frac{1}{2}R\,g_{\mu\nu}=\kappa\mathcal{T}_{\mu\nu}, \tag{3.33}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and the Ricci scalar of the metric $g_{\mu\nu}$ respectively, and $\mathcal{T}_{\mu\nu}$ will be some a second rank tensor which depends on the metric, ${\Delta_\lambda}^{\mu\nu}$ and $T_{\mu\nu}$. The expression for $\mathcal{T}_{\mu\nu}$ in terms of these three quantities is rather lengthy and we will refrain from writing it here. However, it should already be clear that the theory described by eq. (3.33) is General Relativity with modified matter interactions. For fields for which the hypermomentum vanishes, $\mathcal{T}_{\mu\nu}=T_{\mu\nu}$.

3.4 Higher orders

We can now move on to higher orders. Since the connection has three indices and the derivative one index, there is no $[l^{-3}]$ scalar quantity one can construct out of them. Similarly, one cannot construct an $[l^{-3}]$ scalar quantity using curvature invariants. The next order is $[l^{-4}]$. The terms that could straightforwardly lead to invariants after (several) contractions with the metric are

$$\begin{aligned}
&\mathcal{R}^{\alpha}{}_{\beta\gamma\delta}\mathcal{R}^{\mu}{}_{\nu\lambda\sigma}, \qquad \nabla_\mu\nabla_\nu\mathcal{R}^{\alpha}{}_{\beta\gamma\delta}, \qquad \mathcal{R}^{\alpha}{}_{\beta\gamma\delta}S_{\mu\nu}{}^{\lambda}S_{\tau\omega}{}^{\rho}, \qquad \mathcal{R}^{\alpha}{}_{\beta\gamma\delta}\nabla_\rho S_{\mu\nu}{}^{\lambda}\\
&S_{\mu\nu}{}^{\lambda}\nabla_\rho\mathcal{R}^{\alpha}{}_{\beta\gamma\delta}, \qquad S_{\mu\nu}{}^{\lambda}S_{\alpha\beta}{}^{\sigma}S_{\gamma\delta}{}^{\kappa}S_{\tau\omega}{}^{\rho}, \qquad S_{\mu\nu}{}^{\lambda}S_{\alpha\beta}{}^{\sigma}\nabla_\rho S_{\gamma\delta}{}^{\kappa}\\
&S_{\mu\nu}{}^{\lambda}\nabla_\rho\nabla_\kappa S_{\alpha\beta}{}^{\sigma}, \qquad \nabla_\rho S_{\mu\nu}{}^{\lambda}\nabla_\kappa S_{\alpha\beta}{}^{\sigma}, \qquad \nabla_\mu\nabla_\nu\nabla_\rho S_{\alpha\beta}{}^{\sigma}, \qquad (3.34)
\end{aligned}$$

Clearly each of these terms can lead to various invariants. It goes beyond the purpose of this thesis to list all possible terms.[2] However, before going further, the following subtle points are worth mentioning:

1. Due to the symmetries (or lack thereof) of the Riemann tensor when constructed with an independent connection, there are more invariants than in the purely metric case. For example $\mathcal{R}_{\mu\nu}$ is not symmetric and hence $\mathcal{R}_{\mu\nu}\mathcal{R}_{\kappa\lambda}g^{\mu\lambda}g^{\nu\kappa}$ and $\mathcal{R}_{\mu\nu}\mathcal{R}_{\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda}$ are not equal.

2. ∇_μ is constructed with the independent connection and, hence, total divergences such as $\nabla_\mu u^\mu$ do not lead to pure surface terms and cannot be discarded.

3. Since the metric is not covariantly conserved by the independent connection taking the covariant derivatives first and contracting, or contracting first and then taking a derivative does not lead to the same result. For example the terms $g^{\mu\nu}g^{\alpha\beta}\nabla_\mu\nabla_\nu\mathcal{R}_{\alpha\beta}$ and $g^{\mu\nu}\nabla_\mu\nabla_\nu\mathcal{R}$ differ.

Regarding point (ii) one could propose to split the covariant derivative into a metric covariant derivative, which is a surface term, and the rest, such as in (3.15). However, writing the rest explicitly would require the use of metric derivatives through the use of the Levi-Civita connection, as discussed above. Something similar can be said about point (iii). The two terms given as an example differ by a term including a covariant derivative of the metric. This raises the question of whether both of them should be considered. As mentioned earlier, whether terms including derivatives of the metric should be included is really a matter of choice that can be answered only

[2] An exhaustive list of all possible second and fourth order invariants one can construct in the more limiting case where the non-metricity vanishes can be found in [38]. Given that our minimal coupling assumption prevents us from the using of the non-metricity to construct invariants (see also below), this list should cover our case as well.

by invoking some minimal coupling principle. If one wants to use the metric purely for contracting indices as suggested previously, then the terms including derivatives of the metric should be suppressed.

Let us now move on and consider the effect of the higher order terms on the dynamics of the connection. Considering the most general fourth order action is formidable due to the vast number of invariant one would have to include. However, carefully considering isolated terms of different type can still reveal the complete picture.

Clearly there are term in eq. (3.34) that would not introduce new degrees of freedom if they were added to action (3.16) as they do not contain extra derivatives, such as the S^4 term (indices suppressed). Such terms exist at all even orders, e.g. S^{2n} (again indices suppressed). On the other hand $[l^{-4}]$ terms which contain two derivatives of the Cartan torsion tensor, such as $(\nabla S)^2$ (indices suppressed) would inevitable make the torsion dynamical.

What about fourth order curvature invariants? Let us for the moment set aside the term $\mathcal{R}^2$, since it belongs to the general $f(\mathcal{R})$ class, which we will discuss extensively later, and as we will see it constitutes a rather special case. A much more characteristic example to consider, which is simple enough to keep calculations tractable and yet general enough to give us the bigger picture is the following

$$S=\frac{1}{16\,\pi\,l_p^2}\int dx^4\sqrt{-g}\left[\mathcal{R}+l_p^2\mathcal{R}_{\mu\nu}\mathcal{R}_{\kappa\lambda}(ag^{\mu\kappa}g^{\nu\lambda}+bg^{\mu\lambda}g^{\nu\kappa})\right]+S_M \tag{3.35}$$

As mentioned earlier, when $\mathcal{R}_{\mu\nu}$ is not symmetric, as in our case, the 2 terms in the parenthesis will not lead to the same invariant. In fact, the action can be re-written as

$$S=\frac{1}{16\,\pi\,l_p^2}\int dx^4\sqrt{-g}\left[\mathcal{R}+l_p^2c_1\mathcal{R}_{(\mu\nu)}\mathcal{R}^{(\mu\nu)}+l_p^2c_2\mathcal{R}_{[\mu\nu]}\mathcal{R}^{[\mu\nu]}\right]+S_M \tag{3.36}$$

where $c_1 = a + b$ and $c_2 = a - b$. This latter form of the action makes the variation easier. The field equations for the metric and the connection are respectively

$$
\begin{aligned}
&\mathcal{R}_{(\mu\nu)} - \frac{1}{2}\left(\mathcal{R} + l_p^2 c_1 \mathcal{R}_{(\alpha\beta)}\mathcal{R}^{(\alpha\beta)} + l_p^2 c_2 \mathcal{R}_{[\alpha\beta]}\mathcal{R}^{[\alpha\beta]}\right) g_{\mu\nu} \\
&\qquad +2l_p^2\, c_1 R_{(\alpha\mu)} R_{(\beta\nu)} g^{\alpha\beta} + 2l_p^2\, c_2 R_{[\alpha\mu]} R_{[\beta\nu]} g^{\alpha\beta} = \kappa T_{\mu\nu}\,, \qquad (3.37)\\
&\frac{1}{\sqrt{-g}}\Big\{ -\nabla_\lambda \left[\sqrt{-g}g^{\mu\nu} + 2\sqrt{-g}\left(l_p^2 c_1 \mathcal{R}^{(\mu\nu)} + l_p^2 c_2 \mathcal{R}^{[\mu\nu]}\right)\right] + \\
&\qquad +\nabla_\sigma \left[\sqrt{-g}g^{\mu\sigma} + 2\sqrt{-g}\left(l_p^2 c_1 \mathcal{R}^{(\mu\sigma)} + l_p^2 c_2 \mathcal{R}^{[\mu\sigma]}\right)\right] \delta^\nu{}_\lambda \Big\} + \\
&\qquad +2S_{\lambda\sigma}{}^\sigma \left[g^{\mu\nu} + 2\left(l_p^2 c_1 \mathcal{R}^{(\mu\nu)} + l_p^2 c_2 \mathcal{R}^{[\mu\nu]}\right)\right] \\
&\qquad -2S_{\alpha\sigma}{}^\sigma \delta^\nu{}_\lambda \left[g^{\mu\alpha} + 2\left(l_p^2 c_1 \mathcal{R}^{(\mu\alpha)} + l_p^2 c_2 \mathcal{R}^{[\mu\alpha]}\right)\right] + \\
&\qquad +4\left(l_p^2 c_1 \mathcal{R}^{(\mu\alpha)} + l_p^2 c_2 \mathcal{R}^{[\mu\alpha]}\right) S_{\alpha\lambda}{}^\nu = \kappa \Delta_\lambda{}^{\mu\nu}\,. \qquad (3.38)
\end{aligned}
$$

In the previous section we were able to use the field equation for the connection in order to algebraically express the latter in terms of the metric and the matter fields. Inspecting eq. (3.38), however, one easily realized that, unlike eq. (3.18), it appears to include derivatives of the connection due to the presence of $\mathcal{R}_{\mu\nu}$. One could think to use eq. (3.37) in order to algebraically express $\mathcal{R}_{\mu\nu}$ (at least at component level) in terms of the metric and the matter fields (this idea is actually inspired by the specific case of $f(R)$ actions in the more restricted setting of the Palatini formalism where the connection does not couple to the matter — this will be discussed below). If this were the case, one could eliminate $\mathcal{R}_{\mu\nu}$ from eq. (3.38) and turn it again into an algebraic equation for the connection.

However, this is not possible for generic values of c_1 and c_2, or a and b for the following simple reasons:

- $\mathcal{R}_{\mu\nu}$ is not necessarily symmetric and, therefore, has 16 independent components, whereas eq. (3.37) leads to only 10 components equation as it is symmetric in μ and ν. Therefore, it cannot be used to determine $\mathcal{R}_{\mu\nu}$ fully, in terms of the metric and the components of $T_{\mu\nu}$.

- $T_{\mu\nu}$ is not necessarily independent of the connection, as it may include covariant derivatives of certain matter fields. Therefore, even if one would impose such conditions so that eq. (3.37) could be solved algebraically to give $\mathcal{R}_{\mu\nu}$ in terms of the metric and $T_{\mu\nu}$, e.g. impose the constraint $\mathcal{R}_{[\mu\nu]} = 0$ *a priori*, that would not actually help in algebraically expressing the connection as a function of the matter fields and the metric (at least for generic matter fields).

It should then be clear that the independent connection cannot be eliminated in metric-affine gravity once generic higher order curvature invariants have been added.

The same issue has been considered in Ref. [34] for the simpler case of generalized Palatini gravity, i.e. under the assumption that connection does not enter the matter action. This would corresponds to a vanishing ${\Delta_\lambda}^{\mu\nu}$. The first of the difficulties just discussed is still present in this case when trying to eliminate the connection algebraically by the procedure described above. However, since $T_{\mu\nu}$ is independent of the connection in generalized Palatini gravity, the second difficulty raised here is not really an issue. Hence, it is easier in this framework to write down exceptional Lagrangians for which the connection can be eliminated (it is just an auxiliary field). We refer the reader to Ref. [34] for more details. We refrain here from discussing similar exceptions or special cases for metric-affine gravity, as this would require severe fine tuning and/or *a priori* constraints.

Also, we shall not consider explicitly the effect of the mixed terms which include both the Cartan torsion tensor and the Riemann or the Ricci tensor, as this would not add anything new to the qualitative understanding we presented so far. What should be clear by now is that the presence of terms including derivatives of the Cartan torsion tensor or higher order curvature invariants generically leads to a dynamical connection. Therefore, higher than second order actions generically lead to new dynamical degrees of freedom.

3.5 Metric-affine $f(\mathcal{R})$ gravity as a special case

Metric-affine $f(\mathcal{R})$ theories of gravity have been extensively studied lately [31]. They constitute a distinct class within higher order actions, in the sense that they allows one to treat terms of different and arbitrarily high order on the same footing. Therefore, even though within the metric-affine setup there is no reason to single out $f(\mathcal{R})$ actions as better motivated ones — on the contrary, restricting an action to be of this type requires fine tuning — their simplicity is indeed a good argument for adopting them as toy-models from which to extract general lessons. On the other hand, exactly because they are so special, it is dubious whether $f(\mathcal{R})$ actions can be considered as representative higher order metric-affine theories from the point of view of their dynamics. This is something that is worth exploring further, and is part of our motivation for considering them separately here.

The other part comes from the observation that in the simpler setting of generalized Palatini gravity, where the connection does not enter the matter action, the whole $f(\mathcal{R})$ class constitutes an exception for which the independent connection does not carry dynamics and can be algebraically eliminated [12, 14]. This is true even if the connection is not assumed to be symmetric [14]. It is, hence, worth exploring in detail what happens in the more general metric-affine framework, in order to avoid confusion and misconceptions.

The action for $f(\mathcal{R})$ theories reads

$$S = \frac{1}{16\,\pi\,l_p^2}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M \tag{3.39}$$

This action as it stands cannot lead to consistent field equations in the presence of matter, as the gravity part of the action has a symmetry that is not shared by the matter action. Indeed, the Ricci scalar of the connection $\mathcal{R}$ remain invariant under the projective transformation

$$\Gamma^{\rho}{}_{\mu\nu} \to \Gamma^{\rho}{}_{\mu\nu} + \delta^{\rho}{}_{\mu}\xi_{\nu} \tag{3.40}$$

(ξ_μ being an arbitrary covariant vector field). Consequently any function $f(\mathcal{R})$ and any action of the $f(\mathcal{R})$ type will also be projective invariant. However, matter actions that depend on the connection will not be projective invariant. This has been discussed several times in the literature [32, 42, 43, 31, 4].

To resolve the inconsistency one needs to somehow break the projective invariance in the gravity sector. The only way to do that, given that we do not want to alter the form of the action, is to constraint the connection to some extent. The meaning of projective invariance is very similar to usual gauge invariance, in the sense that it implies that the connection can be determined only up to a projective transformation. So, to break gauge invariance we need a constraint that that would act as "gauge fixing". Clearly, given the nature of the projective transformation we essentially need to fix a vector. It has been argued in Refs. [31, 4] that the best choice for $f(\mathcal{R})$ gravity is to set

$$S_\mu \equiv S_{\alpha\mu}{}^{\alpha} = 0 \tag{3.41}$$

This constraint can be imposed implicitly, but also explicitly by adding to the action the Lagrange multiplier

$$S_{LM} = \int d^4x\sqrt{-g}B^\mu S_\mu \,. \tag{3.42}$$

Varying the total action with respect to metric $g^{\mu\nu}$, connection $\Gamma^\rho{}_{\mu\nu}$ and Lagrange Multiplier B^μ lead, after some simple manipulations [3, 31], to the following set of field equations

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu} \,, \tag{3.43}$$

$$\begin{aligned} -\nabla_\lambda(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) + \nabla_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\sigma\mu}\right)\delta^\nu{}_\lambda + 2\sqrt{-g}f'(\mathcal{R})(g^{\mu\nu}S_{\lambda\sigma}{}^\sigma \\ -g^{\mu\rho}\delta^\nu{}_\lambda S_{\rho\sigma}{}^\sigma + g^{\mu\sigma}S_{\sigma\lambda}{}^\nu) = \kappa\sqrt{-g}\left(\Delta_\lambda{}^{\mu\nu} - \frac{2}{3}\Delta_\sigma{}^{\sigma[\nu}\delta^{\mu]}{}_\lambda\right), \end{aligned} \tag{3.44}$$

$$S_{\alpha\mu}{}^\alpha = 0 \,. \tag{3.45}$$

where a prime denotes differentiation with respect to the argument.

We now check whether it is possible to eliminate algebraically the connection from the field equations. This can be done following a similar procedure as the one used in section 3.3. A contraction of eq. (3.44) yields

$$\nabla_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\sigma\mu}\right) = \kappa\frac{2}{3}\sqrt{-g}\Delta_\lambda{}^{(\mu\lambda)} \,. \tag{3.46}$$

We can use this equation in order to eliminate the second term in (3.44) to get

$$\begin{aligned} -\nabla_\lambda(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) + 2\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}S_{\sigma\lambda}{}^\nu = \\ = \kappa\sqrt{-g}\left(\Delta_\lambda{}^{\mu\nu} - \frac{2}{3}\Delta_\sigma{}^{\sigma[\nu}\delta^{\mu]}{}_\lambda - \frac{2}{3}\Delta_\sigma{}^{(\mu\sigma)}\delta^\nu{}_\lambda\right) \,. \end{aligned} \tag{3.47}$$

Using the identity

$$g_{\mu\nu}\partial_\lambda(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) = 4\sqrt{-g}\partial_\lambda f'(\mathcal{R}) + 2f'(\mathcal{R})\sqrt{-g}\partial_\lambda \ln\sqrt{-g} \,, \tag{3.48}$$

and after contracting eq. (3.47) with the metric in the μ and ν indices one gets

$$\partial_\lambda \ln\sqrt{-g} = \frac{1}{2}\left[-\frac{\kappa}{f'}\left(g_{\mu\nu}\Delta_\lambda{}^{\mu\nu} - \frac{2}{3}g_{\mu\lambda}\Delta_\sigma{}^{(\mu\sigma)}\right) - 4\frac{\partial_\lambda f'}{f'} + 2\Gamma^\sigma{}_{\sigma\lambda}\right] \,. \tag{3.49}$$

Eliminating the density related terms and lowering the indices as we did in the previous section eq. (3.47) yields

$$-\partial_\lambda g_{\alpha\beta} - g_{\alpha\beta}\frac{\partial_\lambda f'}{f'} + \Gamma^\mu{}_{\beta\lambda} g_{\mu\alpha} + \Gamma^\mu{}_{\lambda\alpha} g_{\mu\beta} = \frac{\kappa}{f'}\Bigg[\frac{1}{2} g_{\alpha\beta} g_{\mu\nu} \Delta_\lambda{}^{\mu\nu} \tag{3.50}$$
$$-\frac{1}{3} g_{\alpha\beta} g_{\mu\lambda} \Delta_\sigma{}^{(\mu\sigma)} - g_{\mu\alpha} g_{\nu\beta} \Delta_\lambda{}^{\mu\nu} + \frac{1}{3}\left(g_{\alpha\lambda} g_{\nu\beta} \Delta_\sigma{}^{\sigma\nu} + g_{\lambda\beta} g_{\mu\alpha} \Delta_\sigma{}^{\mu\sigma}\right)\Bigg] .$$

Adding suitable permutations of eq. (3.50) one gets

$$\Gamma^\rho{}_{\alpha\beta} = \left\{{}^{\rho}{}_{\alpha\beta}\right\} + \frac{1}{2f'}\left[\partial_\alpha f' \delta^\rho{}_\beta + \partial_\beta f' \delta^\rho{}_\alpha - g^{\rho\lambda} g_{\alpha\beta} \partial_\lambda f'\right] + \frac{\kappa}{f'} W_{\alpha\beta}{}^\rho , \tag{3.51}$$

where $\left\{{}^{\rho}{}_{\alpha\beta}\right\}$ is the usual Levi-Civita connection associated with the metric $g_{\mu\nu}$ and $W_{\alpha\beta}{}^\rho$ is a tensor encompassing all the hypermomenta terms

$$W_{\alpha\beta}{}^\rho = \quad -\frac{1}{2}\Bigg\{\frac{1}{2} g_{\alpha\beta} g_{\mu\nu} \Delta^{\rho\mu\nu} - g_{\mu\nu} \delta^\rho{}_{(\alpha} \Delta_{\beta)}{}^{\mu\nu} + \Delta_\beta{}^\rho{}_\alpha + \Delta_{\alpha\beta}{}^\rho \tag{3.52}$$
$$-\Delta^\rho{}_{\alpha\beta} - g_{\alpha\beta} \Delta_\sigma{}^{(\rho\sigma)} + \frac{1}{3}\delta^\rho{}_\alpha g_{\mu\beta}\left(2\Delta_\sigma{}^{[\sigma\mu]} + \Delta_\sigma{}^{(\sigma\mu)}\right)$$
$$+\frac{1}{3}\delta^\rho{}_\beta g_{\mu\alpha}\left(2\Delta_\sigma{}^{[\sigma\mu]} + \Delta_\sigma{}^{(\sigma\mu)}\right)\Bigg\} .$$

Eq. (3.51) provides an expression for the connection in terms of the metric, the hypermomentum but also $\mathcal{R}$, via the presence of f. So, we essentially run into the same difficulties we faced in the previous section when trying to eliminate the connection. However, here $\mathcal{R}$ is just a scalar quantity. Consider the trace of eq. (3.43)

$$\mathcal{R} f'(\mathcal{R}) - 2f(\mathcal{R}) = \kappa T . \tag{3.53}$$

For a given function f this is an algebraic equation in $\mathcal{R}$. Setting aside pathological cases in which this equation has no root, and the exceptional case where $f(\mathcal{R}) \propto \mathcal{R}^2$, which corresponds to a conformally invariant gravitational action (see Refs. [9, 17, 31] for more details), eq. (3.53) can be used to express $\mathcal{R}$ as an algebraic function of T. This expression can in turn be used to eliminate $\mathcal{R}$ in favour of T in the f terms in eq. (3.51). Therefore, from now on we can be thinking of eq. (3.51) as expressing the affine connection as a function of just derivatives of metric, $T_{\mu\nu}$ and the hypermomentum. This mean we are clear of the first difficulty encountered for generic fourth order actions.

