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Laboratoire Paul Painlevé
UNIVERSITÉ LILLE 1

Some topics in the geometry of framed sheaves and their moduli spaces

Supervisors
Prof. Ugo Bruzzo
Prof. Dimitri Markushevich

Candidate
Francesco Sala

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Contents

Conventions

Let E be a coherent sheaf on a Noetherian scheme Y. The *support of* E is the closed set $\mathrm{Supp}(E) := \{x \in Y \mid E_x \neq 0\}$. Its dimension is called the *dimension of* E and is denoted by $\dim(E)$. The sheaf E is *pure* if for all nontrivial coherent subsheaves $E' \subset E$, we have $\dim(E') = \dim(E)$. Let us denote by $T(E)$ the *torsion subsheaf* of E, i.e., the maximal subsheaf of E of dimension less or equal to $\dim(E) - 1$.

Let Y be a projective scheme over a field. Recall that the Euler characteristic of a coherent sheaf E is $\chi(E) := \sum_i (-1)^i \dim \mathrm{H}^i(Y, E)$. Fix an ample line bundle $\mathcal{O}(1)$ on Y. Let $P(E, n) := \chi(E \otimes \mathcal{O}(n))$ be the *Hilbert polynomial of* E and $\det(E)$ its *determinant line bundle* (cf. Section 1.1.17 in [**35**]). The *degree of* E, $\deg(E)$, is the integer $c_1(\det(E)) \cdot H^{d-1}$, where $H \in |\mathcal{O}(1)|$ is a hyperplane section.

By Lemma 1.2.1 in [**35**], the Hilbert polynomial $P(E)$ can be uniquely written in the form

$$P(E, n) = \sum_{i=0}^{\dim(E)} \beta_i(E) \frac{n^i}{i!},$$

where $\beta_i(E)$ are rational coefficients. Moreover for $E \neq 0$, $\beta_{\dim(E)}(E) > 0$.

Let $Y \to S$ be a morphism of finite type of Noetherian schemes. If $T \to S$ is an S-scheme, we denote by Y_T the fibre product $T \times_S Y$ and by $p_T \colon Y_T \to T$ and $p_Y \colon Y_T \to Y$ the natural projections. If E is a coherent sheaf on Y, we denote by E_T its pull-back to Y_T. For $s \in S$ we denote by Y_s the fibre $\mathrm{Spec}(k(s)) \times_S Y$. For a coherent sheaf E on Y, we denote by E_s its pull-back to Y_s. Often, we shall think of E as a collection of sheaves E_s parametrized by $s \in S$.

Whenever a scheme has a base field, we assume that the latter is an algebraically closed field k of characteristic zero.

A *polarized variety of dimension* d is a pair $(X, \mathcal{O}_X(1))$, where X is a nonsingular, projective, irreducible variety of dimension d, defined over k, and $\mathcal{O}_X(1)$ a very ample line bundle. The *canonical line bundle of* X is denoted by ω_X and its associated divisor by K_X.

Let E be a coherent sheaf on X. By Hirzebruch-Riemann-Roch theorem the coefficients of the Hilbert polynomial of E are polynomial functions of its Chern classes, in particular

$$P(E, n) = \deg(X)\mathrm{rk}(E)\frac{n^d}{d!} + \left(\deg(E) - \mathrm{rk}(E)\frac{\deg(\omega_X)}{2}\right)\frac{n^{d-1}}{(d-1)!} + \text{terms of lower order in } n. \qquad (1)$$

CHAPTER 1

Introduction

1. Historical background

This dissertation is primarily concerned with the study of *framed sheaves* on nonsingular projective varieties and the geometrical properties of the moduli spaces of these objects. In particular, we deal with a generalization to the framed case of known results for (semi)stable torsion free sheaves, such as *(relative) Harder-Narasimhan filtration*, *Mehta-Ramanathan restriction theorems*, *Uhlenbeck-Donaldson compactification*, *Atiyah class* and *Kodaira-Spencer map*. The main motivations for the study of these moduli spaces come from physics, in particular, *gauge theory*, as we shall explain in the following.

Gauge theory and instantons. There have been over the last 30 years remarkable instances where physical theories provided a formidable input to mathematicians, offering the stimulus to the creation of new mathematical theories, and supplying strong evidence for highly nontrivial theorems. An example of this kind of interaction between mathematics and physics is gauge theory. A first example of gauge theory can be found in the electromagnetic theory, in particular *Maxwell equations*. The fields entering the Maxwell equations, the electric and magnetic fields, may be written in a suitable way as derivatives of two potentials, the scalar and the vector potential. However, these potentials are defined up to a suitable combination of the derivatives of another scalar field; this is the *gauge invariance* of electromagnetism. Now, the essence of gauge theory, from the physical viewpoint, is that this gauge invariance dictates the way matter interacts via the electromagnetic fields.

The first workable gauge theory after electromagnetism is Yang-Mills theory (see [**80**], for a more general approach see also [**78**]). However gauge theory entered the mathematical scene only when it was realized that a gauge field may be interpreted as a connection on a fibre bundle. In a modern mathematical formulation, the gauge potential A is described as a connection on a principal G-bundle P defined over a four-dimensional (Euclidean) space-time X. In the physics literature, the Lie group G is called *gauge group*. In the absence of matter fields the Lagrangian of the theory is proportional to the L^2 norm of the curvature, or strength field, $Tr(F_A \wedge F_A^*)$, thus yielding nonlinear second-order ODEs as equations of motion for the potential (*YM equations*, from Yang-Mills). A remarkable breakthrough in solving $SU(2)$ YM equations, hence finding non trivial vacuum states of the theory, came in [**7**]. The *pseudoparticle solutions* (or *instantons*), introduced there, correspond to Hodge anti-selfdual (ASD) connections A whose curvature satisfies $F_A^* = -F_A$, and are classified by the instanton number n, geometrically identified with the second Chern number of the bundle $n = c_2(P)$. The original physical theory is defined over $X = \mathbb{R}^4$, but the requirement to consider only *finite energy fields* translates into working with bundles over S^4. We restrict our attention to the case of G equals to $SU(r)$.

In [**4**], Atiyah and Ward study in details $SU(2)$-instantons on S^4, and in particular they prove the existence of a correspondence between gauge-equivalence classes of instantons on S^4 and isomorphism classes of locally free sheaves on $\mathbb{CP}^3$ satisfying some properties. These locally free sheaves are characterized by some cohomological properties and in the literature they are called *mathematical instantons*. The geometrical properties of moduli spaces of *mathematical instantons* are widely studied in algebraic geometry (see, e. g., [**30**] , [**31**], in which Hartshorne studies mathematical instantons as a first step to a full understanding of the geometrical properties of locally free sheaves on projective spaces; for the irreducibility of the moduli spaces see, e.g., [**6**], [**21**]; for the smoothness of the moduli spaces see, e.g., [**43**], [**17**]).

Independently, Drinfeld and Manin ([**20**]) and Atiyah ([**2**]) prove the existence of the same correspondence for $SU(r)$-instantons on S^4. The existence of such a correspondence between these *analytical* objects and *algebro-geometric* objects is due to the fact that one can always associate to a $SU(r)$-instanton some linear data. This idea is well-explained in the article [**3**], in which the authors prove that one can associate to an $SU(r)$-instanton of charge n on S^4 the linear datum (B_1, B_2, i, j), where $B_1, B_2 \in \mathrm{End}(\mathbb{C}^n)$, $i \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ and $j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$, satisfying the following properties

(i) $[A, B] + ij = 0$,
(ii) there exists no proper subspace $V \subsetneq \mathbb{C}^n$ such that $B_i(V) \subseteq V$ for $i = 1, 2$ and $\mathrm{Im}\, i \subset V$ (*stability condition*),
(iii) there exists no nonzero subspace $W \subset \mathbb{C}^n$ such that $B_i(W) \subseteq W$ for $i = 1, 2$ and $W \subset \ker j$ (*costability condition*).

In the literature, (B_1, B_2, i, j) is called an *ADHM datum*. Moreover, two gauge-equivalence instantons correspond to two ADHM data (B_1, B_2, i, j) and (B_1', B_2', i', j') equivalent with respect to the following relation: for $g \in GL(\mathbb{C}, n)$, (B_1, B_2, i, j) and (B_1', B_2', i', j') are equivalent if and only if

$$gB_ig^{-1} = B_i' \text{ for } i = 1, 2;\ gi = i';\ jg^{-1} = j'. \tag{2}$$

Framed instantons. One can also define the so-called *framed instantons*. In the principal bundle picture, these are pairs (A, ϕ) where A is an anti-selfdual connection on a principal $SU(r)$-bundle P on X, and ϕ is a point in the fibre P_x over a fixed point $x \in X$, i.e., a "frame". Correspondingly, one restricts to considering gauge transformations that fix the frame. The framing has a meaning in physical theories: an instanton is invariant with respect to *global rotations* of $\mathbb{R}^4$, on the other hand a framed instanton is invariant only with respect to *local rotations*. In the supersymmetric setting, this means that while the moduli space parametrizing $SU(r)$-instantons with charge n represents the space of classical vacua of a quantized gauge theory, the framing has the meaning of a vacuum expectation value of some fields (technically, the scalar fields in the $\mathcal{N} = 2$ vector multiplet).

In [**18**], by using Atiyah-Ward correspondence, Donaldson proves that gauge-equivalence classes of framed $SU(r)$-instantons with instanton number n on S^4 are in one-to-one correspondence with isomorphism classes of locally free sheaves on $\mathbb{CP}^2$ of rank r and second Chern class n that are trivial along a fixed line l_∞, and have a fixed trivialization there. Moreover these objects can be described by ADHM data.

Afterwards, King proves the same correspondence between framed $SU(r)$-instantons on the projective plane with reverse orientation $\overline{\mathbb{CP}^2}$ and locally free sheaves on the blow up of $\mathbb{CP}^2$ at a point, trivial along a fixed line and with a fixed trivialization there ([**38**]). Buchdahl generalizes this result for framed $SU(r)$-instantons on the connected sum of n copies of $\mathbb{CP}^2$ and locally free sheaves on the blow up of $\mathbb{CP}^2$ at n points, trivial along a fixed line and with a fixed trivialization there ([**16**]). Moreover, Nakajima proves a similar correspondence for framed $SU(r)$-instantons on the so-called *Asymptotically Locally Euclidean spaces* ([**61**]). This kind of correspondences provides some first tools to translate problems coming from gauge theories into a mathematical language.

If we consider a datum (B_1, B_2, i, j) satisfying only the conditions (i) and (ii) written above, one can prove that it corresponds to a pair (E, α), where E is a torsion free sheaf on $\mathbb{CP}^2$, locally trivial in a neighborhood of a fixed line l_∞, and α is an isomorphism $E|_{l_\infty} \xrightarrow{\sim} O_{l_\infty}^{\oplus r}$. In the literature these objects are called *framed sheaves* on $\mathbb{CP}^2$ and α *framing at infinity*. Moreover, equivalence classes of (B_1, B_2, i, j), with respect to the relation (2), correspond to isomorphism classes of framed sheaves on $\mathbb{CP}^2$. Thus, by using these linear data, we construct a moduli space $M(r, n)$ that parametrizes isomorphism classes of framed sheaves (E, α) on $\mathbb{CP}^2$ with E of rank r and second Chern class n, that is, a nonsingular quasi-projective variety of dimension $2rn$. Moreover the open subset $M^{reg}(r, n)$ consisting of isomorphism classes of *framed vector bundles*, i.e., framed sheaves (E, α) with E locally free, is isomorphic to the moduli space of gauge-equivalence classes of framed $SU(r)$-instantons of charge n on S^4, by Donaldson's result. In some sense we can look at $M(r, n)$ as a partial compactification of $M^{reg}(r, n)$. A detailed explanation of this construction of $M(r, n)$ is in Chapter 2 in [**60**]. In Chapter 3 of the same book, Nakajima explains another way to construct $M(r, n)$ as the hyper-Kähler quotient

$$M(r,n) = \{(B_1, B_2, i, j) \,|\, \text{condition (i) holds, } [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j = \zeta \mathrm{id}\} / U(n),$$

where $(\cdot)^\dagger$ is the Hermitian adjoint and ζ is a fixed positive real number. Moreover, Nakajima constructs another type of partial compactification $M^{Uh}(r, n)$ of $M^{reg}(r, n)$, called *Uhlenbeck-Donaldson compactification*, as the affine algebro-geometric quotient

$$M^{Uh}(r,n) := \{(B_1, B_2, i, j) \,|\, [B_1, B_2] + ij = 0\} // GL(\mathbb{C}, n).$$

By these descriptions via linear data, we have a projective morphism

$$\pi_r \colon M(r,n) \to M^{Uh}(r,n), \tag{3}$$

such that the restriction to the *locally free part* is an isomorphism with its image.

In [**63**], Nakajima and Yoshioka conjecture that, by using Uhlenbeck-Donaldson's theory of ideal $SU(r)$-instantons on S^4 (see Section 4.4 in [**19**]), one can give a topology to the set

$$\coprod_{i=0}^{n} M^{reg}(r, n-i) \times Sym^i(\mathbb{C}^2)$$

and prove that this latter space is homeomorphic to $M^{Uh}(r, n)$. Moreover, in analogy with the construction of the Uhlenbeck-Donaldson compactification for μ-stable locally free sheaves on nonsingular projective surfaces (see [**47**], [**55**] for the rank two case, [**48**] and [**35**], Section 8.2, for the general case), Nakajima and Yoshioka conjecture that sheaf-theoretically the morphism π_r is

$$[(E, \alpha)] \in M(r,n) \xmapsto{\pi_r} \left([(E^{\vee\vee}, \alpha)], \operatorname{supp}\left(E^{\vee\vee}/E\right)\right) \in M^{reg}(r, n-i) \times Sym^i(\mathbb{C}^2) \subset M^{Uh}(r,n),$$

where we denote by $E^{\vee\vee}$ the double dual of E, $\operatorname{supp}({}^{E^{\vee\vee}}/_{E})$ is the support of the zero-dimensional sheaf ${}^{E^{\vee\vee}}/_{E}$ counted with multiplicities and i the length of it.

Instanton counting. In 1994 N. Seiberg and E. Witten ([**71**], [**72**]) state an ansatz for the exact prepotential of $\mathcal{N}=2$ Yang-Mills theory in four dimensions with gauge group $SU(2)$. This solution has been extended to $SU(r)$ and to theories with matter, has been rederived in the context of string theory. One challenge has been to find a field theory derivation of this result, or at least to verify it with usual quantum field theory methods. The Seiberg-Witten prepotential can be computed with instanton calculus, and there has been much work devoted to developing this calculus in order to test their ansatz. Unfortunately, it has been difficult to obtain explicit results beyond instanton number two due to the complexity of the ADHM construction. One of the results of the research on instanton calculus has been, however, that the coefficients of the Seiberg-Witten prepotential can be computed as the *equivariant integral* of an equivariant differential form on the moduli space of framed instantons. In [**64**], Nekrasov produces explicit formulae for the Seiberg-Witten prepotential for gauge group $SU(r)$ and general matter content. Moduli spaces of framed sheaves on $\mathbb{CP}^2$ represent the natural ambient spaces on which one computes these integrals. More precisely, let us fix $l_\infty = \{[z_0 : z_1 : z_2] \mid z_0 = 0\}$. Let T_e be the maximal torus of $\mathrm{GL}(\mathbb{C}, r)$ consisting of diagonal matrices and let $T := \mathbb{C}^* \times \mathbb{C}^* \times T_e$. We define an action on $M(r,n)$ as follows: for $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$, let F_{t_1,t_2} be the automorphism on $\mathbb{P}^2$ defined as

$$F_{t_1,t_2}([z_0 : z_1 : z_2]) := [z_0 : t_1 z_1 : t_2 z_2].$$

For $\operatorname{diag}(e_1, e_2, \dots, e_r) \in T_e$, let $G_{(e_1,\dots,e_r)}$ denote the isomorphism of $\mathcal{O}_{l_\infty}^{\oplus r}$ given (locally) by $(s_1, \dots, s_r) \mapsto (e_1 s_1, \dots, e_r s_r)$. Then for a point $[(E,\alpha)] \in M(r,n)$ we define

$$(t_1, t_2, e_1, \dots e_r) \cdot [(E,\alpha)] := [((F_{t_1,t_2}^{-1})^*(E), \alpha')],$$

where α' is the composite of morphisms

$$(F_{t_1,t_2}^{-1})^*(E)|_{l_\infty} \xrightarrow{(F_{t_1,t_2}^{-1})^*(\alpha|_{l_\infty})} (F_{t_1,t_2}^{-1})^*(\mathcal{O}_{l_\infty}^{\oplus r}) \longrightarrow \mathcal{O}_{l_\infty}^{\oplus r} \xrightarrow{G_{(e_1,\dots,e_r)}} \mathcal{O}_{l_\infty}^{\oplus r}$$

where the middle arrow is the morphism given by the action.

For $k = 1, \dots, r$, let e_k be the 1-dimensional T-module given by

$$(t_1, t_2, e_1, \dots, e_r) \mapsto e_k.$$

In the same way consider the 1-dimensional T-modules t_1, t_2. Let $\varepsilon_1, \varepsilon_2$ and a_k be the second Chern classes of t_1, t_2 and e_k, $k = 1, 2, \dots r$. Thus the T-equivariant cohomology of a point is $\mathbb{C}[\varepsilon_1, \varepsilon_2, a_1, \dots, a_r]$. From a geometric viewpoint, Nekrasov's partition function (or, more precisely, its *instanton part*) is the generating function

$$Z(\varepsilon_1, \varepsilon_2, a_1, \dots, a_r; q) := \sum_{n=0}^{\infty} q^n \int_{M(r,n)} 1,$$

where 1 is the equivariant fundamental class of $\mathrm{H}_T^*(M(r,n))$. Hence it is a function of the equivariant parameters $\varepsilon_1, \varepsilon_2, a_1, \dots, a_r$ and a formal variable q. In the case of gauge theories with *masses*, one can define Nekrasov's partition function as the generating function of the integral of an equivariant cohomology class depending on the equivariant parameters and the masses. Actually, the moduli space is not compact (it is only quasi-projective) and therefore, strictly speaking, the integral is not defined. However one can formally apply the localization formula in equivariant cohomology, and the resulting expression is a rational function in the equivariant parameters. Nekrasov's partition function is explicitly computed in [**13**] for framed sheaves on $\mathbb{CP}^2$. Nakajima and Yoshioka computed Nekrasov's partition formula for the blow-up of $\mathbb{CP}^2$ at a point (see [**63**], [**62**]). A general computation for toric surfaces is given in [**24**]. These computations are done by looking at the finite set of fixed points of the toric action on the moduli space. The tangent spaces at the fixed points provide representations of the acting torus, and one can compute the characters of the representations. This allows one to compute the "right-hand side" of the localization formula, and therefore, to compute Nekrasov's partition function. The identification of the fixed points, and the calculations of the characters, is done with some combinatorial computations, using Young tableaux. This is what is meant (at least by mathematicians) by *instanton counting*.

Moduli spaces of framed sheaves on nonsingular projective varieties. One can generalize the notion of framed sheaves on $\mathbb{CP}^2$ to a nonsingular projective variety. Let X be a nonsingular projective variety, $D \subset X$ an effective divisor and F_D a locally free sheaf on it.

Definition 1.1. A *framed vector bundle* on X is a pair (E, α) where E is a locally free sheaf on X, locally free on a neighborhood of D, and α is an isomorphism $E|_D \xrightarrow{\sim} F_D$. We call α *framing* and F_D *framing sheaf*.

For arbitrary D and F_D, the family of framed vector bundles is *too big*, hence we have to choose *good* D and F_D to *restrict* it.

Definition 1.2. An effective divisor D on X is called *a good framing divisor* if we can write $D = \sum_i n_i D_i$, where D_i are prime divisors and $n_i > 0$, and there exists a nef and big divisor of the form $\sum_i a_i D_i$, with $a_i \geq 0$. For a coherent sheaf F on X supported on D, we shall say that F is a *good framing sheaf on* D, if it is locally free of rank r and there exists a real number A_0, $0 \leq A_0 < \frac{1}{r} D^2$, such that for any locally free subsheaf $F' \subset F$ of constant positive rank, $\frac{1}{\mathrm{rk}(F')} \deg(F') \leq \frac{1}{\mathrm{rk}(F)} \deg(F) + A_0$.

Definition 1.3. The framing sheaf F_D is *simplifying* if for any two framed vector bundles (E, α) and (E', α') on X, the group $\mathrm{H}^0(X, \mathcal{H}om(E, E')(-D))$ vanishes.

In [**46**], Lehn proves that if the divisor D is good and the framing sheaf F_D is good and simplifying, there exists a fine moduli space of framed vector bundles on X in the category of separated algebraic spaces. In [**49**], Lübke proves a similar result: if X is a compact complex manifold, D a smooth hypersurface (not necessarily "good") and if F_D is simplifying, then the moduli space of framed vector bundles on X exists as a Hausdorff complex space.

From now on, let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d. Let F be a coherent sheaf on X. In [**33**] and [**34**] Huybrechts and Lehn generalize the previous definition of framed sheaves, in the following way:

Definition 1.4. A *framed module*[1] on X is a pair $\mathcal{E} := (E, \alpha)$, where E is a coherent sheaf on X and $\alpha \colon E \to F$ is a morphism of coherent sheaves. We call α *framing* of E.

Huybrechts and Lehn define a generalization of Gieseker semistability (resp. μ-semistability) for framed sheaves that depends on a rational polynomial, that we call *stability polynomial*.

[1] In [**33**], Huybrechts and Lehn call this object *stable pair*.

More precisely, let δ be a rational polynomial of degree $d-1$ with positive leading coefficient δ_1.

Definition 1.5. A framed module (E,α) of positive rank is said to be *(semi)stable* with respect to δ if and only if the following conditions are satisfied:

(i) $\mathrm{rk}(E)P(E')\ (\leq)\ \mathrm{rk}(E')(P(E)-\delta)$ for all subsheaves $E' \subset \ker\alpha$,
(ii) $\mathrm{rk}(E)(P(E')-\delta)\ (\leq)\ \mathrm{rk}(E')(P(E)-\delta)$ for all subsheaves $E' \subset E$.

Definition 1.6. A framed module (E,α) of positive rank is μ-(semi)stable with respect to δ_1 if and only if $\ker\alpha$ is torsion free and the following conditions are satisfied:

(i) $\mathrm{rk}(E)\deg(E')\ (\leq)\ \mathrm{rk}(E')(\deg(E)-\delta_1)$ for all subsheaves $E' \subset \ker\alpha$,
(ii) $\mathrm{rk}(E)(\deg(E')-\delta_1)\ (\leq)\ \mathrm{rk}(E')(\deg(E)-\delta_1)$ for all subsheaves $E' \subset E$ with $\mathrm{rk}(E') < \mathrm{rk}(E)$.

One has the usual implications among different stability properties of a framed module of positive rank:

$$\mu-\text{stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu-\text{semistable}.$$

Let us denote by $\underline{\mathcal{M}}^{ss}_{\delta}(X;F,P)$ (resp. $\underline{\mathcal{M}}^{s}_{\delta}(X;F,P)$) the contravariant functor from the category of Noetherian k-schemes of finite type to the category of sets, that associates to every scheme T the set of isomorphism classes of families of semistable (resp. stable) framed sheaves on X with Hilbert polynomial P parametrized by T. The main result in Huybrechts and Lehn's papers is the following:

Theorem 1.7. *There exists a projective scheme $\mathcal{M}^{ss}_{\delta}(X;F,P)$ that corepresents the functor $\underline{\mathcal{M}}^{ss}_{\delta}(X;F,P)$. Moreover there is an open subscheme $\mathcal{M}^{s}_{\delta}(X;F,P)$ of $\mathcal{M}^{ss}_{\delta}(X;F,P)$ that represents the functor $\underline{\mathcal{M}}^{s}_{\delta}(X;F,P)$, i.e., $\mathcal{M}^{s}_{\delta}(X;F,P)$ is a fine moduli space for stable framed sheaves.*

The theory developed by Huybrechts and Lehn covers not only the case of framed vector bundles à la Lehn, but also other kinds of additional structures on coherent sheaves. Let $F=\mathcal{O}_X$ and consider the framed module $(E,\alpha\colon E\to F)$ with E a locally free sheaf. By dualizing, one get a locally free sheaf $G=E^{\vee}$ together with a morphism $\phi=\alpha^{\vee}\colon \mathcal{O}_X\to G$, hence a pair $(G,\phi\in \mathrm{H}^0(X,G))$. In the literature this object is called a *Higgs pair* on X. Higgs pairs yield solutions of so-called *vortex equations* (see, e.g., [**11**], [**12**],[**22**], [**23**]). Moreover, the stability condition for Higgs pairs (see [**8**], [**75**]) coincides with the stability condition above.

Moduli spaces of framed sheaves on nonsingular projective surfaces. There is another way to extend the original definition of framed sheaves on $\mathbb{CP}^2$ with framing along a fixed line to arbitrary nonsingular projective surfaces. Huybrechts and Lehn's theory provides new tools for constructing moduli spaces of framed sheaves on nonsingular projective surfaces.

Let X be a nonsingular projective surface over $\mathbb{C}$, D a big and nef curve and F_D a good framing sheaf on it.

Definition 1.8. A *framed sheaf* on X is a pair (E,α) where E is a torsion free sheaf, E is locally free in a neighborhood of D and α is a morphism from E to F_D such that $\alpha|_D$ is an isomorphism.

In [**14**], Bruzzo and Markushevich prove the following result.

Theorem 1.9. *There exists a very ample line bundle on X and a positive rational number δ_1 such that every framed sheaf on X is a μ-stable framed module with respect to δ_1. In particular there exists a fine moduli space $\mathcal{M}^*(X;F_D,P)$ for framed sheaves on X with fixed Hilbert polynomial P, that is, an open subscheme of $\mathcal{M}^s_\delta(X;F_D,P)$, for any rational polynomial $\delta(n)=\delta_1 n+\delta_0$.*

This result improves and extends to torsion free sheaves Lehn's result for framed vector bundles. Moreover, if D is a smooth connected curve of genus zero with positive self-intersection and F_D is a Gieseker-semistable coherent $\mathcal{O}_D$-module, then the moduli space of framed sheaves on X is a nonsingular quasi-projective variety.

There are other results about the construction of moduli spaces of framed sheaves on nonsingular projective surfaces over $\mathbb{C}$ in which Huybrechts and Lehn's theory of framed modules is not used. For example, in [**65**], Nevins proves that if D is a smooth connected curve with positive self-intersection and F_D is a semistable locally free sheaf, there exists a moduli space of framed sheaves on X, that is a scheme. On the other hand, for the blowup of $\mathbb{CP}^2$ at a finite number of points and for Hirzebruch surfaces, there are constructions of moduli spaces of framed sheaves using some generalizations of the ADHM data (see, respectively, [**32**], [**68**]).

Symplectic structures on moduli spaces of framed sheaves. Let l_∞ be a line in the complex projective plane $\mathbb{CP}^2$. As described in Chapter 3 of Nakajima's book [**60**], the moduli space $M(r,n)$ of framed sheaves on $\mathbb{CP}^2$ of rank r and second Chern class n is a hyper-Kähler quotient. On the other hand it is possible to define a hyper-Kähler structure by using the theory of $SU(r)$-framed instantons. It was proved by Kronheimer and Nakajima ([**40**]), and by Maciocia ([**51**]) that these two structures are isomorphic. By fixing a complex structure on $M(r,n)$, the hyper-Kähler structure induces a holomorphic symplectic form on $M(r,n)$.

Leaving aside these results for framed sheaves on $\mathbb{CP}^2$, the only relevant result in the literature for framed sheaves on arbitrary nonsingular projective surfaces is due to Bottacin (see [**10**]). Let X be a complex nonsingular projective surface, D an effective divisor such that $D=\sum_{i=1}^n C_i$, where C_i is an integral curve for $i=1,\ldots,n$, and F_D a locally free $\mathcal{O}_D$-module. Fix a Hilbert polynomial P. Bottacin constructs Poisson brackets on the moduli space $\mathcal{M}^*_{lf}(X;F_D,P)$ of framed vector bundles on X, induced by global sections of the line bundle $\omega_X^{-1}(-2\,D)$. In particular, when X is the complex projective plane, $D=l_\infty$ and F_D the trivial vector bundle of rank r on l_∞, he provides a symplectic structure on the moduli space $M^{reg}(r,n)$ of framed vector bundles on $\mathbb{CP}^2$, induced by the standard holomorphic symplectic structure of $\mathbb{C}^2=\mathbb{CP}^2\setminus l_\infty$. It is not known if this symplectic structure is equivalent to that given by the ADHM construction.

Bottacin's result can be seen as a generalization to the framed case of the construction of Poisson brackets and holomorphic symplectic two-forms on the moduli spaces of Gieseker-stable torsion free sheaves on X. We recall briefly the main results for torsion free sheaves. In [**56**], Mukai proved that any moduli space of simple sheaves on a $K3$ surface or abelian surface has a non-degenerate holomorphic two-form. Its closedness was proved later independently by

Mukai ([**57**]), O'Grady ([**66**]) and Ran ([**67**]) for vector bundles, by Bottacin ([**9**]) for Gieseker-stable torsion free sheaves. Mukai's result was extended by Tyurin ([**76**]) to moduli spaces of Gieseker-stable vector bundles over surfaces of general type and over Poisson surfaces; a more thorough study of the Poisson case was accomplished by Bottacin in [**9**]. In all these situations, the symplectic two-form is defined in terms of the Atiyah class. Moreover, the two-form or a Poisson bivector on the moduli space of Gieseker-stable torsion free sheaves is induced by the one on the base space of the sheaves.

2. My work

Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d, F a coherent sheaf on X and δ a rational polynomial of degree $d-1$ and positive leading coefficient δ_1. Leaving aside the results on the representability of the moduli functor $\underline{\mathcal{M}}_\delta^{(s)s}(X; F, P)$ discussed, a complete theory of framed modules and a study of the geometry of their moduli spaces is missing in the literature.

From now on we call *framed sheaves* Huybrechts and Lehn's framed modules, (D, F)-*framed sheaves* the pairs (E, α) where E is a coherent sheaf on a nonsingular projective variety X, locally free in a neighborhood of a divisor D, and α is a isomorphism $E|_D \xrightarrow{\sim} F$, where F is a locally free $\mathcal{O}_D$-module, and (D, F)-*framed vector bundles* the (D, F)-framed sheaves in which the underlying coherent sheaf is locally free.

In this thesis we provide a complete study of the properties of the (μ)-(semi)stability conditions for framed sheaves and their behaviour with respect to restrictions to hypersurfaces of X. Moreover, we extend to (D, F)-framed sheaves the notion of *Atiyah class* and generalize the definition of *Kodaira-Spencer map* and the construction of closed two-forms via the Atiyah class.

Even if we want to work with torsion free framed sheaves, torsion may appear in the graded objects of the Harder-Narasimhan and Jordan-Hölder filtrations. For this reason, we choose Definition 1.5 as the definition of (semi)stability for framed sheaves of positive rank and we give a *new* definition for the (semi)stability of framed sheaves of rank zero. The latter is different from that given by Huybrechts and Lehn in their papers (see Section 2).

Definition 1.10. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\mathrm{rk}(E) = 0$. If α is injective, we say that $\mathcal{E}$ is *semistable*[2]. Moreover, if $P(E) = \delta$ we say that $\mathcal{E}$ is *stable* with respect to δ.

This definition singles out exactly those objects which may appear as torsion components of the Harder-Narasimhan and Jordan-Hölder filtrations.

In the case of Gieseker semistability, to verify if a torsion free sheaf E is (semi)stable or not, one can restrict oneself to the family of *saturated* subsheaves of E. In the absolute case, one can find inside this family the maximal destabilizing subsheaf of E. In the relative case, one can consider a flat family E of coherent sheaves on the fibres of a projective morphism $X \to S$, where S is an integral k-scheme of finite type, and one can study the *behaviour* of the semistability condition while moving along the base scheme. More precisely, by using the boundedness of the family of torsion free quotients of the sheaves $E|_{X_s}$ on the fibres X_s, for $s \in S$ (cf. Lemma 3.12), one can find a *generically* minimal destabilizing quotient.

[2] For torsion sheaves, the definition of semistability of the corresponding framed sheaves does not depend on δ.

In the framed case the situation is more complicated. We need to introduce a generalization of the notion of saturation:

Definition 1.11. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Let E' be a subsheaf of E. The *framed saturation* $\bar{E}'$ of E' is the saturation of E' as subsheaf of

- $\ker \alpha$, if $E' \subset \ker \alpha$.
- E, if $E' \not\subset \ker \alpha$.

In this way, we get the following characterization:

Proposition 1.12. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Then the following conditions are equivalent:*

(a) *$\mathcal{E}$ is semistable with respect to δ.*
(b) *For any framed saturated subsheaf $E' \subset E$ one has $P(E', \alpha') \leq \mathrm{rk}(E')p(\mathcal{E})$.*
(c) *For any surjective morphism of framed sheaves $\varphi \colon \mathcal{E} \to (Q, \beta)$, where $\alpha = \beta \circ \varphi$ and Q is one of the following:*
   - *Q is a coherent sheaf of positive rank with nonzero framing β such that $\ker \beta$ is nonzero and torsion free,*
   - *Q is a torsion free sheaf with zero framing β,*
   - *$Q = {}^{E}/_{\ker \alpha}$,*

   one has $\mathrm{rk}(Q)p(\mathcal{E}) \leq P(Q, \beta)$.

As one can see from the previous proposition, in the framed case it may happen that rank zero subsheaves destabilize a framed sheaf of positive rank. This phenomenon does not happen in the nonframed case. Inside the family of framed saturated subsheaves of a framed sheaf of positive rank, one can find the maximal one, with respect to the inclusion. Moreover, by using this subsheaf one can construct the Harder-Narasimhan filtration, as explained in Theorem 2.33.

Now consider the relative case: let $f \colon X \to S$ be a projective flat morphism, where S is an integral k-scheme of finite type, and an flat family F of coherent sheaves of rank zero on the fibres of f has been chosen as a framing sheaf. If we define flat families of framed sheaves of positive rank on the fibres of f as pairs $\mathcal{E} = (E, \alpha \colon E \to F)$, where E is an flat flat family of coherent sheaves of positive rank on the fibres of f, we can encounter a problem of *jumping* of the framing when moving along the base scheme, i.e., the possibility that there exists a nonempty open subset U of S and two points $s_1, s_2 \in U$ such that the restriction of α to the fibre at s_1 is zero and the restriction of α to the fibre at s_2 is nonzero. Moreover, we do not want that there exists a point $s \in S$ such that $\ker \alpha_s$ is a framed-destabilizing subsheaf of $\mathcal{E}_s$, because we would like only to deal with destabilizing quotients of $\mathcal{E}_s$ of positive rank. To avoid these situations, we give the following definition of families of framed sheaves:

Definition 1.13. A *flat family of framed sheaves of positive rank on the fibres of the morphism* f consists of a framed sheaf $\mathcal{E} = (E, \alpha \colon E \to F)$ on X, where $\alpha_s \neq 0$ and $\mathrm{rk}(E_s) > 0$ for all $s \in S$ and E and $\mathrm{Im}\, \alpha$ are flat families of coherent sheaves on the fibres of f.

We prove that the family of saturated subsheaves of the fibres of $\mathcal{E}$ is bounded (cf. Proposition 3.14 and 3.15), hence, as in the nonframed case, we can construct a *generically* minimal destabilizing quotient of the framed sheaves $\mathcal{E}|_{X_s}$, for $s \in S$, and a relative version of the Harder-Narasimhan filtration with respect to a stability polynomial $\bar{\delta}$ such that $\bar{\delta} \leq P(\operatorname{Im} \alpha_s)$ for any point $s \in S$ (cf. Section 4). To avoid the problem of the jumping of the framings of the minimal destabilizing quotient when moving along the base scheme, we construct this quotient by using the *relative framed Quot scheme*, that parametrizes only quotients of $\mathcal{E}|_{X_s}$, for $s \in S$, with nonzero induced framings, and the open subset of the relative Grothendieck Quot scheme parametrizing only quotients with generically zero framings. In this way we get a quotient with a framing that generically does not jump.

Regarding the study of Gieseker's stability for torsion free sheaves, one can define the so-called *socle* and *extended socle*. These are "special" saturated subsheaves of a semistable torsion free sheaf E: the socle is the sum of all destabilizing subsheaves of E with the same reduced Hilbert polynomial of E, while the extended socle plays the same role as the maximal destabilizing subsheaf in the stable case. Unfortunately, we cannot introduce framed analogues of these objects in general because the sum of two framed saturated subsheaves of a fixed framed sheaf may not be framed saturated. Thus we generalize the socle and the extended socle only for semistable (D, F)-framed sheaves, where D is a divisor and F a locally free $\mathcal{O}_D$-module. On the other hand, we define in general the notion of a Jordan-Hölder filtration, leading to the notions of S-equivalence and polystability, that play a key role in the construction of the moduli space of (semi)stable framed sheaves with fixed Hilbert polynomial.

Also in the case of μ-semistability, we give two different definitions: Definition 1.6 for framed sheaves of positive rank and a definition for framed sheaves of rank zero similar to that given before. All the previous results hold also for the μ-semistability condition.

By using the relative Harder-Narasimhan and Jordan-Hölder filtrations, we fill one more gap of theory of framed sheaves, by providing a generalization of the Mehta-Ramanathan theorems:

Theorem 1.14. *Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d. Let F be a coherent sheaf on X supported on a divisor D_F. Let $\mathcal{E} = (E, \alpha\colon E \to F)$ be a framed sheaf on X of positive rank with nontrivial framing. If $\mathcal{E}$ is μ-semistable with respect to δ_1, then there is a positive integer a_0 such that for all $a \geq a_0$ there is a dense open subset $U_a \subset |\mathcal{O}_X(a)|$ such that for all $D \in U_a$ the divisor D is smooth and meets transversally the divisor D_F, and $\mathcal{E}|_D$ is μ-semistable with respect to $a\delta_1$. Moreover, a_0 depends only on the Chern character of E.*

The same statement holds with "μ-semistable" replaced by "μ-stable" under the following additional assumptions: the framing sheaf F is a locally free O_{D_F}-module and $\mathcal{E}$ is a (D_F, F)-framed sheaf on X.

Mehta-Ramanathan theorems are very useful as they often allow one to reduce a problem from a higher-dimensional variety to a surface or even to a curve, as for example happens with the proof of Hitchin-Kobayashi correspondence (see Chapter VI in [**39**]).

The classical Mehta-Ramanathan theorems are also used in the algebro-geometric construction of the Uhlenbeck-Donaldson compactification of moduli space of μ-stable vector bundles on a nonsingular projective surface ([**47**] and [**35**], Section 8.2). In the same way, our framed version of these theorems is used in a work of Bruzzo, Markushevich and Tikhomirov in the construction of the Uhlenbeck-Donaldson compactification for framed sheaves.

We briefly recall the construction of this compactification. Let $(X, \mathcal{O}_X(1))$ be a polarized surface, $D \subset X$ a big and nef curve and F a coherent $\mathcal{O}_D$-module. Fix a stability polynomial δ and a numerical polynomial P of degree two. One can define a scheme $R^{\mu ss}(c, \delta)$ that, roughly speaking, parametrizes framed sheaves on X with topological invariants defined by a numerical K-theory class $c \in K(X)_{num}$ that are μ-semistable with respect to δ_1. On $R^{\mu ss}(c, \delta) \times X$ we can define a *universal family* $\mathcal{E} = (E, \alpha \colon E \to p_X^*(F))$, where p_X is the projection from the product to X.

Let $a \gg 0$ and $C \in |\mathcal{O}_X(a)|$ a general curve. Then C is smooth and transversal to D. By using the μ-semistable part of Theorem 1.14, the restriction of $\mathcal{E}$ to $R^{\mu ss}(c, \delta) \times C$ produces a family of generically μ-semistable framed sheaves of positive rank on C and therefore a rational map $R^{\mu ss}(c, \delta) \dashrightarrow \mathcal{M}^{ss}_{\delta_1}(C; F|_C, c|_C)$ from $R^{\mu ss}(c, \delta)$ to the moduli space $\mathcal{M}^{ss}_{\delta_1}(C; F|_C, c|_C)$ of semistable framed sheaves of topological invariant $c|_C$ on C (on a curve semistability coincides with μ-semistability). In Chapter 6 we prove that there exists a line bundle $\mathcal{L}_1$ on $R^{\mu ss}(c, \delta)$ such that the pullback of an ample line bundle of $\mathcal{M}^{ss}_{\delta_1}(C; F|_C, c|_C)$ is isomorphic to $\mathcal{L}_1^{\otimes \nu}$ for some positive integer ν. In this way we obtain that $\mathcal{L}_1$ is generated by global sections.

By taking the projective spectrum of the direct sum of the spaces of global sections of suitable powers of $\mathcal{L}_1$ (as it is explained in Chapter 6), we can define a projective scheme $\mathcal{M}^{\mu ss}_{\delta}$ and projective morphism

$$\pi \colon \mathcal{M}_\delta(X; F, P) \longrightarrow \mathcal{M}^{\mu ss}_{\delta}.$$

As it is proved in [**15**], $\mathcal{M}^{\mu ss}_{\delta}$ is, in a naive sense, a moduli space of μ-semistable framed sheaves.

Let F be a locally free $\mathcal{O}_D$-module. If we restrict ourselves to the open subset $\mathcal{M}_{X,D}(r, \mathcal{A}, n)$ consisting of μ-stable (D, F)-framed sheaves of rank r, determinant line bundle $\mathcal{A}$ and second Chern class n, we obtain a map

$$\begin{aligned} \pi_r := \pi|_{\mathcal{M}_{X,D}(r,\mathcal{A},n)} \colon \mathcal{M}_{X,D}(r, \mathcal{A}, n) \quad &\longrightarrow \quad \coprod_{l \geq 0} \mathcal{M}_{X,D}(r, \mathcal{A}, n-l) \times Sym^l(X \setminus D) \\ (E, \alpha) \quad &\longmapsto \quad \left((E^{\vee\vee}, \alpha^{\vee\vee}), \operatorname{supp}\left({}^{E^{\vee\vee}}/_{E} \right) \right). \end{aligned}$$

Moreover, the restriction of π_r to the open subset consisting of μ-stable (D, F)-framed vector bundles is a bijection onto the image.

This result follows from Theorem 4.6 in [**15**], where the μ-stable part of Theorem 1.14 is used, and generalizes a similar construction for $(l_\infty, \mathcal{O}_{l_\infty}^{\oplus r})$-framed sheaves on $\mathbb{CP}^2$ (see Chapter 3 in [**60**], see also formula (3)).

