Detecting an immersed body in a fluid. Stability and reconstruction.

Andrea Ballerini

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To my sister

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Abstract

We consider the inverse problem of the detection of a single body, immersed in a bounded container filled with a fluid which obeys the stationary Stokes or Navier Stokes equations, from a single measurement of force and velocity on a portion of the boundary. Under appropriate a priori hypotheses we obtain an estimate of stability of log-log type for both cases. We then present a numerical method for the reconstruction of the body using a boundary elements representation of the solutions, combined with the iteratively regularized Gauss-Newton method, and present some partial numerical results in this direction.

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Chapter 1

Introduction

The inverse problem of detecting an inaccessible part of a boundary of a domain with the knowledge of measurements taken on an accessible part of the boundary arises from countless real life type situations. Possibly in any sort of non destructive testing, one wants to recover information about a part of a body that is not accessible to direct measurements using data available on a part of that body that is accessible. To name a few concrete examples, tomography, geological prospection, quality testing (detection of impurities or inhomogeneities in a sample). This field of research started a new burst of interest since the introduction of the *electrical impedance tomography* and the seminar paper by Calderòn in 1980 ([22]). Before stating the problem examined in this thesis, we will outline two inverse problems already studied in literature. Even if these are not the topic of concern of the thesis, their discussion may turn out to be very instructive: the bases laid along their treatment will clarify the approach we take for the inverse problem for the stationary Stokes and Navier Stokes equation (which is introduced in the next section, as well as at the beginning of each chapter) and highlight their common features and the difficulties encountered in the analysis.

1.1 The inverse problem with unknown boundaries

One of the best known, as well as most thoroughly studied inverse problems of the kind is the Calderòn problem also known as electrical impedence tomography. Consider a bounded domain Ω with sufficiently smooth boundary $\partial\Omega$. Suppose that Ω represents an electrically conducting body, whose conductivity is given by the scalar function $\gamma = \gamma(x)$. The problem is to determine γ from measurements of potential and voltage on the boundary alone. Let us formulate the question in mathematical terms.

Suppose that a boundary potential $g \in \mathbf{H}^{1/2}(\partial\Omega)$ is assigned on $\partial\Omega$. The induced potential $u \in \mathbf{H}^1(\Omega)$ is the weak solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\gamma\nabla u) &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We then measure the induced current $\psi = \gamma\nabla u \cdot \nu|_{\partial\Omega}$, where ν is the outer normal field to $\partial\Omega$. The inverse problem is: *determine γ, given a set of boundary measurements (g, ψ)* . We can formulate the analogous inverse problem when Neumann type boundary data are assigned: that is, given $\psi \in \mathbf{H}^{-1/2}(\partial\Omega)$, one measures the values at the boundary $g = u|_{\partial\Omega}$ of the solution of the Neumann problem (with an additional normalization condition, needed for uniqueness):

$$\begin{cases} \operatorname{div}(\gamma\nabla u) &= 0 \quad \text{in } \Omega, \\ \gamma\nabla u \cdot \nu &= \psi \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega} u &= 0 \quad , \end{cases} \tag{1.2}$$

and tries to recover γ from the knowledge of the analogous set of boundary measurements. To fix the ideas, we shall consider the Dirichlet setup. The first works on the subject assumed that all possible measurements are available, that is, that the Dirichlet-to-Neumann map

$$\begin{aligned} \Lambda_\gamma : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) &\to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \\ g &\mapsto \gamma\nabla u \cdot \nu\,, \end{aligned}$$

is known. It is well known, as Hadamard first showed [34], that the Cauchy problem for the harmonic equations is severely ill posed: hence, stability can only be obtained by adding a priori hypotheses on the unknown variable γ (hence, *conditional stability*). The analysis of the uniqueness and the conditional stability of the inverse problem in this setup corresponds to the injectivity and continuity of the map $\gamma \mapsto \Lambda_\gamma$ for γ belonging to some appropriate function space. The uniqueness issue was treated by Kohn and Vogelius [42] and Sylvester and Uhlmann [67] when $n \geq 3$ and by Nachman [60] and Astala and Päivärinta [11] for $n = 2$. The stability, on the other hand, was examined by Alessandrini [2] for $n \geq 3$, and by Barceló, Barceló and Ruiz [15] and Barceló, Faraco and Ruiz [16] for $n = 2$.

All the approaches just listed require the knowledge of the whole set of possible measurements, as well as some degree of regularity for the coefficient γ (except the uniqueness result for $n = 2$ by Astala and Päivärinta [11], which only requires γ to be in L^∞). In a real life type situation, however, these requests cannot be met. First, the available actual measurements are necessarily finite in number. Secondly, the unknown parameter γ may fail to be regular, or continuous, or even bounded. Furthermore, in many model problems, one cannot access the whole boundary but only a portion of it. This is for example the very widely studied case of the detection of unknown boudaries we introduced at the beginning of the chapter: we assume that $\partial\Omega$ is split into two disjoint portions, I and A, where I is not accessible directly, so that the boundary data have to be assigned on A and perhaps measured on a yet smaller portion $\Gamma \subset A$. The inaccessible portion I could represent a corrosion in a conductor, or an impurity in an elastic material, and so on. A very popular example of such problems is the problem of the detection of inclusions: by *inclusion* we mean a body $D \subset\subset \Omega$ having different properties from its surroundings. This would be the case, for example, of a material with conductivity γ of the form

$$\gamma = 1 + (k-1)\chi_D, \tag{1.3}$$

representing a homogeneous conductor Ω with constant condictivity 1 containing an inclusion D of conductivity $k \neq 1$. The function γ is clearly discontinuous; moreover, in some realistic physical models it could also attain extreme values ($k = 0$ represents a cavity, $k = \infty$ a perfectly conducting inclusion). In light of the practical interest of the problem, there is in literature a line of research regarding the detection of unknown inclusions from the knowledge of *one* boundary measurement, that is one couple (g, ψ), measured on a portion of the boundary $\Gamma \subset \partial\Omega$.

Let us now consider D_1, D_2 two subsets of Ω, with conductivity given by γ_1, γ_2 respectively of the form (1.3). Consider the domains $\Omega_i = \Omega \setminus \overline{D_i}$ for $i = 1, 2$. We consider the solutions u_i to (1.1) with conductivities $\gamma_{D_1}, \gamma_{D_2}$ respectively, with the same boundary data g on the common outer boundary $\partial\Omega$ and $u_i = 0$ on ∂D_i. We obtain the Cauchy data (g, ψ_1) and (g, ψ_2), respectively. The uniqueness and stability issues in this context may be formulated as:

1. *Uniqueness: Under what hypotheses* $\psi_1 = \psi_2$ *on* Γ *imply that* $D_1 = D_2$?

2. *Stability: Under what hypotheses does* $\|\psi_1 - \psi_2\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \leq \epsilon$ *imply that (for a suitable choice of a distance)* $\operatorname{dist}(D_1, D_2) \leq \omega(\epsilon)$*, with* $\omega(t) \to 0$ *as* $t \to 0$ *? What is then the rate of convergence of* ω*?*

Another very important question, naturally arising from the applications, is how to perform the actual reconstruction in a numerical fashion. Since an ill posed problem is numerically very challenging, this issue is subjected to the taming of the ill posedness, which we will discuss first.

We point out that the issue of uniqueness is still open and there are only partial results available (see for example [41]) for general conductivities of the form (1.3); on the other hand, general uniqueness results have been shown in the extreme case of cavities (see [7], [19] and [17]). The main ingredient used in the proofs is a form of the unique continuation property for harmonic functions. The basic idea behind the proof is to show that if the solutions u_1 and u_2 defined before yield the same boundary measurement $\psi_1 = \psi_2$, then their difference $w = u_1 - u_2$ can be extended to an identically null solution inside the domain, which then would imply $u_1 = u_2$ on $\Omega_1 \triangle \Omega_2$. If, for example, $\Omega_1 \setminus \Omega_2 \neq \emptyset$, an ellipticity argument shows that $u_1 = 0$ in $\Omega_1 \setminus \Omega_2$. If one can apply unique continuation, then $u_1 = 0$ in Ω_1, which is a contradiction if the boundary data is assumed non trivial. See for example [19], [7].

Given the severely ill posed nature of the problem, one should expect that only a weak rate of stability can be achieved, even with strong a priori hypotheses: namely, as the examples given by Di Cristo and Rondi show ([25]) - a conditional stability rate of logarithmic type with respect to the norm of the Cauchy data. This rate was indeed estabilished under additional assumptions of a priori regularity on the unknown boundary and on the boundary data g: see [4], for the problem with unknown boundaries; see also [7], [19]. Besides the result itself, the scheme used in the proofs is particularly relevant to our purposes: in fact, the approach taken in the proof can be adapted to other problems, such as the Lamè system of linearized elasticity and the stationary Stokes and Navier Stokes equations. Following [8], we outline the main steps of a stability proof as follows:

1. We obtain a *three spheres inequality* for the equation. This type of results concerns a form of quantitative estimate of the unique continuation property, generalizing Hadamard's three circle theorem for harmonic functions on the plane (which is considered the first result of such kind). In the context of elliptic equations with variable coefficients, relevant results are those of Landis [50], Garofalo and Lin [30], Brummelhuis [20] and Kukavica [47].

2. An iteration of the three spheres inequality yields an *estimate of propagation of smallness*. Basically, given a solution u assumed to be small (in the norm sense) on a small ball $B_\rho \subset \Omega$, this allows to estimate how the solution remains small on a larger domain $G \subset \Omega$, bounding $\|u\|_{L^2(G)}$ or $\|\nabla u\|_{L^2(G)}$ from above in terms of $\|u\|_{L^2(B_\rho)}$.

3. Since the solutions are only measured on the boundary $\partial\Omega$ (or on a portion of it) we need to estimate how much u is affected in the interior, when the boundary data is perturbed by error. In order to do this, we extend the solution to an open set containing Ω and whose boundary agrees with $\partial\Omega$ on a subset of Γ. The extended solution has to solve an inhomogeneous equation, but with a right hand side whose norm can be controlled by the Cauchy data on Γ. These estimates are called *stability estimates of continuation from Cauchy data*. General results of this type were first obtained by Payne [61, 62] and later developed by Trytten [69]. See also [8] for an account on the later techniques developed in this topic.

4. Combine the previous steps to apply the estimates of propagation of smallness in the interior of the extended domain, to obtain an estimate of stability in the interior from the Cauchy data. These have to be combined with geometric arguments in order to derive information on the distance between the two sets D_1 and D_2.

In addition to these basic steps, one can use the *doubling inequalities* in the interior and at the boundary to improve the estimate. These are quantitative estimates related to the *strong* unique continuation principle. The doubling inequalities in the interior (first introduced by Garofalo and Lin [30]) allow one to estimate the local vanishing rate of a solution on balls of decreasing radii, which turns out to be helpful to obtain a quantitative estimate of the rate of stability of the inverse problem in the interior.
Doubling inequalities at the boundary, on the other hand, enable to translate the estimates of the vanishing rate of u in the interior into similar estimates near the boundary, and have been shown for elliptic equations with variable coefficients by Adolfsson and Escauriaza [1] under the hypothesis of $C^{1,\alpha}$ regularity on the boundary (see also Kukavica and Nyström [48]). As it turns out, this result plays a crucial role to obtain the best possible (log type) stability estimate.

We would like to stress once again how the above listed elements constitute a general procedure for a proof of a stability result. In the scalar case, these facts are all known to hold. In recent times, however, an analogous theory for vector valued equations has been developed for inverse problems of the same kind and have been successfully applied in numerous variations of the inverse problem associated to (1.1) to obtain the best rate of stability possible. As one could expect, the theory is not quite as complete as it is for scalar equations: this is the case, for example, of the inverse problem associated to linearized elasticity as well as the problem treated in this thesis of the stationary Stokes and Navier Stokes equations. These problems share a good amount of similarities: it is then useful to briefly state the former and outline the methods exploited for its treatment, highlight how the approaches to the proofs remain essentially the same as in the scalar case and what needs to be changed in the multi dimensional setting.

We consider an elastic body Ω contained in $\mathbb{R}^2$ or $\mathbb{R}^3$, which may contain an unknown rigid inclusion D. The problem is to identify D by applying a traction field $\phi \in \mathbf{H}^{-1/2}(\partial\Omega)$ at the boundary and measuring the induced displacement field on an accessible portion of the boundary $\Gamma \subset \partial\Omega$. Assuming that the elasticity tensor $\mathbb{C}$ of the body Ω is known, one has that the displacement field u obeys the following Lamé system of equations of linearized elasticity (see [23] for a detailed derivation of the system):

$$\begin{cases} \operatorname{div}(\mathbb{C}\nabla u) &= 0 & \text{in } \Omega \setminus \overline{D}, \\ (\mathbb{C}\nabla u)\nu &= \varphi & \text{on } \partial\Omega, \\ u|_{\partial D} &\in \mathcal{R}, & \\ \int_{\partial D} (\mathbb{C}\nabla u)\nu \cdot r &= 0, & \text{for all } r \in \mathcal{R}, \\ u &= 0 & \text{on } \partial D, \end{cases} \tag{1.4}$$

where $\mathcal{R}$ is the linear space of infinitesimal rigid displacements, that is linear functions of the form $r(x) = Wx + c$ and W is any constant skew matrix, and the last condition is a normalization condition needed to guarantee uniqueness. Under the compatibility condition $\int_{\partial\Omega} \varphi \cdot r = 0$ for all $r \in \mathcal{R}$, a weak solution $u \in \mathbf{H}^1(\Omega \setminus \overline{D})$ exists. The inverse problem is to determine D from a single pair of Cauchy data $(u, (\mathbb{C}\nabla u)\nu)$ measured on $\Gamma \subset \partial\Omega$. Again, the key issues to the study of the inverse problem are uniqueness and stability. Given two solutions u_i of (1.4) for $D = D_i$, with $i = 1, 2$, satisfying $(\mathbb{C}\nabla u_i)\nu = \varphi$ on $\partial\Omega$:

1. *Uniqueness: When does*

$$(u_1 - u_2)|_\Gamma \in \mathcal{R},$$

   imply that $D_1 = D_2$?

2. *Stability : If*

$$\min_{r \in \mathcal{R}} \|(u_1 - u_2) - r\|_{\mathbf{L}^2(\Gamma)} \le \epsilon,$$

   is it true that $\operatorname{dist}(D_1, D_2) \le \omega(\epsilon)$, *with $\omega(t) \to 0$ as $t \to 0$? What is then the rate of convergence of ω?*

These issues have been examined in [59] for the problem described above (see also [58] for a similar treatment of the general problem of the detection of unknown boundaries). The uniqueness result given there requires C^1 regularity of the boundary ∂D, and is again based on the (weak) unique continuation principle in the context of the Lamé system (first proved by Weck [71]). The presence of the infinitesimal rigid displacements space $\mathcal{R}$ adds technical difficulties, since different space dimensions require different treatments (that is, two different proofs for the two and three dimensional settings).
Regarding the stability issue, we expect again nothing better than a weak stability rate, since this problem is severely ill posed as well. The techniques exploited for the proof are, in principle, the same as for the scalar case: namely, quantitative estimates of unique continuation. However, at the present time the theory for the Lamé system (and for systems of equations in general) is far less complete than the scalar case. In particular, no doubling inequalities at the boundary are known for this setup. To overcome this difficulty, it is then necessary to prove a finer version of the estimates

of propagation of smallness, which, once combined with the steps presented before, yields an estimate of log-log type stability. It is worth noting that, if a doubling inequality at the boundary were to be proved for these equations, then one could apply the same reasoning used for the scalar equations (like in [4]) and prove a log type stability (once again, the best possible).
To conclude, we would like to outline a general approach to the problem of the reconstruction. A very fruitful type of technique used to tackle the problem of reconstructing unknown boundaries is the boundary elements method. This has been exploited for a variety of problems in numerical analysis and engineering involving elliptic partial differential equations, in various fields of science such as elasticity, geomechanics, structural mechanics, electromagnetism, acoustics, hydrodynamics. These techniques are based upon the reduction of the differential equations to boundary integral equations which can be solved for a boundary distribution, so that it is not necessary to compute the solution in the whole domain. Once the unknown boundary distribution is available, the value of the solution at any point may be computed by direct evaluation. The key advantage of the boundary elements method is that they reduce the dimension of the solution space with respect to physical dimensions by one. This is often accomplished by representing the solution using single or double layer potentials (or a combination of both) with unknown densities, which may be solved by specifying the boundary conditions. A very instructive model problem of this type is described (and solved) by Kress [43], in the context of the Laplace equation. We refer to the aforementioned [43] for a useful primer on layer potential techniques; [21] (and references therein) for the treatment of a related problem in the context of the equation of conductivity. In the context of the Stokes equations, see [10] for a different approach using domain differentiation, [44] for the analysis of a similar problem regarding the Oseen equations exploiting a boundary layer potential representation technique.

1.2 The stationary Stokes and Navier Stokes equations case

In this thesis we examine the analogous problem of the detection of an inclusion in a stationary Stokes fluid and in a stationary Navier-Stokes fluid. Let $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$. We will consider the physically relevant cases $n = 2$ or $n = 3$. Assume that Ω is a container filled with a Stokes fluid, and we want to detect an object D which is possibly immersed in this container, by collecting measurements of the velocity of the fluid motion and of the boundary forces, but we only have access to a portion Γ of the boundary $\partial\Omega$. Once a suitable $g \in \mathbf{H}^{1/2}(\Gamma)$ is assigned, the fluid will obey the following system:

$$\begin{cases} \operatorname{div} \sigma(u,p) &= 0 \quad \text{in } \Omega \backslash \overline{D}, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \backslash \overline{D}, \\ u &= g \quad \text{on } \Gamma, \\ u &= 0 \quad \text{on } \partial D. \end{cases} \tag{1.5}$$

Here,

$$\sigma(u,p) = \mu(\nabla u + \nabla u^T) - p\,\mathbb{I}$$

is the *stress tensor*, where $\mathbb{I}$ denotes the $n \times n$ identity matrix, and $\mu > 0$ is the viscosity. The last request in (1.5) is the so called "no-slip condition". Let us denote by ν the outer normal vector field to $\partial\Omega$. Once g is assigned on Γ, we measure on Γ the induced normal component of the stress tensor

$$\sigma(u,p) \cdot \nu = \psi, \tag{1.6}$$

and try to recover D from a single pair of Cauchy data (g, ψ) known on the accessible part of the boundary Γ. Let us mention that a different inverse problem, regarding the identification of the viscosity from boundary measurements, has been also studied recently by Li and Wang [51]. The inverse problem described above is severely ill posed. The uniqueness issue has been solved, see [9], under the hypothesis of $\partial\Omega$ being of Lipschitz class and slightly stronger regularity hypotheses on the boundary data g, by means, again, of an *ad hoc* unique continuation principle. We have that: if (u_1, p_1) and (u_2, p_2) are two solutions of (1.5) corresponding to an assigned boundary data g, for $D = D_1$ and $D = D_2$ respectively, and $\sigma(u_1, p_1) \cdot \nu = \sigma(u_2, p_2) \cdot \nu$ on Γ, then $D_1 = D_2$. An analogous uniqueness result for the corresponding Neumann problem has been recently given by Badra, Caubet and Dambrine [12]. In this thesis we treat the problem of the stability of the inverse problem associated to (1.5):

Given two solutions (u_i, p_i) to (1.5) for two different D_i, for $i = 1, 2$, with the same boundary data g, if

$$\|\sigma(u_1, p_1) \cdot \nu - \sigma(u_2, p_2) \cdot \nu\| \le \epsilon,$$

what is the rate of convergence of $\mathrm{d}_{\mathcal{H}}(D_1, D_2)$ as $\epsilon \to 0$?
(We denote by $\mathrm{d}_{\mathcal{H}}$ the Hausdorff distance). There were only some partial "directional stability" type result, given in [26] and [9]. These are essentially conditional stability results restricted to a specific family of domains D_τ, obtained from a reference domain D_0 by applying a family of diffeomorphisms of one real parameter τ, of the form $\phi_\tau = \tau\phi_0 + Id$. This type of result, however, would not guarantee an a priori uniform stability estimate for the distance between two domains that yield boundary measurement that are close to each other, because all the constants involved are not bounded from below and depend upon the diffeomorphism ϕ_0, which cannot be known in

advance. In light of Rondi and Di Cristo's examples mentioned before, we only can expect a weak rate of stability even with strong additional a priori hypotheses. Furthermore, as it was the case for linearized elasticity, no doubling inequality at the boundary for (1.5) are known to hold.

Chapter 2 of the thesis is devoted to the proof of a log-log type conditional stability for the Hausdorff distance between the boundaries of the inclusions in the Stokes equations setting [13]. The approach to the proof mimicks, in principle, the one we outlined for linear elasticity in the previous section. The starting point, as we said, is estabilishing a three spheres inequality. We use the one proved by Nagayasu, Lin and Wang [53] for systems of equations with iterated laplacian as principal part, which we adapt to the Stokes equations, see Section 4. In this case, the a priori hypotheses of regularity (all found in Section 2) on the boundary data and on the unknown domain are essentially the minimal ones required for uniqueness Then, we prove a refined version estimate of propagation of smallness analogous to that in [59] and [58] (Section 4), and a stability estimate of continuation from Cauchy data (Section 5). These, combined, allow us to prove the stability result (Section 3).

Once the stability result for the Stokes equations is obtained, it came natural to ask ourselves what would change in the stationary Navier Stokes setting. We consider an inclusion D immersed in Ω, which is in turn filled with a fluid obeying the stationary Navier Stokes equations

$$\begin{cases} \operatorname{div} \sigma(u,p) &= (u\cdot\nabla)u & \text{in } \Omega\backslash\overline{D}, \\ \operatorname{div} u &= 0 & \text{in } \Omega\backslash\overline{D}, \\ u &= g & \text{on } \Gamma, \\ u &= 0 & \text{on } \partial D, \end{cases} \tag{1.7}$$

and the same inverse problem of detecting D with only one boundary measurement of the normal component of the stress tensor. The uniqueness result from [9] also applies on this inverse problem, thus, in Chapter 3, we examine the issue of stability.

The starting point is a recently appeared paper by Lin, Uhlmann and Wang [54], stating a three spheres inequality for a linearized version of the Navier Stokes equations, also admitting terms with lower order derivatives. In order to adapt the latter to the inverse problem, however, there appears the necessity of stronger a priori bounds: namely, we impose bound the $\mathbf{C}^{1,\alpha}$ norm of the solution, by an a priori limitation on the $\mathbf{C}^{1,\alpha}$ norm of the boundary data. This additional regularity request implies also that we deal with classical solutions (whereas we could consider weak solutions in the Stokes equations case). Also, since the difference of two solutions to the Navier Stokes equations is not itself a solution (because of nonlinearity), we had to write another three spheres inequality for such functions. Aside from these additional requests, and technical complications due to the presence of the nonlinearity, the basic outline of the analysis remains the same:

using the three spheres inequality we prove a refined version estimate of propagation of smallness analogous to that in [59] and [58] (Section 4), and a stability estimate of continuation from Cauchy data (Section 5). These, combined, allow us to prove the main result of Chapter 3, namely a log-log stability estimate (Section 3). The results of this study is contained in [14]. In Chapter 4 we examine the problem of reconstruction in the case of stationary Stokes fluid in the plane. Since the identifiability result by Alvarez et al. [9], more investigation were conducted in this direction. Alvarez et. al. in [10] show a numerical method for the reconstruction of some parameters of immersed bodies. We write the inverse problem as the problem of inversion of a non linear integral operator. Nonlinearity and ill posedness make the operator very difficult to invert, especially because, in real life type situations, we have to take into account the presence of noise in the data which, because of the ill posedness, may seriously alter the solution. The strategy is then twofold: we apply an iterative Newton method, which consists in replacing the nonlinear equation with its linearized version; then, to tame the ill posedness, we use on the linearized problem a regularization method such as Tikhonov regularization, with a suitable choice of the regularization parameter: such a choice is fundamental, since a parameter that is too large yields a poor approximation of the solution, on the other hand if the parameter is too small the stability is compromised. Since the quality of the iterates deteriorates in the presence of noise as the number of iterations increases, the choice of an appropriate stopping rule is essential to the regularization method. We refer to Kaltenbacher's monograph [40], and also to Engl, Hanke and Neubauer [27] for a topic review on regularization techniques for non linear and ill posed inverse problems. We point out that iterative regularization methods as such have been successfully exploited for the solution of inverse obstacle problems for elliptic equations, see, to name a few, Hettlich and Rundell [35], Kress and Rundell [45], as well as for inverse obstacle scattering problems, see, among others, Colton and Kress [24] and the bibliography therein, and Hohage [36, 37].

In Section 1, we introduce iterative regularization methods, in particular the iteratively regularized Gauss Newton method, or IRGNM. Following along the lines of Kress and Meyer [44], who studied a similar inverse problem for the Oseen equations, we apply the theory of layer potential for the Stokes equations (described in depth in [70], see also [49]): using the technique of single layer potentials, we formulate the Stokes equations as a system of boundary integral equations, then show how to write them in a discrete fashion in Section 3. These steps allow one to numerically evaluate the solution of the direct problem, thus solving it. Once the direct problem is solved, we show how to apply IRGNM to the inverse problem of the reconstruction, in Section 4.

While we are yet unable to present numerical results for the inverse problem for the Stokes equation, which is still work in progress, we present in Section 5 numerical result for a preparatory work on a simpler problem of the same kind. These results should be seen as a preliminary test of the iterative method applied to the reconstruction, and should nonetheless give partial indication on the results we could expect to obtain.

Chapter 2

Stability for Stokes fluids

2.1 Introduction

We consider an inverse problem associated to the Stokes system. Let $\Omega \subset \mathbb{R}^n$ (we assume $n = 2, 3$, which are indeed the physically relevant cases) with a sufficiently smooth boundary $\partial\Omega$. We want to detect an object D immersed in this container, by collecting measurements of the velocity of the fluid motion and of the boundary forces, but we only have access to a portion Γ of the boundary $\partial\Omega$. The fluid obeys the Stokes system in $\Omega\backslash\overline{D}$:

$$\begin{cases} \operatorname{div}\sigma(u,p) &= 0 \quad \text{in } \Omega\backslash\overline{D}, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega\backslash\overline{D}, \\ u &= g \quad \text{on } \Gamma, \\ u &= 0 \quad \text{on } \partial D. \end{cases} \tag{2.1}$$

Here,

$$\sigma(u,p) = \mu(\nabla u + \nabla u^T) - p\,\mathbb{I}$$

is the *stress tensor*, where $\mathbb{I}$ denotes the $n \times n$ identity matrix, and μ is the viscosity. The last request in (2.1) is the so called "no-slip condition". We will always assume constant viscosity, $\mu(x) = 1$, for all $x \in \Omega\backslash\overline{D}$. We observe that if $(u,p) \in \mathbf{H}^1(\Omega\backslash\overline{D}) \times L^2(\Omega\backslash\overline{D})$ solves (2.1), then it also satisfies

$$\triangle u - \nabla p = 0.$$

Call ν the outer normal vector field to $\partial\Omega$. The ideal experiment we perform is to assign $g \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ and measure on Γ the normal component of the stress tensor it induces,

$$\sigma(u,p)\cdot\nu = \psi, \tag{2.2}$$

and try to recover D from a single pair of Cauchy data (g, ψ) known on the accessible part of the boundary Γ. Under the hypothesis of $\partial\Omega$ being of Lipschitz class and slightly stronger regularity hypotheses on the boundary data g, the uniqueness for this inverse problem has been shown to hold (see

[9]) by means of unique continuation techniques. By uniqueness we mean the following fact: if u_1 and u_2 are two solutions of (2.1) corresponding to an assigned boundary data g, for $D = D_1$ and $D = D_2$ respectively, and $\sigma(u_1, p_1) \cdot \nu = \sigma(u_2, p_2) \cdot \nu$ on Γ, then $D_1 = D_2$.
In this chapter we investigate the problem of stability, which we may roughly state as follows:
Given two solutions (u_i, p_i) *to* (2.1) *for two different* D_i*, for* $i = 1, 2$*, with the same boundary data* g*, if*

$$\|\sigma(u_1, p_1) \cdot \nu - \sigma(u_2, p_2) \cdot \nu\| \leq \epsilon,$$

what is the rate of convergence of $\mathrm{d}_{\mathcal{H}}(D_1, D_2)$ *as* $\epsilon \to 0$*?*
(We denote by $\mathrm{d}_{\mathcal{H}}$ the Hausdorff distance). There are some partial "directional stability" type result, given in [26] and [9]. This type of result, however, would not guarantee an a priori uniform stability estimate for the distance between two domains that yield boundary measurement that are close to each other. In the general case, even if we add some a priori information on the regularity of the unknown domain, we can only obtain a weak rate of stability. As for the Stokes problem, even if we add some a priori information on the regularity of the unknown domain, we can only obtain a weak rate of stability. This does not come unexpected since, even for much simpler problems of the same kind, the dependence of D from the Cauchy data is at most of logarithmic type. See, for example, [4] for a similar problem on electric conductivity, or [58], [59] for an inverse problem regarding elasticity. There are, in fact, several counterexamples showing that the optimal rate of convergence for the inverse conductivity problem is no better than of log type (see [3] and [25]).
The aim of our analysis is thus to prove a log-log type stability for the Hausdorff distance between the boundaries of the inclusions, assuming a $C^{2,\alpha}$ regularity bound. Such estimates have been estabilished for various kinds of elliptic equations, for example, [4], [7], for the electric conductivity equation, [58], [59] for the elasticity system and the detection of cavities or rigid inclusions, and [13] for the Stokes equation. The main tool used to prove stability here and in the aforementioned papers [4], [58], [59], [13] is a quantitative estimate of continuation from boundary data, in the interior, in the form of a three spheres inequality (see Theorem 2.9) and its main consequences. However, while in [4] the estimates are of log type for a scalar equation, here, and in [58], [59] and [13], only an estimate of log-log type could be obtained for a system of equations. To improve that, one would need a doubling inequality at the boundary for systems of equations, which basically would allow to extend the reach of the unique continuation property up to the boundary. Unfortunately, to the present time, none are available; on the other hand they are known to hold in the scalar case.
The basic steps of the present paper closely follows [58], [59], and are the following:

1. *An estimate of propagation of smallness from the interior.* The proof of this estimate relies essentially on the three spheres inequality for solutions of the bilaplacian system. Since both the Lamé system and the Stokes system can be represented as solutions of such equations (at least locally and in the weak sense, see [6] for a derivation of this for the elasticity system), we expected the same type of result to hold for both cases.