This is not the case for the second point we made previously though, *i.e.* that $T_{\mu\nu}$, and hence T, can generically depend on the connection. Even though the requirement that the matter satisfies second order differential equations of motion essentially implies that the dependence of $T_{\mu\nu}$ on the connection will be algebraic (there can be only first covariant derivatives of the matter fields is S_M), the fact that there are first order derivatives of f' in eq. (3.51) is enough to give derivatives of the connection. Therefore, in metric-affine $f(\mathcal{R})$ the connection satisfies a dynamical equation in general.

A remarkable observation is the following. Taking the antisymmetric part of (3.51) in its lower indices we get

$$\begin{aligned} \Gamma^{\rho}{}_{[\alpha\beta]} \equiv S_{\alpha\beta}{}^{\rho} &= \Delta_{[\beta}{}^{\rho}{}_{\alpha]} + \Delta_{[\alpha\beta]}{}^{\rho} - \Delta^{\rho}{}_{[\alpha\beta]} \\ &= g^{\rho\lambda}\left(\Delta_{\beta[\lambda\alpha]} + \Delta_{\alpha[\beta\lambda]} - \Delta_{\lambda[\alpha\beta]}\right) . \end{aligned} \tag{3.54}$$

This implies that the torsion is still non-dynamical and vanishes for matter fields with vanishing $\Delta_{\gamma}{}^{[\alpha\beta]}$. It is only the symmetric part of the connection that carries dynamics. As already stressed in [31] torsion is non-propagating in metric-affine $f(\mathcal{R})$ and it is introduced by matter fields having $\Delta_{\gamma}{}^{[\alpha\beta]} \neq 0$.

The fact that the connection appears to satisfy a first order differential equation (namely eq. (3.51), given the presence of derivatives of $f(\mathcal{R})$), at least if one assumes that T does not include any derivatives of the connection, seems worrying. However, it is very difficult to tell if this is indeed a problem. Neither do we have the exact form of the equation, nor do we know which degrees of freedom hiding in the connection will actually be excited.

Of course, for matter fields which do not couple to the connection (scalar field, gauge fields) or if one imposes that the independent connection does not enter the matter action S_M, $T_{\mu\nu}$ is independent of the connection as well and $\Delta_{\lambda}{}^{\mu\nu} = 0$. In this case the connection can indeed be eliminated and one recovers the results of Palatini $f(\mathcal{R})$ gravity [14]. Another special case is the one where $f(\mathcal{R}) = \mathcal{R}$, as is this case $f' = 1$ and $\mathcal{R}$ is no longer present in the eq. (3.51), which now takes the form

$$\Gamma^{\rho}{}_{\alpha\beta} = \left\{{}^{\rho}{}_{\alpha\beta}\right\} + \kappa\, W_{\alpha\beta}{}^{\rho} . \tag{3.55}$$

One can then write

$$\begin{aligned}\mathcal{R}_{(\alpha\beta)} &\equiv \partial_\rho \Gamma^\rho{}_{(\alpha\beta)} - \partial_{(\beta}\Gamma^\rho{}_{\alpha)\rho} + \Gamma^\rho{}_{\sigma\rho}\Gamma^\sigma{}_{(\alpha\beta)} - \Gamma^\rho{}_{\sigma(\beta}\Gamma^\sigma{}_{\alpha)\rho} = \\ &= R_{\alpha\beta} + \kappa\left[\bar{\nabla}_\rho W_{(\alpha\beta)}{}^\rho - \bar{\nabla}_{(\beta} W_{\alpha)\rho}{}^\rho + W_{\sigma\rho}{}^\rho W_{(\alpha\beta)}{}^\sigma - W_{\sigma(\beta}{}^\rho W_{\alpha)\rho}{}^\sigma\right],\end{aligned} \tag{3.56}$$

where $R_{\mu\nu}$ is the Ricci tensor of the metric $g_{\mu\nu}$ and $\bar{\nabla}_\mu$ is the covariant derivative defined with the Levi-Civita connection of the same metric. Contracting with the metric one gets

$$\mathcal{R} = R + \kappa\left[2\bar{\nabla}_{[\rho} W^\mu{}_{\mu]}{}^\rho + W_{\sigma\rho}{}^\rho W_\mu{}^{\mu\sigma} - W_\sigma{}^{\mu\rho} W_{\mu\rho}{}^\sigma\right]. \tag{3.57}$$

We can now use eqs. (3.55), (3.56) and (3.57) in order to completely eliminate the connection and end up with the single field equation for the metric

$$\begin{aligned}G_{\alpha\beta} &= \kappa\, T_{\alpha\beta} + \frac{\kappa}{2} g_{\alpha\beta}\left\{2\bar{\nabla}_{[\rho} W^\mu{}_{\mu]}{}^\rho + W_{\sigma\rho}{}^\rho W_\mu{}^{\mu\sigma} - W_\sigma{}^{\mu\rho} W_{\mu\rho}{}^\sigma\right\} \\ &\quad -\kappa\left\{\bar{\nabla}_\rho W_{(\alpha\beta)}{}^\rho - \bar{\nabla}_{(\beta} W_{\alpha)\rho}{}^\rho + W_{\sigma\rho}{}^\rho W_{(\alpha\beta)}{}^\sigma - W_{\sigma(\beta}{}^\rho W_{\alpha)\rho}{}^\sigma\right\},\end{aligned} \tag{3.58}$$

where, as usual,

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta}, \tag{3.59}$$

is the Einstein tensor of the metric $g_{\mu\nu}$. Therefore, $f(\mathcal{R}) = \mathcal{R}$ metric-affine gravity reduces to General Relativity with extra matter interactions. This is anyway clearly just a subcase of the most general second order action we examined in section 3.3 with vanishing a_i's.

However, we have shown for any other function $f(\mathcal{R})$ the connection cannot be algebraically eliminated in the presence of matter fields that couple to it.

3.6 Summary

Metric-affine theories of gravity provide an interesting alternative to General Relativity: in such an approach, the (gravitational) dynamical fields are pairs consisting of a pseudo–Riemannian metric and an independent connection, not necessarily symmetric, on the space-time manifold. Furthermore, the action could include covariant derivatives of the matter fields, with the covariant derivative naturally defined using the independent connection. As a result, in these theories a direct coupling involving matter and connection is also present. In this chapter we have explored the role and the dynamics of the connection in such theories.

We have employed power counting in order to construct the action and search for the minimal requirements it should satisfy for the connection to be dynamical. We found that, for the most general action containing lower order invariants of the curvature and the torsion, the independent connection does not carry any dynamics. It actually reduces to the role of an auxiliary field and can be completely eliminated algebraically in favour of the metric and the matter field, introducing extra interactions with respect to general relativity. However, we have also showed that including higher order terms in the action radically changes this picture and excites new degrees of freedom in the connection, making it (or parts of it) dynamical. Constructing actions that constitute exceptions to this rule requires significant fine tuning and/or extra a priori constraints on the connection. We have also considered $f(\mathcal{R})$ actions as a particular example in order to show that they constitute a distinct class of metric-affine theories with special properties, and as such they cannot be used as representative toy theories to study the properties of metric-affine gravity.

Chapter 4

Cosmology beyond the Standard Model

4.1 Backreaction problem: suggested solutions

In order to explain the puzzling cosmological observations without using dark energy, many efforts have been done in the context of inhomogeneous models, where the full effects of General Relativity come into play. Two main, different approaches have been outlined to solve such problem.

In one approach exact inhomogeneous cosmological models can be utilised. It has been shown that the Lemaitre-Tolman-Bondi (LTB) solution can be used to fit the observed data without the need of dark energy, although this comes to the price of placing the observer in a preferred location.

A second approach, and the one of interest in this thesis, is backreactions through averaging. The averaging problem in cosmology is of considerable importance for the correct interpretation of cosmological data. The correct equations on cosmological scales are obtained by averaging the Einstein field equations of GR (eventually supplemented by a theory of photon propagation; i.e., information on what trajectories actual particles follow). By assuming spatial homogeneity and isotropy on the largest scales, the inhomogeneities affect the dynamics through correction (backreaction) terms, which can lead to behaviour qualitatively and quantitatively different from the FLRW models; in particular, the expansion rate may be significantly affected. In the next three subsections we will see three examples of some different approaches to solve the problem.

4.1.1 Exact inhomogeneous solutions to Einstein equations: LTB and Swiss-cheese models

The Lemaitre-Tolman-Bondi solution [47, 48, 49] describes a spherically symmetric spacetime filled with an irrotational pressureless ideal fluid (matter, dust). The matter particles are in a free fall under their own gravity tracing geodesics and the vanishing rotation of the geodesic congruence assures that the geodesics are orthogonal to spatial hypersurfaces. The corresponding family of such hypersurfaces define a convenient foliation and coordinate system on the spacetime. With the resulting synchronous and matter-comoving coordinates, the stress-energy tensor is diagonal $T_{\mu\nu} = \mathrm{diag}(\rho(r,t), 0, 0, 0)$ while the metric can be recasted as

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+2E(r)}dr^2 + R^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{4.1}$$

where a prime denotes a partial derivative with respect to the radial coordinate r. The arbitrary integration function $E(r)$ results from integrating the Bondi condition $G_{tr} = 0$ and determines the local 3-curvature of the spatial slices. The areal radius $R(r,t)$ determines the area of a sphere of radius r and the time coordinate t measures the proper time of the comoving matter.

We demand that $R'(t,r) > 0$ for any r and t in order to avoid shell crossing, namely that an outer shell, with a larger value of r, does not have a larger area radius R than an inner shell. In this way, the condition of neglecting dust pressure is also fullfilled.

Integrating the $G_{rr} = 0$ Einstein equation leads to the evolution equation

$$\dot{R}^2(r,t) = 2E(r) + \frac{2GM(r)}{R(r,t)}\,, \tag{4.2}$$

with the dot here referring to partial derivatives with respect to t. $M(r)$ is an integration function that defines a Euclidean mass, connected to the comoving matter density ρ by the other Einstein equations

$$\rho(r,t) = \frac{M'(r)}{4\pi R^2(t,r)R'(t,r)} \Longrightarrow M(r) = \int_0^r 4\pi R^2(t,\tilde{r})R'(t,\tilde{r})\rho(\tilde{r},t)d\tilde{r}\,; \tag{4.3}$$

interestingly, $M(r)$ is different from the comoving rest mass by a factor $(1+2E(r))^{-\frac{1}{2}}$ in the integral.

To fully specify the LTB model, one has to choose three functions, $E(r)$, $M(r)$ and an integration function coming from (4.2), the big bang time $t_B(r)$. One of these three functions can be fixed via a rescaling of the coordinate r, leaving the problem with two physically different free functions.

It goes without saying that the LTB proposal has the flaw to describe unreliable spherical symmetric configurations. As a first step to go beyond this main limitation of LTB solutions and in order to attempt at understanding the role of large-scale non-linear cosmic inhomogeneities in the interpretation of observable data, many possible manipulations of LTB models were explored in the past. Among these, Swiss-cheese models had a leading role in the literature.

A Swiss-cheese model (see [55] and references therein)is a cosmological model where "cheese" regions, described by a FLRW metric, are surrounded by several spherically symmetric holes, which on the reverse are modelled by LTB solutions. So, a Swiss-cheese is a foam of spherical symmetric holes, but it is not a spherical symmetric model as a whole. The parameters of the LTB model for the round hole, namely the matter density ρ and the function $E(r)$, must be chosen in order to match the FLRW metric on the boundary of the sphere, *i.e.* at this border the density has to match the FLRW density and $2E(r)$ has to go to $-kr^2$ with constant k. Under such condition, a realistic physical picture of a Swiss-cheese model becomes a configuration where, given a sphere, all the matter in the inner region is pushed to the border of the sphere while the quantity of matter inside the sphere does not change. With the density chosen with this shape, an observer outside the hole will not feel the presence of the hole as far as local physics is concerned (this does not apply to global quantities), the cheese is evolving as a FLRW universe while the holes evolve in a different way. The cheese can be filled with as many holes as possible, even with different sizes and density profiles, and still be described by an exact solution of the Einstein equations (as long as there is no superposition among the holes and the correct matching is achieved).

4.1.2 Averaging *à la* Buchert

Let us now briefly review the Buchert formalism in GR for a universe filled with an irrotational dust. As in the case of a LTB model, it is possible to choose a foliation of spacetime with spacelike hypersurfaces orthogonal to the flow at any event. We will then apply the averaging procedure with respect to a family of observers comoving with the dust and characterized by a four-velocity field u^μ, thus avoiding gauge complications related to the choice of an arbitrary set of observers tilted with respect to the cosmological matter fluid [76]. Actually, in an inhomogeneous universe the four-velocity of these observers is not simply $u^\mu = \delta^{0\mu}$ but there are also local fluctuations δu^μ, so that $u^\mu = \delta^{0\mu} + \delta u^\mu$ corresponding to the possible choices of time on the inhomogeneous hypersurfaces. Therefore, the procedure adopted here of projecting the Einstein equations onto u^μ and then averaging is not free of ambiguities and gauge-dependence issues. This projection and the spatial average do not commute. With this caveat in mind, we proceed as is usually done in the literature by choosing Gaussian normal coordinates (see below).

It is also convenient to define a template metric mimicking the main properties of a FLRW universe on large scales [77, 78] but encoding the small scale lumpy structures. In this way the averaged quantities will assume the usual meaning as in the traditional cosmological framework. The scale of the domain used in the averaging procedure is chosen as the cosmological volume over which it would be reasonable to recover homogeneity, *i.e.*, somehow larger than 100 h^{-1} Mpc.

Let us briefly recall the essential points of Buchert's averaging approach, referring the reader to [79] for details. For the sake of simplicity we turn our attention to Buchert's original model (see [56] for a comprehensive review). This consists of a universe filled with an irrotational dust as the material source, with energy density ρ and four-velocity u^μ satisfying $u_\mu u^\mu = -1$. In the real universe, the matter cannot locally be treated as dust everywhere, but the deviations are unlikely to be relevant for quantities integrated over large scales, which is what enters into the observations. For treatment of non-dust matter, see [61, 60]. The corresponding Einstein equations and stress-energy covariant conservation equation read

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 8\pi G\, \rho\, u_\mu u_\nu - \Lambda g_{\mu\nu}\;, \tag{4.4}$$

$$\nabla_\mu \left(\rho\, u^\mu u^\nu\right) = 0\,, \tag{4.5}$$

where $\rho \equiv T_{\mu\nu} u^\mu u^\nu$. By adopting Gaussian normal coordinates it is possible to apply the standard ADM procedure for the 3+1 splitting of spacetime [75]. In these coordinates the spacetime manifold can be foliated with spacelike Cauchy hypersurfaces parametrized by the proper time t. In this framework the surfaces are comoving with the fluid in such a way that, casting the metric in the form

$$ds^2 = -dt^2 + g_{ij}\left(t, X^k\right) dX^i \otimes dX^j \qquad\qquad (i,j,k = 1,2,3), \tag{4.6}$$

we have $u^\mu = (1,0,0,0)$ and $u^\nu \nabla_\nu u^\mu = 0$. The second fundamental form (extrinsic curvature) $K_{\mu\nu}$ of the geodesic normal slicing of spacetime is introduced as follows: Let $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ be the induced metric on the 3-surfaces. Then $K_{\mu\nu}$ is defined as the Lie derivative of this Riemannian metric in the time direction,

$$K_{\mu\nu} = -\frac{1}{2}\pounds_u h_{\mu\nu} = -\nabla_\mu u_\nu = -\frac{1}{2}\partial_t h_{\mu\nu}\,. \tag{4.7}$$

Given the form of the metric (4.6), K_{00} and K_{0i} vanish while K_{ij} can be expressed in terms of the expansion tensor θ_{ij}, the expansion scalar $\theta \equiv \theta^i{}_i$, and the traceless shear tensor σ_{ij} as

$$K_{ij} = -\theta_{ij} = -\left(\sigma_{ij} + \frac{\theta}{3}\, g_{ij}\right)\,, \quad K \equiv K_i{}^i = -\,\theta \qquad (i,j = 1,2,3). \tag{4.8}$$

For an infinitesimal fluid element, θ indicates how its volume changes in time, keeping the shape and the orientation fixed, while shear changes the shape. In the FLRW case, the volume expansion rate is just 3H, where H is the Hubble parameter.

Denoting with D_μ the derivative operator associated with the metric $h_{\mu\nu}$, it is possible to derive the Gauss–Codazzi relations between the curvature of the 3-surface, the extrinsic curvature and the spacetime curvature [75]:

$${}^{(3)}R_{\mu\nu\rho\sigma} = {}^{(4)}R_{\alpha\beta\gamma\delta} h^\alpha{}_\mu h^\beta{}_\nu h^\gamma{}_\rho h^\delta{}_\sigma - K_{\mu\rho} K_{\nu\sigma} + K_{\mu\sigma} K_{\nu\rho}\,, \tag{4.9}$$

$$D_\rho K^\rho{}_\nu - D_\nu K = h^\mu{}_\nu R_{\mu\rho} u^\rho \,. \tag{4.10}$$

Saturating indices with the induced metric $h_{\mu\nu}$, it is possible to rearrange eq. (4.9) as

$$G_{\mu\nu} u^\mu u^\nu = \frac{1}{2}\left({}^{(3)}\mathcal{R} + K^2 - K_{ij}K^{ij}\right) \,, \tag{4.11}$$

where ${}^{(3)}\mathcal{R}$ is the scalar 3-curvature, *i.e.*, the projection of the Ricci scalar onto the spatial hypersurface. On the other hand, using the definition of the Riemann tensor it follows that

$$R_{\mu\nu} u^\mu u^\nu = K^2 - K_{\mu\nu}K^{\mu\nu} - \nabla_\mu \left(u^\mu \nabla_\nu u^\nu\right) + \nabla_\nu \left(u^\mu \nabla_\mu u^\nu\right) \,, \tag{4.12}$$

with the last term vanishing because of the geodesic equation obeyed by the four-velocity of the dust. By combining (4.12) with (4.11) and taking into account the definition (4.7) of extrinsic curvature, we are able to express the scalar curvature of spacetime as

$${}^{(4)}R = {}^{(3)}\mathcal{R} + K^2 + K_{ij}K^{ij} - 2\pounds_u K \,. \tag{4.13}$$

The Hamiltonian or energy constraint and the evolution equation for the expansion scalar (Raychaudhuri equation) can be derived from appropriate contractions of the Einstein equations: the Hamiltonian constraint is obtained by doubly contracting eq. (4.4) with u^μ and using eq. (4.11),

$$\frac{1}{2}\left({}^{(3)}\mathcal{R} + K^2 - K_{ij}K^{ij}\right) = 8\pi G\rho + \Lambda \,, \tag{4.14}$$

while the equation for the scalar expansion is found by tracing the Einstein equation. Taking into account eq. (4.13) and the fact that $\pounds_u K = \partial_t K$, it follows that

$${}^{(3)}\mathcal{R} + K^2 + K_{ij}K^{ij} - 2\partial_t K = 8\pi G\rho + 4\Lambda \,. \tag{4.15}$$

The scheme proposed by Buchert involves scalar quantities averaged over a compact domain D with volume $V_D \equiv \int_D d^3X\, \sqrt{{}^{(3)}g}$,

$$\langle \psi(t, X_i) \rangle_D \equiv \frac{1}{V_D} \int_D d^3X\, \sqrt{{}^{(3)}g}\, \psi\left(t, X_i\right) \,. \tag{4.16}$$

Hence, in order to apply the averaging procedure, it is useful to re-arrange eqs. (4.14) and (4.15) taking into account the relations (4.8). In this way, we find the scalar equations [2]

$$\frac{1}{2}\left({}^{(3)}\mathcal{R}+\frac{2}{3}\,\theta^2-2\sigma^2\right)=8\pi G\rho+\Lambda\,, \tag{4.17}$$

$$ {}^{(3)}\mathcal{R}+\frac{4}{3}\,\theta^2+2\sigma^2+2\dot{\theta}=8\pi G\rho+4\Lambda\,, \tag{4.18}$$

where we have defined the shear scalar as $\sigma^2\equiv\frac{1}{2}\sigma_{ij}\sigma^{ij}$.

