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A PROPOSAL FOR A
VIRTUAL FUNDAMENTAL CLASS
FOR ARTIN STACKS

PhD thesis of:

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Introduction

Virtual fundamental classes are a major technical tool in enumerative geometry. They were introduced with different approaches in [17] and in [4] in order to get classes in the Chow groups of the moduli spaces of stable maps $\overline{M}_{g,n}(V,\beta)$; in turn, these classes made possible to define well behaved (e.g. deformation invariant) Gromov-Witten invariants, see [5], [3] and [2].
In general, in the language of [4], a virtual fundamental class $[X]^{\mathrm{vir}} \in A_d X$ is defined in the Chow group with rational coefficients, for a DM stack X endowed with a perfect obstruction theory (d being the virtual dimension of X).

1. Aim The aim of this work is to propose a definition of the virtual fundamental class for Artin stacks, see chapter 2.
We'll also give a generalization to Artin stacks of the infinitesimal criterion for obstruction theories, see chapter 3.

2. Possible applications We have applications in two different scenarios:

- Moduli spaces of stable maps: definition of Gromov-Witten invariants in non zero characteristic.
- Moduli spaces of semistable sheaves: definition of Donaldson and Donaldson-Thomas invariants in presence of semistable sheaves that are not stable.

3. Remark The definition of virtual fundamental class for Artin stacks was not possible at the beginning because of the absence (for Artin stacks) of two useful technical devices: Chow groups and the cotangent complex. We now have these devices at our disposal. Chow groups and intersection theory for Artin stacks are provided by [15]. A working theory for the cotangent complex of a morphism of Artin stacks is provided by [16], [22]. Nonetheless the presence of these tools is not enough to overcome all the difficulties. The existing construction relies on the correspondence between Picard stacks and 2-term complexes of abelian sheaves, [1, Deligne, Exp. XVIII]. For instance, a global resolution of an obstruction theory over a DM stack is given by a 2-term complex $[\mathcal{E}^{-1} \longrightarrow \mathcal{E}^0]$ of locally free sheaves. Part of the original construction in [4] is to build a cone stack inside the associated vector bundle stack $[E_1/E_0]$. Since the cotangent complex of an Artin stack has a relevant piece also in degree 1 then a global resolution of an obstruction theory has the form $[\mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E}^1]$, and one cannot exploit directly the above correspondence.

4. Moduli of stable maps Given a smooth projective variety V over a field $\mathbb{K}$, a homology class $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\, V, \mathbb{Z})$, and two integers $g, n \geq 0$, we can construct the moduli space $\overline{\mathcal{M}}_{g,n}(V,\beta)$ of genus g n-marked stable maps of class β in V, see [5]. This is

an algebraic stack, which is a proper DM stack by [5, Th. 3.14] if (g, n, β) is bounded by the characteristic of $\mathbb{K}$, i.e. if $\operatorname{ch}\mathbb{K} = 0$ or if $\operatorname{ch}\mathbb{K} > 0$ and there exists an ample $\mathcal{L} \in \operatorname{Pic} V$ s.t. $\beta(\mathcal{L}) < \operatorname{ch}\mathbb{K}$.
$\overline{\mathcal{M}}_{g,n}(V, \beta)$ can be endowed with a perfect obstruction theory (relative to $\mathfrak{M}_{g,n}$, the stack of genus g n-marked prestable curves). If (g, n, β) is bounded by the characteristic of $\mathbb{K}$ then the machinery of [4] gives a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(V, \beta)]^{\mathrm{vir}}$ of dimension $3g - 3 + n + (1 - g)\mathrm{dim}V - \beta(\omega_V)$, where ω_V is the canonical line bundle of V, see [3]. After choosing a suitable cohomology theory $H^\bullet$, e.g. Chow cohomology with coefficients in $\mathbb{Q}$, we get a system of Gromov-Witten invariants

$$\langle I_{g,n}(V, \beta)\rangle : H^\bullet(V)^{\otimes n} \longrightarrow \mathbb{Q}$$

by

$$\langle I_{g,n}(V, \beta)\rangle(\alpha) := \deg(\mathrm{ev}^*(\alpha) \cap [\overline{\mathcal{M}}_{g,n}(V, \beta)]^{\mathrm{vir}}),$$

where $\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(V, \beta) \longrightarrow V^n$ is the evaluation map. Notice that we used that $\overline{\mathcal{M}}_{g,n}(V, \beta)$ is DM to get the virtual class and that $\overline{\mathcal{M}}_{g,n}(V, \beta)$ is proper to have the possibility to take the degree of a class.
If (g, n, β) is not bounded by $\operatorname{ch}\mathbb{K}$ then the above argument doesn't work. In particular $\overline{\mathcal{M}}_{g,n}(V, \beta)$ can have non-reduced stabilizers and hence it's no more a DM stack. For instance this happens for $p = \operatorname{ch}\mathbb{K} > 0$ and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d)$, where $d > 0$ is an integer and we made the identification

$$\mathbb{Z} \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic}\mathbb{P}^1, \mathbb{Z})$$

by sending d to the class β_d defined by $\beta_d(\mathcal{O}_{\mathbb{P}^1}(r)) := dr$. We have

$$(0, 0, d) \text{ bounded by } \operatorname{ch}\mathbb{K} \qquad \Longleftrightarrow \qquad d < p.$$

The map $f_d \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d)$ defined by

$$[x, y] \in \mathbb{P}^1 \mapsto [x^d, y^d] \in \mathbb{P}^1$$

has automorphism group

$$\operatorname{Aut}(f_d) = \operatorname{Spec} \frac{\mathbb{K}[z]}{(z^d - 1)},$$

where $z = x/y$. If, for instance, $d = p$ then

$$\operatorname{Aut}(f_p) = \operatorname{Spec} \frac{\mathbb{K}[z]}{(z - 1)^p},$$

which is non-reduced.
The existence of virtual fundamental classes for Artin stacks would be an essential tool for extending the definition of Gromov-Witten invariants in case where (g, n, β) is not bounded by $\operatorname{ch}\mathbb{K}$.

5. Moduli of semistable sheaves The tool of the virtual fundamental class has also been used to associate numerical invariants to (suitable kinds of) smooth projective varieties V via the moduli spaces of (semi)stable coherent sheaves over the variety itself. In [25] the Donaldson-Thomas invariants are defined for V a Calabi-Yau or Fano 3-fold; in [20] (an algebraic and generalized version of the) Donaldson invariants are defined for V a surface. In both cases the author had to do something to avoid Artin stacks,

for which the virtual fundamental class didn't exist. In the Donaldson-Thomas case the assumption that any semistable sheaf is stable was added. In the Donaldson case proper DM "approximations" of the Artin stacks involved were considered. The definition of virtual fundamental classes for Artin stacks could improve the situation allowing to remove that assumption in the DT case and allowing a simpler approach in the Donaldson case. Actually, since the stacks involved are not proper there is a problem to face in order to get a number. One could try to exploit the existence of a proper coarse moduli space.
We'll briefly sketch the situation in the DT case.
Given a smooth projective variety V over an algebraically closed field $\mathbb{K}$ of characteristic zero, two invertible sheaves $\mathcal{O}_V(1)$ and $\mathcal{D}$ over V with $\mathcal{O}_V(1)$ ample, and a polynomial $P \in \mathbb{Q}[z]$, we denote by $\mathcal{M}(V, P, \mathcal{D})$ the moduli space of coherent sheaves over V, semistable and of Hilbert polynomial P with respect to $\mathcal{O}_V(1)$, and with determinant $\mathcal{D}$. $\mathcal{M}(V, P, \mathcal{D})$ is an algebraic stack, (cf. for instance [10]; cf. [14] for a general study of the moduli spaces of sheaves). Denote by $\mathcal{M}^s(V, P, \mathcal{D}) \subseteq \mathcal{M}(V, P, \mathcal{D})$ the open substack of stable sheaves, and by $M^s(V, P, \mathcal{D}) \subseteq M(V, P, \mathcal{D})$ the corresponding coarse moduli spaces. In [25] the so called Donaldson-Thomas invariants are defined, working with the coarse moduli spaces, for V a Calabi-Yau or Fano 3-fold. $M^s(V, P, \mathcal{D})$ is endowed with a perfect obstruction theory for which the virtual dimension of the moduli space is 0. On the other side $M(V, P, \mathcal{D})$ is proper. Making the additional assumption that any semistable sheaf is in fact stable, i.e. $M^s(V, P, \mathcal{D}) = M(V, P, \mathcal{D})$, we can take the degree $\deg\,[M^s(V, P, \mathcal{D})]^{\mathrm{vir}}$ of the virtual class, getting the invariant. In the language of stacks $\mathcal{M}^s(V, P, \mathcal{D})$ is a DM stack endowed with a perfect obstruction theory. If any semistable sheaf is stable then it is proper and we can take the degree of the virtual class, as before. We would like to remove the additional assumption. None of the stacks $\mathcal{M}^s(V, P, \mathcal{D})$ and $\mathcal{M}(V, P, \mathcal{D})$ is proper, but the latter has a proper coarse moduli space. In view of getting the invariant we should work with $\mathcal{M}(V, P, \mathcal{D})$, which is an Artin stack, and get a virtual fundamental class for it.

6. Outline of the thesis

`(Chapter 1)` We give the basic references, we recall some known results and we prove some technical lemmas.
`(Chapter 2)` We propose a definition of virtual fundamental class for Artin stacks. Given an algebraic stack $\mathcal{X}$, and an obstruction theory $\mathcal{D}^\bullet \longrightarrow \mathcal{L}^\bullet_{\mathcal{X}}$ over it that admits a global resolution $\mathcal{E}^\bullet$, we'll define a class in the Chow group of $\mathcal{X}$:

$$[\mathcal{X}]^{\mathrm{vir}} \in A_{\mathrm{rk}\,\mathcal{D}^\bullet}\,\mathcal{X},$$

by intersecting the zero section of $E_1 := Spec\,Sym^\bullet \mathcal{E}^{-1} \longrightarrow \mathcal{X}$ with the so called global normal cone $C(\mathcal{E}^\bullet) \hookrightarrow E_1$. The proof of the independence of the resolution of the class so defined is reduced to a conjecture, see (57).
In the introduction to the chapter we give a more detailed sketch of the construction.
In the first section we give the definition of obstruction theories for Artin stacks, we define the local normal cone and prove the basic fact of its well behavior with respect to smooth base changes.
In the second section we define the global normal cone by gluing local normal cones.
In the third section we study the relation between the global normal cones associated with different resolutions.
`(Chapter 3)` Lemma 4.6 of [4] provides an infinitesimal criterion to check if a given morphism $\phi : \mathcal{E}^\bullet \longrightarrow \mathcal{L}^\bullet_X$ is an obstruction theory for the DM stack X. This criterion works thanks to the existence of a deformation theoretic sequence associated with a pointed

scheme (or DM stack). In order to get an analogous criterion in the case of Artin stacks, we describe a generalization of the deformation sequence, see (64).
In the first section we discuss how the deformation sequence is used to get the criterion.
In the second section we recall the construction of the sequence for the scheme case.
In the third section we state the main result (deformation sequence for a pointed algebraic stack $\mathcal{X}$) and reduce the proof of it to the construction of an analogous sequence for a smooth pointed groupoid $X_\bullet$ that presents $\mathcal{X}$. In particular we have to define deformation functors associated to $\mathcal{X}$ and $X_\bullet$ and prove that they are equivalent, see (70).
In the fourth section we construct the sequence for a smooth pointed groupoid and prove that it satisfies the required properties.

7. Notation for the references References to the bibliography are in square brackets, e.g. [13]. References to chapters, sections and subsections are given without parentheses, e.g. chapter 1, section 1.3 and subsection 2.1.1. References to equations and diagrams are given in round parentheses and contain the number to the chapter, e.g. the n-th equation/diagram of chapter m is $(m.n)$. References to an "elementary" paragraph are given with round brackets and don't contain the number of the chapter, e.g. we are in (7).

Contents

Chapter 1

Preliminaries

In this chapter we'll recall and give references for the known results that we need in the sequel. We'll also prove some technical lemmas.

8. Basic references This is a work in algebraic geometry. The theory of schemes provides a solid technical background for the modern algebraic geometry. Possible basic references for *scheme theory* are [11], [13] and [21]. In modern algebraic geometry the functorial point of view plays a prominent role. We assume known the main notions of *category theory*, see for instance [18]. Working at this abstract level we should face some foundational problems of set theoretic nature. The author finds this kind of problems interesting and fascinating in their own but, for simplicity, he decided to neglect them in this work. We also give basic references for *commutative algebra*: [6], [19], and *homological algebra*: [26], [9] and [12].

9. Schemes and stacks We assume schemes and algebraic stacks to be of finite type over an algebraically closed field $\mathbb{K}$. For the theory of stacks we refer for instance to [16], [22], [7, Part 1].

1.1 Stacks and groupoids

10. Groupoids in a category Given a category $\mathcal{C}$ we denote a groupoid in this category by (X_1, X, s, t, e, i, m), where the symbols stay for: morphisms, objects, source, target, identity, inverse and multiplication respectively. We'll also use the notation $X_\bullet$ or $X_1 \rightrightarrows X$. Sources and targets of the structure morphisms of the groupoid are given by:

$$X_2 \xrightarrow{m} X_1 \xrightarrow{i} X_1 \xrightarrow{s,t} X \xrightarrow{e} X_1,$$

where we used the notation:

$$\begin{array}{ccc} X_2 & \xrightarrow{\nu_t} & X_1 \\ {\scriptstyle \nu_s}\downarrow & \square & \downarrow{\scriptstyle s} \\ X_1 & \xrightarrow[t]{} & X. \end{array}$$

These data satisfy the axioms for a groupoid.
A morphism of groupoids $X_\bullet \xrightarrow{f_\bullet} Y_\bullet$ is given by a pair of maps

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ X & \xrightarrow[f]{} & Y \end{array} \tag{1.1}$$

that commute with the structure maps of the groupoids.
We say that a morphism of groupoids is *cartesian* and we use the notation

$$X_\bullet \xrightarrow{\Box} Y_\bullet$$

if the squares in the diagram (1.1) are cartesian.
We say that a morphism of groupoids is *fully faithful* if the following diagram is cartesian:

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ X \times X & \xrightarrow[f\times f]{} & Y \times Y. \end{array}$$

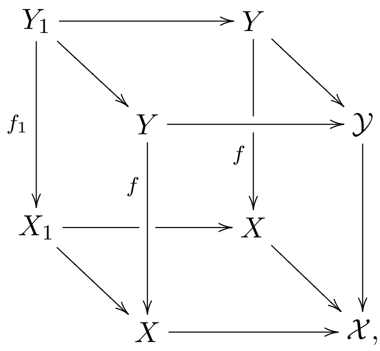
We say that a groupoid in the category of schemes is *smooth* if the source and target maps are smooth.
A morphism of groupoids in the category of schemes is said to be *Morita* if it's fully faithful and f is smooth and surjective.

11. Algebraic stacks Vs smooth groupoids Given a smooth groupoid $X_\bullet = X_1 \rightrightarrows X$ in the category of schemes, there is an associated algebraic stack $[X_\bullet]$, and this comes with an atlas $X \longrightarrow [X_\bullet]$.
Conversely, given an algebraic stack $\mathcal{X}$ and a morphism from a scheme $X \xrightarrow{\pi} \mathcal{X}$, there is an associated groupoid $X_\bullet$ by taking

$$\begin{array}{ccc} X_1 & \xrightarrow{t} & X \\ {\scriptstyle s}\downarrow & \Box & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\pi]{} & \mathcal{X}, \end{array}$$

and inducing the other structure maps of the groupoid thanks to the universal property of this (2-)cartesian square. If π is smooth then the induced groupoid is smooth, and we have a map from the induced stack and the given one: $[X_\bullet] \longrightarrow \mathcal{X}$. If moreover π is surjective than this is an isomorphism of stacks.
Moreover, starting with a smooth groupoid $X_\bullet$, the groupoid associated to the natural atlas $X \longrightarrow [X_\bullet]$ is naturally isomorphic to $X_\bullet$.
This says that it's equivalent to talk about smooth groupoids and to talk about algebraic stacks endowed with an atlas.
If a morphism of groupoids $Y_\bullet \longrightarrow X_\bullet$ is Morita then the induced map of stacks $[Y_\bullet] \longrightarrow [X_\bullet]$ is an isomorphism. Conversely if the algebraic stacks associated with two smooth groupoids $X_\bullet$ and $Z_\bullet$ are isomorphic then the groupoids themselves are Morita-equivalent,

i.e. there are Morita morphisms $Y_\bullet \longrightarrow X_\bullet$ and $Y_\bullet \longrightarrow Z_\bullet$ from a third smooth groupoid. Given a representable morphism $\mathcal{Y} \longrightarrow \mathcal{X}$ of algebraic stacks with a property p stable undere base change, and an atlas $X \longrightarrow \mathcal{X}$, we can construct the cartesian cube



and we get a cartesian morphism of groupoids $Y_\bullet \xrightarrow{f_\bullet} X_\bullet$ in which f and f_1 have the property p. Conversely, given such a morphism of groupoids, the associated morphism of stacks $[Y_\bullet] \longrightarrow [X_\bullet]$ is representable and has p.

1.2 Linear spaces and cones

In this section we'll give a brief introduction to linear spaces (abelian cones in the notation of [4]) and cones. We assume these and the related notions to be known, and we refer, for instance, to [4], [8] and [24].

12. Affine morphisms and quasi coherent sheaves of algebras Given a scheme X we have an equivalence of categories between the category Aff_X of schemes affine over X and the category qc-$\mathcal{O}_X$-Alg of quasi-coherent sheaves of $\mathcal{O}_X$-algebras; this is well explained in [11, I.9.1].
The objects of Aff_X are affine morphisms of schemes $p : Y \longrightarrow X$. i.e. morphisms such that for any affine open subscheme $U \subseteq X$ the preimage $p^{-1}(U) \subseteq Y$ is also affine. We have a natural contravariant functor

$$\begin{array}{ccc} \mathrm{Aff}_X & \xrightarrow{\mathcal{A}} & \text{qc-}\mathcal{O}_X\text{-Alg} \\ Y \xrightarrow{p} X & \longmapsto & p_*\mathcal{O}_Y. \end{array}$$

The claim is that this functor has an inverse (in the sense that it induces an equivalence of categories):

$$\mathit{Spec} : \text{qc-}\mathcal{O}_X\text{-Alg} \longrightarrow \mathrm{Aff}_X.$$

This functor can be characterized as follows: given a quasi coherent sheaf of algebras $\mathcal{Q}$ over X and an open affine $\operatorname{Spec} A = U \subseteq X$, denote by Q the A-algebra for which the associated sheaf $\widetilde{Q}$ over U is $\mathcal{Q}_{|U}$. Then we have a canonical cartesian diagram:

$$\begin{array}{ccc} \operatorname{Spec} Q & \longrightarrow & \mathit{Spec}\,\mathcal{Q} \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & X. \end{array}$$

13. Cones Given a quasi coherent sheaf of algebras $\mathcal{S}$ over a scheme X we can ask for the additional structure of an $\mathbb{N}$-grading: $\mathcal{S}^\bullet = \oplus_{n\in\mathbb{N}}\mathcal{S}^n$. Given a grading we can ask for it to satisy the conditions: $\mathcal{S}^0 = \mathcal{O}_X$, $\mathcal{S}^\bullet$ is locally generated (as algebra) in degree 1, and $\mathcal{S}^1$ is coherent. By definition, a *cone* over X is the spectrum of such an algebra $\mathcal{S}^\bullet$:

$$C := Spec\,\mathcal{S}^\bullet \xrightarrow{p} X.$$

C is a scheme affine over X and the additional structure and hypotheses over $\mathcal{S}$ induce some additional structure on X, namely a section of p:

$$X \xrightarrow{0} C,$$

called the zero section, and a map (over X):

$$\begin{array}{ccc} \mathbb{A}^1_X \times C & \longrightarrow & C \\ (\lambda, c) & \longmapsto & \lambda \cdot c, \end{array}$$

called the multiplication by a scalar. These satisfy the axioms

$$\begin{aligned} \lambda \cdot (\mu \cdot c) &= (\lambda\mu) \cdot c \\ 1 \cdot c &= c \\ 0 \cdot c &= 0 \end{aligned}$$

for any $\lambda, \mu \in \mathbb{A}^1_X$ and $c \in C$. Moreover locally over X the cone C can be embedded (as a closed subcone) in $\mathbb{A}^n_X$ for some n. Notice that $\mathbb{A}^1_X$ is a ring object over X and, for any $n \geq 0$, $\mathbb{A}^n_X$ is a cone over X (in fact a linear space).
The algebraic description of a cone (spectrum of a graded algebra) is in fact equivalent to the geometric one (affine scheme over X plus zero section and multiplication by a scalar).

