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Hunting colored (quantum) butterflies

A geometric derivation of the TKNN-equations

Thesis submitted for the degree of Doctor Philosophiæ

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To my grandfathers Giuseppe and Giovanni,
who watch me from the sky;
to my grandmothers Maria and Anna,
who support me on earth.

Acknowledgments

Many people deserve my acknowledgments for the help and encouragement that they gave me along the way.

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Last but not least, I would like to thank my parents for encouraging my studies and my wonderful wife Raffaella for the big help in creating the images of this thesis and mainly for her endless emotional support.

It is a truth universally acknowledged, that a (almost!) young man can not achieve a Ph.D. without a *Good Fortune*. The name that I give to my fortune is *God*!

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Trieste,
October 2010

Giuseppe De Nittis

Preface

This Ph.D. thesis concludes a research accomplished within the Mathematical Physics Sector of SISSA, International School for Advanced Studies of Trieste, from November 2007 to September 2010. The work has been carried out under the constant supervision of Prof. Gianfausto Dell'Antonio and in collaboration with Gianluca Panati.

This dissertation is structured according to a fictitious tripartition.

- The first part (Chapters 1 and 2) aims at introducing the reader into the subject and presenting in a concise but exhaustive way main results achieved and techniques employed. Chapter 1 starts with Avron's tale about the story of the quantum Hall effect (QHE). These introductory pages, aimed to fix the basic physical notions and the nomenclature of the QHE, can be skipped by the reader expert in the field. The rest of Chapter 1 is devoted to a general and non technical exposition of the initial motivations (open problems) that inspired this work and of the main results achieved (solution of the problems). Therefore, Chapter 1 fixes precisely the scope of this thesis. In Chapter 2, the "Ariadne's thread" of our research project is unrolled. This chapter contains the rigorous statements of our main results, as well a consistent presentation of needful mathematical tools. Reading these first two chapters should be enough to have a detailed knowledge about scopes and results of the thesis.

- The second part (Chapters 3, 4 and 5) contains the technical aspects, that is the proofs of the main theorems, as well the "paraphernalia" of lemmas, proposition, notions, needful to build the proofs. Chapter 3 and 4 are largely based on two papers:

  - (De Nittis and Panati 2010): *"Effective models for conductance in magnetic fields: derivation of Harper and Hofstadter models"*. Available as preprint at *http://arxiv.org/abs/1007.4786.*
  - (De Nittis and Panati 2009): *"The geometry emerging from the symmetries of a quantum system"*. Available as preprint at *http://arxiv.org/abs/0911.5270.*

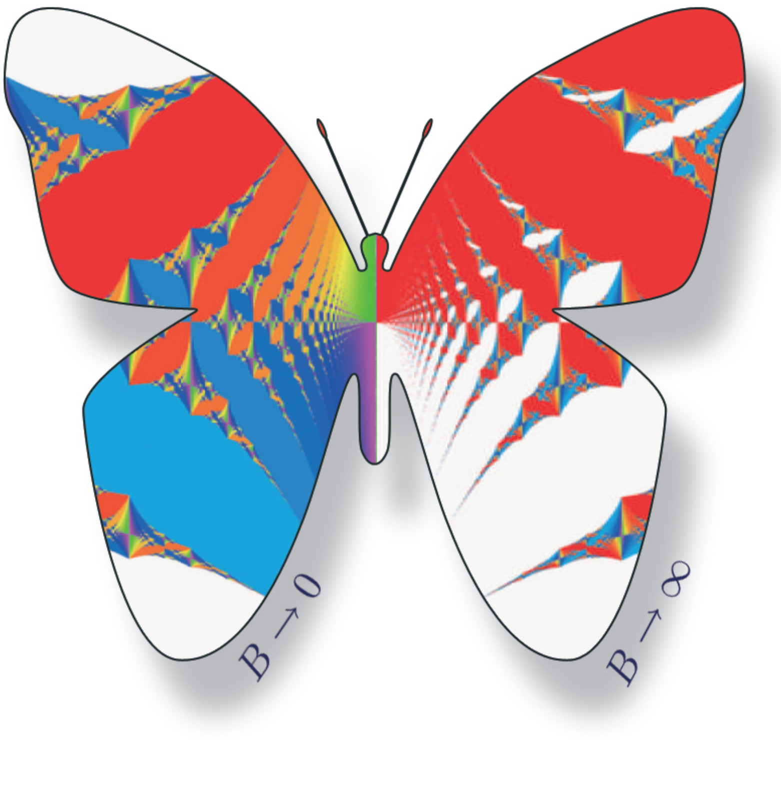
  A third paper, containing a compendium of Chapter 2 and Chapter 5, is in preparation with Gianluca Panati and Frédéric Faure.

- The third part (Appendices A, B and C), containing auxiliary material, aims to make this dissertation as much self-consistent as possible.

In order to help the reader to “navigate” the text, each chapter has been equipped with a small abstract which describes the content of the sections.

Trieste, Giuseppe De Nittis
October 2010

Hunting colored (quantum) butterflies



Contents

Prologue

Despite of its title, this dissertation is not supposed to be a compendium of a young entomologist's research on new exotic species of colorful butterflies. The following pages will not either tell the story of a hunter determined to catch unknown specimen of enigmatic multicolored insects living in the forest in the heart of Africa or South America.

The unwary reader, possibly intrigued by the title, would be somehow surprised to realized that this work is actually a Ph.D. thesis in Mathematical Physics.

Nevertheless, this ambiguity hides some truths. Mathematical ideas fly light with "butterfly wings" in mathematician's mind. They are painted with the "gaudy colors" of intuition and imagination. The mathematician spends his time "hunting for new problems" just like the entomologist does for his preys. Tools he uses to get his "hunting trophy" are theories, theorems, proofs and so on.

Keeping in mind this analogy, the reader may consider this thesis as the story of my personal hunt to unveil the secrets of *quantum butterflies*.

Well, It is time to cry aloud: - *the hunt begins!* -.

Chapter 1

Introduction

On ne connaît pas complètement une science tant qu'on n'en sait pas l'histoire.

(One does not know completely a science as long as one does not know its history.)

Auguste Comte
Cours de philosophie positive, 1830-1842

Abstract

The aim of this introductory chapter is to present the scope of this thesis fixing basic notions and terminology, as well to provide a complete, but non technical, exposition of the main results. Section 1.1 is devoted to a historical review of the quantum Hall effect (QHE), trough main steps that lead to its "topological interpretation". The notions of topological quantization and topological quantum numbers are expounded using the Dirac's monopole as a paradigm. This first section is "borrowed" from (Avron et al. 2003, Avron et al. 2001). Thouless et al. showed in the seminal paper (Thouless et al. 1982) that the quantized values of the Hall conductance are topological quantum numbers. The content of the paper by Thouless et al. is discussed in Section 1.2. A special attention is paid to the TKNN-equations which are Diophantine equations for the values of the quantized conductance. TKNN-equations play a crucial rôle in this thesis. Some assertions in (Thouless et al. 1982) are lacking of rigorous justifications. These "gaps" in the mathematical structure of the work of Thouless et al. are listed in Section 1.3. One of the goal of this thesis is to fill such mathematical gaps and so Section 1.3 can be considered as our "operation plan". In this section we present (quite informally) the main results of this thesis. Quantum butterflies are diagrammatic representations of the TKNN-equations. Section 1.4 contains a description of the main features of these charming pictures. A geometric justification of the TKNN-equations is needed to provide quantum butterflies with a rigorous geometric meaning. This is the main goal of this thesis.

1.1 *Ante factum:* phenomenology of the QHE and topological quantum numbers

The *quantum Hall effect* (QHE) is the central argument of this thesis. Therefore, it could be appropriate to start with a brief review of the phenomenology and the theory of QHE, in order to provide the inexpert reader with basic notions and needed terminology. The long story of the QHE, from the first experiments up to the brilliant intuition of its "topological interpretation", has been excellently narrated by J. E. Avron in the beautiful introductions of (Avron et al. 2003, Avron et al. 2001). Due to the completeness, the synthesis and the charm of Avron's presentations, I realized that it was quite impossible (at least for me!) to expose in a better way the story of the QHE. Therefore, I considered more "honest" to borrow the Avron's tale, offering to the reader a moment of quality literature. The reader expert in QHE is advised to skip directly to Sections 1.2 and 1.3.

The beginning of the story

The story of *Hall effect* begins with a blunder made by J. C. Maxwell. In the first edition of his book, *A treatise on Electricity and Magnetism*, discussing about the deflection of a current by a magnetic field, Maxwell wrote: *"It must be carefully remembered, that the mechanical force*[1] *which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it. [...] The only force which acts on the electric currents is the electromotive force, which must be distinguished from mechanical force [...]."* (Maxwell 1873, pp. 144-145). Such an assertion should sound odd to a modern reader, but at that time it was not so obvious to doubt the Maxwell's words.

In 1878, E. H. Hall, student at Johns Hopkins University, was studying the Maxwell's treatise for a class by H. A. Rowland and being puzzled by the above Maxwell's remark, he queried his teacher. Rowland's answer was that *"[...] he doubted the truth of Maxwell's statement and had sometime before made a hasty experiment for the purpose of detecting, if possible, some action of the magnet on the current itself, though without success. Being very busy with other matters however, he had no immediate intention of carrying the investigation further."* (Hall 1879, p. 288). Figure 1.1 shows a sketch of the experimental setup proposed by Rowland.

At first attempt, possibly because of the failure of Rowland's experiment, Hall decided to undertake a new experiment aimed at measuring the *magnetoresistence*[2]. Nowadays we know that this is a much harder experiment and indeed it failed, in accordance with Maxwell's prevision. At this point Hall, following an intuition of Rowland, repeated the initial experiment made by his mentor, replacing the original thick metal bar with a thin ($d \ll w$ in Figure 1.1) gold leaf. The thinness of the sample should compensate for the weakness of the available magnetic fields. The result was that the magnetic field deflected the galvanometer needle showing that the magnetic field permanently altered the charge distribution, contrarily to Maxwell's prediction. The transverse potential difference between the edges, V_{H} in Figure 1.1, is called *Hall voltage* and the *Hall conductance*[3] is the longitudinal current I divided by V_H.

[1]The mechanical force which is observed acting on the conductor is known as the *ponderomotive force*.

[2]The *magnetoresistence* is the variation of the electrical resistance due to the magnetic field.

[3]Some authors use the terminology *Hall conductivity* instead *Hall conductance*. The two expressions are both correct. Indeed in two spatial dimensions ($d \ll w$ in Figure 1.1) the conductivity (microscopic quantity)

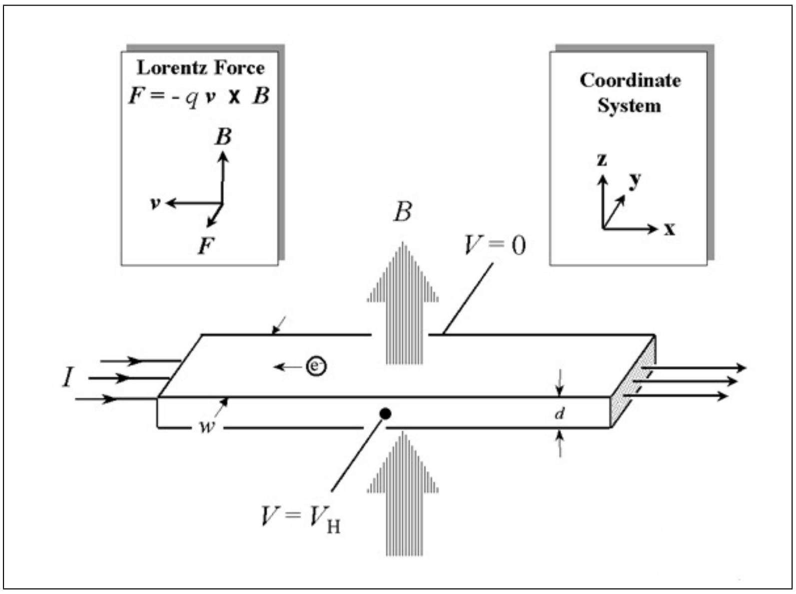


Figure 1.1: *Sketch of the experimental setup for the Hall effect. A magnetic field B normal to a thin bar of a (conductor) sample induces a Lorentz force on a current I which flows in the x-direction. This force separates charges and leads to the emergence of a voltage V_H in the (transverse) y-direction. This is the* Hall voltage *detected by Hall in 1878 and erroneously predicated to be zero by Maxwell.*

As a consequence of the discovery of this new effect, known as *(classical) Hall effect* (CHE), Hall obtained a position at Harvard. His paper (Hall 1879) was published in 1879, the year of Maxwell's death. In the second edition of Maxwell's book, which appeared posthumously in 1881, there is a polite footnote by the editor saying: *"Mr. Hall has discovered that a steady magnetic field does slightly alter the distribution of currents in most conductors so that the statement [...] must be regarded as only approximately true."*

1929 traces a second remarkable year for the story of Hall effect. Since the early experiments, it was clear that the magnitude, and even the sign of the Hall voltage depends on the material properties of the conductor. Although this peculiarity made the Hall effect an important diagnostic tool for investigating the carriers of electric current, the fact that the Hall voltage was found to be positive for some conductors and negative for others opened a new problem. One sign is what one would expect for (free) electrons moving under the combined action of mutually perpendicular electric and magnetic fields. The unexpected sign, instead suggested the disconcerting idea that the charge of electrons was wrong! This phenomenon was called the *anomalous Hall effect*.

R. Peierls, at that time student of W. Heisenberg, was challenged by his mentor to solve the problem of the anomalous Hall effect. The right tool was provided by the

$\sigma_H := {}^{j}/_{E_H}$ coincides with the conductance (macroscopic quantity) $\Sigma_H := {}^{I}/_{V_H}$ since the *longitudinal density of current* is $j := {}^{I}/_{w}$ and the *transverse electric field* is given by $E_H = {}^{V_H}/_{w}$ (Morandi 1988). In this sense the quantization of the Hall conductivity is therefore a macroscopic quantum phenomenon.

new (quantum) mechanics of which Heisenberg was one of the founding father. In fact, Peierls was enlightened by the results of F. Bloch (also Heisenberg's student) concerning the quantum mechanical behavior of electrons in a periodic crystalline field. Peierls realized that when the conduction band is only partially full, the electrons behaves as free particles, and the Hall response is consequently normal (right). However, when the conduction band is completely full the electrons move in the wrong way because of diffraction through the lattice. The conductance turns out to be determined by the missing electrons, i.e. the *holes*, and the anomaly is solved since the charge of a hole is opposite (wrong) to the charge of an electron (Peierls 1985, pp. 36-38).

The third step in the story of the Hall effect begins a century after Hall's discovery. In 1980, performing experiments at the Grenoble High Magnetic Field Laboratory (France) on the Hall conductance of a two-dimensional gas at very low temperature, K. von Klitzing discovered that the Hall conductance, as a function of the strength of the external magnetic field, exhibited a staircase sequence of wide plateaus. Moreover the values of the *Hall resistance*[4] turn out to be integer multiples of a basic constant (the *von Klitzing constant*)

$$R_{\mathrm{K}} := \frac{h}{e^2} \simeq \ 25812.807557\ \Omega, \tag{1.1}$$

where $h \simeq 6.62606896 \times 10^{-34}$ J · s is the Planck constant and $e \simeq 1.602176487 \times 10^{-19}$ C the elementary electron charge. Von Klitzing was awarded the Nobel prize in 1985 for the discovery of this new effect (von Klitzing et al. 1980), today named *quantum Hall effect* (QHE). The surprising precision in the (measured) quantization of the values of the resistance during experiments of QHE has provided metrologiste a superior standard of electrical resistance.

The most remarkable features of the QHE is that the quantization takes place with extraordinary precision in systems that are imprecisely characterized on the microscopic scale. Different samples have different distributions of impurities, different geometry and different concentrations of electrons. Nevertheless, whenever their Hall conductance is quantized, the quantized values mutually agree with an astonishing precision. How to explain the robustness of this phenomenon of quantization?

The first attempt in this direction was made in 1981 by R. Laughlin. In his paper (Laughlin 1981), the author considered a cold two-dimensional electron gas such that the thermal agitation can be neglected (the free particle approximation). In this regime the time evolution of the system is recovered by the knowledge of the wavefunction of a single electron. Laughlin suggested to interpret the QHE as the effect of a *quantum pump*. He assumed that the electron gas was confined on a cylindrical surface with a strong magnetic field applied in the normal direction as shown in Figure 1.3. The two opposite edges of the surface are connected to distinct electron reservoirs R_1 and R_2. The pump effect, which transfers charges from R_1 to R_2, is driven by a magnetic

[4]The Hall resistance is defined as $R_{\mathrm{H}} := {}^{V_{\mathrm{H}}}/_{I}$. According to Footnote 3, R_{H} is the inverse of the Hall conductance Σ_{H} and, due to the two-dimensional geometry, it coincides also with the inverse of the Hall conductivity. The latter is by definition the *Hall resistivity* $\rho_{\mathrm{H}} := {}^{E_{\mathrm{H}}}/_{j}$.

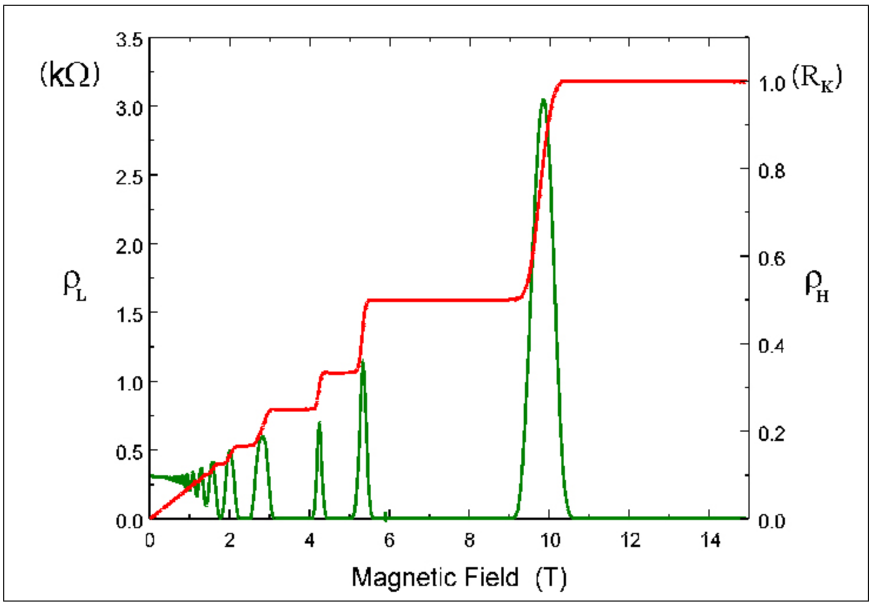


Figure 1.2: *The figure shows (in red) the quantization of the Hall resistivity ρ_H in a GaAs-GaAlAs heterojunction, recorded at a temperature of* 30 *mK. The diagram includes (in green) also the longitudinal component of resistivity ρ_L, which shows regions of zero resistance corresponding to each QHE plateau. The horizontal axis shows the values of the magnetic field in units of tesla (T). The values of the Hall resistivity are recorded on the left vertical axis in unit of von Klitzing constant ($R_K = {}^h/_{e^2}$). It is apparent the quantization of the plateaus at $\rho_H = \frac{1}{n} R_K$ with $n = 1, 2, 3, 4, 6, 8, \ldots$. The right vertical axis shows the values of the longitudinal resistivity in units of kilo-ohm (kΩ).*

flux Φ through the ring which can be controlled by an external operator. Changing the intensity of Φ, an electromotive force is generated around the cylinder and, by Hall effect, one observes a transfer of charge from one reservoir to the other. The Hamiltonian for the system is gauge invariant under flux changes by integral multiples of the *magnetic flux quantum* $\Phi_0 := {}^{hc}/_{e}$ (Aharonov-Bohm principle, cf. (Schwarzschild 1986)) where $c \simeq 299792458\ {}^{\mathrm{m}}/_{\mathrm{s}}$ is the speed of light in vacuum. Therefore, a cycle of the pump corresponds to a (adiabatic) change of the flux from Φ to $\Phi + \Phi_0$. A simple calculation shows that the Hall conductance of the system (measured in units ${}^{e^2}/_{h}$) is given by the number of electrons transported between R_1 and R_2 in a cycle of the pump. Using Laughlin's words, “ *Since, by gauge invariance, adding Φ_0 maps the system back into itself, the energy increase due to it results from the net transfer of n electrons [...] from one edge to the other* ” (Laughlin 1981, p. 5633). The quantization of the Hall conductance follows as a simple consequence of the electric charge quantization.

Nevertheless, the above explanation contains a subtle gap. Admittedly, the measurement of the number of electrons in a reservoir, as well as the number of electrons transferred from R_1 to R_2, must be an integer accordingly to the basic principles of quantum mechanics. However, there is no “a priori” reason why each cycle of the pump

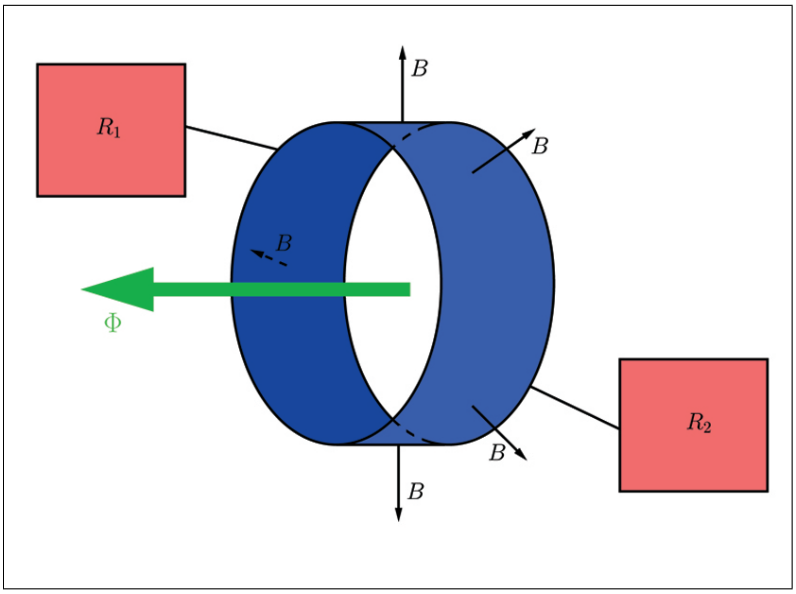


Figure 1.3: *Schematic representation of the Laughlin's gedanken experiment. A two dimensional surface with a cylindrical geometry contains a cold gas of electrons. The two opposite edges of the surface are connected to distinct electron reservoirs R_1 and R_2. A strong magnetic field B acts orthogonally to the surface. Φ denotes a time-dependent magnetic flux through the loop formed by the surface.*

should transfer the *same* number of particles[5]. In a quantum theory the measured Hall conductance is the *average* number of particles transferred in a cycle of the pump. Since in general this number is a fluctuating integer, then its average does not need to be quantized.

Laughlin's work played a fundamental rôle in the development of the theory of QHE. However, to fill the gap in his explanation, one has to explain why averages are also quantized. The "magic tools" which quantize averages are *topological quantum numbers* (Thouless 1998).

However, we are now far from the beginning of the story ... and it is time to go beyond.

Topological quantization: the missing tool

There are two distinct mechanisms that force physical quantities to assume quantized values. The first mechanism is the *orthodox quantization*, namely the quantization emerging from the basic principles of quantum mechanics, according to the original for-

[5] Gauge invariance requires that, after a cycle, the pump (i.e. the electron gas without the reservoirs) is back in its original state. Nevertheless, in a quantum theory, this does not imply that the transported charge in different cycles must be the same. While in classical mechanics reproducing the state of a system necessarily implies reproduction of the outcomes of a measure, the same is no longer true in the quantum world. So the gauge invariance is not sufficient to state that the number of electrons transferred in every cycle of the pump is constant.

mulation given by W. Heisenberg, E. Schrödinger, M. Born, etc. Essentially, the orthodox quantization is a consequence of the fact that observables are represented by matrices, and a measurement always yields an eigenvalue of the matrix as outcome. For instance, the number of charged particles that one finds in an electrometer is a quantized quantity since the operator "number of particles" (which can be thought as an infinite matrix) associated to this observable possesses a spectrum (set of eigenvalues) given by the set of integers $\{0, 1, 2, \ldots\}$.

Topological quantization is a more arcane and deep form of quantization, rather different than the orthodox quantization. The first pioneering work, which opened the exploration of this new paradigm for the quantization, was done by P. A. M. Dirac (Dirac 1931) with his attempt to explain the quantization of the charge. Dirac proposed a theory to prove that the existence of a *quantum of charge* naturally follows from the first principles of quantum mechanics.

He considered a *magnetic monopole* (i.e. a point-like magnetic charge) whose magnetic field behaves as $B(r) = q_m({}^{r}/_{|r|^3})$ with $r \in \mathbb{R}^3$ and q_m the *magnetic charge* of the monopole. Due to the divergence of B to be equal to zero almost everywhere except for the locus of the magnetic monopole at $r = 0$, one can locally define a vector potential A such that $B(r) = \nabla_r \times A(r)$. Nevertheless , the vector potential cannot be defined globally just because the divergence of the magnetic field is singular (proportional to the Dirac delta function) at the origin. With respect to a spherical coordinate system, one has to define a set of functions for the vector potential on the northern hemisphere, and one for the southern hemisphere. These two vector potentials are matched at the equator, and the change between the two functions corresponds to a *gauge transformation*. The wave function of a *probe* charge (i.e. an electrically charged particle) that orbits along the equator sets a phase shift $\Delta\phi$ as in the Aharonov-Bohm effect. $\Delta\phi$ is proportional to the electric charge q_e of the probe particle, as well as to the magnetic charge q_m of the source. As the global phase $\mathrm{e}^{i\phi(r)}$ of the probe charge wave function should not change after the full trip around the equator, the extra-phase $\Delta\phi$ added in the wave function has to be a multiple of 2π, i.e. $\mathbb{Z} \ni \frac{\Delta\phi}{2\pi} = C\frac{q_e q_m}{2\pi} = q_e C_m$ where C is a suitable dimensional constant and $C_m := C\frac{q_m}{2\pi}$. This is known as the *Dirac quantization condition*. The possible existence of even a single magnetic monopole in the universe would imply $q_e = n{}^{1}/_{C_m}$, that is the quantization of the electric charge in units of C_m^{-1}.

From a topological point of view, if one tries to write the vector potential for the magnetic monopole as a single function in the whole space one finds a singularity on a string (called *Dirac string*) that starts on the monopole and goes off to infinity. The string behaves as a thin solenoid carrying a magnetic flux. Hence, if the flux is quantized according to the Dirac quantization condition, the singularity of the vector potential can be removed by a gauge transformation. Since only the modulus of the wave function (rather than its phase) and the electromagnetic fields (rather than the potentials) have direct physical meaning, the singularity is only apparent, as it can be removed by a gauge transform. The string is invisible to a quantum particle, and the magnetic monopole is all that remains.

For various theoretical and experimental reasons, Dirac's theory is not a completely satisfactory solution of the charge quantization problem. However, it is a paradigm of an interesting mechanism of quantization that has a topological origin. In Dirac's scenario the quantization of the charge q_m is not a consequence of the fact that the extra-phase $\Delta\phi$ is associated to an operator with a discrete set of eigenvalues. In fact, q_e and q_m play the rôle of ordinary numerical parameters in the theory. Since the quantization of q_m has a topological origin, one refers to it as a *topological quantum number* (TQN).

A consequence of the Dirac's theory is that every measurement of the charge q_e yields the same value n (in units of C_m^{-1}), and not different multiples of a basic unit. Thus, both the single measurement and the average are quantized with same value n. This is why topological quantum numbers are responsible for the quantization of expectation values.

The arcane has been revealed ... and now we know the way to go beyond.

1.2 *Factum:* topological interpretation of the QHE by Thouless *et al.*

Nowadays, topological quantum numbers play a prominent rôle in many problems arising in solid-state physics (Thouless 1998). Just to mention few examples, they appear in the contexts of adiabatic evolutions (Berry 1984, Simon 1983), macroscopic polarization (Thouless 1983, King-Smith and Vanderbilt 1993, Resta 1992, Panati et al. 2009) and quantum pumps (Avron et al. 2004, Graf and Ortelli 2008). However, for the purposes of this thesis we are mainly interested in the application of the topological quantization in the context of QHE (cf. (Graf 2007) for a recent review).

B. A. Dubrovin and S. P. Novikov discovered that a two dimensional system of non-interacting electrons in a periodic potential exhibits an interesting topology (Dubrovin and Novikov 1980). Novikov refers he queried his colleagues at the Landau Institute about the physical interpretation of the topological invariants he discovered, but nobody provided him with a useful insight[6].

Only later in 1982 D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs (TKNN), studying independently the same model considered by Dubrovin and Novikov, realized that the emerging topological quantum numbers are related with the Hall conductance (Thouless et al. 1982).

As the work of Thouless *et al.* is the "starting point" for this thesis, it is worth to go through the major ideas contained in the "TKNN-paper" (Thouless et al. 1982). The strategy of their proof can be divided into four fundamental steps.

[6]The reader has to note that the paper of Dubrovin and Novikov was submitted on February 1980, two months before the submission of the seminal paper of von Klitzing *et al.* (May 1980). It is not surprising that nobody in Landau Institute was able, at that time, to recognize the link between the Novikov's work and the QHE.

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Quantized Hall Conductance in a Two-Dimensional Periodic Potential

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(Received 30 April 1982)

The Hall conductance of a two-dimensional electron gas has been studied in a uniform magnetic field and a periodic substrate potential U. The Kubo formula is written in a form that makes apparent the quantization when the Fermi energy lies in a gap. Explicit expressions have been obtained for the Hall conductance for both large and small $U/\hbar\omega_c$.

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Figure 1.4: *Header of the seminal paper of Thouless* et al. *that contains the first topological explanation for the QHE.*

(I) The analysis of the QHE requires the study of the spectral properties of the two dimensional single particle Schrödinger operator, given (in suitable units) by

$$H_{\rm BL} = \frac{1}{2}\left(-i\frac{\partial}{\partial x_1} + Bx_2\right)^2 + \frac{1}{2}\left(-i\frac{\partial}{\partial x_2} - Bx_1\right)^2 + V_\Gamma(x_1, x_2),$$

where the potential V_Γ is periodic with respect to $\Gamma \simeq \mathbb{Z}^2$ and B is the strength of an orthogonal uniform magnetic field. However, the analysis of such an operator is a formidable task. The first intuition of Thouless *et al.* was that the relevant physical features of the system can be captured by simpler (effective) models in suitable physical regimes, as the (*Harper*) *regime* of *strong* magnetic field (i.e. $V_\Gamma/B \ll 1$) or the (*Hofstadter*) *regime* of *weak* magnetic field (i.e. $V_\Gamma/B \gg 1$). In particular, in (Thouless et al. 1982) the authors considered explicitly the Harper regime (equivalently the weak periodic potential limit) which leads to study a simpler Hamiltonian $H_{\rm Har}$ (*Harper Hamiltonia*) and its eigenvalues equation known as *Harper's equation* (cf. Figure 1.5).

(II) The geometry of the crystalline structure and the strength of the magnetic field are the two specifications of any apparatus for the detection of the Hall effect. Thus, the flux of the magnetic field through the fundamental cell of the crystal lattice (conventionally denoted by h_B^{-1}) is the natural parameter in the description of the QHE. When h_B takes rational values, the system shows a $\mathbb{Z}^2$-symmetry, i.e. there exists a pair of commuting unitary operators T_1 and T_2 such that $[T_j, H_{\rm BL}] = 0$ for $j = 1, 2$. The same is true for the effective Harper Hamiltonian $H_{\rm Har}$. Thouless *et al.* used this information to decompose the operator $H_{\rm Har}$ in a family of Hamiltonians $\hat{H}(k)$ parametrized by the points $k := (k_1, k_2)$ of a two dimensional torus $\mathbb{T}^2$ (cf. Figure 1.6). The decomposition procedure follows by a simultaneous diagonalization of $H_{\rm Har}$ together with the unitaries T_1 and T_2 that implement the $\mathbb{Z}^2$-symmetry. Any spectral projection P of $H_{\rm Har}$ decomposes as a k-dependent family of spectral projections $P(k)$ of $\hat{H}(k)$. The *range* of $P(k)$, denoted with ${\rm Im}P(k)$, defines a k-dependent family of vector spaces. The collection (disjoint union) $\bigsqcup {\rm Im}P(k)$ was interpreted by Thouless *et al.* as the total space of an "emerging" vector bundle over the base space $\mathbb{T}^2$.

(III) The third step consists in the use of the *Kubo formula* (linear response theory) to compute the Hall conductance (cf. Figure 1.7). In virtue of the decomposition induced by the symmetry (step II), Thouless *et al.* showed that Kubo formula reduces to the following integral:

$$\sigma_H^{B\to\infty}(P) = \alpha\, \frac{i}{(2\pi)} \int_{\mathbb{T}^2} \underbrace{\mathbf{Tr}\Big(P(k)\ [\partial_{k_1}P(k); \partial_{k_2}P(k)]\Big)\ dk_1 \wedge dk_2}_{=:\mathrm{Tr}\ K(P)},$$

where $\sigma_H^{B\to\infty}(P)$ denotes the Hall conductance associated to the spectral projection P of the Harper Hamiltonian, α is a dimensional constant ($\alpha = {e^2}/{h}$ in usual units) and $K(P)$ is a *curvature* for the vector bundle associated to P via decomposition. The integral of the trace of $\frac{i}{2\pi}K(P)$ (two-form) over the two-dimensional manifold $\mathbb{T}^2$ defines an integer $C_1(P)$ called (first) *Chern number*. Since it is known that the Chern numbers are integer topological invariants (topological quantum numbers), one has $\sigma_H^{B\to\infty}(P) \in \alpha\mathbb{Z}$ for any spectral projection, that is the quantization (in units of α) of the Hall conductance.

(IV) This is the most interesting step contained in the TKNN-paper, at least for the purposes of this thesis. Thouless *et al.*, on the basis of (quite obscure) theoretical motivations, deduced the existence of a duality between the opposite regimes of strong and weak magnetic field. In particular, they claimed that in both regimes the Hall conductance is (up to a dimensional factor $\alpha = {e^2}/{h}$) the Chern number of a suitable vector bundle defined by a spectral projection. Assume that the effective models for the Hofstadter regime and Harper regimes have the same spectral structure. Let P_j be the spectral projection defined by the energy spectrum up to the gap G_j and denote with $t(G_j) := \sigma_H^{B\to\infty}(P_j)$ (resp. $s(G_j) := \sigma_H^{B\to 0}(P_j)$) the Hall conductance (i.e. Chern number) in the Harper (resp. Hofstadter) regime. According to the content of the TKNN-paper, the integers $t(G_j)$ and $s(G_j)$ are related by means of a Diophantine equation (cf. Figure 1.8)

$$N\ t(G_j) + M\ s(G_j) = j \qquad\qquad j = 1, \ldots, N \tag{1.2}$$

where the integers M and N are fixed by the condition of rationality $h_B = {M}/{N}$ (c.f. Section 2.5). We refer to (1.2) as the system of *TKNN-equations*. The formula (1.2) is the manifestation of a "mysterious" geometric duality connecting the opposite regimes of strong and weak magnetic field. It is quite surprising that very different physical regimes are related by a so simple and elegant algebraic formula.

The problem of evaluating this quantum number remains. We have considered the potential

$$U(x,y)=U_1\cos(2\pi x/a)+U_2\cos(2\pi y/b), \tag{6}$$

both in the limit of a weak periodic potential ($|U|\ll\hbar\omega_c$) and in the tight-binding limit of a strong periodic potential. In the weak-potential limit the wave function can be written as a superposition of the nearly degenerate Landau functions in the same Landau level N:

$$u_{k_1k_2}=\sum_{n=1}^{p}d_n\sum_{l=-\infty}^{\infty}\chi_N\left(x-\frac{\hbar k_2}{eB}-lqa-\frac{nqa}{p}\right)\exp\left[-ik_1\left(x-lqa-\frac{nqa}{p}\right)+2\pi iy\frac{(lp+n)}{b}\right], \tag{7}$$

where χ_N is the appropriate oscillator wave function. Since the term $U_1\cos(2\pi x/a)$ is diagonal in the Landau functions and $U_2\cos(2\pi y/b)$ changes the value of n by unity, the amplitudes d_n satisfy the secular equation of the form

$$V\exp(-iqak_1/p)d_{n-1}+2V'\cos(qbk_2/p+2\pi nq/p)d_n+V\exp(iqak_1/p)d_{n+1}=Ed_n, \tag{8}$$

with V and V' proportional to U_2 and U_1,[7] and with $d_{n+p}=d_n$. This is known as Harper's equation.[9] Its spectrum has been studied in detail by Hofstadter[10] for the iostropic case $V=V'$. We have made considerable use of his results.

Figure 1.5: *In the limit of a strong magnetic field (i.e. weak periodic potential limit), relevant features of the dynamics of the system are captured by a simple effective model known as Harper's equation.*

We consider electrons in a potential $U(x,y)$ which is periodic in x,y with periods a, b, and in a uniform magnetic induction B perpendicular to the plane of the electrons. The band structure of such a system depends critically on $\varphi=abeB/h$, which is the number of flux quanta per unit cell.

We take φ to be a rational number p/q; the behavior for irrational values of φ can be deduced by taking an appropriate limit. We use the Landau gauge in which the vector potential has components $(0,eBx)$. In this gauge the eigenfunctions of the Schrödinger equation can be chosen to satisfy the generalized Bloch condition

$$\psi_{k_1k_2}(x+qa,y)\exp(-2\pi ipy/b-ik_1qa)=\psi_{k_1k_2}(x,y+b)\exp(-ik_2b)=\psi_{k_1k_2}(x,y), \tag{1}$$

where k_1 (modulo $2\pi/aq$) and k_2 (modulo $2\pi/b$) are good quantum numbers.[8] We can now define functions $u_{k_1k_2}=\psi_{k_1k_2}\exp(-ik_1x-ik_2y)$ which satisfy the generalized periodic boundary conditions

$$u_{k_1k_2}(x+qa,y)e^{-2\pi ipy/b}=u_{k_1k_2}(x,y+b)=u_{k_1k_2}(x,y), \tag{2}$$

and are eigenfunctions of a Hamiltonian

$$\hat{H}(k_1,k_2)=\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}+\hbar k_1\right)^2+\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y}+\hbar k_2-eBx\right)^2+U(x,y). \tag{3}$$

The components of the velocity operator are then given by $\hbar^{-1}$ times the partial derivatives of $\hat{H}$ with respect to k_1, k_2.

Figure 1.6: *The system exhibits a* $\mathbb{Z}^2$*-symmetry under the assumption that the flux of the magnetic field through the fundamental cell of the crystal lattice takes rational values. This symmetry is used to decompose the Hamiltonian of the system by means of a simultaneous diagonalization procedure.*

An alternative approach is to use the Kubo formula for a bulk two-dimensional conductor.

Because of the relation between the velocity operator and the derivatives of $\hat{H}$, the Kubo formula can be written as

$$\sigma_H = \frac{ie^2}{A_0\hbar} \sum_{\epsilon_\alpha < E_F} \sum_{\epsilon_\beta > E_F} \frac{(\partial\hat{H}/\partial k_1)_{\alpha\beta}(\partial\hat{H}/\partial k_2)_{\beta\alpha} - (\partial\hat{H}/\partial k_2)_{\alpha\beta}(\partial\hat{H}/\partial k_1)_{\beta\alpha}}{(\epsilon_\alpha - \epsilon_\beta)^2}, \quad (4)$$

where A_0 is the area of the system and $\epsilon_\alpha, \epsilon_\beta$ are eigenvalues of the Hamiltonian. This can be related to the partial derivatives of the wave functions u, and gives

$$\sigma_H = \frac{ie^2}{2\pi h} \sum \int d^2k \int d^2r \left(\frac{\partial u^*}{\partial k_1}\frac{\partial u}{\partial k_2} - \frac{\partial u^*}{\partial k_2}\frac{\partial u}{\partial k_1} \right)$$
$$= \frac{ie^2}{4\pi h} \sum \oint dk_j \int d^2r \left(u^* \frac{\partial u}{\partial k_j} - \frac{\partial u^*}{\partial k_j} u \right), \quad (5)$$

where the sum is over the occupied electron subbands and the integrations are over the unit cells in r and k space.

The integral over the k-space unit cell has been converted to an integral around the unit cell by Stokes's theorem. For nonoverlapping subbands ψ is a single-valued analytic function everywhere in the unit cell, which can only change by an r-independent phase factor θ when k_1 is changed by $2\pi/aq$ or k_2 by $2\pi/b$. The integrand reduces to $\partial\theta/\partial k_j$. The integral is $2i$ times the change in phase around the unit cell and must be an integer multiple of $4\pi i$.

Figure 1.7: *The* Kubo formula *is used to compute the Hall conductance. The decomposition induced by the symmetry reduces the Kubo formula to an integral of a curvature, namely a* Chern number. *The latter is an integer topological invariant*

The value of s_r is independent of k_2 and determined by the Diophantine equation

$$r = s_r q + t_r p, \quad (9)$$

where $|s| \leq p/2$. The rth gap is of order $(V/V')^{|s_r|}$.

The total Hall current carried by the rth band is quantized according to $\sigma_H = (e^2/h)(t_r - t_{r-1})$. If the Fermi surface is located in the rth gap of the Nth Landau level, the total Hall conductance is equal to

$$\sigma_H = (e^2/h)(t_r + N - 1), \quad (11)$$

with t_r the solution of Eq. (9).

In the opposite limit of a strong potential U the same Eq. (8) can be obtained, where V and V' are the tight-binding matrix elements that take an electron from a site to its neighbors in the x and y directions, and φ is replaced by $1/\varphi$, so that p and q are interchanged.[9,10] The result is that the Hall conductance σ_H is equal to te^2/h. Again, t is given by Eq. (9), but now s is unconstrained and t must lie between $-\frac{1}{2}q$ and $+\frac{1}{2}q$.

Figure 1.8: *The* TKNN-equation *(9) connects the Hall conductance (read Chern number) in the strong magnetic field regime (t_r) with the one in the weak magnetic field regime (s_r). TKNN-equations are the manifestation of a* geometric duality *between the two opposite regimes.*

The strategy used in (Thouless et al. 1982) can be followed for a wide class of periodic potentials. This explains why the QHE is insensitive to the fine details of the microscopic structure of the sample used in the experiment. However, the theory of Thouless *et al.* does not explain the quantization of the Hall conductance either in the case electron-electron interaction is taken into account, or in the case of presence of disorder. Both factors play a rôle in the real Hall effect. Much progresses have been made in understanding this issue, (Laughlin 1983, Kunz 1987, Bellissard 1988b, Bellissard et al. 1994, Kellendonk and Schulz-Baldes 2004, Combes et al. 2006) but this is out from the scope of this thesis.

Finally, it is interesting to see how the theory of Thouless *et al.* has been experimentally verified (Albrecht et al. 2001). The experimental confirmation testifies once again the relevance of the TKNN-paper.

1.3 Overview of the results

Although the TKNN-paper is a milestone in the way towards a theoretical explanation of the QHE, the structure of the proof contains many mathematical "gaps". In order to make the theory of Thouless *et al.* rigorous, one needs to complete some missing "mid-steps" between each of the four steps described in Section 1.2.

In this thesis we propose a rigorous "reinterpretation" of the work of Thouless *et al.* Using various mathematical tools (adiabatic theory, differential geometry, non-commutative geometry, etc.), we derive a series of new and rigorous results which improve and generalize the theory sketched in (Thouless et al. 1982). For convenience we follow the above steps subdivision to list the main results of this thesis.

(I) SAPT-type and algebraic-type results

During the last decades, there were many works aiming to a rigorous derivation of the effective models for the Hamiltonian H_{BL} in the limits of strong and weak magnetic fields (Bellissard 1988a, Helffer and Sjöstrand 1989b). However, all the previous derivations are based on a notion of "approximate model" which turns out to be too weak for our purpose. As discussed in Section 2.2, a rigorous procedure is needed to obtain effective models that are *unitarily* equivalent (in a suitable *asymptotic* sense) to the original model. This is a relevant property which assures that the content of physical information of the original Hamiltonian H_{BL} is fully preserved by the effective Hamiltonians. *Space-adiabatic perturbation theory* (SAPT) provides the appropriate mathematical machinery for a unitarily equivalent derivation of the effective models.

A self-consistent presentation of the results concerning the *adiabatic* derivation of the effective Hamiltonians ("SAPT-type" results) is postponed to Section 2.1. Chapter 3 contains the technicality concerning the derivation of the effective models by means of the mathematical apparatus of SAPT. In particular **Theorem 3.3.14** concerns the rigorous derivation of the *Hofstadter Hamiltonian* (effective model for the weak magnetic field limit), while the *Harper Hamiltonian* (effective model for the strong magnetic field limit) is derived in **Theorem 3.4.8**. The content of our SAPT-type results is summarized in the following diagram

$$
\begin{array}{ccccc}
 & & H_{\text{Hof}} := \wp(\mathcal{U}_0, \mathcal{V}_0) & & L^2(\mathbb{T}^2) =: \mathcal{H}_0 \\
 & \overset{B\to 0}{\nearrow} & \Uparrow & & \\
\mathcal{H}_{\text{phy}} := L^2(\mathbb{R}^2) \quad & H_{\text{BL}} & \text{isospectrality} & & \\
 & \underset{B\to\infty}{\searrow} & \Downarrow & & \\
 & & H_{\text{Har}} := \wp(\mathcal{U}_\infty, \mathcal{V}_\infty) & & L^2(\mathbb{R}) =: \mathcal{H}_\infty .
\end{array}
$$

In Hofstadter regime ($B \to 0$), the original Hamiltonian H_{BL}, defined on the *physical* Hilbert space $\mathcal{H}_{\text{phy}} := L^2(\mathbb{R}^2)$, is asymptotically unitarily equivalent to an effective operator $H_{\text{Hof}} := \wp(\mathcal{U}_0, \mathcal{U}_0)$ (*Hofstadter-like* Hamiltonian) defined on the *reference* Hilbert space $\mathcal{H}_0 := L^2(\mathbb{T}^2)$. The unitary operators $\mathcal{U}_0$ and $\mathcal{V}_0$ act on $\mathcal{H}_0$ according to equation (2.7), while $\wp$ denotes a formal polynomial in two variables containing also negative powers; for instance $\wp(x, y) = x + x^{-1} + y + y^{-1}$. Similarly, the asymptotically unitarily equivalent effective model $H_{\text{Har}} := \wp(\mathcal{U}_\infty, \mathcal{V}_\infty)$ (*Harper-like* Hamiltonian) for the Harper regime ($B \to \infty$), acts on the *reference* Hilbert space $\mathcal{H}_\infty := L^2(\mathbb{R})$ and it is given in terms of a polynomial combination of the unitaries $\mathcal{U}_\infty$ and $\mathcal{V}_\infty$ defined by equation (2.15).

From the above diagram some relevant consequences emerge ("algebraic-type" results). Up to a special condition on the values of the magnetic fields in the strong and weak regimes (Assumption 2.3.6), Hofstadter-like Hamiltonians and Harper-like Hamiltonians share the same algebraic structure, which is the structure of the Non-Commutative Torus (NCT). This *algebraic duality* is analyzed in Section 2.3 (**Theorem 2.3.7**) and its main consequence is the isospectrality between H_{Hof} and H_{Har} (arrow $\Leftarrow - \Rightarrow$ in the diagram).

(II) Spectral decomposition and emerging geometry

It is well known that, if a Hamiltonian operator commutes with a family of operators (symmetries) then their simultaneous diagonalization leads to a decomposition of the original Hamiltonian into a family of (generally simpler) operators parametrized by a spectral parameter (e.g. the eigenvalues of the operators that implement the symmetries). From a mathematical point of view, this is a sophisticated version of the *spectral decomposition theory* by von Neumann (Maurin 1968, Dixmier 1981). The so-called *Bloch-Floquet theory* (Wilcox 1978, Kuchment 1993) is one of the more fruitful application of the above idea. In the TKNN-paper the authors used a similar "decomposition strategy", provided that the magnetic flux per unit cell of the lattice takes a rational value. However, the subtle point which needs more care is the association of the *spectral-type* decomposition coming from the von Neumann theory with a vector bundle structure. Indeed, it is no obvious that a spectral-type decomposition which is based on a measure-theoretic structure, can be related in a *natural* and *unique* way with a topological object like a vector bundle. Thus, the following questions arise: how does the topology (and the geometry) of the decomposition emerge? To which extent is this topological information independent by the specific decomposition procedure?

In Chapter 4 we provide a complete answer to these questions in a quite general framework. We introduce the notion of *physical frame* (Definition 4.1.2), i.e. a triple $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ with $\mathcal{H}$ a separable Hilbert space which corresponds to the set of physical states, $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ a C^*-algebra of bounded operators on $\mathcal{H}$ which contains the relevant physical models, ; $\mathfrak{S} \subset \mathfrak{A}'$ ($\mathfrak{A}'$ is the commutant $\mathfrak{A}$) a commutative unital C^*-algebra which describes a set of simultaneously implementable physical symmetries. Assuming that $\mathfrak{S}$ is a $\mathbb{Z}^d$*-algebra* (i.e. it is generated by d unitaries $U_1, \ldots, U_d$ according to Definition 4.1.3) with the *wandering property* (Definition 4.5.1), we provide a "recipe" (*generalized Bloch-Floquet transform*) to realize "by hand" the von Neumann spectral decomposition (**Theorem 4.6.4**). The underlying vector bundle structure is recovered at an algebraic level and it is *uniquely* specified by the triple $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ (**Theorem 4.7.9**). The element of the C^*-algebra $\mathfrak{A}$ (up to some extra conditions) are mapped in continuous sections of the endomorphism bundle (**Theorem 4.7.15**) providing a unitarily equivalent *bundle representation* for $\mathfrak{A}$ (Definition 2.7.2). In Sections 5.2 we apply the general theory of Chapter 4 to Hofstadter-like and Harper-like models. The bundle decomposition of the Hofstadter-like Hamiltonians, as well as that of the Harper-like Hamiltonians, is established in **Theorem 2.7.4**.

(III) Kubo-Chern equivalence

The rigorous justification of the Kubo formula is generally a hard problem. Some rigorous results have been obtained in the context of QHE models (Bellissard et al. 1994, Elgart and Schlein 2004, Bouclet et al. 2005). However, for a rigorous justification of results in the TKNN-paper, one has to derive the Kubo formula and prove the equivalence between transverse conductance and Chern numbers in the Hofstadter regime, as well as in the Harper regime. This is still an open problem, out of the scope of this thesis. In the following part of this thesis we assume the pragmatic position that: *Chern numbers associated to spectral projections of a Hamiltonian are, by definition, the values of the transverse conductance* (Kubo-Chern equivalence).

(IV) Geometric duality and generalized TKNN-equations

The TKNN-paper contains no prove of the remarkable TKNN-equations. Nevertheless, according to the interpretation of the authors, the TKNN-equations establish a (algebraic) duality between Chern numbers of different vector bundles. Although in the last decades many works have been aimed to a rigorous derivation of TKNN-equations (Středa 1982, MacDonald 1984, Dana et al. 1985, Avron and Yaffe 1986), none of these results consider to look at the integers s_r and t_r (cf. Figure 1.8) as Chern numbers of suitable vector bundles. One of the main result of this thesis is the realization of a purely geometric proof of the TKNN-equations.

If a Hamiltonians admits a bundle decomposition, then its spectral projections (into the gaps) define vector subbundles (Lemma 2.7.3). The isospectrality between Hofstadter-like and Harper-like Hamiltonians implies a one-to-one correspondence between the spectral projections of H_{Hof} and H_{Har}. Let $\mathscr{L}_0(P)$ (resp. $\mathscr{L}_\infty(P)$) be the vector bundle associated with the spectral (gap) projection P of the Hofstadter-like (resp. Harper-like) Hamiltonian H_{Hof} (resp. H_{Hof}). We prove (**Theorem 2.8.1**) that there exists an isomorphism of vector bundles between $\mathscr{L}_0(P)$ and $\mathscr{L}_\infty(P)$ established by the formula

$$f_1^* \mathscr{L}_\infty(P) \simeq f_2^* \mathscr{L}_0(P) \otimes \mathscr{I}$$

where $f_j : \mathbb{T}^2 \to \mathbb{T}^2$, $j = 1, 2$, are suitable continuous maps, $f_j^* \mathscr{L}_\sharp(P)$ denotes the pull-back vector bundle of $\mathscr{L}_\sharp(P)$ ($\sharp = 0, \infty$) via f_j and $\mathscr{I}$ is a suitable *line bundle* which introduces an extra-twist. The above formula is a manifestation of a deep *geometric duality* which relates the opposite regimes of strong and weak magnetic field. The TKNN-equations are a straightforward consequence of such a geometric duality (**Corollary 2.8.2**). Non-ommutative geometry provide the appropriate "language" to explain the geometric meaning of TKNN-equations and to generalize them to the case of an irrational magnetic flux (Section 2.9).

In particular, the set of results presented above fills the mathematical gaps contained in TKNN-paper. From this point of view, one of the merits of this thesis is that it endows the powerful theory of Thouless *et al.* with the mathematical exactness it deserves.

1.4 Why quantum butterflies?

The *Hofstadter butterfly* or *(black and white) quantum butterfly* (Figure 1.9) is a fractal-type diagram showing the collection of the energy spectra of a family of bounded operators $\mathfrak{h}_\theta$, parametrized by $\theta \in \mathbb{R}$ (*universal Hofstadter operators*, equation (2.29)).

Figure 1.9 was firstly described by D. Hofstadter in 1976, in his Ph.D. thesis under the supervision of G. Wannier (Hofstadter 1976). Hofstadter was fascinated by M. Azbel's suggestion that under certain circumstances the energy spectrum of such quantum systems can be a fractal set. Indeed, the self-similar character of the Hofstadter butterfly turned out to be closely related to the fractal nature of its spectrum (for irrational values of the parameter θ).

The importance of the Figure 1.9 for this thesis is due to the fact that the spectrum of $\mathfrak{h}_\theta$ describes the spectrum of both the *Hofstadter Hamiltonian* and the *Harper Hamiltonian*. In the first case the parameter θ is proportional to B, while in the latter to $1/B$. Therefore in both limits θ plays the rôle of a small (*adiabatic*) parameter. The exact relation between $\mathfrak{h}_\theta$, the Hofstadter Hamiltonian and the Harper Hamiltonian, as well the the meaning of θ, are clarified in Chapter 2 (in particular in Sections 2.1 and 2.3).

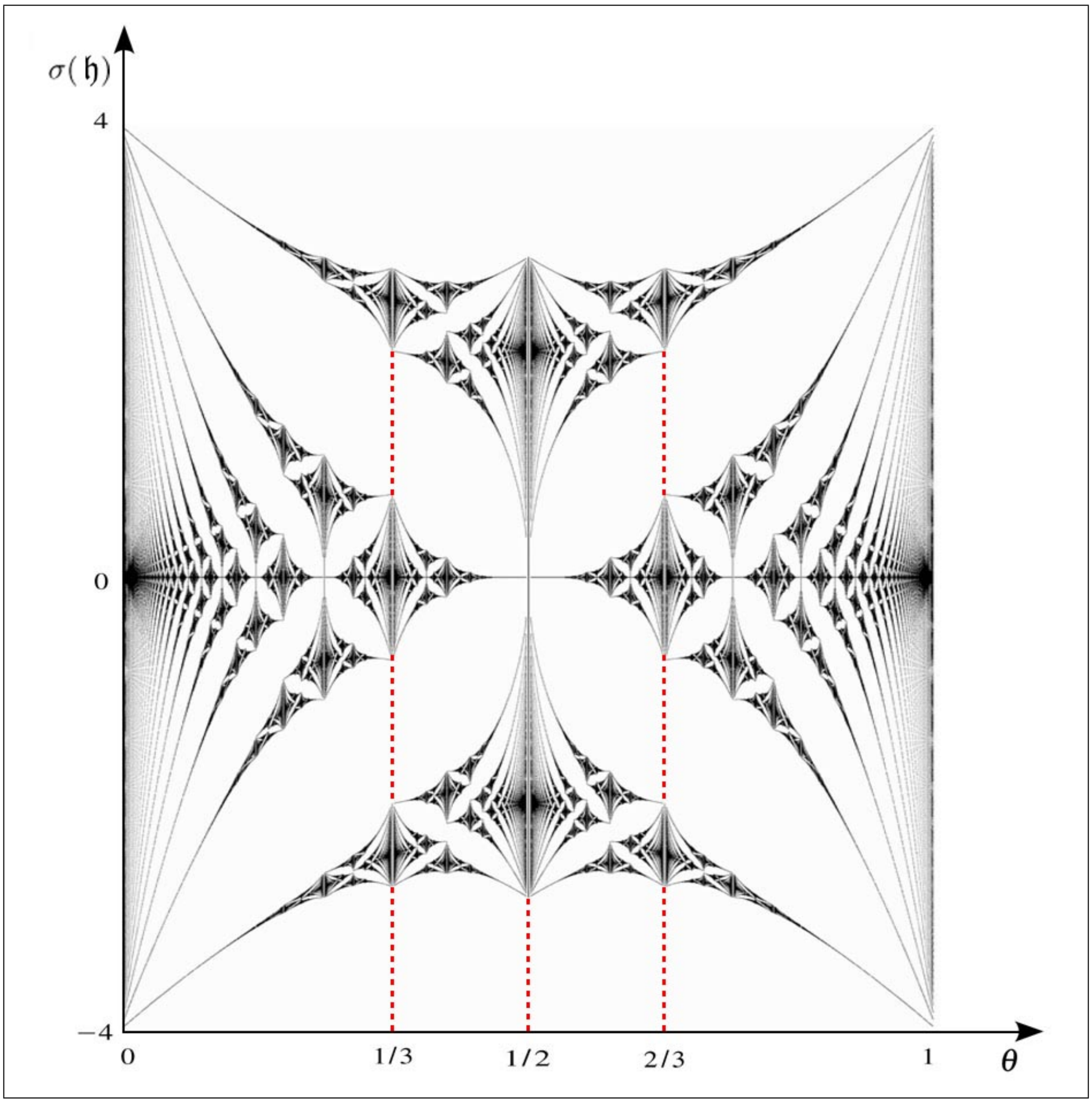


Figure 1.9: *The* (black and white) quantum butterfly *is the collection of spectra of a one-parameter family of bounded operators* $\mathfrak{h}_\theta$ *with* $\theta \in [0,1]$*. If* $\theta = {}^M\!/_N$ *(rational condition) then the set* $\sigma(\mathfrak{h}_{M/N})$ *(vertical axis) is made up by* N energy bands *(black segments) if* N *is odd (e.g.* $\theta = {}^1\!/_3$ *or* ${}^2\!/_3$*) and by* $N-1$ *if* N *is even (e.g.* $\theta = {}^1\!/_2$*). In the latter case the two central segments touch at* 0*. For* $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *the spectrum is of* Cantor type *and has zero Lebesgue measure (hence it is not possible to visualize it).*

Interestingly, the history of the model that gives rise to the Hofstadter butterfly goes back to R. Peierls who proposed it as a thesis problem to P. G. Harper. Nevertheless, neither Peierls nor Hofstadter considered this model in relation with the Hall effect. Instead, they were interested in its intriguing quantum mechanical spectral features.

In order to understand the structure of the "black-butterfly" (i.e. the black part of Figure 1.9) we summarize the main results concerning the dependence of the spectrum $\sigma(\mathfrak{h}_\theta)$ on the parameter θ.

(BH-1) For any $\theta \in \mathbb{R}$, $\|\mathfrak{h}_\theta\| \leqslant 4$ which implies $\sigma(\mathfrak{h}_\theta) \subseteq [-4, 4]$.

(BH-2) For any $\theta \in \mathbb{R}$ the spectrum is symmetric with respect to the zero energy, i.e. $\sigma(\mathfrak{h}_\theta) = \sigma(-\mathfrak{h}_\theta)$.

(BH-3) Since $\sigma(\mathfrak{h}_{\theta+n}) = \sigma(\mathfrak{h}_\theta)$ for any integer $n \in \mathbb{Z}$, one has that it is sufficient to study the spectrum for $\theta \in [0, 1]$. Furthermore, the equality $\sigma(\mathfrak{h}_{-\theta}) = \sigma(\mathfrak{h}_\theta)$ also implies isospectrality between $\mathfrak{h}_\theta$ and $\mathfrak{h}_{1-\theta}$ and then the symmetry of Figure 1.9 with respect to $\theta = 1/2$.

(BH-4) Assume $\theta \in \mathbb{Q}$ with $\theta = M/N$, $M \in \mathbb{Z}$, $N \in \mathbb{N} \setminus \{0\}$ and M and N coprime. The spectrum $\sigma(\mathfrak{h}_{M/N})$ is made up by N (resp. $N - 1$) *energy bands* if N is odd (resp. even). Note that in the case of a even N the central gap is "closed" (that is *empty*). The "open" (that is *non-empty*) gaps between two consecutive energy bands have width larger than 8^{-N} (von Mouche 1989, Choi et al. 1990).

(BH-5) If $\theta \in \mathbb{R} \setminus \mathbb{Q}$ then $\sigma(\mathfrak{h}_\theta)$ is of *Cantor-type* (c.f. Definition 2.4.1) and has zero Lebesgue measure.

(BH-6) Let θ and θ' be such that $|\theta - \theta'| < C$. For any $\epsilon \in \sigma(\mathfrak{h}_\theta)$ there exists a $\epsilon' \in \sigma(\mathfrak{h}_{\theta'})$ such that $|\epsilon - \epsilon'| < 6\sqrt{2|\theta - \theta'|}$ (Avron et al. 1990).

We provide a justification of (BH-1), (BH-2) and (BH-3) at the end of Section 2.3.

Property (BH-5) has been, for a long time, a conjecture known as *Ten Martini Problem*[7] (TMP). The proof was established only recently by A. Avila and S. Jitomirskaya (Avila and Jitomirskaya 2009). For a review on the history of the "long way" to the solution of TMP we refer to (Last 2005, Section 3).

From (BH-4) and (BH-5), it follows that the butterfly (i.e. the black part) in Figure 1.9 has zero Lebesgue measure as subset of the rectangle $[0, 1] \times [-4, 4]$. This suggests that all the information is encoded in the *gap structure* (i.e. the white part).

Point (BH-6) states that the spectrum has a Hölder continuous dependence of order $1/2$ on the parameter θ. In particular, this implies that for every gap in the spectrum of $\mathfrak{h}_\theta$ of length ℓ and for any θ' such that $12\sqrt{2|\theta - \theta'|} < \ell$, there is a corresponding gap in the spectrum of $\mathfrak{h}_{\theta'}$ of length bigger than $\ell - 12\sqrt{2|\theta - \theta'|}$. In other words, the gap structure of the Hofstadter's butterfly is locally continuous, i.e. any point in the plane (θ, σ) of Figure 1.9 which is in a gap has an open neighborhood entirely contained in a gap zone. This means that the gap structure of Figure 1.9 is made up by "open islands" containing no spectral points.

[7] The proof of the Ten Martini Problem (TMP) establishes the topological structure of the spectrum of $\mathfrak{h}_\theta$ when $\theta \in \mathbb{R} \setminus \mathbb{Q}$. However, a stronger version of this conjecture, the *Strong Ten Martini Problem* (STMP), is still open. The question is to prove that all the gaps prescribed by the *Gap Labelling Theory* (GLT) are "open" (i.e. non-empty). The interested reader can find in (Shubin 1994, Section 5) a complete explanation of the relations between GLT, TMP and STMP (and also *Super-Strong Ten Martini Problem* (SSTMP), a very strong version of the problem still unsolved). For GLT the reader can refer to the review (Simon 1982, and references therein).

Finally, when $\mathfrak{h}_\theta$ is rational, it follows from the theory of periodic Schrödinger operators that $\sigma(\mathfrak{h}_\theta)$ is purely absolutely-continuous. Otherwise, when $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\sigma(\mathfrak{h}_\theta)$ is supported on an uncountable set of zero Lebesgue measure, i.e. it is singular-continuous.

Figure 1.9 gives a complete description of the structure of the energy spectrum of $\mathfrak{h}_\theta$ but it does not provide more detailed spectral information such as the degree of degeneracy of the eigenspaces (i.e. the *density of the states*). Such information turns out to be necessary in the analysis of the QHE.

The *colored quantum butterflies* (Figure 1.10), are due to J. E. Avron and D. Osadchy (Avron 2004, Osadchy and Avron 2001) and interpreted by the authors as "thermodynamic" phase diagram for the Hall conductance (c.f. Section 2.5). Colors represent the quantized values of the Hall conductance. Warm colors (like red) correspond to positive values for the Hall conductance, while cold colors (like blue) correspond to negative values. White means zero Hall conductance.

Diagram (A) in Figure 1.10 describes the situation in the regime of weak magnetic field (Hofstadter regime). In this case, the external magnetic field acts as a perturbation of the band spectrum structure of the periodic *Bloch Hamiltonian* which describes the interaction with the crystal. The effect of this perturbation is the creation of new gaps. When the Fermi energy is placed in one of these gaps the Hall conductance is an integer which can be coded by a color. In this way any gap is associated to a color as showed in diagram (A). In this regime the colored butterfly repeats periodically on the θ-axis, with unit period. The white horizontal margins which flank the colored butterflies in (A) mean that the Hall conductance vanishes if the energy band associated to the crystalline structure is either empty or completely full. This is what Peierls expected, that is - *insulators should have vanishing Hall conductance!* -

Diagram (B) in Figure 1.10 describes the situation in the regime of strong magnetic field (Harper regime). In this case, the periodic potential due to the crystalline structure acts as a perturbation of the *Landau Hamiltonian*. It is well known that the spectrum of the Landau Hamiltonian is a collection of equally spaced infinitely degenerate points, known as Landau levels. The weak periodic potential splits each of the Landau levels creating new gaps. Diagram (B) describes the Hall conductance when the Fermi energy sits within the gaps. Note that, contrary to the weak field regime, the color coding of diagram (B) is not periodic with respect to θ. Moreover each butterfly in (A) exhibits inversion symmetry, while butterflies in (B) do not have such a symmetry.

Apparent differences between diagrams (A) and (B) in Figure 1.10 suggest that the regimes of weak and strong magnetic field give rise to very different physical scenarios.

Colored quantum butterflies play a relevant rôle for this thesis. The reason is that the color-coding of the butterflies has been computed by Avron using the Diophantine TKNN-equations. In other words Figure 1.10 is simply a graphic representation of the duality which relates the opposite regimes of weak and strong magnetic field.

The TKNN-equations are the foundation of the arcane beauty of the colored quantum butterflies. A purely geometric derivation of the TKNN-equations is needed to "capture" these "flashy exotic mathematical insects".

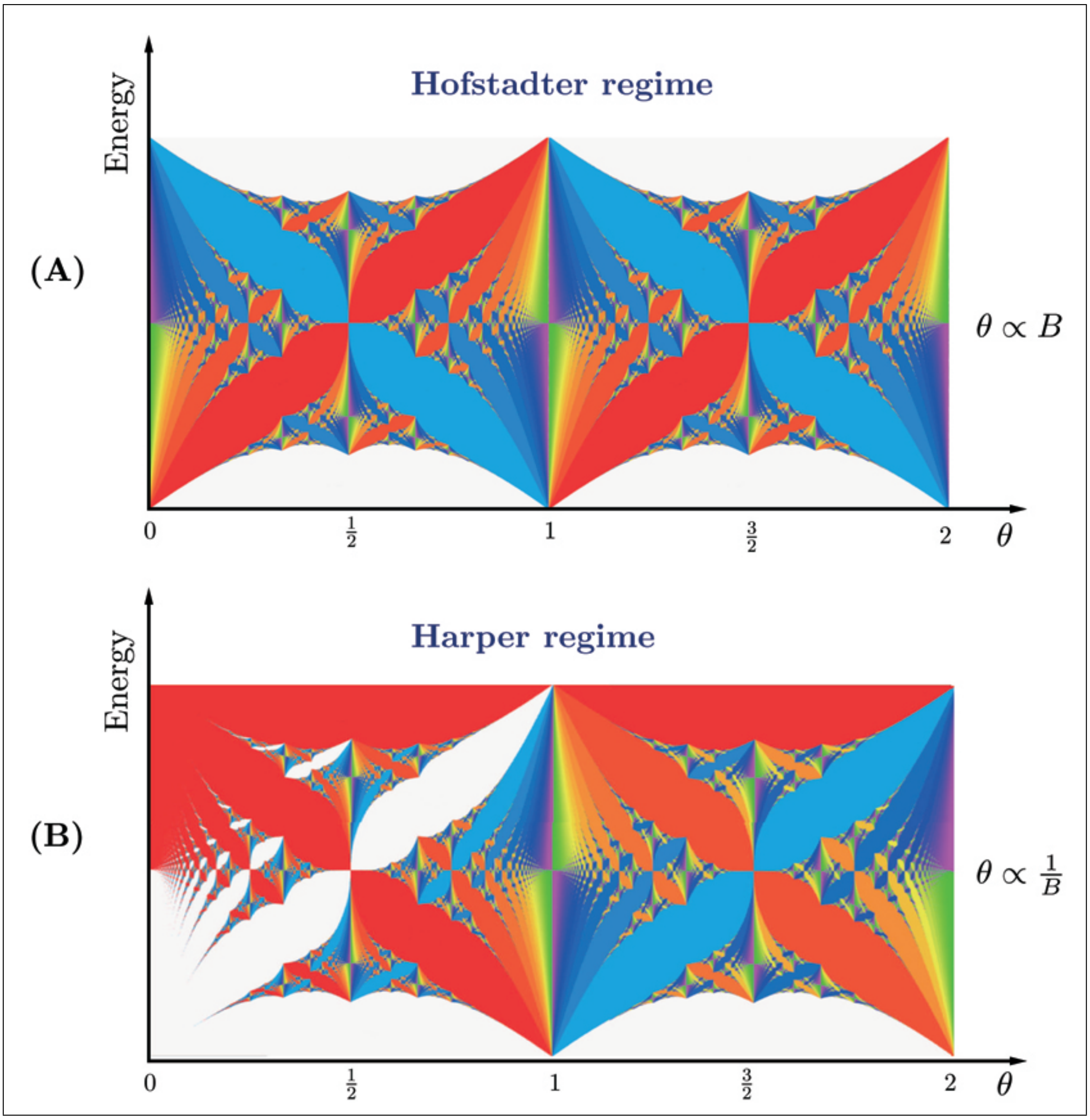


Figure 1.10: [Our elaboration of pictures in (Avron 2004)]. *The colored quantum butterflies are graphic representations of the TKNN-equations. Diagram (A) is the "phase diagram" for the QHE in the regime of a weak magnetic field (Hofstadter regime). Colors represent different values of the "thermodynamic function" Hall conductance. The parameter on the horizontal axis is proportional to the strength of the magnetic field ($\theta \propto B$). Diagram (B) shows the same phase diagram for the strong magnetic field regime (Harper regime). In this case the parameter on the horizontal axis is proportional to the inverse of the strength of the magnetic field ($\theta \propto 1/B$). Butterfly in (A) repeats periodically in θ while butterfly in (B) changes colors at any unit step in θ.*

Chapter 2

Results and techniques

Pour connaître la rose, quelqu'un emploie la géométrie et un autre emploie le papillon.

(To know the rose, someone uses the geometry and another uses the butterfly.)