Another main result of this thesis consists of the generalization to the framed case of the notion of the Atiyah class. Let $(X, \mathcal{O}_X(1))$ be a polarized surface and S a Noetherian k-scheme of finite type. Let E be a flat family of coherent sheaves on the fibres of the projection morphism $p_S \colon S \times X \to S$. The Atiyah class of E is the element $at(E)$ in $\operatorname{Ext}^1(E, \Omega^1_{S\times X} \otimes E)$ that represents the obstruction for the existence of an algebraic connection on E. The Atiyah class was introduced in [**1**] for the case of vector bundles and in [**36**] and [**37**] for any complex of coherent sheaves. One way to define the Atiyah class $at(E)$ is by using the so-called *sheaf of first jets* $J^1(E)$ (see, e.g., [**50**]).

Let $D \subset X$ be a divisor and F a locally free $\mathcal{O}_D$-module. We introduce the following definition of a S-flat family of (D, F)-framed sheaves:

Definition 1.15. A *flat family of (D,F)-framed sheaves parametrized by S* is a pair $\mathcal{E} = (E,\alpha)$ where E is a coherent sheaf on $S \times X$, flat over S, $\alpha\colon E \to p_X^*(F)$ is a morphism such that for any $s \in S$ the sheaf $E|_{\{s\}\times X}$ is locally free in a neighborhood of $\{s\} \times D$ and $\alpha|_{\{s\}\times D}\colon E|_{\{s\}\times D} \to p_X^*(F)|_{\{s\}\times D}$ is an isomorphism.

Let $\mathcal{E} = (E,\alpha)$ be a S-flat family of (D,F)-framed sheaves. We introduce the *framed sheaf of first jets* $J^1_{fr}(\mathcal{E})$ as the subsheaf of the sheaf of first jets $J^1(E)$ consisting of those sections whose $p_S^*(\Omega^1_S)$-part vanishes along $S \times D$. We define the *framed Atiyah class* $at(\mathcal{E})$ of $\mathcal{E}$ by using $J^1_{fr}(\mathcal{E})$ as a class in

$$\mathrm{Ext}^1(E, \left(p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \oplus p_X^*(\Omega^1_X)\right) \otimes E).$$

Consider the induced section $\mathcal{A}t(\mathcal{E})$ under the global-relative map

$$\begin{aligned}&\mathrm{Ext}^1\left(E, \left(p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \oplus p_X^*(\Omega^1_X)\right) \otimes E\right) \longrightarrow \\ &\longrightarrow \mathrm{H}^0(S, \mathcal{E}xt^1_{p_S}(E, \left(p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \oplus p_X^*(\Omega^1_X)\right) \otimes E)),\end{aligned}$$

coming from the relative-to-global spectral sequence

$$\begin{aligned}&\mathrm{H}^i(S, \mathcal{E}xt^j_{p_S}(E, \left(p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \oplus p_X^*(\Omega^1_X)\right) \otimes E)) \Rightarrow \\ &\Rightarrow \mathrm{Ext}^{i+j}(E, \left(p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \oplus p_X^*(\Omega^1_X)\right) \otimes E).\end{aligned}$$

By considering the S-part $\mathcal{A}t_S(\mathcal{E})$ of $\mathcal{A}t(\mathcal{E})$ in

$$\mathrm{H}^0(S, \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E)),$$

we define the framed version of the Kodaira-Spencer map.

Definition 1.16. The *framed Kodaira-Spencer map* associated to the family $\mathcal{E}$ is the composition

$$\begin{aligned}KS_{fr}\colon (\Omega^1_S)^\vee &\xrightarrow{\mathrm{id}\otimes\mathcal{A}t_S(\mathcal{E})} (\Omega^1_S)^\vee \otimes \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E) \to \\ &\longrightarrow \mathcal{E}xt^1_{p_S}(E, p_S^*((\Omega^1_S)^\vee \otimes \Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E) \to \\ &\longrightarrow \mathcal{E}xt^1_{p_S}(E, p_X^*(\mathcal{O}_X(-D)) \otimes E).\end{aligned}$$

This framed Atiyah class allows one to get some new results.

Let $\delta \in \mathbb{Q}[n]$ be a stability polynomial and P a numerical polynomial of degree two. Let $\mathcal{M}^*_\delta(X;F,P)$ be the moduli space of (D,F)-framed sheaves on X with Hilbert polynomial P that are stable with respect to δ. This is an open subset of the fine moduli space $\mathcal{M}_\delta(X;F,P)$ of stable framed sheaves with Hilbert polynomial P. Let us denote by $\mathcal{M}^*_\delta(X;F,P)^{sm}$ the smooth locus of $\mathcal{M}^*_\delta(X;F,P)$. Let us denote by $\tilde{\mathcal{E}} = (\tilde{E},\tilde{\alpha})$ the universal objects of $\mathcal{M}^*_\delta(X;F,P)^{sm}$. Let p be the projection from $\mathcal{M}^*_\delta(X;F,P)^{sm} \times X$ to $\mathcal{M}^*_\delta(X;F,P)^{sm}$.

It is a known fact that the Kodaira-Spencer map is an isomorphism on the smooth locus of the moduli space of Gieseker-stable torsion free sheaves on X (cf. Theorem 10.2.1 in [**35**]). We have proved the framed version of this result.

Theorem 1.17. *The framed Kodaira-Spencer map defined by $\tilde{\mathcal{E}}$ induces a canonical isomorphism*

$$KS_{fr}\colon T\mathcal{M}^*_\delta(X;F,P)^{sm} \xrightarrow{\sim} \mathcal{E}xt^1_p(\tilde{E}, \tilde{E} \otimes p_X^*(\mathcal{O}_X(-D))).$$

From this theorem it follows that for any point $[(E,\alpha)]$ of $\mathcal{M}^*_\delta(X;F,P)^{sm}$, the vector space $\mathrm{Ext}^1(E,E(-D))$ is naturally identified with the tangent space $T_{[(E,\alpha)]}\mathcal{M}^*_\delta(X;F,P)$.

For any $\omega \in \mathrm{H}^0(X,\omega_X(2\,D))$, we can define a skew-symmetric bilinear form

$$\begin{aligned}&\mathrm{Ext}^1(E,E(-D))\times \mathrm{Ext}^1(E,E(-D)) \xrightarrow{\circ} \mathrm{Ext}^2(E,E(-2\,D))\\ &\xrightarrow{tr} \mathrm{H}^2(X,\mathcal{O}_X(-2\,D)) \xrightarrow{\cdot\omega} \mathrm{H}^2(X,\omega_X)\cong k.\end{aligned}$$

When varying $[(E,\alpha)]$, these forms fit into an exterior two-form $\tau(\omega)$ on $\mathcal{M}^*_\delta(X;F,P)^{sm}$. We proved that $\tau(\omega)$ is a closed form (cf. Theorem 7.15) and provided a criterion of its non-degeneracy (cf. Proposition 7.17). In particular, if the line bundle $\omega_X(2\,D)$ is trivial, the two-form $\tau(1)$ induced by $1\in \mathrm{H}^0(X,\omega_X(2\,D))\cong \mathbb{C}$ defines a holomorphic symplectic structure on $\mathcal{M}^*_\delta(X;F,P)^{sm}$. As an application, in Section 6 of Chapter 7, we show that the moduli space of (D,F)-framed sheaves on the second Hirzebruch surface $\mathbb{F}_2$ has a symplectic structure, where D is a *conic* on $\mathbb{F}_2$ and F a Gieseker-semistable locally free $\mathcal{O}_D$-module.

Our results about restriction theorems for framed sheaves have appeared in [**69**]. Symplectic structures on moduli spaces of framed sheaves are a subject of a forthcoming paper ([**70**]).

3. Contents by chapters

This dissertation is structured as follows. In Chapter 2, we define the notion of framed sheaf and morphisms of framed sheaves. Moreover, we give a definition of the (μ)-semistability for framed sheaves: we give a characterization of the semistability condition, introduce the notion of framed saturation and construct the maximal framed-destabilizing subsheaf. In this chapter we point out that in the framed case there may exist destabilizing subsheaves of rank zero. Finally, we construct Harder-Narasimhan and Jordan-Hölder filtrations. In a last part of the chapter we prove that the family of (μ)-semistable framed sheaves with fixed Hilbert polynomial is bounded.

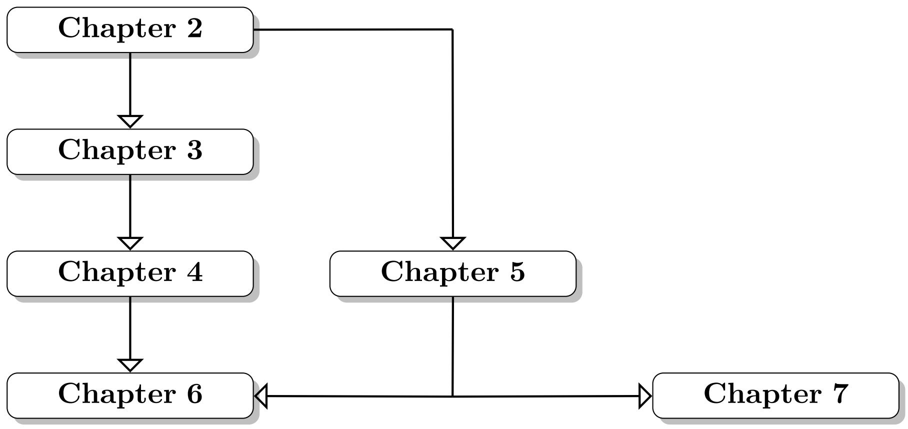
In Chapter 3, we define the notion of families of framed sheaves and construct a *framed* version of the Grothendieck Quot scheme. By using a *boundedness* result for the family of destabilizing subsheaves of a framed sheaf, we obtain the main result of the chapter, that is, the construction of the relative Harder-Narasimhan filtration. In Chapter 4, we provide a generalization of Mehta-Ramanathan theorems for framed sheaves ([**53**], [**54**]) by using *framed* versions of the techniques developed in Chapter 7 of [**35**].

In Chapter 5 we explain the construction of the moduli space of (semi)stable framed sheaves on nonsingular projective varieties, by following the work of Huybrechts and Lehn in [**34**]. We use the definition of a family given by Huybrechts and Lehn in [**33**], which is somehow different from that given in Chapter 3, as we explained in Remark 5.3. Moreover, we construct the moduli space of (D,F)-framed sheaves on a nonsingular projective surface X as an open subset of the moduli space of μ-stable framed sheaves by a suitable choice of a very ample line bundle on X and a stability polynomial.

In Chapter 6, we deal with a generalization of the *Le Potier determinant line bundles* to the framed case and the construction of the Uhlebenck-Donaldson compactification for framed sheaves, where our results on restriction theorems are applied. In particular, we provide the proof of Proposition 6.3, that state that a suitable *framed* Le Potier determinant line bundle is semiample, in which is deeply used the first part of Theorem 1.14.

In Chapter 7, we generalize to the framed case most of the results explained in Chapter 10 in [**35**]. After briefly recalling of the classical theory of Atiyah class for families of coherent sheaves, we introduce the notion of the *framed* sheaf of first jets and, in terms of it, we define the *framed* Atiyah class. Moreover, we introduce the *framed* Kodaira-Spencer map and prove that this map is an isomorphism on the smooth locus of the moduli space of stable (D, F)-framed sheaves. Finally, we show how one constructs closed two-forms by using the Atiyah class and nonzero global sections of the line bundle $\omega_X(2\,D)$ and give a criterion for their non-degeneracy. As an application, we provide a symplectic structure on the moduli spaces of (D, F)-framed sheaves on the second Hirzebruch surface $\mathbb{F}_2$, for D a *conic* on $\mathbb{F}_2$ and F a Gieseker-semistable locally free $\mathcal{O}_D$-module.

4. Interdependence of the Chapters



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CHAPTER 2

Framed sheaves on smooth projective varieties

This chapter provides the basic definitions of the theory of framed sheaves. After introducing the notions of framed sheaves and morphisms of such objects, we define a generalization of Gieseker's semistability condition for framed sheaves (see Section 2) and give a characterization of this condition. Harder-Narasimhan and Jordan-Hölder filtrations are defined in Sections 5 and 6, respectively. In Section 7 we give the definition of μ-semistability for framed sheaves. We conclude the chapter by recalling the notion of bounded families and the Mumford-Castelnuovo regularity. Moreover we show the boundedness of the family of (μ)-semistable framed sheaves of positive rank.

Each Section of the chapter starts with a summary which describes when the results in the framed case coincide with the corresponding ones in the nonframed case or when there are unexpected phenomena. We refer to the book [**35**] of Huybrechts and Lehn for the nonframed case.

1. Preliminaries on framed sheaves

In this section we introduce the notions of *framed sheaf* and *morphism of framed sheaves.* Moreover for such objects we introduce some invariants, such as the *framed Hilbert polynomial* and the *framed degree.* When the framing is zero, a framed sheaf is just its underlying coherent sheaf and these notions coincide with the classical ones (see Section 1.2 of [**35**]).

Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d. Fix a coherent sheaf F on X and a polynomial $\delta \in \mathbb{Q}[n]$ with positive leading coefficient δ_1. We call F *framing sheaf* and δ *stability polynomial.*

Definition 2.1. A *framed sheaf* on X is a pair $\mathcal{E} := (E, \alpha)$, where E is a coherent sheaf on X and $\alpha\colon E \to F$ is a morphism of coherent sheaves. We call α *framing* of E.

For any framed sheaf $\mathcal{E} = (E, \alpha)$, we define the function $\epsilon(\alpha)$ by

$$\epsilon(\alpha) := \begin{cases} 1 & \text{if } \alpha \neq 0, \\ 0 & \text{if } \alpha = 0. \end{cases}$$

The *framed Hilbert polynomial of* $\mathcal{E}$ is

$$P(\mathcal{E}, n) := P(E, n) - \epsilon(\alpha)\delta(n),$$

and the *framed degree of* $\mathcal{E}$ is

$$\deg(\mathcal{E}) := \deg(E) - \epsilon(\alpha)\delta_1.$$

We call *Hilbert polynomial of* $\mathcal{E}$ the Hilbert polynomial $P(E)$ of E. If E is a d-dimensional coherent sheaf, we define the *rank of* $\mathcal{E}$ as the rank of E. The *reduced framed Hilbert polynomial of* $\mathcal{E}$ is

$$p(\mathcal{E}, n) := \frac{P(\mathcal{E}, n)}{\mathrm{rk}(\mathcal{E})}$$

and the *framed slope of* $\mathcal{E}$ is

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}.$$

If E' is a subsheaf of E with quotient $E'' = {}^{E}/_{E'}$, the framing α induces framings $\alpha' := \alpha|_{E'}$ on E' and α'' on E'', where the framing α'' is defined in the following way: $\alpha'' = 0$ if $\alpha' \neq 0$, else α'' is the induced morphism on E''. With this convention the framed Hilbert polynomial of $\mathcal{E}$ behaves additively:

$$P(\mathcal{E}) = P(E', \alpha') + P(E'', \alpha'') \tag{4}$$

and the same happens for the framed degree:

$$\deg(\mathcal{E}) = \deg(E', \alpha') + \deg(E'', \alpha''). \tag{5}$$

Notation: If $\mathcal{E} = (E, \alpha)$ is a framed sheaf on X and E' is a subsheaf of E, then we denote by $\mathcal{E}'$ the framed sheaf (E', α') and by ${}^{\mathcal{E}}/_{E'}$ the framed sheaf (E'', α'').

Thus we have a canonical framing on subsheaves and on quotients. The same happens for subquotients, indeed we have the following result.

Lemma 2.2 (Lemma 1.12 in [**34**])**.** *Let $H \subset G \subset E$ be coherent sheaves and α a framing of E. Then the framings induced on ${}^{G}/_{H}$ as a quotient of G and as a subsheaf of ${}^{E}/_{H}$ agree. Moreover*

$$P\left(\frac{{}^{\mathcal{E}}/_{H}}{{}^{G}/_{H}}\right) = P\left({}^{\mathcal{E}}/_{G}\right) \quad \textit{and} \quad \deg\left(\frac{{}^{\mathcal{E}}/_{H}}{{}^{G}/_{H}}\right) = \deg\left({}^{\mathcal{E}}/_{G}\right).$$

Now we introduce the notion of a morphism of framed sheaves.

Definition 2.3. Let $\mathcal{E} = (E, \alpha)$ and $\mathcal{G} = (G, \beta)$ be framed sheaves. A *morphism of framed sheaves* $\varphi\colon \mathcal{E} \to \mathcal{G}$ between $\mathcal{E}$ and $\mathcal{G}$ is a morphism of the underlying coherent sheaves $\varphi\colon E \to G$ for which there is an element $\lambda \in k$ such that $\beta \circ \varphi = \lambda\alpha$. We say that $\varphi\colon \mathcal{E} \to \mathcal{G}$ is injective (surjective) if the morphism $\varphi\colon E \to G$ is injective (surjective).

Remark 2.4. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf. If E' is a subsheaf of E with quotient $E'' = {}^{E}/_{E'}$, then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r, "i"] \arrow[d, "\alpha'"] & E \arrow[r, "q"] \arrow[d, "\alpha"] & E'' \arrow[r] \arrow[d, "\alpha''"] & 0 \\
 & F \arrow[r, "\cdot\lambda"] & F \arrow[r, "\cdot\mu"] & F &
\end{tikzcd}$$

where $\lambda = 0, \mu = 1$ if $\alpha' = 0$, $\lambda = 1, \mu = 0$ if $\alpha' \neq 0$. Thus the inclusion morphism i (the projection morphism q) induces a morphism of framed sheaves between $\mathcal{E}'$ and $\mathcal{E}$ ($\mathcal{E}$ and ${}^{\mathcal{E}}/_{E'}$). Note that in general an injective (surjective) morphism $E \to G$ between the underlying sheaves of two framed sheaves $\mathcal{E} = (E, \alpha)$ and $\mathcal{G} = (G, \beta)$ does not lift to a morphism $\mathcal{E} \to \mathcal{G}$ of the corresponding framed sheaves. $\triangle$

Lemma 2.5 (Lemma 1.5 in [**34**]). *Let $\mathcal{E} = (E, \alpha)$ and $\mathcal{G} = (G, \beta)$ be framed sheaves. The set* $\mathrm{Hom}(\mathcal{E}, \mathcal{G})$ *of morphisms of framed sheaves is a linear subspace of* $\mathrm{Hom}(E, G)$. *If $\varphi \colon \mathcal{E} \to \mathcal{G}$ is an isomorphism, then the factor λ in the definition can be taken in k^*. In particular, the isomorphism $\varphi_0 = \lambda^{-1}\varphi$ satisfies $\beta \circ \varphi_0 = \alpha$. Moreover, if $\mathcal{E}$ and $\mathcal{G}$ are isomorphic, then their framed Hilbert polynomials and their framed degrees coincide.*

Proposition 2.6. *Let $\mathcal{E} = (E, \alpha)$ and $\mathcal{G} = (G, \beta)$ be framed sheaves. If φ is a nontrivial morphism of framed sheaves between $\mathcal{E}$ and $\mathcal{G}$, then*

$$P\left({}^{E}\!/_{\ker\varphi}, \alpha''\right) \leq P(\operatorname{Im}\varphi, \beta') \quad \text{and} \quad \deg\left({}^{E}\!/_{\ker\varphi}, \alpha''\right) \leq \deg(\operatorname{Im}\varphi, \beta').$$

Proof. Consider a morphism of framed sheaves $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{G})$, $\varphi \neq 0$. There exists $\lambda \in k$ such that $\beta \circ \varphi = \lambda\alpha$. Note that ${}^{E}\!/_{\ker\varphi} \simeq \operatorname{Im}\varphi$ hence their Hilbert polynomials and their degree coincide. It remains to prove that $\epsilon(\alpha'') \geq \epsilon(\beta')$. If $\lambda = 0$, then $\beta' = 0$ and therefore $\epsilon(\beta') = 0 \leq \epsilon(\alpha'')$. Assume now $\lambda \neq 0$: $\alpha = 0$ if and only if $\beta|_{\operatorname{Im}\varphi} = 0$, hence $\epsilon(\beta') = 0 = \epsilon(\alpha'')$. If $\alpha \neq 0$, then also $\alpha'' \neq 0$. Indeed if $\alpha'' = 0$, then $\alpha|_{\ker\varphi} \neq 0$; this implies that $\lambda(\alpha|_{\ker\varphi}) = (\beta \circ \varphi)|_{\ker\varphi} = 0$ and therefore $\lambda = 0$, but this is in contradiction with our previous assumption. Thus, if $\lambda \neq 0$ and $\alpha \neq 0$ then we obtain $\epsilon(\beta') = 1 = \epsilon(\alpha'')$. □

Remark 2.7. Let $\mathcal{E} = (E, \alpha)$ and $\mathcal{G} = (G, \beta)$ be framed sheaves and $\varphi \colon \mathcal{E} \to \mathcal{G}$ a nontrivial morphism of framed sheaves. By the previous proposition, we get

$$P(\mathcal{E}) = P(\ker\varphi, \alpha') + P\left({}^{E}\!/_{\ker\varphi}, \alpha''\right) \leq P(\ker\varphi, \alpha') + P(\operatorname{Im}\varphi, \beta').$$

The inequality may be strict. This phenomenon does not appear in the nonframed case and it depends on the fact that in general we do not know if the isomorphism ${}^{E}\!/_{\ker\varphi} \cong \operatorname{Im}\varphi$ induces an isomorphism ${}^{\mathcal{E}}\!/_{\ker\varphi} \cong (\operatorname{Im}\varphi, \beta')$. △

2. Semistability

In this section we give a generalization to framed sheaves of Gieseker's (semi)stability condition for coherent sheaves (see Definition 1.2.4 in [**35**]). Comparing to the classical case, the (semi)stability condition for framed sheaves has an additional parameter δ, which is a polynomial with rational coefficients. The definition belongs to Huybrechts and Lehn's article [**33**]; we only had to modify it for the case of torsion sheaves. The necessity to handle torsion sheaves is due to the fact that even if we want to work only with torsion free ones, the graded factors of the *framed* Harder-Narasimhan or Jordan-Hölder filtrations may be torsion. We will also present examples where the underlying coherent sheaf of a semistable framed sheaf is not necessarily torsion free, and examples of non-semistable framed sheaves (E, α) with E Gieseker-semistable (see Example 2.10).

Recall that there is a natural ordering of rational polynomials given by the lexicographic order of their coefficients. Explicitly, $f \leq g$ if and only if $f(m) \leq g(m)$ for $m \gg 0$. Analogously, $f < g$ if and only if $f(m) < g(m)$ for $m \gg 0$.

We shall use the following convention: if the word "(semi)stable" occurs in any statement in combination with the symbol $(\leq)$, then two variants of the statement are asserted at the same time: a "semistable" one involving the relation "$\leq$" and a "stable" one involving the relation "$<$".

We now give a definition of semistability for framed sheaves $\mathcal{E} = (E, \alpha)$ of positive rank.

Definition 2.8. A framed sheaf $\mathcal{E} = (E, \alpha)$ of positive rank is said to be *(semi)stable* with respect to δ if and only if the following conditions are satisfied:

(i) $\mathrm{rk}(E)P(E') \ (\leq)\ \mathrm{rk}(E')P(\mathcal{E})$ for all subsheaves $E' \subset \ker\alpha$,
(ii) $\mathrm{rk}(E)(P(E') - \delta) \ (\leq)\ \mathrm{rk}(E')P(\mathcal{E})$ for all subsheaves $E' \subset E$.

Lemma 2.9 (Lemma 1.2 in [**34**]). *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf of positive rank. If $\mathcal{E}$ is (semi)stable with respect to δ, then* $\ker\alpha$ *is torsion free.*

Proof. Let $T(\ker\alpha)$ denote the torsion subsheaf of $\ker\alpha$. By the semistability condition, we get

$$\mathrm{rk}(E)P(T(\ker\alpha), n) \ (\leq)\ \mathrm{rk}(T(\ker\alpha))\,(P(E, n) - \delta(n)) \ \text{ for } n \gg 0.$$

Since $\mathrm{rk}(T(\ker\alpha)) = 0$, we get $P(T(\ker\alpha), n) \ (\leq)\ 0$ for $n \gg 0$. On the other hand, if $T(\ker\alpha) \neq 0$, then the leading coefficient of $P(T(\ker\alpha), n)$ is positive. Thus we get a contradiction and therefore $T(\ker\alpha) = 0$. $\square$

Example 2.10. Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d and $D = D_1 + \cdots + D_l$ an effective divisor on X, where $D_1, \ldots, D_l$ are distinct prime divisors. Consider the short exact sequence associated to the line bundle $\mathcal{O}_X(-D)$:

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \xrightarrow{\alpha} i_*(\mathcal{O}_Y) \longrightarrow 0,$$

where $Y = \mathrm{supp}(D) = D_1 \cup \cdots \cup D_l$. Recall that

$$P(i_*(\mathcal{O}_Y)) = \deg(Y)\frac{n^{d-1}}{(d-1)!} + \text{ terms of lower degree in } n.$$

Let $\delta(n) \in \mathbb{Q}[n]$ be a polynomial of degree $d-1$ such that $\delta > P(i_*(\mathcal{O}_Y))$. Then we get

$$P(\mathcal{O}_X, n) - \delta(n) < P(\mathcal{O}_X, n) - P(i_*(\mathcal{O}_Y), n) = P(\mathcal{O}_X(-D), n) < P(\mathcal{O}_X, n).$$

Thus we obtain that the framed sheaf $(\mathcal{O}_X, \alpha\colon \mathcal{O}_X \to i_*(\mathcal{O}_Y))$ is not semistable with respect to δ.

We thus have obtained an example of a framed sheaf which is not semistable with respect to a fixed δ but the underlying coherent sheaf is Gieseker-semistable. It is possible to construct examples of semistable framed sheaves whose underlying coherent sheaves are not Gieseker-semistable, how we will see in Example 2.50.

On the other hand, it is easy to check that the framed sheaf

$$(\mathcal{O}_X(-D) \oplus i_*(\mathcal{O}_Y), \alpha\colon \mathcal{O}_X(-D) \oplus i_*(\mathcal{O}_Y) \to i_*(\mathcal{O}_Y))$$

is semistable with respect to $\delta := P(i_*(\mathcal{O}_Y))$ and the underlying coherent sheaf has a nonzero torsion subsheaf. $\triangle$

Definition 2.11. A framed sheaf $\mathcal{E} = (E, \alpha)$ of positive rank is *geometrically stable* with respect to δ if for any base extension $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \xrightarrow{f} X$, the pull-back $f^*(\mathcal{E}) := (f^*(E), f^*(\alpha))$ is stable with respect to δ.

In general, a stable framed sheaf is not geometrically stable. The two notions coincide only for a particular class of framed sheaves of positive rank, as we will show in Section 6.

Lemma 2.12 (Lemma 1.7 in [**34**]). *If $\deg(\delta) \geq d$ and $\mathrm{rk}(F) > 0$, then for any semistable framed sheaf $\mathcal{E} = (E, \alpha)$ of positive rank the framing α is zero or injective. Moreover, every semistable framed sheaf is stable.*

PROOF. Assume that $\alpha \neq 0$. Let $E' \neq 0$ be a subsheaf of $\ker \alpha$. By the semistability of $\mathcal{E}$, we get

$$\mathrm{rk}(E)P(E') - \mathrm{rk}(E')P(E) \leq -\mathrm{rk}(E')\delta.$$

The two polynomials on the left-hand side are of degree d and have the same leading coefficient. If $\deg(\delta) \geq d$, this yields a contradiction. Thus α is injective. Moreover the condition (ii) in the Definition 2.8 is strictly satisfied because of the dominance of δ. □

In the case $\mathrm{rk}(F) = 0$, for $\deg(\delta) \geq d$ the framing of any semistable framed sheaf of positive rank is zero, hence this case is not interesting. Moreover, the last lemma shows that when $\mathrm{rk}(F) > 0$, the discussion of semistable framed sheaves of positive rank reduces to the study of subsheaves of F, which is covered by Grothendieck's theory of the Quot scheme, if $\deg(\delta) \geq d$. For these reasons, as it is done in [**33**], we assume that δ has degree $d-1$ and write:

$$\delta(n) = \delta_1 \frac{n^{d-1}}{(d-1)!} + \delta_2 \frac{n^{d-2}}{(d-2)!} + \cdots + \delta_d \in \mathbb{Q}[n]$$

with $\delta_1 > 0$.

We have the following characterization of the semistability condition in terms of quotients:

Proposition 2.13. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf of positive rank. Then the following conditions are equivalent:*

(a) *$\mathcal{E}$ is semistable with respect to δ.*
(b) *For any surjective morphism of framed sheaves $\varphi \colon \mathcal{E} \to (Q, \beta)$, one has $\mathrm{rk}(Q)p(\mathcal{E}) \leq P(Q, \beta)$.*

PROOF. Let E' be the kernel of φ. By using the equation

$$P(\mathcal{E}') - \mathrm{rk}(E')p(\mathcal{E}) = \mathrm{rk}(E/E')p(\mathcal{E}) - P(\mathcal{E}/E'), \tag{6}$$

and Proposition 2.6, we get the assertion. □

In the papers by Huybrechts and Lehn, one finds two different definitions of the (semi)stability of rank zero framed sheaves. In [**33**], they use the same definition for the framed sheaves of positive or zero rank, and with that definition, all framed sheaves of rank zero are automatically semistable but not stable (with respect to any stability polynomial δ). According to Definition 1.1 in [**34**], the semistability of a rank zero framed sheaf depends on the choice of a stability polynomial δ, but all semistable framed sheaves of rank zero are automatically stable. Now we give a new definition of the (semi)stability for rank zero framed sheaves which singles out exactly those objects which may appear as torsion components of the Harder-Narasimhan and Jordan-Hölder filtrations.

Definition 2.14. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\mathrm{rk}(E) = 0$. If α is injective, we say that $\mathcal{E}$ is *semistable*[1]. Moreover, if $P(E) = \delta$ we say that $\mathcal{E}$ is *stable* with respect to δ.

[1]For torsion sheaves, the definition of semistability of the corresponding framed sheaves does not depend on δ.

Remark 2.15. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\mathrm{rk}(E) = 0$. Assume that $\mathcal{E}$ is stable with respect to δ. If G is a subsheaf of E, then $P(G) < P(E) = \delta$; on the other side since the framing is injective, $\alpha|_G \neq 0$ and therefore $P(G, \alpha|_G) = P(G) - \delta < 0 = P(E) - \delta = P(\mathcal{E})$. Hence any subsheaf G of E satisfies an inequality condition, similar to inequality (ii) of Definition 2.8. $\triangle$

Lemma 2.16 (Lemma 2.1 in [**33**]). *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker \alpha$ is nonzero and α is surjective. If $\mathcal{E}$ is (semi)stable with respect to δ, then*

$$\delta \ (\leq) \ P(E) - \frac{\mathrm{rk}(E)}{\mathrm{rk}(\ker \alpha)}(P(E) - P(F)).$$

If F is a torsion sheaf, then $\delta \ (\leq) \ P(F)$ and in particular $\delta_1 \ (\leq) \ \deg(F)$.

Proof. By the (semi)stability condition, we get

$$\mathrm{rk}(E)P(\ker \alpha) \ (\leq) \ \mathrm{rk}(\ker \alpha)P(\mathcal{E}) = \mathrm{rk}(\ker \alpha)\left(P(E) - \delta\right).$$

Since $\mathrm{rk}(\ker \alpha) > 0$ by Lemma 2.9, we obtain

$$\delta \ (\leq) \ P(E) - \frac{\mathrm{rk}(E)}{\mathrm{rk}(\ker \alpha)} P(\ker \alpha).$$

Since $P(E) - P(\ker \alpha) = P(\mathrm{Im}\,\alpha) = P(F)$, we obtain the assertion. Moreover, if F is a torsion sheaf, then $\mathrm{rk}(\mathrm{Im}\,\alpha) = 0$. Therefore $\mathrm{rk}(\ker \alpha) = \mathrm{rk}(E)$ and

$$\delta \ (\leq) \ P(E) - P(\ker \alpha) = P(F).$$

In particular, by formula (1) we obtain $\delta_1 \ (\leq) \ \deg(F)$. $\square$

3. Characterization of semistability

Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf, and assume that $\ker \alpha$ is nonzero and torsion free. In this section we would like to answer the following question: to verify if $\mathcal{E}$ is (semi)stable or not, do we need to check the inequalities (i) and (ii) in the Definition 2.8 for all subsheaves of E? Or, can we restrict our attention to a smaller family of subsheaves of E? For Gieseker's (semi)stability condition, this latter family consists of *saturated* subsheaves of E (see Proposition 1.2.6 in [**35**]). In the framed case, we need to enlarge this family because of the framing, as we explain in what follows.

Definition 2.17. Let E be a coherent sheaf. The *saturation* of a subsheaf $E' \subset E$ is the minimal subsheaf $\bar{E}' \subset E$ containing E' such that the quotient $E/\bar{E}'$ is pure of dimension $\dim(E)$ or zero.

Now we generalize this definition to framed sheaves:

Definition 2.18. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Let E' be a subsheaf of E. The *framed saturation* $\bar{E}'$ of E' is the saturation of E' as subsheaf of

- $\ker \alpha$, if $E' \subset \ker \alpha$.
- E, if $E' \not\subset \ker \alpha$.

Remark 2.19. Let $\bar{E}'$ be the framed saturation of $E' \subset E$. In the first case described in the definition, if $\mathrm{rk}(E') < \mathrm{rk}(\ker\alpha)$, then the quotient $Q = {}^{E}/_{\bar{E}'}$ is a coherent sheaf of positive rank, with nonzero induced framing β, which fits into an exact sequence

$$0 \longrightarrow Q' \longrightarrow Q \xrightarrow{\beta} \mathrm{Im}\,\alpha \longrightarrow 0, \tag{7}$$

where $Q' = \ker\beta$ is a torsion free quotient of $\ker\alpha$. If $\mathrm{rk}(E') = \mathrm{rk}(\ker\alpha)$, then $\bar{E}' = \ker\alpha$ and $Q = {}^{E}/_{\ker\alpha}$. In the second case, Q is a torsion free sheaf with zero induced framing. Moreover

- $\mathrm{rk}(E') = \mathrm{rk}(\bar{E}')$, $P(E') \leq P(\bar{E}')$ and $\deg(E') \leq \deg(\bar{E}')$,
- $P(\mathcal{E}') \leq P(\bar{\mathcal{E}}')$ and $\deg(\mathcal{E}') \leq \deg(\bar{\mathcal{E}}')$.

$\triangle$

Example 2.20. Let us consider the framed sheaf $(\mathcal{O}_X, \alpha\colon \mathcal{O}_X \to i_*(\mathcal{O}_Y))$ on X, defined in Example 2.10. Since $\ker\alpha = \mathcal{O}_X(-D)$, the saturation of $\mathcal{O}_X(-D)$ (as subsheaf of $\mathcal{O}_X$) is $\mathcal{O}_X$ but the framed saturation of $\mathcal{O}_X(-D)$ is $\mathcal{O}_X(-D)$. $\triangle$

We have the following characterization:

Proposition 2.21. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker\alpha$ is nonzero and torsion free. Then the following conditions are equivalent:*

(a) *$\mathcal{E}$ is semistable with respect to δ.*
(b) *For any framed saturated subsheaf $E' \subset E$ one has $P(E', \alpha') \leq \mathrm{rk}(E')p(\mathcal{E})$.*
(c) *For any surjective morphism of framed sheaves $\varphi\colon \mathcal{E} \to (Q, \beta)$, where $\alpha = \beta \circ \varphi$ and Q is one of the following:*
  - *Q is a coherent sheaf of positive rank with nonzero framing β such that $\ker\beta$ is nonzero and torsion free,*
  - *Q is a torsion free sheaf with zero framing β,*
  - *$Q = {}^{E}/_{\ker\alpha}$,*

  one has $\mathrm{rk}(Q)p(\mathcal{E}) \leq P(Q, \beta)$.

Proof. The implication $(a) \Rightarrow (b)$ is obvious. By Remark 2.19, $P(\mathcal{E}') \leq P(\bar{\mathcal{E}}') \leq \mathrm{rk}(\bar{E}')p(\mathcal{E}) = \mathrm{rk}(E')p(\mathcal{E})$, where $\bar{E}'$ is the framed saturation of E', thus $(b) \Rightarrow (a)$. Finally, the framed sheaf $\mathcal{Q}$ has the properties stated in condition (c) if and only if $\ker\varphi$ is a framed saturated subsheaf of $\mathcal{E}$, hence $(b) \iff (c)$. $\square$

Corollary 2.22. *Let $\mathcal{E} = (E, \alpha)$ and $\mathcal{G} = (G, \beta)$ be framed sheaves of positive rank with the same reduced framed Hilbert polynomial p.*

(1) *If $\mathcal{E}$ is semistable and $\mathcal{G}$ is stable, then any nontrivial morphism $\varphi\colon \mathcal{E} \to \mathcal{G}$ is surjective.*
(2) *If $\mathcal{E}$ is stable and $\mathcal{G}$ is semistable, then any nontrivial morphism $\varphi\colon \mathcal{E} \to \mathcal{G}$ is injective.*
(3) *If $\mathcal{E}$ and $\mathcal{G}$ are stable, then any nontrivial morphism $\varphi\colon \mathcal{E} \to \mathcal{G}$ is an isomorphism. Moreover, in this case $\mathrm{Hom}(\mathcal{E}, \mathcal{G}) \simeq k$. If in addition $\alpha \neq 0$, or equivalently, $\beta \neq 0$, then there is a unique isomorphism φ_0 with $\beta \circ \varphi_0 = \alpha$.*

Proof. Let $\varphi\colon \mathcal{E} \to \mathcal{G}$ be a nontrivial morphism of framed sheaves. Suppose that φ is not surjective. If $\mathrm{rk}(\mathrm{Im}\,\varphi) > 0$, then by the (semi)stability condition we get

$$p = p(\mathcal{E}) \leq p({}^{\mathcal{E}}/_{\ker\varphi}) \leq p(\mathrm{Im}\,\varphi, \beta') < p(\mathcal{G}) = p,$$

which is impossible. If $\mathrm{rk}(\mathrm{Im}\,\varphi) = 0$, then we obtain $P({}^{\mathcal{E}}/_{\ker\varphi}) \leq P(\mathrm{Im}\,\varphi, \beta') < 0$, hence $p < p(\ker\varphi, \alpha')$, but this contradicts the semistability of $\mathcal{E}$. Thus we proved the statement (1). In the same way one can prove statement (2) and the first part of statement (3). In order to prove the remaining statements it is enough to show $\mathrm{End}(\mathcal{E}) = k \cdot \mathrm{id}_E$. Suppose that φ is an automorphism of $\mathcal{E}$. Choose $x \in \mathrm{supp}(E)$ and let μ be an eigenvalue of φ restricted to the fiber E_x. Then $\varphi - \mu\mathrm{id}_E$ is not surjective at x and hence not an isomorphism, which implies $\varphi - \mu\mathrm{id}_E = 0$. □

Definition 2.23. Let $\mathcal{E}$ be a framed sheaf. We say that $\mathcal{E}$ is *simple* if $\mathrm{Aut}(\mathcal{E}) = k^* \cdot \mathrm{id}_E$.

4. Maximal framed-destabilizing subsheaf

Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker\alpha$ is nonzero and torsion free. If $\mathcal{E}$ is not semistable with respect to δ, then there exist destabilizing subsheaves of $\mathcal{E}$. In this section we would like to find the maximal one (with respect to the inclusion) and show that it has some interesting properties. Because of the framing, it is possible that this subsheaf has rank zero or it is not saturated and we want to emphasize that this kind of situations are not possible in the nonframed case (see Lemma 1.3.5 in [**35**]).

Proposition 2.24. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker\alpha$ is nonzero and torsion free. If $\mathcal{E}$ is not semistable with respect to δ, then there is a subsheaf $G \subset E$ such that for any subsheaf $E' \subseteq E$ one has*

$$\mathrm{rk}(E')P(\mathcal{G}) \geq \mathrm{rk}(G)P(\mathcal{E}')$$

and in case of equality, one has $E' \subset G$.

Moreover, the framed sheaf $\mathcal{G}$ is uniquely determined and is semistable with respect to δ.

Proof. On the set of nontrivial subsheaves of E we define the following order relation $\preceq$: let G_1 and G_2 be nontrivial subsheaves of E, $G_1 \preceq G_2$ if and only if $G_1 \subseteq G_2$ and $\mathrm{rk}(G_2)P(\mathcal{G}_1) \leq \mathrm{rk}(G_1)P(\mathcal{G}_2)$. Since any ascending chain of subsheaves stabilizes, for any subsheaf E', there is a subsheaf G' such that $E' \subseteq G' \subseteq E$ and G' is maximal with respect to $\preceq$.

First assume that there exists a subsheaf E' of rank zero with $P(\mathcal{E}') > 0$, that is, $P(E') > \delta$. Let $T(E)$ be the torsion subsheaf of E. Then $P(T(E)) \geq P(E') > \delta$. Hence $E' \preceq T(E)$. Moreover, there are no subsheaves $G \subset E$ of positive rank such that $T(E) \preceq G$. Indeed, should that be the case, by the definition of $\preceq$, we would obtain $P(T(E)) - \delta \leq 0$, in contradiction with the previous inequality. Thus we choose $G := T(E)$. Since $\alpha|_G = 0$, G is semistable.

From now on assume that for every rank zero subsheaf $T \subset E$ we have $P(T, \alpha') \leq 0$. Let $G \subset E$ be a $\preceq$-maximal subsheaf with minimal rank among all $\preceq$-maximal subsheaves. Note that $\mathrm{rk}(G) > 0$. Suppose there exists a subsheaf $H \subset E$ with $\mathrm{rk}(H)p(\mathcal{G}) < P(\mathcal{H})$. By hypothesis we have $\mathrm{rk}(H) > 0$. From $\preceq$-maximality of G we get $G \not\subseteq H$ (in particular $H \neq E$). Now we want to show that we can assume $H \subset G$ by replacing H with $G \cap H$.

If $H \not\subseteq G$, then the morphism $\varphi\colon H \to E \to {}^{E}/_{G}$ is nonzero. Moreover $\ker\varphi = G \cap H$. The sheaf $I = \mathrm{Im}\,\varphi$ is of the form ${}^{J}/_{G}$ with $G \subsetneq J \subset E$ and $\mathrm{rk}(J) > 0$. By the $\preceq$-maximality of G we have $p(\mathcal{J}) < p(\mathcal{G})$, hence we obtain

$$\mathrm{rk}(G)P(\mathcal{I}) = \mathrm{rk}(G)(P(\mathcal{J}) - P(\mathcal{G})) < \mathrm{rk}(J)P(\mathcal{G}) - \mathrm{rk}(G)P(\mathcal{G}) = \mathrm{rk}(I)P(\mathcal{G}),$$

and therefore

$$\mathrm{rk}(G)P(\mathcal{I}) < \mathrm{rk}(I)P(\mathcal{G}). \tag{8}$$

Now we want to prove the following:

Claim: The sheaf $G \cap H$ is a nontrivial subsheaf of positive rank of E.