14. Linear spaces Given a coherent sheaf $\mathcal{F}$ over a scheme X, the symmetric algebra $Sym^\bullet\mathcal{F}$ over it is a graded sheaf of $\mathcal{O}_X$-algebras of the kind described in (13). By definition, a *linear space* over X is the spectrum of such a sheaf of algebras:

$$F := Spec\,Sym^\bullet\mathcal{F} \longrightarrow X.$$

In particular a linear space is a cone. The hypothesis on the kind of algebra for which we take the spectrum in order to produce a linear space induce some additional structure on F. In particular we have maps (over X):

$$\begin{array}{ccc} F \times F & \longrightarrow & F \\ (f, g) & \longmapsto & f + g, \end{array}$$

and

$$\begin{array}{ccc} F & \longrightarrow & F \\ f & \longmapsto & -f. \end{array}$$

The whole bunch of structure maps of F satisfies the axioms that make F an $\mathbb{A}^1_X$-module over X. The algebraic description of a linear space (spectrum of the symmetric algebra

of a sheaf) is in fact equivalent to the geometric one ($\mathbb{A}^1_X$-module over X). The functor $Spec\,Sym^{\bullet}$ defines a contravariant equivalence of categories between the category of coherent sheaves over X and the category of linear spaces over X.
We remark that the category of linear spaces over X is abelian.
A *vector bundle* of rank r over X is a linear space that is locally isomorphic to $\mathbb{A}^r_X$. Vector bundles corresponds to locally free sheaves under the correspondence between linear spaces and coherent sheaves.

15. E-cones Given a morphism $E_0 \longrightarrow E_1$ of linear spaces over a scheme X and a closed subcone $C \subseteq E_1$ we say that C is an E_0-*cone* if it's invariant under the additive action of E_0 on E_1.

16. Exact sequences of cones Given morphisms of cones $E \longrightarrow B \longrightarrow C$ we say that

$$0 \longrightarrow E \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of cones if E is a vector bundle and there exists a commutative diagram diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \square & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0,
\end{array}$$

where the lower row is exact of linear spaces and B is an E-cone (Cf. [24, Def 1.2, Rem 1.3]).

17. Normal cone and normal space associated with a closed embedding Let $X \hookrightarrow Y$ be a closed embedding of schemes, and let I be the ideal sheaf of X in Y. We define

$$C_{X/M} := Spec \bigoplus_{n\in\mathbb{N}} \frac{I^n}{I^{n+1}},$$

the *normal cone* of X in Y, and

$$N_{X/M} := Spec\,Sym^{\bullet} I/I^2,$$

the *normal space* of X in Y. The normal space is the abelian hull of the normal cone, so that in particular we have a closed embeddings of cones over X:

$$C_{X/M} \hookrightarrow N_{X/M}.$$

18. Remark Given a closed embedding of schemes $X \hookrightarrow M$ with M smooth, we have a natural morphism of linear spaces over X:

$$T_{M|X} \longrightarrow N_{X/M},$$

with $T_{M|X}$ a vector bundle. Moreover, the normal cone $C_{X/M} \subseteq N_{X/M}$ is a $T_{M|X}$-cone, see [24, Exm 1.4].

19. Proposition *Let be given a square of scheme morphisms*

$$\begin{array}{ccc} X & \hookrightarrow & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \hookrightarrow & W \end{array}$$

where the hooked arrows are closed embeddings. Then the following holds.
If the square is commutative then we have a map $\alpha : C_X Z \longrightarrow g^* C_Y W$.
If the square is cartesian then α *is a closed embedding, see [8, B.6.1].*
If the square is cartesian and f *is flat then* α *is an isomorphism.*
The same is true with normal spaces in place of normal cones.

20. Lemma *Let be given a surjective ring morphism* $A \longrightarrow B$ *with finitely generated kernel* J. *Let* $t = (t_1, ..., t_p)$ *be a finite set of variables. Build the commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & J' & \longrightarrow & A[[t]] & \longrightarrow & B & \longrightarrow & 0, \end{array}$$

where in the bottom row $t \mapsto 0$. *Then:*

$$\bigoplus_n \frac{J'^n}{J'^{n+1}} = \left(\bigoplus_n \frac{J^n}{J^{n+1}} \right) [\overline{t}],$$

as graded B*-algebras, with* $\deg(\overline{t}_i) = 1$.
`(Proof)` Choose a finite set $f = (f_1, ..., f_k)$ of generators for J: $J = (f)$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & B[\overline{f}] & \longrightarrow & \bigoplus_n \frac{J^n}{J^{n+1}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & B[\overline{f}, \overline{t}] & \longrightarrow & \bigoplus_n \frac{J'^n}{J'^{n+1}} & \longrightarrow & 0, \end{array}$$

where $\overline{f} \mapsto f$, $\overline{t} \mapsto t$ in degree 1 (notice that $J' = J[[t]] + (t)$). The thesis follows if we show that $L' = L[\overline{t}]$. Clearly $L[\overline{t}] \subseteq L'$. To show the converse we'll use multi-indices

$$\begin{aligned} \lambda &= (\lambda_1, ..., \lambda_p) \in \mathbb{N}^p \\ \mu &= (\mu_1, ..., \mu_k) \in \mathbb{N}^k. \end{aligned}$$

Take an element of $B[\overline{f}, \overline{t}]$:

$$p = \sum_{\lambda,\mu} p_{\lambda,\mu} \overline{f}^\mu \overline{t}^\lambda = \sum_\lambda p_\lambda(\overline{f}) \overline{t}^\lambda,$$

with $p_\lambda(\overline{f}) = \sum_\mu p_{\lambda,\mu} \overline{f}^\mu$. Assume that $p \in L'$ i.e. that for all $n \geq 0$

$$\sum_{\substack{\lambda,\, \mu \\ |\lambda| + |\mu| = n}} p_{\lambda,\mu} f^\mu t^\lambda \in J'^{n+1}.$$

We have to show that for all λ and for all $m \geq 0$

$$\sum_m \sum_{\mu:|\mu|=m} p_{\lambda,\mu} f^\mu \in J^{m+1}.$$

Since $J' = J[[t]] + (t)$ then the hypothesis implies that for all $n \geq 0$ and for all λ

$$\sum_{\mu:|\lambda|+|\mu|=n} p_{\lambda,\mu} f^\mu \in J^{n-\lambda+1},$$

and this is equivalent to the thesis by the identification, for any λ, $m = n - |\lambda|$. □

1.3 Going up and going down for E-cones

Going up and going down for E-cones are basic technical tools in the construction of virtual fundamental classes. In this section we'll recall the basic results about them from [24]. The complexes we are going to write are complexes of linear spaces concentrated in degrees 0 and 1.

In particular, given a two term complex $E_\bullet = E_0 \longrightarrow E_1$ of linear spaces over a scheme X, we are going to consider E_0-cones inside E_1 i.e. closed subcones $C \subseteq E_1$ that are invariant under the action of E_0. We are interested in making these cones go up and down via morphisms of complexes:

$$\begin{array}{ccc} E_\bullet & \qquad & E_0 \xrightarrow{d} E_1 \\ \Phi \big\downarrow & & \Phi_0 \big\downarrow \qquad \big\downarrow \Phi_1 \\ F_\bullet & & F_0 \xrightarrow[d]{} F_1, \end{array}$$

and studying properties of these operations. What going up and going down do can be visualized in the diagram below. In the rest of the section we'll give the definitions and the properties we need.

$$\begin{array}{ccc} \Big\{ E_0\text{-cones in } E_1 \Big\} & & \Big\{ F_0\text{-cones in } F_1 \Big\} \\ \Phi^!(C) \subseteq E_1 & \longleftarrow & C \subseteq F_1 \\ B \subseteq E_1 & \longmapsto & \Phi_!(B) \subseteq F_1 \\ & \text{if going down} & \\ & \text{is applicable} & \end{array}$$

21. Going-up The going up operation is defined by means of a pullback and as such it can always be performed. For the results of this paragraph see [24, Def. 2.2, Prop. 2.3, Prop 2.4].

Given a morphism of complexes $\Phi : E_\bullet \longrightarrow F_\bullet$ and an F_0-cone $C \subseteq F_1$, we get an E_0-cone $\Phi^!(C) \subseteq E_1$ by

$$\Phi^!(C) := \Phi_1^{-1}(C).$$

If $\Psi : E_\bullet \longrightarrow F_\bullet$ is another morphism of complexes and Ψ is homotopic to Φ then

$$\Phi^!(C) = \Psi^!(C).$$

If $\Theta : D_\bullet \longrightarrow E_\bullet$ is another morphism of complexes then

$$(\Phi\Theta)^!(C) = \Theta^!\Phi^!(C).$$

If $C' \subseteq F_1$ is another F_0-cone then

$$C \subseteq C' \qquad \Longrightarrow \qquad \Phi^!(C) \subseteq \Phi^!(C').$$

22. Going down The going down operation is a quotient of a cone by a vector bundle and as such need some hypotheses to be performed. For the results of this paragraph see [24, Def. 2.6, Prop. 2.9, Prop 2.11].

Given a morphism of complexes $\Phi : E_\bullet \longrightarrow F_\bullet$ we say that *going down is applicable* to Φ if E_0 is a vector bundle, $h^0(\Phi)$ is an isomorphism and $h^1(\Phi)$ is a closed embedding. Let $B \subseteq E_1$ be an E_0-cone.

If going down is applicable to Φ then an F_0-cone $\Phi_!(B) \subseteq F_1$ is uniquely determined by the diagram

$$\begin{array}{ccccccc} & & & & B \oplus F_0 & \longrightarrow & \Phi_!(B) \\ & & & & \downarrow & \square & \downarrow \\ 0 & \longrightarrow & E_0 & \longrightarrow & E_1 \oplus F_0 & \longrightarrow & F_1 \end{array} \tag{1.2}$$

and by the condition

$$\Phi_!(B) \subseteq \mathrm{Im}(E_1 \oplus F_0).$$

The conditions on the cohomology of Φ are equivalent to the exactness of the sequence of linear spaces in the lower row of the above diagram; the condition on E_0 ensures the possibility to perform the quotient, see [24, Prop 1.6].

If $\Psi : E_\bullet \longrightarrow F_\bullet$ is another morphism of complexes and Ψ is homotopic to Φ then going down is applicable to Ψ if and only if it's applicable to Φ and in this case

$$\Phi_!(B) = \Psi_!(B).$$

If $\Theta : F_\bullet \longrightarrow G_\bullet$ is another morphism of complexes and going down is applicable to Φ and Θ then going down is applicable to $\Theta\Phi$ and

$$(\Theta\Phi)_!(C) = \Theta_!\Phi_!(C).$$

If $B' \subseteq E_1$ is another E_0-cone and going down is applicable to Φ then

$$B \subseteq B' \qquad \Longleftrightarrow \qquad \Phi_!(B) \subseteq \Phi_!(B').$$

For the definition of going down in the derived category see Def. 2.13 of [24].

23. Going up Vs going down Assume to have a morphism of complexes $\Phi : E_\bullet \longrightarrow F_\bullet$ s.t going down is applicable. We have some results that relate going up and going down, see [24, Prop 2.10, Prop. 2.12].
For any E_0-cone $B \subseteq E_1$

$$\Phi^! \Phi_! (B) = B.$$

For any F_0-cone $C \subseteq F_1$

$$\Phi_! \Phi^! (C) = C \cap \Phi_! (E_1).$$

If Φ is a quasi-isomorphism then

$$\Phi_! \Phi^! (C) = C,$$

indeed the morphism $E_1 \oplus F_0 \longrightarrow F_1$ of (1.2) is surjective and $\Phi_!(E_1) = F_1$.

24. Lemma *Let*

$$\begin{array}{ccc} A_\bullet & \xrightarrow{\alpha} & B_\bullet \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ C_\bullet & \xrightarrow[\delta]{} & D_\bullet \end{array}$$

be a square of two term complexes of linear spaces over X that commutes up to homotopy. Assume that: going down is applicable to β and γ, $A_1 \longrightarrow B_1$ is surjective and $h^1(C) \longrightarrow h^1(D)$ is injective. Then:

$$\beta_! \alpha^! = \delta^! \gamma_!.$$

(Proof) By $\gamma\alpha = \delta\beta$ up to homotopy it follows that $\alpha^! \gamma^! = \beta^! \delta^!$. Applying $\beta_!$ to the left and $\gamma_!$ to the right we get $\beta_! \alpha^! = \beta_! \beta^! \delta^! \gamma_!$. We'll show below that $\delta^! \gamma_! (B_1) \subseteq \beta_!(A_1)$. It follows that for any B_0-cone $H \subseteq B_1$ we have

$$\begin{aligned} \beta_! \alpha^! (H) &= \beta_! \beta^! \delta^! \gamma_! (H) = \delta^! \gamma_! (H) \cap \beta_! (A_1) = \\ &= \delta^! \gamma_! (H), \end{aligned}$$

where we used $\delta^! \gamma_! (H) \subseteq \delta^! \gamma_! (B_1)$.
It remains to show that $\delta^! \gamma_! (B_1) \subseteq \beta_! (A_1)$. Given the commutative diagram of linear spaces with exact rows, which is nothing but the morphism between the mapping cones of β and γ induced by α and δ:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_0 & \longrightarrow & A_1 \oplus C_0 & \longrightarrow & C_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_0 & \longrightarrow & B_1 \oplus D_0 & \longrightarrow & D_1, \end{array}$$

and noticing that

$$\begin{aligned} \beta_!(A_1) &= \mathrm{Im}(A_1 \oplus C_0) \subseteq C_1 \\ \delta^! \gamma_! (B_1) &= \delta^{-1}(\mathrm{Im}(B_1 \oplus D_0)) \subseteq C_1, \end{aligned}$$

the thesis is that for all $c_1 \in C_1$, if there exist $b_1 \in B_1$ and $d_0 \in D_0$ s.t. $\delta(c_1) = \gamma(b_1) + \mathrm{d}(d_0)$, then there exist $a_1 \in A_1$ and $c_0 \in C_0$ s.t. $c_1 = \beta(a_1) + \mathrm{d}(c_0)$. This follows by an easy diagram chasing in the diagram below using the hypotheses on $A_1 \longrightarrow B_1$ and $h^1(C) \longrightarrow h^1(D)$:

$$\begin{array}{ccccc} A_1 & \longrightarrow & C_1 & \longrightarrow & h^1(C_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & D_1 & \longrightarrow & h^1(D_\bullet); \end{array}$$

we have to be little care due to the commutativity only up to homotopy of the square in the statement; for instance in the diagram above the left square is not commutative but the right one and the big rectangle are. □

25. Lemma *Let $\Phi : E_\bullet \longrightarrow F_\bullet$ be a morphism of 2-term complexes of linear spaces over X s.t. going down is applicable, and let $f : Y \longrightarrow X$ be a flat morphism. Then going down is applicable to $f^*\Phi : f^*E_\bullet \longrightarrow f^*F_\bullet$ and*

$$f^* \circ \Phi_! = (f^*\Phi)_! \circ f^*$$

as maps from E_0-cones in E_1 (over X) to f^*F_0-cones in f^*F_1 (over Y).
`(Proof)` Since f is flat then f^* is exact. This implies that going down is applicable to $f^*\Phi$. In order to prove the formula we take the definition of $\Phi_!(B)$ for any E_0-cone $B \subseteq E_1$, i.e. the diagram

$$\begin{array}{ccccccc} & & & & B \oplus F_0 & \longrightarrow & \Phi_!(B) \\ & & & & \big\downarrow & \square & \big\downarrow \\ 0 & \longrightarrow & E_0 & \longrightarrow & E_1 \oplus F_0 & \longrightarrow & F_1 \end{array}$$

and the condition

$$\Phi_!(B) \subseteq \mathrm{Im}(E_1 \oplus F_0).$$

Taking f^* of these we get

$$\begin{array}{ccccccc} & & & & f^*B \oplus f^*F_0 & \longrightarrow & f^*(\Phi_!(B)) \\ & & & & \big\downarrow & \square & \big\downarrow \\ 0 & \longrightarrow & f^*E_0 & \longrightarrow & f^*E_1 \oplus f^*F_0 & \longrightarrow & f^*F_1 \end{array}$$

and the condition

$$f^*(\Phi_!(B)) \subseteq f^*\mathrm{Im}(E_1 \oplus F_0) = \mathrm{Im}(f^*E_1 \oplus f^*F_0).$$

This implies the thesis (exactness of f^* is used again). □

1.4 Intersection theory

We'll refer to [8] and [15] for the intersection theory for schemes and algebraic stacks respectively. Now we'll very briefly recall the main results that we need in the sequel.

For any scheme X of finite type over a field $\mathbb{K}$, there is an associated group $A_\bullet X$, the Chow group of X, graded by the integers, with $A_i X = 0$ if $i < 0$ or $i > \dim X$. Elements

of A_iX are given by finite formal linear combinations with integer coefficients of closed i-dimensional subvarieties of X modulo an equivalence relation. We can think of A_iX as some kind of algebraic analogue of the singular homology group (with integer coefficients) $H_i(Y)$ associated with a topological space Y.
The first main property we need is that $A_\bullet X$ is functorial for flat morphisms of some relative dimension. More precisely, if $f : X \longrightarrow Y$ is a flat morphism of schemes of relative dimension d, then there is a graded morphism

$$f^* : A_\bullet Y \longrightarrow A_{\bullet+d}X$$

that sends the class of a closed subvariety V of Y to the class of $f^{-1}V$ in X, [8, Sec. 1.7]. Moreover, given f, g two of such morphisms that are composable then

$$(gf)^* = f^* g^*.$$

The second main property we need is that if $E \xrightarrow{\pi} X$ is a vector bundle of $\mathrm{rk}\, r$ (and hence π is flat of relative dimension r), then the pullback morphism

$$\pi^* : A_\bullet X \longrightarrow A_{\bullet+r}E$$

is an isomorphism, [8, Th. 3.3].
There is a generalization of the construction of $A_\bullet X$ for X an algebraic stack, and the above two properties still hold, [15, Th. 2.1.12]. Moreover, the homotopy invariance holds for vector bundle stacks over stacks that admit a stratification by global quotients, [15, Prop 4.3.2]. For reference, we'll write these facts in a statement:

26. Theorem *For any algebraic stack $\mathcal{X}$ of finite type over a field there is an associated $\mathbb{Z}$-graded group $A_\bullet \mathcal{X}$ that is (contra)functorial for flat morphisms of some fixed relative dimension. Moreover, if $E \xrightarrow{\pi} \mathcal{X}$ is a vector bundle of rank r then the pullback morphism*

$$\pi^* : A_\bullet \mathcal{X} \longrightarrow A_{\bullet+r}E$$

is an isomorphism. The same is true if $\mathcal{X}$ admits a stratification by global quotients and E is a vector bundle stack.

Chapter 2

Virtual fundamental classes for Artin stacks

Virtual fundamental classes are a major technical tool in enumerative geometry. They were introduced with different approaches in [17] and in [4] in order to get classes in the Chow groups of the moduli spaces of stable maps $\overline{M}_{g,n}(V,\beta)$; in turn, these classes allow to define well behaved (e.g. deformation invariant) Gromov-Witten invariants, see [5], [3] and [2]. We'll heavily refer to the presentation of the subject given in [24].
The existing construction of the virtual fundamental class works for DM stacks. In this chapter we propose a way to generalize the construction to Artin stacks.

Any scheme X (of finite type over a field) has a fundamental class $[X] \in A_\bullet X$ in the Chow group. The need for a 'virtual' fundamental class can be explained as follows. Assume that X is defined as the zero scheme of a section s of a vector bundle E over a smooth scheme W. Since X is defined by $\operatorname{rk} E$ equations in a $(\dim W)$-dimensional ambient, the expected dimension of X is $\dim W - \operatorname{rk} E$. It can happen that the actual dimension of X is bigger than the expected one, but we can always build a class

$$[X]^{\mathrm{vir}} \in A_{\dim W - \operatorname{rk} E}\, X$$

of the right dimension by defining $[X]^{\mathrm{vir}} := 0^![W]$, where 0 is the zero section of E:

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & \square & \downarrow s \\ W & \xrightarrow[0]{} & E. \end{array}$$

This is the right class to consider because its numerical data are invariant under deformations: see [8, Ch. 10] for the basic principle and, for instance, [2, Pag. 46] for a discussion in the context we are working with.
We notice that the class so defined is not intrinsic to X but it depends on how X is presented as a zero scheme of a section. This is a general fact: virtual fundamental classes are not intrinsic to the space X in which they live but depends on an extra geometrical datum to be attached to X: a perfect obstruction theory, in the language of [4]. In the example above the class $[X]^{\mathrm{vir}}$ is the virtual fundamental class associated with the obstruction

theory:

$$\begin{array}{ccc} E^\vee_{|X} & \longrightarrow & \Omega_{W|X} \\ \downarrow & & \downarrow \mathrm{id} \\ I/I^2 & \longrightarrow & \Omega_{W|X}, \end{array}$$

where I is the ideal sheaf of X in W. The datum of an of an obstruction theory over X is what is needed to get a virtual fundamental class without necessarily present X as the zero scheme of a section of a vector bundle.

We'll now briefly describe how the virtual fundamental class is gotten out of an obstruction theory in [4]. For X a scheme (or DM stack) of finite type over a field $\mathbb{K}$, endowed with a perfect obstruction theory $\phi : \mathcal{E}^\bullet \longrightarrow \mathcal{L}^\bullet_X$, we can define a closed embedding of the intrinsic normal cone of X inside a vector bundle stack:

$$\mathfrak{C}_X \longrightarrow h^1/h^0(\mathcal{E}^{\bullet\vee}).$$

We define the virtual fundamental class

$$[X]^{\mathrm{vir}} \in A_{\mathrm{rk}\,\mathcal{E}^\bullet} X$$

associated with the given obstruction theory by intersecting the zero section of this vector bundle stack with the intrinsic normal cone (see also [15, Par. 5.2]).
If $\mathcal{E}^\bullet$ has a global resolution $\mathcal{F}^\bullet = \mathcal{F}^{-1} \longrightarrow \mathcal{F}^0$, then we can form the fibered product

$$\begin{array}{ccc} C(\mathcal{F}^\bullet) & \longrightarrow & \mathfrak{C}_X \\ \downarrow & & \downarrow \\ F_1 & \longrightarrow & [F_1/F_0] = h^1/h^0(\mathcal{E}^{\bullet\vee}), \end{array}$$

and we have a closed embedding of a cone inside a vector bundle:

$$C(\mathcal{F}^\bullet) \longrightarrow F_1.$$

Intersecting the zero section of this vector bundle with the cone we get a class that coincide with the above one (in particular it's independent of the choice of the resolution).
The global resolution $\mathcal{F}^\bullet$ of the obstruction theory contains all the informations needed to construct the closed embedding $C(\mathcal{F}^\bullet) \longrightarrow F_1$ and hence the virtual fundamental class. In [24] we find a detailed description of this construction that avoids the construction of the intrinsic normal cone. It's this last point of view that we want to exploit in order to define a virtual fundamental class for Artin stacks, as outlined below.