Paul Claudel
L'Oiseau noir dans le soleil levant, 1927

Abstract

The present chapter aims to expose the main results of this thesis in a rigorous way. We introduce the principal notions and techniques which are indispensable for a self-consistent technical presentation of the arguments developed in this work. The first part of this thesis (adiabatic-part) concerns a series of results coming from the application of the space-adiabatic perturbation theory (SAPT) (c.f. Chapter 3). These results are presented in the first two sections of this chapter. Section 2.1 is devoted to introduce the Hofstadter-like and Harper-like Hamiltonians which are the effective models for the QHE in the limit of weak magnetic field (Hofstadter regime) or strong magnetic field (Harper regime), respectively. Section 2.2 aims to explain the relevance of SAPT for the purposes of this work. In Sections 2.3 and 2.4 we present the algebraic results of this thesis (algebraic-part). The notion of Non-Commutative Torus (NCT) is used to prove the isospectrality between Hofstadter-like and Harper-like Hamiltonians (algebraic duality), as well as to provide a description of the structure of the spectrum of such models in terms of "abstract" spectral projections. Section 2.5 contains a description of the relation between the TKNN-equations and the colored quantum butterflies. In Section 2.6 we justify the difference in the coloring of the two butterflies as a consequence of the absence of a unitary equivalence between Hofstadter and Harper Hamiltonians. The last three sections of this chapter aim to describe the geometric results of this thesis (geometric-part). In Section 2.7 we discuss the relation between "abstract" spectral projections and vector bundles. Section 2.8 is devoted to derive TKNN-equations from a geometric duality between the vector bundles associated to spectral projections of the Hofstadter and Harper Hamiltonians. In Section 2.9 we present a "non-commutative" generalized version of the TKNN-equations. Open problems and possible generalizations are listed in Section 2.10.

2.1 Physical background and relevant regimes for the QHE

Schrödinger operators with periodic potential and magnetic field have been fascinating physicists and mathematicians for the last decades. Due to the competition between the crystal length scale and the magnetic length scale, these operators reveal striking features as fractal spectra (Geyler et al. 2000) or quantization of the Hall conductance (Thouless et al. 1982, Avron et al. 1983, Bellissard et al. 1994, Kellendonk et al. 2002). The colored quantum butterflies (c.f. Figure 1.10) summarize these features in pictorial diagrams.

The mathematical model commonly used for the quantum Hall effect (QHE) (Morandi 1988, Graf 2007) is the two-dimensional *Bloch-Landau Hamiltonian*

$$\mathsf{H}_{\mathrm{BL}} := \frac{1}{2m}\left(-i\hbar\nabla_r - \iota_q \frac{|q|B}{2c} e_\perp \wedge r\right)^2 + \mathsf{V}_\Gamma\,(r)\,, \tag{2.1}$$

acting in the Hilbert space $\mathcal{H}_{\mathrm{phy}} = L^2(\mathbb{R}^2, d^2r)$, $r = (r_1, r_2) \in \mathbb{R}^2$. Here c is the speed of light, $h := 2\pi\hbar$ is the Planck constant, m is the mass and q the charge (positive if $\iota_q = 1$ or negative if $\iota_q = -1$) of the charge carrier, B is the strength of the external uniform time-independent magnetic field, $e_\perp = (0, 0, 1)$ is a unit vector orthogonal to the sample, and V_Γ is a periodic potential which describes the interaction of the carrier with the ionic cores of the crystal. For the sake of a simpler notation, in this introduction we assume that the periodicity lattice Γ is simply $\mathbb{Z}^2$.

While extremely interesting, a direct analysis of the fine properties of the operator H_{BL} is a formidable task. Thus the need to study simpler effective models which capture the main features of (2.1) in suitable physical regimes, as for example in the limit of weak (resp. strong) magnetic field. The relevant dimensionless parameter appearing in the problem is $h_B := {}^{\Phi_0}/_{Z\Phi_B} \propto {}^1/_B$, where $\Phi_0 = {}^{hc}/_e$ is the *magnetic flux quantum*, $\Phi_B = \Omega_\Gamma B$ is the flux of the external magnetic field through the unit cell of the periodicity lattice Γ (whose area is Ω_Γ) and $Z = {}^{|q|}/_e$ is the *magnitude* of the charge q of the carrier in units of e (the *positron* charge). It is also useful to introduce the reduced constant $\hbar_B := {}^{h_B}/_{2\pi}$.

Hofstadter regime, Hofstadter-like Hamiltonians, Hofstadter unitaries

We refer to the limit of weak magnetic field as *Hofstadter regime*. In this limit, corresponding to $\hbar_B \to \infty$, one expects that the relevant features are captured by the well-known *Peierls' substitution* (Peierls 1933, Harper 1955, Hofstadter 1976), thus yielding to consider, for each *Bloch band* $\mathcal{E}_* = \mathcal{E}_*(k_1, k_2)$ of interest, the following effective model: in the Hilbert space[1] $\mathcal{H}_0 := L^2(\mathbb{T}^2, d^2k)$, k being the Bloch momentum and $\mathbb{T}^2$ the two-dimensional torus (c.f. Convention 2.7.1), one considers the Hamiltonian operator

$$H_{\mathsf{eff}}^{B\to 0}\ \varphi\ = \mathcal{E}_*\left(k - \left(\frac{\iota_q}{\hbar_B}\right)\frac{1}{2}\,e_\perp \wedge i\nabla_k\right)\ \varphi, \qquad \varphi \in \mathcal{H}_0. \tag{2.2}$$

In physicists' words, the above Hamiltonian corresponds to replace the variables k_1 and k_2 in $\mathcal{E}_*$ with the symmetric operators (*kinetic momenta*)

$$\mathscr{K}_1 := k_1 + \frac{i}{2}\left(\frac{\iota_q}{\hbar_B}\right)\frac{\partial}{\partial k_2}, \qquad \mathscr{K}_2 := k_2 - \frac{i}{2}\left(\frac{\iota_q}{\hbar_B}\right)\frac{\partial}{\partial k_1}. \tag{2.3}$$

[1]We use the special symbol $\mathcal{H}_0$ to point out that this is the appropriate Hilbert space to describe the physics of the QHE in the Hofstadter regime $B \to 0$.

Since (formally) $[\mathscr{K}_1, \mathscr{K}_2] = i(\iota_q/\hbar) \neq 0$, the latter prescription is formal and (2.2) must be defined by an appropriate variant of the Weyl quantization.

The rigorous justification of the Peierls' substitution and the definition and the derivation of the Hamiltonian (2.2) are the content of Section 3.3.

We call *Hofstadter-like Hamiltonian* any operator in the form (2.2). It is evident that any effective Hamiltonian $H_{\text{eff}}^{B\to 0}$, as well the magnetic momenta $\mathscr{K}_1$ and $\mathscr{K}_2$, depend on the value of the magnetic field through the dimensionless *adiabatic parameter* $\epsilon_0(B) := 1/2\pi\hbar_B$. The name *Hofstadter Hamiltonian* is used only when the energy band has the special form $\mathcal{E}_*(k_1, k_2) = 2\cos k_1 + 2\cos k_2$. From (2.2) it follows that the explicit form of the Hofstadter Hamiltonian is

$$(H_{\text{Hof}}^{\epsilon_0}\,\varphi)(k_1, k_2) := \sum_{\sigma\in\{+,-\}} \mathrm{e}^{i\sigma k_1}\varphi(k_1\ ,\ k_2 - \pi\sigma\iota_q\epsilon_0) + \mathrm{e}^{i\sigma k_2}\ \varphi(k_1 + \pi\sigma\iota_q\epsilon_0\ ,\ k_2). \tag{2.4}$$

The Bloch band $\mathcal{E}_* = \mathcal{E}_*(k_1, k_2)$ which defines the effective Hamiltonian (2.2) is a smooth function $\mathcal{E}_* : \mathbb{T}^2 \to \mathbb{R}$ and we denote by

$$\mathcal{E}_*(k_1, k_2) = \sum_{n,m\in\mathbb{Z}} e_{n,m}\ \mathrm{e}^{i(nk_1+mk_2)}, \tag{2.5}$$

its Fourier series. We introduce the *Hofstadter unitaries*

$$\mathcal{U}_0 := \mathrm{e}^{i\mathscr{K}_1}, \qquad \mathcal{V}_0 := \mathrm{e}^{i\mathscr{K}_2}, \qquad \mathcal{U}_0\mathcal{V}_0 = \mathrm{e}^{-i2\pi\ (\iota_q\epsilon_0)}\mathcal{V}_0\mathcal{U}_0 \tag{2.6}$$

which act on $\mathcal{H}_0$ as

$$\begin{cases} (\mathcal{U}_0\varphi)(k_1, k_2) := \mathrm{e}^{ik_1}\ \varphi(k_1\ ,\ k_2 - \pi\ \iota_q\epsilon_0) \\ \\ (\mathcal{V}_0\varphi)(k_1, k_2) := \mathrm{e}^{ik_2}\ \varphi(k_1 + \pi\ \iota_q\epsilon_0\ ,\ k_2) \end{cases} \qquad \varphi \in \mathcal{H}_0. \tag{2.7}$$

Via Peierls' substitution, one obtains that the Hofstadter-like operator $H_{\text{eff}}^{B\to 0}$ associated to the energy band (2.5) can be written as

$$H_{\text{eff}}^{B\to 0} = \sum_{n,m\in\mathbb{Z}} e_{n,m}\ \mathrm{e}^{i\pi nm(\iota_q\epsilon_0)}\ \mathcal{U}_0^n\mathcal{V}_0^m. \tag{2.8}$$

In particular, for the Hofstadter Hamiltonian (2.4) one has the compact expression

$$H_{\text{Hof}}^{\epsilon_0} = \mathcal{U}_0 + \mathcal{U}_0^{-1} + \mathcal{V}_0 + \mathcal{V}_0^{-1}. \tag{2.9}$$

Harper regime, Harper-like Hamiltonians, Harper unitaries

We use the name *Harper regime* for the limit of a strong magnetic field. In this limit, corresponding to $\hbar_B \to 0$, the periodic potential can be considered a small perturbation of the *Landau Hamiltonian*[2], which provides the leading order approximation of H_{BL}.

[2]The *Landau Hamiltonian* H_{L} is defined by equation (2.1) when $\mathsf{V}_\Gamma = 0$. In the same way one obtains the *Bloch Hamiltonian* (or *periodic Hamiltonian*) H_{B} setting $B = 0$ in (2.1).

To the next order of accuracy in $\hbar_B$, to each Landau level there corresponds an effective Hamiltonian, acting on the Hilbert space[3] $\mathcal{H}_\infty := L^2(\mathbb{R}, dx)$, given (up to a suitable rescaling of the energy) by

$$H_{\text{eff}}^{B\to\infty}\ \psi = \mathsf{V}_\Gamma \left(-i(\iota_q \hbar_B) \frac{\partial}{\partial x}, x \right) \psi, \qquad \psi \in \mathcal{H}_\infty, \tag{2.10}$$

where the right-hand side refers to the ordinary $(\iota_q \hbar_B)$-*Weyl quantization* of the $\mathbb{Z}^2$-periodic function $\mathsf{V}_\Gamma : \mathbb{R}^2 \to \mathbb{R}$. We refer to Section 3.4 for a rigorous derivation of the effective Hamiltonian (2.10).

Each effective Hamiltonian $H_{\text{eff}}^{B\to\infty}$ depends on the value of the magnetic field through the dimensionless *adiabatic parameter* $\epsilon_\infty(B) := 2\pi\hbar_B$. We call *Harper-like Hamiltonian* any operator of the form (2.10), using the name *Harper Hamiltonian* for the special case $\mathsf{V}_\Gamma(p, x) = 2\cos(2\pi p) + 2\cos(2\pi x)$. From equation (2.10), it follows that the Harper Hamiltonian acts as

$$(H_{\text{Har}}^{\epsilon_\infty}\ \psi)(x) := \psi(x - \epsilon_\infty) + \psi(x + \epsilon_\infty) + 2\cos(2\pi x)\ \psi(x). \tag{2.11}$$

The function $\mathsf{V}_\Gamma : \mathbb{R}^2 \to \mathbb{R}$ which defines the effective Hamiltonian (2.10) is a smooth function $\mathsf{V}_\Gamma : \mathbb{R}^2 \to \mathbb{R}$ which is $\mathbb{Z}^2$-periodic with Fourier series denoted by

$$\mathsf{V}_\Gamma(p, q) = \sum_{n,m\in\mathbb{Z}} v_{n,m}\ \mathrm{e}^{i2\pi(np+mq)}. \tag{2.12}$$

The effective Hamiltonian $H_{\text{eff}}^{B\to\infty}$ is obtained from V_Γ via the usual Weyl quantization which agrees with the formal rule $(p, q) \mapsto (P, Q)$ where

$$Q := \text{multiplication by } x, \qquad P := -\frac{i}{2\pi}(\iota_q \epsilon_\infty)\frac{\partial}{\partial x}, \qquad [Q; P] = \frac{i}{2\pi}(\iota_q \epsilon_\infty). \tag{2.13}$$

We introduce the Harper unitaries

$$\mathcal{U}_\infty := \mathrm{e}^{i2\pi Q}, \qquad \mathcal{V}_\infty := \mathrm{e}^{i2\pi P}, \qquad \mathcal{U}_\infty \mathcal{V}_\infty = \mathrm{e}^{-i2\pi\ (\iota_q \epsilon_\infty)} \mathcal{V}_\infty \mathcal{U}_\infty, \tag{2.14}$$

explicitly defined by

$$\begin{cases} (\mathcal{U}_\infty \psi)(x) := \mathrm{e}^{i2\pi x}\ \psi(x) \\ \\ (\mathcal{V}_\infty \psi)(x) := \psi(x + \iota_q \epsilon_\infty) \end{cases} \qquad \psi \in \mathcal{H}_\infty. \tag{2.15}$$

The Harper-like operator $H_{\text{eff}}^{B\to\infty}$ associated to the periodic function (2.12) can be written in terms of the Harper unitaries as

$$H_{\text{eff}}^{B\to\infty} = \sum_{n,m\in\mathbb{Z}} v_{n,m}\ \mathrm{e}^{i\pi nm(\iota\epsilon_\infty)}\ \mathcal{V}_\infty^n \mathcal{U}_\infty^m. \tag{2.16}$$

In particular, the Harper Hamiltonian (2.11) reads

$$H_{\text{Har}}^{\epsilon_\infty} = \mathcal{U}_\infty + \mathcal{U}_\infty^{-1} + \mathcal{V}_\infty + \mathcal{V}_\infty^{-1}. \tag{2.17}$$

[3] As in Note 1, we use the special symbol $\mathcal{H}_\infty$ to point out that this is the appropriate Hilbert space in the Harper regime $B \to \infty$.

The choice of the nomenclature: an historical review

The use of operators of the form (2.2) as *tight binding models* for electrons in a crystal traces back to the pioneering works of R. Peierls (Peierls 1933) and P. G. Harper (Harper 1955). However, the study of the spectral properties of such operators began with the seminal paper (Hofstadter 1976) in which D. Hofstadter described the spectrum of the Hamiltonian (2.4) producing the beautiful picture known as Hofstadter butterfly (Figure 1.9). In view of that, we call Hofstadter Hamiltonian[4] the operator (2.4) and, more generally, Hofstadter-like Hamiltonian any operator in the form (2.2). Hofstadter-like operators are the effective models for the regime of weak magnetic field. This justifies the name *Hofstadter regime* for the limit of zero magnetic field.

The regime of strong magnetic field was originally investigated by A. Rauh (Rauh 1974, Rauh 1975). However the correct effective model, the operator (2.10), was derived firstly (but not rigorously) by M. Wilkinson in (Wilkinson 1987). In a remarkable series of papers (Helffer and Sjöstrand 1988, Helffer and Sjöstrand 1990, Helffer and Sjöstrand 1989a) B. Helffer and J. Sjöstrand studied the spectrum of the operator (2.10) and its relation with the spectra of a one-parameter family of one-dimensional operators on $\ell^2(\mathbb{Z})$ defined by

$$(h^{\theta}_{\beta}u)_n := u_{n-1} + u_{n+1} + 2\cos(2\pi\theta n + \beta)u_n, \qquad \{u_n\}_n \in \ell^2(\mathbb{Z}), \tag{2.18}$$

where $\theta \in \mathbb{R}$ is a fixed number (*deformation parameter*) and $\beta \in [0, 2\pi)$ is the parameter of the family. In the work of the French authors, operator (2.18) is called *Harper operator* (and indeed it was introduced by Harper in (Harper 1955)). However, in the last three decades, operator (2.18) has been extensively studied by many authors (see (Last 2005, Last 1994) for an updated review) with the name of *almost-Mathieu operator*. To avoid confusion and make the nomenclature clear, we chose to adhere to the most recent convention, using the name *almost-Mathieu operator* for (2.18). We thus decided to give credits to Harper's work by associating his name to operators of type (2.10). Consequently we refer to the limit of strong magnetic field as *Harper regime*.

The first rigorous derivation of the effective models (2.2) and (2.10) was obtained by J. Bellissard in an algebraic context in (Bellissard 1988a) and subsequently by B. Helffer and J. Sjöstrand in (Helffer and Sjöstrand 1989b), inspired by the latter paper. In particular, in (Helffer and Sjöstrand 1989b) it is proven that $H^{B\to\infty}_{\text{eff}}$ (resp. $H^{B\to 0}_{\text{eff}}$) has, locally on the energy axis, the same spectrum and the same density of states of H_{BL}, in the appropriate limit. However, although relevant, this property of isospectrality is weaker than the notion of unitary equivalence.

[4]We insist on the fact that this nomenclature is far to be unique. For instance, the operator (2.4) (up to a Fourier transform) is called *discrete magnetic Laplacian* by M. A. Shubin in (Shubin 1994)

2.2 Why is space-adiabatic perturbation theory required?

Beyond the spectrum and the density of states, there are other mathematical properties of H_{BL} which reveal interesting physical features, as for example the orbital magnetization (Gat and Avron 2003, Thonhauser et al. 2005, Ceresoli et al. 2006) or the Hall conductance. These properties are not invariant under a loose equivalence relation as isospectrality, then it is important to show that $H_{\mathrm{eff}}^{B\to\infty}$ and $H_{\mathrm{eff}}^{B\to 0}$ are approximately *unitarily equivalent* to H_{BL} in the appropriate limits. This is one of the main goals of this thesis. The problem is not purely academic, since it is not hard to produce examples of isospectral operators which are however *not unitarily equivalent* and which exhibit differences in the values of the physical observables. In particular, in Sections 2.3 and 2.6, we show that $H_{\mathrm{Hof}}^{\epsilon_0}$ and $H_{\mathrm{Har}}^{\epsilon_\infty}$ are isospectral but not unitarily equivalent. Moreover the TKNN-equations are a fingerprint of this lack of unitary equivalence. One concludes that, in the study of phenomena like the conductance, it is not enough to prove that the effective models are isospectral to the original Hamiltonians.

We thus introduce the stronger notion of unitarily effective model, referring to the concept of almost-invariant subspace introduced by G. Nenciu (Nenciu 2002) and to the related notion of effective Hamiltonian, which we shortly review. The common mathematical background for the aforementioned notions is the *space-adiabatic perturbation theory* (SAPT) developed by G. Panati, H. Spohn and S. Teufel in (Panati et al. 2003b, Panati et al. 2003a, Teufel 2003).

Let us focus on the regime of weak (resp. strong) magnetic field and define $\varepsilon := 2\pi\epsilon_0 = 1/\hbar_B$ (resp. $\epsilon_\infty/2\pi = \hbar_B$) so that $\varepsilon \to 0$ in the relevant limit. Let Π_ε be an orthogonal projection in $\mathcal{H}_{\mathrm{phy}}$ such that, for any $N \in \mathbb{N}, N \le N_0$ there exist a constant C_N such that

$$\|[\mathsf{H}_{\mathrm{BL}}; \Pi_\varepsilon]\| \le C_N\, \varepsilon^N \tag{2.19}$$

for ε sufficiently small. Then $\mathrm{Im}\Pi_\varepsilon$ is called an *almost-invariant subspace* (Nenciu 2002, Teufel 2003) at accuracy N_0, since it follows by a Duhammel's argument that

$$\|(1 - \Pi_\varepsilon)\, \mathrm{e}^{-is\,\mathsf{H}_{\mathrm{BL}}}\, \Pi_\varepsilon\| \le C_N\, \varepsilon^N\, |s|$$

for every $s \in \mathbb{R}$, $N \le N_0$. Granted the existence of such a subspace, we call (*unitarily*) *effective Hamiltonian* a self-adjoint operator $H_{\mathrm{eff}}^{\varepsilon}$ acting on a Hilbert space $\mathcal{H}_{\mathrm{ref}}$, such that there exists a unitary $U_\varepsilon : \mathrm{Im}\Pi_\varepsilon \to \mathcal{H}_{\mathrm{ref}}$ such that for any $N \in \mathbb{N}, N \le N_0$, one has

$$\|\left(\Pi_\varepsilon\, \mathsf{H}_{\mathrm{BL}} - U_\varepsilon^{-1} H_{\mathrm{eff}}^{\varepsilon} U_\varepsilon\right) \Pi_\varepsilon\| \le C'_N\, \varepsilon^N. \tag{2.20}$$

The estimates (2.19) and (2.20) imply that

$$\|\left(\mathrm{e}^{-is\,\mathsf{H}_{\mathrm{BL}}} - U_\varepsilon^{-1}\, \mathrm{e}^{-is\, H_{\mathrm{eff}}^{\varepsilon}}\, U_\varepsilon\right) \Pi_\varepsilon\| \le C''_N\, \varepsilon^N |s|. \tag{2.21}$$

When the macroscopic time-scale $t = \varepsilon s$ is physically relevant, the estimate above is simply rescaled. The triple $(\mathcal{H}_{\mathrm{ref}}, U_\varepsilon, H_{\mathrm{eff}})$ is, by definition, a *unitarily effective model* for H_{BL}. To our purposes, it is important to notice that the asymptotic unitary equivalence in (2.20) assures that the topological quantities related with the spectral projections of $\Pi_\varepsilon \, \mathbf{H}^{\varepsilon}_{\mathrm{BL}} \, \Pi_\varepsilon$ (K-theory, Chern numbers, ...) are equal to those of $H^{\varepsilon}_{\mathrm{eff}}$, for ε sufficiently small. This claim follows by observing that the topological content of a system is preserved by (symmetry preserving) unitary equivalences (cf. Chapter 4) and small perturbation (robustness of the topological invariants).

In Chapter 3 we prove that in the limit $\hbar_B \to \infty$ the Hofstadter-like Hamiltonian (2.2) provides a unitarily effective model for $\mathbf{H}_{\mathrm{BL}}$ with accuracy $N_0 = 1$, and we exhibit an iterative algorithm to construct an effective model at any order of accuracy $N_0 \in \mathbb{N}$ (Theorem 3.3.14). As for the limit $\hbar_B \to 0$, up to a rescaling of the energy, the non-trivial leading order (accuracy $N_0 = 1$) for the effective Hamiltonian is given by the Haper-like Hamiltonian (2.10). We also exhibit the effective Hamiltonian with accuracy $N_0 = 2$, i.e. up to errors of order $\mathcal{O}(\varepsilon^2)$ (Theorem 3.4.8). Moreover, due to the robustness of the adiabatic techniques, we can generalize the simple model described by (2.1) to include other potentials, like a periodic vector potential $\mathbf{A}_\Gamma$, as in (3.1). This terms produces interesting consequences especially in the Harper regime (c.f. Section 3.4.8) and it could play a relevant rôle in the theory of orbital magnetization. This kind of generalization is new with respect to both (Bellissard 1988a) and (Helffer and Sjöstrand 1989b).

A numerical simulation of the spectrum of the Hofstadter operator $H^{\epsilon_0}_{\mathrm{Hof}}$ (resp. Harper operator $H^{\epsilon_\infty}_{\mathrm{Har}}$), as a function of the adiabatic parameter $\epsilon_0 = 1/h_B$ (resp. $\epsilon_\infty = h_B$), leads to a fascinating Hofstadter butterfly (Figure 1.9). This claim will be clarified at the end of Section 2.3. As discussed in Section 1.4, the spectral structure of the butterfly (i.e. the black part) has zero measure as a subset of the square and so the physically relevant object is its complement, namely the gap structure (i.e. the white part). As pointed out by J. Avron and D. Osadchy (Avron 2004, Osadchy and Avron 2001), to each open connected white region (*island*) can be associated a color which codes the value of the transverse conductance and which can be considered as a label for a thermodynamic phase of the system. The correspondence color-gap depends on the particular regime, therefore one has a colored quantum butterfly for the Hofstadter Hamiltonian (diagram (A) of Figure 1.9) and similarly, a colored quantum butterfly for the Harper Hamiltonian (diagram (B) of Figure 1.9). With this language in mind and assuming the interpretation of the colors as topological quantum numbers in the spirit of (Thouless et al. 1982), the main result of Chapter 3 can be reformulated by saying that the Hofstadter-like and Harper-like Hamiltonians are "colour-preserving effective models" for the original Bloch-Landau Hamiltonian $\mathbf{H}_{\mathrm{BL}}$. Thus they describe, though in a distorted and approximated way, some aspects of the thermodynamics of the original system. SAPT is a "colour-preserving" perturbation theory and for this reason a SAPT-type derivation of the effective models is need to provide a mathematical foundation for colored quantum butterflies.

2.3 Algebraic duality and isospectrality

The family of the Hofstadter-like Hamiltonians and that of the Harper-like Hamiltonians have a common algebraic structure. To explore this algebraic analogy we need to introduce a C^*-algebraic concept, that is the *Non-Commutative Torus* (NCT).

Basic notions about Non-Commutative Torus

The NCT (also called *rotation C^*-algebra*) was introduced by Connes in (Connes 1980) as a simple example of a non-commutative manifold and in the last decades it has been extensively studied by many authors. We refer to the comprehensive monographes (Boca 2001) and (Gracia-Bondía et al. 2001).

Due to the relevant rôle that the NCT plays in this thesis, we consider appropriate to introduce this object rigorously. Following (Boca 2001, pp. 1-2), we define the C^*-algebra of the NCT in a formal (and universal) way starting from two "abstract" elements $\mathfrak{u}$ and $\mathfrak{v}$ which are unitary with respect to a formal involution $*$,

$$\mathfrak{u} = \mathfrak{u}^*, \qquad \mathfrak{v} = \mathfrak{v}^*, \qquad \mathfrak{u}\mathfrak{v} = \mathrm{e}^{i2\pi\theta}\, \mathfrak{v}\mathfrak{u}, \quad \theta \in \mathbb{R}. \tag{2.22}$$

Last equation in (2.22) says that $\mathfrak{u}$ and $\mathfrak{v}$ commute up to a phase and θ is called *deformation parameter*. The set $\mathfrak{L}_\theta$ of the finite complex linear combinations of the monomials $\mathfrak{u}^n\mathfrak{v}^m$, $n, m \in \mathbb{Z}$, has the structure of a unital $*$-algebra with unit $\mathfrak{u}^0 = \mathbb{1} = \mathfrak{v}^0$. The *NCT-algebra* with deformation parameter θ, denoted by $\mathfrak{A}_\theta$, is the C^*-algebra generated by the closure of $\mathfrak{L}_\theta$ with respect the *universal* norm

$$\|\mathfrak{a}\| := \sup_\pi\{\|\pi(\mathfrak{a})\|_{\mathscr{B}(\mathcal{H})} \ : \ \pi : \mathfrak{L}_\theta \to \mathscr{B}(\mathcal{H}) \ \text{ is a } *\text{-representation}\}.$$

When $\theta \in \mathbb{Q}$ then $\mathfrak{A}_\theta$ is called *rational* NCT-algebra.

The NCT has a "universal behavior" as far as the representation theory is concerned. Two properties are of particular relevance for our aims (Boca 2001, Remark 1.2):

- **Surjective representation property:** Let $\mathcal{U}$ and $\mathcal{V}$ be two unitary operators acting on the Hilbert space $\mathcal{H}$ such that $\mathcal{U}\mathcal{V} = \mathrm{e}^{i2\pi\theta}\mathcal{V}\mathcal{U}$ and denote by $C^*(\mathcal{U},\mathcal{V})$ the C^*-algebra generated by them in $\mathscr{B}(\mathcal{H})$ (the algebra of bounded operators). The mapping $\pi(\mathfrak{u}) = \mathcal{U}$, $\pi(\mathfrak{v}) = \mathcal{V}$ extends algebraically to a *representation* (i.e. $*$-morphism) $\pi : \mathfrak{A}_\theta \to C^*(\mathcal{U},\mathcal{V}) \subset \mathscr{B}(\mathcal{H})$ which is *surjective*.

- **Universal property:** If $\mathfrak{B}_\theta$ is a C^*-algebra generated by two unitaries $\mathfrak{u}'$ and $\mathfrak{v}'$ such that $\mathfrak{u}'\mathfrak{v}' = \mathrm{e}^{i2\pi\theta}\mathfrak{v}'\mathfrak{u}'$, and if $\mathfrak{B}_\theta$ has the *surjective representation property*, then the mapping $\mathfrak{u} \mapsto \mathfrak{u}'$ and $\mathfrak{v} \mapsto \mathfrak{v}'$ extends to a $*$-isomorphism between $\mathfrak{A}_\theta$ and $\mathfrak{B}_\theta$.

REMARK 2.3.1 (Frame and isomorphisms of NCT-algebra)**.** Even though the definition of the algebra $\mathfrak{A}_\theta$ is subordinate to a choice of a pair of generators $(\mathfrak{u},\mathfrak{v})$, the algebraic structure of $\mathfrak{A}_\theta$ is independent of such a choice. Let $(\mathfrak{u}',\mathfrak{v}')$ be a *frame*, namely any pair of unitaries in $\mathfrak{A}_\theta$ such that $\mathfrak{u}'\mathfrak{v}' = \mathrm{e}^{i2\pi\theta}\,\mathfrak{v}'\mathfrak{u}'$. Each frame defines an equivalent system of generators for $\mathfrak{A}_\theta$ and the universal property assures that there exists an automorphism $\alpha \in \mathrm{Aut}(\mathfrak{A}_\theta)$ such that $\alpha(\mathfrak{u}') = \mathfrak{u}$ and $\alpha(\mathfrak{v}') = \mathfrak{v}$. This means that there is no canonical choice for the system of generators of the NCT-algebra. Having this in mind, we will refer to the algebra $\mathfrak{A}_\theta$ assuming a "a priori" privileged choice for the generators $(\mathfrak{u},\mathfrak{v})$.

Another consequence of the universal property is that the C^*-algebras $\mathfrak{A}_\theta$ and $\mathfrak{A}_{n+\theta}$ are mutually $*$-isomorphic for any $n \in \mathbb{Z}$. Moreover, the map $c(\mathfrak{u}) = \mathfrak{v}$ and $c(\mathfrak{v}) = \mathfrak{u}$ defines a relevant $*$-ismorphism, called *charge-conjugation*, between $\mathfrak{A}_\theta$ and $\mathfrak{A}_{-\theta}$. One can prove that these two mappings are the only isomorphisms between NCT-algebras with different deformation parameters. In other words, if $\theta, \theta' \in [0, 1/2]$ with $\theta \neq \theta'$, then $\mathfrak{A}_\theta$ and $\mathfrak{A}_{\theta'}$ are not $*$-isomorphic (Gracia-Bondía et al. 2001, Corollary 12.7 and following comments).

When the deformation parameter is an integer $N \in \mathbb{Z}$ then $\mathfrak{A}_N$ is $*$-isomorphic to the commutative C^*-algebra $C(\mathbb{T}^2)$ of the continuous function on the two dimensional torus. In this sense $\mathfrak{A}_\theta$, for $\theta \notin \mathbb{Z}$, is the natural *non commutative* generalization of the algebra $C(\mathbb{T}^2)$. To prove the latter claim let $\jmath_i(z) := z_i$, $i = 1, 2$ for any $z := (z_1, z_2) \in \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$ (coordinate functions). The *Stone-Weierstrass Theorem* (Reed and Simon 1973, Theorem IV.10) implies $C^*(\jmath_1, \jmath_2) = C(\mathbb{T}^2)$. If $\theta = N \in \mathbb{Z}$ then $\mathfrak{u}$ and $\mathfrak{v}$ commute and $\pi(\mathfrak{u}) = \jmath_1$, $\pi(\mathfrak{v}) = \jmath_2$ defines a surjective representation $\pi : \mathfrak{A}_N \to C(\mathbb{T}^2)$ (surjective representation property). On the other hand the *Gel'fand-Naĭmark Theorem* (Bratteli and Robinson 1987, Theorem 2.1.11A) states that $\mathfrak{A}_N$ is $*$-isomorphic to $C(X) \hookrightarrow C(\mathbb{T}^2)$ where X denotes the *Gel'fand spectrum* of $\mathfrak{A}_N$ and the injection follows observing that $X \subseteq \sigma(\mathfrak{u}) \times \sigma(\mathfrak{v}) \subseteq \mathbb{S}^1 \times \mathbb{S}^1$. ♦◇

The *smooth NCT-algebra* $\mathfrak{A}^\infty_\theta$ is defined by

$$\mathfrak{A}^\infty_\theta := \left\{ \mathfrak{a} = \sum_{n,m\in\mathbb{Z}} a_{n,m}\mathfrak{u}^n\mathfrak{v}^m \ : \ \{a_{n,m}\} \in \mathcal{S}(\mathbb{Z}^2) \right\} \tag{2.23}$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the space of *rapidly decreasing* double sequences. This means that $\mathfrak{a} \in \mathfrak{A}^\infty_\theta$ if for any $k \in \mathbb{N} \setminus \{0\}$ one has bounded semi-norms

$$\|\mathfrak{a}\|_k = \sup_{m,n\in\mathbb{Z}} |a_{n,m}| \big(1 + |n| + |m|\big)^k < \infty. \tag{2.24}$$

$\mathfrak{A}^\infty_\theta$ is a dense unital $*$-algebra in $\mathfrak{A}_\theta$, stable under the holomorphic functional calculus, i.e. it is a Fréchet unital pre-C^*-algebra (Gracia-Bondía et al. 2001, Definition 12.6). Obviously $\mathfrak{A}^\infty_\theta \simeq C^\infty(\mathbb{T}^2)$ if $\theta \in \mathbb{Z}$.

Hofstadter representation

The universal property of the NCT-algebra and equation (2.6) imply that the mapping

$$\pi_0(\mathfrak{u}) := \mathcal{U}_0, \qquad\qquad \pi_0(\mathfrak{v}) := \mathcal{V}_0, \tag{2.25}$$

defines a representation, named *Hofstadter representation*, of $\mathfrak{A}_\theta$ on the Hilbert space $\mathcal{H}_0$. The representation

$$\pi_0 : \mathfrak{A}_\theta \to C^*(\mathcal{U}_0, \mathcal{V}_0) \subset \mathscr{B}(\mathcal{H}_0), \qquad \theta(B) := -\iota_q \epsilon_0(B) = -\frac{\iota_q}{h_B}$$

is surjective and sometime we use (with a little abuse of nomenclature) the name *Hofstadter C^*-algebra* to denote $\pi_0(\mathfrak{A}_\theta) = C^*(\mathcal{U}_0, \mathcal{V}_0)$. The representation π_0 encodes the full algebraic structure of the universal NCT-algebra, indeed

LEMMA 2.3.2. *For any value of the deformation parameter θ, the Hofstadter representation is* faithful*, i.e. π_0 is a $*$-isomorphism between $\mathfrak{A}_\theta$ and $C^*(\mathcal{U}_0, \mathcal{V}_0)$.*

The proof of Lemma 2.3.2 is based on the GNS construction and it is postponed to Section 5.1.2. The above lemma, together with smoothness of the Bloch band $\mathcal{E}_*$ which enters into the definition of the Hofstadter-like operators imply

COROLLARY 2.3.3. *Any Hofstadter-like operator* (2.2) *is realized as $\pi_0(\mathfrak{d})$ with $\mathfrak{d}$ a self-adjoint element (i.e. $\mathfrak{d} = \mathfrak{d}^*$) in the smooth algebra $\mathfrak{A}_\theta^\infty$.*

Harper representation

From (2.14) and the universal property of the NCT-algebra it follows that the mapping

$$\pi_\infty(\mathfrak{u}) := \mathcal{U}_\infty, \qquad \pi_\infty(\mathfrak{v}) := \mathcal{V}_\infty, \tag{2.26}$$

defines a representation, named *Harper representation*, of the NCT-algebra on the Hilbert space $\mathcal{H}_\infty$. The representation

$$\pi_\infty : \mathfrak{A}_\theta \to C^*(\mathcal{U}_\infty, \mathcal{V}_\infty) \subset \mathscr{B}(\mathcal{H}_\infty), \qquad \theta(B) := -\iota_q \epsilon_\infty(B) = -\iota_q h_B$$

is surjective and sometime we use (with a little abuse of nomenclature) the name *Harper C^*-algebra* to denote $\pi_\infty(\mathfrak{A}_\theta) = C^*(\mathcal{U}_\infty, \mathcal{V}_\infty)$. Similarly to the Hofstadter case, the representation π_∞ encodes the full algebraic structure of the universal NCT-algebra, indeed

LEMMA 2.3.4. *For any value of the deformation parameter θ, the Harper representation is* faithful*, i.e. π_∞ is a $*$-isomorphism between $\mathfrak{A}_\theta$ and $C^*(\mathcal{U}_\infty, \mathcal{V}_\infty)$.*

The proof of Lemma 2.3.4 is postponed to Section 5.1.3. The assumption on the regularity of the periodic function V_Γ (c.f. Assumption 3.2.1) together with the above lemma imply

COROLLARY 2.3.5. *Any Harper-like operator* (2.10) *is realized as $\pi_\infty(\mathfrak{d})$ with $\mathfrak{d}$ a self-adjoint element (i.e. $\mathfrak{d} = \mathfrak{d}^*$) in the smooth algebra $\mathfrak{A}_\theta^\infty$.*

Algebraic duality and isospectrality

The algebraic structure of the NCT-algebra is common both to the Hofstadter and the Harper regimes. Indeed, Corollary 2.3.3 states that any Hofstadter-like Hamiltonian is an element of $\pi_0(\mathfrak{A}_\theta)$ and similarly Corollary 2.3.5 states that any Harper-like Hamiltonian is an element of $\pi_\infty(\mathfrak{A}_{\theta'})$, where the deformation parameters θ and θ' are related to the adiabatic parameters ϵ_0 and ϵ_∞, respectively. To compare the Hofstadter and Harper representations, looking for similarities and differences, one needs to assume $\theta = \theta'$. This condition can be established in terms of the strength of the magnetic fields which characterize the Hofstadter and Harper regimes.

ASSUMPTION 2.3.6 (Algebraic duality condition)**.** *Let B_1 (resp. B_2) be the strength of a weak (resp. strong) magnetic field, i.e. $0 < B_1 \ll 1 \ll B_2$. Let $\epsilon_0(B_1) = 1/h_{B_1}$ be the adiabatic parameter for the Hofstadter regime and $\epsilon_\infty(B_2) = h_{B_2}$ the adiabatic parameter for the Harper regime. The condition $\epsilon_0(B_1) = \epsilon_\infty(B_2) = -\iota_q\, \theta$, for some $\theta \in \mathbb{R}$, is called* algebraic duality condition. *In terms of the strength of the magnetic fields it is equivalent to set $B_1 B_2 = (\Phi_0/ZS_\Gamma)^2$, with $|\theta| = \sqrt{B_1/B_2}$.*

Observe that the sign of θ depends on the sign of the charge of the carriers. In particular, one has $\theta > 0$ for *electrons* ($\iota_q = -1$) and $\theta < 0$ for *holes* ($\iota_q = +1$). In view of that, one calls *charge-conjugation transform* any operation which induces the change $\theta \mapsto -\theta$ (cf. Remark 2.3.1).

If Assumption 2.3.6 holds true, then the mutual relations between the universal C^*-algebra $\mathfrak{A}_\theta$, its Hofstadter realization $\pi_0(\mathfrak{A}_\theta) \subset \mathscr{B}(\mathcal{H}_0)$ and its Harper realization $\pi_\infty(\mathfrak{A}_\theta) \subset \mathscr{B}(\mathcal{H}_\infty)$ are summarized in the diagram below:

$$\begin{array}{ccc} & \mathfrak{A}_\theta & \\ {}^{\mathsf{A}}\swarrow\!\!\!\nearrow & & \nwarrow\!\!\!\searrow^{\mathsf{B}} \\ \pi_0(\mathfrak{A}_\theta) & \underset{\mathsf{C}}{\longleftrightarrow} & \pi_\infty(\mathfrak{A}_\theta). \end{array} \tag{2.27}$$

Here, the double arrow " $\longleftrightarrow$ " denotes an *isomorphism* of C^*-algebras. Arrows A and B express the content of Lemmas 2.3.2 and 2.3.4, respectively. The existence of arrow C follows by composition of A and B. In summary, the previous lemmas imply the following:

THEOREM 2.3.7 (Algebraic duality)**.** *If Assumption 2.3.6 holds true, the Hofstadter C^*-algebra $C^*(\mathfrak{U}_0, \mathfrak{V}_0)$ and the Harper C^*-algebra $C^*(\mathfrak{U}_\infty, \mathfrak{V}_\infty)$ are isomorphic.*

The first consequence of the above result concerns the relations between the spectra of elements of $\mathfrak{A}_\theta$ and the spectra of the related representatives via π_0 and π_∞. Given a $\mathfrak{a} \in \mathfrak{A}_\theta$, its *algebraic spectrum* is defined as $\sigma(\mathfrak{a}) := \mathbb{C} \setminus \rho(\mathfrak{a})$ where $\rho(\mathfrak{a}) := \{\lambda \in \mathbb{C} \ : \ (\mathfrak{a} - \lambda \mathbb{1})^{-1} \in \mathfrak{A}_\theta\}$ denotes the *resolvent set*. Similarly, if A is a linear (not necessarily bounded) operator on the Hilbert space $\mathcal{H}$, then the *Hilbertian spectrum* is defined as $\sigma_{\mathcal{H}}(A) := \mathbb{C} \setminus \rho_{\mathcal{H}}(A)$ where $\rho_{\mathcal{H}}(A) := \{\lambda \in \mathbb{C} \ : \ (A - \lambda \mathbb{1}_{\mathcal{H}})^{-1} \in \mathscr{B}(\mathcal{H})\}$.

COROLLARY 2.3.8 (Isospectrality). *Let Assumption 2.3.6 hold true. Then for any* $\mathfrak{a} \in \mathfrak{A}_\theta$

$$\sigma_{\mathcal{H}_0}(\pi_0(\mathfrak{a})) = \sigma(\mathfrak{a}) = \sigma_{\mathcal{H}_\infty}(\pi_\infty(\mathfrak{a})). \tag{2.28}$$

Proof. The claim follows from two simple observations: 1) in the definition of $\rho(\mathfrak{a})$ (resp. $\rho_{\mathcal{H}}(A)$) one can change $\mathfrak{A}_\theta$ with $C^*(\mathfrak{a})$ which is the commutative C^*-subalgebra generated in $\mathfrak{A}_\theta$ by $\mathfrak{a}$ (resp. $\mathscr{B}(\mathcal{H})$ with $C^*(A)$) (Bratteli and Robinson 1987, Proposition 2.2.7); 2) if $\pi : \mathfrak{A}_\theta \to \mathscr{B}(\mathcal{H})$ is a faithful representation then $C^*(\mathfrak{a}) \simeq C^*(\pi(\mathfrak{a}))$, which implies $\rho(\mathfrak{a}) = \rho_{\mathcal{H}}(\pi(\mathfrak{a}))$. ■

Universal Hofstadter operator and Hofstadter butterfly

The *universal Hofstadter operator* is the element of the universal algebra $\mathfrak{A}_\theta$ defined by

$$\mathfrak{h}_\theta := \mathfrak{u} + \mathfrak{u}^* + \mathfrak{v} + \mathfrak{v}^*. \tag{2.29}$$

A comparison between equations (2.9), (2.17) and (2.29) shows that $H_{\text{Hof}}^{\epsilon_0} = \pi_0(\mathfrak{h}_\theta)$ and $H_{\text{Har}}^{\epsilon_\infty} = \pi_\infty(\mathfrak{h}_\theta)$, provided that the algebraic duality condition $-\iota_q\, \epsilon_0 = \theta = -\iota_q\, \epsilon_\infty$ holds true. Corollary 2.3.8 states the isospectrality of these operators, namely

$$\sigma_{\mathcal{H}_0}(H_{\text{Hof}}^{\epsilon_0}) = \sigma(\mathfrak{h}_\theta) = \sigma_{\mathcal{H}_\infty}(H_{\text{Har}}^{\epsilon_\infty}) \qquad \text{if} \quad \epsilon_0 = \epsilon_\infty = -\iota_q\, \theta. \tag{2.30}$$

The spectrum[5] of $\mathfrak{h}_\theta$ (when θ takes values in $[0,1]$) is described by the Hofstadter butterfly showed in Figure 1.9, and equation (2.30) proves the claim stated in Section 1.4 concerning the isospectrality between the Hofstadter and the Harper Hmiltonians.

We are now in position to justify properties (HB-1), (HB-2) and (HB-3) listed in Section 1.4. The first follows from (2.29), indeed

$$\|\mathfrak{h}_\theta\| \leqslant \|\mathfrak{u}\| + \|\mathfrak{u}^*\| + \|\mathfrak{v}\| + \|\mathfrak{v}^*\| = 4.$$

Property (HB-2) follows from the fact that $(-\mathfrak{u}, -\mathfrak{v})$ is a frame for $\mathfrak{A}_\theta$ and the mapping $(\mathfrak{u}, \mathfrak{v}) \mapsto (-\mathfrak{u}, -\mathfrak{v})$ extends to an automorphism of $\mathfrak{A}_\theta$. Property (HB-3) is a consequence of the isomorphism between $\mathfrak{A}_\theta$ and $\mathfrak{A}_{\theta+n}$ and the fact that the mapping $(\mathfrak{u}, \mathfrak{v}) \mapsto (\mathfrak{v}, \mathfrak{u})$ extends to an isomorphism between $\mathfrak{A}_\theta$ and $\mathfrak{A}_{-\theta}$ (cf. Remark 2.3.1).

2.4 Band spectrum, gap projections and gap labeling

The main consequence of Corollaries 2.3.3 and 2.3.5 is that one can investigate the spectral structure of Hofstadter-like and Harper-like Hamiltonians by looking at the universal representatives in the algebra $\mathfrak{A}_\theta$. In particular, we are interested in *self-adjoint smooth* elements, namely in operators $\mathfrak{d} \in \mathfrak{A}_\theta^\infty$ such that $\mathfrak{d} = \mathfrak{d}^*$. Two types of spectral structures are important for the aims of this thesis.

[5] In order to fully appreciate the rich and elegant structure of the spectrum of $\mathfrak{h}_\theta$, we suggest to look at some of its numerical drawings like Figure 1.9. A nice selection of numerically computed illustrations has been provided in (Guillement et al. 1989).

DEFINITION 2.4.1 (Band spectrum vs. Cantor spectrum). *Let $\mathfrak{d} \in \mathfrak{A}_\theta$ be a self-adjoint element and denote with $\sigma(\mathfrak{d})$ its spectrum. A closed interval $I \subset \mathbb{R}$ is an* energy band *for $\mathfrak{d}$ if $I \subseteq \sigma(\mathfrak{d})$ and $I' \cap (\mathbb{R} \setminus \sigma(\mathfrak{d})) \neq \emptyset$ for any open interval $I' \supset I$. An open (non-empty) interval $G \subset \mathbb{R} \setminus \sigma(\mathfrak{d})$, delimited by two spectral points is called an* ("open"=non-empty) gap *for $\mathfrak{d}$. We will say that $\mathfrak{d}$ has a* band spectrum *if $\sigma(\mathfrak{d})$ is a locally finite union of energy bands*[6]*, while $\mathfrak{d}$ has a* Cantor spectrum *if $\sigma(\mathfrak{d})$ is a closed, nowhere dense set, which has no isolated points (Cantor structure).*

An element $\mathfrak{p} \in \mathfrak{A}_\theta$ which is a self-adjoint idempotent, i.e. $\mathfrak{p}^* = \mathfrak{p} = \mathfrak{p}^2$, is called an *orthogonal projection*. We will denote by $\text{Proj}(\mathfrak{A}_\theta)$ the set of the orthogonal projections in $\mathfrak{A}_\theta$ and with $\text{Proj}(\mathfrak{A}_\theta^\infty) = \text{Proj}(\mathfrak{A}_\theta) \cap \mathfrak{A}_\theta^\infty$ the set of *smooth projections*. Obviously $\text{Proj}(\mathfrak{A}_\theta)$ is non-empty since the *trivial projections* 0 and $\mathbb{1}$ are always elements of $\mathfrak{A}_\theta^\infty$. Generally, $\mathfrak{A}_\theta$ has also non-trivial projections[7]. There exists a deep relation between the structure of the spectrum of self-adjoint elements of $\mathfrak{A}_\theta$ and the existence of non-trivial projections.

LEMMA 2.4.2 (Spectral projection). *Let $\mathfrak{d} \in \mathfrak{A}_\theta$ (resp. $\mathfrak{d} \in \mathfrak{A}_\theta^\infty$) be a self-adjoint element and $\sigma(\mathfrak{d}) \subset \mathbb{R}$ its spectrum. For any $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \sigma(\mathfrak{d})$ (with $\lambda_1 < \lambda_2$) there exists a* spectral projection *$\mathfrak{p}_{[\lambda_1,\lambda_2]} \in \text{Proj}(\mathfrak{A}_\theta)$ (resp. $\mathfrak{p}_{[\lambda_1,\lambda_2]} \in \text{Proj}(\mathfrak{A}_\theta^\infty)$). Moreover $\mathfrak{p}_{[\lambda_1,\lambda_2]} = 0$ (resp. $= \mathbb{1}$) if and only if $[\lambda_1, \lambda_2] \cap \sigma(\mathfrak{d}) = \emptyset$ (resp. $\sigma(\mathfrak{d}) \subset [\lambda_1, \lambda_2]$).*

Proof. let Λ be a closed rectifiable path in $\mathbb{C}$ which intersects the real axis in λ_1 and λ_2 (see Figure 2.1). The projection $\mathfrak{p}_{[\lambda_1,\lambda_2]}$ is defined via holomorphic functional calculus using the *Riesz formula*

$$\mathfrak{p}_{[\lambda_1,\lambda_2]} := \frac{1}{i2\pi} \oint_\Lambda (\lambda \mathbb{1} - \mathfrak{d})^{-1} \, d\lambda.$$

Obviously $\mathfrak{p}_{[\lambda_1,\lambda_2]} \in \mathfrak{A}_\theta$ since $\mathfrak{A}_\theta$ is closed under holomorphic calculus and $\mathfrak{p}_{[\lambda_1,\lambda_2]} = 0$ (resp. $\mathbb{1}$) if and only if $[\lambda_1, \lambda_2] \cap \sigma(\mathfrak{d}) \neq \emptyset$ (resp. $\sigma(\mathfrak{d}) \subset [\lambda_1, \lambda_2]$) since it depends only on the germ of the constant function 1 on $\sigma(\mathfrak{d})$. Finally, $\mathfrak{d} \in \mathfrak{A}_\theta^\infty$ implies $\mathfrak{p}_{[\lambda_1,\lambda_2]} \in \mathfrak{A}_\theta^\infty$ since the smooth algebra is stable under holomorphic calculus (Gracia-Bondía et al. 2001, Definitions 3.25 and 3.26, Proposition 3.45). ■

For the purposes of this thesis we are mainly interested in the spectral structure of self-adjoint elements in the rational NCT-algebra. We fix the following:

CONVENTION 2.4.3 (rationality condition). *When $\theta \in \mathbb{Q}$, its representative is uniquely fixed as $\theta = {}^M/_N$ with $M \in \mathbb{Z}$, $N \in \mathbb{N} \setminus \{0\}$ and M, N coprime, namely* $\text{g.c.d}(N, M) = 1$. *According to this convention* $\text{sign}(\theta) := {}^M/_{|M|}$.

In order to proceed with the analysis of the spectral structure of elements in $\mathfrak{A}_\theta$, we need an additional structure on the NCT. Let $(\mathfrak{u}, \mathfrak{v})$ be a (fixed) system of generators for $\mathfrak{A}_\theta$. The linear map $⨍ \colon \mathfrak{A}_\theta \to \mathbb{C}$ defined on the monomials by

$$⨍ \, (\mathfrak{u}^n \mathfrak{v}^m) := \delta_{n,0} \, \delta_{n,0} \tag{2.31}$$

[6]Since a single point is a (trivial) closed interval, the *pure point spectrum* fits in the definition of band spectrum.

[7]This claim is no longer true if $\theta \in \mathbb{Z}$. In fact, the integrality of θ implies $\mathfrak{A}_\theta \simeq C(\mathbb{T}^2)$ and the latter C^*-algebra has only trivial projections since $\mathbb{T}^2$ is connected.

extends to $\mathfrak{A}_\theta$ by linearity. It is by now a known fact that $⨍$ is a *faithful state* (c.f. Appendix B.1) on $\mathfrak{A}_\theta$ with the *trace property* $⨍\,(\mathfrak{a}\mathfrak{b}-\mathfrak{b}\mathfrak{a})=0$ for any $\mathfrak{a},\mathfrak{b}\in\mathfrak{A}_\theta$ (Boca 2001, pp. 1-5) . We refer to $⨍$ as the *non-commutative integral*[8] over $\mathfrak{A}_\theta$. In general the definition of $⨍$ is not canonical since it is subordinate to the choice of a system of generators $(\mathfrak{u},\mathfrak{v})$. It is canonical for irrational $\theta\in\mathbb{R}\setminus\mathbb{Q}$ since in this case there exists a unique tracial state on $\mathfrak{A}_\theta$. In the rational case $\theta={}^M\!/{}_N$ one can prove that (Boca 2001, Corollary 1.22)

$$⨍\colon \mathrm{Prj}(\mathfrak{A}_{M/N})\to\left\{0,\frac{1}{N},\ldots,\frac{N-1}{N},1\right\}. \tag{2.32}$$

Moreover, the faithfulness of $⨍$ implies that $⨍\,(\mathfrak{p})=0$ (resp. 1) if and only if $\mathfrak{p}=0$ (resp. $\mathbb{1}$). The number $\mathrm{Rk}(\mathfrak{p}):=N\,⨍\,(\mathfrak{p})$ is called *rank* or *dimension* of the projection $\mathfrak{p}$.

In the rational case $\theta={}^M\!/{}_N$, property (2.32) entails

$$\mathcal{K}_\theta:=\inf\{⨍\,(\mathfrak{p})\ :\ 0\neq\mathfrak{p}\in\mathrm{Proj}(\mathfrak{A}_\theta)\}=\frac{1}{N}>0\qquad \text{if}\quad \theta=\frac{M}{N}. \tag{2.33}$$

The number $\mathcal{K}_\theta$ is called *Kadison constant* of the pair $(\mathfrak{A}_\theta,⨍)$ and $\mathcal{K}_\theta>0$ (called *Kadison property*) is the main ingredient to prove that self-adjoint elements in $\mathfrak{A}_\theta$ have a band spectrum (Gruber 2001, Theorem 7).

PROPOSITION 2.4.4 (Gap structure). *Let $\mathfrak{d}\in\mathfrak{A}_\theta$ (resp. $\mathfrak{d}\in\mathfrak{A}_\theta^\infty$) be a self-adjoint element and assume rationality condition $\theta={}^M\!/{}_N$.*

(i) *The spectrum of $\mathfrak{d}$ consists of at most N intervals, hence it admits at most $N+1$ gaps (counting also the unlimited* top *and* bottom *gaps).*

Let $\{G_0,\ldots,G_{N_0}\}$, with $N_0\leqslant N$, be the family of "open" gaps in the spectrum of $\mathfrak{d}$. The labeling of the gaps is fixed in agreement with the ordering $\prec$ on the family of gaps corresponding to their order of occurrence on $\mathbb{R}$, namely $G_0\prec\ldots\prec G_{N_0}$.

(ii) *To each gap is associated a* gap projection $\mathfrak{P}_j\in\mathrm{Proj}(\mathfrak{A}_\theta)$ *(resp. $\mathfrak{P}_j\in\mathrm{Proj}(\mathfrak{A}_\theta^\infty)$) with the convention that $\mathfrak{P}_0=0$ and $\mathfrak{P}_{N_0}=\mathbb{1}$.*

Proof. Let $[a;b]$ be any closed interval with $a,b\in\mathbb{R}\setminus\sigma(\mathfrak{d})$ and $\{\lambda_1,\ldots,\lambda_r\}\subset[a;b]\setminus\sigma(\mathfrak{d})$ such that $\lambda_0:=a<\lambda_1<\ldots<\lambda_r<b=:\lambda_{r+1}$ with $[\lambda_j,\lambda_{j+1}]\cap\sigma(\mathfrak{d})\neq\emptyset$. Following Lemma 2.4.2 one has that $\mathfrak{p}_{[a,b]}=\bigoplus_{j=0}^r\mathfrak{p}_{[\lambda_j,\lambda_{j+1}]}$. The Kadison property (2.33) implies that $1\geqslant⨍\,(\mathfrak{p}_{[a,b]})\geqslant(r+1)\mathcal{K}_\theta=\frac{r+1}{N}$ which forces $r\leqslant N-1$. This means that $\sigma(\mathfrak{d})\cap[a,b]$ is at most union of N disjoint energy bands for any pair $a,b\in\mathbb{R}\setminus\sigma(\mathfrak{d})$. Point (i) follows by observing that $\sigma(\mathfrak{d})$ is a bounded set.

Now let $N_0\leqslant N$ be the total number of disjoint energy bands in $\sigma(\mathfrak{d})$. We label the energy bands according to their order on $\mathbb{R}$, i.e. $I_1\prec I_2\prec\ldots\prec I_{N_0}$, as showed in Figure 2.1. Each of these intervals defines, via holomorphic calculus, a *band projection* $\mathfrak{p}_j\in\mathrm{Proj}(\mathfrak{A}_{M/N})$, $1\leqslant j\leqslant N_0$. Moreover, since the energy bands are mutually non intersecting,

[8] When $\theta\in\mathbb{Z}$, then $\mathfrak{A}_\theta\simeq C(\mathbb{T}^2)$ and $⨍$ coincides with the ordinary integration on $\mathbb{T}^2$ with respect to the normalized Haar measure.

$\mathfrak{p}_j \mathfrak{p}_k = 0$ whenever $j \neq k$. The *gap structure* of $\sigma(\mathfrak{d})$ consists of N_0+1 *"open"* gaps counting also the (unlimited) *bottom gap* which is the open interval from $-\infty$ to the minimum of $\sigma(\mathfrak{d})$ and the (unlimited) *top gap* which is the open interval from the maximum of $\sigma(\mathfrak{d})$ to $+\infty$. We can label the gaps by their order of occurrence on the real line, $G_0 \prec G_1 \prec G_2 \prec \ldots \prec G_{N_0}$ with G_0 the bottom gap and G_{N_0} the top gap. To any gap it is associated the *gap projection* $\mathfrak{P}_j := \bigoplus_{i=0}^{j} \mathfrak{p}_i$. The trivial projection $0 (=: \mathfrak{P}_0)$ is associated to the bottom gap and the trivial projection $\mathbb{1} (=: \mathfrak{P}_{N_0})$ is associated to the top gap. ■

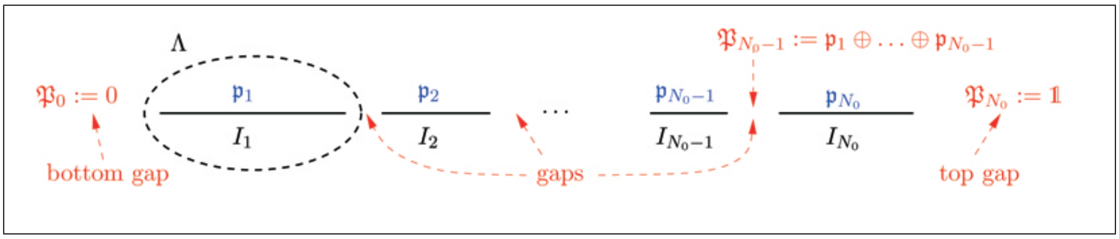


Figure 2.1: *Band structure of the spectrum of a self-adjoint element $\mathfrak{d} \in \mathfrak{A}_\theta$ for a rational value $\theta = {}^{M}\!/_{N}$ of the deformation parameter.*

The number

$$\mathscr{N}(G_j) := \mathrm{Rk}(\mathfrak{P}_j) = N \,⨍\, (\mathfrak{P}_j), \tag{2.34}$$

which coincides with the dimension of $\mathfrak{P}_j$, provide an alternative increasing labeling for the gaps of the spectrum of $\mathfrak{d}$ in the spirit of the celebrated *gap labelling Theorem* (Simon 1982, Bellissard 1993).

Remark 2.4.5. (Gap structure for irrational deformation parameter) If the spectrum of $\mathfrak{d} \in \mathfrak{A}_\theta$ is Cantor (which may happen only if $\theta \in \mathbb{R} \setminus \mathbb{Q}$) then any point in $\sigma(\mathfrak{d})$ is an accumulation point and it is not possible to define the band projections via holomorphic calculus. Nevertheless, we can yet define the gap projections. Since $\sigma(\mathfrak{d})$ is bounded we can fix $-\infty < \lambda_0 < \min \sigma(\mathfrak{d})$ and for any $\lambda \in \mathbb{R} \setminus \sigma(\mathfrak{d})$, $\lambda > \lambda_0$ we can define the *gap projection* $\mathfrak{P}_\lambda \in \mathfrak{A}_\theta$ by means of the holomorphic functional calculus. The number $\tilde{\mathscr{N}}(G) := ⨍ (\mathfrak{P}_\lambda)$ defines a label for the gap $G \ni \lambda$ which is independent on λ. A general result shows that that for any non-trivial projection $\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_\theta)$ there exists a $n \in \mathbb{Z} \setminus \{0\}$ such that $⨍ (\mathfrak{p}) = n\theta \mod \mathbb{Z}$ (Boca 2001, Corollary 11.8). If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then n is determined uniquely and $⨍$ maps the set $\mathrm{Proj}(\mathfrak{A}_\theta)$ onto the countable dense set $\mathscr{G}_\theta := (\mathbb{Z} + \theta\mathbb{Z}) \cap [0, 1]$ (Gracia-Bondía et al. 2001, Theorem 12.6). Thus n defines uniquely a *canonical label* for the element of $\mathrm{Proj}(\mathfrak{A}_\theta)$ and $\mathscr{G}_\theta$ coincides with the maximal set of possible *gap-labels* which, however, can be larger than the set of the actual "open" (i.e. non-empty) gaps in $\sigma(\mathfrak{d})$. The *strong ten Martini problem* consists in proving the existence of a bijection between $\mathscr{G}_\theta$ and the set of the "open" gaps in the (Cantor) spectrum of a self-adjoint element of $\mathfrak{A}_\theta$ when θ is irrational. ♦◊

2.5 Colored butterflies as a gap labeling

The properties of the spectrum of the universal Hofstadter operator $\mathfrak{h}_\theta$ (2.29) have been discussed in Section 1.4. We are mainly interested in the case of rational values of the deformation parameter. Let $\theta = {}^M/_N$ according to Convention 2.4.3. In this case the spectral structure of $\mathfrak{h}_{M/N}$ is well known. With the notation introduced in Section 2.4 one has that:

- N **odd:** $\sigma(\mathfrak{h}_{M/N})$ consists of N disjoint energy bands and $N+1$ "open" gaps (the maximum number of disjoint bands and non-empty gaps) ordered as $G_0 \prec G_1 \prec G_2 \prec \ldots \prec G_N$. Any energy band I_j defines a non trivial projection $\mathfrak{p}_j$ such that $⨍\ (\mathfrak{p}_j) = {}^{n_j}/_N$ with $n_j \in \{1, \ldots, N-1\}$ according to (2.32). The equality $\mathbb{1} = \mathfrak{P}_N := \bigoplus_{j=0}^{N} \mathfrak{p}_j$, the normalization, the faithfulness and the linearity of $⨍$ imply $N = \sum_{j=1}^{N} n_j$, namely $n_j = 1$ for any $j = 1, \ldots, N$. In other words any band projection has dimension 1, i.e. $\mathrm{Rk}(\mathfrak{p}_j) = 1$ and the labeling (2.34) for the gaps is given by $\mathscr{N}(G_j) = j$, $0 \leqslant j \leqslant N$.
- N **even:** $\sigma(\mathfrak{h}_{M/N})$ consists of $N-1$ disjoint energy bands and only N open gaps $G_0 \prec G_1 \prec G_2 \prec \ldots \prec G_{N-1}$ since the central gap is "closed". With an argument similar to that for the odd case one shows that, except for a single band projection which has dimension 2, all the other $N-2$ band projections have dimension 1. A symmetry argument shows that the "bigger" projection is the central one, namely $\mathfrak{p}_{N/2}$. The automorphism $\alpha \in \mathrm{Aut}(\mathfrak{A}_\theta)$ defined by $(\mathfrak{u}, \mathfrak{v}) \mapsto (-\mathfrak{u}, -\mathfrak{v})$ maps $\mathfrak{h}_\theta$ in $-\mathfrak{h}_\theta$ showing that the spectrum of $\mathfrak{h}_\theta$ is symmetric with respect to a reflection around the zero energy. Obviously $\alpha^2 = \mathrm{id}$. The application of α to Riesz formula shows that $\alpha(\mathfrak{p}_j) = \mathfrak{p}_{N-j}$ for any $j = 1, \ldots, N-1$. In particular one deduces the invariance of the central band projection, i.e. $\alpha(\mathfrak{p}_{N/2}) = \mathfrak{p}_{N/2}$. The final argument is the invariance of the noncommutative integral with respect to α, namely $⨍ \circ\alpha = ⨍$. This property can be checked directly on the monomials $\mathfrak{u}^n\mathfrak{v}^m$. The final conclusion is that the labeling of the "open" gaps is given by $\mathscr{N}(G_j) = j$, if $0 \leqslant j \leqslant {}^N/_2 - 1$ and by $\mathscr{N}(G_j) = j + 1$ if ${}^N/_2 \leqslant j \leqslant N - 1$.

The system of TKNN-equations (c.f. Section 1.2) proposed by Thouless *et al.* consists of a family of Diophantine equations labeled by the "open" gaps in the spectrum of $\mathfrak{h}_{M/N}$. With the notation above introduced, the TKNN-equations can be written as[9]

$$M\ s(G_j) + N\ t(G_j) = \mathscr{N}(G_j) \qquad\qquad j = 0, \ldots, N_0 \tag{2.35}$$

with $N_0 = N$ (resp. $N-1$) if N is odd (resp. even). The system is completed by the constraint

$$|s(G_j)| < \frac{N}{2} \qquad\qquad \forall\, j = 0, \ldots, N_0. \tag{2.36}$$

[9]The reader can check directly from Figure 1.8 that the system (2.35) coincides with the equations derived in (Thouless et al. 1982) up to the change of notation $t(G_r) = t_r$, $s(G_r) = s_r$, $N = p$, $M = q$.

Assuming Convention 2.4.3, equation (2.35) is solved by a unique pair of integers $s(G_j)$ and $t(G_j)$ for each fixed gap G_j. The existence of a solution is guaranteed by the assumption g.c.d.$(N, M) = 1$, the uniqueness is a consequence of the constraint (2.36).

The integers $s(G_j)$ and $t(G_j)$ provide two different labelings for the gap G_j. Figure 2.2 shows a diagrammatic representation in which the labels are coded with colors (warm colors like red correspond to positive integers, cold colors like blue correspond to negative integers and white means zero). The butterfly on the left codes the values of $s(G_j)$, while the butterfly on the right codes the values of $t(G_j)$. An immediate consequence of (2.35) and (2.36) is that, independently of the rational value of deformation parameter $\theta = {}^{M}\!/_{N}$, $s(G_0) = t(G_0) = 0$ which explains the white ($= 0$) as bottom color in both butterflies, and $s(G_{N_0}) = 0$, $t(G_{N_0}) = 1$ which explains the difference between the top colors.

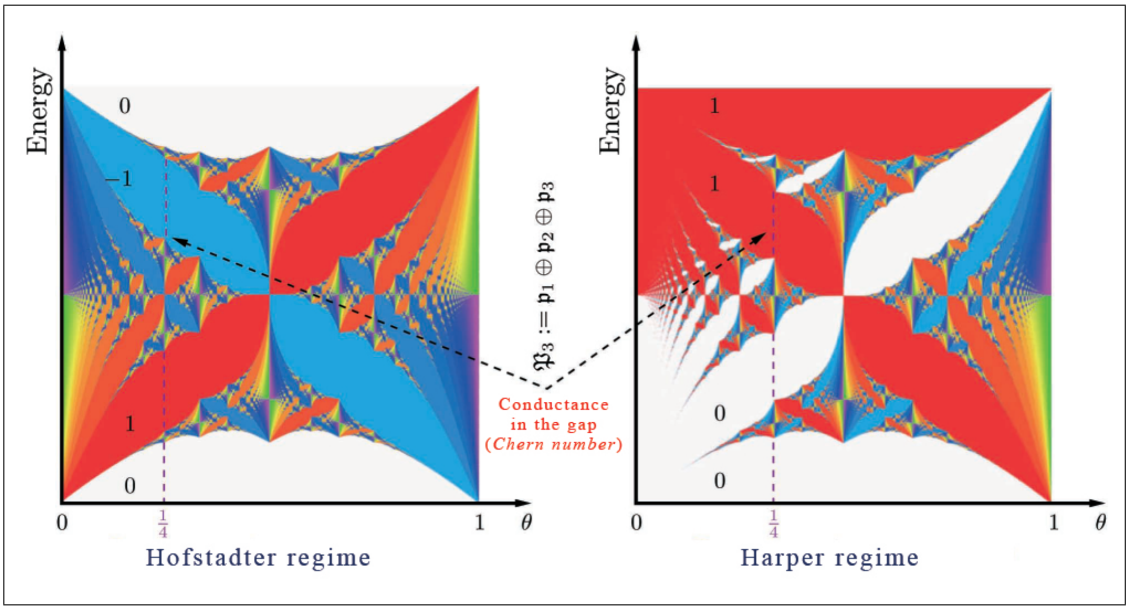


Figure 2.2: [Our elaboration of pictures in (Avron 2004)]. *The two diagrams showed in Figure 2.2 are the colored versions of the Hofstadter's butterfly realized by J. E. Avron and D. Osadchy (Avron 2004, Osadchy and Avron 2001) (cf. Section 1.4). They are known as* colored quantum butterflies. *The butterfly on the left (resp. right) is a thermodynamic phase diagram for the Hall conductance in the Hofstadter (resp. Harper) regime. Any color codes an integer number which describes the quantized values of the Hall conductance and gives a labeling for the gaps in the spectrum of* $\mathfrak{h}_\theta$. *To any "open" gap* G_j *is associated the gap projection* $\mathfrak{P}_j$. *Under the Hofstadter and Harper representations* π_0 *and* π_∞, *the projection* $\mathfrak{P}_j$ *defines two distinct vector bundles over the torus* $\mathbb{T}^2$. *The colors code the Chern numbers of these vector bundles, in accordance with the topological interpretation of the TKNN-equations.*

Let $\mathfrak{P}_j$ be the gap projection associated with the gap G_j (Proposition 2.4.4). It is reasonable to consider the numbers $s(G_j)$ and $t(G_j)$ as quantities depending on $\mathfrak{P}_j$. Moreover, the physical interpretation proposed in (Thouless et al. 1982) is that $s(G_j)$ is the Hall conductance related to the energy spectrum up to the gap G_j in the limit of a weak magnetic field (Hofstadter regime), while $t(G_j)$ is the Hall conductance related to the same energy spectrum but in the opposite limit of a strong magnetic field (Harper regime). Thus, one can infer that the relation between $s(G_j)$ and $\mathfrak{P}_j$ depends on the Hofstadter representation π_0, namely $s(G_j) := C(\pi_0(\mathfrak{P}_j))$. Similarly, one can infer that $t(G_j) := C(\pi_\infty(\mathfrak{P}_j))$. The physical interpretation of the integers $s(\cdot)$ and $t(\cdot)$ leads to consider the colored butterflies in Figure 2.2 (c.f. Section 1.4) as thermodynamic phase diagrams (Avron 2004, Osadchy and Avron 2001). In particular, the butterfly on the left shows the various thermodynamic phases for the Hall conductance (assumed as thermodynamic function) in the Hofstadter regime. In this case the thermodynamic coordinates are the energy (proportional to the *chemical potential*) on the vertical axis and the magnetic field ($\theta \propto B$) on the horizontal axis. For a fixed value of the magnetic field, the color associated to the gap G_j codes (in unit of $^{e^2}/_h$) the value of the Hall conductance when the chemical potential lies in the gap G_j. Similarly, the butterfly on the right shows the various phases for the Hall conductance in the Harper regime. In this case the thermodynamic coordinates are the energy and the inverse of the magnetic field ($\theta \propto {}^1/_B$).