Proof. Assume that $G \cap H = 0$. In this case, we get $H \cong I$; moreover this isomorphism lifts to an isomorphism $\mathcal{H} \cong \mathcal{I}$ of the corresponding framed sheaves and therefore φ lifts to a morphism of framed sheaves φ between $\mathcal{H}$ and $({}^{E}/{}_{G}, \beta)$. By Proposition 2.6, $P(\mathcal{H}) \leq P(\mathcal{I})$ and using formula (8) one has

$$\mathrm{rk}(H)P(\mathcal{G}) < \mathrm{rk}(G)P(\mathcal{H}) \leq \mathrm{rk}(G)P(\mathcal{I}) < \mathrm{rk}(H)P(\mathcal{G}),$$

which is absurd.

The rank of $G \cap H$ is positive, indeed if we assume that $\mathrm{rk}(G \cap H) = 0$, then we have $\mathrm{rk}(I) = \mathrm{rk}(H)$ and again by Proposition 2.6 and formula (8) we get

$$\begin{aligned} \mathrm{rk}(G)P(G \cap H, \alpha') &= \mathrm{rk}(G)P(\mathcal{H}) - \mathrm{rk}(G)P({}^{H}/{}_{G \cap H}, \alpha'') \\ &\geq \mathrm{rk}(G)P(\mathcal{H}) - \mathrm{rk}(G)P(\mathcal{I}) > \mathrm{rk}(G)P(\mathcal{H}) - \mathrm{rk}(H)P(\mathcal{G}) > 0 \end{aligned}$$

hence $G \cap H$ is a rank zero subsheaf of E with $P(G \cap H, \alpha') > 0$, but this is in contradiction with the hypothesis. $\square$

By the following computation:

$$\begin{aligned} \mathrm{rk}(G \cap H)\left(p(G \cap H, \alpha') - p(\mathcal{H})\right) &= \mathrm{rk}({}^{H}/{}_{G \cap H})\left(p(\mathcal{H}) - p({}^{H}/{}_{G \cap H}, \alpha')\right) \\ &> \mathrm{rk}(I)\left(p(\mathcal{H}) - p(\mathcal{I})\right) > \mathrm{rk}(I)\left(p(\mathcal{H}) - p(\mathcal{G})\right) > 0 \end{aligned}$$

we get $p(\mathcal{H}) < p(G \cap H, \alpha')$, hence from now on we can consider a subsheaf $H \subset G$ such that H is $\preceq$-maximal in G, $\mathrm{rk}(H) > 0$ and

$$p(\mathcal{G}) < p(\mathcal{H}).$$

Let $H' \subset E$ be a sheaf that contains H and is $\preceq$-maximal in E. In particular, one has

$$p(\mathcal{G}) < p(\mathcal{H}) \leq p(\mathcal{H}').$$

By $\preceq$-maximality of H and G, we have $H' \nsubseteq G$. Then the morphism $\psi\colon H' \to E \to {}^{E}/{}_{G}$ is nonzero and $H \subset \ker\psi$. As before

$$p(\mathcal{H}') < p(\ker\psi, \alpha').$$

Thus we have $H \subset H' \cap G = \ker\psi$ and $p(\mathcal{H}) < p(\ker\psi, \alpha')$, hence $H \preceq \ker\psi$. This contradicts the $\preceq$-maximality of H in G. Thus for all subsheaves $H \subseteq E$, we have $\mathrm{rk}(H)p(\mathcal{G}) \geq P(\mathcal{H})$.

If there is a subsheaf $H \subset E$ of rank zero such that $P(\mathcal{H}) = 0$ and $H \nsubseteq G$, then by using the same argument as before, we get $P(H \cap G, \alpha') > 0$, but this is in contradiction with the hypothesis. So there are no subsheaves $H \subset E$ of rank zero such that $P(\mathcal{H}) = 0$ and $H \nsubseteq G$. If there is a subsheaf $H \subset E$ of positive rank such that $p(\mathcal{G}) = p(\mathcal{H})$, then $H \subset G$. In fact, if $H \nsubseteq G$ then we can replace H by $G \cap H$ and using the same argument as before we obtain $p(\mathcal{G}) = p(\mathcal{H}) < p(H \cap G, \alpha')$ and this is absurd. $\square$

Definition 2.25. We call G *the maximal framed-destabilizing subsheaf of* $\mathcal{E}$.

Remark 2.26. Note that if G is the maximal framed-destabilizing subsheaf of $\mathcal{E}$, then it is framed saturated. $\triangle$

We give now a criterion to find the maximal framed-destabilizing subsheaf that will be useful later.

Proposition 2.27. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Assume that $\mathcal{E}$ is not semistable with respect to δ but there are no rank zero framed-destabilizing subsheaves. If $G \subset E$ is a positive rank subsheaf with positive rank quotient $G' = {}^{E}\!/_{G}$ such that*

(i) *the framed sheaves $\mathcal{G}$ and $\mathcal{G}'$ are semistable with respect to δ,*
(ii) *$p(\mathcal{G}) > p(\mathcal{G}')$,*

then G is the maximal framed-destabilizing subsheaf of $\mathcal{E}$.

Proof. Let H be a subsheaf of E.

Case 1: $H \subsetneq G$. By semistability we get $P(\mathcal{H}) \leq \mathrm{rk}(H) p(\mathcal{G})$.

Case 2: $G \subsetneq H$. By properties (i) and (ii) one has $\mathrm{rk}({}^{H}\!/_{G}) p(\mathcal{G}) > \mathrm{rk}({}^{H}\!/_{G}) p(\mathcal{G}') \geq P({}^{H}\!/_{G}, \gamma)$, where γ is the induced framing on ${}^{H}\!/_{G}$, and therefore

$$P(\mathcal{H}) < P({}^{H}\!/_{G}, \gamma) + P(\mathcal{G}) = \mathrm{rk}({}^{H}\!/_{G}) p(\mathcal{G}) + \mathrm{rk}(G) p(\mathcal{G}) = \mathrm{rk}(H) p(\mathcal{G}).$$

Consider now the case in which $G \not\subseteq H$ and $H \not\subseteq G$. The morphism $\varphi \colon H \to E \to {}^{E}\!/_{G}$ is nonzero. Moreover $\ker \varphi = H \cap G$.

Case 3: $H \cap G = 0$. In this case the morphism φ is injective. If $\mathrm{rk}(H) = 0$, then by hypothesis $P(\mathcal{H}) \leq 0 = \mathrm{rk}(H) p(\mathcal{G})$. Assume that $\mathrm{rk}(H) > 0$. Then φ induces a morphism of framed sheaves $\varphi \colon \mathcal{H} \to \mathcal{G}'$, hence by Proposition 2.6 we obtain $p(\mathcal{H}) \leq p(\mathrm{Im}\, \varphi, \beta') \leq p(\mathcal{G}') < p(\mathcal{G})$, where β is the induced framing on G'.

Case 4: $H \cap G \neq 0$. From the hypothesis follows that $P(H \cap G, \alpha') \leq \mathrm{rk}(H \cap G) p(\mathcal{G})$ and $P({}^{H}\!/_{\ker \varphi}, \alpha'') \leq P(\mathrm{Im}\, \varphi, \beta') \leq \mathrm{rk}({}^{H}\!/_{\ker \varphi}) p(\mathcal{G}') < \mathrm{rk}({}^{H}\!/_{\ker \varphi}) p(\mathcal{G})$, hence we get

$$P(\mathcal{H}) = P(\ker \varphi, \alpha') + P({}^{H}\!/_{\ker \varphi}, \alpha'') < (\mathrm{rk}(\ker \varphi) + \mathrm{rk}({}^{H}\!/_{\ker \varphi})) p(\mathcal{G}) = \mathrm{rk}(H) p(\mathcal{G}). \qquad \square$$

If the rank of the framing sheaf F is zero, then we have this additional characterization:

Proposition 2.28. *Let F be a coherent sheaf of rank zero and $\mathcal{E} = (E, \alpha \colon E \to F)$ a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Assume that $\mathcal{E}$ is not semistable with respect to δ. Then $\ker \alpha$ is the maximal framed-destabilizing subsheaf of $\mathcal{E}$ if and only if it is Gieseker-semistable and $P({}^{E}\!/_{\ker \alpha}, \beta) < 0$, where β is the induced framing.*

Proof. This follows from the same arguments as in the previous proposition. $\square$

4.1. Minimal framed-destabilizing quotient. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Suppose that $\mathcal{E}$ is not semistable with respect to δ.

Remark 2.29. If the rank of the framing sheaf F is zero, we further assume that $\ker \alpha$ is not the maximal framed-destabilizing subsheaf.

Let T_1 be the set consisting of the quotients $E \xrightarrow{q} Q \to 0$ such that

- Q is torsion free,
- the induced framing on $\ker q$ is nonzero,

- $p(\mathcal{Q}) < p(\mathcal{E})$.

Let T_2 be the set consisting of the quotients $E \xrightarrow{q} Q \to 0$ such that

- Q has positive rank,
- the induced framing on $\ker q$ is zero,
- Q fits into an exact sequence of the form (7),
- $p(\mathcal{Q}) < p(\mathcal{E})$.

By Proposition 2.21, the set $T_1 \cup T_2$ is nonempty. For $i = 1, 2$ define an order relation on T_i as follows: if $Q_1, Q_2 \in T_i$, we say that $Q_1 \sqsubseteq Q_2$ if and only if $p(\mathcal{Q}_1) \leq p(\mathcal{Q}_2)$ and $\mathrm{rk}(Q_1) \leq \mathrm{rk}(Q_2)$ in the case $p(\mathcal{Q}_1) = p(\mathcal{Q}_2)$.

Let us consider the relation $\sqsubset$ defined in the following way: for $Q_1, Q_2 \in T_i$, we have $Q_1 \sqsubset Q_2$ if and only if $Q_1 \sqsubseteq Q_2$ and $p(\mathcal{Q}_1) < p(\mathcal{Q}_2)$ or $\mathrm{rk}(Q_1) < \mathrm{rk}(Q_2)$ in the case $p(\mathcal{Q}_1) = p(\mathcal{Q}_2)$. Let Q^i_- be a $\sqsubset$-minimal element in T_i, for $i = 1, 2$. Define

$$Q_- := \begin{cases} Q^1_- & \text{if } p(\mathcal{Q}_1) < p(\mathcal{Q}_2) \text{ or if } p(\mathcal{Q}_2) = p(\mathcal{Q}_1) \text{ and } \mathrm{rk}(Q_1) \leq \mathrm{rk}(Q_2), \\ Q^2_- & \text{if } p(\mathcal{Q}_2) < p(\mathcal{Q}_1) \text{ or if } p(\mathcal{Q}_2) = p(\mathcal{Q}_1) \text{ and } \mathrm{rk}(Q_2) < \mathrm{rk}(Q_1). \end{cases}$$

By easy computations one can prove the following:

Proposition 2.30. *The sheaf $G := \ker(E \to Q_-)$ is the maximal framed-destabilizing subsheaf of $\mathcal{E}$.*

5. Harder-Narasimhan filtration

In this section we construct the Harder-Narasimhan filtration for a framed sheaf. We adapt the techniques used by Harder and Narasimhan in the case of vector bundles on curves (see [**27**]). When the framing sheaf has rank zero, the rank of the kernel of the framing is equal to the rank of the sheaf and because of this fact we get a more involved characterization of the Harder-Narasimhan filtration than in the nonframed case (as one can see in Proposition 2.35). The characterization of the Harder-Narasimhan filtration when the framing sheaf has positive rank is similar to the nonframed case (see Theorem 1.3.4 in [**35**]).

In this section we consider separately the case in which the rank of the framing sheaf F is zero and the case in which $\mathrm{rk}(F)$ is positive.

In the first case, we can have two types of torsion sheaves as graded factors of the Harder-Narasimhan filtration of a framed sheaf (E, α): the torsion subsheaf $T(E)$ of E and the quotient ${}^{E}/_{\ker \alpha}$. In the second case, the only torsion sheaf that can appear as a graded factor of the Harder-Narasiham filtration is the torsion subsheaf.

Consider first the case $\mathrm{rk}(F) = 0$.

Definition 2.31. Let F be a coherent sheaf of rank zero and $\mathcal{E} = (E, \alpha\colon E \to F)$ a framed sheaf where $\ker \alpha$ is nonzero and torsion free. A *Harder-Narasimhan filtration* for $\mathcal{E}$ is an increasing filtration of framed saturated subsheaves

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) \subset \cdots \subset \mathrm{HN}_l(\mathcal{E}) = E \tag{9}$$

which satisfies the following conditions

(A) the quotient sheaf $gr_i^{\mathrm{HN}}(\mathcal{E}) := {}^{\mathrm{HN}_i(\mathcal{E})}/_{\mathrm{HN}_{i-1}(\mathcal{E})}$ with the induced framing α_i is a semistable framed sheaf with respect to δ for $i = 1, 2, \ldots, l$.

(B) The quotient ${}^{E}/_{\mathrm{HN}_{i-1}(\mathcal{E})}$ has positive rank, the kernel of the induced framing is nonzero and torsion free and the subsheaf $gr_i^{\mathrm{HN}}(\mathcal{E})$ is the maximal framed-destabilizing subsheaf of $({}^{E}/_{\mathrm{HN}_{i-1}(\mathcal{E})}, \alpha'')$ for $i = 1, 2, \ldots, l-1$.

Lemma 2.32. *Let F be a coherent sheaf of rank zero and $\mathcal{E} = (E, \alpha\colon E \to F)$ a framed sheaf where $\ker\alpha$ is nonzero and torsion free. Suppose that $\mathcal{E}$ is not semistable (with respect to δ). Let G be the maximal framed-destabilizing sheaf of $\mathcal{E}$. If $G \neq \ker\alpha$, then for every rank zero subsheaf T of ${}^{E}/_{G}$, we get $P(T, \beta') \leq 0$, where β is the induced framing on ${}^{E}/_{G}$.*

Proof. If the quotient ${}^{E}/_{G}$ is torsion free then the condition is trivially satisfied. Otherwise let $T \subset {}^{E}/_{G}$ be a rank zero subsheaf with $P(T, \beta') > 0$. The sheaf T is of the form ${}^{E'}/_{G}$, where $G \subset E'$ and $\mathrm{rk}(E') = \mathrm{rk}(G)$, hence we obtain $p(\mathcal{E}') > p(\mathcal{G})$, therefore E' contradicts the maximality of G. □

Theorem 2.33. *Let F be a coherent sheaf of rank zero and $\mathcal{E} = (E, \alpha\colon E \to F)$ a framed sheaf where $\ker\alpha$ is nonzero and torsion free. Then there exists a unique Harder-Narasimhan filtration for $\mathcal{E}$.*

Proof. Existence. If $\mathcal{E}$ is a semistable framed sheaf with respect to δ, then we put $l = 1$ and a Harder-Narasimhan filtration is

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) = E$$

Else there exists a subsheaf $E_1 \subset E$ such that E_1 is the maximal framed-destabilizing subsheaf of $\mathcal{E}$. If $E_1 = \ker\alpha$, then a Harder-Narasimhan filtration is

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \ker\alpha \subset \mathrm{HN}_2(\mathcal{E}) = E$$

Otherwise, by Lemma 2.32 $({}^{E}/_{E_1}, \alpha'')$ is a framed sheaf with $\ker\alpha'' \neq 0$ torsion free and no rank zero framed-destabilizing subsheaves. If $({}^{E}/_{E_1}, \alpha'')$ is a semistable framed sheaf, then a Harder-Narasimhan filtration is

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset E_1 \subset \mathrm{HN}_2(\mathcal{E}) = E$$

Else there exists a subsheaf $E_2' \subset {}^{E}/_{E_1}$ of positive rank such that E_2' is the maximal framed-destabilizing subsheaf of $({}^{E}/_{E_1}, \alpha'')$. We denote by E_2 its pre-image in E. Now we apply the previous argument to E_2 instead of E_1. Thus we can iterate this procedure and we obtain a finite length increasing filtration of framed saturated subsheaves of E, which satisfies conditions (A) and (B).

Uniqueness. The uniqueness of the Harder-Narasimhan filtration follows from the uniqueness of the maximal framed-destabilizing subsheaf. □

Remark 2.34. By construction, for $i > 0$ at most one of the framings α_i is nonzero and all but possibly one of the factors $gr_i^{\mathrm{HN}}(\mathcal{E})$ are torsion free. In particular if $\mathrm{rk}(gr_1^{\mathrm{HN}}(\mathcal{E})) = 0$, then $gr_1^{\mathrm{HN}}(\mathcal{E}) = T(E)$ and $\alpha_1 \neq 0$; if $\mathrm{rk}(gr_l^{\mathrm{HN}}(\mathcal{E})) = 0$, then $gr_l^{\mathrm{HN}}(\mathcal{E}) = {}^{E}/_{\ker\alpha}$ and $\alpha_l \neq 0$. △

Now we want to relate condition (B) in Definition 2.31 with the framed Hilbert polynomials of the pieces of the Harder-Narasimhan filtration. In particular we get the following.

Proposition 2.35. *Let F be a coherent sheaf of rank zero and $\mathcal{E} = (E, \alpha\colon E \to F)$ a framed sheaf where $\ker\alpha$ is nonzero and torsion free. Suppose there exists a filtration of the form (9) satisfying condition (A). Then condition (B) is equivalent to the following:*

(B') *the quotient $({}^{E}\!/_{\mathrm{HN}_j(\mathcal{E})}, \alpha'')$ is a framed sheaf where $\ker\alpha''$ is nonzero and torsion free for $j = 1, 2, \ldots, l-2$, it has no rank zero framed-destabilizing subsheaves, and*

$$\mathrm{rk}(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}))P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i) > \mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}), \alpha_{i+1}) \tag{10}$$

for $i = 1, \ldots, l-1$.

Proof. The arguments used to prove this proposition are similar to the one used in the proof of the analogous result for vector bundles on curves (see Lemma 1.3.8 in [**27**]). For completeness, we give all the details of the proof.

Suppose that there is an increasing filtration (9) such that conditions (A) and (B) are satisfied. Consider the following short exact sequence

$$0 \longrightarrow gr_i^{\mathrm{HN}}(\mathcal{E}) \longrightarrow {}^{\mathrm{HN}_{i+1}(\mathcal{E})}\!/_{\mathrm{HN}_{i-1}(\mathcal{E})} \longrightarrow gr_{i+1}^{\mathrm{HN}}(\mathcal{E}) \longrightarrow 0.$$

The subsheaf $gr_i^{\mathrm{HN}}(\mathcal{E})$ is the maximal framed-destabilizing subsheaf of the framed sheaf $({}^{E}\!/_{\mathrm{HN}_{i-1}(\mathcal{E})}, \alpha'')$. By using Lemma 2.2 and formula (6), we get

$$\mathrm{rk}(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}))P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i) > \mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}), \alpha_{i+1}).$$

Vice versa, suppose now that (9) satisfies conditions (A) and (B'). First we prove that $gr_{l-1}^{\mathrm{HN}}(\mathcal{E})$ is the maximal framed-destabilizing subsheaf of $({}^{E}\!/_{\mathrm{HN}_{l-2}(\mathcal{E})}, \alpha'')$. Consider the short exact sequence

$$0 \longrightarrow {}^{\mathrm{HN}_{l-1}(\mathcal{E})}\!/_{\mathrm{HN}_{l-2}(\mathcal{E})} \longrightarrow {}^{E}\!/_{\mathrm{HN}_{l-2}(\mathcal{E})} \longrightarrow {}^{E}\!/_{\mathrm{HN}_{l-1}(\mathcal{E})} \longrightarrow 0.$$

By condition (B') we get

$$\mathrm{rk}(gr_l^{\mathrm{HN}}(\mathcal{E}))P(gr_{l-1}^{\mathrm{HN}}(\mathcal{E}), \alpha_{l-1}) > \mathrm{rk}(gr_{l-1}^{\mathrm{HN}}(\mathcal{E}))P(gr_l^{\mathrm{HN}}(\mathcal{E}), \alpha_l).$$

Moreover, by condition (A) we have that $(gr_{l-1}^{\mathrm{HN}}(\mathcal{E}), \alpha_{l-1})$ and $(gr_l^{\mathrm{HN}}(\mathcal{E}), \alpha_l)$ are semistable framed sheaves. If $\mathrm{rk}(gr_l^{\mathrm{HN}}(\mathcal{E}))$ is positive, then by Lemma 2.2 and Proposition 2.27 the sheaf $gr_{l-1}^{\mathrm{HN}}(\mathcal{E})$ is the maximal framed-destabilizing subsheaf of $({}^{E}\!/_{\mathrm{HN}_{l-2}(\mathcal{E})}, \alpha'')$.

Otherwise, if $\mathrm{rk}(gr_l^{\mathrm{HN}}(\mathcal{E})) = 0$, then by relation (10) follows that $P(gr_l^{\mathrm{HN}}(\mathcal{E}), \alpha_l) < 0$. Since $gr_l^{\mathrm{HN}}(\mathcal{E}) \neq 0$, we get $\alpha_l \neq 0$, hence $\mathrm{HN}_{l-1}(\mathcal{E}) \subset \ker\alpha$ and $\mathrm{rk}(\mathrm{HN}_{l-1}(\mathcal{E})) = \mathrm{rk}(\ker\alpha) = \mathrm{rk}(E)$. Thus by definition of framed saturation, we get $\mathrm{HN}_{l-1}(\mathcal{E}) = \ker\alpha$, hence $gr_l^{\mathrm{HN}}(\mathcal{E}) = {}^{E}\!/_{\ker\alpha}$; by Proposition 2.28 we obtain that $gr_{l-1}^{\mathrm{HN}}(\mathcal{E})$ is the maximal framed-destabilizing subsheaf of $({}^{E}\!/_{\mathrm{HN}_{l-2}(\mathcal{E})}, \alpha'')$.

We proceed to prove that condition (B) is satisfied by downward induction on i. Fix $i > 1$ and consider the exact sequences

$$0 \longrightarrow {}^{\mathrm{HN}_i(\mathcal{E})}\!/_{\mathrm{HN}_{i-1}(\mathcal{E})} \longrightarrow {}^{E}\!/_{\mathrm{HN}_{i-1}(\mathcal{E})} \longrightarrow {}^{E}\!/_{\mathrm{HN}_i(\mathcal{E})} \longrightarrow 0$$

and

$$0 \longrightarrow {}^{\mathrm{HN}_{i+1}(\mathcal{E})}\!/_{\mathrm{HN}_i(\mathcal{E})} \longrightarrow {}^{E}\!/_{\mathrm{HN}_i(\mathcal{E})} \longrightarrow {}^{E}\!/_{\mathrm{HN}_{i+1}(\mathcal{E})} \longrightarrow 0.$$

By inductive hypothesis, we know that $gr_{i+1}^{\mathrm{HN}}(\mathcal{E})$ is the maximal framed-destabilizing subsheaf of $({}^{E}\!/_{\mathrm{HN}_i(\mathcal{E})}, \alpha'')$. Since $1 < i < l-2$, the framed sheaf $({}^{E}\!/_{\mathrm{HN}_i(\mathcal{E})}, \alpha'')$ has no framed-destabilizing subsheaves of rank zero, hence $\mathrm{rk}(gr_{i+1}^{\mathrm{HN}}(\mathcal{E})) > 0$. We prove now that $gr_i^{\mathrm{HN}}(\mathcal{E})$ is the maximal framed-destabilizing subsheaf of $({}^{E}\!/_{\mathrm{HN}_{i-1}(\mathcal{E})}, \alpha'')$. Let Q be a coherent subsheaf of ${}^{E}/_{\mathrm{HN}_{i-1}(\mathcal{E})}$. As usual, we denote by $\mathcal{Q}$ the associated framed sheaf with the induced framing. Note that Q is of the form ${}^{H}/_{\mathrm{HN}_{i-1}(\mathcal{E})}$ with $\mathrm{HN}_{i-1}(\mathcal{E}) \subset H$ and $\mathrm{rk}(\mathrm{HN}_{i-1}(\mathcal{E})) \leq \mathrm{rk}(H)$. If $Q \subset gr_i^{\mathrm{HN}}(\mathcal{E})$, then by condition (A) we get

$$\mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P(\mathcal{Q}) \leq \mathrm{rk}(Q)P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i).$$

If $\mathrm{HN}_i(\mathcal{E}) \subset H$, then by inductive hypothesis and by condition (B') we get

$$\mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P({}^{\mathcal{H}}/_{\mathrm{HN}_i(\mathcal{E})}) < \mathrm{rk}({}^{H}/_{\mathrm{HN}_i(\mathcal{E})})P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i).$$

Therefore $\mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P(\mathcal{Q}) < \mathrm{rk}(Q)P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i)$. This also happens for $H = E$, too. So the framed sheaf ${}^{\mathcal{E}}/_{\mathrm{HN}_i(\mathcal{E})}$ is not semistable.

We still need to check the case when $H \nsubseteq \mathrm{HN}_i(\mathcal{E})$ or $\mathrm{HN}_i(\mathcal{E}) \nsubseteq H$. In this case the morphism $\varphi\colon H \to E \to {}^{E}/_{\mathrm{HN}_i(\mathcal{E})}$ is nonzero and $\ker\varphi = H \cap \mathrm{HN}_i(\mathcal{E}) \neq 0$. By condition (A) we get

$$\mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P({}^{\ker\varphi}/_{\mathrm{HN}_{i-1}(\mathcal{E})}, \beta) \leq \mathrm{rk}({}^{\ker\varphi}/_{\mathrm{HN}_{i-1}(\mathcal{E})})P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i),$$

where β is the induced framing on ${}^{\ker\varphi}/_{\mathrm{HN}_{i-1}(\mathcal{E})}$. Moreover by Proposition 2.6 one has

$$P\Big(\frac{\mathcal{Q}}{{}^{\ker\varphi}/_{\mathrm{HN}_{i-1}(\mathcal{E})}}\Big) = P({}^{\mathcal{H}}/_{\ker\varphi}) \leq P(\mathrm{Im}\varphi, \alpha'') \leq \mathrm{rk}({}^{H}/_{\ker\varphi})p(gr_{i+i}^{\mathrm{HN}}(\mathcal{E}), \alpha_{i+i}),$$

hence $\mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P\Big(\frac{\mathcal{Q}}{{}^{\ker\varphi}/_{\mathrm{HN}_{i-1}(\mathcal{E})}}\Big) < \mathrm{rk}({}^{H}/_{\ker\varphi})P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i)$. Therefore

$$\begin{aligned}
\mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P(\mathcal{Q}) &= \mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))\Big(P\Big(\frac{\mathcal{Q}}{{}^{\ker\varphi}/_{\mathrm{HN}_{i-1}(\mathcal{E})}}\Big) + P\Big({}^{\ker\varphi}/_{\mathrm{HN}_{i-1}(\mathcal{E})}, \beta\Big)\Big) = \\
&< \mathrm{rk}({}^{\ker\varphi}/_{\mathrm{HN}_{i-1}(\mathcal{E})})P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i) + \mathrm{rk}({}^{H}/_{\ker\varphi})P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i) \\
&< \mathrm{rk}(Q)P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i).
\end{aligned}$$

Thus the sheaf $gr_i^{\mathrm{HN}}(\mathcal{E})$ is the maximal framed-destabilizing subsheaf of $({}^{E}/_{\mathrm{HN}_{i-1}(\mathcal{E})}, \alpha'')$.

For $i = 1$, if $\mathrm{HN}_1(\mathcal{E})$ has positive rank, we can apply the same argument as before; if $\mathrm{rk}(\mathrm{HN}_1(\mathcal{E})) = 0$, then by relation (10) it follows $P(\mathrm{HN}_1(\mathcal{E}), \alpha_1) > 0$, thus by the definition of the maximal framed-destabilizing subsheaf, we get $\mathrm{HN}_1(\mathcal{E}) = T(E)$. □

Now we turn to the case in which the rank of F is positive. First, we give the following definition.

Definition 2.36. Let F be a coherent sheaf of positive rank and $\mathcal{E} = (E, \alpha\colon E \to F)$ a framed sheaf where $\ker\alpha$ is nonzero and torsion free. A *Harder-Narasihman filtration* for $\mathcal{E}$ is an increasing filtration of framed saturated subsheaves

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) \subset \cdots \subset \mathrm{HN}_l(\mathcal{E}) = E$$

which satisfies the following conditions

(A) the quotient sheaf $gr_i^{\mathrm{HN}}(\mathcal{E}) := {}^{\mathrm{HN}_i(\mathcal{E})}/_{\mathrm{HN}_{i-1}(\mathcal{E})}$ with the induced framing α_i is a semistable framed sheaf with respect to δ for $i = 1, 2, \ldots, l$.

(B) the quotient $({}^{E}/_{\mathrm{HN}_i(\mathcal{E})}, \alpha'')$ is a framed sheaf where $\ker\alpha''$ is nonzero and torsion free for $i = 1, \ldots, l-1$, it has no rank zero framed-destabilizing subsheaves, and

$$\mathrm{rk}(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}))P(gr_i^{\mathrm{HN}}(\mathcal{E}), \alpha_i) > \mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}), \alpha_{i+1}).$$

In this case one can prove results, similar to those stated in Lemma 2.32, Theorem 2.33 and Proposition 2.35. In particular we get the following:

Theorem 2.37. *Let F be a coherent sheaf of positive rank and $\mathcal{E} = (E, \alpha\colon E \to F)$ a framed sheaf where* $\ker\alpha$ *is nonzero and torsion free. Then there exists a unique Harder-Narasimhan filtration for $\mathcal{E}$.*

We conclude this section by proving a result for the maximal framed-destabilizing subsheaf of a framed sheaf. This result holds for a framing sheaf of any rank.

Let B be a torsion free sheaf on X. We denote by $p_{max}(B)$ the *maximal reduced Hilbert polynomial* of B, that is, the reduced Hilbert polynomial of the maximal destabilizing subsheaf of B (see Section 1.3 in [**35**]).

Lemma 2.38. *Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank and B a torsion free sheaf with zero framing. Suppose that $p(\mathcal{E}) > p_{max}(B)$. Then* $\mathrm{Hom}(\mathcal{E}, (B, 0)) = 0$.

Proof. Let $\varphi \in \mathrm{Hom}(\mathcal{E}, (B, 0))$, $\varphi \neq 0$. Let j be minimal such that $\varphi(E) \subset HN_j(B)$. Then there exists a nontrivial morphism of framed sheaves $\bar{\varphi}\colon \mathcal{E} \to gr_j^{HN}(B)$. By Propositions 2.6 and 2.21 we get

$$p(\mathcal{E}) \leq p(E/\ker\bar{\varphi}, \alpha'') \leq p(\mathrm{Im}\,\bar{\varphi}) \leq p(gr_j^{HN}(B)) \leq p_{max}(B)$$

and this is a contradiction with our assumption. □

Proposition 2.39. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where* $\ker\alpha$ *is nonzero and torsion free. Assume that $\mathcal{E}$ is not semistable with respect to δ. Let G be the maximal framed-destabilizing subsheaf of $\mathcal{E}$. Then*

$$\mathrm{Hom}\,(\mathcal{G}, \mathcal{E}/G) = 0.$$

Proof. We have to consider separately four different cases.

Case 1: $G = \ker\alpha$. In this case by definition of morphism of framed sheaves, we get $\mathrm{Hom}\,(\mathcal{G}, \mathcal{E}/G) = 0$.

Case 2: $\alpha|_G = 0$ and $\mathrm{rk}(G) < \ker\alpha$. In this case $\mathrm{Hom}\,(\mathcal{G}, \mathcal{E}/G) = \mathrm{Hom}(G, \ker\alpha/G)$. Recall that G is a Gieseker-semistable sheaf and $\ker\alpha/G$ is a torsion free sheaf; moreover from the maximality of G follows that $p_G > p(T/G)$ for all subsheaves $T/G \subset \ker\alpha/G$, hence $p_{min}(G) = p(G) > p_{max}(\ker\alpha/G)$ and by Lemma 1.3.3 in [**35**] we obtain the assertion.

Case 3: $\alpha|_G \neq 0$ and $\mathrm{rk}(G) > 0$. In this case E/G is a torsion free sheaf and the induced framing is zero. From the maximality of $\mathcal{G}$ it follows that $p(\mathcal{G}) > p(T/G)$ for all subsheaves $T/G \subset E/G$, so we can apply Lemma 2.38 and we get the assertion.

Case 4: $G = T(E)$. Let $\varphi\colon T(E) \to E/T(E)$. Since $\mathrm{rk}(\mathrm{Im}\,\varphi) = 0$ and $E/T(E)$ is torsion free, we have $\mathrm{Im}\,\varphi = 0$ and therefore we obtain the assertion. □

5.1. Base field extension. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf on X where $\ker\alpha$ is nonzero and torsion free. Let K be an extension of k. Consider the following cartesian diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\phi}"] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ \mathrm{Spec}(K) \arrow[r, "\phi"] & \mathrm{Spec}(k) \end{tikzcd}$$

Put $\tilde{E} := \tilde{\phi}^*(E)$, $\tilde{F} := \tilde{\phi}^*(F)$, $\tilde{\alpha} := \tilde{\phi}^*(\alpha)$ and $\tilde{\mathcal{E}} := (\tilde{E}, \tilde{\alpha})$.

Now we describe the behaviour of the semistability condition with respect to the base field extension. In particular, we give a generalization of Proposition 3 in [**41**]:

Theorem 2.40. *A subsheaf $G \subset E$ is the maximal framed-destabilizing subsheaf of $\mathcal{E}$ if and only if $\tilde{\phi}^*(G)$ is so for $\tilde{\mathcal{E}}$.*

Proof. Note that since $\tilde{\phi}$ is a flat morphism, the sheaf $\ker \tilde{\alpha} = \tilde{\phi}^*(\ker \alpha)$ is torsion free. The Hilbert polynomial is preserved under base extensions, so the framed Hilbert polynomial is preserved. If $E' \subset E$ is a framed-destabilizing subsheaf, then so is $\tilde{\phi}^*(E') \subset \tilde{E}$. Hence if $\tilde{\mathcal{E}}$ is semistable, then $\mathcal{E}$ is semistable. So it suffices to prove that if G_K is the maximal framed-destabilizing subsheaf of $\tilde{\mathcal{E}}$, then there is $G \subset E$ such that $\tilde{\phi}^*(G) = G_K$.

Since G_K is finitely presented, it is defined over some field L, $k \subset L \subset K$, which is finitely generated over k, so we can suppose that $K = k(x)$ for some single element $x \in K$ and ${}^{K}/_{k}$ is a purely trascendental or separable extension. Note that there do not exist field extensions of k which are purely inseparable, because k is a perfect field. Let $\sigma \in \mathrm{Gal}({}^{K}/_{k})$, we denote by $\sigma_{\tilde{X}}$ the automorphism of $\tilde{X}$ over X induced by σ. Since $\sigma^*_{\tilde{X}}(G_K)$ has the same Hilbert polynomial of G_K and $\epsilon(\tilde{\alpha}|_{G_K}) = \epsilon(\tilde{\alpha}|_{\sigma^*_{\tilde{X}}(G_K)})$, we must have $\sigma^*_{\tilde{X}}(G_K) = G_K$. Hence by descent theory (see [**25**], p. 22) there exists a subsheaf $G \subset E$ such that $\tilde{\phi}^*(G) = G_K$. Since the framed Hilbert polynomial of G coincides with the one of G_K, we get that G is the maximal framed-destabilizing subsheaf of $\mathcal{E}$. □

6. Jordan-Hölder filtration

By analogy to the study of Gieseker-semistable coherent sheaves we will define Jordan-Hölder filtrations for framed sheaves. Because of the framing, one needs to use Lemma 2.2 in the construction of the filtration. Moreover, in general we cannot extend the notions of *socle* and the *extended socle* for stable torsion free sheaves to the framed case, because, for example, the sum of two framed saturated subsheaves may not be framed saturated, hence we construct these objects only for a smaller family of framed sheaves with extra properties.

Definition 2.41. Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank r. A Jordan-Hölder filtration of $\mathcal{E}$ is a filtration

$$E_\bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

such that all the factors ${}^{E_i}/_{E_{i-1}}$ together with the induced framings α_i are stable with framed Hilbert polynomial $P({}^{E_i}/_{E_{i-1}}, \alpha_i) = \mathrm{rk}({}^{E_i}/_{E_{i-1}}) p(\mathcal{E})$.

Proposition 2.42 (Proposition 1.13 in [**34**])**.** *Jordan-Hölder filtrations always exist. The framed sheaf*

$$gr(\mathcal{E}) := (gr(E), gr(\alpha)) = \bigoplus_i ({}^{E_i}/_{E_{i-1}}, \alpha_i)$$

does not depend on the choice of the Jordan-Hölder filtration.

Proof. If $\mathcal{E}$ is not stable, then there exists a proper subsheaf $E' \subset E$ such that $P(\mathcal{E}') = \mathrm{rk}(E') p(\mathcal{E})$. Let E' be the maximal subsheaf with this property. Then E' is framed saturated, $\mathcal{E}'$ is semistable and ${}^{\mathcal{E}}/_{E'}$ is stable. Inductively, we can construct a finite length descending sequence of subsheaves, that will be a Jordan-Hölder filtration of $\mathcal{E}$. Let $E_\bullet$ and $E'_\bullet$ be two such filtrations. Let j the smallest index such that $E_1 \subset E'_j$. The morphism $\varphi\colon E_1 \to E'_j \to E'_j/E'_{j-1}$ is nontrivial and induces a morphism between the corresponding framed sheaves. Since (E_1, α_1) and $(E'_j/E'_{j-1}, \alpha'_j)$ are stable, by Corollary 2.22 we get that φ is an isomorphism of framed sheaves. Moreover the morphism $E'_{j-1} \to E/E_1$ is injective. Therefore we obtain an exact sequence of framed sheaves

$$0 \longrightarrow \mathcal{E}'_{j-1} \longrightarrow \mathcal{E}/E_1 \longrightarrow \mathcal{E}/E'_j \longrightarrow 0.$$

By Lemma 2.2, the induced Jordan-Hölder filtrations on $\mathcal{E}/E'_j$ and $\mathcal{E}'_{j-1}$ by $E'_\bullet$ give rise to a Jordan-Hölder filtration of $\mathcal{E}/E_1$, whose graded object by induction on the rank of E is isomorphic to the graded object of the filtration $E_\bullet/E_1$. Therefore we get the assertion. □

Remark 2.43. By construction, for $i > 0$ all subsheaves E_i are framed saturated and the framed sheaves (E_i, α') are semistable with framed Hilbert polynomial $\mathrm{rk}(E_i)p(\mathcal{E})$. In particular (E_1, α') is a stable framed sheaf. Moreover at most one of the framings α_i is nonzero and all but possibly one of the factors E_i/E_{i-1} are torsion free. △

Now we introduce an equivalence relation that will be important in the construction of *moduli spaces* of semistable framed sheaves of positive rank (cf. Chapter 5), because these spaces parametrizes the equivalence classes of this relation.

Definition 2.44. Two semistable framed sheaves $\mathcal{E}$ and $\mathcal{G}$ of positive rank with reduced Hilbert polynomial p are called *S-equivalent* if their associated graded objects $gr(\mathcal{E})$ and $gr(\mathcal{G})$ are isomorphic.

Obviously, if an S-equivalence class contains a stable framed sheaf then it does not contain nonisomorphic framed sheaves.

Definition 2.45. A framed sheaf $\mathcal{E} = (E, \alpha)$ of positive rank is *polystable* if E has a filtration $E_\bullet\colon 0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$ such that

(i) for $i = 2, \ldots, n$, every exact sequence

$$0 \longrightarrow E_{i-1} \longrightarrow E_i \longrightarrow E_i/E_{i-1} \longrightarrow 0$$

splits,

(ii) $E_\bullet$ is a Jordan-Hölder filtration of $\mathcal{E}$.

As we saw above, every S-equivalence class of semistable framed sheaves contains exactly one polystable framed sheaf up to isomorphism. Thus, the moduli space of semistable framed sheaves of positive rank in fact parametrizes polystable framed sheaves.

Lemma 2.46. *Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank r. Then there exists at most one subsheaf $E' \subset E$ such that $\alpha|_{E'} \neq 0$, $\mathcal{E}'$ is a stable framed sheaf and $P(\mathcal{E}') = \mathrm{rk}(E')p(\mathcal{E})$.*

Proof. Suppose that there exist E_1 and E_2 subsheaves of E such that $\alpha|_{E_i} \neq 0$, the framed sheaf $\mathcal{E}_i$ is stable (with respect to δ) and $P(\mathcal{E}_i) = r_i p(\mathcal{E})$, where $r_i = \mathrm{rk}(E_i)$, for $i = 1, 2$. So we have $P(E_i) = r_i p(\mathcal{E}) + \delta$ for $i = 1, 2$. Let $E_{12} = E_1 \cap E_2$. Suppose that

$E_{12} \neq 0$ and $\alpha|_{E_{12}} \neq 0$. Denote by r_{12} the rank of E_{12}. Since $\mathcal{E}_i$ is stable, we have that $P(E_{12}) - \delta < r_{12}p(\mathcal{E})$. Consider the exact sequence

$$0 \longrightarrow E_{12} \longrightarrow E_1 \oplus E_2 \longrightarrow E_1 + E_2 \longrightarrow 0.$$

The induced framing on $E_1 + E_2$ is nonzero; we denote it by β.

$$\begin{aligned} P(E_1 + E_2) &= P(E_1) + P(E_2) - P(E_{12}) = r_1p(\mathcal{E}) + \delta + r_2p(\mathcal{E}) + \delta - P(E_{12}) \\ &> \operatorname{rk}(E_1 + E_2)p(\mathcal{E}) + \delta \end{aligned}$$

and therefore

$$P(E_1 + E_2, \beta) = P(E_1 + E_2) - \delta > \operatorname{rk}(E_1 + E_2)p(\mathcal{E}),$$

but this is a contradiction, because $\mathcal{E}$ is semistable. Now consider the case $\alpha|_{E_{12}} = 0$. By similar computations, we obtain

$$P(E_1 + E_2, \beta) = P(E_1 + E_2) - \delta > \operatorname{rk}(E_1 + E_2)p(\mathcal{E}) + \operatorname{rk}(E_1 + E_2)\delta > \operatorname{rk}(E_1 + E_2)p(\mathcal{E}),$$

but this is absurd. Thus $E_{12} = 0$ and therefore $E_1 + E_2 = E_1 \oplus E_2$. In this case we get

$$\begin{aligned} P(E_1 + E_2, \beta) &= P(E_1 + E_2) - \delta = P(E_1) + P(E_2) - \delta \\ &= r_1p(\mathcal{E}) + \delta + r_2p(\mathcal{E}) + \delta - \delta \\ &= \operatorname{rk}(E_1 + E_2)p(\mathcal{E}) + \delta > \operatorname{rk}(E_1 + E_2)p(\mathcal{E}), \end{aligned}$$

but this is not possible. □

Remark 2.47. Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank r. If there exists $E' \subset E$ such that $\operatorname{rk}(E') = 0$ and $P(E') = \delta$, then $E' = T(E)$, indeed from $P(T(E)) \geq P(E')$ follows that $P(T(E)) \geq \delta$. Since $\mathcal{E}$ is semistable, we have $P(T(E)) = \delta$ and so $E' = T(E)$. △

By using similar computations as before, one can prove:

Lemma 2.48. *Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank. Let E_1 and E_2 be two different subsheaves of E such that $P(\mathcal{E}_i) = \operatorname{rk}(E_i)p(\mathcal{E})$ for $i = 1, 2$. Then $P(E_1 + E_2, \alpha') = \operatorname{rk}(E_1 + E_2)p(\mathcal{E})$ and $P(E_1 \cap E_2, \alpha') = \operatorname{rk}(E_1 \cap E_2)p(\mathcal{E})$.*

6.1. Framed sheaves that are locally free along the support of the framing sheaf. In this section we assume that F is supported on a divisor D and is a locally free $\mathcal{O}_D$-module.