27. Global normal cone: outline of the construction Fix an algebraic stack $\mathcal{X}$ and an obstruction theory $\mathcal{D}^\bullet \xrightarrow{\psi} \mathcal{L}^\bullet_{\mathcal{X}}$ over it, see (30). Choose a global resolution $\mathcal{E}^\bullet \xrightarrow{\theta} \mathcal{D}^\bullet$, see (31), and get a globally resolved obstruction theory $\phi : \mathcal{E}^\bullet \longrightarrow \mathcal{L}^\bullet_{\mathcal{X}}$, see (32).
The first step is to construct what we call, miming the nomenclature in [24], the *local normal cone*. Given a local embedding

$$\begin{array}{ccc} U & \hookrightarrow & M \\ {\scriptstyle\pi}\downarrow & & \\ \mathcal{X} & & \end{array}$$

(π smooth, U and M schemes with M smooth, $U \hookrightarrow M$ a closed embedding, see (38)), and a lifting α for the chart U relative to ϕ, see (33), we can build a cone inside a vector bundle over U, see (39):

$$C(U, \alpha, M) \hookrightarrow \pi^* E_1.$$

We call this cone the local normal cone associated with the data above. Here

$$E_1 := \operatorname{Spec} \operatorname{Sym}^\bullet \mathcal{E}^{-1}$$

is a vector bundle over $\mathcal{X}$.

The second step is to study the behavior of the local normal cone under smooth base change. Given compatible local embeddings

$$\begin{array}{ccccc} & & V & \hookrightarrow & P \\ & \swarrow^{\rho} & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xleftarrow[\pi]{} & U & \hookrightarrow & M, \end{array}$$

(p, q smooth), and compatible liftings α, β for U, V respectively, then we have, see (40):

$$C(V, \beta, P) = p^* C(U, \alpha, M).$$

The third step is to show that $C(U, \alpha, M)$ is independent of the chosen embedding in M. In particular, given a chart $X \xrightarrow{\pi} \mathcal{X}$ and a lifting α for X relative to ϕ we get a cone inside a vector bundle over X, see (48):

$$C(X, \alpha) \hookrightarrow \pi^* E_1.$$

$C(X, \alpha)$ is well behaved under smooth base change too: given two compatible charts (with p smooth):

$$\begin{array}{ccc} Y & \xrightarrow{p} & X \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle \pi} \\ & & \mathcal{X}, \end{array}$$

and compatible liftings α and β for X and Y, then we have, see (49):

$$C(Y, \beta) = p^* C(X, \alpha).$$

The fourth step is to make the comparison between two charts. Given two charts X and Z with liftings α and γ, and built the 2-cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{r} & Z \\ {\scriptstyle p}\downarrow & \searrow^{\rho} & \downarrow{\scriptstyle \omega} \\ X & \xrightarrow[\pi]{} & \mathcal{X}, \end{array}$$

then there is a lifting β for Y compatible both with α and γ, see (35).

We apply this fact twice. The first time with $Z = X$ and $\gamma = \alpha$, and we get a diagram of groupoids, see (50):

$$\begin{array}{ccc} C_\bullet(X, \alpha) & \overset{\square}{\hookrightarrow} & \pi_\bullet^* E_1 \\ & {\scriptstyle\square}\searrow & \downarrow{\scriptstyle \square} \\ & & X_\bullet. \end{array}$$

Taking the associated diagram of stacks and assuming that X is an atlas we get a cone inside a vector bundle over $\mathcal{X}$:

$$[C_\bullet(X,\alpha)] \hookrightarrow E_1.$$

Applying the comparison between charts a second time we can show that $[C_\bullet(X,\alpha)]$ is in fact independent of the atlas X and the lifting α, see (51). Since an atlas with lifting always exists, see (34), then we get a cone inside a vector bundle over $\mathcal{X}$:

$$C(\mathcal{E}^\bullet) \hookrightarrow E_1,$$

which we call the *global normal cone* associated with the given resolution of the obstruction theory. $C(\mathcal{E}^\bullet)$ is of pure dimension $\operatorname{rk}\mathcal{E}^0 - \operatorname{rk}\mathcal{E}^1$, as remarked in (52).

28. The virtual fundamental class Given an algebraic stack $\mathcal{X}$, an obstruction theory $\mathcal{D}^\bullet \longrightarrow \mathcal{L}^\bullet_{\mathcal{X}}$ over it, and a global resolution $\mathcal{E}^\bullet \longrightarrow \mathcal{D}^\bullet$, we get a cone inside a vector bundle over $\mathcal{X}$, as outlined in (27):

$$C(\mathcal{E}^\bullet) \hookrightarrow E_1 \xrightarrow{\pi} \mathcal{X}.$$

We now apply [15, Th. 2.1.12 (vi)], see (26), to get a class in the Chow group of $\mathcal{X}$:

$$[\mathcal{X}]^{\mathrm{vir}} := (\pi^*)^{-1}[C(\mathcal{E}^\bullet)] \in A_{\operatorname{rk}\mathcal{D}^\bullet}\mathcal{X}, \tag{2.1}$$

where $\operatorname{rk}\mathcal{D}^\bullet := \operatorname{rk}\mathcal{E}^\bullet = -\operatorname{rk}\mathcal{E}^1 + \operatorname{rk}\mathcal{E}^0 - \operatorname{rk}\mathcal{E}^{-1}$ is independent of the choice of the resolution. If $\mathcal{X}$ admits a stratification by global quotients then we can apply [15, Prop. 4.3.2], see (26), and get an equivalent definition of the virtual class by intersecting the global normal cone stack $\mathcal{C}(\mathcal{E}^\bullet)$, see (53), with the zero section of $\underline{\pi} : [E_1/E_0] \longrightarrow \mathcal{X}$:

$$[\mathcal{X}]^{\mathrm{vir}} = (\underline{\pi}^*)^{-1}[\mathcal{C}(\mathcal{E}^\bullet)] \in A_{\operatorname{rk}\mathcal{D}^\bullet}\mathcal{X}.$$

29. Independence of the resolution *If $\mathcal{X}$ admits a stratification by global quotients and conjecture (57) is true, then the virtual fundamental class defined in (2.1) is independent of the resolution, see (58).*

2.1 Local normal cone

In this section we construct the local normal cone $C(U,\alpha,M)$, see (39) and compare with the outline (27), and we study its behavior under smooth base change, see (40). In order to do this we have to give the definition of obstruction theory for an Artin stack, see (30), and introduce the notion of chart with lifting, see (33).

30. Definition *An* `obstruction theory` *for an algebraic stack $\mathcal{X}$ is a morphism in the derived category:*

$$\psi : \mathcal{D}^\bullet \longrightarrow \mathcal{L}^\bullet_{\mathcal{X}},$$

where $\mathcal{L}^\bullet_{\mathcal{X}}$ is the cotangent complex of $\mathcal{X}$ (see [16], [22]), s.t. $h^1(\psi)$ and $h^0(\psi)$ are isomorphisms and $h^{-1}(\psi)$ is surjective.

31. Definition *A* `global resolution` *for a complex* $\mathcal{D}^\bullet$ *is an isomorphism in the derived category:*

$$\theta : \mathcal{E}^\bullet \longrightarrow \mathcal{D}^\bullet,$$

with $\mathcal{E}^\bullet = [\mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E}^1]$ *and the* $\mathcal{E}^i$*'s locally free.*

32. Notation The composition $\mathcal{E}^\bullet \xrightarrow{\theta} \mathcal{D}^\bullet \xrightarrow{\psi} \mathcal{L}^\bullet_{\mathcal{X}}$ of a global resolution and an obstruction theory will be called a *globally resolved obstruction theory.* This has to be meant as fixing an obstruction theory and then choosing a global resolution; the construction we are going to perform will depend on the obstruction theory but not on the global resolution chosen (which we have to assume the existence of). We'll denote such a composition by:

$$\phi : \mathcal{E}^\bullet \longrightarrow \mathcal{L}^\bullet_{\mathcal{X}}.$$

In this section we are going to fix an algebraic stack $\mathcal{X}$ and a globally resolved obstruction theory ϕ over it.

33. Charts with lifting Let be given an algebraic stack $\mathcal{X}$ and a globally resolved obstruction theory $\phi : \mathcal{E}^\bullet \longrightarrow \mathcal{L}^\bullet_{\mathcal{X}}$ over it. A chart over the stack is a smooth morphism $U \xrightarrow{\pi} \mathcal{X}$ from a scheme U. A *lifting* for the chart U relative to ϕ is a morphism

$$\alpha : \Omega_\pi \longrightarrow \pi^*\mathcal{E}^1$$

(where Ω_π is the sheaf of relative differentials, which is locally free by the smoothness of π) of sheaves over X s.t the following diagram commutes in the derived category:

$$\begin{array}{ccc} \Omega_\pi & \xrightarrow{\alpha} & \pi^*\mathcal{E}^\bullet[1] \\ & \searrow & \downarrow {\scriptstyle \pi^*\phi[1]} \\ & & \pi^*\mathcal{L}^\bullet_{\mathcal{X}}[1]. \end{array}$$

If such an α exists, we say that the chart U *admits a lifting* (relative to ϕ). A *chart with lifting* is a pair (U, α), i.e. a chart with a given choice of a lifting. If π turns to be surjective we call (U, α) an *atlas* with lifting.
Fix a 2-commutative diagram in which the maps are smooth:

$$\begin{array}{ccc} V & \xrightarrow{p} & U \\ & {\scriptstyle \rho}\searrow & \downarrow {\scriptstyle \pi} \\ & & \mathcal{X}. \end{array}$$

Given liftings α, β for U and V respectively, we say that the chosen liftings are *compatible* if the following diagram commutes:

$$\begin{array}{ccc} p^*\Omega_\pi & \xrightarrow{p^*\alpha} & p^*\pi^*\mathcal{E}^1 \\ \downarrow & & \downarrow \wr \\ \Omega_\rho & \xrightarrow[\beta]{} & \rho^*\mathcal{E}^1. \end{array}$$

If the map p is étale then $p^*\Omega_\pi \longrightarrow \Omega_\rho$ is an isomorphism. Hence, if U admits a lifting then so V does; moreover, a choice of a lifting α for U induces a choice of a lifting for V, which we denote by α_V.

34. Proposition *Given $\mathcal{X}$ and ϕ as in (32), there exists an altas with lifting (X, α) for $\mathcal{X}$ relative to ϕ.*

`(Proof)` Take an atlas $X' \xrightarrow{\pi'} \mathcal{X}$ for $\mathcal{X}$. Let $\mathcal{U}$ be an open cover of X' that trivializes $\Omega_{\pi'}$. Defining

$$X := \bigvee_{U \in \mathcal{U}} U,$$

we get commutative diagrams (one for each $U \in \mathcal{U}$):

$$\begin{array}{ccc} U & \longrightarrow & X' \\ \downarrow & \searrow^{\pi_U} & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow[\pi]{} & \mathcal{X}, \end{array}$$

where X is an atlas for $\mathcal{X}$.

In the following diagrams with complexes of sheaves we'll use solid arrows to denote morphisms of complexes and dashed arrows to denote morphisms in the derived category. We have to build an arrow α that makes the following diagram (over X) commute:

$$\begin{array}{ccc} & & \pi^*\mathcal{E}^\bullet[1] \\ & \nearrow^{\alpha} & \downarrow{\scriptstyle \pi^*\phi[1]} \\ \Omega_\pi & \dashrightarrow & \pi^*\mathcal{L}^\bullet_{\mathcal{X}}[1]. \end{array}$$

Restricting to U, for any $U \in \mathcal{U}$, we are left with finding an arrow α_U that makes the following diagram (over U) commute:

$$\begin{array}{ccc} & & \pi_U^*\mathcal{E}^\bullet[1] \\ & \nearrow^{\alpha_U} & \downarrow{\scriptstyle \pi_U^*\phi[1]} \\ \Omega_{\pi_U} & \dashrightarrow & \pi_U^*\mathcal{L}^\bullet_{\mathcal{X}}[1]. \end{array}$$

Choosing representatives for the morphisms in the derived category we get a diagram

$$\begin{array}{ccccc} & & \mathcal{F}^\bullet & \xrightarrow{\text{qis}} & \pi_U^*\mathcal{E}^\bullet[1] \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \pi_U^*\phi[1]} \\ \Omega_{\pi_U} & \xrightarrow[\gamma]{} & \mathcal{G}^\bullet & \xleftarrow[\text{qis}]{} & \pi_U^*\mathcal{L}^\bullet_{\mathcal{X}}[1], \end{array}$$

where we can assume $\mathcal{F}^i = \mathcal{G}^i = 0$ for $i > 0$ (by suitably taking the truncation $\tau_{\leq 0}$), and where $\Omega_{\pi_U} = \Omega_{\pi'|U}$ is free. We are left with finding a map α that makes the following diagram commute up to homotopy:

$$\begin{array}{ccc} & & \mathcal{F}^\bullet \\ & \nearrow^{\alpha} & \downarrow{\scriptstyle \beta} \\ \Omega_{\pi_U} & \xrightarrow[\gamma]{} & \mathcal{G}^\bullet. \end{array}$$

The map α is determined by a morphism of sheaves $\Omega_{\pi_U} \longrightarrow \mathcal{F}^0$ and since Ω_{π_U} is free this is determined by global sections $f_1^0, ..., f_r^0$ of $\mathcal{F}^0$ (where r is the rank of Ω_{π_U}). An homotopy between $\beta\alpha$ and γ is determined by a morphism of sheaves $\Omega_{\pi_U} \stackrel{k}{\longrightarrow} \mathcal{G}^{-1}$ s.t. $\gamma = \beta\alpha + \mathrm{d}k$, i.e. by global sections $g_1^{-1}, ..., g_r^{-1}$ of $\mathcal{G}^{-1}$ s.t. for all $j = 1, ..., r$ we have $g_j^0 = \beta(f_j^0) + \mathrm{d}g_j^{-1}$, where $g_1^0, ..., g_r^0$ are the global sections of $\mathcal{G}^0$ that determine γ. Given g_j^0, the existence of g_j^{-1} and f_j^0 with the claimed property follows from the fact that $h^0(\beta)$ is surjective (because ϕ is an obstruction theory). □

35. Proposition *Given $\mathcal{X}$ and ϕ as in (32), assume to have two charts with lifting (X, α) and (Z, γ), and a 2-cartesian square*

$$\begin{array}{ccc} Y & \xrightarrow{r} & Z \\ {\scriptstyle p}\downarrow & \searrow^{\rho} & \downarrow{\scriptstyle \omega} \\ X & \xrightarrow[\pi]{} & \mathcal{X}. \end{array}$$

Then there is a canonical lifting β for the chart Y, that is compatible with α and γ via the maps p and r respectively.

`(Proof)` We draw the relative cotangent sequences of the morphisms p and r:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & r^*\Omega_\omega & & & & \\ & & & & \downarrow & \searrow^{\sim} & & & \\ 0 & \longrightarrow & p^*\Omega_\pi & \longrightarrow & \Omega_\rho & \longrightarrow & \Omega_p & \longrightarrow & 0 \\ & & & \searrow_{\sim} & \downarrow & & & & \\ & & & & \Omega_r & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & . \end{array}$$

From this we get an isomorphism $p^*\Omega_\pi \oplus r^*\Omega_\omega \stackrel{\sim}{\longrightarrow} \Omega_\rho$. Taking p^* of the compatibility diagram for the lifting α:

$$\begin{array}{ccc} \Omega_\pi & \xrightarrow{\alpha} & \pi^*\mathcal{E}^\bullet[1] \\ & \searrow & \downarrow{\scriptstyle \pi^*\phi[1]} \\ & & \pi^*\mathcal{L}^\bullet_{\mathcal{X}}[1]; \end{array}$$

and taking r^* of the compatibility diagram for the lifting γ:

$$\begin{array}{ccc} \Omega_\omega & \xrightarrow{\gamma} & \omega^*\mathcal{E}^\bullet[1] \\ & \searrow & \downarrow{\scriptstyle \omega^*\phi[1]} \\ & & \omega^*\mathcal{L}^\bullet_{\mathcal{X}}[1]; \end{array}$$

we get an induced commutative square:

$$
\begin{array}{ccc}
p^*\Omega_\pi \oplus r^*\Omega_\omega & \longrightarrow & \rho^*\mathcal{E}^\bullet[1] \\
\wr\downarrow & & \downarrow{\scriptstyle \rho^*\phi[1]} \\
\Omega_\rho & \longrightarrow & \rho^*\mathcal{L}^\bullet_{\mathcal{X}}[1].
\end{array}
$$

Inverting the vertical isomorphism we get the induced lifting β proving the statement. □

36. Obstruction theory over a chart with lifting Given $\mathcal{X}$ and ϕ as in (32), and a chart with lifting (U, α), see (33), we are going to describe how to get a globally resolved obstruction theory over U. Notice that since U is a scheme we require the resolution to have non trivial entries only in degrees -1 and 0, as in the classical theory.
Indeed, with the above data we get a commutative diagram in the derived category over U, where the lower row is the exact triangle given by the relative cotangent sequence of the smooth morphism $\pi : U \longrightarrow \mathcal{X}$ [22, eq. 8.1.5]:

$$
\begin{array}{ccccccc}
\pi^*\mathcal{E}^\bullet & & & & \Omega_\pi & \xrightarrow{\alpha} & \pi^*\mathcal{E}^\bullet[1] \\
{\scriptstyle \pi^*\phi}\downarrow & & & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi^*\phi[1]} \\
\pi^*\mathcal{L}^\bullet_{\mathcal{X}} & \longrightarrow & \mathcal{L}^\bullet_U & \longrightarrow & \Omega_\pi & \longrightarrow & \pi^*\mathcal{L}^\bullet_{\mathcal{X}}[1].
\end{array}
$$

We denote by $\mathcal{C}^\bullet(\alpha)$ the cone of the morphism α. By pure homological algebra arguments we get a commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
\pi^*\mathcal{E}^\bullet & \longrightarrow & \mathcal{C}^\bullet(\alpha)[-1] & \longrightarrow & \Omega_\pi & \xrightarrow{\alpha} & \pi^*\mathcal{E}^\bullet[1] \\
{\scriptstyle \pi^*\phi}\downarrow & & {\scriptstyle \psi}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi^*\phi[1]} \\
\pi^*\mathcal{L}^\bullet_{\mathcal{X}} & \longrightarrow & \mathcal{L}^\bullet_U & \longrightarrow & \Omega_\pi & \longrightarrow & \pi^*\mathcal{L}^\bullet_{\mathcal{X}}[1].
\end{array}
\tag{2.2}
$$

It's shown in Lemma (37) that the morphism ψ is an obstruction theory over X.
The next step is to get a resolution of this obstruction theory. We have

$$
\mathcal{C}^\bullet(\alpha)[-1] = [\pi^*\mathcal{E}^{-1} \xrightarrow{\binom{0}{\mathrm{d}}} \Omega_\pi \oplus \pi^*\mathcal{E}^0 \xrightarrow{(\alpha,\mathrm{d})} \pi^*\mathcal{E}^1],
$$

(with $\pi^*\mathcal{E}^1$ in degree 1) and since this complex has no cohomolgy in degree 1 then the map (α, d) is surjective. We thus get the following quasi-isomorphism of complexes in which all the sheaves are locally free:

$$
\begin{array}{ccccccc}
\mathcal{E}^\bullet_{U,\alpha} & & \pi^*\mathcal{E}^{-1} & \longrightarrow & \ker(\alpha, \mathrm{d}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \text{q.iso}} & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}^\bullet(\alpha)[-1] & & \pi^*\mathcal{E}^{-1} & \longrightarrow & \Omega_\pi \oplus \pi^*\mathcal{E}^0 & \xrightarrow[(\alpha,\mathrm{d})]{} & \pi^*\mathcal{E}^1.
\end{array}
\tag{2.3}
$$

Composing with ψ we thus get the claimed globally resolved obstruction theory over U:

$$
\phi_{U,\alpha} : \mathcal{E}^\bullet_{U,\alpha} \longrightarrow \mathcal{L}^\bullet_U.
$$

Taking *Spec Sym*$^\bullet$ we get the above obstruction theory in the form of linear spaces:

$$\phi^{U,\alpha} : L_\bullet^U \longrightarrow E_\bullet^{U,\alpha},$$

in which:

$$E_1^{U,\alpha} = \pi^* E_1.$$

Given another chart with lifting (V,ρ), compatible with (U,α) in the sense of (33), we get a commutative diagram in the derived category over V:

$$\begin{array}{ccc} L_\bullet^V & \longrightarrow & p^* L_\bullet^U \\ \downarrow & & \downarrow \\ E_\bullet^{V,\beta} & \longrightarrow & p^* E_\bullet^{U,\alpha}. \end{array}$$

We notice that if the map $p : V \longrightarrow U$ is étale then $L_\bullet^V \longrightarrow L_\bullet^U{}_{|V}$ is an isomorphism (in the derived category) and $E_\bullet^{V,\beta} \longrightarrow E_\bullet^{U,\alpha}{}_{|V}$ is an isomorphism (of complexes), cf. (44); here we denoted the pullback via p with "restriction" to V.

