On the Approximation of Conservation Laws by Vanishing Viscosity

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Contents

Abstract

In this thesis I collect some recent results on the approximation of conservation laws by vanishing viscosity. The exposition is organized as follows.

In the first chapter I provide a general introduction to the motivations of my work and a short overview of the results I obtained, I recall the tools and the main ideas involved in the proves, and I give the main references.

Chapters 2 and 3 are devoted to present the results I obtained in collaboration with prof. A. Bressan, dealing with the viscous approximation of a scalar conservation law, [10], [11]. We start by considering a piecewise smooth solution to a scalar conservation law,

$$u_t + f(u)_x = 0, \qquad u : I\!R^+ \times I\!R \mapsto I\!R, \tag{1}$$

with fixed, bounded initial data of the form

$$u(0, x) = \bar{u}(x),$$

and strictly convex flux f of class $\mathcal{C}^2$. As shown by the analysis of Goodman and Xin, [22], in the case the solution u of an hyperbolic system of conservation laws contains finitely many non-interacting entropic shocks, its viscous approximations u^ε, defined as the solutions of the family of Cauchy problems

$$\begin{cases} u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}, \\ u^\varepsilon(0, x) = \bar{u}(x), \end{cases} \tag{2}$$

admit a singular perturbation expansion. In particular, they admit expansions in terms of powers of ε both in the region where u is smooth and near the discontinuities. My joint work with prof. A.Bressan, [11], presented here in Chapter 3, provides a positive answer to the question whether a similar inner and outer expansion can still be performed for $t > \tau$, in the case the solution u to the scalar conservation law (1) contains arbitrarily many shock interactions, until at a certain time τ an isolated shock emerges. An important point in our proof is the estimate on the time needed for the viscous shock to appear. This is an application of the main result in [10], which will be presented in details in Chapter 2. There we consider a viscous scalar conservation law with smooth, possibly non-convex flux

$$u_t + f(u)_x = u_{xx}, \tag{3}$$

Assume that the (arbitrarily large) initial data remains in a small neighborhood of given states u^-, u^+ as $x \to \pm\infty$, with u^-, u^+ connected by a stable shock profile. We then show that the solution eventually forms a viscous shock. The process of shock formation is described through three phases and we are able to estimate the time needed of each of them to be completed.

Chapter 4 deals with the vanishing viscosity approximation for genuinely nonlinear hyperbolic system of conservation laws, in one space dimension,

$$u_t + f(u)_x = 0, \qquad u : I\!R^+ \times I\!R \to I\!R^n. \tag{4}$$

As a first step in the study of the convergence of viscous approximations I focus my attention on the special situation in which the solution u is self similar and contains exactly one interaction of shock waves. I adapt the approximation technique used in [13] to obtain sharper bounds from above on the convergence rate of the viscous approximations, u^ε.

In the second part of the chapter, which is a work in progress, I consider a weakly coupled system of the form

$$\begin{cases} u_t + \left(u^2/2\right)_x + \rho f(u, v)_x = u_{xx}\,, \\ v_t + \left(v^2/2\right)_x + \rho g(u, v)_x = v_{xx}\,, \end{cases}$$

and I study how the solution changes as the parameter ρ varies in $I\!R^+$. This would lead me to a better description of the large time behavior of vanishing viscosity approximations when the inviscid solution u is allowed to contain interactions between shock waves of different families.

Chapter 1.
Introduction

1.1. Motivations and references

The Cauchy problem for a $n \times n$ system of conservation laws in one space dimension takes the form

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,x) = \bar{u}(x). \end{cases} \tag{1.1.1}$$

Here $u = (u_1, \ldots, u_n)$ is the vector of conserved quantities, while $f = (f_1, \ldots, f_n)$ are the fluxes. In the following, if not otherwise specified, we will always assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class $\mathcal{C}^2$ and that the system is strictly hyperbolic, i.e. for all values of u in the range, the Jacobian matrix $Df(u)$ has n distinct eigenvalues

$$\lambda_1(u) < \ldots < \lambda_n(u).$$

We call $r_i(u)$ the right eigenvector of the matrix $Df(u)$ associated with the eigenvalue $\lambda_i(u)$.

Even for smooth initial data, the solution u can develop shock discontinuities in finite time. This accounts for most of the difficulties in the theoretical and numerical study of the problem. In the literature various approximating algorithms have been used to prove existence, uniqueness and stability of entropy admissible solutions, under suitable assumptions on the initial data $\bar{u}$. Namely: the Glimm scheme [20], the wave-front tracking, [23], [7], [1], and the method of vanishing viscosity, [6].

In my research activity I had the opportunity to familiarize with two of these techniques, the front tracking algorithm and the vanishing viscosity method. The second method is the one used in this Ph.D dissertation, while in my master thesis, under the supervision of prof. Alberto Bressan and prof. Andrea Marson, I studied the stability of the front tracking solution to the initial and boundary value problem for a system of conservation laws in one space dimension, [17].

The front tracking algorithm gives piecewise constant approximations u^ν of the solution u. The building block in this approach is the Riemann problem, i.e. a Cauchy problem of the form

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0, \end{cases} \end{cases}$$

where u^- and u^+ are constant states. The first step in this algorithm consists in approximating the initial data $\bar{u}$ by a piecewise constant function $u^\nu(0,\cdot)$, such that

$$\|\bar{u} - u^\nu(0,\cdot)\|_{\mathbf{L}^1} \leq \nu, \qquad \nu > 0,$$

then, at each point of jump a piecewise constant solution of the Riemann problem corresponding to the states on the left and on the right of the jump is constructed. Piecing together these local solutions we obtain an approximate solution u^ν which is well defined until two of the line discontinuities interact. Then we solve the Riemann problem generated by the interaction and we extend the solution until the next interaction takes place. Under suitable hypothesis on the Cauchy problem (1.1.1) this procedure can by iterated to obtain a global approximate solution in $I\!R^+ \times I\!R$.

When dealing with a front tracking approximation u^ν, we are able to describe the profile of u^ν at any instant t in $I\!R^+$. This property is the main advantage of the front tracking algorithm with respect to the otherwise more general vanishing viscosity method, and made possible to obtain important results on the structure and asymptotic behavior of the solutions, [28], [7]. The vanishing viscosity method allows to obtain more general results on existence and stability of entropy weak solutions. To apply the front tracking algorithm and the Glimm scheme, one usually assumes the additional hypothesis

(H) For all i in $\{1, \dots, n\}$ the i^{th} characteristic field is either linearly degenerate, i.e.

$$D\lambda_i(u) \cdot r_i(u) = 0,$$

for all possible states u, or genuinely nonlinear, i.e.

$$D\lambda_i(u) \cdot r_i(u) > 0,$$

for all possible u.

This assumption greatly simplifies the solution of the Riemann problem and, when it can be avoided, its absence leads to heavy technicalities in the proofs. Since the Riemann problem plays no special role in the vanishing viscosity approach, the hypothesis (H) loses its function in this framework and is not assumed. Moreover vanishing viscosity gives global existence and uniqueness of solutions for the whole class of strictly hyperbolic systems, not necessarily in conservation form.

When these techniques are both applicable, the solution obtained by the method of vanishing viscosity coincides with the one obtained by the front tracking algorithm or the Glimm scheme. In fact, the main uniqueness result in this framework states that every small BV entropy admissible solution of (1.1.1) coincides with the limit the front tracking algorithm, see [7].

The vanishing viscosity approach is ideally justified by a physical remark. On one hand systems of conservation laws have been introduced to model in an effective way physical phenomena, like the motion of fluids. On the other hand it is clear that in a physical situation some diffusive effects, neglected by our model, should be present. As a consequence, it is natural to suppose that the entropy solution u of (1.1.1) should be close to the solution of

$$\begin{cases} u_t + f(u)_x = \varepsilon \left(B(u)\, u_x\right)_x, \\ u(0,x) = \bar{u}(x), \end{cases}$$

where B is a suitable viscosity matrix, depending on the problem, and $\varepsilon > 0$ is small enough, [15], [55]. Following this idea, we consider the system with artificial viscosity associated with (1.1.1)

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx}, \\ u(0,x) = \bar{u}(x), \end{cases} \tag{1.1.2}$$

and conjecture that the unique entropy admissible solution of Eq. (1.1.1) can be singled out through a limiting process for $\varepsilon \to 0$. The vanishing viscosity method was applied very early in the study of hyperbolic equations. In particular, existence, uniqueness and global stability of the solution of a scalar conservation law in one space dimension have been proved by O. Oleinik, [39], in 1959, by means of comparison principles for parabolic equations. In a similar way, S. Kruzhkov extended these results for the whole class of $\mathbf{L}^\infty$ solution in several space dimensions in 1970, [25]. For what concerns vanishing viscosity in the vectorial case, different techniques have been employed. The result on convergence of vanishing viscosity approximations obtained by S. Bianchini and A. Bressan in [6] relies on *a priori BV* bounds, uniform with respect to t and ε, on the viscous approximations $u^\varepsilon(t,\cdot)$, while earlier results were obtained by compensated compactness, [16] and singular perturbation, [22].

The rates of convergence associated with the different methods used in the study of hyperbolic systems of conservation laws are all rather slow. An estimate on the rate of convergence of the Glimm scheme at time fixed was established by A. Bressan and A. Marson in [14], under the hypothesis (H). Calling $\nu = \Delta x$ the length of the mesh grid, they find

$$\|u^\nu(t,\cdot) - u(t,\cdot)\|_{\mathbf{L}^1} = o(1)\cdot\sqrt{\Delta x}|\ln(\Delta x)|.$$

For what concerns the vanishing viscosity method, under the assumption that all characteristic fields are genuinely nonlinear, A. Bressan and T. Young obtained the following estimate, [13]

$$\|u^\varepsilon(T,\cdot) - u(T,\cdot)\|_{\mathbf{L}^1} \le C\cdot(1+T)\ \text{Tot.Var.}\{\bar{u}\}\cdot\sqrt{\varepsilon}\,(1+|\ln\varepsilon|)\ .$$

The right hand side of this estimate depends linearly on time and this is not optimal. A refinement of this dependence is provided in Chapter 4. The estimate we obtain, under the same hypothesis as in [13], is

$$\|u^\varepsilon(T,\cdot) - u(T,\cdot)\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\text{Tot.Var.}\{\bar{u}\}\cdot\sqrt{\varepsilon T}\,(1+\ln T+|\ln\varepsilon|)\ .$$

In the literature, the only earlier result concerning the description of the transient behavior of a vanishing viscosity approximation in terms of the viscous parameter, which is the main focus of this thesis, is contained in the work by J. Goodman and Z. Xin, [22]. Under the hypothesis that the solution u is piecewise smooth with a finite number of noninteracting, entropic shocks, the authors show that it is possible to find smooth functions v_j such that in the regions far from the discontinuities u^ε admits the asymptotic expansion in terms of power of ε

$$u^\varepsilon = u + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots + \varepsilon^k v_k + \cdots. \tag{1.1.3}$$

To represent u^ε near each shock discontinuity $s(t)$ the authors introduce a shock layer, described in terms of a stretched variable

$$\eta = \frac{(x - s(t))}{\varepsilon} + \delta(\varepsilon, t)\,,$$

where δ is the shift in the position of the center of the viscous shock, and takes the form

$$\sum_{i=0}^{\infty} \varepsilon^i \delta_i(t)\,.$$

Then they prove that there exist some smooth functions U_j such that

$$u(t,\eta) = U_0(t,\eta) + \varepsilon U_1(t,\eta) + \varepsilon^2 U_2(t,\eta) + \cdots . \tag{1.1.4}$$

The leading term $U_0(t,\cdot)$ turns out to be the unique viscous shock profile connecting the states $u(t,s(t)^-)$, $u(t,s(t)^+)$ to the right and to the left of the shock $s(t)$. The functions v_j, U_j and δ_{j+1} are also asked to satisfy suitable matching conditions on the zone of possible overlapping, so that, in [22], a converging sequence of smooth approximate solutions of (1.1.2), $\{v^\varepsilon\}_\varepsilon$, can be constructed by patching together the outer expansion, (1.1.3), with the inner expansion, (1.1.4). Using v^ε as an intermediate comparison term the authors are able to prove the convergence of the viscous approximations u^ε at an optimal rate. Given any μ in $[0,1]$, there holds

$$\sup_A |u(t,x) - u^\varepsilon(t,x)| \le C_\mu \varepsilon,$$

where C_μ is a positive constant depending only on μ and the set A is defined as

$$A \doteq \{(t,x);\quad t \in [0,T],\ |x - s(t)| \ge \mu\}.$$

Since the analysis in [22], which I sketched above, only applies to isolated, non-interacting entropic shocks, it is of interest to understand whether a similar inner and outer expansion can still be performed after several shock interactions have occurred. In Chapter 3, based on a joint work with prof. A. Bressan, [11], a positive answer in the case of a scalar conservation law with strictly convex flux is provided. A detailed introduction to this work can be found in Section 3. A major issue in the proof of our result is the estimate on the time needed for a viscous shock to emerge in the solution u^ε after all interactions have taken place. In Chapter 2 I describe the formation of a scalar viscous shock profile under suitable assumptions on the initial data. The result, in collaboration with prof. A. Bressan, is obtained in the more general case of possibly non convex flux, and is published in [10]. Section 2 gives a global overview of the theorem and its proof.

In Chapter 4 the discussion focuses on the viscous approximation for systems of conservation laws. Some of the results I present there, concerning the rate of convergence of vanishing viscosity approximations and the study of the structure of viscous approximations when one shock interaction is present in u, are still works in progress. A unitary introduction to this topic is provided in Section 4.

1.2. On the formation of scalar viscous shocks

Consider a scalar conservation law with viscosity

$$u_t + f(u)_x = u_{xx}, \tag{1.2.1}$$

where the flux function f is smooth but not necessarily convex. If the initial data

$$u(0,x) = \bar u(x), \tag{1.2.2}$$

has well defined limits as $x \to \pm\infty$, then the asymptotic behavior of the solution as $t \to \infty$ is strongly related to the Riemann problem

$$u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} u^- \doteq \lim_{x\to-\infty} \bar u(x) & \text{if } x < 0, \\ u^+ \doteq \lim_{x\to+\infty} \bar u(x) & \text{if } x > 0. \end{cases}$$

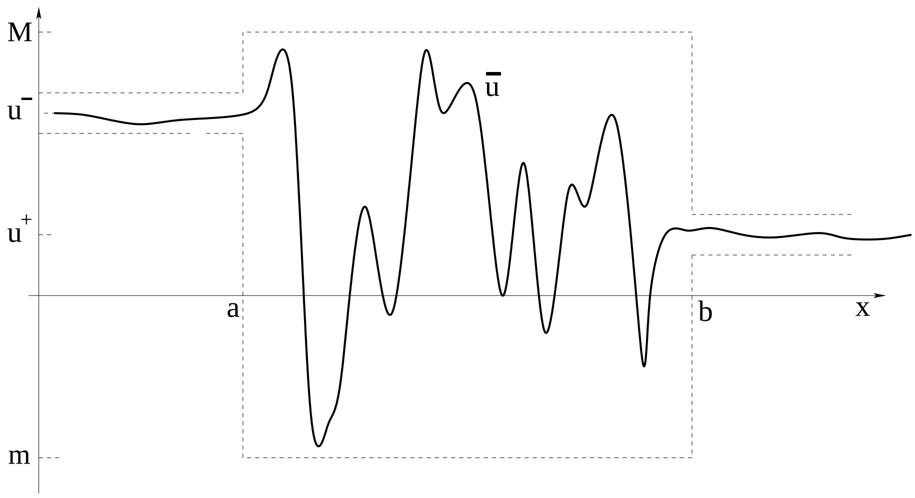

Fig. 1.2.1: The initial condition.

In particular, if this Riemann problem admits a solution consisting of a single, stable shock, then as $t \to \infty$ the solution of the Cauchy problem (1.2.1)-(1.2.2) approaches a viscous shock profile. The stability of travelling shock profiles is a topic of major interest, both in the scalar case as well as for hyperbolic systems with viscosity. The main results for the scalar case are due to J. Goodman [21], P. Howard [24], C. K. Jones, G. Gardner and T. Kapitula [26], A. Matsumura [31], A. Matsumura and K. Nishihara [32] and K. Nishihara [35], [36]. Viscous travelling waves for strictly hyperbolic systems were studied by S. Bianchini and A. Bressan [6], T. P. Liu [29], [30], R. L. Pego [40] and A. Szepessy and Z. P. Xin [48]. Detailed estimates on the convergence rates to this stable shock profile have been established by K. Nishihara and H. Zhao [37] and by M. Nishikawa [38].

Our main interest here is in the transient behavior of the solution, rather than in the asymptotic limit. We consider a general class of initial data, as shown in Fig. 1.2.1. For some small constant $\delta_0 > 0$, we assume that

$$\begin{aligned} &|\bar{u}(x) - u^-| \le \delta_0, && x \le a\,, \\ &|\bar{u}(x) - u^+| \le \delta_0, && x \ge b\,, \\ &\bar{u}(x) \in [m, M]\,, && x \in I\!R\,. \end{aligned} \tag{1.2.3}$$

Here the constants $m < M$ and $a < b$ can be arbitrary, possibly very large. Assuming that there exists a viscous travelling profile $\phi(\cdot)$ connecting the states u^-, u^+, we wish to estimate how long does it take for the solution to develop a single viscous shock. Given $\kappa > \delta_0$, we seek a time T such that, for all $t \ge T$, there exists a suitable shifted profile $\phi(\,\cdot - c(t))$ with

$$\sup_x \; |u(t,x) - \phi(x - c(t))| \le \kappa\,.$$

Notice that the assumptions in (1.2.3) do not imply the existence of the limits of $\bar{u}(x)$ as $x \to \pm\infty$, and hence do not guarantee the asymptotic convergence to any viscous shock profile. The time T after which the viscous shock is formed will be estimated in terms of the upper and lower bounds m, M, and the length $b-a$ of the interval where $\bar{u}$ can be far from a viscous profile.

We remark that the techniques used in the present analysis are substantially different from most of the literature. Indeed, due to the absence of any limit as $x \to \pm\infty$, one cannot use any integral norm to estimate the distance between the solution $u(t, \cdot)$ and a travelling shock profile. Instead of energy methods, we rely on classical comparison arguments with upper and lower solutions, and on the variable transformation used by S. Bianchini and A. Bressan in [4], [6]. We recall here the main ideas. Let $u = u(t,x)$ be a solution of the viscous conservation law (1.2.1). Calling $w \doteq f(u) - u_x$, at each time t we consider the curve in the $u-w$ plane

$$x \mapsto \gamma(t,x) \doteq \begin{pmatrix} u(t,x) \\ f(u(t,x)) - u_x(t,x) \end{pmatrix}.$$

The motion of this curve occurs in the direction of curvature. Indeed, writing $w = w(t,u)$, each branch where $w - f(u)$ has a constant sign evolves according to

$$w_t = (w - f(u))^2 w_{uu}.$$

It is well known that, if $u(t,x) = \phi(x - \lambda t)$ is a viscous shock profile, then the corresponding curve $\tilde{\gamma}(t, \cdot)$ is constant in time and coincides with the segment connecting the points

$$P^- \doteq \left(u^-, f\left(u^-\right)\right), \qquad P^+ \doteq \left(u^+, f\left(u^+\right)\right).$$

According to the Rankine-Hugoniot condition, the slope of this segment is precisely the speed of the shock, namely

$$\lambda = \frac{f\left(u^+\right) - f\left(u^-\right)}{u^+ - u^-}.$$

In the following, for sake of definiteness we take $u^- > u^+$. We assume that the states u^-, u^+ are connected by a stable shock, satisfying

$$f'\left(u^+\right) < \frac{f\left(u^+\right) - f\left(u^-\right)}{u^+ - u^-} < f'\left(u^-\right), \tag{1.2.4}$$

$$f\left(\theta u^+ + (1-\theta) u^-\right) < \theta f\left(u^+\right) + (1-\theta) f\left(u^-\right), \qquad 0 < \theta < 1. \tag{1.2.5}$$

Our main result estimates the time at which one single large shock emerges in the solution profile. The first step in the proof of this result consists in establishing a relation between the distance of the curves γ and $\tilde{\gamma}$, associated respectively to the solution u and the viscous profile, ϕ, connecting the states u^- and u^+, and the $\mathbf{L}^\infty$ distance between u and $\phi(\cdot, \cdot - c(\cdot))$. Once this relation is proved, see Lemma 2.2 in Chapter 2, we can focus on the description of the evolution of γ in the plane $u - w$. By means of a comparison argument we are able to show that the curve γ enters a fixed neighborhood of $\tilde{\gamma}$ in time T. The estimate on T is the main result of this work.

1.3. On the convergence of viscous approximations after shock interactions

Consider a scalar conservation law in one space dimension,

$$u_t + f(u)_x = 0, \tag{1.3.1}$$

together with its viscous approximations

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon\,.$$

For a fixed initial data with small total variation

$$u(0,\cdot) = \bar{u}(\cdot)\,, \tag{1.3.2}$$

detailed results on the estimate of the convergence rate can be found in the work by H. Nessyahu and E. Tadmor, [34], E. Tadmor and T. Tang, [50], and Z. H. Teng and P. Zhang, [52]. Also for computational purposes, it is interesting to examine whether viscous approximations admit a power series expansion in the viscosity coefficient ε. In the case of a Hamilton-Jacobi equation on a bounded open domain $\Omega \subset I\!R^m$, W. H. Fleming and P. E. Souganidis, [18], showed that the solutions of the elliptic problem

$$\begin{cases} -\varepsilon\Delta u^\varepsilon + H(x, Du^\varepsilon) + u^\varepsilon = 0 & \text{for} \quad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad u^\varepsilon(x) = 0 & \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{1.3.3}$$

admit an asymptotic expansion of the form

$$u^\varepsilon = u + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots + \varepsilon^k v_k + o(\varepsilon^k). \tag{1.3.4}$$

Here the leading term u is the viscosity solution of the first order equation, formally obtained by setting $\varepsilon = 0$ in (1.3.3). The expansion (1.3.4) is valid restricted to suitable subsets of the domain Ω, where the limit solution u is smooth and can be constructed by the method of characteristics. This result was later used by A. Szpiro and P. Dupuis, [49], to derive a higher order numerical method for Hamilton-Jacobi equations.

A recent paper by W. Shen and Z. Xu, [45], has established a similar result in the context of a scalar conservation law. Namely, assume that the limit solution u of the Cauchy problem (1.3.1)-(1.3.2) is smooth on a region Ω in the $t-x$ plane bounded by two characteristics, say,

$$\Omega \doteq \{\, (t,x)\,;\quad t \in [0,T]\,,\quad a + f'(\bar{u}(a))\, t < x < b + f'(\bar{u}(b))\, t\}\,,$$

with $a < b$. Then one can determine functions v_j such that the expansion (1.3.4) is uniformly valid on every compact subset of Ω. Indeed, the analysis on [45] shows that the presence of an arbitrary number of (possibly interacting) shocks outside the domain Ω does not affect the validity of the expansion in the region where u is smooth.

For discontinuous solutions, the viscous approximations clearly cannot converge uniformly on a neighborhood of a shock. If the solution u contains only a finite number of isolated, non-interacting, entropic shocks, then the analysis of J. Goodman and Z. Xin, [22], applies.

The result presented in Chapter 3 shows that for a scalar conservation law a similar inner and outer expansion can still be performed after several shock interactions have occurred. More precisely, we consider a solution u to the conservation law (1.3.1) which contains arbitrarily many shock interactions, until at a certain time τ an isolated shock emerges. In addition, we consider a second solution $\tilde{u}$ containing one single shock, choosing the initial data $\tilde{u}(0,\cdot)$ in such a way that $\tilde{u} = u$ for $t > \tau$. Then we show that for $t > \tau$ the viscous approximations u^ε become exponentially close to $\tilde{u}^\varepsilon$ as $\varepsilon \to 0$. Indeed,

$$\|u^\varepsilon(t,\cdot) - \tilde{u}^\varepsilon(t,\cdot)\|_{\mathcal{C}^\nu} = o(\varepsilon^k),$$

for every $k, \nu \geq 1$. As a corollary, since $\tilde{u}^\varepsilon(t)$ admits a singular perturbation expansion, so does $u^\varepsilon(t)$ for all $t > \tau$. The proof relies on a homotopy method. We define the 1-parameter family of Cauchy problems

$$\begin{cases} u_t^{\varepsilon,\theta} + f(u^{\varepsilon,\theta})_x = \varepsilon u_{xx}^{\varepsilon,\theta}, \\ u^{\varepsilon,\theta}(0,x) = \theta u^\varepsilon(0,x) + (1-\theta)\,\tilde{u}^\varepsilon(0,x)\,, \end{cases}$$

then we study the decay of the infinitesimal perturbation

$$z^{\varepsilon,\theta} \doteq \frac{\partial}{\partial\theta}\, u^{\varepsilon,\theta}\,.$$

In particular, in the proof, we need to show that after a certain time τ_1, which does not depend on θ, each $u^{\varepsilon,\theta}$ contains a viscous shock profile. This can be done by applying the result presented in Chapter 2.