Definition 2.49. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf on X. We say that $\mathcal{E}$ is (D, F)*-framable* if E is locally free in a neighborhood of D and $\alpha|_D$ is an isomorphism. We call $\mathcal{E}$ also (D, F)*-framed sheaf.*

Recall that in general for a framed sheaf $\mathcal{E} = (E, \alpha)$ where $\ker \alpha$ is nonzero and torsion free, the torsion subsheaf of E is supported on $\operatorname{Supp}(F)$. Therefore if $\mathcal{E}$ is (D, F)-framable, E is torsion free.

Example 2.50. Let $\mathbb{CP}^2$ be the complex projective plane and $\mathcal{O}_{\mathbb{CP}^2}(1)$ the hyperplane line bundle. Let l_∞ be a line in $\mathbb{CP}^2$ and $i\colon l_\infty \to \mathbb{CP}^2$ the inclusion map. The torsion free sheaves of rank r on $\mathbb{CP}^2$, trivial along the line l_∞ are — in the language we introduced before — $(l_\infty, \mathcal{O}_{l_\infty}^r)$-framed sheaves of rank r on $\mathbb{CP}^2$. Let $M(r, n)$ be the moduli space of $(l_\infty, \mathcal{O}_{l_\infty}^r)$-framed sheaves of rank r and second Chern class n on $\mathbb{CP}^2$. This moduli space is nonempty for $n \geq 1$ as one can see from the description of this space through ADHM data (see, e.g., Chapter 2 in [**60**]). Let $[(E,\alpha)]$ be a point in $M(r,1)$: the sheaf E is a torsion free sheaf of rank r and second Chern class one. By Proposition 9.1.3 in [**45**], E is not Gieseker-semistable. On the other hand, the framed sheaf (E,α) is stable with respect to a suitable choice of δ (cf. Theorem 5.13). Thus we have proved that there exist semistable framed sheaves such that the underlying coherent sheaves are not Gieseker-semistable. $\triangle$

Lemma 2.51. *Let $\mathcal{E} = (E,\alpha)$ be a semistable (D,F)-framed sheaf. Let E_1 and E_2 be two different framed saturated subsheaves of E such that $p(\mathcal{E}_i) = p(\mathcal{E})$, for $i = 1,2$. Assume that $\alpha|_{E_1} = 0$. Then $E_1 + E_2$ is a framed saturated subsheaf of E such that $gr(E_1 + E_2, \alpha') = gr(\mathcal{E}_1) \oplus gr(\mathcal{E}_2)$.*

Proof. Since $\mathcal{E}$ is (D,F)-framable, the quotient ${}^{E}/_{E_i}$ is torsion free for $i = 1,2$, hence ${}^{E}/_{(E_1+E_2)}$ is torsion free as well and therefore $E_1 + E_2$ is framed saturated. By Lemma 2.48, $p(E_1 + E_2, \alpha') = p(\mathcal{E})$. Moreover we can always start with a Jordan-Hölder filtration of $\mathcal{E}_i$ and complete it to one of $(E_1 + E_2, \alpha')$, hence we get $gr(\mathcal{E}_i) \subset gr(E_1 + E_2, \alpha')$ (as framed sheaves) for $i = 1,2$. Let $G_\bullet : 0 = G_0 \subset G_1 \subset \cdots \subset G_{l-1} \subset G_l = E_1$ be a Jordan-Hölder filtration for $\mathcal{E}_1$ and $H_\bullet : 0 = H_0 \subset H_1 \subset \cdots \subset H_{s-1} \subset H_s = E_2$ a Jordan-Hölder filtration for $\mathcal{E}_2$. Consider the filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{l-1} \subset G_l = E_1 \subset E_1 + H_p \subset \cdots \subset E_1 + H_{t-1} \subset H_t = E_1 + E_2$$

where $p = \min\{i \mid H_i \not\subset E_1\}$. We want to prove that this is a Jordan-Hölder filtration for $(E_1 + E_2, \alpha')$. It suffices to prove that ${}^{E_1+H_j}/_{E_1+H_{j-1}}$ with its induced framing γ_j is stable for $j = p,\dots,t$ (we put $H_{p-1} = 0$). First note that by Lemma 2.48, we get $P(E_1 + H_j, \alpha') = \mathrm{rk}(E_1 + H_j)p(\mathcal{E})$ and $P(E_1 + H_{j-1}, \alpha') = \mathrm{rk}(E_1 + H_{j-1})p(\mathcal{E})$, hence

$$P({}^{E_1+H_j}/_{E_1+H_{j-1}}, \gamma_j) = \mathrm{rk}({}^{E_1+H_j}/_{E_1+H_{j-1}})p(\mathcal{E}).$$

Since ${}^{E}/_{E_1+H_{j-1}}$ is torsion free, $\mathrm{rk}({}^{E_1+H_j}/_{E_1+H_{j-1}}) > 0$. Let ${}^{T}/_{E_1+H_{j-1}}$ be a subsheaf of ${}^{E_1+H_j}/_{E_1+H_{j-1}}$. We have

$$\begin{aligned} P({}^{T}/_{E_1+H_{j-1}}, \gamma_j') &= P(T,\alpha') - P(E_1 + H_{j-1}, \alpha') \leq \mathrm{rk}(T)p(\mathcal{E}) - \mathrm{rk}(E_1 + H_{j-1})p(\mathcal{E}) \\ &= \mathrm{rk}({}^{T}/_{E_1+H_{j-1}})p(\mathcal{E}) = \mathrm{rk}({}^{T}/_{E_1+H_{j-1}})p({}^{E_1+H_j}/_{E_1+H_{j-1}}, \gamma_j), \end{aligned}$$

so the framed sheaf $({}^{E_1+H_j}/_{E_1+H_{j-1}}, \gamma_j)$ is semistable. Moreover we can construct the following exact sequence of coherent sheaves

$$0 \longrightarrow {}^{E_1\cap H_j}/_{E_1\cap H_{j-1}} \longrightarrow {}^{H_j}/_{H_{j-1}} \xrightarrow{\varphi} {}^{E_1+H_j}/_{E_1+H_{j-1}} \longrightarrow 0.$$

Recall that the induced framing on E_1 is zero, hence the induced framing on ${}^{E_1\cap H_j}/_{E_1\cap H_{j-1}}$ is zero as well and therefore the morphism φ induces a surjective morphism between framed sheaves

$$\varphi \colon ({}^{H_j}/_{H_{j-1}}, \beta_j) \longrightarrow ({}^{E_1+H_j}/_{E_1+H_{j-1}}, \gamma_j).$$

Since $({}^{H_j}/_{H_{j-1}}, \beta_j)$ is stable, by Corollary 2.22 the morphism φ is injective, hence it is an isomorphism. $\square$

Now we introduce the *extended framed socle* of a semistable (D,F)-framed sheaf, that plays a similar role of the maximal destabilizing subsheaf of a framed sheaf of positive rank.

Definition 2.52. Let $\mathcal{E} = (E, \alpha)$ be a semistable (D, F)-framed sheaf. We call *framed socle* of $\mathcal{E}$ the subsheaf of E given by the sum of all framed saturated subsheaves $E' \subset E$ such that the framed sheaf $\mathcal{E}' = (E', \alpha|_{E'})$ is stable with reduced framed Hilbert polynomial $p(\mathcal{E}') = p(\mathcal{E})$.

Let $\mathcal{E} = (E, \alpha)$ be a semistable (D, F)-framed sheaf. Consider the following two conditions on framed saturated subsheaves $E' \subset E$:

(a) $p(\mathcal{E}') = p(\mathcal{E})$,
(b) each component of $gr(\mathcal{E}')$ is isomorphic (as a framed sheaf) to a subsheaf of E.

Let E_1 and E_2 be two different framed saturated subsheaves of E satisfying conditions (a) and (b). By previous lemmas the subsheaf $E_1 + E_2$ is a framed saturated subsheaf of E satisfying conditions (a) and (b) as well.

Definition 2.53. For a semistable (D, F)-framed sheaf $\mathcal{E} = (E, \alpha)$, we call *extended framed socle* the maximal framed saturated subsheaf of E satisfying the above conditions (a) and (b).

Proposition 2.54. *Let G be the extended framed socle of a semistable (D, F)-framed sheaf $\mathcal{E} = (E, \alpha)$. Then*

(1) *G contains the framed socle of $\mathcal{E}$.*
(2) *If $\mathcal{E}$ is simple and not stable, then G is a proper subsheaf of E.*

Proof. (1) It follows directy from the definition.

(2) Let $E_\bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$ be a Jordan-Hölder filtration of $\mathcal{E}$. If $E = G$, the framed sheaf $(E/E_{l-1}, \alpha_l)$ is isomorphic (as framed sheaf) to a proper subsheaf $E' \subset E$ with induced framing α'. The composition of morphisms of framed sheaves

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "\alpha"] & E/E_{l-1} \arrow[r, "\sim"] \arrow[d, "\alpha_l"] & E' \arrow[r, "i"] \arrow[d, "\alpha'"] & E \arrow[d, "\alpha"] \\ F \arrow[r, "\cdot\nu"] & F \arrow[r, "\cdot\lambda"] & F \arrow[r, "\cdot\mu"] & F \end{tikzcd}$$

induces a morphism $\varphi\colon \mathcal{E} \to \mathcal{E}$ that is not a scalar endomorphism of $\mathcal{E}$. $\square$

Corollary 2.55. *A (D, F)-framed sheaf $\mathcal{E} = (E, \alpha)$ is stable with respect to δ if and only if it is geometrically stable.*

Proof. Assume $\mathcal{E}$ is stable but not geometrically stable. Let K be a field extension of k. According to the previous lemma, the extended framed socle G of $f^*(\mathcal{E})$, where $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \xrightarrow{f} X$, is a proper subsheaf of $f^*(E)$. Since the extended framed socle is invariant under all automorphisms in $\mathrm{Gal}(K/k)$, it is already defined over k, thus we get a contradiction (cf. the arguments in the proof of Theorem 2.40). On the other hand, since the framed Hilbert polynomial is preserved under base extensions, if $\mathcal{E}$ is not stable, then it is not geometrically stable. $\square$

7. Slope-(semi)stability

In this section we give a generalization to framed sheaves of the Mumford-Takemoto (semi)stability condition for torsion free sheaves (see Definition 1.2.12 in [**35**]). Also in this case one can construct examples of framed sheaves that are semistable with respect to this new condition but the underlying coherent sheaves are not μ-semistable and vice versa.

Definition 2.56. A framed sheaf $\mathcal{E} = (E, \alpha)$ of positive rank is μ-(semi)stable with respect to δ_1 if and only if $\ker \alpha$ is torsion free and the following conditions are satisfied:

(i) $\mathrm{rk}(E)\deg(E')\ (\leq)\ \mathrm{rk}(E')\deg(\mathcal{E})$ for all subsheaves $E' \subset \ker\alpha$,
(ii) $\mathrm{rk}(E)(\deg(E')-\delta_1)\ (\leq)\ \mathrm{rk}(E')\deg(\mathcal{E})$ for all subsheaves $E' \subset E$ with $\mathrm{rk}(E') < \mathrm{rk}(E)$.

One has the usual implications among different stability properties of a framed module of positive rank:

$$\mu-\text{stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu-\text{semistable}.$$

Definition 2.57. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\mathrm{rk}(E) = 0$. If α is injective, we say that $\mathcal{E}$ is μ-semistable[2]. Moreover, if the degree of E is δ_1, we say that $\mathcal{E}$ is μ-stable with respect to δ_1.

All the previous results hold also for μ-(semi)stability.

For $i \geq 0$, let us denote by $\mathrm{Coh}_i(X)$ the full subcategory of $\mathrm{Coh}(X)$ whose objects are sheaves of dimension less or equal to i. Let $\mathrm{Coh}_{d,d-1}(X)$ be the quotient category $\mathrm{Coh}_d(X)/\mathrm{Coh}_{d-1}(X)$. In Section 1.6 of [**35**], Huybrechts and Lehn define the notion of μ-Jordan-Hölder filtration for μ-semistable sheaves E in the category $\mathrm{Coh}_{d,d-1}(X)$. For a μ-semistable torsion free sheaf E, the graded object associated to a μ-Jordan-Hölder filtration is uniquely determined only in codimension one.

In our case, we define μ-Jordan-Hölder filtrations by using filtrations in which every term is a framed saturared subsheaf of the next term. In this way, the graded object is uniquely determined. The notions we gave in Section 6 of this chapter for semistable framed sheaves of positive rank will be extended in this section to μ-semistable framed sheaves of positive rank by using this definition of μ-Jordan-Hölder filtration. Thus, when the framing of a μ-semistable framed sheaf is zero, our definition of μ-Jordan-Hölder filtration does not coincide with the nonframed one given by Huybrechts and Lehn (cf. Section 1.6 in [**35**]).

8. Boundedness I

In order to construct moduli spaces one first has to ensure that the set of sheaves one wants to parametrize is not too big. Indeed the family of semistable framed sheaves is *bounded*, i.e., it is reasonably small. In this section, we introduce the notion of bounded family and give some characterizations, by using the so-called *Mumford-Castelnuovo regularity.* Thanks to a trick that allows one to use results about torsion free sheaves in the framed case, we give a proof of the boundedness of the family of (μ)-semistable framed sheaves with a fixed Hilbert polynomial P.

Let Y be a projective scheme over k and $\mathcal{O}_Y(1)$ a very ample line bundle.

[2]For torsion sheaves, the definition of μ-semistability of the corresponding framed sheaves does not depend on δ_1.

Definition 2.58. A family of isomorphism classes of coherent sheaves on Y is bounded if there is a k-scheme S of finite type and a coherent $\mathcal{O}_{S\times Y}$-sheaf G such that the given family is contained in the set $\{G|_{\mathrm{Spec}(k(s))\times Y} \,|\, s \text{ a closed point in } S\}$.

Later we use the word *family* in a different setting (cf. Chapter 3). Now we give two characterizations of this notion.

Proposition 2.59 (Proposition 1.2 in [**26**]). *Let $\mathcal{G}$ and $\mathcal{G}'$ be two bounded families of coherent sheaves on Y. Then*

(1) *the families of kernels, cokernels and images of morphisms $G \to G'$, where $G \in \mathcal{G}$ and $G' \in \mathcal{G}'$, are bounded.*
(2) *the family of extensions of an element of $\mathcal{G}$ by an element of $\mathcal{G}'$ is bounded.*

Before giving the second characterization, we need to introduce the notion of *Mumford-Castelnuovo regularity.*

Definition 2.60. Let m be an integer. A coherent sheaf G is said to be *m-regular*, if

$$\mathrm{H}^i(X, G(m-i)) = 0 \text{ for all } i > 0.$$

Because of Serre's vanishing theorem, for any sheaf G there is an integer m such that G is m-regular. It is possible to prove that if G is m-regular, then $G(m)$ is globally generated. Moreover, if G is m-regular then G is m'-regular for all integers $m' \geq m$. Because of this fact, the following definition makes sense.

Definition 2.61. The *Mumford-Castelnuovo regularity* of a coherent sheaf G is the number $\mathrm{reg}(G) = \inf\{m \in \mathbb{Z} \,|\, G \text{ is } m\text{-regular}\}$.

The regularity of G is $-\infty$ if and only if G is a zero-dimensional sheaf.

Proposition 2.62. *The following properties of family of sheaves $\{G_\iota\}_{\iota\in I}$ are equivalent:*

(i) *The family is bounded.*
(ii) *The set of Hilbert polynomials $\{P(G_\iota)\}_{\iota\in I}$ is finite and there is a uniform bound $\mathrm{reg}(G_\iota) \leq \rho$ for all $\iota \in I$.*
(iii) *The set of Hilbert polynomials $\{P(G_\iota)\}_{\iota\in I}$ is finite and there is a coherent sheaf G such that all G_ι admit surjective morphisms $G \to G_\iota$.*

Proof. See Lemma 1.7.6 in [**35**] and Theorem 2.1 in [**26**]. □

Now we would like to prove that the family of μ-semistable framed sheaves of positive rank on a polarized variety $(X, \mathcal{O}_X(1))$ of dimension d is bounded. To do this, we want to use the following result due to Maruyama:

Theorem 2.63 ([**52**]). *Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d. Let P be a numerical polynomial and C a constant. Then the family of torsion free sheaves G on X with Hilbert polynomial P and $\mu_{max}(G) \leq C$ is bounded, where $\mu_{\max}(G)$ is the maximal slope of G, that is, the slope of the maximal destabilizing subsheaf of G.*

To apply this result in the study of framed sheaves, we need to use the following trick. For the framing sheaf F, choose once and for all a fixed locally free sheaf $\hat{F}$ and a surjective morphism $\phi\colon \hat{F} \to F$. We denote by B the kernel of ϕ. Then to each framed sheaf $\mathcal{E} = (E, \alpha)$ of positive rank we can associate a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] \\
0 \arrow[r] & \ker\alpha \arrow[r] \arrow[d, equal] & \hat{E} \arrow[r, "\hat{\alpha}"] \arrow[d, "\hat{\phi}"] & \hat{F} \arrow[d, "\phi"] \\
0 \arrow[r] & \ker\alpha \arrow[r] & E \arrow[r, "\alpha"] \arrow[d] & F \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

The second row of the diagram shows that $\hat{E}$ is torsion free if the kernel of α is torsion free, hence in particular this happens if (E, α) is μ-semistable.

Proposition 2.64. *Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d. The family of framed sheaves of positive rank on X, μ-semistable with respect to δ_1 and with fixed Hilbert polynomial P, is bounded.*

Proof. Let $\mathcal{E} = (E, \alpha)$ be a μ-semistable framed sheaf of positive rank on X. Let us consider the torsion free sheaf $\hat{E}$ associated to E, given by the previous diagram. Since $\hat{F}$ and ϕ are fixed, the Hilbert polynomial $P(\hat{E}) = P + P(B)$ of $\hat{E}$ does not depend on $\mathcal{E}$. Let G be a nonzero subsheaf of $\hat{E}$. Let us denote by G its image through $\hat{\phi}$ and by B_G the kernel of the restriction morphism $\phi|_{\hat{G}}$. By the μ-semistability of $\mathcal{E}$, we get $\deg(G) \leq \mathrm{rk}(G)(\mu(E) + \delta_1)$. Hence

$$\mu(\hat{G}) = \frac{\deg(G) + \deg(B_G)}{\mathrm{rk}(\hat{G})} \leq \frac{\mathrm{rk}(G)(\mu(E) + \delta_1) + \mathrm{rk}(B_G)\mu_{max}(B)}{\mathrm{rk}(\hat{G})}.$$

This show that $\mu_{max}(\hat{E})$ is uniformly bounded from above. Therefore, by Theorem 2.63, the family of sheaves $\hat{E}$ associated to μ-semistable framed sheaves $\mathcal{E}$ is bounded. Since B is fixed and the sheaves E are quotients of the sheaves $\hat{E}$, the family of sheaves E associated to μ-semistable framed sheaves (E, α) of positive rank with fixed polynomial P is bounded by Proposition 2.59. □

By using the same argument, one can prove the following.

Proposition 2.65. *Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d. The family of framed sheaves of positive rank on X, semistable with respect to δ and with fixed Hilbert polynomial P, is bounded.*

CHAPTER 3

Families of framed sheaves

In the first chapter we proved some elementary properties of framed sheaves related to semistability. In this chapter we see how these properties vary in algebraic families. The main result of the chapter is the construction of the relative Harder-Narasimhan filtration. In Section 1 we recall the notion of flatness for coherent sheaves and define the notion of flat family of framed sheaves. In Section 2 we construct a framed version of the Grothendieck Quot scheme and as a byproduct we obtain a *universal quotient* (with fixed Hilbert polynomial) of a family of framed sheaves such that the induced framing is either nonzero at each fibre or zero at each fibre. In this way not only the Hilbert polynomial of that quotient but also its framed Hilbert polynomial is constant along the fibres. In Section 3 we introduce the notion of hat-slope of a coherent sheaf and provide a boundedness result for families of quotients of a given family of framed sheaves. That will be useful in the constructions of the minimal framed-destabilizing quotient of a fixed family of framed sheaves and the relative Harder-Narasimhan filtration given in Section 4.

All the results of this chapter hold also for the μ-(semi)stability condition.

1. Flat families

In this section we recall the definition of flatness. Moreover, we state some properties that we shall use in the following. Finally we introduce the notion of families of framed sheaves of positive rank.

Let $g\colon Y \to S$ be a morphism of finite type of Noetherian schemes.

Definition 3.1. A *flat family of coherent sheaves on the fibres of the morphism* $g\colon Y \to S$ is a coherent sheaf G on Y, which is flat over S.

This means that for each point $y \in Y$ the stalk G_y is flat over the local ring $\mathcal{O}_{S,f(y)}$. If G is S-flat, G_T is T-flat for any base change $T \to S$. If $0 \to G' \to G \to G'' \to 0$ is a short exact sequence of coherent $\mathcal{O}_Y$-sheaves and if G'' is S-flat then G' is S-flat if and only if G is S-flat. If $Y \cong S$, G is S-flat if and only if G is locally free.

Assume that g is a projective morphism and consider a g-ample line bundle $\mathcal{O}_Y(1)$ on Y, that is, a line bundle on Y such that the restriction to any fibre Y_s is ample for $s \in S$. Let G be a coherent $\mathcal{O}_Y$-sheaf. Let us consider the following assertions:

(1) G is S-flat,
(2) for all sufficiently large m the sheaves $g_*(G \otimes \mathcal{O}_Y(m))$ are locally free,
(3) the Hilbert polynomial $P(G_s)$ is locally constant as a function of $s \in S$.

Proposition 3.2 (Theorem III 9.9 in [**29**])**.** *There are implications* $1 \Leftrightarrow 2 \Rightarrow 3$. *If* S *is reduced then also* $3 \Rightarrow 1$.

Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d, S an integral k-scheme of finite type and $f\colon X \to S$ a projective flat morphism. It is easy to prove that $\mathcal{O}_X(1)$ is a f-ample line bundle on X.

Let us denote by $\bar{d}$ the dimension of the fibre X_s for $s \in S$. Fix a flat family of sheaves F of rank zero on the fibres of f and a rational polynomial $\bar{\delta}$ of degree $\bar{d} - 1$ and positive leading coefficient $\bar{\delta}_1$.

Now we introduce the notion of *flat families of framed sheaves*. We want to deal with families parametrizing framed sheaves of positive rank with nonzero framings. Moreover, we want to avoid the possibility that in some fibre the kernel of the framing destabilizes the corresponding framed sheaf. For these reasons, we give the following *ad hoc* definition.

Definition 3.3. A *flat family of framed sheaves of positive rank on the fibres of the morphism* f consists of a framed sheaf $\mathcal{E} = (E, \alpha\colon E \to F)$ on X, where $\alpha_s \neq 0$ and $\mathrm{rk}(E_s) > 0$ for all $s \in S$ and E and $\mathrm{Im}\,\alpha$ are flat families of coherent sheaves on the fibres of f.

Remark 3.4. By flatness of E and $\mathrm{Im}\,\alpha$, we have that also $\ker\alpha$ is S-flat.

Let us consider a flat family $\mathcal{E} = (E, \alpha)$ of framed sheaves of positive rank r on the fibres of f such that $P(\mathrm{Im}\,\alpha_s) \geq \bar{\delta}$ for $s \in S$. From now on we fix S, $f\colon X \to S$, F, $\bar{\delta}$ and $\mathcal{E} = (E, \alpha)$ as introduced above, unless otherwise stated.

2. Relative framed Quot scheme

In this section we introduce the notions of representability and (universal) corepresentability for a contravariant functor. We recall the construction of the relative Quot scheme and construct the relative framed Quot scheme as a closed subscheme of it.

Let $\mathcal{C}$ be a category, $\mathcal{C}^\circ$ the opposite category, i.e., the category with the same objects and reversed arrows, and let $\mathcal{C}'$ be the functor category whose objects are the functors $\mathcal{C}^\circ \to (Sets)$ and whose morphisms are the natural transformations between functors. The Yoneda Lemma (weak version) states that the functor $\mathcal{C} \to \mathcal{C}'$ which associates to $M \in \mathrm{Ob}(\mathcal{C})$ the functor $\mathrm{Mor}_{\mathcal{C}}(\cdot, M)\colon T \to \mathrm{Mor}_{\mathcal{C}}(T, M)$ embeds $\mathcal{C}$ as a full subcategory into $\mathcal{C}'$. A functor in $\mathcal{C}'$ of the form $\mathrm{Mor}_{\mathcal{C}}(\cdot, M)$ is said to be *represented* by the object M.

Definition 3.5. Let $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$ be a functor. A *universal object* for $\mathcal{F}$ is a pair (M, ξ) consisting of an object M of $\mathcal{C}$, and an element $\xi \in \mathcal{F}(M)$, with the property that for each object U of $\mathcal{C}$ and each $\sigma \in \mathcal{F}(U)$, there is a unique arrow $g\colon U \to M$ such that $(\mathcal{F}(g))(\xi) = \sigma \in \mathcal{F}(U)$.

By the Yoneda lemma, there is a bijective correspondence $\mathrm{Mor}_{\mathcal{C}'}(\mathrm{Mor}_{\mathcal{C}}(\cdot, M), \mathcal{F}) \cong \mathcal{F}(M)$. From this fact, we get the following result.

Proposition 3.6. *A functor $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$ is representable if and only if it has an universal object.*

Also, if $\mathcal{F}$ has a universal object (M, ξ), $\mathcal{F}$ is represented by M.

Definition 3.7. A functor $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$ is *corepresented* by $F \in \mathrm{Ob}(\mathcal{C})$ if there is a $\mathcal{C}'$-morphism $\varphi\colon \mathcal{F} \to \mathrm{Mor}_{\mathcal{C}}(\cdot, F)$ such that any morphism $\varphi'\colon \mathcal{F} \to \mathrm{Mor}_{\mathcal{C}}(\cdot, F')$ factors through a unique morphism $f_{\mathcal{C}'}\colon \mathrm{Mor}_{\mathcal{C}}(\cdot, F) \to \mathrm{Mor}_{\mathcal{C}}(\cdot, F')$. A functor $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$ is *universally corepresented* by $\varphi\colon \mathcal{F} \to \mathrm{Mor}_{\mathcal{C}}(\cdot, F)$ if for any morphism $\mathrm{Mor}_{\mathcal{C}}(\cdot, T) \to \mathrm{Mor}_{\mathcal{C}}(\cdot, F)$, the fiber product $\mathcal{T} = \mathrm{Mor}_{\mathcal{C}}(\cdot, T) \times_{\mathrm{Mor}_{\mathcal{C}}(\cdot, F)} \mathcal{F}$ is corepresented by T. Finally, $\mathcal{F}$ is *represented* by F if $\varphi\colon \mathcal{F} \to \mathrm{Mor}_{\mathcal{C}}(\cdot, F)$ is a $\mathcal{C}'$-isomorphism.

If F represents $\mathcal{F}$ then it also universally corepresents $\mathcal{F}$, and if F corepresents $\mathcal{F}$ then it is unique up to a unique isomorphism. This follows directly from the definition.

Let us recall the definition of the relative Quot scheme. Let $\mathcal{C} = (Sch/S)$ be the category of Noetherian S-schemes of finite type. Let E be a coherent $\mathcal{O}_X$-module and $P \in \mathbb{Q}[n]$ a numerical polynomial, i.e., a rational polynomial such that for any $n \in \mathbb{Z}$, $P(n) \in \mathbb{Z}$. We define the functor

$$\underline{Quot}_{X/S}(E, P)\colon \mathcal{C}^\circ \longrightarrow (Sets)$$

as follows: if $T \to S$ is an object in $\mathcal{C}$, let $\underline{Quot}_{X/S}(E, P)(T)$ be the set of all T-flat coherent quotient sheaves $E_T \to Q$ with $P(Q_t) = P$ for all $t \in T$, modulo isomorphism. If $g\colon T' \to T$ is an S-morphism, let $\underline{Quot}_{X/S}(E, P)(g)$ be the map that sends $E_T \to Q$ to $E_{T'} \to g_X^* Q$, where $g_X\colon X_{T'} \to X_T$ is the induced morphism by g.

Theorem 3.8 (Theorem 2.2.4 in [**35**])**.** *The functor $\underline{Quot}_{X/S}(E, P)$ is represented by a projective S-scheme $\pi\colon Quot_{X/S}(E, P) \to S$.*

In the following we call *Quot scheme* the scheme $Quot_{X/S}(E, P)$.

Now we introduce the *framed version* of the Quot scheme. Let $\mathcal{E} = (E, \alpha)$ be a S-flat family of framed sheaves of positive rank and P a numerical polynomial. Define the functor

$$\underline{FQuot}_{X/S}(E, \alpha, P)\colon \mathcal{C}^\circ \longrightarrow (Sets)$$

in the following way:

- For an object $T \to S$, $\underline{FQuot}_{X/S}(E, \alpha, P)(T \to S)$ is the set consisting of coherent quotient sheaves (modulo isomorphism) $E_T \xrightarrow{q} Q \to 0$ such that
  (i) Q is T-flat,
  (ii) the Hilbert polynomial of Q_t is P for all $t \in T$,
  (iii) there is a induced morphism $\widetilde{\alpha}\colon Q \to F_T$ such that $\widetilde{\alpha} \circ q = \alpha_T$.
- For a S-morphism $g\colon T' \to T$, $\underline{FQuot}_{X/S}(E, \alpha, P)(g)$ is $\underline{Quot}_{X/S}(E, P)(g)$.

Obviously, this functor is a subfunctor of $\underline{Quot}_{X/S}(E, P)$. We have the following result.

Theorem 3.9. *The functor $\underline{FQuot}_{X/S}(E, P)$ is represented by a projective S-scheme*

$$\pi_{fr}\colon FQuot_{X/S}(E, \alpha, P) \to S,$$

that is a closed subscheme of $Quot_{X/S}(E, P)$.

Proof. The property (iii) in the definition is closed, hence one can construct a closed subscheme $FQuot_{X/S}(E, \alpha, P) \subset Quot_{X/S}(E, P)$ that represents the functor $\underline{FQuot}_{X/S}(E, \alpha, P)$, by using the same arguments of the proof of Theorem 1.6 in [**73**]. Moreover the composition of morphisms

$$\pi_{fr}\colon FQuot_{X/S}(E, \alpha, P) \hookrightarrow Quot_{X/S}(E, P) \xrightarrow{\pi} S$$

makes $FQuot_{X/S}(E, \alpha, P)$ a projective S-scheme. □

Roughly speaking, $FQuot_{X/S}(E, \alpha, P)$ parametrizes all the quotients $E_s \xrightarrow{q} Q$, for $s \in S$, such that the induced framing on $\ker q$ is zero.

The *universal object* on $FQuot_{X/S}(E, \alpha, P) \times_S X$ is the pull-back of the universal object on $Quot_{X/S}(E, P) \times_S X$ with respect to the morphism $FQuot_{X/S}(E, \alpha, P) \times_S X \to Quot_{X/S}(E, P) \times_S X$, induced by the closed embedding $FQuot_{X/S}(E, \alpha, P) \hookrightarrow Quot_{X/S}(E, P)$.

Let $s \in S$ and $q \in \pi_{fr}^{-1}(s)$ be k-rational points corresponding to the commutative diagram on X_s

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] & E_s \arrow[r, "q"] \arrow[d, "\alpha_s"] & Q \arrow[r] \arrow[dl, "\tilde{\alpha}"] & 0 \\ & & F_s & & \end{tikzcd}$$

One has the following result about the tangent space of $\pi_{fr}^{-1}(s)$ at q:

Proposition 3.10. *The kernel of the linear map* $(\mathrm{d}\pi_{fr})_q \colon T_q FQuot_{X/S}(E, \alpha, P) \to T_s S$ *is isomorphic to the linear space* $\mathrm{Hom}(K, {}^{\ker \alpha_s}/_K) = \mathrm{Hom}(\mathcal{K}, \mathcal{Q})$.

Proof. It suffices to readapt the techniques used in the proof of the corresponding result for π (see Proposition 4.4.4 in [**74**]). □

Now we have a tool for constructing a flat family of quotients (with a fixed Hilbert polynomial) of $\mathcal{E}$ such that the induced framing is nonzero in each fibre. Using the relative Quot scheme associated to E, one can construct a flat family of quotients such that the induced framing is *generically* zero.

3. Boundedness II

In this section we characterize the families of quotient sheaves of a family of framed sheaves. In particular we prove that these families are bounded if the *hat-slopes* of their elements are bounded from above. As an application of this result, we prove that the property of being (semi)stable is open in families of framed sheaves.

Definition 3.11. Let E a coherent sheaf. We call *hat-slope* the rational number

$$\hat{\mu}(E) = \frac{\beta_{\dim(E)-1}(E)}{\beta_{\dim(E)}(E)}.$$

For a polynomial $P(n) = \sum_{i=0}^{t} \beta_i n^i/i!$ we define $\hat{\mu}(P) = \beta_{t-1}/\beta_t$.

Lemma 3.12 (Lemma 2.5 in [**26**])**.** *Let $Y \to S$ be a projective morphism of Noetherian schemes and denote by $\mathcal{O}_Y(1)$ a line bundle on Y, which is very ample relative to S. Let L be a coherent sheaf on Y and $\mathscr{E}$ the set of isomorphism classes of quotient sheaves G of L_s for s running over the points of S. Suppose that the dimension of Y_s is $\leq r$ for all s. Then the coefficient $\beta_r(G)$ is bounded from above and from below, and $\beta_{r-1}(G)$ is bounded from below. If $\beta_{r-1}(G)$ is bounded from above, then the family of sheaves ${}^{G}/_{T(G)}$ is bounded.*

Corollary 3.13. *Let E be a flat family of coherent sheaves on the fibres of the morphism $f \colon X \to S$. Then the family of torsion free quotients Q of E_s for $s \in S$ with hat-slopes bounded from above is a bounded family.*

From this result and Lemma 2.62 it follows that there are only a finite number of rational polynomials corresponding to Hilbert polynomials of destabilizing quotients Q of E_s for $s \in S$. Thus it is possible to find the "minimal" polynomial that will be the Hilbert polynomial of the minimal destabilizing quotient of E_s for a generic point $s \in S$. Now we want to use the same argument in the framed case.

Let $\mathscr{F}_1$ be a family of quotients $E_s \xrightarrow{q} Q$, for $s \in S$, such that

- $\ker \alpha_s$ is torsion free,
- $\ker q \not\subseteq \ker \alpha_s$,
- Q is torsion free and $\hat{\mu}(Q) < \hat{\mu}(E_s)$.

Proposition 3.14. *The family $\mathscr{F}_1$ is bounded.*

Proof. The family $\mathscr{F}_1$ is contained in the family of torsion free quotients of E, with hat-slopes bounded from above, hence it is bounded by Corollary 3.13 and Proposition 2.62. $\square$

Let $\mathscr{F}_2$ be a family of quotients

$$\begin{tikzcd} E_s \arrow[r, "q"] \arrow[d, "\alpha_s"] & Q \arrow[r] \arrow[dl, "\tilde{\alpha}"] & 0 \\ F_s & & \end{tikzcd}$$

for which

- $\ker \alpha_s$ is a torsion free sheaf,
- Q fits into a exact sequence
$$0 \longrightarrow Q' \longrightarrow Q \xrightarrow{\tilde{\alpha}} \operatorname{Im} \alpha_s \longrightarrow 0$$
where $Q' = \ker \tilde{\alpha}$ is a nonzero torsion free quotient of $\ker \alpha_s$,
- $\hat{\mu}(Q) < \hat{\mu}(E_s) + \bar{\delta}_1$.

Proposition 3.15. *The family $\mathscr{F}_2$ is bounded.*

Proof. Since a family given by extensions of elements from two bounded families is bounded (cf. Proposition 2.59), it suffices to prove that every element in $\mathscr{F}_2$ is an extension of two elements that belong to two bounded families. By definition of flat family of framed sheaves, the families $\{\ker \alpha_s\}_{s \in S}$ and $\{\operatorname{Im} \alpha_s\}_{s \in S}$ are bounded. So it remains to prove that the family of quotients Q' is bounded. Since the family $\{\ker \alpha_s\}$ is bounded, there exists a coherent sheaf T on X such that $\ker \alpha_s$ admits a surjective morphism $T_s \to \ker \alpha_s$ (see Lemma 2.62), hence the quotient Q' admits a surjective morphism $T_s \to Q'$. By Lemma 3.12, the coefficient $\beta_{\bar{d}}(Q')$ is bounded from above and from below and the coefficient $\beta_{\bar{d}-1}(Q')$ is bounded from below. Moreover, since $\{E_s\}$ and $\{\operatorname{Im} \alpha_s\}$ are bounded families, the coefficients of their Hilbert polynomials are uniformly bounded from above and from below, hence $\hat{\mu}(E_s)$ is uniformly bounded from above and from below and since $\hat{\mu}(Q) < \hat{\mu}(E_s) + \bar{\delta}_1$, we obtain that $\hat{\mu}(Q)$ is uniformly bounded from above. By a simple computation we obtain that $\hat{\mu}(Q') \leq A\hat{\mu}(Q) + B$ for some constants A, B, hence we get that $\hat{\mu}(Q')$ is uniformly bounded from above and by Lemma 3.12 we obtain the assertion. $\square$

As an application of the previous results we obtained the following.

Proposition 3.16. *Let S, $f\colon X \to S$, F, $\bar{\delta}$ and $\mathcal{E} = (E, \alpha)$ be as before. The set of points $s \in S$ such that (E_s, α_s) is (semi)stable with respect to $\bar{\delta}$ is open in S.*

Proof. In the proof we apply the same arguments as in the nonframed case (see Proposition 2.3.1 in [**35**]).

Let P denote the Hilbert polynomial of E. For $i = 1, 2$ let $A_i \subset \mathbb{Q}[n]$ be the set consisting of polynomials P'' such that there is a point $s \in S$ and a surjection $E_s \to E''$, where $P_{E''} = P''$ and $E'' \in \mathscr{F}_i$. Note that by Propositions 3.14 and 3.15 the sets A_1 and A_2 are finite. Denote by p'' the reduced Hilbert polynomial associated to the rational polynomial P'' and by r'' its leading coefficient.

Semistable case. Define the sets

$$\begin{aligned} T_1 &= \left\{ P'' \in A_1 \,|\, p'' < p - \frac{\bar{\delta}}{r} \right\}, \\ T_2 &= \left\{ P'' \in A_2 \,|\, p'' - \frac{\bar{\delta}}{r''} < p - \frac{\bar{\delta}}{r} \right\}. \end{aligned}$$

For any $P'' \in A_1$ we consider the relative Quot scheme $\pi : Quot_{X/S}(E, P'') \to S$. Since π is projective, the image $S(P'')$ is a closed subset of S. For any $P'' \in A_2$ the image $S_{fr}(P'')$ of $FQuot_{X/S}(E, \alpha, P'')$ through π_{fr} is closed in S. Thus

$$(E_s, \alpha_s) \text{ is semistable if and only if } s \notin \left(\bigcup_{P'' \in T_1} S(P'') \right) \cup \left(\bigcup_{P'' \in T_2} S_{fr}(P'') \right),$$

Note that these unions are finite, hence closed in S.

Stable case. The proof in this case is similar to the previous one, by using the sets

$$\begin{aligned} T_1' &= \left\{ P'' \in A_1 \,|\, p'' \leq p - \frac{\bar{\delta}}{r} \text{ and } P'' < P \right\}, \\ T_2' &= \left\{ P'' \in A_2 \,|\, p'' - \frac{\bar{\delta}}{r''} \leq p - \frac{\bar{\delta}}{r} \text{ and } P'' < P \right\}. \end{aligned}$$

□

4. Relative Harder-Narasimhan filtration

In this section we would like to construct a flat family of minimal framed-destabilizing quotients associated to a framed sheaf. The construction is more complicated than in the nonframed case (see Theorem 2.3.2 in [**35**]). Iterating this construction, we obtain the relative Harder-Narasimhan filtration of a family of framed sheaves of positive rank.

Theorem 3.17. *Let $(X, \mathcal{O}_X(1))$, S, $f\colon X \to S$, F, $\bar{\delta}$ and $\mathcal{E} = (E, \alpha)$ be as before. Then there is an integral k-scheme T of finite type, a projective birational morphism $g\colon T \to S$, a dense open subscheme $U \subset T$ and a flat quotient Q of E_T such that for all points t in U, (E_t, α_t) is a framed sheaf of positive rank where $\ker \alpha_t$ is nonzero and torsion free and Q_t is the minimal framed-destabilizing quotient of $\mathcal{E}_t$ with respect to $\bar{\delta}$ or $Q_t = E_t$.*

Moreover, the pair (g, Q) is universal in the sense that if $g'\colon T' \to S$ is any dominant morphism of k-integral schemes and Q' is a flat quotient of $E_{T'}$ satisfying the same property of Q, there is an S-morphism $h\colon T' \to T$ such that $h_X^*(Q) = Q'$.

Proof. Let P denote the Hilbert polynomial of E. For $i = 1, 2$, let $A_i \subset \mathbb{Q}[n]$ be as in the proof of Proposition 3.16. Let

$$\begin{aligned} B_1 &= \left\{ P'' \in A_1 \,|\, p'' < p - \frac{\bar{\delta}}{r} \right\}, \\ B_2 &= \left\{ P'' \in A_2 \,|\, p'' - \frac{\bar{\delta}}{r''} \leq p - \frac{\bar{\delta}}{r} \right\}. \end{aligned}$$

The set $B_1 \cup B_2$ is nonempty. We define an order relation on B_1: $P_1 \sqsubseteq P_2$ if and only if $p_1 \leq p_2$ and $r_1 \leq r_2$ in the case $p_1 = p_2$. We define an order relation on B_2: $P_1 \sqsubseteq P_2$ if and only if $p_1 - \frac{\bar{\delta}}{r_1} \leq p_2 - \frac{\bar{\delta}}{r_2}$ and $r_1 \leq r_2$ in the case of equality.