It is also useful to recall the energy conservation equation (4.5), which takes the form

$$\dot{\rho}=K\rho=-\theta\rho\,. \tag{4.19}$$

In a spatially homogeneous and isotropic universe with curvature index κ described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric [1]

$$ds^2=-dt^2+a^2(t)\left[\frac{dr^2}{1-\kappa r^2}+r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)\right] \tag{4.20}$$

and dominated by dust, one has

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G\rho}{3}+\frac{\Lambda}{3}-\frac{\kappa}{a^2}\,, \tag{4.21}$$

$$\frac{\ddot{a}}{a}=-\frac{4\pi}{3}G\rho+\frac{\Lambda}{3}\,, \tag{4.22}$$

$$\dot{\rho}+3\,\frac{\dot{a}}{a}\,\rho=0\,. \tag{4.23}$$

Using the averaging procedure, eqs. (4.17)-(4.19) can always be written in the form of a Friedmann-like system of averaged equations, following the operational definition (4.16) and exploiting the non-trivial commutation relation that holds for any scalar quantity $\psi(t,X_i)$ [79]

$$\langle\psi(t,X_i)\rangle^{\cdot}_D-\langle\dot{\psi}(t,X_i)\rangle_D=\langle\psi(t,X_i)\theta\rangle_D-\langle\psi(t,X_i)\rangle_D\langle\theta\rangle_D\,. \tag{4.24}$$

Let us introduce also a dimensionless scale factor normalized by the volume V_{D_i} of the region D at some initial time t_i as $a_D(t)\equiv\left(V_D/V_{D_i}\right)^{1/3}$, with the property that

[2]Hereafter an overdot denotes differentiation with respect to the comoving time t and the Latin indices i and j assume the values 1, 2, and 3.

[1]The Buchert scheme applies to vorticity-free spacetimes. For a more general case see [76].

the averaged expansion rate is written as

$$\langle\theta\rangle_D = \frac{\dot{V}_D}{V_D} = 3\,\frac{\dot{a}_D}{a_D} \equiv 3H_D\,. \tag{4.25}$$

We define a "kinematical backreaction" term, vanishing on a FLRW background, as

$$\mathcal{Q}_D \equiv \frac{2}{3}\left(\langle\theta^2\rangle_D - \langle\theta\rangle_D^2\right) - 2\langle\sigma^2\rangle_D = \frac{2}{3}\langle\theta^2\rangle_D - 2\langle\sigma^2\rangle_D - 6H_D^2\,. \tag{4.26}$$

The Einstein scalar equations and the covariant conservation equation now yield

$$3\left(\frac{\dot{a}_D}{a_D}\right)^2 - 8\pi G\left\langle\rho\right\rangle_D - \Lambda = -\frac{\left\langle{}^{(3)}\mathcal{R}\right\rangle_D + \mathcal{Q}_D}{2}\,, \tag{4.27}$$

$$3\,\frac{\ddot{a}_D}{a_D} + 4\pi G\left\langle\rho\right\rangle_D - \Lambda = \mathcal{Q}_D\,, \tag{4.28}$$

$$\left\langle\dot{\rho}\right\rangle_D + \left\langle\theta\rho\right\rangle_D = \left\langle\rho\right\rangle_D^{\cdot} + 3\,\frac{\dot{a}_D}{a_D}\left\langle\rho\right\rangle_D = 0\,, \tag{4.29}$$

respectively. The energy constraint (4.27) and the Friedmann acceleration law (4.28) lead to a differential integrability condition involving $\mathcal{Q}_D$ and $\left\langle{}^{(3)}\mathcal{R}\right\rangle_D$ that accounts for the coupling between 3-curvature and fluctuations:

$$\frac{1}{a_D^6}\,\partial_t\left(\mathcal{Q}_D\,a_D^6\right) + \frac{1}{a_D^2}\,\partial_t\left(\left\langle{}^{(3)}\mathcal{R}\right\rangle_D\,a_D^2\right) = 0\,. \tag{4.30}$$

The system of averaged equations is not closed because there are only three independent equations for the four unknown functions $a_D, \left\langle\rho\right\rangle_D, \mathcal{Q}_D, \left\langle{}^{(3)}\mathcal{R}\right\rangle_D$. This means that, in principle, different spacetimes could evolve in different ways even when they have the same average initial conditions. Extra assumptions are needed to close the system, for example assuming a certain effective cosmic equation of state, or demanding a particular functional relationship between $\mathcal{Q}_D$ and $\left\langle{}^{(3)}\mathcal{R}\right\rangle_D$ (as it is done in [79, 80] in order to obtain scaling solutions).

4.1.3 Macroscopic Gravity

The Macroscopic Gravity [2] (MG) is the only approach to the averaging problem in GR which gives a prescription for the correlation functions emerging in an averaging

[2]see [50] for a review

of the non-linear field equation (without which the averaging of the Einstein equations simply amount to definitions of the new averaged terms). The MG approach is a non trivial generalization of the metric affine connection geometry providing a fully covariant, gauge independent and non-perturbative scheme. The theory rests on the definition of a covariant 4-volume averaging procedure for tensor fields on a Riemannian spacetime. The formalism uses a bilocal operator $\mathcal{W}^{a'}_{j}(x',x)$ to define the averaging operation in the manifold $\mathcal{M}$. The bivector $\mathcal{W}^{a'}_{j}(x',x)$ transforms as a vector at event x' and as a co-vector at event x. The average of a general tensorial field $P^a_b(x)$ over a finite spacetime domain $\boldsymbol{\Sigma}$ can be defined as

$$\bar{P}^a_b(x) = \langle\ \widetilde{P}^a_b\ \rangle(x) = \frac{1}{V_{\boldsymbol{\Sigma}}}\int_{\boldsymbol{\Sigma}} d^4x'\sqrt{-g'}\tilde{P}^a_b(x',x)\,, \tag{4.31}$$

where $V_{\boldsymbol{\Sigma}} = \int_{\boldsymbol{\Sigma}} d^4x'\sqrt{-g'}$ and $\tilde{P}^a_b(x',x)$ is the bilocal extension (in x and x') of the tensor $P^a_b(x)$ obtained by using the operator $\mathcal{W}^{a'}_{j}(x',x)$

$$\tilde{P}^a_b(x',x) = \mathcal{W}^a_{i'}(x,x')P^{i'}_{j'}(x')\mathcal{W}^{j'}_{b}(x',x)\,. \tag{4.32}$$

Applying opportunely the averaging procedure to the connection Γ^a_{bc} on $\mathcal{M}$ yields to an averaged connection $\bar{\Gamma}^a_{bc}$ which is taken to be the connection on the averaged manifold $\bar{\mathcal{M}}$

$$\langle\ \widetilde{\Gamma}^a_{bc}\ \rangle = \bar{\Gamma}^a_{bc}\,. \tag{4.33}$$

The metric $G_{ab} \equiv \bar{g}_{ab}$ associated with the averaged connection (that is, the metric whose Christoffel symbols are the $\bar{\Gamma}^a_{bc}$) can be assumed to be the average of the inhomogeneous metric g_{ab} on $\mathcal{M}$. Averaging the Einstein equations on $\mathcal{M}$ leads to the equations satisfied by the averaged metric, which can be written as

$$E^a_b = 8\pi G_N\ T^a_b + {}^{(grav)}T^a_b\,, \tag{4.34}$$

where now E^a_b is the Einstein tensor constructed from the metric G_{ab}, T^a_b is the averaged energy-momentum tensor and ${}^{(grav)}T^a_b$ is a tensorial correlation object which acts like an effective gravitational energy-momentum tensor.

For the cosmological problem additional assumptions are required: with reasonable cosmological assumptions, the correlation tensor in Zalaletdinov's scheme takes the form of a spatial curvature and Buchert's scheme can be realized as a consistent limit [62].

4.2 Hybrid models

While the averaging formalism is interesting in itself, and the idea of explaining the cosmological data through backreaction in the context of pure Einstein gravity with no dark energy is very appealing, it has not been demonstrated yet that this idea works in practice. It is undeniable that matter inhomogeneities have a backreaction effect but it is not clear that over/under-densities such as those observed around us are sufficiently large to significantly affect the cosmic dynamics, and are not limited to small perturbative effects. While the jury is still out on whether backreaction explains the observed cosmic acceleration or not, one realizes that virtually all high energy theories attempting to quantize gravity or unifying it with the other interactions predict deviations from GR. As already mentioned in chapter 2, in string theories and supergravity the gravitational field includes a dilaton [3] whose presence is unavoidable and which couples non-minimally to the curvature of spacetime [65]. Such a behaviour is mimicked by scalar-tensor gravity [66, 67] (for example, an early representative of string theories, the bosonic string theory reduces to an $\omega = -1$ Brans–Dicke theory in the low-energy limit [68]).

While scalar-tensor theories are constrained on Solar System scales and by the binary pulsar systems [69], we do not have many constraints on larger scales (except, possibly, those due to the variation of the effective gravitational coupling during Big Bang nucleosynthesis). It is possible, therefore, that the backreaction idea may have to be implemented in alternative theories of gravity. In fact, it could even be that, if backreaction doesn't quite work in GR, it is "helped" by a non-Einsteinian component of gravity. In [70] a formalism that implements Buchert's scheme into models with variable Newton "constant" was already developed, motivated by the non-perturbative renormalization group improvement of the action functional [71]. Here, instead, we restrict our attention to scalar-tensor gravity as the prototypical generalization of GR.

The following observation can be made *a priori*: the Brans–Dicke-like field that necessarily permeates all of spacetime can be described as an effective form of matter by writing the scalar-tensor field equations in the form of effective Einstein equations.

[3] The dilaton field is in addition to the massless, spin-two graviton and to the antisymmetric Kalb–Ramond field.

The effective energy-momentum tensor characterizing this form of ϕ-matter easily violates all the energy conditions and, therefore, is more likely to produce the cosmic acceleration.

Another aspect is worth pointing out: it is widely believed that quantum corrections to the Einstein–Hilbert action introduce quadratic deviations from the usual Lagrangian density R, which may well have propelled the inflationary epoch in the early universe [1], *e.g.* as in Starobinsky's inflation [72]. For a spatially homogeneous and isotropic universe, quadratic corrections die off quickly as the universe expands and R decreases. However, in an inhomogeneous universe, they might help the backreaction mechanism. Now, we have already showed in section 2.3.2, that a theory described by a non-linear Lagrangian density $f(R)$ in the metric formalism is equivalent to an $\omega = 0$ Brans–Dicke theory with a scalar field degree of freedom given by $\phi = f'(R)$ with a suitable scalar field potential. Therefore, by studying scalar-tensor theory, we also catch the effect of the simplest quadratic corrections to GR.

Let us remind the form of the scalar-tensor action expressed in the Jordan frame

$$S_{ST} = \int d^4x\sqrt{-g}\left\{\frac{1}{16\pi}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi)\right] + \alpha_M\mathcal{L}_M\right\}\,, \tag{4.35}$$

where ϕ is the Brans–Dicke-like scalar field with potential $V(\phi)$ and coupling function $\omega(\phi)$, g is the determinant of the metric tensor $g_{\mu\nu}$, R is the Ricci curvature, $\mathcal{L}_M$ is the Lagrangian density describing the ordinary matter sector with coupling costant α_M, and we adopt the notations of Ref. [75].

The conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2\, g_{\mu\nu}\,, \qquad \Omega = \sqrt{G\phi} \tag{4.36}$$

and the scalar field redefinition

$$d\tilde{\phi} = \sqrt{\frac{2\omega(\phi)+3}{16\pi G}}\,\frac{d\phi}{\phi} \tag{4.37}$$

[1]We do not refer here specifically to $f(R)$ theories based on large-scale modifications of gravity [53, 54]. It would be rather pointless to study the backreaction effect in those $f(R)$ theories since it is already known that, in their metric version, they may provide viable models to explain the cosmic acceleration [3, 7, 74].

turn the action (4.35) into its Einstein frame form

$$S_{ST} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \, \tilde{\nabla}_\nu \tilde{\phi} - U\left(\tilde{\phi}\right) + \tilde{\alpha}_M(\tilde{\phi}) \mathcal{L}_M \right] , \qquad (4.38)$$

where a tilde denotes quantities in the rescaled world, and

$$U\left(\tilde{\phi}\right) = \frac{V[\phi(\tilde{\phi})]}{\left[G\phi(\tilde{\phi})\right]^2} , \qquad \tilde{\alpha}_M(\tilde{\phi}) = \frac{\alpha_M}{\left[G\phi(\tilde{\phi})\right]^2} . \qquad (4.39)$$

The "new" scalar field $\tilde{\phi}$ couples minimally to the curvature but non-minimally to the matter fields.

4.3 Averaging procedure for scalar-tensor cosmology

Our goal is studying the backreaction mechanism of spatial inhomogeneities on the cosmic dynamics in the context of scalar-tensor gravity.

It is convenient to write the field equations of scalar-tensor gravity in the form of effective Einstein equations, which allows for the direct application of Buchert's formalism to this class of theories. It must be pointed out that choosing this form of the equations implies that the scalar field ϕ plays the role of the inverse of a Newton "constant" now varying in space and time (the effective gravitational coupling in the action (4.35) is $G_{eff} = \phi^{-1}$, although the coupling in a Cavendish experiment is instead $G_{eff} = \frac{1}{\phi} \frac{2(\omega+2)}{2\omega+3}$ [81]). It is rather simple to notice that the presence of this extra field introduces a new ambiguity with respect to GR due to the non-linearity of the averaging procedure. In fact, the variation of the action (4.35) with respect to $g^{\mu\nu}$ yields the field equations

$$\phi G_{\mu\nu} = 8\pi \left(T_{\mu\nu} + T^{(\phi)}_{\mu\nu} \right) , \qquad (4.40)$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor and

$$T^{(\phi)}_{\mu\nu} = \frac{\omega(\phi)}{\phi} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \phi \nabla_\sigma \phi \right) + \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi - \frac{V(\phi)}{2} g_{\mu\nu} . \qquad (4.41)$$

While it is common to divide by ϕ to put this equation in the form of the effective Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}\,T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\sigma\phi\nabla_\sigma\phi\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) - \frac{V(\phi)}{2\phi}\,g_{\mu\nu}\,, \quad (4.42)$$

this operation does not commute with the spatial average if $\partial\phi/\partial x^i \neq 0$. As a result, once the scalar averaging has been performed, $\langle\phi\ {}^{(4)}R\rangle_D \neq \langle\phi\rangle_D\langle{}^{(4)}R\rangle_D$. This problem does not appear in GR where the coupling is a true constant and is peculiar to scalar-tensor gravity. The outcomes of taking the average of eq. (4.40) or of eq. (4.42) are different. For ease of comparison with GR we choose to proceed by averaging eq. (4.42) but with a second caveat to keep in mind. Further, if one decides to adopt the Einstein conformal frame instead of the Jordan frame, the relevant integro-differential equations can, in principle, have different solutions in the two frames. But this ambiguity remains even if we stay in the Jordan frame, depending on the choice one makes to use the scalar field directly linked to the gravitational sector or, as in our case, to recast the field equations as effective Einstein-like equations.

The variation of the action (4.35) with respect to the scalar field yields the equation of motion for ϕ

$$\Box\phi = \frac{1}{2\omega(\phi)+3}\left[-8\pi\rho - \frac{d\omega}{d\phi}\nabla^\sigma\phi\,\nabla_\sigma\phi + \phi\,\frac{dV(\phi)}{d\phi} - 2V(\phi)\right]. \quad (4.43)$$

The Hamiltonian constraint is obtained by double contraction of the previous equation with u^μ (time-time component of the field equations)

$$\frac{1}{2}\left({}^{(3)}\mathcal{R} + K^2 - K_{ij}K^{ij}\right) = \frac{8\pi\rho}{\phi} + \frac{\omega(\phi)}{2}\,\frac{\dot{\phi}^2}{\phi^2} + \frac{\omega(\phi)}{2\phi^2}\,g^{ij}\partial_i\phi\partial_j\phi + \frac{1}{\phi}\left(\ddot{\phi} + \Box\phi\right) + \frac{V(\phi)}{2\phi}\,, \quad (4.44)$$

while the evolution equation for the expansion scalar now reads

$$\begin{aligned} {}^{(4)}R &= {}^{(3)}\mathcal{R} + K^2 + K_{ij}K^{ij} - 2\partial_t K \\ &= -g^{\mu\nu}\left[\frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\sigma\phi\nabla_\sigma\phi\right)\right. \\ &\qquad \left. + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) - \frac{V(\phi)}{2\phi}g_{\mu\nu}\right] \\ &= 8\pi\frac{\rho}{\phi} + \frac{\omega(\phi)}{\phi^2}\nabla^\mu\phi\nabla_\mu\phi + \frac{3\Box\phi}{\phi} + \frac{2V(\phi)}{\phi}\,. \end{aligned} \tag{4.45}$$

By averaging the last two equations and using both the definition (4.26) of backreaction and the fact that $K^2 - K_{ij}K^{ij} = \frac{2}{3}\theta^2 - 2\sigma^2$, one obtains

$$\begin{aligned} \frac{1}{2}\left\langle {}^{(3)}\mathcal{R}\right\rangle_D + \frac{1}{2}\mathcal{Q}_D + 3H_D^2 &= 8\pi\left\langle\frac{\rho}{\phi}\right\rangle_D + \left\langle\frac{\omega(\phi)}{2}\,\frac{\dot\phi^2 + g^{ij}\partial_i\phi\partial_j\phi}{\phi^2}\right\rangle_D \\ &\qquad + \left\langle\frac{\ddot\phi + \Box\phi}{\phi} + \frac{V(\phi)}{2\phi}\right\rangle_D\,, \end{aligned} \tag{4.46}$$

$$\begin{aligned} \left\langle {}^{(3)}\mathcal{R}\right\rangle_D - \mathcal{Q}_D + 6H_D^2 + 6\,\frac{\ddot a_D}{a_D} &= 8\pi\left\langle\frac{\rho}{\phi}\right\rangle_D + \left\langle\omega(\phi)\left(\frac{-\dot\phi^2 + g^{ij}\partial_i\phi\partial_j\phi}{\phi^2}\right)\right\rangle_D \\ &\qquad + \left\langle\frac{3\Box\phi + 2V(\phi)}{\phi}\right\rangle_D\,. \end{aligned} \tag{4.47}$$

By combining the last two equations and using eq. (4.43) the cosmic acceleration is expressed as

$$\begin{aligned} \frac{\ddot a_D}{a_D} &= -\frac{8\pi}{3}\left\langle\frac{\rho}{\phi}\left(\frac{\omega(\phi)+2}{2\omega(\phi)+3}\right)\right\rangle_D + \frac{\mathcal{Q}_D}{3} - \frac{1}{3}\left\langle\omega(\phi)\left(\frac{\dot\phi}{\phi}\right)^2\right\rangle_D - \frac{1}{3}\left\langle\frac{\ddot\phi}{\phi}\right\rangle_D \\ &\quad - \frac{1}{6}\left\langle\frac{1}{2\omega(\phi)+3}\,\frac{d\omega}{d\phi}\,\nabla^\sigma\phi\,\nabla_\sigma\phi\right\rangle_D + \frac{1}{6}\left\langle\frac{1}{2\omega(\phi)+3}\left(\frac{dV}{d\phi} + (2\omega(\phi)+1)\,\frac{V}{\phi}\right)\right\rangle_D\,. \end{aligned} \tag{4.48}$$

Since $\phi > 0$ and $\omega(\phi) > 0$ in order to keep the gravitational coupling positive, the positive energy density of dust in the first term on the right hand side causes deceleration.

The constraints on the magnitude of the factor $2(\omega+2)/(2\omega+3)$ depend on the range of the ϕ. If the latter is comparable with the size of the solar system then the Cassini bound $\omega > 40000$ [82] applies. However, this bound does not apply if the field is short-ranged or if endowed with a range depending on the environment (chameleon mechanism).