However, there is an open question from a mathematical point of view: what is the meaning of the function C which relates the integers $s(\cdot)$ and $t(\cdot)$ with the gap projections via the representations π_0 and π_∞? One of the main results of this thesis is to prove that the "function" C denotes the *(first) Chern number* of a suitable vector bundle associated to the gap projection $\mathfrak{P}_j$ and depending on the particular (Hofstadter or Harper) representation (cf. Section 2.7). This result leads to a rigorous proof of the topological interpretation of the TKNN-equations.

2.6 Absence of unitary equivalence

Corollary 2.3.8 states that the Hofstadter-like Hamiltonian $\pi_0(\mathfrak{a})$ and the Harper-like Hamiltonian $\pi_\infty(\mathfrak{a})$ associated to the same universal element $\mathfrak{a} \in \mathfrak{A}_\theta$ are *isospectral* with spectrum given by $\sigma(\mathfrak{a})$, provided that the duality condition (Assumption 2.3.6) holds true. However, the relation of *isospectrality* between two operators is quite weak since it concerns only the equality of the spectra as subsets of $\mathbb{C}$; this property is related only to the algebraic structure of the operator algebras and does not depend on the representations. However, for a linear operator acting on a Hilbert space, the notion of *algebraic spectrum* can be enriched by means of the underlying Hilbert space structure. As a matter of fact , the little amount of information in the algebraic spectrum is improved by the determination of the *spectral measure* or by analyzing the *Lebesgue decomposition of the spectrum* or computing the *density of the states*, etc. This information defines the *spectral type* of an operator. Obviously, two operators which are merely isospectral, can have different spectral types, while two operators which are unitarily equivalent have the same algebraic spectrum and the same spectral type. These considerations suggest the following question: under Assumption 2.3.6, do the isospectral operators $\pi_0(\mathfrak{a})$ and $\pi_\infty(\mathfrak{a})$ have the same spectral type?

A hint to answer the above question comes from the analysis of the colored butterflies in Section 2.5. The isospectrality between the effective models $H^{\theta}_{\rm Hof}$ and $H^{\theta}_{\rm Har}$ (with $\theta = -\iota_q\epsilon_0 = -\iota_q\epsilon_\infty$ according to Assumption 2.3.6) leads to the same spectral diagram, the black and white quantum butterfly showed in Figure 1.9. However, if one looks to finer properties like the Hall conductance, then the Hamiltonians $H^{\theta}_{\rm Hof}$ and $H^{\theta}_{\rm Har}$ produce different colored butterflies (cf. Figure 1.10 or 2.2). Since the Hall conductance is a "spectral quantity" (it depends on the spectral projection associated to the gap), one can suspect that $H^{\theta}_{\rm Hof}$ and $H^{\theta}_{\rm Har}$ have different spectral type. This is not so surprising, in fact $H^{\theta}_{\rm Hof}$ and $H^{\theta}_{\rm Har}$ are effective Hamiltonians which describe opposite and very different physical regimes. It is plausible that there exist physical quantities able to discriminate between the two regimes.

The above considerations suggest that the operators $H^{\theta}_{\rm Hof}$ and $H^{\theta}_{\rm Har}$ are not unitary equivalent. A first, but partial, indication is given by the following result:

THEOREM 2.6.1 (Weak "no go"). *Let Assumption 2.3.6 hold true. Then there exists no unitary map* $W : \mathcal{H}_0 \to \mathcal{H}_\infty$ *such that* $W\, \mathcal{U}_0\, W^{-1} = \mathcal{U}_\infty$ *and* $W\, \mathcal{V}_0\, W^{-1} = \mathcal{V}_\infty$.

Proof. We sketch an argument proposed by G. Emch in (Emch 1996). Suppose that such a W exists. Then W implements unitarily the $*$-isomorphism defined by the arrow C of diagram (2.27), i.e. $W\, \pi_0(\mathfrak{A}_\theta)\, W^{-1} = \pi_\infty(\mathfrak{A}_\theta)$. Such a unitary equivalence extends to a unitary equivalence of the related von Neumann algebras (c.f. Section B.1) $\mathfrak{M}_0(\mathfrak{A}_\theta) := \pi_0(\mathfrak{A}_\theta)''$ and $\mathfrak{M}_\infty(\mathfrak{A}_\theta) := \pi_\infty(\mathfrak{A}_\theta)''$. However, this is impossible since $\mathfrak{M}_0(\mathfrak{A}_\theta)$ is a *standard* von Neumann algebra (c.f. Section 5.1.2) while $\mathfrak{M}_\infty(\mathfrak{A}_\theta)$ is *not standard* (c.f. Section 5.1.3) and the property to be standard (or not) is preserved by unitary equivalences. ■

The above "no go" result allows the Hofstadter-like Hamiltonian $\pi_0(\mathfrak{a})$ and the Harper-like Hamiltonian $\pi_\infty(\mathfrak{a})$, related to the same element $\mathfrak{a} \in \mathfrak{A}_\theta$, to have different spectral types, but it does not exclude the existence of a special unitary $\tilde{W}$ which intertwines only $\pi_0(\mathfrak{a})$ and $\pi_\infty(\mathfrak{a})$ and not the full C^*-algebras. To exclude the unitary equivalence between each pair of operators we need a stronger version of the above "no go" theorem.

The main result of this thesis is to provide a geometric and generalized version of TKNN-equations which relate some "spectral fingerprints" of $\pi_0(\mathfrak{a})$ and $\pi_\infty(\mathfrak{a})$ for any (self-adjoint) $\mathfrak{a} \in \mathfrak{A}_\theta$. These spectral quantities can be defined in terms of topological invariants (TQN) of suitable geometric structures emerging from the spectral decomposition of the algebras $\pi_0(\mathfrak{A}_\theta)$ and $\pi_\infty(\mathfrak{A}_\theta)$ (see Section 2.7). The difference between the topological invariants associated to $\pi_0(\mathfrak{a})$ and $\pi_\infty(\mathfrak{a})$ it is enough to exclude the existence of a special intertwiner $\tilde{W}$ (strong "no go" result). The existence of a formula which relates these invariants is a consequence of a deep duality between $\pi_0(\mathfrak{A}_\theta)$ and $\pi_\infty(\mathfrak{A}_\theta)$ which is of geometric type. The proof of this *geometric duality* is the principal aim of this thesis.

2.7 From projections to vector bundles: the “two-fold way”

Section 2.5 ended with the assertion that there exists a relation between gap projections of the Hofstadter (resp. Harper) Hamiltonian and vector bundles. The present section aims to explain the nature of such a relation.

Firstly, we need to introduce some notations concerning the theory of *vector bundles*. For a detailed and complete exposition about this subject, we refer to (Lang 1985, Chapter III) or (Gracia-Bondía et al. 2001, Chapter 2). Fundamental notions useful for the purpose of this thesis are sketched in Appendix C.

We use the symbol $\iota : \mathscr{E} \to \mathbb{T}^2$ to denote a Hermitian vector bundle with two-dimensional torus as base space. To simplify notation we will occasionally use only the symbol $\mathscr{E}$. The vector bundle has *rank* N if $\iota^{-1}(z) \simeq \mathbb{C}^N$ for any $z \in \mathbb{T}^2$. The symbol $\Gamma(\mathscr{E})$ denotes the (finitely generated and projective) $C(\mathbb{T}^2)$-module of continuous sections of the vector bundle. The endomorphism bundle (c.f. Proposition 4.7.13) associated to the rank N vector bundle $\mathscr{E} \to \mathbb{T}^2$ is the rank N^2-vector bundle $\mathrm{End}(\mathscr{E}) \to \mathbb{T}^2$ with typical fiber $\mathrm{End}(\mathbb{C}^N) \simeq \mathrm{Mat}_N(\mathbb{C})$ and transition functions induced by the adjoint action of those of $\mathscr{E}$. The symbol $\Gamma(\mathrm{End}(\mathscr{E}))$ denotes the corresponding $C(\mathbb{T}^2)$-module of sections.

CONVENTION 2.7.1 (n-dimesional torus)**.** *The manifold $\mathbb{T}^n := \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ (n times) is parametrized by the points (denoted with k) of the square $[0, 2\pi]^n$ up to the identification of the opposite edges. Such an identification is made explicit by the map $k := (k_1, \ldots, k_n) \mapsto z(k) := (e^{ik_1}, \ldots, e^{ik_n})$. We consider $\mathbb{T}^n$ to be endowed with the flat metric (the product metric of the canonical Riemannian metric on $\mathbb{S}^1$) in such a way that the related volume form agrees with the normalized Haar measure $dz(k) := \frac{d^n k}{(2\pi)^n}$ with $d^n k := dk_1 \wedge \ldots \wedge dk_n$. Sometimes we use the short notation $L^p(\mathbb{T}^n)$ instead $L^p(\mathbb{T}^n, dz)$ to denote the space of the p-summable functions with respect to dz.*

The Hermitian structure on the bundle $\mathscr{E}$ and the existence of a volume form on $\mathbb{T}^2$ allows us to define a scalar product on the space of sections $\Gamma(\mathscr{E})$ as

$$\langle s; r\rangle := \int_{\mathbb{T}^2} (s(z); r(z))_z \, dz, \qquad\qquad s, r \in \Gamma(E),$$

where $(\cdot\, ;\, \cdot)_z$ denotes the scalar product in the fiber space $\iota^{-1}(z) \simeq \mathbb{C}^N$. The completion of $\Gamma(\mathscr{E})$ with respect to the norm $\|s\|_{L^2} := \sqrt{\langle s; s\rangle}$ leads to the Hilbert space $L^2(\mathscr{E})$ of the L^2-*sections* of the vector bundle $\mathscr{E}$. Obviously, $\Gamma(\mathscr{E}) \subset L^2(\mathscr{E})$ due to the compactness of the base manifold. We denote with $\mathscr{B}(L^2(\mathscr{E}))$ the C^*-algebra of the bounded operators on the Hilbert space $L^2(\mathscr{E})$. The fiber metric $(\cdot\, ;\, \cdot)_z$ defines a $C(\mathbb{T}^2)$-valued Hermitian structure on $\Gamma(\mathscr{E})$ through $\{s; r\}(z) := (s(z); r(z))_z$ for $s, r \in \Gamma(E)$. This endows $\Gamma(\mathscr{E})$ with the structure of a Hilbert module over $C(\mathbb{T}^2)$. Let $\mathrm{End}_{C(\mathbb{T}^2)}(\Gamma(\mathscr{E}))$ be the C^*-algebra of the adjontable operators on the $C(\mathbb{T}^2)$-module $\Gamma(\mathscr{E})$ (Boca 2001, Proposition 3.1 and Theorem 3.8). Any element in $\mathrm{End}_{C(\mathbb{T}^2)}(\Gamma(\mathscr{E}))$ extends uniquely to a bounded operator on $L^2(\mathscr{E})$ hence, with a slight abuse of notation, we can write $\mathrm{End}_{C(\mathbb{T}^2)}(\Gamma(\mathscr{E})) \subset \mathscr{B}(L^2(\mathscr{E}))$. The (localization) isomorphism $\Gamma(\mathrm{End}(\mathscr{E})) \simeq \mathrm{End}_{C(\mathbb{T}^2)}(\Gamma(\mathscr{E}))$ shows that $\Gamma(\mathrm{End}(\mathscr{E}))$ is a unital C^*-algebra and justifies the (abuse of) notation $\Gamma(\mathrm{End}(\mathscr{E})) \subset \mathscr{B}(L^2(\mathscr{E}))$.

DEFINITION 2.7.2 (Bundle decomposition[10]). *Let $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ be a C^*-algebra of bounded operators on the Hilbert space $\mathcal{H}$. We say that $\mathfrak{A}$ admits a* bundle decomposition over X *if there exist a vector bundle $\mathscr{E} \to X$ (with X a finite dimensional compact manifold) and a unitary map $\mathcal{F} : \mathcal{H} \to L^2(\mathscr{E})$ such that $\mathcal{F}\, \mathfrak{A}\, \mathcal{F}^{-1} \subset \Gamma(\mathrm{End}(\mathscr{E}))$.*

The mere request of a unitary map between $\mathcal{H}$ and $L^2(\mathscr{E})$ is trivial since all separable Hilbert spaces are unitarily equivalent. Nevertheless, such a map transforms $\mathfrak{A}$ in a subalgebra of $\mathscr{B}(L^2(\mathscr{E}))$. The non-trivial part of the above definition consists in the (rather strong) request that $\mathcal{F}\, \mathfrak{A}\, \mathcal{F}^{-1} \subset \Gamma(\mathrm{End}(\mathscr{E}))$. The unitarity of the map $\mathcal{F}$ assures that spectral properties of operators in $\mathfrak{A}$ are preserved after the decomposition. On the other hand, $\Gamma(\mathrm{End}(\mathscr{E}))$ is a geometric object and its geometry encodes features of the original algebra $\mathfrak{A}$. In other words, an algebra of operators admits a bundle decomposition if it has a hidden geometric structure which emerges up to a unitary transform. A general scenario for the appearance of an emerging geometric structure is the existence of a family of symmetries $\mathfrak{S}$ for the algebra $\mathfrak{A}$. This point of view is developed in Chapter 4. A triple $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ is called a *physical frame* (Definition 4.1.2). The content of Theorems 4.7.9 and 4.7.15 is that (under some technical assumptions) any physical frame with *wandering property* (Definition 4.5.1) induces a bundle decomposition $\mathcal{F}$. In this case we say that the bundle decomposition $\mathcal{F}$ is *subordinate* to the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$. If $\mathfrak{S}$ is maximal commutative inside the commutant $\mathfrak{A}'$, then the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ is called *irreducible* and we refer to the induced $\mathcal{F}$ as an *irreducible* (subordinate) bundle decomposition.

The interpretation of the spectral properties of $\mathfrak{A}$ in terms of emerging geometric quantities is supported by the following general result.

LEMMA 2.7.3. *Let $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ be a C^*-algebra admitting a bundle decomposition over X with (rank N) vector bundle $\iota : \mathscr{E} \to X$. Then, any projection $P \in \mathfrak{A}$ defines a vector subbundle $\mathscr{L}(P) \subset \mathscr{E}$.*

Proof. If $P \in \mathfrak{A}$ is a projection, then $\mathcal{F}\, P\, \mathcal{F}^{-1} =: P(\cdot) \in \Gamma(\mathrm{End}(\mathscr{E}))$ is an *projection-valued section*, namely $P(x)$ is a projection in $\mathrm{End}(\iota^{-1}(x)) \simeq \mathrm{Mat}_N(\mathbb{C})$ for any $x \in X$. Since any element of $\Gamma(\mathrm{End}(\mathscr{E}))$ defines a *bundle map* (Gracia-Bondía et al. 2001, Corollary 2.7), we can build the *image vector bundle* associated to $P(\cdot)$. Let $\mathrm{Im}(P)_x := \{P(x)\mathrm{v}\ ;\ \mathrm{v} \in \iota^{-1}(x)\} \subset \iota^{-1}(x)$ and consider the total space

$$\mathscr{L}(P) := \bigsqcup_{x \in X} \mathrm{Im}(P)_x \simeq \{(x, \mathrm{v}) \in X \times \mathscr{E}\ :\ \iota(\mathrm{v}) = x,\ \ P(x)\mathrm{v} = \mathrm{v}\} \tag{2.37}$$

[10]The space $L^2(\mathscr{E})$ agrees with the *direct integral* (c.f. Appendix B.3) of the Hilbert spaces $\mathcal{H}(z) := \iota^{-1}(z)$, i.e. $L^2(\mathscr{E}) = \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}(z)\, dz$. The unitary map $\mathcal{F} : \mathcal{H} \to L^2(\mathscr{E})$ induces a *direct integral decomposition* or more simply a *fiber decomposition* of the Hilbert space $\mathcal{H}$. Let $\mathscr{D} \subset \mathcal{F}\mathscr{B}(\mathcal{H})\mathcal{F}^{-1}$ be the set of *decomposable operators* (or *operator fields* c.f. Appendix B.3). If $\mathcal{F}\mathfrak{A}\mathcal{F}^{-1} \subset \mathscr{D}$, then the C^*-algebra $\mathcal{A}$ admits a *fiber decomposition*, namely $\mathcal{F}\mathfrak{A}\mathcal{F}^{-1} = \int_{\mathbb{T}^2}^{\oplus} \pi_z(\mathfrak{A})\, dz$ with $\pi_z : \mathfrak{A} \to \mathscr{B}(\mathcal{H}(z))$ is a representation of $\mathfrak{A}$ for any $z \in \mathbb{T}^2$. However, the fiber decomposition of a C^*-algebra is a purely measure-theoretic notion which contains no topological information. On the contrary, the notion of of bundle decomposition is purely topological and is related with the definition of *continuous field of C^*-algebras* (Dixmier 1982). Observing that $\Gamma(\mathscr{E})$ is dense in $L^2(\mathscr{E})$, it follows that the notion of bundle decomposition is stronger than that of fiber decomposition. Loosely speaking, a bundle decomposition is a *continuous* fiber decomposition.

where the symbol $\bigsqcup_{x\in X}$ denotes the disjoint union of sets labeled by X. Equation (2.37) implies that $\mathscr{L}(P) \subset \mathscr{E}$ and $\mathscr{L}(P) \cap \iota^{-1}(x) = \mathrm{Im}(P)_x$ and so defines the total space of a vector subbundle $\iota : \mathscr{L}(P) \to X$ provided that the function $x \mapsto \dim(\mathrm{Im}(P)_x) = \mathrm{Rk}(P(x))$ is constant. The latter claim follows by standard arguments showing that the map $x \mapsto \mathrm{Rk}(P(x))$ is both lower and upper semicontinuous, hence locally constant (Dixmier and Douady 1963, Section 1). If X is connected, then $x \mapsto \mathrm{Rk}(P(x))$ is constant and the common value, denoted with $\mathrm{Rk}(P)$, fixes the rank of the vector subbundle $\mathscr{E}(P)$. The isomorphism which appears in the right-hand side of (2.37) is justified by the fact that the map $(x, \mathrm{v}) \mapsto \mathrm{v}$ defines a linear isomorphism between the fiber spaces $\{x\} \times \mathrm{Im}(P)_x$ and $\mathrm{Im}(P)_x$. ■

The relevance of the Definition 2.7.2 for a geometric derivation of the TKNN-equations (2.35) is related to the following fundamental result:

THEOREM 2.7.4 (Bundle decomposition in Hofstadter and Harper representations). *Let $\mathfrak{A}_{M/N}$ be the the rational NCT-algebra (according to Convention 2.4.3), $\pi_0 : \mathfrak{A}_{M/N} \to \mathcal{H}_0$ the Hofstadter representation and $\pi_\infty : \mathfrak{A}_{M/N} \to \mathcal{H}_\infty$ the Harper representation.*

(i) *The operator algebra $\pi_0(\mathfrak{A}_{M/N}) \subset \mathscr{B}(\mathcal{H}_0)$ admits an* irreducible bundle decomposition *over $\mathbb{T}^2$ with (rank N) Hermitian vector bundle $\mathscr{E}_0 \to \mathbb{T}^2$ (called* Hofstadter vector bundle*) and a unitary transform $\mathcal{F}_0 : \mathcal{H}_0 \to L^2(\mathscr{E}_0)$. The decomposition is* unique *(up to equivalences). The vector bundle $\mathscr{E}_0$ is trivial, hence its (first) Chern number is zero, $C_1(\mathscr{E}_0) = 0$. Finally, the bundle representation $\mathcal{F}_0\ \pi_0(\mathfrak{A}_{M/N})\ {\mathcal{F}_0}^{-1}$ is generated by the endomorphism sections $\mathcal{U}_0(\cdot) := \mathcal{F}_0\ \pi_0(\mathfrak{u})\ {\mathcal{F}_0}^{-1}$ and $\mathcal{V}_0(\cdot) := \mathcal{F}_0\ \pi_0(\mathfrak{v})\ {\mathcal{F}_0}^{-1}$ explicitly defined (in local coordinates according to Convention 2.7.1) by*

$$\mathcal{U}_0(k) \equiv \mathbb{U}(e^{-ik_1}), \qquad \mathcal{V}_0(k) \equiv \mathbb{V}(e^{ik_2}), \qquad k = (k_1, k_2) \in \mathbb{R}^2 \tag{2.38}$$

where $\mathbb{U}$ and $\mathbb{V}$ are defined by (2.40)*.*

(ii) *The operator algebra $\pi_\infty(\mathfrak{A}_{M/N}) \subset \mathscr{B}(\mathcal{H}_\infty)$ admits an* irreducible bundle decomposition *over $\mathbb{T}^2$ with (rank N) Hermitian vector bundle $\mathscr{E}_\infty \to \mathbb{T}^2$ (called* Harper vector bundle*) and unitary transform $\mathcal{F}_\infty : \mathcal{H}_\infty \to L^2(\mathscr{E}_\infty)$. The decomposition is* unique *(up to equivalences). The vector bundle $\mathscr{E}_\infty$ is non trivial with (first) Chern number $C_1(\mathscr{E}_\infty) = 1$. Finally, the* bundle representation *$\mathcal{F}_\infty\ \pi_\infty(\mathfrak{A}_{M/N})\ {\mathcal{F}_\infty}^{-1}$ is generated by the endomorphism sections $\mathcal{U}_\infty(\cdot) := \mathcal{F}_\infty\ \pi_\infty(\mathfrak{u})\ {\mathcal{F}_\infty}^{-1}$ and $\mathcal{V}_\infty(\cdot) := \mathcal{F}_\infty\ \pi_\infty(\mathfrak{v})\ {\mathcal{F}_\infty}^{-1}$ explicitly defined by*

$$\mathcal{U}_\infty(k) \equiv \mathbb{U}(e^{i\frac{M}{N}k_1}), \qquad \mathcal{V}_\infty(k) \equiv \mathbb{V}(e^{ik_2}), \qquad k = (k_1, k_2) \in \mathbb{R}^2. \tag{2.39}$$

To complete definitions (2.38) and (2.39) we need to introduce the $N \times N$ complex matrices

$$\mathbb{U}(\lambda) := \lambda \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \mathrm{e}^{i2\pi\frac{M}{N}} & 0 & \dots & 0 \\ 0 & 0 & \mathrm{e}^{i4\pi\frac{M}{N}} & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \mathrm{e}^{i2\pi(N-1)\frac{M}{N}} \end{pmatrix}, \quad \mathbb{V}(\lambda) := \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \lambda \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{2.40}$$

with $\lambda \in \mathbb{C}$. Obviously, $\mathbb{U}(\lambda)$ and $\mathbb{V}(\lambda)$ are unitary if and only if $|\lambda| = 1$ (i.e. $\lambda \in \mathbb{S}^1$) and

$$\mathbb{U}(\lambda)\mathbb{V}(\lambda') = \mathrm{e}^{i2\pi\frac{M}{N}}\ \mathbb{V}(\lambda')\mathbb{U}(\lambda), \qquad \forall\ \lambda, \lambda' \in \mathbb{C}.$$

The proof of claim (i) is postponed in Section 5.2.1 where the Hofstadter vector bundles is built "by hand". Similarly, claim (ii) is proved in Section 5.2.2. More generally, the proof of Theorem 2.7.4 is based on a general technique developed in Chapter 4. The unitary maps $\mathcal{F}_0$ and $\mathcal{F}_\infty$ are called *generalized Bloch-Floquet transforms*.

The *uniqueness* claimed in (i) of Theorem 2.7.4 means that any other bundle representation $\tilde{\mathcal{F}}_0$ subordinate to an irreducible physical frame with wandering property is unitarily equivalent to $\mathcal{F}_0$ and selects a vector bundle $\mathscr{E}_0'$ which is isomorphic to $\mathscr{E}_0$. In other words, for the Hofstadter representation there exists a unique bundle decomposition (up to equivalences) subordinate to an irreducible physical frame with wandering property. The same holds true for the Harper representation. This form of uniqueness for the bundle decomposition $\mathcal{F}_0$ (resp. $\mathcal{F}_\infty$) implies that the geometry of the Hofstadter (resp. Harper) vector bundle is a fingerprint for the physics (observables + symmetries) of the Hofstadter (resp. Harper) regime.

In the Hofstadter case (c.f. Section 5.2.1) the structure of $\iota : \mathscr{E}_0 \to \mathbb{T}^2$ is defined by means of a system of N orthonormal sections $\boldsymbol{\zeta}_0 := \{\zeta_0^0, \dots, \zeta_0^{N-1}\}$, with $\zeta_0^j : \mathbb{R}^2 \to \Phi_0^*$ (a suitable "ambient" vector space), subjected to *periodic* conditions

$$\boldsymbol{\zeta}_0(k_1 + 2\pi, k_2) = \boldsymbol{\zeta}_0(k_1, k_2 + 2\pi) = \boldsymbol{\zeta}_0(k_1, k_2). \tag{2.41}$$

Equations (2.41) allows $\boldsymbol{\zeta}_0$ to be a *global frame of sections* over the base manifold $\mathbb{T}^2$, hence the resulting vector bundle is trivial, i.e. $\mathscr{E}_0 \simeq \mathbb{T}^2 \times \mathbb{C}^N$. The triviality of the vector bundle implies automatically the vanishing of all the Chern classes and related Chern numbers (Husemoller 1994, Proposition 4.1). The triviality of the vector bundle $\mathscr{E}_0$ implies also the triviality of endomorphism bundle $\mathrm{End}(\mathscr{E}_0)$ and in fact the sections $\mathcal{U}_0(\cdot)$ and $\mathcal{V}_0(\cdot)$ are globally defined, as showed by equation (2.38).

The structure of the Harper vector bundle $\iota : \mathscr{E}_\infty \to \mathbb{T}^2$ is defined by a system of N orthonormal sections $\boldsymbol{\zeta}_\infty := \{\zeta_\infty^0, \dots, \zeta_\infty^{N-1}\}$, with $\zeta_\infty^j : \mathbb{R}^2 \to \Phi_\infty^*$ (a suitable "ambient" vector space), subjected to *covariance* conditions

$$\boldsymbol{\zeta}_\infty(k_1 + 2\pi, k_2) = \mathbb{G}(k_2) \cdot \boldsymbol{\zeta}_\infty(k_1, k_2), \qquad \boldsymbol{\zeta}_\infty(k_1, k_2 + 2\pi) = \boldsymbol{\zeta}_\infty(k_1, k_2) \tag{2.42}$$

where the unitary $N \times N$ matrix $\mathbb{G}(k_1, k_2)$ is defined by

$$\mathbb{G}(k_1,k_2) := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \mathrm{e}^{ik_2} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{2.43}$$

Equations (2.42) show that ζ_∞ defines a frame of sections on the base manifold $\mathbb{T}^2$ which is globally defined only in the k_2-direction, but which is twisted by $\mathbb{G}(\cdot)$ in the k_1-direction. The resulting vector bundle $\mathscr{E}_\infty$ is non-trivial with (non-trivial) transition functions defined by means of the matrix-valued function $\mathbb{G}(\cdot)$ (c.f. Section 5.2.2). The Harper vector bundle admits a curvature $K_{\mathrm{Har}} := \frac{-i}{N}\mathbb{1}_N \; dk_1 \wedge dk_2$, called *Harper curvature*. It follows that the first Chern class of the Harper vector bundle is $c_1(\mathscr{E}_\infty) = \frac{1}{(2\pi)^2}\, dk_1 \wedge dk_2$ which implies $C_1(\mathscr{E}_\infty) = 1$.

Any abstract projection $\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_\theta)$ defines a projection $P_0 := \pi_0(\mathfrak{p})$ in the Hofstadter representation and a projection $P_\infty := \pi_\infty(\mathfrak{p})$ in the Harper representation. Let $\theta = {}^M/_N$. Theorem 2.7.4 assures that $P_0(\cdot) := \mathscr{F}_0 \pi_0(\mathfrak{p}) \mathscr{F}_0{}^{-1}$ is an orthogonal projection in $\Gamma(\mathrm{End}(\mathscr{E}_0))$ and $P_\infty(\cdot) := \mathscr{F}_\infty \pi_\infty(\mathfrak{p}) \mathscr{F}_\infty{}^{-1}$ is an orthogonal projection in $\Gamma(\mathrm{End}(\mathscr{E}_\infty))$. According to Lemma 2.7.3, $P_0(\cdot)$ defines a vector subbundle of the Hofstadter vector bundle, denoted by $\mathscr{L}_0(\mathfrak{p}) \subset \mathscr{E}_0$, and similarly $P_\infty(\cdot)$ defines a vector subbundle of the Harper vector bundle, denoted by $\mathscr{L}_\infty(\mathfrak{p}) \subset \mathscr{E}_\infty$. Then, to any projection in $\mathfrak{A}_{M/N}$ we can associate in two ways a vector bundle over the base manifold $\mathbb{T}^2$. This *"two-fold way"* is summarized by the following diagram

$$\begin{array}{ccccccccc} & & P_0 & \xrightarrow{\mathscr{F}_0 \dots \mathscr{F}_0^{-1}} & P_0(\cdot) & \xrightarrow{\mathrm{Im}} & \mathscr{L}_0(\mathfrak{p}) & \xrightarrow{C_1} & C_0(\mathfrak{p}) \\ & \overset{\pi_0}{\nearrow} & & & & & & & \\ \mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_\theta) & & \mathscr{B}(\mathcal{H}) & & \Gamma(\mathrm{End}(\mathscr{E})) & & \mathscr{E} \to \mathbb{T}^2 & & \mathbb{Z} \\ & \underset{\pi_\infty}{\searrow} & & & & & & & \\ & & P_\infty & \xrightarrow[\mathscr{F}_\infty \dots \mathscr{F}_\infty^{-1}]{} & P_\infty(\cdot) & \xrightarrow[\mathrm{Im}]{} & \mathscr{L}_\infty(\mathfrak{p}) & \xrightarrow[C_1]{} & C_\infty(\mathfrak{p}). \end{array} \tag{2.44}$$

The last arrows the diagram (2.44), denoted with C_1, associate the first Chern number to vector bundles over $\mathbb{T}^2$. We use the short notation $C_\sharp(\mathfrak{p}) := C_1(\mathscr{L}_\sharp(\mathfrak{p}))$ to denote the first Chern number of the vector bundle $\mathscr{L}_\sharp(\mathfrak{p})$, with $\sharp = 0, \infty$.

Which kind of relation there exists between the two vector bundles $\mathscr{L}_0(\mathfrak{p})$ and $\mathscr{L}_\infty(\mathfrak{p})$ associated to the same $\mathfrak{p}$? Can such a relation imply any kind of dependence between the related Chern numbers?

Our next goal is to provide an answer to these questions.

2.8 From geometric duality to TKNN-equations

In this section we present one of the main results of this thesis which address the question stated at the end of Section 2.7.

We denote by $\tilde{\iota} : \mathscr{I} \to \mathbb{T}^2$ the *determinant line bundle* associated to the Harper vector bundle $\iota : \mathscr{E}_\infty \to \mathbb{T}^2$, namely $\mathscr{I}$ is the rank 1 Hermitian vector bundle with transition functions given by the determinant of the transition functions of $\mathscr{E}_\infty$. One has that $C_1(\mathscr{I}) = C_1(\mathscr{E}_\infty) = 1$ (c.f. Appendix C and equation (C.9)).

THEOREM 2.8.1 (Geometric duality). *Let $\mathfrak{A}_{M/N}$ be the rational NCT-algebra (as in Convention 2.4.3) and $\mathfrak{p} \in \text{Proj}(\mathfrak{A}_{M/N})$.*

(i) *The vector bundles $\mathscr{L}_0(\mathfrak{p})$ and $\mathscr{L}_\infty(\mathfrak{p})$ have the same rank given by the number* $\text{Rk}(\mathfrak{p})$ *which is the dimension of $\mathfrak{p}$ (c.f. equation* (2.34)*).*

(ii) *Let $f_{(n,m)} : \mathbb{T}^2 \to \mathbb{T}^2$, with $n, m \in \mathbb{Z}$, be the continuous map defined by*

$$f_{(n,m)}(e^{ik_1}, e^{ik_1}) = (e^{ink_1}, e^{imk_2}). \tag{2.45}$$

Then

$$f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p}) \;\simeq\; f^*_{(-M,1)}\mathscr{L}_0(\mathfrak{p}) \otimes \mathscr{I} \tag{2.46}$$

where $f^*_{(n,m)}\mathscr{L}_\sharp(\mathfrak{p})$, $\sharp = 0, \infty$, denotes the pullback of the vector bundle $\mathscr{L}_\sharp(\mathfrak{p})$ induced by the map $f_{(n,m)}$, and $\mathscr{I}$ is the determinant bundle of $\mathscr{E}_\infty$ (c.f. Appendix C).

The proof of the duality relation (2.46) is postponed to Section 5.3. Claim (i) follows from (ii) and Proposition 5.2.2.

The *geometric duality* between the vector bundles $\mathscr{L}_0(\mathfrak{p})$ and $\mathscr{L}_\infty(\mathfrak{p})$ is the core of a geometric derivation of the TKNN-equations.

COROLLARY 2.8.2 (Generalized TKNN-equations). *Let $\mathfrak{A}_{M/N}$ be the rational NCT-algebra (according to Convention 2.4.3). Any $\mathfrak{p} \in \text{Proj}(\mathfrak{A}_{M/N})$ defines a* TKNN-equation

$$N\ C_\infty(\mathfrak{p}) + M\ C_0(\mathfrak{p}) = \text{Rk}(\mathfrak{p}) \tag{2.47}$$

which relates the Chern numbers $C_\infty(\mathfrak{p})$ and $C_0(\mathfrak{p})$.

Proof. The derivation of equation (2.47) from the duality formula (2.46) is straightforward. Observing that isomorphic vector bundles have same characteristic classes, one has

$$c_1(f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p})) = c_1(f^*_{(-M,1)}\mathscr{L}_0(\mathfrak{p}) \otimes \mathscr{I}).$$

Applying formula (C.10) to the right-hand side, one has

$$c_1(f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p})) = c_1(f^*_{(-M,1)}\mathscr{L}_0(\mathfrak{p})) + \text{Rk}(\mathfrak{p})\ c_1(\mathscr{I}), \tag{2.48}$$

since the rank of $\mathscr{I}$ is 1 and the rank of $f^*_{(-M,1)}\mathscr{L}_0(\mathfrak{p})$ coincides with the rank of $\mathscr{L}_0(\mathfrak{p})$ which is $\text{Rk}(\mathfrak{p})$.

To complete the proof we need to integrate over $\mathbb{T}^2$ both sides of (2.48). From the left-hand side one obtains

$$C_1(f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p})) = \int_{\mathbb{T}^2} c_1(f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p})) \stackrel{I}{=} \int_{\mathbb{T}^2} f^*_{(N,1)}c_1(\mathscr{L}_\infty(\mathfrak{p})) \stackrel{II}{=} NC_\infty(\mathfrak{p})$$

where equality I follows from the *functoriality* of the Chern class (i.e. $c_1(f^*\mathscr{E}) = f^*c_1(\mathscr{E})$) and equality II follows from Lemma C.0.2. The same computation for the right-hand side shows that $C_1(f^*_{(-M,1)}\mathscr{L}_\infty(\mathfrak{p})) = -MC_0(\mathfrak{p})$. Finally, $C_1(\mathscr{I}) = C_1(\mathscr{E}_\infty) = 1$. ■

If $\mathfrak{p} = \mathbb{1}$, then $\mathscr{L}_0(\mathbb{1}) = \mathscr{E}_0$, $\mathscr{L}_\infty(\mathbb{1}) = \mathscr{E}_\infty$, Rk$(\mathfrak{p}) = N$ and equation (2.47) is compatible with the results obtained in Theorem 2.7.4 , namely $C_0(\mathbb{1}) := C_1(\mathscr{E}_0) = 0$ and $C_\infty(\mathbb{1}) := C_1(\mathscr{E}_\infty) = 1$.

Equation (2.47) defines a geometric generalized version of the TKNN-system (2.35). With the notation introduced in Section 2.5, let $\mathfrak{P}_j \in$ Proj$(\mathfrak{A}^\infty_{M/N})$ the gap projection corresponding to the j-th gap G_j ($0 \leqslant j \leqslant N_0$) in the spectrum of the universal Hofstadter operator $\mathfrak{h}_{M/N} \in \mathfrak{A}^\infty_{M/N}$. With the identification $t(G_j) := C_\infty(\mathfrak{P}_j)$ and $s(G_j) := C_0(\mathfrak{P}_j)$ and using the labeling $\mathscr{N}(G_j) :=$ Rk$(\mathfrak{P}_j)$ defined by (2.34), then equations (2.47) reduce exactly to the system of Diophantine equations (2.35) proposed by Thouless *et al.* in (Thouless et al. 1982).

2.9 A “non-commutative look” to TKNN-equations

The geometric generalization of the TKNN-equations (2.47) has been derived under the assumption of rationality for the deformation parameter $\theta = M/N$. A natural question is whether it is possible to give any meaning to (2.47) also in the irrational case $\theta \in \mathbb{R}\setminus\mathbb{Q}$. To provide an answer we need to introduce new structures on the NCT-algebra.

Canonical derivations and universal Chern number

In view of Remark 2.3.1, the map $\rho_z(\mathfrak{u}) = z_1\mathfrak{u}$, $\rho_z(\mathfrak{v}) = z_2\mathfrak{v}$ extends to an automorphism for any $z = (z_1, z_2) \in \mathbb{T}^2$. The map $\mathbb{T}^2 \ni z \mapsto \rho_z \in$ Aut$(\mathfrak{A}_\theta)$ defines a *strongly continuous action* of the commutative group $\mathbb{T}^2$ on $\mathfrak{A}_\theta$. Let $z(k) = (\mathrm{e}^{ik_1}, \mathrm{e}^{ik_2}) \in \mathbb{T}^2$. For any $\mathfrak{a}$ in the dense set $\mathfrak{L}_\theta$ one defines $\partial\!\!\!\mathscr{T}_j\,(\mathfrak{a}) \in \mathfrak{L}_\theta$, $j = 1, 2$, as

$$\partial\!\!\!\mathscr{T}_j\,(\mathfrak{a}) := 2\pi\, \frac{d}{dk_j}\rho_{z(k)}(\mathfrak{a})\Big|_{k=0}. \tag{2.49}$$

It follows that $\partial\!\!\!\mathscr{T}_1$ and $\partial\!\!\!\mathscr{T}_2$: (i) are $\mathbb{C}$-linear $\mathfrak{A}_\theta$-valued maps; (ii) satisfy the *Leibniz's law* $\partial\!\!\!\mathscr{T}_j\,(\mathfrak{a}\mathfrak{b}) = \partial\!\!\!\mathscr{T}_j\,(\mathfrak{a})\,\mathfrak{b} + \mathfrak{a}\,\partial\!\!\!\mathscr{T}_j\,(\mathfrak{b})$; (iii) are *symmetric*, $\partial\!\!\!\mathscr{T}_j\,(\mathfrak{a}^*) = \partial\!\!\!\mathscr{T}_j\,(\mathfrak{a})^*$; (iv) *commute*, $\partial\!\!\!\mathscr{T}_1 \circ \partial\!\!\!\mathscr{T}_2 = \partial\!\!\!\mathscr{T}_2 \circ \partial\!\!\!\mathscr{T}_1$. We refer to $\partial\!\!\!\mathscr{T}_1$ and $\partial\!\!\!\mathscr{T}_2$ as *canonical derivations*. Let $(\mathfrak{u}, \mathfrak{v})$ be a frame for $\mathfrak{A}_\theta$. A simple computation shows that

$$\partial\!\!\!\mathscr{T}_1\,(\mathfrak{u}^n\mathfrak{v}^m) := i2\pi n\; \mathfrak{u}^n\mathfrak{v}^m, \qquad \partial\!\!\!\mathscr{T}_2\,(\mathfrak{u}^n\mathfrak{v}^m) := i2\pi m\; \mathfrak{u}^n\mathfrak{v}^m \tag{2.50}$$

which proves that $\partial\!\!\!\mathscr{T}_1$ and $\partial\!\!\!\mathscr{T}_2$ are unbounded. The maximal invariant domain of definition for $\partial\!\!\!\mathscr{T}_1$ and $\partial\!\!\!\mathscr{T}_2$ coincides with the smooth algebra $\mathfrak{A}^\infty_\theta$

The *universal Chern number* is the map

$$\mathfrak{C}_1 : \mathrm{Proj}(\mathfrak{A}_\theta^\infty) \to \mathbb{Z}$$

defined by

$$\mathfrak{C}_1(\mathfrak{p}) := \frac{1}{i2\pi} ⨍ (\mathfrak{p}[\partial_1 (\mathfrak{p}); \partial_2 (\mathfrak{p})]) \tag{2.51}$$

where $[\partial_1 (\mathfrak{p}); \partial_2 (\mathfrak{p})] := \partial_1 (\mathfrak{p})\, \partial_2 (\mathfrak{p}) - \partial_2 (\mathfrak{p})\, \partial_1 (\mathfrak{p})$. Equation (2.51) is called *Connes formula*. The integrality of $\mathfrak{C}_1$ is proved in (Connes 1980), but more details can be found in (Connes 1994, Chapter III).

The relevance of the map $\mathfrak{C}_1$ for our purposes is related to the following result (c.f. Proposition 5.2.2):

PROPOSITION 2.9.1. *For any* $\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_{M/N}^\infty)$ *one has* $C_0(\mathfrak{p}) = \mathfrak{C}_1(\mathfrak{p})$.

Constraint for the TKNN-equations of the Hofstadter operator

The first interesting consequence of Proposition 2.9.1 concerns the bound (2.36) which completes the TKNN-system proposed in (Thouless et al. 1982). Let $\mathfrak{P}_j \in \mathrm{Proj}(\mathfrak{A}_{M/N}^\infty)$ the gap projection corresponding to the j-th gap G_j $(0 \leqslant j \leqslant N_0)$ in the spectrum of the universal Hofstadter operator $\mathfrak{h}_{M/N} \in \mathfrak{A}_{M/N}^\infty$. With the identification $s(G_j) = C_0(\mathfrak{P}_j)$, Proposition 2.9.1 leads to rewrite (2.36) as $|\mathfrak{C}_1(\mathfrak{P}_j)| < {}^N\!/2$. The latter bound has been proved in (Choi et al. 1990, Corollary 3.4). Then, Proposition 2.9.1 provides a purely geometric justification for the bound (2.36) and, together with Corollary 2.8.2, completes our purpose to find a rigorous geometric derivation of the result claimed in (Thouless et al. 1982). However, the bound $|\mathfrak{C}_1(\mathfrak{P}_j)| < {}^N\!/2$ holds true only for the spectral projections of the universal Hofstadter operator. In principle, the choice of a different operator leads to a different bound and each bound depends on the form of the related operator.

Generalization to irrational values of the deformation parameter

By means of the equality $C_0 = \mathfrak{C}_1$, the definition $\mathrm{Rk}(\mathfrak{p}) := N ⨍ (\mathfrak{p})$ and $\theta = {}^M\!/N$, one can rewrite the TKNN-equation (2.47) in the following form

$$C_\infty(\mathfrak{p}) \, = ⨍ (\mathfrak{p}) - \theta\, \mathfrak{C}_1(\mathfrak{p}). \tag{2.52}$$

Formally, equation (2.52) contains only quantity defined in terms of the abstract algebra $\mathfrak{A}_\theta$, hence it has a perfect meaning for any $\theta \in \mathbb{R}$.

The above formula has an interesting application. Let $\mathfrak{d}_\theta \in \mathfrak{A}_\theta^\infty$ be a self-adjoint element (not necessarily the universal Hofstadter operator $\mathfrak{h}_\theta$) and $\mathfrak{P}_G$ the gap projection associated to the gap $G \subset \mathbb{R} \setminus \sigma(\mathfrak{d}_\theta)$ and defined via Riesz formula according to Lemma 2.4.2. If $\theta = {}^M\!/N$, then the integers $C_\infty(\mathfrak{P}_G)$ and $C_0(\mathfrak{P}_G)$ defines two distinct numerical labels for the gap G. An interesting question is whether these labels, initially defined for rational values of θ, are stable for small perturbations of the deformation parameter

θ. A good criterion of stability is given in the following terms. Consider a family of self-adjoint elements $\mathfrak{d}_\theta := f(\mathfrak{u},\mathfrak{v})$ with $(\mathfrak{u},\mathfrak{v})$ a frame for $\mathfrak{A}_\theta$, $f \in \mathbb{C}^\infty(\mathbb{T}^2)$ a fixed real smooth functions and $\theta \in I$, where I is an interval of $\mathbb{R}$. The functional expression of $\mathfrak{d}_\theta$ is fixed and it depends on θ only through the fundamental commutation relation between the generators of $\mathfrak{A}_\theta$. Moreover $\mathfrak{d}_\theta \in \mathfrak{A}_\theta^\infty$ for any $\theta \in I$. Suppose that $\lambda_G \in \mathbb{R} \setminus \sigma(\mathfrak{d}_\theta)$ for any $\theta \in I$ and denote by $\mathfrak{P}_G^\theta \in \mathrm{Proj}(\mathfrak{A}_\theta^\infty)$ the related gap projection corresponding to $(-\infty, \lambda_G]$. The functions $\theta \mapsto \mathfrak{C}_1(\mathfrak{P}_G^\theta)$ is constant in I (Boca 2001, Proposition 11.11). Moreover, as a consequence of the description of the group $K_0(\mathfrak{A}_\theta)$ given in (Pimsner and Voiculescu 1980), one deduces that

$$⨍\, (\mathfrak{P}_G^\theta) = m(\mathfrak{P}_G^\theta) + \theta\mathfrak{C}_1(\mathfrak{P}_G^\theta) \tag{2.53}$$

(Boca 2001, p. 145), where the integer $m(\cdot) \in \mathbb{Z}$ is uniquely determined by the condition $0 \leqslant ⨍\,(\cdot) \leqslant 1$. Since $\mathfrak{C}_1(\mathfrak{P}_G^\theta)$ is constant in I, one infers that also $m(\mathfrak{P}_G^\theta)$ is constant under small perturbations of θ. Since $C_0 = \mathfrak{C}_1$ in view of Proposition 2.9.1 and $C_\infty = ⨍ - \theta\mathfrak{C}_1 = m$ (comparison between (2.52) and (2.53)), it follows that the integers C_0 and C_∞ are "stable labels" for "stable gaps". In other words equation (2.52) is meaningful also for irrational values of the deformation parameter provided that there exists a stable "open" gap under small perturbation of θ.

Cohomological interpretation

A final remark concerns the link between the "non-commutative version" of the TKNN-equation (2.52) and the geometry of the NCT-algebra. Indeed equation (2.52) is related to the *periodic cyclic cohomology* of the algebra $\mathfrak{A}_\theta^\infty$ which is the $\mathbb{Z}_2$-graded group $PH^\bullet(\mathfrak{A}_\theta^\infty) := PH^{\mathrm{ev}}(\mathfrak{A}_\theta^\infty) \oplus PH^{\mathrm{od}}(\mathfrak{A}_\theta^\infty)$ with $PH^\sharp(\mathfrak{A}_\theta^\infty) \simeq \mathbb{C}^2$, $\sharp = \mathrm{ev}, \mathrm{od}$ (Connes 1994) (c.f. Appendix B.4 for the basic definitions). Since the two independent generators of $PH^{\mathrm{ev}}(\mathfrak{A}_\theta^\infty)$ are exactly the noncommutative trace $⨍$ and the universal Chern number $\mathfrak{C}_1$, it follows that the integer valued function C_∞ is an element of the even part of the periodic cyclic cohomology group of $\mathfrak{A}_\theta^\infty$.

2.10 Prospectives and open problems

As usual, the solution of a problem opens the way to new speculations and stimulating challenges. This is true also for this thesis. During our investigation, we collected interesting problems closely related with the arguments of this thesis, that we have not had the time to examine. At the same time, we realized that some of our results could have relevant generalizations by means of interactions with other mathematical fields. We sketch below a short list containing the main open problems and the more stimulating prospectives of further generalizations.

- Proof of the *Kubo-Chern equivalence* for the Harper and Hofstadter regimes (c.f. Section 1.3), supposedly by means of SAPT techniques.

- Derivation of the effective model for the Bloch-Landau Hamiltonian H_{BL} (2.1) with a perturbed magnetic field $B+\delta B$ in the limit $\delta B \to 0$. This problem is related with the non-triviality of the Bloch vector bundle in view of the absence of time-reversal symmetry.

- Derivation of the effective models for the Harper and Hofstadter regimes taking into account also of the effects due to disorder (random or aperiodic potential). The first step should be the generalization of SAPT to the cases of aperiodic or random potentials using non-commutative tools *à la Bellissard* (Bellissard et al. 1994).

- Generalization of the TKNN-equations to higher dimensional Harper representations (work in progress with Giovanni Landi) and possible applications to the coupling of Landau bands in presence of a periodic vector potential (c.f. Section 3.4.8) and to the study of hexagonal lattices (graphene) (Bellissard et al. 1991).

Chapter 3

Derivation of Harper and Hofstadter models

Pour bien savoir les choses, il en faut savoir le détail; et comme il est presque infini, nos connaissances sont toujours superficielles et imparfaites.

(To understand matters rightly we should understand their details; and as that knowledge is almost infinite, our knowledge is always superficial and imperfect.)

François de La Rochefoucauld,
Réflexions ou sentences et maximes morales, 1665-1678

Abstract

Some relevant transport properties of solids do not depend only on the spectrum of the electronic Hamiltonian, but on finer properties preserved only by unitary equivalence, the most striking example being the conductance. When interested in such properties, and aiming to a simpler model, it is mandatory to check that the simpler effective Hamiltonian is approximately unitarily equivalent to the original one, in the appropriate asymptotic regime. In this chapter, we prove that the Hamiltonian for the QHE is approximately unitarily equivalent to a Hofstadter-like (resp. Harper-like) Hamiltonian, in the limit of weak (resp. strong) magnetic field. Section 3.1 provides a brief compendium of the SAPT "philosophy" while Section 3.2 aims to fix the mathematical description of the model. Section 3.3 is devoted to the adiabatic theory in the Hofstadter regime *while the Harper regime is expounded in Section 3.4. Finally, in Section 3.4.8 we show that an additional periodic magnetic potential induces in the Harper regime a non-trivial coupling of the Landau bands.*

3.1 An insight to space-adiabatic perturbation theory

The results obtained in this chapter are based on the observation that both the Hofstadter and the Harper regime are space-adiabatic limits, and can be treated in the framework of *space-adiabatic perturbation theory*, (SAPT) (Panati et al. 2003b, Panati et al. 2003a), see also (Teufel 2003). As for the Hofstadter regime, the proof follows ideas similar to the ones in (Panati et al. 2003a). Our generalization allows however to consider a constant magnetic field (while in (Panati et al. 2003a) the vector potential is assumed in $C^\infty_{\mathrm{b}}(\mathbb{R}^d)$) and to include a periodic vector potential. Moreover the proof extends the one in (Panati et al. 2003a), in view of the use of the special symbol classes defined in Section 3.3.4. On the contrary, from the discussion of the Harper regime $\hbar_B \to 0$ some new mathematical problems emerge. Then, although the "philosophy" of the proof of Theorem 3.4.8 is of SAPT-type, the technical part is new as it will be explained in Section 3.4. Notice that the regime of weak magnetic field can also be conveniently approached by using the *magnetic Weyl quantization* (Măntoiu and Purice 2004, Măntoiu et al. 2005, Iftimie et al. 2007, Iftimie et al. 2009), a viewpoint which is investigated in (De Nittis and Lein 2010).

For the sake of completeness, we summarize some salient aspects of the SAPT. We refer to (Teufel 2003) for a complete exposition. Let H be the Hamiltonian of a generic physical system which acts on the total (or physical) Hilbert space $\mathcal{H}_{\text{phy}}$. For the SAPT to be applicable, three important ingredients needs:

(i) a distinction between *fast* and *slow degrees of freedom* which is mathematically expressed by a unitary decomposition of the physical space $\mathcal{H}_{\text{phy}}$ into a product space $\mathcal{H}_{\text{s}} \otimes \mathcal{H}_{\text{f}}$ (or, more generally, a direct integral), the first factor being the space of slow degrees of freedom and the second the space of fast degrees of freedom; $\mathcal{H}_{\text{s}} \cong L^2(\mathcal{M})$ for suitable measure space $\mathcal{M}$ is also required;

(ii) a dimensionless *adiabatic parameter* $\varepsilon \ll 1$ that quantifies the separation of scales between the fast and slow degrees of freedom and which measures how far are the slow degrees of freedom to be "classical" in terms of some process of quantization;

(ii) a *relevant part of the spectrum* for the fast dynamics which remains separated from the rest of the spectrum under the perturbation caused by the slow degrees of freedom.

3.2 Description of the model

The Hamiltonian (2.1) describes the dynamics of particle with mass m and charge q which interacts with the ionic structure of a two dimensional crystal and with an external orthogonal uniform magnetic field. A more general model is provided by the operator

$$\mathsf{H}_{\text{BL}} := \frac{1}{2m}\left[-i\hbar\nabla_r - \frac{q}{c}\mathbf{A}_\Gamma(r) - \frac{q}{c}\mathbf{A}(r)\right]^2 + \mathsf{V}_\Gamma(r) + q\,\Phi(r) \tag{3.1}$$

still called *Bloch-Landau Hamiltonian* and, with an abuse of notation, still denoted with the same symbol used in (2.1). The vector-valued function $\mathbf{A} := (\mathsf{A}_1, \mathsf{A}_2)$ is a vector potential corresponding to an (orthogonal) *external* magnetic field $\mathbf{B} = \nabla_r \wedge \mathbf{A} = (\partial_1\mathsf{A}_2 - \partial_2\mathsf{A}_1)\ e_\perp$, Φ is a scalar potential corresponding to a (parallel) *external* electric field $\mathbf{E} = -\nabla_r\Phi$ and $\mathbf{A}_\Gamma$ and V_Γ are *internal* periodic potentials which describe the electromagnetic interaction with the ionic cores of the crystal lattice. The external vector potential is assumed to have the following structure

$$\mathbf{A}(r) = \mathbf{A}_0(r) + \mathbf{A}_B(r), \tag{3.2}$$

where $\mathbf{A}_0$ is a bounded function and $\mathbf{A}_B$ describes a uniform orthogonal magnetic field of strenght B, i.e. in the *symmetric gauge*

$$\mathbf{A}_B(r) = \frac{B}{2}\,e_\perp \wedge r = \left(-\frac{B}{2}r_2, \frac{B}{2}r_1\right), \qquad \nabla_r \wedge \mathbf{A}_B = B\,e_\perp, \qquad \nabla_r \cdot \mathbf{A}_B = 0. \tag{3.3}$$

The evolution of the system is prescribed by the Schrödinger equation

$$i\hbar \frac{d}{ds}\psi(r,s) = \mathbf{H}_{\mathrm{BL}}\, \psi(r,s), \tag{3.4}$$

where s corresponds to a *microscopic* time-scale.

Mathematical description of the crystal structure

The periodicity of the crystal is described by a two dimensional *lattice* $\Gamma \subset \mathbb{R}^2$ (i.e. a discrete subgroup of maximal dimension of the addictive group $\mathbb{R}^2$), thus $\Gamma \simeq \mathbb{Z}^2$. Let $\{a,b\} \subset \mathbb{R}^2$ be two generators of Γ, i.e.

$$\Gamma = \{\gamma \in \mathbb{R}^2 \ : \ \gamma = n_1 a + n_2 b, \quad n_1, n_2 \in \mathbb{Z}\}.$$

The *fundamental* or *Voronoi cell* of Γ is $M_\Gamma := \{r \in \mathbb{R}^2 \mid r = l_1\, a + l_2\, b, \quad l_1, l_2 \in [0,1]\}$ and its area is given by $\Omega_\Gamma = |a \wedge b|$. We fix the orientation of the lattice in such a way that $\Omega_\Gamma = (a_1 b_2 - a_2 b_1) > 0$. We say that a function $f_\Gamma : \mathbb{R}^2 \to \mathbb{C}$ is Γ-*periodic* if $f_\Gamma(r+\gamma) = f_\Gamma(r)$ for all $\gamma \in \Gamma$ and all $r \in \mathbb{R}^2$. The electrostatic and magnetostatic crystal potentials V_Γ and A_Γ are assumed to be Γ-periodic according to the previous definition.

An important notion is that of *dual lattice* Γ^* which is the set of the vectors $\gamma^* \in \mathbb{R}^2$ such that $\gamma^* \cdot \gamma \in 2\pi\mathbb{Z}$ for any $\gamma \in \Gamma$. Let $\{a^*, b^*\} \subset \mathbb{R}^2$ be defined by the relations $a^* \cdot a = b^* \cdot b = 1$ and $a^* \cdot b = b^* \cdot a = 0$; these vectors are the generators of the lattice Γ^*, i.e.

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^2 \ : \ \gamma^* = m_1\, 2\pi a^* + m_2\, 2\pi b^*, \quad m_1, m_2 \in \mathbb{Z}\}.$$

The *Brillouin zone*

$$M_{\Gamma^*} := \{k \in \mathbb{R}^2 \mid k = k_1\, a^* + k_2\, b^*, \quad k_1, k_2 \in [0, 2\pi]\}$$

is the fundamental cell of the dual lattice Γ^*. The explicit expressions for the dual generators $\{a^*, b^*\}$ in terms of the basis $\{a,b\}$ is

$$a^* = \frac{e_\perp \wedge b}{|a \wedge b|} = \frac{1}{\Omega_\Gamma}(b_2, -b_1), \qquad b^* = -\frac{e_\perp \wedge a}{|a \wedge b|} = \frac{1}{\Omega_\Gamma}(-a_2, a_1). \tag{3.5}$$

It follows from (3.5) that the surface of the Brillouin zone is $\Omega_{\Gamma^*} = (2\pi)^2 |a^* \wedge b^*| = {}^{(2\pi)^2}/_{\Omega_\Gamma}$.

Given a Γ-periodic function f_Γ, we denote its Fourier decomposition as

$$f_\Gamma(r) = \sum_{\gamma^* \in \Gamma^*} f(\gamma^*)\, \mathrm{e}^{i\gamma^* \cdot r} = \sum_{m_1, m_2 \in \mathbb{Z}} f_{m_1, m_2}\, \mathrm{e}^{i2\pi(m_1\, a^* + m_2\, b^*)\cdot r}. \tag{3.6}$$

A $\mathbb{Z}^2$-*periodic* function $f : \mathbb{R}^2 \to \mathbb{C}$ is a function periodic with respect to an orthonormal lattice, namely such that $f(x_1 + 1, x_2) = f(x_1, x_2 + 1) = f(x_1, x_2)$ for any $x_1, x_2 \in \mathbb{R}$. If one changes the variables as $x_1 := a^* \cdot r$ and $x_2 := b^* \cdot r$ one has that $f_\Gamma(r) := f(a^* \cdot r, b^* \cdot r)$ is Γ-periodic in r. Every Γ-periodic function can be obtained in this way.

Assumptions on the regularity of the potentials and self-adjointness

We denote by $C_{\mathrm{b}}^{n}(\mathbb{R}^2, \mathbb{R})$ the space of real-valued n-times differentiable functions (smooth functions if $n = \infty$) with continuous and bounded derivatives up to order n. Concerning the internal potentials A_Γ and V_Γ we need to assume that:

ASSUMPTION 3.2.1 (Internal potentials, strong form)**.** *The Γ-periodic potential V_Γ and the two components of the Γ-periodic vector potential A_Γ are functions of class $C_{\mathrm{b}}^{\infty}(\mathbb{R}^2, \mathbb{R})$.*

Sometime we can relax the above assumption and we can consider the weaker version:

ASSUMPTION 3.2.2 (Internal potentials, weak form)**.** *The two components of the Γ-periodic vector potential A_Γ are in $C_{\mathrm{b}}^{1}(\mathbb{R}^2, \mathbb{R})$. The Γ-periodic potential V_Γ verifies the condition $\int_{M_\Gamma} |\mathsf{V}_\Gamma(r)|^2 \, d^2 r < +\infty$.*

Assumption 3.2.2 implies that V_Γ is *uniformly locally* L^2 and this implies also that V_Γ is infinitesimally bounded with respect to $-\Delta_r$ (Reed and Simon 1978, Theorem XIII.96). Concerning the external potentials A and Φ, we need to assume that:

ASSUMPTION 3.2.3 (External potentials)**.** *The scalar potential Φ is of class $C_{\mathrm{b}}^{\infty}(\mathbb{R}^2, \mathbb{R})$. The vector potential A consists of a linear term A_B of the form (3.3) plus a bounded term A_0 which is of class $C_{\mathrm{b}}^{\infty}(\mathbb{R}^2, \mathbb{R})$.*

When the external potentials A and Φ vanish, the Bloch-Landau Hamiltonian (3.1) reduces to the *periodic Hamiltonian* (or *Bloch Hamiltonian*)

$$\mathsf{H}_{\mathrm{per}} := \frac{1}{2m}\left[-i\hbar\nabla_r - \frac{q}{c}\mathsf{A}_\Gamma(r)\right]^2 + \mathsf{V}_\Gamma(r). \tag{3.7}$$

The domains of self-adjointness of H_{BL} and $\mathsf{H}_{\mathrm{per}}$ are described in the following proposition. Its proof, together with some basic notion about the *Sobolev space* $\mathcal{H}^2(\mathbb{R}^2)$ and the *magnetic-Sobolev space* $\mathcal{H}^2_{\mathrm{M}}(\mathbb{R}^2)$, is postponed to Section A.1.1.

PROPOSITION 3.2.4. *Let Assumptions 3.2.2 and 3.2.3 hold true. Then both H_{BL} and $\mathsf{H}_{\mathrm{per}}$ are essentially self-adjoint operators on $L^2(\mathbb{R}^2, d^2r)$ with common domain of essential self-adjointness the space of smooth functions with compact support $C_{\mathrm{c}}^{\infty}(\mathbb{R}^2, \mathbb{C})$. Moreover the domain of self-adjointness of $\mathsf{H}_{\mathrm{per}}$ is $\mathcal{H}^2(\mathbb{R}^2)$ while the domain of self-adjointness of H_{BL} is $\mathcal{H}^2_{\mathrm{M}}(\mathbb{R}^2)$.*

3.3 Space-adiabatic theory for the Hofstadter regime

3.3.1 Adiabatic parameter for weak magnetic fields

The SAPT for a Bloch electron developed in (Panati et al. 2003a) is based on the existence of a separation between the *microscopic space scale* fixed by the lattice spacing $\ell := \sqrt{\Omega_\Gamma}$, and a *macroscopic space scale* fixed by the scale of variation of the "slowly varying" external potentials. The existence of such a separation of scales is expressed by introducing a dimensionless parameter $\varepsilon \ll 1$ (*adiabatic parameter*) to control the scale of variation of the vector potential and the scalar potential Φ appearing in (3.1), namely by setting $\mathsf{A} = \mathsf{A}(\varepsilon r)$ and $\Phi = \Phi(\varepsilon r)$. In particular the external magnetic and electric fields are weak compared to the fields generated by the ionic cores.

It is useful to rewrite the (ε-dependent) Hamiltonian (3.1) in a dimensionless form. The microscopic unit of length being ℓ, we introduce the dimensionless position vector $x := r/\ell$ and the dimensionless gradient $\nabla_x = \ell \nabla_r$. Moreover, since the vector potential has the dimension of a length times a magnetic field, then $A(\varepsilon x) := \varepsilon/\ell B \, \mathsf{A}(\varepsilon \ell x)$ is a dimensionless function, with B a dimensional constant which fixes the order of magnitude of the magnetic field due to A. Similarly for A_Γ (with $\varepsilon = 1$). Factoring out the dimensional constants one finds

$$H_{\mathrm{BL}} := \frac{1}{\mathcal{E}_0}\mathsf{H}_{\mathrm{BL}} = \frac{1}{2}\left[-i\nabla_x - \underbrace{\frac{q\Omega_\Gamma B_\Gamma}{c\hbar}}_{=:\hbar_\Gamma^{-1}} A_\Gamma(x) - \iota_q \underbrace{\frac{|q|\Omega_\Gamma B}{c\hbar}}_{=\hbar_B^{-1}} \frac{1}{\varepsilon} A(\varepsilon x)\right]^2 + V_\Gamma(x) + \phi(\varepsilon x), \quad (3.8)$$

where $\mathcal{E}_0 := \hbar^2/m\Omega_\Gamma$ is the natural *unit of the energy* fixed by the problem, $V_\Gamma(x) := 1/\mathcal{E}_0 \, \mathsf{V}_\Gamma(\ell x)$ and $\phi(\varepsilon x) := q/\mathcal{E}_0 \, \Phi(\varepsilon \ell x)$ are both dimensionless quantities. The constant $\hbar_\Gamma$ will play no particular rôle in the rest of this paper, so it is reabsorbed into the definition of the dimensionless vector potential A_Γ, i.e. formally $\hbar_\Gamma = 1$.

Comparing the dimensional Hamiltonian (3.8) with the original Hamiltonian (3.1), or observing that the strenght of the magnetic field goes to zero (at least linearly) with ε, it is physically reasonable to estimate $\varepsilon \hbar_B \propto 1$. This is rigorously true in the case of a uniform external magnetic field.

The external force due to A and ϕ are of order of ε and therefore have to act over a time of order ε^{-1} to produce a finite change, which defines the macroscopic time-scale. The *macroscopic* (slow) dimensionless time-scale is fixed by $t := \varepsilon \frac{\mathcal{E}_0}{\hbar} s$ where s is the dimensional microscopic (fast) time-scale. With this change of scale the Schrödinger equation (3.4) reads

$$i\varepsilon \frac{d}{dt}\psi = H_{\mathrm{BL}}\,\psi \quad (3.9)$$

with H_{BL} given by equation (3.8).

REMARK 3.3.1. Observe that from the definition of the dimensionless periodic potential A_Γ and V_Γ it follows that they are periodic with respect to the transform $x \mapsto x + \gamma/\ell$. This means that A_Γ and V_Γ are periodic with respect to a "normalized" lattice whose fundamental cell has surface 1. ♦◊

3.3.2 Separation of scales: the Bloch-Floquet transform

To make explicit the presence of the linear term of the external vector potential, we can rewrite the (3.8) as follows

$$H_{\mathrm{BL}} = \frac{1}{2}\left[-i\nabla - A_\Gamma(x) - A_0(\varepsilon x) - \iota_q \frac{1}{2} e_\perp \wedge \varepsilon x\right]^2 + V_\Gamma(x) + \phi(\varepsilon x), \quad (3.10)$$

where the adiabatic parameter ε expresses the separation between the macroscopic length-scale, defined by the external potentials, and the microscopic length-scale, defined by the internal Γ-periodic potentials. The separation between slow and fast degrees of freedom can be expressed decomposing the physical Hilbert space $\mathcal{H}_{\text{phy}} = L^2(\mathbb{R}^2, d^2x)$ into a product of two Hilbert spaces or, more generally, into a direct integral. To this end, we use the Bloch-Floquet transform (Kuchment 1993). As in (Panati et al. 2003a) we define the *(modified) Bloch-Floquet transform* $\mathcal{Z}$ of a function $\psi \in \mathcal{S}(\mathbb{R}^2)$ to be

$$(\mathcal{Z}\psi)(k,\theta) := \sum_{\gamma\in\Gamma} \mathrm{e}^{-i(\theta+\gamma)\cdot k}\psi(\theta+\gamma), \qquad (k,\theta) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{3.11}$$

Directly from the definition one can check the following periodicity properties:

$$(\mathcal{Z}\psi)(k,\theta+\gamma) = (\mathcal{Z}\psi)(k,\theta) \qquad \forall\, \gamma \in \Gamma, \tag{3.12}$$

and

$$(\mathcal{Z}\psi)(k+\gamma^*,\theta) = \mathrm{e}^{-i\theta\cdot\gamma^*}(\mathcal{Z}\psi)(k,\theta) \qquad \forall\, \gamma^* \in \Gamma^*. \tag{3.13}$$

Equation (3.12) shows that for any fixed $k \in \mathbb{R}^2$, $(\mathcal{Z}\psi)(k,\cdot)$ is a Γ-periodic function and can be seen as an element of $\mathcal{H}_{\mathrm{f}} := L^2(\mathbb{V}, d^2\theta)$ with $\mathbb{V} := \mathbb{R}^2/\Gamma$ a two-dimensional slant torus (*Voronoi torus*). The torus $\mathbb{V}$ coincides with the the fundamental cell M_Γ endowed with the identification of the opposite edges and $d^2\theta$ denotes the (normalized) measure induced on $\mathbb{V}$ by the identification with M_Γ. The Hilbert space $\mathcal{H}_{\mathrm{f}}$ is the *space of fast degrees of freedom*, corresponding to the microscopic scale. Equation (3.13) involves a unitary representation of the group of the (dual) lattice translations Γ^* on the Hilbert space $\mathcal{H}_{\mathrm{f}}$, namely

$$\Gamma^* \ni \gamma^* \xrightarrow{\ \tau\ } \tau(\gamma^*) \in \mathscr{U}(\mathcal{H}_{\mathrm{f}})$$

where $\tau(\gamma^*)$ is the multiplication with $\mathrm{e}^{i\theta\cdot\gamma^*}$. It will be convenient to introduce the Hilbert space

$$\mathcal{H}_\tau := \left\{\psi \in L^2_{\mathrm{loc}}\left(\mathbb{R}^2, d^2\underline{k}, \mathcal{H}_{\mathrm{f}}\right) \;:\; \psi(k-\gamma^*,\cdot) = \tau(\gamma^*)\,\psi(k,\cdot)\right\} \tag{3.14}$$

equipped with the inner product

$$\langle\psi;\varphi\rangle_{\mathcal{H}_\tau} := \int_{M_{\Gamma^*}} (\psi(k);\varphi(k))_{\mathcal{H}_{\mathrm{f}}}\, d^2\underline{k}$$

where $d^2\underline{k} := \frac{d^2k}{(2\pi)^2}$ is the normalized measure. There is a natural isomorphism from $\mathcal{H}_\tau$ to $L^2\left(M_{\Gamma^*}, d^2\underline{k}, \mathcal{H}_{\mathrm{f}}\right)$ given by restriction from $\mathbb{R}^2$ to M_{Γ^*}, and with inverse given by τ-covariant continuation, as suggested by (3.13). The Bloch-Floquet transform (3.11) extends to a unitary map

$$\mathcal{Z} : \mathcal{H}_{\text{phy}} \longrightarrow \mathcal{H}_\tau \simeq L^2\left(M_{\Gamma^*}, d^2\underline{k}, \mathcal{H}_{\mathrm{f}}\right) \simeq L^2(M_{\Gamma^*}, d^2\underline{k}) \otimes \mathcal{H}_{\mathrm{f}}. \tag{3.15}$$

The Hilbert space $L^2(M_{\Gamma^*}, d^2\underline{k})$ can be seen as the *space of slow degrees of freedom* and in this sense the transform $\mathcal{Z}$ produces a decomposition of the physical Hilbert space according to the existence of fast and slow degrees of freedom.