2. *A stability estimate of continuation from the Cauchy data.* This result also relies heavily on the three spheres inequality, but in order to obtain a useful estimate of continuation near the boundary, we need to extend a given solution of the Stokes equation a little outside the domain, so that the extended solution solves a similar system of equation. Once the solution has been properly extended, we may apply the stability estimates from the interior to the extended solution and treat them like estimates near the boundary for the original solution.

3. *An extension lemma for solutions to the Stokes equations.* This step requires finding appropriate conditions on the velocity field u as well as for the pressure p, in order for the boundary conditions to make sense.

The chapter is structured as follows. We start by stating the apriori hypotheses we will need throughout the paper and the main result, Theorem 2.1. Then we present the estimates of continuation from the interior, Propositions 2.4, 2.5, and Propositions 2.6 and 2.7 which deal, in turn, with the stability estimates of continuation from Cauchy data and a better version of the latter under some additional regularity hypotheses, and we use them for the proof of Theorem 2.1. In the subsequent sections we prove Proposition 2.4 and 2.5 using the three spheres inequality (Theorem 2.9). Then we prove Proposition 2.6, which will use an extension argument (Proposition 2.14), which will in turn be proven in the last section.

2.2 The stability result

2.2.1 Notations and definitions

In this section we introduce the notations we will use all along.
Let $x \in \mathbb{R}^n$. We will denote by $B_\rho(x)$ the ball in $\mathbb{R}^n$ centered in x of radius ρ. We will indicate $x = (x_1, \dots, x_n)$ as $x = (x', x_n)$ where $x' = (x_1 \dots x_{n-1})$. Accordingly, $B'_\rho(x')$ will denote the ball of center x' and radius ρ in $\mathbb{R}^{n-1}$. When referring to regularity of a domain, it will be understood according to the following definition.

Definition Let $\Omega \subset \mathbb{R}^n$ a bounded domain. We say $\Gamma \subset \partial\Omega$ is of class $C^{k,\alpha}$ with constants ρ_0, $M_0 > 0$, where k is a nonnegative integer, $\alpha \in [0, 1)$ if, for any $P \in \Gamma$ there exists a rigid transformation of coordinates in which $P = 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{(x', x_n) \in B_{\rho_0}(0) \text{ s.t. } x_n > \varphi(x')\}, \tag{2.3}$$

where φ is a real valued function of class $C^{k,\alpha}(B'_{\rho_0}(0))$ such that

$$\begin{aligned} \varphi(0) &= 0, \\ \nabla\varphi(0) &= 0, \text{ if } k \geq 1 \\ \|\varphi\|_{C^{k,\alpha}(B'_{\rho_0}(0))} &\leq M_0\rho_0. \end{aligned}$$

When $k = 0$, $\alpha = 1$ we will say that Γ is *of Lipschitz class with constants* ρ_0, M_0.

Remark We normalize all norms in such a way they are all dimensionally equivalent to their argument and coincide with the usual norms when $\rho_0 = 1$. In this setup, the norm taken in the previous definition is intended as follows:

$$\|\varphi\|_{C^{k,\alpha}(B'_{\rho_0}(0))} = \sum_{i=0}^{k} \rho_0^i \|D^i\varphi\|_{L^\infty(B'_{\rho_0}(0))} + \rho_0^{k+\alpha}|D^k\varphi|_{\alpha, B'_{\rho_0}(0)},$$

where $|\cdot|$ represents the α-Hölder seminorm

$$|D^k\varphi|_{\alpha, B'_{\rho_0}(0)} = \sup_{x', y' \in B'_{\rho_0}(0), x' \neq y'} \frac{|D^k\varphi(x') - D^k\varphi(y')|}{|x' - y'|^\alpha},$$

and $D^k\varphi = \{D^\beta\varphi\}_{|\beta|=k}$ is the set of derivatives of order k. Similarly we set

$$\|u\|^2_{\mathbf{L}^2(\Omega)} = \frac{1}{\rho_0^n}\int_\Omega u^2$$

$$\|u\|^2_{\mathbf{H}^1(\Omega)} = \frac{1}{\rho_0^n}\Big(\int_\Omega u^2 + \rho_0^2\int_\Omega |\nabla u|^2\Big).$$

The same goes for the trace norms $\|u\|_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}$ and the dual norms $\|u\|_{\mathbf{H}^{-1}(\Omega)}$, $\|u\|_{\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)}$ and so forth.

We will sometimes use the following notation, for $h > 0$:

$$\Omega_h = \{x \in \Omega \text{ such that } d(x, \partial\Omega) > h\}.$$

2.2.2 A priori information

We now state the a priori hypotheses needed to prove the stability result.
(1) *A priori information on the domain*
We assume $\Omega \subset \mathbb{R}^n$ to be a bounded domain, such that

$$\partial\Omega \text{ is connected,} \tag{2.4}$$

and it has a sufficiently smooth boundary, i.e.,

$$\partial\Omega \text{ is of class } C^{2,\alpha} \text{ of constants } \rho_0,\ M_0, \tag{2.5}$$

where $\alpha \in (0,1]$ is a real number, $M_0 > 0$, and $\rho_0 > 0$ is what we shall treat as our dimensional parameter. By ν we will denote the outer normal vector field to a domain.

$$|\Omega| \leq M_1 \rho_0^n, \tag{2.6}$$

where $M_1 > 0$.
In our setup, we choose a special open and connected portion $\Gamma \subset \partial\Omega$ as being the accessible part of the boundary, where, ideally, all measurements are taken. We assume that there exists a point $P_0 \in \Gamma$ such that

$$\partial\Omega \cap B_{\rho_0}(P_0) \subset \Gamma. \tag{2.7}$$

(2) *A priori information about the obstacle*
We consider $D \subset \Omega$, which represents the obstacle we want to detect from the boundary measurements, on which we require that

$$\Omega \setminus \overline{D} \text{ is connected,} \tag{2.8}$$

$$\partial D \text{ is connected.} \tag{2.9}$$

We require the same regularity on D as we did for Ω, that is,

$$\partial D \text{ is of class } C^{2,\alpha} \text{ with constants } \rho_0,\ M_0. \tag{2.10}$$

In addition, we suppose that the obstacle is "well contained" in Ω, meaning

$$d(D, \partial\Omega) \geq \rho_0. \tag{2.11}$$

(3) *A priori information about the boundary data*
For the Dirichlet-type data g we assign on the accessible portion of the boundary Γ, we assume that

$$\begin{aligned} g \in \mathbf{H}^{\frac{3}{2}}(\partial\Omega), \quad g \not\equiv 0, \\ \operatorname{supp} g \subset\subset \Gamma. \end{aligned} \tag{2.12}$$

The divergence free equation implies the following necessary condition on the boundary data:

$$\int_{\partial\Omega} g \,\mathrm{d}s = 0. \tag{2.13}$$

We also ask that, for a given constant $F > 0$, we have

$$\frac{\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}}{\|g\|_{\mathbf{L}^2(\Gamma)}} \leq F. \tag{2.14}$$

Under the above conditions on g, one can prove that there exists a constant $c > 0$, only depending on M_0, such that the following equivalence relation holds:

$$\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq \|g\|_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)} \leq c\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}. \tag{2.15}$$

2.2.3 The main result

Let $\Omega \subset \mathbb{R}^n$, and $\Gamma \subset \partial\Omega$ satisfy (2.5)-(2.7). Let $D_i \subset \Omega$, for $i = 1, 2$, satisfy (2.8)-(2.11), and let us denote by $\Omega_i = \Omega \setminus \overline{D_i}$. We may state the main result as follows.

Theorem 2.1 (Stability). *Let $g \in \mathbf{H}^{\frac{3}{2}}(\Gamma)$ be the assigned boundary data, satisfying (2.12)-(2.14). Let $u_i \in \mathbf{H}^1(\Omega_i)$ solve (2.1) for $D = D_i$. If, for $\epsilon > 0$, we have*

$$\rho_0 \|\sigma(u_1, p_1) \cdot \nu - \sigma(u_2, p_2) \cdot \nu\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \leq \epsilon, \tag{2.16}$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \rho_0 \omega \left(\frac{\epsilon}{\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}} \right), \tag{2.17}$$

where $\omega : (0, +\infty) \to \mathbb{R}^+$ is an increasing function satisfying, for all $0 < t < \frac{1}{e}$:

$$\omega(t) \leq C(\log|\log t|)^{-\beta}. \tag{2.18}$$

The constants $C > 0$ and $0 < \beta < 1$ only depend on n, M_0, M_1 and F.

2.2.4 The Helmholtz-Weyl decomposition

We find it convenient to recall a classical result which will come in handy later on. A basic tool in the study of the Stokes equations (2.1) is the Helmholtz-Weyl decomposition of the space $\mathbf{L}^2(\Omega)$ in two orthogonal spaces:

$$\mathbf{L}^2(\Omega) = H \oplus H^{\perp}, \tag{2.19}$$

where

$$H = \{u \in \mathbf{L}^2(\Omega) \ : \text{div}\, u = 0,\ u|_{\partial\Omega} = 0\}$$

and

$$H^{\perp} = \{u \in \mathbf{L}^2(\Omega) \ : \exists\, p \in \mathbf{H}^1(\Omega) \ : \ u = \nabla p\,\}.$$

This decomposition is used, for example, to prove the existence of a solution of the Stokes system (among many others, see [49]).

From this, and using a quite standard "energy estimate" reasoning, one can prove the following (see [49], [29], or [68]):

Theorem 2.2 (Regularity for the direct Stokes problem.). *Let $m \geq -1$ an integer number and let $E \subset \mathbb{R}^n$ be a bounded domain of class C^r , with $r = \max\{m+2, 2\}$. Let us consider the following problem:*

$$\begin{cases} \operatorname{div} \sigma(u,p) & = f \quad \text{in } E, \\ \operatorname{div} u & = 0 \quad \text{in } E, \\ u & = g \quad \text{on } \partial E, \end{cases} \tag{2.20}$$

where $f \in \mathbf{H}^m(E)$ and $g \in \mathbf{H}^{m+\frac{3}{2}}(E)$. Then there exists a weak solution $(u,p) \in \mathbf{H}^{m+2}(E) \times H^{m+1}(E)$ and a constant c_0, only depending on the regularity constants of E such that

$$\|u\|_{\mathbf{H}^{m+2}(E)} + \rho_0 \|p - p_E\|_{H^{m+1}(E)} \leq c_0 \big(\rho_0 \|f\|_{\mathbf{H}^m(E)} + \|g\|_{\mathbf{H}^{m+\frac{3}{2}}(\partial E)} \big), \tag{2.21}$$

where p_E denotes the average of p in E, $p_E = \frac{1}{|E|}\int_E p$.

Finally, we would like to recall the following version of Poincaré inequality, dealing with functions that vanish on an open portion of the boundary (see [55], or [5] for a precise evaluation of the constants in terms of the Poincaré constant of the domain and the measure of the portion of the boundary of the domain where the function vanishes on).

Proposition 2.3 (Poincaré inequality). *Let $E \subset \mathbb{R}^n$ be a bounded domain with boundary of Lipschitz class with constants ρ_0, M_0 and satisfying* (3.6). *Then for every $u \in \mathbf{H}^1(E)$ such that*

$$u = 0 \text{ on } \partial E \cap B_{\rho_0}(P),$$

where P is some point in ∂E, we have

$$\|u\|_{\mathbf{L}^2(E)} \leq C\rho_0 \|\nabla u\|_{\mathbf{L}^2(E)}, \tag{2.22}$$

where C is a positive constant only depending on M_0 and M_1.

2.3 Proof of Theorem 2.1

The proof of Theorem 2.1 relies on the following sequence of propositions.

Proposition 2.4 (Lipschitz propagation of smallness). *Let E be a bounded Lipschitz domain with constants ρ_0, M_0, satisfying (2.6). Let u be a solution to the following problem:*

$$\begin{cases} \operatorname{div} \sigma(u,p) & = 0 \quad \text{in } E, \\ \operatorname{div} u & = 0 \quad \text{in } E, \\ u & = g \quad \text{on } \partial E, \end{cases} \tag{2.23}$$

where g satisfies

$$g \in \mathbf{H}^{\frac{3}{2}}(\partial E), \quad g \not\equiv 0, \tag{2.24}$$

$$\int_{\partial E} g \, \mathrm{d}s = 0, \tag{2.25}$$

$$\frac{\|g\|_{\mathbf{H}^{\frac{1}{2}}(\partial E)}}{\|g\|_{\mathbf{L}^2(\partial E)}} \le F, \tag{2.26}$$

for a given constant $F > 0$. Also suppose that there exists a point $P \in \partial E$ such that

$$g = 0 \;\; on \;\; \partial E \cap B_{\rho_0}(P). \tag{2.27}$$

Then there exists a constant $s > 1$, depending only on n and M_0 such that, for every $\rho > 0$ and for every $\bar{x} \in E_{s\rho}$, we have

$$\int_{B_\rho(\bar{x})} |\nabla u|^2 dx \ge C_\rho \int_E |\nabla u|^2 dx. \tag{2.28}$$

Here $C_\rho > 0$ is a constant depending only on n, M_0, M_1, F, ρ_0 and ρ. The dependence of C_ρ from ρ and ρ_0 can be traced explicitly as

$$C_\rho = \frac{C}{\exp\left[A\left(\frac{\rho_0}{\rho}\right)^B\right]} \tag{2.29}$$

where A, B, $C > 0$ only depend on n, M_0, M_1 and F.

Proposition 2.5 (Lipschitz propagation of smallness up to boundary data)**.** *Under the hypotheses of Theorem 2.1, for all $\rho > 0$, if $\bar{x} \in (\Omega_i)_{(s+1)\rho}$, we have for $i = 1, 2$:*

$$\frac{1}{\rho_0^{n-2}} \int_{B_\rho(\bar{x})} |\nabla u_i|^2 dx \ge C_\rho \|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}, \tag{2.30}$$

where C_ρ is as in (2.29) (with possibly a different value of the term C), and s is given by Proposition 2.4.

Proposition 2.6 (Stability estimate of continuation from Cauchy data)**.** *Under the hypotheses of Theorem 2.1 we have*

$$\frac{1}{\rho_0^{n-2}} \int_{D_2 \setminus D_1} |\nabla u_1|^2 \le C \|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \, \omega\left(\frac{\epsilon}{\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}}\right) \tag{2.31}$$

$$\frac{1}{\rho_0^{n-2}} \int_{D_1 \setminus D_2} |\nabla u_2|^2 \le C \|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \, \omega\left(\frac{\epsilon}{\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}}\right) \tag{2.32}$$

where ω is an increasing continuous function, defined on $\mathbb{R}^+$ and satisfying

$$\omega(t) \le C\big(\log|\log t|\big)^{-c} \tag{2.33}$$

for all $t < e^{-1}$, where C only depends on n, M_0, M_1, F, and $c > 0$ only depends on n.

Proposition 2.7 (Improved stability estimate of continuation)**.** *Let the hypotheses of Theorem 2.1 hold. Let G be the connected component of $\Omega_1 \cap \Omega_2$ containing Γ, and assume that ∂G is of Lipschitz class of constants $\tilde{\rho}_0$ and $\tilde{M}_0$, where $M_0 > 0$ and $0 < \tilde{\rho}_0 < \rho_0$. Then (2.31) and (2.32) both hold with ω given by*

$$\omega(t) = C|\log t|^{\gamma}, \tag{2.34}$$

defined for $t < 1$, where $\gamma > 0$ and $C > 0$ only depend on M_0, $\tilde{M}_0$, M_1 and $\frac{\rho_0}{\tilde{\rho}_0}$.

Proposition 2.8. *Let Ω_1 and Ω_2 two bounded domains satisfying (2.5). Then there exist two positive numbers d_0, $\tilde{\rho}_0$, with $\tilde{\rho}_0 \leq \rho_0$, such that the ratios $\frac{\rho_0}{\tilde{\rho}_0}$, $\frac{d_0}{\rho_0}$ only depend on n, M_0 and α such that, if*

$$d_{\mathcal{H}}(\overline{\Omega_1}, \overline{\Omega_2}) \leq d_0, \tag{2.35}$$

then there exists $\tilde{M}_0 > 0$ only depending on n, M_0 and α such that every connected component of $\Omega_1 \cap \Omega_2$ has boundary of Lipschitz class with constants $\tilde{\rho}_0$, $\tilde{M}_0$.

We postpone the proofs of Propositions 2.4, 2.5, 2.6 and 2.7 until later. The proof of Proposition 2.8 is purely geometrical and can be found in [4].

Proof of Theorem 2.1. Let us call

$$d = d_{\mathcal{H}}(\partial D_1, \partial D_2).$$

Let η be the quantity on the right hand side of (2.31) and (2.32), so that

$$\int_{D_2 \setminus D_1} |\nabla u_1|^2 \leq \eta,$$
$$\int_{D_1 \setminus D_2} |\nabla u_2|^2 \leq \eta.$$

We can assume without loss of generality that there exists a point $x_1 \in \partial D_1$ such that $\mathrm{dist}(x_1, \partial D_2) = d$. That being the case, we distinguish two possible situations:
(i) $B_d(x_1) \subset D_2$,
(ii) $B_d(x_1) \cap D_2 = \emptyset$.
In case (i), by the regularity assumptions on ∂D_1, we find a point $x_2 \in D_2 \setminus D_1$ such that $B_{td}(x_2) \subset D_2 \setminus D_1$, where t is small enough ($t = \frac{1}{1+\sqrt{1+M_0^2}}$ suffices). Using (2.30), with $\rho = \frac{td}{s}$ we have

$$\int_{B_\rho(x_2)} |\nabla u_1|^2 dx \geq \frac{C\rho_0^{n-2}}{\exp\left[A\left(\frac{s\rho_0}{td}\right)^B\right]} \|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}. \tag{2.36}$$

By Proposition 2.6, we have:

$$\omega\left(\frac{\epsilon}{\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}}\right) \geq \frac{C}{\exp\left[A\left(\frac{s\rho_0}{td}\right)^B\right]}, \tag{2.37}$$

and solving for d we obtain an estimate of log-log-log type stability:

$$d \leq C\rho_0\left\{\log\left[\log\left|\log\frac{\epsilon}{\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}}\right|\right]\right\}^{-\frac{1}{B}}, \tag{2.38}$$

provided $\epsilon < e^{-e}\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}$: this is not restrictive since, for larger values of ϵ, the thesis is trivial. If we call d_0 the right hand side of (2.38), we have that there exists ϵ_0 only depending on n, M_0, M_1 and F such that, if $\epsilon \leq \epsilon_0$ then $d \leq d_0$. Proposition 2.8 then applies, so that G satisfies the hypotheses of Proposition 2.7. This means that we may choose ω of the form (2.34) in (2.37), obtaining (2.31). Case (ii) can be treated analogously, upon substituting u_1 with u_2. □

2.4 Proof of Proposition 2.4

The main idea of the proof of Proposition 2.4 is a repeated application of a three-spheres type inequality. Inequalities as such play a crucial role in almost all stability estimates from Cauchy data, thus they have been adapted to a variety of elliptic PDEs: in the context of the scalar elliptic equations (see [4]), then in the determination of cavities or inclusions in elastic bodies ([59], [58]) and more in general, for scalar elliptic equations ([8]) as well as systems ([53]) with suitably smooth coefficients. We recall in particular the following estimate, which is a special case of a result of Nagayasu, Lin and Wang ([53]), dealing with systems of differential inequalities of the form:

$$|\triangle^l u^i| \leq K_0 \sum_{|\alpha|\leq\left[\frac{3l}{2}\right]} |D^\alpha u| \quad i = 1,\dots,n. \tag{2.39}$$

Then the following holds (see [53]):

Theorem 2.9 (Three spheres inequality.)**.** *Let $E \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary with constants ρ_0, M_0. Let $B_R(x)$ a ball contained in E, and let $u \in \mathbf{H}^{2l}(E)$ be a solution to (2.39). Then there exists a real number $\vartheta^* \in (0, e^{-1/2})$, depending only on n, l and K_0 such that, for all $0 < r_1 < r_2 < \vartheta^* r_3$ with $r_3 \leq R$ we have:*

$$\int_{B_{r_2}} |u|^2 dx \leq C\left(\int_{B_{r_1}} |u|^2 dx\right)^\delta \left(\int_{B_{r_3}} |u|^2 dx\right)^{1-\delta} \tag{2.40}$$

where $\delta \in (0,1)$ and $C > 0$ are constants depending only on n, l, K_0, $\frac{r_1}{r_3}$ and $\frac{r_2}{r_3}$, and the balls B_{r_i} are centered in x.

First, we show that Proposition 2.5 follows from Proposition 2.4:

Proof of Proposition 2.5. From Proposition 2.4 we know that

$$\int_{B_\rho(x)} |\nabla u_i|^2 dx \geq C_\rho \int_{\Omega\setminus\overline{D_i}} |\nabla u_i|^2 dx,$$

where C_ρ is given in (2.29). We have, using Poincaré inequality (2.22) and the trace theorem,

$$\int_{\Omega\setminus\overline{D_i}} |\nabla u_i|^2 dx \geq C\rho_0^{n-2}\|u_i\|^2_{\mathbf{H}^1(\Omega\setminus\overline{D_i})} \geq C\rho_0^{n-2}\|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}. \tag{2.41}$$

Applying the above estimate to (2.28) and using (2.15) will prove our statement. □

Next, we introduce a lemma we shall need later on:

Lemma 2.10. *Let the hypotheses of Proposition 2.4 be satisfied. Then*

$$\|u\|_{\mathbf{L}^2(E)} \geq \frac{C}{F^2}\rho_0\|\nabla u\|_{\mathbf{L}^2(E)} \tag{2.42}$$

where $C > 0$ only depends on n, M_0 and M_1.

The proof is obtained in [58], with minor modifications. We report it here for the sake of completeness.

Proof. Assume $\rho_0 = 1$, otherwise the thesis follows by scaling. The following trace inequality holds (see [33, Theorem 1.5.1.10]):

$$\|u\|_{\mathbf{L}^2(\partial E)} \leq C(\|\nabla u\|_{\mathbf{L}^2(E)}\|u\|_{\mathbf{L}^2(E)} + \|u\|^2_{\mathbf{L}^2(E)}), \tag{2.43}$$

where C only depends on M_0 and M_1. Using the Poincaré inequality (2.22), we have

$$\frac{\|\nabla u\|_{\mathbf{L}^2(E)}}{\|u\|_{\mathbf{L}^2(E)}} \leq C\frac{\|\nabla u\|^2_{\mathbf{L}^2(E)}}{\|u\|^2_{\mathbf{L}^2(\partial E)}}. \tag{2.44}$$

This, together with (2.21), immediately gives the thesis. □

A proof of Proposition 2.4 has already been obtained in [58] dealing with linearized elasticity; we outline it here with the due adaptations.

Proof of Proposition 2.4. We outline the main steps taken in the proof. First, we show that the three spheres inequality (2.40) applies to ∇u. Then, the goal is to estimate $\|\nabla u\|_{\mathbf{L}^2(E)}$ by covering the set E with a sequence of cubes Q_i with center q_i of "relatively small" size. Each of these cubes is contained in a sphere S_i, thus we estimate the norm of ∇u in every sphere of center q_i, by connecting q_i with x with a continuous arc, and apply an iteration of the three spheres inequality to estimate $\|\nabla u\|_{\mathbf{L}^2(S_i)}$ in terms of $\|\nabla u\|_{\mathbf{L}^2(B_\rho(x))}$. However, the estimates deteriorate exponentially as we increase the number of spheres (or equivalently, if the radius ρ is comparable with the distance of x from the boundary) giving an exponentially worse estimate of the constant C_ρ. To solve this problem, the idea is to distinguish two areas within $E_{s\rho}$, which we shall call A_1, A_2. We consider A_1 as the set of points $y \in E_{s\rho}$ such that $\operatorname{dist}(y, \partial E)$ is sufficiently large, whereas A_2 is given as the complement in $E_{s\rho}$ of A_1. Then, whenever we need to compare the norm of ∇u on two balls whose centers lie in A_2, we reduce the number of spheres by iterating the three spheres inequality over a sequence of balls with increasing radius, exploiting the Lipschitz character of ∂E by building a cone to which all the balls are internall tangent to. Once we have reached a sufficiently large distance from the boundary, we are able to pick a chain of larger balls, on which we can iterate the three speres inequality again without deteriorating the estimate too much. This line of reasoning allows us to estimate the norm of ∇u on any sphere contained in $E_{s\rho}$, thus the whole $\|\nabla u\|_{\mathbf{L}^2(E)}$.

Step 1. *If $u \in \mathbf{H}^1(E)$ solves (2.23) then the three spheres inequality (2.40) applies to ∇u.*

Proof of Step 1. We show that u can be written as a solution of a system of the form (2.39). By Theorem 2.2, we have $u \in \mathbf{H}^2(E)$ so that we may take the laplacian of the second equation in (2.1):

$$\triangle \operatorname{div} u = 0.$$

Commuting the differential operators, and recalling the first equation in (2.1),

$$\triangle p = 0$$

thus p is harmonic, which means that, if we take the laplacian of the first equation in (2.1) we get

$$\triangle^2 u = 0,$$

so that ∇u is also biharmonic, hence the thesis. □

We closely follow the geometric construction given in [58]. In the aforementioned work the object was to estimate $\|\widehat{\nabla} u\|$, by applying the three spheres inequality to $\widehat{\nabla} u = \frac{1}{2}(\nabla u + \nabla u^T)$ (the symmetrized gradient of u); in order to relate it to the boundary data, this step had to be combined with Korn and Caccioppoli type inequalities. Here the estimates are obtained for $\|\nabla u\|$.

From now on we will denote, for $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ such that $|\xi| = 1$, and $\vartheta > 0$,

$$C(z, \xi, \vartheta) = \left\{ x \in \mathbb{R}^n \text{ s.t. } \frac{(x - z) \cdot \xi}{|x - z|} > \cos \vartheta \right\} \tag{2.45}$$

the cone of vertex z, direction ξ and width 2ϑ.
Exploiting the Lipschitz character of ∂E, we can find $\vartheta_0 > 0$ depending only on M_0, $\vartheta_1 > 0$, $\chi > 1$ and $s > 1$ depending only on M_0 and n, such that the following holds (we refer to [58] for the explicit expressions of the constants ϑ_0, ϑ_1, χ, s, and for all the detailed geometric constructions).
Step 2. *Choose $0 < \vartheta^* \leq 1$ according to Theorem 2.9 .There exists $\overline{\rho} > 0$, only depending on M_0, M_1 and F, such that:*
If $0 < \rho \leq \bar{\rho}$, and $x \in E$ is such that $s\rho < \operatorname{dist}(x, \partial E) \leq \frac{\vartheta^*}{4}$, then there exists $\hat{x} \in E$ satisfying the following conditions:

(i) $B_{\frac{5\chi\rho}{\vartheta^*}}(x) \subset C(\hat{x}, e_n = \frac{x-\hat{x}}{|x-\hat{x}|}, \vartheta_0) \cap B_{\frac{\vartheta^*}{8}}(\hat{x}) \subset E,$

(ii) Let $x_2 = x + \rho(\chi+1)e_n$. Then the balls $B_\rho(x)$ and $B_{\chi\rho}(x_2)$ are internally tangent to the cone $C(\hat{x}, e_n, \vartheta_1)$.