1.4. On the vanishing viscosity approximation in the vectorial case

Consider the system of conservation laws

$$u_t + f\,(u)_x = 0, \qquad\qquad u : I\!R^+ \times I\!R \to \Omega \subset I\!R^n, \tag{1.4.1}$$

and assume that its solution u contains one interaction between shock waves of different families. We want to study the behavior of the solution u^ε of the corresponding system with artificial viscosity

$$u_t^\varepsilon + f\,(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon,$$

in a neighborhood of the interaction. In addition to the usual regularity assumptions on the flux function f, we suppose that the Jacobian matrix $Df(u)$ is uniformly strictly hyperbolic, i.e. it has n distinct eigenvalues $\lambda_j(u)$, $j = 1, \ldots, n$, for all u in Ω and there exists $c > 0$, so that

$$\sup_{u\in\Omega} \lambda_j\,(u) + 2c < \inf_{u\in\Omega} \lambda_{j+1}\,(u) - 2c, \qquad\qquad \text{for all } \; j \leq p-1.$$

Without loss of generality we can assume the interaction to take place at the origin, then for each $j = 1, \ldots, n$ we call j^{th} primary region, Ω_j, the portion of the $t-x$ plane defined as follows. Let $\delta = c/3$, then

$$\Omega_j = \{(t,x); \quad -N + (\lambda_j^- - \delta)t < x < N + (\lambda_j^+ + \delta)t\}.$$

It is clear that the primary regions are well separated for all $t > \bar{t} = 2N/\delta$. One expects that all the waves of the j^{th} family generated by the interaction enter Ω_j in finite time. Then, if we were dealing with a diagonal or a Temple system, [51], the formation of a viscous shock profile would take place, as we saw for the scalar case, in each primary region. To understand what happens in a general situation it is useful to recall the analysis done by T.P. Liu in [28] on the large time behavior of the solution of a generalized Riemann problem, i.e. a Cauchy problem given by the system (1.4.1) together with an initial datum of the form

$$u(0,x) = \bar{u}(x) = \begin{cases} u_l & \text{for } x < -N, \\ u_0(x) & \text{for } -N \le x \le N, \\ u_r & \text{for } x > N. \end{cases} \tag{1.4.2}$$

Here u_0 is a measurable function, u_l and u_r are constant states and $N > 0$. The result obtained by T.P. Liu says that the solution of (1.4.1)-(1.4.2), u^*, converges asymptotically, as t goes to $+\infty$, to the solution of the Riemann problem with initial data

$$u(0,x) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0. \end{cases} \tag{1.4.3}$$

The system is assumed to be coupled, then the waves generated at time $t = 0$ can interact with each other and produce new waves before reaching the primary region associated with their family. The new waves are much weaker than the previous ones, since they are of second order with respect to the total variation of the system, but they are spread on an interval larger than $[-N, N]$, then the rate of uncoupling slows down. As a consequence, the rates of convergence obtained in [28] are only algebraic in the norm of total variation. They also depend on the structure of the solution of (1.4.3) and in particular, if u only contains shocks and contact discontinuities, the solution u^* will converges to a superposition of shock waves and travelling profiles at rate t^{-2}.

For the same reasons we do not expect to recover exponential rates of convergence in time for our problem and, as a consequence, we think that it will not be possible to generalize the result by J. Goodman and Z. Xin, [22], to the case in which the solution u of the original system is allowed to contain shock interactions. In fact, the generalization has been fully achieved for the scalar case, as I outlined in the two previous section, relying on two main results. On one hand W. Shen and Z. Xu proved in [45] that the outer expansion can be performed in all regions where the value of u can be obtained by the method of characteristics. When u is a vector, the domain of dependence of an interaction point does not enjoy this property, unless the system is diagonal. On the other hand my first result in collaboration with A. Bressan shows that the formation of the outgoing shock profile after one interaction is completed in time $T < \infty$ in the scalar case. This allows for the cancellation of all small perturbations in finite time. In particular, in all compact subsets of the half-plane $(\tau, +\infty) \times I\!R$, our solution u^ε coincides, up to an exponentially small error, with the viscous approximation of a solution $\tilde{u}$, which satisfies the conditions required in [22]. If the rate of decoupling after interaction is algebraic for systems, as the result by T.P. Liu suggests, none of these results would be generalized.

My approach to the problem goes in two directions. First I consider a self similar solution of (1.4.1) containing exactly one interaction between shocks of different families and I refine the convergence estimate for the vanishing viscosity approximations in [13] to this very simple situation. This gives me an upper bound of the convergence rate. Then I study the system of two parabolic equations

$$\begin{cases} u_t + \left(u^2/2\right)_x + \rho f\,(u,v)_x = u_{xx}\,, \\ v_t + \left(v^2/2\right)_x + \rho g\,(u,v)_x = v_{xx}\,. \end{cases}$$

The coupling parameter $\rho \geq 0$ is supposed to be small, then we say that the system is weakly coupled. I want to describe the change in the solution $U_\rho = (u,v)^t$ as ρ varies. In particular, writing U_ρ in the form

$$U_\rho = U_0 + \rho W,$$

I am interested in the time behavior of the term W. Since the presence of ρ changes the flux function of the system, it also changes the asymptotic profile of the solution, and I expect W to converge to a piecewise constant function different from zero as t goes to $\pm\infty$. The rate of this convergence should give me a lower bound on the time needed for the outgoing viscous shocks to form after interaction.

Chapter 2.
On the Formation of Scalar Viscous Shocks

2.1. Introduction

Consider a scalar conservation law with viscosity

$$u_t + f(u)_x = u_{xx}\,, \tag{2.1.1}$$

where the flux function f is smooth but not necessarily convex. To this equation we associate an initial datum $\bar{u}$ of the following form, see Fig. 2.2.1. For some small constant $\delta_0 > 0$, we assume that

$$\begin{aligned} &|\bar{u}(x) - u^-| \leq \delta_0, && x \leq a\,, \\ &|\bar{u}(x) - u^+| \leq \delta_0, && x \geq b\,, \\ &\bar{u}(x) \in [m, M]\,, && x \in \mathbb{R}\,. \end{aligned} \tag{2.1.2}$$

Here the constants $m < M$ and $a < b$ can be arbitrary, possibly very large. Assuming that there exists a viscous travelling profile $\phi(\cdot)$ connecting the states u^-, u^+, we wish to estimate how long does it take for the solution to develop a single viscous shock. Given $\kappa > \delta_0$, we seek a time T such that, for all $t \geq T$, there exists a suitable shifted profile $\phi(\,\cdot - c(t))$ with

$$\sup_x\; |u(t,x) - \phi(x - c(t))| \leq \kappa\,.$$

Notice that the assumptions in (2.1.2) do not imply the existence of the limits of $\bar{u}(x)$ as $x \to \pm\infty$, and hence do not guarantee the asymptotic convergence to any viscous shock profile. Our result can be stated as follows.

Theorem 2.1. *Let f be a smooth flux function, and assume that the states $u^+ < u^-$ are connected by a stable viscous shock profile $\phi(\cdot)$, i.e. the conditions (1.2.4)–(1.2.5) are satisfied. Then there exists a constant C such that the following holds. For every $\delta_0 > 0$ sufficiently small, if $\bar{u}$ is an initial condition satisfying (2.1.2), then the corresponding solution $u = u(t,x)$ satisfies*

$$\|u(t,\cdot) - \varphi(t, \cdot - c(t))\|_{\mathbf{L}^\infty} \leq C\delta_0\,, \tag{2.1.3}$$

for a suitable shift $c(t)$ and all times t sufficiently large.

The time T after which the viscous shock is formed will be estimated in terms of the upper and lower bounds m, M, and the length $b - a$ of the interval where $\bar{u}$ can be far from a viscous profile. Due to the absence of any limit as $x \to \pm\infty$, one cannot use any integral norm to estimate the distance between the solution $u(t,\cdot)$ and a travelling shock profile. Our proof relies on classical comparison arguments with upper and lower solutions, and on the variable transformation used by S. Bianchini and A. Bressan in [4], [6], the next section is devoted to the description of these tools.

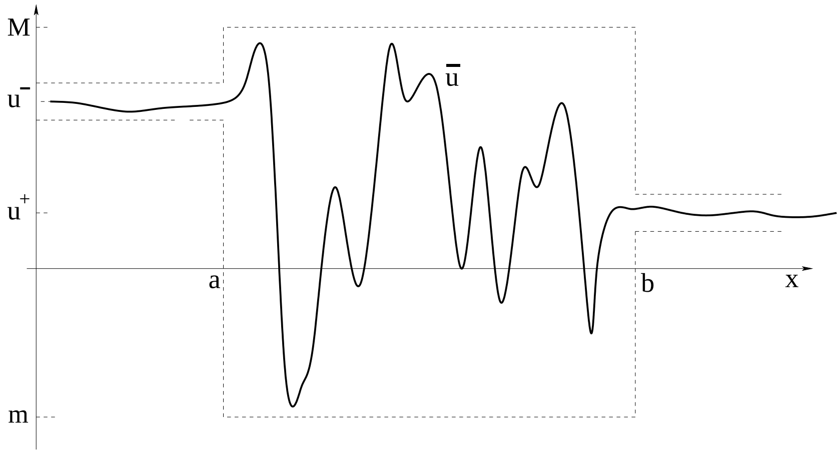

Fig. 2.2.1: The initial condition.

2.2. Comparison results

Consider a bounded solution $u = u(t,x)$ of Eq. (2.1.1). Calling $w \doteq f(u) - u_x$, at each time t we consider the curve in the u-w plane

$$x \mapsto \gamma(t,x) \doteq \begin{pmatrix} u(t,x) \\ f(u(t,x)) - u_x(t,x) \end{pmatrix}. \tag{2.2.1}$$

The motion of this curve occurs in the direction of curvature. Indeed, writing $w = w(t,u)$, each branch where $w - f(u)$ has a constant sign evolves according to

$$w_t = (w - f(u))^2 \, w_{uu} \,. \tag{2.2.2}$$

It is well known that, if $u(t,x) = \phi(x - \lambda t)$ is a viscous shock profile, then the corresponding curve $\gamma(t,\cdot)$ is constant in time and coincides with the segment connecting the points

$$P^- \doteq \big(u^-, f(u^-)\big) \quad , \qquad P^+ \doteq \big(u^+, f(u^+)\big) \,. \tag{2.2.3}$$

According to the Rankine-Hugoniot condition, the slope of this segment is precisely the speed of the shock, namely

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} \,. \tag{2.2.4}$$

To study how the curve γ evolves in time, we consider suitable upper and lower solutions.

Definition 2.1. A $\mathcal{C}^2$ function $w^+(t,u)$, defined for $u \in [u_1, u_2]$, is an **upper solution** of Eq. (2.2.2) if

$$w_t^+ \geq \big(f(u) - w^+\big)^2 \, w_{uu}^+ \,. \tag{2.2.5}$$

Similarly, a function w^- is a **lower solution** if

$$w_t^- \leq \big(f(u) - w^-\big)^2 \, w_{uu}^- \,. \tag{2.2.6}$$

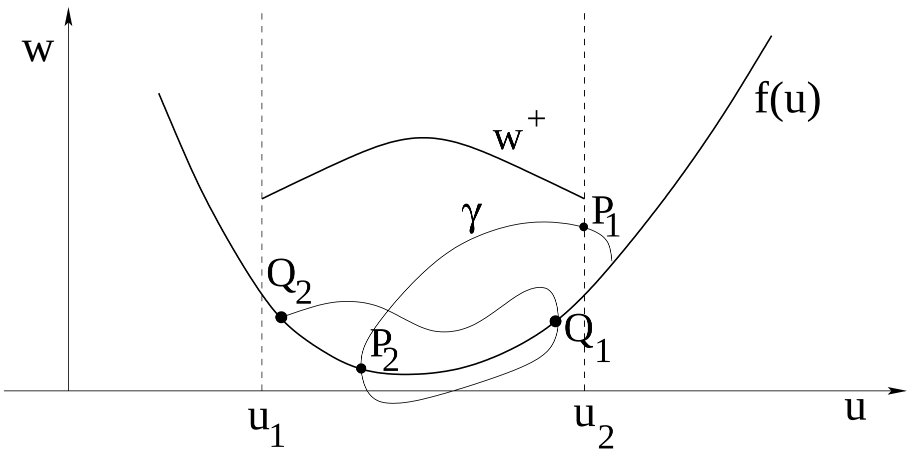

Fig. 2.2.2: The two branches of γ, between the points P_1, P_2 and between the points Q_1, Q_2, lie below the upper solution w^+.

The hypo-graph of $w^+(t)$ is defined as

$$W^+_{hypo}(t) \doteq \{(u,w)\ ;\quad w \le w^+(t,u)\ ,\quad u \in [u_1,\, u_2]\} \cup \{(u,w)\ ;\quad u \notin [u_1,\, u_2]\}\,.$$

Similarly, if w^- is defined for $u \in [u_1, u_2]$, the epi-graph of $w^-(t)$ is defined as

$$W^-_{epi}(t) \doteq \{(u,w)\ ;\quad w \ge w^-(t,u)\ ,\quad u \in [u_1,\, u_2]\} \cup \{(u,w)\ ;\quad u \notin [u_1,\, u_2]\}\,.$$

By standard techniques in the theory of parabolic equations, one obtains the following comparison results.

Lemma 2.1. *Let $w^+ : [0,T] \times [u_1, u_2] \mapsto I\!R$ be an upper solution of Eq. (2.2.2), with*

$$w^+(t,u) > f(u)\ , \qquad \text{for all } t, u\,. \tag{2.2.7}$$

Let $u = u(t,x)$ be a solution of Eq. (2.1.1), and let $\gamma = (u,w) = (u, f(u) - u_x)$ be the corresponding curve in Eq. (2.2.1). Consider a portion of γ, say

$$\{\gamma(t,x)\ ;\quad t \in [0,T]\ ,\quad x \in [\alpha(t)\,,\, \beta(t)]\}\,,$$

where $\alpha(\cdot)$, $\beta(\cdot)$ are continuous functions. Assume that

$$\gamma(0,x) \in W^+_{hypo}(0)\ , \qquad x \in I\!R\,, \tag{2.2.8}$$

$$\gamma(t,\alpha(t)) \in W^+_{hypo}(t)\ , \qquad \gamma(t,\beta(t)) \in W^+_{hypo}(t)\ , \tag{2.2.9}$$

and moreover, for all $x \in [\alpha(t)\,,\ \beta(t)]$, one has

$$f(u(t,x)) - u_x(t,x) \le w^+(t,u_1)\ , \qquad \text{whenever } \ u(t,x) = u_1\,, \tag{2.2.10}$$

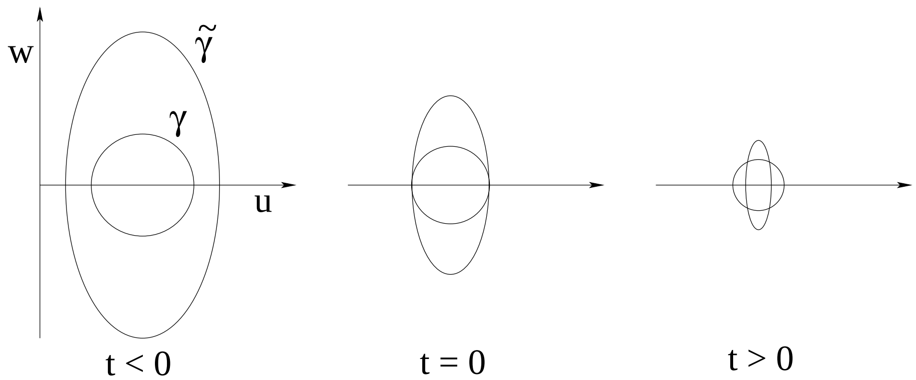

Fig. 2.2.3: The number of intersections between two curves $\gamma, \tilde{\gamma}$ can increase in time, but only starting at points where $w = f(u)$.

$$f\left(u\left(t,x\right)\right) - u_x\left(t,x\right) \leq w^+\left(t,u_2\right)\,, \qquad \textit{whenever} \quad u\left(t,x\right) = u_2\,. \tag{2.2.11}$$

Then

$$\gamma\left(t,x\right) \in W^+_{hypo}\left(t\right)\,, \qquad \textit{for all} \quad t \in [0,T]\,, \quad x \in \left[\alpha\left(t\right)\,,\ \beta\left(t\right)\right]\,. \tag{2.2.12}$$

Of course, an entirely similar result holds in the case of a lower solution w^-.

The proof is obtained by applying standard comparison results, see the books by M. H. Protter and H. F. Weinberger [41], and R. P. Sperb [47], to every branch $w = w(t,u)$ of the curve γ lying above the graph of f (hence with $u_x < 0$), with $u_1 \leq u(t,x) \leq u_2$. Notice that the endpoints of each branch either lie on the graph of f, or on one of the vertical lines $u = u_1$, $u = u_2$, or else correspond to the endpoints $x = \alpha(t)$ or $x = \beta(t)$. Our assumptions state precisely that $w \leq w^+$ in each of these cases.

Remark 2.1. Assume that f is convex, with $f''(u) \geq k > 0$. Taking

$$w^-\left(t,u\right) \doteq f\left(u\right) - \frac{1}{kt}\,,$$

one checks that

$$w^-_t = \frac{1}{kt^2} \leq \left(f\left(u\right) - w^-\right)^2 w^-_{uu} = \frac{1}{k^2t^2}\, f''\left(u\right).$$

Hence, by a comparison argument, every branch of the curve γ with $w < f(u)$ satisfies $w^-(t,u) \leq w(t,u)$. In turn this implies the well known Oleinik's estimate

$$u_x\left(t,x\right) = f\left(u\left(t,x\right)\right) - w\left(t,\, u\left(t,x\right)\right) \leq \frac{1}{kt} = f\left(u\left(t,x\right)\right) - w^-\left(t,\, u\left(t,x\right)\right)\,. \tag{2.2.13}$$

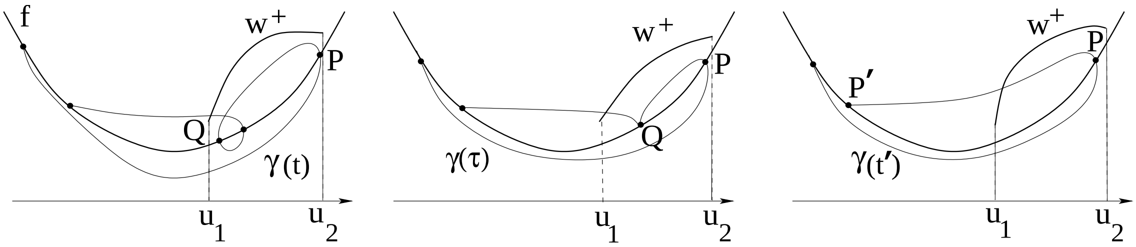

Fig. 2.2.4: The function w^+ provides an upper bound for the branch PQ, for $t < \tau$. After the lower branch collapses, w^+ is not an upper bound for the branch PP', because the boundary condition in (2.2.10) does not hold.

Remark 2.2. The above comparison method is effective when we need to estimate the derivative u_x along one single branch of γ, for example on the interval $x \in [\alpha(t)\,,\ \beta(t)]$ between a point of local maximum and a point of local minimum of the solution $u(t, \cdot)$. However, these estimates do not tell us how fast a local maximum decreases, or how fast a local minimum increases.

Example 2.1. Assume $f \equiv 0$, so that Eq. (2.1.1) reduces to the heat equation. Consider the solutions

$$u\,(t, x) = e^{-t} \cos x\,, \qquad\qquad \tilde{u}\,(t, x) = e^{-4t} \cos 2x\,.$$

As shown in Fig. 2.2.3, in the u-w plane, the corresponding curves $\gamma(t, \cdot)$ are circles, while the curves $\tilde{\gamma}(t, \cdot)$ are ellipses:

$$\gamma\,(t, x) = e^{-t} \left(\cos x\,,\quad \sin x\right)\,, \qquad\qquad \tilde{\gamma}\,(t, x) = e^{-4t} \left(\cos 2x\,,\ \ 2 \sin 2x\right)\,.$$

If we consider the upper branches $w, \tilde{w}$ of these curves, we see that

$$\tilde{w}\,(t, u) \geq w\,(t, u)\,, \tag{2.2.14}$$

for $t \leq 0$ and every u in the common domain of $w, \tilde{w}$. In this case, the comparison Lemma 2.1 can be applied with $\tilde{w} = w^+$, $\alpha(t) \equiv 0$, $\beta(t) \equiv \pi/2$, on any time interval of the form $[-T, 0]$. However, the conclusion fails for $t > 0$. Indeed, when $t > 0$ the boundary condition in (2.2.9) is not satisfied.

Remark 2.3. For two solutions $w, \tilde{w}$ of the same parabolic equation, the number of intersections between the graphs of $w(t, \cdot)$ and $\tilde{w}(t, \cdot)$ does not increase in time. However, for the curves $\gamma(t, \cdot), \tilde{\gamma}(t, \cdot)$ in the above example, this number of intersections does increase. A new pair of intersections can be created only at a point (u, w) with $w = f(u)$, i.e. where the curves do not locally define functions $w = w(u)$.

If $u = u(t, x)$ is a solution of the Cauchy problem

$$\begin{cases} u_t + f\,(u)_x = u_{xx}, \\ u\,(0, x) = \bar{u}\,(x)\,, \end{cases} \tag{2.2.15}$$

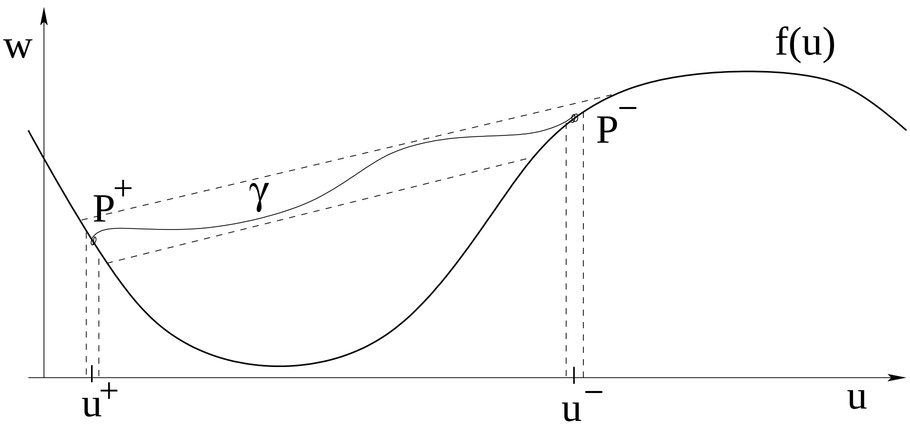

Fig. 2.2.5: If the curve $\gamma(t,\cdot)$ lies sufficiently close to the segment joining $P^+ = (u^+,\ f(u^+))$ with $P^- = (u^-,\ f(u^-))$, then the profile of $u(t,\cdot)$ is close to a travelling viscous shock.

Lemma 2.1 provides a priori bounds on the norm of the derivative $\|u_x(t,\cdot)\|_{\mathbf{L}^\infty}$, for $t > 0$. Indeed, assume that the initial data satisfies $m \le \bar{u}(x) \le M$. Choose two quadratic polynomials of the form

$$p\,(u) \doteq A + Bu + \frac{u^2}{2}\,, \qquad\qquad q\,(u) \doteq A' + B'u - \frac{u^2}{2}\,,$$

such that

$$p\,(u) \le f\,(u) \le q\,(u)\ , \qquad\qquad \text{for all}\ \ u \in [m, M]\ . \tag{2.2.16}$$

Then

$$w^-\,(t,u) = p\,(u) - \frac{1}{t}\,, \qquad\qquad w^+\,(t,u) = q\,(u) + \frac{1}{t}\,,$$

are respectively a lower solution and an upper solution of Eq. (2.2.2). Therefore, any branch of the curve γ corresponding to the solution $u = u(t,x)$ of Eq. (2.1.1) will satisfy

$$w^-\,(t,u) \le w\,(t,u) \le w^+\,(t,u)\ .$$

In turn this implies the pointwise estimates

$$\inf_{u\in[m,M]} \left[f\,(u) - q\,(u) - \frac{1}{t}\right] \le u_x\,(t,x) \le \sup_{u\in[m,M]} \left[f\,(u) - p\,(u) + \frac{1}{t}\right]\ . \tag{2.2.17}$$

2.3. Distance from a shock profile

Let $\varphi(x)$ be a travelling viscous shock, connecting the states u^-, u^+. Then the corresponding curve $\tilde{\gamma}$ is precisely the portion of the straight line

$$f\left(u^-\right) + \lambda\left(u - u^-\right) \doteq \tilde{w}\left(u\right), \tag{2.3.1}$$

with $u \in [u^+, u^-]$. As mentioned in the previous section, here

$$\lambda = \frac{f\left(u^-\right) - f\left(u^+\right)}{u^- - u^+}, \tag{2.3.2}$$

is the slope of the line, and also the speed of the shock.

Now assume that $x \mapsto u(x)$ is a smooth function such that the corresponding curve $x \mapsto (u(x),\ f(u(x)) - u_x(x))$ is close to the line (2.3.1). More precisely, assume that

$$\left| f\left(u\left(x\right)\right) - u_x\left(x\right) - \tilde{w}\left(u\left(x\right)\right)\right| \leq \delta\,, \tag{2.3.3}$$

for some $\delta > 0$ small and all $x \in I\!R$. We wish to estimate how much the function $u(\cdot)$ differs from a viscous shock profile. In other words, we seek

$$\inf_{c \in I\!R} \left\| u\left(\cdot\right) - \varphi\left(\cdot - c\right)\right\|_{\mathbf{L}^\infty}. \tag{2.3.4}$$

Notice that here we need to shift the shock profile by a constant c, in order to fit the graph of u as closely as possible. As shown in Fig. 2.2.5, the condition (2.3.3) implies that the curve γ has a branch described as a function $w = w(u)$, for u bounded away from u^-, u^+. However, when u is close to either u^- or u^+, several values of w may correspond to a single u.

In order to provide an upper bound on (2.3.4), we fix the middle value $u^\dagger \doteq (u^+ + u^-)/2$, and consider the unique point $x^\dagger$ such that $u(x^\dagger) = u^\dagger$. We then shift the profile φ in such a way that

$$\varphi\left(x^\dagger\right) = u\left(x^\dagger\right) = u^\dagger\,. \tag{2.3.5}$$

For notational convenience, we can assume that this shift corresponds to $c = 0$. The two functions u, φ thus provide solutions to the Cauchy problems

$$\frac{du}{dx} = f\left(u\right) - w\left(u\right), \qquad \frac{d\varphi}{dx} = f\left(\varphi\right) - \tilde{w}\left(\varphi\right), \tag{2.3.6}$$

with the same initial condition (2.3.5) at $x = x^\dagger$. We recall that $\tilde{w}$ is the linear function introduced by (2.3.1). Solving (2.3.6) by separation of variables we obtain

$$\int_{u^\dagger}^{u} \frac{ds}{f\left(s\right) - w\left(s\right)} = x - x^\dagger\,, \qquad \int_{u^\dagger}^{\varphi} \frac{ds}{f\left(s\right) - \tilde{w}\left(s\right)} = x - x^\dagger\,. \tag{2.3.7}$$

We now fix $u^+ < \omega^+ < \omega^- < u^-$ such that the following conditions hold.