We need to discuss how differential and multiplication operators behave under $\mathcal{Z}$. Let $Q = (Q_1, Q_2)$ be the multiplication by $x = (x_1, x_2)$ defined on its maximal domain and $P = (P_1, P_2) = -i\nabla_x$ with domain the Sobolev space $\mathcal{H}^1(\mathbb{R}^2)$, then from (3.11) it follows:

$$\mathcal{Z}\, P\, \mathcal{Z}^{-1} = k \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} + \mathbb{1}_{L^2(M_{\Gamma^*})} \otimes -i\nabla_\theta, \qquad \mathcal{Z}\, Q\, \mathcal{Z}^{-1} = i\nabla_k^\tau \tag{3.16}$$

where $-i\nabla_\theta$ acts on the domain $\mathcal{H}^1(\mathbb{V})$ while the domain of the differential operator $i\nabla_k^\tau$ is the space $\mathcal{H}_\tau \cap \mathcal{H}^1_{\mathrm{loc}}(\mathbb{R}^2, \mathcal{H}_\mathrm{f})$, namely it consists of vector-valued distributions which are in $\mathcal{H}^1(M_{\Gamma^*}, \mathcal{H}_\mathrm{f})$ and satisfy the θ-dependent boundary condition associated with (3.13). The central feature of the Bloch-Floquet transform is, however, that multiplication operators corresponding to Γ-periodic functions like A_Γ or V_Γ are mapped into multiplication operators corresponding to the same function, i.e.

$$\mathcal{Z}\, A_\Gamma(x)\, \mathcal{Z}^{-1} = \mathbb{1}_{L^2(M_{\Gamma^*})} \otimes A_\Gamma(\theta) \qquad \mathcal{Z}\, V_\Gamma(x)\, \mathcal{Z}^{-1} = \mathbb{1}_{L^2(M_{\Gamma^*})} \otimes V_\Gamma(\theta). \tag{3.17}$$

Let $H^{\mathcal{Z}} := \mathcal{Z}\, H_{\mathrm{BL}}\, \mathcal{Z}^{-1}$ be the Bloch-Floquet transform of the Bloch-Landau Hamiltonian (3.10). According to relations (3.16) and (3.17) one obtains from (3.10) that

$$H^{\mathcal{Z}} = \frac{1}{2}\left[-i\nabla_\theta + k - A_\Gamma(\theta) - A_0(i\varepsilon\nabla_k^\tau) - \iota_q \frac{1}{2} e_\perp \wedge (i\varepsilon\nabla_k^\tau)\right]^2 + V_\Gamma(\theta) + \phi(i\varepsilon\nabla_k^\tau). \tag{3.18}$$

The domain of self-adjointness of $H^{\mathcal{Z}}$ is $\mathcal{Z}\mathcal{H}^2_{\mathrm{M}}(\mathbb{R}^2) \subset \mathcal{H}_\tau$, i.e. the image under $\mathcal{Z}$ of the second magnetic-Sobolev space.

3.3.3 The periodic Hamiltonian and the gap condition

When $\varepsilon = 0$ the Bloch-Landau Hamiltonian (3.10) reduces to the periodic Hamiltonian

$$H_{\mathrm{per}} = \frac{1}{2}\left[-i\nabla_x - A_\Gamma(x)\right]^2 + V_\Gamma(x). \tag{3.19}$$

According to (3.18) the Bloch-Floquet transform maps H_{per} into a fibered operator. In other words, denoting $H^{\mathcal{Z}}_{\mathrm{per}} := \mathcal{Z}\, H_{\mathrm{per}}\, \mathcal{Z}^{-1}$, one has $H^{\mathcal{Z}}_{\mathrm{per}} = \int^{\oplus}_{M_{\Gamma^*}} H_{\mathrm{per}}(k)\, d^2\underline{k}$ where, for each $k \in M_{\Gamma^*}$

$$H_{\mathrm{per}}(k) = \frac{1}{2}\left[-i\nabla_\theta + k - A_\Gamma(\theta)\right]^2 + V_\Gamma(\theta). \tag{3.20}$$

The operator $H_{\mathrm{per}}(k)$ acts on $\mathcal{H}_\mathrm{f} = L^2(\mathbb{V}, d^2\theta)$ with self-adjointness domain $\mathcal{D} := \mathcal{H}^2(\mathbb{V})$ (the second Sobolev space) independent of $k \in M_{\Gamma^*}$. Moreover it is easy to check that the Bloch-Floquet transform induces the following property of periodicity, called *τ-equivariance*:

$$H_{\mathrm{per}}([k] - \gamma^*) = \tau(\gamma^*)\, H_{\mathrm{per}}([k])\, \tau(\gamma^*)^{-1} \in \Gamma^* \qquad \forall \gamma^* \in \Gamma^*. \tag{3.21}$$

where the notation $k := [k] - \gamma^*$ denotes the a.e.-unique decomposition of $k \in \mathbb{R}^2$ as a sum of $[k] \in M_{\Gamma^*}$ and $\gamma^* \in \Gamma^*$.

Remark 3.3.2 (Analiticity)**.** For any $k \in \mathbb{R}^2$, let $I(k)$ be the unitary operator acting on $\mathcal{H}_\mathrm{f}$ as the multiplication by $\mathrm{e}^{-i\theta \cdot k}$. Obviously $I(k) = I([k] - \gamma^*) = I([k])\tau(\gamma^*)^{-1}$. A simple computation shows that

$$H_{\mathrm{per}}(k) = I(k)\, H_{\mathrm{per}}(0)\, I(k)^{-1} \tag{3.22}$$

where the equality holds on the fixed domain of self-adjointness $\mathcal{D} = \mathcal{H}^2(\mathbb{V})$. The τ-equivariance property (3.21) follows immediately from (3.22). Moreover from (3.22) is evident that $H_{\text{per}}(k)$ defines an *analytic family (of type A) in the sense of Kato* (Reed and Simon 1978, Chapter XII). Finally a short computation shows

$$(\partial_{k_j} H_{\text{per}})(k) = -iI(k)\,[\theta_j; H_{\text{per}}(0)]\,I(k)^{-1} = I(k)\,\left(-i\nabla_\theta - A_\Gamma(\theta)\right)_j\,I(k)^{-1}$$

and $(\partial^2_{k_j} H_{\text{per}})(k) = \mathbb{1}_{\mathcal{D}}$, $(\partial^2_{k_1,k_2} H_{\text{per}})(k) = 0$ on the domain $\mathcal{D}$. ♦◊

The spectrum of H_{per}, which coincides with the spectrum of $H^{\mathcal{Z}}_{\text{per}}$, is given by the union of all the spectra of $H_{\text{per}}(k)$. The following classical results hold true:

PROPOSITION 3.3.3. *Let V_Γ and A_Γ satisfy Assumption 3.2.2, then:*

(i) *for any $k \in \mathbb{R}^2$ the operator $H_{\text{per}}(k)$ defined by* (3.19) *is self-adjoint with domain $\mathcal{D} = \mathcal{H}^2(\mathbb{V})$ and is bounded below;*

(ii) *$H_{\text{per}}(k)$ has compact resolvent and its spectrum is purely discrete with eigenvalues $\mathcal{E}_n(k) \to +\infty$ as $n \to +\infty$;*

(iii) *let the eigenvalues be arranged in increasing order and repeated according to their multiplicity for any $k \in M_{\Gamma^*}$, i.e. $\mathcal{E}_1(k) \leqslant \mathcal{E}_2(k) \leqslant \mathcal{E}_3(k) \leqslant \ldots$ then $\mathcal{E}_n(k)$ is a continuous Γ^*-periodic function of k.*

The above result differs from the standard theory of periodic Schrödinger operators just for the presence of a periodic vector potential A_Γ. Since we were no able to find a suitable reference in the literature, we sketch its proof in Appendix A.1.1.

We call $\mathcal{E}_n(\cdot)$ the n-th *Bloch band* or *energy band*. The corresponding normalized eigenstates $\{\varphi_n(k)\}_{n\in\mathbb{N}} \subset \mathcal{D}$ are called *Bloch functions* and form, for any $k \in M_{\Gamma^*}$, an orthonormal basis of $\mathcal{H}_{\text{f}}$. Notice that, with this choice of the labelling, $\mathcal{E}_n(\cdot)$ and $\varphi_n(\cdot)$ are continuous in k, but generally they are not smooth functions if *eigenvalue crossings* are present.

We say that a family of Bloch bands $\{\mathcal{E}_n(\cdot)\}_{n\in\mathcal{I}}$, with $\mathcal{I} := [I_+, I_-] \cap \mathbb{N}$, is *isolated* if

$$\inf_{k\in M_{\Gamma^*}} \text{dist}\left(\bigcup_{n\in\mathcal{I}}\{\mathcal{E}_n(k)\}, \bigcup_{j\notin\mathcal{I}}\{\mathcal{E}_j(k)\}\right) = C_g > 0. \tag{3.23}$$

The existence of an isolated part of the spectrum is a necessary ingredient for an adiabatic theory. We introduce the following:

ASSUMPTION 3.3.4 (Constant gap condition)**.** *The spectrum of H_{per} admits a family of Bloch bands $\{\mathcal{E}_n(\cdot)\}_{n\in\mathcal{I}}$ which is isolated in the sense of* (3.23)*.*

Let $P_{\mathcal{I}}(k)$ be the spectral projector of $H_{\text{per}}(k)$ corresponding to the family of eigenvalues $\{\mathcal{E}_n(k)\}_{n\in\mathcal{I}}$, then $P^{\mathcal{Z}}_{\mathcal{I}} := \int^{\oplus}_{M_{\Gamma^*}} P_{\mathcal{I}}(k)\, d^2\underline{k}$ is the projector on the isolated family of Bloch bands labeled by $\mathcal{I}$. In terms of Bloch functions (using the Dirac notation), one has that

$P_{\mathcal{I}}(\cdot) = \sum_{n\in\mathcal{I}} |\varphi_n(\cdot)\rangle\langle\varphi_n(\cdot)|$. However, in general, $\varphi_n(\cdot)$ are not smooth functions of k at eigenvalue crossing, while $P_{\mathcal{I}}(\cdot)$ is a smooth function of k because of the gap condition. Moreover, from the periodicity of $H_{\rm per}(\cdot)$, one argues $P_{\mathcal{I}}([k]-\gamma^*) = \tau(\gamma^*)\, P_{\mathcal{I}}([k])\, \tau(\gamma^*)^{-1}$. In general the smoothness of $P_{\mathcal{I}}(\cdot)$ is not enough to assure the existence of family of orthonormal basis for the subspaces $\mathrm{Im}P_{\mathcal{I}}(\cdot)$ which varies smoothly (or only continuously) with respect to $k \in M_{\Gamma^*}$. Then we need the following assumption.

ASSUMPTION 3.3.5 (Continuous frame)**.** *Let $\{\mathcal{E}_n(\cdot)\}_{n\in\mathcal{I}}$ be a family of Bloch bands ($|\mathcal{I}| = m > 1$). We assume that there exists an orthonormal basis $\{\psi_n(\cdot)\}_{j=1}^m$ of $\mathrm{Im}P_{\mathcal{I}}(\cdot)$ whose elements are smooth and (left) τ-covariant with respect to k, i.e. $\psi_j(\cdot-\gamma^*) = \tau(\gamma^*)\psi_j(\cdot)$ for all $j = 1,\ldots,m$ and $\gamma^* \in \Gamma^*$.*

Note that it is not required that $\psi_j(k)$ is an eigenfunction of $H_{\rm per}(k)$. However, in the special but important case in which the family of bands consist of a single isolated m-fold degenerate eigenvalue, i.e. $\mathcal{E}_n(k) = \mathcal{E}_*(k)$ for every $n = 1,\ldots,m$, then the Assumption 3.3.5 is equivalent to the existence of an orthonormal basis consisting of smooth and τ-covariant Bloch functions.

REMARK 3.3.6 (Time-reversal symmetry breaking)**.** As far as low dimensional models are concerned ($d \leq 3$), Theorem 1 in (Panati 2007) assures that Assumption 3.3.5 is true whenever the Hamiltonian $H_{\rm per}$ is invariant with respect to the *time-reversal symmetry*, which is implemented in the Schrödinger representation by the complex conjugation operator. However, the term $A_\Gamma \neq 0$ in $H_{\rm per}$ generically breaks the time reversal symmetry. Therefore, to consider also the effects due to a periodic vector potential, we need to assume the existence of a smooth family of frames. Anyway is opinion of the authors that the result in (Panati 2007) can be extended to the case of a periodic vector potential, at least assuming that A_Γ is small in a suitable sense. ♦◊

Let k_0 be a fixed point in M_{Γ^*} and define the projection $\pi_{\rm r} := P_{\mathcal{I}}(k_0)$. If the Assumption 3.3.4 holds true then the dimension of $\pi_{\rm r}$ agrees with the dimension of $P_{\mathcal{I}}(k)$ for all $k \in \mathbb{R}^2$. Let $\{\chi_n\}_{j=1}^m$ be an orthonormal basis for Im $\pi_{\rm r}$ and define a unitary map

$$u_0(k) := \widetilde{u}_0(k) + u_0^\perp(k), \qquad \text{with} \qquad \widetilde{u}_0(k) := \sum_{1\leq j\leq \ell} |\chi_j\rangle\langle\psi_j(k)|, \tag{3.24}$$

which maps Im $P_{\mathcal{I}}(k)$ in Im $\pi_{\rm r}$. The definition of this unitary is not unique because the freedom in the choice of the frame and of the orthogonal complement $u_0^\perp(k)$. From the definition and the τ-covariance of $\psi_j(\cdot)$ one has that $u_0(k)\; P_{\mathcal{I}}(k)\; u_0(k)^{-1} = \pi_{\rm r}$ and $u_0([k]-\gamma^*) = u_0([k])\; \tau(\gamma^*)^{-1}$ (*right τ-covariance*).

3.3.4 τ-equivariant and special τ-equivariant symbol classes

Proposition 3.3.3 shows that for any $k \in \mathbb{R}^2$, the operator $H_{\rm per}(k)$ defines an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}_{\rm f}$ with dense domain $\mathcal{D} := \mathcal{H}^2(\mathbb{V})$. However the domain $\mathcal{D}$ can be considered itself as a Hilbert space with respect to the Sobolev norm $\|\cdot\|_{\mathcal{D}} := \|(\mathbb{1}_{\mathcal{H}_{\rm f}} - \Delta_\theta)\cdot\|_{\mathcal{H}_{\rm f}}$ and so $H_{\rm per}(k)$ can be seen as a bounded linear operator from $\mathcal{D}$ to $\mathcal{H}_{\mathrm{f}}$, i.e. as an element of the Banach space $\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}})$. The map $\mathbb{R}^2 \ni k \mapsto H_{\mathrm{per}}(k) \in \mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}})$ is a special example of a *operator-valued symbol*. For a summary about the theory of the Weyl quantization of vector-valued symbols, we refer to Appendices A and B in (Teufel 2003). In what follows we will need the following definition.

DEFINITION 3.3.7 (Hörmander symbol classes). *A* symbol *is any map F from the (cotangent) space $\mathbb{R}^2 \times \mathbb{R}^2$ to the Banach space $\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}})$, i.e. $\mathbb{R}^2 \times \mathbb{R}^2 \ni (k,\eta) \mapsto F(k,\eta) \in \mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}})$. A function $w : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,+\infty)$ is said to be an* order function *if there exists constants $C_0 > 0$ and $N_0 > 0$ such that*

$$w(k,\eta) \leqslant C_0 \left(1 + |k-k'|^2 + |\eta-\eta'|^2\right)^{\frac{N_0}{2}} w(k',\eta') \tag{3.25}$$

for every $(k,\eta),(k',\eta') \in \mathbb{R}^2 \times \mathbb{R}^2$. A symbol $F \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2, \mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}}))$ is an element of the (Hörmander) symbol class *$S^w(\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}}))$ with order function w, if for every $\alpha,\beta \in \mathbb{N}^2$ there exists a constant $C_{\alpha,\beta} > 0$ such that $\|(\partial_k^\alpha \partial_\eta^\beta F)(k,\eta)\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}})} \leqslant C_{\alpha,\beta}\, w(k,\eta)$ for every $(k,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2$.*

According to the previous definition, the vector-valued map $H_{\mathrm{per}}(\cdot)$ defines a Hörmander symbol constant in the η-variables and with order function $v(k,\eta) := 1 + |k|^2$ (see the proof of Proposition 3.3.12 below). However, as showed by equation (3.21), the symbol $H_{\mathrm{per}}(\cdot)$ satisfies an extra condition of periodicity.

DEFINITION 3.3.8 (τ-equivariant symbols). *Let Γ^* be a two dimensional lattice (the dual lattice defined in Section 3.2 for our aims) and $\tau : \Gamma^* \to \mathscr{U}(\mathcal{H}_{\mathrm{f}})$ the unitary representation defined in Section 3.3.2. Denote by $\widetilde{\tau} := \tau|_{\mathcal{D}}$ the bounded-operator [1] representation of Γ^* in $\mathcal{D}$. A symbol $F \in S^w(\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}}))$ is said to be* τ-equivariant *if*

$$F(k-\gamma^*,\eta) = \tau(\gamma^*)\ F(k,\eta)\ \widetilde{\tau}(\gamma^*)^{-1} \qquad \forall\, \gamma^* \in \Gamma^*, k \in \mathbb{R}^2.$$

The space of τ-equivariant symbols is denoted as $S^w_\tau(\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}}))$.

For the purposes of this work, it is convenient to focus on special classes of symbols. By considering the *kinetic momentum function*$\mathbb{R}^2 \times \mathbb{R}^2 \ni (k,\eta) \overset{\kappa}{\longmapsto} \kappa(k,\eta) := k - A(\eta) \in \mathbb{R}^2$, with A fulfilling Assumption (B), one defines the *minimal coupling map* by

$$(k,\eta) \overset{\jmath_\kappa}{\longmapsto} \jmath_\kappa(k,\eta) := (\kappa(k,\eta),\eta) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{3.26}$$

DEFINITION 3.3.9 (Special τ-equivariant symbols). *Let w be an order function, in the sense of (3.25). We define*

$$S^w_{\kappa;\tau}(\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}})) := \{\widetilde{F} = F \circ \jmath_\kappa\ :\ F \in S^w_\tau(\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}}))\}.$$

[1] Clearly, $\tau(\gamma^*)$ acts as an invertible bounded operator on the space $\mathcal{D}$, but it is no longer unitary with respect to the Sobolev-norm defined on $\mathcal{D}$.

We refer to $S^{w}_{\kappa;\tau}(\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f}))$ as the class of *special τ-equivariant symbols*. The following result shows that special symbols can be considered as genuine τ-equivariant symbols with respect to a modified order function. The key ingredient is the linear growth of the kinetic momentum.

LEMMA 3.3.10. *With the above notations $S^{w}_{\kappa;\tau}(\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f})) \subset S^{w'}_{\tau}(\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f}))$ where $w' := w \circ \jmath_\kappa$.*

Proof. If $F \in S^{w}_{\tau}(\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f}))$ then also $F \circ \jmath_\kappa$ is τ-equivariant, indeed $\kappa(k-\gamma^*,\eta) = \kappa(k,\eta)-\gamma^*$ and $(F\circ\jmath_\kappa)(k-\gamma^*,\eta) = \tau(\gamma^*)\,(F\circ\jmath_\kappa)(k,\eta)\,\widetilde{\tau}(\gamma^*)^{-1}$. Since $\jmath_\kappa$ is a smooth function, then also the composition $F \circ \jmath_\kappa$ is a smooth function. Observing that $(F \circ \jmath_\kappa)(k,\eta) = F(k - A(\eta),\eta)$ it follows that

$$
\begin{aligned}
(\partial_{k_j}(F\circ\jmath_\kappa))(k,\eta) &= ((\partial_{k_j}F)\circ\jmath_\kappa)(k,\eta)\\
(\partial_{\eta_j}(F\circ\jmath_\kappa))(k,\eta) &= ((\partial_{\eta_j}F)\circ\jmath_\kappa)(k,\eta) + \sum_{i=1}^{2}(\partial_{\eta_j}\kappa_i)(k,\eta)\,((\partial_{k_i}F)\circ\jmath_\kappa)(k,\eta)
\end{aligned}
$$

where $\partial_{\eta_j}\kappa_i$ are bounded functions in view of Assumption 3.2.3. From the first equation it follows that

$$
\|\partial_{k_j}(F\circ\jmath_\kappa)(k,\eta)\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f})} \leqslant C_{j,0}\;(w\circ\jmath_\kappa)(k,\eta) \qquad j=1,2
$$

for suitable positive constants $C_{j,0}$. Similarly the second equation implies

$$
\|\partial_{\eta_j}(F\circ\jmath_\kappa)(k,\eta)\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f})} \leqslant [C_{0,j} + K(C_{1,0}+C_{2,0})](w\circ\jmath_\kappa)(k,\eta).
$$

where $K > 0$ is a bound for the functions $\partial_{\eta_j}\kappa_i$. By an inductive argument on the number of the derivatives one can proof that the derivatives of $F\circ\jmath_\kappa$ are bounded by the function $w' := w\circ\jmath_\kappa$. To complete the proof we need to show that w' is an order function according to Definition 3.3.7. This follows by a simple computation using the fact that κ has a linear growth in k and η. ■

In view of Lemma 3.3.10, all the results of Appendix B of (Teufel 2003) hold true for symbols in $S^{w}_{\kappa;\tau}(\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f}))$ and in particular the quantization of a symbol in $S^{w}_{\kappa;\tau}(\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f}))$ preserves the τ-equivariance. Moreover, the pointwise product or the Moyal product of two symbols of order w_1 and w_2 produce a symbol of order $w_1 w_2$ (Teufel 2003, Propositions B.3 and B.4).

REMARK 3.3.11 (Notation)**.** In what follows we use the short notation $F(\kappa;\eta) := (F \circ \jmath_\kappa)(k,\eta)$ to denote the special symbol $F\circ\jmath_\kappa \in S^{w}_{\kappa;\tau}(\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f}))$ related to the τ-equivariant symbols $F \in S^{w}_{\tau}(\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f}))$. We emphasize on the use of the semicolon ";" instead the comma "," and of the symbol of the kinetic momentum κ instead the Bloch-momentum k.

♦◊

3.3.5 Semiclassics: quantization of equivariant symbols

As explained in Section 3.3.2, the Bloch-Floquet transform $\mathcal{Z}$ provides the separation between the fast degrees of freedom, associated to the Hilbert space $\mathcal{H}_{\mathrm{f}} = L^2(\mathbb{V}, d^2\theta)$, and the slow degrees of freedom, associated to the Hilbert $L^2(M_{\Gamma^*}, d^2\underline{k})$. A fruitful point of view is to consider the slow degrees of freedom "classical" with respect to the "quantum" fast degrees of freedom. Mathematically, this is achived by recognizing that the Hamiltonian $H^{\mathcal{Z}}$ defined in (3.18) is the *Weyl quantization* of an operator-valued "semiclassical" symbol over the classical phase space $\mathbb{R}^2 \times \mathbb{R}^2$. As explained rigorously in the Appendices A and B of (Teufel 2003), the quantization procedure maps an operator-valued symbol $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathscr{B}(\mathcal{D}, \mathcal{H}_{\mathrm{f}})$ into a linear operator $\mathrm{Op}_\varepsilon(F) : \mathcal{S}(\mathbb{R}^2, \mathcal{D}) \to \mathcal{S}(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$, where $\mathcal{S}(\mathbb{R}^2, \mathcal{H})$ denotes the space of $\mathcal{H}$-valued Schwartz functions. The quantization procedure concerns only the slow degrees of freedom and at a formal level can be identified with the prescription

$$k \longmapsto \mathrm{Op}_\varepsilon(k) := \text{multiplication by } k \otimes \mathbb{1}_{\mathcal{D}}; \qquad \eta \longmapsto \mathrm{Op}_\varepsilon(\eta) := i\varepsilon\nabla_k \otimes \mathbb{1}_{\mathcal{D}}. \tag{3.27}$$

Let us consider the operator-valued symbol $H_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathscr{B}(\mathcal{D}, \mathcal{H}_{\mathrm{f}})$ defined by

$$H_0(k,\eta) := \frac{1}{2}\left[-i\nabla_\theta + k - A_\Gamma(\theta) - A_0(\eta) - \iota_q \frac{1}{2} e_\perp \wedge \eta\right]^2 + V_\Gamma(\theta) + \phi(\eta). \tag{3.28}$$

The symbol H_0 does not depend on ε and in view of Proposition 3.3.3 it defines an unbounded operator on $\mathcal{H}_{\mathrm{f}}$ with domain of self-adjointness $\mathcal{D} = \mathcal{H}^2(\mathbb{V})$ for any choice of $(k,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2$. According to the notation of Section 3.3.4, and comparing (3.28) with (3.20) we can write

$$H_0(k,\eta) = H_{\mathrm{per}}\left(\kappa(k,\eta)\right) + \phi(\eta) = (H_\phi \circ \jmath_\kappa)(k,\eta). \tag{3.29}$$

where $H_\phi(k,\eta) := H_{\mathrm{per}}(k) + \phi(\eta)$. As suggested by equation (3.21), H_ϕ is a τ-equivariant symbol. Thus the symbol H_0 is τ-equivariant with respect to the kinetic momentum κ. The following result establishes the exact symbol class for H_0.

PROPOSITION 3.3.12. *If Assumption 3.2.2 and 3.2.3 hold true then* $H_0 \in S^v_{\kappa;\tau}(\mathscr{B}(\mathcal{D}; \mathcal{H}_{\mathrm{f}}))$ *with order function* $v(k,\eta) := 1 + |k|^2$.

Proof. Using the result of Lemma 3.3.10, we only need to show that $H_\phi \in S^v(\mathscr{B}(\mathcal{D}, \mathcal{H}_{\mathrm{f}}))$. The later claim is easy to verify, indeed the derivative in η are bounded functions, the second derivative in k is a constant and the derivatives of higher order in k are zero. Then, we have to check only the growth of the first derivative in k. A simple computation shows that

$$\|(\partial_{k_j} H_\phi)(k,\eta)\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}})} = \|(\partial_{k_j} H_{\mathrm{per}})(k)\ (\mathbb{1}_{\mathcal{H}_{\mathrm{f}}} - \Delta_\theta)^{-1}\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})}$$

and since $\partial_{k_j} H_{\mathrm{per}}$ is τ-equivariant (see Remark 3.3.2), then

$$\|(\partial_{k_j} H_\phi)(k,\eta)\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\mathrm{f}})} = \|(\partial_{k_j} H_{\mathrm{per}})([k])\tau(\gamma^*)^{-1}(\mathbb{1}_{\mathcal{H}_{\mathrm{f}}} - \Delta_\theta)^{-1}\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})}.$$

Observing that $\tau(\gamma^*)$ is the multiplication by $\mathrm{e}^{i\theta\cdot\gamma^*}$ in $\mathcal{H}_\mathrm{f}$ and by a simple computation that $(\partial_{k_j} H_\mathrm{per})([k])\tau(\gamma^*)^{-1} = \tau(\gamma^*)^{-1}[-2\gamma^*_j + (\partial_{k_j} H_\mathrm{per})([k])]$ one has

$$\|(\partial_{k_j} H_\phi)(k,\eta)\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_\mathrm{f})} \leqslant C_1|\gamma^*_j| + \|(\partial_{k_j} H_\mathrm{per})([k])\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_\mathrm{f})} \leqslant C_1(|k| + C_3) + C_2$$

where $C_1 = 2\|(\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \Delta_\theta)^{-1}\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})}$, $C_2 := \max_{k\in M_{\Gamma^*}} \|(\partial_{k_j} H_\mathrm{per})([k])\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_\mathrm{f})}$ and $|\gamma^*_j| \leqslant |\gamma^*| = |k - [k]| \leqslant |k| + C_3$ with $C_3 := \max_{k\in M_{\Gamma^*}} |k|$. The claim follows observing that $1 + |k| \leqslant 2(1 + |k|^2)$. ■

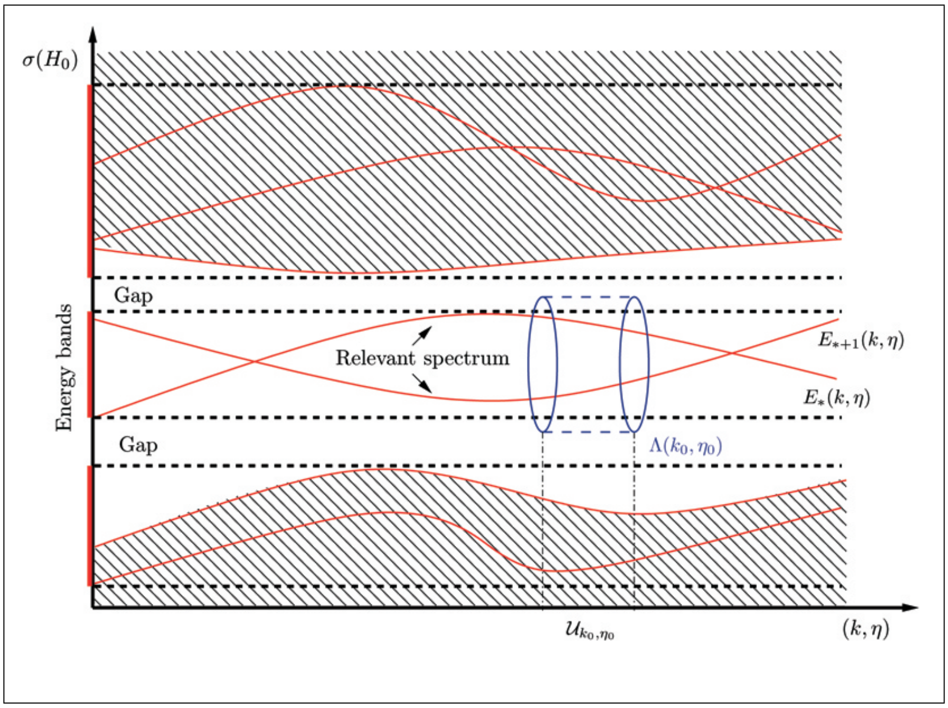


Figure 3.1: *Structure of the spectrum of $H_0(k,\eta)$. The picture shows schematically a "relevant part of the spectrum", consisting of two energy bands $\{E_*, E_{*+1}\}$, with $E_{*+j}(k,\eta) = \mathcal{E}_{*+j}(\kappa(k,\eta)) + \phi(\eta)$. Notice that we assume only a local gap condition, as stated in (3.30), while in the picture a stronger condition is satisfied: a gap exists when projecting the relevant bands on the vertical axis.*

Equation (3.29) provides information about the dependence on k and η of the spectrum of H_0. The n^th eigenvalue $E_n(k,\eta)$ of the operator $H_0(k,\eta)$ is related to the n^th eigenvalue $\mathcal{E}_n(k)$ of the periodic Hamiltonian $H_\mathrm{per}(k)$ by the relation $E_n(k,\eta) = \mathcal{E}_n(\kappa(k,\eta)) + \phi(\eta)$. The function $E_n : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is still Γ^*-periodic in k but only oscillating with bounded variation in η. Assumption 3.3.4 for the family of Bloch bands $\{\mathcal{E}_n(\cdot)\}_{n\in\mathcal{I}}$ immediately implies that

$$\inf_{(k,\eta)\in M_{\Gamma^*}\times\mathbb{R}^2} \mathrm{dist}\left(\bigcup_{n\in\mathcal{I}} \{E_n(k,\eta)\}, \bigcup_{j\notin\mathcal{I}} \{E_j(k,\eta)\} \right) = C_g > 0. \tag{3.30}$$

This is the relevant part of the spectrum of H_0 which we are interested in.

According to the general theory (Teufel 2003, Appendices A and B), one has that

$$\mathrm{Op}_{\varepsilon}(H_0) = \frac{1}{2}\left[-i\nabla_{\theta} + k - A_{\Gamma}(\theta) - A_0\left(i\varepsilon\nabla_k\right) - \iota_q \frac{1}{2} e_{\perp} \wedge \left(i\varepsilon\nabla_k\right)\right]^2 + V_{\Gamma}(\theta) + \phi\left(i\varepsilon\nabla_k\right) \quad (3.31)$$

defines a linear operator from $\mathcal{S}(\mathbb{R}^2, \mathcal{D})$ in $\mathcal{S}(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$ and by duality it extends to a continuous mapping $\mathrm{Op}_{\varepsilon}(H_0) : \mathcal{S}'(\mathbb{R}^2, \mathcal{D}) \to \mathcal{S}'(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$ (with an abuse of notation we use the same symbol for the extended operator). The τ-equivariance assures that $\mathrm{Op}_{\varepsilon}(H_0)\varphi([k] - \gamma^*) = \tau(\gamma^*)\mathrm{Op}_{\varepsilon}(H_0)\varphi([k])$ (Teufel 2003, Proposition B.3). Since $\mathrm{Op}_{\varepsilon}(H_0)$ preserves τ-equivariance, then it can be restricted to an operator on the domain $\mathcal{Z}\mathcal{H}^2_{\mathbf{M}}(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2, \mathcal{D})$ which is the domain of self-adjointness of $H^{\mathcal{Z}}$, according to (3.18). To conclude that $\mathrm{Op}_{\varepsilon}(H_0)$, restricted to $\mathcal{Z}\mathcal{H}^2_{\mathbf{M}}(\mathbb{R}^2)$, agrees with $H^{\mathcal{Z}}$ it is enough to recall that $i\nabla^{\tau}_k$ is defined as $i\nabla_k$ restricted to its natural domain $\mathcal{H}^1\left(\mathbb{R}^2, \mathcal{D}\right) \cap \mathcal{H}_{\tau}$ and to use the spectral calculus. These arguments justify the following:

PROPOSITION 3.3.13. *The Hamiltonian $H^{\mathcal{Z}}$, defined by* (3.18)*, agrees on its domain of definition with the Weyl quantization of the operator-valued symbol H_0 defined by* (3.28)*.*

With a little abuse of notation, we refer to this result by writing $H^{\mathcal{Z}} = \mathrm{Op}_{\varepsilon}(H_0)$.

3.3.6 Main result: effective dynamics for weak magnetic fields

Let A_{ε} and B_{ε} be ε-dependent (possibly unbounded) linear operators in $\mathcal{H}$. We write $A_{\varepsilon} = B_{\varepsilon} + \mathcal{O}_0(\varepsilon^{\infty})$ if: for any $N \in \mathbb{N}$ there exist a positive constant C_N such that

$$\|A_{\varepsilon} - B_{\varepsilon}\|_{\mathscr{B}(\mathcal{H})} \leq C_N\, \varepsilon^N \quad (3.32)$$

for every $\varepsilon \in [0, \varepsilon_0)$. Notice that, though the operators are unbounded, the difference is required to be a bounded operator.

We refer to Appendices A and B of (Teufel 2003) for the basic terminology concerning pseudodifferential operators, and in particular as for the notions of *principal symbol*, *asymptotic expansion*, *resummation*, *Moyal product*.

THEOREM 3.3.14. *Let Assumptions 3.2.2, 3.2.3, 3.3.4 and 3.3.5 be satisfied and let $\{E_n(\cdot)\}_{n\in\mathcal{I}}$ (with $|\mathcal{I}| = m$) be an isolated family of energy bands for H_0 satisfying condition* (3.30)*. Then:*

1. Almost-invariant subspace: *there exist an orthogonal projection $\Pi_{\varepsilon} \in \mathscr{B}(\mathcal{H}_{\tau})$, with $\Pi_{\varepsilon} = \mathrm{Op}_{\varepsilon}(\pi) + \mathcal{O}_0(\varepsilon^{\infty})$ and the symbol $\pi(k, \eta) \asymp \sum_{j=0}^{\infty} \varepsilon^j\ \pi_j(k, \eta)$ having principal part $\pi_0(k, \eta) = P_{\mathcal{I}}(k - A(\eta))$, so that*

$$[H^{\mathcal{Z}}; \Pi_{\varepsilon}] = \mathcal{O}_0(\varepsilon^{\infty}).$$

In particular for any $N \in \mathbb{N}$ there exist a C_N such that

$$\|(1 - \Pi_{\varepsilon})\, e^{-i\frac{t}{\varepsilon}H^{\mathcal{Z}}}\ \Pi_{\varepsilon}\| \leq C_N\ \varepsilon^N\ |t| \quad (3.33)$$

for ε sufficiently small, $t \in \mathbb{R}$.

2. Effective dynamics: *let $\mathcal{H}_{\mathrm{ref}} = L^2(M_{\Gamma^*}, d^2\underline{k}) \otimes \mathcal{H}_{\mathrm{f}}$, π_{r} as defined above (3.24) and $\Pi_{\mathrm{r}} = \mathbb{1}_{L^2(M_{\Gamma^*})} \otimes \pi_{\mathrm{r}} \in \mathscr{B}(\mathcal{H}_{\mathrm{ref}})$. Then there exist a unitary operator*

$$U_\varepsilon : \mathcal{H}_\tau \to \mathcal{H}_{\mathrm{ref}}$$

such that

(i) $U_\varepsilon = \mathrm{Op}_\varepsilon(u) + \mathcal{O}_0(\varepsilon^\infty)$, *where the symbol $u \asymp \sum_{j=0}^{\infty} \varepsilon^j u_j$ has principal part u_0 given by (3.24) with k replaced by $\kappa(k,\eta)$;*

(ii) $\Pi_{\mathrm{r}} = U_\varepsilon \, \Pi_\varepsilon \, U_\varepsilon{}^{-1}$;

(iii) *posing $\mathcal{K} := \Pi_{\mathrm{r}}\mathcal{H}_{\mathrm{ref}}$, one has*

$$U_\varepsilon \, \Pi_\varepsilon \, H^{\mathcal{Z}} \, \Pi_\varepsilon \, U_\varepsilon^{-1} = H^\varepsilon_{\mathrm{eff}} + \mathcal{O}_0(\varepsilon^\infty) \in \mathscr{B}(\mathcal{K})$$

with $H^\varepsilon_{\mathrm{eff}} = \mathrm{Op}_\varepsilon(h)$ and h a resummation of the formal symbol $u \,\sharp\, \pi \,\sharp\, H_0 \,\sharp\, \pi \sharp u^{-1}$ (thus algorithmically computable at any finite order). Moreover,

$$\|(e^{-i\frac{t}{\varepsilon}H^{\mathcal{Z}}} - U_\varepsilon^{-1} \, e^{-i\frac{t}{\varepsilon}H^\varepsilon_{\mathrm{eff}}} \, U_\varepsilon) \, \Pi_\varepsilon\| \le C'_N \;\; \varepsilon^N \; (\varepsilon + |t|). \qquad (3.34)$$

REMARK 3.3.15. The previous theorem and the following proof generalize straightforwardly to any dimension $d \in \mathbb{N}$. We prefer to state it only in the case $d = 2$ in view of the application to the QHE and of the comparison with the results in Section 3, the latter being valid only for $d = 2$. ♦◊

Proof of Theorem 3.3.14

Step 1. Almost-invariant subspace

The proof of the existence of the super-adiabatic projection is very close to the proof of Proposition 1 of (Panati et al. 2003a), so we only sketch the strategy and emphasize the main differences with respect to that proof.

First of all, one constructs a formal symbol $\pi \asymp \sum_{j=0}^{\infty} \varepsilon^j \pi_j$ (the *Moyal projection*) such that: (i) $\pi \sharp \pi \asymp \pi$; (ii) $\pi^\dagger = \pi$; (iii) $H_0 \sharp \pi \asymp \pi \sharp H_0$ where $\asymp$ denotes the asymptotic equivalence of formal series.

The symbol π is constructed recursively at any order $j \in \mathbb{N}$ starting from π_0 and H_0. One firstly show the uniqueness of π (Panati et al. 2003b, Lemma 2.3). The uniqueness allows us to construct π locally, i.e. in a neighborhood of some point $z_0 := (k_0, \eta_0) \in \mathbb{R}^2 \times \mathbb{R}^2$. From the continuity of the map $k \mapsto H_{\mathrm{per}}(k)$ and the condition (3.23) it follows that there exists a neighborhood $\mathcal{U}_{k_0}$ of k_0 such that for every $k \in \mathcal{U}_{k_0}$ the set $\{\mathcal{E}_n(k)\}_{n \in \mathcal{I}}$ can be enclosed by a positively-oriented circle $\Sigma(k_0) \subset \mathbb{C}$ independent of k. Moreover it is possible to choose $\Sigma(k_0)$ in such a way that: it is symmetric with respect to the real axis;

$\mathrm{dist}(\Sigma(k_0), \sigma(H_{\mathrm{per}}(k))) \geqslant \frac{1}{4}C_g$ for all $k \in \mathcal{U}_{k_0}$; Radius$(\Sigma(k_0)) \leqslant C_r$ with C_r independent of k_0; $\Sigma(k_0 - \gamma^*) = \Sigma(k_0)$ for any $\gamma^* \in \Gamma^*$.
With the notation of Section 3.3.4 we have $H_0 = H_\phi \circ \jmath_\kappa$ with $H_\phi(k, \eta) := H_{\mathrm{per}}(k) + \phi(\eta)$. Let $\widetilde{\Lambda}(k_0, \eta_0) := \Sigma(k_0) + \phi(\eta_0)$ denote the translation of the circle $\Sigma(k_0)$ by $\phi(\eta_0) \in \mathbb{R}$ and pose $\Lambda := \widetilde{\Lambda} \circ \jmath_\kappa$. From the smoothness of ϕ it follows that there exists a neighborhood $\mathcal{U}_{z_0} \subset \mathbb{R}^2 \times \mathbb{R}^2$ of z_0 such that $\mathrm{dist}(\Lambda(z_0), \sigma(H_0(z))) \geqslant \frac{1}{4}C_g$ for all $z \in \mathcal{U}_{z_0}$. Moreover $\Lambda(z_0)$ is symmetric with respect to the real axis, has radius bounded by C_r and is Γ^*-periodic in the variable $\kappa = k - A(\eta)$ (see Figure 3.1).

We proceed by using the *Riesz formula*, namely by posing

$$\pi_j(z) := \frac{i}{2\pi} \oint_{\Lambda(z_0)} d\lambda \; R_j(\lambda, z) \qquad \text{on} \quad \mathcal{U}_{z_0}$$

where $R_j(\lambda, \cdot)$ denotes the j-th term in the *Moyal resolvent* $R(\lambda, \cdot) = \sum_{j=0}^{\infty} \varepsilon^j R_j(\lambda, \cdot)$ (also known as the *parametrix*), defined by the request that

$$(H_0(\cdot) - \lambda \mathbb{1}_{\mathcal{D}}) \sharp R(\lambda, \cdot) = \mathbb{1}_{\mathcal{H}_{\mathrm{f}}}, \qquad R(\lambda, \cdot) \sharp (H_0(\cdot) - \lambda \mathbb{1}_{\mathcal{D}}) = \mathbb{1}_{\mathcal{D}} \qquad \text{on} \quad \mathcal{U}_{z_0}.$$

Each term R_j is computed by a recursive procedure starting from $R_0(\lambda, \cdot) := (H_0(\cdot) - \lambda \mathbb{1}_{\mathcal{D}})^{-1}$, as illustrated in (Gérard et al. 1991). Following (Panati et al. 2003a, equations (30) and (31)) one obtains that

$$R_j(\lambda, z) = -R_0(\lambda, z) \; L_j(\lambda, z) \tag{3.35}$$

where L_j is the $(j-1)$-th order obstruction for R_0 to be the Moyal resolvent, i.e.

$$(H_0(\cdot) - \lambda \mathbb{1}_{\mathcal{D}}) \sharp \left(\sum_{n=0}^{j-1} \varepsilon^n \; R_j(\lambda, \cdot) \right) = \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} + \varepsilon^j \; L_j(\cdot) + \mathcal{O}(\varepsilon^{j+1}). \tag{3.36}$$

At the first order $L_1 = -\frac{i}{2} \{H_0, R_0\}_{k,\eta}$, with $\{\cdot, \cdot\}_{k,\eta}$ the Poisson brackets.

The technical (and crucial) part of the proof is to show that

$$\pi_j \in S^v_{\kappa;\tau}\left(\mathscr{B}(\mathcal{H}_{\mathrm{f}}, \mathcal{D})\right) \cap S^1_{\kappa;\tau}\left(\mathscr{B}(\mathcal{H}_{\mathrm{f}})\right)$$

for all $j \in \mathbb{N}$, with $v(k, \eta) := (1 + |k|^2)$. By means of the recursive construction each $R_j(\lambda, \cdot)$ inherits the special τ-equivariance from the principal symbol $R_0(\lambda, \cdot) = ((H_\phi \circ \jmath_\kappa)(\cdot) - \lambda \mathbb{1}_{\mathcal{D}})^{-1}$. The special periodicity in κ of the domain of integration $\Lambda(\cdot)$ which appears in the Riesz formula assures also the special τ-equivariance of each $\pi_j(\cdot)$.
Since $\|(\partial_z^\alpha \pi_j)(z)\|_\flat \leqslant 2\pi C_r \; \sup_{\lambda \in \Lambda(z_0)} \|\partial_z^\alpha (R_j)(\lambda, z)\|_\flat$ ($\flat$ means either $\mathscr{B}(\mathcal{H}_{\mathrm{f}})$ or $\mathscr{B}(\mathcal{H}_{\mathrm{f}}; \mathcal{D})$, $\alpha \in \mathbb{N}^4$ is a multiindex and $\partial_z^\alpha := \partial_{k_1}^{\alpha_1} \partial_{k_2}^{\alpha_2} \partial_{\eta_1}^{\alpha_3} \partial_{\eta_2}^{\alpha_4}$), we need only to prove that $R_j(\lambda, \cdot) \in S^v_{\kappa;\tau}\left(\mathscr{B}(\mathcal{H}_{\mathrm{f}}, \mathcal{D})\right) \cap S^1_{\kappa;\tau}\left(\mathscr{B}(\mathcal{H}_{\mathrm{f}})\right)$ uniformly in λ. This is the delicate point of the proof.

First of all, from the definition of $\Lambda(z_0)$ it follows that

$$\|R_0(\lambda, z)\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})} = [\mathrm{dist}(\lambda, \sigma(H_0(z)))]^{-1} \leqslant {}^{4}\!/\!{}_{C_g}$$

uniformly in λ. Let $\sigma \in \mathbb{N}^4$, with $|\sigma| = 1$. One observes that $\partial_z^\sigma R_0(\lambda, z) = -R_0(\lambda, z) N_z^\sigma(\lambda, z)$ with $N_z^\alpha(\lambda, z) := \partial_z^\alpha H_0(z)\ R_0(\lambda, z)$. From the relation

$$\partial_z^\sigma N_z^\alpha = N_z^{\alpha+\sigma} - N_z^\alpha N_z^\sigma$$

and an inductive argument, it follows the *chain rule*

$$\partial_z^\alpha R_0 = R_0 \ \sum \omega_{\beta_1 \ldots \beta_{|\alpha|}} N_z^{\beta_1} \ldots N_z^{\beta_{|\alpha|}}$$

where $\beta_1, \ldots, \beta_{|\alpha|} \in \mathbb{N}^4$, $|\alpha| := \alpha_1 + \ldots + \alpha_4$, $\omega_{\beta_1 \ldots \beta_{|\alpha|}} = \pm 1$ is a suitable sign function and the sum runs over all the combinations of multiindices such that $\beta_1 + \ldots + \beta_{|\alpha|} = \alpha$ with the convention $N_z^0 = \mathbb{1}$. The chain rule implies that $R_0 \in S^1_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ provided that

$$\|N_z^\alpha\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} = \|\partial_z^\alpha H_0\ R_0\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant C_\alpha \qquad \text{uniformly in } \lambda.$$

The latter condition is true since $\|(\partial_z^\alpha H_0)(k, \eta)\ R_0(\lambda, k, \eta)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant (g \circ \jmath_\kappa)(k, \eta)$, for a suitable $g(k, \eta)$, Γ^*-periodic in k and bounded in η; the latter claim can be checked as in Proposition 3.3.12.

Similarly, to prove that $R_0 \in S^v_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_\mathrm{f}, \mathcal{D}))$ we need to show that

$$\|R_0\ N_z^\alpha\|_{\mathscr{B}(\mathcal{H}_\mathrm{f}, \mathcal{D})} = \|(\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \Delta_\theta)\ R_0\ N_z^\alpha\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant C_\alpha\ v'(\cdot)$$

uniformly in λ. Since N_z^α is bounded on $\mathcal{H}_\mathrm{f}$ it is sufficient to show that $\|(\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \Delta_\theta) R_0(\lambda; z)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant C'_\alpha\ v'(z)$. Observe that $\|(\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \Delta_\theta) R_0(\lambda, [\kappa] - \gamma^*; \eta)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} = \|(\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \Delta_\theta)\tau(\gamma^*)^{-1} R_0(\lambda, [\kappa]; \eta)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})}$. The commutation relation

$$-\Delta_\theta \tau(\gamma^*)^{-1} = \tau(\gamma^*)^{-1} \left(|\gamma^*|^2 + i2\gamma^* \cdot \nabla_\theta - \Delta_\theta \right)$$

and the straightforward bound

$$\| \left(|\gamma^*|^2 + i2\gamma^* \cdot \nabla_\theta - \Delta_\theta \right) (\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \Delta_\theta)^{-1} \|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant C(1 + |\gamma^*|^2) \leqslant C'(1 + |\kappa(k, \eta)|^2)$$

imply

$$\|(\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \Delta_\theta) R_0(\lambda, z)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant C'_\alpha\ v'(z) \|(\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \Delta_\theta) R_0(\lambda, [\kappa]; \eta)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})}$$

with $v' := v \circ \jmath_\kappa$. Finally observe that

$$\|(\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \Delta_\theta) R_0(\lambda, [\kappa]; \eta)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant C([\kappa]; \eta) \leqslant C''. \tag{3.37}$$

The first inequality above follows by an expansion on the Fourier basis, for fixed $[\kappa]$ and η; the second follows from the fact that $[\kappa]$ takes values on a compact set and the explicit dependence on η is through the bounded function ϕ. The bound (3.37) implies that $R_0 \in S^v_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_\mathrm{f}, \mathcal{D})) \cap S^1_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ uniformly in λ.

To prove that $R_j \in S^1_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_\mathrm{f}))$, we observe that for any $\alpha \in \mathbb{N}^d$ one has

$$\partial_z^\alpha R_j(\lambda, z) = R_0(\lambda, z)\ M_{z;j}^\alpha(\lambda, z)$$

where $M_{z;j}^\alpha$ is a linear combination of terms which are product of N_z^β and $\partial_z^\delta L_j$ with $|\beta|, |\delta| \leqslant |\alpha|$. Thus, it is sufficient to prove that $L_j \in S^1_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ for every $j \in \mathbb{N}$. The latter claim is proved by induction on $j \in \mathbb{N}$. Referring to (3.35), one has trivially that $L_1 \in S^1_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$. L_{j+1} is a linear combination of products of N^α_z (with $0 \leqslant |\alpha| \leqslant j+1$) and $M^\beta_{z,i}$ (with $|\beta| + i = j+1$ and $0 \leqslant i \leqslant j$). Then the induction hypothesis on L_i for all $i = 1, \ldots, j$ implies that L_{j+1} is in $S^1_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$.

Finally observing that $\|\partial^\alpha_z R_j\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}},\mathcal{D})} \leqslant \|M^\alpha_{z;j}\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})}\ \|R_0\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}},\mathcal{D})}$ and using the fact that $R_0 \in S^v_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_{\mathrm{f}},\mathcal{D}))$ it follows that $R_j \in S^v_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_{\mathrm{f}},\mathcal{D})) \cap S^1_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$ uniformly in λ, for all $j \in \mathbb{N}$.

As explained in Section 3.3.4, we can apply the result of Proposition B.4 in (Teufel 2003) to special τ-equivariant symbols obtaining $H_0 \sharp \pi \in S^{v^2}_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$. However the τ-equivariance of $H_0\sharp\pi$ and its derivatives implies that the norms are bounded in z, hence $H_0\sharp\pi \in S^1_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$ which implies by adjointness also $\pi\sharp H_0 \in S^1_{\kappa;\tau}(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$. By construction $[H^{\mathcal{Z}}; \mathrm{Op}_\varepsilon(\pi)] = \mathrm{Op}_\varepsilon(H_0\sharp\pi - \pi\sharp H_0) = \mathcal{O}_0(\varepsilon^\infty)$ where the remainder is bounded in the norm of $\mathscr{B}(\mathcal{H}_\tau)$.

The operator $\mathrm{Op}_\varepsilon(\pi)$ is only approximately a projection, since $\mathrm{Op}_\varepsilon(\pi)^2 = \mathrm{Op}_\varepsilon(\pi\sharp\pi) = \mathrm{Op}_\varepsilon(\pi) + \mathcal{O}_0(\varepsilon^\infty)$. We obtain the super-adiabatic projection Π_ε by using the trick in (Nenciu and Sordoni 2004). Indeed, one notices that, for ε sufficiently small, the spectrum of $\mathrm{Op}_\varepsilon(\pi)$ does not contain e.g. the points $\{1/2\}$ and $\{3/2\}$. Thus, the formula

$$\Pi_\varepsilon = \frac{i}{2\pi} \oint_{|z-1|=1/2} (\mathrm{Op}_\varepsilon(\pi) - z)^{-1}. \tag{3.38}$$

yields an orthogonal projector such that $\Pi_\varepsilon = \mathrm{Op}_\varepsilon(\pi) + \mathcal{O}_0(\varepsilon^\infty)$.

Finally, equation (3.33) follows by observing that $[H^{\mathcal{Z}}; \Pi_\varepsilon] = \mathcal{O}_0(\varepsilon^\infty)$ implies

$$[\mathrm{e}^{-i\frac{t}{\varepsilon}H^{\mathcal{Z}}}; \Pi_\varepsilon] = \mathcal{O}_0(\varepsilon^\infty |t|)$$

as proved in (Teufel 2003, Corollary 3.3).

Step 2. Construction of the intertwining unitary

The construction of the intertwining unitary follows as in the proof of Proposition 2 of (Panati et al. 2003a). Firstly one constructs a formal symbol $u \asymp \sum_{j=0}^\infty \varepsilon^j u_j$ such that: (i) $u^\dagger\sharp u = u\sharp u^\dagger = \mathbb{1}_{\mathcal{H}_{\mathrm{f}}}$; (ii) $u\sharp\pi\sharp u^\dagger = \pi_{\mathrm{r}}$.

The existence of such a symbol follows from a recursive procedure starting from u_0 and using the expansion of $\pi \asymp \sum_{j=0}^\infty \varepsilon^j \pi_j$ obtained above. However, the symbol u which comes out of this procedure is not unique.

Since u_0 is *right τ-covariant* (c.f. end of Section 3.3.3) in κ, then one can prove by induction that the same is also true for all the symbols u_j and hence for the full symbol u. Finally, since $u_0 \in S^1(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$ one deduces by induction also $u_j \in S^1(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$ for all $j \in \mathbb{N}$. The quantization of this symbol is an element of $\mathscr{B}(\mathcal{H}_\tau, \mathcal{H}_{\mathrm{ref}})$ satisfying the following properties:

(i) $\mathrm{Op}_\varepsilon(u)\mathrm{Op}_\varepsilon(u)^\dagger = \mathbb{1}_{\mathcal{H}_{\mathrm{ref}}} + \mathcal{O}_0(\varepsilon^\infty)$,

(ii) $\mathrm{Op}_\varepsilon(u)^\dagger\mathrm{Op}_\varepsilon(u) = \mathbb{1}_{\mathcal{H}_\tau} + \mathcal{O}_0(\varepsilon^\infty)$,

(iii) $\mathrm{Op}_\varepsilon(u)\Pi_\varepsilon\mathrm{Op}_\varepsilon(u)^\dagger = \Pi_\mathrm{r} + \mathcal{O}_0(\varepsilon^\infty)$.

Nevertheless $\mathrm{Op}_\varepsilon(u)$ can be modified by an $\mathcal{O}_0(\varepsilon^\infty)$ term using the same technique of Lemma 3.3 (Step II) in (Panati et al. 2003b) to obtain the true unitary U_ε.

Step 3. Effective dynamics

The last step of the proof is identical to the corresponding part of (Panati et al. 2003a, Proposition 3).

3.3.7 Hofstadter-like Hamiltonians

We now focus on the special case of a single isolated energy band E_*, i.e. $m = 1$, and we comment on the relation between the effective Hamiltonian, the celebrated Peierls' substitution and Hofstadter-like Hamiltonians (c.f. Sections 2.1 and 2.3).

In this special case, $\pi_0(\kappa) = |\psi_*(\kappa)\rangle\langle\psi_*(\kappa)|$ and $u_0(\kappa) = |\chi\rangle\langle\psi_*(\kappa)| + u_0^\perp$ where $\psi_*(k)$ is the eigenvector of $H_\mathrm{per}(k)$ corresponding to the eigenvalue $\mathcal{E}_*(k)$. Let $h \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ be a resummation of the formal symbol $u\sharp\pi\sharp H_0\sharp\pi\sharp u^{-1}$. A straightforward computation yields

$$h_0 = u_0\,\pi_0\,H_0\,\pi_0\,u_0^\dagger = |\chi\rangle\langle\psi_*|\;|\psi_*\rangle\langle\psi_*|\;H_0\;|\psi_*\rangle\langle\psi_*|\;|\psi_*\rangle\langle\chi| = E_*\,\pi_\mathrm{r}.$$

Since π_r is one-dimensional, h_0 can be regarded as a scalar-valued symbol with explicit expression

$$h_0(k,\eta) = E_*(k,\eta) = \mathcal{E}_*\left(k - A(\eta)\right) + \phi(\eta).$$

By considering the quantization of the latter, the effective one-band Hamiltonian reads

$$\mathrm{Op}_\varepsilon(h_0) = E_*(k, i\varepsilon\nabla_k) = \mathcal{E}_*\left(k - A(i\varepsilon\nabla_k)\right) + \phi(i\varepsilon\nabla_k). \tag{3.39}$$

The latter formula corresponds to the momentum-space reformulation of the well-known Peierls' substitution (Peierls 1933, Ashcroft and Mermin 1976).

To illustrate this point, we specialize to the case of a uniform external magnetic field and zero external electric field, setting $\phi = 0$ and $A_0 = 0$ in (3.28). The leading order contribution to the dynamics in the almost invariant subspace is therefore given by a bounded operator, acting on the reference Hilbert space $L^2(M_{\Gamma^*}, d^2\underline{k})$, defined as the quantization (in the sense of Section 3.3.5) of the function $\mathcal{E}_* \circ \jmath_\kappa : (k,\eta) \mapsto \mathcal{E}_*(k - A(\eta))$, defined on $\mathbb{T}^d \times \mathbb{R}^d$.

Loosely speaking, the above procedure corresponds to the following "substitution rule": one may think to quantize the smooth function $\mathcal{E}_* : \mathbb{T}^d \to \mathbb{R}$ by formally replacing the variables (k_1, k_2) with the operators $(\mathscr{K}_1', \mathscr{K}_2')$ defined by

$$\mathscr{K}_1' := k_1 + \frac{i}{2}(\iota_q\varepsilon)\frac{\partial}{\partial k_2}, \qquad \mathscr{K}_2' := k_2 - \frac{i}{2}(\iota_q\varepsilon)\frac{\partial}{\partial k_1}, \tag{3.40}$$

regarded as unbounded operators acting on $L^2(M_{\Gamma^*}, d^2\underline{k})$. To make this procedure rigorous, one can expand $\mathcal{E}_*$ in its Fourier series, i.e. $\mathcal{E}_*(k) = \sum_{n,m\in\mathbb{Z}} c_{n,m}\ \mathrm{e}^{i2\pi(na+mb)\cdot k}$ and define the *Peierls quantization* of $\mathcal{E}_*$ as the operator obtained by the same series expansion with the phases $e^{i2\pi(na+mb)\cdot k}$ replaced by the unitary operators $e^{i2\pi(na+mb)\cdot \mathscr{K}'}$ (the series is norm-convergent, in view of the regularity of $\mathcal{E}_*$). This fixes uniquely the prescription for the quantization.

To streamline the notation, one introduces new coordinates $\xi_1 := 2\pi(a \cdot k)$ and $\xi_2 := 2\pi(b \cdot k)$ such that the function $\mathcal{E}'_*$, $\mathcal{E}'_*(\xi_1, \xi_2) := \mathcal{E}_*(k(\xi))$ becomes $(2\pi\mathbb{Z})^2$-periodic. The change of variables induces a unitary map from $L^2(M_{\Gamma^*}{}^2, d^2\underline{k})$ to $L^2(\mathbb{T}^2, d^2\xi)$ which intertwines the operators (3.40) with the operators (recall $\varepsilon = {}^{2\pi}/_{h_B}$)

$$\mathscr{K}_1 := \xi_1 + i\pi \left(\frac{\iota_q}{h_B}\right) \frac{\partial}{\partial \xi_2}, \qquad \mathscr{K}_2 := \xi_2 - i\pi \left(\frac{\iota_q}{h_B}\right) \frac{\partial}{\partial \xi_1}, \tag{3.41}$$

so that $2\pi(a \cdot \mathscr{K}') \mapsto \mathscr{K}_1$ and $2\pi(b \cdot \mathscr{K}') \mapsto \mathscr{K}_2$.

Let $F : \mathbb{T}^2 \to \mathbb{C}$ be sufficiently regular that its Fourier series

$$F(\xi_1, \xi_2) = \sum_{n,m \in \mathbb{Z}} f_{n,m} e^{i(n\xi_1 + m\xi_2)}$$

is uniformly-convergent. We define the Peierls quantization of F as

$$\widehat{F} := \sum_{n,m \in \mathbb{Z}} f_{n,m} e^{i(n\mathscr{K}_1 + m\mathscr{K}_2)}.$$

Let $\mathcal{U}_0 = e^{i\mathscr{K}_1}$ and $\mathcal{V}_0 = e^{i\mathscr{K}_2}$ (*Hofstadter unitaries*), acting on $\mathcal{H}_0 := L^2(\mathbb{T}^2, d^2\xi)$ as

$$(\mathcal{U}_0\psi)(\xi_1, \xi_2) = e^{i\xi_1}\psi(\xi_1, \xi_2 - \pi\iota_q\epsilon_0), \qquad (\mathcal{V}_0\psi)(\xi_1, \xi_2) = e^{i\xi_2}\psi(\xi_1 + \pi\iota_q\epsilon_0, \xi_2). \tag{3.42}$$

where $\epsilon_0(B) := {}^1/_{h_B}$ We regard (3.42) as the definition of the two unitaries, so there is no need to specify the domain of definition of the generators (3.41). Thus the Peierls quantization of the function F defines a bounded operator on $\mathcal{H}_0$ given, in terms of the Hofstadter unitaries, by

$$\widehat{F} = \sum_{n,m=-\infty}^{+\infty} f_{n,m}\, e^{i\pi nm(\iota_q\epsilon_0)}\, \mathcal{U}_0^n\, \mathcal{V}_0^m, \tag{3.43}$$

where the fundamental commutation relation $\mathcal{U}_0\mathcal{V}_0 = e^{-i2\pi(\iota_q\epsilon_0)}\, \mathcal{V}_0\mathcal{U}_0$ has been used. Formula (3.43) defines a *Hofstadter-like Hamiltonian* with *deformation parameter* $-\iota_q\epsilon_0$ (c.f. Section 2.1). Summarizing, we draw the following

CONCLUSION 3.3.16. *Under the assumption of Theorem 3.3.14, for every* $\mathsf{V}_\Gamma \in L^2_{\mathrm{loc}}(\mathbb{R}^2, d^2r)$, *in the Hofstadter regime* ($h_B \to \infty$)*, the dynamics generated by the Hamiltonian* H_{BL} (2.1) *in the subspace related to a single isolated Bloch band, is approximated up to an error of order* $\epsilon_0 = {}^1/_{h_B}$ *(and up to a unitary transform) by the dynamics generated on the reference Hilbert space* $\mathcal{H}_0 := L^2(\mathbb{T}^2, d^2\xi)$ *by a Hofstadter-like Hamiltonian, i.e. by a power series in the Hofstadter unitaries* $\mathcal{U}_0$ *and* $\mathcal{V}_0$ *defined by* (3.42).

3.4 Space-adiabatic theory for the Harper regime

3.4.1 Adiabatic parameter for strong magnetic fields

We now consider the case of a strong external magnetic field. Since we are interested in the limit $B \to +\infty$ we set $\mathbf{A}_0 = 0$ and $\Phi = 0$ in the Hamiltonian (3.1). By exploiting the gauge freedom, we choose

$$\nabla_r \cdot \mathbf{A}_\Gamma = 0, \qquad \int_{M_\Gamma} \mathbf{A}_\Gamma(r)\, d^2 r = 0, \tag{3.44}$$

this choice being always possible (Sobolev 1997). Let us denote by $Q_r = (Q_{r_1}, Q_{r_2})$ the multiplication operators by r_1 and r_2 and with $P_r = (P_{r_1}, P_{r_2}) = -i\hbar\nabla_r$. Taking into account conditions (3.44) and $A_0 = 0$, $\phi = 0$, the Hamiltonian (3.1) is rewritten as

$$\mathbf{H}_{\mathrm{BL}} = \frac{1}{2m}\left[\left(P_{r_1} + \frac{qB}{2c}Q_{r_2}\right)^2 + \left(P_{r_2} - \frac{qB}{2c}Q_{r_1}\right)^2\right] + \widetilde{\mathsf{V}}_\Gamma(Q_r) + \widetilde{\mathsf{W}}(Q_r) \tag{3.45}$$

where

$$\widetilde{\mathsf{V}}_\Gamma(Q_r) = \mathsf{V}_\Gamma(Q_r) + \frac{q^2}{2mc^2}|\mathbf{A}_\Gamma(Q_r)|^2 \tag{3.46}$$

and

$$\widetilde{\mathsf{W}}(Q_r, P_r) = -\frac{q}{mc}(\mathsf{A}_\Gamma)_1(Q_r)\,\left[P_{r_1} + \frac{qB}{2c}Q_{r_2}\right] - \frac{q}{mc}(\mathsf{A}_\Gamma)_2(Q_r)\,\left[P_{r_2} - \frac{qB}{2c}Q_{r_1}\right] \tag{3.47}$$

with $(\mathsf{A}_\Gamma)_1$ and $(\mathsf{A}_\Gamma)_2$ the Γ-periodic components of the vector potential $\mathbf{A}_\Gamma$. The first of (3.44) assures that $\widetilde{\mathsf{W}}$ is a symmetric operator.

Remark 3.4.1. Observe that $\mathbf{H}^0_{\mathrm{BL}} := \mathbf{H}_{\mathrm{BL}} - \widetilde{\mathsf{W}}$ corresponds to a Bloch-Landau Hamiltonian without Γ-periodic vector potential and with a "modified" Γ-periodic scalar potential $\widetilde{\mathsf{V}}_\Gamma$. Then, the presence of a Γ-periodic vector potential $\mathbf{A}_\Gamma$ can be described through a non-periodic "perturbation" $\widetilde{\mathsf{W}}$ of the Bloch-Landau Hamiltonian H^0_{BL}. Obviously $\mathbf{A}_\Gamma = 0$ implies $\widetilde{\mathsf{V}}_\Gamma = \mathsf{V}_\Gamma$ and $\widetilde{\mathsf{W}} = 0$. ♦◊

It is useful to define two new pairs of canonical dimensionless operators:

$$\text{(fast)}\begin{cases} K_1 := -\dfrac{1}{2\delta}\, b^* \cdot Q_r - \iota_q \dfrac{\delta}{\hbar}\, a \cdot P_r \\[2mm] K_2 := \quad\dfrac{1}{2\delta}\, a^* \cdot Q_r - \iota_q \dfrac{\delta}{\hbar}\, b \cdot P_r \end{cases} \qquad \text{(slow)}\begin{cases} G_1 := \dfrac{1}{2}\, b^* \cdot Q_r - \iota_q \dfrac{\delta^2}{\hbar}\, a \cdot P_r \\[2mm] G_2 := \dfrac{1}{2}\, a^* \cdot Q_r + \iota_q \dfrac{\delta^2}{\hbar}\, b \cdot P_r \end{cases} \tag{3.48}$$

where $\delta := \sqrt{\hbar_B} = \sqrt{\Phi_0/2\pi Z\Phi_B}$ according to the notation introduced in Section 3.1. Since $\delta^2 \propto 1/B$, the limit of strong magnetic field corresponds to $\delta \to 0$. We consider δ as the *adiabatic parameter* in the Harper regime. A direct computation shows that

$$[K_1; K_2] = i\iota_q\, \mathbb{1}_{\mathcal{H}_{\mathrm{phy}}}, \qquad [G_1; G_2] = i\iota_q\, \delta^2\, \mathbb{1}_{\mathcal{H}_{\mathrm{phy}}}, \qquad [K_j; G_k] = 0, \qquad j,k = 1,2. \tag{3.49}$$

These new variables are important for three reasons:

(a) they make evident a separation of scales between the slow degrees of freedom related to the the dynamics induced by the periodic potential and the fast degrees of freedom related to the cyclotron motion induced by the external magnetic field. Indeed, for $\mathsf{V}_\Gamma = 0$, the *fast variables* (K_1, K_2) (the *kinetic momenta*) describe the kinetic energy of the cyclotron motion, while the *slow variables* (G_1, G_2) correspond semiclassically to the center of the cyclotron orbit and are conserved quantities.

(b) The new variables are dimensionless. According to the notation used in Section 3.3.1 let $H_{\rm BL} := {}^1\!/\!_{\mathcal{E}_0}\mathbf{H}_{\rm BL}$ be the dimensionless Bloch-Landau Hamiltonian with $\mathcal{E}_0 := {}^{\hbar^2}\!/\!_{m\Omega_\Gamma}$ the *natural unit of energy*.

(c) The use of the new variables simplifies the expression of the Γ-periodic functions appearing in $\mathbf{H}_{\rm BL}$. Indeed, $a^* \cdot Q_r = G_2 + \delta\ K_2$ and $b^* \cdot Q_r = G_1 - \delta\ K_1$, hence if f_Γ is any Γ-periodic function one has

$$f_\Gamma(Q_r) = f(G_2 + \delta\ K_2, G_1 - \delta\ K_1) \tag{3.50}$$

where f is the $\mathbb{Z}^2$-periodic function related to f_Γ.

In terms of the new variables (3.48), the Hamiltonian $H_{\rm BL}$ reads

$$H_{\rm BL} = \frac{1}{\delta^2}\ \Xi(K_1, K_2) + V\left(G_2 + \delta\ K_2, G_1 - \delta\ K_1\right) + \frac{1}{\delta}\ W(K_1, G_1, K_2, G_2) \tag{3.51}$$

where

$$\Xi(K_1, K_2) := \frac{1}{2\Omega_\Gamma}\left[|a|^2 {K_2}^2 + |b|^2 {K_1}^2 - a \cdot b\ \{K_1; K_2\}\right] \tag{3.52}$$

is a quadratic function of the operators K_1 and K_2 ($\{\cdot;\cdot\}$ denotes the anticommutator), V is the $\mathbb{Z}^2$-periodic function related to the Γ-periodic function ${}^1\!/\!_{\mathcal{E}_0}\widetilde{\mathsf{V}}_\Gamma$ and W denotes the function ${}^1\!/\!_{\mathcal{E}_0}\widetilde{\mathsf{W}}$ with respect the new canonical pairs, namely

$$W(K_1, G_1, K_2, G_2) = f_1\left(G_2 + \delta\ K_2, G_1 - \delta\ K_1\right) K_1 - f_2\left(G_2 + \delta\ K_2, G_1 - \delta\ K_1\right) K_2 \tag{3.53}$$

where f_1 and f_2 are the $\mathbb{Z}^2$-periodic dimensionless functions

$$f_1(a^* \cdot r, b^* \cdot r) := 2\pi \frac{Z\Omega_\Gamma}{\Phi_0}(a^* \cdot \mathbf{A}_\Gamma)(r) \qquad \text{and} \qquad f_2(a^* \cdot r, b^* \cdot r) := 2\pi \frac{Z\Omega_\Gamma}{\Phi_0}(b^* \cdot \mathbf{A}_\Gamma)(r).$$

An easy computation shows that the first gauge condition of (3.44) is equivalent to

$$\frac{\partial f_1}{\partial x_1}(x_1, x_2) + \frac{\partial f_2}{\partial x_2}(x_1, x_2) = 0. \tag{3.54}$$

Obviously W is a symmetric operator, since $\widetilde{\mathsf{W}}$ is symmetric.