The idea is now to repeat iteratively the construction made once in Step 2. We define the following sequence of points and radii:

$$\begin{aligned} &\rho_1 = \rho, \quad \rho_k = \chi\rho_{k-1}, \quad \text{for } k \geq 2, \\ &x_1 = x, \quad x_k = x_{k-1} + (\rho_{k-1} + \rho_k)e_n, \qquad \text{for } k \geq 2. \end{aligned}$$

We claim the following geometrical facts (the proof of which can be found again in [58], except the first, which is [8, Proposition 5.5]):

There exist $0 < h_0 < 1/4$ only depending on M_0, $\bar{\rho} > 0$ only depending on M_0, M_1 and F, an integer $k(\rho)$ depending also on M_0 and n, such that, for all $h \leq h_0$, $0 < \rho \leq \bar{\rho}$ and for all integers $1 < k \leq k(\rho) - 1$ we have:

1. *E_h is connected,*
2. *$B_{\rho_k}(x_k)$ is internally tangent to $C(\hat{x}, e_n, \vartheta_1)$,*
3. *$B_{\frac{5\chi\rho_k}{\vartheta^*}}(x_k)$ is internally tangent to $C(\hat{x}, e_n, \vartheta_0)$,*
4. *The following inclusion holds:*
$$B_{\frac{5\rho_k}{\vartheta^*}}(x_k) \subset B_{\frac{\vartheta^*\rho_0}{8}}(\hat{x}), \tag{2.46}$$
5. *$k(\rho)$ can be bounded from above as follows:*
$$k(\rho) \leq \log \frac{\vartheta^* h_0 \rho_0}{5\rho} + 1. \tag{2.47}$$

Call $\rho_{k(\rho)} = \chi^{k(\rho)-1}\rho$; from (2.47) we have that

$$\rho_{k(\rho)} \leq \frac{\vartheta^* h_0 \rho_0}{5}. \tag{2.48}$$

In what follows, in order to ease the notation, norms will be always understood as being $\mathbf{L}^2$ norms, so that $\|\cdot\|_U$ will stand for $\|\cdot\|_{\mathbf{L}^2(U)}$.

Step 3. *For all $0<\rho\le\bar\rho$ and for all $x\in E$ such that $s\rho\le \mathrm{dist}(x,\partial E)\le \frac{\vartheta^*\rho_0}{4}$, the following hold:*

$$\frac{\|\nabla u\|_{B_{\rho_{k(\rho)}}(x_{k(\rho)})}}{\|\nabla u\|_E}\le C\left(\frac{\|\nabla u\|_{B_\rho(x)}}{\|\nabla u\|_E}\right)^{\delta_\chi^{k(\rho)-1}}, \tag{2.49}$$

$$\frac{\|\nabla u\|_{B_\rho(x)}}{\|\nabla u\|_E}\le C\left(\frac{\|\nabla u\|_{B_{\rho_{k(\rho)}}(x_{\rho_{k(\rho)}})}}{\|\nabla u\|_E}\right)^{\delta^{k(\rho)-1}}, \tag{2.50}$$

where $C>0$ and $0<\delta_\chi<\delta<1$ only depend on M_0.

Proof of Step 3. We apply to ∇u the three-spheres inequality, with balls of center x_j and radii $r_1^j=\rho_j$, $r_2^j=3\chi\rho_j$, $r_3^j=4\chi\rho_j$, for all $j=1,\dots,k(\rho)-1$. Since $B_{r_1^{j+1}}(x_{j+1})\subset B_{r_2^j}(x_j)$, by the three spheres inequality, there exists C and δ_χ only depending on M_0, such that:

$$\|\nabla u\|_{B_{\rho_{j+1}}(x_{j+1})}\le C\Big(\|\nabla u\|_{B_{\rho_j}(x_j)}\Big)^{\delta_\chi}\Big(\|\nabla u\|_{B_{4\chi\rho_j}(x_j)}\Big)^{1-\delta_\chi}. \tag{2.51}$$

This, in turn, leads to:

$$\frac{\|\nabla u\|_{B_{\rho_{j+1}}(x_{j+1})}}{\|\nabla u\|_E}\le C\left(\frac{\|\nabla u\|_{B_{\rho_j}(x_j)}}{\|\nabla u\|_E}\right)^{\delta_\chi}, \tag{2.52}$$

for all $j=0,\dots k(\rho)-1$. Now call

$$m_k=\frac{\|\nabla u\|_{B_{\rho_{j+1}}(x_{j+1})}}{\|\nabla u\|_E}.$$

so that (2.52) reads

$$m_{k+1}\le Cm_k^{\delta_\chi}\,\|\nabla u\|_E^{1-\delta_\chi}, \tag{2.53}$$

which, inductively, leads to

$$m_N\le\tilde C m_0^\alpha, \tag{2.54}$$

where $\tilde C=C^{1+\delta_\chi+\dots+\delta_\chi^{k(\rho)-2}}$. Since $0<\delta_\chi<1$, we have $1+\delta_\chi+\dots+\delta_\chi^{k(\rho)-2}\le\frac{1}{1-\delta_\chi}$, and since we may take $C>1$,

$$\tilde C\le C^{\frac{1}{1-\delta_\chi}}. \tag{2.55}$$

Similarly, we obtain (2.50): we find a $0<\delta<1$ such that the three spheres inequality applies to the balls $B_{\rho_j}(x_j)$, $B_{3\rho_j}(x_j)$ $B_{4\rho_j}(x_j)$ for $j=2,\dots,k(\rho)$; observing that $B_{\rho_j(x_{j-1})}\subset B_{3\rho_j}(x_j)$, the line of reasoning followed above applies identically. $\square$

Step 4.
For all $0 < \rho \leq \overline{\rho}$, and for every $\bar{x} \in E_{s\rho}$ we have

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{A+B\log\frac{\rho_0}{\rho}}}. \tag{2.56}$$

Proof. We distinguish two subcases:

(i). $\bar{x}$ is such that $\mathrm{dist}(\bar{x}, \partial E) \leq \frac{\vartheta^*\rho_0}{4}$,

(ii). $\bar{x}$ is such that $\mathrm{dist}(\bar{x}, \partial E) > \frac{\vartheta^*\rho_0}{4}$.

Proof of Case (i). Let us consider δ, δ_χ we introduced in Step 3. Take any point $y \in E$ such that $s\rho < \mathrm{dist}(y, \partial E) \leq \frac{\vartheta^*\rho_0}{4}$. By construction, the set $E_{\frac{5\rho_{k(\rho)}}{\vartheta^*}}$ is connected, thus there exists a continuous path $\gamma : [0,1] \to E_{\frac{5\rho_{k(\rho)}}{\vartheta^*}}$ joining $\bar{x}_{k(\rho)}$ to $y_{k(\rho)}$. We define a ordered sequence of times t_j, and a corresponding sequence of points $x_j = \gamma(t_j)$, for $j = 1, \dots, L$ in the following way: $t_1 = 0$, $t_L = 1$, and

$$t_j = \max\{t \in (0,1] \text{ such that } |\gamma(t) - x_i| = 2\rho_{k(\rho)}\}\ , \text{ if } |x_i - y_{k(\rho)}| > 2\rho_{k(\rho)},$$

otherwise, let $k = L$ and the process is stopped. Now, all the balls $B_{\rho_{k(\rho)}}(x_i)$ are pairwise disjoint, the distance between centers is given by $|x_{j+1} - x_j| = 2\rho_{k(\rho)}$ for all $j = 1 \dots L-1$ and for the last point, $|x_L - y_{k(\rho)}| \leq 2\rho_{k(\rho)}$. The number of points, using (2.6), is at most

$$L \leq \frac{M_1\rho_0^n}{\omega_n\rho_{k(\rho)}^n}. \tag{2.57}$$

Iterating the three spheres inequality over this chain of balls, we obtain

$$\frac{\|\nabla u\|_{B_{\rho_{k(\rho)}}(y_{k(\rho)})}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_{\rho_{k(\rho)}}(\bar{x}_{k(\rho)})}}{\|\nabla u\|_E}\right)^{\delta^L} \tag{2.58}$$

On the other hand, by the previous step we have, applying (2.49) and (2.50) for $x = \bar{x}$ and $x = y$ respectively,

$$\frac{\|\nabla u\|_{B_{\rho_{k(\rho)}}(\bar{x}_{k(\rho)})}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{k(\rho)-1}}, \tag{2.59}$$

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_{\rho_{k(\rho)}}(y_{k(\rho)})}}{\|\nabla u\|_E}\right)^{\delta^{k(\rho)-1}}, \tag{2.60}$$

where C, as before, only depends on n and M_0. Combining (2.58), (2.59) and (2.60), we have

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \le C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{k(\rho)-1}\delta^{k(\rho)+L-1}}, \tag{2.61}$$

for every $y \in E_{s\rho}$ satisfying $\mathrm{dist}(y, \partial E) \le \frac{\vartheta^* \rho_0}{4}$. Now consider $y \in E$ such that $\mathrm{dist}(y, \partial E) > \frac{\vartheta^* \rho_0}{4}$. Call

$$\tilde{r} = \vartheta^* \rho_{k(\rho)}. \tag{2.62}$$

By construction (2.48) and (2.46) we have

$$\mathrm{dist}(\bar{x}_{k(\rho)}, \partial E) \ge \frac{5\rho_{k(\rho)}}{\vartheta^*} > \frac{5}{\vartheta^*}\tilde{r}, \tag{2.63}$$

$$\mathrm{dist}(y, \partial E) \ge \frac{5\rho_{k(\rho)}}{\vartheta^*} > \frac{5}{\vartheta^*}\tilde{r}, \tag{2.64}$$

and again $E_{\frac{5}{\vartheta^*}\tilde{r}}$ is connected, since $\tilde{r} < \rho_{k(\rho)}$. We are then allowed to join $\bar{x}_{k(\rho)}$ to y with a continuous arc, and copy the argument seen before over a chain of at most $\tilde{L}$ balls of centers $x_j \in E_{\frac{5}{\vartheta^*}\tilde{r}}$ and radii $\tilde{r}$, $3\tilde{r}$, $4\tilde{r}$, where

$$\tilde{L} \le \frac{M_1}{\omega_n \tilde{r}^n}. \tag{2.65}$$

Up to possibly shrinking $\overline{\rho}$, we may suppose $\rho \le \tilde{r}$; iterating the three spheres inequality as we did before, we get

$$\frac{\|\nabla u\|_{B_{\tilde{r}}(y)}}{\|\nabla u\|_E} \le C\left(\frac{\|\nabla u\|_{B_{\tilde{r}}(\bar{x}_{k(\rho)})}}{\|\nabla u\|_E}\right)^{\delta^{\tilde{L}}}, \tag{2.66}$$

which, in turn, by (2.59) and since $\rho \le \tilde{r} < \rho_{k(\rho)}$, becomes

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \le C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{k(\rho)-1}\delta^{\tilde{L}}}, \tag{2.67}$$

with C depending only on M_0 and n. The estimate (2.67) holds for all $y \in E$ such that $\mathrm{dist}(y, \partial E) > \frac{\vartheta^*}{4}$. We now put (2.47), (2.67), (2.61), (2.57) (2.65) together, by also observing that $\delta_\chi \le \delta$ and trivially $\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \le 1$, we obtain precisely (2.56), for $\rho \le \overline{\rho}$, where $C > 1$ and $B > 0$ only depend on M_0, while $A > 0$ only depend on M_0 and M_1.

Proof of case (ii). We use the same constants δ and δ_χ introduced in Step 3. Take $\rho \le \bar{\rho}$, then $B_{s\rho}(\bar{x}) \subset B_{\frac{\vartheta^*\rho_0}{16}}(\bar{x})$, and for any point $\tilde{x}$ such that

$|\bar{x}-\tilde{x}| = s\rho$, we have $B_{\frac{\vartheta^*\rho_0}{8}}(\tilde{x}) \subset E$. Following the construction made in Steps 2 and 3, we choose a point $\bar{x}_{k(\rho)} \in E_{\frac{5}{\vartheta^*}\rho_{k(\rho)}}$, such that

$$\frac{\|\nabla u\|_{B_{\rho_{k(\rho)}}(\bar{x}_{k(\rho)})}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{k(\rho)-1}}, \tag{2.68}$$

with $C > 1$ only depending on n, M_0. If $y \in E$ is such that $s\rho < \mathrm{dist}(y, \partial E) \leq \frac{\vartheta^*\rho_0}{4}$, then, by the same reasoning as in Step 4.(i), we obtain

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{k(\rho)-1}\delta^{k(\rho)+L-1}}, \tag{2.69}$$

with $C > 1$ again depending only on M_0. If, on the other hand, $y \in E$ is such that $\mathrm{dist}(y, \partial E) \geq \frac{\vartheta^*\rho_0}{4}$, taking $\tilde{r}$ as in (2.62), using the same argument as in Step 4.(i), we obtain

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{k(\rho)-1}\delta^{\tilde{L}}}, \tag{2.70}$$

where again $C > 1$ only depends on M_0. From (2.69),(2.70), (2.57),(2.65) and (2.47), and recalling that, again, $\delta_\chi \leq \delta$, and $\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq 1$, we obtain

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{A+B\log\frac{\rho_0}{\rho}}}, \tag{2.71}$$

where $C > 1$ and $B > 0$ only depend on M_0, while $A > 0$ only depends on M_0, M_1. $\square$

Step 5. *For every $\rho \leq \bar{\rho}$ and for every $\bar{x} \in E_{s\rho}$ the thesis (2.28) holds.*

Proof of Step 5. Suppose at first that $\bar{x} \in E_{s\rho}$ satisfies $\mathrm{dist}(\bar{x}, \partial E) \leq \frac{\vartheta^*\rho_0}{4}$. We cover $E_{(s+1)\rho}$ with a sequence of non-overlapping cubes of side $l = \frac{2\rho}{\sqrt{n}}$, so that every cube is contained in a ball of radius ρ and center in $E_{s\rho}$. The number of cubes is bounded by

$$N = \frac{|\Omega| n^{\frac{n}{2}}}{(2\rho)^n} \leq \frac{M_1 n^{\frac{n}{2}}\rho_0^n}{(2\rho)^n}.$$

If we then sum over $k = 0$ to N in (2.56) we can write:

$$\frac{\|\nabla u\|_{E_{(s+1)\rho}}}{\|\nabla u\|_E} \leq C\left(\frac{\rho}{\rho_0}\right)^{-\frac{n}{2}}\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{A+B\log\frac{\rho_0}{\rho}}}. \tag{2.72}$$

Here C depends only on M_0 and M_1. Now, we need to estimate the left hand side in (2.72). In order to do so, we start by writing

$$\frac{\|\nabla u\|_{E_{(s+1)\rho}}}{\|\nabla u\|_E} = 1 - \frac{\|\nabla u\|_{E\setminus E_{(s+1)\rho}}}{\|\nabla u\|_E}. \tag{2.73}$$

By Lemma 2.10 and the Hölder inequality,

$$\rho_0^2\|\nabla u\|^2_{E\setminus E_{(s+1)\rho}} \leq CF^2\|u\|^2_{E\setminus E_{(s+1)\rho}} \leq CF^2|E\setminus E_{(s+1)\rho}|^{\frac{1}{n}}\|u\|^2_{\mathbf{L}^{\frac{2n}{n-1}}(E\setminus E_{(s+1)\rho})}. \tag{2.74}$$

On the other hand, by the Sobolev and the Poincaré inequalities:

$$\|u\|_{\mathbf{L}^{\frac{2n}{n-1}}(E)} \leq C\|u\|_{\mathbf{H}^{\frac{1}{2}}(E)} \leq C\|u\|_E \leq C\rho_0\|\nabla u\|_E. \tag{2.75}$$

It can be proven (see [8, Lemma 5.7]) that

$$|E \setminus E_{(s+1)\rho}| \leq C\rho, \tag{2.76}$$

where C depends on M_0, M_1 and n. We thus obtain that

$$\frac{\|\nabla u\|_{E\setminus E_{(s+1)\rho}}}{\|\nabla u\|_E} \leq CF^2|E \setminus E_{(s+1)\rho}|^{\frac{1}{n}}. \tag{2.77}$$

Therefore, combining (2.77) and (2.76), we have that for sufficiently small ρ,

$$\frac{\|\nabla u\|_{E_{(s+1)\rho}}}{\|\nabla u\|_E} \leq \frac{1}{2}, \tag{2.78}$$

thus (2.72) becomes

$$\int_{B_\rho(\bar{x})} |\nabla u|^2 \geq C\left(\frac{\rho}{\rho_0}\right)^{n\delta_\chi^{-A-B\log\frac{\rho_0}{\rho}}} \int_E |\nabla u|^2.$$

Since for all $t > 0$ we have $|\log t| \leq \frac{1}{t}$, it is immediate to verify that (2.28) holds. Now take $\bar{x} \in E_{s\rho}$ such that $\mathrm{dist}(\bar{x}, \partial E) > \frac{\vartheta^*\rho_0}{4}$. Then $B_{s\rho}(\bar{x}) \subset B_{\frac{\vartheta^*\rho_0}{16}}(\bar{x})$, then for any point $\tilde{x}$ such that $|\bar{x} - \tilde{x}| = s\rho$, we have $B_{\frac{\vartheta^*\rho_0}{8}}(\tilde{x}) \subset E$. Following the construction made in Steps 2 and 3, we choose a point $\bar{x}_{k(\rho)} \in E_{\frac{5}{\vartheta^*}\rho_{k(\rho)}}$, such that

$$\frac{\|\nabla u\|_{B_{\rho_{k(\rho)}}(\bar{x}_{k(\rho)})}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{k(\rho)-1}}, \tag{2.79}$$

with $C > 1$ only depends on n, M_0.
If $y \in E$ is such that $s\rho < \mathrm{dist}(y, \partial E) \leq \frac{\vartheta^*\rho_0}{4}$, then, by the same reasoning as in Step 4, we obtain

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq C\left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E}\right)^{\delta_\chi^{k(\rho)-1}\delta^{k(\rho)+L-1}}, \tag{2.80}$$

with $C > 1$ again depending only on n and M_0. If, on the other hand, $y \in E$ is such that $\mathrm{dist}(y, \partial E) \geq \frac{\vartheta^* \rho_0}{4}$, taking $\tilde{r}$ as in (2.62), using the same argument as in Step 4, we obtain

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq C \left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E} \right)^{\delta_\chi^{k(\rho)-1} \delta^{\bar{L}}}, \tag{2.81}$$

where again $C > 1$ only depends on n and M_0. From (2.80),(2.81), (2.57),(2.65) and (2.47), and recalling that, again, $\delta_\chi \leq \delta$, and $\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq 1$, we obtain

$$\frac{\|\nabla u\|_{B_\rho(y)}}{\|\nabla u\|_E} \leq C \left(\frac{\|\nabla u\|_{B_\rho(\bar{x})}}{\|\nabla u\|_E} \right)^{\delta_\chi^{A + B \log \frac{\rho_0}{\rho}}}, \tag{2.82}$$

where $C > 1$ and $B > 0$ only depend on n and M_0, while $A > 0$ only depends on n, M_0, M_1. The thesis follows from the same cube covering argument as in Step 4. $\square$

Conclusion. So far, we have proven (2.28) true for every $\rho \leq \bar{\rho}$, and for every $\bar{x} \in E_{s\rho}$, where $\bar{\rho}$ only depends on M_0, M_1 and F. If $\rho > \bar{\rho}$ and $\bar{x} \in E_{s\rho} \subset E_{s\bar{\rho}}$, then, using what we have shown so far,

$$\|\nabla u\|_{B_\rho(\bar{x})} \geq \|\nabla u\|_{B_{\bar{\rho}}(\bar{x})} \geq \tilde{C} \|\nabla u\|_E, \tag{2.83}$$

where $\tilde{C}$ again only depends on n, M_0, M_1 and F. On the other hand, by the regularity hypotheses on E, it is easy to show that

$$\frac{\rho}{\rho_0} \leq \frac{\mathrm{diam}(E)}{2s} \leq \frac{C^*}{2s} \tag{2.84}$$

thus the thesis

$$\int_{B_\rho(\bar{x})} |\nabla u|^2 \geq \frac{C}{\exp\left[A\left(\frac{\rho_0}{\rho}\right)^B\right]} \int_E |\nabla u|^2$$

is trivial, if we set

$$C = \tilde{C} \exp\left[A\left(\frac{2s}{C^*}\right)^B\right].$$

$\square$

2.5 Stability of continuation from Cauchy data

Throughout this section, we shall again distinguish two domains $\Omega_i = \Omega \setminus \overline{D_i}$ for $i = 1, 2$, where D_i are two subset of Ω satisfying (2.8) to (2.11). We start by putting up some notation. In the following, we shall call

$$U_\rho^i = \{x \in \overline{\Omega_i} \text{ s.t.} \mathrm{dist}(x, \partial\Omega) \leq \rho\}.$$

The following are well known results of interior regularity for the bilaplacian (see, for example, [56], [32]):

Lemma 2.11 (Interior regularity of solutions). *Let u_i be the weak solution to 2.1 in Ω_i. Then for all $0 < \alpha < 1$ we have that $u_i \in C^{1,\alpha}(\overline{\Omega_i \setminus U^i_{\frac{\rho_0}{8}}})$ and*

$$\|u_i\|_{C^{1,\alpha}(\overline{\Omega_i \setminus U^i_{\frac{\rho_0}{8}}})} \leq C\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \tag{2.85}$$

$$\|u_1 - u_2\|_{C^{1,\alpha}(\overline{\Omega_1 \cap \Omega_2})} \leq C\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \tag{2.86}$$

where $C > 0$ only depends on α, M_0.

Proof. Using standard energy estimates, as in Theorem 2.2, it follows that

$$\|u_i\|_{\mathbf{H}^1(\Omega_i)} \leq C\|g\|_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}. \tag{2.87}$$

On the other hand, using interior regularity estimates for biharmonic functions, we have

$$\|u_i\|_{C^{1,\alpha}(\overline{\Omega_i \setminus U^i_{\frac{\rho_0}{8}}})} \leq C\|u_i\|_{\mathbf{L}^\infty(\overline{\Omega_i \setminus U^i_{\frac{\rho_0}{16}}})} \leq \|u_i\|_{\mathbf{L}^2(\Omega_i)}, \tag{2.88}$$

where $C > 0$ only depends on α and M_0. Combining (2.87), (2.88), and recalling (2.15), immediately leads to (2.85). As for (2.86), we observe that $u_1 - u_2 = 0$ on Γ (actually, on $\partial\Omega$); therefore, the $C^{1,\alpha}$ norm of $u_1 - u_2$ in $U^1_{\frac{\rho_0}{2}} \cap U^2_{\frac{\rho_0}{2}}$ can be estimated in the same fashion; using (2.85) in the remaining part, we get (2.86). $\square$

We will also need the following lemma, proved in [4]:

Lemma 2.12 (Regularized domains). *Let Ω be a domain satisfying (2.5) and (2.6), and let D_i, for $i = 1, 2$ be two connected open subsets of Ω satisfying (2.10), (2.11). Then there exist a family of regularized domains $D^h_i \subset \Omega$, for $0 < h < a\rho_0$, with C^1 boundary of constants $\widetilde{\rho}_0$, $\widetilde{M}_0$ and such that*

$$D_i \subset D^{h_1}_i \subset D^{h_2}_i \quad \text{if } 0 < h_1 \leq h_2; \tag{2.89}$$

$$\gamma_0 h \leq \mathrm{dist}(x, \partial D_i) \leq \gamma_1 h \quad \text{for all } x \in \partial D^h_i; \tag{2.90}$$

$$\mathrm{meas}(D^h_i \setminus D_i) \leq \gamma_2 M_1 \rho_0^2 h; \tag{2.91}$$

$$\mathrm{meas}_{n-1}(\partial D^h_i) \leq \gamma_3 M_1 \rho_0^2; \tag{2.92}$$

and for every $x \in \partial D^h_i$ there exists $y \in \partial D_i$ such that

$$|y - x| = \mathrm{dist}(x, \partial D_i), \quad |\nu(x) - \nu(y)| \leq \gamma_4 \frac{h^\alpha}{\rho_0^\alpha}; \tag{2.93}$$

where by $\nu(x)$ we mean the outer unit normal to ∂D^h_i, $\nu(y)$ is the outer unit normal to D_i, and the constants a, γ_j, $j = 0 \ldots 4$ and the ratios $\frac{\widetilde{M}_0}{M_0}$, $\frac{\widetilde{\rho}_0}{\rho_0}$ only depend on M_0 and α.

We shall also need a stability estimate for the Cauchy problem associated with the Stokes system with homogeneous Cauchy data. The proof of the following result, which will be given in the next section, basically revolves around an extension argument. Let us consider a bounded domain $E \subset \mathbb{R}^n$ satisfying hypotheses (2.5) and (2.6), and take $\Gamma \subset \partial E$ a connected open portion of the boundary of class $C^{2,\alpha}$ with constants ρ_0, M_0. Let $P_0 \in \Gamma$ such that (2.7) holds. By definition, after a suitable change of coordinates we have that $P_0 = 0$ and

$$E \cap B_{\rho_0}(0) = \{(x', x_n) \in E \text{ s.t.} \, x_n > \varphi(x')\} \subset E, \tag{2.94}$$

where φ is a $C^{2,\alpha}(B'_{\rho_0}(0))$ function satisfying

$$\begin{aligned} \varphi(0) &= 0, \\ |\nabla\varphi(0)| &= 0, \\ \|\varphi\|_{C^{2,\alpha}(B'_{\rho_0}(0))} &\leq M_0\rho_0. \end{aligned}$$

Define

$$\begin{aligned} \rho_{00} &= \frac{\rho_0}{\sqrt{1+M_0^2}}, \\ \Gamma_0 &= \{(x', x_n) \in \Gamma \text{ s.t. } |x'| \leq \rho_{00}, \ x_n = \varphi(x')\}. \end{aligned} \tag{2.95}$$

Theorem 2.13. *Under the above hypotheses, let (u, p) be a solution to the problem:*

$$\begin{cases} \operatorname{div} \sigma(u,p) &= 0 \quad \text{in } E, \\ \operatorname{div} u &= 0 \quad \text{in } E, \\ u &= 0 \quad \text{on } \Gamma, \\ \sigma(u,p)\cdot\nu &= \psi \quad \text{on } \Gamma, \end{cases} \tag{2.96}$$

where $\psi \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$. Let $P^* = P_0 + \frac{\rho_{00}}{4}\nu$ where ν is the outer normal field to $\partial\Omega$. Then we have

$$\|u\|_{\mathbf{L}^\infty(E\cap B_{\frac{3\rho_{00}}{8}}(P^*))} \leq \frac{C}{\rho_0^{\frac{n}{2}}} \|u\|^{1-\tau}_{\mathbf{L}^2(E)} (\rho_0\|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)})^\tau, \tag{2.97}$$

where $C > 0$ and τ only depend on α and M_0.

Proof of Proposition 2.6. Let $\theta = \min\{a, \frac{7}{8\gamma_1}\frac{1}{2\gamma_0(1+M_0^2)}\}$ where a, γ_0, γ_1 are the constants depending only on M_0 and α introduced in Lemma 2.12, then let $\overline{\rho} = \theta\rho_0$ and fix $\rho \leq \overline{\rho}$. We introduce the regularized domains D_1^ρ, D_2^ρ according to Lemma 2.12.