(i) For all $u \in [\omega^+, \omega^-]$ one has

$$\tilde{w}\left(u\right) - f\left(u\right) > \delta\,. \tag{2.3.8}$$

(ii) The map
$$u \mapsto \tilde{w}(u) - f(u) \,,$$
is monotone increasing on $[u^+, \omega^+]$ and monotone decreasing on $[\omega^-, u^-]$.

(iii) If $u > u^-$ and $f(u) \le \tilde{w}(u) + \delta$ then $u - u^- \le u^- - \omega^-$. Similarly, if $u < u^+$ and $f(u) \le \tilde{w}(u) + \delta$ then $u^+ - u \le \omega^+ - u^+$.

Notice that (2.3.3) and (2.3.8) together imply that the restriction of the curve γ to the subinterval $[\omega^+, \omega^-]$ is the graph of a function $w = w(u)$. Moreover, for $\omega \in [\omega^+, \omega^-]$ there exists a unique point $x(\omega) \in I\!R$ such that $u(x) = \omega$, because $x \mapsto u(x)$ is monotone decreasing when $u \in [\omega^+, \omega^-]$. We claim that, for some constant C_ω independent of δ, there holds
$$\begin{aligned} &\|u - \varphi\|_{\mathbf{L}^\infty(]-\infty, x^\dagger])} \le 2|u^- - \omega^-| + \\ &\quad + \sup_{\omega \in [u^\dagger, \omega^-]} \int_{u^\dagger}^{\omega} \left| \frac{1}{f(s) - \tilde{w}(s)} - \frac{1}{f(s) - w(s)} \right| ds \cdot C_\omega \left(\tilde{w}(\omega) + \delta - f(\omega)\right) . \end{aligned} \tag{2.3.9}$$
To justify the above inequality, consider any $x < x^\dagger$. If $\min\{u(x), \varphi(x)\} \ge \omega^-$, by the assumption (iii) we have
$$|u(x) - \varphi(x)| \le |u(x) - u^-| + |\varphi(x) - u^-| \le 2|u^- - \omega^-| \,,$$
and (2.3.9) trivially follows. Next, assume that $\min\{u(x), \varphi(x)\} < \omega^-$. To fix the ideas, let $\omega \doteq \varphi(x) < u(x)$. Then there will be a unique point $\bar{x} > x$ such that $u(\bar{x}) = \varphi(x)$. We now have
$$\begin{aligned} |u(x) - \varphi(x)| &= |u(x) - u(\bar{x})| \le \\ &\le |\bar{x} - x| \cdot \sup_{x' \ge x} |u_x(x')| \le \\ &\le \int_{u^\dagger}^{\omega} \left| \frac{1}{f(s) - \tilde{w}(s)} - \frac{1}{f(s) - w(s)} \right| ds \cdot C_\omega \left(\tilde{w}(\omega) + \delta - f(\omega)\right) . \end{aligned}$$
In the last inequality we used the assumption (ii) to get rid of the case $u(x) > \omega^-$, and (2.3.3) and (2.3.8) to bound the absolute value of $u_x(x')$ from above and from below, for all x' such that $u(x') \in [\omega^+, \omega^-]$. We obtain
$$0 < |u_x(x')| < \delta + \tilde{w}\left(u(x')\right) - f\left(u(x')\right) < \delta + \sup_{u \in [\omega^+, \omega^-]} \{\tilde{w}(u) - f^-(u)\} \,,$$
where f^- is the convex envelope of f. Then for any fixed value of u, in particular for ω, there exists a constant C_u such that
$$\sup_{x' \ge x} |u_x(x')| < C_u \left(\tilde{w}(u) + \delta - f(u)\right) .$$
Of course, an entirely similar argument yields
$$\begin{aligned} &\|u - \varphi\|_{\mathbf{L}^\infty([x^\dagger, \infty[)} \le 2|\omega^+ - u^+| + \\ &\quad + \sup_{\omega \in [\omega^+, u^\dagger]} \int_{\omega}^{u^\dagger} \left| \frac{1}{f(s) - \tilde{w}(s)} - \frac{1}{f(s) - w(s)} \right| ds \cdot C_\omega \left(\tilde{w}(\omega) + \delta - f(\omega)\right) . \end{aligned} \tag{2.3.10}$$

For every pair ω^-, ω^+ satisfying the above conditions (i)–(iii), we now have the estimates (2.3.9)-(2.3.10). Next, we examine which choice of ω^-, ω^+ yields the best bounds, assuming that $\delta > 0$ is a very small constant. From (2.3.1) it follows

$$\tilde{w}(\omega) + \delta - f(\omega) = \delta + \mathcal{O}(1) \cdot \left(u^- - \omega\right), \tag{2.3.11}$$

where the Landau symbol $\mathcal{O}(1)$ denotes a uniformly bounded quantity. Moreover,

$$\begin{aligned} &\int_{u^\dagger}^{\omega} \left| \frac{1}{f(s) - \tilde{w}(s)} - \frac{1}{f(s) - w(s)} \right| ds \leq \\ &\leq \int_{u^\dagger}^{\omega} \left| \frac{1}{\tilde{w}(s) - f(s) - \delta} - \frac{1}{\tilde{w}(s) - f(s)} \right| ds = \\ &= \mathcal{O}(1) \cdot \int_{u^\dagger}^{\omega} \frac{\delta}{(u^- - s)^2}\, ds = \mathcal{O}(1) \cdot \frac{\delta}{u^- - \omega}. \end{aligned} \tag{2.3.12}$$

Choosing

$$\omega^+ = u^+ + C'\delta, \qquad \omega^- = u^- - C'\delta,$$

for some constant C' sufficiently large, all conditions (i)–(iii) are satisfied. From (2.3.11)-(2.3.12), for another constant C'' we obtain

$$\begin{aligned} &\sup_{\omega \in [u^\dagger, \omega^-]} \left\{ \int_{u^\dagger}^{\omega} \left| \frac{1}{f(s) - \tilde{w}(s)} - \frac{1}{f(s) - w(s)} \right| ds \cdot \left(\tilde{w}(\omega) + \delta - f(\omega)\right) \right\} \leq \\ &\leq \sup_{\omega \in [u^\dagger, \omega^-]} \left\{ \frac{C''\delta}{u^- - \omega} \cdot \left(\delta + C''\left(u^- - \omega\right)\right) \right\} = \\ &= \mathcal{O}(1) \cdot \delta. \end{aligned} \tag{2.3.13}$$

Using (2.3.13) and the analogous estimates on the interval $[\omega^+, u^\dagger]$, we finally obtain

$$\|u - \varphi\|_{\mathbf{L}^\infty(\mathbb{R})} = \mathcal{O}(1) \cdot \delta. \tag{2.3.14}$$

The above analysis can be summarized as follows.

Lemma 2.2. *Let f be a smooth flux function, and assume that the states $u^+ < u^-$ satisfy the conditions (1.2.4)-(1.2.5), and are thus connected by a stable viscous shock profile $\varphi(\cdot)$. Then there exists a constant K_0 such that the following holds. For every small $\delta > 0$, let $x \mapsto u(x)$ be a smooth profile which satisfies (2.3.3), together with*

$$\limsup_{x \to -\infty} u(x) > \frac{u^+ + u^-}{2}, \qquad \liminf_{x \to +\infty} u(x) < \frac{u^+ + u^-}{2}. \tag{2.3.15}$$

Then, for some $c \in \mathbb{R}$ there holds

$$\|u - \varphi(\cdot - c)\|_{\mathbf{L}^\infty(\mathbb{R})} \leq K_0\, \delta. \tag{2.3.16}$$

Notice that the assumption (2.3.15) is needed only to rule out trivial situations such as $u(x) \equiv u^-$ for all $x \in \mathbb{R}$.

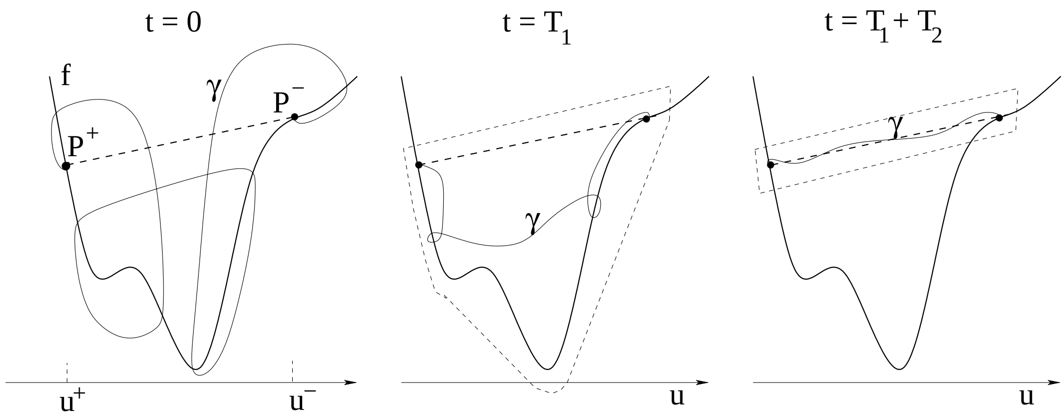

Fig. 2.4.1: The curve $\gamma(t,\cdot)$ approaches the segment joining the points P^+, P^-, in two phases.

2.4. Formation of a shock profile: first phase

According to Lemma 2.2, to prove an estimate of the form (2.1.3) it suffices to show that the graph of the corresponding curve $x \mapsto \gamma(t,x) = (u(t,x)\,,\ f(u(t,x) - u_x(t,x)))$ is entirely contained in a narrow strip around the graph of $\tilde w$ in (2.3.1). We recall that this is the case if and only if (2.3.3) holds.

By the assumptions (2.1.2), the initial curve $\gamma(0,\cdot)$ lies in a small neighborhood of $P^- = (u^-,\ f(u^-))$ as $x \to -\infty$, and in a small neighborhood of $P^+ = (u^+,\ f(u^+))$ as $x \to \infty$. In the middle, however, the initial curve $\gamma(0,\cdot)$ can be very far from the segment connecting P^- with P^+. Our main interest is to determine how long does it take for the curve γ to approach this segment. Within the process of shock formation, we distinguish two phases. The first phase is completed after time T_1, the second requires an additional time T_2.

PHASE 1: For $t \geq T_1$, the curve $\gamma(t,\cdot)$ becomes entirely contained in a neighborhood of the convex closure of the graph of f, with $u \in [u^+, u^-]$. Namely, for some small $\delta_1 \geq 2\delta_0 > 0$ one has

$$\gamma\,(t,x) \in \Omega^{\delta_1} \doteq \overline{co}\{(u+\xi\ ,\ \ f\,(u))\ ;\quad u \in \left[u^+ - \delta_1\ ,\ \ u^- + \delta_1\right]\ ,\quad |\xi| \leq \delta_1\}\,. \tag{2.4.1}$$

Moreover, the function $x \mapsto u(t,x)$ is "almost decreasing", namely

$$x < y \qquad \Longrightarrow \qquad u\,(t,x) \geq u\,(t,y) - \delta_1\,. \tag{2.4.2}$$

PHASE 2: For all $t \geq T_1 + T_2$ the curve $\gamma(t,\cdot)$ becomes entirely contained in a small strip around the segment connecting $P^+ = (u^+,\ f(u^+))$ and $P^- = (u^-,\ f(u^-))$, namely

$$|f\,(u\,(t,x)) - u_x\,(t,x) - \tilde w\,(u\,(t,x))\,| \leq \delta_2\,, \tag{2.4.3}$$

for some small constant $\delta_2 > \delta_1$ and all $x \in I\!R$. We recall that $u \mapsto \tilde w(u)$ is the linear function defined at (2.3.1). Its graph is the straight line through the points P^+, P^-.

The position of the curve $x \mapsto \gamma(t,x)$ at the initial time, and after the completion of Phases 1 and 2, is shown in Fig. 2.4.1. In this section we examine how long it takes for Phase 1 to be completed. The time needed to achieve Phase 2 will be analyzed in the subsequent section. As shown in Section 2.3, at the end of Phase 2 the solution $u(t,\cdot)$ is close to a travelling shock profile, as desired.

Clearly, a bound on T_1 must depend on the initial conditions at (2.1.2), through the values of $m \le u^+ < u^- \le M$ and the length of the interval $[a,b]$.

Lemma 2.3. *Let $u = u(t,x)$ be a solution of Eq. (2.1.1) with initial data as in (2.1.2). Then, for any $\delta_1 \ge 2\delta_0$, the corresponding curve $\gamma(t,\cdot)$ is entirely contained inside the convex set Ω^{δ_1} in (2.4.1) as soon as $t \ge \tau_1 + \tau_2$, with*

$$\tau_1 \doteq \frac{(b-a)^2}{4\pi\delta_0^2} \cdot \max\{\big(M-u^-\big)^2,\, \big(u^+-m\big)^2\}, \tag{2.4.4}$$

$$\tau_2 \doteq \frac{\big(u^- - u^+ + 2\delta_1\big)^2}{8\delta_1}. \tag{2.4.5}$$

Proof. 1. Set $\omega_0 \doteq u^- + \delta_0$ and define

$$h(t) \doteq \max_x\{u(t,x) - \omega_0\}. \tag{2.4.6}$$

Let $v = v(t,x)$ be the solution of

$$v_t + f(v+\omega_0)_x = v_{xx}, \tag{2.4.7}$$

with initial data

$$v(0,x) = \bar v(x) \doteq (\bar u(x) - \omega_0)_+ . \tag{2.4.8}$$

Here and in the sequel, $(s)_+ = \max\{s,0\}$ denotes the positive part of a number s. By a straightforward comparison argument we have $\omega_0 + v(t,x) \ge u(t,x)$ for all t,x. Therefore

$$h(t) \le \max_x v(t,x) . \tag{2.4.9}$$

If the flux f is linear, say $f(u) = r+su$, then the Cauchy problem defined by (2.4.8)-(2.4.9) can be explicitly written in terms of a convolution with the Gaussian kernel $G(t,x) = (4\pi t)^{-1/2}e^{-x^2/4t}$, namely

$$v(t,x) = \int G(t,x-y)\,\bar v(y-st)\,dy. \tag{2.4.10}$$

Hence

$$h(t) \le \|v(t,\cdot)\|_{\mathbf{L}^\infty} \le \|G(t,\cdot)\|_{\mathbf{L}^\infty} \cdot \|v(0,\cdot)\|_{\mathbf{L}^1} \le \frac{\big(M-u^- - \delta_0\big)(b-a)}{2\sqrt{\pi t}}. \tag{2.4.11}$$

We claim that the same estimate (2.4.11) holds also for a general flux function f. This will be proved by comparing the solution v of (2.4.7)-(2.4.8) with a suitable radially symmetric rearrangement. For each $t, h > 0$, define

$$\psi^v(t,h) \doteq \mathrm{meas}\{x\,;\;\; v(t,x) \geq h\}\,.$$

We then define the radially symmetric, radially decreasing rearrangement $\hat{v}$ as the unique function such that

$$\hat{v}(t,x) = \hat{v}(t,-x)\,, \qquad \hat{v}(x) \geq \hat{v}(y)\,, \qquad \text{if} \quad 0 \leq x < y\,, \tag{2.4.12}$$

$$\mathrm{meas}\{x\,;\;\hat{v}(t,x) \geq h\} = \mathrm{meas}\{x\,;\;v(t,x) \geq h\}\,, \qquad \text{for all} \;\; t, h > 0\,. \tag{2.4.13}$$

Let $w = w(t,x)$ be the solution of the Cauchy problem

$$w_t = w_{xx}\,, \qquad w(0,x) = \hat{v}(0,x)\,.$$

Then $w(t,\cdot)$ is radially symmetric, radially decreasing, and satisfies

$$\|w(t,\cdot)\|_{\mathbf{L}^\infty} \leq \frac{\big(M - u^- - \delta_0\big)\,(b-a)}{2\sqrt{\pi t}}\,. \tag{2.4.14}$$

We claim that, for every $t \geq 0$, one has

$$\psi^v(t,h) \leq \psi^w(t,h)\,, \qquad h > 0\,. \tag{2.4.15}$$

Indeed, assume that, for some t, h, there holds

$$\psi^v(t,h) = \psi^w(t,h)\,, \qquad \frac{\partial \psi^v}{\partial h}(t,h) = \frac{\partial \psi^w}{\partial h}(t,h)\,. \tag{2.4.16}$$

We need to show that (2.4.16) implies

$$\frac{\partial \psi^v}{\partial t}(t,h) \leq \frac{\partial \psi^w}{\partial t}(t,h)\,. \tag{2.4.17}$$

Toward this goal, consider the point

$$x^* \doteq \inf\{x > 0\,;\quad w(t,x) < h\}\,,$$

and call $x_1, \ldots, x_\nu$ the points where $v(t,x) = h$ and $v_x(t,x) \neq 0$. We then have

$$\frac{\partial \psi^w}{\partial t}(t,h) = -2|w_x(t,x^*)|\,, \qquad \frac{\partial \psi^v}{\partial t}(t,h) = -\sum_{i=1}^{\nu} |v_x(t,x_i)|\,. \tag{2.4.18}$$

We now use an elementary inequality. If $a_1, a_2, \ldots, a_\nu > 0$ and $b > 0$ are numbers such that

$$\sum_{i=1}^{\nu} \frac{1}{a_i} = \frac{2}{b}\,, \qquad \nu \geq 2\,,$$

then

$$\sum_i a_i \geq 2b\,. \tag{2.4.19}$$

Observing that

$$\sum_{i=1}^{\nu} \frac{1}{|v_x(t,x_i)|} = -\frac{\partial \psi^v}{\partial h}(t,h) = -\frac{\partial \psi^w}{\partial h}(t,h) = \frac{2}{|w_x(t,x^*)|}\,, \tag{2.4.20}$$

from the inequality (2.4.19) we obtain (2.4.17).

A comparison argument now yields $\psi^v(t,h) \leq \psi^w(t,h)$, for all $t, h > 0$. In particular, the same bound (2.4.14) holds for v as well:

$$\|v(t,\cdot)\|_{\mathbf{L}^\infty} \leq \frac{\big(M - u^- - \delta_0\big)(b-a)}{2\sqrt{\pi t}}\,. \tag{2.4.21}$$

Hence, for all $x \in I\!R$ and $t > 0$ we have

$$u(t,x) \leq u^- + \delta_0 + \frac{\big(M - u^- - \delta_0\big)(b-a)}{2\sqrt{\pi t}}\,. \tag{2.4.22}$$

An entirely similar argument yields

$$u(t,x) \geq u^+ - \delta_0 - \frac{\big(u^+ - \delta_0 - m\big)(b-a)}{2\sqrt{\pi t}}\,. \tag{2.4.23}$$

The bounds (2.4.22)-(2.4.23) together imply that, for every $\delta_1 \geq 2\delta_0$, there holds

$$u^+ - \delta_1 \leq u(t,x) \leq u^- + \delta_1\,, \tag{2.4.24}$$

for all $x \in I\!R$ and $t \geq \tau_1$, with τ_1 as defined at (2.4.4).

2. Consider the concave and the convex envelopes of f restricted to the interval $I \doteq [u^+ - 2\delta_1,\ u^- + 2\delta_1]$, i.e.

$$f^+(u) \doteq \inf\{g(u)\ ;\quad g''(v) \leq 0\,,\ \ g(v) \geq f(v)\ ,\quad \text{for all } \ v \in I\}\,,$$

$$f^-(u) \doteq \sup\{g(u)\ ;\quad g''(v) \geq 0\,,\ \ g(v) \leq f(v)\ ,\quad \text{for all } \ v \in I\}\,.$$

In this second step we consider an initial condition $u(0,\cdot)$ such that the corresponding curve $\gamma(0,\cdot)$ lies entirely inside the strip where $u \in [u^+ - \delta_1\,,\ \ u^- + \delta_1]$. We seek an estimate on the time τ_2 that one has to wait until the curve γ lies entirely inside the convex set Ω^{δ_1} defined at (2.4.1).

Consider the polynomial $p(u) = A + Bu - (u^2/2)$, choosing the constants A, B so that

$$p\big(u^+ - \delta_1\big) = p\big(u^- + \delta_1\big) = 0.$$

Then the concave function

$$w^+(t,u) = f^+(u) + \frac{p(u)}{t}\,,$$

is an upper solution to Eq. (2.2.2). By the comparison lemma stated in Section 2.2, for every $t > 0$ the curve $\gamma(t,\cdot)$ lies entirely below the graph of the function $w^+(t,\cdot)$. Similarly, the convex function

$$w^-(t,u) = f^-(u) - \frac{p(u)}{t},$$

is a lower solution to Eq. (2.2.2). For every $t > 0$ the curve $\gamma(t,\cdot)$ lies entirely above the graph of the function $w^-(t,\cdot)$. When

$$t \geq \tau_2 \doteq \frac{1}{\delta_1} \cdot \max_{u \in I\!R} p(u) = \frac{\left(u^- - u^+ + 2\delta_1\right)^2}{8\delta_1}, \tag{2.4.25}$$

the entire curve $\gamma(t,\cdot)$ is contained in the region

$$\{(u,w)\,; \quad u \in \left[u^+ - \delta_1,\ u^- + \delta_1\right], \quad w \in \left[f^-(u) - \delta_1,\ f^+(u) + \delta_1\right]\}. \tag{2.4.26}$$

Together, (2.4.24) and (2.4.26) yield the desired conclusion.

It is worth noticing that the same rearrangement techniques used above yield an estimate on the possible oscillation of the solution, between $u^+ + \delta_0$ and $u^- - \delta_0$ (see Fig. 2.4.2). The following lemma gives an explicit estimate of the time needed to achieve the estimate (2.4.2).

Lemma 2.4. *Let $u = u(t,x)$ be a solution of (2.2.15), with initial data satisfying (2.1.2). Then, for any $t > 0$ and any two points $x < y$ such that*

$$u(t,x),\, u(t,y) \in \left[u^+ + \delta_0, \quad u^- - \delta_0\right],$$

one has

$$u(t,x) \geq u(t,y) - \frac{(M-m)(b-a)}{\sqrt{\pi t}}. \tag{2.4.27}$$

Proof. If u is monotone decreasing for a fixed $t > 0$, the inequality (2.4.27) is clearly satisfied. If u is not monotone, consider any value $\omega \in [u^+ + \delta_0, \quad u^- - \delta_0]$. For $t \geq 0$ define the points

$$\alpha(t) = \inf\{x\,; \quad u(t,x) \leq \omega\},$$

$$\beta(t) = \sup\{x\,; \quad u(t,x) \geq \omega\}.$$

Consider the initial data $\bar{v}$ defined as

$$\bar{v}(x) = \omega, \qquad \text{if} \quad x \notin [\alpha(0), \beta(0)], \tag{2.4.28}$$

$$\bar{v}(x) = \min\{\bar{u}(x), \omega\}, \qquad \text{if} \quad x \in [\alpha(0), \beta(0)]. \tag{2.4.29}$$

Let $v = v(t,x)$ be the solution of (2.1.1) with $v(0,x) = \bar{v}(x)$. By a maximum principle, it is clear that $v(t,x) \leq \omega$.

Now consider the domain

$$D_\omega \doteq \{(t,x)\,; \quad \alpha(t) < x < \beta(t), \quad t > 0\},$$

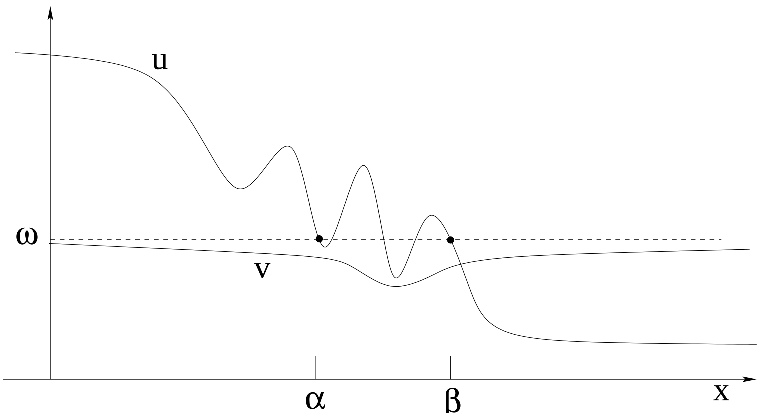

Fig. 2.4.2: Estimating the decay of oscillations in the wave profile.

By the definition of D_ω and the continuity of u, we have $u \leq v$ on the parabolic boundary of D_ω. A comparison argument yields

$$v\,(t,x) \leq u\,(t,x)\ , \qquad\qquad (t,x) \in D_\omega\,. \tag{2.4.30}$$

On the other hand, since

$$\int |\omega - \bar{v}\,(x)\,|\,dx \leq (b-a)\,(\omega - m) \leq (b-a)\,(M-m)\ ,$$

we conclude, by an argument similar to the one used in Lemma 2.3 to prove (2.4.22), that

$$v\,(t,x) \geq \omega - \frac{(b-a)\,(M-m)}{2\sqrt{\pi t}}\,. \tag{2.4.31}$$

If now

$$u\,(t,x) < u\,(t,y) = \omega\,,$$

for some $x < y$, then (t,x) belongs to D_ω, and (2.4.30)-(2.4.31) imply

$$u\,(t,y) - \frac{(b-a)\,(M-m)}{2\sqrt{\pi t}} \leq v\,(t,x) \leq u\,(t,x)\ .$$

Since $x < y$ and $\omega = u(t,y)$ were arbitrary, this proves the lemma.

2.5. Formation of a shock profile: second phase

In this section we consider a solution of Eq. (2.1.1) with initial condition satisfying

$$\begin{aligned} &|\bar{u}(x) - u^-| \le \delta_0, && x \le a', \\ &|\bar{u}(x) - u^+| \le \delta_0, && x \ge b', \\ &\bar{u}(x) \in \left[u^+ - \delta_1\,,\ u^- + \delta_1\right], && x \in I\!R. \end{aligned} \tag{2.5.1}$$

Moreover, we assume that this solution satisfies the estimates (2.4.1)–(2.4.3) for all $t \ge 0$. Our goal is to estimate how long one has to wait until the solution u satisfies also the additional inequality (2.4.4).