In an optimistic view, the backreaction term $\mathcal{Q}_D$ is positive and contributes to acceleration, as generally argued in GR. However, this is not necessarily the case: in fact, prior to the 1998 discovery of the cosmic acceleration, the same backreaction term, with negative sign, was proposed as a solution to the dark matter problem (see [83] and Sec. 5.5.2 of [84]). This shows that the sign of $\mathcal{Q}_D$ is highly uncertain. The third term on the right hand side of eq. (4.48) is definitely negative and contributes to decelerate the universe, while the signs of the fourth and fifth terms are undetermined.

There is little doubt that the terms involving the first and second derivatives of ϕ are small and, at best (*i.e.*, when $\langle\ddot{\phi}\rangle_D < 0$) their effects conflict. However, the constraints on the temporal and spatial variation of ϕ after nucleosynthesis are rather poor. While the time variation of the gravitational coupling is constrained as $\left|\frac{\dot{G}}{G}\right| \simeq \left|\frac{\dot{\phi}}{\phi}\right| < H_0^{-1}$ (where H_0 is the present value of the Hubble parameter) [69], there is basically no constraint on the second time derivative of ϕ.

The last term including the potential and its derivative is novel with respect to GR and could significantly affect the acceleration. While this could be interpreted as an obvious consequence of the fact that a potential can mimic a cosmological constant, we show later (see the case of $f(R)$ gravity discussed below) that it can be important and positive even in cases for which late time acceleration cannot be *a priori* expected from the form of the Lagrangian.

In summary, while no definitive conclusion can be reached on whether the inclusion of backreaction induces late time acceleration (as in the GR case), nonetheless there are encouraging new terms in scalar-tensor cosmology. Unfortunately no definitive answer on the relative magnitude and sign of the specific terms can be provided in such a general framework. Hence, in the following we shall consider specific implementation of the theory in which eq. (4.48) simplifies.

4.3.1 Brans–Dicke cosmology

As an example of the procedure developed, let us specialize the whole formalism to a true Brans–Dicke theory (*i.e.*, $V \equiv 0$ and $\omega(\phi) \equiv \omega_0 =$ constant) and let us also assume the scalar field to be spatially smooth on the scales of interest, $\phi = \phi(t)$. This is clearly an oversimplification but serves the purpose of illustration. This assumption implies that all the averages involving the scalar field ϕ are domain-independent. In this context, the ambiguity in the choice of the representation described in the previous section is no longer present. Then, eqs. (4.46) and (4.47) become

$$\frac{1}{2}\left\langle {}^{(3)}\mathcal{R}\right\rangle_D + \frac{1}{2}\mathcal{Q}_D + 3H_D^2 = 8\pi\frac{\langle\rho\rangle_D}{\phi} + \frac{\omega_0}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 - 3H_D\frac{\dot{\phi}}{\phi}\,, \tag{4.49}$$

$$\frac{6\ddot{a}_D}{a_D} = -\left\langle {}^{(3)}\mathcal{R}\right\rangle_D + \mathcal{Q}_D - 6H_D^2 + 8\pi\frac{\langle\rho\rangle_D}{\phi} - \omega_0\frac{\dot{\phi}^2}{\phi^2} - 3\,\frac{\left(\ddot{\phi} + 3H_D\dot{\phi}\right)}{\phi}\,. \tag{4.50}$$

The consistency relation between the Hamiltonian constraint and the Raychaudhuri equation can now be derived by differentiating the latter with respect to time and then substituting the result, the Hamiltonian constraint, and the equation of motion for the scalar field in the former. The result is

$$\begin{aligned}&\frac{1}{a_D^6}\,\partial_t\left(\mathcal{Q}_D\,a_D^6\right) + \frac{1}{a_D^2}\,\partial_t\left(\left\langle {}^{(3)}\mathcal{R}\right\rangle_D\,a_D^2\right) = \\ &= \frac{2}{a_D^{\frac{6\omega_0+12}{2\omega_0+3}}}\,\partial_t\left[8\pi\frac{\langle\rho\rangle_D}{\phi}\,a_D^{\frac{6\omega_0+12}{2\omega_0+3}}\right] + \frac{1}{a_D^6}\,\partial_t\left[\frac{\omega_0\,\dot{\phi}^2}{\phi^2}\,a_D^6\right] - \frac{6}{a_D^4}\,\partial_t\left[\frac{\dot{\phi}}{\phi}\,H_D\,a_D^4\right]\,.\end{aligned} \tag{4.51}$$

As a consistency check, one can notice that this equation reduces to the corresponding eq. (4.30) in the limit $\omega_0 \to \infty$, $\phi \approx \text{const.} + \mathrm{O}\left(\frac{1}{\omega_0}\right)$ in which Brans–Dicke theory reduces to GR [3] (this can be seen by using the form of the solution of eq. (4.29), $\langle\rho\rangle_D \propto a_D^{-3}$, in the first term on the right hand side of eq. (4.52)).

Let us consider a class of solutions in which the scalar field has the form

$$\phi(t) = \phi_0 + \phi_1 \mathrm{e}^{-\beta t}\,, \tag{4.52}$$

[3] In the case of a massive dust, the limit of Brans–Dicke theory to GR is free of the ambiguities arising when $T = 0$ and the expansion $\phi = \text{const.} + \mathrm{O}\left(\frac{1}{\omega_0}\right)$ is indeed correct (see [85] and references therein).

where the requirement of a positive, non-vanishing scalar field implies $\phi_0\,,\beta > 0$ and $\phi_1 > -\phi_0$. Using the general solution of eq. (4.29) we can express the averaged energy density as $\langle\rho\rangle_D(t) = \langle\rho\rangle_D^0\, a_D^{-3}(t)$, where the scale factor has been normalized at the starting time of the growth of structures (in our notation, $a_D(t=0)=1$ where $t=0$ corresponds to the last scattering surface). Inserting this relationship into the equation of motion for ϕ, it is possible to solve with respect to $a(t)$. The effective gravitational coupling is finite for both small and large times t, and the corresponding averaged scale factor is

$$a_D(t) = \mathrm{e}^{\frac{\beta t}{3}}\,(1-\gamma t)^{1/3} \tag{4.53}$$

with

$$\gamma = \frac{8\pi\langle\rho\rangle_D^0}{\beta\phi_1(2\omega+3)}\,. \tag{4.54}$$

It is an easy task to show that late time accelerated solutions can be found for suitable values of the parameters. However, the physically motivated requirement that the backreaction is negligible at early stages further restricts the allowed range.[4]

The following expressions for the averaged scalar 3-curvature, $\langle\mathcal{R}\rangle_D$, and the backreaction term $\mathcal{Q}_D$ defined in (4.26), are immediately obtained:

$$\langle\mathcal{R}\rangle_D = \frac{\beta\phi_1\gamma - 24\pi\langle\rho\rangle_D^0 - 2\beta e^{\beta t}\phi_0[\gamma(2+\beta t)-\beta]}{2\,(\phi_1+e^{\beta t}\phi_0)\,(\gamma t-1)}\,, \tag{4.55}$$

$$\mathcal{Q}_D = \frac{\beta^2\phi_1^2\omega}{\phi_1+e^{\beta t}\phi_0} + \frac{-8\pi\langle\rho\rangle_D^0 + \beta\phi_1[\gamma(2\beta t-1)-2\beta]}{2\,(\phi_1+e^{\beta t}\phi_0)\,(\gamma t-1)} + \\ +\frac{1}{3}\left[\beta^2 + \frac{2\beta\gamma}{\gamma t-1} - \frac{2\gamma^2}{(\gamma t-1)^2}\right]\,. \tag{4.56}$$

The initial value of the backreaction term $\mathcal{Q}_D$ could be different from zero (albeit small), as long as we assume a perturbed FLRW universe at the last scattering epoch. Furthermore, $\mathcal{Q}_D$ approaches the asymptotic value $\beta^2/3$, giving a positive contribution to the acceleration.

[4] An example of such a solution can be found for the set of values $(\beta,\phi_0,\phi_1,\omega,\langle\rho\rangle_D^0) = (0.002, 750, -1, 40000, 1)$.

4.3.2 Metric $f(R)$ gravity

We now consider the case of metric $f(R)$ gravity, described by the action

$$S' = \frac{1}{16\pi}\int d^4x\sqrt{-g}f(R) + S_M\,, \tag{4.57}$$

where $f(R)$ is a non linear function of its argument [3]. We know that this theory is equivalent to an $\omega = 0$ Brans–Dicke theory with Brans–Dicke scalar $\phi \equiv f'(R)$ and potential $V(\phi) = Rf'(R) - f(R)$ [73]. For the sake of illustration, let us take into account the Lagrangian density in the form $f(R) = R + \alpha R^n$ with $n > 1$ and $\alpha > 0$ as required for local stability [86]. Then, the potential can be expressed as

$$V(\phi) = \frac{n-1}{n^{\frac{n}{n-1}}\alpha^{\frac{1}{n-1}}}(\phi-1)^{\frac{n}{n-1}} \tag{4.58}$$

and eq. (4.48) reduces to

$$\frac{\ddot{a}_D}{a_D} = -\frac{16}{9}\left\langle\frac{\rho}{\phi}\right\rangle_D + \frac{\mathcal{Q}_D}{3} - \frac{1}{3}\left\langle\frac{\ddot{\phi}}{\phi}\right\rangle_D + \frac{2n-1}{18\,n^{\frac{n}{n-1}}}\frac{1}{\alpha^{\frac{1}{n-1}}}\left\langle(\phi-1)^{\frac{1}{n-1}}\right\rangle_D\,. \tag{4.59}$$

α arises from quantum corrections and is presumably small, so it would seem that the last term on the right hand side of the previous equation is large. However, this is not the case because $(\phi-1)^{\frac{1}{n-1}}$ is also small and contains the same power of α: in fact, by expressing $(\phi-1)$ as a function of R, the last term of eq. (4.59) is rewritten as $\frac{2n-1}{18n}\langle R\rangle_D$. Nevertheless, it is relevant that this term is not suppressed by positive powers of α, as one might expect, and hence it may contribute significantly to the cosmic acceleration. The third term on the right hand side, for small values of α, is instead

$$-\frac{1}{3}\left\langle\frac{\ddot{\phi}}{\phi}\right\rangle_D \simeq -\frac{1}{3}\alpha n(n-1)\left\langle(n-2)R^{n-3}\dot{R}^2 + R^{n-2}\ddot{R}\right\rangle_D\,. \tag{4.60}$$

For the physically well-motivated case $n = 2$ associated to Starobinsky inflation in the early universe [72], this term reduces to $-\frac{2\alpha}{3}\left\langle\ddot{R}\right\rangle_D$ and hence it is subdominant with respect to the last term of eq. (4.59). Finally for the first two terms on the right hand side of eq. (4.59) the same considerations presented after eq. (4.48) apply.

4.4 Summary

The increasing improvement in quality and quantity of the cosmological data motivates a proper evaluation of the backreaction of matter inhomogeneities. Hence, any test of alternative theories of gravitation will have to take into account possible corrections due to the backreaction mechanism, whether the latter are large or not. For this reason, we analyzed here the possibility of improving the averaging scheme in the prototypical alternative theories of gravity, the scalar-tensor ones.

Keeping this goal in mind and following the path outlined by Buchert and collaborators, we have derived two scalar equations (the Hamiltonian constraint and the equation for the scale factor) from contractions of the field equations written in the form of effective Einstein equations. The more general working frame exposed an intrinsic ambiguity of the averaging proposal related to the scalar degree of freedom in scalar-tensor theories. The ambiguity is twofold as it leads to different averaged equations for different conformal frames and, within a chosen frame, to different results depending on the way the field equations are cast at the beginning of the calculation. We made here the choice of working in the Jordan conformal frame and later on in the calculation the *ansatz* of a domain-independent scalar field allowed us to circumvent the ambiguity linked to the non-commutativity of the operations involved.

As in GR, the system of equations obtained is not closed, hence one extra assumption is needed in order to solve it. The backreaction term $\mathcal{Q}_D$, and other terms as well, have signs that are undetermined and hence cannot be associated to a clear effect. This is not too surprising, considering that a loss of information is unavoidable whenever an average is performed. Averaging makes it impossible to disentangle the individual contributions of inhomogeneities and anisotropies, but here even the collective effects are uncertain. While no definitive conclusion can be reached (as in the GR case), nonetheless there are encouraging new terms in scalar-tensor cosmology. In particular, we noticed that the term including the scalar field potential and its derivative could significantly affect the acceleration.

In order to gain a better understanding of the potentialities of the backreaction terms in eq. (4.48) to contribute significantly to late time acceleration we finally specialized to two specific sub-cases, namely Brans-Dicke and metric $f(R)$ gravity. In the first case we have provided, as a proof of principle, a toy model solution which is accelerated at late times due to the presence of the Brans–Dicke scalar field ϕ. In the second case, we have studied a polynomial Lagrangian using the connection between metric $f(R)$ and scalar-tensor theories. While it is natural to expect that higher order corrections to the Einstein–Hilbert Lagrangian would be suppressed by their small dimensional coefficients, we found that a generic αR^n term contributes via the potential term in eq. (4.48) without showing any suppression in α. Moreover, the fact that this term is now proportional to the averaged Ricci scalar implies that it is not necessarily small at late times.

Chapter 5

Testing alternative theories on large scales

5.1 Challenging GR with cosmological observations

Cosmology is going through a golden age as we can nowadays start reconstructing the expansion history of the universe with unprecedented precision. The huge range of data sets spans a wide realm of observations with heterogeneous nature, providing us with a much more accurate tool for investigating the evolution of the universe [88]. We have already seen that most of the observations agree on the evidence that the universe is undergoing an era of positively accelerated expansion, requiring the existence of a (more or less conservative) source able to produce it. A large set of cosmological models, where the late time acceleration is a by-product of some modified gravitational dynamics, has been investigated. It goes without saying that a sensible test to discriminate among different cosmological evolutions should pass through a proper interpretation of high redshift data. In this context, it is clear that the development of a "gravitational dynamics independent" reconstruction of the expansion history of our universe does play a crucial role.

Cosmography provides such unbiased test of the cosmological history by assuming just homogeneity and isotropy and then use the so obtained Friedman-Lemaître-

Robertson-Walker (FLRW) metric to express the distances[1] of the observed objects as power series in a suitable redshift parameter. The coefficients of such powers, casted into a combination of successive weighted derivatives of the scale factor $a(t)$, contain the relevant information for a kinematic description of the universe [89, 90, 91, 92, 93, 94, 95].

It is quite obvious that by adding higher order powers to the redshift expansions of such scales it is possible to improve the data fitting, since more free parameters are involved. However, for a given data set, there will be an upper bound on the order which is statistically significant in the data analysis. On the other side, it was noticed in ref. [97] that given a data set reaching sufficiently high redshifts, a premature truncation of the cosmographic analysis can lead to wrong estimates for the cosmographic parameters. In this sense it is crucial to always determine the order of the expansion which maximizes the statistical significance of the fit for a given data set or an ensemble of them. We shall hence determine such order by performing suitable F-tests depending on the collection of data sets we shall consider. The so obtained parameters will then finally allow to evaluate, in a dynamic independent way, the viability of any theory aiming to explain the current expansion of the universe.

5.2 Cosmographic expansions

As a pedagogical example, we will discuss first the procedure that has been followed in order to obtain the cosmographic expansion for the luminosity distance. As

[1] Note that depending on which physical quantity one is measuring, it could be more convenient to extract from some data set a particular distance indicator than another one. These different quantities have different expressions of the Taylor expansion in redshift, such that it could be more natural to estimate cosmographic parameters, whose expression instead does not depend on the analytic expansion, in one of these particular frameworks. This ambiguity led also to a misconception about the appropriate definition of distance one should investigate. From now on we will refer only to: luminosity distance as the most direct choice in the case of measures of distance for Supernovae Type Ia (SNeIa) and Gamma Ray Bursts (GRBs); volume distance for Baryon Acoustic Oscillations (BAOs); angular diameter distance for the Cosmic Microwave Background (CMB) (see below for their definitions). For a critical discussion about such difficulties, see ref. [89].

already pointed out, we will start from the only assumption that the universe is homogeneous and isotropic, so that the metric describing its properties is the FLRW one

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \; ; \tag{5.1}$$

using this metric, it is possible to express the luminosity distance d_L as a power expansion in the redshift parameter z (or in term of the y-parameter, defined as $y \equiv z/(1+z)$), where the coefficients of the expansion are some functions of the scale factor $a(t)$ and its higher order derivatives.

Following ref. [102], the relation between the apparent luminosity l of an object and its absolute luminosity L defines the luminosity distance d_L

$$l = \frac{L}{4\pi r_1^2 a^2(t_0)(1+z)^2} = \frac{L}{4\pi d_L^2} \, , \tag{5.2}$$

where r_1 is the comoving radius of the light source emitting at time t_1, t_0 is the later time an observer in $r=0$ is catching the photons, and redshift z is, as usual, defined as $1+z = a(t_0)/a(t_1)$. The radial coordinate r_1 in a FLRW universe can be written for small distances as [94]

$$r_1 = \int_{t_1}^{t_0} \frac{c}{a(t)} dt - \frac{k}{3!}\left[\int_{t_1}^{t_0} \frac{c}{a(t)} dt\right]^3 + \mathcal{O}\left(\left[\int_{t_1}^{t_0} \frac{c}{a(t)} dt\right]^5\right) \, , \tag{5.3}$$

with $k = -1, \; 0, \; +1$ respectively for hyperspherical, Euclidean or spherical universe. In such a way, it is possible to recover the expansion of d_L for small z

$$d_L(z) = \frac{c}{H_0}\left\{z + \frac{1}{2}(1-q_0)z^2 - \frac{1}{6}\left(1 - q_0 - 3q_0^2 + j_0 + \frac{kc^2}{H_0^2 a^2(t_0)}\right) z^3 + \mathcal{O}(z^4)\right\} , \tag{5.4}$$

where we have defined the cosmographic parameters as

$$\begin{aligned}
H_0 &\equiv \left.\frac{1}{a(t)}\frac{da(t)}{dt}\right|_{t=t_0} \equiv \left.\frac{\dot{a}(t)}{a(t)}\right|_{t=t_0} \, , \\
q_0 &\equiv -\left.\frac{1}{H^2}\frac{1}{a(t)}\frac{d^2a(t)}{dt^2}\right|_{t=t_0} \equiv -\left.\frac{1}{H^2}\frac{\ddot{a}(t)}{a(t)}\right|_{t=t_0} \, , \\
j_0 &\equiv \left.\frac{1}{H^3}\frac{1}{a(t)}\frac{d^3a(t)}{dt^3}\right|_{t=t_0} \equiv \left.\frac{1}{H^3}\frac{a^{(3)}(t)}{a(t)}\right|_{t=t_0} \, .
\end{aligned} \tag{5.5}$$

A comment is necessary here: as already stressed in ref. [92] the ill-behaviour at high z (close and higher than $z \approx 1$) of the usual redshift expansions strongly affects the results leading in general to an underestimate of the errors. In order to avoid these problems, as well as to control properly the approximation associated with the truncation of the expansion, it is useful to recast all the involved quantities as functions of the improved parameter $y = z/(1+z)$ [92, 96, 97]. In such a way, being $z \in (0, \infty)$ mapped into $y \in (0, 1)$, it becomes possible to retrieve improved convergence properties of the Taylor series at high redshift [92, 98].

If we use the redshift variable $y = z/(1+z)$, the definition of the cosmographic parameters will not be affected, while now the luminosity distance turns out to be

$$d_L(y) = \frac{c}{H_0}\left\{y - \frac{1}{2}(q_0 - 3)y^2 + \frac{1}{6}\left[11 - 5q_0 + 3q_0^2 - j_0 + \Omega_{k_0}\right]y^3 + \mathcal{O}(y^4)\right\}, \tag{5.6}$$

where $\Omega_{k_0} = -kc^2/H_0^2 a^2(t_0)$ is the spatial curvature energy density. For a flat universe, $\Omega_{k_0} = 0$. Since we are interested in spanning the universe at any redshift, in the following we will use only the formulation of the expansion in the variable y. In our analysis we will put constraints up to fourth and fifth order parameters s_0 and c_0:

$$\begin{aligned} s_0 &\equiv \frac{1}{H^4}\frac{1}{a(t)}\frac{d^4a(t)}{dt^4}\bigg|_{t=t_0} \equiv \frac{1}{H^4}\frac{a^{(4)}(t)}{a(t)}\bigg|_{t=t_0}, \\ c_0 &\equiv \frac{1}{H^5}\frac{1}{a(t)}\frac{d^5a(t)}{dt^5}\bigg|_{t=t_0} \equiv \frac{1}{H^5}\frac{a^{(5)}(t)}{a(t)}\bigg|_{t=t_0}. \end{aligned} \tag{5.7}$$

In the Appendix one can find the cosmographic series for all the physical quantities involved in our study.