Let G be the connected component of $\Omega \setminus (\overline{D_1 \cup D_2})$ which contains $\partial\Omega$, and G^ρ be the connected component of $\overline{\Omega} \setminus (D_1^\rho \cup D_2^\rho)$ which contains $\partial\Omega$. We have that

$$D_2 \setminus \overline{D_1} \subset \Omega_1 \setminus \overline{G} \subset \big((D_1^\rho \setminus \overline{D_1}) \setminus \overline{G}\big) \cup \big((\Omega \setminus G^\rho) \setminus D_1^\rho\big)$$

and

$$\partial\big((\Omega \setminus G^\rho) \setminus D_1^\rho\big) = \Gamma_1^\rho \cup \Gamma_2^\rho,$$

where $\Gamma_2^\rho = \partial D_2^\rho \cap \partial G^\rho$ and $\Gamma_1^\rho \subset \partial D_1^\rho$. It is thus clear that

$$\int_{D_2\setminus\overline{D_1}} |\nabla u_1|^2 \le \int_{\Omega_1\setminus\overline{G}} |\nabla u_1|^2 \le \int_{(D_1^\rho\setminus\overline{D_1})\setminus\overline{G}} |\nabla u_1|^2 + \int_{(\Omega\setminus G^\rho)\setminus D_1^\rho} |\nabla u_1|^2. \tag{2.98}$$

The first summand is easily estimated, for using (2.85) and (2.91) we have

$$\int_{(D_1^\rho\setminus\overline{D_1})\setminus\overline{G}} |\nabla u_1|^2 \le C\rho_0^{n-2}\|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\frac{\rho}{\rho_0} \tag{2.99}$$

where C only depends on the M_0, M_1 and α. We call $\Omega(\rho) = (\Omega \setminus G^\rho) \setminus D_1^\rho$. The second term in (2.98), using the divergence theorem twice, becomes:

$$\begin{aligned}
&\int_{\Omega(\rho)} |\nabla u_1|^2 = \int_{\partial\Omega(\rho)} (\nabla u_1 \cdot \nu)u_1 - \int_{\Omega(\rho)} \triangle u_1 \cdot u_1 = \\
&\int_{\partial\Omega(\rho)} (\nabla u_1 \cdot \nu)u_1 - \int_{\Omega(\rho)} \nabla p_1 \cdot u_1 = \int_{\partial\Omega(\rho)} (\nabla u_1 \cdot \nu)u_1 + \int_{\partial\Omega(\rho)} p_1(u_1\cdot\nu) = \\
&\int_{\Gamma_1^\rho} (\nabla u_1 \cdot \nu)u_1 + \int_{\Gamma_2^\rho} (\nabla u_1 \cdot \nu)u_1 + \int_{\Gamma_1^\rho} p_1(u_1\cdot\nu) + \int_{\Gamma_2^\rho} p_1(u_1\cdot\nu).
\end{aligned} \tag{2.100}$$

About the first and third term, if $x \in \Gamma_1^\rho$, using Lemma 2.12, we find $y \in \partial D_1$ such that $|y - x| = d(x, \partial D_1) \le \gamma_1\rho$; since $u_1(y) = 0$, by Lemma 2.11 we have

$$|u_1(x)| = |u_1(x) - u_1(y)| \le C\frac{\rho}{\rho_0}\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}. \tag{2.101}$$

On the other hand, if $x \in \Gamma_2^\rho$, there exists $y \in D_2$ such that $|y - x| = d(x, \partial D_2) \le \gamma_1\rho$. Again, since $u_2(y) = 0$, we have

$$\begin{aligned}
|u_1(x)| &\le |u_1(x) - u_1(y)| + |u_1(y) - u_2(y)| \\
&\le C\big(\frac{\rho}{\rho_0}\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} + \max_{\partial G^\rho\setminus\partial\Omega}|w|\big),
\end{aligned} \tag{2.102}$$

where $w = u_1 - u_2$. Combining (2.101), (2.102) and (2.100) and recalling (2.85) and (2.92) we have:

$$\int_{D_2\setminus D_1} |\nabla u_1|^2 \le C\rho_0^{n-2}\Big(\|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\frac{\rho}{\rho_0} + \|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\max_{\partial G^\rho\setminus\partial\Omega}|w|\Big) \tag{2.103}$$

We now need to estimate $\max_{\partial G^\rho\setminus\partial\Omega}|w|$. We may apply (2.40) to w, since it is biharmonic. Let $x \in \partial G^\rho \setminus \partial\Omega$ and

$$\rho^* = \frac{\rho_0}{16(1+M_0^2)}, \tag{2.104}$$

$$x_0 = P_0 - \frac{\rho_{00}}{16}\nu, \tag{2.105}$$

where ν is the outer normal to $\partial\Omega$ at the point P_0. By construction $x_0 \in \overline{\widetilde{\Omega}_{\frac{\rho^*}{2}}}$. There exists an arc $\gamma : [0,1] \mapsto G^\rho \cap \overline{\widetilde{\Omega}_{\frac{\rho^*}{2}}}$ such that $\gamma(0) = x_0$, $\gamma(1) = x$ and $\gamma([0,1]) \subset G^\rho \cap \overline{\widetilde{\Omega}_{\frac{\rho^*}{2}}}$. Let us define a sequence of points $\{x_i\}_{i=0\ldots S}$ as follows: $t_0 = 0$, and

$$t_i = \max\{t \in (0,1] \text{ such that } |\gamma(t) - x_i| = \frac{\gamma_0\rho\vartheta^*}{2}\} \text{ , if } |x_i - x| > \frac{\gamma_0\rho\vartheta^*}{2},$$

otherwise, let $i = S$ and the process is stopped. Here ϑ^* is the constant given in Theorem 2.9. All the balls $B_{\frac{\gamma_0\rho\vartheta^*}{4}}(x_i)$ are pairwise disjoint, the distance between centers $|x_{i+1} - x_i| = \frac{\gamma_0\rho\vartheta^*}{2}$ for all $i = 1\ldots S-1$ and for the last point, $|x_S - x| \le \frac{\gamma_0\rho\vartheta^*}{2}$. The number of spheres is bounded by

$$S \le C\Big(\frac{\rho_0}{\rho}\Big)^n$$

where C only depends on α, M_0 and M_1. For every $\rho \le \bar{\rho}$, we have that, letting

$$\rho_1 = \frac{\gamma_0\rho\vartheta^*}{4},\ \rho_2 = \frac{3\gamma_0\rho\vartheta^*}{4},\ \rho_3 = \gamma_0\rho\vartheta^*$$

an iteration of the three spheres inequality on a chain of spheres leads to

$$\int_{B_{\rho_2}(x)} |w|^2 dx \le C\Big(\int_G |w|^2 dx\Big)^{1-\delta^S}\Big(\int_{B_{\rho_3}(x_0)} |w|^2 dx\Big)^{\delta^S} \tag{2.106}$$

where $0 < \delta < 1$ and $C > 0$ only depend on M_0 and α. From our choice of $\bar{\rho}$ and ϑ^*, it follows that $B_{\frac{\gamma_0\rho\vartheta^*}{4}}(x_0) \subset B_{\rho^*}(x_0) \subset G \cap B_{\frac{3\rho_1}{4}}(P^*)$, where we follow the notations from Theorem 2.13. Let us call

$$\tilde{\epsilon} = \frac{\epsilon}{\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}}. \tag{2.107}$$

Using (2.97), (2.87) and (2.16) on (2.106) and applying Theorem 2.13 we then have:

$$\int_{B_{\rho_2}(x)} |w|^2 dx \le C\rho_0^{n-2}\|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\tilde{\epsilon}^{2\tau\delta^S}. \tag{2.108}$$

The following interpolation inequality holds for all functions v defined on the ball $B_t(x) \subset \mathbb{R}^n$:

$$\|v\|_{\mathbf{L}^\infty(B_t(x))} \le C\Big(\Big(\int_{B_t(x)} |v|^2\Big)^{\frac{1}{n+2}} |\nabla v|^{\frac{n}{n+2}}_{\mathbf{L}^\infty(B_t(x))} + \frac{1}{t^{n/2}}\Big(\int_{B_t(x)} |v|^2\Big)^{\frac{1}{2}}\Big) \tag{2.109}$$

We apply it to w in $B_{\rho_2}(x)$, using (2.108) and (2.85) we obtain

$$\|w\|_{\mathbf{L}^\infty(B_{\rho_2}(x))} \leq C\Big(\frac{\rho_0}{\rho}\Big)^{\frac{n}{2}}\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)}\widetilde{\epsilon}^{\gamma\delta^S}, \tag{2.110}$$

where $\gamma = \frac{2\tau}{n+2}$. Finally, from (2.110) and (2.103) we get:

$$\int_{D_2\setminus D_1}|\nabla u_1|^2 \leq C\rho_0^{n-2}\|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\Big(\frac{\rho}{\rho_0}+\Big(\frac{\rho_0}{\rho}\Big)^{\frac{n}{2}}\widetilde{\epsilon}^{\gamma\delta^S}\Big) \tag{2.111}$$

At this point we observe that it is sufficient to prove the thesis in a smaller interval $(0,\overline{\mu})$ with $\overline{\mu}\leq e^{-1}$, because for larger values the thesis is trivial. With this in mind we consider

$$\widetilde{\mu} = \exp\Big(-\frac{1}{\gamma}\exp\Big(\frac{2S\log\delta}{\theta^n}\Big)\Big)$$

and $\overline{\mu} = \min\{\widetilde{\mu}, \exp(-\gamma^2)\}$, and only consider $\widetilde{\epsilon}\leq\overline{\mu}$. Choose ρ depending upon $\tilde{\epsilon}$ of the form

$$\rho(\tilde{\epsilon}) = \rho_0\left(\frac{2S\log|\delta|}{\log|\log\widetilde{\epsilon}^{\gamma}|}\right)^{-\frac{1}{n}}.$$

We have that ρ is defined and increasing in the interval $(0,e^{-1})$, and by definition $\rho(\overline{\mu})\leq\rho(\widetilde{\mu}) = \theta\rho = \overline{\rho}$, we apply (2.111) to (2.98) with $\rho = \rho(\widetilde{\epsilon})$ to obtain

$$\int_{D_2\setminus D_1}|\nabla u_1|^2 \leq C\rho_0^{n-2}\|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\log|\log\widetilde{\epsilon}|^{\gamma}, \tag{2.112}$$

and since $\widetilde{\epsilon}\leq\exp(-\gamma^2)$ it is elementary to prove that

$$\log|\log\widetilde{\epsilon}^{\gamma}| \geq \frac{1}{2}\log|\log\widetilde{\epsilon}|,$$

so that (2.112) finally reads

$$\int_{D_2\setminus D_1}|\nabla u_1|^2 \leq C\rho_0^{n-2}\|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\,\omega(\widetilde{\epsilon}),$$

with $\omega(t) = \log|\log t|^{\frac{1}{n}}$ defined for all $0<t<e^{-1}$, and C depends on M_0, M_1 and α. $\square$

Proof of Proposition 2.7. We will prove the thesis for u_1, the case u_2 being completely analogous. First of all, we observe that

$$\int_{D_2\setminus D_1}|\nabla u_1|^2 \leq \int_{\Omega_1\setminus G}|\nabla u_1|^2 = \int_{\partial(\Omega_1\setminus G)}(\nabla u_1\cdot\nu)u_1 + \int_{\partial(\Omega_1\setminus G)}p_1(u_1\cdot\nu) \tag{2.113}$$

and that

$$\partial(\Omega_1\setminus G)\subset\partial D_1\cup(\partial D_2\cap\partial G)$$

and recalling the no-slip condition, applying to (2.113) computations similar to those in (2.98), (2.99), we have

$$\begin{aligned}\int_{D_2\setminus D_1}|\nabla u_1|^2 \leq \int_{\partial D_2\cap\partial G}(\nabla u_1\cdot\nu)w + \int_{\partial D_2\cap\partial G}p_1(w\cdot\nu)\leq\\ \leq C\rho_0^{n-2}\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\max_{\partial D_2\cap\partial G}|w|,\end{aligned}$$

where again $w = u_1 - u_2$ and C only depends on α, M_0 and M_1. Take a point $z \in \partial G$. By the regularity assumptions on ∂G, we find a direction $\xi \in \mathbb{R}^n$, with $|\xi| = 1$, such that the cone (recalling the notations used during the proof of Proposition 2.4) $C(z, \xi, \vartheta_0) \cap B_{\rho_0}(z) \subset G$, where $\vartheta_0 = \arctan\frac{\rho_0}{M_0}$. Again ([8, Proposition 5.5]) G_ρ is connected for $\rho \leq \frac{\rho_0 h_0}{3}$ with h_0 only depending on M_0. Now set

$$\begin{aligned}\lambda_1 &= \min\Big\{\frac{\tilde\rho_0}{1+\sin\vartheta_0}, \frac{\tilde\rho_0}{3\sin\vartheta_0}, \frac{\rho_0}{16(1+M_0^2)\sin\vartheta_0}\Big\},\\ \vartheta_1 &= \arcsin\Big(\frac{\sin\vartheta_0}{4}\Big),\\ w_1 &= z + \lambda_1\xi,\\ \rho_1 &= \vartheta^* h_0\lambda_1\sin\vartheta_1.\end{aligned}$$

where $0 < \vartheta^* \leq 1$ was introduced in Theorem 2.9. By construction, $B_{\rho_1}(w_1) \subset C(z,\xi,\vartheta_1) \cap B_{\tilde\rho_0}(z)$ and $B_{\frac{4\rho_1}{\vartheta^*}}(w_1) \subset C(z,\xi,\vartheta_0) \cap B_{\tilde\rho_0}(z) \subset G$. Furthermore $\frac{4\rho_1}{\vartheta^*} \leq \rho^*$, hence $B_{\frac{4\rho_1}{\vartheta^*}} \subset G$, where ρ^* and x_0 were defined by (2.104) and (2.105) respectively, during the previous proof. Therefore, w_1, $x_0 \in \overline{G_{\frac{4\rho_1}{\vartheta^*}}}$, which is connected by construction. Iterating the three spheres inequality (mimicking the construction made in the previous proof)

$$\int_{B_{\rho_1}(w_1)}|w|^2dx \leq C\Big(\int_G|w|^2dx\Big)^{1-\delta^S}\Big(\int_{B_{\rho_1}(x_0)}|w|^2dx\Big)^{\delta^S} \qquad (2.114)$$

where $0 < \delta < 1$ and $C \geq 1$ depend only on n, and $S \leq \frac{M_1\rho_0^n}{\omega_n\rho_1^n}$. Again, since $B_{\rho^*}(x_0) \subset G \cap B_{\frac{3}{8}\rho_1}(P_0)$, we apply Theorem 2.13 which leads to

$$\int_{B_{\rho_1}(w_1)}|w|^2 \leq C\rho_0^n\|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\tilde\epsilon^{2\beta}, \qquad (2.115)$$

where $0 < \beta < 1$ and $C \geq 1$ only depend on α, M_0, and $\frac{\tilde\rho_0}{\rho_0}$ and $\tilde\epsilon$ was defined in (2.107). So far the estimate we have is only on a ball centered in w_1, we need to approach $z \in \partial G$ using a sequence of balls, all contained in $C(z,\xi,\vartheta_1)$, by suitably shrinking their radii. Take

$$\chi = \frac{1-\sin\vartheta_1}{1+\sin\vartheta_1}$$

and define, for $k \geq 2$,

$$\begin{aligned} \lambda_k &= \chi \lambda_{k-1}, \\ \rho_k &= \chi \rho_{k-1}, \\ w_k &= z + \lambda_k \xi. \end{aligned}$$

With these choices, $\lambda_k = \lambda \chi^{k-1} \lambda_1$, $\rho_k = \chi^{k-1} \rho_1$ and $B_{\rho_{k+1}}(w_{k+1}) \subset B_{3\rho_k}(w_k)$, $B_{\frac{4}{\vartheta^*}\rho_k}(w_k) \subset C(z, \xi, \vartheta_0) \cap B_{\tilde{\rho}_0}(z) \subset G$. Denote by

$$d(k) = |w_k - z| - \rho_k,$$

we also have

$$d(k) = \chi^{k-1} d(1),$$

with

$$d(1) = \lambda_1 (1 - \vartheta^* \sin \vartheta_1).$$

Now take any $\rho \leq d(1)$ and let $k = k(\rho)$ the smallest integer such that $d(k) \leq \rho$, explicitly

$$\frac{\left| \log \frac{\rho}{d(1)} \right|}{\log \chi} \leq k(\rho) - 1 \leq \frac{\left| \log \frac{\rho}{d(1)} \right|}{\log \chi} + 1. \tag{2.116}$$

We iterate the three spheres inequality over the chain of balls centered in w_j and radii ρ_j, $3\rho_j$, $4\rho_j$, for $j = 1, \dots, k(\rho) - 1$, which yields

$$\int_{B_{\rho_{k(\rho)}}(w_{k(\rho)})} |w|^2 \leq C \|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \rho^n \tilde{\epsilon}^{2\beta\delta^{k(\rho)-1}}, \tag{2.117}$$

with C only depending on α, M_0 and $\frac{\tilde{\rho}_0}{\rho_0}$. Using the interpolation inequality (2.109) and (2.86) we obtain

$$\|w\|_{\mathbf{L}^\infty(B_{\rho_{k(\rho)}}(w_{k(\rho)}))} \leq C \|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \frac{\tilde{\epsilon}^{\beta_1 \delta^{k(\rho)-1}}}{\chi^{\frac{n}{2}(k(\rho)-1)}}, \tag{2.118}$$

where $\beta_1 = \frac{2\beta}{n+2}$ depends only on α, M_0, M_1 and $\frac{\tilde{\rho}_0}{\rho_0}$. From (2.118) and (2.86) we obtain

$$|w(z)| \leq C \|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \left(\frac{\rho}{\rho_0} + \frac{\tilde{\epsilon}^{\beta_1 \delta^{k(\rho)-1}}}{\chi^{\frac{n}{2}(k(\rho)-1)}} \right), \tag{2.119}$$

Finally, call

$$\rho(\tilde{\epsilon}) = d(1) |\log \tilde{\epsilon}^{\beta_1}|^{-B},$$

with

$$B = \frac{|\log \chi|}{2 \log |\delta|}.$$

and let $\tilde{\mu} = \exp(-\beta_1^{-1})$. We have that $\rho(\tilde{\epsilon})$ is monotone increasing in the interval $0 < \tilde{\epsilon} < \tilde{\mu}$, and $\rho(\tilde{\mu}) = d(1)$, so $\rho(\tilde{\epsilon}) \leq d(1)$ there. Putting $\rho = \rho(\tilde{\epsilon})$ into (2.119) we obtain

$$\int_{D_2 \setminus D_1} |\nabla u_1|^2 \leq C\rho_0^{n-2} \|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} |\log \tilde{\epsilon}|^{-B}, \tag{2.120}$$

where C only depends on α, M_0 and $\frac{\tilde{\rho}_0}{\rho_0}$. □

2.6 Proof of Theorem 2.13

As already premised, in order to prove Theorem 2.13, we will need to perform an extension argument on the solution to (2.1) we wish to estimate. This has been done for solutions to scalar elliptic equations with sufficiently smooth coefficients ([39]). Here, however, we are dealing with a system: extending u implies finding a suitable extension for the pressure p as well; moreover, both extensions should preserve some regularity they inherit from the original functions. Following the notations given for Theorem 2.13 we define

$$Q(P_0) = B'_{\rho_{00}}(0) \times \Big[-\frac{M_0 \rho_0^2}{\sqrt{1+M_0^2}}, \frac{M_0 \rho_0^2}{\sqrt{1+M_0^2}} \Big].$$

We have:

$$\Gamma_0 = \partial E \cap Q(P_0). \tag{2.121}$$

We then call $E^- = Q(P_0) \setminus E$ and $\widetilde{E} = E \cup E^- \cup \Gamma_0$.

Lemma 2.14 (Extension)**.** *Suppose the hypotheses of Theorem 2.13 hold. Consider the domains E^-, $\widetilde{E}$ as constructed above. Take, furthermore, $g \in \mathbf{H}^{\frac{5}{2}}(\partial E)$. Let (u, p) be the solution to the following problem:*

$$\begin{cases} \operatorname{div} \sigma(u,p) &= 0 \quad \text{in } E, \\ \operatorname{div} u &= 0 \quad \text{in } E, \\ u &= g \quad \text{on } \Gamma, \\ \sigma(u,p) \cdot \nu &= \psi \quad \text{on } \Gamma, \end{cases} \tag{2.122}$$

Then there exist functions $\tilde{u} \in \mathbf{H}^1(\widetilde{E})$, $\widetilde{p} \in L^2(\widetilde{E})$ and a functional $\Phi \in \mathbf{H}^{-1}(\widetilde{E})$ such that $\tilde{u} = u$, $\tilde{p} = p$ in E and $(\widetilde{u}, \widetilde{p})$ solve the following:

$$\begin{aligned} \triangle \widetilde{u} + \nabla \widetilde{p} = \Phi \quad \text{in } \widetilde{E}, \\ \operatorname{div} \widetilde{u} = 0 \quad \text{in } \widetilde{E}. \end{aligned} \tag{2.123}$$

If

$$\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} + \rho_0 \|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} = \eta,$$

then we have

$$\|\Phi\|_{\mathbf{H}^{-1}(\widetilde{E})} \leq C\frac{\eta}{\rho_0}. \tag{2.124}$$

where $C > 0$ only depends on α and M_0.

Proof. From the assumptions we made on the boundary data and the domain, it follows that $(u, p) \in \mathbf{H}^3(E) \times L^2(E)$. We can find (see [57] or [18]) a function $u^- \in \mathbf{H}^3(E^-)$ such that

$$\begin{aligned} \operatorname{div} u^- = 0 \quad \text{in} \quad E^-, \qquad u^- = g \quad \text{on} \quad \Gamma, \\ \|u^-\|_{\mathbf{H}^3(E^-)} \leq C\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}, \end{aligned} \tag{2.125}$$

with C only depending on $|E|$. We now call

$$F^- = \triangle u^-,$$

by our assumptions we have $F^- \in \mathbf{H}^1(E^-)$. Let $p^- \in H^1(E^-)$ be the weak solution to the following Dirichlet problem:

$$\begin{cases} \triangle p^- - \operatorname{div} F^- &= 0 \quad \text{in } E^-, \\ p^- &= 0 \quad \text{on } \partial E^-. \end{cases} \tag{2.126}$$

We now define

$$X^- = F^- - \nabla p^-. \tag{2.127}$$

This field is divergence free by construction, and its norm is controlled by

$$\|X^-\|_{\mathbf{L}^2(E^-)} \leq C\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \tag{2.128}$$

We thus extend (u, p) as follows:

$$\widetilde{u} = \begin{cases} u & \text{in } E, \\ u^- & \text{in } E^-, \end{cases}$$

$$\widetilde{p} = \begin{cases} p & \text{in } E, \\ p^- & \text{in } E^-. \end{cases}$$

Notice that by construction $\widetilde{u} \in \mathbf{H}^1(\widetilde{E})$. Now take any $v \in \mathbf{H}^1_0(\widetilde{E})$, we have

$$\begin{aligned} &\int_{\widetilde{E}} (\nabla\widetilde{u} + (\nabla\widetilde{u})^T - \widetilde{p}\,\mathbb{I}) \cdot \nabla v = \\ &= \int_E (\nabla u + (\nabla u)^T - p\,\mathbb{I}) \cdot \nabla v + \int_{E^-} (\nabla u^- + (\nabla u^-)^T - p^-\,\mathbb{I}) \cdot \nabla v. \end{aligned} \tag{2.129}$$

About the first term, using (2.1) and the divergence theorem we obtain

$$\int_E (\nabla u + (\nabla u)^T - p\,\mathbb{I}) \cdot \nabla v = \int_\Gamma \psi \cdot v. \tag{2.130}$$

Define $\Phi_1(v) = \int_\Gamma \psi \cdot v$ for all $v \in \mathbf{H}^1_0(\widetilde{E})$. Using the decomposition made in (2.127) on the second term, we have

$$\begin{aligned}
&\int_{E^-} (\nabla u^- + (\nabla u^-)^T - p^- \,\mathbb{I}) \cdot \nabla v = \\
&= \int_\Gamma (\nabla u^- + (\nabla u^-)^T - p^- \,\mathbb{I}) \cdot \nu\, v - \int_{E^-} \operatorname{div} \big(\nabla u^- + (\nabla u^-)^T - p^- \,\mathbb{I}\big) \cdot v = \\
&= \int_\Gamma (\nabla u^- + (\nabla u^-)^T) \cdot \nu\, v - \int_{E^-} (\triangle u^- - \nabla p^-) \cdot v = \\
&= \int_\Gamma (\nabla u^- + (\nabla u^-)^T) \cdot \nu\, v - \int_{E^-} X^- \cdot v = \Phi_2(v) + \Phi_3(v),
\end{aligned} \tag{2.131}$$

where we define for all $v \in \mathbf{H}^1_0(\widetilde{E})$ the functionals

$$\begin{aligned}
\Phi_2(v) &= \int_\Gamma (\nabla u^- + (\nabla u^-)^T) \cdot \nu\, v, \\
\Phi_3(v) &= -\int_{E^-} X^- \cdot v
\end{aligned}$$

We can estimate each of the linear functionals Φ_1, Φ_2 and Φ_3 easily, for we have (by (2.130) and the trace theorem):

$$\big|\Phi_1(v)\big| \leq \|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \|v\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq C\rho_0 \|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \|v\|_{\mathbf{H}^1(E^-)}, \tag{2.132}$$

moreover (using (2.131) and (2.125))

$$\big|\Phi_2(v)\big| \leq \|\nabla u\|_{\mathbf{L}^2(\Gamma)} \|v\|_{\mathbf{L}^2(\Gamma)} \leq C\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \|v\|_{\mathbf{H}^1(E^-)}, \tag{2.133}$$

and, at last, by (2.128),

$$\big|\Phi_3(v)\big| \leq \|X^-\|_{\mathbf{L}^2(E^-)} \|v\|_{\mathbf{L}^2(E^-)} \leq C\|g\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \|v\|_{\mathbf{H}^1(E^-)}. \tag{2.134}$$

Then, defining $\Phi(v) = \Phi_1(v) + \Phi_2(v) + \Phi_3(v)$ for all $v \in \mathbf{H}^1_0(\widetilde{E})$, putting together (2.130), (2.131), (2.132), (2.133) and (2.134), we have (2.124). $\square$

Proof of Theorem 2.13. Consider the domain $\widetilde{E}$ built at the beginning of this section, and take $\widetilde{u}$ the extension of u built according to Theorem 2.14. By linearity, we may write $\widetilde{u} = u_0 + w$ where (w, q) solves

$$\operatorname{div} \sigma(w, q) = \widetilde{\Phi} \qquad \text{in} \quad \widetilde{E}, \tag{2.135}$$

and $w \in \mathbf{H}^1(\widetilde{E})$, whereas $u_0 \in \mathbf{H}^1_0(\widetilde{E})$ and (u_0, p_0) solves

$$\begin{cases} \operatorname{div} \sigma(u_0, p_0) &= 0 \quad \text{in } \widetilde{E}, \\ \operatorname{div} u_0 &= 0 \quad \text{in } \widetilde{E}. \end{cases} \tag{2.136}$$

Using well known results about interior regularity of solutions to strongly elliptic equations

$$\|u_0\|_{\mathbf{L}^\infty(B_{\frac{t}{2}}(x))} \leq t^{-\frac{n}{2}}\|u_0\|_{\mathbf{L}^2(B_{\frac{t}{2}}(x))}. \tag{2.137}$$

It is then sufficient to estimate $\|u\|_{\mathbf{L}^2(B(x))}$ for a "large enough" ball near the boundary. Since (see the proof of Proposition 2.4) $\triangle^2 u_0 = 0$, we may apply Theorem 2.9 to u_0. Calling $r_1 = \frac{\rho_{00}}{8}$, $r_2 = \frac{3\rho_{00}}{8}$ and $r_3 = \rho_{00}$ we have (understanding that all balls are centered in P^*)

$$\|u_0\|_{\mathbf{L}^2(B_{r_2})} \leq C\|u_0\|^\tau_{\mathbf{L}^2(B_{r_1})}\|u_0\|^{1-\tau}_{\mathbf{L}^2(B_{r_3})}. \tag{2.138}$$

Let us call $\eta = \rho_0\|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}$. By the triangle inequality, (2.125) and (2.87) we have that

$$\|u_0\|_{\mathbf{L}^2(B_r)} \leq \|\widetilde{u}\|_{\mathbf{L}^2(B_r)} + \|w\|_{\mathbf{L}^2(B_r)} \leq \|\widetilde{u}\|_{\mathbf{L}^2(B_r)} + C\eta, \tag{2.139}$$

for $r = r_1, r_3$; furthermore, we have

$$\|\widetilde{u}\|_{\mathbf{L}^2(B_{r_2}\cap E)} \leq \|u_0\|_{\mathbf{L}^2(B_{r_2})} + \|w\|_{\mathbf{L}^2(B_{r_2})} \leq \|u_0\|_{\mathbf{L}^2(B_{r_2})} + C\eta. \tag{2.140}$$

Putting together (2.138), (2.139), (2.140), and recalling (2.87) and (2.21) we get

$$\begin{aligned} &\|u\|_{\mathbf{L}^2(B_{r_2})} \leq \|\widetilde{u}\|_{\mathbf{L}^2(B_{r_2}\cap E)} \leq \\ \leq &C\eta + C(\|\widetilde{u}\|_{\mathbf{L}^2(B_{r_1})} + C\eta)^\tau(\|\widetilde{u}\|_{\mathbf{L}^2(B_{r_3}\cap E)} + C\eta)^{1-\tau} \leq \\ \leq &C\big(\eta + \eta^\tau(\eta + \|u\|_{\mathbf{L}^2(E)})^{1-\tau}\big) \leq C\eta^\tau\|u\|^{1-\tau}_{\mathbf{L}^2(E)}. \end{aligned} \tag{2.141}$$

□

Chapter 3

The nonlinear stationary case

3.1 Introduction

In this chapter we generalize the study done in the previous chapter to the inverse problem associated to the stationary Navier-Stokes system. We consider a bounded set $\Omega \subset \mathbb{R}^n$ (where again $n = 2, 3$) with a sufficiently smooth boundary $\partial\Omega$, filled with a Navier-Stokes fluid of constant viscosity μ. We want to detect an object D immersed in this container, by collecting measurements of the velocity of the fluid motion and of the boundary forces, but we only have access to a portion Γ of the boundary $\partial\Omega$. Once assigned a boundary condition g on Γ, the velocity $u = (u_1, \ldots, u_n)$ and the pressure p of the fluid will obey the following Navier-Stokes system in $\Omega\backslash\overline{D}$:

$$\begin{cases} \operatorname{div} \sigma(u,p) &= (u \cdot \nabla)u \quad \text{in } \Omega\backslash\overline{D}, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega\backslash\overline{D}, \\ u &= g \quad \text{on } \Gamma, \\ u &= 0 \quad \text{on } \partial D. \end{cases} \tag{3.1}$$

The ith component of the nonlinear term $(u \cdot \nabla)u$ is given by

$$\big((u \cdot \nabla)u\big)_i = \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j}. \tag{3.2}$$

We require again, in the last equation on (3.1), the no-slip condition on the boundary of D. Call ν the outer normal vector field to $\partial\Omega$. Once $g \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ is assigned, we measure on Γ the induced normal component of the stress tensor

$$\psi = \sigma(u,p) \cdot \nu, \tag{3.3}$$

and try to recover D from a single pair of Cauchy data (g, ψ) known on the accessible part of the boundary Γ. Under some additional regularity hypotheses, namely of $\partial\Omega$ being of Lipschitz class, and $g \in \mathbf{H}^{\frac{3}{2}}(\Gamma)$, the uniqueness for this inverse problem has been shown to hold as well (see [9])

by means of unique continuation techniques. This means that if u_1 and u_2 are two solutions of (3.1) corresponding to an assigned boundary data g, for $D = D_1$ and $D = D_2$ respectively, and $\sigma(u_1, p_1) \cdot \nu = \sigma(u_2, p_2) \cdot \nu$ on Γ, then $D_1 = D_2$. We analyze the problem of stability:
Given two solutions (u_i, p_i) to (3.1) for two different D_i, for $i = 1, 2$, with the same boundary data g, if

$$\|\sigma(u_1, p_1) \cdot \nu - \sigma(u_2, p_2) \cdot \nu\| \leq \epsilon,$$

what is the rate of convergence of $\mathrm{d}_{\mathcal{H}}(D_1, D_2)$ *as $\epsilon \to 0$?*
In the previous chapter we proved a rate of convergence of log-log type for the analogous problem in the simpler context of the Stokes system. Here, we will prove an equivalent result for the stationary Navier-Stokes equations. As for the (yet simpler) Stokes problem, we have seen that, even if we add some a priori information on the regularity of the unknown domain, we can only obtain a weak rate of stability. We thus expect- at most- a log-log type stability for the Hausdorff distance between the boundaries of the inclusions, once appropriate a priori hypotheses are made. This will be indeed the main result of this chapter. The structure of the proof is the same as for the Stokes equations: the key result is a three spheres inequality (see Theorem 3.9) and its main consequences. A very recent paper by Lin, Uhlmann and Wang ([54]) extended the validity of the three spheres inequality to linearized Navier-Stokes systems: this allows us to apply it to differences of solutions of (3.1), see Proposition 3.12. In order to adapt this result to the Navier-Stokes equations, however, we are forced to add yet more a priori hypotheses on the solutions, mainly because of the nonlinear character of the equations. Proposition 3.12, in fact, applies to linearized stationary Navier-Stokes systems with coefficients bounded in an appropriate norm. We meet this request by restricting the choice of boundary data, i.e. we require a strong regularity bound- i.e., $\mathbf{C}^{1,\alpha}$-on the boundary data.
The ansatz we use for the proof of the stability theorem is identical to that of the Stokes equations case. Therefore, the structure of this chapter also remains essentially the same as the previous one.