Let C_1 be the set of polynomials $P'' \in B_1$ such that $\pi(Quot_{X/S}(E, P'')) = S$ and for any $s \in S$ one has $\pi^{-1}(s) \not\subset FQuot_{X/S}(E, \alpha, P'')$. Let C_2 be the set of polynomials $P'' \in B_2$ such that $\pi_{fr}(FQuot_{X/S}(E, \alpha, P'')) = S$. Note that $C_1 \cup C_2$ is nonempty. Now we want to find a polynomial P_- in $C_1 \cup C_2$ that is the Hilbert polynomial of the minimal framed-destabilizing quotient of E_s for a general point $s \in S$.

Let us consider the relation $\sqsubset$ defined in the following way: for $P_1, P_2 \in B_1$ we have $P_1 \sqsubset P_2$ if and only if $P_1 \sqsubseteq P_2$ and $p_1 < p_2$ or $r_1 < r_2$ in the case $p_1 = p_2$. In a similar way we can define $\sqsubset$ for polynomials in B_2. Let P_-^i be a $\sqsubset$-minimal polynomial among all polynomials of C_i for $i = 1, 2$. Consider the following cases:

- Case 1: $p_-^1 < p_-^2 - \frac{\bar{\delta}}{r_-^2}$. Put $P_- := P_-^1$.
- Case 2: $p_-^1 > p_-^2 - \frac{\bar{\delta}}{r_-^2}$. Put $P_- := P_-^2$.
- Case 3: $p_-^1 = p_-^2 - \frac{\bar{\delta}}{r_-^2}$. If $r_-^2 < r_-^1$, put $P_- := P_-^2$, otherwise $P_- := P_-^1$.

Note that the set

$$\Big(\bigcup_{\substack{P'' \in B_1 \\ P'' \sqsubset P_-^1}} \pi(Quot_{X/S}(E, P'')) \Big) \cup \Big(\bigcup_{\substack{P'' \in B_2 \\ P'' \sqsubset P_-^2}} \pi_{fr}(FQuot_{X/S}(E, \alpha, P'')) \Big)$$

is a proper closed subscheme of S. Let U_- be its complement. Let U_{tf} be the dense open subscheme of S consisting of points s such that $\ker \alpha_s$ is torsion free. Put $V = U_- \cap U_{tf}$.

Suppose that $P_- \in C_2$, the other case is similar. By definition of P_- the projective morphism $\pi_{fr}\colon FQuot_{X/S}(E, \alpha, P_-) \to S$ is surjective. For any point $s \in S$ the fiber of π_{fr} at s parametrizes possible quotients of E_s with Hilbert polynomial P_-. If $s \in V$, then any such quotient is a minimal framed-destabilizing quotient by construction of V. Recall that the minimal framed-destabilizing quotient is unique by Proposition 2.24: this implies that $\pi_{fr}|_U\colon U := \pi_{fr}^{-1}(V) \to V$ is bijective. By Theorem 2.40, that quotient is defined over the residue field $k(s)$, hence for $t \in U$, $s = \pi_{fr}(t)$ one has $k(s) \simeq k(t)$. Let $t \in \pi_{fr}^{-1}(s)$ be a point corresponding to a diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] & E_t \arrow[r, "q"] \arrow[d, "\alpha_t"] & Q \arrow[r] \arrow[dl, "\tilde{\alpha}"] & 0 \\ & & F_t & & \end{tikzcd}$$

By Proposition 3.10, the Zariski tangent space of $\pi_{fr}^{-1}(s)$ at t is $\mathrm{Hom}(\mathcal{K}, \mathcal{Q})$. Since K is the maximal framed-destabilizing subsheaf of $\mathcal{E}_t$, we have that $\mathrm{Hom}(\mathcal{K}, \mathcal{Q}) = 0$ by Proposition 2.39 and therefore $\Omega_{U/V} = 0$, hence $\pi_{fr}|_U \colon U \to V$ is unramified. Since π_{fr} is projective, we have that $\pi_{fr}|_U$ is a proper morphism. Since V is integral, we obtain that $\pi_{fr}|_U$ is an isomorphism. Now let T be the clousure of U in $FQuot_{X/S}(E, \alpha, P_-)$ with its reduced subscheme structure and $f \colon = \pi|_T \colon T \to S$ is a projective birational morphism. We put Q equal to the pull-back on X_T of the universal quotient on $FQuot_{X/S}(E, \alpha, P_-) \times_S X$.

The proof of the universality of the pair (g, Q) is similar to that for the case of torsion free sheaves (second part of Theorem 2.3.2 in [**35**]), since to prove this part of the theorem we need only the universal property of $FQuot_{X/S}(E, \alpha, P_-)$ or $Quot_{X/S}(E, P_-)$. $\square$

Now we can conclude this section by giving the construction of the relative version of the Harder-Narasimhan filtration.

Theorem 3.18. *Let $(X, \mathcal{O}_X(1))$, S, $f \colon X \to S$, F, $\bar{\delta}$ and $\mathcal{E} = (E, \alpha)$ be as before. There exists an integral k-scheme T of finite type, a projective birational morphism $g \colon T \to S$ and a filtration*

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) \subset \cdots \subset \mathrm{HN}_l(\mathcal{E}) = E_T$$

such that the following holds:

- *The factors ${}^{\mathrm{HN}_i(\mathcal{E})}/_{\mathrm{HN}_{i-1}(\mathcal{E})}$ are T-flat for all $i = 1, \ldots, l$, and*
- *there is a dense open subscheme $U \subset T$ such that $(\mathrm{HN}_\bullet(\mathcal{E}))_t = g_X^* \mathrm{HN}_\bullet(\mathcal{E}_{g(t)})$ for all $t \in U$.*

Moreover, the pair $(g, \mathrm{HN}_\bullet(\mathcal{E}))$ is universal in the sense that if $g' \colon T' \to S$ is any dominant morphism of k-integral schemes and $E'_\bullet$ is a filtration of E_T satisfying these two properties, there is an S-morphism $h \colon T' \to T$ such that $h_X^*(\mathrm{HN}_\bullet(\mathcal{E})) = E'_\bullet$.

Proof. By applying Theorem 3.17 to the pair $(S, \mathcal{E})$ we get a projective birational morphism $g_1 : T_1 \to S$ of integral k-schemes of finite type, a dense open subsheme U_1 and a T_1-flat quotient Q with the properties asserted in that theorem. If $Q_t = E_t$ for all $t \in U_1$, we obtain the trivial relative Harder-Narasimhan filtration:

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 \subset \mathrm{HN}_1(\mathcal{E}) = E_{T_1}$$

Otherwise, Q is a flat family of sheaves of positive rank parametrized by T. If the induced framings on the fibres of Q are nonzero, then Q with the induced framing by α is a flat family of framed sheaves of positive rank parametrized by T and we can apply Theorem 3.17 to the pair $(T, \mathcal{Q})$. If on the contrary the framings of Q_t for $t \in U_1$ are zero, we can apply the nonframed version of the previous theorem (Theorem 2.3.2 in [**35**]) to the pair $(T, \mathcal{Q})$. In this way we obtain a finite sequence of morphisms

$$T_l \longrightarrow T_{l-1} \longrightarrow \cdots \longrightarrow T_1 = T \to S$$

and an associated filtration such that the composition of these morphism and the filtration have the required properties. The universality of the filtration follows from the universality of the relative minimal framed-destabilizing quotient. □

CHAPTER 4

Restriction theorems for μ-(semi)stable framed sheaves

In this chapter we generalize the Mehta-Ramanathan restriction theorems to framed sheaves. We limit our attention to the case in which the framing sheaf F is a coherent sheaf supported on a divisor D_F. In the framed case the results depend also on the parameter δ_1. Moreover the proofs are somehow more complicated than in the nonframed case (see, e.g., Section 7.2 in [**35**]) because of the presence of the framing. In Section 1 we provide the proof for the semistable case, in Section 2 we prove the stable case.

1. Slope-semistable case

In this section we provide a generalization of Mehta-Ramanathan's theorem for μ-semistable torsion free sheaves (Theorem 6.1 in [**53**]).

Theorem 4.1. *Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d. Let F be a coherent sheaf on X supported on a divisor D_F. Let $\mathcal{E} = (E, \alpha\colon E \to F)$ be a framed sheaf on X of positive rank with nontrivial framing. If $\mathcal{E}$ is μ-semistable with respect to δ_1, there exists a positive integer a_0 such that for all $a \geq a_0$ there is a dense open subset $U_a \subset |\mathcal{O}_X(a)|$ such that for all $D \in U_a$ the divisor D is smooth, meets transversally the divisor D_F and $\mathcal{E}|_D$ is μ-semistable with respect to $a\delta_1$.*

In order to prove this theorem, we need some preliminary results: for a positive integer a, let $\Pi_a := |\mathcal{O}_X(a)|$ be the complete linear system of hypersurfaces of degree a in X and let $Z_a := \{(D, x) \in \Pi_a \times X | x \in D\}$ be the *incidence variety* with its natural projections

$$\begin{array}{ccc} Z_a & \xrightarrow{q} & X \\ \big\downarrow{\scriptstyle p} & & \\ \Pi_a & & \end{array}$$

Remark 4.2. It is possible to give a schematic structure on Z_a so that p is a projective flat morphism (see Section 3.1 in [**35**]). Moreover $\mathrm{Pic}(Z_a) = q^*(\mathrm{Pic}(X)) \oplus p^*(\mathrm{Pic}(\Pi_a))$ (see Section 2 in [**53**]).

For all $D \in \Pi_a$, the Hilbert polynomials of the restrictions $E|_D$, $F|_D$ and $\mathrm{Im}\,\alpha|_D$ are indipendent from D, indeed, e.g., the Hilbert polynomial of $E|_D$ is $P(E|_D, n) = P(E, n) - P(E, n-a)$. Since Π_a is a reduced scheme, by Proposition 3.2 q^*F is a flat family of sheaves of rank zero on the fibres of p and $(q^*E, q^*\alpha)$ is a flat family of framed sheaves of positive rank on the fibres of p. For any a and for general $D \in \Pi_a$ the restriction $\ker\alpha|_D$ is torsion free (see Corollary 1.1.14 in [**35**]), hence the set $\{C \in \Pi_a \mid \ker\alpha|_C \text{ is torsion free}\} \subset \Pi_a$ is nonempty. Since $\mathcal{E}$ is μ-semistable with respect to δ_1, we have $\deg(\mathrm{Im}\,\alpha) \geq \delta_1$, hence

$\deg(\operatorname{Im} \alpha|_D) = a \deg(\operatorname{Im} \alpha) \geq a\delta_1$ for an integer $a > 0$. According to Theorem 3.17, which states the existence of the relative minimal μ-framed-destabilizing quotient with respect to $\bar{\delta}_1 = a\delta_1$, there are a dense open subset $V_a \subset \Pi_a$ and a V_a-flat quotient on $Z_{V_a} := V_a \times_{\Pi_a} Z_a$

$$\begin{tikzcd} (q^*E)|_{Z_{V_a}} \arrow[r, "q_a"] \arrow[d, "(q^*\alpha)|_{Z_{V_a}}"] & Q_a \\ (q^*F)|_{Z_{V_a}} & \end{tikzcd}$$

with a morphism $\tilde{\alpha}_a \colon Q_a \to (q^*F)|_{Z_{V_a}}$, such that for all $D \in V_a$ the framed sheaf $(E|_D, \alpha|_D)$ has positive rank, and $\ker \alpha|_D$ is torsion free; moreover, $Q_a|_D$ is a coherent sheaf of positive rank, $\tilde{\alpha}_a|_D$ is the framing induced by $\alpha|_D$ and $(Q_a|_D, \tilde{\alpha}_a|_D)$ is the minimal μ-framed-destabilizing quotient of $(E|_D, \alpha|_D)$. Let Q be an extension of $\det(Q_a)$ to some line bundle on all of Z_a. Then Q can be uniquely decomposed as $Q = q^*L_a \otimes p^*M$ with $L_a \in \operatorname{Pic}(X)$ and $M \in \operatorname{Pic}(\Pi_a)$. Note that $\deg(Q_a|_D) = a \deg(L_a)$ for $D \in V_a$.

Let $U_a \subset V_a$ be the dense open set of points $D \in V_a$ such that D is smooth and meets transversally the divisor D_F.

Let $\deg(a)$, $r(a)$ and $\mu_{fr}(a)$ denote the degree, the rank and the framed slope of the minimal μ-framed-destabilizing quotient of $(E|_D, \alpha|_D)$ for a general point $D \in \Pi_a$. By construction of the relative minimal μ-framed-destabilizing quotient, the quantity $\epsilon(\tilde{\alpha}_a|_D)$ is independent of $D \in V_a$, so we denote it by $\epsilon(a)$. Then we have $1 \leq r(a) \leq \operatorname{rk}(E)$ and

$$\frac{\mu_{fr}(a)}{a} = \frac{\deg L_a - \epsilon(a)\delta_1}{r(a)} \in \frac{\mathbb{Z}}{\delta_1''(\operatorname{rk}(E)!)} \subset \mathbb{Q},$$

where $\delta_1 = {}^{\delta_1'}/_{\delta_1''}$.

Let $l > 1$ be an integer, $a_1, \dots, a_l$ positive integers and $a = \sum_i a_i$. We would like to compare $r(a)$ (resp. ${}^{\mu_{fr}(a)}/_a$) with $r(a_i)$ (resp. ${}^{\mu_{fr}(a_i)}/_{a_i}$) for all $i = 1, \dots, l$. To do this, we use the following result, which allows us to compare the rank and the framed degree of Q_{a_i} in a generic fibre with the same invariants of a "special quotient" of $(q^*E)|_{Z_{V_a}}$.

Lemma 4.3 (Lemma 7.2.3 in [**35**]). *Let $l > 1$ be an integer, $a_1, \dots, a_l$ positive integers, $a = \sum_i a_i$, and $D_i \in U_{a_i}$ divisors such that $D = \sum_i D_i$ is a divisor with normal crossings. Then there is a smooth locally closed curve $C \subset \Pi_a$ containing the point $D \in \Pi_a$ such that $C \setminus \{D\} \subset U_a$ and $Z_C = C \times_{\Pi_a} Z_a$ is smooth in codimension 2.*

Remark 4.4. If $D_1 \in U_{a_1}$ is given, one can always find $D_i \in U_{a_i}$ for $i \geq 2$ such that $D = \sum_i D_i$ is a divisor with normal crossings.

Lemma 4.5. *Let $a_1, \dots, a_l$ be positive integers, with $l > 1$, and $a = \sum_i a_i$. Then $\mu_{fr}(a) \geq \sum_i \mu_{fr}(a_i)$ and in case of equality $r(a) \geq \max\{r(a_i)\}$.*

Proof. Let D_i be divisors satisfying the requirements of Lemma 4.3 and let C be the curve with the properties of 4.3. Over V_a there is the quotient

$$\begin{tikzcd} (q^*E)|_{Z_{V_a}} \arrow[r, "q_a"] \arrow[d, "(q^*\alpha)|_{Z_{V_a}}"] & Q_a \\ (q^*F)|_{Z_{V_a}} & \end{tikzcd} \tag{11}$$

Now we have to consider two cases:

(1) there exists a nonzero framing $\tilde{\alpha}_a$ on Q_a such that $(q^*\alpha)|_{Z_{V_a}} = \tilde{\alpha}_a \circ q_a$,
(2) $\ker q_a|_{D'} \not\subset \ker \alpha|_{D'}$ for all $D' \in V_a$.

For the first case we have that $\tilde{\alpha}_a|_{D'} \neq 0$ for all $D' \in V_a$. The restriction of diagram (11) to $Z_{V_a \cap C}$ is

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & (q^*E)|_{Z_{V_a\cap C}} \arrow[r, "q_a|_{Z_{V_a\cap C}}"] \arrow[d, "(q^*\alpha)|_{Z_{V_a\cap C}}"'] & Q_a|_{Z_{V_a\cap C}} \arrow[r] \arrow[dl, "\tilde{\alpha}_a|_{Z_{V_a\cap C}}"] & 0 \\ & & (q^*F)|_{Z_{V_a\cap C}} & & \end{tikzcd}$$

Since the morphism $Z_{V_a \cap C} \to Z_C$ is flat (because it is an open embedding), we have $\ker(q^*\alpha|_{Z_{V_a\cap C}}) = (\ker q^*\alpha|_{Z_C})|_{Z_{V_a\cap C}}$ and we can extend the inclusion $K \subset \ker q^*\alpha|_{Z_{V_a\cap C}}$ to an inclusion $K_C \subset \ker q^*\alpha|_{Z_C}$ on Z_C. Since $V_a \cap C = C \setminus \{D\}$, in this way we extend $Q_a|_{Z_{V_a\cap C}}$ to a C-flat quotient Q_C of $q^*E|_{Z_C}$ and we get the following commutative diagram

$$\begin{tikzcd} (q^*E)|_{Z_C} \arrow[r, "q_C"] \arrow[d, "(q^*\alpha)|_{Z_C}"'] & Q_C \arrow[dl, "\tilde{\alpha}_C"] \\ (q^*F)|_{Z_C} & \end{tikzcd}$$

and therefore $\tilde{\alpha}_C|_c \neq 0$ for all $c \in C$. By the flatness of Q_C we obtain $P(Q_C|_c, n) = P(Q_C|_D, n)$ for all $c \in C \setminus \{D\}$, hence $\mathrm{rk}(Q_C|_D) = r(a)$ and $\deg(Q_C|_D) = \deg(a)$, therefore $\mu(Q_C|_D, \tilde{\alpha}_C|_D) = \mu_{fr}(a)$. Let $\bar{Q} = {}^{Q_C|_D}/_{T'(Q_C|_D)}$, where $T'(Q_C|_D)$ is the sheaf that to every open subset U associates the set of sections f of $Q_C|_D$ in U such that there exists $n > 0$ for which $I_D^n \cdot f = 0$, where I_D is the ideal sheaf associated to D. Roughly speaking, $T'(Q_C|_D)$ is the part of the torsion subsheaf $T(Q_C|_D)$ of $Q_C|_D$ that is not supported in the intersection $D \cap D_F$. By the transversality of D_i with respect to D_F, we have $T'(Q_C|_D) \subset \ker \tilde{\alpha}_C|_D$, hence there is a nonzero induced framing $\bar{\alpha}$ on $\bar{Q}$. Moreover, $\mathrm{rk}(\bar{Q}|_{D_i}) = \mathrm{rk}(\bar{Q}) = \mathrm{rk}(Q_C|_D) = r(a)$. So

$$\mu_{fr}(a) = \mu(Q_C|_D, \tilde{\alpha}_C|_D) \geq \mu(\bar{Q}, \bar{\alpha}).$$

The sequence

$$0 \longrightarrow \bar{Q} \longrightarrow \bigoplus_i \bar{Q}|_{D_i} \longrightarrow \bigoplus_i \bigoplus_{i<j} \bar{Q}|_{D_i \cap D_j} \longrightarrow 0$$

is exact modulo sheaves of dimension $d-3$ (the kernel of the morphism $\bar{Q} \longrightarrow \bigoplus_i \bar{Q}|_{D_i}$ is zero because the divisors D_i are transversal with respect to the singular set of $\bar{Q}$). By the same computations as in the proof of Lemma 7.2.5 in [**35**] we have

$$\mu(\bar{Q}) = \sum_i \Big(\mu(\bar{Q}|_{D_i}) - \frac{1}{2}\sum_{j\neq i}\Big(\frac{\mathrm{rk}\,\big(\bar{Q}|_{D_i\cap D_j}\big)}{r(a)} - 1\Big)a_i a_j\Big).$$

For every i and $j \neq i$ we define also the sheaf $T_{ij}(\bar{Q}|_{D_i})$ as the sheaf on D_i that to every open subset U associates the set of sections f of $\bar{Q}|_{D_i}$ in U such that there exists $n > 0$ for which $I^n_{D_j} \cdot f = 0$. Note that $T_{ij}(\bar{Q}|_{D_i}) \subset \ker \bar{\alpha}|_{D_i}$. We define $Q_i = \bar{Q}|_{D_i}/\bigoplus_{j\neq i} T_{ij}(\bar{Q}|_{D_i})$. By construction $\mathrm{rk}(Q_i) = \mathrm{rk}(\bar{Q})$, there exists a nonzero induced framing α_i on Q_i, and

$$\mu(Q_i) = \mu(\bar{Q}|_{D_i}) - \sum_{j\neq i}\Big(\frac{\mathrm{rk}\,\big(\bar{Q}|_{D_i\cap D_j}\big)}{r(a)} - 1\Big)a_i a_j.$$

Therefore $\mu(\bar{Q}) \geq \sum_i \mu(Q_i)$, and

$$\mu_{fr}(a) \geq \mu(\bar{Q}, \bar{\alpha}) \geq \sum_i \mu(Q_i, \alpha_i).$$

By definition of minimal framed μ-destabilizing quotient, we have $\mu(Q_i, \alpha_i) \geq \mu_{fr}(a_i)$, hence $\mu_{fr}(a) \geq \sum_i \mu_{fr}(a_i)$.

Consider the second case. On the restriction to $Z_{V_a \cap C}$ we have the quotient:

$$\begin{array}{ccc} (q^*E)|_{Z_{V_a\cap C}} & \xrightarrow{\quad q \quad} & Q_a|_{Z_{V_a\cap C}} \\ \big\downarrow (q^*\alpha)|_{Z_{V_a\cap C}} & & \\ (q^*F)|_{Z_{V_a\cap C}} & & \end{array}$$

By definition of Q_a we get $\ker q|_{D'} \not\subset \ker \alpha|_{D'}$ for all points $D' \in V_a \cap C$, hence $\ker q \not\subset \ker(q^*\alpha)|_{Z_{V_a\cap C}}$. As before, we can extend $Q_a|_{Z_{V_a\cap C}}$ to a C-flat quotient

$$\begin{array}{ccc} (q^*E)|_{Z_C} & \xrightarrow{q_C} & Q_C \\ \big\downarrow (q^*\alpha)|_{Z_C} & & \\ (q^*F)|_{Z_C} & & \end{array}$$

Since $\ker q_C$ and $\ker(q^*\alpha)|_{Z_C}$ are C-flat, also $\ker q_C \cap \ker(q^*\alpha)|_{Z_C}$ is C-flat. Moreover for all points $D' \in V_a \cap C$ we have $(\ker q_C \cap \ker(q^*\alpha)|_{Z_C})|_{D'} = \ker q|_{D'} \cap \ker \alpha|_{D'}$, hence by flatness we get $\ker q_C|_{D'} \not\subset \ker \alpha|_{D'}$ for all points $D' \in C$. As before, by flatness of Q_C we have that $\mathrm{rk}(Q_C|_D) = r(a)$ and $\deg(Q_C|_D) = \deg(a)$; moreover the induced framing on $Q_C|_D$ is zero, hence $\mu(Q_C|_D) = \mu_{fr}(a)$. Let $\bar{Q} = Q_C|_D/T(Q_C|_D)$ and $Q_i = \bar{Q}|_{D_i}/T(\bar{Q}|_{D_i})$. Using the same computations as in the proof of Lemma 7.2.5 in [**35**], we obtain $\mu(\bar{Q}) \geq \sum_i \mu(Q_i)$. As before, we get $\mu_{fr}(a) = \mu(Q_C|_D) \geq \mu(\bar{Q}) \geq \sum_i \mu(Q_i) \geq \sum_i \mu_{fr}(a_i)$.

Now let us consider the case $\mu_{fr}(a) = \sum_i \mu_{fr}(a_i)$. In both cases, if we denote by α_i the induced framing on Q_i; from this equality, it follows that $\mu(Q_i, \alpha_i) = \mu_{fr}(a_i)$ and $\mathrm{rk}(\bar{Q}|_{D_i \cap D_j}) = r(a)$. Since $\mu_{fr}(a_i)$ is the framed slope of the minimal framed μ-destabilizing quotient, we have $r(a) = \mathrm{rk}(Q_i) \geq r(a_i)$ for all i. □

By using the same arguments as in Corollary 7.2.6 in [**35**], we can prove:

Corollary 4.6. *$r(a)$ and ${}^{\mu_{fr}(a)}/_{a}$ are constant for $a \gg 0$.*

If ${}^{\mu_{fr}(a)}/_{a} = {}^{\mu_{fr}(a_i)}/_{a_i}$ and $r(a) = r(a_i)$ for all i, then Q_i is the minimal framed μ-destabilizing quotient of $E|_{D_i}$, hence $Q_C|_{D_i}$ differs from the minimal framed μ-destabilizing quotient of $E|_{D_i}$ only in dimension $d-3$, in particular their determinant line bundles are isomorphic. From this argument it follows:

Lemma 4.7. *There is a line bundle $L \in \mathrm{Pic}(X)$ such that $L_a \simeq L$ for all $a \gg 0$.*

Proof. The proof is similar to that of Lemma 7.2.7 in [**35**]. □

By Corollary 4.6 and Lemma 4.7, $\epsilon(a)$ is constant for $a \gg 0$.

In this way, we proved that for any $a \gg 0$ and V_a-flat family Q_a (introduced before), an extension of the determinant line bundle $\det(Q_a)$ is of the form $q^*L \otimes p^*M$ for some line bundle $M \in \mathrm{Pic}(\Pi_a)$. Moreover $\deg(Q_a|_D) = a \deg(L)$ for any $D \in V_a$.

Proof of Theorem 4.1. Suppose the theorem is false: we have to consider separately two cases: $\epsilon(a) = 1$ and $\epsilon(a) = 0$ for $a \gg 0$. In the first case we have

$$\frac{\deg(L) - \delta_1}{r} < \mu(\mathcal{E})$$

and $1 \leq r \leq \mathrm{rk}(E)$, where $r = r(a)$ for $a \gg 0$. We want to construct a rank r quotient Q of E, with nonzero induced framing β and $\det(Q) = L$. Thus

$$\mu(\mathcal{Q}) < \mu(\mathcal{E})$$

and therefore we obtain a contradiction with the hypothesis of μ-semistability of $\mathcal{E}$ with respect to δ_1. Let a be sufficiently large, $D \in U_a$ and the minimal framed μ-destabilizing quotient

$$\begin{tikzcd} E|_D \arrow[r, "q_D"] \arrow[d, "\alpha|_D"'] & Q_D \arrow[dl, "\beta_D"] \\ F|_D & \end{tikzcd}$$

Put $K_D = \ker \beta_D$ and $L_{K_D} = \det(K_D)$. By Proposition 2.21 (for μ-semistability), Q_D fits into an exact sequence

$$0 \longrightarrow K_D \longrightarrow Q_D \longrightarrow \mathrm{Im}\, \alpha|_D \longrightarrow 0 \tag{12}$$

with K_D torsion free quotient of $\ker\alpha|_D$. So there exists an open subscheme $D' \subset D$ such that $K_D|_{D'}$ is locally free of rank r and $D \setminus D'$ is a closed subset of codimension two in D. Consider the restriction of the sequence (12) on D'

$$0 \longrightarrow K_D|_{D'} \longrightarrow Q_D|_{D'} \longrightarrow \operatorname{Im}\alpha|_{D'} \longrightarrow 0.$$

By Proposition V-6.9 in [**39**], we have a canonical isomorphism

$$L_{K_D}|_{D'} \otimes \det(\operatorname{Im}\alpha|_{D'}) = \det(Q_D|_{D'}) = L|_{D'}.$$

If we denote by $\bar{L}$ the determinant bundle of $\operatorname{Im}\alpha$, we get

$$L_{K_D}|_{D'} = L|_{D'} \otimes \bar{L}^\vee|_{D'} = (L \otimes \bar{L}^\vee)|_{D'}$$

and therefore

$$L_{K_D} = (L \otimes \bar{L}^\vee)|_D$$

So we have a morphism $\sigma_D\colon \Lambda^r \ker\alpha|_D \to (L \otimes \bar{L}^\vee)|_D$ which is surjective on D' and morphisms

$$D' \longrightarrow Grass(\ker\alpha, r) \longrightarrow \mathbb{P}(\Lambda^r \ker\alpha)$$

Consider the exact sequence

$$\begin{aligned}&\operatorname{Hom}(\Lambda^r \ker\alpha, (L \otimes \bar{L}^\vee)(-a)) \longrightarrow \operatorname{Hom}(\Lambda^r \ker\alpha, L \otimes \bar{L}^\vee) \longrightarrow \\ &\longrightarrow \operatorname{Hom}(\Lambda^r \ker\alpha, (L \otimes \bar{L}^\vee)|_D) \longrightarrow \operatorname{Ext}^1(\Lambda^r \ker\alpha, (L \otimes \bar{L}^\vee)(-a))\end{aligned}$$

By Serre's vanishing theorem and Serre duality, one has for $i = 0, 1$

$$\operatorname{Ext}^i(\Lambda^r \ker\alpha, (L \otimes \bar{L}^\vee)(-a)) = \mathrm{H}^{d-i}(X, \Lambda^r \ker\alpha \otimes (L \otimes \bar{L}^\vee)^\vee \otimes \omega_X^\vee(a))^\vee = 0$$

for all $a \gg 0$ (since $d \geq 2$), hence

$$\operatorname{Hom}(\Lambda^r \ker\alpha, L \otimes \bar{L}^\vee) = \operatorname{Hom}(\Lambda^r \ker\alpha|_D, (L \otimes \bar{L}^\vee)|_D).$$

So for a sufficiently large, the morphism σ_D extends to a morphism $\sigma\colon \Lambda^r \ker\alpha \to L \otimes \bar{L}^\vee$. The support of the cokernel of σ meets D in a closed subscheme of codimension two in D, hence there is an open subscheme $X' \subset X$ such that $\sigma|_{X'}$ is surjective, $X \setminus X'$ is a closed subscheme of codimension two and $D' = X' \cap D$. So we have a morphism $i\colon X' \to \mathbb{P}(\Lambda^r \ker\alpha)$ and we want it to factorize through $Grass(\ker\alpha, r)$. The ideal sheaf of $Grass(\ker\alpha, r)$ in $\mathbb{P}(\Lambda^r \ker\alpha)$ is generated by finitely many sheaves $I_\nu \subset S^\nu(\Lambda^r \ker\alpha)$, $\nu \leq \nu_0$. The morphism i factors through $Grass(\ker\alpha, r)$ if and only if the composite maps

$$\phi_\nu\colon I_\nu \longrightarrow S^\nu(\Lambda^r \ker\alpha) \longrightarrow (L \otimes \bar{L}^\vee)^\nu$$

vanish. But we already know that the restriction of ϕ_ν to D vanishes, so that we can consider ϕ_ν as elements in $\operatorname{Hom}(\Lambda^r \ker\alpha, (L \otimes \bar{L}^\vee)(-a))$. Clearly, these groups vanish for $a \gg 0$. Thus the morphism i factorizes and we get a rank r locally free quotient

$$\ker\alpha|_{X'} \longrightarrow K_{X'}$$

such that $\det(K_{X'}) = (L \otimes \bar{L}^\vee)|_{X'}$. So we can extend $K_{X'}$ to a rank r coherent quotient K of $\ker\alpha$ such that $\det(K) = L \otimes \bar{L}^\vee$. Let $G = \ker(\ker\alpha \to K)$. We have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & \ker\alpha \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathrm{Im}\,\alpha \arrow[r] \arrow[d] & \mathrm{Im}\,\alpha \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We have that the determinant of Q is canonically isomorphic to $\det(K) \otimes \bar{L} = L$, so $\mathcal{Q}$ destabilizes $\mathcal{E}$ and this contradicts the hypothesis.

In the second case we have

$$\frac{\deg(L)}{r} < \mu(\mathcal{E}).$$

Let a be sufficiently large, $D \in U_a$ and the minimal framed μ-destabilizing quotient

$$\begin{tikzcd}
E|_D \arrow[r, "q_D"] \arrow[d, "\alpha|_D"'] & Q_D \\
F|_D &
\end{tikzcd}$$

with $\ker q_D \not\subset \ker\alpha|_D$. By Proposition 2.21 (for μ-semistability), Q_D is torsion free, hence there exists an open subscheme $D' \subset D$ such that $D \setminus D'$ is a closed set of codimension two in D and $Q_D|_{D'}$ is locally free of rank r. Moreover $\ker q_D|_{D'} \not\subset \ker\alpha|_{D'}$. Using the same arguments than the previous case, we extend $Q_D|_{D'}$ to a quotient $Q_{X'}$ of X' which is locally free of rank r with $\det(Q_{X'}) = L|_{X'}$. By construction we have $\ker(E|_{X'} \to Q_{X'}) \not\subset \ker\alpha|_{X'}$, hence in this way we obtain a quotient Q of E with $\det(Q) = L$ and zero induced framing, such that $\mathcal{Q}$ destabilizes $\mathcal{E}$. $\square$

2. Slope-stable case

In this section we want to prove the following generalization of Mehta-Ramanathan's theorem for μ-stable torsion free sheaves (Theorem 4.3 in [**54**]).

Theorem 4.8. *Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d. Let F be a coherent sheaf on X supported on a divisor D_F, over which is a locally free $\mathcal{O}_{D_F}$-module. Let $\mathcal{E} = (E, \alpha\colon E \to F)$ be a (D_F, F)-framed sheaf on X. If $\mathcal{E}$ is μ-stable with respect to δ_1, there exists a positive integer a_0 such that for all $a \geq a_0$ there is a dense open subset $W_a \subset |\mathcal{O}_X(a)|$ such that for all $D \in W_a$ the divisor D is smooth, meets transversally the divisor D_F and $\mathcal{E}|_D$ is μ-stable with respect to $a\delta_1$.*

The techniques we need to prove this theorem are quite similar to the ones used before. By Proposition 2.54 a μ-semistable (D_F, F)-framed sheaf which is simple but not μ-stable has a proper extended framed socle. Thus we first show that the restriction of a μ-stable (D_F, F)-framed sheaf is simple and we use the extended framed socle (rather its quotient) as a replacement for the minimal μ-framed-destabilizing quotient.

Proposition 4.9. *Let $\mathcal{E} = (E, \alpha)$ be a μ-stable (D_F, F)-framed sheaf. For $a \gg 0$ and general $D \in |\mathcal{O}_X(a)|$ the restriction $\mathcal{E}|_D = (E|_D, \alpha|_D)$ is simple.*

To prove this result, we need to define the double dual of a framed sheaf. Let $\mathcal{E} = (E, \alpha)$ be a (D_F, F)-framed sheaf; we define a framing $\alpha^{\vee\vee}$ on the double dual of E in the following way: $\alpha^{\vee\vee}$ is the composition of morphisms

$$E^{\vee\vee} \longrightarrow E^{\vee\vee}|_{D_F} \simeq E|_{D_F} \xrightarrow{\alpha|_{D_F}} F|_{D_F}.$$

Then α is the framing induced on E by $\alpha^{\vee\vee}$ by means of the inclusion morphism $E \hookrightarrow E^{\vee\vee}$. We denote the framed sheaf $(E^{\vee\vee}, \alpha^{\vee\vee})$ by $\mathcal{E}^{\vee\vee}$. Note that also $\mathcal{E}^{\vee\vee}$ is a (D_F, F)-framed sheaf.

Lemma 4.10. *Let $\mathcal{E} = (E, \alpha)$ be a μ-stable (D_F, F)-framed sheaf. Then the framed sheaf $\mathcal{E}^{\vee\vee} = (E^{\vee\vee}, \alpha^{\vee\vee})$ is μ-stable.*

Proof. Consider the exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow A \longrightarrow 0$$

where A is a coherent sheaf supported on a closed subset of codimension at least two. Thus $\mathrm{rk}(E^{\vee\vee}) = \mathrm{rk}(E)$ and $\deg(E^{\vee\vee}) = \deg(E)$. Moreover, since $\alpha = \alpha^{\vee\vee}|_E$, we have $\mu(\mathcal{E}^{\vee\vee}) = \mu(\mathcal{E})$. Let G be a subsheaf of $E^{\vee\vee}$ and denote by G' its intersection with E. So $\mathrm{rk}(G) = \mathrm{rk}(G')$, $\deg(G) = \deg(G')$ and $\alpha|_{G'} = \alpha^{\vee\vee}|_G$. Thus we obtain

$$\mu(G, \alpha^{\vee\vee}|_G) = \mu(G', \alpha|_{G'}) < \mu(\mathcal{E}) = \mu(\mathcal{E}^{\vee\vee}).$$

$\square$

Recall that a d-dimensional coherent sheaf G on X is *reflexive* if the natural morphism $G \to G^{\vee\vee}$ is an isomorphism.

Lemma 4.11. *Let G be a reflexive sheaf. For $a \gg 0$ and $D \in |\mathcal{O}_X(a)|$ the homomorphism* $\mathrm{End}(G) \to \mathrm{End}(G|_D)$ *is surjective.*

Proof. Let D be an element in $|\mathcal{O}_X(a)|$. Consider the exact sequence

$$0 \longrightarrow G(-a) \longrightarrow G \longrightarrow G|_D \longrightarrow 0.$$

By applying the functor $\mathrm{Hom}(G, \cdot)$ we obtain

$$0 \longrightarrow \mathrm{Hom}(G, G(-a)) \longrightarrow \mathrm{End}(G) \longrightarrow \mathrm{End}(G|_D) \longrightarrow \mathrm{Ext}^1(G, G(-a)) \to \cdots$$

Recall the Relative-to-Global spectral sequence

$$\mathrm{H}^i(X, \mathcal{E}xt^j(G, G \otimes \omega_X(a))) \Rightarrow \mathrm{Ext}^{i+j}(G, G \otimes \omega_X(a)).$$

For sufficiently large $a \gg 0$ we get

$$\mathrm{Ext}^1(G, G(-a))^\vee \simeq \mathrm{Ext}^{n-1}(G, G \otimes \omega_X(a)) \simeq \mathrm{H}^0(X, \mathcal{E}xt^{n-1}(G, G) \otimes \omega_X(a)).$$

Since G is reflexive, the homological dimension $\mathrm{dh}(G)$ is less or equal to $n-2$ and therefore $\mathcal{E}xt^{n-1}(G, G) = 0$. Hence for a sufficiently large, $\mathrm{End}(G) \longrightarrow \mathrm{End}(G|_D)$ is surjective. $\square$

Proof of Proposition 4.9. For arbitrary a and general $D \in |\mathcal{O}_X(a)|$ the sheaf $E|_D$ is torsion free on D and $E^{\vee\vee}|_D$ is reflexive on D, moreover the double dual of $E|_D$ (as sheaf on D) is $E^{\vee\vee}|_D$ (cf. Section 1.1 in [**35**]). We have injective homomorphisms

$$\begin{aligned} \delta &: \quad \mathrm{End}(E) \longrightarrow \mathrm{End}(E^{\vee\vee}), \\ \delta_D &: \quad \mathrm{End}(E|_D) \longrightarrow \mathrm{End}(E^{\vee\vee}|_D). \end{aligned}$$

Let $\varphi \in \mathrm{End}(\mathcal{E})$: the image $\varphi^{\vee\vee} = \delta(\varphi)$ of φ is an element of $\mathrm{End}(E^{\vee\vee}, \alpha^{\vee\vee})$, indeed if $\alpha \circ \varphi = \lambda\alpha$, then we can define an endomorphism of $\mathcal{E}^{\vee\vee}$ in the following way:

$$\begin{tikzcd}
E^{\vee\vee} \arrow[r, "\varphi^{\vee\vee}"] \arrow[d] \arrow[ddd, bend right=60, "\alpha^{\vee\vee}"'] & E^{\vee\vee} \arrow[d] \arrow[ddd, bend left=60, "\alpha^{\vee\vee}"] \\
E^{\vee\vee}|_{D_F} \arrow[r, "\varphi^{\vee\vee}|_{D_F}"] \arrow[d, "\simeq"] & E^{\vee\vee}|_{D_F} \arrow[d, "\simeq"'] \\
E|_{D_F} \arrow[r, "\varphi|_{D_F}"] \arrow[d, "\alpha|_{D_F}"] & E|_{D_F} \arrow[d, "\alpha|_{D_F}"'] \\
F|_{D_F} \arrow[r, "\cdot\lambda"] & F|_{D_F}
\end{tikzcd}$$

In the same way it is possible to prove that for $\varphi \in \mathrm{End}(\mathcal{E}|_D)$, $\delta_D(\varphi)$ is an element of $\mathrm{End}(\mathcal{E}^{\vee\vee}|_D)$. So the homomorphisms

$$\begin{aligned} \delta &: \quad \mathrm{End}(\mathcal{E}) \longrightarrow \mathrm{End}(\mathcal{E}^{\vee\vee}), \\ \delta_D &: \quad \mathrm{End}(\mathcal{E}|_D) \longrightarrow \mathrm{End}(\mathcal{E}^{\vee\vee}|_D) \end{aligned}$$

are injective. Therefore it suffices to show that $\mathcal{E}^{\vee\vee}|_D$ is simple for $a \gg 0$ and general D. By Lemma 4.10, $\mathcal{E}^{\vee\vee}$ is μ-stable, hence by point (3) of Corollary 2.22 it is simple. By Lemma 4.11, the homomorphism $\chi\colon \mathrm{End}(E^{\vee\vee}) \to \mathrm{End}(E^{\vee\vee}|_D)$ is surjective for $a \gg 0$ and general D. Since for $\varphi \in \mathrm{End}(\mathcal{E}^{\vee\vee})$, $\chi(\varphi)$ is an element of $\mathrm{End}(\mathcal{E}^{\vee\vee}|_D)$, we have that the map

$$\chi|_{\mathrm{End}(\mathcal{E}^{\vee\vee})}\colon \mathrm{End}(\mathcal{E}^{\vee\vee}) \to \mathrm{End}(\mathcal{E}^{\vee\vee}|_D)$$

is also surjective. Thus $\mathrm{End}(\mathcal{E}|_D) = \mathrm{End}(\mathcal{E}^{\vee\vee}|_D) \simeq k$. □

Remark 4.12. Since $\mathcal{E}$ is μ-stable with respect to δ_1, we have $\deg(\mathrm{Im}\,\alpha) > \delta_1$. This implies $\deg(\mathrm{Im}\,\alpha|_D) = a\deg(\mathrm{Im}\,\alpha) > a\delta_1$ for a positive integer, hence $\ker\alpha|_D$ is not μ-framed-destabilizing for all $D \in \Pi_a$. △

Let $a_0 \geq 3$ be an integer such that for all $a \geq a_0$ and a general $D \in \Pi_a$, the restriction $\mathcal{E}|_D$ is μ-semistable with respect to $a\delta_1$ and simple (cf. Proposition 4.9). Suppose that for an integer $a \geq a_0$, the framed sheaf $\mathcal{E}|_D$ is not μ-stable with respect to $a\delta_1$ for a general divisor D. Then $\mathcal{E}|_{D_\eta}$ is not geometrically μ-stable for the divisor D_η associated to the generic point $\eta \in |\mathcal{O}_X(a)|$, i.e., the pull-back to some extension of $k(\eta)$ is not μ-stable (cf. Corollary 2.55). Hence $\mathcal{E}|_{D_\eta}$ is not μ-stable. Since $\mathcal{E}|_{D_\eta}$ is simple, by Proposition 2.54 the extended socle of $\mathcal{E}|_{D_\eta}$ is a proper framed μ-destabilizing subsheaf. Consider the corresponding quotient sheaf Q_η, with induced framing β_η: we can extend it to a coherent quotient $q^*E \to Q_a$ over all of Z_a.