The problem has a natural time-scale which is fixed by the *cyclotron frequency* $\omega_{\rm c} = \frac{|q|B}{mc}$. With respect to the (fast) *ultramicroscopic time-scale* $\tau := \omega_{\rm c} s$, equation (3.4) becomes

$$i \frac{1}{\delta^2} \frac{\partial}{\partial \tau} \psi = H_{\rm BL} \psi, \qquad \qquad \delta^2 = \frac{\mathcal{E}_0}{\hbar \omega_{\rm c}}. \tag{3.55}$$

Thus the physically relevant Hamiltonian is

$$H^\delta_{\rm BL} := \delta^2\ H_{\rm BL} = \Xi(K_1, K_2) + \delta\ W(K_1, G_1, K_2, G_2) + \delta^2\ V\left(G_2 + \delta\ K_2, G_1 - \delta\ K_1\right). \tag{3.56}$$

3.4.2 Separation of scales: the von Neumann unitary

The commutation relations (3.49) show that (K_1, K_2) and (G_1, G_2) are two pairs of canonical conjugate operators. The *Stone-von Neumann uniqueness Theorem* (Bratteli and Robinson 1997, Corollary 5.2.15) assures the existence of a unitary map $\mathcal{W}$ (called *von Neumann unitary*)

$$\mathcal{W} : \mathcal{H}_{\text{phy}} \longrightarrow \mathcal{H}_{\text{w}} := \mathcal{H}_{\text{s}} \otimes \mathcal{H}_{\text{f}} = L^2(\mathbb{R}, dx_{\text{s}}) \otimes L^2(\mathbb{R}, dx_{\text{f}}) \tag{3.57}$$

such that

$$\mathcal{W}G_1\mathcal{W}^{-1} = Q_{\text{s}} = \text{multiplication by } x_{\text{s}}, \qquad \mathcal{W}G_2\mathcal{W}^{-1} = P_{\text{s}} = -i\iota_q\delta^2\frac{\partial}{\partial x_{\text{s}}} \tag{3.58}$$

$$\mathcal{W}K_1\mathcal{W}^{-1} = Q_{\text{f}} = \text{multiplication by } x_{\text{f}}, \qquad \mathcal{W}K_2\mathcal{W}^{-1} = P_{\text{f}} = -i\iota_q\,\frac{\partial}{\partial x_{\text{f}}}. \tag{3.59}$$

The explicit construction of the von Neumann unitary $\mathcal{W}$ is described in Appendix A.2.

Let $X_j := G_j + (-1)^j\delta\, K_j$ with $j = 1, 2$. From (3.58) and (3.59) it follows that

$$X_1' := \mathcal{W}X_1\mathcal{W}^{-1} = Q_{\text{s}} - \delta\, Q_{\text{f}}, \qquad X_2' := \mathcal{W}X_2\mathcal{W}^{-1} = P_{\text{s}} + \delta\, P_{\text{f}}. \tag{3.60}$$

Since X_1 and X_2 commute, one can use the spectral calculus to define any measurable function of X_1 and X_2. For any $f \in L^\infty(\mathbb{R}^2, d^2x)$ one defines

$$f(X_1, X_2) := \int_{\mathbb{R}^2} f(x_1, x_2)\; d\mathbb{E}^{(1)}_{x_1}\; d\mathbb{E}^{(2)}_{x_2}$$

where $d\mathbb{E}^{(j)}$ is the projection-valued measure corresponding to X_j. In view of the unitarity of $\mathcal{W}$, and observing that $d\mathbb{E}'^{(j)} := \mathcal{W}d\mathbb{E}^{(j)}\mathcal{W}^{-1}$ is the projection-valued measure of X_j', one obtains that

$$\mathcal{W}f(X_1, X_2)\mathcal{W}^{-1} = \int_{\mathbb{R}^2} f(x_1, x_2)\; d\mathbb{E}'^{(1)}_{x_1}\; d\mathbb{E}'^{(2)}_{x_2} = f(X_1', X_2').$$

So the effect of the conjugation through $\mathcal{W}$ on a function f of the operators X_1 and X_2 formally amounts to replace the operators X_j with X_j' inside f.

In view of the above remark, one can easily rewrite H^δ_{BL}, making explicit the rôle of the fast and slow variables, obtaining

$$H^{\mathcal{W}} := \mathcal{W}H^\delta_{\text{BL}}\mathcal{W}^{-1} = \mathbb{1}_{\mathcal{H}_{\text{s}}} \otimes \Xi(Q_{\text{f}}, P_{\text{f}}) + \delta\, W(Q_{\text{f}}, Q_{\text{s}}, P_{\text{f}}, P_{\text{s}}) + \delta^2\, V\,(P_{\text{s}} + \delta\, P_{\text{f}}, Q_{\text{s}} - \delta\, Q_{\text{f}}) \tag{3.61}$$

where, according to (3.53),

$$W(Q_{\text{f}}, Q_{\text{s}}, P_{\text{f}}, P_{\text{s}}) = f_1\,(P_{\text{s}} + \delta\, P_{\text{f}}, Q_{\text{s}} - \delta\, Q_{\text{f}})\; Q_{\text{f}} - f_2\,(P_{\text{s}} + \delta\, P_{\text{f}}, Q_{\text{s}} - \delta\, Q_{\text{f}})\; P_{\text{f}}. \tag{3.62}$$

3.4.3 Relevant part of the spectrum: the Landau bands

The existence of a separation between fast and slow degrees of freedom and the decomposition of the physical Hilbert space $\mathcal{H}_{\text{phy}}$ into the product space $\mathcal{H}_{\text{w}} = \mathcal{H}_{\text{s}} \otimes \mathcal{H}_{\text{f}}$ are the first two ingredients to develop the SAPT. According to the general scheme, we "replace" the canonical operators corresponding to the slow degrees of freedom with classical variables which will be re-quantized "a posteriori". Mathematically, we show that the Hamiltonian $H^{\mathcal{W}}$ acting in $\mathcal{H}_{\text{w}}$ is the Weyl quantization of the operator-valued function (symbol) H_δ,

$$H_\delta(p_{\text{s}}, x_{\text{s}}) := \Xi(Q_{\text{f}}, P_{\text{f}}) + \delta \underbrace{W(Q_{\text{f}}, x_{\text{s}}, P_{\text{f}}, p_{\text{s}})}_{=W_\delta(p_{\text{s}}, x_{\text{s}})} + \delta^2 \underbrace{V\left(p_{\text{s}} + \delta\, P_{\text{f}}, x_{\text{s}} - \delta\, Q_{\text{f}}\right)}_{=V_\delta(p_{\text{s}}, x_{\text{s}})}. \tag{3.63}$$

The quantization is defined (formally) by the rules

$$x_{\text{s}} \longmapsto \mathbf{Op}_\delta(x_{\text{s}}) := Q_{\text{s}} \otimes \mathbb{1}_{\mathcal{H}_{\text{f}}}, \qquad p_{\text{s}} \longmapsto \mathbf{Op}_\delta(p_{\text{s}}) := P_{\text{s}} \otimes \mathbb{1}_{\mathcal{H}_{\text{f}}}.$$

For every $(p_{\text{s}}, x_{\text{s}}) \in \mathbb{R}^2$, equation (3.63) defines an unbounded operator $H_\delta(p_{\text{s}}, x_{\text{s}})$ which acts in the Hilbert space $\mathcal{H}_{\text{f}}$. To make the quantization procedure rigorous, as explained in Appendix A of (Panati et al. 2003b), we need to consider H_δ as function from $\mathbb{R}^2$ into some Banach space which is also a domain of self-adjointness for $H_\delta(p_{\text{s}}, x_{\text{s}})$. We take care of this details in the Section 3.4.4.

To complete the list of ingredients needed for the SAPT, we need to analize the spectrum of the principal part of the symbol (3.63) as $(p_{\text{s}}, x_{\text{s}})$ varies in $\mathbb{R}^2$. The principal part of the symbol, denoted by $H_0(p_{\text{s}}, x_{\text{s}})$, is given by (3.63) when $\delta = 0$, so it reads:

$$H_0(p_{\text{s}}, x_{\text{s}}) := \Xi(Q_{\text{f}}, P_{\text{f}}) = \frac{1}{2\Omega_\Gamma}\left[|a|^2 {P_{\text{f}}}^2 + |b|^2 {Q_{\text{f}}}^2 - a \cdot b\ \{Q_{\text{f}}; P_{\text{f}}\}\right]. \tag{3.64}$$

Since the principal symbol is constant on the phase space, i.e. $H_0(p_{\text{s}}, x_{\text{s}}) = \Xi$ for all $(p_{\text{s}}, x_{\text{s}}) \in \mathbb{R}^2$, we are reduced to compute the spectrum of Ξ. As well-known (see Remark 3.4.2 below), the spectrum of Ξ is pure point with $\sigma(\Xi) = \{\lambda_n := (n + 1/2) : n \in \mathbb{N}\}$. We refer to the eigenvalue λ_n as the n-th *Landau level*.

The spectrum of the symbol H_0 consists of a collection of constant functions $\sigma_n : \mathbb{R}^2 \to \mathbb{R}$, $n \in \mathbb{N}$, $\sigma_n(p_{\text{s}}, x_{\text{s}}) \equiv \lambda_n$, which we call *Landau bands*. The band σ_n is separated by the rest of the spectrum by a constant gap. In the gap condition (analogous to (3.23)) one can choose $C_{\text{g}} = 1$. Therefore, each finite family of contiguous Landau bands defines a relevant part of the spectrun appropriate to develop the SAPT.

REMARK 3.4.2 (Domain of self-adjointness)**.** We describe explicitly the domain of self-adjointness of $H_0(p_{\text{s}}, x_{\text{s}})$. Mimicking the standard theory of Landau levels, one introduces operators

$$\mathfrak{a} := \frac{i}{\sqrt{2}} \frac{\ell}{\Omega_\Gamma}\left[(a_1 + i a_2) P_{\text{f}} - (b_1 + i b_2) Q_{\text{f}}\right] = \frac{i}{\sqrt{2}}\left[\overline{z_a}\, P_{\text{f}} - \overline{z_b}\, Q_{\text{f}}\right] \tag{3.65}$$

$$\mathfrak{a}^\dagger := \frac{-i}{\sqrt{2}} \frac{\ell}{\Omega_\Gamma}\left[(a_1 - i a_2) P_{\text{f}} - (b_1 - i b_2) Q_{\text{f}}\right] = \frac{-i}{\sqrt{2}}\left[z_a\, P_{\text{f}} - z_b\, Q_{\text{f}}\right], \tag{3.66}$$

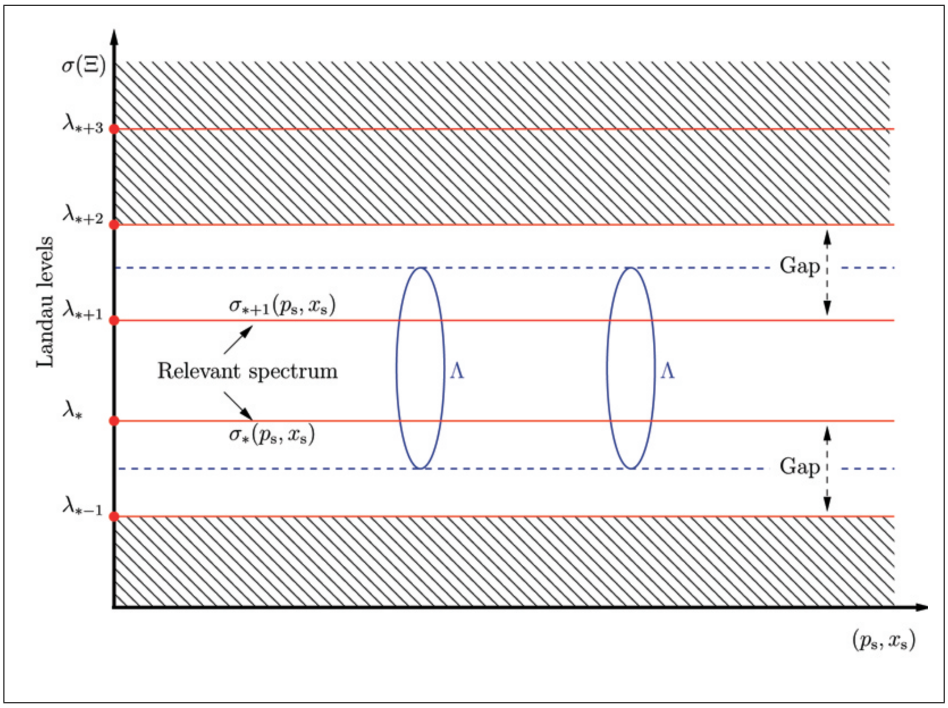


Figure 3.2: *Structure of the spectrum of H_0. The picture shows a "relevant part of the spectrum" consisting of two Landau bands of constant energy λ_* and λ_{*+1}.*

where $z_a := \frac{1}{\ell}(a_1 - ia_2)$ and $z_b := \frac{1}{\ell}(b_1 - ib_2)$. It is easy to check that

$$\mathfrak{a}\mathfrak{a}^\dagger = \Xi(Q_\mathrm{f}, P_\mathrm{f}) + \iota_q \frac{1}{2}\mathbb{1}_{\mathcal{H}_\mathrm{f}}, \qquad \mathfrak{a}^\dagger\mathfrak{a} = \Xi(Q_\mathrm{f}, P_\mathrm{f}) - \iota_q \frac{1}{2}\mathbb{1}_{\mathcal{H}_\mathrm{f}}, \qquad [\mathfrak{a}; \mathfrak{a}^\dagger] = \iota_q \, \mathbb{1}_{\mathcal{H}_\mathrm{f}}. \tag{3.67}$$

Without loss of generality, we suppose that $\iota_q = 1$. Let ψ_0 be the *ground state* defined by $\mathfrak{a}\psi_0 = 0$. A simple computation shows that $\psi_0(x_\mathrm{f}) = C\mathrm{e}^{-(\beta - i\alpha)x_\mathrm{f}^2}$, where $C > 0$ is a normalization constant, and $\alpha \in \mathbb{R}$, $\beta > 0$ are related to the geometry of the lattice Γ by $\alpha := {}^{a\cdot b}/_{2|a|^2}$ and $\beta := {}^{\Omega_\Gamma}/_{2|a|^2}$. Since ψ_0 is a fast decreasing smooth function, the vectors $\psi_n := (n!)^{-\frac{1}{2}}(\mathfrak{a}^\dagger)^n\psi_0$, with $n = 0, 1, \ldots$, are well defined. From the algebraic relations (3.67) it follows straightforwardly that: (i) $\mathfrak{a}\psi_n = \sqrt{n}\psi_{n-1}$; (ii) the family of vectors $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_\mathrm{f}$ called the *generalized Hermite basis*; (iii) $\Xi\psi_n = \lambda_n\psi_n$; (iv) the spectrum of Ξ is pure point with $\sigma(\Xi) = \{\lambda_n \ : \ n \in \mathbb{N}\}$.

Let $\mathcal{L} \subset \mathcal{H}_\mathrm{f}$ be the set of the finite linear combinations of the elements of the basis $\{\psi_n\}_{n\in\mathbb{N}}$. The unbounded operators $\mathfrak{a}$, $\mathfrak{a}^\dagger$ and Ξ are well defined on $\mathcal{L}$ and on this domain $\mathfrak{a}^\dagger$ acts as the adjoint of $\mathfrak{a}$ and Ξ is symmetric. Both $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ are closable and we will denote their closure by the same symbols. The operator Ξ is essentially selfadjoint on the domain $\mathcal{L}$ (the deficiency indices are both zero) and so its domain of selfadjointness $\mathcal{F} := \mathcal{D}(\Xi)$ is the closure of $\mathcal{L}$ with respect to the *graph norm* $\|\psi\|_\Xi^2 := \|\psi\|_{\mathcal{H}_\mathrm{f}}^2 + \|\Xi\psi\|_{\mathcal{H}_\mathrm{f}}^2$. The graph norm is equivalent to the more simple *regularized norm*

$$\|\psi\|_\mathcal{F} := \|\Xi\psi\|_{\mathcal{H}_\mathrm{f}}.$$

The domain $\mathcal{F}$ has the structure of an Hilbert space with Hermitian structure provided by the *regularized scalar product* $(\psi;\varphi)_{\mathcal{F}} := (\Xi\psi;\Xi\varphi)_{\mathcal{H}_{\mathrm{f}}}$. ♦◊

3.4.4 Symbol class and asymptotic expansion

In this section, we firstly identify the Banach space in which the symbol H_δ defined by (3.63) takes values and, secondarily, we explain in which sense H_δ is a "semiclassical symbol" in a suitable Hörmander symbol class. The main results are contained in Proposition 3.4.3. Readers who are not interested in technical details can jump directly to the next section. For the definitions of the Hörmander classes $S^1(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$ and $S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_{\mathrm{f}}))$ we refer to Section 3.3.4.

Proposition 3.4.3. *Assume that Assumption 3.2.1 hold true. Then for any $(p_{\mathrm{s}},x_{\mathrm{s}})\in\mathbb{R}^2$ the operator $H_\delta(p_{\mathrm{s}},x_{\mathrm{s}})$ is essentially self-adjoint on the dense domain $\mathcal{L}\subset\mathcal{H}_{\mathrm{f}}$ consisting of finite linear combinations of generalized Hermite functions, and its domain of self-adjointness is the domain $\mathcal{F}$ on which the operator $H_0=\Xi$ is self-adjoint. Finally, H_δ is in the Hörmander class $S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_{\mathrm{f}}))$.*

In particular, $H_\delta(p_{\mathrm{s}},x_{\mathrm{s}})$ is a bounded operator from the Hilbert space $\mathcal{F}$ to the Hilbert space $\mathcal{H}_{\mathrm{f}}$ for all $(p_{\mathrm{s}},x_{\mathrm{s}})\in\mathbb{R}^2$. The proof of the Proposition 3.4.3 follows from the *Kato-Rellich Theorem* showing that for any $(p_{\mathrm{s}},x_{\mathrm{s}})\in\mathbb{R}^2$ the operator $H_\delta(p_{\mathrm{s}},x_{\mathrm{s}})$ differs from H_0 by a relatively bounded perturbation. The latter claim will be proved in Lemmas 3.4.4 and 3.4.5 below.

In view of Assumption 3.2.1, $\widetilde{\mathsf{V}}_\Gamma\in C^\infty_{\mathrm{b}}(\mathbb{R}^2,\mathbb{R})$ so its Fourier series

$$\widetilde{\mathsf{V}}_\Gamma(r)=\sum_{n,m\in\mathbb{Z}} w_{n,m}\ \mathrm{e}^{i2\pi n\ a^*\cdot r}\mathrm{e}^{i2\pi m\ b^*\cdot r}$$

converges uniformly and

$$\sum_{n,m=-\infty}^{+\infty}|m|^{\alpha_1}|n|^{\alpha_2}\,|w_{n,m}|\leqslant C_\alpha$$

for all $\alpha\in\mathbb{N}^2$ (Walker 1988, Theorem 3.6).

Let V be the $\mathbb{Z}^2$-periodic function related to ${}^1\!/\!\varepsilon_0\widetilde{\mathsf{V}}_\Gamma$, as in Section 3.4.1. In view of (3.50) one has

$$\mathcal{W}\frac{\widetilde{\mathsf{V}}_\Gamma}{\mathcal{E}_0}(X_1,X_2)\mathcal{W}^{-1}=V(P_{\mathrm{s}}+\delta P_{\mathrm{f}},Q_{\mathrm{s}}-\delta P_{\mathrm{f}})=\sum_{n,m=-\infty}^{+\infty}v_{n,m}\ \mathrm{e}^{i2\pi(nP_{\mathrm{s}}+mQ_{\mathrm{s}})}\ \mathrm{e}^{i2\pi\delta(nP_{\mathrm{f}}-mQ_{\mathrm{f}})} \tag{3.68}$$

with $v_{n,m}:={}^1\!/\!\varepsilon_0 w_{n,m}$ and where we used the fact that fast and slow variables commute and $[Q_{\mathrm{s}};P_{\mathrm{s}}]=\delta^2[Q_{\mathrm{f}};P_{\mathrm{f}}]$. The operator (3.68) can be seen as the Weyl quantization of the operator-valued symbol

$$V_\delta(p_{\mathrm{s}},x_{\mathrm{s}}):=\sum_{n,m=-\infty}^{+\infty}v_{n,m}\ \mathrm{e}^{i2\pi(np_{\mathrm{s}}+mx_{\mathrm{s}})}\ \mathrm{e}^{i2\pi\delta(nP_{\mathrm{f}}-mQ_{\mathrm{f}})} \tag{3.69}$$

with quantization rule

$$\mathrm{Op}_\delta\left(\mathrm{e}^{i2\pi(np_{\mathrm{s}}+mx_{\mathrm{s}})}\right) = \mathrm{e}^{i2\pi(nP_{\mathrm{s}}+mQ_{\mathrm{s}})} \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}}. \tag{3.70}$$

LEMMA 3.4.4. *Let Assumption 3.2.1 hold true. Then* $V_\delta \in S^1(\mathscr{B}(\mathcal{H}_f)) \cap S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_{\mathrm{f}}))$*. In particular* $V_\delta(p_{\mathrm{s}}, x_{\mathrm{s}})$ *is a bounded self-adjoint operator on* $\mathcal{H}_{\mathrm{f}}$ *for all* $(p_{\mathrm{s}}, x_{\mathrm{s}}) \in \mathbb{R}^2$.

Proof. It is sufficient to show that $V_\delta \in S^1(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$ since

$$\|\partial^\alpha V_\delta\|_{\mathscr{B}(\mathcal{F};\mathcal{H}_{\mathrm{f}})} = \|(\partial^\alpha V_\delta)\ \Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})} \leqslant 2\|\partial^\alpha V_\delta\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})}$$

in view of $\|\Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})} = 2$. Let $\alpha := (\alpha_1, \alpha_2) \in \mathbb{N}^2$, then

$$\left\|\partial_{p_{\mathrm{s}}}^{\alpha_1}\partial_{x_{\mathrm{s}}}^{\alpha_2} V_\delta(p_{\mathrm{s}}, x_{\mathrm{s}})\right\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})} \leqslant (2\pi)^{|\alpha|} \sum_{n,m+-\infty}^{+\infty} |n|^{\alpha_1}|m|^{\alpha_2}\,|v_{n,m}| \leqslant \frac{(2\pi)^{|\alpha|}}{\mathcal{E}_0} C_\alpha$$

for all $(p_{\mathrm{s}}, x_{\mathrm{s}}) \in \mathbb{R}^2$, as a consequence of the unitarity of $\mathrm{e}^{i2\pi\delta(nP_{\mathrm{f}}-mQ_{\mathrm{f}})}$. The self-adjointness follows by observing that $\{v_{n,m}\}$ are the Fourier coefficients of a real function. ■

Assumption 3.2.1 implies that the Γ-periodic functions $a^*\cdot\mathbf{A}_\Gamma$ and $b^*\cdot\mathbf{A}_\Gamma$ are elements of $C_{\mathrm{b}}^\infty(\mathbb{R}^2,\mathbb{R})$. By the same arguments above, one proves that the operators $f_j(P_{\mathrm{s}} + \delta P_{\mathrm{f}}, Q_{\mathrm{s}} - \delta P_{\mathrm{f}})$, $j = 1, 2$, appearing in (3.62), are the Weyl quantization of the operator-valued functions

$$f_\delta^{(j)}(p_{\mathrm{s}}, x_{\mathrm{s}}) := \sum_{n,m=-\infty}^{+\infty} f_{n,m}^{(j)}\ \mathrm{e}^{i2\pi(np_{\mathrm{s}}+mx_{\mathrm{s}})}\ \mathrm{e}^{i2\pi\delta(nP_{\mathrm{f}}-mQ_{\mathrm{f}})} \qquad j = 1, 2 \tag{3.71}$$

according to (3.70). The coefficients $\frac{1}{2\pi}\frac{\Phi_0}{Z\Omega_\Gamma} f_{n,m}^{(j)}$ are the Fourier coefficients of $a^*\cdot\mathbf{A}_\Gamma$ if $j = 1$ and of $b^*\cdot\mathbf{A}_\Gamma$ if $j = 2$. Thus, equation (3.62) shows that the operator $W(Q_{\mathrm{f}}, Q_{\mathrm{s}}, P_{\mathrm{f}}, P_{\mathrm{s}})$ coincides with the Weyl quantization of the operator-valued symbol

$$W_\delta(p_{\mathrm{s}}, x_{\mathrm{s}}) := f_\delta^{(1)}(p_{\mathrm{s}}, x_{\mathrm{s}})\ Q_{\mathrm{f}} + f_\delta^{(2)}(p_{\mathrm{s}}, x_{\mathrm{s}})\ P_{\mathrm{f}}, \tag{3.72}$$

defined, initially, on the dense domain $\mathcal{L}$.

LEMMA 3.4.5. *Let Assumption 3.2.1 hold true. Then* $f_\delta^{(j)} \in S^1(\mathscr{B}(\mathcal{H}_f)) \cap S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_f))$*, for* $j = 1, 2$*. For any* $(p_{\mathrm{s}}, x_{\mathrm{s}}) \in \mathbb{R}^2$*, the bounded operators* $f_\delta^{(j)}(p_{\mathrm{s}}, x_{\mathrm{s}})$ *are self-adjoint while* $W_\delta(p_{\mathrm{s}}, x_{\mathrm{s}})$ *is symmetric on the dense domain* $\mathcal{L}$ *and infinitesimally bounded with respect to* Ξ*. Finally* $W_\delta \in S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_f))$*.*

Proof. As in the first part of the proof of Lemma 3.4.4, one proves that $f_\delta^{(j)} \in S^1(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))\cap S^1(\mathscr{B}(\mathfrak{F};\mathcal{H}_{\mathrm{f}}))$ and its self-adjointness. The operator $W_\delta(p_{\mathrm{s}}, x_{\mathrm{s}})$ is a linear combination of Q_{f} and P_{f}, which are densely defined on $\mathcal{L}$, multiplied by bounded operators. Using (3.54) one checks by a direct computation that $W_\delta(p_{\mathrm{s}}, x_{\mathrm{s}})$ acts as a symmetric operator on $\mathcal{L}$. Since Q_{f} and P_{f} are infinitesimally bounded with respect to Ξ, then the same holds true for $W_\delta(p_{\mathrm{s}}, x_{\mathrm{s}})$, $(p_{\mathrm{s}}, x_{\mathrm{s}}) \in \mathbb{R}^2$. The last claim follows by observing that

$$\|\partial^\alpha(f_\delta^{(j)}X_{\mathrm{f}})\|_{\mathscr{B}(\mathcal{F};\mathcal{H}_{\mathrm{f}})} = \|(\partial^\alpha f_\delta^{(j)})\ X_{\mathrm{f}}\,\Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})} \leqslant \|X_{\mathrm{f}}\,\Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})}\ \|\partial^\alpha f_\delta^{(j)}\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})},$$

with $j = 1, 2$ and $X_{\mathrm{f}} = Q_{\mathrm{f}}$ or P_{f}. Since $\|X_{\mathrm{f}}\,\Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_{\mathrm{f}})} \leqslant C$ and $f_\delta^{(j)} \in S^1(\mathscr{B}(\mathcal{H}_{\mathrm{f}}))$, the claim is proved. ■

Lemmas 3.4.4 and 3.4.5, together with the fact that $H_0 \equiv \Xi$ is clearly in $S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_{\mathrm{f}}))$ imply the last part of Proposition 3.4.3.

3.4.5 Semiclassics: the $\mathcal{O}(\delta^4)$-approximated symbol

In this section we consider the asymptotic expansion for the symbol H_δ in the parameter δ. The Fourier expansion (3.69) for V_δ and the similar expression for W_δ, namely

$$W_\delta(p_{\mathrm{s}}, x_{\mathrm{s}}) = \sum_{n,m=-\infty}^{+\infty} \mathrm{e}^{i2\pi(np_{\mathrm{s}}+mx_{\mathrm{s}})}\ \mathrm{e}^{i2\pi\delta(nP_{\mathrm{f}}-mQ_{\mathrm{f}})}\left[f^{(1)}_{n,m}\ Q_{\mathrm{f}} + f^{(2)}_{n,m}\ P_{\mathrm{f}}\right], \tag{3.73}$$

suggest a way to expand the symbol H_δ in powers of δ. By inserting the expansion $\mathrm{e}^{i2\pi\delta I_{n,m}} = \sum_{j=0}^{+\infty}\frac{(i2\pi\delta)^j}{j!}{I_{n,m}}^j$, with $I_{n,m} := nP_{\mathrm{f}} - mQ_{\mathrm{f}}$, in (3.69) and (3.73) and by exchanging the order of the series one obtains the formal expansions

$$V_\delta(p_{\mathrm{s}}, x_{\mathrm{s}}) \simeq \sum_{j=0}^{+\infty} \delta^j\ V_{j+2}(p_{\mathrm{s}}, x_{\mathrm{s}}) \qquad W_\delta(p_{\mathrm{s}}, x_{\mathrm{s}}) \simeq \sum_{j=0}^{+\infty} \delta^j\ W_{j+1}(p_{\mathrm{s}}, x_{\mathrm{s}}) \tag{3.74}$$

where

$$V_{j+2}(p_{\mathrm{s}}, x_{\mathrm{s}}) := \frac{(i2\pi)^j}{j!}\sum_{n,m=-\infty}^{+\infty} \mathrm{e}^{i2\pi(np_{\mathrm{s}}+mx_{\mathrm{s}})}\ {I_{n,m}}^j\ v_{n,m} \tag{3.75}$$

and

$$W_{j+1}(p_{\mathrm{s}}, x_{\mathrm{s}}) := \frac{(i2\pi)^j}{j!}\sum_{n,m=-\infty}^{+\infty} \mathrm{e}^{i2\pi(np_{\mathrm{s}}+mx_{\mathrm{s}})}\ {I_{n,m}}^j\ \left[f^{(1)}_{n,m}\ Q_{\mathrm{f}} + f^{(2)}_{n,m}\ P_{\mathrm{f}}\right]. \tag{3.76}$$

In view of (3.54), one easily shows that the operators W_j are (formally) symmetric.

The justification of the formal expansions above requires some cautions: (i) we need to specify the domains of definitions of the unbounded operators ${I_{n,m}}^j$ and consequently the domains of definitions of V_j and W_j; (ii) we need to justify the exchange of the order of the series in the equations (3.69) and (3.73).

As for (i), one notices that

$$I_{n,m} = \alpha_{n,m}\ \mathfrak{a} + \overline{\alpha_{n,m}}\ \mathfrak{a}^\dagger, \qquad \alpha_{n,m} := \frac{nz_b - mz_a}{\sqrt{2}}. \tag{3.77}$$

For all $(n,m) \in \mathbb{Z}^2$ the operators $I_{n,m}$ are essentially self-adjoint on the invariant dense domain $\mathcal{L}$ (their deficiency indices are both zero). The powers ${I_{n,m}}^j$ are also well defined and essentially self-adjoint on $\mathcal{L}$, as consequence of the *Nelson Theorem* (Reed and Simon 1975, Theorem X.39) since the set $\{\psi_n\}_{n\in\mathbb{N}}$ of the generalized Hermite functions is a *total set of analytic vectors* for every $I_{n,m}$ (Reed and Simon 1975, Example 2, Section X.6). The domain of self-adjointness for ${I_{n,m}}^j$ is the closure of $\mathcal{L}$ with respect the corresponding graph norm.

The operator $V_j(p_{\mathrm{s}}, x_{\mathrm{s}})$ defined by equation (3.75) is an homogeneous polynomial of degree $j-2$ in $\mathfrak{a}$ and $\mathfrak{a}^\dagger$. It is symmetric (hence closable) and essentially self-adjoint on the invariant dense domain $\mathcal{L}$. Analogously, the operators

$$M_{n,m}^{j} := {I_{n,m}}^{j} \left[f_{n,m}^{(1)}\, Q_{\mathrm{f}} + f_{n,m}^{(2)}\, P_{\mathrm{f}} \right] = {I_{n,m}}^{j} \left[g_{n,m}\, \mathfrak{a} + \overline{g_{n,m}}\, \mathfrak{a}^{\dagger} \right], \qquad g_{n,m} := \frac{z_a f_{n,m}^{(1)} + z_b f_{n,m}^{(2)}}{\sqrt{2}} \tag{3.78}$$

which appear in the right-hand side of equation (3.76), are essentially self-adjoint on $\mathcal{L}$ since the set of the generalized Hermite functions provides a total set of analytic vectors. Thus we answered to point (i).

Since the generalized Hermite functions are a total set of analytic vectors for any $I_{n,m}$, then the series $\sum_{j=0}^{+\infty} \frac{(i2\pi\delta)^j}{j!} {I_{n,m}}^{j} \psi$ converges in norm for every $\psi \in \mathcal{L}$. From this observation, the fact that the series of coefficients $v_{n,m}$, $f_{n,m}^{(1)}$ and $f_{n,m}^{(2)}$ are absolutely convergent and that Q_{f} end P_{f} leave invariant the domain $\mathcal{L}$, one argues that for all $\psi \in \mathcal{L}$ the double series which defines $V_\delta(p_{\mathrm{s}}, x_{\mathrm{s}})\psi$ and $W_\delta(p_{\mathrm{s}}, x_{\mathrm{s}})\psi$ are absolutely convergent, hence the order of the sums can be exchanged. Thus the series appearing on the right-hand side of (3.74) agrees with V_δ and W_δ respectively on the dense domain $\mathcal{L}$. By a density argument, the equality in (3.74) holds true on the full domain of definition of V_δ (which is $\mathcal{H}_{\mathrm{f}}$) and W_δ respectively.

In view of the above, we write the "semiclassical expansion" of the symbol H_δ as:

$$H_\delta(p_{\mathrm{s}}, x_{\mathrm{s}}) = \Xi + \sum_{j=1}^{+\infty} \delta^j\, H_j(p_{\mathrm{s}}, x_{\mathrm{s}}), \qquad H_j(p_{\mathrm{s}}, x_{\mathrm{s}}) := W_j(p_{\mathrm{s}}, x_{\mathrm{s}}) + V_j(p_{\mathrm{s}}, x_{\mathrm{s}}) \tag{3.79}$$

with $V_1 = 0$.

Proposition 3.4.3 shows that the natural domain for the full symbol $H_\delta(p_{\mathrm{s}}, x_{\mathrm{s}})$ is the domain $\mathcal{F}$ of self-adjointness of Ξ. However, if we want to truncate the series (3.79) at the j-th order, we must be careful in the determination of the domain of definition of the single terms and to control the remainder. Every term in the expansion (3.79) is essentially self-adjoint on $\mathcal{L}$. However, the j-th order term H_j is the sum of two homogeneous polynomials in Q_{f} and P_{f} (or equivalently in $\mathfrak{a}$ and $\mathfrak{a}^\dagger$), W_j of degree j and V_j of degree $j-2$. Since $W_j = 0$ if $\mathbf{A}_\Gamma = 0$, one obtains

$$\deg\, H_j = \begin{cases} j, & \text{if } \mathbf{A}_\Gamma \neq 0 \\ j-2, & \text{if } \mathbf{A}_\Gamma = 0 \end{cases}$$

where deg H_j means the degree of H_j as a polynomial in Q_{f} and P_{f}. If deg $H_j > 2$ then the operator H_j is not bounded by the principal symbol Ξ, and in this sense it cannot be considered as a "small perturbation" in the sense of Kato. Moreover, some other problems appear (see Remark 3.4.9). In order to avoid these problems, we truncate the expansion (3.79) up to the polynomial term of degree 2, i.e. up to order δ^2 if $\mathbf{A}_\Gamma = 0$ and up to order δ^4 if $\mathbf{A}_\Gamma \neq 0$.

Hereafter let $\natural$ be the *indicator function* of the periodic vector potential, defined as

$$\natural = \begin{cases} 0, & \text{if } A_\Gamma \neq 0 \\ 1, & \text{if } A_\Gamma = 0. \end{cases}$$

Let $\widetilde{H}^{\natural}_{\delta}(p_{\rm s},x_{\rm s}) := \Xi + \sum_{j=1}^{2(1+\natural)} \delta^j H_j(p_{\rm s},x_{\rm s})$, namely

$$\widetilde{H}^{1}_{\delta}(p_{\rm s},x_{\rm s}) = \Xi + \delta^2 \sum_{n,m=-\infty}^{+\infty} v_{n,m}\, \mathrm{e}^{i2\pi(np_{\rm s}+mx_{\rm s})} \left(\mathbb{1}_{\mathcal{H}_{\rm f}} + i2\pi\delta I_{n,m} + \frac{1}{2}(i2\pi)^2\delta^2 {I_{n,m}}^2\right) \tag{3.80}$$

$$\widetilde{H}^{0}_{\delta}(p_{\rm s},x_{\rm s}) = \Xi + \delta \sum_{n,m=-\infty}^{+\infty} \mathrm{e}^{i2\pi(np_{\rm s}+mx_{\rm s})} \left[M^0_{n,m} + \delta(i2\pi M^1_{n,m} + v_{n,m}\mathbb{1}_{\mathcal{H}_{\rm f}})\right]. \tag{3.81}$$

We call $\widetilde{H}^{\natural}_{\delta}$ the *approximated symbol* up to order $\delta^{2(1+\natural)}$. As a consequence of the Kato-Rellich theorem we have the following result:

PROPOSITION 3.4.6. *Under Assumption 3.2.1 there exists a constant δ_0 such that for every $\delta < \delta_0$ and for every $(p_s,x_s) \in \mathbb{R}^2$ the operator $\widetilde{H}^{\natural}_{\delta}(p_s,x_s)$ (both for $\natural = 0$ or 1) is self-adjoint on the domain $\mathcal{F}$ and bounded from below. Moreover $\widetilde{H}^{\natural}_{\delta} \in S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_{\rm f}))$.*

Proof. As proved in Lemma A.1.2, $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ are infinitesimally bounded with respect to Ξ. This fact and Assumption 3.2.1, which assures the fast decay of the coefficients $v_{n,m}$ and $g_{n,m}$ (see (3.78)), imply that the operators

$$\sum_{n,m=-\infty}^{+\infty} v_{n,m}\, \mathrm{e}^{i2\pi(np_{\rm s}+mx_{\rm s})} \left(\mathbb{1}_{\mathcal{H}_{\rm f}} + i2\pi\delta I_{n,m}\right), \qquad \sum_{n,m=-\infty}^{+\infty} \mathrm{e}^{i2\pi(np_{\rm s}+mx_{\rm s})} \left[M^0_{n,m} + \delta v_{n,m}\mathbb{1}_{\mathcal{H}_{\rm f}}\right]$$

are infinitesimally bounded with respect to Ξ and are elements of $S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_{\rm f}))$.

The operators ${I_{n,m}}^2$ and $M^1_{n,m}$ are only bounded (and not infinitesimally bounded) with respect to Ξ. First of all it is easy to check that $\|{\mathfrak{a}^\sharp}^2\psi\|_{\mathcal{H}_{\rm f}} \leqslant 3\|\Xi\psi\|_{\mathcal{H}_{\rm f}}$ for every $\psi \in \mathcal{F}$, where $\mathfrak{a}^\sharp$ means $\mathfrak{a}$ or $\mathfrak{a}^\dagger$. Then, for every $\psi \in \mathcal{F}$,

$$\|{I_{n,m}}^2\psi\|^2_{\mathcal{H}_{\rm f}} \leqslant |\alpha_{n,m}|^4 \left(\|\mathfrak{a}^2\psi\|_{\mathcal{H}_{\rm f}} + \|\{\mathfrak{a};\mathfrak{a}^\dagger\}^2\psi\|_{\mathcal{H}_{\rm f}} + \|{\mathfrak{a}^\dagger}^2\psi\|_{\mathcal{H}_{\rm f}}\right)^2 \leqslant 27\frac{d^4}{\Omega^2}(n^2+m^2)^2\, \|\Xi\psi\|^2_{\mathcal{H}_{\rm f}} \tag{3.82}$$

where we used the inequality $(\alpha+\beta+\gamma)^2 \leqslant 3(\alpha^2+\beta^2+\gamma^2)$, the identity $\{\mathfrak{a};\mathfrak{a}^\dagger\} = 2\Xi$ and the bound $|\alpha_{n,m}|^2 \leqslant d^2/\Omega(n^2-m^2)$ with $d^2 := \max\{|a|,|b|\}$. Assumption 3.2.1 assures that the operator $\delta^4 2\pi^2 \sum_{n,m=-\infty}^{+\infty} v_{n,m}\, \mathrm{e}^{i2\pi(np_{\rm s}+mx_{\rm s})} {I_{n,m}}^2$, which appears in (3.80), is bounded by Ξ by a constant $\delta^4 C$, with $C \propto \sum_{n,m=-\infty}^{+\infty} v_{n,m}(n^2+m^2)$, and is in $S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_{\rm f}))$. The claim for $\widetilde{H}^1_{\delta}$ follows from the Kato-Rellich theorem fixing $\delta_0 := C^{-\frac{1}{4}}$.

The claim for $\widetilde{H}^0_{\delta}$ follows in the same way proving an inequality of the type (3.82) for $M^1_{n,m} = c_{n,m}\mathfrak{a}^2 + \overline{c_{n,m}}{\mathfrak{a}^\dagger}^2 + 2\Re(d_{n,m})\Xi + \iota_q\Im(d_{n,m})$ where $c_{n,m} := \alpha_{n,m}g_{n,m}$ and $d_{n,m} := \alpha_{n,m}\overline{g_{n,m}}$. Observe that the series of coefficients $g_{n,m}$ decays rapidly, then also the serie $c_{n,m}$ and $d_{n,m}$ have a fast decay and in particular are bounded. This implies that in the inequality of type (3.82) we can find a global constant which does not depend on n and m. ■

It is useful to have explicit expressions of the first terms H_j, in terms of $\mathfrak{a}$ and $\mathfrak{a}^\dagger$. From equations (3.79), (3.80) and (3.81), using the Fourier expansion of the derivatives of V and $g := 1/\sqrt{2}\,(z_a f_1 + z_b f_2)$, it is easy to check the following:

- Case 1: $\mathsf{A}_\Gamma = 0$ **-** In this situation

$$\widetilde{H}^1_\delta = \Xi + \delta^2 H_2 + \delta^3 H_3 + \delta^4 H_4$$

with

$$H_2(p_{\rm s}, x_{\rm s}) = V(p_{\rm s}, x_{\rm s})\ \mathbb{1}_{\mathcal{H}_{\rm f}} \tag{3.83}$$

$$H_3(p_{\rm s}, x_{\rm s}) = -\frac{1}{\sqrt{2}}\left[D_z(V)\ \mathfrak{a} + D_{\overline{z}}(V)\ \mathfrak{a}^\dagger\right] \tag{3.84}$$

$$H_4(p_{\rm s}, x_{\rm s}) = \frac{1}{4}\left[|D_z|^2(V)\ 2\Xi + D_z^2(V)\ \mathfrak{a}^2 + D_{\overline{z}}^2(V)\ \mathfrak{a}^{\dagger 2}\right] \tag{3.85}$$

where D_z is the differential operator defined by $D_z := \left(z_a \frac{\partial}{\partial x_{\rm s}} - z_b \frac{\partial}{\partial p_{\rm s}}\right)$ and $D_{\overline{z}}$ is obtained by replacing z_a and z_b with $\overline{z}_a$ and $\overline{z}_b$. Since V is real, $D_{\overline{z}}(V) = \overline{D_z(V)}$, which shows that H_3 is symmetric. The explicit expression of the second order differential operator $|D_z|^2 := D_z \circ D_{\overline{z}}$ is

$$|D_z|^2 = \frac{1}{\Omega_\Gamma}\left(|a|^2\ \frac{\partial^2}{\partial x_{\rm s}^2} - 2a\cdot b\ \frac{\partial^2}{\partial x_{\rm s}\partial p_{\rm s}} + |b|^2\ \frac{\partial^2}{\partial p_{\rm s}^2}\right). \tag{3.86}$$

For a square lattice $|D_z|^2$ coincides with the Laplacian $\partial^2_{p_{\rm s}} + \partial^2_{x_{\rm s}}$.

- Case 2: $\mathsf{A}_\Gamma \neq 0$ **-** In this situation

$$\widetilde{H}^0_\delta = \Xi + \delta H_1 + \delta^2 H_2$$

with

$$H_1(p_{\rm s}, x_{\rm s}) = g(p_{\rm s}, x_{\rm s})\ \mathfrak{a} + \overline{g}(p_{\rm s}, x_{\rm s})\ \mathfrak{a}^\dagger \tag{3.87}$$

$$H_2(p_{\rm s}, x_{\rm s}) = V\ \mathbb{1}_{\mathcal{H}_{\rm f}} - \sqrt{2}D_z(\overline{g})\ \Xi - \frac{1}{\sqrt{2}}\left[D_z(g)\ \mathfrak{a}^2 + D_{\overline{z}}(\overline{g})\ \mathfrak{a}^{\dagger 2}\right]. \tag{3.88}$$

In the computation of (3.88) we used the first of the gauge conditions (3.44) which assures that

$$D_z(\overline{g}) = \frac{1}{\sqrt{2}\Omega_\Gamma}\left[|a|^2\ \frac{\partial f_1}{\partial x_{\rm s}} + a\cdot b\left(\frac{\partial f_2}{\partial x_{\rm s}} - \frac{\partial f_1}{\partial p_{\rm s}}\right) - |b|^2\ \frac{\partial f_2}{\partial p_{\rm s}}\right]$$

is a real function. From the definition of g, f_1 and f_2 it follows that

$$g(a^*\cdot r, b^*\cdot r) = \pi\sqrt{2}\frac{Z\ell}{\Phi_0}\left[(\mathsf{A}_\Gamma)_1 - i(\mathsf{A}_\Gamma)_2\right](r), \tag{3.89}$$

namely g is the dimensionless $\mathbb{Z}^2$-periodic function related to the Γ-periodic function $(\mathsf{A}_\Gamma)_1 - i(\mathsf{A}_\Gamma)_2$, up to a multiplicative constant.

3.4.6 Main result: effective dynamics for strong magnetic fields

We need a preliminary estimates on the remainder. The difference $\mathfrak{R}^{\natural}_{\delta} := H_{\delta} - \widetilde{H}^{\natural}_{\delta}$ is a self-adjoint element of $S^1(\mathscr{B}(\mathcal{F};\mathcal{H}_{\rm f}))$, which we call the *remainder symbol*. To develop the SAPT for the Harper regime we need to estimate the order of the remainder symbol. The next result shows essentially that

$$H_{\delta}(p_{\rm s},x_{\rm s}) = \widetilde{H}^{\natural}_{\delta}(p_{\rm s},x_{\rm s}) + \mathcal{O}(\delta^{2(\natural+1)}), \qquad \pi_{\rm r}\, H_{\delta}(p_{\rm s},x_{\rm s})\, \pi_{\rm r} = \pi_{\rm r}\, \widetilde{H}^{\natural}_{\delta}(p_{\rm s},x_{\rm s})\, \pi_{\rm r} + \mathcal{O}(\delta^{2(\natural+1)+1}), \tag{3.90}$$

where

$$\pi_{\rm r} := \sum_{i=1}^{m} |\psi_{k_i}\rangle\langle\psi_{k_i}| \tag{3.91}$$

is the projection on the subspace spanned by the finite family of generalized Hermite functions $\{\psi_{k_i}\}_{i=1}^{m}$. In other words, the error done by replacing the true symbol H_{δ} with the approximated symbol $\widetilde{H}^{\natural}_{\delta}$ (which has order $2(\natural+1)$ in δ) is of the same order of the approximated symbol, so in this sense $\widetilde{H}^{\natural}_{\delta}$ is not a good approximation for H_{δ}. On the other side, what we need to develop the SAPT is to control the operator $\pi_{\rm r}\, H_{\delta}\, \pi_{\rm r}$, which is well approximated by $\pi_{\rm r}\, \widetilde{H}^{\natural}_{\delta}\, \pi_{\rm r}$ up to an error of order $2(\natural+1)+1$ in δ.

PROPOSITION 3.4.7. *Let Assumption 3.2.1 hold true. Then $\mathfrak{R}^{\natural}_{\delta}$ has order $\mathcal{O}(\delta^{2(\natural+1)})$, i. e. there exist a constant C such that $\|\mathfrak{R}^{\natural}_{\delta}(p_s,x_s)\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f})} \leqslant C\delta^{2(\natural+1)}$ for all $(p_s,x_s)\in\mathbb{R}^2$. Moreover $\|\mathfrak{R}^{\natural}_{\delta}\,\pi_r\|_{\mathscr{B}(\mathcal{H}_{\rm f})} = \|\pi_r\,\mathfrak{R}^{\natural}_{\delta}\|_{\mathscr{B}(\mathcal{H}_{\rm f})} \leqslant C\delta^{2(\natural+1)+1}$, for all $(p_s,x_s)\in\mathbb{R}^2$, i.e. $\mathfrak{R}^{\natural}_{\delta}\,\pi_r$, $\pi_r\,\mathfrak{R}^{\natural}_{\delta}$ and $[\mathfrak{R}^{\natural}_{\delta};\pi_r]$ are $\mathscr{B}(\mathcal{H}_{\rm f})$-valued symbols of order $\mathcal{O}(\delta^{2(\natural+1)+1})$.*

Proof. **(Case $\natural=1$)** The explicit expression of the remainder symbol is

$$\mathfrak{R}^{1}_{\delta}(p_{\rm s},x_{\rm s}) = \delta^2 \sum_{n,m=-\infty}^{+\infty} v_{n,m}\, \mathrm{e}^{i2\pi(np_{\rm s}+mx_{\rm s})} \left[\mathrm{e}^{i2\pi\delta I_{n,m}} - \left(\mathbb{1}_{\mathcal{H}_{\rm f}} + i2\pi\delta I_{n,m} + \frac{1}{2}(i2\pi)^2\delta^2 {I_{n,m}}^2\right)\right]. \tag{3.92}$$

and from (3.92) it follows that $\|\mathfrak{R}^{1}_{\delta}(p_{\rm s},x_{\rm s})\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f})} \leqslant \delta^2\sum_{n,m=-\infty}^{+\infty}|v_{n,m}|\Lambda_{n,m}$ with

$$\Lambda_{n,m} := \sup_{\psi\in\mathcal{H}_{\rm f}\setminus\{0\}} \frac{\left\|\left[\mathrm{e}^{i2\pi\delta I_{n,m}} - \left(\mathbb{1}_{\mathcal{H}_{\rm f}} + i2\pi\delta I_{n,m} + \frac{1}{2}(i2\pi)^2\delta^2 {I_{n,m}}^2\right)\right]\Xi^{-1}\psi\right\|_{\mathcal{H}_{\rm f}}}{\|\psi\|_{\mathcal{H}_{\rm f}}} \tag{3.93}$$

since $\|\psi\|_{\mathcal{F}} := \|\Xi\psi\|_{\mathcal{H}_{\rm f}}$ and $\mathcal{F} = \Xi^{-1}\mathcal{H}_{\rm f}$. The operators $I_{n,m}$ are essentially self-adjoint on $\mathcal{L}$ and we denote their closure with the same symbol. Since the operators ${I_{n,m}}^2$ are positive, we can consider the resolvent operators $R_{n,m} := ({I_{n,m}}^2 + \mathbb{1}_{\mathcal{H}_{\rm f}})^{-1}$. Let suppose that

$$\zeta_{n,m}(\delta) := \left\|\left[\mathrm{e}^{i2\pi\delta I_{n,m}} - \left(\mathbb{1}_{\mathcal{H}_{\rm f}} + i2\pi\delta I_{n,m} + \frac{1}{2}(i2\pi)^2\delta^2 {I_{n,m}}^2\right)\right] R_{n,m}\right\|_{\mathscr{B}(\mathcal{H}_{\rm f})} \leqslant \zeta(\delta), \tag{3.94}$$

for all $n,m\in\mathbb{Z}$, with $\sup_{\delta}\zeta(\delta) < +\infty$. Then equation (3.93) would imply

$$\Lambda_{n,m} \leqslant \zeta(\delta)\,\left\|({I_{n,m}}^2 + \mathbb{1}_{\mathcal{H}_{\rm f}})\,\Xi^{-1}\right\|_{\mathscr{B}(\mathcal{H}_{\rm f})}.$$

Noticing that ${I_{n,m}}^2 = {\alpha_{n,m}}^2\ \mathfrak{a}^2 + \overline{\alpha_{n,m}}^2\ {\mathfrak{a}^\dagger}^2 + 2|\alpha_{n,m}|^2\ \Xi$ and observing that $\|\Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_{\rm f})} = 2$, $\|\mathfrak{a}^2\Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_{\rm f})} = 1$ and $\|{\mathfrak{a}^\dagger}^2\Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_{\rm f})} = 2\sqrt{2}$, one deduces from inequality (3.94) that

$$\|\mathfrak{R}^1_\delta(p_{\rm s},x_{\rm s})\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f})} \leqslant C_1 \left(\sum_{n,m=-\infty}^{+\infty} |v_{n,m}|(|n|+|m|)^2\right) \delta^2\zeta(\delta) \leqslant C_2\delta^2\zeta(\delta)$$

for suitable positive constants C_1 and C_2.

It remains to prove the inequality (3.94) and the estimate on $\zeta(\delta)$. By spectral calculus one has that $\zeta_{n,m}(\delta) = \sup_{t\in\sigma(I_{n,m})} |Z_\delta(t)| \leqslant \sup_{t\in\mathbb{R}} |Z_\delta(t)| =: \zeta(\delta)$ where

$$Z_\delta(t) := 4\pi^2\delta^2\ \frac{\mathrm{e}^{i2\pi\delta t} - \left(1 + i2\pi\delta t - \frac{1}{2}(2\pi\delta t)^2\right)}{(2\pi\delta t)^2 + 4\pi^2\delta^2}.$$

After some manipulations and the change of variable $\tau := 2\pi\delta t$ one has that

$$G_\delta(\tau) := \frac{1}{4\pi^4\delta^4}\left|Z_\delta\left(\frac{\tau}{2\pi\delta}\right)\right|^2 \leqslant \frac{\tau^4 + 4\tau^2\cos(\tau) - 8\tau\sin(\tau) - 8\cos(\tau) + 8}{\tau^4} < C_3.$$

Thus $\zeta(\delta)^2 = 4\pi^4\delta^4\ \sup_{\tau\in\mathbb{R}} G_\delta(\tau) \leqslant 4\pi^4 C_3\delta^4$, hence $\|\mathfrak{R}^1_\delta(p_{\rm s},x_{\rm s})\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f})} \leqslant C\delta^4$. This concludes the first part of the proof.

Since $\|\mathfrak{R}^1_\delta\pi_{\rm r}\|_{\mathscr{B}(\mathcal{H}_{\rm f})} \leqslant \sum_{i=1}^m \|\mathfrak{R}^1_\delta|\psi_{k_i}\rangle\langle\psi_{k_i}|\|_{\mathscr{B}(\mathcal{H}_{\rm f})}$, then it is enough to show that for any Hermite vector ψ_k the inequality $\|\mathfrak{R}^1_\delta|\psi_k\rangle\langle\psi_k|\|_{\mathscr{B}(\mathcal{H}_{\rm f})} \leqslant C_k\delta^5$ holds true. Observing that $\|\mathfrak{R}^1_\delta|\psi_k\rangle\langle\psi_k|\|_{\mathscr{B}(\mathcal{H}_{\rm f})} = \|\mathfrak{R}^1_\delta\psi_k\|_{\mathcal{H}_{\rm f}}$, one deduces

$$\lim_{\delta\to 0}\delta^{-5}\|\mathfrak{R}^1_\delta|\psi_k\rangle\langle\psi_k|\|_{\mathscr{B}(\mathcal{H}_{\rm f})} = \lim_{\delta\to 0}\left\|\sum_{n,m=-\infty}^{+\infty} v_{n,m}\mathrm{e}^{i2\pi(np_{\rm s}+mx_{\rm s})}\left(\sum_{j=3}^{+\infty}\frac{(i2\pi)^j\delta^{j-3}}{j!}{I_{n,m}}^j\right)\psi_k\right\|_{\mathcal{H}_{\rm f}}$$

$$\leqslant \frac{4}{3}\pi^3\sum_{n,m=-\infty}^{+\infty}|v_{n,m}|\ \|{I_{n,m}}^3\psi_k\|_{\mathcal{H}_{\rm f}} \leqslant \frac{32}{3}\pi^3 C'\|{\mathfrak{a}^\dagger}^3\psi_k\|_{\mathcal{H}_{\rm f}} = \frac{32}{3}\sqrt{(k+3)!}\ \pi^3 C' =: C_k$$

where $C' := \sum_{n,m=-\infty}^{+\infty}|v_{n,m}|\ |\alpha_{n,m}|^3$ is finite in view of Assumption 3.2.1.

This shows that for all $\delta\in[0,\delta_0)$ (for a suitable $\delta_0 > 0$) the norm $\|\mathfrak{R}^1_\delta|\psi_k\rangle\langle\psi_k|\|_{\mathscr{B}(\mathcal{H}_{\rm f})}$ is bounded by $C_k\delta^5$ and so it follows that $\|\mathfrak{R}^1_\delta\pi_{\rm r}\|_{\mathscr{B}(\mathcal{H}_{\rm f})} \leqslant mC\delta^5$ with $C := \max_{1,\ldots,m}\{C_{k_i}\}$. Finally $\|\pi_{\rm r}\mathfrak{R}^1_\delta\|_{\mathscr{B}(\mathcal{H}_{\rm f})} = \|(\mathfrak{R}^1_\delta\pi_{\rm r})^\dagger\|_{\mathscr{B}(\mathcal{H}_{\rm f})} = \|\mathfrak{R}^1_\delta\pi_{\rm r}\|_{\mathscr{B}(\mathcal{H}_{\rm f})}$.

(Case $\natural = 0$) The proof proceeds as in the previous case. Divide the remainder symbol in two terms $\mathfrak{R}^0_\delta = \mathfrak{R}^0_0 + \mathfrak{R}^0_1$ where:

$$\mathfrak{R}^0_0(p_{\rm s},x_{\rm s}) := \delta\sum_{n,m=-\infty}^{+\infty}\mathrm{e}^{i2\pi(np_{\rm s}+mx_{\rm s})}\left(\mathrm{e}^{i2\pi\delta I_{n,m}} - \mathbb{1}_{\mathcal{H}_{\rm f}} - i2\pi\delta I_{n,m}\right)M^0_{n,m}$$

$$\mathfrak{R}^0_1(p_{\rm s},x_{\rm s}) := \delta^2\sum_{n,m=-\infty}^{+\infty} v_{n,m}\ \mathrm{e}^{i2\pi(np_{\rm s}+mx_{\rm s})}\left(\mathrm{e}^{i2\pi\delta I_{n,m}} - \mathbb{1}_{\mathcal{H}_{\rm f}}\right).$$

The control of $\mathfrak{R}^0_1$ is easy, indeed $\|\mathfrak{R}^0_1\|_{\mathscr{B}(\mathcal{D},\mathcal{H}_{\rm f})} \leqslant 2\|\mathfrak{R}^0_1\|_{\mathscr{B}(\mathcal{H}_{\rm f})} \leqslant 4C\delta^2$ where $C := \sum_{n,m=-\infty}^{+\infty}|v_{n,m}|$. Moreover (with the same technique used for the case $\natural = 1$), one can check that for any

Hermite vector ψ_k the function $t_1(\delta) := \frac{1}{\delta^3}\|\mathfrak{R}_1^0(p_s, x_s)\psi_k\|_{\mathcal{H}_f}$ is bounded by a constant $C_k > 0$ in a suitable interval $[0, \delta_0)$. This assures that $\|\mathfrak{R}_1^0\, \pi_r\|_{\mathscr{B}(\mathcal{H}_f)}$ is of order $\mathcal{O}(\delta^3)$.

To control $\mathfrak{R}_0^0$ we need to estimate $\Sigma_{n,m} := \|\left(e^{i2\pi\delta I_{n,m}} - \mathbb{1}_{\mathcal{H}_f} - i2\pi\delta I_{n,m}\right) M_{n,m}^0 \Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_f)}$. Let $R'_{n,m}$ be the resolvent $(I_{n,m}+i\mathbb{1}_{\mathcal{H}_f})^{-1}$. It is easy to check that $\|(I_{n,m}+i\mathbb{1}_{\mathcal{H}_f})M_{n,m}^0\Xi^{-1}\|_{\mathscr{B}(\mathcal{H}_f)}$ is bounded by a linear expression in $|n|$ and $|m|$. Indeed, as proved in Proposition 3.4.6, both $M_{n,m}^0$ and $M_{n,m}^1$ are bounded by Ξ. Finally, by means of spectral calculus $\|\left(e^{i2\pi\delta I_{n,m}} - \mathbb{1}_{\mathcal{H}_f} - i2\pi\delta I_{n,m}\right) R'_{n,m}\|^2_{\mathscr{B}(\mathcal{H}_f)}$ is bounded by the maximum in τ of the function $F_\delta(\tau) := 4\pi^2\delta^2\, \frac{\tau^2 - 2\tau\sin\tau - 2\cos\tau + 2}{\tau^2}$.

The last part follows observing that $M_{n,m}^0$ is a linear combinations of $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ and so they act splitting a Hermite vector ψ_k as $c_{n,m}^k\psi_{k-1} + d_{n,m}^k\psi_{k+1}$ where, for a fixed k, the coefficients depend on $f_{n,m}^{(j)}$. To conclude the proof it is sufficient to notice that $t_0(\delta) := \frac{1}{\delta^2}\|\left(e^{i2\pi\delta I_{n,m}} - \mathbb{1}_{\mathcal{H}_f} - i2\pi\delta I_{n,m}\right)\psi_k\|_{\mathcal{H}_f}$ is bounded by a constant $C_k > 0$ in a suitable interval $[0, \delta_0)$. ■

We are now in position to derive the adiabatically decoupled effective dynamics We recall that the Weyl quantization of the symbol H_δ is the Hamiltonian (3.61), namely $\mathrm{Op}_\delta(H_\delta) = H^{\mathcal{W}}$. As for the approximated symbol $\widetilde{H}_\delta^\natural$, we pose $\widetilde{H}^\natural := \mathrm{Op}_\delta(\widetilde{H}_\delta^\natural)$. Both $H^{\mathcal{W}}$ and $\widetilde{H}^\natural$ are bounded operators from $L^2(\mathbb{R}, dx_s)\otimes\mathcal{F}$ to $\mathcal{H}_w := L^2(\mathbb{R}, dx_s)\otimes\mathcal{H}_f$.

THEOREM 3.4.8. *Let Assumption 3.2.1 be satisfied. Let $\{\sigma_n(\cdot)\}_{n\in\mathcal{I}}$, with $\mathcal{I} = \{n, \dots, n+m-1\}$, be a family of Landau bands for Ξ and let $\pi_r := \sum_{n\in\mathcal{I}}|\psi_n\rangle\langle\psi_n|$ be the spectral projector of $H_0 = \Xi$ corresponding to the set $\{\sigma_n(p_s, x_s)\}_{n\in\mathcal{I}}$. Then:*

1. Almost-invariant subspace: *there exists an orthogonal projection $\Pi_\delta^\natural \in \mathscr{B}(\mathcal{H}_w)$, with $\Pi_\delta^\natural = \mathrm{Op}_\delta(\pi) + \mathcal{O}_0(\delta^\infty)$, $\pi(p_s, x_s) \asymp \sum_{j=0}^\infty \delta^j\, \pi_j(p_s, x_s)$, and $\pi_0(p_s, x_s) \equiv \pi_r$, such that*

$$[\widetilde{H}^\natural; \Pi_\delta^\natural] = \mathcal{O}_0(\delta^\infty), \qquad [H^{\mathcal{W}}; \Pi_\delta^\natural] = \mathcal{O}_0(\delta^{2(\natural+1)+1}). \tag{3.95}$$

2. Effective dynamics: *let $\Pi_r := \mathbb{1}_{\mathcal{H}_s}\otimes\pi_r \in \mathscr{B}(\mathcal{H}_w)$ and $\mathcal{K} := \mathrm{Im}\,\Pi_r \simeq L^2(\mathbb{R}, dx_s)\otimes\mathbb{C}^m$. Then there exists a unitary operator $U_\delta^\natural \in \mathscr{B}(\mathcal{H}_w)$ such that*

(i) $U_\delta^\natural = \mathrm{Op}_\delta(u) + \mathcal{O}_0(\delta^\infty)$, *where the symbol $u \asymp \sum_{j=0}^\infty \delta^j u_j$ has principal part $u_0 \equiv \mathbb{1}_{\mathcal{H}_f}$;*

(ii) $\Pi_r = U_\delta^\natural\, \Pi_\delta^\natural\, {U_\delta^\natural}^{-1}$;

(iii) *Let $h^\natural$ in $S^1(\mathscr{B}(\mathcal{H}_f))$ be a resummation of the formal symbol $u\sharp\pi\sharp\widetilde{H}_\delta^\natural\sharp\pi\sharp u^\dagger$ and define the* effective Hamiltonian *by $H_{\text{eff}}^\delta := \mathrm{Op}_\delta(h^\natural)$. Since $[H_{\text{eff}}^\delta; \Pi_r] = 0$, H_{eff}^δ is a bounded operator on $\mathcal{K}$. Then*

$$U_\delta^\natural\, \Pi_\delta^\natural\, H^{\mathcal{W}}\, \Pi_\delta^\natural\, {U_\delta^\natural}^{-1} = H_{\text{eff}}^\delta + \mathcal{O}_0(\delta^{2(\natural+1)+1}) \in \mathscr{B}(\mathcal{K}). \tag{3.96}$$

2'. Effective dynamics for a single Landau band when $\mathsf{A}_\Gamma = 0$: *Consider a single Landau band $\sigma_*(\cdot) = \lambda_*$, so that $\pi_r = |\psi_*\rangle\langle\psi_*|$. Then, up to the order δ^4, one has that*

$$H_{\text{eff}}^\delta = \lambda_*\mathbb{1}_{\mathcal{H}_s} + \delta^2\, V(P_s, Q_s) + \delta^4\, \frac{\lambda_*}{2}\, Y(P_s, Q_s) + \mathcal{O}_0\left(\delta^5\right) \tag{3.97}$$

where $V(P_{\rm s}, Q_{\rm s}) := \mathrm{Op}_\delta(V)$ *is the Weyl quantization of the* $\mathbb{Z}^2$*-periodic function* $V(p_{\rm s}, x_{\rm s})$ *related to the* Γ*-periodic potential* V_Γ*, while* $Y(P_{\rm s}, Q_{\rm s}) := \mathrm{Op}_\delta(|D_z|^2(V))$ *is the Weyl quantization of the function* $|D_z|^2(V)(p_{\rm s}, x_{\rm s})$ *defined through the differential operator* (3.86).

The derivation of the effective dynamics when $\mathbf{A}_\Gamma \neq 0$ will be considered in Section 3.4.8.

Proof of Theorem 3.4.8

Step 1. Almost-invariant subspace

As explained in the first part of proof of the Theorem 3.3.14 one constructs a formal symbol π (the *Moyal projection*) such that: (i) $\pi\sharp\pi \asymp \pi$; (ii) $\pi^\dagger = \pi$; (iii) $\widetilde{H}^\natural_\delta \sharp \pi \asymp \pi \sharp \widetilde{H}^\natural_\delta$. Such a symbol $\pi \asymp \sum_{j=0}^{\infty} \delta^j \pi_j$ is constructed recursively order by order starting from $\pi_0 = \pi_{\rm r}$ and $\widetilde{H}^\natural_\delta$ and it is unique (Panati et al. 2003b, Lemma 2.3). The recursive relations are

$$\pi_n := \pi_n^{\rm D} + \pi_n^{\rm OD} \tag{3.98}$$

where the *diagonal part* is $\pi_n^{\rm D} := \pi_{\rm r} G_n \pi_{\rm r} + (\mathbb{1}_{\mathcal{H}_{\rm f}} - \pi_{\rm r}) G_n (\mathbb{1}_{\mathcal{H}_{\rm f}} - \pi_{\rm r})$ with

$$G_n := \left[\left(\sum_{j=0}^{n-1} \delta^j \pi_j \right) \sharp \left(\sum_{j=0}^{n-1} \delta^j \pi_j \right) - \left(\sum_{j=0}^{n-1} \delta^j \pi_j \right) \right]_n . \tag{3.99}$$

The *off-diagonal part* is defined by the implicit relation $[H_0; \pi_n^{\rm OD}] = -F_n$ where

$$F_n := \left[\widetilde{H}^\natural_\delta \sharp \left(\sum_{j=0}^{n-1} \delta^j \pi_j + \delta^n \pi_n^{\rm D} \right) - \left(\sum_{j=0}^{n-1} \delta^j \pi_j + \delta^n \pi_n^{\rm D} \right) \sharp \widetilde{H}^\natural_\delta \right]_n . \tag{3.100}$$

The uniqueness allows us to construct π only locally and this local construction is explained in the second part of Lemma 2.3 in (Panati et al. 2003b). In our case we can choose a $(p_{\rm s}, x_{\rm s})$-independent positively oriented complex circle $\Lambda \subset \mathbb{C}$, symmetric with respect to the real axis, which encloses the family of (constant) spectral bands $\{\sigma_n(\cdot) = \lambda_n\}_{n\in\mathcal{I}}$ and such that $\mathrm{dist}(\Lambda, \sigma(H_0)) \geqslant \frac{1}{2}$ (see Figure 3.2). For all $\lambda \in \Lambda$ we construct recursively the *Moyal resolvent* (or *parametrix*) $R^\natural(\lambda;\cdot) := \sum_{j=0}^{\infty} \delta^j R^\natural_j(\lambda;\cdot)$ of $\widetilde{H}^\natural_\delta$, following the same technique explained during the proof of Theorem 3.3.14. The approximants of the symbol π are related to the approximants of the Moyal resolvent by the usual Riesz formula $\pi_j(z) := \dfrac{i}{2\pi} \oint_\Lambda d\lambda\ R^\natural_j(\lambda; z)$ where $z := (p_{\rm s}, x_{\rm s}) \in \mathbb{R}^2$. Some care is required to show (iii) since, by construction, $\widetilde{H}^\natural_\delta \sharp \pi$ takes values in $\mathscr{B}(\mathcal{H}_{\rm f})$ while $\pi \sharp \widetilde{H}^\natural_\delta$ takes values in $\mathscr{B}(\mathcal{F})$. To solve this problem one can use the same argument proposed in Lemma 7 of (Panati et al. 2003a).

The technical and new part of the proof consist in showing that $\pi \in S^1(\mathscr{B}(\mathcal{H}_{\rm f})) \cap S^1(\mathscr{B}(\mathcal{H}_{\rm f}, \mathcal{F}))$. The Riesz formula implies $\|(\partial_z^\alpha \pi_j)(z)\|_\flat \leqslant 2\pi\ \sup_{\lambda\in\Lambda} \|\partial_z^\alpha R^\natural_j(\lambda; z)\|_\flat$ for all $\alpha \in \mathbb{N}^2$ ($\flat$ means either $\mathscr{B}(\mathcal{H}_{\rm f})$ or $\mathscr{B}(\mathcal{H}_{\rm f}; \mathcal{F})$ and $\partial_z^\alpha := \partial_{p_{\rm s}}^{\alpha_1} \partial_{x_{\rm s}}^{\alpha_2}$) since Λ does not depend on z. Then we need only to show that $R^\natural_j(\lambda;\cdot) \in S^1(\mathscr{B}(\mathcal{H}_{\rm f})) \cap S^1(\mathscr{B}(\mathcal{H}_{\rm f}, \mathcal{F}))$. The choice of Λ assures $\|R_0^\natural(\lambda;z)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} = \|(\Xi - \lambda \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1}\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant 2$. Moreover $\partial_z^\alpha R_0^\natural(\lambda;z) = 0$ for all $\alpha \neq 0$ and this implies that $R_0^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ uniformly in λ. Since $\|R_0^\natural(\lambda;z)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f},\mathcal{F})} = \|\Xi(\Xi - \lambda \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1}\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant \infty$ one concludes that $R_0^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f},\mathcal{F}))$ uniformly in λ.

By means of equation (3.35), one has $R_j^\natural = -R_0^\natural L_j^\natural$ where $L_j^\natural$ is the *j-th order obstruction* for $R_0^\natural$ to be the Moyal resolvent. In view of this recursive relation, the proof of $R_j^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ for all $j \in \mathbb{N}$ is reduced to show that $L_j^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ for all $j \in \mathbb{N}$.