37. Lemma *With the notation of (36), ψ is an obstruction theory over U.*
`(Proof)` We have to show that $h^0(\psi)$ is an isomorphism, $h^{-1}(\psi)$ is surjective and $h^1(\mathcal{C}^\bullet) = 0$, where we have shortened the notation by putting $\mathcal{C}^\bullet := \mathcal{C}^\bullet(\alpha)[-1]$. We perform the proof by taking the long exact sequence of cohomology of the diagram (2.2). Notice that since π is smooth then its cotangent complex is concentrated in degree zero; moreover, since π is flat then π^* is exact and we can take the pullback of complexes componentwise and π^* commutes with cohomology.
In degree -1 the morphism of long sequences reads

$$\begin{array}{ccccccc} 0 & \longrightarrow & h^{-1}(\pi^*\mathcal{E}^\bullet) & \longrightarrow & h^{-1}(\mathcal{C}^\bullet) & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle h^{-1}(\pi^*\phi)} & & \downarrow{\scriptstyle h^{-1}(\psi)} & & \downarrow \\ 0 & \longrightarrow & h^{-1}(\pi^*\mathcal{L}_\mathcal{X}^\bullet) & \longrightarrow & h^{-1}(\mathcal{L}_U^\bullet) & \longrightarrow & 0. \end{array}$$

The horizontal arrows of the middle square are isomorphisms, and $h^{-1}(\pi^*\phi) = \pi^* h^{-1}(\phi)$ is surjective by hypothesis on ϕ. Hence $h^{-1}(\psi)$ is surjective as well.
In degrees 0 and 1 the morphism of long sequences reads

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & h^0(\pi^*\mathcal{E}^\bullet) & \longrightarrow & h^0(\mathcal{C}^\bullet) & \longrightarrow & \Omega_\pi & \longrightarrow & h^1(\pi^*\mathcal{E}^\bullet) & \longrightarrow & h^1(\mathcal{C}^\bullet) & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle h^0(\pi^*\phi)} & & \downarrow{\scriptstyle h^0(\psi)} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h^1(\pi^*\phi)} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & h^0(\pi^*\mathcal{L}_\mathcal{X}^\bullet) & \longrightarrow & h^0(\mathcal{L}_U^\bullet) & \longrightarrow & \Omega_\pi & \longrightarrow & h^1(\pi^*\mathcal{L}_\mathcal{X}^\bullet) & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Since $h^0(\pi^*\phi)$ and $h^1(\pi^*\phi)$ are isomorphisms by hypothesis on ϕ then we can conclude that $h^0(\psi)$ is an isomorphism and $h^1(\mathcal{C}^\bullet) = 0$. □

38. Local embeddings A *local embedding* (U, M) of the algebraic stack $\mathcal{X}$ is a diagram

$$\begin{array}{ccc} U & \hookrightarrow & M \\ {\scriptstyle \pi}\downarrow & & \\ \mathcal{X} & & \end{array}$$

where π is smooth, U and M are schemes with M smooth, and the hooked arrow is a closed embedding.

39. Local normal cone Fix $\mathcal{X}$ and ϕ as in (32). Given a triple (U, α, M) where (U, α) is a chart with lifting, see (33), and (U, M) is a local embedding, see (38), we are going to construct an $E_0^{U,\alpha}$-cone, see (15), inside $E_1^{U,\alpha} = \pi^* E_1$ over U:

$$C(U, \alpha, M) \hookrightarrow \pi^* E_1.$$

Indeed, consider the composition (in the derived category)

$$\phi^{U,\alpha,M} : L_\bullet^{U,M} \xrightarrow{\sim} \tau_{\leq 1} L_\bullet^U \xrightarrow{\tau_{\leq 1}\phi^{U,\alpha}} \tau_{\leq 1} E_\bullet^{U,\alpha} = E_\bullet^{U,\alpha},$$

for which going down is applicable, see (22), where $\phi^{U,\alpha}$ is defined in (36) and where the complex

$$L_\bullet^{U,M} := [T_{M|U} \longrightarrow N_{U/M}]$$

has $T_{M|U}$ in degree 0. Notice that the normal cone $C_{U/M}$ of U in M is a $T_{M|U}$-cone inside $N_{U/M}$, see (18). Thus it makes sense to define

$$C(U, \alpha, M) := (\phi^{U,\alpha,M})_!(C_{U/M}).$$

We call $C(U, \alpha, M)$ the *local normal cone* associated with the data above.
Notice that in particular $C(U, \alpha, M)$ is a $\pi^* E_0$-cone via the map induced by taking *Spec Sym*$^\bullet$ of the composition:

$$\mathcal{E}^0_{U,\alpha} \hookrightarrow \Omega_\pi \oplus \pi^* \mathcal{E}^0 \longrightarrow \pi^* \mathcal{E}^0.$$

40. Proposition *Let be given triples (U, α, M) and (V, β, P) as in (39). Assume to have a commutative diagram*

$$\begin{array}{ccccc} & & V & \hookrightarrow & P \\ & \swarrow^{\rho} & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xleftarrow[\pi]{} & U & \hookrightarrow & M, \end{array}$$

with p, q smooth, and assume that α and β are compatible in the sense of (33). Then

$$C(V, \beta, P) = p^* C(U, \alpha, M)$$

as subcones of $\rho^* E_1$, over V.
`(Proof)` The proof will be done in subsection 2.1.1, see (42). A little bit more than stated is true: see remark (43). □

41. Remark We can visualize the thesis of proposition (40) by means of the cartesian diagram

$$\begin{array}{ccc} \rho^* E_0 & \longrightarrow & \pi^* E_0 \\ \downarrow & & \downarrow \\ C(V,\beta,P) & \longrightarrow & C(U,\alpha,M) \\ \downarrow & & \downarrow \\ \rho^* E_1 & \longrightarrow & \pi^* E_1 \\ \downarrow & & \downarrow \\ V & \longrightarrow & U, \end{array}$$

or by means of the commuative diagram (over V)

$$\begin{array}{ccc} \rho^* E_0 & \xrightarrow{\sim} & p^*\pi^* E_0 \\ \downarrow & & \downarrow \\ C(V,\beta,P) & \xrightarrow{\sim} & p^*C(U,\alpha,M) \\ \downarrow & & \downarrow \\ \rho^* E_1 & \xrightarrow{\sim} & p^*\pi^* E_1. \end{array}$$

2.1.1 Base change of the local normal cone

We'll prove proposition (40), from which we'll take the notation, except for the following simplifications:

$$\begin{aligned} L_\bullet^U &= [T_{M|U} \longrightarrow N_{U/M}] \\ L_\bullet^V &= [T_{P|V} \longrightarrow N_{V/P}] \end{aligned}$$

(the proper notation would be $L_\bullet^{U,M}$ and $L_\bullet^{V,P}$) and

$$\begin{array}{lll} \Phi & := \phi^{U,\alpha,M} & : L_\bullet^U \longrightarrow E_\bullet^U \\ \Psi & := p^*\Phi & : p^* L_\bullet^U \longrightarrow p^* E_\bullet^U \\ \Theta & := \phi^{V,\beta,P} & : L_\bullet^V \longrightarrow E_\bullet^V. \end{array}$$

We'll work with complexes of linear spaces concentrated in degrees $[0,1]$, as described in section 1.3.
We have a commutative diagram in the derived category of complexes over V:

$$\begin{array}{ccc} L_\bullet^V & \xrightarrow{a} & p^* L_\bullet^U \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Psi} \\ E_\bullet^V & \xrightarrow[b]{} & p^* E_\bullet^U. \end{array}$$

The vertical arrows are obstruction theories, namely the h^0's are isomorphisms and the h^1's are closed embeddings (recall that p is flat and hence p^* preserves these properties).

In wiew of applying going up and going down for E-cones we notice that the degree 0 part of the complexes in the diagram are vector bundles. The horizontal arrows are morphisms of complexes. The morphism a is the differential of p. The following proposition is a formal statement for (40).

42. Proposition

$$\Theta_!(C_{V/P}) = b^!(p^*\Phi_!(C_{U/M})).$$

`(Proof)` Apply lemmas (46), (45) and (25) to get:

$$\Theta_!(C_{V/P}) = \Theta_! a^!(p^*C_{U/M}) = b^!\Psi_!(p^*C_{U/M}) = b^!(p^*\Phi_!(C_{U,M})).$$

□

43. Remark The degree 1 part of the morphism b can be canonically identified with the identity of ρ^*E_1, see (44). Hence $\Theta_!(C_{V/P}) = p^*\Phi_!(C_{U/M})$ as subcones of ρ^*E_1. The equality written in proposition (42) is an isomorphism of E_0^V-cones, and hence in particular of ρ^*E_0-cones.

44. Remark on the morphism b The morphism $b : E_\bullet^V \longrightarrow p^*E_\bullet^U$ has the form:

$$\begin{array}{ccc} E_0^V & \twoheadrightarrow & p^*E_0^U \\ \downarrow & & \downarrow \\ \rho^*E_1 & \xrightarrow[\text{id}]{} & \rho^*E_1, \end{array}$$

and in particular $h^1(b)$ is an isomorphism.
To see this we first write the morphism $p^*\mathcal{C}^\bullet(\alpha)[-1] \longrightarrow \mathcal{C}^\bullet(\beta)[-1]$, see (36):

$$\begin{array}{ccccc} \rho^*\mathcal{E}^{-1} & \longrightarrow & p^*\Omega_\pi \oplus \rho^*\mathcal{E}^0 & \xrightarrow{(p^*\alpha, p^*\mathrm{d})} & \rho^*\mathcal{E}^1 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \rho^*\mathcal{E}^{-1} & \longrightarrow & \Omega_\rho \oplus \rho^*\mathcal{E}^0 & \xrightarrow{(\beta, \mathrm{d})} & \rho^*\mathcal{E}^1. \end{array}$$

Then we notice that by definition (and exploiting that p is flat) $\mathcal{E}_V^0$ and $p^*\mathcal{E}_U^0$ fit into the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{E}_U^0 & \longrightarrow & p^*\Omega_\pi \oplus \rho^*\mathcal{E}^0 & \longrightarrow & \rho^*\mathcal{E}^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathcal{E}_V^0 & \longrightarrow & \Omega_\rho \oplus \rho^*\mathcal{E}^0 & \longrightarrow & \rho^*\mathcal{E}^1 & \longrightarrow & 0. \end{array}$$

Since the central vertical map is injective so is the left vertical one, and we conclude by taking $Spec\, Sym^\bullet$.

45. Lemma *For any $p^*T_{M|U}$-cone $C \subseteq p^*N_{U/M}$ we have:*

$$\Theta_! a^!(C) = b^!\Psi_!(C).$$

`(Proof)` We want to apply lemma (24). We have to deal with the fact that Θ and Ψ are morphisms in the derived category, i.e. we have to handle a diagram of the form

$$\begin{array}{ccc} L^V_\bullet & \xrightarrow{a} & p^*L^U_\bullet \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ G_\bullet & \xrightarrow{e} & H_\bullet \\ {\scriptstyle \sigma}\uparrow\wr & & \wr\uparrow{\scriptstyle \tau} \\ E^\rho_\bullet & \xrightarrow[b]{} & p^*E^\pi_\bullet \end{array}$$

where all the arrows are morphisms of complexes, the squares commute up to homotopy, σ and τ are quasi-isomorphisms, in degree 0 all the complexes have a vector bundle, going down is applicable to c and d.
In order to apply the lemma we notice that in degree 1 the morphism $a_1 : N_{V/P} \longrightarrow p^*N_{U/M}$ is surjective by (2.4); moreover $h^1(e)$ is an isomorphism because $h^1(b)$ is by (44). Hence we can apply the lemma and get $c_! a^! = e^! d_!$.
Since $\Theta_! = \sigma^! c_!$ and $\Psi_! = \tau^! d_!$ we have that $\Theta_! a^! = \sigma^! c_! a^! = \sigma^! e^! d_! = b^! \tau^! d_! = b^! \Psi_!$. □

46. Lemma

$$a^!(p^*C_{U/M}) = C_{V/P}.$$

`(Proof)` We'll define a vector bundle $T_{p/q}$ over V fitting into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{p/q} & \longrightarrow & N_{V/P} & \xrightarrow{a_1} & p^*N_{U/M} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & T_{p/q} & \longrightarrow & C_{V/P} & \longrightarrow & p^*C_{U/M} & \longrightarrow & 0, \end{array} \tag{2.4}$$

and we'll prove that the rows of this diagram are exact sequences. From these facts it follows that the right square of the diagram is cartesian, which is the thesis.
We call J the ideal sheaf of V in P, and I the ideal sheaf of U in P. We have the following diagram of sheaves where all the rows and columns are exact:

$$\begin{array}{ccccccccccc} & & & & & & 0 & & 0 & & \\ & & & & & & \uparrow & & \uparrow & & \\ & & & & & & \Omega_{q|V} & \longrightarrow & \Omega_p & & \\ & & & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Omega_V^{-1} & \longrightarrow & J/J^2 & \longrightarrow & \Omega_{P|V} & \longrightarrow & \Omega_V & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & p^*\Omega_U^{-1} & \longrightarrow & p^*I/I^2 & \longrightarrow & p^*(\Omega_{M|U}) & \longrightarrow & p^*\Omega_U & \longrightarrow & 0 \\ & & & & & & \uparrow & & \uparrow & & \\ & & & & & & 0 & & 0 & & . \end{array}$$

It follows that $\Omega_{q|V} \longrightarrow \Omega_p$ is surjective and hence, taking the kernel, we have a short exact sequence of locally free sheaves:

$$0 \longrightarrow \Omega_{p/q} \longrightarrow \Omega_{q|V} \longrightarrow \Omega_p \longrightarrow 0.$$

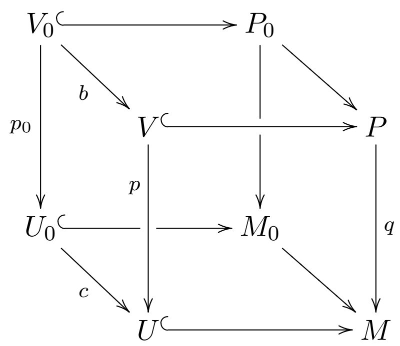
We complete the above diagram to the one below, where a priori all the rows and columns but the one in the box are exact:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \Omega_{p/q} & \longrightarrow & \Omega_{q|V} & \longrightarrow & \Omega_p & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega_V^{-1} & \longrightarrow & J/J^2 & \longrightarrow & \Omega_{P|V} & \longrightarrow & \Omega_V & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & p^*\Omega_U^{-1} & \longrightarrow & p^*I/I^2 & \longrightarrow & p^*(\Omega_{M|U}) & \longrightarrow & p^*\Omega_U & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & .
\end{array}$$

Indeed the exactness of the sequence in the box follows easily by diagram chasing. Taking *Spec Sym*$^\bullet$ of this sequence we get the exactness of the upper row of (2.4).
In order to prove that the lower row of (2.4) is exact we have to prove that locally $C_{V/P} = T_{p/q} \times_V p^* C_{U/M}$. Since this is an isomorphism of cones it's enough to prove it after restricting to the spectrum of the completion of the local ring at the closed points of V, as described in the following. Let $y \in V \subseteq P$ be a closed point of V and put $x := p(y) \in U \subseteq M$. We put

$$\begin{aligned}
V_0 &:= \operatorname{Spec} \widehat{\mathcal{O}}_{y,V} \\
P_0 &:= \operatorname{Spec} \widehat{\mathcal{O}}_{y,P} \\
U_0 &:= \operatorname{Spec} \widehat{\mathcal{O}}_{x,U} \\
M_0 &:= \operatorname{Spec} \widehat{\mathcal{O}}_{x,M}.
\end{aligned}$$

We have a commutative cube:



where the horizontal squares are cartesian and the arrows from the rear square to the front square are flat. Hence $b^*C_{V/P} = C_{V_0/P_0}$ and $c^*C_{U/M} = C_{U_0/M_0}$; moreover $b^*T_{p/q}$ is a trivial vector bundle over V_0. By the smoothness assumptions we have that:

$$\begin{aligned}
\widehat{\mathcal{O}}_{x,M} &= \mathbb{K}[[x_1, ..., x_m]] \\
\widehat{\mathcal{O}}_{x,U} &= \widehat{\mathcal{O}}_{x,M}[[x_1, ..., x_m]]/I \\
\widehat{\mathcal{O}}_{y,P} &= \mathbb{K}[[x_1, ..., x_m, z_1, ..., z_s, \tilde{t}_1, ...\tilde{t}_k]] \\
\widehat{\mathcal{O}}_{y,V} &= \widehat{\mathcal{O}}_{x,U}[[\overline{z}_1, ..., \overline{z}_s]].
\end{aligned}$$

Notice that $\operatorname{rk} T_{p/q} = k$. Applying lemma (47) we can change the identifications of the completed local rings with formal power series ring in such a way to get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \mathbb{K}[[x]] & \longrightarrow & \widehat{\mathcal{O}}_{x,U} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I[[z]] & \longrightarrow & \mathbb{K}[[x,z]] & \longrightarrow & \widehat{\mathcal{O}}_{x,U}[[z]] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I[[z,t]] + (t) & \longrightarrow & \mathbb{K}[[x,z,t]] & \longrightarrow & \widehat{\mathcal{O}}_{x,U}[[z]] & \longrightarrow & 0,
\end{array} \tag{2.5}$$

where in the last row $t \mapsto 0$. Since $\mathbb{K}[[x,z]] \otimes_{\mathbb{K}[[x]]} \widehat{\mathcal{O}}_{x,U} = \widehat{\mathcal{O}}_{x,U}[[z]]$, and since for a noetherian ring A, $A[[z]]$ is flat over A, then we can apply proposition (19) to the upper right square of (2.5) and get:

$$\bigoplus_n \frac{I[[z]]^n}{I[[z]]^{n+1}} = \left(\bigoplus_n \frac{I^n}{I^{n+1}} \right) \otimes_{\widehat{\mathcal{O}}_{x,U}} \widehat{\mathcal{O}}_{y,V}.$$

Applying lemma (20) to the lower right square of (2.5) we get:

$$\bigoplus_n \frac{(I[[z]] + (t))^n}{(I[[z]] + (t))^{n+1}} = \left(\bigoplus_n \frac{I[[z]]^n}{I[[z]]^{n+1}} \right) [t].$$

We conclude that

$$\begin{aligned}
C_{V_0/P_0} &= \operatorname{Spec} \left(\bigoplus_n \frac{(I[[z]] + (t))^n}{(I[[z]] + (t))^{n+1}} \right) \\
&= \operatorname{Spec} \left(\left(\left(\bigoplus_n \frac{I^n}{I^{n+1}} \right) \otimes_{\widehat{\mathcal{O}}_{x,U}} \widehat{\mathcal{O}}_{y,V} \right) \otimes_{\widehat{\mathcal{O}}_{y,V}} \widehat{\mathcal{O}}_{y,V}[\overline{t}_1, ..., \overline{t}_k] \right) \\
&= p_0^* C_{U_0/M_0} \times_{V_0} b^* T_{p/q}.
\end{aligned}$$

□

47. Lemma *Let be given a morphism of algebras*

$$f : \mathbb{K}[[x, z, \tilde{t}]] \longrightarrow \mathbb{K}[[x, \overline{z}]]$$

where the bunches of variables are as in (46), and s.t. $(\partial z/\partial \overline{z})(0)$ *is invertible. Then there exists a commutative square*

$$\begin{array}{ccc} \mathbb{K}[[x,z,\widetilde{t}]] & \xrightarrow{f} & \mathbb{K}[[x,\overline{z}]] \\ h \downarrow \wr & & g \downarrow \wr \\ \mathbb{K}[[x,z,t]] & \longrightarrow & \mathbb{K}[[x,z]] \end{array}$$

where in the bottom row $t \mapsto 0$*.*

`(Proof)` The composition $\mathbb{K}[[x,z]] \longrightarrow \mathbb{K}[[x,z,\widetilde{t}]] \xrightarrow{f} \mathbb{K}[[x,\overline{z}]]$ is invertible by the hypothesis on the Jacobian and by the inverse function theorem for formal power series. We call g the inverse. In order to define h we have to define $\widetilde{t} = \widetilde{t}(x,z,t)$. We put

$$\widetilde{t} = t + (gf)_3(x,z),$$

where the subscript 3 refers to the viarables $\widetilde{t}$ of $(x,z,\widetilde{t})$. □

2.2 Global normal cone

Fix $\mathcal{X}$ and ϕ as in (32). Chosen a chart with lifing (X,α), see (33), we are going first to construct a π^*E_0-cone

$$C(X,\alpha) \hookrightarrow \pi^*E_1$$

over X by gluing the local normal cones of (39), see proposition (48). Then we'll study the behavior of $C(X,\alpha)$ under smooth base change, see proposition (49); this will allow us to define an $X_\bullet$-equivariant structure over $C(X,\alpha)$ in the sense of proposition (50). Provided that X is an atlas this will define an E_0-cone

$$[C_\bullet(X,\alpha)] \hookrightarrow E_1$$

over $\mathcal{X}$, diagram (2.8). This cone is in fact independent of the atlas with lifting chosen, see theorem (51).