Let W_a be the dense open subset of points $D \in \Pi_a$ such that

- D is a smooth divisor, meets transversally the divisor D_F, $E|_D$ is torsion free,

- Q_a is flat over W_a and $\epsilon\left((\tilde{\alpha}_a)|_D\right) = \epsilon(\beta_\eta)$, where we denote by $\tilde{\alpha}_a$ the induced framing on Q_a.

Thus $Q_a|_D$ is a coherent sheaf of positive rank such that with the induced framing is a μ-framed-destabilizing quotient for all $D \in W_a$.

Lemma 4.13. *If there exists a divisor $D_0 \in W_a$, $a \geq a_0$, such that $\mathcal{E}|_{D_0}$ is μ-stable with respect to $a\delta_1$, then for all $D' \in W_{a'}$ the framed sheaf $\mathcal{E}|_{D'}$ is μ-stable with respect to $a'\delta_1$ for all $a' \geq 2a$.*

Proof. If the lemma is false, then there exists $a' \geq 2a$ and a divisor $D \in W_{a'}$ such that $\mathcal{E}|_D$ is not μ-stable with respect to $a'\delta_1$. Choose a divisor $D_1 \in W_{a'-a}$ such that $D = D_0 + D_1$ is a divisor with normal crossings. Let $C \subset \Pi_{a'}$ be a curve with the properties asserted in the Lemma 4.3. Using the same techniques as in the proof of Lemma 4.5, we can extend $q^*E \to Q_{a'}$ to a C-flat quotient $(q^*E)|_{Z_C} \to Q_C$. Using the same notations and computations than before, we have

$$a'\mu(E,\alpha) = \mu(E|_D, \alpha|_D) = \mu(Q_C|_D, \tilde{\alpha}_C|_D) \geq \mu(\bar{Q}, \bar{\alpha}) \geq \mu(Q_0, \alpha_0) + \mu(Q_1, \alpha_1).$$

Since $a' - a \geq a_0$, $(E|_{D_1}, \alpha|_{D_1})$ is μ-semistable, hence $\mu(Q_1, \alpha_1) \geq (a'-a)\mu(E,\alpha)$. Moreover by hypothesis $\mu(Q_0, \alpha_0) > a\mu(E,\alpha)$, hence we have a contradiction. □

Proof of Theorem 4.8. Assume that the theorem is false: for all $a \geq a_0$ and general $D \in \Pi_a$, $\mathcal{E}|_D$ is not μ-stable with respect to $a\delta_1$. Thus one can construct for any $a \geq a_0$ a coherent quotient $q^*E \to Q_a$ and a dense open subset $W_a \subset \Pi_a$ such that $Q_a|_D$, with the induced framing, is a μ-framed-destabilizing for all $D \in W_a$. We denote by $\epsilon(a)$ the quantity $\epsilon(\tilde{\alpha}_a|_D)$ for $D \in W_a$. As before, there are line bundles $L_a \in \mathrm{Pic}(X)$ such that $\det(Q_a|_D) = L_a|_D$ for $D \in W_a$ and all $a \geq a_0$.

Let $N \subset \mathbb{Z}$ be an infinite subset consisting of integers $a \geq a_0$ such that $\mathrm{rk}(Q_a)$ is constant, say r. By Remark 4.12 we have $0 < r < \mathrm{rk}(E)$. By using the same arguments of the proof of the Lemma 4.5, one can prove that if $a_1, a_2, \ldots, a_l$ are integers in N, with $l > 1$ and $a_i \geq a_0$ for $i = 1, \ldots, l$ and $a = \sum a_i$, and D_i are divisors in W_{a_i} such that $D = \sum D_i$ is a divisor with normal crossings, then $L_a|_{D_i}$ is the determinant line bundle of some μ-framed-destabilizing quotient of $\mathcal{E}|_{D_i}$.

Lemma 4.14. *Let $\mathcal{G} = (G, \beta)$ be a framed sheaf of positive rank. If $\mathcal{G}$ is μ-semistable with respect to $\bar{\delta}_1$, then the set T of determinant line bundles of μ-framed-destabilizing quotients of $\mathcal{G}$ is finite and its cardinality is bounded by $2^{\mathrm{rk}(G)}$.*

Proof. Let $gr(\mathcal{G}) \simeq (G_1, \beta_1) \oplus (G_2, \beta_2) \oplus \cdots \oplus (G_l, \beta_l)$ be the grade object associated to a Jordan-Hölder filtration of $\mathcal{G}$. Recall that $\mathcal{G}_i = (G_i, \beta_i)$ is μ-stable with respect to $\bar{\delta}_1$ and $\deg(\mathcal{G}_i) = \mathrm{rk}(G_i)\mu(\mathcal{G})$, for $i = 1, \ldots, l$. Let G' be a subsheaf of G with $\deg(G', \beta') = \mathrm{rk}(G')\mu(\mathcal{G})$. We can start with a stable filtration of $\mathcal{G}'$ and complete it to one of $\mathcal{G}$:

$$0 = G'_0 \subset G'_1 \subset \cdots \subset G'_s = G' \subset \cdots \subset G_l = G.$$

Since $gr(\mathcal{G})$ is indipendent of the filtration, we have that $\det(G')$ has to be isomorphic to one of $\det(G_{i_1}) \otimes \cdots \otimes \det(G_{i_j})$. Thus the set T is finite and its cardinality is bounded by $2^{\mathrm{rk}(G)}$. □

Let $a \geq 2a_0$ and $D \in W_{a_0}$ be an arbitrary point. By Lemma 4.13, we have that $\mathcal{E}|_D$ is not μ-stable with respect to $a_0\delta_1$. If we denote by T_D the set T of the previous lemma associated to $\mathcal{E}|_D$, then $L_a|_D \in T_D$. Consider the function

$$\begin{array}{rcl} \varphi : N_{\geq 2a_0} & \to & \prod_{D \in W_{a_0}} T_D \\ a & \mapsto & (L_a|_D)_D \end{array}$$

Let $\sim$ be the equivalence relation on $N_{\geq 2a_0}$ defined in the following way: $a \sim a'$ if and only if the set $\{s \in W_{a_0} \mid \varphi(a)(s) = \varphi(a')(s)\}$ is dense in W_{a_0}. By using the same arguments as in the nonframed case, one can prove that there are at most $2^{\mathrm{rk}(E)}$ distinct equivalence classes and, in particular, there is at least one infinite class $\tilde{N}$. By this result, we get the following.

Lemma 4.15. *There is a line bundle $L \in \mathrm{Pic}(X)$ such that $L \simeq L_a$ for all $a \in \tilde{N}$. Moreover $\epsilon(a)$ is constant for $a \in \tilde{N}$.*

Proof. Let $a, a' \in \tilde{N}$. $a \sim a'$ means that $\varphi(a)$ and $\varphi(a')$ are equal on a dense subset of W_{a_0}, then $L_a|_D \simeq L_{a'}|_D$ for all D in a dense subset of Π_{a_0}. It suffices to prove that $L_a \simeq L_{a'}$ (see Lemma 7.2.2 in [**35**]). □

Summing up, we have that there is a line bundle L on X and an integer $0 < r < \mathrm{rk}(E)$ such that for $a \gg 0$ and for general $D \in W_a$

$$\mu(Q_a|_D, \tilde{\alpha}_a|_D) = \frac{\deg(L|_D) - \epsilon(a)a\delta_1}{r} = a\Big(\frac{\deg(L) - \epsilon(a)\delta_1}{r}\Big) = \mu(E|_D, \alpha|_D) = a\mu(E, \alpha),$$

hence

$$\frac{\deg(L) - \epsilon(a)\delta_1}{r} = \mu(E, \alpha).$$

Using the arguments at the end of the proof of the restriction theorem for μ-semistable framed sheaves, one can show that this suffices for constructing a framed μ-destabilizing quotient $E \to Q$ for sufficiently large a. This contradicts the assumptions of the theorem. □

Remark 4.16. Since the family of μ-semistable framed sheaves with fixed Chern character is bounded (cf. Proposition 2.64), the positive constant a_0 in the statement of Theorem 4.1 depends only on the Chern character. The same holds for the μ-stable case. △

CHAPTER 5

Moduli spaces of (semi)stable framed sheaves

In this chapter we give a construction of moduli spaces of semistable framed sheaves of positive rank. The contents of this chapter will be useful later on, when we apply our restriction theorems to the definition of *Uhlenbeck-Donaldson compactification* for framed sheaves (see Chapter 6) and when we construct symplectic structures on the moduli spaces of stable framed sheaves (see Chapter 7). If the framing is trivial (i.e. it is the zero morphism), these are just the ordinary moduli spaces of semistable torsion free sheaves (cf. Chapter 4 in [**35**]). Therefore, we shall always assume that the framings are nontrivial, unless the contrary is explicitly stated.

Now we give an overview of the construction by following what Huybrechts and Lehn made in [**33**]; all technical results will be only stated.

1. The moduli functor

In this section we introduce the *moduli functor* associated to (semi)stable framed sheaves.

Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension d. Fix a stability polynomial δ of degree $d-1$ and a framing sheaf F.

Definition 5.1. *A flat family of coherent sheaves on X parametrized by a Noetherian scheme S* consists of a coherent sheaf E on $S \times X$, flat over S.

Definition 5.2. *A flat family of framed sheaves of positive rank on X parametrized by a Noetherian scheme S* is a pair $\mathcal{E} = (E, \alpha)$, consisting of a coherent sheaf E on $S \times X$, flat over S, and a morphism $\alpha\colon E \to p_X^*(F)$ such that $\mathrm{rk}(E_s) > 0$ and $\alpha_s \neq 0$ for every point $s \in S$. An *isomorphism of flat families of framed sheaves* (E, α) and (G, β) parametrized by S is an isomorphism $\varphi\colon E \to G$ for which there exists $\lambda \in \mathcal{O}_S^*$ such that $\beta \circ \varphi = p_S^*(\lambda)\alpha$.

Remark 5.3. In Definition 3.3 we impose the condition that the image sheaf $\operatorname{Im}\alpha$ of the framing α in a flat family must be S-flat because we do not want that the kernel of the framing can destabilize. Since, in this chapter, we will only deal with flat families of (semi)stable framed sheaves of positive rank, in the previous definition we did not assume that $\operatorname{Im}\alpha$ is S-flat. $\triangle$

Let P be a numerical polynomial of degree d. Define the moduli functor from the category of Noetherian k-schemes of finite type to the category of sets

$$\underline{\mathcal{M}}_\delta^{(s)s}(X; F, P)\colon ({}^{Sch}\!/_{k})^\circ \to (Sets)$$

that assigns to any scheme S the set $\underline{\mathcal{M}}_\delta^{(s)s}(X; F, P)(S)$ of isomorphism classes of flat families of (semi)stable framed sheaves on X parametrized by S with Hilbert polynomial P, and to any morphism $f\colon T \to S$, the map $\underline{\mathcal{M}}_{\delta}^{(s)s}(X;F,P)(f)$ obtained by pulling-back sheaves via $f \times id_X$.

Definition 5.4. A scheme is called a *moduli space of (semi)stable framed sheaves* if it corepresents the functor $\underline{\mathcal{M}}_{\delta}^{(s)s}(X;F,P)$.

2. The construction

Now we construct the moduli space of (semi)stable framed sheaves as a GIT quotient of a certain subscheme in the product of a Quot scheme and projective space by a natural group action. Since semistable framed sheaves with fixed Hilbert polynomial form a bounded family, we can choose a suitable sheaf such that the associated Quot scheme parametrizes all their underlying coherent sheaves. On the other hand, roughly speaking, the fixed projective space parametrizes the framings of these framed sheaves. Since the quotient is obtained by using the so-called GIT stability, we need to relate this notion with the stability condition for framed sheaves introduce in Chapter 2 (cf. Proposition 5.6). Finally, in Theorem 5.9 we prove the (co)representability of the moduli functors for (semi)stable framed sheaves of positive rank.

Let P be a numerical polynomial of degree d. According to the Proposition 2.65, the family of semistable framed sheaves of positive rank on X with fixed Hilbert polynomial P is bounded. In particular, by Proposition 2.62 there is an integer m such that any underlying sheaf E of a semistable framed sheaf (E,α) is m-regular. Hence, $E(m)$ is globally generated and $h^0(E(m)) = P(m)$. Thus if we let $V := k^{\oplus P(m)}$ and $\mathcal{H} := V \otimes_k \mathcal{O}_X(-m)$, there is a surjection

$$g\colon \mathcal{H} \longrightarrow E,$$

obtained by composing the canonical evaluation map $\mathrm{H}^0(E(m)) \otimes \mathcal{O}_X(-m) \to E$ with an isomorphism $V \to \mathrm{H}^0(E(m))$. This defines a closed point $[g\colon \mathcal{H} \longrightarrow E] \in Q := Quot_{X/k}(\mathcal{H},P)$. For sufficiently large l the standard maps

$$Q \longrightarrow Grass(V \otimes \mathrm{H}^0(\mathcal{O}_X(l-m)), P(l)) \longrightarrow \mathbb{P}(\Lambda^{P(l)}(V \otimes \mathrm{H}^0(\mathcal{O}_X(l-m))))$$

are well-defined closed immersions. Let $\mathcal{L}$ denote the corresponding very ample line bundle on Q. Let $\mathbb{P} := \mathbb{P}(\mathrm{Hom}(V, \mathrm{H}^0(F(m)))^\vee)$. A point $[a] \in \mathbb{P}$ induces a morphism

$$\mathcal{H} \longrightarrow F$$

defined up to a constant factor. Finally, let $Quot_{X/k}(\mathcal{H},P,F)$ be the closed subscheme of $Q \times \mathbb{P}$ formed by pairs $([g],[a])$ such that there is a morphism $\alpha\colon E \to F$ for which the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "g"] \arrow[dr, "a"'] & E \arrow[d, "\alpha"] \\ & F \end{tikzcd}$$

commutes. Obviously α is uniquely determined by a. Let $\mathcal{O}(1)$ be the pullback of $\mathcal{O}_{\mathbb{P}}(1)$ on $Quot_{X/k}(\mathcal{H},P,F)$ through the natural projection $p_{\mathbb{P}}$.

Remark 5.5. The scheme $Quot_{X/k}(\mathcal{H},P,F)$ is quite different from the *framed Quot scheme* introduced in Section 2, because $Quot_{X/k}(\mathcal{H},P,F)$ identifies pairs with the same underlying coherent sheaf and framings that differ by a nonzero constant. On the other hand, these pairs correspond to different points in the framed Quot scheme. $\triangle$

The universal objects on Q and $\mathbb{P}$ induces a *universal object* on $Quot_{X/k}(\mathcal{H}, P, F) \times X$

$$V \otimes \mathcal{O}_{Quot_{X/k}(\mathcal{H},P,F)\times X} \longrightarrow \tilde{E},$$

with a *universal framing*[1]

$$\alpha_{\tilde{E}} \colon \tilde{E} \longrightarrow p^*_{Quot_{X/k}(\mathcal{H},P,F)}(\mathcal{O}_{\mathbb{P}}(1)) \otimes p_X^*(F).$$

The action of $SL(V)$ on V induces well-defined actions on Q and $\mathbb{P}$ which are compatible, so that one has an action of $SL(V)$ on $Quot_{X/k}(\mathcal{H}, P, F)$. Moreover the ample line bundles

$$\mathcal{L}_{Quot_{X/k}(\mathcal{H},P,F)}(n_1, n_2) := p_Q^*(\mathcal{L})^{\otimes n_1} \otimes p_{\mathbb{P}}^*(\mathcal{O}_{\mathbb{P}}(1))^{\otimes n_2}$$

carry natural $SL(V)$-linearization, where $p_Q, p_{\mathbb{P}}$ are the projections from $Quot_{X/k}(\mathcal{H}, P, F)$ to Q and $\mathbb{P}$, respectively. We choose n_1 and n_2 such that

$$\frac{n_2}{n_1} = A_X(l) := (P(l) - \delta(l))\frac{\delta(m)}{P(m) - \delta(m)} - \delta(l), \tag{13}$$

assuming, of course, that l is chosen large enough so as to make this term positive.

Since torsion freeness is an open property for families of sheaves (cf. Proposition 2.3.1 in [**35**]), we can define an open subscheme $U \subset Quot_{X/k}(\mathcal{H}, P, F)$ consisting of those points that represents framed sheaves with torsion free kernel. If there are any semistable framed sheaves with the given Hilbert polynomial at all (otherwise the present discussion is void), then U is nonempty and we denote by Z its closure in $Quot_{X/k}(\mathcal{H}, P, F)$.

Now we recall a technical result due to Huybrechts and Lehn that relates the (semi)stability of the points of Z with respect to the $SL(V)$-action with the (semi)stability condition of framed sheaves of positive rank.

Proposition 5.6 (Proposition 3.2 in [**34**])**.** *For sufficiently large l, a point $([g], [a]) \in Z$ is (semi)stable with respect to the linearization of $\mathcal{L}_{Quot_{X/k}(\mathcal{H},P,F)}(n_1, n_2)$ if and only if the corresponding framed sheaf (E, α) is (semi)stable with respect to δ and g induces an isomophism $V \to \mathrm{H}^0(E(m))$.*

Let $Z^s \subset Z^{ss} \subset Z$ denote the open subschemes of stable and semistable points of Z with respect to the $SL(V)$-action, respectively. By the previous proposition, a point in $Z^{(s)s}$ corresponds, roughly speaking, to a (semi)stable framed sheaf (E, α) of positive rank together with a choice of a basis in $\mathrm{H}^0(E(m))$.

Now we want to describe what kind of geometrical properties are inherited by the quotient that we shall construct by using the GIT-(semi)stability condition. First recall the following notions.

Definition 5.7. Let G an affine algebraic group over k acting on a k-scheme Y. A morphism $f \colon Y \to M$ is a *good quotient*, if

- f is affine and invariant.

[1] This morphism is not a framing in the sense of Definition 5.2, but it is locally a framing in the way explained by Proposition 1.14 in [**33**].

- f is surjective, and $U \subset M$ is open if and only if $f^{-1}(U) \subset Y$ is open.
- The natural homomorphism $\mathcal{O}_M \to (f_*(\mathcal{O}_Y))^G$ is an isomorphism.
- If W is an invariant closed subset of Y, then $f(W)$ is a closed subset of M. If W_1 and W_2 are disjoint invariant closed subsets of Y, then $f(W_1) \cap f(W_2) = \emptyset$.

The morphism f is said to be a *geometric quotient* if it is a good quotient and the geometric fibres of f are the orbits of geometric points of Y.

By applying Theorem 1.10 and Remark 1.11 of [**58**], we obtain the following result.

Proposition 5.8. *There exists a projective scheme $\mathcal{M}^{ss}$ and a morphism $\pi\colon Z^{ss} \to \mathcal{M}^{ss}$ which is a good quotient for the action of $SL(V)$ on Z^{ss}. Moreover there is an open subscheme $\mathcal{M}^s \subset \mathcal{M}^{ss}$ such that $Z^s = \pi^{-1}(\mathcal{M}^s)$ and $\pi|_{Z^s}\colon Z^s \to \mathcal{M}^s$ is a geometric quotient. Moreover, there is a positive integer ν and a very ample line bundle $\mathcal{O}_{\mathcal{M}^{ss}}(1)$ on $\mathcal{M}^{ss}$ such that $\mathcal{L}_{Quot_{X/k}(\mathcal{H},P,F)}(n_1,n_2)^{\otimes\nu}|_{Z^{ss}} \cong \pi^*(\mathcal{O}_{\mathcal{M}^{ss}}(1))$.*

Now we are ready to prove the main theorem of this chapter.

Theorem 5.9. *Let $\delta \in \mathbb{Q}[n]$ be a polynomial of degree $d-1$ with positive leading coefficient. There is a projective scheme $\mathcal{M}^{ss}_\delta(X;F,P)$ that corepresents the moduli functor $\underline{\mathcal{M}}^{ss}_\delta(X;F,P)$. Moreover, there is an open subscheme $\mathcal{M}^{s}_\delta(X;F,P) \subset \mathcal{M}^{ss}_\delta(X;F,P)$ which represents the moduli functor $\underline{\mathcal{M}}^{s}_\delta(X;F,P)$, i.e. $\mathcal{M}^{s}_\delta(X;F,P)$ is a fine moduli spaces parametrizing stable framed sheaves of positive rank on X. A closed point in $\mathcal{M}^{ss}_\delta(X;F,P)$ represents an S-equivalence class of semistable framed sheaves.*

Proof. Let T be a Noetherian scheme parametrizing a flat family (E,α) of semistable framed sheaves of positive rank. Let m be still the number choose at the beginning of this section. Then $\mathcal{V} = (p_T)_*(E \otimes p_X^*(\mathcal{O}_X(m)))$ is a locally free sheaf of rank $P(m)$ on T and $g\colon p_T^*(\mathcal{V}) \to E$ is surjective. Moreover, the framing α induces a morphism $a\colon \mathcal{V} \to \mathcal{O}_T \otimes \mathrm{H}^0(F(m))$. Covering T by small enough open subschemes T_i, we can find trivializations $V \otimes \mathcal{O}_{T_i} \to \mathcal{V}|_{T_i}$, where V is a vector space. Thus the compositions of g and a with these trivializations gives morphisms $g_i\colon V \otimes \mathcal{O}_{T_i\times X} \to E$ and $a_i\colon V \otimes \mathcal{O}_{T_i} \to \mathrm{H}^0(F(m)) \otimes \mathcal{O}_{T_i}$. Hence we obtain maps $f_i\colon T_i \to Quot_{X/k}(\mathcal{H},P,F) \subset Q \times \mathbb{P}$. Moreover, by Proposition 5.6, $f_i(T_i) \subset Z^{ss} \subset Quot_{X/k}(\mathcal{H},P,F)$. The trivializations of $\mathcal{V}$ over the intersection T_{ij} of two open sets T_i and T_j differ by a morphism $g\colon T_{ij} \to GL(V)$, in the sense that $f_i|_{T_{ij}} = g \cdot f_j|_{T_{ij}}$. Therefore, if π denotes the geometric quotient $Z^{ss} \to \mathcal{M}^{ss}$, the morphisms $\pi \circ f_i$ and $\pi \circ f_j$ coincide on T_{ij} and thus glue to give a morphism $f\colon T \to \mathcal{M}^{ss}$. If the family (E,α) consists of stable framed sheaves of positive rank, obviously $f(T) \subset \mathcal{M}^s$. This gives a natural transformation

$$\underline{\mathcal{M}}^{ss}_\delta(X;F,P) \to \mathrm{Mor}(\cdot, \mathcal{M}^{ss}).$$

Let N be any other scheme with a natural transformation $\underline{\mathcal{M}}^{ss}_\delta(X;F,P) \to \mathrm{Mor}(\cdot, N)$, then the universal family over Z^{ss} defines a $SL(V)$-invariant morphism $Z^{ss} \to N$ which must factor through π and a morphism $\mathcal{M}^{ss} \to N$. This show that $\mathcal{M}^{ss}$ corepresents the functor $\underline{\mathcal{M}}^{ss}_\delta(X;F,P)$.

By taking étale slices to the $SL(V)$-action on Z^s, we can find an étale cover $\mathcal{M}' \to \mathcal{M}^s$ over which a universal family $\mathcal{G} = (G,\beta)$ exists (cf. Luna's Étale Slice Theorem, see Chapter 4 in [**35**]). Let $\mathcal{M}'' = \mathcal{M}' \times_{\mathcal{M}^s} \mathcal{M}'$. Take an isomorphism $\Phi\colon p_1^*(\mathcal{G}) \to p_2^*(\mathcal{G})$, which is normalized by the requirement that $p_1^*(\beta) \circ \Phi = p_2^*(\beta)$. The uniqueness result of Corollary 2.22 implies that

Φ satisfies the cocycle condition of descend theory (cf. Chapter VII in [**59**]). Hence, (G, β) descends to a universal family on $\mathcal{M}^s$ and therefore $\mathcal{M}^s$ represents the functor $\underline{\mathcal{M}}^s_\delta(X; F, P)$. Finally, the assertion about the closed point of $\mathcal{M}^{ss}$ is proved in Proposition 3.3 in [**34**]. $\square$

We conclude this section by stating a smoothness criterion for the fine moduli space of stable framed sheaves.

Theorem 5.10 (Theorem 4.1 in [**34**])**.** *Let $[(E, \alpha)]$ be a point in $\mathcal{M}^s_\delta(X; F, P)$. Consider E and $E \stackrel{\alpha}{\to} F$ as complexes which are concentrated in dimensions zero, and (zero, one), respectively.*

(i) *The Zariski tangent space of $\mathcal{M}^s_\delta(X; F, P)$ at a point $[(E, \alpha)]$ is naturally isomorphic to the first hyper-Ext group $\mathbb{E}\mathrm{xt}^1(E, E \stackrel{\alpha}{\to} F)$.*
(ii) *If the second hyper-Ext group $\mathbb{E}\mathrm{xt}^2(E, E \stackrel{\alpha}{\to} F)$ vanishes, then $\mathcal{M}^s_\delta(X; F, P)$ is smooth at $[(E, \alpha)]$.*

3. An example: moduli spaces of framed sheaves on surfaces

In this section we are dealing with framed sheaves that are locally free along the support of the framing sheaf. In particular, we would like to construct a moduli space parametrizing these objects under some mild conditions on the framing sheaf and its support in the case in which the ambient space is a surface. We follow the work of Bruzzo and Markushevich (see [**14**]).

Let $\mathbb{C}$ be the field of complex numbers and $(X, \mathcal{O}_X(1))$ a polarized variety of dimension d over it. Fix an effective divisor D and a sheaf F on X, supported on D, over which is a locally free $\mathcal{O}_D$-module. Recall that a framed sheaf $\mathcal{E} = (E, \alpha\colon E \to F)$ is called a (D, F)-framed sheaf if E is locally free in a neighborhood of D and $\alpha|_D$ is an isomorphism. From these properties, it follows that E is torsion free.

As we explained before, the boundedness property for a family of geometrical objects is the first step to construct moduli spaces that parametrize such objects. In [**46**], Lehn proved that the family of (D, F)-framed sheaf is bounded under some assumptions on the divisor D and on the framing sheaf F. More precisely, we need to give the following definition.

Definition 5.11. An effective divisor D on X is called *a good framing divisor* if we can write $D = \sum_i n_i D_i$, where D_i are prime divisors and $n_i > 0$, and there exists a nef and big divisor of the form $\sum_i a_i D_i$, with $a_i \geq 0$. For a coherent sheaf F on X supported on D, we shall say that F is a *good framing sheaf on* D, if it is locally free of rank r and there exists a real number A_0, $0 \leq A_0 < \frac{1}{r} D^2$, such that for any locally free subsheaf $F' \subset F$ of constant positive rank, $\frac{1}{\mathrm{rk}(F')} \deg(F') \leq \frac{1}{\mathrm{rk}(F)} \deg(F) + A_0$.

Theorem 5.12. *Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension $d \geq 2$. Let D be a good framing divisor and F a coherent sheaf on X, supported on D, which is a locally free $\mathcal{O}_D$-module. Then for every numerical polynomial $P \in \mathbb{Q}[n]$ of degree d, the family of (D, F)-framed sheaves on X with Hilbert polynomial P is bounded.*

Proof. For locally free (D, F)-framed sheaves, this result is proved in Theorem 3.2.4 of [**46**], but the proof goes through also in the general case. $\square$

Although we have a boundedness result for (D, F)-framed sheaves on varieties of arbitrary dimension, at this moment we are able to construct a moduli space for these objects only in the two-dimensional case. In particular, we will prove that there exists an ample divisor and a positive rational number for which all the (D, F)-framed sheaves with fixed Hilbert polynomial are μ-stable. Hence this moduli space will be an open subscheme of the moduli space of stable framed sheaves, constructed in the previous section.

Theorem 5.13. *Let X be a smooth projective surface, D a big and nef curve, and F a good framing sheaf on D. Then for any numerical polynomial $P \in \mathbb{Q}[n]$ of degree 2, there exists an ample divisor H on X and a positive rational number δ_1 such that all the (D, F)-framed sheaves on X with Hilbert polynomial P are μ-stable with respect to δ_1 and the ample divisor H.*

Proof. Since we are dealing with different ample divisors, for a coherent sheaf G of positive rank on X, we denote its slope with respect to an ample divisor C by $\mu_C(G)$.

Let X be a smooth projective surface and C an ample divisor on it. Fix a numerical polynomial $P \in \mathbb{Q}[n]$ of degree 2. Since the family of (D, F)-framed sheaves $\mathcal{E} = (E, \alpha)$ with Hilbert polynomial P is bounded, by Proposition 2.62 and Lemma 3.12, there exists a non-negative constant A_1, independent from E, such that for any (D, F)-framed sheaf $\mathcal{E} = (E, \alpha)$ and for any saturated subsheaf $E' \subset E$ of rank $r' < r = \mathrm{rk}(E)$

$$\mu_C(E') \leq \mu_C(E) + A_1.$$

For $n > 0$, let us denote by H_n the ample divisor $C + nD$. We shall verify that there exists a positive integer n such that the range of positive real numbers δ_1, for which all the (D, F)-framed sheaves of Hilbert polynomial P are μ-stable with respect to δ_1 and H_n, is nonempty.

Let $\mathcal{E} = (E, \alpha)$ be a (D, F)-framed sheaf and E' a nonzero subsheaf of E of rank r'. Assume first that E' is not contained in $\ker \alpha$. Hence $0 < r' < r$. The μ-stability condition with respect to δ_1 and H_n for $\mathcal{E}$ reads

$$\mu_{H_n}(E') < \mu_{H_n}(E) + \left(\frac{1}{r'} - \frac{1}{r}\right)\delta_1. \tag{14}$$

By saturating E', we can make $\mu_{H_n}(E')$ bigger, so we may assume that E' is a saturated subsheaf of $\mathcal{E}$, and hence that it is locally free in a neighborhood of D. Thus $E'|_D \subset E|_D$ and we get

$$\mu_{H_n}(E') = \frac{n}{r'}\deg(E'|_D) + \mu_C(E') \leq \mu_{H_n}(E) + nA_0 + A_1. \tag{15}$$

Thus we see that the inequality (15) implies the inequality (14) whenever

$$\frac{rr'}{r - r'}(nA_0 + A_1) < \delta_1. \tag{16}$$

Let $E' \subset \ker \alpha \cong E \otimes \mathcal{O}_X(-D)$ of rank $r' < r$. As before, we can assume that E' is saturated, hence it is a locally free sheaf on a neighborhood of D and $E'|_D \subset E|_D$. In this case the μ-stability condition for $\mathcal{E}$ is

$$\mu_{H_n}(E') < \mu_{H_n}(E) - \frac{1}{r}\delta_1. \tag{17}$$

The inclusion $E' \otimes \mathcal{O}_X(D) \subset E$ yields

$$\mu_{H_n}(E') < \mu_{H_n}(E) - H_n D + nA_0 + A_1 = \mu_{H_n}(E) - (D^2 - A_0)n + A_1 - DC. \tag{18}$$

We see that the inequality (18) implies the inequality (17) whenever

$$\delta_1 \leq r\left[(D^2 - A_0)n - A_1 + DC\right]. \tag{19}$$

Let us consider $E' \subset \ker\alpha$ of rank r. By framed saturating E', we can take $E' = \ker\alpha = E \otimes \mathcal{O}_X(-D)$. Hence

$$\mu_{H_n}(\ker\alpha) = \mu_{H_n}(E) - H_n D.$$

The inequality (17) is satisfied in this case, whenever $\delta_1 < r\left[D^2 n + CD\right]$; but the inequality (19) trivially implies this latter inequality. Hence the inequalities (16), (19), for all $r' = 1, \ldots, r-1$, have a nonempty interval of common solutions δ_1 if

$$n > \max\left\{\frac{rA_1 - CD}{D^2 - rA_0}, 0\right\}.$$

$\square$

Corollary 5.14. *Let X be a smooth projective surface, D a big and nef curve, and F a good framing sheaf on D. Then for any numerical polynomial $P \in \mathbb{Q}[n]$ of degree 2, there exists a quasi-projective scheme $\mathcal{M}^*(X; F, P)$ which is a fine moduli space of (D, F)-framed sheaves on X with Hilbert polynomial P.*

Remark 5.15. Let D be a smooth irreducible curve with $D^2 > 0$ and F a Gieseker-semistable locally free $\mathcal{O}_D$-module. By Example 1.4.5 and Theorem 2.2.14 in [**42**], D is a big and nef curve. Moreover, F is a good framing sheaf with $A_0 = 0$.

Let us assume that $(K_X + D) \cdot D < 0$. One can prove that $\mathrm{Ext}^2(E, E \xrightarrow{\alpha} F) = 0$ for any (D, F)-framed sheaf (E, α) on X. Thus by Theorem 5.10, $\mathcal{M}^*(X; F, P)$ is smooth for any Hilbert polynomial P.

CHAPTER 6

Uhlenbeck-Donaldson compactification for framed sheaves on surfaces

In this chapter we give an interesting application of the restriction theorems for μ-(semi)stable framed sheaves proved in Chapter 4. In particular, we describe the so-called *Uhlenbeck-Donaldson compactification* $\mathcal{M}^{\mu ss}(c, \delta)$ of the moduli space of μ-stable framed vector bundles on a nonsingular projective surface X. We define a semiample line bundle on the locally closed subscheme of $Quot_{X/k}(\mathcal{H}, P(c), F)$ (introduced in the previous chapter) that parametrizes, roughly speaking, μ-semistable framed sheaves of positive rank on X. In the proof of the semiampleness of this line bundle we heavily use Theorem 4.1. By using this line bundle (or more precisely the spaces of global sections of its powers), we define $\mathcal{M}^{\mu ss}(c, \delta)$ and a projective morphism π from the moduli space of semistable framed sheaves of topological invariants defined by c on X to $\mathcal{M}^{\mu ss}(c, \delta)$. Moreover, by using Theorem 4.8 we give a description of π in the case of (D, F)-framed sheaves.

In Section 1 we recall the construction of the *Le Potier determinant bundles* (see also [**44**]). In Section 2 we define a semiample line bundle that we will use in Section 3 to define the Uhlenbeck-Donalson compactification.

1. Determinant line bundles

Let Y be a Noetherian scheme. The *Grothendieck group* $K^0(Y)$ is the quotient of the free abelian group generated by all the locally free sheaves on Y, by the subgroup generated by all expressions $E - E' - E''$, whenever there is an exact sequence $0 \to E' \to E \to E'' \to 0$ of locally free sheaves on Y. $K^0(Y)$ is a commutative ring with unity $1 = [\mathcal{O}_Y]$ with respect to the operation $[E_1] \cdot [E_2] := [E_1 \otimes E_2]$ for locally free sheaves E_1 and E_2. Since the determinant is multiplicative in short exact sequences, it defines a homomorphism

$$\det\colon K^0(Y) \to \mathrm{Pic}(Y).$$

If one consider all the coherent sheaves on Y, by using the same definition, one can obtain the Grothendieck group $K_0(Y)$. Moreover, we can give to it a structure of $K^0(Y)$-module.

A projective morphism $f\colon Y \to S$ of Noetherian schemes induces a homomorphism $f_!\colon K_0(Y) \to K_0(S)$, by putting $f_!([G]) = \sum_{\nu \geq 0} (-1)^{\otimes \nu} [R^{\otimes \nu} f_*(G)]$. If f is a smooth projective morphism of relative dimension d between schemes of finite type over k, by Proposition 2.1.10 in [**35**], for any flat family G of coherent sheaves on the fibres of f, there is a locally free resolution

$$0 \longrightarrow E_d \longrightarrow E_{d-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow G$$

such that $R^d f_*(E_\nu)$ is locally free for $\nu = 0, \ldots, d$, $R^i f_*(G_\nu) = 0$ for $i \neq d$ and $\nu = 0, \ldots, d$. Thus $[G] \in K^0(Y)$ and $f_![G] \in K^0(S)$. Obviously, we can use the same argument for any locally free sheaf on Y, hence we get a well defined homomorphism $f_!\colon K^0(Y) \longrightarrow K^0(S)$.

Let X be a nonsingular projective variety of dimension d. In this case $K_0(X) = K^0(X)$ and we will denote it by $K(X)$. Two classes u and u' in $K(X)$ are said to be numerically equivalent, and we will denote $u \equiv u'$, if their difference is contained in the radical of the bilinear form $(a, b) \mapsto \chi(a \cdot b)$. Let $K(X)_{num} = {}^{K(X)}/{\equiv}$.

By Hirzebruch-Riemann-Roch theorem, $\chi(a \cdot b)$ depends on the rank and the Chern classes of a and b. Hence the numerical behaviour of $a \in K(X)_{num}$ is determined by its associated rank $\mathrm{rk}(a)$ and Chern classes $c_i(a)$.

Let us fix a very ample line bundle $\mathcal{O}_X(1)$ on X. For any class c in $K(X)_{num}$, we write $c(n) := c \cdot [\mathcal{O}_X(n)]$ and denote by $P(c)$ the associated Hilbert polynomial $P(c, n) = \chi(c(n))$.

A flat family E of coherent sheaves on X parametrized by a Noetherian scheme S defines an element $[E] \in K^0(S \times X)$, and as the projection p_S is a smooth projective morphism, there is a well defined morphism $(p_S)_!\colon K^0(S \times X) \to K^0(S)$.

Definition 6.1. Let E be a family of coherent sheaves on X parametrized by a Noetherian scheme S. Let $\lambda_E\colon K(X) \to \mathrm{Pic}(S)$ be the composition of the homomorphisms

$$\lambda_E\colon K(X) \xrightarrow{p_X^*} K^0(X \times S) \xrightarrow{\cdot [E]} K^0(S \times X) \xrightarrow{(p_S)_!} K^0(S) \xrightarrow{\det} \mathrm{Pic}(S).$$

Lemma 6.2 (Lemma 8.1.2 in [**35**]). *The following properties hold for the homomorphism λ:*

(1) *If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of S-flat families of coherent sheaves, then $\lambda_E(u) \cong \lambda_{E'}(u) \otimes \lambda_{E''}(u)$ for any class $u \in K(X)$,*
(2) *If E is a S-flat family and $f\colon S' \to S$ a morphism, for any $u \in K(X)$ one has $f^*(\lambda_E(u)) = \lambda_{f^*(E)}(u)$.*
(3) *If G is an algebraic group, S a scheme with a G-action and E a G-linearized S-flat family of coherent sheaves on X, then λ_E factors through the group* $\mathrm{Pic}^G(S)$ *of isomorphism classes of G-linearized line bundles on S.*
(4) *Let E be a S-flat family of coherent sheaves of numerical K-theory class $c \in K(X)_{num}$ and $\mathcal{N}$ a locally free $\mathcal{O}_S$-sheaf. Then $\lambda_{E \otimes p_S^*(\mathcal{N})}(u) \cong \lambda_E(u)^{\mathrm{rk}(\mathcal{N})} \otimes \det(\mathcal{N})^{\chi(c \cdot u)}$.*

Let us denote by H the divisor associated to $\mathcal{O}_X(1)$ and let $h = [\mathcal{O}_H]$ be its class in $K(X)$. Let E be a family of coherent sheaves on X parametrized by a Noetherian scheme S, x a point in X and $c \in K(X)_{num}$. Let

$$u_i(c) = -\mathrm{rk}(c) \cdot h^i + \chi(c \cdot h^i) \cdot [\mathcal{O}_x] \ \text{ for } \ i \geq 0.$$

In the following we will consider the line bundles $\lambda_E(u_i(c)) \in \mathrm{Pic}(S)$ for $i \geq 0$.

2. Semiample line bundles

Let $(X, \mathcal{O}_X(1))$ be a polarized surface. Fix a stability polynomial $\delta(n) = \delta_1 n + \delta_0 \in \mathbb{Q}[n]$, with $\delta_1 > 0$, and a framing sheaf F that is a coherent $\mathcal{O}_D$-module, where $D \subset X$ is a fixed big and nef curve. Fix a numerical K-theory class $c \in K(X)_{num}$ with rank r, Chern classes c_1 and c_2, and a line bundle $\mathcal{A}$ with $c_1(\mathcal{A}) = c_1$. Let us denote by $P(c)$ the Hilbert polynomial associated to c.

Let $a \gg 0$ be an integer and $C \in |\mathcal{O}_X(a)|$ a general curve. Then C is smooth and transversal to D. By the boundedness of the family of μ-semistable framed sheaves with Hilbert polynomial $P(c)$ (cf. Proposition 2.64), we can fix a sufficiently large number m such that for each μ-semistable framed sheaf $\mathcal{E} = (E, \alpha)$ with Hilbert polynomial $P(c)$, the sheaf E is m-regular and $\mathrm{h}^1(E(m-a)) = 0$. Let us define $V = k^{\oplus P(c,m)}$ and $\mathcal{H} = V \otimes \mathcal{O}_X(-m)$ and consider the scheme

$$Y := Quot_{X/k}(\mathcal{H}, P(c), F) \subset Quot_{X/k}(\mathcal{H}, P(c)) \times \mathbb{P}(\mathrm{Hom}(V, \mathrm{H}^0(F(m)))^\vee),$$

defined in Chapter 5, Section 2. Put $\mathbb{P} := \mathbb{P}(\mathrm{Hom}(V, \mathrm{H}^0(F(m)))^\vee)$.

Let $R^{\mu ss}(c, \delta)$ be the locally closed subscheme of Y formed by pairs $([g \colon \mathcal{H} \to E], [a \colon \mathcal{H} \to F])$ such that E is a coherent sheaf with Hilbert polynomial $P(c)$ and determinant $\mathcal{A}$, the framed sheaf (E, α) is μ-semistable with respect to δ_1, where the framing α is defined uniquely by the relation $a = \alpha \circ g$, and g induces an isomorphism $V \to \mathrm{H}^0(E(m))$.