The first order obstruction, computed by means of (3.36), is

$$L_1^\natural(\lambda;z) = \delta^{-1}[(\widetilde{H}_\delta^\natural(z) - \lambda \mathbb{1}_{\mathcal{H}_\mathrm{f}})\sharp R_0^\natural(\lambda;z) - \mathbb{1}_{\mathcal{H}_\mathrm{f}}]_1 = H_1(z)\, R_0^\natural(\lambda;z) - \frac{i}{2}\{\Xi; R_0^\natural(\lambda;z)\}_{p_\mathrm{s},x_\mathrm{s}}.$$

Since Ξ and $R_0^\natural$ do not depend on $z \in \mathbb{R}^2$ it follows that $L_1^\natural = H_1 R_0^\natural$. The operator H_1 is linear in $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ (with all its derivative) if $\natural = 0$ or $H_1 = 0$ if $\natural = 1$. In both cases H_1 (with its derivatives) is infinitesimally bounded with respect to Ξ (Lemma A.1.2). This shows that $L_1^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ (but not in $S^1(\mathscr{B}(\mathcal{H}_\mathrm{f},\mathcal{F}))$ if $\natural = 0$).

We proceed by induction assuming that $L_j^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ for all $j \leqslant m \in \mathbb{N}$. The $(m+1)$-th order obstruction $L_{m+1}^\natural$ can be computed by means of equation (3.36) and the Moyal formula for the expansion of $\sharp$ (Teufel 2003, equation (A.9)). After some manipulations, one gets

$$L_{m+1}^\natural(\lambda;z) = \frac{1}{(2i)^{m+1}} \sum_{\substack{\alpha_1+\alpha_2+r+l=m+1 \\ 0\leqslant l\leqslant m,\ 0\leqslant r\leqslant 2(\natural+1)}} \frac{(-1)^{|\alpha|+1}}{\alpha_1!\alpha_2!} \left(\partial_{x_\mathrm{s}}^{\alpha_1}\partial_{p_\mathrm{s}}^{\alpha_2} H_r R_0^\natural\right)(\lambda;z)\ \left(\partial_{p_\mathrm{s}}^{\alpha_1}\partial_{x_\mathrm{s}}^{\alpha_2} L_l^\natural\right)(\lambda;z).$$

Since $H_r R_0^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ uniformly in λ (c.f. Remark 3.4.9) then $L_{m+1}^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$, and this concludes the inductive argument.

Finally to prove $R_j^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f},\mathcal{F}))$, observe that $\|\partial_z^\alpha R_j^\natural\|_{\mathscr{B}(\mathcal{H}_\mathrm{f},\mathcal{F})} = \|\Xi\, R_0^\natural\, (\partial_z^\alpha L_j^\natural)\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant C_\alpha \|\Xi\, R_0^\natural\|_{\mathscr{B}(\mathcal{H}_\mathrm{f})} \leqslant +\infty$ for all $j, \alpha \in \mathbb{N}$.

Remark 3.4.9. It clearly emerges from the proof that the order $\delta^{2(\natural+1)}$ is the best approximation which can be obtained with this technique. The obstruction is the condition $H_r R_0^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$, which can be satisfied by the resolvent $R_0^\natural := (\Xi - \zeta \mathbb{1}_\mathrm{f})^{-1}$ only for $0 \leqslant r \leqslant 2(\natural + 1)$. ♦◇

Proposition A.9 of (Teufel 2003) assures that $\widetilde{H}_\delta^\natural \sharp \pi \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ and, by adjointness, also $\pi \sharp \widetilde{H}_\delta^\natural \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$. By construction $[\widetilde{H}^\natural; \mathrm{Op}_\delta(\pi)] = \mathrm{Op}_\delta([\widetilde{H}_\delta^\natural;\pi]_\sharp) = \mathcal{O}_0(\delta^\infty)$ where $[\widetilde{H}_\delta^\natural;\pi]_\sharp := \widetilde{H}_\delta^\natural \sharp \pi - \pi \sharp \widetilde{H}_\delta^\natural = \mathcal{O}(\delta^\infty)$ denotes the *Moyal commutator*. Observing that $[H_\delta;\pi]_\sharp = [\widetilde{H}_\delta^\natural + \mathfrak{R}_\delta^\natural;\pi]_\sharp = [\mathfrak{R}_\delta^\natural;\pi]_\sharp + \mathcal{O}(\delta^\infty)$ and since Proposition 3.4.7 implies $[\mathfrak{R}_\delta^\natural;\pi]_\sharp = [\mathfrak{R}_\delta^\natural;\pi_\mathrm{r}] + \mathcal{O}(\delta^{2(\natural+1)+1}) = \mathcal{O}(\delta^{2(\natural+1)+1})$, it follows $[H_\delta;\pi]_\sharp = \mathcal{O}(\delta^{2(\natural+1)+1})$ which implies after the quantization $[H^\mathcal{W};\mathrm{Op}_\delta(\pi)] = \mathcal{O}_0(\delta^{2(\natural+1)+1})$.

The last step is to obtain the true projection $\Pi_\delta^\natural$ (the *super-adiabatic projection*) from $\mathrm{Op}_\delta(\pi)$ by means of the formula (3.38). Since $\Pi_\delta^\natural - \mathrm{Op}_\delta(\pi) = \mathcal{O}_0(\delta^\infty)$, one recovers the estimates (3.95).

Step 2. Construction of the intertwining unitary

The construction of the intertwining unitary follows as in the proof of Theorem 3.1 of (Panati et al. 2003b). Firstly one constructs a formal symbol $u \asymp \sum_{j=0}^{\infty} \delta^j u_j$ such that: (i) $u^\dagger \sharp u = u \sharp u^\dagger = \mathbb{1}_{\mathcal{H}_\mathrm{f}}$; (ii) $u \sharp \pi \sharp u^\dagger = \pi_\mathrm{r}$.

The existence of such a symbol follows from a recursive procedure starting from u_0 (which can be fixed to be $\mathbb{1}_{\mathcal{H}_\mathrm{f}}$ in our specific case) and using the expansion of $\pi \asymp \sum_{j=0}^{\infty} \delta^j \pi_j$ obtained above. However, the symbol u which comes out of this procedure is not unique. The recursive relations are

$$u_n := a_n + b_n \qquad \text{with} \qquad a_n := -\frac{1}{2} A_n, \qquad b_n := [\pi_\mathrm{r}; B_n] \tag{3.101}$$

where

$$A_n := \left[\left(\sum_{j=0}^{n-1} \delta^j u_j \right) \sharp \left(\sum_{j=0}^{n-1} \delta^j u_j \right)^\dagger - \mathbb{1}_{\mathcal{H}_\mathrm{f}} \right]_n \tag{3.102}$$

and

$$B_n := \left[\left(\sum_{j=0}^{n-1} \delta^j u_j + \delta^n a_n \right) \sharp \pi \sharp \left(\sum_{j=0}^{n-1} \delta^j u_j + \delta^n a_n \right)^\dagger - \pi_\mathrm{r} \right]_n \tag{3.103}$$

Since $u_0 = \mathbb{1}_{\mathcal{H}_\mathrm{f}} \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$, then it follows by induction that $u_j \in S^1(\mathscr{B}(\mathcal{H}_\mathrm{f}))$ for all $j \in \mathbb{N}$.

The quantization of u is an element of $\mathscr{B}(\mathcal{H}_\mathrm{w})$ but it is not a true unitary. Nevertheless $\mathrm{Op}_\delta(u)$ can be modified by an $\mathcal{O}_0(\delta^\infty)$ term using the same technique of Lemma 3.3 (Step II) in (Panati et al. 2003b) to obtain the true unitary $U_\delta^\natural$.

Step 3. Effective dynamics

By construction $[H^\delta_{\text{eff}}; \Pi_\mathrm{r}] = \mathrm{Op}_\delta([h^\natural; \pi_\mathrm{r}]_\sharp) = [U_\delta^\natural \, \Pi_\delta^\natural \, \widetilde{H}_\delta^\natural \, \Pi_\delta^\natural \, {U_\delta^\natural}^{-1}; \Pi_\mathrm{r}] = 0$ since $\Pi_\mathrm{r} = U_\delta^\natural \, \Pi_\delta^\natural \, {U_\delta^\natural}^{-1}$. Moreover equation (3.96) follows observing that $U_\delta^\natural \, \Pi_\delta^\natural \, H^{\mathcal{W}} \, \Pi_\delta^\natural \, {U_\delta^\natural}^{-1} - H^\delta_{\text{eff}}$ coincides with the quantization of $u \sharp \pi \sharp \mathfrak{R}_\delta^\natural \sharp \pi \sharp u^\dagger$ which is a symbol of order $\mathcal{O}(\delta^{2(\natural+1)+1})$.

Step 4. The case of a single Landau band when $\mathbf{A}_\Gamma = 0$

We need to expand the Moyal product $h^{\natural=1} = u \sharp \pi \sharp \widetilde{H}_\delta^1 \sharp \pi \sharp u^\dagger = \pi_\mathrm{r} \sharp u \sharp \widetilde{H}_\delta^1 \sharp u^\dagger \sharp \pi_\mathrm{r} + \mathcal{O}(\delta^\infty)$ up to the order δ^4. To compute the various terms of the expansion $h^{\natural=1} \asymp \sum_{j=0}^{\infty} \delta^j h_j$ it is useful to define $\chi_j := [u \sharp \widetilde{H}_\delta^1 \sharp u^\dagger]_j$, so that $h_j = \pi_\mathrm{r} \chi_j \pi_\mathrm{r}$. Observing that

$$u \sharp \widetilde{H}_\delta^1 - \left(\sum_{j=0}^{m-1} \delta^j \, \chi_j \right) \sharp u = \left(u \sharp \widetilde{H}_\delta^1 \sharp u^\dagger - \sum_{j=0}^{m-1} \delta^j \chi_j \right) \sharp u + \mathcal{O}(\delta^\infty) = \delta^m \chi_m + \mathcal{O}(\delta^{m+1})$$

one obtains the useful formula

$$\chi_m = \left[u \sharp \widetilde{H}_\delta^1 - \left(\sum_{j=0}^{m-1} \delta^j \, \chi_j \right) \sharp u \right]_m . \tag{3.104}$$

At the zeroth order one finds $h_0 = \pi_0\, u_0\, H_0\, u_0^\dagger\, \pi_0 = \pi_\mathrm{r}\, \Xi\, \pi_\mathrm{r} = \lambda_*\, \pi_\mathrm{r}$ since $u_0 = \mathbb{1}_{\mathcal{H}_\mathrm{f}}$ and $\pi_0 = \pi_\mathrm{r}$. Its quantization is the operator $\mathrm{Op}_\delta(h_0) = \lambda_* \mathbb{1}_{\mathcal{H}_\mathrm{s}}$ acting on $\mathcal{K} = L^2(\mathbb{R}, dx_\mathrm{s})$.

As for the first order ($m = 1$), $\chi_1 = u_1 H_0 + u_0 H_1 - \chi_0 u_1 + [u_0 \sharp H_0]_1 - [\chi_0 \sharp u_0]_1 = [u_1; \Xi]$ since $\chi_0 = u_0 H_0 u_0^{-1} = \Xi$ and $H_1 = 0$. Then $h_1 = \pi_\mathrm{r}[u_1; \Xi]\pi_\mathrm{r} = \lambda_*(\pi_\mathrm{r} u_1 \pi_\mathrm{r} - \pi_\mathrm{r} u_1 \pi_\mathrm{r}) = 0$, hence $\mathrm{Op}_\delta(h_1) = 0$.

At the second order ($m = 2$), one obtains after some manipulations $\chi_2 = H_2 + u_2\, \Xi - \Xi\, u_2 - \chi_1\, u_1$ which implies $h_2 = \pi_\mathrm{r} H_2 \pi_\mathrm{r} - \pi_\mathrm{r} \chi_1 u_1 \pi_\mathrm{r}$. We need to compute u_1. Using equations (3.101), (3.102) and (3.103) one obtains that $-2a_1 := \left[u_0 \sharp u_0{}^\dagger - \mathbb{1}_{\mathcal{H}_\mathrm{f}}\right]_1 = 0$ and $b_1 := [\pi_\mathrm{r}; B_1]$ with $B_1 = [u_0 \sharp \pi \sharp u_0{}^* - \pi_\mathrm{r}]_1 = \pi_1$ since $a_1 = 0$. To compute π_1 we use equations (3.98), (3.99) and (3.100). Since $G_1 = [\pi_\mathrm{r} \sharp \pi_\mathrm{r} - \pi_\mathrm{r}]_1 = 0$ it follows that $\pi_1^\mathrm{D} = 0$. In the case of a single energy band in the relevant part of the spectrum, the implicit relation which defines π_n^OD can be solved, obtaining the useful equation

$$\pi_n^\mathrm{OD} = \pi_\mathrm{r} F_n (\Xi - \lambda_* \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1} (\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \pi_\mathrm{r}) - (\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \pi_\mathrm{r})(\Xi - \lambda_* \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1} F_n \pi_\mathrm{r}. \tag{3.105}$$

Since $F_1 = [\widetilde{H}_\delta^1 \sharp \pi_\mathrm{r} - \pi_\mathrm{r} \sharp \widetilde{H}_\delta^1]_1 = H_1 \pi_\mathrm{r} - \pi_\mathrm{r} H_1 = 0$, being $H_1 = 0$, it follows $B_1 = \pi_1 = \pi_1^\mathrm{OD} = 0$ and consecutively $u_1 = b_1 = 0$. Then $h_2 = \pi_\mathrm{r}\, H_2\, \pi_\mathrm{r} = V\, \pi_\mathrm{r}$, according to (3.83), and its quantization defines on $\mathcal{K}$ the operator $\mathrm{Op}_\delta(h_2) = V(P_\mathrm{s}, X_\mathrm{s})$.

Considering (3.104) at the third order ($m = 3$) and using $u_1 = 0$, one obtains after some computations $\chi_3 = H_3 + u_3\, \Xi - \Xi\, u_3 - \chi_1\, u_2$ which implies $h_3 = \pi_\mathrm{r}\, H_3\, \pi_\mathrm{r} - \pi_\mathrm{r}\, \chi_1\, u_2\, \pi_\mathrm{r}$. Thus we need to compute u_2. Since $u_1 = 0$, it follows $-2a_2 = [u_0 \sharp u_0{}^\dagger - \mathbb{1}_{\mathcal{H}_\mathrm{f}}]_2 = 0$, $B_2 = [u_0 \sharp \pi \sharp u_0{}^\dagger - \pi_\mathrm{r}]_2 = \pi_2$ and $b_2 = [\pi_\mathrm{r}; \pi_2]$. Since $\pi_1 = 0$, one has that $G_2 = [\pi_\mathrm{r} \sharp \pi_\mathrm{r} - \pi_\mathrm{r}]_2 = 0$ which implies $\pi_2^\mathrm{D} = 0$. To compute π_2^OD we need $F_2 = [\widetilde{H}_\delta^1 \sharp \pi_\mathrm{r} - \pi_\mathrm{r} \sharp \widetilde{H}_\delta^1]_2 = [H_2; \pi_\mathrm{r}] = [\mathbb{1}_{\mathcal{H}_\mathrm{f}}; \pi_\mathrm{r}] = 0$, where $H_2 = V\ \mathbb{1}_{\mathcal{H}_\mathrm{f}}$ has been used. Then $B_2 = \pi_2 = \pi_2^\mathrm{OD} = 0$ and consequently $u_2 = b_2 = 0$. Therefore $h_3 = \pi_\mathrm{r} H_3 \pi_\mathrm{r}$, and equation (3.84) implies that $\pi_\mathrm{r} H_3 \pi_\mathrm{r} = 0$ in view of $\pi_\mathrm{r} \mathfrak{a} \pi_\mathrm{r} = \langle \psi_* | \mathfrak{a} | \psi_* \rangle \pi_\mathrm{r} = 0$ and similarly for $\mathfrak{a}^\dagger$. Then $\mathrm{Op}_\delta(h_3) = 0$.

To compute the fourth order, we do not need to compute u_3 and π_3. Indeed, by computing (3.104) at the fourth order ($m = 4$) one finds $\chi_4 = H_4 + u_4\, \Xi - \Xi\, u_4 + u_3\, H_1 - \chi_3\, u_1 = H_4 + u_4\, \Xi - \Xi\, u_4$ since $H_1 = u_1 = u_2 = 0$. Then $h_4 = \pi_\mathrm{r} H_4 \pi_\mathrm{r} = \frac{\lambda_*}{2} |D_z(V)|^2\, \pi_\mathrm{r}$, according to equation (3.85), and its quantization yields $\mathrm{Op}_\delta(h_4) = \frac{\lambda_*}{2} Y(P_\mathrm{s}, Q_\mathrm{s})$.

Remark 3.4.10. In the derivation of the effective dynamics, one realizes that $\pi = \pi_\mathrm{r} + \mathcal{O}(\delta^3)$ and $u = \mathbb{1}_{\mathcal{H}_\mathrm{f}} + \mathcal{O}(\delta^3)$. To find a non trivial correction we need to compute the third order. Let $u_3 = a_3 + b_3$ and $\pi_3 = \pi_3^\mathrm{D} + \pi_3^\mathrm{OD}$. Since $u_1 = u_2 = 0$ then $-2a_3 = 0$. Moreover $b_3 = [\pi_\mathrm{r}; B_3]$ with $B_3 = \pi_3$ since $a_3 = 0$. Since $\pi_1 = \pi_2 = 0$ one has that $G_3 = 0$ which implies $\pi_3^\mathrm{D} = 0$. To compute π_3^OD we need $F_3 = [\widetilde{H}_\delta^1 \sharp \pi_\mathrm{r} - \pi_\mathrm{r} \sharp \widetilde{H}_\delta^1]_3 = [H_3; \pi_\mathrm{r}]$ since H_2 and its derivatives commute with π_r. Now $\pi_\mathrm{r} \pi_3 (\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \pi_\mathrm{r}) = \pi_\mathrm{r} F_3 (\Xi - \lambda_* \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1} (\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \pi_\mathrm{r}) = -\pi_\mathrm{r} H_3 (\Xi - \lambda_* \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1} (\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \pi_\mathrm{r})$ which implies

$$\begin{aligned} \pi_3 = \pi_3^\mathrm{OD} &= -(\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \pi_\mathrm{r})(\Xi - \lambda_* \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1} H_3 \pi_\mathrm{r} - \pi_\mathrm{r} H_3 (\Xi - \lambda_* \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1} (\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \pi_\mathrm{r}) \\ u_3 = b_3 &= (\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \pi_\mathrm{r})(\Xi - \lambda_* \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1} H_3 \pi_\mathrm{r} - \pi_\mathrm{r} H_3 (\Xi - \lambda_* \mathbb{1}_{\mathcal{H}_\mathrm{f}})^{-1} (\mathbb{1}_{\mathcal{H}_\mathrm{f}} - \pi_\mathrm{r}). \end{aligned}$$

To give an explicit representation of $\pi = \pi_\mathrm{r} + \delta^3 \pi_3 + \mathcal{O}(\delta^4)$ and $u = \mathbb{1}_{\mathcal{H}_\mathrm{f}} + \delta^3 u_3 + \mathcal{O}(\delta^4)$ we denote with $\mathcal{H}_*^{(3)} \subset \mathcal{H}_\mathrm{f}$ the three-dimensional vector space spanned by the three

Hermite vectors ψ_{*+1}, ψ_* and ψ_{*-1}. Using the decomposition $\mathcal{H}_\mathrm{f} = \mathcal{H}_*^{(3)} \oplus \mathcal{H}_*^{(3)\perp}$ and setting $\{\psi_{*+1}, \psi_*, \psi_{*-1}\}$ as canonical basis of $\mathcal{H}_*^{(3)} \simeq \mathbb{C}^3$ one has that

$$\pi_3 = \begin{pmatrix} 0 & \overline{\omega_*} & 0 \\ \omega_* & 0 & -\overline{\mu_*} \\ 0 & -\mu_* & 0 \end{pmatrix} \oplus 0, \qquad u_3 = \begin{pmatrix} 0 & -\overline{\omega_*} & 0 \\ \omega_* & 0 & -\overline{\mu_*} \\ 0 & \mu_* & 0 \end{pmatrix} \oplus 0, \tag{3.106}$$

where $\omega_*(p_\mathrm{s}, x_\mathrm{s}) := \frac{\sqrt{2(n_*+1)}}{3} D_z(V)(p_\mathrm{s}, x_\mathrm{s})$ and $\mu_*(p_\mathrm{s}, x_\mathrm{s}) := \sqrt{2n_*} D_z(V)(p_\mathrm{s}, x_\mathrm{s})$ according to the notation of (3.84). ♦◊

3.4.7 Harper-like Hamiltonians

The first term in (3.97) is a multiple of the identity, and therefore does not contribute to the dynamics as far as the expectation values of the observables are concerned. The leading term, providing a non-trivial contribution to the dynamics at the original microscopic time scale $s \propto \delta^2\tau$, is the bounded operator $V(P_\mathrm{s}, Q_\mathrm{s})$ acting on the reference Hilbert space $L^2(\mathbb{R}, dx_\mathrm{s})$. This operator is the Weyl quantization of the $\mathbb{Z}^2$-periodic smooth function V defined on the classical phase space $\mathbb{R}^2$. Hereafter we write $x_\mathrm{s} \equiv x$ to simplify the notation.

The quantization procedure can be reformulated by introducing the unitary operators $\mathcal{U}_\infty := \mathrm{e}^{-i2\pi Q}$ and $\mathcal{V}_\infty := \mathrm{e}^{-i2\pi P}$ (*Harper unitaries*), acting on $\mathcal{H}_\infty := L^2(\mathbb{R}, dx)$ as

$$(\mathcal{U}_\infty\psi)(x) = \mathrm{e}^{-i2\pi x}\psi(x), \qquad (\mathcal{V}_\infty\psi)(x) = \psi(x - \iota_q\epsilon_\infty) \tag{3.107}$$

where $\epsilon_\infty B := h_B = 2\pi\delta^2$.

For any $\mathbb{Z}^2$-periodic function

$$F(p, x) = \sum_{n,m=-\infty}^{+\infty} f_{n,m}\mathrm{e}^{-i2\pi(np+mx)}$$

whose Fourier series is uniformly convergent, the h_B-Weyl quantization of F is given by the formula

$$\widehat{F}(\mathcal{U}_\infty, \mathcal{V}_\infty) = \sum_{n,m=-\infty}^{+\infty} f_{n,m}\mathrm{e}^{-i\pi nm(\iota_q\epsilon_\infty)}\, \mathcal{V}_\infty^n \mathcal{U}_\infty^m. \tag{3.108}$$

where the fundamental commutation relation $\mathcal{U}_\infty\mathcal{V}_\infty = \mathrm{e}^{-i2\pi(\iota_q\epsilon_\infty)}\, \mathcal{V}_\infty\mathcal{U}_\infty$ has been used. Formula (3.108) defines a *Harper-like Hamiltonian* with deformation parameter $-\iota_q\, \epsilon_\infty$ (c.f. Sections 2.1 and 2.3).

In analogy with Section 3.3.7, we summarize the discussion in the following conclusion.

CONCLUSION 3.4.11. *Under the assumptions of Theorem 3.4.8, for every* $\mathsf{V}_\Gamma \in C_\mathrm{b}^\infty(\mathbb{R}^2, \mathbb{R})$, *in the Harper regime* ($h_B \to 0$)*, the dynamics generated by the Hamiltonian* $\mathbf{H}_\mathrm{BL}$ (2.1) *restricted to the spectral subspace corresponding to a Landau level* λ_* *is approximated up* *to an error of order $\epsilon_\infty := h_B$ (and up to a unitary transform and an energy rescaling) by the dynamics generated on the reference Hilbert space $\mathcal{H}_\infty := L^2(\mathbb{R}, dx)$ by a Harper-like Hamiltonian, i.e. by a power series in the Harper unitaries $\mathcal{U}_\infty$ and $\mathcal{V}_\infty$, defined by* (3.107).

3.4.8 Coupling of Landau bands in a periodic magnetic potential

According to Theorem 3.4.8, the first non-trivial term which describes the effective dynamics in the almost invariant subspace related to a single Landau band λ_* is of order $\delta^2 \propto h_B$. An important ingredient in the proof is that $\mathbf{A}_\Gamma = 0$ implies $H_1 = 0$. Moreover, the second non-trivial correction appears at order $\delta^4 \propto {h_B}^2$ although $H_3 \neq 0$. Indeed, the correction at order δ^3 vanishes since H_3, defined by (3.84), is linear in $\mathfrak{a}$ and $\mathfrak{a}^\dagger$, hence $\langle \psi_* | H_3 | \psi_* \rangle = 0$. This observation suggests that for a family of Landau bands which contains two contiguous bands $\{\lambda_*, \lambda_{*+1}\}$ one has, in general, a second non-trivial correction of order $\delta^3 \propto {h_B}^{\frac{3}{2}}$ for the effective dynamics. Indeed, in this case one has $\pi_\mathrm{r} H_3 \pi_\mathrm{r} \neq 0$ since $\langle \psi_* | H_3 | \psi_{*+1} \rangle$ is generally non zero. Nevertheless, also in this case, the first non-trivial correction is of order δ^2.

Is there any mechanism to produce a non-trivial correction in the effective dynamics with leading order $\delta \propto {h_B}^{\frac{1}{2}}$? An affirmative answer requires $H_1 \neq 0$, and the latter condition is satisfied if we include in the Hamiltonian $\mathbf{H}_{\mathrm{BL}}$ the effect of a Γ-periodic vector potential $\mathbf{A}_\Gamma$ (i.e. $\natural = 0$). Since in this situation H_1 is linear in $\mathfrak{a}$ and $\mathfrak{a}^\dagger$, to obtain a non-trivial effect we need to consider a spectral subspace which contains at least two contiguous Landau bands.

Our goal is to derive the (non-trivial) leading order for the effective Hamiltonian in this framework. According to the notation of Theorem 3.4.8, we need to expand the Moyal product $h^{\natural=0} = u \sharp \pi \sharp \widetilde{H}^0_\delta \sharp \pi \sharp u^\dagger = \pi_\mathrm{r} \sharp u \sharp \widetilde{H}^0_\delta \sharp u^\dagger \sharp \pi_\mathrm{r} + \mathcal{O}(\delta^\infty)$ up to the first order δ. The symbols $\pi = \pi_\mathrm{r} + \mathcal{O}(\delta)$ and $u = \mathbb{1}_{\mathcal{H}_\mathrm{f}} + \mathcal{O}(\delta)$ are derived as in the general construction showed in the proof of Theorem 3.4.8. Now $\mathcal{K} := \mathrm{Im}\, \Pi_\mathrm{r} \simeq L^2(\mathbb{R}, dx_\mathrm{s}) \otimes \mathbb{C}^2$.

Expanding at zero order one finds $h_0 = \pi_0\, u_0\, H_0 u_0^{-1}\, \pi_0 = \pi_\mathrm{r}\, \Xi\, \pi_\mathrm{r} = \pi_\mathrm{r}\, \Xi = \Xi\, \pi_\mathrm{r}$ and its quantization is the operator on $\mathcal{K}$ defined by

$$\mathrm{Op}_\delta(h_0) = \begin{pmatrix} \left(n_* + \frac{3}{2}\right) \mathbb{1}_{\mathcal{H}_\mathrm{s}} & 0 \\ 0 & \left(n_* + \frac{1}{2}\right) \mathbb{1}_{\mathcal{H}_\mathrm{s}} \end{pmatrix} = (n_* + 1) \mathbb{1}_{\mathcal{K}} + \begin{pmatrix} \frac{1}{2} \mathbb{1}_{\mathcal{H}_\mathrm{s}} & 0 \\ 0 & -\frac{1}{2} \mathbb{1}_{\mathcal{H}_\mathrm{s}} \end{pmatrix} \tag{3.109}$$

As for the next order, from equation (3.104) it follows $\chi_1 = H_1 + u_1 H_0 - H_0 u_1$ (we use $\chi_0 = H_0$) which implies $h_1 = \pi_\mathrm{r} \chi_1 \pi_\mathrm{r} = \pi_\mathrm{r}\, H_1\, \pi_\mathrm{r} + \pi_\mathrm{r}\, [u_1; \Xi]\, \pi_\mathrm{r}$. To conclude the computation we need u_1 and π_1. Using the recursive formulas (3.98), (3.99), (3.100), (3.101), (3.102) and (3.103), one obtains $-2a_1 := \left[u_0 \sharp u_0{}^\dagger - \mathbb{1}_{\mathcal{H}_\mathrm{f}}\right]_1 = 0$, $b_1 := [\pi_\mathrm{r}; B_1]$ and $B_1 = [u_0 \sharp \pi \sharp u_0{}^* - \pi_\mathrm{r}]_1 = \pi_1$ since $a_1 = 0$. Observing that $G_1 = [\pi_\mathrm{r} \sharp \pi_\mathrm{r} - \pi_\mathrm{r}]_1 = 0$, it follows that $\pi_1^{\mathrm{D}} = 0$ and so $u_1 = [\pi_\mathrm{r}; \pi_1] = [\pi_\mathrm{r}; \pi_1^{\mathrm{OD}}]$ which implies $\pi_\mathrm{r} u_1 \pi_\mathrm{r} = 0$. Finally $\pi_\mathrm{r} [u_1; \Xi] \pi_\mathrm{r} = \pi_\mathrm{r} [u_1; \pi_\mathrm{r} \Xi \pi_\mathrm{r}] \pi_\mathrm{r} = 0$ and so $h_1 = \pi_\mathrm{r} H_1 \pi_\mathrm{r}$. According to (3.87) the quantization of h_1 is an operator which acts on $\mathcal{K}$ as

$$\mathbf{Op}_\delta(h_1) = \sqrt{n_* + 1} \begin{pmatrix} 0 & \mathcal{G}(P_{\rm s}, Q_{\rm s}) \\ \mathcal{G}(P_{\rm s}, Q_{\rm s})^\dagger & 0 \end{pmatrix} \tag{3.110}$$

where the operator $\mathcal{G}(P_{\rm s}, Q_{\rm s})$ is defined on $L^2(\mathbb{R}, dx_{\rm s})$ as the Weyl quantization of the $\mathbb{Z}^2$-periodic function g defined by equation (3.89). Summarizing, we obtained the following result:

THEOREM 3.4.12 (Effective Hamiltonian with a periodic magnetic potential)**.** *Under the assumptions of Theorem 3.4.8, in the case* $\mathsf{A}_\Gamma \neq 0$ *the dynamics in the spectral subspace related to a family of two contiguous Landau bands* $\{\sigma_{*+j}(\cdot) = \lambda_{*+j} \mid j = 0, 1\}$ *is approximated by the effective Hamiltonian* $H^\delta_{\rm eff} := Op_\delta(h^{(\natural=0)})$ *on the reference space* $\mathcal{K} = L^2(\mathbb{R}, dx_s) \otimes \mathbb{C}^2$ *which is given, up to errors of order* δ^2*, by*

$$H^\delta_{\rm eff} = (n_* + 1)\mathbb{1}_\mathcal{K} + \sqrt{n_* + 1} \begin{pmatrix} \frac{1}{2\sqrt{n_*+1}}\mathbb{1}_{\mathcal{H}_s} & \delta\, \mathcal{G}(P_s, Q_s) \\ \delta\, \mathcal{G}(P_s, Q_s)^\dagger & -\frac{1}{2\sqrt{n_*+1}}\mathbb{1}_{\mathcal{H}_s} \end{pmatrix} + \mathcal{O}_0\left(\delta^2\right), \tag{3.111}$$

according to the notation introduced in (3.109) *and* (3.110)*.*

Equation (3.89) shows that $g(p_{\rm s}, x_{\rm s}) = g_1(p_{\rm s}, x_{\rm s}) - i g_2(p_{\rm s}, x_{\rm s})$ where the function g_1 and g_2 are related to the component $(\mathsf{A}_\Gamma)_1$ and $(\mathsf{A}_\Gamma)_2$ of the Γ-periodic vector potential by the relation $g_j(a^* \cdot r, b^* \cdot r) = \pi\sqrt{2}\frac{Z\ell}{\Phi_0}(\mathsf{A}_\Gamma)_j(r)$, $j = 1, 2$. Let $\mathcal{G}_j(P_{\rm s}, Q_{\rm s})$ be the Weyl quantization of g_j. By introducing the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_\perp = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.112}$$

one can rewrite the effective Hamiltonian (3.111) in the form

$$H^\delta_{\rm eff} = \left((n_* + 1)\mathbb{1}_{\mathbb{C}^2} + \frac{1}{2}\sigma_\perp\right) \otimes \mathbb{1}_{\mathcal{H}_{\rm s}} + \delta\sqrt{n_* + 1}\ \sum_{j=1}^{2} \sigma_j \otimes \mathcal{G}_j(P_{\rm s}, Q_{\rm s}) + \mathcal{O}_0\left(\delta^2\right). \tag{3.113}$$

Clearly, the operator $\mathcal{G}_j(P_{\rm s}, Q_{\rm s})$ are Harper-like Hamiltonians and can be represented as a power series of the Harper unitaries $\mathcal{U}_\infty$ and $\mathcal{V}_\infty$ of type (3.108). In this case the coefficients in the expansion are (up to a multiplicative constant) the Fourier coefficients of the components $(\mathsf{A}_\Gamma)_j$ of the Γ-periodic vector potential.

The determination of the spectrum of $H^\delta_{\rm eff}$ can be reduced to the (generally simpler) problem of the computation of the spectrum of $\mathcal{G}\mathcal{G}^\dagger$.

PROPOSITION 3.4.13. *Let* $H^{\delta=1}_{\rm eff}$ *be the first order approximation of the effective Hamiltonian* (3.111) *(or* (3.113)*). Then*

$$\sigma(H^{\delta=1}_{\rm eff}) = (n_* + 1) + S_+ \cup S_-, \quad S_\pm := \{\pm\sqrt{1/4 + \delta^2(n_* + 1)\,\lambda}\ :\ \lambda \in \overline{\sigma}(\mathcal{G}\mathcal{G}^\dagger)\}$$

where $\overline{\sigma}(\mathcal{G}\mathcal{G}^\dagger) = \sigma(\mathcal{G}\mathcal{G}^\dagger) \cup \{0\}$ *if* $\{0\} \in \sigma(\mathcal{G}^\dagger\mathcal{G}) \setminus \sigma(\mathcal{G}\mathcal{G}^\dagger)$ *and* $\overline{\sigma}(\mathcal{G}\mathcal{G}^\dagger) = \sigma(\mathcal{G}\mathcal{G}^\dagger)$ *otherwise.*

Proof. We give only a sketch of the proof. The term $(n_* + 1)\mathbb{1}_{\mathcal{K}}$ shifts the spectrum by a constant value (n_*+1), then we can consider only the spectrum of $\mathcal{B} := H_{\text{eff}}^{\delta=1} - (n_*+1)\mathbb{1}_{\mathcal{K}}$. A simple computation shows that

$$\mathcal{B}^2 = \begin{pmatrix} \frac{1}{2}\mathbb{1}_{\mathcal{H}_s} & \delta\sqrt{n_*+1}\,\mathcal{G} \\ \delta\sqrt{n_*+1}\,\mathcal{G}^\dagger & -\frac{1}{2}\mathbb{1}_{\mathcal{H}_s} \end{pmatrix}^2 = \frac{1}{4}\mathbb{1}_{\mathcal{K}} + \delta^2(n_*+1)\begin{pmatrix} \mathcal{G}\mathcal{G}^\dagger & 0 \\ 0 & \mathcal{G}^\dagger\mathcal{G} \end{pmatrix}$$

which implies that $\sigma(\mathcal{B}^2) = \{1/4 + \delta^2(n_*+1)\lambda \ : \ \lambda \in \sigma(\mathcal{G}\mathcal{G}^\dagger) \cup \sigma(\mathcal{G}^\dagger\mathcal{G})\}$. The operators $\mathcal{G}\mathcal{G}^\dagger$, $\mathcal{G}^\dagger\mathcal{G}$ and $\mathcal{B}$ are bounded and self-adjoint. To show that $\sigma(\mathcal{G}\mathcal{G}^\dagger) \setminus \{0\} = \sigma(\mathcal{G}^\dagger\mathcal{G}) \setminus \{0\}$, let $\lambda \in \sigma(\mathcal{G}\mathcal{G}^\dagger)$ with $\lambda \neq 0$ and $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_s \setminus \text{Ker}(\mathcal{G}^\dagger)$ be a sequence of non zero vectors such that $\|(\mathcal{G}\mathcal{G}^\dagger - \lambda)\psi_n\|_{\mathcal{H}_s} \to 0$ (*Weyl's criterion*), then $\|(\mathcal{G}^\dagger\mathcal{G} - \lambda)\mathcal{G}^\dagger\psi_n\|_{\mathcal{H}_s} \leqslant \|\mathcal{G}^\dagger\|_{\mathscr{B}(\mathcal{H}_s)}\|(\mathcal{G}\mathcal{G}^\dagger - \lambda)\psi_n\|_{\mathcal{H}_s} \to 0$. This implies that $\sigma(\mathcal{G}\mathcal{G}^\dagger) \cup \sigma(\mathcal{G}^\dagger\mathcal{G}) = \overline{\sigma}(\mathcal{G}\mathcal{G}^\dagger)$. Now let $\varepsilon_\pm(\lambda) := \pm\sqrt{1/4 + \delta^2(n_*+1)\,\lambda}$ with $\lambda \in \overline{\sigma}(\mathcal{G}\mathcal{G}^\dagger)$ and $\{\psi_n\}_{n\in\mathbb{N}}$ a sequence of generalized eigenvectors for $\mathcal{G}\mathcal{G}^\dagger$ relative to λ. Then $\Psi_n^{(\pm)} := ((1/2 + \varepsilon_\pm)\psi_n, \delta\sqrt{n_*+1}\mathcal{G}^\dagger\psi_n) \in \mathcal{H}_s \otimes \mathbb{C}^2$ is a sequence of generalized eigenvectors for $\mathcal{B}$ relative to $\varepsilon_\pm$. ■

Chapter 4

Bloch-Floquet transform and emerging geometry

Diviser chacune des difficultés que j'examinerais, en autant de parcelles qu'il se pourrait, et qu'il serait requis pour les mieux résoudre.

(Divide each difficulty into as many parts as is feasible and necessary to resolve it.)

René Descartes
Discours de la méthode, 1637

Abstract

We investigate the relation between the symmetries of a quantum system and its topological quantum numbers, in a general C^*-algebraic framework. We prove that, under suitable assumptions on the symmetry algebra, there exists a generalization of the Bloch-Floquet transform which induces a direct-integral decomposition of the algebra of observables. Such generalized transform selects uniquely the set of "continuous sections" in the direct integral, thus yielding a Hilbert bundle. The proof is constructive and yields an explicit description of the fibers. The emerging geometric structure provides some topological invariants of the quantum system. In greater detail, the content of the paper is the following: Section 4.1 provides the the basic notions of physical frame and $\mathbb{G}$-algebra; Section 4.2 contains some simple guiding examples; Section 4.3 and 4.4 are devoted to review the von Neumann's complete spectral theorem and the Maurin's nuclear spectral theorem; Section 4.5 concerns the notion of wandering property for a commutative C^*-algebra generated by a finite family of operators; Section 4.6 provides a formula which generalizes Bloch-Floquet transform to the case of a $\mathbb{Z}^N$-algebra which satisfies the wandering property; Section 4.7 is devoted to show how a non trivial topology (and geometry) emerges in a canonical and essentially unique way from the decomposition induced by the generalized Bloch-Floquet transform.

4.1 Motivation for a "topological" decomposition

Topological quantum numbers play a prominent rôle in solid-state physics (Thouless 1998) A typical way to compute a topological quantum number in presence of symmetries is to fibrate the C^*-algebra of physical observables, and the Hilbert space where it is represented, with respect to the action of an abelian symmetry group. The prototypical example is provided by periodic systems and the usual Bloch-Floquet transform.

Example 4.1.1 (*Periodic systems, intro*)**.** The electron dynamics in a periodic crystal is generated by

$$H_{\text{per}} := -\frac{1}{2}\Delta + V_\Gamma \tag{4.1}$$

defined on a suitable domain (of essential self-adjointness) in the "physical" Hilbert space $\mathcal{H}_{\text{phy}} := L^2(\mathbb{R}^d, d^d x)$. The periodicity of the crystal is described by the lattice

$$\Gamma := \{\gamma \in \mathbb{R}^d \ : \gamma = \sum_{j=1}^{d} n_j\, \gamma_j,\ n_j \in \mathbb{Z}\} \simeq \mathbb{Z}^d$$

where $\{\gamma_1, \ldots, \gamma_d\}$ is a linear basis of $\mathbb{R}^d$. The potential V_Γ is Γ-periodic, i.e. $V_\Gamma(\,\cdot\, - \gamma) = V_\Gamma(\,\cdot\,)$ for all $\gamma \in \Gamma$. The $\mathbb{Z}^d$-symmetry is implemented by the translation operators $\{T_1, \ldots, T_d\}$, $(T_j\psi)(x) := \psi(x - \gamma_j)$. One defines the *Bloch-Floquet transform*[1] (Kuchment 1993), initially for $\psi \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz space), by posing

$$(\mathcal{U}_{\text{BF}}\psi)(k, \theta) := \sum_{\gamma \in \Gamma} e^{-i\gamma \cdot k} \big(T_j^{n_j} \psi\big)(\theta), \qquad (k, \theta) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{4.2}$$

where $\gamma = \sum_j n_j \gamma_j$. Definition (4.2) extends to a unitary operator

$$\mathcal{U}_{\text{BF}} : \mathcal{H}_{\text{phy}} \longrightarrow \int_{\mathbb{B}}^{\oplus} \mathcal{H}(k)\, d^d\underline{k} \tag{4.3}$$

where $d^d\underline{k} := d^dk/(2\pi)^d$, $\mathbb{B} \simeq \mathbb{T}^d$ is the fundamental cell of the dual lattice Γ^* or *Brillouin zone* and

$$\mathcal{H}(k) := \big\{\varphi \in L^2_{\text{loc}}(\mathbb{R}^d, d^d\theta) : \varphi(\theta + \gamma) = e^{ik\cdot\gamma}\varphi(\theta) \quad \forall \gamma \in \Gamma\big\}.$$

In this representation, the Fermi projector $P_\mu = E_{(-\infty,\mu)}(H_{\text{per}})$ is a decomposable operator, in the sense that $\mathcal{U}_{\text{BF}}\, P_\mu\, \mathcal{U}_{\text{BF}}^{-1} = \int_{\mathbb{B}}^{\oplus} P(k)\, dk$. Thus, under the assumption that μ lies in a spectral gap, the Fermi projector defines (canonically) a complex vector bundle over $\mathbb{B}$, whose fiber at $k \in \mathbb{B}$ is $\text{Im}P(k) \subset \mathcal{H}(k)$ (called *Bloch bundle* in (Panati 2007)). Some geometric properties of this vector bundle are physically measurable: for example, for $d = 2$, the Chern number corresponds to the transverse conductance (measured in suitable units). As far as the time-reversal symmetric Hamiltonian (4.1) is concerned, such Chern number is zero; however, the generalization of this procedure to the case of magnetic translations is relevant in the understanding of the QHE. ◀▷

This chapter addresses the following questions:

(Q-I) to which extent is it possible to generalize the Bloch-Floquet transform? how general is the decomposition procedure outlined above?

(Q-II) how does the topology (geometry) of the decomposition emerge? is there an explicit procedure to construct such geometric structure?

[1] We point on the fact that the *Bloch-Floquet transform* $\mathcal{U}_{\text{BF}}$ defined by equation (4.2) differs from its *modified* version $\mathcal{Z}$ (sometime called *Zack transform*) defined in Section 3.3.2 by equation (3.11). The comparison between $\mathcal{U}_{\text{BF}}$ and $\mathcal{Z}$ is discussed in (Panati 2007).

(Q-III) to which extent is this topological information unique? More precisely, does it depend on the Hilbert space representation of the algebra of observables?

As for question (Q-III), we observe that the datum of a C^*-algebra and a symmetry group does *not* characterize the topological information. For instance the Hofstadter and Harper representation of the NCT-algebra, although isomorphic (Section 2.3) are not unitarily equivalent (Section 2.6). Both representations can be fibered with respect to a $\mathbb{Z}^2$-symmetry, but the corresponding Chern numbers are different (Section 2.8). This observation leads naturally to investigate the last question above, which can be rephrased as: under which conditions two isomorphic representations of a C^*-algebra induce the same topological invariants? The long-term goal is to understand how and under which conditions the symmetries of a physical system are related to observable effects whose origin is geometric (e.g. topological quantum numbers).

Our stage is a general framework: $\mathcal{H}$ is a separable Hilbert space which corresponds to the physical states; $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ is a C^*-algebra of bounded operators on $\mathcal{H}$ which contains the relevant physical models (the self-adjoint elements of $\mathfrak{A}$ can be thought of as Hamiltonians); the commutant $\mathfrak{A}'$ (the set of all the elements in $\mathscr{B}(\mathcal{H})$ which commute with $\mathfrak{A}$) can be thought of as the set of all the physical symmetries with respect to the physics described by $\mathfrak{A}$; any commutative unital C^*-algebra $\mathfrak{S} \subset \mathfrak{A}'$ describes a set of simultaneously implementable physical symmetries.

DEFINITION 4.1.2 (Physical frame). *A* physical frame *is a triple* $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ *where* $\mathcal{H}$ *is a separable Hilbert space,* $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ *is a* C^**-algebra and* $\mathfrak{S} \subset \mathfrak{A}'$ *is a commutative unital* C^**-algebra. The physical frame* $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ *is called* irreducible *if* $\mathfrak{S}$ *is* maximal commutative *(c.f. Appendix B.1). Two physical frames* $\{\mathcal{H}_1, \mathfrak{A}_1, \mathfrak{S}_1\}$ *and* $\{\mathcal{H}_2, \mathfrak{A}_2, \mathfrak{S}_2\}$ *are said* (unitarily) equivalent *if there exists a unitary map* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that* $\mathfrak{A}_2 = U\mathfrak{A}_1U^{-1}$ *and* $\mathfrak{S}_2 = U\mathfrak{S}_1U^{-1}$.

We focus on triples $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ whose C^*-algebra $\mathfrak{S}$ describes symmetries with an intrinsic group structure. In these cases $\mathfrak{S}$ is related to a representation of the group in $\mathcal{H}$, as stated in the following definition.

DEFINITION 4.1.3 ($\mathbb{G}$-algebra). *Let* $\mathbb{G}$ *be a topological group and* $\mathbb{G} \ni g \mapsto U_g \in \mathscr{U}(\mathcal{H})$ *a strongly continuous unitary representation of* $\mathbb{G}$ *in the group* $\mathscr{U}(\mathcal{H})$ *of the unitary operators on* $\mathcal{H}$*. The representation is* faithful *if* $U_g = \mathbb{1}$ *implies* $g = e$ *(*e *is the identity of the group) and is* algebraically compatible *if the operators* $\{U_g \,:\, g \in \mathbb{G}\}$ *are linearly independent in* $\mathscr{B}(\mathcal{H})$*. Let* $\mathfrak{S}(\mathbb{G})$ *be the unital* C^**-algebra generated algebraically by* $\{U_g \,:\, g \in \mathbb{G}\}$ *and closed with respect to the operator norm of* $\mathscr{B}(\mathcal{H})$*. When the representation of* $\mathbb{G}$ *is faithful and algebraically compatible we say that* $\mathfrak{S}(\mathbb{G})$ *is a* $\mathbb{G}$-algebra *in* $\mathcal{H}$*.*

Questions analogous to (Q-I) have been often investigated in the literature, the von Neumann's and Maurin's theorems being the cornerstones in the field (see the short review in Sections 4.3 and 4.4). Topological questions analogous to (Q-II) have been partially investigated in (Godement 1951). Our goal is however slightly different: we aim to obtain an *explicit* description of the decomposition and a *computable recipe* to describe the induced geometry. The price to pay is, of course, to assume stronger hypotheses than the mentioned theorems.

Our explicit answer to question (Q-I) consist of a generalization of the Bloch-Floquet transform, see Section 4.6, with a completely satisfactory answer under the (admittedly strong) hypothesis that $\mathfrak{S}$ is a $\mathbb{Z}^d$-algebra satisfying the wandering property (Theorem 4.6.4). Questions (Q-II) and (Q-III), which are (partially) new with respect to the classical literature, are addressed in Section 4.7, in particular by Theorem 4.7.9. Loosely speaking, the answer is that a physical frame, with $\mathfrak{S}$ a $\mathbb{Z}^d$-algebra satisfying the wandering property, induces canonically a geometric structure (Hilbert bundle), and that equivalent physical frames induce isomorphic Hilbert bundles. Notice that it is crucial that the unitary equivalence intertwines the symmetry algebras.

The previous questions are not purely academic. The technique we develop in this work is the key to prove a geometric duality between the Harper and the Hofstadter models (Theorem 2.8.1), which allows a rigorous proof of the celebrated TKNN formula (Corollary 2.8.2).

4.2 Some guiding examples

A simple prototypical example: symmetries induced by a finite group

It is well known that every finite commutative group is isomorphic to a product group $\mathbb{F} = \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_N}$, where $\mathbb{Z}_{p_j} := \{[0], \ldots, [p_j - 1]\}$ is the *cyclic group* of order $p_j \in \mathbb{N}$. For every $t := (t_1, \ldots, t_N) \in \widehat{\mathbb{F}}$, with $\widehat{\mathbb{F}} := \prod_{j=1}^{N} \{0, \ldots, p_j - 1\}$, let $g_t := ([t_1], \ldots, [t_N])$ be any element in $\mathbb{F}$. The set of indices $\widehat{\mathbb{F}}$ coincides with the dual group of $\mathbb{F}$. The order of the group is equal to the order of its dual, $|\mathbb{F}| = |\widehat{\mathbb{F}}| = p_1 \ldots p_N$. Let $U : \mathbb{F} \to \mathscr{U}(\mathcal{H})$ be a faithful and algebraically compatible unitary representation on a separable Hilbert space $\mathcal{H}$. In particular $U_1 := U_{([1],[0],\ldots,[0])}, \ldots, U_N := U_{([0],[0],\ldots,[1])}$ is a minimal family of generators for the $\mathbb{F}$-algebra $\mathfrak{S}(\mathbb{F})$. Using a multiindex notation we can write $U_{g_t} = U_1^{t_1} \ldots U_N^{t_N} =: U^t$ for all $g_t \in \mathbb{F}$. The condition $U_j^{p_j} = \mathbb{1}$ implies that if U_j has an eigenvalue then it should be a root of the unity of order p_j, i.e. a suitable integer power of $z_j := \exp i(2\pi/p_j)$. Some relevant questions arise in a natural way: is it possible to compute algorithmically the eigenvalues and the eigenspaces of the generators U_j? Is it possible to diagonalize simultaneously the C^*-algebra $\mathfrak{S}(\mathbb{F})$ and to compute its Gel'fand spectrum (the set of the simultaneous eigenvalues)? The answers to these questions are implicit in the following formula:

$$P_t := \frac{1}{|\mathbb{F}|} \sum_{g_n \in \mathbb{F}} z^{-it\cdot n}\, U_{g_n} := \frac{1}{p_1 \ldots p_N} \sum_{n \in \widehat{\mathbb{F}}} (\mathrm{e}^{-i\frac{2\pi}{p_1}t_1})^{n_1} \ldots (\mathrm{e}^{-i\frac{2\pi}{p_N}t_N})^{n_N}\, U_1^{n_1} \ldots U_N^{n_N}. \tag{4.4}$$

For all $t \in \widehat{\mathbb{F}}$ equation (4.4) defines an orthogonal projection; indeed it is immediate to check that: (i) $P_t^\dagger = P_t$ (the adjoint produces a permutation of the indices in the sum); (ii) $P_t P_{t'} = \delta_{t,t'} P_t$ (since $\sum_{0 \leqslant n \leqslant p_j - 1} {z_j}^{t_j n_j} = p_j\, \delta_{t_j,0}$); (iii) from the property of algebraic compatibility it follows that $P_t \neq 0$ for all $t \in \widehat{\mathbb{F}}$; (iv) $\bigoplus_{t \in \widehat{\mathbb{F}}} P_t = P_0 = \mathbb{1}$; (v) $U_j P_t = {z_j}^{t_j}\, P_t$

for all $j = 1, \ldots, N$. We will refer to P_t as the t-th *Bloch-Floquet projection*. The family of the projections $\{P_t\}_{t\in\widehat{\mathbb{F}}}$ induces an orthogonal decomposition of the Hilbert space $\mathcal{H}$ labeled by the set $\widehat{\mathbb{F}}$. Let $\mathcal{H}(t) := \mathrm{Im}(P_t)$, then the map

$$\mathcal{H} \stackrel{\mathcal{U}_{\mathfrak{S}(\mathbb{F})}}{\longrightarrow} \bigoplus_{t\in\widehat{\mathbb{F}}} \mathcal{H}(t) \tag{4.5}$$

defined by $(\mathcal{U}_{\mathfrak{S}(\mathbb{F})}\varphi)(t) := P_t\varphi =: \varphi(t)$ is called *(discrete) Bloch-Floquet transform*. The transform $\mathcal{U}_{\mathfrak{S}(\mathbb{F})}$ is unitary since $\|\varphi\|^2_{\mathcal{H}} = \sum_{t\in\widehat{\mathbb{F}}} \|P_t\varphi\|^2_{\mathcal{H}}$. Every Hilbert space $\mathcal{H}(t)$ is a space of simultaneous eigenvectors for the C^*-algebra $\mathfrak{S}(\mathbb{F})$, and the corresponding eigenvalues are generated as functions of $z_1^{t_1}, \ldots, z_N^{t_N}$. In particular the Gel'fand spectrum of $\mathfrak{S}(\mathbb{F})$ (which coincides with the joint spectrum of the generating family $U_1, \ldots, U_N$) is (homeomorphic to) the dual group $\widehat{\mathbb{F}}$. Finally the transform $\mathcal{U}_{\mathfrak{S}(\mathbb{F})}$ maps the Hilbert space $\mathcal{H}$ into a "fibered" space over the discrete set $\widehat{\mathbb{F}}$. The Hilbert structure is obtained "gluing" the fiber spaces $\mathcal{H}(t)$ by the counting measure defined on $\widehat{\mathbb{F}}$ (direct integral, Appendix B.3). The canonical projection $\bigoplus_{t\in\widehat{\mathbb{F}}} \mathcal{H}(t) \xrightarrow{\pi} \widehat{\mathbb{F}}$ endows the fibered space with the structure of vector bundle (with 0-dimensional base). ◀▷

Two examples of physical frame

In the rest of this work we will generalize the previous decomposition to cases in which the C^*-algebra of the symmetries is more complicated than the one generated by a finite group. However, this simple example encodes already many relevant aspects which appear in the general cases.

EXAMPLE 4.2.1 (*Periodic systems, part one*)**.** Let H_{per} be the operator defined by (4.1). The *Gel'fand-Naĭmark Theorem* (c.f. Appendix B.2) shows that there exists a isomorphism between the commutative C^*-algebra $C_0(\sigma(H_{\mathrm{per}}))$ and a commutative non-unital C^*-algebra $\mathfrak{A}_0(H_{\mathrm{per}})$ of bounded operators in $\mathcal{H}$. The elements of $\mathfrak{A}_0(H_{\mathrm{per}})$ are the operators $f(H_{\mathrm{per}}) \in \mathscr{B}(\mathcal{H})$, for $f \in C_0(\sigma(H_{\mathrm{per}}))$, obtained via the spectral theorem. Let $\mathfrak{A}(H_{\mathrm{per}})$ be the multiplier algebra of $\mathfrak{A}_0(H_{\mathrm{per}})$ in $\mathscr{B}(\mathcal{H})$. This is a unital commutative C^*-algebra which contains $\mathfrak{A}_0(H_{\mathrm{per}})$ (as an essential ideal), its Gel'fand spectrum is the (Stone-Čech) compactification of $\sigma(H_{\mathrm{per}})$ and the Gel'fand isomorphism maps $\mathfrak{A}(H_{\mathrm{per}})$ into the unital C^*-algebra of the continuous and bounded functions on $\sigma(H_{\mathrm{per}})$ denoted by $C_{\mathrm{b}}(\sigma(H_{\mathrm{per}}))$ (c.f. Section B.2 for details). We assume that $\mathfrak{A}(H_{\mathrm{per}})$ is the C^*-algebra of physical models.

Let $\{T_1, \ldots, T_d\}$ be the translation operators corresponding to $\{\gamma_1, \ldots, \gamma_d\}$, defined in Example 4.1.1. Since $[T_i; T_j] = 0$ for any i, j, it follows that the unital C^*-algebra $\mathfrak{S}_T$ generated by the translations, their adjoints and the identity operator is commutative. Moreover, since $[H_{\mathrm{per}}; T_j] = 0$ it follows that $\mathfrak{S}_T \subset \mathfrak{A}(H_{\mathrm{per}})'$. Then the translations $\mathfrak{S}_T$ are simultaneously implementable physical symmetries for the physics described by the Hamiltonian (4.1). Thus $\{\mathcal{H}_{\mathrm{phy}}, \mathfrak{A}(H_{\mathrm{per}}), \mathfrak{S}_T\}$ is a physical frame. It is a convenient model to study the properties of an electron in a periodic medium. ◀▷

EXAMPLE 4.2.2 (*Mathieu-like Hamiltonians, part one*)**.** Let $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ be the one-dimensional torus. In the Hilbert space $\mathcal{H}_{\mathrm{M}} := L^2(\mathbb{T}, ds)$ consider the Fourier orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ defined by $e_n(s) := (2\pi)^{-\frac{1}{2}}\mathrm{e}^{ins}$. Let u and v be the unitary operators defined, for any $g \in \mathcal{H}_{\mathrm{M}}$, by

$$(ug)(s) := g(s + 2\pi\theta), \qquad (vg)(s) := \mathrm{e}^{is}\, g(s), \qquad uv = \mathrm{e}^{i2\pi\theta}\, vu \tag{4.6}$$

with $\theta \in \mathbb{R}$ the *deformation parameter*. The commutation relation in (4.6) shows that the unitaries u and v define a representation (called *Mathieu representation*) of the the NCT-algebra $\mathfrak{A}_\theta$ on the Hilbert space $\mathcal{H}_{\mathrm{M}}$ (c.f. Section 2.3). We denote with $\mathfrak{A}^\theta_{\mathrm{M}} := \pi_{\mathrm{M}}(\mathfrak{A}_\theta)$ the unital C^*-subalgebra of $\mathscr{B}(\mathcal{H}_{\mathrm{M}})$ generated by $u := \pi_{\mathrm{M}}(\mathfrak{u})$ and $v := \pi_{\mathrm{M}}(\mathfrak{v})$. With an innocent abuse of nomenclature, we refer to $\mathfrak{A}^\theta_{\mathrm{M}}$ as the *Mathieu C^*-algebra* and we call *Mathieu-like operator* any element in $\mathfrak{A}^\theta_{\mathrm{M}}$. This name is due to the fact that the universal Hofstadter operator $\mathfrak{h}_\theta \in \mathfrak{A}_\theta$ defined by equation (2.29) is mapped by π_{M} in the *almost-Mathieu operator*

$$(h^\theta_0 g)(s) := g(s - 2\pi\theta) + g(s + 2\pi\theta) + 2\cos(s)g(s). \tag{4.7}$$

The action of u and v on the elements of the Fourier basis is given explicitly by $ue_n = \mathrm{e}^{i2\pi\theta n}e_n$ and $ve_n = e_{n+1}$ for all $n \in \mathbb{Z}$ and a simple computation shows that equation (4.7) coincides with equation (2.18) (with $\beta = 0$) up to a Fourier transform. The Mathieu representation π_{M} is faithful only when $\theta \notin \mathbb{Q}$.

We focus now on the commutant ${\mathfrak{A}^\theta_{\mathrm{M}}}'$. Let $h \in \mathscr{B}(\mathcal{H}_{\mathrm{M}})$ be a bounded operator such that $[h; u] = 0 = [h; v]$ and let $he_n = \sum_{m\in\mathbb{Z}} h_{n,m}\, e_m$, $h_{n,m} \in \mathbb{C}$, be the action of h on the basis vectors. The relation $[h; v] = 0$ implies $h_{n+1,m+1} = h_{n,m}$ and the relation $[h; u] = 0$ implies $\mathrm{e}^{i2\pi(n-m)\theta}h_{n,m} = h_{n,m}$ for all $n, m \in \mathbb{Z}$. If $\theta \notin \mathbb{Q}$ then $\mathrm{e}^{i2\pi(n-m)\beta} \neq 1$ unless $n = m$, hence $h_{n,m} = 0$ if $n \neq m$ and the condition $h_{n+1,n+1} = h_{n,n}$ implies that $h = \alpha_h\mathbb{1}$ with $\alpha_h \in \mathbb{C}$. This shows that in the irrational case $\theta \notin \mathbb{Q}$ the commutant of the Mathieu C^*-algebra is trivial.

To have a non trivial commutant we need to assume that $\theta := {}^{p}/_{q}$ with p, q non zero integers such that $\gcd(q, p) = 1$. In this case the condition $h \in {\mathfrak{A}^{p/q}_{\mathrm{M}}}'$ implies that $h_{n,m} \neq 0$ if and only if $m - n = kq$ for some $k \in \mathbb{Z}$, moreover $h_{n,n+kq} = h_{0,kq} =: h'_k$ for all $n \in \mathbb{Z}$. Let w be the unitary operator defined on the orthonormal basis by $we_n := e_{n+q}$, namely $w = v^q$. The relations for the commutant imply that $h \in {\mathfrak{A}^{p/q}_{\mathrm{M}}}'$ if and only if $h = \sum_{k\in\mathbb{Z}} h'_k\, w^k$. Then in the rational case the commutant of the Mathieu C^*-algebra is the von Neumann algebra generated in $\mathscr{B}(\mathcal{H}_{\mathrm{M}})$ as the strong closure of the family of finite polynomials in w. We denote by $\mathfrak{S}^q_{\mathrm{M}} := C^*(w)$ the unital commutative C^*-algebra generated by w. Observe that it does not depend on p. The triple $\{\mathcal{H}_{\mathrm{M}}, \mathfrak{A}^{q/p}_{\mathrm{M}}, \mathfrak{S}^q_{\mathrm{M}}\}$ is an example of physical frame.
◀▷

4.3 The complete spectral theorem by von Neumann

The complete spectral theorem is a useful generalization of the usual spectral decomposition of a normal operator on a Hilbert space. It shows that symmetries reduce the description of the full algebra $\mathfrak{A}$ to a family of simpler representations. The main tool used in the theorem is the notion of the direct integral of Hilbert spaces (c.f. Appendix B.3). The "spectral" content of the theorem amounts to the characterization of the base space for the decomposition (the "set of labels") and of the measure which glues together the spaces so that the Hilbert space structure is preserved. These information emerges essentially from the Gel'fand theory (c.f. Appendix B.2). The definitions of decomposable and continuously diagonal operator are reviewed in Appendix B.3.

THEOREM 4.3.1 (von Neumann's complete spectral theorem)**.** *Let $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ be a physical frame and μ the basic measure carried by the spectrum X of $\mathfrak{S}$ (c.f. Appendix B.2). Then there exist*

a) *a direct integral $\mathfrak{H} := \int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ with $\mathcal{H}(x) \neq \{0\}$ for all $x \in X$,*

b) *a unitary map $\mathcal{F}_{\mathfrak{S}} : \mathcal{H} \to \mathfrak{H}$, called* $\mathfrak{S}$-Fourier transform[2]*,*

such that:

(i) *the unitary map $\mathcal{F}_{\mathfrak{S}}$ intertwines the Gel'fand isomorphism $C(X) \ni f \xrightarrow{\mathscr{G}} A_f \in \mathfrak{S}$ and the canonical isomorphism of $C(X)$ onto the continuously diagonal operators $C(\mathfrak{H})$, i.e. the following diagram commutes*

$$\begin{array}{ccc} & f \in C(X) & \\ \swarrow^{\mathscr{G}} & & \searrow \\ \mathfrak{S} \ni A_f & \xrightarrow[\mathcal{F}_{\mathfrak{S}} \ldots \mathcal{F}_{\mathfrak{S}}{}^{-1}]{} & M_f(\cdot) \in C(\mathfrak{H}) \end{array}$$

(ii) *the unitary conjugation $\mathcal{F}_{\mathfrak{S}} \ldots \mathcal{F}_{\mathfrak{S}}^{-1}$ maps the elements of $\mathfrak{A}$ in decomposable operators on $\mathfrak{H}$; more precisely there is a measurable family $x \mapsto \pi_x$ of representations of $\mathfrak{A}$ on $\mathcal{H}(x)$ such that $\mathcal{F}_{\mathfrak{S}}\, \mathfrak{A}\, \mathcal{F}_{\mathfrak{S}}{}^{-1} = \int_X^{\oplus} \pi_x(\mathfrak{A})\, d\mu(x)$;*

(iii) *the representations π_x are irreducible if and only if the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ is irreducible.*

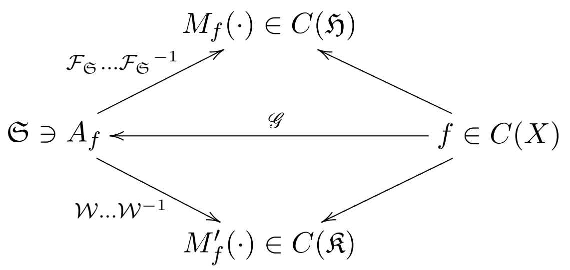
REMARK 4.3.2. For a complete proof of the above theorem one can see (Maurin 1968, Theorem 25, Chapter I and Theorem 2, Chapter V) or (Dixmier 1981, Theorem 1, Part II Chapter 6). For our purposes it is interesting to recall how the fiber Hilbert spaces $\mathcal{H}(x)$ are constructed. For $\psi, \varphi \in \mathcal{H}$ let $\mu_{\psi,\varphi} = h_{\psi,\varphi}\, \mu$ the relation which links the spectral measure $\mu_{\psi,\varphi}$ with the basic measure μ. For μ-almost every $x \in X$ the value of the Radon-Nikodym derivative $h_{\psi,\varphi}$ in x defines a semi-definite sesquilinear form on $\mathcal{H}$, i.e. $(\psi;\varphi)_x := h_{\psi,\varphi}(x)$. Let $\mathcal{I}_x := \{\psi \in \mathcal{H} \;:\; h_{\psi,\psi}(x) = 0\}$. Then the quotient space $\mathcal{H}/\mathcal{I}_x$ is a pre-Hilbert space and $\mathcal{H}'(x)$ is defined to be the its completion. By construction $\mathcal{H}'(x) \neq \{0\}$ for μ-almost every $x \in X$. Let $N \subset X$ be the μ- negligible set on which $\mathcal{H}'(x)$ is trivial or not well defined. Then $\mathfrak{H} := \int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ with $\mathcal{H}(x) := \mathcal{H}'(x)$ if $x \in X \setminus N$ and $\mathcal{H}(x) := H$ if $x \in N$ where H is an arbitrary non trivial Hilbert space. ♦◇

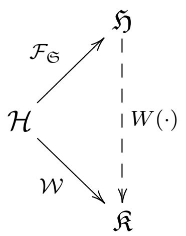
[2] According to the terminology used in (Maurin 1968).

The previous theorem provides only a partial answer to our motivating questions. Firstly, it provides only a partial answer to question (Q-I), since no explicit and computable "recipe" to construct the fiber Hilbert spaces is given. More importantly, Theorem 4.3.1 concerns a measure-theoretic decomposition of the Hilbert space, but it does not select a topological structure, yielding no answer to question (Q-II). In more geometric terms, the elements of $\int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ can be regarded as L^2-*sections* of a fibration over X, while the topological structure is encoded by the (still not defined) space of *continuous sections*. We will show in Section 4.7 that the Bloch-Floquet transform provides a natural choice of a subspace of $\int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ which can be interpreted as the subspace of continuous sections, thus yielding a topological structure.

Given the triple $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$, the direct integral decomposition invoked in the statement of Theorem 4.3.1 is essentially unique in measure-theoretic sense. The space X is unique up to homeomorphism: it agrees with the spectrum of $C(\mathfrak{H})$ in such a way that the canonical isomorphism of $C(X)$ onto $C(\mathfrak{H})$ may be identified with the Gel'fand isomorphism. As for the uniqueness of the direct integral decomposition, the following result holds true (Dixmier 1981, Theorem 3, Part II Chapter 6).

THEOREM 4.3.3 (Uniqueness). *With the notation of Theorem 4.3.1, let ν be a positive measure with support X, $\prod_{x \in X} \mathcal{K}(x)$ a field of non-zero Hilbert spaces over X endowed with a measurable structure, $\mathfrak{K} := \int_X^{\oplus} \mathcal{K}(x)\, d\nu(x)$, $C(\mathfrak{K})$ the commutative unital C^*-algebra of continuously diagonal operators on $\mathfrak{K}$ and $C(X) \to C(\mathfrak{K})$ the canonical isomorphism. Let $\mathcal{W}$ be a unitary map from $\mathcal{H}$ onto $\mathfrak{K}$ transforming $A_f \in \mathfrak{S}$ into $M'_f(\cdot) \in C(\mathfrak{K})$ for all $f \in C(X)$, i.e. such that the first diagram commutes.*





Then, μ and ν are equivalent measures (so one can assume that $\mu = \nu$ up to a rescaling isomorphism). Moreover there exists a decomposable unitary $W(\cdot)$ from $\mathfrak{H}$ onto $\mathfrak{K}$, such that $W(x) : \mathcal{H}(x) \to \mathcal{K}(x)$ is a unitary operator μ-almost everywhere and $\mathcal{W} = W(\cdot) \circ \mathcal{F}_{\mathfrak{S}}$, i.e. the second diagram commutes.

COROLLARY 4.3.4 (Unitary equivalent triples). *Let $\{\mathcal{H}_1, \mathfrak{A}_1, \mathfrak{S}_1\}$ and $\{\mathcal{H}_2, \mathfrak{A}_2, \mathfrak{S}_2\}$ be two equivalent physical frames and U the unitary map which intertwines between them. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ denote the direct integral decomposition of the two triples and let $\mathcal{F}_{\mathfrak{S}_1}$ and $\mathcal{F}_{\mathfrak{S}_2}$ be the two $\mathfrak{S}$-Fourier transforms. Then $W(\cdot) := \mathcal{F}_{\mathfrak{S}_2} \circ U \circ \mathcal{F}_{\mathfrak{S}_1}{}^{-1}$ is a decomposable unitary operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, so that $W(x) : \mathcal{H}_1(x) \to \mathcal{H}_2(x)$ is a unitary map for μ-almost every $x \in X$.*

4.4 The nuclear spectral theorem by Maurin

The complete spectral theorem by von Neumann shows that any physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ admits a representation in which the Hilbert space is decomposed (in a measure-theoretically unique way) in a direct integral $\mathcal{F}_{\mathfrak{S}} : \mathcal{H} \to \int_X^{\oplus} \mathcal{H}(x)\ d\mu(x)$, the elements of $\mathfrak{S}$ are simultaneously diagonalized and the C^*-algebra $\mathfrak{A}$ is decomposed on the fibers. In this sense the map $\mathcal{F}_{\mathfrak{S}}|_x$ restricted to the point $x \in X$ generalizes the rôle of the projection (4.4). The contribution of Maurin is a characterization of the fiber spaces $\mathcal{H}(x)$ as common generalized eigenspaces for $\mathfrak{S}$.

A key ingredient of the Maurin's theorem is the notion of *(nuclear) Gel'fand triple*. The latter is a triple $\{\Phi, \mathcal{H}, \Phi^*\}$ with $\mathcal{H}$ a separable Hilbert space, $\Phi \subset \mathcal{H}$ a norm-dense subspace such that Φ has a topology for which it is a nuclear space and the inclusion map $\imath : \Phi \hookrightarrow \mathcal{H}$ is continuous, and Φ^* is the topological dual of Φ. Identifying $\mathcal{H}$ with its dual space $\mathcal{H}^*$, one gets an *antilinear* injection $\imath^* : \mathcal{H} \hookrightarrow \Phi^*$. The duality pairing between Φ and Φ^* is compatible with the scalar product on $\mathcal{H}$, namely $\langle \imath^*(\psi_1), \psi_2 \rangle = (\psi_1, \psi_2)_{\mathcal{H}}$ whenever $\psi_1 \in \mathcal{H}$ and $\psi_2 \in \Phi$. Hereafter we write $\langle \psi_1, \psi_2 \rangle$ for $\langle \imath^*(\psi_1), \psi_2 \rangle$. If A is a bounded operator on $\mathcal{H}$ such that $A^\dagger$ leaves invariant Φ and $A^\dagger : \Phi \to \Phi$ is continuous with respect to the nuclear topology of Φ, one defines $\hat{A} : \Phi^* \to \Phi^*$ by posing $\langle \hat{A}\,\eta; \varphi \rangle := \langle \eta; A^\dagger\,\varphi \rangle$ for all $\eta \in \Phi^*$ and $\varphi \in \Phi$. Then $\hat{A}$ is continuous and is an extension of A, defined on $\mathcal{H}$, to Φ^*. References about the theory of Gel'fand triples can be found in (de la Madrid 2005).