48. Proposition *Given* $\mathcal{X}, \phi$ *and a chart with lifting* (X,α) *as above (with* $X \xrightarrow{\pi} \mathcal{X}$*), there exists a unique* π^*E_0*-cone*

$$C(X,\alpha) \hookrightarrow \pi^*E_1$$

over X *s.t. for all open embeddings* $U \longrightarrow X$ *and all closed embeddings* $U \hookrightarrow M$ *with* M *smooth we have*

$$C(X,\alpha)_{|U} = C(U,\alpha_U,M).$$

as $\pi^*E_{0|U}$*-cones, over* U*.*

`(Proof)` Once we have the obstruction theory over X described in (36) the result is standard. We'll recall the proof. We have to show that given two of such local embeddings of X:

$$\begin{array}{ccccccc} U & \hookrightarrow & M & \qquad & U' & \hookrightarrow & M' \\ \downarrow & & & & \downarrow & & \\ X & & & & X & & , \end{array}$$

then $C(U, \alpha_U, M)_{|U\cap U'} = C(U', \alpha_{U'}, M')_{|U\cap U'}$. In order to do this, we notice that we have opens $\overline{M}$ and $\overline{M}'$ of M and M' respectively such that we have the diagram

$$\begin{array}{ccc} U\cap U' & \hookrightarrow & \overline{M}\times\overline{M}' \\ \downarrow & & \downarrow \\ U\cap U' & \hookrightarrow & \overline{M} \\ \downarrow & & \downarrow \\ U & \hookrightarrow & M \end{array}$$

and a similar one with U' and M', in which the lower square is cartesian and the upper square is commuative. We now apply (40) to get:

$$C(U, \alpha_U, M)_{|U\cap U'} = C(U\cap U', \alpha_{U\cap U'}, \overline{M}\times\overline{M}') = C(U', \alpha_{U'}, M')_{|U\cap U'}.$$

□

49. Proposition *Given $\mathcal{X}, \phi$ as above, a 2-commutative diagram with smooth maps*

$$\begin{array}{ccc} Y & \xrightarrow{p} & X \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\pi} \\ & & \mathcal{X}, \end{array}$$

and compatible liftings α and β for X and Y as described in (33), then

$$C(Y, \beta) = p^* C(X, \alpha)$$

as $\rho^* E_0$-cones over Y, i.e. we have the cartesian diagram

$$\begin{array}{ccc} \rho^* E_0 & \longrightarrow & \pi^* E_0 \\ \downarrow & & \downarrow \\ C(Y,\beta) & \longrightarrow & C(X,\alpha) \\ \downarrow & & \downarrow \\ \rho^* E_1 & \longrightarrow & \pi^* E_1 \\ \downarrow & & \downarrow \\ Y & \xrightarrow[p]{} & X. \end{array}$$

`(Proof)` Let $\mathcal{U}$ be an open affine cover of X. For all $U \in \mathcal{U}$ let $\mathcal{V}_U$ be an open affine cover of $p^{-1}(U)$. Since p is of finite type then for any $U \in \mathcal{U}$ and for any $V \in \mathcal{V}_U$ we get a commutative diagram

$$\begin{array}{ccc} V & \hookrightarrow & \mathbb{A}^{n+r} \\ {\scriptstyle p_V}\downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{A}^n. \end{array}$$

Applying definition (39) of the local normal cone and proposition (40) we have:

$$(p^*C(X,\alpha))_{|V} = p_V^*(C(X,\alpha)_{|U}) = p_V^*C(U,\alpha_U,\mathbb{A}^n) = C(V,\beta_V,\mathbb{A}^{n+r}) = C(Y,\beta)_{|V}.$$

□

50. Proposition *Let $X_\bullet$ be the groupoid associated with the chart X, see (11). Then there exists a commutative diagram of groupoids:*

$$\begin{array}{ccc} C_\bullet(X,\alpha) & \overset{\square}{\hookrightarrow} & \pi_\bullet^*E_1 \\ & {\scriptstyle \square}\searrow & \downarrow{\scriptstyle \square} \\ & & X_\bullet, \end{array}$$

where $C_\bullet(X,\alpha)$ is a $\pi_\bullet^*E_0$-cone, in the sense of diagram (2.7).
`(Proof)` We first apply proposition (35) to the cartesian diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{s} & X \\ {\scriptstyle t}\downarrow & \overset{\pi_1}{\searrow} & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\pi]{} & \mathcal{X} \end{array} \tag{2.6}$$

to get a chart with lifting (X_1,α_1). Then we apply proposition (49) to s and t getting:

$$\begin{array}{ccc} \pi_1^*E_0 & \rightrightarrows & \pi^*E_0 \\ \downarrow & & \downarrow \\ C(X_1,\alpha_1) & \rightrightarrows & C(X,\alpha) \\ \downarrow & & \downarrow \\ \pi_1^*E_1 & \rightrightarrows & \pi^*E_1 \\ \downarrow & & \downarrow \\ X_1 & \rightrightarrows & X. \end{array} \tag{2.7}$$

Putting $C_1(X,\alpha) := C(X_1,\alpha_1)$ and exploiting the fact that $C(X,\alpha)$ is a subobject of π^*E_1, for which we already have the groupoid structure, the thesis follows. □

51. Theorem *Let $\mathcal{X}$ be an algebraic stack and $\phi : \mathcal{E}^\bullet \longrightarrow \mathcal{L}_\mathcal{X}^\bullet$ be a globally resolved obstruction theory over it, see (32). Then there exists a unique E_0-cone*

$$C(\mathcal{E}^\bullet) \hookrightarrow E_1 = \operatorname{Spec}\operatorname{Sym}\mathcal{E}^{-1}$$

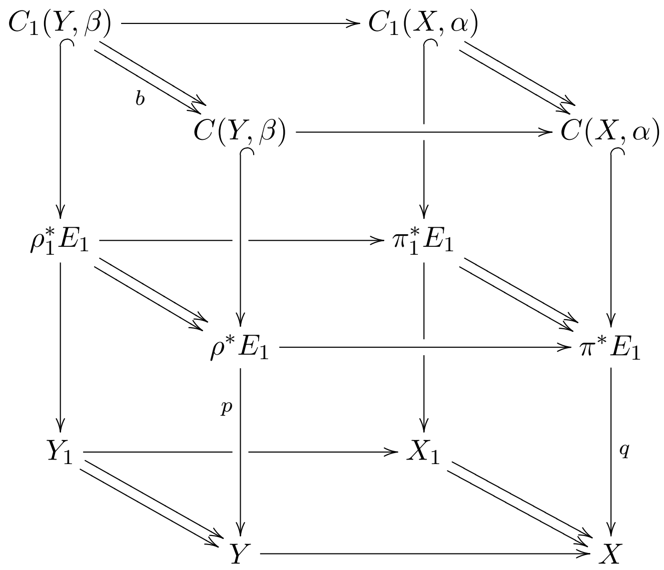
over $\mathcal{X}$ s.t. for all altases with lifting (X, α) we have

$$C(X, \alpha) = \pi^* C(\mathcal{E}^\bullet).$$

`(Proof)` Choose an atlas with lifting (X, α), which exists by (34). By proposition (50) we have

$$\begin{array}{ccccc} [C_\bullet(X,\alpha)] & \hookrightarrow & [\pi_\bullet^* E_1] & \xrightarrow{\sim} & E_1 \\ & \searrow & \downarrow & & \downarrow \\ & & [X_\bullet] & \xrightarrow{\sim} & \mathcal{X}, \end{array} \tag{2.8}$$

and $[C_\bullet(X, \alpha)]$ has the properties claimed for $\mathcal{C}(\mathcal{E}^\bullet)$, compare with (11).
In order to show that this is independent of the choice of the atlas we take another atlas with lifting (Z, ω), and associate to these data a third atlas with lifting (Y, β) as described in (35). We have a diagram



where the vertical rectangles are given by proposition (49), and the ground floor and first floor squares are Morita morphisms of groupoids. Since the vertical arrows from the top floor square are in particular monomorphisms then the top floor is Morita too. It follows that we have:

$$\begin{array}{ccc} [C_\bullet(Y,\beta)] & \xrightarrow{\sim} & [C_\bullet(X,\alpha)] \\ \downarrow & & \downarrow \\ [\rho_\bullet^* E_1] & \xrightarrow{\sim} & [\pi_\bullet^* E_1] \\ \downarrow & & \downarrow \\ [Y_\bullet] & \xrightarrow{\sim} & [X_\bullet]. \end{array}$$

□

52. Remark: the dimension of the global normal cone The global normal cone $\mathcal{C}(\mathcal{E}^\bullet)$ is of pure dimension $\operatorname{rk}\mathcal{E}^0 - \operatorname{rk}\mathcal{E}^1$. Indeed, by [4, Sec. 5] (or [24, Th. 3.3]), and by the diagram (2.3), it follows that $C(X,\alpha)$ is of pure dimension

$$\operatorname{rk}\ker(\alpha,\mathrm{d}) = \operatorname{rk}\mathcal{E}^0 + \operatorname{rk}\Omega_\pi - \operatorname{rk}\mathcal{E}^1.$$

By the cartesianity of the diagrams (2.6) and (2.7), it follows that $\mathcal{C}(\mathcal{E}^\bullet) = [C_\bullet(X,\alpha)]$ is of pure dimenstion

$$\operatorname{rk}\ker(\alpha,\mathrm{d}) - \operatorname{rk}\Omega_\pi = \operatorname{rk}\mathcal{E}^0 - \operatorname{rk}\mathcal{E}^1.$$

53. Global normal cone stack Defining

$$\mathcal{C}(\mathcal{E}^\bullet) := [C(\mathcal{E}^\bullet)/E_0]$$

we get a cone stack inside a vector bundle stack over $\mathcal{X}$ that we call the *global normal cone stack* associated with the given globally resolved obstruction theory. We have a cartesian diagram:

$$\begin{array}{ccc} C(\mathcal{E}^\bullet) & \longrightarrow & \mathcal{C}(\mathcal{E}^\bullet) \\ \big\downarrow & & \big\downarrow \\ E_1 & \longrightarrow & [E_1/E_0]. \end{array}$$

2.3 Dependence of the resolution

We want to study the dependence of the resolution of the virtual class. We assume to be given an algebraic stack $\mathcal{X}$, an obstruction theory $\mathcal{D}^\bullet$, and two global resolutions $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ as in the diagram:

$$\begin{array}{ccccc} \mathcal{E}^\bullet & \xrightarrow[\text{qis}]{\phi} & \mathcal{D}^\bullet & \xleftarrow[\text{qis}]{\psi} & \mathcal{F}^\bullet \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{L}^\bullet_{\mathcal{X}} & & \end{array} \qquad (2.9)$$

,

where the dashed arrows are morphisms in the derived category and the solid arrows are (without loss of generality) morphisms of complexes. Moreover we can assume that $\mathcal{D}^i = 0$ for $i \notin [-1,1]$. By (53) we have two pairs of a cone stack inside a vector bundle stack over $\mathcal{X}$:

$$\mathcal{C}(\mathcal{E}^\bullet) \hookrightarrow [E_1/E_0]$$
$$\mathcal{C}(\mathcal{F}^\bullet) \hookrightarrow [F_1/F_0].$$

We are going to consider diagrams of the form:

$$\begin{array}{ccc} \mathcal{E}^1 & & \\ {\scriptstyle j}\big\downarrow & \searrow^{\phi^1} & \\ \mathcal{E}^\bullet & \xrightarrow[\phi]{} & \mathcal{D}^\bullet, \end{array}$$

where j is the identity of $\mathcal{E}^1$ in degree 1. Since ϕ^1 induces a surjection $\mathcal{E}^1 \longrightarrow h^1(\mathcal{D}^\bullet)$ then we have quasi-isomorphisms

$$[\mathcal{E}^{-1} \longrightarrow \mathcal{E}^0] \longrightarrow \mathcal{C}^\bullet(j) \longrightarrow \mathcal{C}^\bullet(\phi^1),$$

which in turn induce an isomorphism of vector bundle stacks:

$$\mathcal{V}(\phi^1) := h^1/h^0(\mathcal{C}^\bullet(\phi^1)^\vee) \xrightarrow{\sim} [E_1/E_0], \tag{2.10}$$

where we took the notation h^1/h^0 from [4].

54. Proposition *Given data as above, assume to have a morphism $\theta^1 : \mathcal{E}^1 \longrightarrow \mathcal{F}^1$ s.t. the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \mathcal{E}^1 & \xrightarrow{\theta^1} & \mathcal{F}^1 \\ {\scriptstyle \phi^1}\downarrow & \swarrow {\scriptstyle \psi^1} & \\ \mathcal{D}^\bullet & & \end{array} .$$

Then there is a canonical cartesian diagram of cone stacks over X:

$$\begin{array}{ccc} \mathcal{C}(\mathcal{F}^\bullet) & \longrightarrow & \mathcal{C}(\mathcal{E}^\bullet) \\ \downarrow & & \downarrow \\ [F_1/F_0] & \longrightarrow & [E_1/E_0]. \end{array}$$

(Proof) Cover $\mathcal{X}$ by smooth maps $U \xrightarrow{\pi} \mathcal{X}$ s.t. $\pi^*\mathcal{E}^i$ and Ω_π are free, and there is a closed embedding $U \hookrightarrow M$ with M smooth. By abuse of notation we denote with ϕ and ψ also the globally obstruction theories induced by the global resolutions ϕ and ψ.
There exists a diagram that commutes up to homotopy:

$$\begin{array}{ccc} \pi^*\mathcal{E}^\bullet & \xrightarrow{\overline{\theta}} & \pi^*\mathcal{F}^\bullet \\ {\scriptstyle \pi^*\phi}\downarrow & \swarrow {\scriptstyle \pi^*\psi} & \\ \pi^*\mathcal{D}^\bullet & & \end{array} ,$$

with $\overline{\theta}^1 = \pi^*\theta^1$, and we have a lifting α for U relative to ϕ. Defining β by composition as in the diagram:

$$\begin{array}{ccc} \Omega_\pi & \xrightarrow{\alpha} & \pi^*\mathcal{E}^1 \\ & {\scriptstyle \beta}\searrow & \downarrow {\scriptstyle \pi^*\theta^1} \\ & & \pi^*\mathcal{F}^1, \end{array}$$

we have that β is a lifting for U relative to ψ. We have to show that we have a cartesian diagram (over U):

$$\begin{array}{ccc} C(U,\beta,M) & \longrightarrow & C(U,\alpha,M) \\ \downarrow & & \downarrow \\ \pi^*F_1 & \longrightarrow & \pi^*E_1. \end{array}$$

Assume to have shown that $\overline{\theta}$ induces a map $\hat{\theta}$ s.t. going down is applicable to $\hat{\theta}$:

$$\begin{array}{ccc} L_\bullet^{U,M} & \xrightarrow{\phi^{U,\alpha,M}} & E_\bullet^{U,\alpha} \\ & \searrow^{\psi^{U,\beta,M}} & \uparrow \hat{\theta} \\ & & F_\bullet^{U,\beta}; \end{array}$$

then what we have to show is that $\hat{\theta}^! C(U,\alpha,M) = C(U,\beta,M)$. This follows by:

$$\begin{aligned} \hat{\theta}^! C(U,\alpha,M) &= \hat{\theta}^! (\phi^{U,\alpha,M})_! C_{U/M} \\ &= \hat{\theta}^! \hat{\theta}_! (\psi^{U,\beta,M})_! C_{U/M} \\ &= (\psi^{U,\beta,M})_! C_{U/M} \\ &= C(U,\beta,M). \end{aligned}$$

We are left to show the claim about $\hat{\theta}$. This is equivalent (before taking $Spec\, Sym^\bullet$) to show that $\overline{\theta}$ induces a map

$$\tilde{\theta} : \mathcal{E}^\bullet_{U,\alpha} \longrightarrow \mathcal{F}^\bullet_{U,\beta}$$

s.t. $h^0(\tilde{\theta})$ is an isomorphism and $h^{-1}(\tilde{\theta})$ is surjective. Indeed there is a morphism of complexes

$$\begin{array}{ccccc} \pi^*\mathcal{F}^{-1} & \xrightarrow{\binom{0}{\mathrm{d}}} & \Omega_\pi \oplus \pi^*\mathcal{F}^0 & \xrightarrow{(\beta,\mathrm{d})} & \pi^*\mathcal{F}^1 \\ \uparrow & & \uparrow & & \uparrow \\ \pi^*\mathcal{E}^{-1} & \xrightarrow[\binom{0}{\mathrm{d}}]{} & \Omega_\pi \oplus \pi^*\mathcal{E}^0 & \xrightarrow[(\alpha,\mathrm{d})]{} & \pi^*\mathcal{E}^1 \end{array}$$

from which we get the morphism $\widetilde{\theta}$, which we write with the corresponding cohomology in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h^{-1}(\mathcal{F}^\bullet_{U,\beta}) & \longrightarrow & \pi^*\mathcal{F}^{-1} & \longrightarrow & \ker(\beta,\mathrm{d}) & \longrightarrow & h^0(\mathcal{F}^\bullet_{U,\beta}) & \longrightarrow & 0 \\ & & {\scriptstyle h^{-1}(\tilde{\theta})}\uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle h^0(\tilde{\theta})} & & \\ 0 & \longrightarrow & h^{-1}(\mathcal{E}^\bullet_{U,\alpha}) & \longrightarrow & \pi^*\mathcal{E}^{-1} & \longrightarrow & \ker(\alpha,\mathrm{d}) & \longrightarrow & h^0(\mathcal{E}^\bullet_{U,\alpha}) & \longrightarrow & 0. \end{array}$$

From this we see that $h^{-1}(\tilde{\theta}) = h^{-1}(\overline{\theta})$. Morevoer, by an easy diagram chasing, one can show that if $h^0(\overline{\theta})$ is surjective and $h^1(\overline{\theta})$ is injective then $h^0(\tilde{\theta})$ is surjective; if $h^0(\overline{\theta})$ is injective then $h^0(\tilde{\theta})$ is injective. Since $\overline{\theta}$ is a quasi-isomorphism then we conclude. □

55. Definition *Let be given* $\mathcal{E}^1 \xrightarrow{\phi^1} \mathcal{D}^\bullet$ *as above (*$\mathcal{E}^1$ *locally free and* ϕ^1 *induces a surjection* $\mathcal{E}^1 \longrightarrow h^1(\mathcal{D}^\bullet)$*). We say that* ϕ^1 `factorizes through a resolution` *if there is a global resolution through which* ϕ^1 *factorizes:*

$$\begin{array}{ccc} \mathcal{E}^1 & & \\ {\scriptstyle j}\downarrow & \searrow^{\phi^1} & \\ \mathcal{E}^\bullet & \xrightarrow[\phi]{\mathrm{qis}} & \mathcal{D}^\bullet. \end{array}$$

56. Corollary *If in the situation of (54), if θ^1 is an isomorphism then the horizontal arrows of the induced diagram are isomorphisms:*

$$\begin{array}{ccc} \mathcal{C}(\mathcal{F}^\bullet) & \xrightarrow{\sim} & \mathcal{C}(\mathcal{E}^\bullet) \\ \downarrow & & \downarrow \\ [F_1/F_0] & \xrightarrow{\sim} & [E_1/E_0]. \end{array}$$

In particular, given $\mathcal{E}^1 \xrightarrow{\phi^1} \mathcal{D}^\bullet$ that factorizes through a resolution, see (55), then there is a cone stack

$$\mathcal{C}(\phi^1) \hookrightarrow \mathcal{V}(\phi^1)$$

that is independent of the resolution $\mathcal{E}^\bullet$ chosen.

57. Conjecture Given two resolutions (2.9), we can compare them with a third one:

$$\begin{array}{ccccc} \mathcal{E}^1 & \longrightarrow & \mathcal{G}^1 & \longleftarrow & \mathcal{F}^1 \\ & \searrow^{\phi^1} & \downarrow^{\rho^1} & \swarrow^{\psi^1} & \\ & & \mathcal{D}^\bullet & & \end{array},$$

s.t. ρ^1 factorizes through a resolution, see (55), and the induced maps of vector bundle stacks, see (2.10), are flat:

$$\begin{array}{ccc} \mathcal{V}(\rho^1) & \longrightarrow & \mathcal{V}(\phi^1) \\ \downarrow & & \\ \mathcal{V}(\psi^1) & & \end{array}.$$

`(Remark)` A possible try is with $\mathcal{G}^1 = \mathcal{E}^1 \oplus \mathcal{F}^1$ and $\rho^1 = (\phi^1, \psi^1)$.