3.2 The stability result

3.2.1 A priori information

In this section we introduce again the a priori hypotheses needed to prove the stability results. As we pointed out in the introduction to this chapter, the a priori hypotheses on the domains Ω and D will be essentially the same, whereas we will require stronger conditions on the boundary data.
(1) *A priori information on the domain.*
We assume $\Omega \subset \mathbb{R}^n$ to be a bounded domain, such that

$$\partial\Omega \text{ is connected,} \tag{3.4}$$

with a sufficiently smooth boundary, i.e.,

$$\partial\Omega \text{ is of class } C^{2,\alpha} \text{ of constants } \rho_0,\ M_0, \tag{3.5}$$

where $\alpha \in (0,1]$ is a real number, $M_0 > 0$, and $\rho_0 > 0$ is what we shall treat as our dimensional parameter. In what follows ν is the outer normal vector field to $\partial\Omega$. We also require that

$$|\Omega| \leq M_1 \rho_0^n, \tag{3.6}$$

where $M_1 > 0$.
We choose an open and connected portion $\Gamma \subset \partial\Omega$ as being the accessible part of the boundary. We assume that there exists a point $P_0 \in \Gamma$ such that

$$\partial\Omega \cap B_{\rho_0}(P_0) \subset \Gamma. \tag{3.7}$$

(2) *A priori information about the obstacles.*
We consider $D \subset \Omega$, which represents the obstacle we want to detect from the boundary measurements, on which we require that

$$\Omega \setminus \overline{D} \text{ is connected,} \tag{3.8}$$

$$\partial D \text{ is connected.} \tag{3.9}$$

We require the same regularity on D as we did for Ω, that is,

$$\partial D \text{ is of class } C^{2,\alpha} \text{ with constants } \rho_0,\ M_0. \tag{3.10}$$

In addition, we suppose that

$$d(D, \partial\Omega) \geq \rho_0. \tag{3.11}$$

(3) *A priori information about the boundary data.*
For the Dirichlet-type data g we assign on the accessible portion of the boundary Γ, we assume that

$$\begin{gathered} g \in \mathbf{C}^{1,\alpha}(\partial\Omega), \quad g \not\equiv 0, \\ \operatorname{supp} g \subset\subset \Gamma. \end{gathered} \tag{3.12}$$

We prescribe the following compatibility condition (which is necessary for the existence of the solution, and is a consequence of the incompressibility condition):

$$\int_{\partial\Omega} g \,\mathrm{d}s = 0. \tag{3.13}$$

We shall also assume an apriorie bound on the regularity of the flow, by requiring that for a given constant $\mathcal{E}$ we have

$$\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)} \leq \mathcal{E}. \tag{3.14}$$

We also specify a bound on the oscillation of the boundary data g by requiring that, for a given constant $F > 0$,

$$\|g\|_{\mathbf{L}^2(\Gamma)} \geq F. \tag{3.15}$$

Note that (3.14) and (3.15) combined yield an a priori frequency type limitation of the form

$$\frac{\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)}}{\|g\|_{\mathbf{L}^2(\Gamma)}} \leq \frac{\mathcal{E}}{F}.$$

Under the above conditions on g, one can prove that there exists a constant $c > 0$, only depending on M_0, such that the following equivalence relation holds:

$$\frac{1}{c}\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)} \leq \|g\|_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)} \leq c\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)}. \tag{3.16}$$

3.2.2 The direct problem

In this section we collect some useful results regarding the direct problem of finding weak solutions of (3.1). We begin by an existence result (which is a classical result and can be found in [29], [68]) :

Theorem 3.1. *Let* $g \in \mathbf{C}^{1,\alpha}(\partial\Omega)$ *satisfy* (3.12) *and* (3.13)*, let* $\Omega \subset \mathbb{R}^n$ *a bounded set satisfying* (3.4)-(3.7)*, and let* $D \subset \Omega$ *satisfy* (3.8)-(3.11)*. Then for every* $\mu > 0$ *there exists at least one solution* $(u, p) \in \mathbf{H}^1(\Omega \setminus D) \times \mathbf{L}^2(\Omega \setminus D)$ *of* (3.1)*, and*

$$\|u\|_{\mathbf{H}^1(\Omega\setminus D)} \leq C\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)}, \tag{3.17}$$

with C *only depending on* μ*,* M_0*,* M_1*.*

The following is a consequence of the above theorem combined with a global regularity result for solutions of the Navier-Stokes equations, see, for example, [29][VIII, Corollary 5.2]. We point out that an a priori higher degree of integrability of weak solutions has to be assumed.

Theorem 3.2 (Regularity of solutions)**.** *Assume that the hypotheses of the previous theorem are satisfied and suppose, in addition, that* (3.14) *is satisfied. Let* u *be the weak solution to* (3.1) *in* $\Omega \setminus D$*, and suppose that* $u \in \mathbf{H}^1(\Omega \setminus D) \cap \mathbf{L}^n(\Omega \setminus D)$*. Then for all* $0 < \alpha < 1$ *we have that* $u \in C^{1,\alpha}(\overline{\Omega \setminus D})$ *and*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega\setminus D})} \leq C\mathcal{E}, \tag{3.18}$$

where $C > 0$ *only depends on* μ*,* α*,* M_0*.*

Remark The requirement that $u \in \mathbf{H}^1(\Omega \setminus D) \cap \mathbf{L}^n(\Omega \setminus D)$ is actually redundant (in the sense that it follows from $u \in \mathbf{H}^1(\Omega \setminus D)$) when $n \leq 4$, due to the Sobolev embedding theorems.

The uniqueness issue for the direct problem is more subtle. Unlike the linear Stokes equations, here uniqueness is not guaranteed in general. In fact, several examples can be built even in low space dimensions (see [68]). As far as the inverse problem is concerned, whether or not the solution of the direct problem is unique is not relevant, for by formulating the inverse problem we implicitly select one particular solution of the direct problem to work with. To guarantee uniqueness, one can either a priori bound the norm of the solution (see [29], Theorem VIII.2.1), or take "not too large" boundary data g and viscosity μ, as stated by the following (which we will state, for simplicity, only for $n \leq 4$; see [68], Theorem 1.6, pg. 120 for a proof):

Theorem 3.3 (Uniqueness for small data). *Let Ω and g satisfy the same hypotheses of Theorem 3.1, and let u be a solution of* (3.1) *given by Theorem 3.1. Then:*

1. *There exists a constant $C_1 = C_1(\mu, \Omega)$ such that if w is another solution of* (3.1) *and $\|u\|_{\mathbf{H}^1(\Omega\setminus D)} \leq C_1$ then $w = u$.*

2. *There exists a constant $C_2 = C_2(\mu, \Omega, n)$ such that, if $\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)} \leq C_2$, then u is the unique solution of* (3.1).

3. *There exists a constant $C_3 = C_3(\Omega, n, \|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)})$ such that if $\mu \geq C_3$ then u is the unique solution of* (3.1).

3.2.3 The main result

Let $\Omega \subset \mathbb{R}^n$, and $\Gamma \subset \partial\Omega$ satisfy (3.5)-(3.7). Let $D_i \subset \Omega$, for $i = 1, 2$, satisfy (3.8)-(3.11), and let us denote by $\Omega_i = \Omega \setminus \overline{D_i}$. We may state the main result, analogous to Theorem 2.1, as follows.

Theorem 3.4 (Stability). *Let $g \in \mathbf{C}^{1,\alpha}(\Gamma)$ be the assigned boundary data, satisfying* (3.12)-(3.15). *Let $u_i \in \mathbf{H}^1(\Omega_i) \cap \mathbf{L}^n(\Omega_i)$ solve* (3.1) *for $D = D_i$. If, for $\epsilon > 0$, we have*

$$\rho_0 \|\sigma(u_1, p_1) \cdot \nu - \sigma(u_2, p_2) \cdot \nu\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \leq \epsilon, \tag{3.19}$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \rho_0 \omega(\epsilon), \tag{3.20}$$

where $\omega : (0, +\infty) \to \mathbb{R}^+$ is an increasing function satisfying, for all $0 < t < \frac{1}{e}$:

$$\omega(t) \leq C(\log|\log t|)^{-\beta}. \tag{3.21}$$

The constants $C > 0$ and $0 < \beta < 1$ only depend on μ, n, M_0, M_1, $\mathcal{E}$ and F.

3.3 Proof of Theorem 3.4

The proof of Theorem 3.4 relies on the following sequence of propositions.

Proposition 3.5 (Lipschitz propagation of smallness)**.** *Let E be a bounded Lipschitz domain with constants ρ_0, M_0, satisfying* (3.6)*. Let u be a solution to the following problem:*

$$\begin{cases} \operatorname{div} \sigma(u,p) & = (u\cdot\nabla)u & \text{in } E, \\ \operatorname{div} u & = 0 & \text{in } E, \\ u & = g & \text{on } \partial E, \end{cases} \tag{3.22}$$

where g satisfies

$$g \in \mathbf{C}^{1,\alpha}(\partial E), \quad g \not\equiv 0, \tag{3.23}$$

$$\int_{\partial E} g \,\mathrm{d}s = 0, \tag{3.24}$$

$$\|g\|_{\mathbf{C}^{1,\alpha}(\partial E)} \leq \mathcal{E}, \tag{3.25}$$

$$\|g\|_{\mathbf{L}^2(\partial E)} \geq F, \tag{3.26}$$

for given constants $\mathcal{E} > 0$, $F > 0$. Also suppose that there exists a point $P \in \partial E$ such that

$$g = 0 \ \text{ on } \ \partial E \cap B_{\rho_0}(P). \tag{3.27}$$

Then there exists a constant $s > 1$, depending only on n and M_0 such that, for every $\rho > 0$ and for every $\bar{x} \in E_{s\rho}$, we have

$$\int_{B_\rho(\bar{x})} |\nabla u|^2 dx \geq C_\rho \int_E |\nabla u|^2 dx. \tag{3.28}$$

Here $C_\rho > 0$ is a constant depending only on n, M_0, M_1, F, $\mathcal{E}$, ρ_0 and ρ. The constant C_ρ can be written as

$$C_\rho = \frac{C}{\exp\left[A\left(\frac{\rho_0}{\rho}\right)^B\right]} \tag{3.29}$$

where A, B, $C > 0$ only depend on n, M_0, M_1, F and $\mathcal{E}$.

Proposition 3.6 (Lipschitz propagation of smallness up to boundary data)**.** *If the hypotheses of Theorem 3.4 hold, for all $\rho > 0$, if $\bar{x} \in (\Omega_i)_{(s+1)\rho}$, we have for $i = 1, 2$:*

$$\frac{1}{\rho_0^{n-2}} \int_{B_\rho(\bar{x})} |\nabla u_i|^2 dx \geq C_\rho \|g\|^2_{\mathbf{C}^{1,\alpha}(\Gamma)}, \tag{3.30}$$

where C_ρ is as in (3.29) *(with possibly a different value of the term C), and s is given as in Proposition 3.5.*

Proposition 3.7 (Stability estimate of continuation from Cauchy data)**.** *If the hypotheses of Theorem 3.4 hold then we have*

$$\frac{1}{\rho_0^{n-2}}\int_{D_2\setminus D_1}|\nabla u_1|^2 \leq C\omega(\epsilon) \tag{3.31}$$

$$\frac{1}{\rho_0^{n-2}}\int_{D_1\setminus D_2}|\nabla u_2|^2 \leq C\omega(\epsilon) \tag{3.32}$$

where ω is an increasing continuous function, defined on $\mathbb{R}^+$ and satisfying

$$\omega(t) \leq C\big(\log|\log t|\big)^{-c} \tag{3.33}$$

for all $t < e^{-1}$, where C only depends on μ, n, M_0, M_1 and $\mathcal{E}$, and $c > 0$ only depends on n.

Proposition 3.8 (Improved stability estimate of continuation)**.** *Let the hypotheses of Theorem 3.4 hold. Let G be the connected component of $\Omega_1\cap\Omega_2$ containing Γ, and assume that ∂G is of Lipschitz class of constants $\tilde{\rho}_0$ and $\tilde{M}_0$, where $M_0 > 0$ and $0 < \tilde{\rho}_0 < \rho_0$. Then* (3.31) *and* (3.32) *both hold with ω given by*

$$\omega(t) = C|\log t|^{-\gamma}, \tag{3.34}$$

defined for $t < 1$, where $\gamma > 0$ and $C > 0$ only depend on μ, M_0, $\tilde{M}_0$, M_1, $\mathcal{E}$ and $\frac{\rho_0}{\tilde{\rho}_0}$.

We delay the proofs of Propositions 3.5, 3.6, 3.7 and 3.8 until the next sections. For now, we apply them to prove Theorem 3.4.

Proof of Theorem 3.4. Let us call

$$d = d_{\mathcal{H}}(\partial D_1, \partial D_2). \tag{3.35}$$

Let η be the quantity on the right hand side of (3.31) and (3.32), so that

$$\begin{aligned}\int_{D_2\setminus D_1}|\nabla u_1|^2 &\leq \eta,\\ \int_{D_1\setminus D_2}|\nabla u_2|^2 &\leq \eta.\end{aligned} \tag{3.36}$$

Without loss of generality, assume that there exists a point $x_1 \in \partial D_1$ such that $\mathrm{dist}(x_1, \partial D_2) = d$. We distinguish two possible situations:
(i) $B_d(x_1) \subset D_2$,
(ii) $B_d(x_1) \cap D_2 = \emptyset$.
In case (i), by the regularity assumptions on ∂D_1, we find a point $x_2 \in$

$D_2 \setminus D_1$ such that $B_{td}(x_2) \subset D_2 \setminus D_1$, where t is small enough (for example, $t = \frac{1}{1+\sqrt{1+M_0^2}}$ suffices). Using (3.30), with $\rho = \frac{td}{s}$ we have

$$\int_{B_\rho(x_2)} |\nabla u_1|^2 dx \geq \frac{C\rho_0^{n-2}}{\exp\left[A\left(\frac{s\rho_0}{td}\right)^B\right]} \|g\|^2_{\mathbf{C}^{1,\alpha}(\Gamma)}. \tag{3.37}$$

By Proposition 3.7, we have:

$$\omega(\epsilon) \geq \frac{C\|g\|^2_{\mathbf{C}^{1,\alpha}(\Gamma)}}{\exp\left[A\left(\frac{s\rho_0}{td}\right)^B\right]}, \tag{3.38}$$

which, once we recall (3.15) and solve for d, yields this estimate of log-log-log type stability:

$$d \leq C\rho_0 \left\{ \log\left[\log\left|\log \epsilon\right|\right]\right\}^{-\frac{1}{B}}, \tag{3.39}$$

provided $\epsilon < e^{-e}$: this is not restrictive since, for larger values of ϵ, the thesis is trivial. If we call d_0 the right hand side of (3.39), we have that there exists ϵ_0 only depending on n, M_0, M_1 and F such that, if $\epsilon \leq \epsilon_0$ then $d \leq d_0$. Proposition 2.8 from the previous chapter then applies, so that G satisfies the hypotheses of Proposition 3.8. This means that we may choose ω of the form (3.34) in (3.38), obtaining (3.31). Case (ii) can be treated analogously, upon substituting u_1 with u_2. □

3.4 Proof of Proposition 3.5

In a recent paper by Lin, Uhlmann and Wang ([54]), the validity of the three spheres inequality has been extended to solutions $u = (u_1, \dots, u_n)$ of Stokes systems of the form

$$\begin{cases} \triangle u + A(x)\cdot\nabla u + B(x)u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{3.40}$$

where $A(x)$ is a measurable vector, $B(x)$ is a measurable matrix, both satisfying appropriate bounds, and it is agreed that $A\cdot\nabla u = (A\cdot\nabla u_1, \dots, A\cdot\nabla u_n)$ (the $\cdot$ on the right hand side being the ordinary vector dot product). In what follows, it shall be convenient to write out the first equation in (3.40) component-wise (here and everywhere else, we use the convention of summation over repeated indexes):

$$\triangle u_i + a_j \frac{\partial u_i}{\partial x_j} + b_{ij}u_j + \frac{\partial p}{\partial x_i} = 0, \text{ for } i = 1, \dots, n. \tag{3.41}$$

An analogous result when $B = 0$ was obtained by Fabre and Lebeau [28] and Regbaoui [64]. The result in its most general form allows A and B to have some type of singularity; we shall only state a simplified version: we will assume that

$$\|A\|_{C^{1,\alpha}(B_R)} + \|B\|_{C^{0,\alpha}(B_R)} \leq \mathcal{E}. \tag{3.42}$$

Then we have:

Theorem 3.9 (Three spheres inequality.)**.** *Let $u \in \mathbf{H}^1(B_R)$ be a solution to* (3.40) *in a ball B_R. Suppose that the functions $A(x)$, $B(x)$ are measurable and such that* (3.42) *holds. Then there exists a real number $\vartheta^* \in (0, e^{-1/2})$, depending only on n, such that, for all $0 < r_1 < r_2 < \vartheta^* r_3$ with $r_3 \leq R$ we have:*

$$\int_{B_{r_2}} |u|^2 \leq C\Big(\int_{B_{r_1}} |u|^2\Big)^{\delta}\Big(\int_{B_{r_3}} |u|^2\Big)^{1-\delta} \tag{3.43}$$

where the balls B_{r_i} are concentric with B_R and $\delta \in (0,1)$ and $C > 0$ are constants depending only on $\mathcal{E}$, n, $\frac{r_1}{r_3}$ and $\frac{r_2}{r_3}$.

We will also need to formulate a three spheres type inequality for the first derivatives of u:

Corollary 3.10. *Let $u \in \mathbf{H}^1(B_R)$ be a solution to* (3.40)*, and suppose that* (3.42) *holds. Assume furthermore that $B(x) \equiv 0$. Then we have that for all $0 < r_1 < r_2 < \vartheta^* r_3$ with $r_3 \leq R$:*

$$\int_{B_{r_2}} |\nabla u|^2 \leq C\Big(\int_{B_{r_1}} |\nabla u|^2\Big)^{\delta}\Big(\int_{B_{r_3}} |\nabla u|^2\Big)^{1-\delta} \tag{3.44}$$

where ϑ^* is the same as in Theorem 3.9, $\delta \in (0,1)$ and $C > 0$ are constants depending only on $\mathcal{E}$, n, $\frac{r_1}{r_3}$ and $\frac{r_2}{r_3}$, and the balls B_{r_i} are concentric with B_R.

Remark We point out that the first equation (3.1) may be written in the form (3.41) by writing the nonlinear term as in (3.2) and calling $(A(x))_j = u_j$ and $(B(x))_{ij} = 0$. Therefore, the three spheres inequalities (3.43) and (3.44) may be applied to solutions of (3.1) in a domain E as long as (3.42) holds in E with the aforementioned choices of A and B.

We now recall the following Caccioppoli-type inequality for (3.40), which can be found in [31].

Proposition 3.11 (Caccioppoli inequality)**.** *Let $u \in \mathbf{H}^1(B_R)$ be a solution of* (3.40) *in B_R, and suppose that* (3.42) *holds. Then there exists $C > 0$ depending only on n,μ and $\mathcal{E}_1$ such that, for every r with $0 < r < R$ we have*

$$\int_{B_r} |\nabla u|^2 \leq \frac{C}{(R-r)^2}\int_{B_R} |u|^2. \tag{3.45}$$

Combining this with the Poincaré inequality (Proposition 2.3) we can derive Corollary 3.10 from Theorem 3.9:

Proof of Corollary 3.10. We notice that if u is a solution to (3.40) and $B(x) = 0$, then $u - u_E$ is a solution as well, where u_E denotes the average of u in E. Since the classical Poincaré-Wirtinger inequality (see [55]) holds for $u - u_E$, we apply it together with the Caccioppoli inequality and the three spheres inequalities (3.43), (3.44) to obtain:

$$\begin{aligned}\int_{B_{r_2}} |\nabla u|^2 \leq \frac{C_1}{\rho_0^2 (r_3 - r_2)^2} \int_{B_{r_2}} |u - u_E|^2 \leq \\ \leq \frac{C_2}{\rho_0^2 (r_3 - r_2)^2} \Big(\int_{B_{r_1}} |u - u_E|^2 \Big)^\delta \Big(\int_{B_{r_3}} |u - u_E|^2 \Big)^{1-\delta} \leq \\ \leq \frac{C_3}{(r_3 - r_2)^2} \Big(\int_{B_{r_1}} |\nabla u|^2 \Big)^\delta \Big(\int_{B_{r_3}} |\nabla u|^2 \Big)^{1-\delta},\end{aligned}$$

and C_1, C_2, C_3 only depend on $\mathcal{E}$, n, $\frac{r_1}{r_3}$ and $\frac{r_2}{r_3}$. □

We can now show that Proposition 3.6 follows from Proposition 3.5:

Proof of Proposition 3.6. From Proposition 3.5 we know that

$$\int_{B_\rho(x)} |\nabla u_i|^2 \geq C_\rho \int_{\Omega \setminus \overline{D_i}} |\nabla u_i|^2,$$

where C_ρ is given in (3.29). We have, using Poincaré inequality (2.22) and the trace theorem,

$$\int_{\Omega \setminus \overline{D_i}} |\nabla u_i|^2 \geq C \rho_0^{n-2} \|u_i\|^2_{\mathbf{H}^1(\Omega \setminus \overline{D_i})} \geq C \rho_0^{n-2} \|g\|^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}. \tag{3.46}$$

Applying the above estimate to (3.28) and using (3.16) will prove our statement. □

Finally, we will need a three spheres inequality for functions that can be written as differences of solutions of the Navier-Stokes equations (due to nonlinearity, this does not follow the previous remarks).

Proposition 3.12. *Let u_1 and u_2 be solutions of*

$$\begin{cases} \operatorname{div} \sigma(u_i, p_i) &= (u_i \cdot \nabla) u_i \quad \text{in } E, \\ \operatorname{div} u_i &= 0 \quad \text{in } E, \end{cases} \tag{3.47}$$

for $i = 1, 2$. Suppose that $\|u_1\|_{\mathbf{C}^{1,\alpha}(E)} + \|u_2\|_{\mathbf{C}^{0,\alpha}(E)} \leq \mathcal{E}$. Let $B_R(x) \subset E$. Then there exists a real number $\vartheta^* \in (0, e^{-1/2})$, depending only on n, such that, for all $0 < r_1 < r_2 < \vartheta^* r_3$ with $r_3 \leq R$ we have, calling $w = u_1 - u_2$:

$$\int_{B_{r_2}} |w|^2 \leq C \Big(\int_{B_{r_1}} |w|^2 \Big)^\delta \Big(\int_{B_{r_3}} |w|^2 \Big)^{1-\delta} \tag{3.48}$$

where the balls B_{r_i} are centered in x, and $\delta \in (0,1)$ and $C > 0$ are constants depending only on $\mathcal{E}$, n, $\frac{r_1}{r_3}$ and $\frac{r_2}{r_3}$.

Proof. In view of Theorem (3.9) (and the subsequent remarks), it is enough to show that w can be written as a solution to a system of the form (3.40). This is readily done, by subtracting from each other (3.47) for $i = 1, 2$. We may write, in components for $j = 1, \dots, n$:

$$\mu \triangle w_j - (u_2)_i \frac{\partial w_j}{\partial x_i} + \frac{\partial (u_1)_j}{\partial x_i} w_i + \frac{\partial (p_1 - p_2)}{\partial x_j} = 0.$$

Calling $(A(x))_i = (-u_2)_i$ and $(B(x))_{ij} = \frac{\partial (u_1)_j}{\partial x_i}$, we have that (3.42) holds in E, so the hypotheses of Theorem 3.9 hold for w. □

Remark We observe that, since the identically zero function solves (3.47) in E, we can also apply the three spheres inequality to each u_i separately (as we already pointed out in the previous remark).

The proof of Proposition 3.5 is now a consequence of the work done so far.

Proof of Proposition 3.5. The proof is based upon the validity of the three spheres inequality for (3.1). The proof is almost identical to that of Proposition 2.4 in the previuos chapter, which can be applied here with only slight modifications, as it only requires (3.43) and (3.44) and some geometric constructions which exploit the regularity of ∂E. □

3.5 Stability of continuation from Cauchy data

The proof the stability estimate of continuation from the Cauchy data heavily relies upon the upcoming result, which deals with the estimation of the stability of the stationary Navier-Stokes equations with homogeneous Cauchy data, the proof of which, in turn, is based upon an extension argument. We pospone the proof to the next section. Here we will recall the notations from the previous chapter and state the theorem. Let us consider a bounded domain $E \subset \mathbb{R}^n$ satisfying hypotheses (3.5) and (3.6), and take $\Gamma \subset \partial E$ a connected open portion of the boundary of class $C^{2,\alpha}$ with constants ρ_0, M_0. Let $P_0 \in \Gamma$ such that (3.7) holds. By definition, after a suitable change of coordinates we have that $P_0 = 0$ and

$$E \cap B_{\rho_0}(0) = \{(x', x_n) \in E \text{ s.t.} \, x_n > \varphi(x')\} \subset E, \tag{3.49}$$

where φ is a $C^{2,\alpha}(B'_{\rho_0}(0))$ function satisfying

$$\begin{aligned} \varphi(0) &= 0, \\ |\nabla \varphi(0)| &= 0, \\ \|\varphi\|_{C^{2,\alpha}(B'_{\rho_0}(0))} &\le M_0 \rho_0. \end{aligned}$$

Define

$$\begin{aligned} \rho_{00} &= \frac{\rho_0}{\sqrt{1+M_0^2}}, \\ \Gamma_0 &= \{(x', x_n) \in \Gamma \text{ s.t. } |x'| \leq \rho_{00},\ x_n = \varphi(x')\}. \end{aligned} \tag{3.50}$$

In what follows we shall consider (u_i, p_i), for $i = 1, 2$, which are solutions to the following problems:

$$\begin{cases} \operatorname{div} \sigma(u_i, p_i) & = (u_i \cdot \nabla) u_i \quad \text{in } E, \\ \operatorname{div} u_i & = 0 \quad \text{in } E, \\ u_i & = g \quad \text{on } \Gamma, \\ \sigma(u_i, p_i) \cdot \nu & = \psi_i \quad \text{on } \Gamma, \end{cases} \tag{3.51}$$

where g satisfies (3.23)-(3.26), $\psi_i \in \mathbf{C}^{1,\alpha}(\Gamma)$ and we use the same notations as in the proof of Proposition 3.5 (which are also to be understood in what follows). Define $w = u_1 - u_2$, $q = p_1 - p_2$, these will solve a system of the following form:

$$\begin{cases} \operatorname{div} \sigma(w, q) + A \cdot \nabla w + Bw & = 0 \quad \text{in } E, \\ \operatorname{div} w & = 0 \quad \text{in } E, \\ w & = 0 \quad \text{on } \Gamma, \\ \sigma(w, q) \cdot \nu & = \psi_0 \quad \text{on } \Gamma, \end{cases} \tag{3.52}$$

where A and B were explicitated in (3.41) and $\psi_0 = \psi_1 - \psi_2$. We have the following estimate for a solution of systems of the form (3.52), the proof of which is delayed to the next section:

Theorem 3.13. *Let A be a vector of $C^{0,\alpha}(E)$ of constants ρ_0, M_0, let B a matrix of class $C^{0,\alpha}(E)$ of constants ρ_0, M_0, satisfying*

$$\|A\|_{C^{0,\alpha}} + \|B\|_{C^{0,\alpha}} = \mathcal{E}, \tag{3.53}$$

and let (w, q) be a solution of class $\mathbf{C}^{1,\alpha}(E) \times C^0(E)$ to the problem:

$$\begin{cases} \operatorname{div} \sigma(w, q) + A \cdot \nabla w + Bw & = 0 \quad \textit{in } E, \\ \operatorname{div} w & = 0 \quad \textit{in } E, \\ w & = 0 \quad \textit{on } \Gamma, \\ \sigma(w, q) \cdot \nu & = \psi \quad \textit{on } \Gamma, \end{cases} \tag{3.54}$$

where $\psi \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and $\Gamma \subset \partial E$ is of class $C^{1,\alpha}$. Then there exists $\widehat{\rho}$, only depending on M_0, α and $\mathcal{E}$, such that, letting $P^* = P_0 + \frac{\widehat{\rho}}{4}\nu$ where ν is the outer normal field to ∂E, we have:

$$\|w\|_{\mathbf{L}^\infty(E \cap B_{\frac{3\widehat{\rho}}{8}}(P^*))} \leq \frac{C}{\rho_0^{\frac{n}{2}}} \|w\|^{1-\tau}_{\mathbf{L}^2(E)} (\rho_0 \|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)})^\tau, \tag{3.55}$$

where τ only depends on α and M_0 and $C > 0$ also depends on $\mathcal{E}$.