Assume the notation of Theorem 4.3.1. Let $\{\xi_k(\cdot)\ :\ k \in \mathbb{N}\}$ be a *fundamental family* of orthonormal measurable vector fields (see Appendix B.3) for the direct integral $\mathfrak{H}$ defined by the $\mathfrak{S}$-Fourier transform $\mathcal{F}_{\mathfrak{S}}$. Any square integrable vector field $\varphi(\cdot)$ can be written in a unique way as $\varphi(\cdot) = \sum_{k \in \mathbb{N}} \widehat{\varphi}_k(\cdot)\ \xi_k(\cdot)$ where $\widehat{\varphi}_k \in L^2(X, d\mu)$ for all $k \in \mathbb{N}$. Equipped with this notation, the scalar product in $\mathfrak{H}$ reads

$$\langle \varphi(\cdot); \psi(\cdot) \rangle_{\mathfrak{H}} = \int_X \sum_{k=1}^{\dim\ \mathcal{H}(x)} \overline{\widehat{\varphi}_k(x)}\ \widehat{\psi}_k(x)\ d\mu(x).$$

For any $\varphi \in \mathcal{H}$ let $\varphi(\cdot) := \mathcal{F}_{\mathfrak{S}}\varphi$ be the square integrable vector field obtained from φ by the $\mathfrak{S}$-Fourier transform. Denote with $A_f \in \mathfrak{S}$ the operator associated with $f \in C(X)$ through the Gel'fand isomorphism. One checks that

$$(\widehat{\mathcal{F}_{\mathfrak{S}} A_f \varphi})_k(x) = (\xi_k(x); f(x)\varphi(x))_x = f(x)\ \widehat{\varphi}_k(x) \qquad k = 1, 2, \ldots, \dim\ \mathcal{H}(x). \tag{4.8}$$

Suppose that $\{\Phi, \mathcal{H}, \Phi^*\}$ is a Gel'fand triple for the space $\mathcal{H}$. If $\varphi \in \Phi$ then the map $\Phi \ni \varphi \mapsto \widehat{\varphi}_k(x) := (\xi_k(x); \varphi(x))_x \in \mathbb{C}$ is linear; moreover it is possible to show that it is continuous with respect to the nuclear topology of Φ, for an appropriate choice of Φ. This means that there exists $\eta_k(x) \in \Phi^*$ such that

$$\langle \eta_k(x); \varphi \rangle := \widehat{\varphi}_k(x) = (\xi_k(x); \varphi(x))_x \qquad k = 1, 2, \ldots, \dim\ \mathcal{H}(x). \tag{4.9}$$

Suppose that $A_f : \Phi \to \Phi$ is continuous with respect to the nuclear topology for every $f \in C(X)$. Then from equations (4.8) and (4.9) one has that the extended operator

$\hat{A}_f : \Phi^* \to \Phi^*$, namely $\langle \hat{A}_f\, \eta; \varphi\rangle := \langle \eta; A_{\overline{f}}\, \varphi\rangle$ for all $\eta \in \Phi^*$ and $\varphi \in \Phi$, satisfies

$$\langle \hat{A}_f\, \eta_k(x); \varphi\rangle = \langle \eta_k(x); A_{\overline{f}}\, \varphi\rangle = \overline{f}(x)\, \widehat{\varphi}_k(x) = \langle f(x)\eta_k(x); \varphi\rangle \qquad k = 1, 2, \ldots, \dim \mathcal{H}(x) \tag{4.10}$$

for all $\varphi \in \Phi$. Hence,

$$\hat{A}_f\, \eta_k(x) = f(x)\, \eta_k(x) \qquad \text{in } \Phi^*.$$

In this sense $\eta_k(x)$ is a *generalized eigenvector* for A_f. These claims are made precise in the following statement.

THEOREM 4.4.1 (Maurin's nuclear spectral theorem)**.** *With the notation and the assumptions of Theorem 4.3.1, let $\{\Phi, \mathcal{H}, \Phi^*\}$ be a nuclear Gel'fand triple for the space $\mathcal{H}$ such that Φ is $\mathfrak{S}$-invariant, i.e. each $A \in \mathfrak{S}$ is a continuous linear map $A : \Phi \to \Phi$. Then:*

(i) *for all $x \in X$ the $\mathfrak{S}$-Fourier transform $\mathcal{F}_{\mathfrak{S}}|_x : \Phi \to \mathcal{H}(x)$ such that $\Phi \ni \varphi \mapsto \varphi(x) \in \mathcal{H}(x)$ is continuous with respect to the nuclear topology for μ-almost every $x \in X$;*

(ii) *there is a family of linear functionals $\{\eta_k(x) \;:\; k = 1, 2, \ldots, \dim \mathcal{H}(x)\} \subset \Phi^*$ such that equations* (4.9) *and* (4.10) *hold true for μ-almost all $x \in X$;*

(iii) *with the identification $\eta_k(x) \leftrightarrow \xi_k(x)$ the Hilbert space $\mathcal{H}(x)$ is (isomorphic to) a vector subspace of Φ^*; with this identification the $\mathcal{F}_{\mathfrak{S}}$-Fourier transform is defined on the dense set Φ by*

$$\Phi \ni \varphi \overset{\mathcal{F}_{\mathfrak{S}}|_x}{\longmapsto} \sum_{k=1}^{\dim \mathcal{H}(x)} \langle \eta_k(x); \varphi\rangle\, \eta_k(x) \in \Phi^* \tag{4.11}$$

and the scalar product in $\mathcal{H}(x)$ is formally defined by posing $(\eta_k(x); \eta_j(x))_x := \delta_{k,j}$;

(iv) *under the identification in* (iii) *the spaces $\mathcal{H}(x)$ become the* generalized common eigenspaces *of the operators in $\mathfrak{S}$ in the sense that if $A_f \in \mathfrak{S}$ then $\hat{A}_f\, \eta_k(x) = f(x)\, \eta_k(x)$ for μ-almost every $x \in X$ and all $k = 1, 2, \ldots, \dim \mathcal{H}(x)$.*

For a proof we refer to (Maurin 1968, Chapter II). The identification at point (iii) of the Theorem 4.4.1 depends on the choice of a fundamental family of orthonormal measurable vector fields $\{\xi_k(\cdot) \;:\; k \in \mathbb{N}\}$ for the direct integral $\mathfrak{H}$, which is clearly not unique. If $\{\zeta_k(\cdot) \;:\; k \in \mathbb{N}\}$ is a second fundamental family of orthonormal measurable vector fields for $\mathfrak{H}$, then there exists a decomposable unitary map $W(\cdot)$ such that $W(x)\xi_k(x) = \zeta_k(x)$ for μ-almost every $x \in X$ and every $k \in \mathbb{N}$. The composition $U := \mathcal{F}_{\mathfrak{S}}{}^{-1} \circ W(\cdot) \circ \mathcal{F}_{\mathfrak{S}}$ is a unitary isomorphism of the Hilbert space $\mathcal{H}$ which induces a linear isomorphism between the Gel'fand triples $\{\Phi, \mathcal{H}, \Phi^*\}$ and $\{\Psi, \mathcal{H}, \Psi^*\}$ where $\Psi := U\Phi$. One checks that Ψ is a nuclear space in $\mathcal{H}$ with respect to the topology induced from Φ by the map U (i.e. defined by the family of seminorms $p'_\alpha := p_\alpha \circ U^{-1}$). Ψ^*, the topological dual of Ψ, is $\hat{U}\Phi^*$, in view of the continuity of $U^{-1} : \Psi \to \Phi$. The isomorphism of the Gel'fand triples is compatible with the direct integral decomposition. Indeed if $\vartheta_k(x) \leftrightarrow \zeta_k(x)$ is the identification between the new orthonormal basis $\{\zeta_k(x) \;:\; k = 1, 2, \ldots, \dim \mathcal{H}(x)\}$ of $\mathcal{H}(x)$ and a family of linear functionals $\{\vartheta_k(x) \ : \ k = 1, 2, \ldots, \dim \mathcal{H}(x)\} \subset \Psi^*$ then equation (4.9) implies that for any $\varphi \in \Psi$

$$\langle \vartheta_k(x); \varphi \rangle := (\zeta_k(x); \varphi(x))_x = \left(\xi_k(x); W(x)^{-1}\varphi(x)\right)_x = \langle \eta_k(x); U^{-1}\varphi \rangle = \langle \hat{U}\, \eta_k(x); \varphi \rangle. \tag{4.12}$$

Proposition 4.4.2. *Up to a canonical identification of isomorphic Gel'fand triples the realization* (4.11) *of the fiber spaces $\mathcal{H}(x)$ as generalized common eigenspaces is canonical in the sense that it does not depend on the choice of a fundamental family of orthonormal measurable fields.*

From Proposition 4.4.2 and Corollary 4.3.4 and it follows that:

Corollary 4.4.3. *Up to a canonical identification of isomorphic Gel'fand triples the realization* (4.11) *of the fiber spaces $\mathcal{H}(x)$ as generalized common eigenspaces is preserved by a unitary transform of the triple $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$.*

Theorem 4.4.1 assumes the existence of a $\mathfrak{S}$-invariant nuclear space and the related Gel'fand triple. If $\mathfrak{S}$ is generated by a countable family, such nuclear space does exist and there is an algorithmic procedure to construct it (Maurin 1968, Chapter II, Theorem 6).

Theorem 4.4.4 (Existence of the nuclear space)**.** *Let $\{A_1, A_2, \ldots\}$ a countable family of commuting bounded normal operators on the separable Hilbert space $\mathcal{H}$ which generate (together with their adjoints and the identity) the commutative C^*-algebra $\mathfrak{S}$. Then there exists a countable $\mathfrak{S}$-cyclic system $\{\psi_1, \psi_2, \ldots\}$ which generates a nuclear space $\Phi \subset \mathcal{H}$ such that:* a) *Φ is dense in $\mathcal{H}$;* b) *the embedding $\imath : \Phi \hookrightarrow \mathcal{H}$ is continuous;* c) *the maps $A_j^m : \Phi \to \Phi$ are continuous for all $j, m \in \mathbb{N}$.*

Remark 4.4.5. We recall that a countable (or finite) family $\{\psi_1, \psi_2, \ldots\}$ of orthonormal vectors in $\mathcal{H}$ is a $\mathfrak{S}$-*cyclic system* for $\mathfrak{S}$ if the set $\{{A^\dagger}^b A^a \psi_k \ : \ k \in \mathbb{N}, \ \ a, b \in \mathbb{N}^\infty_{\text{fin}}\}$ is dense in $\mathcal{H}$, where $\mathbb{N}^\infty_{\text{fin}}$ is the space of $\mathbb{N}-$valued sequences which are definitely zero (i.e. $a_n = 0$ for any $n \in \mathbb{N} \setminus I$ with $|I| < +\infty$) and $A^a := A_1^{a_1} A_2^{a_2} \ldots A_N^{a_N}$ for some integer N.

Any C^*-algebra $\mathfrak{S}$ (not necessarily commutative) has many $\mathfrak{S}$-cyclic systems. Indeed one can start from any normalized vector $\psi_1 \in \mathcal{H}$ to build the closed subspace $\mathcal{H}_1$ spanned by the action of $\mathfrak{S}$ on ψ_1. If $\mathcal{H}_1 \neq \mathcal{H}$ one can choose a second normalized vector ψ_2 in the orthogonal complement of $\mathcal{H}_1$ to build the closed subspace $\mathcal{H}_2$. Since $\mathcal{H}$ is separable, this procedure produces a countable (or finite) family $\{\psi_1, \psi_2, \ldots\}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots$. Obviously this construction is not unique. The nuclear space Φ claimed in Theorem 4.4.4 depends on the choice of a $\mathfrak{S}$-cyclic system and generically many inequivalent choices are possible. ♦◊

4.5 The wandering property

An interesting and generally unsolved problem is the construction of the invariant subspaces of an operator or of a family of operators. Let $\mathfrak{S}$ be a C^*-algebra contained in $\mathscr{B}(\mathcal{H})$. If $\psi \in \mathcal{H}$ then the subspace $\mathfrak{S}[\psi]$ generated by the action of $\mathfrak{S}$ on the vector ψ is an invariant subspace for the C^*-algebra. The existence of a particular decomposition of the Hilbert space in invariant subspaces depends on the nature of the C^*-algebra. The problem is reasonably simple to solve for the C^*-algebras which satisfy the wandering property.

DEFINITION 4.5.1 (wandering property). *Let $\mathfrak{S}$ be a commutative unital C^*-algebra generated by the countable family $\{A_1, A_2, \ldots\}$ of commuting bounded normal operators and their adjoints (with the convention $A_j^0 := \mathbb{1}$) in a separable Hilbert space $\mathcal{H}$. We will say that $\mathfrak{S}$ has the* wandering property *if there exists a (at most) countable family $\{\psi_1, \psi_2, \ldots\} \subset \mathcal{H}$ of orthonormal vectors which is $\mathfrak{S}$-cyclic (according to Remark 4.4.5) and such that*

$$(\psi_k; {A^\dagger}^b A^a \psi_h)_{\mathcal{H}} = \|A^a \psi_k\|^2_{\mathcal{H}}\ \delta_{k,h}\ \delta_{a,b} \qquad \forall\, h, k \in \mathbb{N},\ \ \forall\, a, b \in \mathbb{N}^\infty_{\text{fin}}, \tag{4.13}$$

where $A^a := A_1^{a_1} A_2^{a_2} \ldots A_N^{a_N}$, $\delta_{k,h}$ is the usual Kronecker delta and $\delta_{a,b}$ is the Kronecker delta for the multiindices a and b.

Let $\mathcal{H}_k := \mathfrak{S}[\psi_k]$ be the Hilbert subspace generated by the action of $\mathfrak{S}$ on the vector ψ_k. If $\mathfrak{S}$ has the wandering property then the Hilbert space decomposes as $\mathcal{H} = \bigoplus_{k\in\mathbb{N}} \mathcal{H}_k$ and each $\mathcal{H}_k$ is a $\mathfrak{S}$-invariant subspace. We will refer to $\mathcal{H}_k$ as a *wandering subspace* and to $\{\psi_1, \psi_2, \ldots\}$ as the *wandering system*. In these subspaces each operator A_j acts as a unilateral weighted shift and this justifies the use of the adjective "wandering" (Nagy and Foias 1970, Chapter 1, Sections 1 and 2). The wandering property implies many interesting consequences.

PROPOSITION 4.5.2. *Let $\mathfrak{S}$ be a commutative unital C^*-algebra generated by the (at most) countable family $\{A_1, A_2, \ldots\}$ of commuting bounded normal operators and their adjoints in a separable Hilbert space $\mathcal{H}$. Suppose that $\mathfrak{S}$ has the wandering property with respect to the family of vectors $\{\psi_1, \psi_2, \ldots\}$, then:*

(i) *the generators can not be selfadjoint, and $A_j^n \neq \mathbb{1}$ for every $n \in \mathbb{N} \setminus \{0\}$;*

(ii) *every generator which is unitary has no eigenvectors;*

(iii) *if $\mathfrak{S}$ is generated by N unitary operators then $\mathfrak{S}$ is a $\mathbb{Z}^N$-algebra.*

Proof. To prove (i) observe that the condition $A_j = A_j^\dagger$ implies that $A_j \psi_k = 0$ for all ψ_k in the system and the $\mathfrak{S}$-cyclicity imposes $A_j = 0$. As for the second claim, by setting $b = 0$ and $h = k$ in equation (4.13) one sees that $A^a = \mathbb{1}$ implies $a = 0$.

To prove (ii) observe that if $\{U, A_1, A_2, \ldots\}$ is a set of commuting generators for $\mathfrak{S}$ with U unitary, then each vector $\varphi \in \mathcal{H}$ can be written as $\varphi = \sum_{n\in\mathbb{Z}} U^n \chi_n$ where $\chi_n = \sum_{k\in\mathbb{N},\, a\in\mathbb{Z}^N} \alpha_{k,a}\, A^a \psi_k$. Clearly $U\varphi = \sum_{n\in\mathbb{Z}} U^n \chi_{n-1}$ and equation (4.13) implies that $\|\varphi\|^2_{\mathcal{H}} = \sum_{n\in\mathbb{Z}} \|\chi_n\|^2_{\mathcal{H}}$. If $U\varphi = \lambda\varphi$, with $\lambda \in \mathbb{S}^1$, then a comparison between the components provides $\chi_{n-1} = \lambda \chi_n$, i.e. $\chi_n = \lambda^{-n} \chi_0$ for all $n \in \mathbb{Z}$. This contradicts the convergence of the series expressing the norm of φ.

To prove (iii) observe that the map $\mathbb{Z}^N \ni a := (a_1, \ldots, a_N) \mapsto U^a = U_1^{a_1} \ldots U_N^{a_N} \in \mathscr{U}(\mathcal{H})$ is a unitary representation of $\mathbb{Z}^N$ on $\mathcal{H}$ which is also (strongly) continuous since $\mathbb{Z}^N$ is discrete. To show that the representation is algebraically compatible, suppose that $\sum_{a\in\mathbb{Z}^N} \alpha_a U^a = 0$; then from equation (4.13) it follows that $0 = (U^b\psi_k; \sum_{a\in\mathbb{Z}^N} \alpha_a U^a \psi_k)_{\mathcal{H}} = \alpha_b$ for all $b \in \mathbb{Z}^N$, and this concludes the proof. ■

Proposition 4.5.2 shows that the wandering property forces a commutative C^*-algebra generated by a finite number of unitary operators to be a $\mathbb{Z}^N$-algebra. This is exactly what happens in the cases in which we are mostly interested

EXAMPLE 4.5.3 (*Periodic systems, part two*)**.** The commutative unital C^*-algebra $\mathfrak{S}_T$ defined in Example 4.2.1 is generated by a unitary faithful representation of $\mathbb{Z}^d$ on $\mathcal{H}_{\text{phy}}$, given by $\mathbb{Z}^d \ni m \mapsto T^m \in \mathscr{U}(\mathcal{H}_{\text{phy}})$ where $m := (m_1, \ldots, m_d)$ and $T^m := T_1^{m_1} \ldots T_d^{m_d}$. The C^*-algebra $\mathfrak{S}_T$ has the wandering property. Indeed let $\mathcal{Q}_0 := \{x = \sum_{j=1}^d x_j\gamma_j \ : -1/2 \leqslant x_j \leqslant 1/2, \ \ j = 1, \ldots, d\}$ the *fundamental unit cell* of the lattice Γ and $\mathcal{Q}_m := \mathcal{Q}_0 + m$ its translated by the lattice vector $m := \sum_{j=1}^d m_j\gamma_j$. Let $\{\psi_k\}_{k\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$ be a family of functions with support in $\mathcal{Q}_0$ providing an orthonormal basis of $L^2(\mathcal{Q}_0)$ up to the natural inclusion $L^2(\mathcal{Q}_0) \hookrightarrow \mathcal{H}_{\text{phy}}$. This system is $\mathfrak{S}_T$-cyclic since $\mathcal{H}_{\text{phy}} = \bigoplus_{m\in\mathbb{Z}^d} L^2(\mathcal{Q}_m)$. Moreover, it is wandering under the action of $\mathfrak{S}_T$ since the intersection $\mathcal{Q}_0 \cap \mathcal{Q}_m$ has zero measure for every $m \neq 0$. The cardinality of the wandering system is $\aleph_0$. Proposition 4.5.2 assures that $\mathfrak{S}_T$ is a $\mathbb{Z}^d$-algebra. Moreover, as a consequence of Proposition 4.5.7 below, the Gel'fand spectrum of $\mathfrak{S}_T$ is homeomorphic to the d-dimensional torus $\mathbb{T}^d$ and the normalized basic measure is the Haar measure dz on $\mathbb{T}^d$. ◀▷

EXAMPLE 4.5.4 (*Mathieu-like Hamiltonians, part two*)**.** The unital commutative C^*-algebra $\mathfrak{S}^q_{\text{M}} \subset \mathscr{B}(\mathcal{H}_{\text{M}})$ defined in Example 4.2.2 is generated by a unitary faithful representation of the group $\mathbb{Z}$ on the Hilbert space $\mathcal{H}_{\text{M}}$. Indeed, the map $\mathbb{Z} \ni k \mapsto w^k \in \mathscr{U}(\mathcal{H}_{\text{M}})$ is an injective group homomorphism. The set of vectors $\{e_0, \ldots, e_{q-1}\} \subset \mathcal{H}_{\text{M}}$ shows that the C^*-algebra $\mathfrak{S}^q_{\text{M}}$ has the wandering property. In this case the cardinality of the wandering system is q. Proposition 4.5.2 assures that $\mathfrak{S}^q_{\text{M}}$ is a $\mathbb{Z}$-algebra. Moreover, Proposition 4.5.7 below will show that the Gel'fand spectrum of $\mathfrak{S}^q_{\text{M}}$ is homeomorphic to the 1-dimensional torus $\mathbb{T}$ and the normalized basic measure on the spectrum coincide with the Haar measure dz on $\mathbb{T}$. The first claim agrees with the fact that the Gel'fand spectrum of $\mathfrak{S}^q_{\text{M}}$ coincides with the (Hilbert space) spectrum of w, the generator of the C^*-algebra, and $\sigma(w) = \mathbb{T}$. The claim about the basic measure agrees with the fact that the vector e_0 is cyclic for the commutant of $\mathfrak{S}^q_{\text{M}}$ (which is the von Neumann algebra generated by $\mathfrak{A}^{p/q}_{\text{M}}$). Indeed, a general result (c.f. Appendix B.2) assures that the spectral measure μ_{e_0,e_0} provides the basic measure. To determine μ_{e_0,e_0} let $F(w) := \sum_{k\in\mathbb{Z}} \alpha_k w^k$ be any element of $\mathfrak{S}^q_{\text{M}}$. From the definition of spectral measure it follows

$$\alpha_0 = (e_0; F(w)e_0) = \int_{\mathbb{T}} F(z)\ d\mu_{e_0,e_0}(z) = \sum_{k\in\mathbb{Z}} \alpha_k \int_0^{2\pi} \mathrm{e}^{ikt}\ d\widetilde{\mu}_{e_0,e_0}(t). \tag{4.14}$$

where the measure $\widetilde{\mu}_{e_0,e_0}$ is related to μ_{e_0,e_0} by the change of variables $z(t) := \mathrm{e}^{it}$ (c.f.

Convention 2.7.1). Equation (4.14) implies that $\widetilde{\mu}_{e_0,e_0}$ agrees with ${dt}/{2\pi}$ on $C(\mathbb{T})$, namely the basic measure μ_{e_0,e_0} is the normalized Haar measure. ◀▷

It is easy to provide examples of commutative unital C^*-algebras which have the wandering property but which are generated by a family of non unitary or non invertible operators.

EXAMPLE 4.5.5. With the notation of Example 4.2.2 let $\widetilde{w}$ be the operator defined on the Fourier basis $\{e_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}_{\mathrm{M}}$ by $\widetilde{w}\ e_n = w_{[n]}\ e_{n+q}$ where $[n]$ means n modulo q and $w_{[n]} \in \mathbb{C}$. The operator $\widetilde{w}$ is a bilateral weighted shift completely characterized by the fundamental weights $w_0, \ldots, w_{q-1}$. The adjoint of $\widetilde{w}$ is defined by $\widetilde{w}^\dagger\ e_n = \overline{w}_{[n-q]}\ e_{n-q} = \overline{w}_{[n]}\ e_{n-q}$ and an easy computation shows that $\widetilde{w}$ is normal, indeed $\widetilde{w}^\dagger\widetilde{w}\ e_n = \widetilde{w}\widetilde{w}^\dagger\ e_n = |w_{[n]}|^2 e_n$. If $|w_{[j]}| \neq 1$ for some $j = 0, \ldots, q-1$ then $\widetilde{w}$ is not unitary. However, the commutative unital C^*-algebra $\widetilde{\mathfrak{S}}^q_{\mathrm{M}}$ generated by $\widetilde{w}$ has the wandering property with respect to the finite system of vectors $\{e_0, \ldots, e_{q-1}\}$. ◀▷

EXAMPLE 4.5.6. Let $\mathcal{H}_{2\mathrm{M}} := \mathcal{H}_{\mathrm{M}} \oplus \mathcal{H}_{\mathrm{M}}$, $e^{(1)}_n := e_n \oplus 0$ and $e^{(2)}_n := 0 \oplus e_n$ where $\{e_n\}_{n\in\mathbb{Z}}$ is the Fourier basis of $\mathcal{H}_{\mathrm{M}}$ according to the notations of Example 4.2.2. Obviously $\{e^{(1)}_n, e^{(2)}_n\}_{n\in\mathbb{Z}}$ is a basis for $\mathcal{H}^{(2)}_{\mathrm{M}}$. The operators $w^{(1)} := w \oplus 0$ and $w^{(2)} := 0 \oplus w$ are not invertible, are normal and commute. Let $\mathfrak{S}^q_{2\mathrm{M}}$ be the commutative C^*-algebra generated by $\mathfrak{w}^{(1)}$, $\mathfrak{w}^{(2)}$. It is immediate to check that $\mathfrak{S}^q_{2\mathrm{M}}$ has the wandering property with respect to the finite system of vectors $\{e^{(1)}_n, e^{(2)}_n\}_{n=0,\ldots,q-1}$. ◀▷

In the relevant cases of commutative unital C^*-algebras generated by a finite set of unitary operators the wandering property provides a useful characterization of the Gel'fand spectrum and the basic measure. We firstly introduce some notation and terminology. Let $\mathbb{G}$ be a discrete group and $\ell^1(\mathbb{G})$ be the set of sequences $c = \{c_g\}_{g\in\mathbb{G}}$ such that $\|c\|_{\ell^1} = \sum_{g\in\mathbb{G}} |c_g| < +\infty$. Equipped with the convolution product $(c * d)_g := \sum_{h\in\mathbb{G}} c_h d_{g-h}$ and involution $c^\dagger := \{\overline{c}_{-g}\}_{g\in\mathbb{G}}$, $\ell^1(\mathbb{G})$ becomes a unital Banach $*$-algebra called the *group algebra* $\mathbb{G}$. The latter is not a C^*-algebra since the norm $\|\cdot\|_{\ell^1}$ does not verify the C^*-condition $\|c * c^*\|_{\ell^1} = \|c\|^2_{\ell^1}$. In general there exist several inequivalent ways to complete $\ell^1(\mathbb{G})$ to a C^*-algebra by introducing suitable C^*-norms. Two of these C^*-extensions are of particular interest. The first is obtained as the completion of $\ell^1(\mathbb{G})$ with respect to the *universal enveloping norm*

$$\|c\|_u := \sup\{\|\pi(c)\|_{\mathcal{H}}\ :\ \pi : \ell^1(\mathbb{G}) \to \mathscr{B}(\mathcal{H}) \text{ is a } * -\text{representation}\}.$$

The resulting abstract C^*-algebra, denoted by $C^*(\mathbb{G})$, is called the *group C^*-algebra* of $\mathbb{G}$ (or *enveloping* C^*-algebra).

The second relevant extension is obtained by means of the concrete representation of the elements $\ell^1(\mathbb{G})$ as (convolution) multiplicative operators on the Hilbert space $\ell^2(\mathbb{G})$. In other words, for any $\xi = \{\xi_g\}_{g\in\mathbb{G}} \in \ell^2(\mathbb{G})$ and $c = \{c_g\}_{g\in\mathbb{G}} \in \ell^1(\mathbb{G})$ one defines the representation $\pi_r : \ell^1(\mathbb{G}) \to \mathscr{B}(\ell^2(\mathbb{G}))$ as

$$\pi_r(c)\xi := c * \xi = \left\{\sum_{h\in\mathbb{G}} c_h \xi_{g-h}\right\}_{g\in\mathbb{G}}.$$

The representation π_r, known as *left regular representation*, is injective. The norm $\|c\|_r := \|\pi_r(c)\|_{\mathscr{B}(\ell^2(\mathbb{G}))}$ defines a new C^*-norm on $\ell^1(\mathbb{G})$, called *reduced norm*, and a new C^*-extension denoted by $C^*_r(\mathbb{G})$ and called *reduced group C^*-algebra*. Since $\|\cdot\|_r \leq \|\cdot\|_u$ it follows that $C^*_r(\mathbb{G})$ is $*$-isomorphic to a quotient C^*-algebra of $C^*(\mathbb{G})$. Nevertheless, if the group $\mathbb{G}$ is abelian, one has the relevant characterization $C^*_r(\mathbb{G}) = C^*(\mathbb{G}) \simeq C(\widehat{\mathbb{G}})$ where $\widehat{\mathbb{G}}$ denotes the dual (or character) group of $\mathbb{G}$. For more details the reader can refer to (Dixmier 1982, Chapter 13) or (Davidson 1996, Chapter VII).

PROPOSITION 4.5.7. *Let $\mathcal{H}$ be a separable Hilbert space and $\mathfrak{S} \subset \mathscr{B}(\mathcal{H})$ a unital commutative C^*-algebra generated by a finite family $\{U_1, \ldots, U_N\}$ of unitary operators. Assume the wandering property. Then:*

(i) *the Gel'fand spectrum of $\mathfrak{S}$ is homeomorphic to the N-dimensional torus $\mathbb{T}^N$;*

(ii) *the basic measure of $\mathfrak{S}$ is the normalized Haar measure dz on $\mathbb{T}^N$.*

Proof. We use the short notation $U^a = U_1^{a_1} \ldots U_N^{a_N}$ for any $a = (a_1, \ldots, a_N) \in \mathbb{Z}^N$.

To prove (i) one notices that the map $F : \ell^1(\mathbb{Z}^N) \to \mathscr{B}(\mathcal{H})$, defined by $F(c) := \sum_{a\in\mathbb{Z}^N} c_a\, U^a$, is a $*$-representation of $\ell^1(\mathbb{Z}^N)$ into $\mathscr{B}(\mathcal{H})$. As in the proof of Proposition 4.5.2, one exploits the wandering property to see that for any $c \in \ell^1(\mathbb{Z}^N)$, $\sum_a c_a U^a = 0$ implies $c = 0$. Thus F is a faithful representation. Moreover $\|F(c)\|_{\mathscr{B}(\mathcal{H})} \leqslant \|c\|_{\ell^1}$ for all $c \in \ell^1(\mathbb{C})$. Finally, the unital $*$-algebra $\mathfrak{L}^1(\mathbb{Z}^N) := F(\ell^1(\mathbb{Z}^N)) \subset \mathscr{B}(\mathcal{H})$ is dense in $\mathfrak{S}$ (with respect to the operator norm), since it does contain the polynomials in $U_1, \ldots, U_N$, which are a dense subset of $\mathfrak{S}$.

In view of the fact that $\mathbb{Z}^N$ is abelian, to prove (i) it is sufficient to show that $\mathfrak{S} \simeq C^*_r(\mathbb{Z}^N)$. Since $\ell^1(\mathbb{Z}^N)$ and $\mathfrak{L}^1(\mathbb{Z}^N)$ are isomorphic Banach $*$-algebras, and $\mathfrak{L}^1(\mathbb{Z}^N)$ is dense in $\mathfrak{S}$, the latter claims follows if one proves that $\|c\|_r = \|F(c)\|_{\mathscr{B}(\mathcal{H})}$ for any $c \in \ell^1(\mathbb{Z}^N)$. Let $\{\psi_k\}_{k\in\mathbb{N}}$ be the wandering system of vectors for $\mathfrak{S}$. The wandering property assures that the closed subspace $\mathfrak{S}[\psi_k] =: \mathcal{H}_k \subset \mathcal{H}$ is isometrically isomorphic to $\ell^2(\mathbb{Z}^N)$, with unitary isomorphism given by $\mathcal{H}_k \ni \sum_{a\in\mathbb{Z}^N} \xi_a U^a \psi_k \mapsto \{\xi_a\}_{a\in\mathbb{Z}^N} \in \ell^2(\mathbb{Z}^N)$. Then, due to the mutual orthogonality of the spaces $\mathcal{H}_k$, there exists a unitary map $\mathcal{R} : \mathcal{H} \to \bigoplus_{k\in\mathbb{N}} \ell^2(\mathbb{Z}^N)$ which extends all the isomorphisms above. A simple computation shows that $\mathcal{R}F(c)\mathcal{R}^{-1} = \bigoplus_{k\in\mathbb{N}} \pi_r(c)$ for any $c \in \ell^1(\mathbb{Z}^N)$. Since $\mathcal{R}$ is isometric, it follows that $\|F(c)\|_{\mathscr{B}(\mathcal{H})} = \|\bigoplus_{k\in\mathbb{N}} \pi_r(c)\|_{\bigoplus_{k\in\mathbb{N}} \ell^2} = \|\pi_r(c)\|_{\ell^2}$, which is exactly the definition of the norm $\|c\|_r$.

To prove (ii) let $\mu_k := \mu_{\psi_k,\psi_k}$ be the spectral measure defined by the vector of the wandering system ψ_k. The Gel'fand isomorphism identifies the generator $U_j \in \mathfrak{S}$ with $z_j \in C(\mathbb{T}^N)$. It follows that for every $a \in \mathbb{Z}^N$ one has

$$\delta_{a,0} = (\psi_k; U^a\psi_k) = \int_{\mathbb{T}^N} z^a\, d\mu_k(z) := \int_0^{2\pi} \ldots \int_0^{2\pi} z_1^{a_1}(t) \ldots z_N^{a_N}(t)\, d\widetilde{\mu}_k(t), \tag{4.15}$$

where the measure $\widetilde{\mu}_k$ is related to μ_k by the change of variables $z(t) := \mathrm{e}^{it}$ (c.f. Convention 2.7.1). Equation (4.15) shows that for any $k \in \mathbb{N}$ the spectral measure $\widetilde{\mu}_k$ agrees with $dz(t) := {dt_1 \ldots dt_N}/{(2\pi)^N}$.

Let A_f be the element of $\mathfrak{S}$ whose image via the Gel'fand isomorphism is the function $f \in C(\mathbb{T}^N)$. Then

$$(U^b\psi_j; A_f U^a \psi_k)_{\mathcal{H}} = \delta_{j,k}(\psi_k; A_f U^{a-b}\psi_k)_{\mathcal{H}} = \int_{\mathbb{T}^N} f(z)\, \delta_{j,k}\, z^{a-b}\, dz.$$

So the spectral measure $\mu_{U^b\psi_j, U^a\psi_k}$ is related to the Haar measure dz by the function $\delta_{j,k}\, z^{a-b}$. Let $\varphi := \sum_{k\in\mathbb{N}, a\in\mathbb{N}^N} \alpha_{a,k}\, U^a\psi_k$ be any vector in $\mathcal{H}$. Notice that, in view of the wandering property, one has $\alpha_{k,a} \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}^N)$. Then a direct computation shows that $\mu_{\varphi,\varphi}(z) = h_{\varphi,\varphi}(z)\, dz$, where $h_{\varphi,\varphi}(z) = \sum_{k\in\mathbb{N}} |F^{(k)}_\varphi(z)|^2$ with $F^{(k)}_\varphi(z) := \sum_{a\in\mathbb{N}^N} \alpha_{k,a} z^a$. Since $F^{(k)}_\varphi \in L^2(\mathbb{T}^N)$, one has $|F^{(k)}_\varphi|^2 \in L^1(\mathbb{T}^N)$. Let $h^{(M)}_{\varphi,\varphi}(z) = \sum_{k=0}^M |F^{(k)}_\varphi(z)|^2$. Since $h^{(M+1)}_{\varphi,\varphi} \geqslant h^{(M)}_{\varphi,\varphi} \geqslant 0$ and $\int_{\mathbb{T}^N} h^{(M)}_{\varphi,\varphi}(z)\, dz = \sum_{k=0}^M \sum_{a\in\mathbb{N}^N} |\alpha_{k,a}|^2 \leqslant \|\varphi\|_{\mathcal{H}}$ for all M, one concludes by the monotone convergence theorem that $h_{\varphi,\varphi} \in L^1(\mathbb{T}^N)$. ■

Not every commutative C^*-algebra generated by a faithful unitary representation of $\mathbb{Z}^N$ has a wandering system. In this situation, even if the spectrum is still a torus, the basic measure can be inequivalent to the Haar measure.

EXAMPLE 4.5.8. Let R_α the unitary operator on $L^2(\mathbb{R}^2)$ which implements a rotation around the origin of the angle α, with $\alpha \notin 2\pi\mathbb{Q}$. Clearly $R_\alpha^N = R_{N\alpha} \neq \mathbb{1}$ for every integer N, hence the commutative unital C^*-algebra $\mathfrak{R}_\alpha$ generated by R_α is a unitary faithful representation of $\mathbb{Z}$. The Gel'fand spectrum of $\mathfrak{R}_\alpha$, which coincides with the spectrum of R_α, is $\mathbb{T}$. Indeed, the vector $\psi_N(\rho,\phi) := \mathrm{e}^{iN\phi} f(\rho)$ (in polar coordinates) is an eigenvector corresponding to the eigenvalue $\mathrm{e}^{iN\alpha}$. The spectrum of R_α is the closure of $\{\mathrm{e}^{iN\alpha}\ :\ N \in \mathbb{Z}\}$, which is $\mathbb{T}$ in view of the irrationality of α. The existence of eigenvectors excludes the existence of a wandering system (see Proposition 4.5.2). Moreover, since R_α has point spectrum it follows that the basic measure is not the Haar measure. Indeed, the spectral measure μ_{ψ_N,ψ_N} corresponding to the eigenvector ψ_N is the *Dirac measure* concentrated in $\{\mathrm{e}^{iN\alpha}\} \subset \mathbb{C}$. ◀▷

4.6 The generalized Bloch-Floquet transform

The aim of this section is to provide a general algorithm to construct the direct integral decomposition of a commutative C^*-algebra which appears in the von Neumann's complete spectral theorem. The idea is to generalize the construction of the Bloch-Floquet projections (4.4) by a consistent reinterpretation of it. In the spirit of Maurin's theorem, Bloch-Floquet projections should be reinterpreted as "projectors on an appropriate distributional space". In this approach a relevant rôle will be played by the wandering property. We consider a commutative unital C^*-algebra $\mathfrak{S}$ on a separable Hilbert space $\mathcal{H}$ generated by the finite family $\{U_1, U_2, \ldots, U_N\}$ of unitary operators admitting a wandering system $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$. According to the results of Section 4.5, $\mathfrak{S}$ is a $\mathbb{Z}^N$-algebra with Gel'fand spectrum $\mathbb{T}^N$ and with the Haar measure dz as basic measure.

Construction of the nuclear space

Consider the orthonormal basis $\{U^a\psi_k\}_{k\in\mathbb{N},a\in\mathbb{Z}^N}$, where $\{\psi_k\}_{k\in\mathbb{N}}$ is the wandering system, and denote by $\mathcal{L}\subset\mathcal{H}$ the family of all finite linear combinations of the vectors of such basis. For every integer $m\geqslant 0$ denote by $\mathcal{H}_m$ the finite dimensional Hilbert space generated by the finite set of vectors $\{U^a\psi_k\ :\ 0\leqslant k\leqslant m,\ 0\leqslant|a|\leqslant m\}$, where $|a|:=|a_1|+\ldots+|a_N|$. Obviously $\mathcal{H}_m\subset\mathcal{L}$. Let denote by D_m the dimension of the space $\mathcal{H}_m$. If $\varphi=\sum_{k\in\mathbb{N},a\in\mathbb{Z}^N}\alpha_{k,a}U^a\psi_k$ is any element of $\mathcal{H}$ then the formula

$$p_m^2(\varphi):=D_m\sum_{\substack{0\leqslant k\leqslant m\\ 0\leqslant|a|\leqslant m}}|(U^a\psi_k;\varphi)_{\mathcal{H}}|^2=D_m\sum_{\substack{0\leqslant k\leqslant m\\ 0\leqslant|a|\leqslant m}}|\alpha_{k,a}|^2, \tag{4.16}$$

defines a seminorm for every $m\geqslant 0$. From (4.16) it follows that $p_m\leqslant p_{m+1}$ for all m. The countable family of seminorms $\{p_m\}_{m\in\mathbb{N}}$ provides a locally convex topology for the vector space $\mathcal{L}$. Let denote by Σ the pair $\{\mathcal{L},\{p_m\}_{m\in\mathbb{N}}\}$, i.e. the vector space $\mathcal{L}$ endowed with the locally convex topology induced by the seminorms (4.16). Σ is a complete and metrizable (i.e. Fréchet) space. However, for our purposes, we need a topology on $\mathcal{L}$ which is strictly stronger than the metrizable topology induced by the seminorms (4.16).

The quotient space $\Phi_m:=\mathcal{L}/\mathcal{N}_m$, with $\mathcal{N}_m:=\{\varphi\in\mathcal{L}\ :\ p_m(\varphi)=0\}$, is isomorphic to the finite dimensional vector space $\mathcal{H}_m$, hence it is nuclear and Fréchet. This follows immediately observing that the norm p_m on Φ_m coincides, up to the positive constant $\sqrt{D_m}$, with the usual Hilbert norm. Obviously $\Phi_m\subset\Phi_{m+1}$ for all $m\geqslant 0$ and the topology of Φ_m agrees with the topology inherited from Φ_{m+1}, indeed $p_{m+1}|_{\Phi_m}=\sqrt{\frac{D_{m+1}}{D_m}}\,p_m$. We define Φ to be $\bigcup_{m\in\mathbb{N}}\Phi_m$ (which is $\mathcal{L}$ as a set) endowed with the *strict inductive limit topology* which is the stronger topology which makes continuous all the injections $\imath_m:\Phi_m\hookrightarrow\Phi$. The space Φ is called a *LF-space* according to the definition in (Treves 1967, Chapter 13) and it is a nuclear space since it is the strict inductive limit of nuclear spaces (Treves 1967, Proposition 50.1). We will say that Φ is the *wandering nuclear space* defined by the $\mathbb{Z}^N$-algebra $\mathfrak{S}$ on the wandering system $\{\psi_k\}_{k\in\mathbb{N}}$.

PROPOSITION 4.6.1. *The wandering nuclear space Φ defined by the $\mathbb{Z}^N$-algebra $\mathfrak{S}$ on the wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ verifies all the properties stated in Theorem 4.4.4.*

Proof. A linear map $\jmath:\Phi\to\Psi$, with Ψ an arbitrary locally convex topological vector space, is continuous if and only if the restriction $\jmath|_{\Phi_m}$ of $\jmath$ to Φ_m is continuous for each $m\geqslant 0$ (Treves 1967, Proposition 13.1). This implies that the canonical embedding $\imath:\Phi\hookrightarrow\mathcal{H}$ is continuous, since its restrictions are linear operators defined on finite dimensional spaces. The linear maps $U^a:\Phi\to\Phi$ for all $a\in\mathbb{N}^N$ are also continuous for the same reason. Finally Φ is norm-dense in $\mathcal{H}$ since as a set it is the dense domain $\mathcal{L}$. ■

The strict inductive limit topology which defines Φ is stronger than the topology induced by the seminorms (4.16) which defines the Fréchet space Σ. The space Φ is complete but not metrizable since every Φ_m is closed in the topology of Φ_{m+1} (Treves 1967, Theorem 13.1).

The transform

We are now in position to define the *generalized Bloch-Floquet transform* $\mathcal{U}_{\mathfrak{S}}$ for the C^*-algebra $\mathfrak{S}$. The Gel'fand spectrum of $\mathfrak{S}$ is $\mathbb{T}^N$ and the Gel'fand isomorphism associates to the generator U_j the function $z_j \in C(\mathbb{T}^N)$. For any $t \in [0, 2\pi)^N$ and for any $\varphi \in \Phi$ we define (formally for the moment) the Bloch-Floquet transform of φ at point t as

$$\Phi \ni \varphi \overset{\mathcal{U}_{\mathfrak{S}}|_t}{\longmapsto} (\mathcal{U}_{\mathfrak{S}}\varphi)(t) := \sum_{a \in \mathbb{Z}^N} z^{-a}(t)\, U^a \varphi \tag{4.17}$$

where $z^a(t) := e^{ia_1t_1} \ldots e^{ia_Nt_N}$ and $U^a := U_1^{a_1} \ldots U_N^{a_N}$. The structure of equation (4.17) suggests that $(\mathcal{U}_{\mathfrak{S}}\varphi)(t)$ is a common generalized eigenvector for the elements of $\mathfrak{S}$, indeed a formal computation shows that

$$U_j(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = z_j(t) \sum_{a \in \mathbb{Z}^N} z_j^{-1}(t) z^{-a}(t) U_j\, U^a \varphi = \mathrm{e}^{it_j}\ (\mathcal{U}_{\mathfrak{S}}\varphi)(t). \tag{4.18}$$

This guess is clarified by the following result.

THEOREM 4.6.2 (Generalized Bloch-Floquet transform)**.** *Let $\mathfrak{S}$ be a $\mathbb{Z}^N$-algebra in the separable Hilbert space $\mathcal{H}$ with generators $\{U_1, \ldots, U_N\}$ and wandering system $\{\psi_k\}_{k\in\mathbb{N}}$, and let Φ be the corresponding nuclear space. Under these assumptions the* generalized Bloch-Floquet transform (4.17) *defines an injective linear map from the nuclear space Φ into its topological dual Φ^* for every $t \in [0, 2\pi)^N$. More precisely, the transform $\mathcal{U}_{\mathfrak{S}}|_t$ maps Φ onto a subspace $\Phi^*(t) \subset \Phi^*$ which is a common generalized eigenspace for the commutative C^*-algebra $\mathfrak{S}$, i. e. $\hat{U}_j\, (\mathcal{U}_{\mathfrak{S}}\varphi)(t) = e^{it_j}\, (\mathcal{U}_{\mathfrak{S}}\varphi)(t)$ in Φ^*. The map $\mathcal{U}_{\mathfrak{S}}|_t : \Phi \to \Phi^*(t) \subset \Phi^*$ is a continuous linear isomorphism, provided Φ^* is endowed with the $*$-weak topology.*

Proof. We need to verify that the right-hand side of (4.17) is well defined as a linear functional on Φ. Any vector $\varphi \in \Phi$ is a finite linear combination $\varphi = \sum_{k\in\mathbb{N}}^{\text{fin}} \sum_{b\in\mathbb{Z}^N}^{\text{fin}} \alpha_{k,b}\, U^b\psi_k$ (the complex numbers $\alpha_{k,b}$ are different from zero only for a finite set of the values of the index k and the multiindex b). Let $\phi = \sum_{h\in\mathbb{N}}^{\text{fin}} \sum_{c\in\mathbb{N}^N}^{\text{fin}} \beta_{h,c}\, U^c\psi_h$ be another element in Φ. The linearity of the dual pairing between Φ^* and Φ and the compatibility of the pairing with the Hermitian structure of $\mathcal{H}$ imply

$$\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi\rangle := \sum_{k\in\mathbb{N}}^{\text{fin}} \sum_{b,c\in\mathbb{Z}^N}^{\text{fin}} \overline{\alpha}_{k,b}\beta_{k,c} \left(\sum_{a\in\mathbb{Z}^N} z^a(t)\ (U^{a+b}\psi_k; U^c\psi_k)_{\mathcal{H}} \right) \tag{4.19}$$

where in the right-hand side we used the orthogonality between the spaces generated by ψ_k and ψ_h if $k \neq h$. Without further conditions equation (4.19) is a finite sum in k, b, c (this is simply a consequence of the fact that φ and ϕ are "test functions") but it is an infinite sum in a which generally does not converge. However, in view of the wandering property one has that $(U^{a+b}\psi_k; U^c\psi_k)_{\mathcal{H}} = \delta_{a+b,c}$, so that (4.19) reads

$$\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi\rangle = \sum_{k\in\mathbb{N}}^{\text{fin}} \sum_{b,c\in\mathbb{Z}^N}^{\text{fin}} \overline{\alpha}_{k,b}\beta_{k,c}\ z^c(t) z^{-b}(t). \tag{4.20}$$

Let $C_{\varphi;k} := \sum_{b\in\mathbb{Z}^N}^{\text{fin}} |\alpha_{k,b}|$ and $C_\varphi := \max_{k\in\mathbb{N}}\{C_{\varphi;k}\}$ (which is well defined since the set contains only a finite numbers of non-zero elements). An easy computation shows that

$$|\langle(\mathcal{U}_{\mathfrak{S}}\varphi)(t);\phi\rangle| \leqslant \sum_{k\in\mathbb{N}}^{\text{fin}} C_{\varphi,k}\left(\sum_{c\in\mathbb{Z}^N}^{\text{fin}} |\beta_{k,c}|\right) \leqslant C_\varphi \sum_{k\in\mathbb{N}}^{\text{fin}}\sum_{c\in\mathbb{Z}^N}^{\text{fin}} |\beta_{k,c}|\,.$$

Let $m \geqslant 0$ be the smallest integer such that $\phi \in \Phi_m$. The number of the coefficients $\beta_{k,c}$ different from zero is smaller than the dimension D_m of Φ_m. Using the Cauchy-Schwarz inequality one has

$$|\langle(\mathcal{U}_{\mathfrak{S}}\varphi)(t);\phi\rangle| \leqslant C_\varphi\sqrt{D_m}\left(\sum_{k\in\mathbb{N}}^{\text{fin}}\sum_{c\in\mathbb{N}^N}^{\text{fin}} |\beta_{k,c}|^2\right)^{\frac{1}{2}} = C_\varphi\, p_m(\phi). \tag{4.21}$$

The inequality (4.21) shows that the linear map $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) : \Phi \to \mathbb{C}$ is continuous when it is restricted to each finite dimensional space Φ_m. Since Φ is endowed with the strict inductive limit topology, this is enough to assure that $(\mathcal{U}_{\mathfrak{S}}\varphi)(t)$ is a continuous linear functional on Φ. So, in view of (4.21), $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) \in \Phi^*$ for all $t \in [0, 2\pi)^N$ and for all $\varphi \in \Phi$.

The linearity of the map $\mathcal{U}_{\mathfrak{S}}|_t : \Phi \to \Phi^*$ is immediate and from equation (4.20) it follows that $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = 0$ (as functional) implies that $\alpha_{k,b} = 0$ for all k and b, hence $\varphi = 0$. This prove the injectivity. To prove the continuity of the map $\mathcal{U}_{\mathfrak{S}}|_t : \Phi \to \Phi^*$, in view of the strict inductive topology on Φ, we need only to check the continuity of the maps $\mathcal{U}_{\mathfrak{S}}|_t : \Phi_m \to \Phi^*$ for all $m \geqslant 0$. Since Φ_m is a finite dimensional vector space with norm p_m, it is sufficient to prove that the norm-convergence of the sequence $\varphi_n \to 0$ in Φ_m implies the $*$-weak convergence $(\mathcal{U}_{\mathfrak{S}}\varphi_n)(t) \to 0$ in Φ^*, i.e. $|\langle(\mathcal{U}_{\mathfrak{S}}\varphi_n)(t);\phi\rangle| \to 0$ for all $\phi \in \Phi$. As inequality (4.21) suggests, it is enough to show that $C_{\varphi_n} \to 0$. This is true since $\varphi_n := \sum_{0<k,|b|\leqslant m} \alpha_{k,b}^{(n)}\, U^b\psi_k \to 0$ in Φ_m implies $\alpha_{k,b}^{(n)} \to 0$.

Finally, since the map $U^{-a} = (U^a)^\dagger$ is continuous on Φ for all $a \in \mathbb{Z}^N$ then $(\hat{U^a}) : \Phi^* \to \Phi^*$ defines a continuous map which extends the operator U^a originally defined on $\mathcal{H}$. In this context the equation (4.18) is meaningful and shows that $\Phi^*(t) := \mathcal{U}_{\mathfrak{S}}|_t(\Phi) \subset \Phi^*$ is a space of common generalized eigenvectors for the elements of $\mathfrak{S}$. ■

The decomposition

The wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ generates under the Bloch-Floquet transform a special family of elements of Φ^*, denoted as

$$\zeta^k(t) := (\mathcal{U}_{\mathfrak{S}}\psi_k)(t) = \sum_{a\in\mathbb{Z}^N} z^{-a}(t)\, U^a\psi_k \qquad \forall\, k \in \mathbb{N}. \tag{4.22}$$

The injectivity of the map $\mathcal{U}_{\mathfrak{S}}$ implies that the functionals $\{\zeta^k(t)\}_{k\in\mathbb{N}}$ are linearly independent for every t. If $\varphi = \sum_{k\in\mathbb{N}}^{\text{fin}}\sum_{b\in\mathbb{Z}^N}^{\text{fin}} \alpha_{k,b}\, U^b\psi_k$ is any element in Φ then a simple computation shows that

$$(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = \sum_{k\in\mathbb{N}}^{\text{fin}}\sum_{b\in\mathbb{Z}^N}^{\text{fin}} \alpha_{k,b}\sum_{a\in\mathbb{N}^N} z^{-a}(t)\, U^{a+b}\psi_k = \sum_{k\in\mathbb{N}}^{\text{fin}} f_{\varphi;k}(t)\, \zeta^k(t) \tag{4.23}$$

where $f_{\varphi;k}(t) := \sum_{b\in\mathbb{Z}^N}^{\text{fin}} \alpha_{k,b}\ z^b(t)$. The equalities in (4.23) should be interpreted in the sense of "distributions", *i.e.* elements of Φ^*. The functions $f_{\varphi;k} : \mathbb{T}^N \to \mathbb{C}$, for all $k \in \mathbb{N}$, are finite linear combination of continuous functions, hence continuous. Equation (4.23) shows that any subspace $\Phi^*(t)$ is generated by finite linear combinations of the functionals (4.22). For every $t \in [0, 2\pi)^N$ we denote by $\mathcal{K}(t)$ the space of the elements of the form $\sum_{k\in\mathbb{N}} \alpha_k\ \zeta^k(t)$ with $\{\alpha_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$. This is a Hilbert space with the inner product induced by the isomorphism with $\ell^2(\mathbb{N})$. In other words the inner product is induced by the "formal" conditions $(\zeta^k(t); \zeta^h(t))_t := \delta_{k,h}$. All the Hilbert spaces $\mathcal{K}(t)$ have the same dimension which is the cardinality of the system $\{\psi_k\}_{k\in\mathbb{N}}$.

PROPOSITION 4.6.3. *For all $t \in [0, 2\pi)^N$ the inclusions $\Phi^*(t) \subset \mathcal{K}(t) \subset \Phi^*$ holds true. Moreover the generalized Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}}|_t$ extends to a unitary isomorphism between the Hilbert space $\mathbb{H} \subset \mathcal{H}$ spanned by the orthonormal system $\{\psi_k\}_{k\in\mathbb{N}}$ and the Hilbert space $\mathcal{K}(t) \subset \Phi^*$ spanned by the* orthonormal frame $\boldsymbol{\zeta}(t) := \{\zeta_k(t)\}_{k\in\mathbb{N}}$.

Proof. The first inclusion $\Phi^*(t) \subset \mathcal{K}(t)$ follows from the definition. For the second inclusion we need to prove that $\omega(t) := \sum_{k\in\mathbb{N}} \alpha_k\ \zeta^k(t)$ is a continuous functional if $\{\alpha_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$. Let $\phi = \sum_{0\leqslant h,|c|\leqslant m} \beta_{h,c}\ U^c\psi_h$ be an element of $\Phi_m \subset \Phi$ then, from the sesquilinearity of the dual pairing and the Cauchy-Schwarz inequality it follows that

$$|\langle\omega(t);\phi\rangle|^2 \leqslant \left(\sum_{k\in\mathbb{N}} |\alpha_k|\ |\langle(\mathcal{U}_{\mathfrak{S}}\psi_k)(t);\phi\rangle|\right)^2 \leqslant \|\alpha\|^2_{\ell^2} \sum_{k\in\mathbb{N}} |\langle(\mathcal{U}_{\mathfrak{S}}\psi_k)(t);\phi\rangle|^2 \tag{4.24}$$

where $\|\alpha\|^2_{\ell^2} = \sum_{k\in\mathbb{N}} |\alpha_k|^2 < \infty$. From equation (4.19) it is clear that $\langle(\mathcal{U}_{\mathfrak{S}}\psi_k)(t);\phi\rangle = 0$ if $\psi_k \notin \Phi_m$, then equation (4.21) and $C_{\psi_k} = 1$ imply $|\langle\omega(t);\phi\rangle| \leqslant \|\alpha\|_{\ell^2}\sqrt{m}\ p_m(\phi)$. This inequality shows that $\omega(t)$ is a continuous functional when it is restricted to each subspace Φ_m and, because the strict inductive limit topology, this proves that $\omega(t)$ lies in Φ^*.

As for the second claim, consider $\omega_n(t) := \sum_{0\leqslant k\leqslant n} \alpha_k\ \zeta^k(t)$. Obviously one has that $\omega_n(t) = (\mathcal{U}_{\mathfrak{S}}\varphi_n)(t) \in \Phi^*(t)$ since $\varphi_n := \sum_{0\leqslant k\leqslant n} \alpha_k\ \psi_k \in \Phi$. Moreover the inequality (4.24) can be used to show that $(\mathcal{U}_{\mathfrak{S}}\varphi_n)(t) \to \omega(t)$ when $n \to \infty$ with respect to the $*$-weak topology of Φ^*. This enables us to define $\omega(t) := (\mathcal{U}_{\mathfrak{S}}\varphi)(t)$ for all $\varphi := \sum_{k\in\mathbb{N}} \alpha_k\ \psi_k \in \mathbb{H}$. The generalized Bloch-Floquet transform acts as a unitary isomorphism between $\mathbb{H}$ and $\mathcal{K}(t)$ with respect to the Hilbert structure induced in $\mathcal{K}(t)$ by the orthonormal basis $\{\zeta^k(t)\}_{k\in\mathbb{N}}$. ■

THEOREM 4.6.4 (Bloch-Floquet spectral decomposition)**.** *Let $\mathfrak{S}$ be a $\mathbb{Z}^N$-algebra in the separable Hilbert space $\mathcal{H}$ with generators $\{U_1, \ldots, U_N\}$, wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ and wandering nuclear space Φ. The generalized Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}}$, defined on Φ by equation* (4.17)*, induces a direct integral decomposition of the Hilbert space $\mathcal{H}$ which is equivalent (in the sense of Theorem 4.3.3) to the decomposition of the von Neumann's theorem 4.3.1. Moreover, the spaces $\mathcal{K}(t)$ spanned in Φ^* by the functionals* (4.22) *provide an explicit realization for the family of common eigenspaces of $\mathfrak{S}$ appearing in Maurin's theorem 4.4.1.*

Proof. Proposition 4.5.7 assures that the Gel'fand spectrum of $\mathfrak{S}$ is the N-dimensional torus $\mathbb{T}^N$ and the basic measure agrees with the normalized Haar measure dz. On the field of Hilbert spaces $\prod_{t\in\mathbb{T}^N}\mathcal{K}(t)$ we can introduce a measurable structure by the fundamental family of orthonormal vector fields $\boldsymbol{\zeta}(\cdot) := \{\zeta_k(\cdot)\}_{k\in\mathbb{N}}$ defined by (4.22). For all $\varphi\in\Phi$ the generalized Bloch-Floquet transform defines a square integrable vector field $(\mathcal{U}_{\mathfrak{S}}\varphi)(\cdot)\in\mathfrak{K} := \int_{\mathbb{T}^N}^{\oplus}\mathcal{K}(t)\,dz(t)$. Indeed equation (4.23) shows that $(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\in\mathcal{K}(t)$ for any t and $\|(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\|_t^2 = \sum_{k\in\mathbb{N}}^{\text{fin}}|f_{\varphi;k}(t)|^2$ is a continuous function (finite sum of continuous functions) hence integrable on $\mathbb{T}^N$. In particular

$$\|(\mathcal{U}_{\mathfrak{S}}\varphi)(\cdot)\|_{\mathfrak{K}}^2 = \int_{\mathbb{T}^N}\|(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\|_t^2\,dz(t) = \sum_{k\in\mathbb{N}}\int_{\mathbb{T}^N}\underbrace{\left(\sum_{b,c\in\mathbb{Z}^N}^{\text{fin}}\overline{\alpha}_{k,b}\alpha_{k,c}\,z^{c-b}(t)\right)}_{=|f_{\varphi;k}(x)|^2}dz(t) = \|\varphi\|_{\mathcal{H}}^2.$$

In view of the density of Φ, $\mathcal{U}_{\mathfrak{S}}$ can be extended to an isometry from $\mathcal{H}$ to $\mathfrak{K}$.

It remains to show that $\mathcal{U}_{\mathfrak{S}}$ is surjective. Any square integrable vector field $\varphi(\cdot)\in\mathfrak{K}$ is uniquely characterized by its expansion on the frame $\boldsymbol{\zeta}(\cdot)$, i.e. $\varphi(\cdot) = \sum_{k\in N}\widehat{\varphi}_k(\cdot)\,\zeta^k(\cdot)$ where $\{\widehat{\varphi}_k(t)\}_{k\in\mathbb{N}}\in\ell^2(\mathbb{N})$ for all $t\in[0,2\pi)^N$. The condition

$$\|\varphi(\cdot)\|_{\mathfrak{K}}^2 = \int_{\mathbb{T}^N}\sum_{k\in\mathbb{N}}|\widehat{\varphi}_k(t)|^2\,dz(t) < +\infty$$

shows that $\widehat{\varphi}_k\in L^2(\mathbb{T}^N)$ for all $k\in\mathbb{N}$. Let $\widehat{\varphi}_k(t) = \sum_{b\in\mathbb{Z}^N}\alpha_{k,b}z^b(t)$ be the Fourier expansion of $\widehat{\varphi}_k$. Since

$$\sum_{k\in\mathbb{N}}\sum_{b\in\mathbb{Z}^N}|\alpha_{k,b}|^2 = \sum_{k\in\mathbb{N}}\|\widehat{\varphi}_k\|_{L^2(\mathbb{T}^N)}^2 = \|\varphi(\cdot)\|_{\mathfrak{K}}^2 < +\infty$$

it follows that $\{\alpha_{k,b}\}_{k\in\mathbb{N},b\in\mathbb{Z}^N}$ is an ℓ^2-sequences and the mapping

$$\varphi(\cdot) = \sum_{k\in\mathbb{N}}\sum_{b\in\mathbb{Z}^N}\alpha_{k,b}\,z^b(\cdot)\,\zeta^k(\cdot)\quad\stackrel{\mathcal{U}_{\mathfrak{S}}^{-1}}{\longmapsto}\quad\varphi := \sum_{k\in\mathbb{N}}\sum_{b\in\mathbb{Z}^N}\alpha_{k,b}\,U^b\psi_k \tag{4.25}$$

defines an element $\varphi\in\mathcal{H}$ starting from the vector field $\varphi(\cdot)\in\mathfrak{K}$. It is immediate to check that $\mathcal{U}_{\mathfrak{S}}$ maps φ in $\varphi(\cdot)$, hence $\mathcal{U}_{\mathfrak{S}}$ is surjective.

If $A_f\in\mathfrak{S}$ is an operator associated with the continuous function $f\in C(\mathbb{T}^N)$ via the Gel'fand isomorphism, then $\mathcal{U}_{\mathfrak{S}}A_f\mathcal{U}_{\mathfrak{S}}^{-1}\varphi(\cdot) = f(\cdot)\varphi(\cdot)$, i.e. $\mathcal{U}_{\mathfrak{S}}$ maps $A_f\in\mathfrak{S}$ in $M_f(\cdot)\in C(\mathfrak{K})$. This allows us to apply the Theorem 4.3.3 which assures that the direct integral $\mathfrak{K}$ coincides, up to a decomposable unitary transform, with the spectral decomposition of $\mathfrak{S}$ established in Theorem 4.3.1. ■

Remark 4.6.5 (Generalized Bloch-Floquet transform and $\mathfrak{S}$-Fourier transform)**.** The generalized Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}}$ can be seen as a "computable" realization of the abstract $\mathfrak{S}$-Fourier transform $\mathcal{F}_{\mathfrak{S}}$ in the von Neumann's theorem 4.3.1. This allows us to interchange the symbols $\mathcal{U}_{\mathfrak{S}}$ and $\mathcal{F}_{\mathfrak{S}}$, when necessary. From Proposition 4.6.3 and from general results about direct integrals (Dixmier 1981, Part II, Chapter 1, Section 8) one obtains the following identifications:

$$\mathcal{H} \xrightarrow{\mathcal{U}_{\mathfrak{S}}\ldots\mathcal{U}_{\mathfrak{S}}^{-1}} \int_{\mathbb{T}^N}^{\oplus} \mathcal{K}(t)\, dz(t) \simeq \int_{\mathbb{T}^N}^{\oplus} \mathbb{H}\, dz(t) \simeq L^2(\mathbb{T}^N, \mathbb{H}). \tag{4.26}$$

♦◇

EXAMPLE 4.6.6 (*Periodic systems, part three*). In the case of Example 4.2.1 the generalized Bloch-Floquet transform reduces to the usual one given by (4.2)

$$(\mathcal{U}_{\mathfrak{S}_T}\varphi)(t,\theta) := \sum_{\gamma\in\Gamma} z^{-\gamma}(t)\, T^{\gamma}\varphi(\theta) = \sum_{\gamma\in\Gamma} \mathrm{e}^{-im_1 t_1}\ldots\mathrm{e}^{-im_d t_d}\varphi(\theta-\gamma),$$

where $\gamma := \sum_{j=1}^{d} m_j\gamma_j$, for all φ in the wandering nuclear space $\Phi \subset \mathcal{H}_{\text{phy}}$, built according to Proposition 4.6.1 from any orthonormal basis of $L^2(\mathcal{Q}_0)$. The fiber spaces in the direct integral decomposition are all unitarily equivalent to $L^2(\mathcal{Q}_0)$ hence the Hilbert space decomposition is

$$\mathcal{H}_{\text{phy}} \xrightarrow{\mathcal{U}_{\mathfrak{S}_T}\ldots\mathcal{U}_{\mathfrak{S}_T}^{-1}} \int_{\mathbb{T}^d}^{\oplus} L^2(\mathcal{Q}_0)\, dz(t)$$

and the dimension of any fiber is $\aleph_0$. The above equation agrees with the decomposition (4.3). ◄▷

EXAMPLE 4.6.7 (*Mathieu-like Hamiltonians, part three*). In this case the wandering nuclear space Φ_{M} is the set of the finite linear combinations of the Fourier basis $\{e_n\}_{n\in\mathbb{Z}}$ and for all $g(s) = \sum_{n\in\mathbb{Z}}^{\text{fin}} \alpha_n \mathrm{e}^{ins}$ in Φ_{M} the Bloch-Floquet transform is

$$(\mathcal{U}_{\mathfrak{S}^q_{\mathrm{M}}}\, g)(t,\vartheta) := \sum_{m\in\mathbb{Z}} \mathrm{e}^{-imt}\, w^m g(\vartheta) = \sum_{n\in\mathbb{Z}}^{\text{fin}} \alpha_n \left(\sum_{m\in\mathbb{Z}} \mathrm{e}^{i[n\vartheta+m(q\vartheta-t)]}\right).$$

The collection $\boldsymbol{\zeta}_{\mathrm{M}}(t) := \{\zeta^0_{\mathrm{M}}(t,\cdot),\ldots,\zeta^{q-1}_{\mathrm{M}}(t,\cdot)\} \subset \Phi^*_{\mathrm{M}}$, with

$$\zeta^k_{\mathrm{M}}(t,\vartheta) := (\mathcal{U}_{\mathfrak{S}^q_{\mathrm{M}}}\, e_k)(t,\vartheta) = \mathrm{e}^{ik\vartheta}\sum_{m\in\mathbb{Z}} \mathrm{e}^{im(q\vartheta-t)},$$

defines a fundamental family of orthonormal fields (or frames). The fiber spaces in the direct integral decomposition are all unitarily equivalent to $\mathbb{C}^q$ hence the Hilbert space decomposition is

$$\mathcal{H}_{\mathrm{M}} \xrightarrow{\mathcal{U}_{\mathfrak{S}^q_{\mathrm{M}}}\ldots\mathcal{U}_{\mathfrak{S}^q_{\mathrm{M}}}^{-1}} \int_{\mathbb{T}}^{\oplus} \mathbb{C}^q\, dz(t).$$

The images of the generators u and v under the map $\mathcal{U}_{\mathfrak{S}^q_{\mathrm{M}}}\ldots\mathcal{U}_{\mathfrak{S}^q_{\mathrm{M}}}^{-1}$ are the two t dependent $q\times q$ matrices

$$u(t) := \begin{pmatrix} 1 & & & \\ & \mathrm{e}^{i2\pi\frac{p}{q}} & & \\ & & \ddots & \\ & & & \mathrm{e}^{i2\pi\frac{p}{q}(q-1)} \end{pmatrix} \qquad v(t) := \begin{pmatrix} 0 & & \mathrm{e}^{it} \\ 1 & \ddots & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}.$$

For every $t\in\mathbb{R}$ the matrices $u(t)$ and $v(t)$ generate an irreducible representation of the NCT-algebra $\mathfrak{A}_{p/q}$ on the Hilbert space $\mathbb{C}^q$ (c.f. Section 5.1.1). ◄▷

Equivalence of physical frames

The dimension of the vector space $\mathbb{H}$ in the decomposition (4.26) is fixed by the cardinality of the wandering system chosen to define the Bloch-Floquet transform. Observing that Theorem 4.3.3 assures that the direct integral decomposition is essentially unique (in measure theoretic sense), one has the following corollary of Theorem 4.6.4.

COROLLARY 4.6.8. *Any two wandering systems associated with a given $\mathbb{Z}^N$-algebra $\mathfrak{S}$ have the same cardinality. Any two wandering systems for $\mathfrak{S}$ are intertwined by a unitary operator which commutes with $\mathfrak{S}$.*

The above result shows that the choice of a wandering system for a $\mathbb{Z}^N$-algebra is essentially unique (i.e. up to unitary equivalence).

The uniqueness of the spectral decomposition (Theorem 4.3.3) together with the identification between Generalized Bloch-Floquet transform and $\mathfrak{S}$-Fourier transform (Remark 4.6.5) imply that the generalized Bloch-Floquet transform (Theorem 4.6.2) and the related direct integral decomposition (Theorem 4.6.4) depend (up to a decomposable unitary) only on the equivalence class of physical frames. We can use this fact to prove the following result.

PROPOSITION 4.6.9. *Let $\{\mathcal{H},\mathfrak{A},\mathfrak{S}_1\}$ and $\{\mathcal{H},\mathfrak{A},\mathfrak{S}_2\}$ be two physical frames where $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two $\mathbb{Z}^d$-algebras having wandering systems of the same cardinality N. Then the two physical frames are in the same equivalence class.*

Proof. We need to prove that there exists a unitary map $W:\mathcal{H}\to\mathcal{H}$ such that $\mathfrak{A} = W\mathfrak{A}W^{-1}$ and $\mathfrak{S}_2 = W\mathfrak{S}_1W^{-1}$. Let $\{U_1,\ldots,U_d\}$ and $\{\psi_1,\ldots,\psi_N\}$ be the generating system and the wandering system for $\mathfrak{S}_1$ and let $\{V_1,\ldots,V_d\}$ and $\{\widetilde{\psi}_1,\ldots,\widetilde{\psi}_N\}$ be the same for $\mathfrak{S}_2$. Since the two wandering systems have the same cardinality N, one can define a one-to one correspondence between the two systems by posing $\psi_j \leftrightarrow \widetilde{\psi}_{\sigma(j)}$, with σ any permutation of the indexes $\{1,\ldots,N\}$. Fixed one of this $N!$ possible correspondences, we define a unitary map W_σ on the space $\mathcal{H}$ which intertwines the two orthonormal basis generated by the wandering systems, namely $W_\sigma(U^a\psi_j) := V^a\widetilde{\psi}_{\sigma(j)}$ for all $a\in\mathbb{N}^d$ and $j\in\{1,\ldots,N\}$. A simple computation shows that $(U_i - W_\sigma^{-1}\,V_i\,W_\sigma)U^a\psi_j = 0$ for all $a\in\mathbb{N}^d$ and $j\in\{1,\ldots,N\}$ which implies, in view of the completeness of the system $U^a\psi_j$, that $U_i = W_\sigma^{-1}\,V_i\,W_\sigma$ for all $i=1,\ldots,d$, thus $\mathfrak{S}_2 = W_\sigma\,\mathfrak{S}_1\,W_\sigma^{-1}$.