58. Proposition *If $\mathcal{X}$ admits a stratification by global quotients and conjecture (57) is true, then the virtual fundamental class defined in (28) is independent of the resolution.*
`(Proof)` By (54) we have a commutative diagram with f flat:

$$\begin{array}{ccc} \mathcal{C}(\mathcal{G}^\bullet) & \longrightarrow & \mathcal{C}(\mathcal{E}^\bullet) \\ \downarrow & \square & \downarrow \\ [G_1/G_0] & \xrightarrow[f]{} & [E_1/E_0] \\ \downarrow^{\sigma} & \swarrow_{\pi} & \\ \mathcal{X} & & \end{array},$$

and $[E_1/E_0] = \mathcal{V}(\phi^1)$ and $[G_1/G_0] = \mathcal{V}(\rho^1)$ by (56). Then we apply [15, Th. 2.1.12], see (26), and get:

$$(\pi^*)^{-1}[\mathcal{C}(\mathcal{E}^\bullet)] = (\sigma^*)^{-1} f^*[\mathcal{C}(\mathcal{E}^\bullet)] = (\sigma^*)^{-1}[\mathcal{C}(\mathcal{G}^\bullet)].$$

Repeating the argument with F in place of E we conclude. □

Chapter 3

Deformation sequences

Lemma 4.6 of [4] provides an infinitesimal criterion to check if a given morphism $\phi : \mathcal{E}^\bullet \longrightarrow \mathcal{L}_X^\bullet$ is an obstruction theory for the DM stack X. This criterion works thanks to the existence of a deformation theoretic sequence associated with a pointed scheme (or DM stack). In order to get an analogous criterion in the case of Artin stacks, in this chapter we'll describe a generalization of the deformation sequence, see theorem (64). See the introduction for a brief summary of the content of this chapter.

3.1 The infinitesimal criterion

We are going to describe how the infinitesimal criterion for obstruction theories works. Let X be a DM stack of finite type over an algebraically closed field $\mathbb{K}$ and let $\phi : \mathcal{E}^\bullet \longrightarrow \mathcal{L}_X^\bullet$ be a morphism in the derived category with $\mathcal{E}^\bullet$ satisfying the condition of [4, Def. 2.3] (the cotangent complex automatically does, see [4, Prop. 2.4]). In particular these complexes have coherent cohomology in degrees -1 and 0. By definition, ϕ is an obstruction theory if and only if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.
The first think we have to observe is that we can check whether ϕ is an obstruction theory or not 'fiberwise' on the points of X. By a point $x \in X$ we'll mean a $\mathbb{K}$-valued point, i.e. a morphism $\operatorname{Spec}\mathbb{K} \xrightarrow{x} X$. Given a point $x \in X$, the fibers of $h^i(\phi)$ at x give morphisms of finite dimensional vector spaces:

$$x^*(h^0\phi) : x^*(h^0\mathcal{E}^\bullet) \longrightarrow x^*(h^0\mathcal{L}_X^\bullet)$$
$$x^*(h^{-1}\phi) : x^*(h^{-1}\mathcal{E}^\bullet) \longrightarrow x^*(h^{-1}\mathcal{L}_X^\bullet),$$

Since $h^0(\phi)$ and $h^{-1}(\phi)$ are coherent over a noetherian DM stack then, essentially by Nakayama's lemma, ϕ is an obstruction theory if and only if for any $x \in X$ the above maps are an isomorphism and surjective respectively.
Now deformation theory enters the game. For $i = 0, 1$ we denote:

$$T_x^i X := (h^{-i}(\mathrm{L}x^*\mathcal{L}_X^\bullet))^\vee$$
$$E_x^i := (h^{-i}(\mathrm{L}x^*\mathcal{E}^\bullet))^\vee.$$

As it will be recalled in section 3.2, the tangent spaces to X at x fit into the exact sequence

$$0 \longrightarrow T_x^0 X \otimes_{\mathbb{K}} I \longrightarrow D(B) \longrightarrow D(A) \longrightarrow T_x^1 X \otimes_{\mathbb{K}} I, \tag{3.1}$$

for any small extension (3.2) of Artinian algebras. It's a fact (see for instance [7, Th. 6.2.4]) that if the vector spaces E^0_x, E^1_x fit into the same exact sequence:

$$0 \longrightarrow E^0_x \otimes_{\mathbb{K}} I \longrightarrow D(B) \longrightarrow D(A) \longrightarrow E^1_x \otimes_{\mathbb{K}} I,$$

then we have a canonical isomorphism $T^0_x X \longrightarrow E^0_x$ and a canonical injection $T^1_x X \longrightarrow E^1_x$.
What happens in some practical cases is that X is a moduli stack and for any $x \in X$ deformation theory gives spaces E^0_x and E^1_x fitting into sequences as above. Usually one can also find a global morphism ϕ that induces the above maps and hence we get morphisms of finite dimensional vector spaces:

$$h^0(\mathrm{L}x^*\phi) : h^0(\mathrm{L}x^*\mathcal{E}^\bullet) \longrightarrow h^0(\mathrm{L}x^*\mathcal{L}^\bullet_X)$$
$$h^{-1}(\mathrm{L}x^*\phi) : h^{-1}(\mathrm{L}x^*\mathcal{E}^\bullet) \longrightarrow h^{-1}(\mathrm{L}x^*\mathcal{L}^\bullet_X),$$

that are an isomorphism and a surjection respectively.
We have to show that, for any $x \in X$, $x^*(h^0\phi)$ is an isomorphism and $x^*(h^{-1}\phi)$ is surjective. Since the question is local we may assume the following things: X is an affine noetherian scheme; there is a closed embedding $X \hookrightarrow M$ with M smooth, $T_xX = T_xM$, and Ω_M free; $\tau_{\geq 1}\phi$ is represented by a morphism of 2-term complexes of coherent sheaves:

$$\begin{array}{ccc} \mathcal{E}^{-1} & \xrightarrow{a} & \mathcal{E}^0 \\ \downarrow & & \downarrow \\ \mathcal{L}^{-1} & \xrightarrow[b]{} & \mathcal{L}^0, \end{array}$$

with $\mathcal{L}^0$ free and $x^*b = 0$ (in fact $\mathcal{L}^{-1} = I/I^2$ with I the ideal sheaf of X in M, and $\mathcal{L}^0 = \Omega_{M|X}$).
Since the above complexes have 0 in positive degrees then $h^0(\mathrm{L}x^*\phi) = x^*(h^0\phi)$. Since by hypothesis $h^0(\mathrm{L}x^*\phi)$ is an isomorphism, then so $x^*(h^0\phi)$ is. Moreover we can restrict to a neighborhood of x in such a way that $h^0(\phi)$ is an isomorphism.
By the diagram with exact rows:

$$\begin{array}{ccccccc} x^*\mathcal{E}^{-1} & \longrightarrow & x^*\mathcal{E}^0 & \longrightarrow & x^*(h^0\mathcal{E}^\bullet) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \wr\downarrow x^*(h^0\phi) & & \\ x^*\mathcal{L}^{-1} & \xrightarrow[0]{} & x^*\mathcal{L}^0 & \longrightarrow & x^*(h^0\mathcal{L}^\bullet) & \longrightarrow & 0, \end{array}$$

we see that $x^*\mathcal{E}^0 \longrightarrow x^*\mathcal{L}^0$ is surjective and hence we can restrict the neighborhood of x further and assume that $\mathcal{E}^0 \longrightarrow \mathcal{L}^0$ is surjective. Since $\mathcal{L}^0$ is free then we can split $\mathcal{E}^0 \longrightarrow \mathcal{L}^0$ and by the diagram:

$$\begin{array}{ccc} \mathcal{D}^{-1} & \longrightarrow & \mathcal{L}^0 \\ \downarrow & \square & \downarrow \\ \mathcal{E}^{-1} & \longrightarrow & \mathcal{E}^0 \\ \downarrow & & \downarrow \\ \mathcal{L}^{-1} & \longrightarrow & \mathcal{L}^0, \end{array}$$

we can assume that ϕ^0 is an isomorphism. It follows that $x^*a = 0$.
Since $\mathcal{L}^0, \mathcal{E}^0$ are free and $x^*b = x^*a = 0$ then $h^{-1}(\mathrm{L}x^*\phi) = h^{-1}(x^*\phi)$, and moreover $h^{-1}(x^*\phi) = x^*\phi^{-1}$. Since by hypothesis $h^{-1}(\mathrm{L}x^*\phi)$ is surjective then so $x^*\phi^{-1}$ is. The images under the differential of $\mathcal{L}^{-1}$ and $\mathcal{E}^{-1}$ identify under the isomorphism ϕ^0, and we use the notation Im for this image. From the diagram with exact rows:

$$\begin{array}{ccccccccc} \mathrm{Tor}_1(Im, \mathbb{K}) & \longrightarrow & x^*(h^{-1}\mathcal{E}^\bullet) & \longrightarrow & x^*\mathcal{E}^{-1} & \longrightarrow & x^*Im & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x^*(h^{-1}\phi)} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \mathrm{Tor}_1(Im, \mathbb{K}) & \longrightarrow & x^*(h^{-1}\mathcal{L}^\bullet) & \longrightarrow & x^*\mathcal{L}^{-1} & \longrightarrow & x^*Im & \longrightarrow & 0 \end{array}$$

we see that $x^*(h^{-1}\phi)$ is an isomorphism.
This concludes the discussion of the infinitesimal criterion for obstruction theories.

59. Remark In case X is an Artin stack the sequence (3.1) is no longer exact and this makes things no longer work. In order to solve this problem we have to prolongue it.

3.2 Deformation sequence for a pointed scheme

In this section we are going to recall the deformation sequence associated with a pointed scheme, which plays a role in the infinitesimal criterion for obstruction theories as it's described in section 3.1. A description of what we call the deformation sequence is given in [7, Ch. 6]; the reader may want to look at [23] for the basic concepts of deformation theory.

60. Notations and assumptions Fix an algebraically closed field $\mathbb{K}$. Schemes (and algebraic stacks) are assumed to be of finite type over $\mathbb{K}$. The word *point* will mean $\mathbb{K}$-valued point and we use notations $x \in X$ and

$$* := \operatorname{Spec} \mathbb{K} \xrightarrow{x} X.$$

Let $Alg^\wedge$ be the category of Noetherian local complete $\mathbb{K}$-algebras with residue field $\mathbb{K}$, and local morphisms. We'll denote the maximal ideal of a local algebra A by m_A. Let Art be the full subcategory of $Alg^\wedge$ given by the Artinian algebras.
By a *small extension* in Art will mean a short exact sequence

$$0 \longrightarrow I \longrightarrow B \xrightarrow{\varphi} A \longrightarrow 0 \tag{3.2}$$

where φ is a morphism in Art and $I \cdot m_B = 0$. Notice that since any $A \in Art$ has a natural map $A \longrightarrow A/m_A = \mathbb{K}$ then the functor Spec sends an Artinian algebra to a pointed scheme:

$$\mathrm{Spec} : Art^\circ \longrightarrow Sch_*.$$

We'll use the notation:

$$\operatorname{Spec}(A \longrightarrow \mathbb{K}) = * \xrightarrow{j_A} \operatorname{Spec} A.$$

61. Deformation functor Given a pointed scheme (X, x) we have an associated functor

$$D = D_x = D(X, x) : Art \longrightarrow Set$$

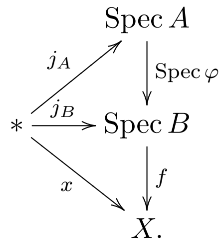
that sends an algebra B to

$$D(B) := \{f : \operatorname{Spec} A \longrightarrow X : f j_B = x\},$$

and a sends a morphism $\varphi : B \longrightarrow A$ of algebras to:

$$D(\varphi) := (\operatorname{Spec} \varphi)^*,$$

where the upper star means composition on the right; the following diagram makes what's happening clearer:



We notice that $D(\mathbb{K})$ is a one point set and that $D(\mathbb{K}[\varepsilon]/(\varepsilon^2))$ is in canonical bijection with the Zariski tangent space $T_x X := (m_{X,x}/m_{X,x}^2)^\vee$ of X at x.

62. Deformation sequence For any pointed scheme (X, x) and for any small extension in *Art*

$$0 \longrightarrow I \longrightarrow B \stackrel{\varphi}{\longrightarrow} A \longrightarrow 0$$

there is an exact sequence

$$0 \longrightarrow T_x^0 X \otimes_{\mathbb{K}} I \stackrel{\gamma}{\longrightarrow} D(B) \stackrel{D(\varphi)}{\longrightarrow} D(A) \stackrel{\text{ob}}{\longrightarrow} T_x^1 X \otimes_{\mathbb{K}} I$$

where γ is an action of the additive group underlying $T_x^0 X \otimes_{\mathbb{K}} I$ on $D(B)$ and exactness means that γ is simply transitive on the fibers of $D(\varphi)$ and $\operatorname{Ker} \text{ob} = \operatorname{Im} D(\varphi)$. Moreover the sequence is simultaneously functorial in morphisms of small extensions and of pointed schemes.

63. Algebraic description We can get the deformation sequence above by means of an algebraic description of the object involved. Let $R := \widehat{\mathcal{O}}_{X,x} \in Alg^\wedge$ be the completion (wrt the maximal ideal) of the local ring of X at x. The deformation sequence reads:

$$0 \longrightarrow \operatorname{Der}_{\mathbb{K}}(R, I) \stackrel{\gamma}{\longrightarrow} \operatorname{Hom}(R, B) \stackrel{\varphi_*}{\longrightarrow} \operatorname{Hom}(R, A) \stackrel{\text{ob}}{\longrightarrow} \operatorname{Ex}_{\mathbb{K}}(R, I),$$

where Hom means homomorphism in $Alg^\wedge$.
We define the action

$$\gamma : \operatorname{Der}_{\mathbb{K}}(R, I) \times \operatorname{Hom}(R, B) \longrightarrow \operatorname{Hom}(R, B)$$

by $\gamma(v, b) := b + v$.
We define the image under the map ob of a morphism $a : R \longrightarrow A$ by

$$\text{ob}(a) := [0 \longrightarrow I \longrightarrow B \times_A R \longrightarrow R \longrightarrow 0].$$

It's easy to check the validity of the claimed properties.

3.3 Deformation sequence for a pointed stack

In this section we'll describe a generalization of the sequence (62) in case X is an algebraic stack: see theorem (64). Notations and assumptions from (60) are still valid.

Let $(\mathcal{X}, \kappa)$ be a pointed algebraic stack:

$$* \xrightarrow{\kappa} \mathcal{X}.$$

The first thing we have to do is to find the right object that replaces the deformation functor D in the new context. Indeed, we can associate a functor to $(\mathcal{X}, \kappa)$, see (66):

$$\mathcal{D} = \mathcal{D}(\mathcal{X}, \kappa) : Art \longrightarrow Grpd.$$

In second place we notice that we have three interesting vector spaces associated with the pointed stack:

$$T^i_\kappa \mathcal{X} = (h^{-i}(\mathrm{L}\kappa^* \mathcal{L}^\bullet_{\mathcal{X}}))^\vee \qquad i = -1, 0, 1.$$

The natural generalization of the deformation sequence would be

$$0 \longrightarrow T^0_\kappa \mathcal{X} \otimes_{\mathbb{K}} I \xrightarrow{\gamma} \pi_0\mathcal{D}(B) \xrightarrow{\pi_0\mathcal{D}(\varphi)} \pi_0\mathcal{D}(A) \xrightarrow{\mathrm{ob}} T^1_\kappa \mathcal{X} \otimes_{\mathbb{K}} I,$$

but in general this is not exact at $T^0_\kappa \mathcal{X} \otimes_{\mathbb{K}} I$ as the action γ is not free on the fibers of $\pi_0\mathcal{D}(\varphi)$. The reader may want to work out the elementary example suggested in (65) to see this fact.
In order to solve this problem we can drop the zero on the left and prolongue the sequence, and the final result is described in the following theorem.

64. Theorem *Let $(\mathcal{X}, \kappa)$ be a pointed algebraic stack. Assume that $\mathcal{X}$ has quasi-affine diagonal. Let*

$$0 \longrightarrow I \longrightarrow B \xrightarrow{\varphi} A \longrightarrow 0$$

be a small extension in Art. Then for any $b \in \mathrm{ob}\,\mathcal{D}(B)$ we have an exact sequence:

$$\begin{array}{llllllll} 0 \longrightarrow & T^{-1}_\kappa \mathcal{X} \otimes I & \xrightarrow{\alpha} & \pi_1(\mathcal{D}(B), b) & \xrightarrow{\pi_1\mathcal{D}\varphi} & \pi_1(\mathcal{D}(A), (\mathcal{D}\varphi)b) & \xrightarrow{\beta} & \\ \xrightarrow{\beta} & T^0_\kappa \mathcal{X} \otimes I & \xrightarrow{\gamma} & \pi_0(\mathcal{D}(B)) & \xrightarrow{\pi_0\mathcal{D}\varphi} & \pi_0(\mathcal{D}(A)) & \xrightarrow{\delta} & T^1_\kappa \mathcal{X} \otimes I, \end{array}$$

where α and γ are actions of the underlying additive groups of the vector spaces to the left on the sets to the right, γ and δ are independent of b, and exactness means:
α is free (exactness at $T^{-1}_\kappa \mathcal{X} \otimes I$);
α acts transitively on the fibers of $\pi_1\mathcal{D}\varphi$ (exactness at $\pi_1(\mathcal{D}(B), b)$);
$\ker \beta = \mathrm{Im}\, \pi_1\mathcal{D}\varphi$ *(exacntess at $\pi_1(\mathcal{D}(A))$);*
$\mathrm{Im}\, \beta =$ *stabilizer of γ at $[b]$ (exactness at $T^0_\kappa \mathcal{X} \otimes I$);*
γ acts transitively on the fibers of $\pi_0\mathcal{D}\varphi$ (exactness at $\pi_0(\mathcal{D}(B))$);
$\ker \delta = \mathrm{Im}\, \pi_0\mathcal{D}\varphi$ *(exactness at $\pi_0(\mathcal{D}(A))$).*
Moreover the above sequence is simultaneously functorial in morphisms of small extensions and of pointed stacks.
`(Proof)` Choose a presentation of $(\mathcal{X}, \kappa)$ by a smooth pointed groupoid $(\mathcal{U}, x)$ as explained in (69). If $\mathcal{D}$ is the deformation functor associated with $(\mathcal{X}, \kappa)$ (see (66)) and $\mathcal{C}$ is the

deformation functor associated with $(\mathcal{U}, x)$ (see (67)) then we have a morphism between these two functors (see (69)) that is an equivalence by (70):

$$\mathcal{C} \simeq \mathcal{D}.$$

In proving this equivalence it's explained the meaning and the role of the assumption about the diagonal of $\mathcal{X}$.
The next step is the construction of a sequence with the claimed properties with $\mathcal{C}$ in place of $\mathcal{D}$. This is done in section 3.4.
Thanks to the equivalence between $\mathcal{C}$ and $\mathcal{D}$ we get a sequence as in the statement of the theorem. The is independent of the presentation.
The bifunctoriality of the sequence is proved by remarking that the deformation functors $\mathcal{D}$ and $\mathcal{C}$ have the right bifunctorial behavior as it's explained in (68). □

65. Example Let $r \in \mathbb{Z}$. Put $X = \operatorname{Spec} \mathbb{K}[x]$, $G = \mathbb{G}_m = \operatorname{Spec} \mathbb{K}[\lambda]_\lambda$, G acts on X by $(\lambda, x) \mapsto \lambda^r x$. Let $* \xrightarrow{x} X$ be given by $x_0 \in \mathbb{K}$. Put $\mathcal{X} := [X/G]$. For $n \geq 1$ we put $A_n = \mathbb{K}[\varepsilon]/(\varepsilon^n)$ and consider the small extension:

$$0 \longrightarrow (\varepsilon^n) \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow 0.$$

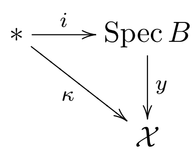
The reader is advised to play with the associated deformation sequence as r, x_0, n vary.

3.3.1 Deformation functors

66. Deformation functor associated with a pointed stack Given a pointed algebraic stack $(\mathcal{X}, \kappa)$ we'll define a functor:

$$\mathcal{D} = \mathcal{D}(\mathcal{X}, \kappa) : Art \longrightarrow Grpd.$$

This is the natural generalization of the deformation functor associated with a pointed scheme described in (61). We'll now describe the action of $\mathcal{D}$ on objects and morphisms of Art. To any $B \in Art$ we have to associate a groupoid $\mathcal{D}(B)$, and to any morphism $\varphi : B \longrightarrow A$ in Art we have to associate a functor $\mathcal{D}(\varphi) : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$.
`(Action on objects)` Given $B \in Art$, the objects of $\mathcal{D}(B)$ are given by the pairs (y, β) where:



and $\beta : yi \longrightarrow \kappa$ (a 2-morphism).
A morphism $\delta : (y, \beta) \longrightarrow (y', \beta')$ is given by a 2-morphism $\delta : y \longrightarrow y'$ s.t. $\beta' \circ (\delta * 1_i) = \beta$, where $*$ is horizontal composition and $\circ$ is vertical composition, cf. [18].
`(Action on morphisms)` Given a morphism $\varphi : B \longrightarrow A$ in Art, the functor $\mathcal{D}(\varphi)$ sends an object $(y, \beta) \in \mathcal{D}(B)$ to $(yj, \beta) \in \mathcal{D}(A)$; the following diagram, where $jl = i$, makes

this clearer:

$$\begin{array}{ccc} & & \operatorname{Spec} A \\ & \overset{l}{\nearrow} & \downarrow j \\ * & \xrightarrow{i} & \operatorname{Spec} B \\ & \underset{\kappa}{\searrow} & \downarrow y \\ & & \mathcal{X}. \end{array}$$

Moreover, $\mathcal{D}(\varphi)$ sends a morphism $\delta : (y, \beta) \longrightarrow (y', \beta')$ in $\mathcal{D}(B)$ to $\delta * 1_j : (yj, \beta) \longrightarrow (y'j, \beta')$ in $\mathcal{D}(A)$.