5.3 Observational data sets

Some recent papers handle the problem of interpreting the data under a cosmographic perspective using different probes [97, 99, 100, 101]. In this thesis we are going to explore the whole ensemble of data sets and use it to constrain the parameters appearing in the expansions of the characteristic scales associated to these indicators: Supernovae and Gamma Ray Bursts, Baryon Acoustic Oscillations, Hubble parameter (Hub) and Cosmic Microwave Background.

The SNIa distance moduli provide the luminosity distance as a function of redshift $D_{\rm L}(z)$. In this thesis we will use the latest SNIa data sets from the Supernova Cosmology Project, "Union2 Compilation" which consists of 557 samples and spans the redshift range $0 \lesssim z \lesssim 1.55$ [103]. In this data set, they improved the data analysis method by using and refining the approach of their previous work [104]. When comparing with the previous "Union Compilation", they extended the sample with the supernovae from refs. [103, 105]. The authors also provide the covariance matrix of data with and without systematic errors and, in order to be conservative, we include systematic errors in our calculations.

In addition, we also consider another luminosity distance indicator provided by GRBs, that can potentially be used to measure the luminosity distance out to higher redshift than SNIa. GRBs are not standard candles since their isotropic equivalent energetics and luminosities span $3-4$ orders of magnitude. However, similarly to SNIa it has been proposed to use correlations between various properties of the prompt emission and also of the afterglow emission to standardize GRB energetics (e.g. ref. [106]). Recently, several empirical correlations between GRB observables were reported, and these findings have triggered intensive studies on the possibility of using GRBs as cosmological "standard" candles. However, due to the lack of low-redshift long GRB data to calibrate these relations, in a cosmology-independent way, the parameters of the reported correlations are given assuming an input cosmology and obviously depend on the same cosmological parameters that we would like to constrain. Thus, applying such relations to constrain cosmological parameters leads to biased results. In ref. [107] this "circular problem" is naturally eliminated by marginalizing over the free parameters involved in the correlations; in addition, some results show that these correlations do not change significantly for a wide range of cosmological parameters [108, 109]. Therefore, in this thesis we use the 69 GRBs over a redshift range $z \in [0.17, 6.60]$ presented in ref. [109], but we keep into account in our statistical analysis the issues related to the circular problem that are more extensively discussed in ref. [107] and also the fact that all the correlations used to standardize GRBs have scatter and are poorly understood under the physical point of view. For a more extensive discussion and for a full presentation of a GRB Hubble

Diagram with the same sample that we used we refer the reader to section 4 of ref. [109].

In the calculation of the likelihood from SNIa and GRBs, we have marginalized over the absolute magnitude M which is a nuisance parameter, as done in refs. [110, 111]

$$\bar{\chi}^2 = A - \frac{B^2}{C} + \ln\left(\frac{C}{2\pi}\right) , \tag{5.8}$$

where

$$A = \sum_i \frac{(\mu^{\rm data} - \mu^{\rm th})^2}{\sigma_i^2} , \; B = \sum_i \frac{\mu^{\rm data} - \mu^{\rm th}}{\sigma_i^2} , \; C = \sum_i \frac{1}{\sigma_i^2} . \tag{5.9}$$

BAOs have been detected in the current galaxy redshift survey data from the SDSS and the Two-degree Field Galaxy Redshift Survey (2dFGRS) [117, 118, 119]. The BAO can directly measure not only the angular diameter distance, $D_{\rm A}(z)$, but also the expansion rate of the universe, $H(z)$, which is powerful for studying dark energy [120]. Since current BAO data are not accurate enough for extracting the information of $D_{\rm A}(z)$ and $H(z)$ separately [121], one can only determine an effective "volume" distance [117]

$$D_{\rm V}(z) \equiv \left[(1+z)^2 D_{\rm A}^2(z) \frac{cz}{H(z)}\right]^{1/3} . \tag{5.10}$$

In this thesis we use the Gaussian priors on the distance ratio of the volume distances as recently extracted from the SDSS and 2dFGRS surveys [119] at $z = 0.35$ and at $z = 0.2$ (the two mean redshifts of the surveys)

$$\frac{D_{\rm V}(z=0.35)}{D_{\rm V}(z=0.2)} = 1.736 \pm 0.065 \; (1\,\sigma) . \tag{5.11}$$

The χ^2 of BAO data used in the Monte Carlo Markov Chain analysis will thus be

$$\chi^2_{\rm BAO} = \frac{(D_{\rm V}(z=0.35)/D_{\rm V}(z=0.2) - 1.736)^2}{0.065^2} . \tag{5.12}$$

It is worth stressing here that both the physics and the data of BAOs depend on the content in matter of the universe $\Omega_{\rm m}$. Hence, they are *a priori* dependent on a chosen dynamical framework (see also ref. [122] for a review). This issue is usually ignored in the data analyses performed in the literature. However, such an approximation turns out to be valid if one does not range far away from the typical fiducial model firstly used in the determination of the physical data points. Indeed, the deviation of different models from the fiducial one can be parametrized and estimated by the ratio D_V(new model)/D_V(fiducial model). The impact of the spacetime priors on the power spectrum measurement was analyzed in ref. [123] and led to the conclusion that the ratio eq. (5.11) is only weakly dependent on dynamical features. Hence, we can safely use BAOs as a further tool to constrain the cosmographic parameters.

Next step in our analysis is the inclusion of the CMB measurement which is sensitive to the distance from the last scattering surface via the locations of peaks and troughs of the acoustic oscillations. This data constrains the curve of the cosmological history at very high redshift, $z \simeq 1100$, and hence could be very helpful to discriminate among competing theoretical models for dark energy, as they necessarily have to coincide at $z \leq 1$ – see for example ref. [131]. The sound horizon at the decoupling[2], $r_s(z_*)$, sets a physical scale for the baryon-photon oscillations depending on the baryon density, the photon energy density, and the cold dark matter density. Now, it is known that the angular diameter distance $D_A(z)$ describes the ratio between the proper size of an object at a certain redshift z and the correlated observed angular size. The angle θ_A under which the sound horizon is observed today is given by

$$\theta_A \equiv \pi l_A^{-1} \equiv r_s(z_*)/D_A(z_*) = 0.593^\circ \pm 0.001^\circ \ (1\,\sigma)\,, \tag{5.13}$$

where l_A denotes the location of the first peak in the multipole space. As for BAOs, the dependence on the cosmological density parameters would not allow the use of CMB observables in a fully cosmographic approach. Following [124] is anyway possible to give model-independent cosmological constraints if one clarifies some extra physical assumptions to be fullfilled by cosmological models. The CMB power spectrum today (apart from the low multipoles shape) is shared by models having the same primordial perturbation spectra and the same value of $\Omega_{\rm CDM}$ and $\Omega_{\rm baryon}$. For this reason a model-independent approach can be handle by restricting our analysis to models having a standard physics up to the decoupling era; asking that

[2] In our calculation, we choose $z_* = 1091.3$, the best fit value obtained by the WMAP group [132].

new physics after decoupling only modifies the small angle spectrum by changing the overall amplitude and $D_A(z_*)$; requiring that any multipole-dependent effect at late time remain small. Such assumptions, while is cutting away some models like $f(R)$ models with no Dark Matter [125] or models with new radiation degrees of freedom, are still general enough to cover most of the cosmological models on the market.

Finally we add the direct determinations of the Hubble parameter $H(z)$ to constrain the cosmographic expansion. Since the Hubble parameter depends on the differential age of the Universe as a function of redshift,

$$H(z) = -\frac{1}{1+z}\frac{dz}{dt} \ , \tag{5.14}$$

measuring the dz/dt could directly estimate $H(z)$. Ref. [112] used the Sloan Digital Sky Survey data and obtained a measurement of $H(z)$ at the redshift $z \simeq 0$. In ref. [113], the public data of Gemini Deep Survey (GDDS) survey [114] and archival data [115] were used in order to get the differential ages of galaxies. In practice, they selected samples of passively evolving galaxies with high-quality spectroscopy, and then used stellar population models to constrain the age of the oldest stars in these galaxies (we refer to their paper for a more exhaustive explanation of the method used). After that, they computed differential ages at different redshift bins and obtained eight determinations of the Hubble parameter $H(z)$ in the redshift range $z \in [0.1, 1.8]$. We calculate the χ^2 value of this Hubble parameter data by using

$$\chi^2_{\rm Hub} = \sum_{i=1}^{9} \frac{(H^{\rm th}(z_i) - H^{\rm obs}(z_i))^2}{\sigma^2_H(z_i)} \ , \tag{5.15}$$

where $H^{\rm th}(z)$ and $H^{\rm obs}(z)$ are the theoretical and observed values of Hubble parameter, and σ_H denotes the error bar of observed data. We also make use of the newly released prior on the Hubble parameter H_0, which consists of a measurement of the Hubble parameter obtained by the Near Infrared Camera and Multi-Object Spectrometer (NICMOS) Camera 2 of the Hubble Space Telescope (HST).

These observations fix the parameter $H_0 = 100h_0\,$(km/s)/Mpc by a Gaussian likelihood function centered around $H_0 = 74.2\,$(km/s)/Mpc and with a standard deviation $\sigma = 3.8\,$(km/s)/Mpc [116]. We stress that all the mentioned methods for determining $H(z)$ are "gravitation theory independent".

An important point must be underlined: the Taylor series of the Hubble parameter already includes into the coefficient of the n-th y-power the same number of cosmographic parameters of the other series expanded up to the $(n+1)$-th y-power (see Appendix). This is due, in comparison with the other distance definitions above, to an extra derivative with respect to time included in the definition of the Hubble parameter (see also [101]).

For this reason, and for the different nature of the Hubble data, we will initially consider constraints (based on standard candles and rulers) of the expansion coefficients associated to different notions of distances; at the end, we will add the Hubble data using one order less in the y-power expansion with respect to the distance data in order to constrain the same set of parameters.

In order to compute the likelihood, we use a Monte Carlo Markov Chain technique as it is usually done in order to explore efficiently a multi-dimensional parameter space in a Bayesian framework. For each Monte Carlo Markov Chain calculation, we run four independent chains that consist of about 300,000 − 500,000 chain elements each. We test the convergence of the chains by using the Gelman and Rubin criterion [126] with $R-1$ of order 0.01, which is more conservative than the often used and recommended value $R-1<0.1$ for standard cosmological calculations.

5.4 Data analysis

In this section we present our main results on the constraints for the cosmographic expansion from the current observational data sets.

With the accumulations of new data and the improvements of their quality, it is of great interest to estimate the free parameters in the polynomial terms of highest order. We have already showed in the past [97] the inconsistency of the results in the analysis of the cosmographic expansion caused by early truncations of the power series. For these reasons we will now present the results obtained for the most meaningful term of the expansion. In order to find out which is the most viable truncation of the series for a given data set, one can use a test comparing two nested models (in this case, two different truncations of the Taylor series). The F-test provides exactly this criterion of comparison, identifying which of two alternatives fits better, and in the more statistically significant way, the data.

Table 5.1: Constraints on the cosmography parameters up to fifth order term from different data combinations.

Data	SNIa				
Parameter	q_0	j_0	s_0	c_0	H_0
Best Fit	-0.41	-1.99	–	–	–
Mean	-0.41 ± 0.16	-1.99 ± 1.36	–	–	–
$\chi^2_{\rm min}$/d.o.f.	549.69/555				
Data	SNIa+GRB				
Parameter	q_0	j_0	s_0	c_0	H_0
Best Fit	-0.78	5.03	50.18	–	–
Mean	-0.76 ± 0.26	4.82 ± 4.07	53.57 ± 46.38	–	–
$\chi^2_{\rm min}$/d.o.f.	628.70/623				
Data	SNIa+GRB+BAO+CMB 4^{th} order				
Parameter	q_0	j_0	s_0	c_0	H_0
Best Fit	-0.32	-2.57	-18.40	–	–
Mean	-0.28 ± 0.17	-2.88 ± 1.64	-17.61 ± 2.56	–	–
$\chi^2_{\rm min}$/d.o.f.	633.33 / 625				
Data	SNIa+GRB+BAO+CMB 5^{th} order				
Parameter	q_0	j_0	s_0	c_0	H_0
Best Fit	-0.17	-6.92	-74.18	-10.58	–
Mean	-0.49 ± 0.29	-0.50 ± 4.74	-9.31 ± 42.96	126.67 ± 190.15	–
$\chi^2_{\rm min}$/d.o.f.	627.61/624				
Data	SNIa+GRB+BAO+Hub+CMB (5^{th} order) +Hub (4^{th} order)				
Parameter	q_0	j_0	s_0	c_0	H_0
Best Fit	-0.24	-4.82	-47.87	-49.08	71.65
Mean	-0.30 ± 0.16	-4.62 ± 1.74	-41.05 ± 20.90	-3.50 ± 105.37	71.16 ± 3.08
$\chi^2_{\rm min}$/d.o.f.	639.81/633				

In such test, one assumes the correctness of one of the models (the one with less parameters), and then assesses the probability for the alternative model to fit the data as well. If this probability is high, then no statistical benefit comes from the extra degrees of freedom associated to the new model. Hence, the smaller the probability, the more significant the data fitting of the second model against the first one will be. Quantitatively, the F-ratio among the two polynomials is defined as

$$F \equiv \frac{(\chi_1^2 - \chi_2^2)}{\chi_2^2} \, \frac{N - n_2}{n_2 - n_1}\,, \tag{5.16}$$

where N is the number of data points, and n_i represent the number of parameters of the i-model. The P-value, *i.e.* the area subtended by the F-distribution curve delimited from the F-ratio point, quantifies the viability of matching models as already mentioned. We use the threshold of 5% as the significance level on the P-value under which the model with one more parameter fits the data better than the other one.

We already found in ref. [97] that variations of the total energy density of the universe $\Omega_0 = 1 - \Omega_{k_0}$, with the spatial curvature parameter ranging between -1 and 1, have a negligible effect on the cosmographic constraints. This is basically due to the fact that the error bars for the cosmographic parameters are still quite large in comparison with the best fit values. Nonetheless, it is worth noting that this will not be necessarily the case when future data, especially at moderate or high redshift, will improve the constraints. It is then possible that future cosmographic analysis might have to include the spatial curvature effects in the reconstruction of the overall history of the universe. This would be the cosmographic expansion counterpart of the strong sensitivity on Ω_{k_0} showed by the reconstruction of $w(z)$ [127].

We then assume $\Omega_{k_0} = 0$ in our analysis and only present the results for the cosmographic parameters, instead of their combinations with Ω_{k_0}, since the effect of curvature can be safely neglected. Table 5.1 shows the constraints on the cosmography parameters as obtained from different data combinations.

We start performing the data analysis with the SNIa data only. We find that already at the fourth order term in the expansion, the minimal χ^2 is 549.59. This is not reduced significantly when compared with the constraint of the third order case, which has $\chi^2_{\rm min} = 549.69$. Hence, introducing the snap free parameter s_0 does

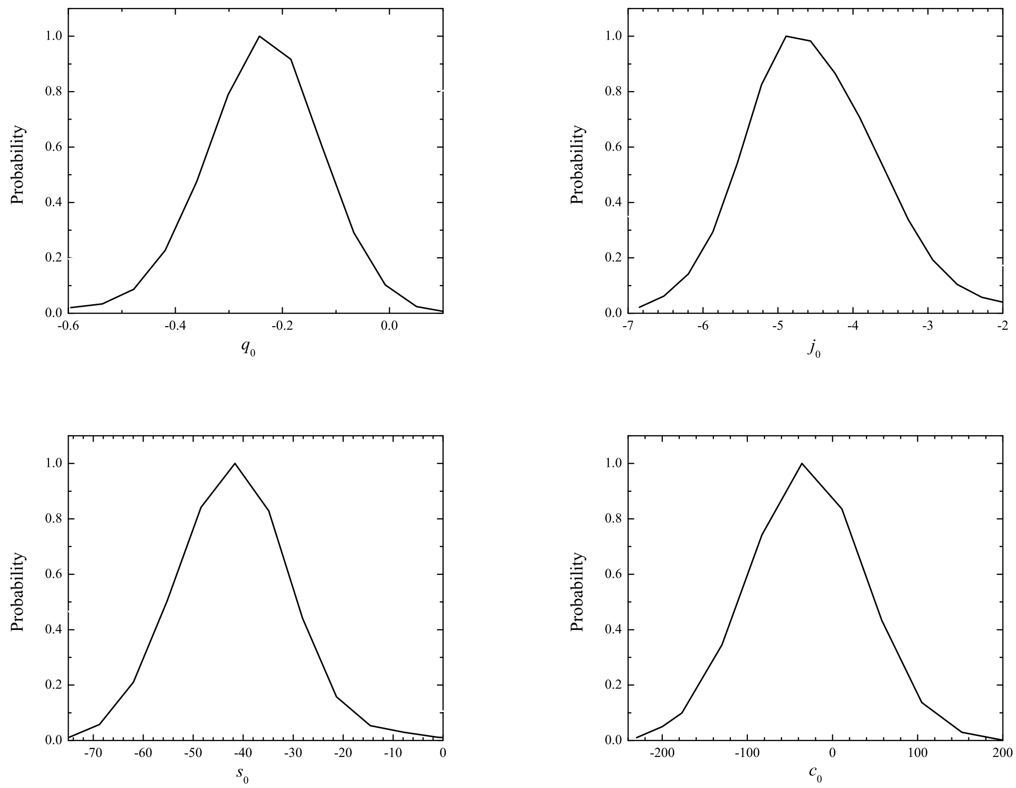

Figure 5.1: One-dimensional likelihood distributions for the parameters q_0, j_0, s_0 and c_0 for the data combinations SNIa+GRB+BAO+CMB+Hub.

not improve the constraints. Indeed, using the F-test, we find a F-ratio of 0.11 and a P-value of 73.93%. Therefore, cosmography up to the fourth order term does not fit the SNIa data significantly better: the cosmographic expansion up to the jerk term j_0 (third order) is enough.

After adding the GRB data, the fourth order case could give a better constraint than third order only. When comparing the SNIa+GRB results, the minimal χ^2 has been reduced by about five ($\chi^2 = 628.70$ instead of $\chi^2 = 633.32$). Using one more time the F-test to contrast third and fourth order expansions, we find F-ratio $= 4.59$, P-value $= 3.26\%$. Thus in this case the fourth order term indeed helps to fit the observed data significantly better. The inclusion of GRBs was found to constrain the deceleration parameter q_0 as $q_0 = -0.76 \pm 0.26$, so confirming that our universe is undergoing an accelerated expansion with a confidence level which is marginally at $3\,\sigma$ [97][3]. The F-test does not suggest to further improve the expansion up to fifth order.

Including the data point related to BAO does not improve significantly the constraints. The constraining power of BAO is rather weak since there is only one BAO data point and its redshift is much lower than those of SN and GRB data. For this reason we will consider directly the data set that includes both BAO and CMB.

When the CMB angle θ_A defined in eq. (5.13) is added into our analysis, the cosmographic curve is constrained at very high redshift, $z_* \simeq 1100$ (namely $y \simeq 1$). Even though the CMB observable is providing just one data point, due to its high redshift it is in principle very helpful for discriminating among competing theoretical models producing late time accelerated expansion, since these necessarily converge to the same cosmological history at small z (see for example [131]). The large difference between the two $\chi^2_{\rm min}$ of the fourth and fifth order expansion in powers of y of the distances, implies that the latter is the (statistically) more reliable parametrization ($F_{\rm ratio} = 5.69$ and $P_{\rm value} = 0.02\%$), giving a result very close to the ΛCDM prediction. Sixth order expansions does not give any substantial statistical improvement.

As already stated at the end of the previous section, Hubble data must be added and analyzed cautiously, since they are inhomogeneous with respect to the previous data sets both in nature and mathematical handling. In Table I we present directly the results for the constraints up to the c_0 cosmographic parameter, since this truncation turns out to be strongly favored with respect to the previous nested model ($F_{\rm ratio} = 19.77$ and $P_{\rm value} < 0.01\%$).