Notice that, by the analysis in Section 2.4, the above assumptions are satisfied by a general solution of Eq. (2.1.1)–(2.1.2) at the end of Phase 1, i.e. for $t \ge T_1$. For notational convenience, we make here a time shift and set $T_1 = 0$.

To check if the additional inequality (2.4.3) holds, the key idea is the following. Let $\hat{\psi}(\cdot)$ be a travelling wave profile, with

$$\lim_{\xi \to -\infty} \hat{\psi}(\xi) = v^-, \qquad \lim_{\xi \to +\infty} \hat{\psi}(\xi) = v^+, \qquad u^+ < v^+ < v^- < u^-. \tag{2.5.2}$$

The corresponding curve $\hat{\gamma}$ in the u-w plane is the segment joining the points

$$\widehat{P}^+ = \left(v^+,\ f(v^+)\right), \qquad \widehat{P}^- = \left(v^-,\ f(v^-)\right).$$

In analogy with (2.3.1), we set

$$\hat{w}(u) = f\left(v^-\right) + \mu\left(u - v^-\right), \qquad \mu = \frac{f\left(v^-\right) - f\left(v^+\right)}{v^- - v^+}. \tag{2.5.3}$$

We now observe that, at any time t, the curve $\gamma(t,\cdot)$ lies above $\hat{\gamma}$ (in the region where $v^+ < u < v^-$) provided that the following Unique Intersection Property holds:

(UIP) For every $c \in I\!R$, there is a unique point $x^* = x^*(c)$ such that

$$u(t, x^*) = \hat{\psi}(x^* - c).$$

In the remainder of this section we thus seek conditions which guarantee that after a time T_2 sufficiently large, u and $\hat{\psi}$ satisfy the condition above.

Let

$$u^+(t,x) = \psi^+\left(x - \lambda^+ t\right), \qquad u^-(t,x) = \psi^-\left(x - \lambda^- t\right), \tag{2.5.4}$$

be two solutions of Eq. (2.1.1), in the form of travelling wave profiles, with speeds $\lambda^- < \lambda < \lambda^+$, such that

$$u^-(0,x) = \psi^-(x) \le u(0,x) \le \psi^+(x) = u^+(0,x). \tag{2.5.5}$$

Notice that the first inequality in (2.5.5) can be achieved by constructing a travelling profile $\tilde{\psi}^-$ with

$$\tilde{\psi}^-(-\infty) = u^- - 2\delta_1, \qquad \tilde{\psi}^-(+\infty) = u^+ - 2\delta_1,$$

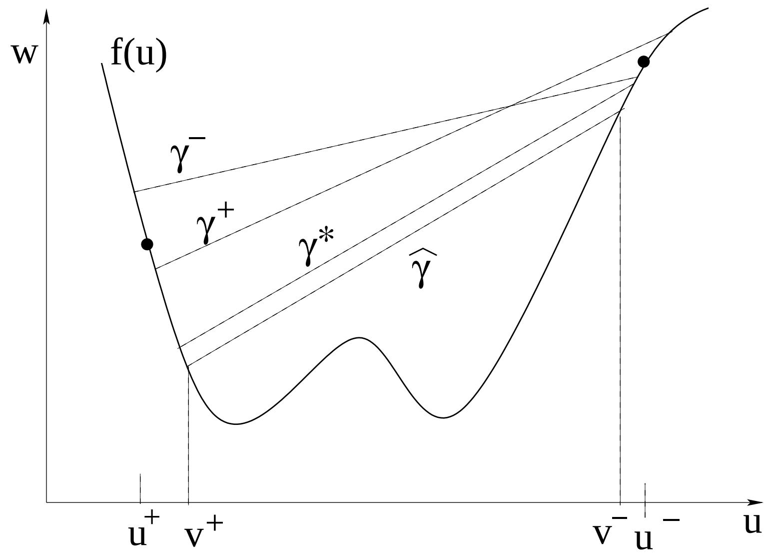

Fig. 2.5.1: In the u-w plane, the four travelling wave profiles correspond to four segments. The segments $\hat{\gamma}$ and γ^* are parallel, both having slope $= \mu$.

and then setting $\psi^-(s) = \tilde{\psi}^-(s+c)$ for c large enough. Similarly, the second inequality can be achieved by constructing a travelling profile $\tilde{\psi}^+$ with

$$\tilde{\psi}^+(-\infty) = u^- + 2\delta_1\,, \qquad\qquad \tilde{\psi}^+(+\infty) = u^+ + 2\delta_1\,,$$

and then setting $\psi^+(s) = \tilde{\psi}^+(s-c)$ for c large enough. By a comparison argument, the assumptions (2.5.5) imply

$$u^-(t,x) \le u(t,x) \le u^+(t,x)\ , \tag{2.5.6}$$

for all $t \ge 0$, $x \in I\!R$.

We then consider a third travelling profile, of the form

$$v(t,x) = \hat{\psi}(x - \mu t)\ ,$$

with

$$v^- = \hat{\psi}(-\infty)\,, \qquad\qquad v^+ = \hat{\psi}(+\infty)\,,$$

assuming that

$$\mu > \lambda^+\,. \tag{2.5.7}$$

Finally, we construct a fourth profile

$$v^*(t,x) = \psi^*(x - \mu t)\ ,$$

travelling exactly with the same speed as $\hat{\psi}$, but with limits

$$u^+ + 2\delta_1 < \psi^*(+\infty) < \hat{\psi}(+\infty) - 2\delta_1 < \hat{\psi}(-\infty) + 2\delta_1 < \psi^*(-\infty) < u^- - 2\delta_1\,. \tag{2.5.8}$$

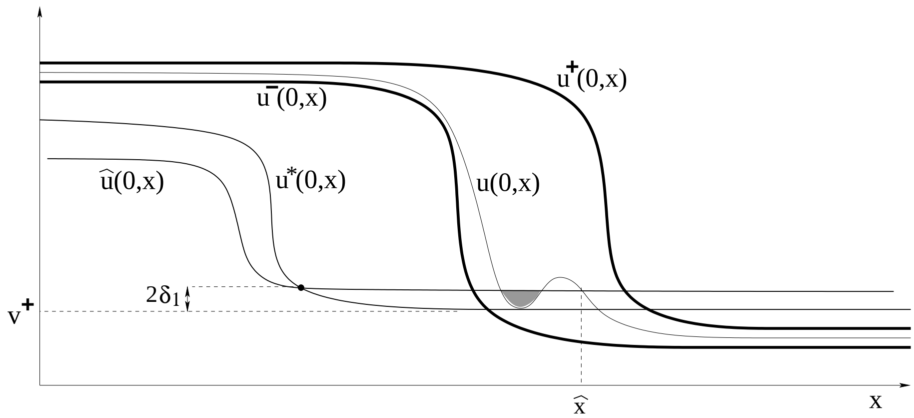

Fig. 2.5.2: The solution u (dashed line) and the four travelling wave solutions, at time t.

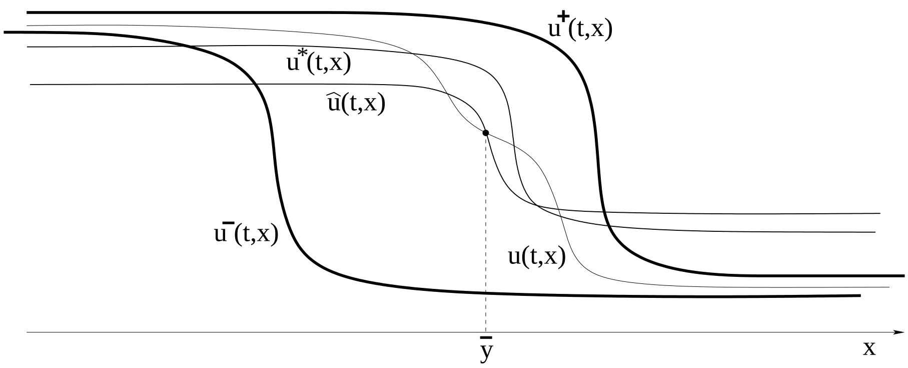

Fig. 2.5.3: The initial position of u, and of the four travelling wave solutions.

By possibly shifting them backwards, we can assume that the travelling profiles $\hat{\psi}, \psi^*$ satisfy the following conditions. There exists a point ξ^* such that

$$\hat{\psi}\left(\xi^*\right) = \psi^*\left(\xi^*\right) = v^+ + 2\delta_1 < \psi^-\left(\xi^*\right) \,, \tag{2.5.9}$$

$$\hat{\psi}\left(\xi\right) \geq \psi^-\left(\xi\right) \quad \Longrightarrow \quad \hat{\psi}\left(\xi\right) - \psi^*\left(\xi\right) > \delta_1 \,. \tag{2.5.10}$$

We now choose a time T_2 large enough so that, for all $t \geq T_2$ one has

$$\hat{\psi}\left(x - \mu t\right) \leq \psi^+\left(x - \lambda^+ t\right) \quad \Longrightarrow \quad \hat{\psi}\left(x - \mu t\right) \geq v^- - \delta_1 \,. \tag{2.5.11}$$

Notice that (2.5.11) can certainly be achieved for t large enough, because $\mu > \lambda^+$.

Now assume that, for some $t \geq T_2$ and some point $\bar{y} \in I\!R$ one has

$$u_x\left(t, \bar{y}\right) > f\left(u\left(t, \bar{y}\right)\right) - \hat{w}\left(u\left(t, \bar{y}\right)\right) \,, \tag{2.5.12}$$

with

$$v^+ + 2\delta_1 < u(t,\bar{y}) < v^- - 2\delta_1 \,. \tag{2.5.13}$$

We claim that this leads to a contradiction. Choose the constant $c > 0$ such that the travelling wave solution

$$\hat{u}(t,x) = \hat{\psi}(x - \mu t + c) \;,$$

satisfies

$$\hat{u}(t,\bar{y}) = u(t,\bar{y}) \;. \tag{2.5.14}$$

By (2.5.12), the graphs of $u(t,\cdot)$ and $\hat{u}(t,\cdot)$ have at least three intersections, as shown in Fig. 2.5.2. Hence the same must be true at time $t = 0$. The initial conditions are illustrated in Fig. 2.5.3. By the choice of $t \geq T_2$, and the assumption (2.5.10) any intersection between $u(0,\cdot)$ and $\hat{u}(0,\cdot)$ must occur at a point y^* such that

$$u(0,y^*) = \hat{u}(0,y^*) \geq \psi^*(y^* + c) + \delta_1. \tag{2.5.15}$$

Since we are assuming the bound (2.4.2) on the upward oscillation of the maps $x \mapsto u(t,x)$, the region between the graphs of u and $\hat{u}$ within the intersections lies entirely above the graph of $\psi^*(\cdot + c)$. By a comparison argument,

$$u(t,x) \geq u^*(t,x) \;,$$

where $u^*(t,x) = \psi^*(x - \mu t + c)$, for all t and all $x \leq \hat{x}(t)$, with

$$\hat{x}(t) = \inf\{x\,; \quad u\left(t,x'\right) < \hat{u}\left(t,x'\right), \quad \text{for all} \;\; x' > x\}\,.$$

In particular,

$$u(t,\bar{y}) = \hat{u}(t,\bar{y}) \geq u^*(t,\bar{y}) \;.$$

This lead to a contradiction since $\hat{u}(t,x) < u^*(t,x)$ unless $\hat{u}(t,x) \leq v^+ + 2\delta_1$ and we assume $u(t,\bar{y}) > v^+ + 2\delta_1$.

It now remains to estimate the time T_2 in terms of a', b' and δ_1, δ_2. If we require that the line $\hat{\psi}$ remains within a δ_2-neighborhood of the segment P^+P^-, the difference in speeds between the travelling wave profiles must be

$$\mu - \lambda^+ = \mathcal{O}(1) \cdot \delta_2 \,. \tag{2.5.16}$$

On the other hand, the distance between the point ξ^* and a' will be

$$a' - \xi^* = \mathcal{O}(1) \cdot \ln \delta_1 \,. \tag{2.5.17}$$

Together, the two above estimates yield

Lemma 2.5. *There exists constants C, C' such that the following holds. Let $u = u(t,x)$ be a solution of Eq. (2.1.1), whose initial data satisfies (2.5.1) and such that (2.4.1)–(2.4.3) hold for all $t \geq 0$. Then, for any $\delta_2 \geq C'\delta_1$, the corresponding curve γ satisfies the inequality (2.4.4), for every $x \in I\!\!R$ and $t \geq T_2$, with*

$$T_2 \doteq C\,\frac{\ln \delta_1 + (b' - a')}{\delta_2} \,. \tag{2.5.18}$$

Combining Lemma 2.3 and Lemma 2.5 we conclude that, after a finite interval of time, the curve $\gamma(t,\cdot)$ becomes sufficiently close to the segment with endpoints $(u^+,\, f(u^+))$, $(u^-,\, f(u^-))$. Using Lemma 2.2, we thus obtain a proof of Theorem 2.1.

Chapter 3.
On the Convergence of Viscous Approximations after Shock Interactions

3.1. The main result

Let the scalar conservation law in one space dimension,

$$u_t + f(u)_x = 0, \tag{3.1.1}$$

have a smooth, convex flux, so that $f''(u) \geq k > 0$ for all u. For a given time $\tau > 0$, consider a bounded solution $u = u(t,x)$ which contains an arbitrary number of interacting shocks for $t < \tau$, but is piecewise smooth with one single shock for $t > \tau$, say located along the curve $x = \xi(t)$. We claim that for $t > \tau$ the solution u admits inner and outer expansions similar to the ones in (1.1.3), (1.1.4). This extends the result obtained by J. Goodman and Z. Xin in [22], since in that paper the solution u was assumed to contains only a finite number of non interacting shocks for all $t > 0$.

We assume that u is piecewise smooth outside a triangular domain, Λ, bounded by the two backward characteristics impinging on the shock at time $t = \tau$. More precisely (see Fig. 3.1.1), we write

$$u^{\pm}(t) \doteq \lim_{x\to\xi(t)\pm} u(t,x),$$

for the left and right limits of u across this shock, and let

$$x^-(t) = \xi(\tau) - f'\big(u^-(\tau)\big)(\tau - t), \qquad x^+(t) = \xi(\tau) - f'\big(u^+(\tau)\big)(\tau - t),$$

be the minimal and maximal backward characteristics through the point $(\tau,\, \xi(\tau))$. Then Λ is defined as

$$\Lambda \doteq \{ (t,x)\,; \quad 0 \leq t \leq \tau\,,\ \ x^-(t) < x < x^+(t) \}.$$

By suitably changing the initial data, we can then construct a second solution $\tilde u$ which is piecewise smooth with one single shock for all times $t \geq 0$, and moreover it coincides with u for $t > \tau$. Indeed, this can be achieved by choosing a suitable piecewise smooth initial condition $\tilde u(0,x)$ such that

$$\tilde u(0,x) = u(0,x), \qquad x \notin \big[x^-(0)\,,\ x^+(0)\big], \tag{3.1.2}$$

$$\int_{x^-(0)}^{x^+(0)} \tilde u(0,x)\,dx = \int_{x^-(0)}^{x^+(0)} u(0,x)\,dx. \tag{3.1.3}$$

The following theorem shows that, for any time $t > \tau$, the viscous approximations to the two solutions u and $\tilde u$ are extremely close. In particular, any singular perturbation expansion valid for $\tilde u^\varepsilon$ remains valid for u^ε as well.

Theorem 3.1 *In the above setting, let u^ε and $\tilde u^\varepsilon$ be the solutions to the viscous conservation law*

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \tag{3.1.4}$$

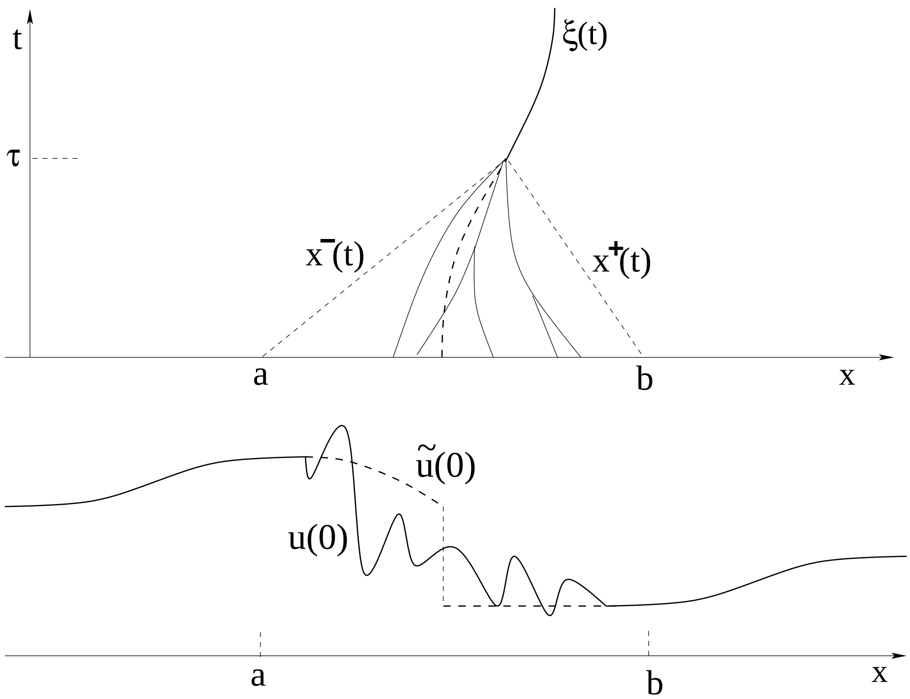

Fig. 3.1.1: The solutions with initial data $u(0,\cdot)$ and $\tilde{u}(0,\cdot)$ coincide after time $t=\tau$.

with initial data

$$u^\varepsilon(x,0) = u(x,0)\ , \qquad\qquad \tilde{u}^\varepsilon(x,0) = \tilde{u}(x,0)\ ,$$

*related as in (*3.1.2*)- (*3.1.3*). Let τ be the time when the single shock forms in the limit solution u. Then, for any integers $k,\nu\geq 0$, one has the high order convergence*

$$\lim_{\varepsilon\to 0+}\ \varepsilon^{-k}\cdot \|u^\varepsilon - \tilde{u}^\varepsilon\|_{\mathcal{C}^\nu(\Omega)} = 0, \tag{3.1.5}$$

uniformly on every compact domain $\Omega \subset\subset \{(t,x)\,;\quad t>\tau\,,\ \ x\in I\!\!R\}$.

We sketch here the main ideas in the proof. Details will be worked out in Section 3. Call

$$U^- \doteq u(\tau,\,\xi(\tau)-)\ , \qquad\qquad U^+ \doteq u(\tau,\,\xi(\tau)+)\,, \tag{3.1.6}$$

the left and right limits of the non-viscous solution u across the shock, at time $t=\tau$. By possibly performing the linear rescaling of coordinates

$$x' = x - \frac{f\left(U^+\right) - f\left(U^-\right)}{U^+ - U^-}\,t\,,$$

and adding a constant to the flux f, it is not restrictive to assume that

$$f\left(U^+\right) = f\left(U^-\right) = 0\,, \tag{3.1.7}$$

so that the velocity of the shock at time $t = \tau$ is $\dot{\xi}(\tau) = 0$. In the t-x plane we consider a rectangle of the form

$$Q = [\tau, \quad \tau_4] \times [\xi(\tau) - \delta_0\,, \ \ \xi(\tau) + \delta_0]\ ,$$

with $\tau_\ell = \tau + \ell \cdot C_0\,\delta_0$, for $\ell = 1, 2, 3, 4$. Notice that, since $\dot{\xi}(\tau) = 0$, given any constant $C_0 > 0$ we can choose $\delta_0 > 0$ small enough so that

$$a \doteq \xi(\tau) - \delta_0 < \xi(t) < \xi(\tau) + \delta_0 \doteq b\,, \tag{3.1.8}$$

for all $t \in [\tau, \ \ \tau + 4C_0\delta_0]$. We recall that the asymptotic convergence result proved in [45] shows that the solutions u^ε and $\tilde{u}^\varepsilon$ are extremely close, away from the shock. In particular, for every $\nu, k \geq 1$ one has

$$\sup_{t \in [\tau, \tau_4]} \|u^\varepsilon(t, \cdot) - \tilde{u}^\varepsilon(t, \cdot)\|_{\mathcal{C}^\nu(I\!R \setminus [a,b])} \quad = \quad \mathcal{O}(1) \cdot \varepsilon^k. \tag{3.1.9}$$

To estimate the distance $u^\varepsilon - \tilde{u}^\varepsilon$ inside the interval $[a, b]$, we shall use a homotopy method. Define $u^{\varepsilon,\theta}$ as the solution of (3.1.4) with interpolated initial data

$$u^{\varepsilon,\theta}(0, x) = \theta u^\varepsilon(0, x) + (1 - \theta)\ \tilde{u}^\varepsilon(0, x)\ .$$

Moreover, call

$$z^{\varepsilon,\theta} \doteq \frac{\partial}{\partial \theta}\, u^{\varepsilon,\theta}\ .$$

A key step in the proof is to establish the asymptotic estimates

$$\int_a^b |z^{\varepsilon,\theta}(\tau_3, x)|\, dx \leq C_k \varepsilon^k, \tag{3.1.10}$$

where $\tau_3 = \tau + 3c_0\delta_0$ and $k \geq 1$. Integrating w.r.t. $\theta \in [0, 1]$, from (3.1.10) it follows

$$\begin{aligned} \int_a^b |u^\varepsilon(\tau_3, x) - \tilde{u}^\varepsilon(\tau_3, x)|\, dx &\leq \int_a^b \left| \int_0^1 \frac{\partial}{\partial \theta} u^{\varepsilon,\theta}(\tau_3, x)\ d\theta \right| dx \leq \\ &\leq \sup_{\theta \in [0,1]} \int_a^b |z^{\varepsilon,\theta}(\tau_3, x)|\, dx \leq C_k \varepsilon^k. \end{aligned}$$

Using the regularity of the solutions $u^{\varepsilon,\theta}$, from the family of integral estimates (3.1.10), at the later time $\tau_4 > \tau_3$ one can derive pointwise estimates of the form

$$\|z^{\varepsilon,\theta}(\tau_4, \cdot)\|_{\mathcal{C}^\nu([a,b])} = \mathcal{O}(1) \cdot \varepsilon^k,$$

for every $k, \nu \geq 1$. Again integrating w.r.t. $\theta \in [0, 1]$, these bounds in turn imply

$$\|u^\varepsilon(\tau_4, \cdot) - \tilde{u}^\varepsilon(\tau_4, \cdot)\|_{\mathcal{C}^\nu([a,b])} = \mathcal{O}(1) \cdot \varepsilon^k\,. \tag{3.1.11}$$

Given the compact domain Ω in the t-x plane, we can now choose $\delta_0 > 0$ so that

$$\Omega \subset [\tau_4, \infty[\, \times I\!R\,.$$

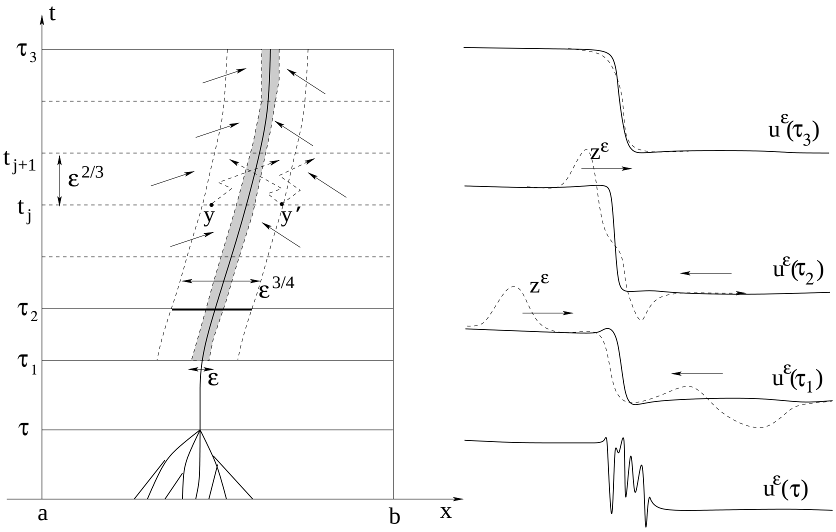

Fig. 3.1.2: A viscous shock solution $u^{\varepsilon,\theta}$ and an infinitesimal perturbation $z^{\varepsilon,\theta}$. At time $t = \tau_1$ a viscous shock has formed. At $t = \tau_2$ most of the perturbation lies inside a small interval $I(\tau_2)$ of length $2\varepsilon^\gamma$. When $t = \tau_3$ nearly all the positive part of the perturbation $z^{\varepsilon,\theta}$ has cancelled with the negative part.

The bounds

$$\|u^\varepsilon(\tau_4,\cdot) - \tilde{u}^\varepsilon(\tau_4,\cdot)\|_{\mathcal{C}^\nu(I\!R)} = \mathcal{O}(1)\cdot\varepsilon^k,$$

which follow from (3.1.9) and (3.1.11), will finally imply (3.1.5).

Observing that each $z^{\varepsilon,\theta}$ provides a solution to the linearized conservation law

$$z_t + [f'(u^{\varepsilon,\theta})z]_x = \varepsilon\, z_{xx}, \tag{3.1.12}$$

to prove the key estimate (3.1.10) we consider the time intervals, with extremal points $\tau < \tau_1 < \tau_2 < \tau_3$, as illustrated in Fig. 3.1.2.