`(Remark)` The construction is functorial in the sense that for $C \xrightarrow{\psi} B \xrightarrow{\varphi} A$ we have $\mathcal{D}(\varphi\psi) = \mathcal{D}(\varphi)\mathcal{D}(\psi)$. Hence $\mathcal{D}$ is a functor of 1-categories where we forget the 2-arrows of *Grpd*.

67. Deformation functor associated with a pointed groupoid Given a smooth pointed groupoid $(\mathcal{U}, x)$ in the category of schemes, we'll define a functor

$$\mathcal{C} = \mathcal{C}(\mathcal{U}, x) : Art \longrightarrow Grpd.$$

Recall from (71) the notation for groupoids in a category. We'll now describe the action of $\mathcal{C}$ on objects and morphisms of *Art*. To any $B \in Art$ we have to associate a groupoid $\mathcal{C}(B)$, and to any morphism $\varphi : B \longrightarrow A$ in *Art* we have to associate a functor $\mathcal{C}(\varphi) : \mathcal{C}(B) \longrightarrow \mathcal{C}(A)$.

`(Action on objects)` Given $B \in Art$, the set of objects of $\mathcal{C}(B)$ is given by

$$\operatorname{Ob} \mathcal{C}(B) := \operatorname{Hom}_*(Z, X),$$

where $Z := \operatorname{Spec} B$ (a pointed scheme), and Hom_* stands for morphisms of pointed schemes.

The set of morphisms of $\mathcal{C}(B)$ is given by

$$\operatorname{Mor} \mathcal{C}(B) := \operatorname{Hom}_*(Z, U).$$

The structure morphisms of the groupoid $\mathcal{C}(B)$ are given by the pushforward (composition on the left) with the structure morphisms of $\mathcal{U}$:

$$\operatorname{Hom}_*(Z,U) \times_{t_*, \operatorname{Hom}_*(Z,X), s_*} \operatorname{Hom}_*(Z,U) = \operatorname{Hom}_*(Z, U \times_{t,X,s} U) \xrightarrow{m_*} \operatorname{Hom}_*(Z,U)$$
$$\operatorname{Hom}_*(Z,U) \xrightarrow{i_*} \operatorname{Hom}_*(Z,U) \xrightarrow{s_*, t_*} \operatorname{Hom}_*(Z,X) \xrightarrow{e_*} \operatorname{Hom}_*(Z,U).$$

`(Action on morphisms)` Given a morphism $\varphi : B \longrightarrow A$ in *Art*, the functor $\mathcal{C}(\varphi) := (\operatorname{Spec} \varphi)^*$ is defined as pullback (composition on the right) with $\operatorname{Spec} \varphi : Y := \operatorname{Spec} A \longrightarrow \operatorname{Spec} B =: Z$. Diagrammatically:

$$\begin{array}{ccc} \operatorname{Hom}_*(Z,U) & \xrightarrow{(\operatorname{Spec}\varphi)^*} & \operatorname{Hom}_*(Y,U) \\ \downarrow\downarrow & & \downarrow\downarrow \\ \operatorname{Hom}_*(Z,X) & \xrightarrow[(\operatorname{Spec}\varphi)^*]{} & \operatorname{Hom}_*(Y,X). \end{array}$$

(Remark) The construction is functorial in the sense that for $C \xrightarrow{\psi} B \xrightarrow{\varphi} A$ we have $\mathcal{C}(\varphi\psi) = \mathcal{C}(\varphi)\mathcal{C}(\psi)$. Hence $\mathcal{C}$ is a functor of 1-categories where we forget the 2-arrows of *Grpd*.

68. Bifunctoriality The construction of the deformation functors associated with a pointed algebraic stack and with a smooth pointed groupoid can be actually carried over at a pure categorical level and in a way that enlights the bifunctorial nature of the deformation sequence. Given a category $\mathscr{C}$ that we assume to have terminal object an fibered products, we denote by $Cfg(\mathscr{C})$ the 2-category of categories fibered in groupoids over $\mathscr{C}$ and by $Grpd(\mathscr{C})$ the 2-category of groupoids in $\mathscr{C}$. We have two functors

$$\begin{aligned} \mathrm{Hom}^{\mathrm{f}}: &\quad \mathscr{C}^{\circ} \times Cfg(\mathscr{C}) \longrightarrow Grpd \\ \mathrm{Hom}^{\mathrm{g}}: &\quad \mathscr{C}^{\circ} \times Grpd(\mathscr{C}) \longrightarrow Grpd \end{aligned}$$

the definition of which is suggested by their names (the superscript f and g just makes the distinction between the fibered and the groupoid case) and should be clear from the ones of $\mathcal{D}$ in (66) and of $\mathcal{C}$ in (67) respectively.
In the application we are interested in, the category we work with is $\mathscr{C} = Sch_*$, the category of pointed schemes. We just restrict our attention to the full 2-subcategory of $Cfg(Sch_*)$ given by the algebraic stacks (with respect to the étale topology), and to the full 2-subcategory of $Grpd(Sch_*)$ given by smooth groupoids.
In particular given a pointed algebraic stack $(\mathcal{X}, \kappa)$ and a smooth pointed groupoid $(\mathcal{U}, x)$ we recover $\mathcal{D}$ and $\mathcal{C}$ by:

$$\begin{aligned} \mathcal{D}(\mathcal{X}, \kappa) &:= \mathrm{Hom}^{\mathrm{f}}(\mathrm{Spec}(\_), (\mathcal{X}, \kappa)) : Art \longrightarrow Grpd \\ \mathcal{C}(\mathcal{U}, x) &:= \mathrm{Hom}^{\mathrm{g}}(\mathrm{Spec}(\_), (\mathcal{U}, x)) : Art \longrightarrow Grpd. \end{aligned}$$

Essentially, what makes the deformation sequence bifunctorial is the bifunctoriality of the functor Hom.

69. Deformation functors and presentation Given a pointed algebraic stack $(\mathcal{X}, \kappa)$ we'll describe how to get a presentation by a smooth pointed groupoid $(\mathcal{U}, x)$ and then how to get a natural transformation $\mathcal{C}(\mathcal{U}, x) \longrightarrow \mathcal{D}(\mathcal{X}, \kappa)$ between the associated deformation functors.
First choose an atlas for $\mathcal{X}$, i.e. a smooth surjective morphism π from a scheme X, and choose a lifting (x, η) of the distinguished point:

$$\begin{array}{ccc} & & X \\ & \overset{x}{\nearrow} & \downarrow{\scriptstyle \pi} \\ * & \xrightarrow[\kappa]{} & \mathcal{X}, \end{array}$$

where $\eta : \pi x \longrightarrow \kappa$ is a 2-morphism making the diagram 2-commute. Then take the 2-fibered product

$$\begin{array}{ccc} U & \xrightarrow{t} & X \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\pi]{} & \mathcal{X}, \end{array}$$

where we denote by $\mu : \pi s \longrightarrow \pi t$ the 2-arrow making the diagram 2-commute. We can induce all the structure morphisms making $\mathcal{U} := U \rightrightarrows X$ a smooth groupoid in the category of schemes. The scheme X being pointed by x, we have in fact a pointed groupoid $(\mathcal{U}, x)$. Notice that U can be canonically made pointed by using the identity of the groupoid and all the structure morphisms become morphisms of pointed schemes, i.e. we can really interpret $(\mathcal{U}, x)$ as a groupoid in the category of pointed schemes.
We now describe how to get a morphism

$$\pi_* : \mathrm{Hom}^{\mathrm{g}}(\_\,, (\mathcal{U}, x)) \longrightarrow \mathrm{Hom}^{\mathrm{f}}(\_\,, (\mathcal{X}, \kappa)).$$

between functors $Sch_* \longrightarrow Grpd$. For any pointed scheme (Y, y) we have to define a functor

$$\pi_*(Y) : \mathrm{Hom}^{\mathrm{g}}((Y, y), (\mathcal{U}, x)) \longrightarrow \mathrm{Hom}^{\mathrm{f}}((Y, y), (\mathcal{X}, \kappa)).$$

`(Action on objects)` To any $f \in \mathrm{Hom}_*(Y, X) = \mathrm{Ob}\,\mathrm{Hom}^{\mathrm{g}}((Y, y), (\mathcal{U}, x))$ we associate $\pi_*(Y)f := (\pi f, \eta) \in \mathrm{Ob}\,\mathrm{Hom}^{\mathrm{f}}((Y, y), (\mathcal{X}, \kappa))$:

$$\begin{array}{ccc} & & Y \\ & \overset{y}{\nearrow} & \downarrow f \\ * & \xrightarrow{x} & X \\ & \underset{\kappa}{\searrow} & \downarrow \pi \\ & & \mathcal{X}. \end{array}$$

`(Action on morphisms)` A morphism $h : f \longrightarrow f'$ in $\mathrm{Hom}^{\mathrm{g}}((Y, y), (\mathcal{U}, x))$ is just an $h \in \mathrm{Hom}_*(Y, U)$ s.t. $sh = f$ and $th = f'$. We define $\pi_*(Y)h := \mu * 1_h : (\pi f, \eta) \longrightarrow (\pi f', \eta)$.
`(Remark)` Composing on the right with the functor Spec we get a morphism

$$\pi_* : \mathcal{C}(\mathcal{U}, x) \longrightarrow \mathcal{D}(\mathcal{X}, \kappa).$$

70. Proposition *With the notation of (69), if $\mathcal{X}$ has quasi affine diagonal then for any $Y \in Sch_*$ the functor $\pi_*(Y)$ is fully faithful. If moreover $Y = \mathrm{Spec}\,A$ with $A \in Art$, then $\pi_*(Y)$ is essentially surjective, and hence an equivalence of categories. In paticular we have an equivalence*

$$\pi_* : \mathcal{C}(\mathcal{U}, x) \longrightarrow \mathcal{D}(\mathcal{X}, \kappa).$$

`(Proof)` There is an algebraic stack $[\mathcal{U}]$ associated with the presentation of $\mathcal{X}$. If we add the datum of the distinguished point (and we consider the fiber over the fixed scheme Y) we get that $[\mathcal{U}]$ is described by the groupoid $[\mathcal{U}, x]$ defined below. The fact that $[\mathcal{U}]$ so described is isomorphic to $\mathcal{X}$ relies on a descent argument for affine morphisms that requires the hypothesis that $\mathcal{X}$ has quasi-affine diagonal; in term of the presentation this means that the preimage of any open affine via the map

$$U \xrightarrow{s,t} X \times X$$

is contained in an open affine. This is a technical point that avoids the use of algebraic spaces.

Recalling that an object of $\mathrm{Hom}^{\mathrm{f}}(Y,(\mathcal{X},\kappa))$ is described by the 2-commutative diagram

$$\begin{array}{ccc} * & \xrightarrow{i} & Y \\ & \searrow^{\kappa} & \downarrow \\ & & \mathcal{X} \end{array}$$

and using the 2-Yoneda lemma one can prove that the category $\mathrm{Hom}^{\mathrm{f}}(Y,(\mathcal{X},\kappa))$ is equivalent to the fiber $\mathcal{X}(Y)$ (plus the extra datum of the distinguished point).
Hence what we have to do is to prove the statement with the category $\mathrm{Hom}^{\mathrm{f}}(Y,(\mathcal{X},\kappa))$ and the functor $\pi_*(Y)$ replaced by the category $[\mathcal{U},x]$ and the functor Φ defined below.
`(Objects of` $[\mathcal{U},x]$`)`: they are given by triples (p,ρ,θ) where $p : P \longrightarrow Y$ is a morphism of schemes with étale-local sections, $\rho = (\rho_0,\rho_1)$ is a cartesian morphism between the groupoid associated to p and $\mathcal{U}$:

$$\begin{array}{ccc} P \times_Y P & \xrightarrow{\rho_1} & U \\ \downarrow\downarrow & \square & \downarrow\downarrow \\ P & \xrightarrow[\rho_0]{} & X, \end{array}$$

and θ is a morphism fitting in the diagram

$$\begin{array}{ccc} U_x & \xrightarrow{\theta} & P \\ {\scriptstyle s_x}\downarrow & \square & \downarrow{\scriptstyle s} \\ * & \xrightarrow[j_Y]{} & Y. \end{array}$$

These data have to satisfy the condition: $\rho\mathscr{G}(j_Y,\theta) = \omega\mathscr{G}(x,x_s)$, where $\mathscr{G}$ is the functor that sends a morphism $P \longrightarrow Y$ to the groupoid $P \times_Y P \longrightarrow P$, and ω is

$$\begin{array}{ccc} U \times_{s,X,s} U & \xrightarrow{\omega_1} & U \\ \downarrow\downarrow & \square & \downarrow\downarrow \\ U & \xrightarrow[\omega_0=t]{} & X, \end{array}$$

with $\omega_1(u_1,u_2) = u_1u_2^{-1}$.
`(Morphisms in` $[\mathcal{U},x]$`)` A morphism $(p,\rho,\theta) \xrightarrow{\varphi} (p',\rho',\theta')$ is given by $\varphi = (\varphi_0,\varphi_1)$ fitting in the diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi_1} & P' \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow[\varphi_0]{} & Y' \end{array}$$

s.t. $\rho'\mathscr{G}(\varphi) = \rho$ and $\varphi\theta = \theta'$.
`(Action of` Φ `on objects)` Given $f \in \mathrm{ob}\,\mathrm{Hom}^{\mathrm{g}}(Y,(\mathcal{U},x))$, i.e. $Y \xrightarrow{f} X$ s.t $fj_Y = x$, we get $\Phi(f) = (p,\rho,\theta)$ by the following construction. p is defined by the diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & U \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle s} \\ Y & \xrightarrow[f]{} & X; \end{array}$$

ρ is defined by $\rho = \omega\mathscr{G}(f, F)$; $\theta : U_x \longrightarrow P = Y \times_X U$ is induced by $(j_y s_x, x_s)$.
Notice that the map $\sigma_p : Y \longrightarrow P = Y \times_X U$ induced by $(1_Y, ef)$ is a section of p ($p\sigma_p = 1_Y$) s.t. $\sigma_p j_Y = \theta\sigma_{s_x}$.
`(Action of Φ on morphisms)` given a morphism $f \xrightarrow{a} f'$ in $\mathrm{Hom}^{\mathrm{g}}(Y, (\mathcal{U}, x))$, i.e. $a : Y \longrightarrow U$ s.t $sa = f$ and $ta = f'$, its image $\Phi(a) : \Phi(f) = (p, \rho, \theta) \longrightarrow \Phi(f') = (p', \rho', \theta')$ is defined by the following construction. Letting $\varepsilon : P \longrightarrow U \times_{s,X,t} U$ induced by (F, iap), then $\Phi(a) : P \longrightarrow P' = Y' \times_{f',X,s} U$ is induced by $(p, m\varepsilon)$.
`(Φ is fully faithful)` Given $f, f' \in \mathrm{ob}\,\mathrm{Hom}^{\mathrm{g}}(Y, (\mathcal{U}, x))$ and a morphism $\varphi : \Phi(f) = (p, \rho, \theta) \longrightarrow \Phi(f') = (p', \rho', \theta')$ between their images, we associate $\Psi(\varphi) : f \longrightarrow f'$ defining $\Psi(\varphi) = iF'\varphi\sigma_p$.
`(Essential image of Φ)` We claim that the essential image of Φ is given by the objects (p, ρ, θ) of $[\mathcal{U}, x]$ s.t. p has a (global) section σ s.t. $\sigma j_Y = \theta\sigma_{s_x}$.
Given an object in the image we have already observed that p has such a section σ_p.
Conversely, given (p, ρ, θ) s.t. such a σ exists, we can construct back f defining $f = \rho_0\sigma$. In order to prove thata $\Phi(f)$ is isomorphic to the object we started with we define $F = \rho_1\epsilon$ where $\epsilon : P \longrightarrow P \times_Y P$ is induced by $(\sigma p, 1_P)$. The resulting square

$$\begin{array}{ccc} P & \xrightarrow{F} & U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow[f]{} & X \end{array}$$

turns out to be cartesian. Moreover $F\theta = x_x$ and $p\theta = j_Y s_x$; and $\omega\mathscr{G}(f, F) = \rho$.
`(Φ is essentially surjective)` We have to prove that if $Y = \mathrm{Spec}\, A$ with $A \in Art$ and (p, ρ, θ) is an object of $[\mathcal{U}, x]$ then p has a section σ s.t. $\sigma j_Y = \theta$. Since p has étale-local sections and since any étale morhpism to Y has itself a section (here we use the particular form of Y) then p admits a global section σ'. Using the same argument as in the computation of the essential image with σ' in place of σ, we can fit p in the cartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow[f]{} & X. \end{array}$$

Once we have this we see that sections of p are in bijection with the maps $a : Y \longrightarrow U$ s.t. $sa = f$. The sections that have the wanted properties correspond the a's s.t. $aj_Y = F\theta\sigma_{s_x}$. Using the fact that s is smooth and Y is the spectrum of an Artinian algebra and applying the infinitesimal criterion for smoothness we see that such an a actually exists; look at:

$$\begin{array}{ccc} * & \xrightarrow{F\theta\sigma_{s_x}} & U \\ {\scriptstyle j_Y}\downarrow & \overset{a}{\nearrow} & \downarrow{\scriptstyle s} \\ Y & \xrightarrow[f]{} & X. \end{array}$$

□

3.4 Deformation sequence for a groupoid

Let $(\mathcal{U}, x)$ be a smooth pointed groupoid in the category of schemes; $\mathcal{U} = U \rightrightarrows X$, $* \xrightarrow{x} X$. We have an associated deformation functor, see (67):

$$\mathcal{C} = \mathcal{C}(\mathcal{U}, x) : Art \longrightarrow Grpd,$$

and associated tangent spaces, see (74):

$$T^i_x \mathcal{U} \qquad i = -1, 0, 1.$$

In this section will build a sequence

$$\begin{array}{llllll} 0 \longrightarrow & T^{-1}_x \mathcal{U} \otimes I & \xrightarrow{\alpha} & \pi_1(\mathcal{C}(B), b) & \xrightarrow{\pi_1 \mathcal{C}\varphi} & \pi_1(\mathcal{C}(A), (\mathcal{C}\varphi)b) \quad \xrightarrow{\beta} \\ \xrightarrow{\beta} & T^0_x \mathcal{U} \otimes I & \xrightarrow{\gamma} & \pi_0(\mathcal{C}(B)) & \xrightarrow{\pi_0 \mathcal{C}\varphi} & \pi_0(\mathcal{C}(A)) \qquad\qquad \xrightarrow{\delta} T^1_x \mathcal{U} \otimes I, \end{array}$$

satisfying the properties claimed in theorem (64). The construction of the sequence is done in subsection 3.4.1 and it relies, as for the scheme case, on an algebraic description of the object involved, which is done in the first part of this section.

71. Notation for pointed groupoids in schemes A *groupoid* in the category of schemes will be denoted by (U, X, s, t, e, i, m) where: $s, t : U \longrightarrow X$ are source and target, $e : X \longrightarrow U$ is the identity, $i : U \longrightarrow U$ is the inverse, $m : U \times_{s,X,t} U \longrightarrow U$ is the multiplication. We denote:

$$\begin{array}{ccc} U \times_{s,X,t} U & \xrightarrow{\nu_s} & U \\ {\scriptstyle \nu_t}\downarrow & & \downarrow{\scriptstyle t} \\ U & \xrightarrow[s]{} & X. \end{array}$$

A groupoid is called *smooth* iff s is smooth (iff t is).
Given $* \xrightarrow{x} X$ we can promote the previous groupoid to a one in the category of pointed schemes; we call such an object a *pointed groupoid*.
Given a pointed groupoid we put:

$$\begin{array}{ccc} U_x & \xrightarrow{x_s} & U \\ {\scriptstyle s_x}\downarrow & \square & \downarrow{\scriptstyle s} \\ * & \xrightarrow[x]{} & X. \end{array}$$

U_x is pointed by the map $\sigma_{s_x} : * \longrightarrow U_x = * \times_X U$ induced by $(1_*, ex)$.
We put in evidence a map which will play a role:

$$U_x \xrightarrow{tx_s} X.$$

We denote by $Grpd(Sch_*)$ the 2-category of pointed groupoids; this is a strict 2-category in which all the 2-arrows are invertible.