The theoretical curves of $\mu(z)$ and $H(z)$ are in good agreement with the observed cosmological data, as shown in figure 5.2. The constraint on H_0 is close to the usually quoted value, namely at 68% confidence level is $H_0 = 71.16 \pm 3.08$ (km/s)/Mpc. One can see that the addition of the Hubble data leads to relevant improvements in the determination of the other cosmographic parameters with the exception of c_0, which is still basically unconstrained. We also checked whether the next cosmographic parameter had to be included. We find that, for the richest combination SN+GRB+BAO+CMB+Hub, the new $\chi^2_{\rm min}$, is extremely close to the value in Table

[3]Note that our best fit here is different from the one reported in ref. [97] due to our use of the improved SN catalogue "Union2 Compilation".

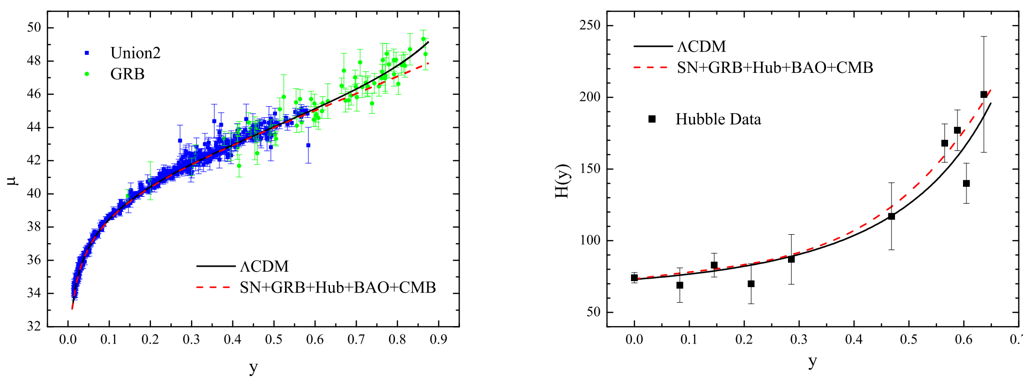

Figure 5.2: Theoretical predictions of distance moduli (left panel) and Hubble parameter (right panel) from the best fit model with the full data combination, together with the observed data sets. We also show the curves obtained in the ΛCDM framework for comparison (thin black solid lines).

5.1. Therefore, we stop our analysis here.

It is here interesting to underline the power of the y-expanded series (convergent as long as $y < 1$) allowing us to describe the whole cosmological history with the use of relatively few parameters. This circumstance becomes for example evident in the left panel of figure 5.2, where the furthest GRB data point reaches the distance of $y \simeq 0.87$ (corresponding to $z \simeq 6.6$).

5.5 Forecasting

Since the present data do not give yet very stringent constraints on the parameters of cosmography, especially for the parameter of fifth order term, it is worthwhile discussing whether future data could determine these parameters more effectively. For this purpose in what follows we shall perform new analysis of possible future constraints by choosing, as a fiducial model, the best fit parameter set for the cosmographic expansion up to the fifth order term as fixed by the combination of all the previously considered data sets.

The projected satellite SNAP (Supernova / Acceleration Probe) would be a space based telescope with a one square degree field of view with 10^9 pixels. It aims at increasing the discovery rate for SNIa to about 2000 per year in the redshift range $0.2 < z < 1.7$. In this thesis we simulate about 2000 SNIa according to the forecast distribution of the SNAP. For the error, we follow the ref. [128] which takes the magnitude dispersion 0.15 and the systematic error $\sigma_{\rm sys} = 0.02 \times z/1.7$. The whole error for each data is given by $\sigma_{\rm mag}(z_i) = \sqrt{\sigma_{\rm sys}^2(z_i) + 0.15^2/n_i}$, where n_i is the number of Supernovae of the i-th redshift bin. Furthermore, we add as an external data set a mock data set of 400 GRBs, in the redshift range $0.4 < z < 6.4$ with an intrinsic dispersion in the distance modulus of $\sigma_\mu = 0.16$ and with a redshift distribution very similar to that of figure 1 of ref. [129].

Regarding a future BAO data set, we adopt the predicted performance of the BOSS survey in SDSS III, which will measure the angular diameter distance $d_A(z)$ and the Hubble expansion rate $H(z)$ of the Universe over a broad range of redshifts. The measurement precision for $d_A(z)$ is 1.0%, 1.1%, and 1.5% at $z = 0.35$, 0.6, and 2.5, respectively, and the forecast precision for the $H(z)$ is 1.8%, 1.7%, and 1.2% at the same redshifts [130]. We also impose a Gaussian prior for the current Hubble parameter H_0 with the error of 1% provided by a future direct measurement.

Next coming CMB measurement, mainly via the Planck satellite, could give quite accurate constraints on the cosmological parameters. The error bar of θ_A could be shrunk by a factor of 3, namely, the standard derivation $\sigma_{\theta_A} = 0.0003°$.

Using all these future mock data, we get the standard derivations of cosmographic parameters: $\sigma_{q_0} = 0.02$, $\sigma_{j_0} = 0.08$, $\sigma_{s_0} = 1.95$, $\sigma_{c_0} = 14.20$ and $\sigma_{H_0} = 0.48$, respectively. We hence see that the constraints on the parameters provided by the future mock data can be strongly improved in comparison with the current constraints in Table 5.1.

5.6 Cosmographic selection of viable cosmological models

In the case of the standard flat ΛCDM model (namely a model described by Cold Dark Matter with the adding of a cosmological parameter) the set of cosmographic parameters results to be (up to fifth order)

$$\begin{aligned} q_0 &= \frac{3}{2}\Omega_{m_0} - 1 \\ j_0 &= 1 \\ s_0 &= 1 - \frac{9}{2}\Omega_{m_0} \\ c_0 &= 1 + 3\Omega_{m_0} + \frac{27}{2}\Omega_{m_0}^2 \, . \end{aligned} \tag{5.17}$$

We can use independent probes to constrain the free parameters of the cosmological model, in this case, for example, the WMAP estimates of Ω_{m_0} for the ΛCDM model.

The theoretical predictions of the cosmographic parameters in the standard ΛCDM model are: $q_0 = -0.588$, $j_0 = 1$, $s_0 = -0.238$ and $c_0 = 2.846$, where we set the current matter density to be the best fit value $\Omega_{m_0} = 0.275$ [4] obtained by the WMAP group [132]. Future experiments, in this perspective, will give stricter constraints on the validity of such hypothesis.

Another example is provided by the Dvali-Gabadadze-Porrati (DGP) self-accelerating braneworld model [133]. The presence of the infinite-volume extra dimension modifies the Friedmann equation as:

$$\frac{H^2}{H_0^2} = \Omega_k (1+z)^2 + \left(\sqrt{\Omega_{r_c}} + \sqrt{\Omega_{r_c} + \Omega_{m_0}(1+z)^3}\right)^2 \, , \tag{5.18}$$

with $\Omega_{r_c} = 1/4 r_c^2 H_0^2$ accounting for the fractional contribution of the bulk-induced term with respect to the crossover radius r_c. In a spatially flat universe, $\Omega_k = 0$ and $\Omega_{r_c} = (1 - \Omega_{m_0})^2/4$, the previous equation reads

$$\frac{H^2}{H_0^2} = \left[\frac{1-\Omega_{m_0}}{2} + \sqrt{\frac{(1-\Omega_{m_0})^2}{4} + \Omega_{m_0}(1+z)^3}\right]^2 \, , \tag{5.19}$$

so, expanding both the sides of eq. (5.19) and equating terms of the same power, we obtains the following expressions for the cosmographic coefficients as functions of the free parameter Ω_{m_0} (see also ref. [134])

---

[4] The estimate of Ω_{m_0} is of course known within a certain error. From now on, for illustrative purposes, we will retain the best fit values of the free parameters, independently estimated in every single cosmological model, as the fiducial ones, without taking into account the associated errors.

$$
\begin{aligned}
q_0 &= \frac{-1+2\Omega_{m_0}}{1+\Omega_{m_0}} \\
j_0 &= \frac{1+3\Omega_{m_0}-6\Omega_{m_0}^2+10\Omega_{m_0}^3}{(1+\Omega_{m_0})^{\,3}} \\
s_0 &= \frac{1-4\Omega_{m_0}+19\Omega_{m_0}^2-134\Omega_{m_0}^3+86\Omega_{m_0}^4-80\Omega_{m_0}^5}{(1+\Omega_{m_0})^{\,5}} \\
c_0 &= \frac{1+13\Omega_{m_0}-141\Omega_{m_0}^2+1259\Omega_{m_0}^3-1996\Omega_{m_0}^4+3828\Omega_{m_0}^5-1604\Omega_{m_0}^6+880\Omega_{m_0}^7}{(1+\Omega_{m_0})^{\,7}}\,.
\end{aligned}
\tag{5.20}
$$

In ref. [135], the DGP model has been constrained starting from gravitational lensing statistics; considering the fractional amount of matter obtained therein, $\Omega_{m_0} = 0.30$, we obtain the following set of values for the previous parameters: $q_0 = -0.308$, $j_0 = 0.742$, $s_0 = -0.432$, $c_0 = 2.926$.

We will now take into account the so-called Cardassian cosmology [136], a model whose modification with respect to standard ΛCDM cosmology consists in the introduction of an additional term ρ^n in the matter source of the Friedmann equation, so that now it can be written in term of the fractional matter density as:

$$
\frac{H^2}{H_0^2} = \Omega_{m_0}(1+z)^3 + (1-\Omega_{m_0})\,(1+z)^{3n}\,. \tag{5.21}
$$

Performing one more time the expansion of both sides of the equation, the first four cosmographic parameters can now be expressed as functions of the two parameters Ω_{m_0} and n

$$
\begin{aligned}
q_0 &= \frac{1}{2}+\frac{3}{2}(1-n)\,(\Omega_{m_0}-1) \\
j_0 &= \frac{1}{2}\left[2+9n\,(\Omega_{m_0}-1)+9n^2\,(1-\Omega_{m_0})\right] \\
s_0 &= \frac{1}{4}\left[4-18\Omega_{m_0}-9n\left(4-7\Omega_{m_0}+3{\Omega_{m_0}}^2\right)+9n^2\left(11-17\Omega_{m_0}+6{\Omega_{m_0}}^2\right)-\right. \\
&\left.\quad -27n^3\left(3-4\Omega_{m_0}+{\Omega_{m_0}}^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
c_0 \;=\; & \left(1+3\Omega_{m_0}+\frac{117\Omega_{m_0}^2}{2}-\frac{243\Omega_{m_0}^3}{2}+\frac{1215\Omega_{m_0}^4}{16}\right)- \\
& -\frac{3}{4}\left(32-80\Omega_{m_0}+291\Omega_{m_0}^2-648\Omega_{m_0}^3+405\Omega_{m_0}^4\right)n+ \\
& +\frac{9}{8}\left(136-242\Omega_{m_0}+349\Omega_{m_0}^2-648\Omega_{m_0}^3+405\Omega_{m_0}^4\right)n^2- \\
& -\frac{27}{4}\left(46-73\Omega_{m_0}+54\Omega_{m_0}^2-72\Omega_{m_0}^3+45\Omega_{m_0}^4\right)n^3+ \\
& +\frac{81}{16}\left(39-56\Omega_{m_0}+26\Omega_{m_0}^2-24\Omega_{m_0}^3+15\Omega_{m_0}^4\right)n^4\,, \qquad (5.22)
\end{aligned}
$$

and for the referring values $\Omega_{m_0} = 0.271$ and $n = 0.035$ [137], eq. (5.22) reads $q_0 = -0.555$, $j_0 = 0.890$, $s_0 = -0.384$, $c_0 = 3.660$.

Finally, we want to show the coefficients in the cosmographic approach of the CPL parametrization [138] for the equation of state of Dark Energy. If we suppose to be in a flat universe, then the Friedmann equation is:

$$
\frac{H^2}{H_0^2} = \Omega_{m_0}(1+z)^3 + (1-\Omega_{m_0})\,(1+z)^{3(1+w_0+w_a)}e^{-\frac{3w_a z}{1+z}}\,, \qquad (5.23)
$$

and the related cosmographic terms result to be (confront also with ref. [139])

$$
\begin{aligned}
q_0 \;=\; & 1+\frac{3}{2}w_0\,(1-\Omega_{m_0}) \\
j_0 \;=\; & 1+\frac{3}{2}\left(3w_0+3w_0^2+w_a\right)(1-\Omega_{m_0}) \\
s_0 \;=\; & -\frac{7}{2}-\frac{33}{4}\,(1-\Omega_{m_0})\,w_a-\frac{9}{4}\,(1-\Omega_{m_0})\,[9+(7-\Omega_{m_0})\,w_a]\,w_0- \\
& -\frac{9}{4}\,(1-\Omega_{m_0})\,(16-3\Omega_{m_0})\,w_0^2-\frac{27}{4}\,(1-\Omega_{m_0})\,(3-\Omega_{m_0})\,w_0^3 \\
c_0 \;=\; & \frac{1}{4}\,(70+3w_a\,(-71+3w_a\,(-7+\Omega_{m_0}))\,(-1+\Omega_{m_0}))+ \\
& +\frac{3}{4}\,(-1+\Omega_{m_0})\,(-163+3w_a\,(-82+21\Omega_{m_0}))\,w_0+ \\
& +\frac{9}{4}\,(-1+\Omega_{m_0})\,(-134-69w_a+3\,(14+11w_a)\,\Omega_{m_0})\,w_0^2+ \\
& +\frac{1}{4}\left(1269-1917\Omega_{m_0}+648\Omega_{m_0}^2\right)w_0^3+\frac{1}{4}\left(486-810\Omega_{m_0}+324\Omega_{m_0}^2\right)w_0^4\,; \qquad (5.24)
\end{aligned}
$$

assuming the values suggested by the seventh-year-release of WMAP [132] for the three free parameters, $\Omega_{m_0} = 0.275$, $w_0 = -0.93$ and $w_a = -0.41$, we get $q_0 =$

-0.511, $j_0 = 0.342$, $s_0 = -2.260$, $c_0 = 1.383$. Table 5.2 shows the values of the cosmographic parameters in the different models taken into account.

It is interesting to note a couple of issues in the comparison of cosmological models with our best fits. Firstly, the best fit for the cosmographic parameters of the SN+GRB+BAO+CMB data set is perfectly compatible with the estimates of the cosmological parameters for a broad variety of models. However, it is easy to realize that the currently available data sets would not allow yet to distinguish among the different cosmological models. In fact, Table 5.2 shows that the error bars are still too large with respect to the differences among the cosmographic parameters of the cosmological models. Nonetheless, the previously discussed forecasted improvement in the quality and the quantity of such data (ameliorating by at least a factor ten the error bars on the cosmographic parameter) should be able to discriminate among competing models.

On the contrary, the best fit of the widest combination of data (that is with the inclusion of the Hubble parameter determinations via the differential age technique), seems to exclude, at a 3-σ around the jerk mean value j_0, almost all cosmological models, including ΛCDM (with the only exception of the CPL modelization, that is still marginally compatible). We have already discussed the intrinsic difference of the Hubble data and why their use should be taken cautiously. It seems clear that this data set while being very powerful in reducing the error bars, is simultaneously introducing strong deviations from the mean values determined via standard candles and rulers. This puzzling discrepancy in the results does not seem related to the order of the truncation: we observed a similar behavior even for (statistically not favored) early or late truncations of the series.

However, it is also true that the high redshift measurements of the Hubble parameter are based on fitting galaxy spectra. As such, this data set is strongly dependent on this fitting procedure which may introduce systematic effects. For this reason, we deem estimates based on the Hubble data currently less robust than those based on standard candles and rulers. Nonetheless, their use here serves to show the possible key role these data could play in the future of Cosmography as they appear to be very effective in reducing the error bars and very sensitive tracers of the cosmological history. We hence conclude that our analysis strongly suggests further investigation of this apparent tension between the Hubble data and ΛCDM (and many competing

Table 5.2: Comparison among cosmographic parameters of different cosmological models. For every model, the evaluation of the cosmographic parameters, for a pedagogical issue, is based on the best fit values of the free parameters introduced in the dynamics and measured with independent probes. However, the value of the jerk parameter for ΛCDM model is an exact value, as can be seen from equations (5.17). The values of the cosmographic parameters are compared with our best fits for the series truncations studied in the last two lines of Table I. "Data set A" includes, up to 5^{th} order, the proper distances indicators, namely SNIa, GRB, BAO, CMB. "Data set B" is the complete data set, obtained adding Hubble data up to the 4^{th} order (for further details, see section 5.6).

Parameter	q_0	j_0	s_0	c_0
ΛCDM	-0.588	1	-0.238	2.846
DGP	-0.308	0.742	-0.432	2.926
Cardassian	-0.555	0.890	-0.384	3.660
CPL Paramet.	-0.511	0.342	-2.260	1.383
Best fit				
Data set A	-0.49 ± 0.29	-0.50 ± 4.74	-9.31 ± 42.96	126.67 ± 190.15
Data set B	-0.30 ± 0.16	-4.62 ± 1.74	-41.05 ± 20.90	-3.50 ± 105.37

models) via a refinement of the determination methods developed in [112, 113].

5.7 Summary

Reaching the highest possible redshift allowed by data is a fundamental tool to discriminate among competing cosmological models. Given that most of the models are built in order to recover Dark Energy at low redshift, their expansion histories are degenerate at late times. To break such a degeneracy, the main requirement is having the knowledge of the early universe expansion curve: this aim can be achieved only by an accurate determination of the higher order parameters, and higher terms in the cosmographic expansion can be consistently reached only using (very) high redshift data.

In the data set we took into account, apart for Supernovae and GRBs, we considered Baryonic Acoustic Oscillations (that are distance indicators at $z \sim 0.3$), a compilation of high redshift Hubble parameter measurements and, at least for a wide gamut of cosmological models, CMB data about the angular size of the sound horizon. This improved data set is helpful in that, apart from better constraining the previously studied cosmographic parameters, it also allows to cast constraints on the next order, so far unbound, expansion coefficient.

The analysis is performed by using Monte Carlo Markov Chains in the multidimensional parameter space to derive the likelihood. As a first step, we consider the most recent catalogs of standard candles, namely Supernovae Type Ia and (properly standardized, see discussion in section 5.3) GRBs. A combination of such data gives constraints up to the fourth order parameter s_0 in the cosmographic series. We have also used the BAO (albeit they mildly improve the cosmographic series fitting) discussing the reliability of such tools in this context.

Secondly, we add data at higher redshift from different probes to further improve the constraints. The CMB data account for a very stable and well determined scale. It is worth noting here, anyway, that on the contrary of the other probes, CMB data provide the problem of a lack of universality in the cosmographic approach. Unfortunately, the set of parameters extracts from CMB observations is not truly independent from the dynamics of the underlying gravitational theory. Its definition, in fact, strictly depends on the assumption of a cosmological model that behaves as General Relativity plus a content of matter of arbitrary nature. It is hence impossible to use it straightforwardly within a purely cosmographic analysis which wants to apply also to non-standard cosmologies (based on exotic modified gravity theories)[5]. In this thesis we proposed CMB data constraints on the cosmographic series by restricting the results to a slightly smaller variety of models. A desirable full solution to this problem can be achieved "standardizing" somehow the CMB parameters or alternatively identifying other CMB observables which could be used

[5]Of course, CMB observables can be used within a given gravitational dynamics to fix the free variables of a cosmological model [140] and hence calculate the corresponding cosmographic parameters to be confronted with those determined purely on the base of standard candles and rulers, as we also showed as application to the evaluation of cosmographic parameters in several cosmological models.

as standard rulers (at least approximately, as for BAOs). We leave this to future investigations.

We then added the high redshift measurements of the Hubble parameter. We found that thanks to these data and the CMB one, it is possible to ameliorate the knowledge of the cosmographic expansion up to the c_0 parameter.

As a completion of our analysis, we have also discussed foreseeable constraints from futuristic data sets provided by projected experiments. We showed that a strong reduction of the typical errors on the parameters estimates is a realistic goal: future surveys, indeed, do have a solid possibility to sufficiently reduce the uncertainties on the lowest order parameters by a factor ten at least, gaining a concrete chance to assess the viability of alternative cosmological models (possibly based on different dynamics).

Finally, we calculated the cosmographic parameter sets for a sample of cosmological models with alternative dynamics (using the so far available best fits for their free parameters). We showed that, while the data set including "standard" distance indicators gives a best fit with which all the cosmological models are still compatible, the inclusion of the Hubble data introduces a tension between the observed cosmographic parameters and the parameters calculated for different models and in particular with ΛCDM which appears to be ruled out at 3-σ due to the jerk best fit value. We have discussed the reliability of such observation taking into account the inhomogeneity of the Hubble data set with respect to the distance indicators ones. While there might be systematic uncertainties in this data due to their complex determination method, we stressed that our analysis strongly suggests they might play a key role in reducing the errors on the estimates of the cosmographic parameters and hence in making Cosmography effective in discriminating among competing cosmological models and gravitational theories.