During the first interval $[\tau,\, \tau_1]$, following the analysis in Chapter 2, we show that a viscous shock is formed. Hence, for all $t \in [\tau_1, \tau_3]$ and $\varepsilon > 0$ small enough, each solution $u^{\varepsilon,\theta}(t,\cdot)$ already contains one large viscous shock, say located around the point $\xi^{\varepsilon,\theta}(t)$. We can identify a thin region around the shock, of the form

$$\Lambda_{\varepsilon,\theta} \doteq \{(t,x)\,;\quad t \in [\tau_1, \tau'],\quad x \in [\xi^{\varepsilon,\theta}(t) - C\varepsilon\ ,\quad \xi^{\varepsilon,\theta}(t) + C\varepsilon]\},$$

such that, for $(t,x) \in [\tau,\, \tau_3] \times [a,b]$, outside this region we have

$$|u^{\varepsilon,\theta}(t,x) - U^-| \le \frac{|U^- - U^+|}{7} \qquad \text{if} \quad x < \xi^{\varepsilon,\theta}(t) - C\varepsilon, \tag{3.1.13}$$

$$|u^{\varepsilon,\theta}(t,x) - U^+| \le \frac{|U^- - U^+|}{7} \qquad \text{if} \quad x > \xi^{\varepsilon,\theta}(t) + C\varepsilon. \tag{3.1.14}$$

Next, we examine the behavior of the perturbation $z = z^{\varepsilon,\theta}$ during the remaining time interval $[\tau_1, \tau_3]$. By (3.1.13)- (3.1.14), the characteristics point strictly toward the strip $\Lambda_{\varepsilon,\theta}$. Indeed,

$$f'(u^{\varepsilon,\theta}(t,x)) \approx f'(U^-) > 0 \qquad \text{for} \quad x < \xi^{\varepsilon,\theta}(t) - C\varepsilon\,,$$

$$f'(u^{\varepsilon,\theta}(t,x)) \approx f'(U^+) < 0 \qquad \text{for} \quad x > \xi^{\varepsilon,\theta}(t) + C\varepsilon\,.$$

After some time, for $t \geq \tau_2$, we can show that almost all the perturbation is contained inside a strip of width $2\varepsilon^\gamma$ around the viscous shock, with $\gamma = 3/4$. Namely, introducing the interval

$$I(t) \doteq [\xi^{\varepsilon,\theta}(t) - \varepsilon^\gamma\,,\ \ \xi^{\varepsilon,\theta}(t) + \varepsilon^\gamma]\,,$$

around the point $\xi^{\varepsilon,\theta}$, for any $k \geq 1$ we have

$$\int_{I\!R\setminus I(t)} |z(t,x)|\,dx = \mathcal{O}(1)\cdot\varepsilon^k. \tag{3.1.15}$$

It now remains to understand what happens inside the interval $I(t)$ containing the shock. According to (3.1.2)- (3.1.3), the difference between the two solutions u^ε and $\tilde{u}^\varepsilon$ has zero total mass. This implies

$$\int_{-\infty}^{\infty} z(t,x)\,dx = 0\,.$$

We claim that, during the interval $[\tau_2, \tau_3]$, almost all the positive mass in $z = z^{\varepsilon,\theta}$ gets cancelled with the negative mass. To prove this, we divide $[\tau_2, \tau_3]$, into equal subintervals of length $\varepsilon^{2/3}$, inserting the points

$$t_j = \tau_2 + j\cdot\varepsilon^{2/3}\,, \qquad\qquad j = 0, 1, \ldots, N_\varepsilon\,.$$

A key step in the proof is to show that

$$\int_a^b |z(t_{j+1},x)|\,dx \leq \alpha\cdot\int_a^b |z(t_j,x)|\,dx\,, \tag{3.1.16}$$

for some constant $\alpha < 1$ and all $j = 0, 1, \ldots, N_\varepsilon - 1$. From (3.1.16) it follows

$$\begin{aligned}\int_a^b |z(\tau_3,x)|\,dx \leq \alpha^{N_\varepsilon}\cdot \int_a^b |z(\tau_2,x)|\,dx \leq& \\ \leq \alpha^{N_\varepsilon}\cdot\ &\|u(0,\cdot) - \tilde{u}(0,\cdot)\|_{L^1(I\!R)} \\ =\mathcal{O}&(1)\cdot\varepsilon^k,\end{aligned}$$

for every $k \geq 1$. Indeed, $N_\varepsilon = (\tau_3 - \tau_2)/\varepsilon^{2/3}$, hence α^{N_ε} is an infinitesimal of higher order w.r.t. ε^k for any $k \geq 1$.

We conclude this section with some intuitive explanation about the inequalities (3.1.16). Calling $\Gamma(t,x,s,y)$ the fundamental solution of the linear parabolic equation (3.1.12), we can write

$$z(t_{j+1},x) = \int \Gamma(t_{j+1},x,t_j,y)\,z(t_j,y)\,dy\,.$$

Notice that $\Gamma(t,\cdot,s,y)$ can be interpreted as the probability density at time t of a random particle which is located at the point y at the initial time s. The motion of the particle is governed by the stochastic diffusion process

$$dY = f'\Big(u^{\varepsilon,\theta}(t,Y(t))\Big)\,dt + \sqrt{2\varepsilon}\,dB\,, \tag{3.1.17}$$

where B denotes a Brownian motion. Consider the two sets

$$A_j^+ \doteq \{x \in I(t_j)\,, \quad z(t_j,x) > 0\}\,, \qquad\qquad A_j^- \doteq \{x \in I(t_j)\,, \quad z(t_j,x) < 0\}\,.$$

Since $z(t_j,\cdot)$ has zero total mass, and almost all of this mass is concentrated inside $I(t_j)$, we can write

$$\begin{aligned} z(t_{j+1},x) \approx \int_{A_j^+} &\Gamma(t_{j+1},x,t_j,y)\,|z(t_j,y)|\,dy+ \\ -\int_{A_j^-}&\Gamma(t_{j+1},x,t_j,y')\,|z(t_j,y')|\,dy'\,. \end{aligned} \tag{3.1.18}$$

For any two points $y,y' \in I(t_j)$ we now have the key inequality

$$\begin{aligned} &\int \Big|\Gamma\left(t_{j+1},x,t_j,y\right) - \Gamma\left(t_{j+1},x,t_j,y'\right)\Big|\,dx \le \\ &\quad\le 2\left(1 - \text{Prob.}\{Y\,(t) = Y'\,(t) \quad \text{for some } \ t \in [t_j,\, t_{j+1}]\,\}\right) \le \\ &\quad\le 2\alpha, \end{aligned} \tag{3.1.19}$$

for some constant $\alpha < 1$. Here Y, Y' are two independent random paths of the diffusion process (3.1.17), starting from the points $y, y' \in I(t_j)$ respectively. Applying (3.1.19) to the case where $y \in A_j^+$ and $y' \in A_j^-$, from (3.1.18) we see that a nontrivial amount of cancellation occurs within each time interval $[t_j,\, t_{j+1}]$. Indeed, neglecting terms which are exponentially small as $\varepsilon \to 0$, we have

$$\int |z\,(t_{j+1},x)\,|\,dx \le \alpha\left(\int_{A_j^+} |z\,(t_j,y)\,|\,dy + \int_{A_j^-} |z\left(t_j,y'\right)|\,dy'\right)\,.$$

Together with (3.1.15), this yields the estimate (3.1.16).

3.2. Proof of the theorem

The proof of Theorem 1 will be given in several steps. As remarked in the previous section, we can assume that (3.1.7) holds, so that the shock has zero speed at the initial time $t = \tau$ when it is formed.

1. Fix times $\tau_\ell = \tau + \ell T$, with $\ell = 1,2,3,4$, choosing $T = C_0\delta_0 > 0$ so that

$$\tau < \tau_4 < \min\ \{t\,; \quad (t,x) \in \Omega \quad \text{for some } \ x \in I\!\!R\}\,.$$

The precise values of the constants C_0, δ_0 will be determined later.

It is convenient to rescale coordinates, and consider $t' = (t-\tau)/\varepsilon$, $x' = (x-\xi(\tau))/\varepsilon$. Observe that the function $v^\varepsilon(t,x) \doteq u^\varepsilon(\tau+\varepsilon t\,,\ \xi(\tau)+\varepsilon x)$ provides a solution to the uniformly parabolic Cauchy problem

$$\begin{cases} v_t + f(v)_x = v_{xx}\,, \\ v^\varepsilon(0,x) = u^\varepsilon(\tau\,,\ \xi(\tau)+\varepsilon x)\,. \end{cases} \tag{3.2.1}$$

It is useful to keep in mind that, as $\varepsilon \to 0$, the derivatives of the functions u^ε become arbitrarily large: $\|u^\varepsilon_x\|_{L^\infty}\,,\ \|u^\varepsilon_{xx}\|_{L^\infty} \to \infty$. However, the derivatives of the rescaled functions v^ε remain uniformly bounded.

2. As in [4], [6], in connection with any solution of (3.2.1) one can consider the planar curve

$$\gamma(t,x) = \begin{pmatrix} v(t,x) \\ w(t,v) \end{pmatrix} \doteq \begin{pmatrix} v(t,x) \\ f(v(t,x)) - v_x(t,x) \end{pmatrix}. \tag{3.2.2}$$

This curve evolves in time, moving in the direction of its curvature. Indeed, along each branch where $v_x = f(v) - w$ has constant sign, the function $w = w(t,v)$ satisfies the parabolic equation

$$w_t = (w - f(v))^2\, w_{vv}\,. \tag{3.2.3}$$

Observe that, if v is a viscous travelling wave solution for the equation (3.2.1), then the corresponding curve γ is a straight line, and does not vary in time. The speed of the travelling wave is given by the constant slope $\partial w/\partial v$. More generally, given any solution $v = v(t,x)$ of (3.2.1), for a fixed value v_0, the speed of the level set $t \mapsto x_0(t)$ implicitly defined by

$$v(t, x_0(t)) = v_0,$$

is given by

$$\frac{d}{dt}x_0(t) = \frac{\partial}{\partial v}w(t,v_0)\,.$$

3. As in (3.1.6), let U^-, U^+ be the left and right limits of the inviscid solution u across the shock, at time $t=\tau$. Since we are assuming that the flux function is strictly convex, we can find intermediate states

$$U^+ < V^+ < V_0 < V^- < U^-,$$

and a constant $\eta_0 > 0$ such that

$$f'(U_0) = 0\,, \qquad \begin{cases} f'(u) \le -2\eta_0 & \text{if} \quad u \le V^+\,, \\ f'(u) \ge 2\eta_0 & \text{if} \quad u \ge V^-\,. \end{cases} \tag{3.2.4}$$

Since the equation (3.2.3) is uniformly parabolic when w is bounded away from $f(v)$, we can find $\eta_1 > 0$ such that the following holds. If $w = w(t,v)$ is any solution of (3.2.3) such that

$$|w(t',v)| \le \eta_1 \qquad \text{for all } t' \in [t-1,\, t]\,, \quad v \in [V^+,\, V^-]\,, \tag{3.2.5}$$

then

$$\left|\frac{\partial}{\partial v}w(t,V_0)\right| \le \eta_0\,. \tag{3.2.6}$$

4. Given two families of viscous solutions u^ε, $\tilde{u}^\varepsilon$, to estimate the distance between the corresponding rescaled solutions $v^\varepsilon, \tilde{v}^\varepsilon$ we shall use a homotopy method. Define $v^{\varepsilon,\theta}$ as the solution of (3.2.1) with interpolated initial data

$$v^{\varepsilon,\theta}(0,x) = \theta u^\varepsilon(\tau,\ \xi(\tau)+\varepsilon x) + (1-\theta)\ \tilde{u}^\varepsilon(\tau,\ \xi(\tau)+\varepsilon x)\ .$$

Moreover, call

$$z^{\varepsilon,\theta} \doteq \frac{\partial}{\partial\theta} v^{\varepsilon,\theta}\,.$$

Then $z = z^{\varepsilon,\theta}$ satisfies the linear equation

$$z_t + [f'(v^{\varepsilon,\theta})\, z]_x = z_{xx}, \tag{3.2.7}$$

together with the initial condition (independent of θ)

$$z(0,x) = u^\varepsilon(\tau,\ \xi(\tau)+\varepsilon x) - \tilde{u}^\varepsilon(\tau,\ \xi(\tau)+\varepsilon x)\,.$$

Observe that, for all $t \geq 0$ and all ε, θ, the assumptions (3.1.2)-(3.1.3) imply

$$\int_{-\infty}^{\infty} z^{\varepsilon,\theta}(t,x)\,dx \;=\; \int_{-\infty}^{\infty} z^{\varepsilon,\theta}(0,x)\,dx \;=\; 0\,.$$

5. We recall that the stretching of time and space variables defined in Step 1 transforms the domain

$$\{(t,x)\,;\quad t\in[\tau,\tau+4T]\,,\qquad x\in[\xi(\tau)-\delta_0\,,\ \xi(\tau)+\delta_0]\},$$

into the domain

$$\left\{(t,x)\,;\quad t\in[0,4T_\varepsilon]\,,\qquad |x|\leq\frac{\delta_0}{\varepsilon}\right\},$$

with $T_\varepsilon \doteq T/\varepsilon$.

Away from the shock, i.e. for $|x|\geq\delta_0/\varepsilon$ in the stretched coordinates, the result in [45] guarantees the high order convergence $v^\varepsilon - \tilde{v}^\varepsilon = \mathcal{O}(\varepsilon^k)$ for every $k\geq 1$. The heart of the matter is to show that

$$\int_{-\delta_0/\varepsilon}^{\delta_0/\varepsilon} |z^{\varepsilon,\theta}(3T_\varepsilon,x)|\,dx = \mathcal{O}(\varepsilon^k), \tag{3.2.8}$$

for every integer $k\geq 1$ and uniformly for $\theta\in[0,1]$. As soon as the integral estimate (3.2.8) is achieved, one can easily achieve similar pointwise estimates, because the coefficients $f'(v^{\varepsilon,\theta})$ of the equation are uniformly smooth. The strategy for proving the bounds (3.2.8) can be outlined as follows.

(i) At time $t = T_\varepsilon \doteq T/\varepsilon$ each solution $v^{\varepsilon,\theta}$ develops a large viscous shock. In the (t,v) variables, the graph of the corresponding curve γ at (3.2.2) becomes very close to a straight segment. More precisely, for $t > T_\varepsilon$ the functions $w^{\varepsilon,\theta}(t,v)$ satisfy

$$|w^{\varepsilon,\theta}(t,v)|\leq\eta_1 \qquad\qquad \text{for all } t\geq T_\varepsilon\,,\quad v\in[V^+,\,V^-]\,.$$

Hence, by (3.2.5)–(3.2.6)

$$\left|\frac{\partial}{\partial v} w^{\varepsilon,\theta}(t, V_0)\right| \leq \eta_0 \,. \tag{3.2.9}$$

We can thus define the approximate location $\xi^{\varepsilon,\theta}(t)$ of the viscous shock, in terms of the identity

$$v^{\varepsilon,\theta}\left(t,\ \xi^{\varepsilon,\theta}(t)\right) = V_0.$$

According to (3.2.9), this viscous shock moves with speed

$$|\dot{\xi}^{\varepsilon,\theta}(t)| \leq \eta_0 \,.$$

(ii) At time $t \geq 2T_\varepsilon$, nearly all the mass in $z^{\varepsilon,\theta}$ is located within the strip

$$I^{\varepsilon,\theta}(t) \doteq \left[\xi^{\varepsilon,\theta}(t) - \varepsilon^{-1/4}\ ,\ \xi^{\varepsilon,\theta}(t) + \varepsilon^{-1/4}\right] .$$

(iii) During the time interval $[2T_\varepsilon\,,\ 3T_\varepsilon]$ nearly all the positive mass of $z^{\varepsilon,\theta}$ is cancelled with the negative mass. As a consequence, at time $t = 3T_\varepsilon$ the asymptotic estimates (3.2.8) hold.

6. We show here that each solution $v^{\varepsilon,\theta}$ develops one large viscous shock within time T_ε. This process of shock formation has been analyzed in detail in Chapter 2. The main differences between the situation analyzed there and the present one are the following: (i) Here we are assuming a strictly convex flux. This simplifies the proof, because we can use the classical one-sided Oleinik estimates on the gradient of the solution. (ii) We are not assuming anything about the behavior of the solution $v^{\varepsilon,\theta}(t, x)$ for $x \to \pm\infty$. Instead, we know that, at the endpoints of the interval $[-\delta_0/\varepsilon\,,\ \delta_0/\varepsilon]$, the function $v^{\varepsilon,\theta}$ takes values very close to U^-, U^+. Moreover its derivative is $v_x^{\varepsilon,\theta} = \varepsilon\, u_x^{\varepsilon,\theta} = \mathcal{O}(1)\cdot\varepsilon\,.$

In the following, η_1 is the constant introduced at (3.2.5). We choose $\delta_1 > 0$ small enough so that

$$|w| \leq \frac{\eta_1}{4},$$

whenever $|w - f(u)| \leq \delta_1$ for some u such that either $|u - U^-| \leq \delta_1$ or $|u - U^+| \leq \delta_1$. Of course this is possible because $f(U^-) = f(U^+) = 0$.

As in (3.1.8), consider an interval $[a, b]$ containing the point $\xi(\tau)$ in its interior, and define the stretched interval $I_\varepsilon = [a_\varepsilon,\, b_\varepsilon]$ according to

$$a_\varepsilon \doteq \frac{a - \xi(\tau)}{\varepsilon} = -\frac{\delta_0}{\varepsilon}\,, \qquad\qquad b_\varepsilon \doteq \frac{b - \xi(\tau)}{\varepsilon} = \frac{\delta_0}{\varepsilon}\,.$$

Choosing $\delta_2 > 0$ and the interval $[a, b]$ small enough, in the rescaled variables we shall have

$$|v^{\varepsilon,\theta}(t, a_\varepsilon) - U^-| + |v^{\varepsilon,\theta}(t, b_\varepsilon) - U^+| \leq \frac{\delta_1}{2}\,, \qquad\qquad \text{for all } t \in [0, \delta_2/\varepsilon]\,. \tag{3.2.10}$$

Since we are assuming $f'' \geq \kappa > 0$, after time τ in the original variables the function u^ε satisfies $u_x^{\varepsilon,\theta} \leq (\kappa\tau)^{-1}$. Hence, in the stretched variables, $v_x^{\varepsilon,\theta} \leq \varepsilon(\kappa\tau)^{-1}$. Together with (3.2.10), this yields

$$U^+ - \frac{\delta_1}{2} - \frac{2\delta_0}{\kappa\tau} \leq v^{\varepsilon,\theta}(t,x) \leq U^- + \frac{\delta_1}{2} + \frac{2\delta_0}{\kappa\tau},$$

valid for $0 \leq t \leq \delta_2/\varepsilon$ and $|x| \leq \delta_0/\varepsilon$. Choosing δ_0 sufficiently small, we can thus achieve

$$v^{\varepsilon,\theta}(t,x) \in \big[U^+ - \delta_1\,,\ U^- + \delta_1\big]\,. \tag{3.2.11}$$

Next, we claim that, if (3.1.8) holds, then the curve $\gamma = \gamma^{\varepsilon,\theta}$ in (3.2.2) corresponding to $v^{\varepsilon,\theta}$ satisfies

$$\gamma(t,x) \in \Lambda_{\delta_1} \doteq \overline{co}\Big\{(u\,,\ f(u)+\xi)\,;\quad u \in \big[U^+ - \delta_1\,,\ U^- + \delta_1\big]\,,\quad |\xi| \leq \frac{\eta_1}{3}\Big\}, \tag{3.2.12}$$

whenever

$$t \in \left[\frac{C_0\delta_0}{2\varepsilon}\,,\ \frac{C_0\delta_0}{\varepsilon}\right], \qquad x \in I_\varepsilon \doteq \left[-\frac{\delta_0}{\varepsilon}\,,\ \frac{\delta_0}{\varepsilon}\right],$$

and $\varepsilon > 0$ is sufficiently small.

Indeed, for $x = \pm\delta_0/\varepsilon$ we have $v_x^{\varepsilon,\theta} = \mathcal{O}(\varepsilon)$, and the estimate (3.2.12) follows from (3.2.11). To prove that (3.2.12) holds for all intermediate values of x, we need to construct suitable upper and lower solutions for the parabolic equation (3.2.3).

We first observe that, since $f'' \geq \kappa > 0$, the function

$$w^-(t,v) \doteq f(v) - \frac{1}{\kappa t},$$

is a lower solution of (3.2.3). Hence every branch of the curve γ satisfies

$$w(t,v) \geq w^-(t,v) = f(v) - \frac{1}{\kappa t} \geq f(v) - \frac{\eta_1}{4},$$

for $t \geq (\kappa\eta_1)^{-1}$. For any choice of C_0, δ_0, this is certainly true when $t \geq C_0\delta_0/2\varepsilon$, with ε sufficiently small.

Next, let f^+ be the affine function which coincides with f at the two points $v = U^+ - \delta_1$ and $v = U^- + \delta_1$. Moreover, consider the polynomial $p(v) = A + Bv - (v^2/2)$, choosing the constants A, B so that

$$p\big(U^+ - \delta_1\big) = p\big(U^- + \delta_1\big) = 1\,.$$

For $v \in [U^+ - \delta_1\,,\ U^- + \delta_1]$, consider a function of the form

$$w^+(t,v) = f^+(v) + \frac{\eta_1}{4} + \beta(t)\,p(v)\,.$$

Computing

$$w_t^+ = \dot\beta(t)\,p(v)\,, \qquad \big(w^+ - f(v)\big)^2 w_{vv}^+ \leq -\left(\beta(t)\,p(v)\right)^2\beta(t)\,,$$

we deduce that the function w^+ is an upper solution of (3.2.3) provided that

$$\dot{\beta}(t)\, p(v) \geq -\beta^3(t)\, p^2(v) \qquad t \geq 0\,, \quad v \in \left[U^+ - \delta_1\,,\ U^- + \delta_1\right]\,.$$

Since $p(v) \geq 1$, this is certainly the case if $\dot{\beta} \geq -\beta^3$, hence if $\beta(t) = t^{-1/2}$. Concerning the endpoints, when $x = \pm\delta_0/\varepsilon$ we already know that $f(v) - w = v_x^{\varepsilon,\theta} = \mathcal{O}(1)\cdot\varepsilon$. By a comparison argument, the portion of the curve $w = w(t,v)$ corresponding to the solution $v^{\varepsilon,\theta}$ as $x \in I_\varepsilon$ lies entirely below the upper solution w^+. For t sufficiently large we thus have

$$w(t,v) \leq w^+(t,v) \leq f^+(v) + t^{-1/2}\cdot \max\{p(u)\ ;\ u \in \mathbb{R}\} + \frac{\eta_1}{4}\,.$$

In particular, this is true when $t \geq C_0\delta_0/2\varepsilon$, for ε sufficiently small. This achieves the proof of (3.2.12).

We now analyze the second phase of shock formation. We claim that, if (3.1.8) continues to hold, then the curve $\gamma = \gamma^{\varepsilon,\theta}$ corresponding to $v^{\varepsilon,\theta}$ satisfies

$$\gamma(t,x) \in \Lambda'_{\delta_1} \doteq \{\,(u,w)\ ;\quad u \in \left[U^+ - \delta_1\,,\ U^- + \delta_1\right]\,,\quad |w| \leq \eta_1\}\,. \tag{3.2.13}$$

for all $t \geq [C_0\delta_0/\varepsilon\,,\ 4C_0\delta_0/\varepsilon]$ and $x \in I_\varepsilon \doteq [-\delta_0/\varepsilon\,,\ \delta_0/\varepsilon]$. Indeed, this result follows from the analysis in Chapter 2, which we briefly recall here. Let $\eta_1 > 0$ be given and assume that the curve γ already lies in the convex set Λ_{δ_1} at (3.2.12) and that the values of γ at the endpoints $x = \pm\delta_0/\varepsilon$ are sufficiently close to $(U^\pm, f(U^\pm))$. Then, according to Lemma 2.5 in Chapter 2, the additional length of time Δt needed to achieve the inclusion (3.2.13) grows linearly with the length of the interval $I_\varepsilon = [-\delta_0/\varepsilon\,,\ \delta_0/\varepsilon]$, say $\Delta t \leq C\cdot 2\delta_0/\varepsilon$.

To achieve the desired estimate (3.2.13) for all $\varepsilon > 0$ sufficiently small, we thus choose the constants in the following order: η_1, δ_1, C_0, and finally δ_0, in such a way that (3.1.8) is satisfied for all $t \in [\tau\,,\ \tau + 4C_0\delta_0]$.

For future purpose, it is convenient to choose here the constant C_0 large enough so that it satisfies the additional inequality

$$C_0 \geq 8/\eta_0\,. \tag{3.2.14}$$

7. Having proved that, after time $T_\varepsilon = C_0\delta_0/\varepsilon$, each solution $v^{\varepsilon,\theta}$ contains a large viscous shock, we now study the behavior of the first order perturbations $z^{\varepsilon,\theta}$. The solution of the linear equation (3.2.7) can be expressed in term of the fundamental solutions. Indeed, for $0 < s < t$ one has

$$z(t,x) = \int \Gamma^{\varepsilon,\theta}(t,x,s,y)\, z(s,y)\, dy\,. \tag{3.2.15}$$

Here $\Gamma^{\varepsilon,\theta}(t,x,s,y)$ is the fundamental solution of (3.2.7) corresponding to a Dirac mass initially located the point y at time s. It is useful here to observe that, for $t \in [\tau,\ \tau+4C_0\delta_0]$, the location of the shock in the original solutions u and $\tilde{u}$ remains strictly inside the fixed interval $[-\delta_0, \delta_0]$. By the analysis in [45] we know that, for $t \in [0, 4T_\varepsilon]$, nearly all the perturbation lies within a bounded interval:

$$\int_{|x|>(\delta_0-c)/\varepsilon} |z^{\varepsilon,\theta}(t,x)|\, dx = o(\varepsilon^k), \tag{3.2.16}$$

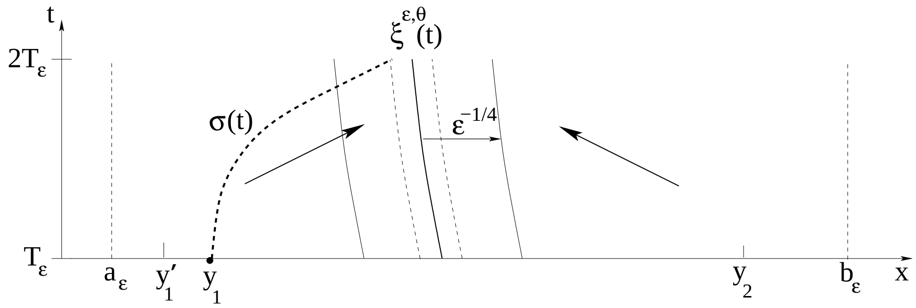

Fig. 2.2.3: Outside a small strip of width $\mathcal{O}(1)$ the characteristic speed points toward the shock.

for every $k \geq 1$. We can thus assume that, in all rescaled solutions $v^{\varepsilon,\theta}$, the viscous shocks are centered at points $\xi^{\varepsilon,\theta}$ such that

$$a_\varepsilon + c\varepsilon^{-1} \;=\; -\frac{\delta_0 - c}{\varepsilon} \;<\; \xi^{\varepsilon,\theta}(t) < \frac{\delta_0 - c}{\varepsilon} \;=\; b_\varepsilon - c\varepsilon^{-1},$$

for some constant $0 < c < \delta_0$. In this step we show that, as $\varepsilon \to 0$, for $t \in [2T_\varepsilon\,,\; 4T_\varepsilon]$ we have the stronger asymptotic estimate

$$\int_{|x-\xi^{\varepsilon,\theta}(t)|>\varepsilon^{-1/4}} |z^{\varepsilon,\theta}(t,x)|\,dx = o(\varepsilon^k)\,. \tag{3.2.17}$$

This shows that nearly all of the perturbation $z^{\varepsilon,\theta}$ is concentrated in a narrow strip around the viscous shock in $v^{\varepsilon,\theta}$. To establish (3.2.17), consider any point $y \in [a_\varepsilon, b_\varepsilon]$. Because of (3.2.15)-(3.2.16), it suffices to show that, as $\varepsilon \to 0$,

$$\int_{|x-\xi^{\varepsilon,\theta}(t)|>\varepsilon^{-1/4}} \Gamma^{\varepsilon,\theta}(2T_\varepsilon, x, T_\varepsilon, y)\,dx = o(\varepsilon^k)\,. \tag{3.2.18}$$

uniformly as the initial point y varies in the interval $[a_\varepsilon + c\varepsilon^{-1}\,,\;\; b_\varepsilon - c\varepsilon^{-1}]$.