Let us denote by p_1, p_2 the projections from $R^{\mu ss}(c, \delta)$ to Y and $\mathbb{P}$, respectively. Let $\tilde{E}$ denote the universal quotient of Y (cf. Chapter 5, Section 2). Define the line bundle on $R^{\mu ss}(c, \delta)$

$$\mathcal{L}_1(n_1, n_2) = p_1^*(\lambda_{\tilde{E}}(u_1(c)))^{\otimes n_1} \otimes p_2^*(\mathcal{O}_{\mathbb{P}}(n_2)).$$

where we set

$$\frac{n_2}{n_1} = A_X(l),$$

where $A_X(l)$ is defined by formula (13) and l is a sufficiently large positive integer such that $A_X(l) > 0$.

Now we want to prove the following result.

Proposition 6.3. *There exists a positive integer l_X such that the line bundle $\mathcal{L}(n_1, n_2)^{\otimes \nu}$ is generated by its $SL(V)$-invariant sections, for $\nu \gg 0$ and $n_2/n_1 = A_X(l_X)$.*

Proof. Let $S = R^{\mu ss}(c, \delta)$. The pullback of the universal quotient of Y to S gives us a flat family $\mathcal{E}_S = (\mathbf{E}, \alpha_{\mathbf{E}})$ of μ-semistable framed sheaves $\mathcal{E} = (E, \alpha \colon E \to F)$ on X with Hilbert polynomial $P(c)$ and determinant $\mathcal{A}$. Moreover, the restriction of $(\mathbf{E}, \alpha_{\mathbf{E}})$ to $S \times C$ yields a family $(\mathbf{E}_C, \alpha_{\mathbf{E}_C})$ of framed sheaves $(E_C, \alpha_C \colon E_C \to F|_C)$ on C, where $i \colon C \hookrightarrow X$ is the inclusion map. By Theorem 4.1, the general[1] element in this family is μ-semistable with respect to $\delta_C := a\delta_1$.

The K-theory class $i^*(c) \in K(C)$ is uniquely determined by r and $\mathcal{A}|_C$. Let $m' = a \deg(X) m$, $V_C = k^{\oplus P(i^*(c), m')}$ and $\mathcal{H}_C = V_C \otimes \mathcal{O}_C(-m')$. Let $Q_C \subset Quot_{C/k}(\mathcal{H}_C, P(i^*(c)))$ be the closed subset parametrizing quotients of $\mathcal{H}_C$ with determinant $\mathcal{A}|_C$. Let us denote by $\tilde{E}_C$ the universal quotient of Q_C. Furthermore, let $\mathbb{P}_C := \mathbb{P}(\mathrm{Hom}(V_C, \mathrm{H}^0(F|_C(m')))^\vee)$, so that a point $[a] \in \mathbb{P}_C$ induces a morphism $\mathcal{H}_C \to F|_C$ defined up to a constant factor. Consider the closed subscheme $Y_C := Quot_{C/k}(\mathcal{H}_C, P(i^*(c)), F|_C)$ of $Q_C \times \mathbb{P}_C$ defined similarly to the scheme Y above. Clearly the group $SL(V_C)$ acts on Y_C.

Let us denote by $p_{1,C}, p_{2,C}$ the projections from Y_C to Q_C and $\mathbb{P}_C$, respectively, and $\deg C = C \cdot H$. Consider the line bundle on Y_C

$$\mathcal{L}'_0(n_1, n_2) = p_{1,C}^*(\lambda_{\tilde{E}_C}(u_0(i^*(c)))^{\otimes n_1} \otimes p_{2,C}^*(\mathcal{O}_{\mathbb{P}_C}(n_2)).$$

Choose n_1, n_2 in a way that there exists a sufficiently large integer l for which

$$\frac{n_2}{n_1} = A_C(l) > 0,$$

[1]By *general* we mean that the property holds true for all closed points in a nonempty open subset.

where $A_C(l)$ is the rational function defined in (13).

Proposition 6.4. *Let $a \gg 0$. There exist m and l_a, depending on a, $\deg(X)$, and c, such that for all $l \geq l_a$ the following statements hold:*

(1) *the line bundle $\mathcal{L}'_0(n_1, n_2)$ is very ample on Y_C,*
(2) *Given a point $([g\colon \mathcal{H}_C \to E_C], [a\colon \mathcal{H}_C \to F|_C]) \in Y_C$, the following assertions are equivalent:*
  - *the corresponding framed sheaf (E_C, α_C) is μ-semistable with respect to δ_C and g induces an isomorphism $V_C \to \mathrm{H}^0(E(m))$,*
  - *the point $([g], [a])$ is semistable for the action of $SL(V_C)$ with respect to the linearization of $\mathcal{L}'_0(n_1, n_2)$,*
  - *there is an integer ν and an $SL(V_C)$-invariant section σ of $\mathcal{L}'_0(n_1, n_2)^{\otimes \nu}$ such that $\sigma([g], [a]) \neq 0$.*
(3) *Two points $([g_i], [a_i])$, $i = 1, 2$, are separated by invariant sections in some tensor power of $\mathcal{L}'_0(n_1, n_2)$ if and only if either both are semistable points with respect to the $SL(V_C)$ action but the corresponding framed sheaves are not S-equivalent, or one of them is semistable but the other is not.*

Proof. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf on X corresponding to a point in Y. First recall that for any general curve $C \in |\mathcal{O}_X(a)|$, the family of subsheaves E'_C of $E|_C$ generated by all subspaces W of V_C is bounded, so the set $\mathcal{N}_{\mathcal{E}|_C}$ of their polynomials $P(E'_C)$ is finite. Since the family $\mathcal{S}^{\mu ss}(c, \delta)$ of μ-semistable framed sheaves of numerical K-theory class c is bounded, the set

$$\mathcal{N}_C(c, \delta_C) := \bigcup_{\mathcal{E} \in \mathcal{S}^{\mu ss}(c,\delta)} \mathcal{N}_{\mathcal{E}|_C}$$

is finite. Hence also the set

$$\mathcal{N}(c, \delta_C) := \bigcup_{C \in |\mathcal{O}_X(a)|} \mathcal{N}_C(c, \delta_C)$$

is finite. For a polynomial $B \in \mathcal{N}(c, \delta_C)$, define

$$G_B(l) := \dim(V_C)\left(n_1 B(l) + n_2 \epsilon(\alpha_{E'_C})\right) - \dim(W)(n_1 P(c, l) + n_2).$$

where E'_C is a subsheaf of some μ-semistable framed sheaf $\mathcal{E} = (E, \alpha)$ of numerical K-theory class c, defined by a subspace $W \subset V_C$, and $\alpha_{E'_C}$ is the induced framing on E'_C. Since the set $\{G_B(l) \,|\, B \in \mathcal{N}(c, \delta_C)\}$ is finite, there exists a number l_a, depending only on a, such that for any $l \geq l_a$ the implication

$$G_B(l) > 0 \Rightarrow G_B(l) \text{ is positive for } l \gg 0,$$

is true for all $B \in \mathcal{N}(c, \delta_C)$. Thus by combining the following argument with Proposition 3.1 in [**34**], we obtain that statement (1) follows from the same arguments of the proof of Theorem 8.1.11 in [**35**], and statement (2) by Proposition 5.6. The assertion (3) follows from the third statement of Theorem 5.9. □

Choose the positive integer l_X such that the following equality holds

$$A_X(l_X) = A_C(l_a).$$

Thus we take $\dfrac{n_2}{n_1} = A_C(l_a) = A_X(l_X)$.

By the choice of m', we can construct a linear map

$$\mathrm{Hom}(V, \mathrm{H}^0(F(m))) \longrightarrow \mathrm{Hom}(V_C, \mathrm{H}^0(F|_C(m'))),$$

that induces a rational map $f \colon \mathbb{P} \dashrightarrow \mathbb{P}_C$. Since f is induced by a linear map, we get

$$f^*(\mathcal{O}_{\mathbb{P}_C}(1)) \cong \mathcal{O}_{\mathbb{P}}(1). \tag{20}$$

Consider now the exact sequence

$$0 \longrightarrow \mathbf{E} \otimes (p_S^*(\mathcal{O}_S) \otimes p_X^*(\mathcal{O}_X(-a)) \longrightarrow \mathbf{E} \longrightarrow \mathbf{E}_C \longrightarrow 0.$$

Assume that m is big enough so that, not only the results in Proposition 6.4 hold, but one also has:

$$(\mathbf{E}_C)_s \text{ is } m'\text{-regular for all } s \in S.$$

The sheaf $p_*(\mathbf{E}_C(-m'))$ is a locally free $\mathcal{O}_S$-module of rank $P(i^*(c), m')$, where $\mathbf{E}_C(-m') = (\mathbf{E}_C \otimes p_S^*(\mathcal{O}_S) \otimes p_C^*(\mathcal{O}_C(m')))$ and $p \colon S \times C \to S$ is the projection. Let $\pi \colon \tilde{S} \to S$ be the *projective frame bundle* associated to $p_*(\mathbf{E}_C(-m'))$ (cf. Examples 4.2.3 and 4.2.6 in [**35**]). By construction there is a *universal* $GL(P(i^*(c), m'))$-equivariant isomorphism

$$V_C \otimes \mathcal{O}_{\tilde{S}} \to \pi^*(p_*(\mathbf{E}_C(-m'))) \otimes \mathcal{O}_{\tilde{S}}(1), \tag{21}$$

that induces a family $(\pi \times \mathrm{id}_X)^*(\mathbf{E}_C)$ of coherent sheaves on X parametrized by $\tilde{S}$ with a surjective morphism

$$p_{\tilde{S}}^*(\mathcal{H}_C) \longrightarrow (\pi \times \mathrm{id}_X)^*(\mathbf{E}_C) \otimes p_{\tilde{S}}^*(\mathcal{O}_{\tilde{S}}(1)).$$

This induces a $SL(P(i^*(c), m'))$-invariant morphism $\Phi_{\mathbf{E}_C} \colon \tilde{S} \to Q_C$. Moreover, we get a $\tilde{S}$-flat family of framed sheaves $\mathcal{E}_{\tilde{S}} := ((\pi \times \mathrm{id}_X)^*(\mathbf{E}_C), (\pi \times \mathrm{id}_X)^*(\alpha_{\mathbf{E}_C}))$. For any $s \in \tilde{S}$, the composition of morphisms

$$p_X^*(\mathcal{H}_C)|_{\{s\} \times C} \xrightarrow{q_s} (\pi \times \mathrm{id}_X)^*(\mathbf{E}_C)|_{\{s\} \times C} \xrightarrow{(\pi \times \mathrm{id}_X)^*(\alpha_{\mathbf{E}_C})} p_C^*(F|_C)|_{\{s\} \times C}$$

gives a morphism $V_C \otimes \mathcal{O}_C(-m') \to F|_C$, and, by passing to cohomology, a morphism $V_C \to \mathrm{H}^0(F|_C(m'))$. Since q_s is uniquely defined by the isomorphism (21) and the framing $(\pi \times \mathrm{id}_X)^*(\alpha_{\mathbf{E}_C})|_{\{s\} \times C}$ is defined up to a nonzero constant factor, we obtain a morphism

$$g_{\mathcal{E}_{\tilde{S}}} \colon \tilde{S} \longrightarrow \mathbb{P}_C.$$

By construction the morphism $\Phi_{\mathbf{E}_C} \times g_{\mathcal{E}_{\tilde{S}}} \colon \tilde{S} \to Q_C \times \mathbb{P}_C$ factors through the closed embedding $Y_C \hookrightarrow Q_C \times \mathbb{P}_C$ (cf. Section 1.3 in [**33**]), hence we obtain an $SL(P(i^*(c), m'))$-invariant morphism

$$\Psi_{\mathcal{E}_{\tilde{S}}} \colon \tilde{S} \longrightarrow Y_C.$$

The group $SL(V)$ acts on S, hence also on $\tilde{S}$. Thus we have an action of $SL(V) \times SL(V_C)$ on $\tilde{S}$ such that π and $\Psi_{\mathcal{E}_{\tilde{S}}}$ are both equivariant for $SL(V) \times SL(V_C)$. By using the same arguments of the proof of Proposition 8.2.3 in [**35**], in particular formula (8.2), we get

$$\lambda_{\mathbf{E}_C}(u_0(i^*(c)))^{a \deg(X)} \cong \lambda_{\mathbf{E}}(u_1(c))^{a^2 \deg(X)}.$$

Thus, from formula (20), it follows

$$\Psi_{\mathcal{E}_{\tilde{S}}}^* \left(\mathcal{L}_0'(n_1, an_2)^{\deg(C)} \right) \cong \pi^* \left(\mathcal{L}_1(n_1, n_2)^{a^2 \deg(X)} \right).$$

Take an arbitrary $SL(V_C)$-invariant section σ in $\mathcal{L}_0'(n_1, an_2)^{\deg(C)}$. Then $\Psi^*_{\mathcal{E}_{\tilde{S}}}(\sigma)$ is a $SL(V) \times SL(V_C)$ -invariant section and therefore descends to a $SL(V)$-invariant section of the line bundle $\mathcal{L}_1(n_1, n_2)^{a^2 \deg(X)}$. In this way we get a linear map

$$s_{\mathcal{E}_S} \colon \mathrm{H}^0\left(Y_C, \mathcal{L}_0'(n_1, an_2)^{\deg(C)}\right)^{SL(V_C)} \longrightarrow \mathrm{H}^0\left(S, \mathcal{L}_1(n_1, n_2)^{a^2 \deg(X)}\right)^{SL(V)}$$

By the definition of GIT semistability with respect to a linearized line bundle (see, e.g., Definition 4.2.9 in [**35**]) and by Proposition 5.6, for any point $s \in S$ such that $(\mathcal{E}_S)|_{\{s\}\times C}$ is semistable, then there is an integer $\nu > 0$ and a $SL(V)$-invariant section $\bar{\sigma}$ in $\mathcal{L}_1(n_1, n_2)^{\otimes \nu}$ such that $\bar{\sigma}(s) \neq 0$. Therefore we get the assertion. □

3. Compactification for framed sheaves

In this section we perform the construction of the Uhlenbeck-Donaldson compactification.

First we need to recall the following result, that is a straightforward generalization of Langton's Theorem (Theorem 2.B.1 in [**35**]):

Theorem 6.5. *Let X be a smooth projective variety over an algebraically closed field k. Let (R, m) be a discrete valuation ring with residue field k and quotient field K. Let $\mathcal{E}$ be an* $\mathrm{Spec}(R)$*-flat family of framed sheaves of positive rank on X such that the pullback $\mathcal{E}_K$ of it in $X_K = \mathrm{Spec}(K) \times X$ is a μ-semistable framed sheaf in X_K. Then there exists a coherent subsheaf $G \subset E$ such that $\mathcal{G}_K = \mathcal{E}_K$ and the pullback $\mathcal{G}_k$ of $\mathcal{G}$ in $\mathrm{Spec}(k) \times X \cong X$ is a μ-semistable framed sheaf in X, where the framed sheaf $\mathcal{G}$ consists of G with the induced framing by $\mathcal{E}$.*

Corollary 6.6. *If T is a separated scheme of finite type over k and if $\phi\colon R^{\mu ss}(c, \delta) \to T$ is any $SL(V)$-invariant morphism, the image of ϕ is proper.*

Proof. The proof of the corollary goes as for Proposition 8.2.5 in [**35**]. □

By Proposition 6.3, the line bundle $\mathcal{L}(n_1, n_2)^{\otimes \nu}$ is generated by its $SL(V)$-invariant section. Thus we can find a finite-dimensional subspace

$$W \subset W_\nu := \mathrm{H}^0(R^{\mu ss}(c, \delta), \mathcal{L}_1(n_1, n_2)^{\otimes \nu})^{SL(V)},$$

that generates $\mathcal{L}(n_1, n_2)^{\otimes \nu}$. Let $\phi_W \colon R^{\mu ss}(c, \delta) \to \mathbb{P}(W)$ be the induced $SL(P(c, m))$-invariant morphism. By the previous corollary, we get that $M_W := \phi(R^{\mu ss}(c, \delta))$ is a projective scheme. By proceeding as in the proof of Proposition 8.2.6 in [**35**], we can prove the following result.

Proposition 6.7. *There is an integer $N > 0$ such that $\oplus_{l \geq 0} W_{lN}$ is a finitely generated graded ring.*

We can eventually define the *Uhlenbeck-Donaldson compactification.*

Definition 6.8. Let N is a positive integer as in the proposition above. Let $\mathcal{M}^{\mu ss}(c, \delta)$ be the projective scheme

$$\mathcal{M}^{\mu ss}(c, \delta) = \mathrm{Proj}\left(\bigoplus_{k \geq 0} \mathrm{H}^0(R^{\mu ss}(c, \delta), \mathcal{L}_1(n_1, n_2)^{\otimes kN})^{SL(P(c,m))}\right),$$

and let $\gamma\colon R^{\mu ss}(c, \delta) \to \mathcal{M}^{\mu ss}(c, \delta)$ be the canonically induced morphism.

As it is proved in Section 4 of [**15**], the morphism π *descends* to a projective morphism

$$\pi\colon \mathcal{M}_\delta^{ss}(X;F,P(c)) \to \mathcal{M}^{\mu ss}(c,\delta).$$

From now on the framing sheaf F is a locally free $\mathcal{O}_D$-module.

Let $R^{\mu ss}(c,\delta)^*$ be the open subset of $R^{\mu ss}(c,\delta)$ consisting of pairs $([g\colon \mathcal{H}\to E],[a\colon \mathcal{H}\to F])$ such that the associated framed sheaf (E,α) is a μ-semistable (D,F)-framed sheaf on X. Let

$$\mathcal{M}^{\mu ss}(c,\delta)^* := \gamma(R^{\mu ss}(c,\delta)^*) \ \text{ and } \ \mathcal{M}_\delta(X;F,P(c))^* := \pi^{-1}(\mathcal{M}^{\mu ss}(c,\delta)^*).$$

These are open subsets of $\mathcal{M}^{\mu ss}(c,\delta)$ and $\mathcal{M}_\delta^{ss}(X;F,P(c))$, respectively. Now we would like to give an explicit description of the morphism

$$\pi_{\mathrm{rk}(c)} := \pi|_{\mathcal{M}_\delta(X;F,P(c))^*}\colon \mathcal{M}_\delta(X;F,P(c))^* \longrightarrow \mathcal{M}^{\mu ss}(c,\delta)^*.$$

Let $\mathcal{E}=(E,\alpha)$ be a μ-semistable (D,F)-framed sheaf. The graded object $gr^\mu(\mathcal{E}) = (gr^\mu(E), gr^\mu(\alpha))$ associated to a μ-Jordan-Hölder filtration of $\mathcal{E}$ is a μ-polystable framed sheaf. By applying the definition of μ-semistability to $E(-D)=\ker\alpha\subset E$, we conclude that $\delta_1\leq r\deg D$. Moreover, in the case of equality, $\ker\alpha\subset E$ is the upper level of a Jordan-Hölder filtration, hence in the associated graded object there is a rank zero quotient $E/_{\ker\alpha}$. Since E is torsion free, this is the only possible torsion sheaf in the graded object associated to a Jordan-Hölder filtration. To avoid this possibility, from now on we impose the following additional hypothesis

$$\delta_1 < r\deg D.$$

Thus the sheaf $gr^\mu(E)$ is torsion free, hence the double dual $(gr^\mu(\mathcal{E}))^{\vee\vee}$ of $gr^\mu(\mathcal{E})$ is a μ-polystable *framed vector bundle*, i.e., a μ-polystable framed sheaf such that the underlying coherent sheaf is locally free (cf. Lemma 4.10). Let us consider the function

$$\begin{array}{rcl} l_{\mathcal{E}}\colon X & \longrightarrow & Sym^l(X\setminus D), \\ x & \longmapsto & \displaystyle\sum_x \mathrm{length}\left(\left(gr^\mu(\mathcal{E})^{\vee\vee}/gr^\mu(\mathcal{E})\right)_x\right)[x], \end{array}$$

where $l=c_2(E)-c_2(gr^\mu(E)^{\vee\vee})$. Both $gr^\mu(\mathcal{E})^{\vee\vee}$ and $l_{\mathcal{E}}$ are well-defined invariants of $\mathcal{E}$, i.e, they do not depend on the choice of the μ-Jordan-Hölder filtration (cf. Proposition 2.42).

By using Theorem 4.8 and the same techiques as the nonframed case (cf. Theorem 8.2.11 in [**35**]), we obtain the following result.

Theorem 6.9 (Theorem 4.6 in [**15**]). *Assume that $\delta_1 < r\deg D$. Two μ-semistable (D,F)-framed sheaves $\mathcal{E}_1=(E_1,\alpha_1)$ and $\mathcal{E}_2=(E_2,\alpha_2)$ of numerical K-theory class c on X define the same closed point in $\mathcal{M}^{\mu ss}(c,\delta)^*$ if and only if*

$$gr^\mu(\mathcal{E}_1)^{\vee\vee}\cong gr^\mu(\mathcal{E}_2)^{\vee\vee} \ \text{ and } \ l_{\mathcal{E}_1}=l_{\mathcal{E}_2}.$$

Remark 6.10. Assume, as before, that $\delta_1 < r\deg D$. Let c be a numerical K-theory class of X with rank r, Chern classes c_1 and c_2, and a line bundle $\mathcal{A}$ with $c_1(\mathcal{A})=c_1$. By the previous theorem, we can define the subset $\mathcal{M}^{\mu-poly}(r,\mathcal{A},c_2,\delta)\subset\mathcal{M}_\delta(X;F,P(c))^*$ consisting of μ-polystable framed vector bundles. Moreover, set-theoretically, there is a stratification

$$\mathcal{M}^{\mu ss}(c,\delta)^* = \coprod_{l\geq 0}\mathcal{M}^{\mu-poly}(r,\mathcal{A},c_2-l,\delta)\times Sym^l(X\setminus D).$$

$\triangle$

From now on assume that X is a nonsingular projective surface over $\mathbb{C}$, D a big and nef curve and F a good framing sheaf on D. By Theorem 5.13, for any numerical K-theory class $c \in K(X)_{num}$ with rank r, Chern classes c_1 and c_2, and a line bundle $\mathcal{A}$ with $c_1(\mathcal{A}) = c_1$, there exists a fine moduli space $\mathcal{M}_{X,D}(r, \mathcal{A}, n)$ of (D, F)-framed sheaves (E, α) on X where E is a torsion free sheaf of rank r, first Chern class c_1, second Chern class c_2 and determinant line bundle $\mathcal{A}$. It is an open subset of the moduli space $\mathcal{M}_{\delta}^{\mu-stable}(X; F, P(c), \mathcal{A})$ of μ-stable framed sheaves on X with the same *topological invariants*, for a suitable choices of a very ample line bundle on X and a stability polynomial δ.

Since the graded object of a μ-stable framed sheaf coincides with the framed sheaf itself, we get the following map

$$\begin{aligned} \pi_r := \pi|_{\mathcal{M}_{X,D}(r,\mathcal{A},n)} \colon \mathcal{M}_{X,D}(r, \mathcal{A}, n) &\longrightarrow \coprod_{l \geq 0} \mathcal{M}_{X,D}(r, \mathcal{A}, n-l) \times Sym^l(X \setminus D) \\ (E, \alpha) &\longmapsto \left((E^{\vee\vee}, \alpha^{\vee\vee}), \operatorname{supp}\left({}^{E^{\vee\vee}}\!/_{E} \right) \right). \end{aligned}$$

Moreover the restriction of π_r to the open subset consisting of (D, F)-framed vector bundles is a bijection onto the image.

Remark 6.11. Let X be the complex projective plane $\mathbb{CP}^2$ and l_∞ a line. Fix positive integer numbers r, n. The open subset in $\mathcal{M}_{\mathbb{CP}^2, l_\infty}(r, \mathcal{O}_{\mathbb{CP}^2}, n)$ consisting of $(l_\infty, \mathcal{O}_{l_\infty}^{\oplus r})$-framed vector bundles on $\mathbb{CP}^2$ is isomorphic to the moduli space of framed $SU(r)$-instantons with instanton number n on S^4 (cf. [**18**]). By using Theorem A' in [**79**] and Uhlenbeck's removable singularities Theorem (see [**77**]), we expect that it is possible to define a topology on the set

$$\coprod_{l=0}^{n} \mathcal{M}_{\mathbb{CP}^2, l_\infty}(r, \mathcal{O}_{\mathbb{CP}^2}, n-l) \times Sym^l(\mathbb{C}^2)$$

such that π_r is a proper map. Moreover we expect that by using Buchdal's work for framed $SU(r)$-instantons on the connected sum of n copies of $\mathbb{CP}^2$ (see [**16**]) it is possible to generalize this result to the moduli spaces of $(l_\infty, \mathcal{O}_{l_\infty}^{\oplus r})$-framed sheaves on the blow up of $\mathbb{CP}^2$ at n points.

CHAPTER 7

Symplectic structures

A *symplectic structure* on a non-singular variety M is by definition a non-trivial regular two-form, i.e., a global section $0 \neq w \in \mathrm{H}^0(M, \Omega_M^2)$, which is non-degenerate and closed. In this chapter we give a construction of closed two-forms on the moduli spaces of stable (D, F)-framed sheaves by using a framed version of the Atiyah class. In particular, in Section 2 we recall the definition of Atiyah class for a flat family of coherent sheaves and describe some geometric constructions one can do by using it, as for example the Kodaira-Spencer map. In Section 3 we give the definitions of the *framed* version of the Atiyah class and of the Kodaira-Spencer map. In Section 4 we prove that the *framed* Kodaira-Spencer map is an isomorphism for the moduli space of stable (D, F)-framed sheaves and, in Section 5, we construct closed two-forms on it. Finally, in Section 6 we apply our results when the ambient space is the second Hirzebruch surface and provide a symplectic structure on the moduli spaces of stable (D, F)-framed sheaves on it.

1. Yoneda pairing and trace map

In this section we introduce the notions of Yoneda pairing (or cup product) for hyper-Ext groups of complexes of sheaves and the trace map. These are some of the technical tools we need to obtain the geometric results of the following sections.

Let Y be a k-scheme of finite type. Let $E^\bullet$ and $G^\bullet$ be finite complexes of locally free sheaves. We define the complex of coherent sheaves $\mathcal{H}om^\bullet(E^\bullet, G^\bullet)$ with components

$$\mathcal{H}om^n(E^\bullet, G^\bullet) = \bigoplus_i \mathcal{H}om(E^i, G^{i+n}),$$

and differential

$$d(\varphi) = d_{G^\bullet} \circ \varphi - (-1)^{\deg \varphi} \cdot \varphi \circ d_{E^\bullet}.$$

If $L^\bullet$ is another finite complex of locally free sheaves, composition yields a morphism

$$\mathcal{H}om^\bullet(G^\bullet, L^\bullet) \otimes \mathcal{H}om^\bullet(E^\bullet, G^\bullet) \longrightarrow \mathcal{H}om^\bullet(E^\bullet, L^\bullet), \tag{22}$$

such that $d(\psi \circ \varphi) = d(\psi) \circ \varphi + (-1)^{\deg \psi} \psi \circ d(\varphi)$ for homogeneous elements φ and ψ.

Recall that the hyper-Ext group $\mathbb{E}\mathrm{xt}^i(E^\bullet, G^\bullet)$ is the hypercohomology of the complex $\mathcal{H}om^\bullet(E^\bullet, G^\bullet)$, that is, the direct limit, over the open coverings $\mathcal{U}$ of Y, of the cohomology of the total complex associated to the Čech complex $C^\bullet(\mathcal{H}om^\bullet(E^\bullet, G^\bullet), \mathcal{U})$. The product (22) induces a product in hypercohomology

$$\mathbb{E}\mathrm{xt}^i(G^\bullet, L^\bullet) \otimes \mathbb{E}\mathrm{xt}^j(E^\bullet, G^\bullet) \longrightarrow \mathbb{E}\mathrm{xt}^{i+j}(E^\bullet, L^\bullet).$$

This is the *Yoneda pairing* for hyper-Ext groups of finite complexes of locally free sheaves.

For any locally free sheaf E, let $tr_E\colon \mathcal{E}nd(E) \to \mathcal{O}_Y$ denote the trace map, which can be defined as the pairing between $E^\vee$ and E, since $\mathcal{E}nd(E) \cong E^\vee \otimes E$. More generally, if $E^\bullet$ is a finite complex of locally free sheaves, define the trace

$$tr_{E^\bullet}\colon \mathcal{H}om^\bullet(E^\bullet, E^\bullet) \longrightarrow \mathcal{O}_Y,$$

by setting $tr_{E^\bullet}|_{\mathcal{H}om(E^i,E^j)} = 0$, except in the case $i = j$, when we put $tr_{E^\bullet}|_{\mathcal{E}nd(E^i)} = (-1)^i tr_{E^i}$.

Let us consider the morphism

$$\begin{array}{rcl} i_{E^\bullet}\colon \mathcal{O}_Y & \longrightarrow & \mathcal{H}om^0(E^\bullet, E^\bullet), \\ 1 & \longmapsto & \sum_i \mathrm{id}_{E^i}. \end{array}$$

Clearly, $tr_{E^\bullet}(i_{E^\bullet}(1)) = \sum_i (-1)^i \mathrm{rk}(E^i)$, which is the *rank* $\mathrm{rk}(E^\bullet)$ of $E^\bullet$.

Both $i_{E^\bullet}$ and $tr_{E^\bullet}$ are chain morphism (where $\mathcal{O}_Y$ is a complex concentrated in degree zero) and induce homomorphisms

$$tr\colon \mathbb{E}\mathrm{xt}^j(E^\bullet, E^\bullet) \to \mathrm{H}^j(Y, \mathcal{O}_Y) \ \text{ and } \ i\colon \mathrm{H}^j(Y, \mathcal{O}_Y) \to \mathbb{E}\mathrm{xt}^j(E^\bullet, E^\bullet).$$

An easy modification of the previous construction leads to homomorphisms

$$tr\colon \mathbb{E}\mathrm{xt}^j(E^\bullet, E^\bullet \otimes \mathcal{N}) \to \mathrm{H}^j(Y, \mathcal{N}) \ \text{ and } \ i\colon \mathrm{H}^j(Y, \mathcal{N}) \to \mathbb{E}\mathrm{xt}^j(E^\bullet, E^\bullet \otimes \mathcal{N}),$$

for any coherent sheaf $\mathcal{N}$ on Y.

Now we turn to the relative version of these constructions. Let S be a k-scheme of finite type and $p\colon Y \to S$ a smooth projective morphism. Any S-flat family E of coherent sheaves admits a finite locally free resolution $E^\bullet \to E$ (see, e.g., Proposition 2.1.10 in [**35**]). Recall that the sheaf of $\mathcal{O}_S$-modules $\mathcal{E}xt^j_p(E, \cdot)$ is the derived functor of $\mathcal{H}om_p(E, \cdot) := p_* \circ \mathcal{H}om(E, \cdot)$. It is easy to see that $\mathcal{E}xt^j_p(E, G)$ is the sheafification of the presheaf defined by

$$U \mapsto \mathrm{Ext}^j(E|_{p^{-1}(U)}, G|_{p^{-1}(U)}),$$

for any open subset $U \subset S$.

Since $E^\bullet$ is a resolution of E, they are quasi-isomorphic, hence $\mathbb{E}\mathrm{xt}^j(E^\bullet, E^\bullet) \cong \mathrm{Ext}^j(E, E)$. Thus by sheafifying the Yoneda pairing and the maps i and tr defined for $E^\bullet$, we get morphisms

$$\mathcal{E}xt^i_p(E, E) \otimes \mathcal{E}xt^j_p(E, E) \longrightarrow \mathcal{E}xt^{i+j}_p(E, E),$$

and

$$\begin{array}{rcl} tr\colon \mathcal{E}xt^j_p(E, E) & \longrightarrow & R^j p_*(\mathcal{O}_Y), \\ i\colon R^j p_*(\mathcal{O}_Y) & \longrightarrow & \mathcal{E}xt^j_p(E, E). \end{array}$$

2. The Atiyah class

In this section we define the *Atiyah class* for flat families of coherent sheaves. The Atiyah class was introduced in [**1**] for the case of vector bundles and in [**36**] and [**37**] for any complex of coherent sheaves. For the definition of the Atiyah class, we will follow the approach of Maakestad which involves the notion of the *sheaf of first jets* (see [**50**]) and, at the same time, Huybrechts and Lehn's description of the Atiyah class in terms of finite locally free resolutions (see Section 10.1.5 in [**35**]).

Let $p_1, p_2\colon Y \times Y \to Y$ be the projections to the two factors. Let $\mathcal{I}$ be the ideal sheaf of the diagonal $\Delta \subset Y \times Y$ and let $\mathcal{O}_{2\Delta} = {}^{\mathcal{O}_{Y\times Y}}/_{\mathcal{I}^2}$ denote the structure sheaf of the first infinitesimal neighborhood of Δ. Note that ${}^{\mathcal{I}}/_{\mathcal{I}^2} \cong \mathcal{N}^{\vee}_{\Delta/Y\times Y} \cong \Omega^1_\Delta$, hence we have the following exact sequence

$$0 \longrightarrow \Omega^1_\Delta \longrightarrow \mathcal{O}_{2\Delta} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \tag{23}$$

The corresponding class $at \in \operatorname{Ext}^1(\mathcal{O}_\Delta, \Omega^1_\Delta)$ is called the *universal Atiyah class* of Y.

Let E be a locally free sheaf on Y. Since $\mathcal{O}_\Delta$ is p_2-flat, after tensorizing with $p_2^*(E)$ the sequence (23) remains exact. By applying the functor $(p_1)_*$, we get a short exact sequence

$$0 \longrightarrow \Omega^1_Y \otimes E \longrightarrow (p_1)_*(\mathcal{O}_{2\Delta} \otimes p_2^*(E)) \longrightarrow E \longrightarrow 0$$

whose extension class $at(E) \in \operatorname{Ext}^1(E, \Omega^1_Y \otimes E)$ is called the *Atiyah class* of E. As it is proved in Proposition 3.4 in [**50**], the Atiyah class $at(E)$ is the obstruction for the existence of an algebraic connection on E.

The sheaf $(p_1)_*(\mathcal{O}_{2\Delta} \otimes p_2^*(E))$ is called the *sheaf of first jets* of E and it is denoted by $J^1(E)$. As it is explained in Section 3 of [**50**], one can describe it in the following way: it is the sheaf of abelian groups $(\Omega^1_Y \otimes E) \oplus E$, with the following left $\mathcal{O}_Y$-module structure: for an open subset U of Y, $a \in \mathcal{O}_Y(U)$ and $(z \otimes e, f) \in J^1(E)(U)$, define

$$a(z \otimes e, f) = (az \otimes e + d(a) \otimes f, af),$$

where d is the exterior differential of Y.

In [**50**], Maakestad constructs the sheaf of first jets $J^1(E)$ for any coherent sheaf E by using the same definition as before. In this way, she obtains an extension

$$0 \longrightarrow \Omega^1_Y \otimes E \longrightarrow J^1(E) \longrightarrow E \longrightarrow 0.$$

The corresponding extension class $at(E) \in \operatorname{Ext}^1(E, \Omega^1_Y \otimes E)$ is called the *Atiyah class* of E.

There is another equivalent way to construct the Atiyah class of a coherent sheaf E. Let $E^\bullet$ be a finite complex of locally free sheaves. One has a short exact sequence

$$0 \longrightarrow \Omega^1_Y \otimes E^\bullet \longrightarrow (p_1)_*(\mathcal{O}_{2\Delta} \otimes p_2^*(E^\bullet)) \longrightarrow E^\bullet \longrightarrow 0$$

defining a class $at(E^\bullet) \in \mathbb{E}\mathrm{xt}^1(E^\bullet, \Omega^1_Y \otimes E^\bullet)$.

A quasi-isomorphism $E^\bullet \to G^\bullet$ of finite complexes of locally free sheaves induces an isomorphism $\mathbb{E}\mathrm{xt}^1(E^\bullet, \Omega^1_Y \otimes E^\bullet) \cong \mathbb{E}\mathrm{xt}^1(G^\bullet, \Omega^1_Y \otimes G^\bullet)$ which identifies $at(E^\bullet)$ and $at(G^\bullet)$. In particular, if E is a coherent sheaf that admits a finite locally free resolution $E^\bullet \to E$, then $at(E^\bullet)$ is independent of the resolution and coincides with the class $at(E)$ defined before.

2.1. Newton polynomials. Let $E^\bullet$ be a finite complex of locally free sheaves on Y. Let $at(E^\bullet)^i$ denote the image in $\mathbb{E}\mathrm{xt}^i(E^\bullet, \Omega^i_Y \otimes E^\bullet)$ of the i-th product

$$at(E^\bullet) \circ \cdots \circ at(E^\bullet) \in \mathbb{E}\mathrm{xt}^i(E^\bullet, (\Omega^1_Y)^{\otimes i} \otimes E^\bullet),$$

under the morphism induced by $(\Omega^1_Y)^{\otimes i} \to \Omega^i_Y$.

Definition 7.1. The i-th *Newton polynomial* of $E^\bullet$ is

$$\gamma^i(E^\bullet) := tr(at(E^\bullet)^i) \in \mathrm{H}^i(Y, \Omega^i_Y).$$

In the same way, one can define the i-th *Newton polynomial* $\gamma^i(E)$ of a coherent sheaf E by using $at(E)$. If $E^\bullet$ is a finite locally free resolution of E, clearly $\gamma^i(E) = \gamma^i(E^\bullet)$.

The de Rham differential $d\colon \Omega^i_Y \to \Omega^{i+1}_Y$ induces k-linear maps

$$d\colon \mathrm{H}^j(Y, \Omega^i_Y) \to \mathrm{H}^j(Y, \Omega^{i+1}_Y).$$

Proposition 7.2. *The i-th* Newton polynomial *of $E^\bullet$ is d-closed.*

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of Y. The trace map only depends on the components with $p = i, q = 0$ in

$$\prod_{p+q=i} C^p(\mathcal{H}om^q(E^\bullet, \Omega^i_Y \otimes E^\bullet), \mathcal{U}).$$

In particular, $\gamma^n(E^\bullet) = \sum_l (-1)^l \gamma^n(E^l)$.

Let us assume that E is a locally free sheaf. Since γ^i is additive with respect to short exact sequences, by using the splitting principe we can assume that E is a line bundle. If $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ are the transition functions of E, $dg_{ij}g_{ij}^{-1}$ is a cocycle representing $at(E)$ (see, e.g., Section 4 in [**1**]). Thus $at(E)$ clearly vanishes under d. □

2.2. The Kodaira-Spencer map. Let $(X, \mathcal{O}_X(1))$ be a polarized surface and S a Noetherian scheme of finite type over k.

Let E be an S-flat family of coherent sheaves on X and $at(E) \in \mathrm{Ext}^1(E, \Omega^1_Y \otimes E)$ its Atiyah class. Consider the induced section $\mathcal{A}t(E)$ under the global-relative map

$$\mathrm{Ext}^1(E, \Omega^1_Y \otimes E) \longrightarrow \mathrm{H}^0(S, \mathcal{E}xt^1_{p_S}(E, \Omega^1_Y \otimes E)),$$

coming from the relative-to-global spectral sequence

$$\mathrm{H}^i(S, \mathcal{E}xt^j_{p_S}(E, \Omega^1_Y \otimes E)) \Rightarrow \mathrm{Ext}^{i+j}(E, \Omega^1_Y \otimes E). \tag{24}$$

The direct sum decomposition $\Omega^1_Y = p_S^*(\Omega^1_S) \oplus p_X^*(\Omega^1_X)$ leads to an analogous decomposition $\mathcal{A}t(E) = \mathcal{A}t_S(E) + \mathcal{A}t_X(E)$.

Definition 7.3. The *Kodaira-Spencer map* associated to the family E is the composition

$$\begin{aligned} KS\colon (\Omega^1_S)^\vee \xrightarrow{\mathrm{id}\otimes \mathcal{A}t_S(E)} & (\Omega^1_S)^\vee \otimes \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S) \otimes E) \to \\ \longrightarrow \quad & \mathcal{E}xt^1_{p_S}(E, p_S^*((\Omega^1_S)^\vee \otimes \Omega^1_S) \otimes E) \to \mathcal{E}xt^1_{p_S}(E, E). \end{aligned}$$

3. The Atiyah class for framed sheaves

In this section we turn to the framed case. Our goal is to define for the case of framed sheaves all the geometric notions introduced in the previous sections. In particular, first we give a definition of the *framed Atiyah class* for flat families of (D, F)-framed vector bundles by using a framed version of the sheaf of first jets. For a flat family (E, α) of (D, F)-framed sheaves we give two equivalent definitions. The first one is given in terms of the *framed* sheaf of first jets. For the second definition, we consider a finite locally free resolution of E and, *locally over the base*, we define a framing on each element of the resolution in a way that the latter becomes a flat family of (D, F)-vector bundles. By using the framed Atiyah class of each element in the resolution, we define the framed Atiyah class of (E, α) locally over the base.

Let $(X, \mathcal{O}_X(1))$ be a polarized surface, $D \subset X$ a divisor and F a locally free $\mathcal{O}_D$-module.

Let S be a Noetherian k-scheme of finite type and p_S, p_X the projections from $Y = X \times S$ to S and X respectively. Let us denote by $\mathbb{D}$ the divisor $S \times D$.

Definition 7.4. A *locally free family of (D,F)-framed sheaves parametrized by S* is a pair $\mathcal{E} = (E, \alpha)$ where E is a locally free sheaf on Y and $\alpha\colon E \to p_X^*(F)$ is a morphism such that $\alpha|_{\{s\}\times D}\colon E|_{\{s\}\times D} \to p_X^*(F)|_{\{s\}\times D}$ is an isomorphism for any $s \in S$.

Remark 7.5. For any point $s \in S$, $\mathcal{E}|_{\{s\}\times X}$ is a (D,F)-framed vector bundle.