In conclusion, the search for high redshift standard rulers and most of all the improvement of the data coming from galaxy surveys seem to be what could possibly bring cosmographic studies into a mature stage and make them powerful, gravity theory independent, tools for selecting among theoretical scenarios. We hence hope that these considerations will further strengthen the case for proposed experiments aimed at improving our knowledge of the cosmic evolution of the high redshift universe (e.g. ref. [141]).

Chapter 6

Discussion and conclusions

In this thesis we have studied some aspects regarding the phenomenology and viability of theories beyond General Relativity.

After a brief review of modified gravity models, we have studied (in section 2.3 and chapter 3) the dynamics of theories of gravity in which the metric and the connection are independent quantities. Such theories are dubbed theories of gravity in the Palatini formalism.

Palatini approach have gained particular attention for what concern the phenomenology of $f(\mathcal{R})$ models, *i.e.* actions where the Lagrangian is some algebraic function of the Ricci scalar of the independent connection, $\mathcal{R}$. Such actions have recently attracted a lot of interest as possible infrared modifications of General Relativity. However, Palatini $f(\mathcal{R})$ gravity models with infrared corrections with respect to GR, have been shown to be non-viable for several reasons.

Generalized Palatini theories of gravity have also been considered. Unlike the exceptional case of the Einstein–Hilbert action, these theories are distinct from the theories one would get starting from the same action (formally) and applying standard metric variation. One cannot say that their dynamics has been well understood in general. That is because the dynamics of the most well studied class, $f(\mathcal{R})$, is rather peculiar and not representative. Indeed, in Palatini $f(\mathcal{R})$ gravity the independent (eventually non-symmetric) connection does not carry any dynamics and can be algebraically eliminated in favour of the metric and the matter fields. The lack of extra dynamics with respect to General Relativity can also be seen by the fact that Palatini $f(\mathcal{R})$ gravity has been shown to be dynamically equivalent to a

Brans–Dicke theory with Brans–Dicke parameter $\omega_0 = -3/2$. This is a particular theory within the Brans–Dicke class in which the scalar does not carry any dynamics and can be algebraically eliminated in favour of the matter fields.

The algebraic elimination of the connection (or the corresponding scalar field in the Brans–Dicke representation) will introduce extra matter interactions making Palatini $f(\mathcal{R})$ theories essentially equivalent to General Relativity with modified source terms. In fact, this property is what lies in the heart of all the viability issues mentioned earlier. However, this is not a generic property of generalized Palatini gravity, as it has been demonstrated in section 2.3, but just a peculiarity of $f(\mathcal{R})$ actions. Generic higher order actions lead to extra dynamical degrees of freedom. We also gave, as a simple example, the specific choice of action which is dynamically equivalent to an Einstein–Proca action (Einstein gravity plus a massive vector field). Moreover, we also identified some specific actions which constitute exceptions, and for which the independent connection can indeed be algebraically eliminated.

In order to move beyond the limits of the Palatini approach, in chapter 3 we considered metric-affine theories of gravity, namely modified theories in which, not only metric and connection are supposed to be independent, but the independent connection is also allowed to enter in the matter action. Instead of restricting ourselves to a specific action, which would inevitably affect the generality of our conclusions, we chose to follow an approach inspired by effective field theory and attempt to understand how are the dynamics of the theory affected when increasing the order of the invariants included in the action.

To this end we first considered the most general action formed by second order invariants and then moved on to examine how these would be modified by including different types of higher order terms in the action. In both cases we imposed a generalized minimal coupling principle in order to reduce the number of terms to be considered, which excludes invariants constructed with the non-metricity or the metric curvature.

We found that even for the most general action one can construct with second order invariants, the connection does not carry any dynamics and can always be algebraically eliminated. That is, at this order, metric-affine gravity can always be written as General Relativity with a modified source term or extra matter interactions. No extra degrees of freedom are excited.

Including higher order terms in the action changes the situation radically. The connection (or parts of it) becomes dynamical and so, it cannot be eliminated algebraically. The theory now propagates more degrees of freedom than General Relativity. Thus, seen as an effective field theory, metric-affine gravity is rather peculiar and its dynamics can deceive: at the lowest order the extra degrees of freedom appear to lose their dynamics and become auxiliary fields, but once higher order terms are taken into account the extra degrees of freedom do propagate. To avoid exciting extra degrees of freedom significant fine tuning and extra *a priori* constraints are required.

Let us also stress that $f(\mathcal{R})$ actions, which have been previously considered in metric-affine gravity, appear to constitute a distinct class with special properties. Even though the connection does carry dynamics in the presence of fields coupling to it — unlike the simplified case of Palatini $f(\mathcal{R})$ gravity — torsion remains non-propagating. The propagating degrees of freedom reside only in the symmetric part of the connection. In this sense, $f(\mathcal{R})$ actions cannot be considered representative examples of generic higher order metric-affine theories.

From an effective field theory perspective it seems that there are dynamical degrees of freedom in metric-affine gravity which appear to "freeze" at low energies and can be eliminated in favour of extra matter interaction. This implies that a possible low energy manifestation of metric-affine gravity could be revealed in matter experiments in terms of such interactions, but the phenomenology of metric-affine theories is not limited to that. It is much richer and it includes extra propagating degrees of freedom, which can potentially be detected. A typical, but certainly not the only, example would be the presence of propagating torsion, whose consequences have been studied in a limiting setting in [44] (See also Ref. [45] and references therein).

As already stressed in the introduction, a conceptually different explanation for the puzzling phenomenology of the actual universe, is that in the context of General Relativity the cosmic acceleration is due to the backreaction of inhomogeneities on the dynamics of an averaged background. In chapter 4 we reviewed some of the possible approaches to reach this aim. Then we analyzed the possibility of improving the averaging scheme in the prototypical alternative theories of gravity, the scalar-tensor ones. In scalar-tensor models, it has been adopted a field permeating the whole spacetime and that can be described as an effective form of matter by writing the field equations in the form of effective Einstein equations. The effective energy-momentum tensor characterizing this form of matter easily violates all the energy conditions and, therefore, is more likely to produce the cosmic acceleration. The backreaction of inhomogeneities on the cosmic dynamics has been studied in the context of the scalar-tensor gravity. Due to terms of indefinite sign in the non-canonical effective energy tensor of the Brans–Dicke-like scalar field, extra contributions to the cosmic acceleration can arise.

In chapter 5 we constrained the parameters describing the kinematical state of the universe using a cosmographic approach, which is fundamental in that it requires a very minimal set of assumptions (namely to specify a metric) and does not rely on the dynamical equations for gravity. On the data side, we considered the most recent compilations of Supernovae and Gamma Ray Bursts catalogues, assisted by a set of high redshift data, namely the Hubble parameter as measured from surveys of galaxies, the luminosity distance from Supernovae and Gamma Ray Bursts data and the Baryonic Acoustic Oscillations as seen in the power spectra of the distribution of galaxies. In order to reliably control the cosmographic approach at high redshifts, we have adopted the expansion in the improved parameter $y = z/(1+z)$. This series has the great advantage to hold also for $z > 1$ and hence it is the appropriate tool for handling data including non-nearby distance indicators.

The data set involved in the analysis presented allows to put constraints on the cosmographic expansion up to fifth order. In any case, as we already showed also in [97], it is worth noting that the order of the truncation of the series must be chosen carefully. In the present case, even though the statistical F-test suggests the fifth order as the most significant truncation, the error bar on the last parameter is indeed so large to make the data-fitting potentially unreliable. This aspect deserves much attention in the future cosmographic studies.

We then derived the set of the cosmographic parameters for several cosmological models (including ΛCDM) in order to compare them with our best fit set. Current data do not allow to discriminate among these competing models; nonetheless the upcoming large scale structure probes may substantially improve the precision with which the lowest cosmographic parameters will be determined so that the degeneracy among alternative cosmological frameworks will be relatively smaller. This seems to suggest, anyway, that obtaining standard candles/rulers from very high redshift data (*e.g.* using different cosmological observables from the spectrum of the cosmic microwave background) will be of crucial importance for the viability of cosmographic tests.

There are a variety of directions in which future research on these subjects can proceed. In particular, due to the wide fields of interests they span, metric-affine theories and generalized modified gravities surely need a deeper understanding.

From the physical point of view, it would be interesting to study some concrete examples of matter fields coupled to connection, namely those fields for which the hypermomentum does not vanish. A typical example is the Dirac field or any massive vector field or tensor field that, having an explicit dependence on the covariant derivative, leads to $\Delta_{\lambda}{}^{\mu\nu}$. In those cases, the fields are potentially able to induce both non-metricity and torsion (remember that those fields that do not introduce torsion because not coupled to the connection, also will not be affected by torsion even if other matter fields produce it). The same property holds for semiclassical spinning dust matter distributions, a generalization of the perfect fluid in the case of nonvanishing spin, a fluid otherwise dubbed Weyssenhoff [46] fluid; even though such kind of matter is an interesting toy model, it has an unsatisfactory theoretical formulation, since there is no unambiguous Lagrangian able to describe it. Instead, one has to postulate some convective forms for the energy-momentum and spin-angular momentum tensors, plus some restrictions to the fluid spin in order to insure the integrabilities conditions of the equations of motion of the particles.

The treatment of macroscopic matter configurations also leaves some important insights to discuss. If torsion is allowed, then the spin of the particles composing a perfect fluid must be taken into account. However, in most of the macroscopic situations particle spins are randomly oriented and not polarized, so that the first-order contribution in spin on the modified field equations vanishes when an averaging on a macroscopic space-time region is performed. Nonetheless, the modified total stress-energy tensor contains quadratic spin corrections that do not average to zero, hence also in the macroscopic limit of the metric-affine theories some non-trivial deviations from GR should be expected. As a last comment on this topic, it is worth mentioning that we should also contemplate some matter configurations involving imperfect fluids (with non vanishing viscosity and heat flux), systems with a certain relevance in the domain of relativistic astrophysics whose matter action is not expected to be independent from the connection, even (differently from perfect fluids) for the simpler case of a symmetric connection.

It would be also very interesting to understand in more detail if large deviations from General Relativity can be achieved when the extra degrees of freedom become active in the strong gravity regime, when higher order terms cannot be neglected. In the meantime, it should be properly studied how exactly these degrees of freedom modify matter interactions at low energies. It is also crucial to examine the predictions of such theories for energy conservation and violations of the various formulations of the equivalence principle. Such considerations would allow us to place constraints on metric-affine theories.

A possible point of concern can be our use of the generalized minimal coupling principle. One could argue that this is not compatible with our effective field theory perspective as radiative corrections would not respect such a principle. One could also feel uneasy treating non-metricity and torsion on a different footing. Indeed, the minimal coupling principle is used here mostly as a way to reduce the number of terms to be taken into consideration and it should not necessarily be considered as a fundamental principle. Abandoning it and considering the most general action possible would be the next step.

A closing remark: clearly, one might question how fundamental is the geometrical interpretation of metric-affine gravity. In fact, since for second order actions one can always eliminate the independent connection, the latter can be regarded as an auxiliary field. Even for actions with higher order terms though, where degrees of freedom residing in the connection will be excited, one could have an equivalent representation without an independent affine connection (recall that an independent connection can always be written as the Levi-Civita connection plus a tensor). Indeed, which representation one choose is a matter of preference, at least at a classical level, as the dynamical content of the theory is one and the same. On the other hand, it is worth pointing out that the choice of representations becomes a factor when constructing the action of the theory. It influences our choices regarding the presence of some terms by making some exclusion principles, such as minimal coupling and its generalizations, more or less appealing (see also Ref. [13] for a more general discussion on this issue). This is a subtle point that needs to be taken seriously into account when performing such studies.

Appendix A

Series expansions for cosmography

We present here more extensively the expansions used in chapter 5. A flat universe, $k = 0$, is assumed in all the expressions below.

Hubble parameter:

$$H[z(y)] = H_0 \left[1 + (q_0+1)y + y^2\left(\frac{j_0}{2} - \frac{q_0^2}{2} + q_0 + 1\right) + \right.$$
$$+\frac{1}{6}y^3\left(-4j_0q_0 + 3j_0 + 3q_0^3 - 3q_0^2 + 6q_0 - s_0 + 6\right) +$$
$$+\frac{1}{24}y^4\left(-4j_0^2 + 25j_0q_0^2 - 16j_0q_0 + 12j_0 + c_0 - 15q_0^4 + \right.$$
$$\left.\left. +12q_0^3 - 12q_0^2 + 7q_0s_0 + 24q_0 - 4s_0 + 24\right)\right] + \mathcal{O}[y^5]\,; \tag{A.1}$$

Luminosity distance:

$$d_L[z(y)] = \frac{1}{H_0}\left[y + \left(\frac{3}{2} - \frac{q_0}{2}\right)y^2 + y^3\left(\frac{q_0^2}{2} - \frac{5q_0}{6} - \frac{j_0}{6} + \frac{11}{6}\right) + \right.$$
$$+y^4\left(\frac{5j_0q_0}{12} - \frac{7j_0}{24} - \frac{5q_0^3}{8} + \frac{7q_0^2}{8} - \frac{13q_0}{12} + \frac{s_0}{24} + \frac{25}{12}\right) +$$
$$+y^5\left(\frac{j_0^2}{12} - \frac{7j_0q_0^2}{8} + \frac{3j_0q_0}{4} - \frac{47j_0}{120} - \frac{c_0}{120} + \frac{7q_0^4}{8} - \frac{9q_0^3}{8} + \frac{47q_0^2}{40} - \right.$$
$$\left.\left. -\frac{q_0s_0}{8} - \frac{77q_0}{60} + \frac{3s_0}{40} + \frac{137}{60}\right)\right] + \mathcal{O}[y^6]\,; \tag{A.2}$$

Angular distance:

$$d_A[z(y)] = \frac{1}{H_0}\left[y - \left(\frac{q_0}{2}+\frac{1}{2}\right)y^2 + y^3\left(\frac{q_0^2}{2}+\frac{q_0}{6}-\frac{j_0}{6}-\frac{1}{6}\right)+\right.$$
$$+y^4\left(\frac{5j_0q_0}{12}+\frac{j_0}{24}-\frac{5q_0^3}{8}-\frac{q_0^2}{8}+\frac{q_0}{12}+\frac{s_0}{24}-\frac{1}{12}\right)+$$
$$+y^5\left(\frac{j_0^2}{12}-\frac{c_0}{120}-\frac{7j_0q_0^2}{8}-\frac{j_0q_0}{12}+\frac{j_0}{40}+\frac{7q_0^4}{8}+\frac{q_0^3}{8}-\frac{3q_0^2}{40}-\frac{q_0s_0}{8}+\right.$$
$$\left.\left.+\frac{q_0}{20}-\frac{s_0}{120}-\frac{1}{20}\right)\right]+\mathcal{O}[y^6]\,; \tag{A.3}$$

Volume distance:

$$d_V[z(y)] = \frac{1}{H_0}\left[y + \left(\frac{1}{3}-\frac{2q_0}{3}\right)y^2 + y^3\left(\frac{29q_0^2}{36}-\frac{5q_0}{18}-\frac{5j_0}{18}+\frac{7}{36}\right)+\right.$$
$$+y^4\left(\frac{43j_0q_0}{54}-\frac{13j_0}{108}-\frac{94q_0^3}{81}+\frac{13q_0^2}{36}-\frac{19q_0}{108}+\frac{s_0}{12}+\frac{11}{81}\right)+$$
$$+y^5\left(\frac{59j_0^2}{324}-\frac{605j_0q_0^2}{324}+\frac{233j_0q_0}{648}-\frac{32j_0}{405}-\frac{7c_0}{360}+\frac{7079q_0^4}{3888}-\frac{523q_0^3}{972}+\right.$$
$$\left.\left.+\frac{773q_0^2}{3240}-\frac{5q_0s_0}{18}-\frac{623q_0}{4860}+\frac{13s_0}{360}+\frac{2017}{19440}\right)\right]+\mathcal{O}[y^6]\,. \tag{A.4}$$

Bibliography

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (Freeman and Co., San Francisco, 1973)

[2] V. Faraoni, *Cosmology in scalar-tensor gravity*, (Kluwer Academic Publishers, Dordrecht, 2004)

[3] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010).

[4] T. P. Sotiriou, *Modified Actions for gravity: Theory and Phenomenology*, PhD thesis (SISSA, Trieste, 2007) arXiv: 0710.4438 [gr-qc].

[5] T. P. Sotiriou, J. Phys. Conf. Ser. **189**, 012039 (2009), arXiv:0810.5594 [gr-qc].

[6] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4** 115 (2007).

[7] A. De Felice and S. Tsujikawa, Living Rev. Rel. **13**, 3 (2010).

[8] D. Barraco, V. H. Hamity and H. Vucetich, Gen. Rel. Grav. **34**, 533 (2002); D. N. Vollick, Phys. Rev. D **68**, 063510 (2003); G. Allemandi, A. Borowiec and M. Francaviglia, Phys. Rev. D **70**, 043524 (2004); G. Allemandi, A. Borowiec, M Francaviglia and S. D. Odintsov, Phys. Rev. D **72**, 063505 (2005); T. P. Sotiriou, Phys. Rev. D **73**, 063515 (2006); M. Amarzguioui, O. Elgaroy, D. F. Mota and T. Multamaki, Astron. Astrophys. **454**, 707 (2006); T. P. Sotiriou, Class. Quant. Grav. **23**, 1253 (2006); T. Koivisto and H. Kurki-Suonio, Class. Quant. Grav. **23**, 2355 (2006) T. Koivisto, Phys. Rev. D **73**, 083517 (2006); T. P. Sotiriou, Phys. Lett. B **645**, 389 (2007); S. Capozziello and M. Francaviglia, Gen. Rel. Grav. **40**, 357 (2008).

[9] M. Ferraris, M. Francaviglia and I. Volovich, Class. Quant. Grav. **11**, 1505 (1994).

[10] K. S. Thorne and C. M. Will, Astrophys. J. **163**, 595 (1971).

[11] T. Koivisto, Class. Quant. Grav. **23**, 4289 (2006).

[12] T. P. Sotiriou, *Proceedings of the Eleventh Marcel Grossmann Meeting on General Relativity*, edited by H. Kleinert, R.T. Jantzen and R. Ruffini, p. 1223-1226, (World Scientific, Singapore, 2008), arXiv: gr-qc/0611158.

[13] T. P. Sotiriou, V. Faraoni and S. Liberati, *Int. J. Mod. Phys. D* **17**, 399 (2008).

[14] T. P. Sotiriou, Class. Quant. Grav. **26**, 152001 (2009).

[15] E. E. Flanagan, Phys. Rev. Lett. **92**, 071101 (2004); Class. Quant. Grav. **21**, 3817 (2004).

[16] G. J. Olmo, Phys. Rev. Lett. **95**, 261102 (2005).

[17] T. P. Sotiriou, Class. Quant. Grav. **23**, 5117 (2006).

[18] E. Barausse, T. P. Sotiriou and J. C. Miller, Class. Quant. Grav. **25**, 062001 (2008); Class. Quant. Grav. **25**, 105008 (2008); EAS Publ. Ser. **30**, 189 (2008).

[19] A. Iglesias, N. Kaloper, A. Padilla and M. Park, Phys. Rev. D **76**, 104001 (2007).

[20] T. P. Sotiriou, Gen. Rel. Grav. **38**, 1407 (2006).

[21] G. J. Olmo, Phys. Rev. D **78**, 104026 (2008).

[22] G. Allemandi, A. Borowiec and M. Francaviglia, Phys. Rev. D **70**, 103503 (2004).

[23] B. Li, J. D. Barrow and D. F. Mota, Phys. Rev. D **76**, 104047 (2007).

[24] G. J. Olmo, H. Sanchis-Alepuz and S. Tripathi, Phys. Rev. D **80**, 024013 (2009).

[25] C. Barragan and G. J. Olmo, arXiv:1005.4136v1 [gr-qc].

[26] F. Bauer, arXiv:1007.2546v1 [gr-qc].

[27] T. P. Sotiriou and S. Liberati, Ann. Phys. **322**, 935 (2007); J. Phys. Conf. Ser. **68**, 012022 (2007)

[28] V. Vitagliano, T. P. Sotiriou and S. Liberati, arXiv:1008.0171 [gr-qc] .

[29] H. A. Buchdahl, J. Phys. A: Math. Gen. **12**, 1235 (1979).

[30] R. M. Wald, *General Relativity*, (The University of Chicago Press, Chicago, 1984).