We now recall that, by the choice of V^+, V^- and η_0, η_1 at (3.2.4)–(3.2.6), we have

$$|\dot\xi^{\varepsilon,\theta}(t)| \leq \eta_0\,,$$

$$\lambda^{\varepsilon,\theta}(t,x) \doteq f'(v^{\varepsilon,\theta}(t,x)) \begin{cases} \geq \;\; 2\eta_0 & \text{if} \quad x \in [a_\varepsilon\,,\;\; \xi^{\varepsilon,\theta} - \rho]\,, \\[2ex] \leq -2\eta_0 & \text{if} \quad x \in [\xi^{\varepsilon,\theta} + \rho\,,\;\; b_\varepsilon]\,. \end{cases}$$

for some constant ρ and all $t \in [T_\varepsilon\,,\;\; 4T_\varepsilon]$.

To prove (3.2.18), we use the representation $\Gamma^{\varepsilon,\theta} = Z_x$, where Z provides the solution to the linear parabolic Cauchy problem

$$Z_t + \lambda^{\varepsilon,\theta}(t,x)\,Z_x = Z_{xx}, \qquad\qquad Z(0,x) = \begin{cases} 0 & \text{if} \quad x < y\,, \\ 1 & \text{if} \quad x > y\,. \end{cases} \tag{3.2.19}$$

We begin by examining the special case where $y = y_1 \doteq a_\varepsilon + c\varepsilon^{-1}$. Thanks to (3.2.14), we can construct a smooth path $t \mapsto \sigma(t)$ such that (see Fig. 3.2.1)

$$\sigma(T_\varepsilon) = y_1 , \quad \sigma(2T_\varepsilon) = \xi^{\varepsilon,\theta}(2T_\varepsilon) - \rho ,$$

$$\dot\sigma(t) \in [0 , \ \eta_0/2] \quad \text{for all } \ t \in [T_\varepsilon, \ , \ 2T_\varepsilon] .$$

For $t \in [T_\varepsilon , \ 2T_\varepsilon]$, consider the two functions (see Fig. 3.2.2)

$$Z_1(t,x) = \begin{cases} e^{(\eta_0/2)(x-\sigma(t))} + \beta T_\varepsilon & \text{if} \quad x < \sigma(t) , \\ 1 + \beta T_\varepsilon & \text{if} \quad x \geq \sigma(t) , \end{cases}$$

$$Z_2(t,x) = \begin{cases} \beta(t - T_\varepsilon) & \text{if} \quad x \leq a_\varepsilon , \\ \beta(t - T_\varepsilon) + \beta(x - a_\varepsilon)^2/2 & \text{if} \quad x > a_\varepsilon , \end{cases}$$

with

$$\beta \doteq \exp\Big\{ -\frac{\eta_0}{2} \cdot \frac{c}{2\varepsilon} \Big\} .$$

Set $y_1' \doteq (a_\varepsilon + y_1)/2$. In connection with the parabolic equation in (3.2.19), a straightforward computation now shows that

- Z_1 is an upper solution for $x \in [a_\varepsilon , \ \infty[$,
- Z_2 is an upper solution for $x \in] - \infty , \ y_1]$,
- $Z_2(t, a_\varepsilon) < Z_1(t, a_\varepsilon)$, while $Z_1(t, y_1') < Z_2(t, y_1')$.

We conclude that the function

$$Z^+(t,x) \doteq \min\, [Z_1(t,x) , \ Z_2(t,x)],$$

is an upper solution of the Cauchy problem (3.2.19). In particular, as $\varepsilon \to 0$, it satisfies the asymptotic estimate

$$\begin{aligned} Z(2T_\varepsilon , \ \xi^{\varepsilon,\theta}(2T_\varepsilon) - \rho - \varepsilon^{-1/4}) \ &\leq \ Z^+(2T_\varepsilon , \ \sigma(2T_\varepsilon) - \varepsilon^{-1/4}) = \\ = \ \exp\Big\{ -\frac{\eta_0}{2} \cdot \varepsilon^{-1/4} \Big\} + \exp\Big\{ -\frac{\eta_0}{2} \cdot \frac{c}{2\varepsilon} \Big\} \cdot \frac{C_0 \delta_0}{\varepsilon} \ &= \ o(\varepsilon^k) , \end{aligned}$$

for any positive integer k. Next, consider the other extreme case where $y = y_2 \doteq b_\varepsilon - c\varepsilon^{-1}$. An entirely similar estimate yields

$$Z(2T_\varepsilon , \ \xi^{\varepsilon,\theta}(2T_\varepsilon) + \rho + \varepsilon^{-1/4}) \geq 1 - o(\varepsilon^k) ,$$

for any $k \geq 1$.

Finally, consider any initial point $y \in [y_1, \, y_2]$. By comparison with the two above cases, we conclude that the corresponding solution of the Cauchy problem (3.2.19) satisfies

$$\begin{aligned} &Z(2T_\varepsilon , \ \xi^{\varepsilon,\theta}(2T_\varepsilon) - \rho - \varepsilon^{-1/4}) = o(\varepsilon^k) , \\ &Z(2T_\varepsilon , \ \xi^{\varepsilon,\theta}(2T_\varepsilon) + \rho + \varepsilon^{-1/4}) \geq 1 - o(\varepsilon^k) . \end{aligned} \tag{3.2.20}$$

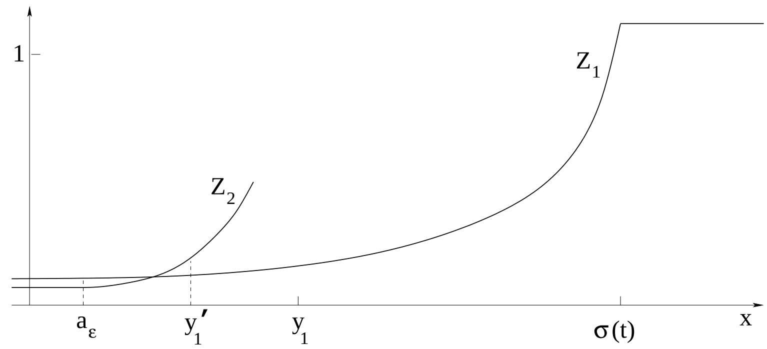

Fig. 3.2.2: Two upper solutions for the parabolic equation (3.2.19).

Recalling that $\Gamma^{\varepsilon,\theta} = Z_x$, from (3.2.20) we deduce (3.2.18), as claimed.

8. In preparation for the next comparison estimate, we study a specific Cauchy problem. Let $M > \eta_0 > 0$ and $\rho > 0$ be given. Define the points

$$P_\varepsilon \doteq \varepsilon^{-1/4}, \qquad Q_\varepsilon = 2\varepsilon^{-1/4} + \rho,$$

and consider the equation

$$W_t + \lambda(x)\, W_x = W_{xx}, \qquad \lambda(x) = \begin{cases} \eta_0 & \text{if} \quad x \in [0,\, 2P_\varepsilon], \\ -M & \text{if} \quad x \notin [0,\, 2P_\varepsilon], \end{cases} \tag{3.2.21}$$

with initial condition

$$W(0,x) = \begin{cases} 0 & \text{if} \quad x < P_\varepsilon, \\ 1 & \text{if} \quad x > P_\varepsilon. \end{cases} \tag{3.2.22}$$

We claim that, as $\varepsilon \to 0$, at time $t = \varepsilon^{-1/3}$ the solution satisfies

$$W(\varepsilon^{-1/3},\, 0) = o(\varepsilon^k), \qquad W(\varepsilon^{-1/3},\, Q_\varepsilon) \le \alpha < 1, \tag{3.2.23}$$

for some constant α independent of ε and any $k \ge 1$.

The first estimate in (3.2.23) is proved by constructing a suitable upper solution, as in the previous step. Set

$$\sigma(t) \doteq \begin{cases} P_\varepsilon + (\eta_0 t/2) & \text{if} \quad t \in [0,\, 2P_\varepsilon/\eta_0], \\ 2P_\varepsilon & \text{if} \quad t \in [2P_\varepsilon/\eta_0\,,\ \varepsilon^{-1/3}]. \end{cases}$$

For $t \in [0, ,\ \varepsilon^{-1/3}]$, consider the two functions (see Fig. 3.2.2)

$$W_1(t,x) = \begin{cases} e^{(\eta_0/2)(x-\sigma(t))} + \beta\varepsilon^{-1/3} & \text{if} \quad x < \sigma(t), \\ 1 + \beta\varepsilon^{-1/3} & \text{if} \quad x \ge \sigma(t), \end{cases}$$

$$W_2(t,x) = \begin{cases} \beta t & \text{if} \quad x \le 0, \\ \beta t + \beta x^2/2 & \text{if} \quad x > 0, \end{cases}$$

with

$$\beta \doteq \exp\Big\{ -\frac{\eta_0}{2} \cdot \varepsilon^{-1/4} \Big\} .$$

In connection with the parabolic equation (3.2.21), a straightforward computation now shows that

- W_1 is an upper solution for $x \in [0\,,\ \infty[\,$,
- W_2 is an upper solution for $x \in]-\infty\,,\ 2P_\varepsilon]$,
- $W_2(t,0) < W_1(t,0)$, while $W_1(t,P_\varepsilon) < W_2(t,P_\varepsilon)$.

We conclude that the function

$$W^+(t,x) \doteq \min\{W_1(t,x)\,,\ W_2(t,x)\},$$

is an upper solution of the Cauchy problem (3.2.19). Therefore, as $\varepsilon \to 0$ we have the asymptotic estimate

$$W(\varepsilon^{-1/3}\,,\ 0) \le W^+(\varepsilon^{-1/3}\,,\ 0) \ = \ \varepsilon^{-1/3} \cdot \exp\Big\{ -\frac{\eta_0}{2} \cdot \varepsilon^{-1/4} \Big\} \ = \ o(\varepsilon^k)\,,$$

for any positive integer k.

To prove the second inequality in (3.2.23) we observe that, when $t \ge 2P_\varepsilon/\eta_0$ and $\sigma(t) = 2P_\varepsilon$, one has

$$W^+(t,\, 2P_\varepsilon - 1) \ \le \ W_2(t,\, 2P_\varepsilon - 1) \ = \ e^{-\eta_0/2} + \beta\varepsilon^{-1/3} \ < \ \frac{1+e^{-\eta_0/2}}{2}\,,$$

for all $\varepsilon > 0$ sufficiently small. On the domain

$$\mathcal{D} \doteq \Big\{ (t,x)\,; \quad t \in [2P_\varepsilon/\eta_0\,,\ \varepsilon^{-1/3}]\,, \quad x \ge 2P_\varepsilon - 1 + \frac{M}{2}(t-\varepsilon^{-1/3}) \Big\}\,,$$

consider the function

$$W^\sharp(t,x) \doteq 1+\varepsilon - \beta^* \exp\Big\{ -2M\Big(x - (2P_\varepsilon - 1) - \frac{M}{2}(t-\varepsilon^{-1/3}) \Big) \Big\}\,,$$

with $\beta^* = (1-e^{-\eta_0/2})/4$. For $\varepsilon > 0$ sufficiently small, one checks that the function $W^\sharp$ provides an upper solution to (3.2.21) on the domain $\mathcal{D}$. Moreover, $W^\sharp > W^+$ on the parabolic boundary of $\mathcal{D}$. We thus conclude that

$$W(t,x) \le \min\{W^+(t,x)\,,\ W^\sharp(t,x)\},$$

for all $(t,x) \in \mathcal{D}$. In particular, this implies

$$W(\varepsilon^{-1/3}, Q_\varepsilon) \ \le \ W^\sharp(\varepsilon^{-1/3}, Q_\varepsilon) \ = \ 1+\varepsilon - \beta^* \cdot e^{-2M(1+\rho)} \ \le \ \alpha\,, \tag{3.2.24}$$

with

$$\alpha = 1 - \frac{1-e^{-\eta_0/2}}{5}\, e^{-2M(1+\rho)} \ < \ 1,$$

and for all $\varepsilon > 0$ sufficiently small.

9. We now divide the time interval $[2T_\varepsilon,\, 3T_\varepsilon]$ into equal subintervals, inserting the times

$$t_j \doteq 2T_\varepsilon + j\cdot\varepsilon^{-1/3}, \qquad\qquad j = 0,1,\ldots,N_\varepsilon\,.$$

We also define the intervals

$$I_j \doteq [\xi^{\varepsilon,\theta}(t_j) - \rho - \varepsilon^{-1/4}\,,\ \ \xi^{\varepsilon,\theta}(t_j) + \rho + \varepsilon^{-1/4}]\,. \tag{3.2.25}$$

We claim that, for each j and every couple of points $y, y' \in I_{j-1}$ one has

$$\int_{I\!R\setminus I_j} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y)\,dx = o(\varepsilon^k), \tag{3.2.26}$$

for all $k \geq 1$, and moreover

$$\int_{I\!R} \Big|\Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y) - \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y')\Big|dx \ \ \leq\ \ 2\alpha\,. \tag{3.2.27}$$

To prove (3.2.26), define

$$Y(t,x) \ \ \doteq\ \ \int_{-\infty}^{x+\xi^{\varepsilon,\theta}(t)-\rho-2\varepsilon^{-1/4}} \Gamma^{\varepsilon,\theta}(t_{j-1}+t\,,\ \ x'\,,\ \ t_{j-1}\,,\ \ y)\,dx'.$$

Observe that Y is a lower solution of the Cauchy problem (3.2.21)-(3.2.22). Hence $Y \leq W$. In particular, taking $x = 0$ we conclude

$$\int_{-\infty}^{\xi^{\varepsilon,\theta}(t)-\rho-2\varepsilon^{-1/4}} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y)\,dx \ \ =\ \ Y(\varepsilon^{-1/3}, 0) \leq W(\varepsilon^{-1/3}, 0) = o(\varepsilon^k),$$

for any $k \geq 1$. By reversing the direction of the x-axis we obtain the symmetric estimate

$$\int_{\xi^{\varepsilon,\theta}(t)+\rho+2\varepsilon^{-1/4}}^{\infty} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y)\,dx = o(\varepsilon^k)\,.$$

Together, the two above estimates yield (3.2.26).

To prove (3.2.27), assume $y < y'$ and consider the function

$$\Gamma^*(t,x) \doteq \Gamma^{\varepsilon,\theta}(t_{j-1}+t, x, t_{j-1}, y') - \Gamma^{\varepsilon,\theta}(t_{j-1}+t, x, t_{j-1}, y).$$

Observe that

$$\int_{-\infty}^{\infty} \Gamma^*(t,x)\,dx = 0.$$

Moreover, for each $t > 0$ this function has exactly one intersection with the x-axis, say located at $x = \zeta(t)$, so that

$$\begin{cases} \Gamma^*(t,x) > 0 & \text{if} \quad x > \zeta(t_{j-1}+t)\,, \\ \Gamma^*(t,x) < 0 & \text{if} \quad x < \zeta(t_{j-1}+t)\,. \end{cases}$$

At time $t = \varepsilon^{-1/3}$ we consider two cases. If $\zeta(t_j) \le \xi^{\varepsilon,\theta}(t_j)$, then

$$\begin{aligned}\int \left|\Gamma^*(\varepsilon^{-1/3}, x)\right| dx \ &= \ 2\int_{-\infty}^{\zeta(t_j)} \left|\Gamma^*(\varepsilon^{-1/3}, x)\right| dx \ \le \\ &\le 2\int_{-\infty}^{\zeta(t_j)} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y')\, dx \le \ 2\int_{-\infty}^{\xi^{\varepsilon,\theta}(t_j)} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y')\, dx \le \\ &\le \ 2Y(\varepsilon^{-1/3},\ \rho + 2\varepsilon^{-1/4}) \le \ 2W(\varepsilon^{-1/3},\ \rho + 2\varepsilon^{-1/4}) \ \le \ 2\alpha\,,\end{aligned}$$

because of (3.2.24). The alternative case, where $\zeta(t_j) \ge \xi^{\varepsilon,\theta}(t_j)$, can be handled in an entirely similar way, reversing the direction of the x-axis.

Because of the representation

$$z^{\varepsilon,\theta}(t_j, x) = \int \Gamma(t_j, x, t_{j-1}, y)\, dy\,,$$

the two estimates (3.2.26)- (3.2.27) show that, during each time interval $[t_{j-1}\,,\ t_j]$, the amount of mass $z^{\varepsilon,\theta}$ that creeps out of the interval I_j at (3.2.25) is asymptotically $o(\varepsilon^k)$, for every $k \ge 1$. Moreover,

$$\int_{I_j} z^{\varepsilon,\theta}(t_j,\, x)\, dx \le \alpha \int_{I_{j-1}} z^{\varepsilon,\theta}(t_{j-1},\, x)\, dx\,.$$

Since the total number of subintervals is $N_\varepsilon \sim \varepsilon^{-2/3}$, we conclude that at time $t = 3T_\varepsilon$ one has the asymptotic estimate

$$\int_{-\infty}^{\infty} |z^{\varepsilon,\theta}(3T_\varepsilon,\, x)|\, dx = o(\varepsilon^k),$$

for any $k \ge 1$.

10. Working still in the stretched variables, from the representation formula

$$z(t+1, x) = \int \Gamma^{\varepsilon,\theta}(t+1, x, t, y)\, z(t, y)\, dy,$$

it follows the estimate

$$\left\|\frac{\partial^{m+n}}{\partial x^m \partial t^n} z^{\varepsilon,\theta}(t+1, x)\right\|_{L^\infty(I\!R)} \le \left\|\frac{\partial^{m+n}}{\partial x^m \partial t^n}\Gamma^{\varepsilon,\theta}(t+1, \cdot, t, y)\right\|_{L^\infty(I\!R)} \cdot \|z^{\varepsilon,\theta}(t, \cdot)\|_{L^1(I\!R)}\,.$$

Observing that the map $t \mapsto \|z^{\theta,\varepsilon}(t, \cdot)\|_{L^1}$ is non-increasing, and using the uniform bounds

$$\left|\frac{\partial^{m+n}}{\partial x^m \partial t^n}\Gamma^{\varepsilon,\theta}(t+1, x, t, y)\right| \le C_{m,n},$$

for suitable constants $C_{m,n}$, we deduce

$$\left|\frac{\partial^{m+n}}{\partial x^m \partial t^n} v^\varepsilon(t, x) - \frac{\partial^{m+n}}{\partial x^m \partial t^n}\tilde v^\varepsilon(t, x)\right| \ \le \ C_{m,n}\, \|z^{\varepsilon,\theta}(3T_\varepsilon, \cdot)\|_{L^1(I\!R)} \ = \ \mathcal{O}(1)\cdot \varepsilon^k\,,$$

for $t \ge 4T_\varepsilon > 3T_\varepsilon + 1$ and for any positive integer k. Returning to the original variables, for $t \ge \tau_4$ we have

$$\left|\frac{\partial^{m+n}}{\partial x^m \partial t^n} u^\varepsilon(t, x) - \frac{\partial^{m+n}}{\partial x^m \partial t^n}\tilde u^\varepsilon(t, x)\right| = \mathcal{O}(1)\cdot \varepsilon^k\, \varepsilon^{-(m+n)}\,.$$

Since the integers $k, m, n \ge 0$ are arbitrary, this achieves the proof. □

Chapter 4. The vanishing viscosity approximation in the vectorial case

4.1. Convergence estimates for self similar solutions

Assume that the solution u of the genuinely nonlinear system

$$u_t + f(u)_x = 0, \tag{4.1.1}$$

is self similar and contains exactly one pair of interacting shock waves. In this section we adapt the approximation technique used by A. Bressan and T. Yang in [13] to obtain sharper bounds from above on the convergence rate of the viscous approximations

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon\, u_{xx}^\varepsilon. \tag{4.1.2}$$

All the computations in this section will be done for $\varepsilon = 1$. The estimates corresponding to all other possible choices of ε can be obtained by rescaling.

We can distinguish two cases. In the first case the solution of the Riemann problem generated by the shock interaction contains only shocks, then we claim that

$$\|u^1(T,\cdot) - u(T,\cdot)\|_{\mathbf{L}^1} \leq C\,(\mathrm{Tot.Var.}\{\bar{u}\})\ . \tag{4.1.3}$$

In the second case the solution of the Riemann problem contains also centered rarefaction waves. Hence the estimate (4.1.3) must be replaced by something weaker, where the error term increases in time. We obtain

$$\|u^1(T,\cdot) - u(T,\cdot)\|_{\mathbf{L}^1} \leq \begin{cases} C\,(\mathrm{Tot.Var.}\{\bar{u}\}) & \text{if} \quad t \leq 0\,, \\ C\,(\mathrm{Tot.Var.}\{\bar{u}\})\,(1 + \ln T) & \text{if} \quad t \leq 0\,. \end{cases} \tag{4.1.4}$$

It can be useful to recall here the technique used in [13] to obtain the error estimate

$$\|u^\varepsilon(T,\cdot) - u(T,\cdot)\|_{\mathbf{L}^1} \leq C \cdot (1+T)\ \mathrm{Tot.Var.}\{\bar{u}\} \cdot \sqrt{\varepsilon}\,(1 + |\ln\varepsilon|)\ . \tag{4.1.5}$$

First of all the authors reduce themselves to consider the case in which u is piecewise constant. To do that, once ε is fixed, it is sufficient to choose a constant $\nu \ll \varepsilon$, for example $\nu = e^{-1/\varepsilon}$, and substitute u in all the estimates by its front tracking approximation u^ν. By definition u^ν satisfies the inequalities

$$\|u^\nu(0) - \bar{u}\|_{\mathbf{L}^1} < \nu,$$

$$\|u^\nu(T) - \bar{u}(T)\|_{\mathbf{L}^1} < \nu,$$

and the total strength of all non-physical fronts in u^ν is strictly smaller than ν. As a consequence, the error done by substituting u by u^ν is of order ν and can be neglected. In particular it is possible to assume $\bar{u}$ to be piecewise constant. In the present situation we cannot take advantage of this substitution to be able to consider general piecewise smooth solutions because such a substitution does not keep constant the number of shock interactions.