Proof of Proposition 3.7. Let $\theta = \min\{a, \frac{7}{8\gamma_1}\frac{1}{2\gamma_0(1+M_0^2)}\}$ where a, γ_0, γ_1 are the constants depending only on M_0 and α introduced in Lemma 2.12, then let $\overline{\rho} = \theta\rho_0$ and fix $\rho \leq \overline{\rho}$. Let D_1^ρ, D_2^ρ be the regularized domain associated with D_1, D_2 respectively, built according to Lemma 2.12. Let G be the connected component of $\Omega \setminus (\overline{D_1 \cup D_2})$ containing $\partial\Omega$, and G^ρ be the connected component of $\overline{\Omega} \setminus (D_1^\rho \cup D_2^\rho)$ which contains $\partial\Omega$. We have that

$$D_2 \setminus \overline{D_1} \subset \Omega_1 \setminus \overline{G} \subset \big((D_1^\rho \setminus \overline{D_1}) \setminus \overline{G}\big) \cup \big((\Omega \setminus G^\rho) \setminus D_1^\rho\big)$$

and

$$\partial\big((\Omega \setminus G^\rho) \setminus D_1^\rho\big) = \Gamma_1^\rho \cup \Gamma_2^\rho, \tag{3.56}$$

where $\Gamma_2^\rho = \partial D_2^\rho \cap \partial G^\rho$ and $\Gamma_1^\rho \subset \partial D_1^\rho$. Then

$$\int_{D_2\setminus\overline{D_1}} |\nabla u_1|^2 \leq \int_{\Omega_1\setminus\overline{G}} |\nabla u_1|^2 \leq \int_{(D_1^\rho\setminus\overline{D_1})\setminus\overline{G}} |\nabla u_1|^2 + \int_{(\Omega\setminus G^\rho)\setminus D_1^\rho} |\nabla u_1|^2. \tag{3.57}$$

The first term can be estimated directly, using (3.18) and (2.91) we have

$$\int_{(D_1^\rho\setminus\overline{D_1})\setminus\overline{G}} |\nabla u_1|^2 \leq C\rho_0^{n-2}\|g\|^2_{\mathbf{C}^{1,\alpha}(\Gamma)}\frac{\rho}{\rho_0} \tag{3.58}$$

where C only depends on the M_0, M_1, α and μ. We call $\Omega(\rho) = (\Omega\setminus G^\rho)\setminus D_1^\rho$. We use the first equation in (3.1), multiply it by u_1 and integrate over $\Omega(\rho)$ to derive the following identity:

$$\begin{aligned}
0 &= \int_{\Omega(\rho)} \mu u_1 \cdot \triangle u_1 - u_1(u_1\cdot\nabla)u_1 - u_1\cdot\nabla p_1 = \\
&= \int_{\Omega(\rho)} \mu u_1 \cdot \triangle u_1 - \frac{1}{2}\mathrm{div}\,(u_1|u_1|^2) - u_1\cdot\nabla p_1 = \\
&= \int_{\partial\Omega(\rho)} \mu(\nabla u_1\cdot\nu)u_1 - \frac{1}{2}u_1\cdot\nu|u_1|^2 - (u_1\cdot\nu)p - \mu\int_{\Omega(\rho)}|\nabla u_1|^2,
\end{aligned}$$

which yields, recalling (3.56),

$$\begin{aligned}
\mu\int_{\Omega(\rho)}|\nabla u_1|^2 = &\int_{\Gamma_1^\rho} \mu(\nabla u_1\cdot\nu)u_1 - \frac{1}{2}u_1\cdot\nu|u_1|^2 - (u_1\cdot\nu)p + \\
&+\int_{\Gamma_2^\rho} \mu(\nabla u_1\cdot\nu)u_1 - \frac{1}{2}u_1\cdot\nu|u_1|^2 - (u_1\cdot\nu)p.
\end{aligned} \tag{3.59}$$

We start by estimating the first integral on the right hand side of (3.59). If $x \in \Gamma_1^\rho$, by Theorem 2.12, we find $y \in \partial D_1$ such that $|y - x| = d(x, \partial D_1) \leq \gamma_1\rho$; since $u_1(y) = 0$, by Lemma 3.2 we have

$$|u_1(x)| = |u_1(x) - u_1(y)| \leq C\frac{\rho}{\rho_0}\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)}. \tag{3.60}$$

On the other hand, if $x \in \Gamma_2^\rho$, there exists $y \in D_2$ such that $|y-x| = d(x, \partial D_2) \le \gamma_1 \rho$. Again, since $u_2(y) = 0$, we have

$$\begin{aligned} |u_1(x)| &\le |u_1(x) - u_1(y)| + |u_1(y) - u_2(y)| \\ &\le C\Big(\frac{\rho}{\rho_0}\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)} + \max_{\partial G^\rho \setminus \partial\Omega} |w|\Big), \end{aligned} \tag{3.61}$$

where $w = u_1 - u_2$. Combining (3.60), (3.61) and (3.59) and recalling (3.18) and (2.92) we have:

$$\int_{D_2\setminus D_1} |\nabla u_1|^2 \le \frac{C}{\mu}\rho_0^{n-2}(\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)}+1+\mu)\Big(\|g\|^2_{\mathbf{C}^{1,\alpha}(\Gamma)}\frac{\rho}{\rho_0}+\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)}\max_{\partial G^\rho\setminus\partial\Omega}|w|\Big) \tag{3.62}$$

We now need to estimate $\max_{\partial G^\rho \setminus \partial\Omega} |w|$. We will do so by means of Proposition 3.12. Take $x \in \partial G^\rho \setminus \partial\Omega$ and define

$$\rho^* = \min\left\{\frac{\rho_0}{16(1+M_0^2)}, \frac{\widehat{\rho}}{16}\right\}, \tag{3.63}$$

$$x_0 = P_0 - \frac{\min\{\rho_{00}, \widehat{\rho}\}}{16}\nu, \tag{3.64}$$

where ν is the outer normal to $\partial\Omega$ at the point P_0. By construction $x_0 \in \overline{\widetilde{\Omega}_{\frac{\rho^*}{2}}}$. There exists an arc $\gamma : [0,1] \mapsto G^\rho \cap \overline{\widetilde{\Omega}_{\frac{\rho^*}{2}}}$ such that $\gamma(0) = x_0$, $\gamma(1) = x$ and $\gamma([0,1]) \subset G^\rho \cap \overline{\widetilde{\Omega}_{\frac{\rho^*}{2}}}$. Let us define

$$\rho_3 = \min\{\gamma_0 \rho \vartheta^*, \widehat{\rho}\},\ \rho_2 = \frac{3}{4}\rho_3,\ \rho_1 = \frac{1}{4}\rho_3, \tag{3.65}$$

where ϑ^* is the constant given in Theorem 3.9. We pick a sequence of $S+1$ times t_i and points $x_i = \gamma(t_i)$, $i = 0 \dots S$, defined by the following construction. Call $t_0 = 0$, then:

$$\begin{aligned} &t_i = \max\big\{t \in (0,1] \text{ s.t. } |\gamma(t) - x_i| = \frac{\rho_3}{2}\big\}, \text{ if } |x_i - x| > \frac{\rho_3}{2}, \\ &\quad \text{otherwise, } i = S, \end{aligned}$$

and stop the process. The number of spheres is bounded by

$$S \le C\Big(\frac{\rho_0}{\rho}\Big)^n$$

where C only depends on α, M_0, M_1 and $\mathcal{E}$. The balls $B_{\frac{\rho_3}{2}}(x_i)$ are pairwise disjoint, the distance between two consecutive centers is given by

$$|x_{i+1} - x_i| = \frac{\rho_3}{2},\ i = 0 \dots S-1,\ |x_S - x| \le \frac{\rho_3}{2}.$$

We iterate the three spheres inequality (3.48) on a chain of spheres with radii ρ_1, ρ_2 and ρ_3, this leads us to

$$\int_{B_{\rho_2}(x)} |w|^2 dx \le C\Big(\int_G |w|^2 dx\Big)^{1-\delta^S}\Big(\int_{B_{\rho_3}(x_0)} |w|^2 dx\Big)^{\delta^S} \tag{3.66}$$

where $0<\delta<1$ and $C>0$ only depend on M_0, α and $\mathcal{E}$. From our choice of $\bar{\rho}$ and ϑ^*, it follows that $B_{\rho_1}(x_0) \subset B_{\rho^*}(x_0) \subset G \cap B_{\frac{3\rho_1}{4}}(P^*)$, where we follow the notations from Theorem 3.13. We may apply Theorem 3.13 to w; thus, using (3.55), (3.17), (3.19) and (3.14) on (3.66) we have:

$$\int_{B_{\rho_2}(x)} |w|^2 dx \le C\rho_0^{n-2}\epsilon^{2\tau\delta^S}. \tag{3.67}$$

We apply again the interpolation inequality (2.109) to w in $B_{\rho_2}(x)$, using (3.67) and (3.18) we obtain

$$\|w\|_{\mathbf{L}^\infty(B_{\rho_2}(x))} \le C\Big(\frac{\rho_0}{\rho}\Big)^{\frac{n}{2}}\epsilon^{\gamma\delta^S}, \tag{3.68}$$

where $\gamma = \frac{2\tau}{n+2}$. Finally, from (3.68) and (3.62), and recalling (3.14) we get:

$$\int_{D_2\setminus D_1} |\nabla u_1|^2 \le C\rho_0^{n-2}\Big(\frac{\rho}{\rho_0} + \Big(\frac{\rho_0}{\rho}\Big)^{\frac{n}{2}}\epsilon^{\gamma\delta^S}\Big), \tag{3.69}$$

with C only depending on M_0, α, μ and $\mathcal{E}$. Let us now choose ρ depending upon ϵ, of the form

$$\rho(\epsilon) = \rho_0\left(\frac{2S\log|\delta|}{\log|\log\epsilon^\gamma|}\right)^{-\frac{1}{n}}.$$

We have that ρ is defined and increasing in the interval $(0, e^{-1})$. Call $\overline{\zeta}$ the number such that $\rho(\overline{\zeta}) = \min\{\overline{\rho}, \widehat{\rho}\}$, and let $\widetilde{\zeta} = \min\{\overline{\zeta}, \exp(-\gamma^2)\}$. Since the thesis is trivial for larger values of ϵ, it is not restrictive to prove it only in the smaller interval $(0, \tilde{\zeta})$. We are able to apply (3.69) to (3.57) with $\rho = \rho(\epsilon)$ for $\epsilon \in (0, \tilde{\zeta})$ to obtain

$$\int_{D_2\setminus D_1} |\nabla u_1|^2 \le C\rho_0^{n-2}\log|\log\epsilon|^\gamma, \tag{3.70}$$

and since $\epsilon \le \exp(-\gamma^2)$ it is elementary to prove that

$$\log|\log\epsilon^\gamma| \ge \frac{1}{2}\log|\log\epsilon|,$$

so that (3.70) finally reads

$$\int_{D_2\setminus D_1} |\nabla u_1|^2 \le C\rho_0^{n-2}\,\omega(\epsilon),$$

with $\omega(t) = \log|\log t|^{\frac{1}{n}}$ defined for all $0<t<e^{-1}$, and C depends on M_0, M_1, α, μ and $\mathcal{E}$. $\square$

Proof of Proposition 3.8. We will prove the thesis for u_1, the case u_2 being completely analogous. First of all, we observe that

$$\int_{D_2\setminus D_1}\mu|\nabla u_1|^2 \leq \int_{\Omega_1\setminus G}\mu|\nabla u_1|^2 = \int_{\partial(\Omega_1\setminus G)}\mu(\nabla u_1\cdot\nu)u_1 - p_1(u_1\cdot\nu) - \frac{1}{2}u_1\cdot\nu|u_1|^2, \tag{3.71}$$

and that

$$\partial(\Omega_1\setminus G)\subset\partial D_1\cup(\partial D_2\cap\partial G).$$

If we recall the no-slip condition, apply to (3.71) computations similar to those in (3.57), (3.58), we get

$$\int_{D_2\setminus D_1}|\nabla u_1|^2 \leq \frac{C}{\mu}\rho_0^{n-2}(\|g\|_{\mathbf{C}^{1,\alpha}(\Gamma)}+1+\mu)\max_{\partial D_2\cap\partial G}|w|, \tag{3.72}$$

where again $w = u_1 - u_2$, and C only depends on α, M_0 and M_1. Take a point $z\in\partial G$. To evaluate $\max|w|$ on $\partial D_2\cap\partial G$, we start by choosing a point $z\in\partial G$ and estimating $\|\nabla u\|$ on a ball centered in z, in terms of $\|\nabla u\|$ evaluated on a ball centered in x_0. We will do so by applying iteratevely the three spheres inequality, twice. By exploiting the regularity assumptions on ∂G, we find a cone centered in z, which we denote by $C(z,\xi,\vartheta_0)$ (where $\vartheta_0 = \arctan\frac{\rho_0}{M_0}$ is half the aperture of the cone and $\xi\in\mathbb{R}^n$ is a unit vector representing the direction of the cone), such that $C(z,\xi,\vartheta_0)\cap B_{\tilde\rho_0}(z)\subset G$. It can be shown ([8, Proposition 5.5]) that G_ρ is connected for $\rho\leq\frac{\tilde\rho_0 h_0}{3}$ with h_0 only depending on M_0. We now claim (without proof, see [13] and [58] for the detailed constructions in the same context) that we may build $\lambda_1>0$ and $\theta_1>0$, such that, if we define

$$\begin{aligned} w_1 &= z+\lambda_1\xi,\\ \rho_1 &= \vartheta^* h_0\lambda_1\sin\vartheta_1. \end{aligned}$$

where $0<\vartheta^*\leq 1$ was introduced in Theorem 3.9, then the following claims hold: $B_{\rho_1}(w_1)\subset C(z,\xi,\vartheta_1)\cap B_{\tilde\rho_0}(z)$ and $B_{\frac{4\rho_1}{\vartheta^*}}(w_1)\subset C(z,\xi,\vartheta_0)\cap B_{\tilde\rho_0}(z)\subset G$. Furthermore $\frac{4\rho_1}{\vartheta^*}\leq\rho^*$, hence $B_{\frac{4\rho_1}{\vartheta^*}}(x_0)\subset G$, where ρ^* and x_0 were defined by (3.63) and (3.64) respectively. It follows that w_1, $x_0\in\overline{G_{\frac{4\rho_1}{\vartheta^*}}}$, which is connected by construction. Iterating the three spheres inequality (3.48) (mimicking the construction made in the previous proof)

$$\int_{B_{\rho_1}(w_1)}|w|^2dx \leq C\Big(\int_G|w|^2dx\Big)^{1-\delta^S}\Big(\int_{B_{\rho_1}(x_0)}|w|^2dx\Big)^{\delta^S} \tag{3.73}$$

where $0<\delta<1$ and $C\geq 1$ depend only on n and $\mathcal{E}$, and $S\leq\frac{M_1\rho_0^n}{\omega_n\rho_1^n}$. We apply Theorem 3.13 in the same fashion as the previous proof, which leads to

$$\int_{B_{\rho_1}(w_1)}|w|^2\leq C\rho_0^n\epsilon^{2\beta},$$

where $0 < \beta < 1$ and $C > 0$ only depend on α, M_0, $\mathcal{E}$ and $\frac{\tilde{\rho}_0}{\rho_0}$. So far the estimate we have is only on a ball centered in w_1, we need to approach $z \in \partial G$ using a sequence of balls, all contained in $C(z, \xi, \vartheta_1)$, by suitably shrinking their radii. Take

$$\chi = \frac{1 - \sin\vartheta_1}{1 + \sin\vartheta_1}$$

and define, for $k \geq 2$,

$$\begin{aligned} \lambda_k &= \chi\lambda_{k-1}, \\ \rho_k &= \chi\rho_{k-1}, \\ w_k &= z + \lambda_k\xi. \end{aligned}$$

With these choices, $\lambda_k = \lambda\chi^{k-1}\lambda_1$, $\rho_k = \chi^{k-1}\rho_1$ and $B_{\rho_{k+1}}(w_{k+1}) \subset B_{3\rho_k}(w_k)$, $B_{\frac{4}{\vartheta^*}\rho_k}(w_k) \subset C(z, \xi, \vartheta_0) \cap B_{\tilde{\rho}_0}(z) \subset G$. Denote by

$$d(k) = |w_k - z| - \rho_k,$$

we also have

$$d(k) = \chi^{k-1}d(1),$$

with

$$d(1) = \lambda_1(1 - \vartheta^* \sin\vartheta_1).$$

Now take any $\rho \leq d(1)$ and let $k = k(\rho)$ the smallest integer such that $d(k) \leq \rho$, explicitly

$$\frac{\left|\log\frac{\rho}{d(1)}\right|}{\log\chi} \leq k(\rho) - 1 \leq \frac{|\log\frac{\rho}{d(1)}|}{\log\chi} + 1. \tag{3.74}$$

We iterate the three spheres inequality (3.12) over the chain of balls centered in w_j and radii ρ_j, $3\rho_j$, $4\rho_j$, for $j = 1, \ldots, k(\rho) - 1$, which yields

$$\int_{B_{\rho_{k(\rho)}}(w_{k(\rho)})} |w|^2 \leq C\rho^n \epsilon^{2\beta\delta^{k(\rho)-1}}, \tag{3.75}$$

with C only depending on α, M_0, $\mathcal{E}$ and $\frac{\tilde{\rho}_0}{\rho_0}$. Using the interpolation inequality (2.109) and (3.18) we obtain

$$\|w\|_{\mathbf{L}^\infty(B_{\rho_{k(\rho)}}(w_{k(\rho)}))} \leq C\frac{\epsilon^{\beta_1\delta^{k(\rho)-1}}}{\chi^{\frac{n}{2}(k(\rho)-1)}}, \tag{3.76}$$

where $\beta_1 = \frac{2\beta}{n+2}$ depends only on α, M_0, M_1, $\mathcal{E}$ and $\frac{\tilde{\rho}_0}{\rho_0}$. From (3.76)) and (3.18) we obtain

$$|w(z)| \leq C\left(\frac{\rho}{\rho_0} + \frac{\epsilon^{\beta_1\delta^{k(\rho)-1}}}{\chi^{\frac{n}{2}(k(\rho)-1)}}\right), \tag{3.77}$$

Finally, call

$$\rho(\epsilon) = d(1)|\log \epsilon^{\beta_1}|^{-B},$$

with

$$B = \frac{|\log \chi|}{2\log|\delta|}.$$

and let $\tilde{\zeta} = \exp(-\beta_1^{-1})$. We have that $\rho(\epsilon)$ is monotone increasing in the interval $0 < \epsilon < \tilde{\zeta}$, and $\rho(\tilde{\zeta}) = d(1)$, so $\rho(\epsilon) \leq d(1)$ there. By choosing $\rho = \rho(\epsilon)$ from (3.77) and (3.72) we obtain

$$\int_{D_2 \setminus D_1} |\nabla u_1|^2 \leq C\rho_0^{n-2}|\log \epsilon|^{-B}, \tag{3.78}$$

where C only depends on μ, α, M_0, $\mathcal{E}$ and $\frac{\widetilde{\rho}_0}{\rho_0}$. □

3.6 Proof of Theorem 3.13

This section is entirely devoted to the proof of Theorem 3.13. Let us start by recalling some notations from the previous chapter that we will need again. We define

$$Q(P_0) = B'_{\rho_{00}}(0) \times \Big[-\frac{M_0\rho_0^2}{\sqrt{1+M_0^2}}, \frac{M_0\rho_0^2}{\sqrt{1+M_0^2}} \Big],$$

and

$$\Gamma_0 = \partial E \cap Q(P_0). \tag{3.79}$$

Finally, let us call $E^- = Q(P_0) \setminus E$ and $\widetilde{E} = E \cup E^- \cup \Gamma_0$. We start the proof by choosing a vector $\widetilde{A}$ and a matrix $\widetilde{B}$ such that $A = \widetilde{A}$, $B = \widetilde{B}$ in E, and

$$\|\widetilde{A}\|_{C^{1,\alpha}(\widetilde{E})} + \|\widetilde{B}\|_{C^{0,\alpha}(\widetilde{E})} \leq C_1\mathcal{E}, \tag{3.80}$$

where $C_1 > 0$ only depends on α and M_0. Our aim is to build an extension $\widetilde{w}$ of w such that it satisfies the extended problem

$$\begin{aligned} \operatorname{div} \sigma(\widetilde{w}, \widetilde{q}) + \widetilde{A} \cdot \nabla \widetilde{w} + \widetilde{B}\widetilde{w} = \Phi \quad &\text{in } \widetilde{E}, \\ \operatorname{div} \widetilde{w} = 0 \quad &\text{in } \widetilde{E}, \end{aligned} \tag{3.81}$$

where $\Phi \in \mathbf{H}^{-1}(\widetilde{E})$ is such that

$$\|\Phi\|_{\mathbf{H}^{-1}(\widetilde{E})} \leq C\|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}. \tag{3.82}$$

Define $\widetilde{w}$, $\widetilde{q}$ as follows:

$$\widetilde{w} = \begin{cases} w & \text{in } E, \\ 0 & \text{in } E^-. \end{cases}$$

$$\widetilde{q} = \begin{cases} q & \text{in } E, \\ 0 & \text{in } E^-. \end{cases}$$

We have that $\widetilde{w} \in \mathbf{H}^1(\widetilde{E})$ and that $\operatorname{div} \widetilde{w} = 0$, in the weak sense in $\widetilde{E}$.

In order to write a system (3.81) for $(\widetilde{w}, \widetilde{q})$, we take any $v \in \mathbf{H}^1_0(\widetilde{E})$ and consider

$$\int_{\widetilde{E}} \sigma(\widetilde{w}, \widetilde{q}) \cdot \nabla v = \int_E \sigma(w, q) \cdot \nabla v + \int_{E^-} \sigma(\widetilde{w}, \widetilde{q}) \cdot \nabla v. \tag{3.83}$$

By the divergence theorem on the first term we obtain

$$\int_E \sigma(w, q) \cdot \nabla v = -\int_E (A \cdot \nabla w) \cdot \nabla v - \int_E Bw \cdot \nabla v + \int_\Gamma \psi \cdot v. \tag{3.84}$$

Define $\Phi(v) = \int_\Gamma \psi \cdot v$ for all $v \in \mathbf{H}^1_0(\widetilde{E})$. Using (3.84) and the trace theorem:

$$\left|\Phi(v)\right| \leq \|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \|v\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq C\rho_0 \|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \|v\|_{\mathbf{H}^1(E^-)}, \tag{3.85}$$

therefore $\widetilde{w}$ satisfies (3.81) in the weak sense, with Φ satisfying (3.82). We now want to apply the three spheres inequality to the inhomogeneous system (3.81). In order to do so, we need to establish local well posedness for the Cauchy problem for the linearized Navier Stokes equations. We claim the following: there exists $\widehat{\rho}$ such that the problem

$$\begin{cases} \operatorname{div} \sigma(w^*, q^*) + \widetilde{A} \cdot \nabla w^* + \widetilde{B} w^* & = \Phi \quad \text{in } B_{\widehat{\rho}}, \\ \operatorname{div} w^* & = 0 \quad \text{in } B_{\widehat{\rho}}, \\ w^* & = 0 \quad \text{on } \partial B_{\widehat{\rho}}. \end{cases} \tag{3.86}$$

admits a weak solution w^* in the ball $B_{\widehat{\rho}}$ such that

$$\|w^*\|_{\mathbf{H}^1_0(B_{\widehat{\rho}})} \leq C\rho_0 \|\Phi\|_{\mathbf{H}^{-1}(E^-)}, \tag{3.87}$$

This can be shown by projecting (3.86) on the space of divergence free function, so that it becomes a pressure free second order partial differential equation in w^* only, with the principal part being the laplacian operator, and the lower order terms are continuous and bounded by a constant depending only on α, M_0 and $\mathcal{E}$. Therefore the existence of a solution and the well posedness of (3.86) is shown by a standard coercivity argument for sufficiently small radii. Using linearity we write $\widetilde{w} = w_0 + w^*$, with $w^* \in \mathbf{H}^1_0(B_{\widehat{\rho}})$ such that (w^*, q^*) solves (3.86) and w_0 solves

$$\begin{cases} \operatorname{div} \sigma(w_0, q_0) + \widetilde{A} \cdot \nabla w_0 + \widetilde{B} w_0 & = 0 \quad \text{in } B_{\widehat{\rho}}, \\ \operatorname{div} w_0 & = 0 \quad \text{on } B_{\widehat{\rho}}. \end{cases} \tag{3.88}$$

Using interior regularity of solutions for elliptic systems we get

$$\|w_0\|_{\mathbf{L}^\infty(B_{\frac{t}{2}}(x))} \leq t^{-\frac{n}{2}} \|w_0\|_{\mathbf{L}^2(B_{\frac{t}{2}}(x))}. \tag{3.89}$$

We will thus need to estimate $\|w_0\|_{\mathbf{L}^2(B(x))}$ on a ball near the boundary. We may apply Theorem 3.9 to w_0, thus, calling and $r_3 = \widehat{\rho}$, $r_1 = \frac{r_3}{8}$, $r_2 = \frac{3\,r_3}{8}$ we have (understanding that all balls are centered in P_0)

$$\|w_0\|_{\mathbf{L}^2(B_{r_2})} \leq C\|w_0\|^{\tau}_{\mathbf{L}^2(B_{r_1})}\|w_0\|^{1-\tau}_{\mathbf{L}^2(B_{r_3})}, \tag{3.90}$$

with $C > 0$ only depending on n and $\mathcal{E}$. Let us call $\eta = \rho_0\|\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}$. By the triangle inequality we have that

$$\|w_0\|_{\mathbf{L}^2(B_r)} \leq \|\widetilde{w}\|_{\mathbf{L}^2(B_r)} + \|w^*\|_{\mathbf{L}^2(B_r)} \leq \|\widetilde{w}\|_{\mathbf{L}^2(B_r)} + C\eta, \tag{3.91}$$

for $r = r_1, r_3$; furthermore,

$$\|\widetilde{w}\|_{\mathbf{L}^2(B_{r_2})} \leq \|w_0\|_{\mathbf{L}^2(B_{r_2})} + \|w^*\|_{\mathbf{L}^2(B_{r_2})} \leq \|w_0\|_{\mathbf{L}^2(B_{r_2})} + C\eta, \tag{3.92}$$

where (in both cases) $C > 0$ only depends on n and $\mathcal{E}$. Putting together (3.90), (3.91), (3.92) we get

$$\begin{aligned}
&\|w\|_{\mathbf{L}^2(B_{r_2}\cap E)} \leq \|\widetilde{w}\|_{\mathbf{L}^2(B_{r_2}\cap E)} \leq \\
\leq &C\eta + C(\|\widetilde{w}\|_{\mathbf{L}^2(B_{r_1})} + C\eta)^{\tau}(\|\widetilde{w}\|_{\mathbf{L}^2(B_{r_3}\cap E)} + C\eta)^{1-\tau} \leq \\
\leq &C\big(\eta + \eta^{\tau}(\eta + \|w\|_{\mathbf{L}^2(E)})^{1-\tau}\big) \leq C\eta^{\tau}\|w\|^{1-\tau}_{\mathbf{L}^2(E)},
\end{aligned} \tag{3.93}$$

which concludes the proof.

Chapter 4

Numerical approaches to reconstruction

In this section we discuss the numerical implementation of the actual reconstruction of an obstacle immersed in a Stokes fluid. We recall that we consider a bounded domain Ω with smooth boundary $\partial\Omega$, containing an unknown inclusion $D \subset \Omega$ and filled with a fluid whose velocity u and pressure p obey the following system of equations and boundary conditions:

$$\begin{cases} \triangle u - \nabla p &= 0, \\ \operatorname{div} u &= 0. \end{cases} \tag{4.1a}$$

$$\begin{cases} u = g, & \text{on } \partial\Omega, \\ u = 0, & \text{on } \partial D, \end{cases} \tag{4.1b}$$

where $g \in \mathbf{H}^{1/2}(\partial\Omega)$ is such that $\int_{\partial\Omega} g \,\mathrm{d}s = 0$. The ideal experiment we perform is to assign a velocity $g \in \mathbf{H}^{1/2}(\partial\Omega)$, and measure on $\partial\Omega$ the resulting normal component of the stress tensor $\sigma(u,p)\cdot\nu$ where

$$\sigma(u,p) = 2\widehat{\nabla} u - p\,\mathbb{I}, \tag{4.2}$$

$\widehat{\nabla} u = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetrized gradient and $\mathbb{I}$ is the 2×2 identity matrix. The inverse problem consists in recovering ∂D with one measurement of $\sigma(u,p)\cdot\nu$ on $\partial\Omega$. This inverse problem is severely ill posed, and we showed that only a weak rate of continuity can be restored by adding a priori hypotheses on the boundary data g and on the unknown boundary ∂D.