Now we would like to introduce a *framed version* of the sheaf of the first jets: we define a subsheaf $J^1_{fr}(\mathcal{E})$ of $J^1(E)$, that we shall call *framed sheaf of first jets* of $\mathcal{E}$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a cover of $\mathbb{D}$ over which $p_X^*(F)|_{\mathbb{D}}$ trivializes, and choose on any U_i a set $\{e_i^0\}$ of basis sections of $\Gamma(p_X^*(F)|_{\mathbb{D}}, U_i)$. Let g_{ij}^0 be transition functions of $p_X^*(F)|_{\mathbb{D}}$ with respect to chosen local basis sections (i.e., $e_i^0 = g_{ij}^0 e_j^0$), constant along S. Let us fix a cover $\mathcal{V} = \{V_i\}_{i\in I}$ of Y over which E trivializes with sets $\{e_i\}$ of basis sections such that $V_i \cap \mathbb{D} = U_i$ for any $i \in I$ and

$$\begin{aligned} e_i|_{\mathbb{D}} &= e_i^0, \\ g_{ij}|_{\mathbb{D}} &= g_{ij}^0. \end{aligned}$$

Let x be a point in Y. If $x \notin \mathbb{D}$, we put $J^1_{fr}(\mathcal{E})_x = J^1(E)_x$. If x is in $\mathbb{D}$, let V_i be an open set of the cover $\mathcal{V}$ that contains x. Then $J^1_{fr}(\mathcal{E})_x \subset J^1(E)_x = (\Omega^1_{Y,x} \otimes E_x) \oplus E_x$ is the $\mathcal{O}_{Y,x}$-module spanned by the basis obtained by tensoring all the elements of the set $\{f_i dz_i^1, \dots, f_i dz_i^s, dz_i^{s+1}, \dots, dz_i^t\}$, where $\{dz_i^1, \dots, dz_i^t\}$ is a basis of $\Omega^1_{Y,x}$, by the elements of the basis $\{e_i\} := \{e_i^1, \dots, e_i^r\}$ of E_x and then adding the elements of $\{e_i\}$, where we denote by $z_i^1, \dots, z_i^s$ and $z_i^{s+1}, \dots, z_i^t$ the local coordinates of S and X on V_i, respectively, and $f_i = 0$ is the local equation of $\mathbb{D}$ on V_i. If x is also a point of the open subset V_j of $\mathcal{V}$, let us denote by $l_{ij} \in \mathcal{O}_Y^*(V_i \cap V_j)$ the transition function on $V_i \cap V_j$ of the line bundle associated to the divisor $\mathbb{D}$ and by J_{ij} the Jacobian matrix of change of coordinates. Let us define the following matrices:

$$L_{ij} := \begin{pmatrix} l_{ij} I_s & 0_{s,t-s} \\ 0_{t-s,s} & I_{t-s} \end{pmatrix}$$

and

$$F_i := \begin{pmatrix} f_i I_s & 0_{s,t-s} \\ 0_{t-s,s} & I_{t-s} \end{pmatrix}$$

where I_k is the identity matrix of order k and $0_{k,l}$ is the k-by-l zero matrix.

The change of basis matrix of the two corresponding bases in $J^1_{fr}(\mathcal{E})_x$ under changes of bases in E_x is:

$$\begin{pmatrix} L_{ij} \otimes g_{ij} & (F_i^{-1} \otimes \mathrm{id}) \cdot dg_{ij} \\ 0 & g_{ij} \end{pmatrix}$$

where the block at the position (1,2) is a regular matrix function, because g_{ij} is constant along $\mathbb{D}$.

The change of basis matrix under changes of local coordinates is:

$$\begin{pmatrix} L_{ij} \cdot J_{ij} \otimes \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

In this way, we get an exact sequence of left $\mathcal{O}_Y$-modules

$$0 \longrightarrow \big(p_S^*(\Omega_S^1)(-\mathbb{D}) \oplus p_X^*(\Omega_X^1)\big) \otimes E \longrightarrow J_{fr}^1(\mathcal{E}) \longrightarrow E \longrightarrow 0,$$

where we denote by $p_S^*(\Omega_S^1)(-\mathbb{D})$ the tensor product $p_S^*(\Omega_S^1) \otimes \mathcal{O}_Y(-\mathbb{D})$.

We call *framed Atiyah class* of $\mathcal{E}$ the class $at(\mathcal{E})$ in $\operatorname{Ext}^1\big(E, \big(p_S^*(\Omega_S^1)(-\mathbb{D}) \oplus p_X^*(\Omega_X^1)\big) \otimes E\big)$ defined by this extension.

Let us consider the short exact sequence

$$0 \longrightarrow p_S^*(\Omega_S^1)(-\mathbb{D}) \oplus p_X^*(\Omega_X^1) \xrightarrow{\ i\ } \Omega_Y^1 \xrightarrow{\ q\ } p_S^*(\Omega_S^1)|_{\mathbb{D}} \longrightarrow 0.$$

After tensoring by E and applying the functor $\operatorname{Hom}(E, \cdot)$, we get the long exact sequence

$$\begin{gathered} \cdots \to \operatorname{Ext}^1(E, \big(p_S^*(\Omega_S^1)(-\mathbb{D}) \oplus p_X^*(\Omega_X^1)\big) \otimes E) \xrightarrow{i_*} \\ \operatorname{Ext}^1(E, \Omega_Y^1 \otimes E) \xrightarrow{q_*} \operatorname{Ext}^1(E, p_S^*(\Omega_S^1)|_{\mathbb{D}} \otimes E) \to \cdots . \end{gathered}$$

By construction, the image of $at(E)$ under i_* is $at(\mathcal{E})$, which is equivalent to saying that we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \big(p_S^*(\Omega_S^1)(-\mathbb{D}) \oplus p_X^*(\Omega_X^1)\big) \otimes E \arrow[r] \arrow[d] & J_{fr}^1(\mathcal{E}) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Omega_Y^1 \otimes E \arrow[r] & J^1(E) \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

Moreover, $q_*(at(E)) = q_*(i_*(at(\mathcal{E}))) = 0$, hence we get the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_Y^1 \otimes E \arrow[r] \arrow[d] & J^1(E) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & p_S^*(\Omega_S^1)|_{\mathbb{D}} \otimes E \arrow[r] & \big(p_S^*(\Omega_S^1)|_{\mathbb{D}} \otimes E\big) \oplus E \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

Example 7.6. Let F be a line bundle on D. Let $\mathcal{L} = (L, \alpha)$ be a locally free family of (D, F)-framed sheaves parametrized by S with L line bundle. As before, choose transition functions g_{ij}^0 and g_{ij} for $p_X^*(F)$ and L, respectively, such that

$$g_{ij}|_{\mathbb{D}} = g_{ij}^0.$$

Recall that $dg_{ij}g_{ij}^{-1}$ is a cocycle representing $at(L)$. By the choice of g_{ij}^0, we get that $d_S(g_{ij})$ vanishes along $\mathbb{D}$, where d_S is the exterior differential of S. Hence $dg_{ij}g_{ij}^{-1}$ can be also interpreted as a cocycle representing $at(\mathcal{L})$. Moreover, it vanishes under the restriction of the de Rham differential $\tilde{d} := d|_{p_S^*(\Omega_S^1)(-\mathbb{D}) \oplus p_X^*(\Omega_X^1)}$. △

Now we want to turn to the non-locally free case. Assume that S is a smooth Noetherian scheme of finite type over k.

Definition 7.7. A *flat family of* (D, F)*-framed sheaves parametrized by* S is a pair $\mathcal{E} = (E, \alpha)$ where E is a coherent sheaf on Y, flat over S, $\alpha\colon E \to p_X^*(F)$ is a morphism such that for any $s \in S$ the sheaf $E|_{\{s\} \times X}$ is locally free in a neighborhood of $\{s\} \times D$ and $\alpha|_{\{s\} \times D}\colon E|_{\{s\} \times D} \to p_X^*(F)|_{\{s\} \times D}$ is an isomorphism.

We would like to define the framed sheaf of first jets $J^1_{fr}(\mathcal{E})$ for $\mathcal{E}$. As before, we set $J^1_{fr}(\mathcal{E})_x = J^1(E)_x$ for $x \notin \mathbb{D}$. Let us fix $x \in \mathbb{D}$; by definition of a flat family of (D, F)-framed sheaves, there exists an open neighborhood $V \subset Y$ of x such that $E|_V$ is a locally free $\mathcal{O}_V$-module. Then we apply the previous construction to the locally free sheaf $E|_V$ and in the same way as before we define $J^1_{fr}(\mathcal{E})_x$. Thus we get an extension

$$0 \longrightarrow \left(p_S^*(\Omega^1_S)(-\mathbb{D}) \oplus p_X^*(\Omega^1_X)\right) \otimes E \longrightarrow J^1_{fr}(\mathcal{E}) \longrightarrow E \longrightarrow 0,$$

and we call the *framed Atiyah class $at(\mathcal{E})$* of $\mathcal{E}$ the corresponding class in

$$\mathrm{Ext}^1(E, \left(p_S^*(\Omega^1_S)(-\mathbb{D}) \oplus p_X^*(\Omega^1_X)\right) \otimes E).$$

There is another way to describe the framed Atiyah class of a flat family of (D, F)-framed sheaves $\mathcal{E} = (E, \alpha)$ by using finite locally free resolutions of E, but in this case the costruction is *local over the base*, as we will explain in the following. First, we recall a result due to Banica, Putinar and Schumacher that will be very useful later on.

Theorem 7.8 (Satz 3 in [**5**]). *Let $p\colon R \to T$ be a flat proper morphism of schemes of finite type over k, T smooth, E and G coherent $\mathcal{O}_R$-modules, flat over T. If the function $y \mapsto \dim \mathrm{Ext}^l(E_y, G_y)$ is constant for l fixed, then the sheaf $\mathcal{E}xt^l_p(E, G)$ is locally free on T and for any $y \in T$ we have*

$$\mathcal{E}xt^i_p(E, G)_y \otimes_{\mathcal{O}_{T,y}} ({}^{\mathcal{O}_{T,y}}/{}_{m_y}) \cong \mathrm{Ext}^i(E_y, G_y) \ \text{ for } \ i = l-1, l.$$

Moreover, the same statement is true for complexes.

Let $\mathcal{E} = (E, \alpha)$ be a flat family of (D, F)-framed sheaves parametrized by S. Since the projection morphism $p_S\colon S \times X \longrightarrow S$ is smooth and projective, there exists a finite locally free resolution $E^\bullet \to E$ of E.

Let us fix a point $s_0 \in S$. By the flatness property, the complex $(E^\bullet)|_{\{s_0\}\times D}$ is a finite resolution of locally free sheaves of $E|_{\{s_0\}\times D} \cong F$. Let us denote by $F^\bullet$ the complex $(E^\bullet)|_{\{s_0\}\times D}$. Define $\mathcal{F}^\bullet := F^\bullet \boxtimes \mathcal{O}_S$.

The complex $\mathcal{F}^\bullet$ is S-flat since $(E^\bullet)|_{\{s_0\}\times D}$ is a complex of locally free $\mathcal{O}_D$-modules and the sheaf $\mathcal{O}_{\mathbb{D}}$ is a S-flat $\mathcal{O}_Y$-module. Moreover, for any $s \in S$, the complex $(\mathcal{F}^\bullet)|_{\{s\}\times X}$ is quasi-isomorphic to F, hence we get

$$\mathrm{Hom}((E^\bullet)|_{\{s\}\times X}, (\mathcal{F}^\bullet)|_{\{s\}\times X}) = \mathrm{Hom}(E|_{\{s\}\times X}, F) \cong \mathrm{End}(F).$$

By applying Theorem 7.8, we get that the the natural morphism of complexes between $E^\bullet$ and $\mathcal{F}^\bullet$ on $\{s_0\} \times X$ extends to a morphism of complexes

$$\alpha_\bullet\colon E^\bullet \longrightarrow \mathcal{F}^\bullet.$$

Let $U \subset S$ be a neighborhood of s_0 such that the following condition holds

$$(\alpha_\bullet)|_{\{s\}\times D} \ \textit{is an isomorphism for any } s \in U. \tag{25}$$

Let $Y_U = U \times X$ and $\mathbb{D}_U = U \times D$. For any i, the pair $(E^i|_{Y_U}, \alpha_i|_{Y_U}\colon E^i|_{Y_U} \to \mathcal{F}^i|_{Y_U})$ is a locally free family $\mathcal{E}^i_U$ of $(D, E^i|_{\{s_0\}\times D})$-framed sheaves parametrized by U. If for any i, we consider the short exact sequence

$$0 \longrightarrow \left(p_U^*(\Omega^1_U)(-\mathbb{D}_U) \oplus p_X^*(\Omega^1_X)\right) \otimes E^i|_{Y_U} \longrightarrow J^1_{fr}(\mathcal{E}^i_U) \longrightarrow E^i|_{Y_U} \longrightarrow 0,$$

defined in Section 2, we get a class $at_U(\mathcal{E})$ in

$$\begin{aligned}&\mathbb{E}\mathrm{xt}^1\left(E^\bullet|_{Y_U}, \left(p_U^*(\Omega_U^1)(-\mathbb{D}_U)\oplus p_X^*(\Omega_X^1)\right)\otimes E^\bullet|_{Y_U}\right)\cong\\ &\cong \mathrm{Ext}^1\left(E|_{Y_U}, \left(p_U^*(\Omega_U^1)(-\mathbb{D}_U)\oplus p_X^*(\Omega_X^1)\right)\otimes E|_{Y_U}\right).\end{aligned}$$

By construction, $at_U(\mathcal{E})$ is independent of the resolution and it is the image of $at(\mathcal{E})$ with respect to the map on Ext-groups induced by the natural morphism $i^*\colon \Omega_S^1 \to \Omega_U^1$, where $i\colon U \hookrightarrow S$ is the inclusion morphism.

3.1. Framed Newton polynomials. Let $\mathcal{E}=(E,\alpha)$ be a flat family of (D,F)-framed sheaves parametrized by S. As we did in Section 2.1 of this chapter, we define

$$at(\mathcal{E})^i \in \mathrm{Ext}^1\left(E, \tilde{\Omega}_Y^i\otimes E\right),$$

where $\tilde{\Omega}_Y^1 := p_S^*(\Omega_S^1)(-\mathbb{D})\oplus p_X^*(\Omega_X^1)$ and $\tilde{\Omega}_Y^i := \Lambda^i(\tilde{\Omega}_Y^1)$ is the i-th exterior power of $\tilde{\Omega}_Y^1$.

Definition 7.9. The i-th *framed Newton polynomial* of $\mathcal{E}$ is

$$\gamma^i(\mathcal{E}) := tr(at(\mathcal{E})^i)\in \mathrm{H}^i(Y,\tilde{\Omega}_Y^i).$$

Let $E^\bullet \to E$ be a finite locally free resolution of E. Let s_0 be a point in S and $U\subset S$ a neighborhood of s_0 satisfying condition (25). We define the i-th *framed Newton polynomial* of $\mathcal{E}$ on U as

$$\gamma_U^i(\mathcal{E}) := tr(at_U(\mathcal{E})^i)\in \mathrm{H}^i(Y_U,\tilde{\Omega}_{Y_U}^i).$$

Moreover, $\gamma^i(\mathcal{E})|_{Y_U}=\gamma_U^i(\mathcal{E})$ by construction.

The restricted de Rham differential $\tilde{d}$ introduced in Example 7.6, induces k-linear maps

$$\tilde{d}\colon \mathrm{H}^i(Y,\tilde{\Omega}_Y^i)\longrightarrow \mathrm{H}^{i+1}(Y,\tilde{\Omega}_Y^i(\mathbb{D})).$$

For any open subset $U\subset S$ the restricted differential $\tilde{d}_U := d|_{p_U^*(\Omega_U^1)(-\mathbb{D}_U)\oplus p_X^*(\Omega_X^1)}$ induces k-linear maps

$$\tilde{d}_U\colon \mathrm{H}^i(Y_U,\tilde{\Omega}_{Y_U}^i)\longrightarrow \mathrm{H}^{i+1}(Y_U,\tilde{\Omega}_{Y_U}^i(\mathbb{D}_U)).$$

Proposition 7.10. *The i-th* framed Newton polynomial *of $\mathcal{E}$ is $\tilde{d}$-closed.*

Proof. Let $U\subset S$ be an open subset satisfying condition (25). The cohomology class $\gamma_U^i(\mathcal{E})$ is $\tilde{d}_U$-closed by the same arguments as in the proof of Proposition 7.2, in particular the splitting principle and Example 7.6. Since the restriction of $\gamma^i(\mathcal{E})$ to Y_U is $\gamma_U^i(\mathcal{E})$ and U is arbitrary, we get that $\gamma^i(\mathcal{E})$ is closed with respect to $\tilde{d}$. □

3.2. The Kodaira-Spencer map for framed sheaves. Let $\mathcal{E}=(E,\alpha)$ be a flat family of (D,F)-framed sheaves parametrized by S. Consider the framed Atiyah class $at(\mathcal{E})$ in $\mathrm{Ext}^1\left(E, \left(p_S^*(\Omega_S^1)(-\mathbb{D})\oplus p_X^*(\Omega_X^1)\right)\otimes E\right)$ and the induced section $\mathcal{A}t(\mathcal{E})$ under the global-relative map

$$\mathrm{Ext}^1\left(E, \left(p_S^*(\Omega_S^1)(-\mathbb{D})\oplus p_X^*(\Omega_X^1)\right)\otimes E\right)\longrightarrow \mathrm{H}^0(S,\mathcal{E}xt^1_{p_S}(E,\left(p_S^*(\Omega_S^1)(-\mathbb{D})\oplus p_X^*(\Omega_X^1)\right)\otimes E)),$$

coming from the relative-to-global spectral sequence

$$\mathrm{H}^i(S,\mathcal{E}xt^j_{p_S}(E,\left(p_S^*(\Omega_S^1)(-\mathbb{D})\oplus p_X^*(\Omega_X^1)\right)\otimes E))\Rightarrow \mathrm{Ext}^{i+j}(E,\left(p_S^*(\Omega_S^1)(-\mathbb{D})\oplus p_X^*(\Omega_X^1)\right)\otimes E).$$

By considering the S-part $\mathcal{A}t_S(\mathcal{E})$ of $\mathcal{A}t(\mathcal{E})$ in

$$\mathrm{H}^0(S,\mathcal{E}xt^1_{p_S}(E,p_S^*(\Omega_S^1)(-\mathbb{D})\otimes E))=\mathrm{H}^0(S,\mathcal{E}xt^1_{p_S}(E,p_S^*(\Omega_S^1)\otimes p_X^*(\mathcal{O}_X(-D))\otimes E)),$$

we define the framed version of the Kodaira-Spencer map.

Definition 7.11. The *framed Kodaira-Spencer map* associated to the family $\mathcal{E}$ is the composition

$$\begin{aligned} KS_{fr}\colon (\Omega^1_S)^\vee &\stackrel{\mathrm{id}\otimes\mathcal{A}t_S(\mathcal{E})}{\longrightarrow} (\Omega^1_S)^\vee \otimes \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S)\otimes p_X^*(\mathcal{O}_X(-D))\otimes E) \to \\ &\longrightarrow \mathcal{E}xt^1_{p_S}(E, p_S^*((\Omega^1_S)^\vee\otimes\Omega^1_S)\otimes p_X^*(\mathcal{O}_X(-D))\otimes E) \to \\ &\longrightarrow \mathcal{E}xt^1_{p_S}(E, p_X^*(\mathcal{O}_X(-D))\otimes E). \end{aligned}$$

3.3. Closed two-forms via the framed Atiyah class. From now on, assume that S is smooth and affine. Let $\mathcal{E}=(E,\alpha)$ be a flat family of (D,F)-framed sheaves parametrized by S.

Let $\gamma^{0,2}$ denote the component of $\gamma^2(\mathcal{E})$ in $\mathrm{H}^0(S,\Omega^2_S)\otimes\mathrm{H}^2(X,\mathcal{O}_X(-2\,D))$.

Definition 7.12. Let τ_S be the homomorphism given by

$$\tau_S\colon \mathrm{H}^0(X,\omega_X(2\,D)) \cong \mathrm{H}^2(X,\mathcal{O}_X(-2\,D))^\vee \stackrel{\cdot\gamma^{0,2}}{\longrightarrow} \mathrm{H}^0(S,\Omega^2_S),$$

where $\cong$ denotes Serre's duality.

Proposition 7.13. *For any $\omega\in\mathrm{H}^0(X,\omega_X(2\,D))$, the associated two-form $\tau_S(\omega)$ on closed in S.*

Proof. We can write

$$\gamma^{0,2} = \sum_l \mu_l\otimes\nu_l,$$

for elements $\mu_l\in\mathrm{H}^0(S,\Omega^2_S)$ and $\nu_l\in\mathrm{H}^2(X,\mathcal{O}_X(-2\,D))$. Since $\tilde{d}(\gamma^2(\mathcal{E}))=0$ (cf. Proposition 7.10), the component of $\tilde{d}(\gamma^{0,2})$ in $\mathrm{H}^0(S,\Omega^3_S)\otimes\mathrm{H}^2(X,\mathcal{O}_X(-2\,D))$ is zero, which means

$$\sum_l d_S(\mu_l)\otimes\nu_l = 0.$$

Therefore

$$d_S(\tau_S(\omega)) = d_S\left(\sum_l \mu_l\cdot\omega(\nu_l)\right) = \sum_l d_S(\mu_l)\cdot\omega(\nu_l) = 0.$$

$\square$

Fix $\omega\in\mathrm{H}^0(X,\omega_X(2\,D))$. For any point $s_0\in S$, we obtained a skew-symmetric bilinear form $\tau_S(\omega)_{s_0}$ on $T_{s_0}S$:

$$\begin{aligned} &T_{s_0}S\times T_{s_0}S \stackrel{KS\times KS}{\longrightarrow} \mathrm{Ext}^1(E|_{\{s_0\}\times X}, E|_{\{s_0\}\times X}(-D))\times\mathrm{Ext}^1(E|_{\{s_0\}\times X}, E|_{\{s_0\}\times X}(-D)) \\ &\stackrel{\circ}{\longrightarrow} \mathrm{Ext}^2(E|_{\{s_0\}\times X}, E|_{\{s_0\}\times X}(-2\,D)) \stackrel{tr}{\longrightarrow} \mathrm{H}^2(X,\mathcal{O}_X(-2\,D)) \stackrel{\cdot\omega}{\longrightarrow} \mathrm{H}^2(X,\omega_X)\cong k. \end{aligned}$$

4. The tangent bundle of moduli spaces of framed sheaves

Let $\mathcal{M}^s(X;P)$ be the moduli space of Gieseker-stable torsion free sheaves on X with Hilbert polynomial P. The open subset $\mathcal{M}_0(X;P) \subset \mathcal{M}^s(X;P)$ of points $[E]$ such that $\mathrm{Ext}^2_0(E,E)$ vanishes is smooth according to Theorem 4.5.4 in [**35**]. Suppose there exists a universal family $\tilde{E}$ on $\mathcal{M}_0(X;P) \times X$. Then it is possible to prove that the Kodaira-Spencer map associated to $\tilde{E}$

$$KS\colon T_{\mathcal{M}_0(X;P)} \longrightarrow \mathcal{E}xt^1_p(\tilde{E},\tilde{E})$$

is an isomorphism, where $p\colon \mathcal{M}_0(X;P) \times X \to \mathcal{M}_0(X;P)$ is the projection (cf. Theorem 10.2.1 in [**35**]). In this section we shall prove the framed analogue of this result for the moduli spaces of stable (D,F)-framed sheaves on X.

Let $\delta \in \mathbb{Q}[n]$ be a stability polynomial and P a numerical polynomial of degree two. Let $\mathcal{M}^*_\delta(X;F,P)$ be the moduli space of (D,F)-framed sheaves on X with Hilbert polynomial P that are stable with respect to δ. This is an open subset of the fine moduli space $\mathcal{M}_\delta(X;F,P)$ of stable framed sheaves with Hilbert polynomial P. Let us denote by $\mathcal{M}^*_\delta(X;F,P)^{sm}$ the smooth locus of $\mathcal{M}^*_\delta(X;F,P)$ and by $\tilde{\mathcal{E}} = (\tilde{E},\tilde{\alpha})$ the universal objects of $\mathcal{M}^*_\delta(X;F,P)^{sm}$. Let p be the projection from $\mathcal{M}^*_\delta(X;F,P)^{sm} \times X$ to $\mathcal{M}^*_\delta(X;F,P)^{sm}$.

Theorem 7.14. *The framed Kodaira-Spencer map defined by $\tilde{\mathcal{E}}$ induces a canonical isomorphism*

$$KS_{fr}\colon T\mathcal{M}^*_\delta(X;F,P)^{sm} \xrightarrow{\sim} \mathcal{E}xt^1_p(\tilde{E},\tilde{E}\otimes p_X^*(\mathcal{O}_X(-D))).$$

Proof. First note that $\mathcal{M}^*_\delta(X;F,P)^{sm}$ is a reduced separated scheme of finite type over k. Hence it suffices to prove that the framed Kodaira-Spencer map is an isomorphism on the fibres over closed points. Let $[(E,\alpha)]$ be a closed point. We want to show that the following diagram commutes

$$\begin{tikzcd} T_{[(E,\alpha)]}\mathcal{M}^*_\delta(X;F,P)^{sm} \arrow[r, "\sim"] \arrow[d, "KS_{fr}([(E,\alpha)])"] & \mathrm{Ext}^1(E,E(-D)) \arrow[dl, equal] \\ \mathrm{Ext}^1(E,E(-D)) & \end{tikzcd}$$

where the horizontal isomorphism comes from deformation theory (see proof of Theorem 4.1 in [**34**]).

Let w be an element in $\mathrm{Ext}^1(E,E(-D))$. Consider the long exact sequence

$$\cdots \to \mathrm{Ext}^1(E,E(-D)) \xrightarrow{j_*} \mathrm{Ext}^1(E,E) \xrightarrow{\alpha_*} \mathrm{Ext}^1(E,F) \to \cdots$$

obtained by applying the functor $\mathrm{Hom}(E,\cdot)$ to the exact sequence

$$0 \longrightarrow E(-D) \xrightarrow{j} E \xrightarrow{\alpha} F \longrightarrow 0.$$

Let $v = j_*(w) \in \mathrm{Ext}^1(E, E)$. We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E(-D) \arrow[r, "\tilde{i}"] \arrow[d, "j"] & \tilde{G} \arrow[r, "\tilde{\pi}"] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r, "i"] & G \arrow[r, "\pi"] & E \arrow[r] & 0
\end{tikzcd}$$

where the first arrow is a representative for w and the second one a representative for v.

Let $S = \mathrm{Spec}(k[\varepsilon])$ be the spectrum of the ring of dual numbers, where $\epsilon^2 = 0$. We can think G as a S-flat family by letting ε act on G as the morphism $i \circ \pi$.

Since $\varepsilon G' = E(-D)$ and $\varepsilon G = E$, by applying snake lemma to the previous diagram we get

$$\begin{tikzcd}
0 \arrow[r] & E(-D) \arrow[r, "\tilde{i}"] \arrow[d] & \tilde{G} \arrow[r, "\tilde{\pi}"] \arrow[d] & E \arrow[r] \arrow[d, "\mathrm{id}_E"] & 0 \\
0 \arrow[r] & E \arrow[r, "i"] \arrow[d] & G \arrow[r, "\pi"] \arrow[d, "\beta"] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varepsilon F \arrow[r] \arrow[d] & \varepsilon F \arrow[r] \arrow[d] & 0 & \\
& 0 & 0 & &
\end{tikzcd}$$

Moreover $\alpha_*(v) = 0$, hence we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "i"] \arrow[d] & G \arrow[r, "\pi"] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varepsilon F \arrow[r] & E \oplus \varepsilon F \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

Thus we get a framing $\gamma\colon G \to F \oplus \varepsilon F$ induced by α and β. Moreover $\gamma|_{\mathbb{D}}$ is an isomorphism. We denote by $\mathcal{G}$ the corresponding S-family of (D, F)-framed sheaves on X.

Since S is affine, the relative-to-global spectral sequence (24) degenerates in the second term, so that we have an isomorphism

$$\mathrm{H}^0(S, \mathcal{E}xt^1_{p_S}(G, \Omega^1_Y \otimes G)) \cong \mathrm{Ext}^1_Y(G, \Omega^1_Y \otimes G).$$

Thus one can see the section $\mathcal{A}t_S(G)$ as an element of

$$\mathrm{Ext}^1_Y(G, p_S^*\Omega^1_S \otimes G) \cong \mathrm{Ext}^1_Y(G, E).$$

Consider the short exact sequence of coherent sheaves over $\mathrm{Spec}\left({}^{k[\varepsilon_1,\varepsilon_2]}/{}_{(\varepsilon_1,\varepsilon_2)^2}\right) \times X$

$$0 \longrightarrow E \xrightarrow{i'} G' \xrightarrow{\pi'} G \longrightarrow 0, \tag{26}$$

where ε_1 and ε_2 act trivially on E and by $i \circ \pi$ on G, and

$$G' \cong {}^{k[\varepsilon_1] \otimes_k G}/{}_{\varepsilon_1\varepsilon_2 G} \cong G \oplus E,$$

with actions

$$\varepsilon_1 = \begin{pmatrix} 0 & \pi \\ 0 & 0 \end{pmatrix} \text{ and } \varepsilon_2 = \begin{pmatrix} i\pi & 0 \\ 0 & 0 \end{pmatrix}.$$

By definition of Atiyah class, $\mathcal{A}t_S(G)$ is precisely the extension class of the short exact sequence (26), considered as a sequence of $k[\varepsilon_1]\otimes\mathcal{O}_X$-modules.

The morphims π induces a pull-back morphism $\pi^*\colon \operatorname{Ext}^1_X(E,E)\to\operatorname{Ext}^1_Y(G,E)$, which is an isomorphism. Moreover $\pi^*(v)=\mathcal{A}t_S(G)$, indeed we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "i'"] \arrow[d, equal] & G' \arrow[r, "\pi'"] \arrow[d, "t'"] & G \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & E \arrow[r, "i"] & G \arrow[r, "\pi"] & E \arrow[r] & 0
\end{tikzcd}$$

Thus G' is the sheaf of first jets of G relative to the quotient $\Omega^1_Y\to p_S^*(\Omega^1_S)\to 0$. By following Maakestad's construction of Atiyah classes of coherent sheaves relative to quotients of Ω^1_Y (cf. Section 3 in [**50**]) and by readapting to this particular case our construction of the framed sheaf of first jets given in Section 3, we can define a *framed sheaf of first jets* $\tilde{G}'$ of the framed sheaf $\mathcal{G}$ relative to $p_S^*(\Omega^1_S)$. Thus we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E(-D) \arrow[r, "\tilde{i}'"] \arrow[d] & \tilde{G}' \arrow[r, "\tilde{\pi}'"] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r, "i'"] & G' \arrow[r, "\pi'"] & G \arrow[r] & 0
\end{tikzcd}$$

The first arrow is a representative for the S-part $\mathcal{A}t_S(\mathcal{G})$ of $\mathcal{G}$ in

$$\operatorname{Ext}^1_Y(G, p_S^*\Omega^1_S(-\mathbb{D})\otimes G)\cong\operatorname{Ext}^1_Y(G,E(-D)).$$

Consider the three-dimensional diagram

$$\begin{tikzcd}
0 \arrow[rr] & & E(-D) \arrow[rr, "\tilde{i}"] \arrow[dr, equal] \arrow[dd, hook] & & \tilde{G}' \arrow[rr, "\tilde{\pi}"] \arrow[dd, hook] & & G \arrow[rr] \arrow[dr, "\pi"] \arrow[dd, equal] & & 0 & \\
& 0 \arrow[rr] & & E(-D) \arrow[rr, "\tilde{i}'"] \arrow[dd, hook] & & \tilde{G} \arrow[rr, "\tilde{\pi}'"] \arrow[dd, hook] & & E \arrow[rr] \arrow[dd, equal] & & 0 \\
0 \arrow[rr] & & E \arrow[rr, "i"] \arrow[dr, equal] \arrow[dd, two heads] & & G' \arrow[rr, "\pi"] \arrow[dr, "t'"] \arrow[dd, two heads] & & G \arrow[rr] \arrow[dr, "\pi"] & & 0 & \\
& 0 \arrow[rr] & & E \arrow[rr, "i'"] \arrow[dd, two heads] & & G \arrow[rr, "\pi'"] \arrow[dd, two heads] & & E \arrow[rr] & & 0 \\
& & F \arrow[rr, equal] \arrow[dr, equal] & & F \arrow[dr, equal] & & & & & \\
& & & F \arrow[rr, equal] & & F & & & &
\end{tikzcd}$$

By diagram chasing, one can define a morphism $\tilde{G}'\to\tilde{G}$ such that the corresponding diagram commutes. Thus the image of w through the map $\operatorname{Ext}^1_X(E,E(-D))\to\operatorname{Ext}^1_Y(G,E(-D))$ is exactly $\mathcal{A}t_S(\mathcal{G})$. This completes the proof. $\square$

5. Closed two-forms on moduli spaces of framed sheaves

In this section we show how to construct closed two-forms on the moduli spaces $\mathcal{M}_\delta^*(X;F,P)^{sm}$ by using global sections of the line bundle $\omega_X(2\,D)$. Moreover, we give a criterion of non-degeneracy for these two-forms.

Let us fix a point $[(E,\alpha)]$ of $\mathcal{M}_\delta^*(X;F,P)^{sm}$. By Theorem 7.14 (also by Theorem 5.10), the vector space $\mathrm{Ext}^1(E,E(-D))$ is naturally identified with the tangent space $T_{[(E,\alpha)]}\mathcal{M}_\delta^*(X;F,P)$. For any $\omega\in\mathrm{H}^0(X,\omega_X(2\,D))$, we can define a skew-symmetric bilinear form

$$\begin{aligned}&\mathrm{Ext}^1(E,E(-D))\times\mathrm{Ext}^1(E,E(-D))\xrightarrow{\circ}\mathrm{Ext}^2(E,E(-2\,D))\\&\xrightarrow{tr}\mathrm{H}^2(X,\mathcal{O}_X(-2\,D))\xrightarrow{\cdot\omega}\mathrm{H}^2(X,\omega_X)\cong k.\end{aligned}$$

By varying of the point $[(E,\alpha)]$, these forms fit into a exterior two-form $\tau(\omega)$ on $\mathcal{M}_\delta^*(X;F,P)^{sm}$.

Theorem 7.15. *For any $\omega\in\mathrm{H}^0(X,\omega_X(2\,D))$, the two-form $\tau(\omega)$ is closed on $\mathcal{M}_\delta^*(X;F,P)^{sm}$.*

Proof. It suffices to prove that given a smooth affine variety S, for any S-flat family $\mathcal{E}=(E,\alpha)$ of (D,F) framed sheaves on X defining a classifying morphism

$$\begin{aligned}\psi\colon S&\longrightarrow\mathcal{M}_\delta^*(X;F,P),\\ s&\longmapsto[\mathcal{E}|_{\{s\}\times X}],\end{aligned}$$

the pullback $\psi^*(\tau(\omega))\in\mathrm{H}^0(S,\Omega_S^2)$ is closed. Since, $\psi^*(\tau(\omega))=\tau_S(\omega)$ by construction, the thesis follows from Proposition 7.13. □

Thus we have constructed closed two-forms $\tau(\omega)$ on the moduli space $\mathcal{M}_\delta^*(X;F,P)^{sm}$ for $\omega\in\mathrm{H}^0(X,\omega_X(2\,D))$. In general, these forms may be degenerate.

Now we want to give a criterion to check when the two-form is non-degenerate. First, we need to recall Serre's duality for bounded complexes of coherent sheaves.

Theorem 7.16 (Serre's duality, cf. [**28**]). *Let M be a smooth projective variety of dimension n and let $A^\bullet$ be a bounded complex of coherent sheaves on M. Then the pairing*

$$\mathbb{E}\mathrm{xt}^{n-i}(A^\bullet,\omega_M)\otimes\mathbb{H}^i(A^\bullet)\longrightarrow\mathrm{H}^n(M,\omega_M)\cong k$$

is perfect.

Proposition 7.17. *Let $\omega\in\mathrm{H}^0(X,\omega_X(2\,D))$ and $[(E,\alpha)]$ a point in $\mathcal{M}_\delta^*(X;F,P)^{sm}$. The closed two-form $\tau(\omega)_{[(E,\alpha)]}$ is non-degenerate at the point $[(E,\alpha)]$ if and only if the multiplication by ω induces an isomorphism*

$$\omega_*\colon\mathrm{Ext}^1(E,E(-D))\longrightarrow\mathrm{Ext}^1(E,E\otimes\omega_X(D)).$$

Proof. Let $E^\bullet$ be a finite locally free resolution of E. Consider the perfect pairing

$$\mathcal{H}om^\bullet(E^\bullet,E^\bullet)\otimes\mathcal{H}om^\bullet(E^\bullet,E^\bullet)\xrightarrow{\circ}\mathcal{H}om^\bullet(E^\bullet,E^\bullet)\xrightarrow{tr}\mathcal{O}_X.$$

If we tensor by $\mathcal{O}_X(-2\,D)$, we get the perfect pairing

$$A^\bullet\otimes A^\bullet\xrightarrow{\circ}\mathcal{H}om^\bullet(E^\bullet,E^\bullet(-2\,D))\xrightarrow{tr}\mathcal{O}_X(-2\,D).$$

where $A^\bullet = \mathcal{H}om^\bullet(E^\bullet, E^\bullet(-D))$. Hence we get an isomorphism $A^\bullet \to \mathcal{H}om^\bullet(A^\bullet, \mathcal{O}_X(-2\,D))$. For any section $\omega\colon \mathcal{O}_X \to \omega_X(2\,D)$, we get a commutative diagram

$$\begin{tikzcd}
(A^\bullet \otimes \omega_X(2\,D)) \otimes A^\bullet \arrow[r, "\sim"] & \mathcal{H}om^\bullet(A^\bullet, \omega_X) \otimes A^\bullet \arrow[r, "eval"] & \omega_X \\
A^\bullet \otimes A^\bullet \arrow[r, "\circ"] \arrow[u, "(1\otimes\omega)\otimes 1"] & \mathcal{H}om^\bullet(E^\bullet, E^\bullet(-2\,D)) \arrow[r, "tr"] & \mathcal{O}_X(-2\,D)) \arrow[u, "\omega\otimes\mathrm{id}_{\mathcal{O}_X(-2\,D)}"']
\end{tikzcd}$$

Passing to cohomology, we get

$$\begin{tikzcd}
\mathrm{Ext}^i(E, E\otimes\omega_X(D)) \otimes \mathrm{Ext}^j(E, E(-D)) \arrow[r, "\sim"] & \mathbb{E}\mathrm{xt}^i(A^\bullet, \omega_X) \otimes \mathbb{H}^j(A^\bullet) \arrow[r] & \mathrm{H}^{i+j}(X, \omega_X) \\
\mathrm{Ext}^i(E, E(-D)) \otimes \mathrm{Ext}^j(E, E(-D)) \arrow[r] \arrow[u, "\omega_*\otimes 1"] & \mathrm{Ext}^{i+j}(E, E(-2\,D)) \arrow[r, "tr"] & \mathrm{H}^{i+j}(X, \mathcal{O}_X(-2\,D)) \arrow[u, "(\omega\otimes\mathrm{id}_{\mathcal{O}_X(-2\,D)})_*"']
\end{tikzcd}$$

For $i = j = 1$, we obtain

$$\begin{tikzcd}
\mathrm{Ext}^1(E, E\otimes\omega_X(D)) \otimes \mathrm{Ext}^1(E, E(-D)) \arrow[r, "\sim"] & \mathbb{E}\mathrm{xt}^1(A^\bullet, \omega_X) \otimes \mathbb{H}^1(A^\bullet) \arrow[r] & \mathrm{H}^2(X, \omega_X) \\
\mathrm{Ext}^1(E, E(-D)) \otimes \mathrm{Ext}^1(E, E(-D)) \arrow[r] \arrow[u, "\omega_*\otimes 1"] & \mathrm{Ext}^2(E, E(-2\,D)) \arrow[r, "tr"] & \mathrm{H}^2(X, \mathcal{O}_X(-2\,D)) \arrow[u, "(\omega\otimes\mathrm{id}_{\mathcal{O}_X(-2\,D)})_*"']
\end{tikzcd}$$

Observe that $\tau(\omega)_{[(E,\alpha)]}$ is the map from the lower left corner of the diagram to the upper right corner. By using Serre's duality for bounded complexes of coherent sheaves (in the form stated in Theorem 7.16), we get that $\tau(\omega)_{[(E,\alpha)]}$ is non-degenerate at the point $[(E,\alpha)]$ if and only if ω_* is an isomorphism. $\square$

Obviously, if the line bundle $\omega_X(2\,D)$ is trivial, for any point $[(E,\alpha)]$ in $\mathcal{M}^*_\delta(X; F, P)^{sm}$ the pairing

$$\tau(1)\colon \mathrm{Ext}^1(E, E(-D)) \times \mathrm{Ext}^1(E, E(-D)) \longrightarrow k$$

is a non-degenerate alternating form.

6. An example of symplectic structure (the second Hirzebruch surface)

We denote by $\mathbb{F}_p$ the p-th Hirzebruch surface $\mathbb{F}_p := \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-p))$, which is the projective closure of the total space of the line bundle $\mathcal{O}_{\mathbb{CP}^1}(-p)$ on $\mathbb{CP}^1$. One can describe explicitly $\mathbb{F}_p$ as the divisor in $\mathbb{CP}^2 \times \mathbb{CP}^1$

$$\mathbb{F}_p := \{([z_0 : z_1 : z_2], [z : w]) \in \mathbb{CP}^2 \times \mathbb{CP}^1 \mid z_1 w^p = z_2 z^p\}.$$

Let us denote by $p\colon \mathbb{F}_p \to \mathbb{CP}^2$ the projection onto $\mathbb{CP}^2$. Let D be the inverse image of a generic line of $\mathbb{CP}^2$ through p. D is a smooth connected curve of genus zero with positive self-intersection.

Let F denote the fibre of the projection $\mathbb{F}_p \to \mathbb{CP}^1$. Then the Picard group of $\mathbb{F}_p$ is generated by D and F. One has

$$D^2 = p,\ D \cdot F = 1,\ F^2 = 0.$$

In particular, the canonical divisor K_p can be expressed as

$$K_p := -2\,D + (p-2)F.$$

Let $X = \mathbb{F}_2$ be the second Hirzebruch surface. In this case X is the projective clousure of the cotangent bundle $T^*\mathbb{CP}^1$ of the complex projective line $\mathbb{CP}^1$.

Let D be as before and F a Gieseker-semistable locally free $\mathcal{O}_D$-module. Note that D is a big and nef curve and F is a good framing sheaf on D. By Corollary 5.14 there exists a fine moduli space $\mathcal{M}^*(X; F, P)$ of (D, F)-framed sheaves on X with Hilbert polynomial P.

The canonical divisor of X is $K_2 = -2\,D$. Since $(K_X + D) \cdot D = -D^2 < 0$, by Remark 5.15 the moduli space $\mathcal{M}^*(X; F, P)$ is smooth. Moreover, the line bundle $\omega_X(2\,D)$ is trivial and, for $1 \in \mathrm{H}^0(X, \omega_X(2\,D)) \cong \mathbb{C}$, the two-form $\tau(1)$ defines a symplectic structure on $\mathcal{M}^*(X; F, P)$.

It is easy to see that our construction provide a generalization to the non-locally free case of Bottacin's construction of symplectic structures on the moduli spaces of (D, F)-framed vector bundles on X with Hilbert polynomial P induced by non-degenerate Poisson structures (cf. [**10**]).

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