[31] T. P. Sotiriou and S. Liberati, Annals Phys. **322** (2007) 935 [arXiv:gr-qc/0604006]; J. Phys. Conf. Ser. **68** (2007) 012022. [arXiv:gr-qc/0611040].

[32] F. W. Hehl and G. D. Kerlicks, Gen. Rel. Grav. **9** (1978) 691.

[33] H. A. Buchdahl, Mon. Not. Roy. Ast. Soc. **150** (1970) 1.

[34] V. Vitagliano, T. P. Sotiriou and S. Liberati, arXiv:1007.3937 [gr-qc].

[35] T. P. Sotiriou, Phys. Lett. B **645** (2007) 389.

[36] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, Rev. Mod. Phys. **48** (1976) 393.

[37] Yu. N. Obukhov, E. J. Vlachynsky, W. Esser, and F. W. Hehl, Phys. Rev. D **56** (1997) 7769.

[38] S. M. Christensen, J. Phys. A **13** (1980) 3001.

[39] A. Papapetrou and J. Stachel, Gen. Rel. Grav. **9** (1978) 1075.

[40] F. W. Hehl, G. D. Kerlick and P. Von Der Heyde, Z. Naturforsch. **31A** (1976) 524

[41] B. Sazdovic, [hep-th/0304086].

[42] E. Schrödinger, *Space-Time Structure*, (Cambridge University Press, Cambridge, 1963).

[43] V. D. Sandberg, Phys. Rev. D **12** (1975) 3013.

[44] S. M. Carroll and G. B. Field, Phys. Rev. D **50** (1994) 3867.

[45] I. L. Shapiro, arXiv:1007.5294 [hep-th].

[46] Y. N. Obukhov, V. A. Korotkii, Class. Quant. Grav. **4** (1987) 1633-1657.

[47] G. Lemaître, *Ann. Soc. Sci. Brussels A* **53**, 51 (1933), reprinted in *Gen. Relat. Gravit.* **29**, 641 (1997).

[48] R.C. Tolman, *Proc. Nat. Acad. Sci U.S.A.* **20**, 169 (1934).

[49] H. Bondi, *Mon. Not. R. Astron. Soc.* **107**, 410 (1947).

[50] R. Zalaletdinov, Int. J. Mod. Phys. **A23** (2008) 1173-1181. [arXiv:0801.3256 [gr-qc]].

[51] A. G. Riess *et al.*, 1998 *Astron. J.* **116** 1009; 1999 *Astron. J.* **118** 2668; 2001 *Astrophys. J.* **560** 49; 2004 *Astrophys. J.* **607** 665; S. Perlmutter *et al.*, 1998 *Nature* **391** 51; 1999 *Astrophys. J.* **517** 565; J. L. Tonry *et al.* 2003 *Astrophys. J.* **594** 1; R. Knop *et al.*, 2003 *Astrophys. J.* **598** 102; B. Barris *et al.* 2004 *Astrophys. J.* **602** 571.

[52] See E. V. Linder, 2008 *Am. J. Phys.* **76** 197 for a comprehensive list of references.

[53] S. Capozziello, S. Carloni and A. Troisi, 2003 *Recent Res. Dev. Astron. Astrophys.* **1** 625 [arXiv:astro-ph/0303041].

[54] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, 2004 *Phys. Rev. D* **70** 043528.

[55] V. Marra, [arXiv:0803.3152 [astro-ph]].

[56] T. Buchert, 2000 *Gen. Rel. Gravit.* **32** 105 [arXiv:gr-qc/9906015].

[57] S. Räsänen, 2004 *J. Cosmol. Astrop. Phys.* **02** 003 [arXiv:astro-ph/0311257].

[58] T. Buchert and M. Carfora, 2008 *Class. Quantum Grav.* **25** 195001 [arXiv:0803.1401]; 2002 *Class. Quantum Grav.* **19** 6109 [arXiv:gr-qc/0210037]; D. L. Wiltshire, 2007 *New J. Phys.* **9** 377 [arXiv:gr-qc/0702082]; 2007 *Phys. Rev. Lett.* **99** 251101; E.W. Kolb, S. Matarrese, A. Riotto, 2006 *New J. Phys.* **8** 322; R. M. Zalaletdinov, 1992 *Gen. Rel. Gravit.* **24** 1015; N. Li and D. J. Schwarz, 2008 *Phys. Rev. D* **78** 083531; 2007 *Phys. Rev. D* **76** 083011; A. Paranjape and T. P. Singh, 2007 *Phys. Rev. D* **76** 044006.

[59] V. Marra, E. Kolb and S. Matarrese, 2008 *Phys. Rev D* **77** 023003 [arXiv:0710.5505], V. Marra, E. Kolb, S. Matarrese and A. Riotto, 2007 *Phys. Rev D* **76** 123004 [arXiv:0708.3622]; V. Marra 2008 [arXiv:0803.3152]; E. Kolb, V. Marra and S. Matarrese, 2009 [arXiv:0901.4566].

[60] S. Räsänen, JCAP **1003** (2010) 018. [arXiv:0912.3370 [astro-ph.CO]].

[61] T. Buchert, Gen. Rel. Grav. **33** (2001) 1381-1405. [gr-qc/0102049].

[62] A. Paranjape, T. P. Singh, Phys. Rev. **D76** (2007) 044006. [gr-qc/0703106].

[63] G. F. R. Ellis, 1984 in *General Relativity and Gravitation*, B. Bertotti ed. (Dordrecht: Reidel).

[64] G. F. R. Ellis and W. Stoeger, 1987 *Class. Quantum Grav.* **4** 1697.

[65] M. B. Green, G. H. Schwarz, and E. Witten, 1987 *Superstring Theory* (Cambridge: Cambridge University Press).

[66] C. H. Brans and R. H. Dicke, 1961 *Phys. Rev.* **124** 925.

[67] P. G. Bergmann, 1968 *Int. J. Theor. Phys.* **1** 25; R. V. Wagoner, 1970 *Phys. Rev. D* **1** 3209; K. Nordvedt, 1970 *Astrophys. J.* **161** 1059.

[68] C. G. Callan, D. Friedan, E. J. Martinez and M. J. Perry, 1985 *Nucl. Phys. B* **262** 593; E. S. Fradkin and A. A. Tseytlin, 1985 *Nucl. Phys. B* **261** 1.

[69] C. M. Will 1993, *Theory and Experiment in Gravitational Physics* (Cambridge: Cambridge University Press);
Ibidem, Living Rev. Rel. **9**, 3 (2005) [arXiv:gr-qc/0510072].

[70] V. Cardone and G. Esposito [arXiv:0805.1203].

[71] M. Reuter and F. Saueressig, 2007 [arXiv:0708.1317]; D. F. Litim, 2008 [arXiv:0810.3675].

[72] A. A. Starobinsky, 1980 *Phys. Lett. B* **91** 99

[73] P. W. Higgs, 1959 *Nuovo Cimento* **11** 816; P. Teyssandier and P. Tourrenc, 1983 *J. Math. Phys.* **24**, 2793; B. Whitt, 1984 *Phys. Lett. B* **145** 176; J. D. Barrow and S. Cotsakis, 1988 *Phys. Lett. B* **214** 515; J. D. Barrow, 1988 *Nucl. Phys. B* **296** 697; D. Wands, 1994 *Class. Quantum Grav.* **11** 269; T. Chiba, 2003 *Phys. Lett. B* **575** 1.

[74] S. Nojiri and S. D. Odintsov, arXiv:0807.0685 [hep-th].

[75] R. M. Wald, 1984 *General Relativity* (Chicago: Chicago University Press).

[76] J. Larena [arXiv:0902.3159]; I. Brown, J. Behrend and K. Malik [arXiv:0903.3264].

[77] J. Larena , J.-M. Alimi, T. Buchert, M. Kunz and P. Corasaniti [arXiv:0808.1161].

[78] A. Paranjape and T. P. Singh, 2008 *Gen. Rel. Gravit.* **40** 139 [arXiv:astro-ph/0609481].

[79] T. Buchert, 2008 *Gen. Rel. Gravit.* **40** 467 [arXiv:0707.2153]; 2007 *AIP Conf. Proc.* **910** 361 [arXiv: gr-qc/0612166].

[80] J. Larena, T. Buchert and J.-M. Alimi, 2006 *Class. Quantum Grav.* **23** 6379.

[81] K. Nordtvedt, 1968 *Phys. Rev. D* **169** 1017.

[82] B. Bertotti, L. Iess and P. Tortora, 2003 *Nature* **425** 374.

[83] T. Buchert, 1996 in *Mapping, Measuring, and Modelling the Universe*, ASP Conference Series vol. 94, P. Coles , V. Martinez and M. J. Pons-Borderia eds.

[84] C. G. Tsagas, A. Challinor and R. Maartens, 2008 *Phys. Repts.* **465** 61

[85] V. Faraoni, 1998 *Phys. Lett. A* **245** 26; 1999 *Phys. Rev. D* **59** 084021.

[86] A. D. Dolgov and M. Kawasaki, Phys. Lett. B **573**, 1 (2003) [arXiv:astro-ph/0307285];
V. Faraoni, Phys. Rev. D **74**, 104017 (2006) [arXiv:astro-ph/0610734];
S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003) [arXiv:hep-th/0307288].

[87] A. Paranjape and T. P. Singh, 2006 *Class. Quantum Grav.* **23** 6955 [arXiv: astro-ph/0605195]; S. Räsänen, 2004 *J. Cosmol. Astrop. Phys.* **11** 010 [arXiv: gr-qc/0408097].

[88] M. Kunz, A. R. Liddle, D. Parkinson and C. Gao, Phys. Rev. D **80**, 083533 (2009); M. Kunz, Phys. Rev. D **80**, 123001 (2009); M. Kunz and D. Sapone, Phys. Rev. Lett. **98**, 121301 (2007); M. J. Mortonson, W. Hu and D. Huterer, Phys. Rev. D **79**, 023004 (2009); M. J. Mortonson, W. Hu and D. Huterer, Phys. Rev. D **81**, 063007 (2010).

[89] C. Cattoen and M. Visser, Phys. Rev. D **78**, 063501 (2008).

[90] T. Chiba and T. Nakamura, Prog. Theor. Phys. **100**, 1077 (1998).

[91] V. Sahni, T. D. Saini, A. A. Starobinsky, U. Alam, JETP Lett. **77**, 201-206 (2003); U. Alam, V. Sahni, T. D. Saini, A. A. Starobinsky, Mon. Not. Roy. Astron. Soc. **344**, 1057 (2003).

[92] C. Cattoen and M. Visser, Class. Quant. Grav. **24**, 5985 (2007).

[93] D. Rapetti, S. W. Allen, M. A. Amin and R. D. Blandford, Mon. Not. Roy. Astron. Soc. **375**, 1510 (2007).

[94] M. Visser, Class. Quant. Grav. **21**, 2603-2616 (2004).

[95] M. Visser, Gen. Rel. Grav. **37**, 1541-1548 (2005).

[96] M. Chevallier, D. Polarski, Int. J. Mod. Phys. D **10**, 213 (2001); E. V. Linder, Phys. Rev. Lett. **90**, 091301 (2003).

[97] V. Vitagliano, J. Q. Xia, S. Liberati and M. Viel, JCAP **03**, 005 (2010).

[98] C. Cattoen and M. Visser, arXiv:gr-qc/0703122.

[99] S. Capozziello and L. Izzo, Astron. Astrophys. **519**, A73 (2010); L. Xu and Y. Wang, arXiv:1007.4734; A. C. C. Guimaraes and J. A. S. Lima, arXiv:1005.2986; N. Liang, P. Wu, S. N. Zhang, Phys. Rev. D **81**, 083518 (2010); L. Xu and Y. Wang, arXiv:1009.0963; H. Gao, N. Liang and Z. H. Zhu, arXiv:1003.5755.

[100] B. Santos, J. C. Carvalho and J. S. Alcaniz, arXiv:1009.2733.

[101] L. Xu and Y. Wang, arXiv:1009.0963.

[102] S. Weinberg, *Cosmology*, Oxford, UK: Oxford Univ. Pr. (2008).

[103] R. Amanullah *et al.*, Astrophys. J. **716**, 712 (2010).

[104] M. Kowalski *et al.*, Astrophys. J. **686**, 749 (2008).

[105] R. Amanullah *et al.*, Astron. Astrophys. **486**, 375 (2008); J. A. Holtzman *et al.*, Astron. J. **136**, 2306 (2008); M. Hicken *et al.*, Astrophys. J. **700**, 331 (2009).

[106] G. Ghirlanda, G. Ghisellini, D. Lazzati and C. Firmani, Astrophys. J. **613**, L13 (2004).

[107] H. Li, J. Q. Xia, J. Liu, G. B. Zhao, Z. H. Fan and X. Zhang, Astrophys. J. **680**, 92 (2008).

[108] C. Firmani, V. Avila-Reese, G. Ghisellini and G. Ghirlanda, Rev. Mex. Astron. Astrofis. **43**, 203 (2007).

[109] B. E. Schaefer, Astrophys. J. **660**, 16 (2007).

[110] M. Goliath, R. Amanullah, P. Astier, A. Goobar and R. Pain, Astron. Astrophys. **380**, 1 (2001).

[111] E. Di Pietro and J. F. Claeskens, Mon. Not. Roy. Astron. Soc. **341**, 1299 (2003).

[112] R. Jimenez, L. Verde, T. Treu and D. Stern, Astrophys. J. **593**, 622 (2003).

[113] J. Simon, L. Verde and R. Jimenez, Phys. Rev. D **71**, 123001 (2005).

[114] R. G. Abraham *et al.*, Astron. J. **127**, 2455 (2004).

[115] T. Treu, M. Stiavelli, S. Casertano, P. Moller and G. Bertin, Mon. Not. Roy. Astron. Soc. **308**, 1037 (1999); T. Treu, M. Stiavelli, P. Moller, S. Casertano and G. Bertin, Mon. Not. Roy. Astron. Soc. **326**, 221 (2001); T. Treu, M. Stiavelli, S. Casertano, P. Moller and G. Bertin, Astrophys. J. **564**, L13 (2002); J. Dunlop, J. Peacock, H. Spinrad, A. Dey, R. Jimenez, D. Stern and R. Windhorst, Nature **381**, 581 (1996); H. Spinrad, A. Dey, D. Stern, J. Dunlop, J. Peacock, R. Jimenez and R. Windhorst, Astrophys. J. **484**, 581 (1997); L. A. Nolan, J. S. Dunlop, R. Jimenez and A. F. Heavens, Mon. Not. Roy. Astron. Soc. **341**, 464 (2003).

[116] A. G. Riess *et al.*, Astrophys. J. **699**, 539 (2009).

[117] D. J. Eisenstein *et al.*, Astrophys. J. **633**, 560 (2005).

[118] S. Cole *et al.*, Mon. Not. Roy. Astron. Soc. **362**, 505 (2005); G. Huetsi, Astron. Astrophys. **449**, 891 (2006); W. J. Percival *et al.*, Mon. Not. Roy. Astron. Soc. **381**, 1053 (2007).

[119] W. J. Percival *et al.*, Mon. Not. Roy. Astron. Soc. **401**, 2148 (2010).

[120] A. Albrecht *et al.*, arXiv:astro-ph/0609591.

[121] T. Okumura, T. Matsubara, D. J. Eisenstein, I. Kayo, C. Hikage, A. S. Szalay and D. P. Schneider, Astrophys. J. **676**, 889 (2008).

[122] B. A. Bassett, R. Hlozek, *Baryon Acoustic Oscillations*, in Dark Energy, Ed. P. Ruiz-Lapuente (2010); arXiv:0910.5224.

[123] M. Tegmark *et al.*, Phys. Rev. D **74**, 123507 (2006).

[124] M. Vonlanthen, S. Rasanen, R. Durrer, JCAP **1008** (2010) 023.

[125] S. Capozziello, V. F. Cardone, A. -Troisi, Mon. Not. Roy. Astron. Soc. **375** (2007) 1423-1440.

[126] A. Gelman and D. Rubin, Statist. Sci. **7**, 457 (1992).

[127] C. Clarkson, M. Cortes, B. A. Bassett, JCAP **0708**, 011 (2007).

[128] A. G. Kim, E. V. Linder, R. Miquel and N. Mostek, Mon. Not. Roy. Astron. Soc. **347**, 909 (2004).

[129] D. Hooper and S. Dodelson, Astropart. Phys. **27**, 113 (2007).

[130] D. Schlegel, M. White and D. Eisenstein, arXiv:0902.4680.

[131] J. Q. Xia and M. Viel, JCAP **0904**, 002 (2009).

[132] E. Komatsu *et al.*, arXiv:1001.4538.

[133] G. R. Dvali, G. Gabadadze, M. Porrati, Phys. Lett. B **485**, 208-214 (2000); C. Deffayet, Phys. Lett. B **502**, 199-208 (2001).

[134] M. Bouhmadi-Lopez, S. Capozziello, V. F. Cardone, Phys. Rev. D **82**, 103526 (2010); F. Y. Wang, Z. G. Dai, S. Qi, Astron. Astrophys. **507**, 53-59 (2009).

[135] Z. -H. Zhu, M. Sereno, Astron. Astrophys. **487**, 831 (2008).

[136] K. Freese, M. Lewis, Phys. Lett. B **540**, 1-8 (2002).

[137] T. Wang, N. Liang, Sci. China G **53**, 1720-1725 (2010).

[138] M. Chevallier, D. Polarski, Int. J. Mod. Phys. D **10**, 213-224 (2001); E. V. Linder, Phys. Rev. Lett. **90**, 091301 (2003).

[139] S. Capozziello and L. Izzo, Astron. Astrophys. **490**, 31 (2008).

[140] O. Elgaroy, T. Multamaki, arXiv:astro-ph/0702343.

[141] J. Liske et al, Mon. Not. Roy. Astron. Soc. **386**, 1192 (2008).

People I want to acknowledge...

Il Capo (alias Stefano Liberati)!!! I know that you don't like it, but I hope next students will continue to dub you in this way... He was for me the eldest brother I never had (teardrop...), it's incredible that he trusted in me during these years.

I really would like to thank Matt Visser, for his kindness and for the suggestions on the possibility to furtherly improve the work of this thesis, and Ravi Sheth, that accepted to review the manuscript even if we sent it to him only few weeks before the discussion.

One thousand of thanks to Valerio Faraoni (the living encyclopaedia of the alternative theories of gravitation) and the -30 C of December 2009. And another special mention to my collaborators Matteo Viel and JQ Xia, for their efforts against everybody and everything in last two years.

Lucio, Jack and Nathan, if I'm so beautifully assuming that round shape it is just thanks to their amazing food, and my girls Lucia, Eleonora, Patrizia and Irene, that never kicked me out from the cafeteria but always offered me a smile!

The Rolling Stones, because I couldn't have driven every day without Jagger&Co. expressing their sympaties... Il Kiwi! Iron-Angus, the man that can climb the Everest standing on his left hand.

The sgarzillous Antonella for her sunny mind, Emir Kusturica and the bloody lively Balkans!, il bivio di Miramare and Barcola with its baracchino, the "Blondie-not-blondie" Gyannet (maybe the only triestina patocca I met in four years here!).

Luca, Claudietta and the guy spitting around at the Metelkova, my (un)official heir Dario, those geniuses of Saramago, Boris Vian and De André. Pilar! (even though she's vegetarian... bah...), Eolo (sitting on the dock of the bay), Piero Angela, le osmize.

Silke Christine Edith (the only one in the world. Luckily.) and Thomas.

Lucaphilippemertens, alias Lucammé, for the bossanova in his blood, my cocola Terrina (missing you!) and my shanti Joana, il Knulp and its amazing world, il Peressini!, this guy has a special soul, my "yellove" Stefania, i Varesotti. Claudia, Teddie and Tommy, they're my family.

Il compagno Zanusso, the autumns on the Carso, il Cinese with his socialist bridges and the importance of eating, Virginia astonished by the spring wind, my dearrrrrrrr Falcolini, Maurizio and his invaluable advise, Ciro and his disproportionately big head, spritz and terán, my didgeri-guru Lorenzo!, Laura (the only princess of our lives) and the Spanish enfante, the Maxmaber Orkestar and all the band of folks gravitating around.

Last remark: *I get high with a little help from my friends*, Lorenzino, Goffredo, Adriano and Luca Jesus.

Che bello! gli amici, una chitarra e lo spinello
e una ragazza giusta che ci sta
e tutto il resto che importanza ha?
Che bello! se piove porteremo anche l'ombrello
in giro per le vie della città
per due boccate di felicità.
Stefano Rosso.