From now on, we will always assume $\partial\Omega$ and ∂D to be C^∞ smooth curves, with outer normal fields ν_Ω and ν_D, respectively, that are star shaped with respect to the origin and are parametrized by $\{z_\Omega(t) : 0 \le t \le 2\pi\}$ and $\{z_D(t) : 0 \le t \le 2\pi\}$ respectively. Moreover, we will assume that the radial functions are of the form

$$z_S(t) = r_S(t)(\cos t, \sin t), \qquad t \in [0, 2\pi], \tag{4.3}$$

for $S = \Omega$ and $S = D$, the functions r_S being strictly positive and smooth. We may formulate the inverse problem as the inversion of the boundary operator described above,

$$\begin{aligned} F : \mathbf{C}^\infty([0, 2\pi]) &\to \mathbf{H}^{-1/2}(\partial\Omega) \\ r_D &\mapsto \sigma(u, p) \cdot \nu|_{\partial\Omega}. \end{aligned} \tag{4.4}$$

The operator F above is nonlinear and ill posed. Furthermore, in any real life experiment, the measurements are never exact, since a certain amount of error due to noise and limited precision is unavoidable, therefore, a direct inversion of F is unfeasible. In this chapter we will first see how to tame the ill posedness of the problem by regularizing it: to this end, we introduce the so called iterative regularization methods. Then, we formulate the Stokes equations (4.1) in a form that is appropriate for numerical implementation, namely, we rewrite them as boundary integral equations; once those are set, we investigate the properties of the integral operators and discretize them. The numerical study is not complete, as the work is still in progress. We would like to remark, however, that the work described in this section is sufficient to numerically implement the problem even though the theory is not complete. We can, nonetheless, get a feel of the numerical efficiency of the procedure by analyzing a closely related, yet simpler, inverse problem of the same kind, this time connected with the Laplace equations. In the last section, after a brief but necessary description of the approach, we show the numerical results of the reconstruction algorithm for the Laplace equation. This should be considered as a preliminary study towards the analysis of the inverse problem (4.1).

4.1 Iterative regularization methods

We consider the following abstract setting. Let $F : D(F) \subset X \to Y$ be a (possibly nonlinear) operator between Hilbert spaces X and Y, and consider the equation

$$F(x) = y. \tag{4.5}$$

We assume that $y \in R(F)$, so that (4.5) admits a solution $x^\dagger$, which we will assume to be unique. As it is always the case in practice, the exact data y will be polluted by noise, so that the right hand side of (4.5) has to be substituted with the noisy data y^δ with the assumption that

$$\|y^\delta - y\| \leq \delta, \tag{4.6}$$

for a fixed parameter δ, called *noise level*. Our model problem is of the form (4.5), and it is ill-posed in the sense that it lacks continuous dependence upon the boundary data. For this reason, the presence of noise makes the direct inversion of F in (4.5) hopeless in practice. To get stable approximations of the solution, one thus needs to exploit different techniques; in particular, the so called *regularization methods*. We refer to the monograph [27] for a thorough discussion on the subject. The main idea of a regularization method is to approximate the inverse of the operator F with a continuous operator. Perhaps the best known among such methods is the *Tikhonov regularization*, which consists on solving the minimization problem

$$\min_{x \in D(F)} \|F(x) - y^\delta\|^2 + \alpha \|x - x_0\|^2, \tag{4.7}$$

where $x_0 \in X$ is an initial guess for $x^\dagger$. When F is a linear operator, the solvability of (4.7) is guaranteed by the following (see for example [27]):

Theorem 4.1. *Let $F = K$ in* (4.5) *be a linear operator. Then the operator $K^*K + \alpha I$ is boundedly invertible, and the Tikhonov functional* (4.7) *has a unique minimum in x_α^δ for all $\alpha > 0$, $y^\delta \in Y$, and $x_0 \in X$, given by*

$$x_\alpha^\delta = (K^*K + \alpha I)^{-1}(K^*y^\delta + \alpha x_0). \tag{4.8}$$

We need to point out that, in general, the convergence of a regularization method to the real solution can be arbitrarily slow. Some additional hypotheses on the exact solution $x^\dagger$ are needed in order to be able to estimate the convergence rate. The most common such requests are of the form

$$x^\dagger = f(K^*K)w, \qquad \|w\| \leq \rho, \tag{4.9}$$

for a continuous function $f : [0, \|K^*K\|] \to [0, \infty)$ with $f(0) = 0$. Conditions as such are called *source condition*. We shall not treat this issue, leaving the reader to [40] for a detailed overview of convergence results for different source conditions applied to various regularization methods.
In the case of nonlinear operator F, stability and convergence cannot be guaranteed in general. We introduce a wider class of regularization methods, called *iterative regularization methods*, according to the following definition.

Definition A sequence

$$x_{n+1}^\delta = G(x_n^\delta, \dots, x_0, y^\delta) \tag{4.10}$$

together with a *stopping rule*, that is an integer $N = N(\delta, y^\delta)$ is an *iterative regularization method* for F if, for all $x^\dagger \in D(F)$, for all y^δ satisfying (4.6) and for all initial guesses x_0 sufficiently close to $x^\dagger$, the following conditions hold:

1. $N(\delta, y^\delta) < \infty$ for $\delta > 0$ and x_n^δ is well defined for all $n \leq N(\delta, y^\delta)$,
2. If the data is exact ($\delta = 0$) then either $N(\delta, y^\delta) < \infty$ and $x_{N(\delta, y^\delta)} = x^\dagger$ or $N(\delta, y^\delta) = \infty$ and $\left\|x_n^\delta - x_0\right\| \to 0$ as $n \to \infty$,

3. the following regularizing property holds:

$$\sup_{\|y^\delta - y\| \leq \delta} \|x^\dagger - x_{N(\delta, y^\delta)}\| \to 0 \text{ as } \delta \to 0. \tag{4.11}$$

The stopping rule has to be chosen appropriately since, typically, the iterations deteriorate after a few steps due to propagation of errors. A very common choice, which we will adopt, is the *discrepancy principle*, which consists in taking $N(\delta, y^\delta)$ as the smallest integer for which

$$\|F(x_N) - y^\delta\| \leq \tau\delta, \tag{4.12}$$

where $\tau > 1$ is a fixed constant.
The iterative method we use belongs to the family of the so called Newton methods. These consist in iteratively computing the update h_k by solving the linearized form of (4.5),

$$F'[x_k^\delta]h_k = y^\delta - F(x_k^\delta), \tag{4.13}$$

which we may equivalently formulate in terms of minimization of the associated energy functional

$$h_k = \min_h \|y^\delta - F(x_k^\delta) - F'[x_k^\delta]h\|^2, \tag{4.14}$$

and then set $x_{k+1}^\delta = x_k^\delta + h_k$. The problem (4.13) will typically inherit the ill posedness from the nonlinear counterpart (4.5); therefore, it will need to be regularized as well. If we apply Tikhonov regularization, we obtain

$$h_k = \min_h \|y^\delta - F(x_k^\delta) - F'[x_k^\delta]h\|^2 + \alpha_k\|h\|^2. \tag{4.15}$$

This leads to the *Levenberg-Marquardt regularization algorithm.* Recalling Theorem 4.1, the $k+1$-th iteration can be computed explicitly as follows:

$$x_{k+1}^\delta = x_k^\delta + (F'[x_k^\delta]^* F'[x_k^\delta] + \alpha_k I)^{-1} F'[x_k^\delta]^*(y^\delta - F(x_k^\delta)). \tag{4.16}$$

We shall actually exploit a related method, due to Bakushinkii, for which the minimization problem (4.15) is modified into

$$\min_h \|y^\delta - F(x_k^\delta) - F'[x_k^\delta]h\|^2 + \alpha_k\|h + x_k^\delta - x_0\|^2, \tag{4.17}$$

to which we may apply again Theorem 4.1 to obtain the explicit unique solution

$$x_{k+1}^\delta = (F'[x_k^\delta]^* F'[x_k^\delta] + \alpha_k I)^{-1}(F'[x_k^\delta]^*(y^\delta - F(x_k^\delta)) + \alpha_k(x_0 - x_k^\delta)). \tag{4.18}$$

This is called the *iteratively regularized Gauss-Newton method.* The modified term now involves the distance of the new iterate x_{k+1}^δ from the initial guess x_0, as opposed to (4.15). This, as it turns out, adds stability to the iterative process, as it prevents errors from noise from adding up along the iterations. Naturally, the convergence of both methods (4.15) and (4.17) will depend upon numerous factors. First of all, the parameters $\{\alpha_k\}$ must converge to 0 as $k \to \infty$. A very common choice is to take $\alpha_k = \alpha_0 q^k$ where $0 < q < 1$. A popular choice for the stopping rule is again the discrepancy principle. Finally, some hypotheses on the behavior of the nonlinearity of F, and source conditions of the form (4.9) must be required. Unfortunately we were not able to prove such conditions for the inverse problem for the Stokes equations. We hope that future research will fill in this gap. We refer again the reader to [40] for a convergence analysis of iterative regularization methods for nonlinear ill posed problems.

4.2 The boundary integral equations method

In this section we formulate the Stokes equations (4.1) using boundary integrals. We start by introducing the fundamental solution of the Stokes equation, or the *Stokes fundamental tensor*, defined for all $x \neq 0$ as follows:

$$
\begin{aligned}
E(x) &= \frac{1}{4\pi}\Big(\frac{x \otimes x}{|x|^2} - \log|x|\,\mathbb{I}\Big), \\
e(x) &= \operatorname{grad}\Phi(x),
\end{aligned}
\tag{4.19}
$$

where $\Phi(x) = -\frac{1}{2\pi}\log|x|$. The fundamental stress tensor associated to the fundamental solution is then given by

$$
\sigma(E, e) = 2\widehat{\nabla} E - e \otimes I = \frac{1}{\pi} \cdot \frac{x \otimes x \otimes x}{|x|^4}. \tag{4.20}
$$

Following the ideas arising from potential theory ([43], [70]), given a smooth bounded domain A with $0 \in \operatorname{int}(A)$, if φ is an integrable vector on ∂A, we define the following integral operators for all points $x \notin \partial A$:

$$
S\varphi(x) = \int_{\partial A} E(x-y)\varphi(y)\,\mathrm{d}s_y, \tag{4.21}
$$

and

$$
P\varphi(x) = \int_{\partial A} e(x-y)\varphi(y)\,\mathrm{d}s_y. \tag{4.22}
$$

The pair $(S\varphi, P\varphi)$ is called the *single layer potential for the Stokes equation* with density φ. One can see that, if A is a bounded domain with smooth boundary ∂A, then the single layer potential will solve (4.1a) for any integrable density φ, both in the interior domain A and the exterior $\mathbb{R}^2 \setminus \overline{A}$. With a slight abuse of notation we shall denote by $\sigma(S\varphi, P\varphi)$ the stress tensor associated with the potentials described above, so that

$$
\sigma(S\varphi, P\varphi)(x) = \int_{\partial A} \sigma(E, e)(x-y)\varphi(y)\,\mathrm{d}s_y. \tag{4.23}
$$

We need to investigate the singular behavior of the single layer potential for the Stokes equation and the normal component of the stress tensor $\sigma(S\varphi, P\varphi)\nu_A$ near the boundary of the domain A, since the integral kernels are singular there. The following continuity and jump relations hold (see [49], [70], [44] and also [63]):

Theorem 4.2. *Let $\varphi \in C^{0,\alpha}(\partial A)$. Let ν_A denote the outer normal field to ∂A. The following hold for all $x \in \partial A$:*

$$\lim_{h\to 0^+} S\varphi(x \pm h\nu_A(x)) = S\varphi(x), \tag{4.24}$$

where the right hand side exists as a finite improper integral, and

$$\lim_{h\to 0^+} P\varphi(x \pm h\nu_A(x)) = P\varphi(x) \pm \frac{1}{2}\varphi(x)\cdot\nu_A(x), \tag{4.25}$$

$$\begin{gathered}\lim_{h\to 0^+} \frac{\partial S\varphi}{\partial \nu_A}(x \pm h\nu_A(x)) = \\ \int_{\partial A} \frac{\partial E}{\partial \nu_A(x)}(x-y)\varphi(y)\,\mathrm{d}s_y \mp \frac{1}{2}[\varphi(x) - \varphi(x)\cdot\nu_A(x)\nu_A(x)],\end{gathered} \tag{4.26}$$

where the terms $P\varphi(x)$ and $\int_{\partial A} \frac{\partial E}{\partial \nu_A(x)}(x-y)\varphi(y)\,\mathrm{d}s_y$ exist as Cauchy principal values. Moreover, the normal component of the stress tensor $\sigma(S\varphi, P\varphi)(x)\nu_A$ is well defined up to $x \in \partial A$, and

$$\lim_{h\to 0^+} \sigma(S\varphi, P\varphi)(x \pm h\nu_A(x))\nu_A(x) = \sigma(S\varphi, P\varphi)(x)\nu_A(x) \mp \frac{1}{2}\varphi(x). \tag{4.27}$$

We also report, without proof (except for the last which we shall examine more closely; see [44], with very slight modifications, for the remaining parts), the following result concerning the operators S and P:

Theorem 4.3. *The following regularity properties hold:*

1. *The single layer potential operator*

$$S : \mathbf{H}^r(\partial A) \to \mathbf{H}^{r+1}(\partial A) \tag{4.28}$$

is bounded for all $r \in \mathbb{R}$.

2. *The single layer potential S is a bounded operator*

$$\begin{aligned} S &: \mathbf{H}^{-1/2}(\partial A) \to \mathbf{H}^1(A), \\ S &: \mathbf{H}^{-1/2}(\partial A) \to \mathbf{H}^1(A_R), \end{aligned} \tag{4.29}$$

and P is a bounded operator

$$\begin{aligned} P &: \mathbf{H}^{-1/2}(\partial A) \to \mathbf{L}^2(A), \\ P &: \mathbf{H}^{-1/2}(\partial A) \to \mathbf{L}^2(A_R), \end{aligned} \tag{4.30}$$

where $A_R = \{x \in \mathbb{R}^2 \setminus \overline{A} : |x| \leq R\}$ for some sufficiently large R.

3. *The kernel of the operator S is given by $N(S) = \mathrm{span}\{\nu_A\}$.*

The operator S is injective only if we restrict it to the smaller quotient space of functions which are orthogonal to ν_A. Since it will be of use later on, let us sketch a proof of the last point.

Suppose $S\varphi = 0$ on ∂A. Then $S\varphi = 0$ in A and from the continuity of S across ∂A it follows that $S\varphi = 0$ everywhere, so from the Stokes equations, we have that $P\varphi$ is constant in A and in $\mathbb{R}^2 \setminus \overline{A}$. From the jump relations on the pressure (4.25) we then have that $\varphi \cdot \nu_A$ is equal to a constant. This combined with (4.26) implies that $\varphi \in \mathrm{span}\{\nu_A\}$. On the other hand, one can see that $S\nu_A = 0$ (see for example [52, Lemma 2.1]).

To deal with this non trivial null space, we can modify the operator S by defining

$$U\varphi = S\varphi + \langle \nu_A, \varphi\rangle \nabla\Phi, \tag{4.31}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket in $\mathbf{H}^{1/2}$, and again Φ is the fundamental solution of the Laplace equation. This operator is injective: in fact, suppose that $U\varphi = 0$ on ∂A. Then, using again the divergence theorem and the fact that $\mathrm{div}\, S\varphi = 0$ we have $\langle \nu_A, S\varphi\rangle = 0$; combining this with the well known fact that $\langle \nu_A, \nabla\Phi\rangle = -1$ (it follows from the residual theorem) we have that $\langle \nu_A, \varphi\rangle = 0$. Going back to (4.31) this implies $S\varphi = 0$, which, in turn, means $\varphi \in \mathrm{span}\{\nu_A\}$. But this together with $\langle \nu_A, \varphi\rangle = 0$ implies $\varphi = 0$, therefore U is injective. Observing that $U - S$ is compact, one can apply Riesz theory ([43]) to U to obtain that U is invertible. The proof is identical to [44], so we omit it.

Our aim is to write the solution of (4.1) in the form of a single layer potential with unknown density φ on the domain $A = \Omega\setminus\overline{D}$. To this extent, we have to write the solution (u, p) of (4.1) as a sum of two single layer potential, each concentrated on one connected component of the boundary, with unknown densities φ, ψ, defined for all points x which do not belong to $\partial\Omega \cup \partial D$ as follows:

$$u(x) = \int_{\partial\Omega} E(x-y)\varphi(y)\,\mathrm{d}s_y + \int_{\partial D} E(x-y)\psi(y)\,\mathrm{d}s_y, \tag{4.32}$$

$$p(x) = \int_{\partial\Omega} e(x-y)\varphi(y)\,\mathrm{d}s_y + \int_{\partial D} e(x-y)\psi(y)\,\mathrm{d}s_y. \tag{4.33}$$

We will later need also the induced stress tensor, given by

$$\sigma(u,p)(x) = \int_{\partial\Omega} \sigma(E,e)(x-y)\varphi(y)\,\mathrm{d}s_y + \int_{\partial D} \sigma(E,e)(x-y)\psi(y)\,\mathrm{d}s_y. \tag{4.34}$$

To assign the boundary conditions on $\partial\Omega$ and ∂D we apply Theorem 4.2, which yields

$$g(x) = \int_{\partial\Omega} E(x-y)\varphi(y)\,\mathrm{d}s_y + \int_{\partial D} E(x-y)\psi(y)\,\mathrm{d}s_y, \quad x \in \partial\Omega, \tag{4.35a}$$

$$0 = \int_{\partial\Omega} E(x-y)\varphi(y)\,\mathrm{d}s_y + \int_{\partial D} E(x-y)\psi(y)\,\mathrm{d}s_y, \;\; x \in \partial D. \tag{4.35b}$$

We introduce a shorthand notation for the integral operators in (4.35). We will denote by S_A the integral operator on ∂A with kernel E evaluated at a point on A (whose kernel will thus be singular), for example for all $x \in \partial\Omega$, we call

$$S_\Omega\varphi(x) = \int_{\partial\Omega} E(x-y)\varphi(y)\,\mathrm{d}s_y. \tag{4.36}$$

When the point of evaluation of the integral operator does not belong to the integration domain, the integral kernel is smooth. We will denote the operator by V with two indexes, the lower one denoting the domain of integration and the upper one the domain of evaluation: for example, for all $x \in \partial D$, we call

$$V_\Omega^D\varphi(x) = \int_{\partial\Omega} E(x-y)\varphi(y)\,\mathrm{d}s_y. \tag{4.37}$$

We also introduce its parametrized version

$$\widetilde{V}_\Omega^D\widetilde{\varphi}(t) = \int_0^{2\pi} E(x - z_\Omega(\tau))\widetilde{\varphi}(\tau)|z'_\Omega(\tau)|\mathrm{d}\tau, \;\; \text{for} \;\; t \in [0, 2\pi], \tag{4.38}$$

where $\widetilde{\varphi}(t) = \varphi(z_\Omega(t))$. The analogous definition is immediate to give also for the other operators. With these notations, we write (4.35) in the form

$$\begin{bmatrix} S_\Omega & V_D^\Omega \\ V_\Omega^D & S_D \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix} \tag{4.39}$$

Let us call

$$\begin{bmatrix} S_\Omega & V_D^\Omega \\ V_\Omega^D & S_D \end{bmatrix} = \mathcal{S},$$

so that (4.39) becomes the operator equation

$$\mathcal{S}\Psi = G, \tag{4.40}$$

with the unknown density $\Psi = \begin{bmatrix} \varphi \\ \psi \end{bmatrix}$, and boundary data $G = \begin{bmatrix} g \\ 0 \end{bmatrix}$.

Recalling the case of one single layer potential, it is almost obvious that the operator $\mathcal{S}$ cannot be injective either. Suppose $\mathcal{S}\Psi = 0$. Following a reasoning identical to the proof of point 3 of lemma 4.3, we find that $\psi \in \operatorname{span}\{\nu_D\}$ and $\varphi \in \operatorname{span}\{\nu_\Omega\}$. The converse also follows in the same way as for the operator S by applying the divergence theorem.

To eliminate this problem, we use the same kind of modification for $\mathcal{S}$ as we did with S. We consider the modified operator

$$\mathcal{U}\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \mathcal{S}\begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} \langle\varphi, \nu_\Omega\rangle \nabla\Phi|_{\partial\Omega} \\ \langle\psi, \nu_D\rangle \nabla\Phi|_{\partial D} \end{bmatrix} \tag{4.41}$$

Using an argument identical to that we used with U to $\mathcal{U}$ on each connected component of $\partial(\Omega \setminus D)$, we can show that $\mathcal{U}\Psi = 0$ implies $\Psi = 0$. Again, from injectivity, by Riesz theory for compact operators, one can obtain that $\mathcal{U}$ is invertible.

Once system (4.40) is solved for Ψ, the direct problem of evaluating the operator F in (4.4) is also solved: the stress tensor given in (4.34) can be evaluated using the density Ψ and recalling the jump relations (4.27).

4.3 Discretization

The integral operator equation (4.41) is of the first kind, and the parametrized version of the integral operator has a weakly singular kernel of logarithmic type. Following the ideas in [43], we adopt a collocation method based on trigonometric interpolation, which we shall briefly outline in what follows.

We choose a discretization level, that is an integer n, and consider $t_i = \frac{2\pi i}{n}$, for $i = 0, \dots, 2n-1$. We represent the curve $\partial\Omega$ with the set of collocation points $\{z_\Omega(t_i)\}_{i=0,\dots,2n-1}$, and we do the same for ∂D. We seek a solution by solving for its nodal value and identifying it by its interpolating trigonometric polynomial in the space $\mathcal{T}_n$, that is the space of functions of the form

$$v(t) = \sum_{m=0}^{n} a_m \cos(mt) + \sum_{m=1}^{n-1} b_m \sin(mt). \tag{4.42}$$

Furthermore, let us call the interpolation operator $Q_n : [0, 2\pi] \to \mathcal{T}_n$. We point out that the trigonometric interpolation just described is well defined for all functions in $\mathbf{H}^p$ for $p > 1/2$, which is a necessary restriction in order for the nodal values to be defined.

The approximation of the operator $\mathcal{S}$ is achieved by discretizing its components. The operators V_D^Ω and V_Ω^D, which are integral with a smooth kernel, can be approximated using the trapezoidal rule. For example, let us consider the parametrized operator $\widetilde{V}_\Omega^D$. Applying the trapezoidal rule to (4.38) yields the finite dimensional operator

$$\widehat{V}_\Omega^D \varphi(z_D(t)) \simeq \frac{\pi}{n} \sum_{j=0}^{2n-1} E(z_D(t) - z_\Omega(t_j)) \widetilde{\varphi}(t_j) |z'_\Omega(t_j)|. \tag{4.43}$$

An analogous formula holds of course for the finite dimensional operator $\widehat{V}_D^\Omega$ which approximates $\widetilde{V}_D^\Omega$ (whose definition is obvious). We point out that the trapezoidal rule is exact on $\mathcal{T}_n$.

The operators S_Ω and S_D, on the other hand, have a logarithmically singular kernel: therefore, we need to exploit a quadrature method to evaluate numerically integral operators having weakly singular kernels. Let us consider

$$\int_0^{2\pi} \Phi(z_\Omega(t) - z_\Omega(\tau))\zeta(\tau)\mathrm{d}\tau, \tag{4.44}$$

and let us split the kernel into

$$\begin{aligned} 4\pi\Phi(z_\Omega(t) - z_\Omega(\tau)) =& \log\left(4\sin^2\frac{t-\tau}{2}\right) - \log\left(\frac{|z_\Omega(t) - z_\Omega(\tau)|^2}{4\sin^2\frac{t-\tau}{2}}\right) = \\ =& w(|t-\tau|) + K_\Omega(t,\tau). \end{aligned} \tag{4.45}$$

We first observe that the second summand is smooth with diagonal values

$$\lim_{\tau\to t} K_\Omega(t,\tau) = \log|z'_\Omega(t)|^2, \tag{4.46}$$

therefore it can be treated with the trapezoidal rule as well.

Following the ideas in [43] (to which we refer for a detailed derivation of the upcoming technique) we approximate

$$W\zeta(t) = \int_0^{2\pi} w(|t-\tau|)\zeta(\tau)\mathrm{d}\tau$$

by a sequence of quadrature rules

$$W_n\zeta(t) = \sum_{j=0}^{2n-1} \alpha_j^{(n)}(t)\zeta(t_j), \tag{4.47}$$

for an appropriate choice of quadrature weights $\alpha_j^{(n)}$, explicitly

$$\alpha_j^{(n)}(t) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_j) - \frac{\pi}{n^2}\cos n(t-t_j), \tag{4.48}$$

for $j = 0,\dots,2n-1$. Let us note that the space $\mathcal{T}_n$ is invariant under W, and that, if $\zeta \in \mathcal{T}_n$, then $W\zeta = W_n\zeta$, that is the quadrature method is exact on the space of trigonometric polynomials.

It can be shown that the quadrature error $|W_nf - Wf|$ for analytic functions f is exponentially decreasing [43, Th. 12.13], uniformly in $[0,2\pi]$. Finally, we observe that the non logarithmic term

$$M(t,\tau) = \frac{(z_\Omega(t) - z_\Omega(\tau)) \otimes (z_\Omega(t) - z_\Omega(\tau))}{|z_\Omega(t) - z_\Omega(\tau)|^2}$$

of the kernel E is also regular, with diagonal values given by

$$\lim_{\tau\to t} M(t,\tau) = \frac{z'_\Omega(t)\otimes z'_\Omega(t)}{|z'_\Omega(t)|^2}. \tag{4.49}$$

We can therefore decompose the operator $\widetilde{\mathcal{U}}$ into $\widetilde{\mathcal{U}} = \mathcal{W}+\mathcal{A}+\mathcal{C}$, where $\mathcal{C}$ is the correction term given in (4.41), $\mathcal{W}$ is the block diagonal operator with 4 copies of W on the diagonal and zero elsewhere,

$$\mathcal{W} = \begin{bmatrix} W & & & \\ & W & & \text{\Large 0} \\ \text{\Large 0} & & W & \\ & & & W \end{bmatrix}$$

and $\mathcal{A}$ is a block operator whose component are integral operators with smooth kernels. We then approximate $\mathcal{W}$ by $\mathcal{W}_n$, which is a block operator of the same structure as $\mathcal{W}$ with W_n as diagonal entries. Then, denoting by $\widetilde{\varphi}_1 = \varphi$ and $\widetilde{\varphi}_2 = \psi$, we call each component of $\mathcal{A}$ by

$$A^{\alpha,\beta}\widetilde{\varphi}_\beta(t) = \int_0^{2\pi} a^{\alpha,\beta}(t,\tau)\widetilde{\varphi}_\beta(\tau)\mathrm{d}\tau,$$

for $\alpha, \beta = 1, 2$, and where each kernel $a^{\alpha,\beta}$ is smooth for all $t, \tau \in [0, 2\pi]$. We then approximate each component with the trapezoidal rule,

$$A_n^{\alpha,\beta}\widetilde{\varphi}_\beta(t) = \frac{\pi}{n}\sum_{j=0}^{2n-1} a^{\alpha,\beta}(t,t_j)\widetilde{\varphi}_\beta(t_j).$$

The operator $\mathcal{C}$ is discretized in an analogous way using the trapezoidal rule, so that we obtain the approximate operator $\widetilde{\mathcal{U}}_n$; we collocate the approximate version of equation (4.40) at the points t_i:

$$\widetilde{\mathcal{U}}_n\widetilde{\Psi}_n(t_i) = \underline{\widetilde{G}}(t_j), \;\; i = 0, \dots, 2n-1, \tag{4.50}$$

a discrete equation in the unknown $\widetilde{\Psi}_n \in \mathcal{T}_n^4$ and right hand side $\underline{\widetilde{G}} = (g(z_\Omega(t_0)), \dots, g(z_\Omega(t_{2n-1})), 0, \dots, 0)^T$. Here and in what follows, underlining a quantity will denote sampling it at the collocation points.

Regarding the solvability of the discretized system (4.50), and the analysis of the convergence of the approximate solutions, we refer to Kress and Sloan [46]. Their study of an analogous problem for scalar valued equations of the first kind with analytic off diagonal terms and weakly singular kernels shows that the convergence of the method is of super-algebraic order, and it translates effortlessly to the vector valued case. It can indeed be shown [46, Th. 2.3] that:

Theorem 4.4. *For sufficiently large n, the discretized system* (4.50) *has a unique solution $\widetilde{\Psi}$. Furthermore, if p, q are such that $1 \leq q \leq p$ and $p > 3/2$ then the following asymptotic error estimate holds:*

$$\|\widetilde{\Psi} - \widetilde{\Psi}_n\|_{\mathbf{H}^q([0,2\pi])} \leq C\left(\frac{\pi}{n}\right)^{p-q}\|\widetilde{\Psi}\|_{\mathbf{H}^p([0,2\pi])} \tag{4.51}$$

for a constant C only depending on q and p.