72. Algebra maps induced by a groupoid Given a pointed groupoid $\mathcal{U} = (U, X, s, t, e, i, m, x)$ as in (71) we put:

$$R := \widehat{\mathcal{O}}_{X,x} \qquad S := \widehat{\mathcal{O}}_{U,ex} \qquad P := \widehat{\mathcal{O}}_{U_x, \sigma_{s_x}}.$$

We have a bunch of maps (in $Alg^\wedge$) induced by the structure morphisms of the groupoid, namely

$$R \xrightarrow{\sigma,\tau} S \xrightarrow{\iota} S \xrightarrow{\varepsilon} R \qquad S \xrightarrow{\zeta} P,$$

where σ is induced by s and so on. We give a special name to the map induced by tx_s, namely

$$\mu := \zeta\tau : R \longrightarrow P.$$

These maps satisfy a bunch of equalities:

$$\varepsilon\sigma = 1_R = \varepsilon\tau \quad \iota\sigma = \tau \quad \varepsilon\iota = \varepsilon \quad \iota\iota = 1_S.$$

We denote:

$$\mathcal{R} = (S, R, \sigma, \tau, \varepsilon, \iota).$$

If the groupoid is *smooth* then there exists an integer $r \geq 0$ and elements $s_1, \cdots, s_r \in m_S$ s.t. the induced R-algebra morphism

$$R[[x_1, \cdots, x_s]] \longrightarrow S$$

is an isomorphism. The induced elements $\zeta(s_i) \in m_P$ give an isomorphism of $\mathbb{K}$ algebras:

$$\mathbb{K}[[x_1, \cdots, x_s]] \longrightarrow P.$$

Under this condition we get another map:

$$S \xrightarrow{\nu} P.$$

73. Algebraic description of the deformation functor Given A in *Art* and a pointed groupoid $\mathcal{U}$ as in (71) we'll give an algebraic description of $\mathrm{Hom}^{\mathrm{g}}(\operatorname{Spec} A, \mathcal{U})$ (cf. (67)). See (72) for the "algebraic side" of the notation.
This description is well captured by the formula:

$$\mathrm{Hom}^{\mathrm{g}}(\operatorname{Spec} A, \mathcal{U}) = \mathrm{Hom}(\mathcal{R}, A).$$

More explicitly

$$\begin{aligned} \operatorname{ob}\mathrm{Hom}^{\mathrm{g}}(\operatorname{Spec} A, \mathcal{U}) &= \mathrm{Hom}(R, A) \\ \operatorname{mor}\mathrm{Hom}^{\mathrm{g}}(\operatorname{Spec} A, \mathcal{U}) &= \mathrm{Hom}(S, A) \end{aligned}$$

and we have

$$\mathrm{Hom}(R, A) \xrightarrow{\varepsilon^*} \mathrm{Hom}(S, A) \xrightarrow{\iota^*} \mathrm{Hom}(S, A) \xrightarrow{\sigma^*,\tau^*} \mathrm{Hom}(R, A).$$

In particular $\pi_0\mathrm{Hom}(\mathcal{R}, A)$ is gotten by $\mathrm{Hom}(R, A)$ quotienting by the equivalence relation

$$a \sim a' \quad \Longleftrightarrow \quad \exists\, h \in \mathrm{Hom}(S, A) \text{ s.t. } \sigma^* h = a \quad \tau^* h = a',$$

and given $a \in \mathrm{Hom}(R, A)$ we have:

$$\pi_1(\mathrm{Hom}(\mathcal{R}, A), a) = \big\{ h \in \mathrm{Hom}(S, A) : \sigma^* h = a = \tau^* h \big\}.$$

An alternative description of $\pi_1(\mathrm{Hom}(\mathcal{R}, A), a)$ is given by the following remarks. If the groupoid we started with is smooth then $\mathrm{Hom}(S, A) = \mathrm{Hom}(P, A) \times \mathrm{Hom}(R, A)$ and the maps $\nu^* : \mathrm{Hom}(S, A) \longrightarrow \mathrm{Hom}(P, A)$, $\sigma^* : \mathrm{Hom}(S, A) \longrightarrow \mathrm{Hom}(R, A)$ correspond to the two projections. In particular the map

$$\nu_a^* : {\sigma^*}^{-1}(a) \longrightarrow \mathrm{Hom}(P, A)$$

is bijective and we have

$$\pi_1(\mathrm{Hom}(\mathcal{R}, A), a) = \{g \in \mathrm{Hom}(P, A) : \tau^* {\nu_a^*}^{-1} g = a\}.$$

74. Tangent spaces to a pointed groupoid in schemes Given a smooth pointed groupoid $\mathcal{U} = (U, X, s, t, e, i, m, x)$ as in (71) we have three finite dimensional $\mathbb{K}$-vector spaces associated to it:

$$T_x^{-1}\mathcal{U} \quad T_x^0\mathcal{U} \quad T_x^1\mathcal{U}.$$

These are defined by the exact sequence

$$0 \longrightarrow T_x^{-1}\mathcal{U} \longrightarrow T_{\sigma_{s_x}} U_x \stackrel{\mathrm{d}(tx_s)}{\longrightarrow} T_x X \longrightarrow T_x^0\mathcal{U} \longrightarrow 0$$

where the central map is the differential of $U_x \stackrel{tx_s}{\longrightarrow} X$ at the distinguished point, and by

$$T_x^1\mathcal{U} = T_x^1 X.$$

`(Algebraic description)` See (72) for the notation. The above sequence reads

$$0 \longrightarrow T_x^{-1}\mathcal{U} \longrightarrow \mathrm{Der}(P, \mathbb{K}) \stackrel{\mu^*}{\longrightarrow} \mathrm{Der}(R, \mathbb{K}) \longrightarrow T_x^0\mathcal{U} \longrightarrow 0$$

and we have

$$T_x^1\mathcal{U} = \mathrm{Ex}(R, \mathbb{K}).$$

Moreover we have the exact sequence

$$0 \longrightarrow \mathrm{Der}(P, \mathbb{K}) \stackrel{\zeta^*}{\longrightarrow} \mathrm{Der}(S, \mathbb{K}) \stackrel{\sigma^*}{\longrightarrow} \mathrm{Der}(R, \mathbb{K}) \longrightarrow 0$$

which is split by $\nu^* : \mathrm{Der}(S, \mathbb{K}) \longrightarrow \mathrm{Der}(P, \mathbb{K})$ and $\varepsilon^* : \mathrm{Der}(S, \mathbb{K}) \longrightarrow \mathrm{Der}(R, \mathbb{K})$ i.e.

$$\sigma^*\varepsilon^* = 1 \quad \nu^*\zeta^* = 1 \quad \zeta^*\nu^* + \varepsilon^*\sigma^* = 1$$

We have another exact sequence:

$$0 \longrightarrow \mathrm{Der}(P, \mathbb{K}) \stackrel{(\zeta\iota)^*}{\longrightarrow} \mathrm{Der}(S, \mathbb{K}) \stackrel{\tau^*}{\longrightarrow} \mathrm{Der}(R, \mathbb{K}) \longrightarrow 0$$

also split by ε^*.
We notice the equalities:

$$\begin{aligned} \sigma^*\zeta^* &= 0 \\ \tau^* &= \sigma^* + (\nu\mu)^* \end{aligned}$$

`(Remark)` Tensoring the above sequences by a finite dimensional vector space I (notice the kernel of a small extension in *Art* is such an object) we get new exact sequences with $T_x^i\mathcal{U}$ replaced by $T_x^i\mathcal{U} \otimes I$ and $\mathrm{Der}(R, \mathbb{K})$ by $\mathrm{Der}(R, I)$ and so on.

3.4.1 Construction of the sequence

Fix $\mathcal{R}$ as in (72) and a small extension in *Art*

$$0 \longrightarrow I \longrightarrow B \xrightarrow{\varphi} A \longrightarrow 0.$$

With the notation of (73) and (74) the sequence we have to construct reads:

$$\begin{array}{llllllll} 0 \longrightarrow & T_x^{-1}\mathcal{U}\otimes I & \stackrel{\alpha}{\dashrightarrow} & \pi_1(\mathrm{Hom}(\mathcal{R},B),b) & \xrightarrow{\pi_1\varphi_*} & \pi_1(\mathrm{Hom}(\mathcal{R},A),\varphi_* b) & \stackrel{\beta}{\dashrightarrow} & \\ & T_x^{0}\mathcal{U}\otimes I & \stackrel{\gamma}{\dashrightarrow} & \pi_0(\mathrm{Hom}(\mathcal{R},B)) & \xrightarrow{\pi_0\varphi_*} & \pi_0(\mathrm{Hom}(\mathcal{R},A)) & \stackrel{\delta}{\dashrightarrow} & \mathrm{Ex}(R,I), \end{array}$$

where $b \in \mathrm{Hom}(R,B)$ will be fixed from (77) on.

75. The map δ and exactness at $\pi_0\mathrm{Hom}(\mathcal{R},A)$ Look at the diagram

$$\begin{array}{ccccc} \mathrm{Hom}(R,B) & \longrightarrow & \mathrm{Hom}(R,A) & \xrightarrow{\mathrm{ob}} & \mathrm{Ex}(R,I) \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \pi_0\mathrm{Hom}(\mathcal{R},B) & \longrightarrow & \pi_0\mathrm{Hom}(\mathcal{R},A) & \underset{\delta}{\dashrightarrow} & \mathrm{Ex}(R,I), \end{array}$$

where the top row is the right part of the deformation sequence for X at x written in algebraic terms. In order to induce the map δ we have to show that if $a, a' \in \mathrm{Hom}(R,A)$ are equivalent then $\mathrm{ob}(a') = \mathrm{ob}(a)$. The exactness at $\pi_0\mathrm{Hom}(\mathcal{R},A)$ follows automatically. Let's proceed to the proof. The assumption is that there exists $h \in \mathrm{Hom}(S,A)$ s.t. $\sigma^* h = a$ and $\tau^* h = a'$. The thesis is that there exists an isomorphism of $\mathbb{K}$-extensions of R by I:

$$\begin{array}{ccccccc} & & & B\times_{\varphi,A,a} R & & & \\ & & \nearrow & \Big\downarrow \Psi & \searrow & & \\ 0 \longrightarrow & I & & & & R & \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & B\times_{\varphi,A,a'} R & & & \end{array}$$

By [23, Par. 1.1.1] it's enough to show that there exists a morphism of $\mathbb{K}$-algebras Ψ that makes the diagram commute.
For bunches of variables:

$$\begin{aligned} x &= (x_1, ..., x_l) \\ y &= (y_1, ..., y_k), \end{aligned}$$

we'll use the multi-index notation; for instance, for $\lambda \in \mathbb{N}^l$ we put

$$x^\lambda = x_1^{\lambda_1} \cdot ... \cdot x_l^{\lambda_l}.$$

We can find an identification $S = R[[x_1, ..., x_l]]$ s.t. the maps $R \xrightarrow{\sigma,\tau} R[[x]] \xrightarrow{\varepsilon} R$ read:

$$\begin{aligned} \sigma(r) &= r x^0 \\ \tau(r) &= \sum_{\lambda\in\mathbb{N}^s} \tau(r)_\lambda x^\lambda \\ \varepsilon(x_i) &= 0. \end{aligned}$$

R is a quotient of a power series ring

$$\widetilde{R} := \mathbb{K}[[y_1, \cdots, y_k]],$$

we can find maps $\widetilde{\sigma}$, $\widetilde{\tau}$, $\widetilde{\varepsilon}$ that makes the following diagram commute

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & \mathbb{K}[[y]] & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \widetilde{\varepsilon}\uparrow\downarrow\widetilde{\sigma},\widetilde{\tau} & & \varepsilon\uparrow\downarrow\sigma,\tau & & \\
0 & \longrightarrow & J[[x]] & \longrightarrow & \mathbb{K}[[y]][[x]] & \longrightarrow & R[[x]] & \longrightarrow & 0,
\end{array}$$

and such that $\widetilde{\varepsilon}\widetilde{\sigma} = \mathrm{id} = \widetilde{\varepsilon}\widetilde{\tau}$.
Putting

$$\widetilde{S} := \mathbb{K}[[x, y]] = \mathbb{K}[[y]][[x]]$$

and choosing also a lifting $\widetilde{h}$ of h we get the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & \widetilde{R} & \xrightarrow{\pi} & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \widetilde{\sigma},\widetilde{\tau}} & & \downarrow{\scriptstyle \sigma,\tau} & & \\
0 & \longrightarrow & J[[x]] & \longrightarrow & \widetilde{S} & \xrightarrow{\pi_S} & S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \widetilde{h}} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & I & \longrightarrow & B & \xrightarrow{\varphi} & A & \longrightarrow & 0.
\end{array}$$

We'll make the following definitions:

$$\begin{array}{rcl}
\widetilde{b} & := & \widetilde{h}\widetilde{\sigma} \\
\widetilde{b}' & := & \widetilde{h}\widetilde{\tau} \\
\widetilde{a} & := & \varphi\widetilde{b} = a\pi \\
\widetilde{a}' & := & \varphi\widetilde{b}' = a'\pi.
\end{array}$$

We'll use a simplified notation for the fibered products:

$$\begin{array}{rcl}
B \times_a R & := & \{(b, r) \in B \times R : \varphi(b) = a(r)\} \\
B \times_{\widetilde{a}} \widetilde{R} & := & \{(b, \widetilde{r}) \in B \times \widetilde{R} : \varphi(b) = \widetilde{a}(r)\}
\end{array}$$

and the same with a' and $\widetilde{a}'$. We'll get Ψ first defining Θ and then making it to descend as in the following diagram:

$$\begin{array}{ccc}
B \times_{\widetilde{a}} \widetilde{R} & \overset{\Theta}{\dashrightarrow} & B \times_{\widetilde{a}'} \widetilde{R} \\
\downarrow & & \downarrow \\
B \times_a R & \underset{\Psi}{\dashrightarrow} & B \times_{a'} R.
\end{array}$$

`(The map Θ)` We define $\widehat{\Theta} : B \times \widetilde{R} \longrightarrow B \times \widetilde{R}$ by:

$$\widehat{\Theta}(b, r) := (b + \widetilde{b}'(r) - \widetilde{b}(r), r).$$

This is a morphism of vector spaces and one can show that $\widehat{\Theta}(B \times_{\widetilde{a}} \widetilde{R}) \subseteq B \times_{\widetilde{a}'} \widetilde{R}$, and that the induced map Θ preserves the product, i.e. it's a morphism of algebras.

`(The map Ψ)` We have to show that, for any $r \in J$, $\widetilde{b}(r) = \widetilde{b}'(r)$. Indeed $\widetilde{b}(r) = \widetilde{h}(rx^0)$. Moreover

$$\widetilde{b}'(r) = \widetilde{h}\left(\sum_{\lambda \in \mathbb{N}^s} \widetilde{\tau}(r)_\lambda x^\lambda\right) = \widetilde{h}(\widetilde{\tau}(r)_0 x^0) + \sum_{\lambda \neq 0} \widetilde{h}(\widetilde{\tau}(r)_\lambda)\widetilde{h}(x^\lambda).$$

Since $\widetilde{\tau}(r) \in J[[x]]$ then for any λ we have $\widetilde{\tau}(r)_\lambda \in J$ and hence $\widetilde{h}(\widetilde{\tau}(r)_\lambda) \in I$. For $\lambda \neq 0$ we have $\widetilde{h}(x^\lambda) \in m_B$. Moreover $\widetilde{\tau}(r)_0 = \widetilde{\varepsilon}\widetilde{\tau}(r) = r$. Since $I \cdot m_B = 0$, from all of this we conclude $\widetilde{b}(r) = \widetilde{b}'(r)$.

76. The map γ and exactness at $\pi_0\mathrm{Hom}(\mathcal{R}, B)$ Look at the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Der}(R,I) & \longrightarrow & \mathrm{Hom}(R,B) & \longrightarrow & \mathrm{Hom}(R,A) \\
& \downarrow & & \downarrow & & \downarrow \\
& T^0\mathcal{U} \otimes I & \overset{}{\underset{\gamma}{\dashrightarrow}} & \pi_0\mathrm{Hom}(\mathcal{R},B) & \longrightarrow & \pi_0\mathrm{Hom}(\mathcal{R},A),
\end{array}$$

where the top row is the left part of the deformation sequence for X at x written in algebraic terms. In order to induce the action γ we have first to show that the action of $\mathrm{Der}(R,I)$ on $\mathrm{Hom}(R,B)$ descend to the quotient, and second to show that the subspace $\mu^*\mathrm{Der}(P,I) \subseteq \mathrm{Der}(R,I)$ acts trivially on the quotient itself. The exactness at $\pi_0\mathrm{Hom}(\mathcal{R},B)$ follows automatically.
First we have to show that for any $h \in \mathrm{Hom}(S,B)$ and $v \in \mathrm{Der}(R,I)$ there exists $h' \in \mathrm{Hom}(S,B)$ s.t. $\sigma^*h' = \sigma^*h + v$ and $\tau^*h' = \tau^*h + v$. Indeed put $h' := h + \varepsilon^*v$.
Second we have to show that for any $w \in \mathrm{Der}(P,I)$ and $b \in \mathrm{Hom}(R,B)$ there exists $h \in \mathrm{Hom}(S,B)$ s.t. $\sigma^*h = b$ and $\tau^*h = b + \mu^*w$. Indeed put $h := \varepsilon^*b + \zeta^*w$.

From now on fix a $b \in \mathrm{Hom}(R,B)$ and define $a := \varphi b \in \mathrm{Hom}(R,A)$.

77. The map β Consider the diagram:

$$\begin{array}{ccc}
\sigma^{*-1}(b) & \xrightarrow{\quad\varphi_*\quad} & \sigma^{*-1}(a) \\
\uparrow & & \uparrow \\
\pi_1(\mathrm{Hom}(\mathcal{R},A),b) & \xrightarrow{\varphi_*} & \pi_1(\mathrm{Hom}(\mathcal{R},A),a)
\end{array}$$

Notice that for any $h \in \sigma^{*-1}(b)$: $\varphi_*h \in \pi_1(\mathrm{Hom}(\mathcal{R},A),a)$ if and only if $\tau^*h - \sigma^*h \in \mathrm{Der}(R,I) \subseteq \mathrm{Hom}_{\mathbb{K}}(R,B)$.
Given $g \in \pi_1(\mathrm{Hom}(\mathcal{R},A),a)$ we have to define $\beta(g) \in T^0_x\mathcal{U} \otimes I$. There exists $h \in \sigma^{*-1}(b)$ s.t. $\varphi_*h = g$. We want to put $\beta(g) := [\tau^*h - \sigma^*h]$.
In order this to make sense we have to show that this is independent of the choice of the lifting h of g. Indeed, given another lifting h' we put $w := \nu^*(h' - h) \in \mathrm{Der}(P,I)$ and we have $\tau^*h' - \sigma^*h' = \tau^*h - \sigma^*h + \mu^*w$.

78. Exactness at $T^0_x\mathcal{U} \otimes I$ Take $v \in \mathrm{Der}(R,I)$. Then $[v] \in \mathrm{Ker}\gamma$ (i.e. the action of $[v]$ leaves $[b]$ fixed) if and only if there exists $h \in \mathrm{Hom}(S,B)$ s.t. $\sigma^*h = b$ and $\tau^*h = b + v$. Moreover $[v] \in \mathrm{Im}\beta$ if and only if there exist $h' \in \mathrm{Hom}(S,B)$ and $w \in \mathrm{Der}(P,I)$ s.t. $\sigma^*h' = b$ and $\tau^*h' - \sigma^*h' = v + \mu^*w$ (notice that this implies $\tau^*\varphi_*h' = a$).

To show that $\mathrm{Im}\beta \subseteq \mathrm{Ker}\gamma$ put $h := h' - \zeta^* w$.
To show that $\mathrm{Ker}\gamma \subseteq \mathrm{Im}\beta$ put $h' := h$ and $w := 0$.

79. Exactness at $\pi_1(\mathrm{Hom}(\mathcal{R}, A), a)$ Take $g \in \pi_1 A$. Then: $g \in \mathrm{Ker}\,\beta$ if and only if there exist $\in \sigma^{*-1}(b)$ and $w \in \mathrm{Der}(P, I)$ s.t. $\varphi^* h = g$ and $\tau^* h - \sigma^* h = \mu^* w$. Moreover: $g \in \mathrm{Im}\pi_1\varphi$ if and only if there exists $h' \in \sigma^{*-1}(b)$ s.t. $\varphi_* h' = h$ and $\tau^* h' = b$.
To show that $\mathrm{Im}\,\pi_1\varphi \subseteq \mathrm{Ker}\,\beta$ put $h := h'$ and $w := 0$.
To show that $\mathrm{Ker}\,\beta \subseteq \mathrm{Im}\,\pi_1\varphi$ put $h' := h - \zeta^* w$.

80. The map α **and exactness at** $T_x^{-1}\mathcal{U} \otimes I$ **and** $\pi_1(\mathrm{Hom}(\mathcal{R}, A), b)$ Look at the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & T_x^{-1}\mathcal{U} \otimes I & \overset{\alpha}{\dashrightarrow} & \pi_1(\mathrm{Hom}(\mathcal{R}, B), b) & \longrightarrow & \pi_1(\mathrm{Hom}(\mathcal{R}, A), a) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Der}(P, I) & \longrightarrow & \mathrm{Hom}(P, B) & \longrightarrow & \mathrm{Hom}(P, A), \end{array}$$

where the bottom row is the left part of the deformation sequence of U_x at σ_{s_x} written in algebraic terms. The map α is induced with the claimed properties of exactness if we show that given $w \in \mathrm{Der}(P, I)$ and $g \in \mathrm{Hom}(P, B)$ s.t $\tau^* \nu_a^{*-1} g = a$ then:

$$\mu^* w = 0 \qquad \text{in and only if} \qquad \tau^* \nu_a^{*-1}(g + w) = a.$$

This follows from $\tau^* \nu_a^{*-1}(g + w) = \tau^* \nu_a^{*-1} g + \mu^* w$. To prove the last formula we observe that

$$\nu^* \zeta^* w = w = g + w - g = \nu^* \nu_a^{*-1}(g + w) - \nu^* \nu_a^{*-1} g$$

which implies that

$$\nu^* \nu_a^{*-1}(g + w) = \nu^*(\nu_a^{*-1} g + \zeta^* w);$$

since $\sigma^*(\nu_a^{*-1} g + \zeta^* w) = a$ then

$$\nu_a^{*-1}(g + w) = \nu_a^{*-1} g + \zeta^* w$$

and applying τ^* to both sides we get the result.

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