It is well known that the system (4.1.2) generates a semigroup of solutions, S_t^ε, Lipschitz continuous with respect to the initial data, [6]. In particular, given an approximate solution w of (4.1.2), the existence of the semigroup can be used to control the error

$$\begin{aligned} \|w(T) - u^\varepsilon(T)\|_{\mathbf{L}^1} &= \|w(T) - S_T^\varepsilon \bar{u}\|_{\mathbf{L}^1} \leq \\ &\leq L \cdot \int_0^T \left\{ \liminf_{h \to 0^+} \frac{\|w(s+h) - S_h^\varepsilon w(s)\|_{\mathbf{L}^1}}{h} \right\} ds. \end{aligned} \tag{4.1.6}$$

In order to obtain the estimate (4.1.5), in [13] the authors construct an approximate solution $t \mapsto w(t, \cdot)$ with the following properties. Let t_i, $i = 1, \ldots, N$, be the interaction times in the front tracking solution u^ν. The approximation w has to be smooth in each strip $[t_i, t_{i+1}[\times I\!R$, and to satisfy

$$\|w(0) - \bar{u}\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \text{Tot.Var.}\{\bar{u}\}\sqrt{\varepsilon}, \qquad \|w(T) - u(T)\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \text{Tot.Var.}\{\bar{u}\}\sqrt{\varepsilon},$$

$$\int_0^T \int |w_t + f(w)_x - w_{xx}|\, dx\, ds = \mathcal{O}(1) \cdot \text{Tot.Var.}\{\bar{u}\}(1+T)\sqrt{\varepsilon}|\ln \varepsilon|,$$

$$\sum_{1 \leq i \leq N} \int |w(t_i+) - w(t_i-)|\, dx = \mathcal{O}(1) \cdot \text{Tot.Var.}\{\bar{u}\}\sqrt{\varepsilon}|\ln \varepsilon|.$$

Then the estimate (4.1.5) can be obtained as follows

$$\begin{aligned} \|u^\varepsilon(T) - u(T)\|_{\mathbf{L}^1} &\leq \\ &\leq \|u^\varepsilon(T) - w(T)\|_{\mathbf{L}^1} + \|w(T) - u(T)\|_{\mathbf{L}^1} \leq \\ &\leq L\|w(0) - \bar{u}\|_{\mathbf{L}^1} + L\int_0^T \int |w_t + f(w)_x - w_{xx}|\, dx\, ds + \\ &\quad + L \sum_{1 \leq i \leq N} \int |w(t_i+) - w(t_i-)|\, dx + \|w(T) - u(T)\|_{\mathbf{L}^1} = \\ &= \mathcal{O}(1) \cdot \text{Tot.Var.}\{\bar{u}\}(1+T)\sqrt{\varepsilon}|\ln \varepsilon|. \end{aligned}$$

The approximate solution w is constructed in two steps. First of all the shock fronts $x_\alpha(t)$, $\alpha = 1, \ldots, N$, with strength larger than a fixed threshold parameter $\theta > 0$, are substituted by inserting in the solution u suitable rescaled viscous travelling profiles. Then, in the regions far from the shock fronts $x_\alpha(t)$, the authors approximate u by mollification. The approximation obtained by mollification would be sufficient to construct a function w with all the properties above if u was a Lipschitz continuous function. The authors show that the additional error terms due to the presence of centered rarefaction waves can be controlled by studying the decay of these waves, [12], while the errors due to shocks with small strength can be estimated by means of suitable Lyapunov functionals. Here we deal with a much simpler situation and we do not need to distinguish between small and large shocks. We start by considering the case in which the solution of the Riemann problem generated by the interaction consists only of shock waves. We can assume the interaction to take place at the origin, and we fix a time instant $t^* > 0$. For every t, such that $|t| < t^*$ we consider the conic region around each shock front delimited by the maximal and the minimal characteristics through the origin. Inside each one of these regions we substitute the shock front by a rescaled viscous shock profiles with suitable boundary data. Outside these regions, since we are considering a piecewise constant solution, our approximate solution w will coincide wit u. More precisely, assume that for $t < 0$ the solution of (4.1.1), u, consists of two approaching shock fronts $\bar{x}_1(t)$, $\bar{x}_2(t)$. At $t = 0$ the interaction takes place and p outgoing shock fronts, x_j, $j = 1, \ldots, p$, are created. Let $\bar{\sigma}_i$, σ_j be the strength of the shock $\bar{x}_i$, x_j and let λ_j^+, λ_j^- be the maximal and the minimal values attained by the j^{th} eigenvalue of $Df(u)$ in the range of u. We also assume the Jacobian matrix $Df(u)$ to be uniformly strictly hyperbolic, so that there exists $c > 0$ such that

$$\lambda_j^+ + 2c < \lambda_{j+1}^- - 2c, \qquad \text{for all } \ j \leq p-1.$$

Given $t^* > 0$, fixed, we construct an approximation of u, w, in the following way. For $t > 0$, $j \neq 1, p$, we define

$$\Phi_j(\xi,t) = \begin{cases} \xi & \text{if} \quad \xi \in [\lambda_j^- t, \ \lambda_j^+ t]\,, \\ \lambda_j^- t + \dfrac{\lambda_j^- t - \xi}{((\lambda_j^- - c)t - \xi)} & \text{if} \quad \xi \in ((\lambda_j^- - c)t, \ \lambda_j^- t)\,, \\ \dfrac{\lambda_j^+ c t^2}{((\lambda_j^+ + c)t - \xi)} & \text{if} \quad \xi \in (\lambda_j^+ t, \ (\lambda_j^+ + c)t)\,. \end{cases} \tag{4.1.7}$$

While for $t < 0$, we have

$$\bar{\Phi}_1(\xi,t) = \begin{cases} \xi & \text{if} \quad \xi \geq \lambda_1^+ t\,, \\ \dfrac{\lambda_1^+ c t^2}{((\lambda_1^+ + c)t - \xi)} & \text{if} \quad \xi \in (\lambda_1^+ t, \ (\lambda_1^+ + c)t)\,. \end{cases} \tag{4.1.8}$$

In a similar way we define Φ_1, Φ_p, and $\bar{\Phi}_2$. Given a shock front x_j, with right and left states u_j^+, u_j^- a we can find a viscous shock profile ω_j satisfying

$$\omega_j'' = (A(\omega_j) - \rho_j)\omega_j', \qquad \lim_{s \to \pm\infty} \omega_j(s) = u_j^{\pm}. \tag{4.1.9}$$

Since (4.1.9) determines ω_j up to a shift and we want it to match x_j as closely as possible, we also require the condition

$$\int_{-\infty}^{0} |\omega_j - u_j^-|\, ds = \int_0^{+\infty} |\omega_j - u_j^+|\, ds. \tag{4.1.10}$$

Then, for $t > t^*$ we define the approximation w as

$$w(t,\xi) = \begin{cases} \omega_1(\Phi_1(t,\xi)) & \text{if} \quad \xi \in (-\infty, x_1(t) + (\lambda_1^+ + c)t)\,, \\ \omega_j(\Phi_j(t,\xi)) & \text{if} \quad \xi \in (x_j(t) - (\lambda_j^- - c)t, x_j(t) + (\lambda_j^+ + c)t), \\ \quad \text{for } j = 2, \dots, p-1\,, & \\ \omega_p(\Phi_p(t,\xi)) & \text{if} \quad \xi \in (x_p(t) - (\lambda_p^- - c)t, +\infty)\,, \\ u(t,x) & \text{otherwise.} \end{cases} \tag{4.1.11}$$

The approximation for $t < -t^*$ is defined in the same way.

For $|t| < t^*$, we fix the size of the interval in which the exact value of $u(t,x)$ has to be substituted with a rescaled viscous shock profile. We use the same definitions as above, but fixing $t = t^*$. Then, if $|t| < t^*$, we can have more than one viscous shock in the same interval of approximation. This means that, to estimate the error for $|t| < t^*$, we have to take into account also the possible interactions between viscous shock in the same interval of approximation.

The estimate (4.1.6) in our case takes the form

$$\begin{aligned} \|w(T) - u^1(T)\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \int_*^T & \|w_t + f(w)_x - \varepsilon w_{xx}\|_{\mathbf{L}^1}\, dt + \\ & + \mathcal{O}(1) \cdot \|w(0-) - w(0+)\|_{\mathbf{L}^1}\,, \end{aligned} \tag{4.1.12}$$

where the second term on the right hand side is due to the fact that w is discontinuous in time when the interaction occurs. By standard interaction estimates we have

$$\|w(t^*+) - w(t^*-)\|_{\mathbf{L}^1} = \mathcal{O}(1)|\bar\sigma_1\bar\sigma_2|. \tag{4.1.13}$$

Following the definitions above, the quantity

$$w_t(t,x) + f(w)_x(t,x) - \varepsilon w_{xx}(t,x),$$

can be non zero only in the regions of the $t - x$ plane where our approximation w is a rescaled viscous shock. These regions are all defined in the same way, so we expect the corresponding errors, $\bar E_1^-$, $\bar E_2^+$, E_1^+, E_2^-, $E_j^\pm$, for $j = 2, \dots, p-1$, to be all of the same order of magnitude. Here we give an explicit estimate of

$$\begin{aligned} E_j^+(t) = \int_{\lambda_j^+ t}^{(\lambda_j^+ + c)t} \Bigg| & (A(\omega_j(\Phi_j(\xi))) - \lambda_j) \frac{\partial \omega_j(\Phi_j(\xi))}{\partial s} \Phi_j'(\xi) + \\ & - \frac{\partial \omega_j(\Phi_j(\xi))}{\partial s} \Phi_j''(\xi) - \frac{\partial^2 \omega_j(\Phi_j(\xi))}{\partial s^2} (\Phi_j'(\xi))^2 \Bigg|\, d\xi\,, \end{aligned} \tag{4.1.14}$$

in the two cases $t > t^*$ and $t < t^*$. Consider first the case $t > t^*$. Recalling the bounds

$$\left| \frac{\partial}{\partial s} \omega_j(s) \right| = \mathcal{O}(1)|\sigma_j|^2 e^{-|\sigma_j s|}, \qquad \left| \frac{\partial^2}{\partial s^2} \omega_j(s) \right| = \mathcal{O}(1)|\sigma_j|^3 e^{-|\sigma_j s|}, \tag{4.1.15}$$

and the definition of Φ_j, we have

$$
\begin{aligned}
E_j^+(t) \le & \int_{\lambda_j^+ t}^{(\lambda_j^+ + c)t} \left| (A(\omega_j(\Phi_j(\xi))) - \lambda_j) \frac{\partial \omega_j(\Phi_j(\xi))}{\partial s} \Phi_j'(\xi) \right| + \\
& + \left| \frac{\partial \omega_j(\Phi_j(\xi))}{\partial s} \Phi_j''(\xi) \right| + \left| \frac{\partial^2 \omega_j(\Phi_j(\xi))}{\partial s^2} (\Phi_j'(\xi))^2 \right| d\xi = \\
& = \mathcal{O}(1) |\sigma_j|^2 K \int_{\lambda_j^+ t}^{(\lambda_j^+ + c)t} e^{-\left| \frac{K\sigma_j}{((\lambda_j^+ + c)t - \xi)} \right|} \left\{ \frac{1}{((\lambda_j^+ + c)t - \xi)^2} + \right. \\
& \left. + \frac{2}{((\lambda_j^+ + c)t - \xi)^3} + \frac{|\sigma_j| K}{((\lambda_j^+ + c)t - \xi)^4} \right\} d\xi,
\end{aligned}
\tag{4.1.16}
$$

where

$$K = \lambda_j^+ c t^2.$$

After the change of variable

$$s = \frac{1}{((\lambda_j^+ + c)t - \xi)},$$

we have

$$E_j^+(t) \le \mathcal{O}(1) |\sigma_j|^2 K \int_{1/ct}^{+\infty} e^{-|K\sigma_j s|} \left(1 + 2s + |\sigma_j| K s^2 \right) ds, \tag{4.1.17}$$

integrating by parts we obtain

$$E_j^+(t) \le \mathcal{O}(1) e^{-|\sigma_j \lambda_j^+ t|} \left(|\sigma_j| + \frac{|\sigma_j|^2 \lambda_j^+}{c} + \frac{4|\sigma_j|}{ct} + \frac{4}{c\lambda_j^+ t^2} \right). \tag{4.1.18}$$

Integrating on $[t^*, +\infty)$ we find the running error corresponding to the cone

$$C_j^+ = \{(t,x); t > t^* \text{ and } x \in (x_j(t) - \lambda_j^+ t, x_j(t) + (\lambda_j^+ + c)t)\},$$

$$
\begin{aligned}
\int_{t^*}^{+\infty} E_j^+(t)\, dt = \mathcal{O}(1) & \int_{t^*}^{+\infty} e^{-|\sigma_j \lambda_j^+ t|} \left(|\sigma_j| + \frac{|\sigma_j|^2 \lambda_j^+}{c} + \frac{4|\sigma_j|}{ct} + \frac{4}{c\lambda_j^+ t^2} \right) dt = \\
& = c_j^+(|\sigma_j|, \lambda_j^+, c, t^*).
\end{aligned}
$$

Consider now the case $t < t^*$. Now we fix $t = t^*$ in the definition of Φ_j and we estimate the double integral

$$
\begin{aligned}
\int_0^{t^*} \int_{\lambda_j^+ t^*}^{(\lambda_j^+ + c)t^*} & \left| (A(\omega_j(\Phi_j(\xi))) - \lambda_j) \frac{\partial \omega_j(\Phi_j(\xi))}{\partial s} \Phi_j'(\xi) + \right. \\
& \left. - \frac{\partial \omega_j(\Phi_j(\xi))}{\partial s} \Phi_j''(\xi) - \frac{\partial^2 \omega_j(\Phi_j(\xi))}{\partial s^2} (\Phi_j'(\xi))^2 \right| d\xi\, dt = \\
& = \mathcal{O}(1) e^{-|\sigma_j \lambda_j^+ t^*|} \left[|\sigma_j| t^* + \frac{|\sigma_j|^2 \lambda_j^+ t^*}{c} + \frac{4|\sigma_j|}{c} + \frac{4}{c\lambda_j^+ t^*} \right] + \\
& + \mathcal{O}(1) t^* \left(\sum_{\alpha,\beta=1}^{p} |\sigma_\alpha \sigma_\beta| - \sum_{\alpha=1}^{p} |\sigma_\alpha|^2 \right) = d_j^+(|\sigma_j|, \lambda_j^+, c, t^*, \text{Tot. Var}\{\bar{u}\}).
\end{aligned}
$$

We remark that the error in the approximation does not depend linearly on the size of the jump in the shock discontinuity, as one could expect. In fact, since the viscous profile associated with a shock front becomes less and less steep as the size of the jump decreases, the area between the two profiles in general does not decrease linearly, as we can see in the following example.

Example, Burgers equation: Consider the family of Riemann problems for the scalar Burgers equation

$$\begin{cases} u_{n,t} + \left(\dfrac{u_n^2}{2}\right)_x = 0, \\ u_n(0,x) = \begin{cases} 2^{-n} & \text{if } x < 0, \\ -2^{-n} & \text{if } x > 0. \end{cases} \end{cases}$$

The solution is a shock front travelling with speed $s_n \equiv 0$. We can approximate this solution by replacing the shock discontinuity with a viscous shock profile connecting the same right and left states. Let w_n be the solution of the boundary value problem

$$w_n'(\xi) = \frac{2^{2n} w_n(\xi)^2 - 1}{2^{2n+1}}, \qquad \lim_{\xi \to \pm\infty} w_n(\xi) = \pm 2^n.$$

It turns out that

$$w_n(\xi) = \frac{1}{2^{2n}} w_0\left(\frac{\xi}{2^{2n}}\right).$$

Then the $\mathbf{L}^1$ distance between u_n and w_n is a constant, $2\ln 4$.

We address now the case in which the solution of the Riemann problem generated by the interaction in u contains also centered rarefaction waves. In this new situation we expect the distance

$$\|u^1(T,\cdot) - u(T,\cdot)\|_{\mathbf{L}^1},$$

to increase in time for $T > 0$.

The approximation w can be constructed exactly as before. In particular w will coincide with u in a neighborhood of the centered rarefaction waves. Thanks to this observation and the estimate above we can say that the estimate in (4.1.4) will be proved once we know what happens when the solution of the Riemann problem consists of a single centered rarefaction wave of the j-th family, $j = 1, \ldots, p$.

Call $\lambda_j(u)$ the j^{th} eigenvalue of $Df(u)$ associated with the eigenvector $r_j(u)$, and let $\omega(s)$ to be the j-th rarefaction curve through u^-, i.e. the solution to the Cauchy problem

$$\begin{cases} \dfrac{d}{d\sigma}\omega(s) = r_j(\omega(s)), \\ \omega(0) = u^-. \end{cases}$$

Let $u^+ = \omega(\sigma)$, then the solution u takes the form

$$u(t,x) = \begin{cases} u^- & \text{if } \; x < \lambda_j(u^-)\,t, \\ \omega(s) & \text{if } \; x = \lambda_j(\omega(s))\,t, \quad \text{for } s \in [0,\sigma], \\ u^+ & \text{if } \; x > \lambda_j(u^+)\,t, \end{cases} \tag{4.1.19}$$

and clearly w coincides with u. We have

$$\begin{aligned}\|u(T)-u^1(T)\|_{\mathbf{L}^1} &= \mathcal{O}(1)\cdot\int_*^T \|u_t+f(u)_x-u_{xx}\|_{\mathbf{L}^1}\,dt = \\ &= \mathcal{O}(1)\cdot\int_*^T \|u_{xx}\|_{\mathbf{L}^1}\,dt.\end{aligned}$$

From (4.1.19) we have

$$u_x(t,x)=\begin{cases}\dot\omega(s)\cdot\dfrac{ds}{dx} & \text{if}\quad x=\lambda_j(\omega(s))\,t,\quad \text{for } s\in[0,\sigma]\ ,\\ 0 & \text{otherwise.}\end{cases}$$

Where u_x is non zero there holds

$$\frac{dx}{ds}=\frac{d\lambda_j(\omega(s))}{ds}\cdot t = r_j\bullet\lambda_j(\omega(s))\cdot t,$$

where $r_j\bullet\lambda_j(\omega(s))$ is the directional derivative

$$r_j\bullet\lambda_j(\omega(s)) \;=\; D\lambda_j(\omega(s))\cdot r_j.$$

Since we assume the system to be genuinely nonlinear we can write

$$\dot\omega(s)\cdot\frac{ds}{dx}=\frac{r_j(\omega(s))}{r_j\bullet\lambda_j(\omega(s))\cdot t}.$$

In the same way we compute

$$\begin{aligned}u_{xx} &= \frac{1}{t}\cdot\frac{d}{dx}\left(\frac{r_j(\omega(s))}{r_j\bullet\lambda_j(\omega(s))}\right)= \\ &= \frac{1}{t}\cdot\frac{r_j\bullet r_j(\omega(s))\cdot r_j\bullet\lambda_j(\omega(s))-r_j(\omega(s))\left(r_j\bullet\lambda_j(\omega(s))\right)_s}{\left(r_j\bullet\lambda_j(\omega(s))\right)^2}\cdot\frac{ds}{dx}.\end{aligned}$$

Then

$$\|u^1(T)-u(T)\|_{\mathbf{L}^1}\mathcal{O}(1)\cdot\int_\star^T\int_{\lambda_j(u^-)t}^{\lambda_j(u^+)t}|u_{xx}|\,dx=\mathcal{O}(1)\int_\star^T\int_0^\sigma\frac{1}{t}ds=\mathcal{O}(1)\left(\ln|T|+1\right)\sigma.$$

4.2. On the dependence of a 2×2 system on the coupling parameter

Consider a weakly coupled system

$$\begin{cases} u_t + \left(u^2/2\right)_x + \rho f\left(u, v\right)_x = u_{xx}\,, \\ v_t + \left(v^2/2\right)_x + \rho g\left(u, v\right)_x = v_{xx}\,. \end{cases} \tag{4.2.1}$$

Let call $U_\rho = (u, v)^t$ the solution of (4.2.1), and $F_\rho(U)$ the flux vector

$$F_\rho\left(U\right) = \begin{pmatrix} \dfrac{u^2}{2} + \rho f\left(u, v\right) \\ \dfrac{v^2}{2} + \rho g\left(u, v\right) \end{pmatrix}. \tag{4.2.2}$$

For $\rho = 0$ we have an explicit solution, given by the superposition of two viscous travelling waves, $V_1(t)$ and $V_2(t)$. We fix the left and right states to be $U^- = (1, 3)^t$ and $U^+ = (-1, 2)^t$, and we choose the initial data in such a way that the viscous profiles cross at $t = 0$. Then for $t < 0$ our solution will be given by the superposition of a wave of the second family joining the states $(1, 3)^t$ with $(1, 2)^t$, with speed $5/2$ and a profile of the first family, joining the states $(1, 2)^t$ with $(-1, 2)^t$, travelling with zero speed. After the crossing, for $t > 0$, we will have one profile of the first family, joining the states $(1, 3)^t$ with $(-1, 3)^t$ with zero speed, and a profile of the second family, with speed $5/2$, joining the states $(-1, 3)^t$ with $(-1, 2)^t$.

As ρ varies the change in the flux implies a change in the asymptotic profile of the solution. The limit states

$$\lim_{x \to \pm\infty} U_\rho\left(t, x\right) = U^\pm,$$

remain the same, but the value of U_ρ in the region between the two shocks depends on ρ. We will call it $U_\rho^m(t)$ and we will write it as a perturbation of the intermediate state in the case $\rho = 0$, U^m. Then for $t < 0$, $U_\rho^m(t) = (1, 2)^t + \rho(a, b)^t$, while for $t > 0$, $U_\rho^m(t) = (-1, 3)^t + \rho(c, d)^t$, for some functions a, b, c, d.

For $t \to -\infty$ the solution U_ρ consists of two approaching viscous profiles, so the intermediate state for $t < 0$ is connected to $U^- = (1, 3)^t$ by a shock of the second family and to $U^+ = (-1, 2)$ by one of the first family. The speeds of the shocks σ_1, σ_2 also depend on ρ. They will be written again as perturbation of the shock speeds in the case $\rho = 0$, so $\sigma_1 = \rho s_1$, $\sigma_2 = 5/2 + \rho s_2$. Solving the system given by the Rankine-Hugoniot conditions

$$\begin{cases} F_\rho\left(U_\rho^m\right) - F_\rho\left(U^-\right) = \sigma_2 \cdot \left(U_\rho^m - U^-\right), \\ F_\rho\left(U^+\right) - F_\rho\left(U_\rho^m\right) = \sigma_1 \cdot \left(U^+ - U_\rho^m\right), \end{cases} \tag{4.2.3}$$

we can obtain a, b, s_1, s_2. We begin by rewriting (4.2.3) in the more explicit form

$$\begin{cases} \dfrac{2\rho a+\rho^2a^2}{2}+\rho\left(f\left(U^m\right)-f\left(U^-\right)\right)+\rho^2\nabla f\left(U^m\right)\cdot(a,\,b)^t=\dfrac{5}{2}\rho a+\rho^2 s_2 a,\\ -\dfrac{5}{2}+2\rho b+\dfrac{\rho^2b^2}{2}+\rho\left(g\left(U^m\right)-g\left(U^-\right)\right)+\rho^2\nabla g\left(U^m\right)\cdot(a,\,b)^t=\left(\dfrac{5}{2}+\rho s_2\right)(-1+\rho b),\\ -\dfrac{2\rho a+\rho^2a^2}{2}+\rho\left(f\left(U^+\right)-f\left(U^m\right)\right)-\rho^2\nabla f\left(U^m\right)\cdot(a,\,b)^t=\rho s_1\left(-2-\rho a\right),\\ -\dfrac{\rho^2b^2+4\rho b}{2}+\rho\left(g\left(U^+\right)-g\left(U^m\right)\right)-\rho^2\nabla g\left(U^m\right)\cdot(a,\,b)^t=-\rho^2 s_1 b. \end{cases}$$

Then we equate coefficients of powers of ρ and we get

$$\begin{aligned} a&=\frac{2}{3}\left(f\left(U^m\right)-f\left(U^-\right)\right)+\mathcal{O}\left(\rho\right),\\ b&=\frac{1}{2}\left(g\left(U^+\right)-g\left(U^m\right)\right)+\mathcal{O}\left(\rho\right),\\ s_1&=\frac{1}{3}\left(f\left(U^m\right)-f\left(U^-\right)\right)-\left(f\left(U^+\right)-f\left(U^m\right)\right)+\mathcal{O}\left(\rho\right),\\ s_2&=\frac{1}{4}\left(g\left(U^+\right)-g\left(U^m\right)\right)-\left(g\left(U^m\right)-g\left(U^-\right)\right)+\mathcal{O}\left(\rho\right). \end{aligned}$$

For $t>0$ we call $\zeta_1=\rho z_1$ $\zeta_2=5/2+\rho z_2$ the vector of shock speeds and we find c, d, z_1, z_2 using the same method as above. We obtain

$$\begin{aligned} c&=-\frac{2}{7}\left(f\left(U^+\right)-f\left(U^m\right)\right)+\mathcal{O}\left(\rho\right),\\ d&=-\frac{1}{3}\left(g\left(U^m\right)-g\left(U^-\right)\right)+\mathcal{O}\left(\rho\right),\\ z_1&=-\frac{1}{7}\left(f\left(U^+\right)-f\left(U^m\right)\right)-\left(f\left(U^m\right)-f\left(U^-\right)\right)+\mathcal{O}\left(\rho\right),\\ z_2&=-\frac{1}{6}\left(g\left(U^m\right)-g\left(U^-\right)\right)-\left(g\left(U^+\right)-g\left(U^m\right)\right)+\mathcal{O}\left(\rho\right). \end{aligned}$$

We are now able to identify the viscous profiles $V_1^{\rho,\pm}$ and $V_2^{\rho,\pm}$ that we expect to appear in the asymptotic profile of U_ρ. However we still have to prove that for any given couple of viscous shocks there exists a solution of (4.2.1) whose asymptotic profile, as $t\to-\infty$, consists exactly of the superposition of the given shock profiles. We postpone the proof of this claim to the next section, where it will be presented for the more general case of $n\times n$ systems.

The remaining part of this section will focus on the change in the solution as ρ varies. Following the analysis above we can assume that, as $t\to-\infty$, U_ρ converges to the superposition of $V_2^{\rho,-}$, connecting the states $(U^-,U^m+\rho(a,b)^t)$ with speed $5/2+\rho s_2$, and $V_1^{\rho,-}$, connecting the states $(U^m+\rho(a,b)^t,U^+)$ with speed ρs_1. Similarly, as $t\to+\infty$, the viscous shock profiles $V_1^{\rho,+}$ and $V_2^{\rho,+}$ are described as perturbations of order ρ of V_1 and V_2. Let define the function $\tilde{V}_j^{\pm} = (\tilde{u}_j^{\pm}, \tilde{v}_j^{\pm})^t$, with $j = 1, 2$, as follows

$$V_j^{\rho,\pm} = V_j + \rho\tilde{V}_j^{\pm}. \tag{4.2.4}$$

The functions $\tilde{V}_j^{\pm}$ satisfy a system of ODE's obtained by inserting (4.2.4) in (4.2.1). As an example we write explicitly how to obtain the system for $\tilde{V}_1^{-}$.