To implement this system, we write it as its equivalent finite linear system

$$\widehat{\mathcal{U}}\underline{\widetilde{\Psi}} = \underline{G}, \tag{4.52}$$

where $\underline{\widetilde{\Psi}} = (\tilde{\varphi}(t_0), \dots, \tilde{\varphi}(t_{2n-1}), \tilde{\psi}(t_0), \dots, \tilde{\psi}(t_{2n-1}))^T$, and $\hat{\mathcal{U}} = \hat{\mathcal{S}} + \hat{\mathcal{C}}$. We call $\hat{\mathcal{S}}$ the $8n \times 8n$ matrix given by

$$\widehat{\mathcal{S}} = \begin{bmatrix} \widehat{S}_\Omega & \widehat{V}_D^\Omega \\ \widehat{V}_\Omega^D & \widehat{S}_D \end{bmatrix}$$

where, with a slight abuse of notation, we identify each finite dimensional operator with the matrix representing it; namely,

$$\widehat{V}_\Omega^D = |z'_\Omega(t_j)| \left(E(z_D(t_i) - z_\Omega(t_j))\right)_{i,j=0,\dots,2n-1} \tag{4.53}$$

and the analogous for $\widehat{V}_D^\Omega$, and

$$\widehat{S}_\Omega = |z'_\Omega(t_j)| \Big(R_{ij} w(|t_i - t_j|)\, \mathbb{I} + \frac{\pi}{n}\left(K_\Omega(t_i, t_j)\, \mathbb{I} + M(t_i, t_j)\right)\Big)_{i,j=0,\dots,2n-1} \tag{4.54}$$

with $R_{ij} = \alpha_j^{(n)}(t_i)$. Finally, $\hat{\mathcal{C}}$ is given by

$$\widehat{\mathcal{C}} = \begin{bmatrix} C_\Omega & 0 \\ 0 & C_D \end{bmatrix},$$

and

$$C_\Omega(t_i, t_j) = \left(\frac{z_\Omega(t_i) \otimes \nu_\Omega(t_j)}{|z_\Omega(t_i)|^2} |z'_\Omega(t_j)|\right)_{i,j=0,\dots,2n-1}.$$

and of course the analogous formula for C_D.

Numerical experiments confirm that high accuracy can be obtained even with a relatively small number of collocation points. As a test example, we considered an outer domain $\Omega = B(0, 2)$ and a squeezed ellipse shaped inclusion D, whose radial function is given by

$$\rho_D(t) = \frac{3}{2}\sqrt{\frac{1}{4}\cos^2(t) + \sin^2(t)}$$

We considered boundary conditions induced by a point source P_0 outside of the domain $\Omega \setminus \overline{D}$, namely $P_0 = (5, 5)^T$. Then we solved the discretized system and used the densities to compute the values of the resulting velocity at a test point inside the domain. Since the real values of the velocity can be computed using the fundamental solution, we were able to estimate the accuracy of the solution, as reported in the table below, reporting the number of collocation points versus the error, evaluated as the standard euclidean norm of the difference between the evaluated solution and the real one.

n	error
6	$3.1587 \cdot 10^{-3}$
12	$1.6341 \cdot 10^{-4}$
24	$4.3634 \cdot 10^{-7}$
48	$5.1606 \cdot 10^{-13}$

Finally, much in the same way we evaluated $u(x)$ on $\partial\Omega$, we can evaluate numerically the normal component of the stress tensor on $\partial\Omega$. The evaluation of $\sigma(u,p)(x)\nu_\Omega(x)$ for $x \in \partial\Omega$ is made by three contributions: one integral $T_\Omega\varphi(x)$ given by

$$T_\Omega\varphi(x) = \frac{1}{\pi}\int_{\partial\Omega} \frac{(x-y)\otimes(x-y)\otimes(x-y)}{|x-y|^4}\varphi(y)\nu_\Omega(x)\,\mathrm{d}s_y. \tag{4.55}$$

for $x \in \partial\Omega$, then $B_\Omega^D\psi(x)$, which is the analogous formula as (4.55) on the domain of integration is ∂D (we use a notation analogous to what we used in (4.36) and (4.37)), and a jump term $-\frac{1}{2}\varphi(x)$ due to the fact that we approach $\partial\Omega$ from the inside. The discretization and evaluation of $B_\Omega^D\psi$ can be accomplished using the trapezoidal rule, since the integral kernel is regular, in a way analogous to (4.43). To evaluate $T_\Omega\varphi(x)$, we take its parametrized version $\widetilde{T}_\Omega\varphi$ and evaluate it at $x = z_\Omega(t)$. The resulting operator has an integral kernel is of the form

$$D(t,\tau) = \frac{(z_\Omega(t)-z_\Omega(\tau))\otimes(z_\Omega(t)-z_\Omega(\tau))}{|z_\Omega(t)-z_\Omega(\tau)|^4}(z_\Omega(t)-z_\Omega(\tau))\cdot\nu_\Omega(z_\Omega(t)).$$

Using (4.49), taking a Taylor expansion up to the second order and recalling that $\tau_\Omega\cdot\nu_\Omega = 0$ (where τ_Ω is the unit tangent vector to $\partial\Omega$), it can be easily shown that the diagonal terms of D exist and are finite, explicitly

$$\lim_{\tau\to t} D(t,\tau) = \frac{z'_\Omega(t)\otimes z'_\Omega(t)}{2|z'_\Omega(t)|^4} z''_\Omega(t)\cdot\nu_\Omega(z_\Omega(t)),$$

so that the integral operator can also be approximated using the trapezoidal rule.

4.4 The inverse problem

Once the direct problem is fully solved, we are in condition to examine the solution of the inverse problem. As we mentioned, a regularization method like IRGNM is needed in order to tame the effect of the ill posedness. As we already pointed out in the previous chapter, the inverse problem enjoys uniqueness, as proved in [9].

To apply IRGNM to F we need explicit knowledge of F as well as its Fréchet derivative F'. Since the operator F is defined on a family of sets, the technique of shape differentiation, developed by Murat and Simon is a convenient way to compute F'. It is beyond the scope of the presentation to give a detailed description of this approach; what is of concern to us is the following result, which allows us to characterize the operator F' as a solution to a boundary value problem of the same type as (4.1).

Theorem 4.5. *Let r_D be the radial function parametrizing ∂D and let (u, p) be the solution to* (4.1)*. The mapping F described in* (4.4) *is Frèchet differentiable, and given a domain variation $w(t) = h(t)(\cos t, \sin t)$, its derivative $F'[r]h$ is given by $\sigma(u', p') \cdot \nu|_{\partial\Omega}$ where (u', p') is the solution to the boundary value problem*

$$\begin{cases} \triangle u' - \nabla p' &= 0, & \text{in } \Omega \setminus \overline{D}, \\ \operatorname{div} u' &= 0, & \text{in } \Omega \setminus \overline{D}, \\ u' &= 0, & \text{on } \partial\Omega, \\ u'(z_D(t)) &= \left(h(t) \dfrac{r_D(t)}{\sqrt{r_D^2(t) + r_D'^2(t)}} \right) \dfrac{\partial u}{\partial \nu}(z_D(t)), & t \in [0, 2\pi]. \end{cases} \tag{4.56}$$

We stress the fact that the computation of the derivative of F can be achieved by solving a problem of the same kind as (4.1), so that the same discretization technique used to evaluate F can be used again to compute F'.

The evaluation of the term $\frac{\partial u}{\partial \nu}\big|_{\partial D}$ can be achieved by either direct computation, using the jump relations (4.26). Alternatively, one can use the following:

Lemma 4.6. *The following identity holds:*

$$\frac{\partial u}{\partial \nu}\Big|_{\partial D} = (\sigma(u, p) + p\,\mathbb{I})\,\nu. \tag{4.57}$$

Proof. From the definition of the stress tensor and the divergence free condition, it is sufficient to prove the following identity: if η is a smooth vector valued function and $\eta = 0$ on ∂D, then

$$(\nabla \eta)^T \cdot \nu = \operatorname{div} \eta.$$

We choose a test point $x_0 \in \partial D$, a neighborhood U of x_0 and a coordinate system in which $\partial D \cap U = \{\gamma(t) : t \in [-1, 1]\}$, $z(0) = x_0 = 0$ and $z'(0) = [k, 0]$, so that the outer normal unit vector at x_0 is $\nu = [0, 1]$. Since $\eta(z(t)) = 0$, by differentiation

$$\nabla \eta(z(t)) z'(t) = 0,$$

which, when we evaluate at $t = 0$, implies $\partial_i \eta_1(x_0) = 0$. Therefore, $\operatorname{div} \eta(z(0)) = \partial_2 \eta_2(x_0)$; on the other hand it also implies $(\nabla \eta)^T(x_0) \cdot \nu = \partial_2 \eta_2(x_0)$. $\square$

4.5 Numerical results

At the present time, the full implementation of the reconstruction for the inverse problem associated to the Stokes equation is still in progress. As a preparatory work, we studied a simpler- yet related- inverse problem for the Laplace equations. The statement of the problem is familiar: given $g \in H^{1/2}(\partial\Omega)$, let $u \in H^1(\Omega \setminus \overline{D})$ be the solution of the following Dirichlet problem:

$$\begin{cases} \triangle u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \\ u &= 0 \quad \text{on } \partial D. \end{cases} \tag{4.58}$$

we want to reconstruct D from the knowledge of the induced normal derivative $\psi = \frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$. This inverse problem is by far better known, and has been well studied throughout literature (see for example Kress' analysis in [43] and [45]): the tool of layer potentials for the Laplace equation is indeed a classical topic in potential theory. For this reason, we shall limit our exposition to setting up the layer potentials and the boundary integral equations, and to presenting the numerical results. The discretization follows the same steps we outlined previously. We refer the reader to the above sources also for the error analysis.

Given a smooth, bounded and open domain A with outer normal ν_A, we define the *single layer potential* of integrable density ζ the integral operator

$$S\zeta(x) = \int_{\partial A} \Phi(x-y)\zeta(y)\mathrm{d}s_y, \tag{4.59}$$

where again $\Phi(x) = -\frac{1}{2\pi}\log|x|$ is the fundamental solution of the Laplace equation. For simplicity of the exposition we shall state the regularity results on a classical setting, understanding that all can be viewed in a weak sense.

Theorem 4.7. *Let $S\zeta$ be defined by* (4.59) *for a Hölder continuous density ζ. Then:*

1. *The operator $S\zeta$ is Hölder continuous throghout $\mathbb{R}^2$,*

2. *Let $x \in \partial A$. For the normal derivative*

$$\frac{\partial S\zeta_\pm}{\partial \nu_A}(x) = \lim_{h\to 0^+} \nabla\Phi(x \pm h\nu_A(x)) \cdot \nu_A(x), \tag{4.60}$$

there hold the following jump relations

$$\frac{\partial S\zeta_\pm}{\partial \nu_A}(x) = \int_{\partial A} \frac{\partial \Phi(x-y)}{\partial \nu_A(x)}\zeta(y)\mathrm{d}s_y \mp \frac{1}{2}\zeta(x), \tag{4.61}$$

where the right hand side exists as an improper integral.

We represent the solutions of (4.58) as a sum of two single layer potentials:

$$u(x) = \int_{\partial\Omega} \Phi(x-y)\varphi_M(y)\mathrm{d}s_y + \int_{\partial D} \Phi(x-y)\psi_M(y)\mathrm{d}s_y. \qquad (4.62)$$

By φ_M we denote

$$\varphi_M = \varphi - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \varphi(y)\mathrm{d}s_y,$$

and the analogous definition for ψ_M. This modification is made to impose the behavior $u(x) = o(1)$ at infinity [43, Th. 6.27]. Passing to the limit in (4.62) for $x \to \partial\Omega$ and $x \to \partial D$, we obtain the system of integral equations in the unknown densities φ and ψ:

$$g(x) = \int_{\partial\Omega} \Phi(x-y)\varphi_M(y)\,\mathrm{d}s_y + \int_{\partial D} \Phi(x-y)\psi_M(y)\,\mathrm{d}s_y, \quad x \in \partial\Omega, \quad (4.63\mathrm{a})$$

$$0 = \int_{\partial\Omega} \Phi(x-y)\varphi_M(y)\,\mathrm{d}s_y + \int_{\partial D} \Phi(x-y)\psi_M(y)\,\mathrm{d}s_y, \quad x \in \partial D. \quad (4.63\mathrm{b})$$

This system of the first kind can then be treated following along the line described before; in particular, a similar convergence analysis can be obtained, see [45]. We observe that in the classical potential approach it is more convenient to use a double layer potential representation to obtain a system of equations of the second kind, which requires less effort to treat numerically. However, in light of the technique we presented for the Stokes equations, we find this approach more relevant to the purpose.

To apply IRGNM to the operator $F : r_D \mapsto \frac{\partial u}{\partial \nu_\Omega}$ other than the knowledge of the values of F itself, we need also its Fréchet derivative F'. This can be done by Theorem 4.5, which applies in this case as well:

Theorem 4.8. *Let r_D be the radial function parametrizing ∂D and let u be the solution to* (4.58)*. The mapping F is Frèchet differentiable, and, given a domain variation $w(t) = h(t)(\cos t, \sin t)$ the derivative $F'[r]h$ is given by $\frac{\partial u'}{\partial \nu_\Omega}$ where u' is the solution to the boundary value problem*

$$\begin{cases} \triangle u' &= 0, & \text{in } \Omega \setminus \overline{D}, \\ u' &= 0, & \text{on } \partial\Omega, \\ u'(z_D(t)) &= \left(h(t)\dfrac{r_D(t)}{\sqrt{r_D^2(t) + r_D'^2(t)}} \right) \dfrac{\partial u}{\partial \nu}(z_D(t)), & t \in [0, 2\pi]. \end{cases} \qquad (4.64)$$

The algorithm used for the inversion is an implementation of IRGNM developed by the research group of professor Thorsten Hohage, whose help is gratefully acknowledged. We applied this routine to our implementation of the boundary operator F. The boundary curves are approximated via a Fourier expansion, and they are considered as $\mathbf{H}^r([0, 2\pi])$ functions with

$r = 1.6$ (which is necessary in order to compute the values of r'_D). We will consider an outer domain given by $\partial\Omega = \partial B(0,2)$. The measured boundary data $\underline{g}$ is measured on the set of equally spaced points. If noise on the boundary data is present, we represent it as

$$\underline{g}_k^\epsilon = \underline{g} + \epsilon_k \|g\|. \tag{4.65}$$

for independent normally distributed random value ϵ_k.

The initial guess was taken as the circle of center 0 and radius 1.2. We choose the parameter α_k in (4.18) of the form $\alpha_k = 0.05\left(\frac{2}{3}\right)^k$. We considered the following curves:

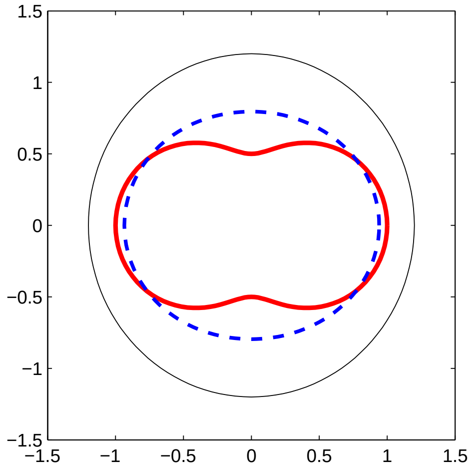
- A peanut shaped curve, parametrized by

$$\rho_D(t) = \frac{1}{2}\sqrt{3\cos^2(t) + 1} \tag{4.66}$$

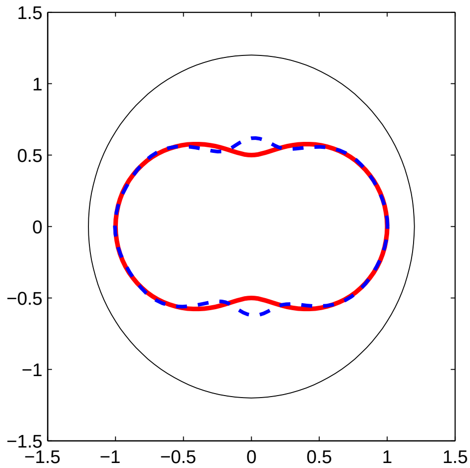
- A bean shaped curve, parametrized by

$$\rho_D(t) = \frac{1 + \frac{9}{10}\cos(t) + \frac{1}{10}\sin(2t)}{1 + \frac{3}{4}\cos(t)}. \tag{4.67}$$

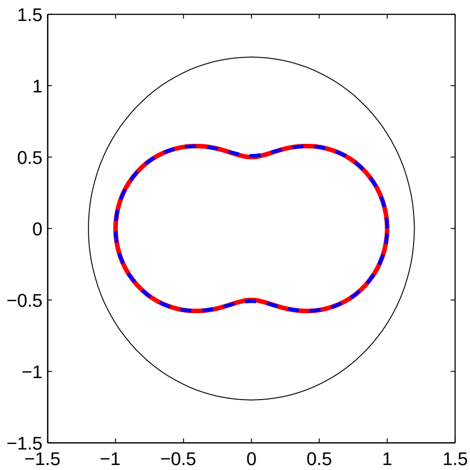
We first analyze the case in which no noise is present. We consider $n = 32$ measurement points for both cases. As boundary data, we consider, at first, the values of the potential induced by a point source at a point P_0, in this example $P_0 = (-10, 0)$. We then tried the reconstruction with boundary data of high frequency, namely $g(z_\Omega(t)) = \sin(4t)$. Since we know that the constants in the stability estimates deteriorate as the frequency of the boundary data increases, we show how this actually affects the reconstruction. Finally, we add noise to the boundary data: first, we consider a 1 % level of noise, then 3 %, 5 %. We found that the method shows a good behavior even in the presence of error, if combined with an appropriate stopping rule, namely we use the discrepancy principle (4.12). If we do not use any, after a few iterations the reconstruction starts to deteriorate. For each case we indicate the error *err* in the reconstruction in terms of the $\mathbf{L}^2$ norm of the difference between the reconstruction and the real solution.



(a) 5 iterations, err=0.24674

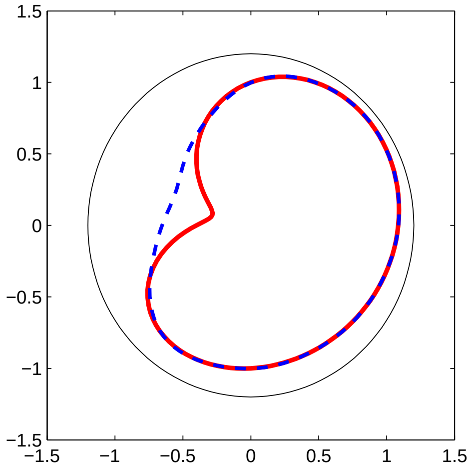


(b) 15 iterations, err=0.08339

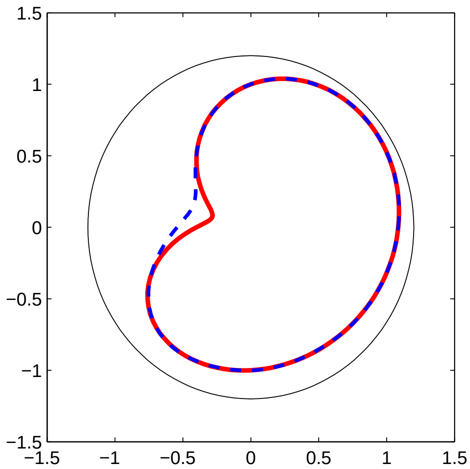


(c) 30 iterations, err=0.00421

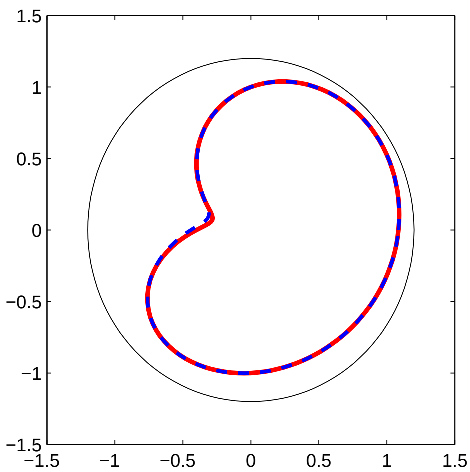
Figure 4.1: Reconstruction of the peanut shaped inclusion with exact low frequency data. The dashed line represents the reconstructions, the thin black circle is the initial guess.



(a) 5 iterations, err=0.39641

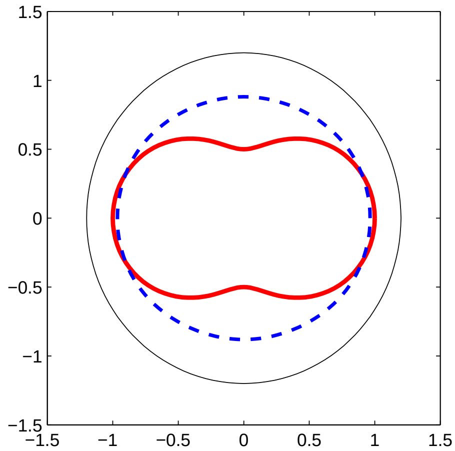


(b) 15 iterations, err=0.19541

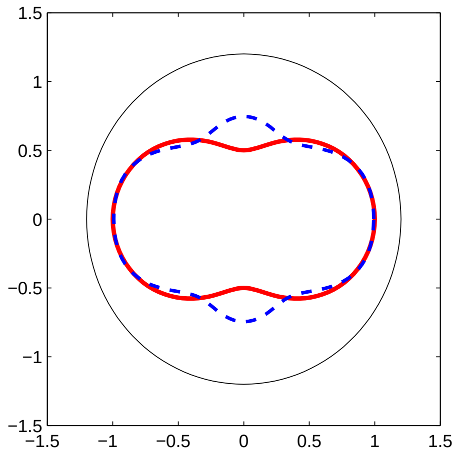


(c) 50 iterations, err=0.002650

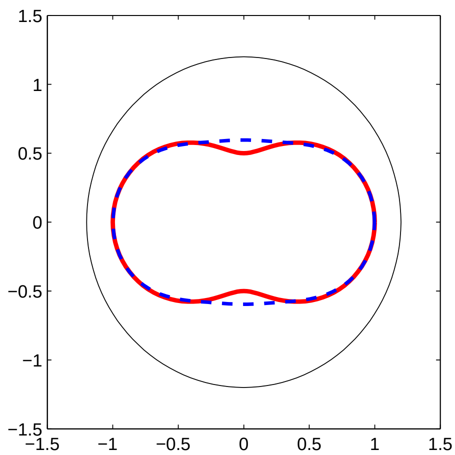
Figure 4.2: Reconstruction of the bean shaped inclusion with exact low frequency data. Same legend as before.



(a) 5 iterations, err=0.39641

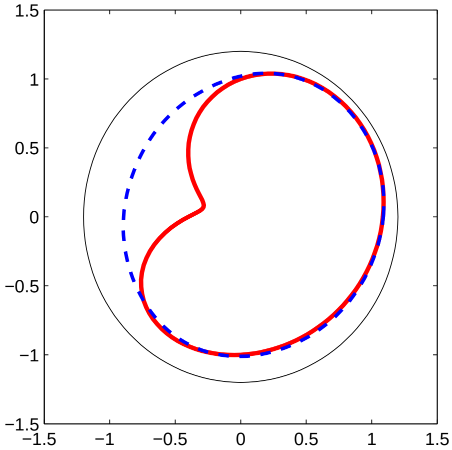


(b) 10 iterations, err=0.39365

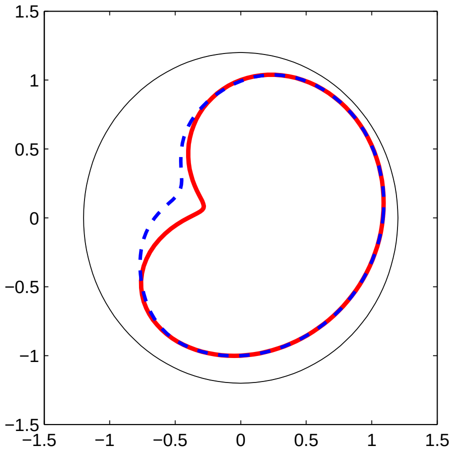


(c) 50 iterations, err=0.06919

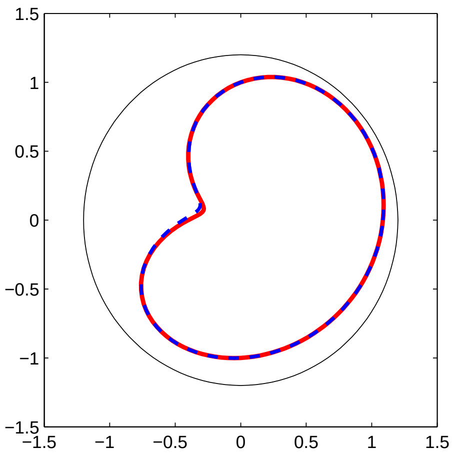
Figure 4.3: Reconstruction of the peanut shaped inclusion with exact high frequency data.



(a) 5 iterations, err=0.69344

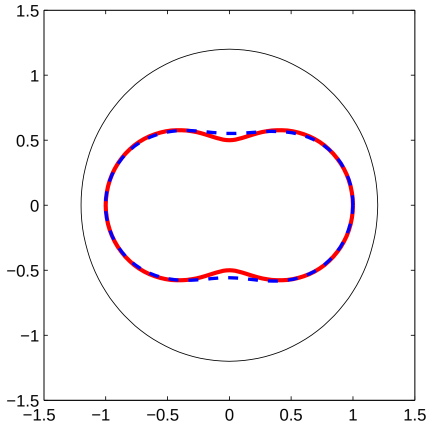


(b) 15 iterations, err=0.33367

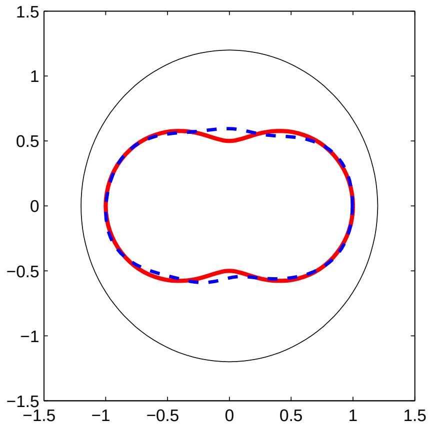


(c) 25 iterations, err=0.02625

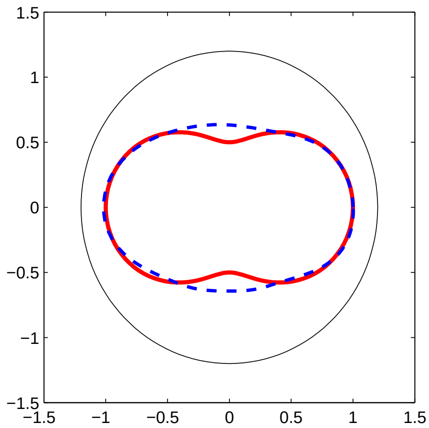
Figure 4.4: Reconstruction of the bean shaped inclusion with exact high frequency data.



(a) 1% noise level, err=0.29103

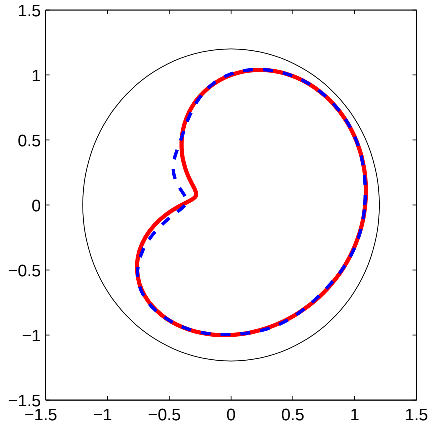


(b) 3% noise level, err=0.31917

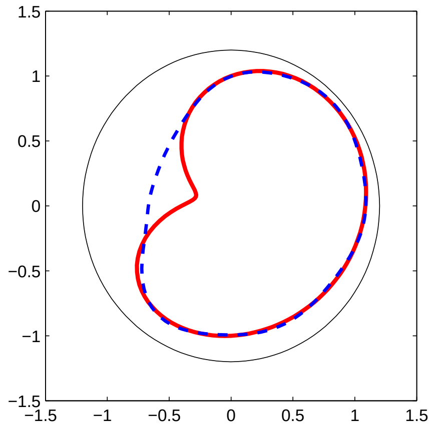


(c) 5% noise level, err=0.42531

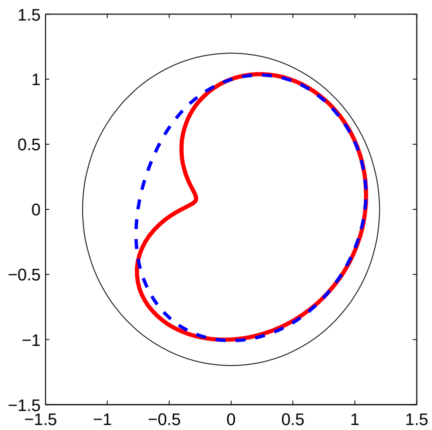
Figure 4.5: Reconstruction of the peanut shaped inclusion with noisy low frequency data, stopped according to the discrepancy principle.



(a) 1% noise level, err=0.29104



(b) 3% noise level, err=0.33261



(c) 5% noise level, err=0.46327

Figure 4.6: Reconstruction of the bean shaped inclusion with noisy low frequency data, stopped according to the discrepancy principle.

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