The system of ODE's satisfied by $V_1^{\rho,-}$ is

$$V_{1,\xi_1}^{\rho,-} = F_\rho(V_1^{\rho,-}) - F_\rho(U^m + \rho(a,b)^t) - \rho s_1(V_1^{\rho,-} - U^m - \rho(a,b)^t). \tag{4.2.5}$$

The variable ξ_1 here is $x - \rho s_1 t$. We notice that the variable in which the equations for V_1 are written is simply x, because this viscous profile has zero speed. However $\partial\xi/\partial x = 1$, so $V_{1,\xi_1} = V_{1,x}$. By substitution we have

$$\begin{aligned} V_{1,\xi_1} + \rho\tilde{V}_{1,\xi_1}^{-} &= F(V_1) - F(U^m) + \rho\tilde{V}_{1,\xi_1}^{-} = \\ &= F_\rho(V_1^{\rho,-}) - F_\rho(U^m + \rho(a,b)^t) - \rho s_1(V_1^{\rho,-} - U^m - \rho(a,b)^t). \end{aligned} \tag{4.2.6}$$

Then the equations satisfied by $\tilde{V}_1^{-}$ are

$$\begin{aligned} \rho\tilde{V}_{1,\xi_1}^{-} &= F(V_1^{\rho,-}) - F(V_1) + F(U^m) - F(U^m + \rho(a,b)^t) + \\ &\quad + \rho\begin{pmatrix} f(V_1^{\rho,-}) - f(U^m + \rho(a,b)^t) \\ g(V_1^{\rho,-}) - g(U^m + \rho(a,b)^t) \end{pmatrix} + \\ &\quad - \rho s_1(V_1 + \rho\tilde{V}_1^{-} - U^m - \rho(a,b)^t) = \\ &= \rho DF(V_1)\cdot\tilde{V}_1^{-} - \rho DF(U^m)\cdot(a,b)^t - \rho s_1(V_1 - U^m) + \\ &\quad + \rho\begin{pmatrix} f(V_1) - f(U^m) \\ g(V_1) - g(U^m) \end{pmatrix} + \mathcal{O}(1)\rho^2. \end{aligned} \tag{4.2.7}$$

Let $V_1 = (u_1,\, v_1)^t$, and rewrite (4.2.7) in components

$$\begin{cases} \tilde{u}_{1,\xi_1}^{-} = u_1(\tilde{u}_1^{-} - a) + f_u(\bar{V})(u_1 - u^m) - (s_1 + a)(u_1 - u^m) + \mathcal{O}(1)\rho, \\ \tilde{v}_{1,\xi_1}^{-} = v_1(\tilde{v}_1^{-} - b) + g_u(\hat{V})(u_1 - u^m) + \mathcal{O}(1)\rho, \end{cases} \tag{4.2.8}$$

where $\bar{V}(t,x)$ and $\hat{V}(t,x)$ are suitable states. Remark that by definition $v_1 - v^m = 0$. The equation for V_2^{-}, V_1^{+} and V_2^{+} are

$$\begin{cases} \tilde{u}_{2,\xi_2}^{-} = \left(u_2 - \dfrac{5}{2}\right)(\tilde{u}_2^{-} - a) + f_v(\bar{V})(v_2 - v^m) + \mathcal{O}(1)\rho, \\ \tilde{v}_{2,\xi_2}^{-} = \left(v_2 - \dfrac{5}{2}\right)(\tilde{v}_2^{-} - b) + g_v(\hat{V})(v_2 - v^m) - (s_2 + b)(v_2 - v^m) + \mathcal{O}(1)\rho. \end{cases} \tag{4.2.9}$$

When $t > 0$, U^m changes but we write it in the same way.

$$\begin{cases} \tilde{u}_{1,\xi_1}^{+} = u_1(\tilde{u}_1^{+} - c) + f_u(\bar{V})(u_1 - u^m) - (z_1 + c)(u_1 - u^m) + \mathcal{O}(1)\rho, \\ \tilde{v}_{1,\xi_1}^{+} = v_1(\tilde{v}_1^{+} - d) + g_u(\hat{V})(u_1 - u^m) + \mathcal{O}(1)\rho, \end{cases} \tag{4.2.10}$$

$$\begin{cases} \tilde{u}^+_{2,\xi_2} = \left(u_2 - \dfrac{5}{2}\right)(\tilde{u}^+_2 - c) + f_v(\bar{V})(v_2 - v^m) + \mathcal{O}(1)\rho, \\ \tilde{v}^+_{2,\xi_2} = \left(v_2 - \dfrac{5}{2}\right)(\tilde{v}^+_2 - d) + g_v(\hat{V})(v_2 - v^m) - (z_2 + d)(v_2 - v^m) + \mathcal{O}(1)\rho. \end{cases} \tag{4.2.11}$$

Define an approximate solution, $\tilde{U}$, given by the superposition of the viscous shocks appearing in the asymptotic profile of U_ρ. For $t < 0$, $\tilde{U}$ takes the form

$$\tilde{U} = V_1^{\rho,-} + V_2^{\rho,-} - U_\rho^{m,-}. \tag{4.2.12}$$

We want to estimate the total error we accumulate in the approximation as t varies in $]-\infty, 0[$. We have to estimate the integral

$$\int_{-\infty}^0 \|U_\rho(s) - \tilde{U}(s)\|_{L^1}\, ds \le L \cdot \int_{-\infty}^0 \int_{-\infty}^s \|\tilde{U}_t + F_\rho(\tilde{U})_x - \tilde{U}_{xx}\|_{L^1}\, d\tau\, ds. \tag{4.2.13}$$

By substitution we obtain

$$\begin{aligned} |\tilde{U}_t - F_\rho(\tilde{U})_x - \tilde{U}_{xx}| &= |F_\rho(\tilde{U})_x - F_\rho(V_1^{\rho.-})_x - F_\rho(V_1^{\rho.-})_x| = \\ &= \Big| V^{\rho,-}_{1,\xi_1} \cdot (V_2^{\rho,-} - U_\rho^m) + V^{\rho,-}_{2,\xi_2} \cdot (V_1^{\rho,-} - U_\rho^m) + \\ &\quad + \rho D \begin{pmatrix} f(\tilde{U}) - f(V_1^{\rho,-}) \\ g(\tilde{U}) - g(V_1^{\rho,-}) \end{pmatrix} \cdot V^{\rho,-}_{1,\xi_1} + \rho D \begin{pmatrix} f(\tilde{U}) - f(V_2^{\rho,-}) \\ g(\tilde{U}) - g(V_2^{\rho,-}) \end{pmatrix} \cdot V^{\rho,-}_{2,\xi_2} \Big| = \\ &= (1 + \rho C) \left| V^{\rho,-}_{1,\xi_1} \cdot (V_2^{\rho,-} - U_\rho^m) + V^{\rho,-}_{2,\xi_2} \cdot (V_1^{\rho,-} - U_\rho^m) \right| = \\ &= \rho \left| u_{1,\xi_1}(\tilde{u}_2 - a) + \tilde{v}_{1,\xi_1}(v_2 - v^m) + \tilde{u}_{2,\xi_2}(u_1 - u^m) + v_{2,\xi_2}(\tilde{v}_1 - b) \right| + \mathcal{O}(\rho^2). \end{aligned} \tag{4.2.14}$$

Consider the term

$$|u_{1,\xi_1}(t,x)(\tilde{u}_2(t,x) - a)|. \tag{4.2.15}$$

Notice that the function $\tilde{u}_2 - a$ is bounded then, far from the center of the viscous profile V_1, the quantity under consideration is exponentially small, since

$$|V'(\xi_1)| = \mathcal{O}(1)|U^m - U^+|^2 e^{-|U^m - U^+|\xi_1}.$$

From the equation (4.2.9) we have that

$$\tilde{u}_2 - a \approx a \cdot \left(e^{-\int_{\xi_2}^{+\infty} |u_2(\xi) - 5/2| d\xi} - 1 \right),$$

where ξ_2 is $x - (5/2 + \rho s_2)t$. Then for $t \ll 0$ the quantity (4.2.15) will be exponentially small for all values of x. The other terms in (4.2.14) have a similar behavior, then the error estimate (4.2.13) gives

$$\int_{-\infty}^0 \|U_\rho(s) - \tilde{U}(s)\|_{L^1}\, ds \le \mathcal{O}(1)\rho.$$

The error terms for this approximation are $\mathcal{O}(\rho)$, hence we can derive a first order expansion of the solution U_ρ. We define the function $W = (w_1, w_2)^t$ from the equality

$$U_\rho(t,x) - \tilde{U}(t,x) = \rho W + \mathcal{O}(1)\rho^2.$$

To obtain linearized equations for the components of W we write here (u_i, v_i) for the components of $V_i^{\rho,-}$, $i = 1, 2$, and (u, v) for the components of U_ρ. We first obtain the equation for w_1.

$$\begin{aligned}(u - u_1 - u_2 - u_\rho^m)_t &= \rho w_{1,t} = \\ &= -uu_x + u_1 u_{1,x} + u_2 u_{2,x} - \rho\Big(f(u,v) - f(u_1,v_1) - f(u_2,v_2)\Big)_x + \rho w_{1,xx},\end{aligned}$$

which, after some easy computations, takes the form

$$\begin{aligned}w_{1,t} + (u_{1,x} + u_{2,x})w_1 + uw_{1,x} - w_{1,xx} &= \\ = -\Big(f(u,v) - f(u_1,v_1) - f(u_2,v_2)\Big)_x &+ \frac{1}{\rho}\Big[(u_1 - u_\rho^m)(u_2 - u_\rho^m)\Big]_x.\end{aligned}$$

By the assumptions on the profile $\tilde{V}_2^{\rho,-}$ we know that $u_2 - u_\rho^m = \rho(\tilde{u}_2 - a)$, then the equation above is homogeneous in ρ. An entirely similar equation holds for the second component w_2.

To go further in our analysis is convenient now to consider an explicit example. The first case to take into consideration is $f(u,v) = uv$ and $g(u,v) = 0$. In this case the equations for W are

$$\begin{cases} w_{1,t} + (u_{1,x} + u_{2,x})w_1 + uw_{1,x} - w_{1,xx} = -[(u_1 - u_\rho^m)(v_2 - v_\rho^m)]_x - [(u_1 - u_\rho^m)(\tilde{u}_2 - a)]_x, \\ w_{2,t} + (v_{1,x} + v_{2,x})w_2 + vw_{2,x} - w_{2,xx} = -[(\tilde{v}_1 - v_\rho^m)(v_2 - v_\rho^m)]_x. \end{cases}$$

4.3. Global solutions $(-\infty < t < +\infty)$ for parabolic systems of conservation laws

Here we present a new proof of the result obtained be D. Serre in [43], on the existence of global solutions $(-\infty < t < +\infty)$ for parabolic systems of conservation laws. In [43] the proof for the vectorial case, the one we are dealing with in the present situation, was obtained by means of entropy estimates. The more recent result in [6], claiming that the system with artificial viscosity

$$U_t + A(U)\,U_x - \varepsilon U_{xx} = 0, \qquad U : I\!R^+ \times I\!R \to I\!R^n,$$

generates a Lipschitz continuous semigroup in $\mathbf{L}^1$, allows us to present here a proof which relies on the same ideas used by D. Serre in [43] for the scalar case.

Consider a system of the form

$$U_t + F(U)_x = U_{xx}, \qquad U : I\!R^+ \times I\!R \to I\!R^n, \tag{4.3.1}$$

with F smooth and such that $DF(U)$ is strictly hyperbolic for all possible U. The following theorem allow us to claim that, given two approaching viscous shock profiles, there exists a unique global solution of (4.3.1) which converges to their superposition as $t \to -\infty$.

Theorem 4.1 *Let V_1 and V_2 be viscous shock profiles connecting respectively the states (U^m, U^+) and (U^-, U^m), with speed σ_1 and σ_2. Assume that the two profiles are approaching, with $\sigma_1 < \sigma_2$. Choosing an intermediate speed $\sigma_1 < \gamma < \sigma_2$, and a function $m : I\!R \mapsto [0,1]$, of class $\mathcal{C}^\infty$, such that*

$$m(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 1, \end{cases}$$

we construct an approximate solution of (4.3.1)

$$\hat{U}(t,x) \doteq m(x-\gamma t)\, V_2(x-\sigma_2 t) + (1 - m(x-\gamma t))\, V_1(x-\sigma_1 t).$$

Then there exists a unique solution of (4.3.1), independent of the choice of γ and m, such that

$$\lim_{t \to -\infty} \|U(t,\cdot) - \hat{U}(t,\cdot)\|_{\mathbf{L}^1} = 0.$$

Proof. In order to fix the ideas we put $\gamma = (\sigma_1 + \sigma_2)/2$. Consider a decreasing sequence $t_n \to -\infty$. For each $n \geq 1$, let $U_n : [t_n, +\infty[\times I\!R \mapsto I\!R^2$ be the exact solution of the Cauchy problem given by (4.3.1) together with the initial data

$$U_n(t_n, x) = \hat{U}(t_n, x).$$

As $n \to \infty$, we claim that the sequence U_n converges to a solution U of (4.3.1) together with all of its derivatives, uniformly on compact sets of $I\!R^2$. Indeed, let $t_n < t_m < 0$. As proved in [6], the equation (4.3.1) generates a Lipschitz continuous semigroup, say with constant L, on a domain $\mathcal{D} \subset \mathbf{L}^1_{loc}(I\!R)$ consisting of functions with sufficiently small total variation. In particular, for any $t \in [t_m, +\infty[$, we have the estimate

$$\|U_m(t) - U_n(t)\|_{\mathbf{L}^1} \leq L \cdot \|U_m(t_m) - U_n(t_m)\|_{\mathbf{L}^1}. \tag{4.3.2}$$

Since U_n is an exact solution on $[t_n, t_m]$ and $U_n(t_n) = \hat{U}(t_n)$, the right hand side of (4.3.2) can be estimated as

$$\|U_n(t_m,\cdot) - U_m(t_m,\cdot)\|_{\mathbf{L}^1} \leq L \cdot \int_{t_n}^{t_m} \|\hat{U}(t,\cdot)_t + F(\hat{U}(t,\cdot))_x - \hat{U}(t,\cdot)_{xx}\|_{\mathbf{L}^1}\, dt.$$

Recalling that $V_1(x-\sigma_1 t)$ and $V_2(x-\sigma_2 t)$ are exact solutions, we obtain

$$\begin{aligned} |\hat{U}_t + F(\hat{U})_x - \hat{U}_{xx}| \;\leq\; & |\gamma m'(V_1 - V_2)| + \\ & + |F'(mV_2 + (1-m)V_1) \cdot m'(V_1 - V_2)| \\ & + |F'(mV_2 + (1-m)V_1) - F'(mV_2)|\,|mV_2'| \\ & + |F'(mV_2 + (1-m)V_1) - F'((1-m)V_1)|\,|(1-m)V_1'| \\ & + |m''(V_2 - V_1)| + 2|m'V_2'| + 2|m'V_1'|. \end{aligned} \tag{4.3.3}$$

Standard estimates on genuinely nonlinear viscous shock profiles yield

$$|V_2(\xi) - U^m| = \mathcal{O}(1)|U^m - U^-|e^{-\kappa|\xi|}, \quad |V_2'(\xi)| = \mathcal{O}(1)|U^m - U^-|^2 e^{-\kappa\xi}, \quad \text{as} \quad \xi \to +\infty\,,$$

$$|V_1(\xi) - U^m| = \mathcal{O}(1)|U^+ - U^m|e^{-\kappa|\xi|}, \quad |V_1'(\xi)| = \mathcal{O}(1)|U^+ - U^m|^2 e^{-\kappa|\xi|}, \quad \text{as} \quad \xi \to -\infty\,,$$

for some $\kappa > 0$ of the same order as $|U^m - U^-|$ and $|U^m - U^+|$. In addition, we recall that both m', m'' vanish outside the interval $[0,1]$. Inserting the above estimates in (4.3.3) we obtain

$$\begin{aligned}\int_{t_n}^{t_m} &\|\hat U(t,\cdot)_t + F(\hat U(t,\cdot))_x - \hat U(t,\cdot)_{xx}\|_{\mathbf{L}^1}\,dt\\ &=\mathcal{O}(1)\cdot\int_{t_n}^{t_m}\left(e^{\kappa|\gamma-\sigma_1|t} + e^{\kappa|\gamma-\sigma_2|t}\right)dt\\ &=\mathcal{O}(1)\cdot e^{\kappa|\sigma_1-\sigma_2|t_m/2}.\end{aligned} \tag{4.3.4}$$

As $t_m \to -\infty$, the right hand side of (4.3.4) approaches zero. Together, (4.3.2) and (4.3.4) imply that the sequence $U_n(t,\cdot)$ is Cauchy in $\mathbf{L}^1_{loc}(I\!\!R)$, for every time $t \in I\!\!R$. Therefore, it converges to a unique limit $U(t,\cdot)$, in the $\mathbf{L}^1$ distance. We now observe that the same sequence U_n is uniformly bounded in $\mathcal{C}^k(I\!\!R^2)$, for every integer $k \geq 1$. By an interpolation argument, we deduce the convergence $U_n \to U$ in $\mathcal{C}^k(\Omega)$ for every $k \geq 1$ and every bounded set $\Omega \subset I\!\!R^2$.

The same kind of estimate gives us the rate at which the solution U approaches the superposition of two viscous travelling profiles as $t \to -\infty$. We have

$$\|U(t,\cdot) - \hat U(t,\cdot)\|_{\mathbf{L}^1} \leq \int_{-\infty}^{t} \|\hat U(t,\cdot)_t + F(\hat U(t,\cdot))_x - \hat U(t,\cdot)_{xx}\|_{\mathbf{L}^1}\,dt = \mathcal{O}(1)e^{-\frac{\kappa|\sigma_1-\sigma_2||t|}{2}}.$$

We remark that here, in the definition of the approximate solution $\hat U$, we introduce a function, m, without physical meaning. For this reason in the previous section we chose to use a different approximation, $\tilde U$.

4.4. On the time dependence of the error

Consider a genuinely nonlinear hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \qquad u : [0,T] \times I\!\!R \to I\!\!R^n, \tag{4.4.1}$$

together with a viscous approximation

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon\, u^\varepsilon_{xx}.$$

For a solution u with small total variation, the convergence rate of u^ε to u, as $\varepsilon \to 0$, has been estimated in [13]. The result found there was

$$\|u^\varepsilon(T,\cdot) - u(T,\cdot)\|_{\mathbf{L}^1} \leq C\cdot(1+T)\,\text{Tot.Var.}\{\bar u\}\cdot\sqrt{\varepsilon}(1 + |\ln\varepsilon|)\,, \tag{4.4.2}$$

where the constant C depends only on the flux function f. In this section we show that the linear dependence on time on the right hand side is not optimal. The new estimate we find takes the form

$$\|u^\varepsilon(T,\cdot) - u(T,\cdot)\|_{\mathbf{L}^1} \leq C'\cdot\text{Tot.Var.}\{\bar u\}\cdot\sqrt{\varepsilon T}(1 + \ln T + |\ln\varepsilon|)\,.$$

Let $(\tau,\xi)=(t/\varepsilon',x/\varepsilon')$, and define the function $v:[0,\varepsilon'T]\times I\!R\to I\!R^n$, as

$$v(t,x)=u^\varepsilon(\frac{t}{\varepsilon'},\frac{x}{\varepsilon'}).$$

From this definition we have

$$v_x(t,x)=u^\varepsilon_\xi(\tau,\xi)\partial_x\xi=\frac{1}{\varepsilon'}u^\varepsilon_\xi(\tau,\xi),$$

$$v_t(t,x)=u^\varepsilon_\tau(\tau,\xi)\partial_t\tau=(-f(u^\varepsilon)_\xi+\varepsilon\, u^\varepsilon_{\xi\xi})\frac{1}{\varepsilon'}=(-f(v)_x\varepsilon'+\varepsilon(\varepsilon')^2v_{xx})\frac{1}{\varepsilon'},$$

then

$$v_t+f(v)_x=\varepsilon\varepsilon'v_{xx}\,. \tag{4.4.3}$$

We consider the distance

$$\|u^\varepsilon(T,\cdot)-u(T,\cdot)\|_{\mathbf{L}^1}=\int_{I\!R}|v(\varepsilon'T,\varepsilon'\xi)-\hat u(\varepsilon'T,\varepsilon'\xi)|\,d\xi=\frac{1}{\varepsilon'}\|v(\varepsilon'T,\cdot)-\hat u(\varepsilon'T,\cdot)\|_{\mathbf{L}^1}.$$

Given the rescaling property of equation (4.4.1), the function $\hat u(\varepsilon'\tau,\varepsilon'\xi)=u(\tau,\xi)$ is still a solution. Then from (4.4.2) and (4.4.3) we have

$$\int_{I\!R}|v(\varepsilon'T,\varepsilon'\xi)-\hat u(\varepsilon'T,\varepsilon'\xi)|\,d\xi=\frac{1}{\varepsilon'}C\cdot(1+\varepsilon'T)\,\mathrm{Tot.Var.}\{\bar u\}\cdot\sqrt{\varepsilon\varepsilon'}(1+|\ln\varepsilon\varepsilon'|)\,,$$

and we would like to optimize this estimate over the choice of ε'. We look for the critical points of the function $\mathcal{F}(\varepsilon')$,

$$\mathcal{F}(\varepsilon')=\frac{1}{\varepsilon'}(1+\varepsilon'T)\,\sqrt{\varepsilon\varepsilon'}(1+|\ln\varepsilon\varepsilon'|)\,.$$

Suppose $\varepsilon\varepsilon'<1$, so that $|\ln\varepsilon\varepsilon'|=-\ln\varepsilon\varepsilon'$.

$$\begin{aligned}\mathcal{F}'(\varepsilon')=&\frac{1}{\varepsilon'^2}\Big\{-(1+\varepsilon'T)\,\sqrt{\varepsilon\varepsilon'}(1+|\ln\varepsilon\varepsilon'|)+\varepsilon'\Big[T\sqrt{\varepsilon\varepsilon'}(1+|\ln\varepsilon\varepsilon'|)+\\&+\frac{\varepsilon(1+\varepsilon'T)}{2\sqrt{\varepsilon\varepsilon'}}(1+|\ln\varepsilon\varepsilon'|)-\frac{\varepsilon}{\sqrt{\varepsilon\varepsilon'}}(1+\varepsilon'T)\Big]\Big\}=\\=&\frac{1}{\varepsilon\varepsilon'^2}\Big[-(\varepsilon+\varepsilon\varepsilon'T)\,\sqrt{\varepsilon\varepsilon'}(1+|\ln\varepsilon\varepsilon'|)+T(\varepsilon\varepsilon')^{3/2}(1+|\ln\varepsilon\varepsilon'|)+\\&+\frac{1}{2}(\varepsilon+\varepsilon\varepsilon'T)\sqrt{\varepsilon\varepsilon'}(1+|\ln\varepsilon\varepsilon'|)-\sqrt{\varepsilon\varepsilon'}(\varepsilon+\varepsilon\varepsilon'T)\Big]=\\=&\frac{1}{\varepsilon\varepsilon'^2}\Big[(\varepsilon+\varepsilon\varepsilon'T)\,\sqrt{\varepsilon\varepsilon'}\Big(-1-|\ln\varepsilon\varepsilon'|+\frac{1}{2}+\frac{|\ln\varepsilon\varepsilon'|}{2}-1\Big)+T(\varepsilon\varepsilon')^{3/2}(1+|\ln\varepsilon\varepsilon'|)\Big]=\\=&\frac{\sqrt{\varepsilon\varepsilon'}}{2\varepsilon\varepsilon'^2}\Big[|\ln\varepsilon\varepsilon'|(\varepsilon\varepsilon'T-\varepsilon)-(3\varepsilon+\varepsilon\varepsilon'T)\Big].\end{aligned}$$

Call $y=\varepsilon\varepsilon'$ and assume $y<1$. We need to find the zeros of the function

$$G(y)=|\ln y|(yT-\varepsilon)-(3\varepsilon+yT).$$

We notice that

$$|\ln y| = \frac{3\varepsilon + yT}{yT - \varepsilon}, \tag{4.4.4}$$

can be satisfied only if $y > \varepsilon/T$. We call $g(y)$ the function on the right hand side of (4.4.4). The function $g(y)$ has a vertical asymptote for $y = \varepsilon/T$, is convex for $y > \varepsilon/T$ and tends to 1 from above as $y \to +\infty$. We observe that for $y = \varepsilon/T$ and $y = 1$, $g(y) > |\ln y|$. We can find critical values for $G(y)$ only if

$$|\ln \bar{y}| \geq g(\bar{y}), \tag{4.4.5}$$

at the point $\bar{y}$ where $|\ln y|$ and $g(y)$ have the same derivative. We have

$$-\frac{1}{\bar{y}} = -\frac{4\varepsilon T}{(T\bar{y} - \varepsilon)^2},$$

for $\bar{y}_{1,2} = (3 \pm 2\sqrt{2})\varepsilon/T$. Since one of the two values is smaller than ε/T we just take into account the other one and call it $\bar{y}$. Inequality (4.4.5) writes explicitly

$$\left| \ln(3 + 2\sqrt{2})\frac{\varepsilon}{T} \right| \geq \frac{3 + \sqrt{2}}{1 + \sqrt{2}},$$

and from it we obtain a condition on the ratio ε/T

$$\frac{\varepsilon}{T} \leq \frac{e^{-\frac{3+\sqrt{2}}{1+\sqrt{2}}}}{3 + 2\sqrt{2}}.$$

If the inequality is strict we expect to find two critical points for $G(y)$, $y_1^\star, y_2^\star$, while if the equality holds the only critical point will be $\bar{y}$. In the first case we have

$$y_1^\star \in \left(\frac{\varepsilon}{T}, \frac{(3 + 2\sqrt{2})\varepsilon}{T} \right), \qquad y_2^\star \in \left(\frac{(3 + 2\sqrt{2})\varepsilon}{T}, e^{-1} \right),$$

where $y_1^\star$ is a local minimum and $y_2^\star$ a local maximum, since

$$G(\varepsilon/T) = -2\varepsilon < 0, \quad G\Big((3 + 2\sqrt{2})\varepsilon/T\Big) \geq 0, \quad G(e^{-1}) = -4\varepsilon < 0.$$

In the second case the derivative of $\mathcal{F}$ is always negative except in $\bar{y}$, then $\bar{y}$ is a saddle point. We are only interested in minima, then from now on we assume the inequality in (4.4.5) to be strict.

As a consequence the critical value of ε' we were looking for lays in the interval $(1/T, (3 + 2\sqrt{2})/T)$, we will call it $\varepsilon^\star = a/T$ and there holds

$$\int_{I\!R} |v(\varepsilon^\star T, \varepsilon^\star \xi) - u(\varepsilon^\star T, \varepsilon^\star \xi)|\, d\xi = C \cdot \frac{(1 + a)}{a} \,\text{Tot.Var.}\{\bar{u}\} \cdot \sqrt{\varepsilon T} \left(1 + \left| \ln \frac{\varepsilon a}{T} \right| \right) =$$

$$= C \cdot \frac{(1 + a)}{a} \,\text{Tot.Var.}\{\bar{u}\} \cdot \sqrt{\varepsilon T} \left(1 + \ln \frac{T}{a} + |\ln \varepsilon| \right).$$

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