Let $\mathcal{U}_{\mathfrak{S}_i}:\mathcal{H}\to\mathfrak{H}_{\mathfrak{S}_i} := \int_{\mathbb{T}^d}^{\oplus}\mathcal{H}(t)\,dz(t)$, $i=1,2$ be the generalized Bloch-Floquet transform related to the physical frame $\{\mathcal{H},\mathfrak{A},\mathfrak{S}_i\}$. In view of Remark 4.6.5, Theorem 4.3.3 assures that $\mathcal{U}_{\mathfrak{S}_2}\,W_\sigma\,\mathcal{U}_{\mathfrak{S}_1}^{-1} = \int_{\mathbb{T}^d}^{\oplus} W_\sigma(t)\,dz(t):\mathfrak{H}_{\mathfrak{S}_1}\to\mathfrak{H}_{\mathfrak{S}_2}$. Finally, the equality

$$\begin{aligned} W_\sigma\,\mathfrak{A}\,W_\sigma^{-1} &= (W_\sigma\mathcal{U}_{\mathfrak{S}_1}^{-1})\left(\int_{\mathbb{T}^d}^{\oplus}\pi_t^{(1)}(\mathfrak{A})\,dz(t)\right)(W_\sigma\mathcal{U}_{\mathfrak{S}_1}^{-1})^{-1} \\ &= \mathcal{U}_{\mathfrak{S}_2}^{-1}\left(\int_{\mathbb{T}^d}^{\oplus}\underbrace{W_\sigma(t)\,\pi_t^{(1)}(\mathfrak{A})\,W_\sigma(t)^{-1}}_{=\pi_k^{(2)}(\mathfrak{A})}\,dz(t)\right)\mathcal{U}_{\mathfrak{S}_2} = \mathfrak{A} \end{aligned}$$

follows from point (ii) of Theorem 4.3.1. ∎

There exists a particular class of C^*-algebras which plays a relevant rôle for the purpose of this thesis.

DEFINITION 4.6.10 (N-homogeneous C^*-algebra (Dixmier 1982))**.** *A C^*-algebra $\mathfrak{A}$ is said* N-homogeneous *if all its irreducible representations are of the same finite dimension N.*

The importance of the above definition lies in the following uniqueness result

PROPOSITION 4.6.11. *Let $\mathfrak{A}$ be a N-homogeneous C^*-algebra, $\pi : \mathfrak{A} \to \mathscr{B}(\mathcal{H})$ a faithful representation and $\{\mathcal{H}, \pi(\mathfrak{A}), \mathfrak{S}_j\}$, $j = 1, 2$ two irreducible physical frames (Definition 4.1.2) with $\mathfrak{S}_j$ maximal commutative $\mathbb{Z}^d$-algebras in the commutant $\pi(\mathfrak{A})$. If $\mathfrak{S}_j$ has a wandering system then its cardinality is N. Assuming that both $\mathfrak{S}_1$ and $\mathfrak{S}_2$ have wandering system, one has that $\{\mathcal{H}, \pi(\mathfrak{A}), \mathfrak{S}_1\}$ and $\{\mathcal{H}, \pi(\mathfrak{A}), \mathfrak{S}_2\}$ are in the same equivalence class.*

Proof. If $\mathfrak{S}_j$ admits a wandering system then one can define the generalized Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}_j} : \mathcal{H} \to \mathfrak{H}_{\mathfrak{S}_j}$ according to Theorem 4.6.4. Since $\mathfrak{R}$ is maximal commutative, $\mathcal{U}_{\mathfrak{S}_j}$ decomposes $\pi(\mathfrak{A})$ in a direct integral of irreducible representations (Theorem 4.3.1). Observing that $\mathfrak{A}$ and $\pi(\mathfrak{A})$ have the same representation theory, since π is faithful and observing that the irreducible representations of $\mathfrak{A}$ are N-dimensional it follows that the fiber Hilbert spaces in the direct integral $\mathfrak{H}_{\mathfrak{S}_j}$ are N-dimensional. The first part of the claim follows by noticing that the dimension of the fiber Hilbert spaces is equal to the cardinality of the wandering system. The second part of the claim follows from Proposition 4.6.9. ■

The above result shows that it is possible to introduce a notion of *standard* physical frame (in any faithful representation) for N-homogeneous C^*-algebras.

Wannier vectors from an algebraic point of view

Equation (4.25) in the proof of Theorem 4.6.4 provides a recipe to invert the unitary map $\mathcal{U}_{\mathfrak{S}} : \mathcal{H} \to \mathfrak{K} := \int_{\mathbb{T}^2}^{\oplus} \mathcal{K}(t)\ dz(t)$. According to (4.25) $\mathcal{U}_{\mathfrak{S}}{}^{-1}$ maps the fundamental vector fields $\zeta^k(\cdot)$ into the wandering vectors ψ_k, and it intertwines multiplication by the exponentials $z^a(\cdot)$ with the unitary operators U^a. We will say that $\mathcal{U}_{\mathfrak{S}}{}^{-1}(\varphi(\cdot))$ is the *Wannier vector* (WV) associated to the vector field $\varphi(\cdot)$.

We denote by $\mathfrak{F} := \prod_{t \in \mathbb{T}^N} \mathcal{K}(t)$ the set of all the vector fields. Let $\mathfrak{F}^\infty$ be the set of square integrable vector fields $\varphi(\cdot) \in \mathfrak{K}$ whose component functions $\widehat{\varphi}_k(\cdot)$ are of class $C^\infty(\mathbb{T}^N)$. Similarly let $\mathfrak{F}^\omega$ be the set of square integrable vector fields which component functions are analytic, i.e. of class $C^\omega(\mathbb{T}^N)$. Obviously

$$\mathcal{U}_{\mathfrak{S}}(\Phi) \subset \mathfrak{F}^\omega \subset \mathfrak{F}^\infty \subset \mathfrak{K} \subset \mathfrak{F}.$$

By ordinary Fourier theory, one observes that if $\varphi(\cdot) \in \mathfrak{F}^\infty$ then the sequence of coefficients $\{\alpha_{k,a}\}_{k \in \mathbb{N}, a \in \mathbb{Z}^N}$ which defines the component functions $\widehat{\varphi}_k(\cdot)$ *decays faster than* *any polynomial*. Similarly if $\varphi(\cdot) \in \mathfrak{F}^\omega$, then the sequence $\{\alpha_{k,a}\}_{k\in\mathbb{N},a\in\mathbb{Z}^N}$ has an *exponential decay*. In analogy with the ordinary Bloch-Floquet theory (Kuchment 1993), these considerations suggest the name of *super-polynomially localized Wannier vectors* for the elements of $\Omega^\infty_{\mathfrak{S}} := \mathcal{U}_{\mathfrak{S}}{}^{-1}(\mathfrak{F}^\infty)$ and *exponentially localized Wannier vectors* for the elements of $\Omega^\omega_{\mathfrak{S}} := \mathcal{U}_{\mathfrak{S}}{}^{-1}(\mathfrak{F}^\omega)$. The above definitions are summarized by the following table.

Symbol	decay of $\{\alpha_{k,a}\}$	Name
Φ	finite	compact supported WV
$\Omega^\omega_{\mathfrak{S}}$	exponential	exponentially localized WV
$\Omega^\infty_{\mathfrak{S}}$	super-polynomial	super-polynomial localized WV
$\Omega_{\mathfrak{S}}$	?	continuous WV
$\mathcal{H}$	ℓ^2	generic

(4.27)

4.7 Emergent geometry

From a geometric viewpoint, the field of Hilbert spaces $\mathfrak{F} := \prod_{x\in X}\mathcal{H}(x)$ can be regarded as a pseudo vector-bundle $\iota : \mathscr{E} \longrightarrow X$, where

$$\mathscr{E} := \bigsqcup_{x\in X} \mathcal{H}(x) \tag{4.28}$$

is the *disjoint union* of the Hilbert spaces $\mathcal{H}(x)$. The use of the prefix "pseudo" refers to the fact that more ingredients are needed to turn $\iota : \mathscr{E} \longrightarrow X$ into a vector bundle. First of all, the map ι must be continuous, which requires a topology on $\mathscr{E}$. As a first attempt, assuming that $\mathcal{H}(x) \subset \Phi^*$ for every $x \in X$, one might consider $\iota : \mathscr{E} \longrightarrow X$ as a sub-bundle of the trivial vector bundle $\iota : X \times \Phi^* \longrightarrow X$, equipped with the topology induced by the inclusion, so that $\iota : \mathscr{E} \longrightarrow X$ becomes a topological bundle whose fibers are Hilbert spaces. However, nothing ensures that the Hilbert space topology defined fiberwise is compatible with the topology of $\mathscr{E}$, a necessary condition to have a meaningful topological theory.

Geometric vs. analytic viewpoint

We begin our analysis with the definition of topological fibration of Hilbert spaces. Following (Fell and Doran 1988, Chapter II, Section 13) we have

DEFINITION 4.7.1 (Geometric viewpoint: Hilbert bundle)**.** *A* Hilbert bundle *is the datum of a topological Hausdorff spaces $\mathscr{E}$ (the* total space*) a compact Hausdorff space X (the* base space*) and a map $\iota : \mathscr{E} \longrightarrow X$ (the* canonical projection*) which is a* continuous open surjection *such that:*

a) *for all $x \in X$ the fiber $\iota^{-1}(x) \subset \mathscr{E}$ is a Hilbert space;*

b) *the application $\mathscr{E} \ni p \longmapsto \|p\| \in \mathbb{C}$ is continuous;*

c) *the operation* $+$ *is continuous as a function on* $\mathscr{S} := \{(p,s) \in \mathscr{E} \times \mathscr{E} \ : \ \iota(p) = \iota(s)\}$ *to* $\mathscr{E}$*;*

d) *for each* $\lambda \in \mathbb{C}$ *the map* $\mathscr{E} \ni p \longmapsto \lambda\, p \in \mathscr{E}$ *is continuous;*

e) *let* 0_x *be the null vector in the Hilbert space* $\iota^{-1}(x)$*; for each* $x \in X$*, the collection of all subsets of* $\mathscr{E}$ *of the form* $\mathscr{U}(O, x, \varepsilon) := \{p \in \mathscr{E} \ : \ \iota(p) \in O, \ \ \|p\| < \varepsilon\}$*, where* O *is a neighborhood of* x *and* $\varepsilon > 0$*, is a basis of neighborhoods of* $0_x \in \iota^{-1}(x)$ *in* $\mathscr{E}$*.*

We will denote by the short symbol $\mathscr{E}$ the Hilbert bundle $\iota : \mathscr{E} \longrightarrow X$, when the base space X and the projection ι are clarified by the context. A *section* of $\mathscr{E}$ is a function $\psi : X \to \mathscr{E}$ such that $\iota \circ \psi = \mathrm{id}_X$. We denote by $\Gamma(\mathscr{E})$ the set of all *continuous sections* of $\mathscr{E}$. As showed in (Fell and Doran 1988), from Definition 4.7.1 it follows that: (i) the scalar multiplication $\mathbb{C} \times \mathscr{E} \ni (\lambda, p) \mapsto \lambda\, p \in \mathscr{E}$ is continuous; (ii) the open sets of $\mathscr{E}$, restricted to a fiber $\iota^{-1}(x)$, generate the Hilbert space topology of $\iota^{-1}(x)$; (iii) the set $\Gamma(\mathscr{E})$ has the structure of a (left) $C(X)$-module. The definition of Hilbert bundle includes all the requests which a "formal" fibration as (4.28) needs to fulfill to be a topological fibration with a topology compatible with the Hilbert structure of the fibers. In this sense the Hilbert bundle is the "geometric object" of our interest.

However, the structure that emerges in a natural way from the Bloch-Floquet decomposition (Theorem 4.6.4) is more easily understood from the analytic viewpoint. Switching the focus from the total space $\mathscr{E}$ to the space of sections $\mathfrak{F}$, the relevant notion is that of *continuous field of Hilbert spaces*, according to (Dixmier 1982, Section 10.1) or (Dixmier and Douady 1963, Section 1).

DEFINITION 4.7.2 (Analytic viewpoint: continuous field of Hilbert spaces)**.** *Let* X *be a compact Hausdorff space and* $\mathfrak{F} := \prod_{x \in X} \mathcal{H}(x)$ *a field of Hilbert spaces. A* continuous structure on $\mathfrak{F}$ *is the datum of a linear subspace* $\Gamma \subset \mathfrak{F}$ *such that:*

a) *for each* $x \in X$ *the set* $\{\sigma(x) \ : \ \sigma(\cdot) \in \Gamma\}$ *is dense in* $\mathcal{H}(x)$*;*

b) *for any* $\sigma(\cdot) \in \Gamma$ *the map* $X \ni x \mapsto \|\sigma(x)\|_x \in \mathbb{R}$ *is continuous;*

c) *if* $\psi(\cdot) \in \mathfrak{F}$ *and if for each* $\varepsilon > 0$ *and each* $x_0 \in X$*, there is some* $\sigma(\cdot) \in \Gamma$ *such that* $\|\sigma(x) - \psi(x)\|_x < \varepsilon$ *on a neighborhood of* x_0*, then* $\psi(\cdot) \in \Gamma$*.*

We will denote by the short symbol $\mathfrak{F}_{\Gamma,X}$ the field of Hilbert spaces $\mathfrak{F}$ endowed with the continuous structure Γ. The elements of Γ are called *continuous vector fields*. The condition b) may be replaced by the requirement that for any $\sigma(\cdot), \varrho(\cdot) \in \Gamma$, the function $X \ni x \mapsto (\sigma(x); \varrho(x))_x \in \mathbb{C}$ is continuous. Condition c) is called *locally uniform closure*. Locally uniform closure is needed in order that the linear space Γ is stable under multiplication by continuous functions on X. This condition implies that Γ is a *(left)* $C(X)$*-module*. A *total set* of continuous vector fields for $\mathfrak{F}_{\Gamma,X}$ is a subset $\Lambda \subset \Gamma$ such that $\Lambda(x) := \{\sigma(x) \ : \ \sigma(\cdot) \in \Lambda\}$ is dense in $\mathcal{H}(x)$ for all $x \in X$. The continuous field of Hilbert spaces is said to be *separable* if it has a countable total set of continuous vector fields.

The link between the notion of continuous field of Hilbert spaces and that of Hilbert bundle is clarified by the following result.

PROPOSITION 4.7.3 (Equivalence between geometric and analytic viewpoint)**.** *Let $\mathfrak{F}_{\Gamma,X}$ be a continuous field of Hilbert spaces over the compact Hausdorff space X. Let $\mathscr{E}(\mathfrak{F}_{\Gamma,X}) := \bigsqcup_{x\in X}\mathcal{H}(x)$ be the disjoint union of the Hilbert spaces $\mathcal{H}(x)$ and ι the canonical surjection of $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ onto X. Then, there exists a unique topology $\mathcal{T}$ on $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ making $\iota : \mathscr{E}(\mathfrak{F}_{\Gamma,X})\longrightarrow X$ a Hilbert bundle over X such that all the continuous vector fields in $\mathfrak{F}_{\Gamma,X}$ are continuous sections of $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$. Moreover, every Hilbert bundle comes from a continuous field of Hilbert spaces.*

Proof. For the details of the proof we refer to (Dixmier and Douady 1963, Section 2) or (Fell and Doran 1988, Chapter II, Theorem 13.18). Here, we only sketch the main ideas.

We use the shorter notation $\mathfrak{F}_{\Gamma,X} \equiv \mathfrak{F}$. One has to equip the set $\mathscr{E}(\mathfrak{F}) := \bigsqcup_{x\in X}\mathcal{H}(x)$ with a topology which satisfies the axioms of Definition 4.7.1. Such a topology $\mathcal{T}$ is generated by the basis of open sets whose elements are the *tubular neighborhoods* $\mathscr{U}(O,\sigma,\varepsilon) := \{p \in \mathscr{E}(\mathfrak{F}) \ : \ \iota(p) \in O, \ \ \|p - \sigma(\iota(p))\| < \varepsilon\}$ for all open sets $O \subseteq X$, all continuous vector fields $\sigma(\cdot) \in \Gamma$ and all positive numbers $\varepsilon > 0$. Since $\iota(\mathscr{U}(O,\sigma,\varepsilon)) = O$ it is clear that with respect to the topology $\mathcal{T}$ the map $\iota : \mathscr{E}(\mathfrak{F}) \to X$ is a continuous open surjection. The topology induced by $\mathcal{T}$ on $\mathcal{H}(x)$ is equivalent to the norm-topology of $\mathcal{H}(x)$. Any vector field $\sigma(\cdot) \in \mathfrak{F}$ can be seen as a map $\sigma : X \to \mathscr{E}(\mathfrak{F})$ such that $\iota\circ\sigma = \mathrm{id}_X$, i.e. it is a section of $\mathscr{E}(\mathfrak{F})$. It follows that $\sigma(\cdot) \in \Gamma$ if and only if σ is a continuous section.

Conversely, let $\iota : \mathscr{E} \to X$ be a Hilbert bundle over the compact Hausdorff space X and let $\Gamma(\mathscr{E})$ be the set of its continuous section. Let $\mathfrak{F}(\mathscr{E}) := \prod_{x\in X}\iota^{-1}(x)$ be the field of Hilbert spaces associated to the bundle $\mathscr{E}$. The compactness of X assures that $\mathscr{E}$ has *enough continuous sections*, i.e. $\{\sigma(x) \ : \ \sigma \in \Gamma(\mathscr{E})\} = \iota^{-1}(x) =: \mathcal{H}(x)$ (Fell and Doran 1988, Douady-Dal Soglio-Hérault theorem, Appendix C). For all $\sigma \in \Gamma(\mathscr{E})$ the map $X \ni x \mapsto \|\sigma(x)\|_x$ is continuous since it is composition of continuous functions. Finally the family $\Gamma(\mathscr{E})$ fulfills the locally uniform closure property (Fell and Doran 1988, Corollary 13.13, Chapter II). This proves that the set of continuous sections $\Gamma(\mathscr{E})$ defines a continuous structure on the field of Hilbert spaces $\mathfrak{F}(\mathscr{E})$. ■

We will say that the set $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ endowed with the topology $\mathcal{T}$ and the canonical surjection ι defines the Hilbert bundle associated with the continuous structure Γ of $\mathfrak{F}$.

Triviality, local triviality and vector bundle structure

A Hilbert bundle is a generalization of a (infinite dimensional) vector bundle, in the sense that some other extra conditions are needed in order to turn it into a genuine vector bundle. For the axioms of vector bundle we refer to (Lang 1985). The most relevant missing condition, is the *local triviality property* (c.f. Section C).

Two Hilbert bundles $\iota_1 : \mathscr{E}_1 :\to X$ and $\iota_2 : \mathscr{E}_2 :\to X$ over the same base space X are said to be *(isometrically) isomorphic* if there exists a homeomorphism $\Theta : \mathscr{E}_1 \to \mathscr{E}_2$ such that a) $\iota_2\circ\Theta = \iota_1$, b) $\Theta_x := \Theta|_{\iota_1^{-1}(x)}$ is a unitary map from the Hilbert space $\iota_1^{-1}(x)$ to the

Hilbert space ${\iota_2}^{-1}(x)$. From the definition it follows that if the Hilbert bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ are isomorphic then the map $\Gamma(\mathscr{E}_1) \ni \sigma \mapsto \Theta \circ \sigma \in \Gamma(\mathscr{E}_2)$ is one to one. A Hilbert bundle is said to be *trivial* if it is isomorphic to the *constant* Hilbert bundle $X \times \mathbb{H} \to X$ where $\mathbb{H}$ is a fixed Hilbert space. It is called *locally trivial* if for every $x \in X$ there is a neighborhood O of x such that the *reduced Hilbert bundle* $\mathscr{E}|_O := \{p \in \mathscr{E} \ : \ \iota(p) \in O\} = \iota^{-1}(O)$ is isomorphic to the constant Hilbert bundle $O \times \mathbb{H} \to O$. Two continuous fields of Hilbert spaces $\mathfrak{F}_{\Gamma,X}$ and $\mathfrak{F}'_{\Gamma',X}$ over the same space X are said to be (isometrically) isomorphic if the associated Hilbert bundles $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ and $\mathscr{E}(\mathfrak{F}'_{\Gamma',X})$ are isomorphic. A continuous field of Hilbert spaces $\mathfrak{F}_{\Gamma,X}$ is said to be trivial (resp. locally trivial) if $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ is trivial (resp. locally trivial).

PROPOSITION 4.7.4. *Let $\mathfrak{F}_{\Gamma,X}$ be a continuous field of Hilbert spaces over the compact Hausdorff space X and $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ the associated Hilbert bundle. Then:*

(i) *if $\mathfrak{F}_{\Gamma,X}$ is separable and X is second-countable (or equivalently metrizable) then the topology defined on the total space $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ is second-countable;*

(ii) *if* dim $\mathcal{H}(x) = \aleph_0$ *for all $x \in X$ and if X is a finite dimensional manifold then the Hilbert bundle $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ is trivial;*

(iii) *if* dim $\mathcal{H}(x) = q < +\infty$ *for all $x \in X$ then the Hilbert bundle $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ is a Hermitian vector bundle with typical fiber $\mathbb{C}^q$.*

Proof. For the proof of (i) we refer to (Fell and Doran 1988, Proposition 13.21, Chapter II). The claim (ii) is proved in (Dixmier and Douady 1963, Theorem 5). We only sketch the proof of (iii).

First of all, we recall that to prove that a Hilbert bundle is a vector bundle we need to prove the local triviality and the continuity of the transition functions (Lang 1985, Chapter III). However, if the fibers are finite dimensional then the continuity of the transition functions follows from the existence of the local trivializations (Lang 1985, Proposition 1, Chapter III).

Let $\mathfrak{F} := \prod_{x\in X} \mathcal{H}(x)$ with dim $\mathcal{H}(x) = q$ for all $x \in X$ and $\{\sigma_1(\cdot), \ldots, \sigma_q(\cdot)\} \subset \Gamma$ such that for a fixed $x_0 \in X$ the collection $\{\sigma_1(x_0), \ldots, \sigma_q(x_0)\}$ is a basis for $\mathcal{H}(x_0)$. Following (Dixmier and Douady 1963, Section 1), we show that $\{\sigma_1(x), \ldots, \sigma_q(x)\}$ is a basis for $\mathcal{H}(x)$ for all x in a suitable neighborhood of x_0. The function $\wp : X \times \mathbb{C}^q \to [0, +\infty)$ defined by $\wp(x, \alpha_1, \ldots, \alpha_q) := |\alpha| \ \| \sum_{j=1}^{q} \frac{\alpha_j}{|\alpha|} \sigma_j(x) \|_x$, with $|\alpha|^2 := \sum_{j=1}^{q} |\alpha_j|^2$, is continuous (composition of continuous function). Then the function $\kappa : X \to [0, +\infty)$ defined by $\kappa(x) := \inf_{|\alpha|=1} \wp(x, \alpha)$ is also continuous since the unit sphere in $\mathbb{C}^q$ is compact. Moreover $\kappa(x_0) > 0$. Since $\{\sigma_1(x), \ldots, \sigma_q(x)\}$ are linearly independent if and only if $\kappa(x) > 0$ it follows that the vectors are linearly independent in a suitable neighborhood O_{x_0} of x_0. In O_{x_0} we can use the Gram-Schmidt formula to obtain a local set of orthonormal continuous vector field $\{\widetilde{\sigma}_1(\cdot), \ldots, \widetilde{\sigma}_q(\cdot)\}$. This local frame enables us to define a map $h_{x_0} : \iota^{-1}(O_{x_0}) \to O_{x_0} \times \mathbb{C}^q$ by $h_{x_0}(p) := (\iota(p), \alpha_1, \ldots, \alpha_q)$ with $\iota(p) = x \in O_{x_0}$ and $p = \sum_{j=1}^{q} \alpha_j \widetilde{\sigma}_j(x)$. The map h_{x_0} is an homeomorphism and for each $x \in O_{x_0}$ is a linear isomorphism between $\mathcal{H}(x)$ and $\mathbb{C}^q$. The collection $\{O_{x_0}\}_{x_0 \in X}$ is a open covering, so we can select by the compactness of X a finite covering $\{O_1, \ldots, O_\ell\}$. The family $\{(O_j, h_j)\}_{j=1,\ldots,\ell}$ is a finite trivializing atlas for the vector bundle. ■

Algebraic viewpoint

Roughly speaking a continuous field of Hilbert spaces is an "analytic object" while a Hilbert bundle is a "geometric object". There is also a third point of view which is of algebraic nature. We introduce an "algebraic object" which encodes all the relevant properties of the set of continuous vector fields (or continuous sections).

DEFINITION 4.7.5 (Algebraic viewpoint: Hilbert module). *A* (left) pre-C^*-module *over a commutative unital C^*-algebra $\mathcal{A}$ is a complex vector space Ω_0 that is also a (left) $\mathcal{A}$-module with a pairing $\{\cdot;\cdot\} : \Omega_0 \times \Omega_0 \to \mathcal{A}$ satisfying, for $\sigma, \varrho, \varsigma \in \Omega_0$ and for $a \in \mathcal{A}$ the following requirements:*

a) $\{\sigma; \varrho + \varsigma\} = \{\sigma; \varrho\} + \{\sigma; \varsigma\}$;

b) $\{\sigma; a\varrho\} = a\{\sigma; \varrho\}$;

c) $\{\sigma; \varrho\}^* = \{\varrho; \sigma\}$;

d) $\{\sigma; \sigma\} > 0$ *if* $\sigma \neq 0$.

The map $|||\cdot||| : \Omega_0 \to [0, +\infty)$ defined by $|||\sigma||| := \sqrt{\|\{\sigma;\sigma\}\|_{\mathcal{A}}}$ is a norm in Ω_0. The completion Ω of Ω_0 with respect to the norm $|||\cdot|||$ is called (left) C^*-module *or* Hilbert module *over $\mathcal{A}$.*

PROPOSITION 4.7.6 (Equivalence between algebraic and analytic viewpoint). *Let $\mathfrak{F}_{\Gamma,X}$ be a continuous field of Hilbert spaces over the compact Hausdorff space X. The set of continuous vector fields Γ has the structure of a Hilbert module over $C(X)$. Conversely any Hilbert module over $C(X)$ defines a continuous field of Hilbert spaces. This correspondence is one-to-one.*

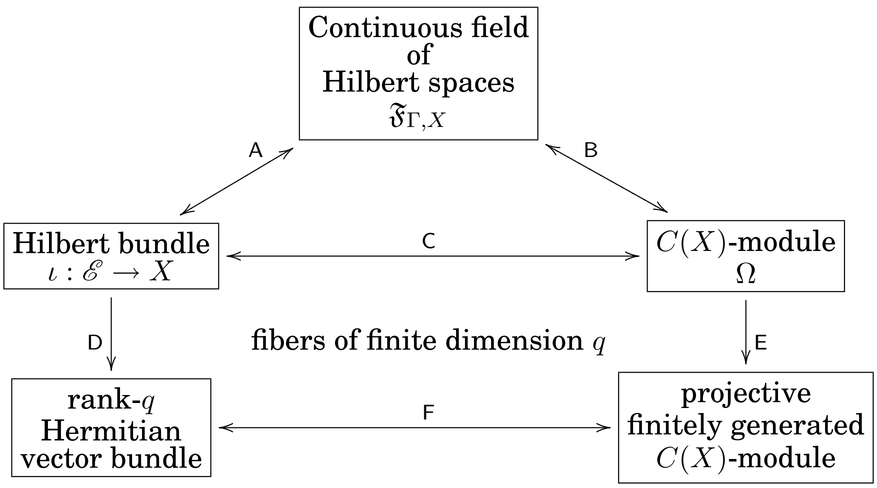
Proof. We shortly sketch the proof, see (Dixmier and Douady 1963, Section 3) for details.

To prove the first part of the statement one observes that for all pairs of continuous vector fields $\sigma(\cdot), \varrho(\cdot) \in \Gamma$ the pairing $\{\cdot;\cdot\} : \Gamma \times \Gamma \to C(X)$ defined fiberwise by the inner product, i.e. by posing $\{\sigma; \varrho\}(x) := (\sigma(x); \varrho(x))_x$, satisfies Definition 4.7.5. The norm is defined by $|||\sigma||| := \sup_{x\in X} \|\sigma(x)\|_x$ and Γ is closed with respect to this norm in view of the property of locally uniform closure.

Conversely let Ω be a C^*-module over $C(X)$. For all $x \in X$ define a pre-Hilbert structure on Ω by $(\sigma; \varrho)_x := \{\sigma; \varrho\}(x)$. The set $\mathcal{I}_x := \{\sigma \in \Omega \ : \ \{\sigma; \sigma\}(x) = 0\}$ is a linear subspace of Ω. On the quotient space $\Omega/\mathcal{I}_x$ the inner product $(\ ;\ )_x$ is a positive definite sesqulinear form and we denote by $\mathcal{H}(x)$ the related Hilbert space. The collection $\{\mathcal{H}(x) : x \in X\}$ defines a field of Hilbert spaces $\mathfrak{F}(\Omega) = \prod_{x\in X} \mathcal{H}(x)$. For all $\sigma \in \Omega$ the canonical projection $\Omega \ni \sigma \overset{\jmath_x}{\longmapsto} \sigma(x) \ni \Omega/\mathcal{I}_x$ defines a vector field $\sigma(\cdot) \in \mathfrak{F}(\Omega)$. It is easy to check that the map $\Omega \ni \sigma \overset{\Gamma}{\longmapsto} \sigma(\cdot) \ni \mathfrak{F}(\Omega)$ is injective. We denote by $\Gamma(\Omega)$ the image of Ω in $\mathfrak{F}(\Omega)$. The family $\Gamma(\Omega)$ defines a continuous structure on $\mathfrak{F}(\Omega)$. Indeed $\{\sigma(x) \ : \ \sigma(\cdot) \in \Gamma(\Omega)\} = \Omega/\mathcal{I}_x$ is dense in $\mathcal{H}(x)$ and $\|\sigma(x)\|^2_x = \{\sigma;\sigma\}(x)$ is continuous. Finally locally uniform closure of $\Gamma(\Omega)$ follows from the closure of Ω with respect to the norm $|||\sigma||| := \sup_{x\in X} \sqrt{\{\sigma;\sigma\}(x)}$ and the existence of a partition of the unit subordinate to a finite cover of X (since X is compact). ■

The Hilbert bundle emerging from the Bloch-Floquet decomposition

We are now in position to provide a complete answer to questions (Q-II) and (Q-III) in Section 4.1. Before proceeding with our analysis, it is useful to summarize in the following diagram the relations between the algebraic, the analytic and the geometric descriptions.



(4.29)

Arrows A and B summarize the content of Propositions 4.7.3 and 4.7.6 respectively, arrow D corresponds to point (iii) of Proposition 4.7.4, and arrow E follows by Proposition 53 in (Landi 1997). Arrow F corresponds to the remarkable *Serre-Swan Theorem* (Landi 1997, Proposition 21), so arrow C can be interpreted as a generalization of the Serre-Swan Theorem.

Coming back to our original problem, let $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ be a physical frame with $\mathcal{H}$ a separable Hilbert space and $\mathfrak{S}$ a $\mathbb{Z}^N$-algebra with generators $\{U_1, \dots, U_N\}$ and wandering system $\{\psi_k\}_{k\in\mathbb{N}}$. The Bloch-Floquet decomposition (Theorem 4.6.4) ensures the existence of a unitary map $\mathcal{U}_{\mathfrak{S}}$, which maps $\mathcal{H}$ into the direct integral $\mathfrak{K} := \int^{\oplus}_{\mathbb{T}^N} \mathcal{K}(t)\ dz(t)$. Let $\mathfrak{F} := \prod_{t\in\mathbb{T}^N} \mathcal{K}(t)$ be the corresponding field of Hilbert spaces. The space $\mathfrak{K}$ is a subset of $\mathfrak{F}$ which has the structure of a Hilbert space and whose elements can be seen as L^2-sections of a "pseudo-Hilbert bundle" $\mathscr{E}(\mathfrak{F}) := \bigsqcup_{t\in\mathbb{T}^N} \mathcal{K}(t)$. This justifies the use of the notation $\mathfrak{K} \equiv L^2(\mathscr{E})$. To answer question (Q-II) in Section 4.1 we need to know how to select *a priori* a continuous structure $\Gamma \subset \mathfrak{K}$ for the field of Hilbert spaces $\mathfrak{F}$. In view of Proposition 4.7.3, this procedure is equivalent to select a priori the family of the continuous section $\Gamma(\mathscr{E})$ of the Hilbert bundle $\mathscr{E}$ inside the Hilbert space $L^2(\mathscr{E})$. We can use the generalized Bloch-Floquet transform to push back this problem at the level of the original Hilbert space $\mathcal{H}$ and to adopt the algebraic viewpoint. With this change of perspective the new, but equivalent, question which we need to answer is: does the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ select a Hilbert module over $C(\mathbb{T}^N)$ inside the Hilbert space $\mathcal{H}$?

Generalizing an idea of (Gruber 2001), we can use the transform $\mathcal{U}_{\mathfrak{S}}$ and the notion of wandering nuclear space Φ to provide a positive answer. The core of our analysis is the following result.

PROPOSITION 4.7.7. *Let $\mathfrak{S}$ be a $\mathbb{Z}^N$-algebra in the separable Hilbert space $\mathcal{H}$ with generators $\{U_1, \ldots, U_N\}$, wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ and wandering nuclear space Φ. Let $\mathfrak{K}$ be the direct integral defined by the Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}} : \mathcal{H} \to \mathfrak{K}$. Then the Bloch-Floquet transform endowes Φ with the structure of a (left) pre-C^*-module over $C(\mathbb{T}^N)$. Let $\Omega_{\mathfrak{S}}$ be the completion of Φ with respect to the C^*-module norm. Then $\Omega_{\mathfrak{S}}$ is a Hilbert module over $C(\mathbb{T}^N)$ such that $\Omega_{\mathfrak{S}} \subset \mathcal{H}$.*

Proof. The set Φ is a complex vector space which can be endowed with the structure of a $C(\mathbb{T}^N)$-module by means of the Gel'fand isomorphism. For any $f \in C(\mathbb{T}^N)$ and $\varphi \in \Phi$ we define the *(left) module product* $\star$ by

$$C(\mathbb{T}^N) \times \Phi \ni (f, \varphi) \longmapsto f \star \varphi := A_f \varphi \in \Phi \tag{4.30}$$

where $A_f \in \mathfrak{S}$ is the operator associated with $f \in C(\mathbb{T}^N)$. The product is well defined since Φ is $\mathfrak{S}$-invariant by construction. The Bloch-Floquet transform allows us also to endow Φ with a pairing $\{\cdot;\cdot\} : \Phi \times \Phi \to C(\mathbb{T}^N)$. Indeed, for any pair $\varphi, \phi \in \Phi$ and for all $t \in \mathbb{T}^n$ we define a sesquilinear form

$$\Phi \times \Phi \ni (\varphi, \phi) \longmapsto \{\varphi, \phi\}(t) := ((\mathcal{U}_{\mathfrak{S}}\varphi)(t); (\mathcal{U}_{\mathfrak{S}}\phi)(t))_t \in \mathbb{C}. \tag{4.31}$$

Moreover $\{\varphi, \phi\}(t)$ is a continuous function of t. Indeed $\varphi, \phi \in \Phi$ means that φ and ϕ are finite linear combinations of the vectors $U^a\psi_k$ and from equation (4.25) and the orthonormality of the fundamental vector fields $\zeta_k(\cdot)$ it follows that $\{\varphi, \phi\}(t)$ consists of a finite linear combination of the exponentials $\mathrm{e}^{it_1}, \ldots, \mathrm{e}^{it_N}$.

Endowed with the operations (4.30) and (4.31), the space Φ becomes a (left) pre-C^*-module over $C(\mathbb{T}^N)$. The Hilbert module $\Omega_{\mathfrak{S}}$ is defined to be the completion of Φ with respect to the norm

$$|||\varphi|||^2 := \sup_{t\in\mathbb{T}^N} \|(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\|_t^2 = \sup_{t\in\mathbb{T}^N} \left(\sum_{k\in\mathbb{N}}^{\text{fin}} |f_{\varphi;k}(t)|^2 \right) \tag{4.32}$$

according to the notation in the proof of Theorem 4.6.4. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in Φ which is Cauchy with respect to the norm $|||\cdot|||$. From (4.32), the unitarity of $\mathcal{U}_{\mathfrak{S}}$ and the normalization of the Haar measure dz on $\mathbb{T}^N$ it follows that $\|\varphi_n - \varphi_m\|_{\mathcal{H}} \leqslant |||\varphi_n - \varphi_m|||$, hence $\{\varphi_n\}_{n\in\mathbb{N}}$ is also Cauchy with respect to the norm $\|\cdot\|_{\mathcal{H}}$, so the limit $\varphi_n \to \varphi$ is an element of $\mathcal{H}$. ■

REMARK 4.7.8 (*Wannier vectors*). The proof of Proposition 4.7.7 provides more information about the relation of $\Omega_{\mathfrak{S}}$ with respect to the relevant families of Wannier vectors. According to table (4.27), we can prove that

$$\Phi \subset \Omega^{\omega}_{\mathfrak{S}} \subset \Omega^{\infty}_{\mathfrak{S}} \subset \Omega_{\mathfrak{S}} \subset \mathcal{H}.$$

Indeed equation (4.32) implies the inequality $\sup_{t\in\mathbb{T}^N} |f_{\varphi_n;k}(t) - f_{\varphi_m;k}(t)| \leqslant |||\varphi_n - \varphi_m|||$ which assures that $f_{\varphi_n;k} \to f_{\varphi;k}$ uniformly with $f_{\varphi;k} \in C(\mathbb{T}^N)$ for all $k \in \mathbb{N}$. Then the components of $(\mathcal{U}_{\mathfrak{S}}\varphi)(\cdot)$ with respect to the fundamental orthonormal frame $\boldsymbol{\zeta}(\cdot) := \{\zeta^k(\cdot)\}_{k\in\mathbb{N}}$ are continuous functions and this justifies the chain of inclusions claimed above. ♦◊

Once selected the Hilbert module $\Omega_{\mathfrak{S}}$, we can use it to define a continuous field of Hilbert spaces as explained in Proposition 4.7.6. It is easy to convince themselves that the abstract construction proposed in Proposition 4.7.6 is concretely implemented by the generalized Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}}$. Then, the set of vector fields $\Gamma_{\mathfrak{S}} := \mathcal{U}_{\mathfrak{S}}(\Omega_{\mathfrak{S}})$ defines a continuous structure on the field of Hilbert spaces $\mathfrak{F} := \prod_{t\in\mathbb{T}^N} \mathcal{K}(t)$ and, in view of Proposition 4.7.3, a Hilbert bundle over the base manifold $\mathbb{T}^N$. This Hilbert bundle, denote by $\mathscr{E}_{\mathfrak{S}}$, is the set $\bigsqcup_{t\in\mathbb{T}^N} \mathcal{K}(t)$ equipped by the topology prescribed by the set of the continuous sections $\Gamma_{\mathfrak{S}}$. The structure of $\mathscr{E}_{\mathfrak{S}}$ depends only on the equivalence class of the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ and we refer to it as the *Bloch-Floquet Hilbert bundle*.

THEOREM 4.7.9 (Emerging geometric structure). *Let $\mathfrak{S}$ be a $\mathbb{Z}^N$-algebra in the separable Hilbert space $\mathcal{H}$ with generators $\{U_1, \ldots, U_N\}$, wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ and wandering nuclear space Φ. Let $\mathfrak{K}$ be the direct integral defined by the Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}} : \mathcal{H} \to \mathfrak{K}$ and $\Omega_{\mathfrak{S}} \subset \mathcal{H}$ the Hilbert module over $C(\mathbb{T}^N)$ defined in Proposition 4.7.7. Then:*

(i) *the family of vector fields $\mathcal{U}_{\mathfrak{S}}(\Omega_{\mathfrak{S}}) =: \Gamma_{\mathfrak{S}}$ defines a continuous structure on $\mathfrak{F} = \Pi_{t\in\mathbb{T}^N}\mathcal{K}(t)$ which realizes the correspondence stated in Proposition 4.7.6;*

(ii) *the Bloch-Floquet Hilbert bundle $\iota : \mathscr{E}_{\mathfrak{S}} \to \mathbb{T}^N$, defined by $\Gamma_{\mathfrak{S}}$ according to Proposition 4.7.3, depends only on the equivalence class of the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$.*

Proof. To prove (i) let $\mathcal{I}_t := \{\varphi \in \Phi \ : \ ((\mathcal{U}_{\mathfrak{S}}\varphi)(t); (\mathcal{U}_{\mathfrak{S}}\varphi)(t))_t = 0\}$. The space $\Phi/\mathcal{I}_t$ is a pre-Hilbert space with respect to the scalar product induced by $\mathcal{U}_{\mathfrak{S}}|_t$. The map $\mathcal{U}_{\mathfrak{S}}|_t : \Phi/\mathcal{I}_t \to \mathcal{K}(t)$ is obviously isometric and so can be extended to a linear isometry from the norm-closure of $\Phi/\mathcal{I}_t$ into $\mathcal{K}(t)$. The map $\mathcal{U}_{\mathfrak{S}}|_t$ is also surjective, indeed $\mathcal{K}(t)$ is generated by the orthonormal basis $\{\zeta^k(t)\}_{k\in\mathbb{N}}$ and ${\mathcal{U}_{\mathfrak{S}}|_t}^{-1}\zeta^k(t) = \psi_k \in \Phi/\mathcal{I}_t$. Then the fiber Hilbert spaces appearing in the proof of Proposition 4.7.6 coincide, up to a unitary equivalence, with the fiber Hilbert spaces $\mathcal{K}(t)$ obtained through the Bloch-Floquet decomposition. Moreover the Bloch-Floquet transform acts as the map defined in the proof of Proposition 4.7.6, which sends any element of the Hilbert module Φ to a continuous section of $\mathfrak{F}$.

To prove (ii) let $\{\mathcal{H}_1, \mathfrak{A}_1, \mathfrak{S}_1\}$ and $\{\mathcal{H}_2, \mathfrak{A}_2, \mathfrak{S}_2\}$ be two physical frames related by a unitary map $W : \mathcal{H}_1 \to \mathcal{H}_2$. If $\mathfrak{S}_1$ is a $\mathbb{Z}^N$-algebra in $\mathcal{H}_1$ then also $\mathfrak{S}_2 = W\mathfrak{S}_1W^{-1}$ is a $\mathbb{Z}^N$-algebra in $\mathcal{H}_2$ and if $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_1$ is a wandering system for $\mathfrak{S}_1$ then $\{\widetilde{\psi}_k := W\psi_k\}_{k\in\mathbb{N}} \subset$

$\mathcal{H}_2$ is a wandering system for $\mathfrak{S}_2$ (with the same cardinality). The two wandering nuclear spaces $\Phi_1 \subset \mathcal{H}_1$ and $\Phi_2 \subset \mathcal{H}_2$ are related by $\Phi_2 = W\Phi_1$. Let $\mathcal{U}_{\mathfrak{S}_j} : \mathcal{H}_j \to \mathfrak{H}_j$, $j = 1, 2$ be the two generalized Bloch-Floquet transforms defined by the two equivalent physical frames. From the explicit expression of $\mathcal{U}_{\mathfrak{S}_2}$ and $\mathcal{U}_{\mathfrak{S}_1}{}^{-1}$, and in accordance with Corollary 4.3.4, one argues that $\mathcal{U}_{\mathfrak{S}_2} \circ W \circ \mathcal{U}_{\mathfrak{S}_1}{}^{-1} =: W(\cdot)$ is a decomposable unitary which is well defined for all t. Let $\varphi, \phi \in \Phi_1$ then

$$\begin{aligned}\{\varphi;\phi\}_1(t) : &= ((\mathcal{U}_{\mathfrak{S}_1}\varphi)(t); (\mathcal{U}_{\mathfrak{S}_1}\phi)(t))_t = (W(t)(\mathcal{U}_{\mathfrak{S}_1}\varphi)(t); W(t)(\mathcal{U}_{\mathfrak{S}_1}\phi)(t))_t \\ &= ((\mathcal{U}_{\mathfrak{S}_2}W\varphi)(t); (\mathcal{U}_{\mathfrak{S}_2}W\phi)(t))_t = \Big((\mathcal{U}_{\mathfrak{S}_2}\widetilde{\varphi})(t); (\mathcal{U}_{\mathfrak{S}_2}\widetilde{\phi})(t)\Big)_t =: \{\widetilde{\varphi};\widetilde{\phi}\}_2(t)\end{aligned}$$

where $\widetilde{\varphi} := W\varphi$ and $\widetilde{\phi} := W\phi$ are elements of Φ_2. This equation shows that Φ_1 and Φ_2 have the same $C(\mathbb{T}^N)$-module structure and so define the same abstract Hilbert module over $C(\mathbb{T}^N)$. The claim follows from the generalization of the Serre-Swan Theorem summarized by arrow C in (4.29). ■

REMARK 4.7.10. With a proof similar to that of point (ii) of Theorem 4.7.9, one deduces also that the Bloch-Floquet-Hilbert bundle $\mathscr{E}_{\mathfrak{S}}$ does not depend on the choice of two unitarily equivalent commutative C^*-algebras $\mathfrak{S}_1$ and $\mathfrak{S}_2$ inside $\mathfrak{A}$. Indeed also in this case the abstract Hilbert module structure induced by the two Bloch-Floquet transforms $\mathcal{U}_{\mathfrak{S}_1}$ and $\mathcal{U}_{\mathfrak{S}_2}$ is the same. ♦◊

Theorem 4.7.9 provides a complete and satisfactory answer to questions (Q-II) and (Q-III) in Section 4.1 for the interesting case of a $\mathbb{Z}^N$-algebras $\mathfrak{S}$ satisfying the wandering property. At this point, it is natural to deduce more information about the topology of the Bloch-Floquet Hilbert bundle from the properties of the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$. An interesting property arises from the cardinality of the wandering system which depends only on the physical frame (c.f. Corollary 4.6.8).

COROLLARY 4.7.11. *The Hilbert bundle $\mathscr{E}_{\mathfrak{S}}$ over the torus $\mathbb{T}^N$ defined by the continuous structure $\Gamma_{\mathfrak{S}}$ is trivial if the cardinality of the wandering system is $\aleph_0$, and is a rank-q Hermitian vector bundle if the cardinality of the wandering system is q. In the latter case the transition functions of the vector bundle can be expressed in terms of the fundamental orthonormal frame of sections $\boldsymbol{\zeta}(\cdot) := \{\zeta_k(\cdot)\}_{k=1,\ldots,q}$, with $\zeta_k(\cdot) := (\mathcal{U}_{\mathfrak{S}}\psi_k)(\cdot)$.*

Proof. The claim follows from Propositions 4.7.4 and 4.7.3 jointly with the fact that the dimension of the fiber spaces $\mathcal{K}(t)$ is the cardinality of the wandering system as proved in Proposition 4.6.3. In the finite dimensional case the fundamental orthonormal frame $\boldsymbol{\zeta}(\cdot)$, defined by (4.22), selects locally a family of frames and so provides the local trivializations for the vector bundle. ■

Decomposition of the observables and endomorphism sections

According to Theorem 4.6.4, the Bloch-Floquet transform (4.17) provides a concrete realization for the unitary map ($\mathfrak{S}$-Fourier transform) whose existence is claimed by the von

Neumann's complete spectral theorem. Let hereafter $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ be a physical frame with $\mathfrak{S}$ a $\mathbb{Z}^N$-algebra admitting a wandering system. Point (ii) of Theorem 4.3.1 implies that under the Bloch-Floquet transform any $O \in \mathfrak{A}$ is mapped into a decomposable operator on the direct integral $\int_{\mathbb{T}^N}^{\oplus} \mathcal{K}(t)\, dz(t)$, i.e. $\mathcal{U}_{\mathfrak{S}} O \mathcal{U}_{\mathfrak{S}}{}^{-1} =: O(\cdot) : t \mapsto O(t) \in \mathscr{B}(\mathcal{K}(t))$ with $O(\cdot)$ weakly mesurable.

The natural question which arises is the following: there exists any topological structure in the C^*-algebra $\mathfrak{A}$ compatible with the Bloch-Floquet Hilbert bundle which emerges from the Bloch-Floquet transform? To answer to this question we firstly analyze the nature of the linear maps which preserve the (Hilbert module) structure of the set of the continuous sections.

DEFINITION 4.7.12 (Hilbert module endomorphism). *Let Ω be a (left) Hilbert module over the commutative unital C^*-algebra $\mathcal{A}$. An* endomorphism *of Ω is a $\mathcal{A}$-linear map $O : \Omega \to \Omega$ which is* adjointable*, i.e. there exists a map $O^\dagger : \Omega \to \Omega$ such that $\{\sigma; O\rho\} = \{O^\dagger\sigma; \rho\}$ for all $\sigma, \rho \in \Omega$. We denote by $\mathrm{End}_{\mathcal{A}}(\Omega)$ the set of all the endomorphisms of Ω.*

As proved in (Gracia-Bondía et al. 2001, Section 2.5) or (Landi 1997, Appendix A), if $O \in \mathrm{End}_{\mathcal{A}}(\Omega)$, then also its adjoint $O^\dagger \in \mathrm{End}_{\mathcal{A}}(\Omega)$ and $\dagger$ is an involution over $\mathrm{End}_{\mathcal{A}}(\Omega)$. Moreover, $\mathrm{End}_{\mathcal{A}}(\Omega)$ endowed with the *endomorphism norm*

$$\|O\|_{\mathrm{End}(\Omega)} := \sup\{|||O(\sigma)||| \ : \ |||\sigma||| \leqslant 1\} \tag{4.33}$$

becomes a C^*-algebra (of bounded operators). For any $\sigma, \rho \in \Omega$ one defines the *rank-1 endomorphism* $|\sigma\}\{\rho| \in \mathrm{End}_{\mathcal{A}}(\Omega)$ by $|\sigma\}\{\rho|(\varsigma) := \{\rho; \varsigma\}\ \sigma$ for all $\varsigma \in \Omega$. The adjoint of $|\sigma\}\{\rho|$ is given by $|\rho\}\{\sigma|$. The linear span of the rank-1 endomorphisms is a selfadjoint two-sided ideal of $\mathrm{End}_{\mathcal{A}}(\Omega)$ (*finite rank endomorphisms*) and its (operator) norm closure is denoted by $\mathrm{End}^0_{\mathcal{A}}(\Omega)$. The elements of the latter are called *compact endomorphisms* of Ω. Since $\mathrm{End}^0_{\mathcal{A}}(\Omega)$ is an *essential ideal* of $\mathrm{End}_{\mathcal{A}}(\Omega)$, it follows that $\mathrm{End}^0_{\mathcal{A}}(\Omega) = \mathrm{End}_{\mathcal{A}}(\Omega)$ if and only if $\mathbb{1}_\Omega \in \mathrm{End}^0_{\mathcal{A}}(\Omega)$.

A remarkable result which emerges from the above theory is the characterization of the compact endomorphisms of the $C(X)$ Hilbert module $\Gamma(\mathscr{E})$ of the continuous sections of a rank-q Hermitian vector bundle $\iota : \mathscr{E} \to X$.

PROPOSITION 4.7.13. *Let $\iota : \mathscr{E} \to X$ be a rank-q Hermitian vector bundle over the compact Hausdorff space X and let $\Gamma(\mathscr{E})$ be the Hilbert module over $C(X)$ of its continuous sections. Then*

$$\mathrm{End}^0_{C(X)}(\Gamma(\mathscr{E})) = \mathrm{End}_{C(X)}(\Gamma(\mathscr{E})) \simeq \Gamma(\mathrm{End}(\mathscr{E})) \tag{4.34}$$

where $\Gamma(\mathrm{End}(\mathscr{E}))$ denotes the continuous sections of the endomorphism vector bundle *$\iota' : \mathrm{End}(\mathscr{E}) \to X$. The* localization isomorphism *appearing in right-hand side of* (4.34) *preserves the composition and the structure of $C(X)$-module.*

Proof. The *localization* (right-hand) *isomorphism* in (4.34) is a consequence of the Serre-Swan Theorem (Gracia-Bondía et al. 2001, Theorems 2.10 and 3.8). Such a theorem states that there exists an equivalence of categories between finite rank Hermitian vector bundles over a compact space X and Hilbert modules over $C(X)$ which are *projective* and *finitely generated*. The latter are modules of the form $\mathfrak{p}\, C(X)^{\oplus M}$, where

$C(X)^{\oplus M} := C(X) \times \ldots \times C(X)$ (M-times), $\mathfrak{p} \in \mathrm{Mat}_M(C(X))$ ($M \times M$ matrices with entries in $C(X)$) is an orthogonal projection ($\mathfrak{p}^2 = \mathfrak{p} = \mathfrak{p}^\dagger$) and M is some positive integer ($M > q$).

The Γ-*functor* sends any (rank q) Hermitian vector bundle $\iota : \mathscr{E} \to X$ in the Hilbert module of its continuous sections $\Gamma(\mathscr{E}) \simeq \mathfrak{p}C(X)^{\oplus M}$ (isomorphism of Hilbert modules) and any *strong bundle map* (i.e. a map which induces the identity in the base space) $f : \mathscr{E} \to \mathscr{E}$ in a Hilbert module endomorphism $\Gamma(f) \in \mathrm{End}_{C(X)}(\Gamma(\mathscr{E})) \simeq \mathrm{End}_{C(X)}(\mathfrak{p}C(X)^{\oplus M})$. Any strong bundle map f can be seen as a section of the *endomorphism bundle* $\iota' : \mathrm{End}(\mathscr{E}) \to X$. This identification defines an isomorphism (of $C(X)$-modules preserving the composition) between the set of the strong bundle maps and $\Gamma(\mathrm{End}(\mathscr{E}))$. The equivalence of categories induced by Γ implies the localization isomorphism $\mathrm{End}_{C(X)}(\Gamma(\mathscr{E})) \simeq \Gamma(\mathrm{End}(\mathscr{E}))$. The isomorphism of $C(X)$-modules (preserving the composition) (Gracia-Bondía et al. 2001, Lemma 2.18)

$$\mathrm{End}^0_{C(X)}(\mathfrak{p}C(X)^{\oplus M}) \simeq \mathfrak{p}\mathrm{Mat}_M(C(X))\mathfrak{p}$$

shows that the identity endomorphism is in $\mathrm{End}^0_{C(X)}(\mathfrak{p}C(X)^{\oplus M})$ since the C^*-algebra $C(X)$ is unital. This implies that

$$\mathrm{End}^0_{C(X)}(\mathfrak{p}C(X)^{\oplus M}) = \mathrm{End}_{C(X)}(\mathfrak{p}C(X)^{\oplus M})$$

since the compact endomorphisms form an essential ideal (Gracia-Bondía et al. 2001, Proposition 3.2). ■

In Proposition 4.7.7 we proved that the Gel'fand isomorphism and the Bloch-Floquet transform equip the wandering nuclear space Φ with the structure of a (left) pre-C^*-module over $C(\mathbb{T}^N)$ by means of the (left) product $\star$ defined by (4.30) and the pairing $\{\ ;\ \}$ defined by (4.31). The closure of Φ with respect to the module norm defines a Hilbert module over $C(\mathbb{T}^N)$ denoted by $\Omega_{\mathfrak{S}} \subset \mathcal{H}$. In this description what is the rôle played by $\mathfrak{A}$? Is it possible, at least under some condition, to interpret the elements of $\mathfrak{A}$ as endomorphism of the Hilbert module $\Omega_{\mathfrak{S}}$? One could try to answer to these questions by observing that for any $O \in \mathfrak{A}$, any $A_f \in \mathfrak{S}$ and any $\varphi \in \Omega_{\mathfrak{S}}$ one has that $O(f \star \varphi) := OA_f\varphi = A_fO\varphi$. The latter might be interpreted as $f \star O(\varphi)$, implying the $C(\mathbb{T}^N)$-linearity of $O \in \mathfrak{A}$ as operator on $\Omega_{\mathfrak{S}}$. However it may happen that $O\varphi \notin \Omega_{\mathfrak{S}}$ which implies that O can not define an endomorphism of $\Omega_{\mathfrak{S}}$. Everything works properly if one consider only elements in the subalgebra $\mathfrak{A}^0 \subset \mathfrak{A}$ defined by

$$\mathfrak{A}^0 := \{O \in \mathfrak{A}\ :\ O : \Omega_{\mathfrak{S}} \to \Omega_{\mathfrak{S}}\}. \tag{4.35}$$

PROPOSITION 4.7.14. *Let $\Omega_{\mathfrak{S}}$ be the Hilbert module over $C(\mathbb{T}^N)$ defined by means of the Bloch-Floquet transform according to Proposition 4.7.7. Let $\mathfrak{A}^0_{\text{s.a.}}$ be the C^*-subalgebra of $\mathfrak{A}$ defined by $\mathfrak{A}^0_{\text{s.a.}} := \{O \in \mathfrak{A}\ :\ O, O^\dagger \in \mathfrak{A}^0\}$ (self-adjoint part of $\mathfrak{A}^0$). Then $\mathfrak{A}^0_{\text{s.a.}} \subset \mathrm{End}_{C(\mathbb{T}^N)}(\Omega_{\mathfrak{S}})$.*

Proof. Let $O \in \mathfrak{A}^0_{\text{s.a.}}$. By definition O is a linear map from $\Omega_{\mathfrak{S}}$ to itself; it is also $C(\mathbb{T}^N)$-linear since $O(f \star \varphi) = OA_f\varphi = A_f O\varphi$ as mentioned. We need to prove that O is bounded with respect to the endomorphism norm (4.33). From the definition (4.32) of the module norm $|||\cdot|||$ it follows that

$$|||O\varphi||| = \sup_{t\in\mathbb{T}^N} \|(\mathcal{U}_{\mathfrak{S}} O\varphi)(t)\|_t = \sup_{t\in\mathbb{T}^N} \|\pi_t(O)(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\|_t \leqslant \|O\|_{\mathscr{B}(\mathcal{H})}\ |||\varphi||| \tag{4.36}$$

where $\pi_t(O) := \mathcal{U}_{\mathfrak{S}}|_t\ O\ \mathcal{U}_{\mathfrak{S}}|_t^{-1}$ defines a representation of the C^*-algebra $\mathfrak{A}$ on the fiber Hilbert space $\mathcal{K}(t)$ and $\|\pi_t(O)\|_{\mathscr{B}(\mathcal{K}(t))} \leqslant \|O\|_{\mathscr{B}(\mathcal{H})}$ since any C^* representation decreases the norm. Thus $\|O\|_{\text{End}(\Omega_{\mathfrak{S}})} \leqslant \|O\|_{\mathscr{B}(\mathcal{H})}$, therefore O defines a continuous $C(\mathbb{T}^N)$-linear map from $\Omega_{\mathfrak{S}}$ to itself. To prove that $O \in \text{End}_{C(\mathbb{T}^N)}(\Omega_{\mathfrak{S}})$ we must show that O is adjointable, which follows from the definition of $\mathfrak{A}^0_{\text{s.a.}}$. ■

It is of particular interest to specialize the previous result to the case of a finite wandering system.

THEOREM 4.7.15 (Bloch-Floquet endomorphism bundle). *Let $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ be a physical frame where $\mathfrak{S}$ is a $\mathbb{Z}^N$-algebra with generators $\{U_1, \ldots, U_N\}$ and wandering system $\{\psi_1, \ldots, \psi_q\}$ of finite cardinality. Then:*

(i) $\mathfrak{A}^0_{\text{s.a.}} = \mathfrak{A}^0$;

(ii) $\mathcal{U}_{\mathfrak{S}}\ \mathfrak{A}^0\ \mathcal{U}_{\mathfrak{S}}^{-1} \subseteq \Gamma(\text{End}(\mathscr{E}_{\mathfrak{S}}))$ *where $\iota : \mathscr{E}_{\mathfrak{S}} \to \mathbb{T}^N$ is the rank q Bloch-Floquet vector bundle defined in Corollary 4.7.11.*

Proof. To prove (i) let $O \in \mathfrak{A}^0$ and observe that if $O\psi_k = \sum_{h=1}^q \sum_{b\in\mathbb{Z}^N} \alpha^{(k)}_{h,b}\ U^b\psi_h$ then $O^\dagger\psi_k = \sum_{h=1}^q \sum_{b\in\mathbb{Z}^N} \overline{\alpha}^{(h)}_{k,b}\ U^{-b}\psi_h$. Since $O\psi_k \in \Omega_{\mathfrak{S}}$, then $f^{(k)}_h(t) := \sum_{b\in\mathbb{Z}^N} \alpha^{(k)}_{h,b}\ z^b(t)$ is a continuous function on $\mathbb{T}^N$ and

$$|||O^\dagger\psi_k|||^2 = \sup_{t\in\mathbb{T}^N} \left(\sum_{h=1}^q |f^{(h)}_k(t)|^2 \right) < +\infty.$$

Then $O^\dagger\psi_k \in \Omega_{\mathfrak{S}}$ for all $k = 1, \ldots, q$. Since $O^\dagger(U^b\psi_k) = U^b(O^\dagger\psi_k) \in \Omega_{\mathfrak{S}}$ for all $b \in \mathbb{Z}^N$ it follows that also $O^\dagger \in \mathfrak{A}^0$. Point (ii) is an immediate consequence of Proposition 4.7.14, Corollary 4.7.11 and Proposition 4.7.13. ■

EXAMPLE 4.7.16 (*Mathieu-like Hamiltonians, part four*). It is immediate to check that both u and v preserve the wandering nuclear space Φ_{M}, so that the full C^*-algebra $\mathfrak{A}^{p/q}_{\text{M}}$ consists of endomorphisms for the Hilbert module realized by means of the Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}^q_{\text{M}}}$. Theorem 4.7.15 claims that $\mathcal{U}_{\mathfrak{S}^q_{\text{M}}}$ maps $\mathfrak{A}^{p/q}_{\text{M}}$ in a subalgebra of the endomorphism bundle associated to the trivial bundle $\mathbb{T}\times\mathbb{C}^q \to \mathbb{T}$. The matrices $u(t)$ and $v(t)$ in Example 4.6.7 are a representation of the generators u and v as elements of $\Gamma(\text{End}(\mathbb{T}\times\mathbb{C}^q)) \simeq C(\mathbb{T}) \otimes \text{Mat}_q(\mathbb{C})$. ◀▷

Chapter 5

The geometry of Hofstadter and Harper models

On se persuade mieux, pour l'ordinaire, par les raisons qu'on a soi-même trouvées, que par celles qui sont venues dans l'esprit des autres.

(People are generally better persuaded by the reasons which they have themselves discovered than by those which have come into the mind of others.)

Blaise Pascal
Pensées, 1670

Abstract

This final chapter aims to present the proofs of the "geometric" results of this thesis, that is Theorem 2.7.4 and Theorem 2.8.1. In Section 5.1 we derive the (standard) physical frames for the Hofstadter and Harper representations. This first step shows that the theory of the (generalized) Bloch-Floquet decomposition, developed in Chapter 4, can be applied successfully to systems in Hofstadter or Harper regime. Employing the machinery of the Bloch-Floquet decomposition, we derive in Section 5.2 the bundle decomposition of the Hofstadter (resp. Harper) representation, as well the geometric structure of the Hofstadter (resp. Harper) vector bundle. The content of these first two sections provides the proof of Theorem 2.7.4. Finally, the proof of Theorem 2.8.1 is achieved is Section 5.3. In this last section, we derive the geometric duality between vector subbundles of the Hofstadter or Harper vector bundles related to a given "abstract" projection, according to the "two-fold way" (2.44).

5.1 Standard physical frames

5.1.1 Representation theory of the NCT-algebra

The representation theory of the NCT-algebra depends heavily on the fact that the deformation parameter is rational or irrational.

Irrational case

The representation theory of the NCT-algebra is particularly simple when the deformation parameter is irrational (Boca 2001, Theorem 1.10):

- **Simplicity and faithfulness:** If $\theta \in \mathbb{R} \setminus \mathbb{Q}$ then the NCT-algebra $\mathfrak{A}_\theta$ is *simple*, i.e. it has no non-trivial closed two-sided ideals. This implies that any (non-trivial) representation $\pi : \mathfrak{A}_\theta \to \mathscr{B}(\mathcal{H})$ is faithful whenever the deformation parameter is irrational.

Rational case

The representation theory in the case of a rational deformation parameter is less simple. First of all, in this case the NCT-algebra has finite dimensional representations. Let $\theta = {}^M\!/_N$ (according to Convention 2.4.3) and define $\underline{\mathbb{U}} := \mathbb{U}(1)$ and $\underline{\mathbb{V}} := \mathbb{V}(1)$ as in (2.40), namely

$$\underline{\mathbb{U}} := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \mathrm{e}^{i2\pi\frac{M}{N}} & 0 & \dots & 0 \\ 0 & 0 & \mathrm{e}^{i4\pi\frac{M}{N}} & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \mathrm{e}^{i2\pi(N-1)\frac{M}{N}} \end{pmatrix}, \quad \underline{\mathbb{V}} := \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \quad (5.1)$$

A straightforward computation shows that $\underline{\mathbb{U}}\underline{\mathbb{V}} = \mathrm{e}^{i2\pi\theta}\underline{\mathbb{V}}\underline{\mathbb{U}}$, moreover $C^*(\underline{\mathbb{U}}, \underline{\mathbb{V}}) = \mathrm{Mat}_N(\mathbb{C})$ (Boca 2001, Lemma 1.8), namely $\underline{\mathbb{U}}$ and $\underline{\mathbb{V}}$ are a frame of generators for $\mathrm{Mat}_N(\mathbb{C})$. For any $z = (z_1, z_2) \in \mathbb{T}^2$ the maps $\pi_z(\mathfrak{u}) = z_1\underline{\mathbb{U}}$, $\pi_z(\mathfrak{v}) = z_2\underline{\mathbb{V}}$ define a surjective representation of $\mathfrak{A}_\theta$ (c.f. Section 2.3, surjective representation property) which is not faithful since $\underline{\mathbb{U}}^N = \underline{\mathbb{V}}^N = \mathbb{1}$. The representation theory of the NCT-algebra for rational values of the deformation parameter is established in (Boca 2001, Theorem 1.9, Proposition 1.11):

- **Irreducible representations and homogeneity:** Let $\theta = {}^M\!/_N$ as in Convention 2.4.3. Each irreducible representation of $\mathfrak{A}_{M/N}$ is unitarily equivalent to one of the representations $\pi_z : \mathfrak{A}_{M/N} \to \mathrm{Mat}_N(\mathbb{C})$, with $z \in \mathbb{T}^2$. Moreover, two irreducible representations π_z and $\pi_{z'}$ are unitarily equivalent if and only if there exist two N-th roots of unity ω_1 and ω_2 such that $z'_j = \omega_j z_j$ with $j = 1, 2$. It follows that $\mathfrak{A}_{M/N}$ is a *N-homogeneous* C^*-algebra (Definition 4.6.10).

- **Faithfulness condition:** A (surjective) representation $\pi : \mathfrak{A}_{M/N} \to C^*(U, V)$ defined by $\pi(\mathfrak{u}) := U$, $\pi(\mathfrak{v}) := V$ is faithful if and only if $C^*(U^N, V^N)$ is isomorphic to $C(\mathbb{T}^2)$.

Canonical bundle representation

Let $\jmath_1(\cdot), \jmath_2(\cdot)$ be the generators of $C(\mathbb{T}^2)$, i.e. $\jmath_i(z) := z_i$, $i = 1, 2$, for any $z = (z_1, z_2) \in \mathbb{T}^2$. Define $\mathcal{U}(\cdot) := \jmath_1(\cdot) \otimes \underline{\mathbb{U}}$ and $\mathcal{V}(\cdot) := \jmath_2(\cdot) \otimes \underline{\mathbb{V}}$. Both $\mathcal{U}(\cdot)$ and $\mathcal{V}(\cdot)$ are elements in $C(\mathbb{T}^2) \otimes \mathrm{Mat}_N(\mathbb{C}) \simeq C(\mathbb{T}^2; \mathrm{Mat}_N(\mathbb{C}))$. Let $\mathbb{T}^2 \times \mathbb{C}^N \to \mathbb{T}^2$ be the rank N trivial vector bundle. The C^*-algebra $C(\mathbb{T}^2; \mathrm{Mat}_N(\mathbb{C}))$ coincides with the family of continuous sections of the related endomorphism bundle, i.e. $C(\mathbb{T}^2; \mathrm{Mat}_N(\mathbb{C})) = \Gamma(\mathrm{End}(\mathbb{T}^2 \times \mathbb{C}^N))$.

- **Canonical bundle representation:** Let $\theta = {}^M\!/_N$ as in Convention 2.4.3. The map $\Pi(\mathfrak{u}) = \mathcal{U}$, $\Pi(\mathfrak{v}) = \mathcal{V}$ defines a $*$-isomorphism between $\mathfrak{A}_{M/N}$ and $C^*(\mathcal{U}(\cdot), \mathcal{V}(\cdot)) \subset$

$\Gamma(\text{End}(\mathbb{T}^2 \times \mathbb{C}^N))$ (Boca 2001, Corollary 1.12). We refer to Π as the *canonical bundle representation*.

Any $\mathfrak{p} \in \text{Proj}(\mathfrak{A}_\theta)$ is mapped by Π in a projection-valued section $\Pi(\mathfrak{p}) := P(\cdot) \in \Gamma(\text{End}(\mathbb{T}^2 \times \mathbb{C}^N))$. According to Lemma 2.7.3, the orthogonal projection $P(\cdot)$ defines a vector subbundle $\mathscr{L}(\mathfrak{p}) \to \mathbb{T}^2$ of the trivial vector bundle, whose total space is given by

$$\mathscr{L}(\mathfrak{p}) = \bigsqcup_{z \in \mathbb{T}^2} \text{Im}(P)_z \simeq \{(z, \text{v}) \in \mathbb{T}^2 \times \mathbb{C}^N \ : \ P(z)\text{v} = \text{v}\}. \tag{5.2}$$

Even if $\mathscr{L}(\mathfrak{p})$ is a subbundle of a trivial vector bundle, its topology can be non-trivial.

PROPOSITION 5.1.1. *Let $\theta = {}^M\!/_N$ as in Convention 2.4.3. For any $\mathfrak{p} \in \text{Proj}(\mathfrak{A}^\infty_{M/N})$ the associated vector bundle $\iota : \mathscr{L}(\mathfrak{p}) \to \mathbb{T}^2$* (5.2) *has rank* $\text{Rk}(\mathfrak{p}) := N ⨍ (\mathfrak{p})$ *and first Chern number*

$$C_1(\mathcal{L}(\mathfrak{p})) = -N\ \mathfrak{C}_1(\mathfrak{p})$$

where $\mathfrak{C}_1$ is defined by equation (2.51).

Proof. Let $z \mapsto \text{Rk}(P(z))$ the function which associates to any $z \in \mathbb{T}^2$ the dimension of the orthogonal projection $P(z) \in \text{Mat}_N(\mathbb{C})$. Since the range of this map is discrete and the domain is connected, to prove that it is constant it is enough to prove that it is locally constant. The latter claim is proved in (Boca 2001, Corollary 1.22). Let Tr_N be the trace on $\text{Mat}_N(\mathbb{C})$ and observe that

$$\frac{1}{N}\int_{\mathbb{T}^2} \text{Tr}_N\left(\mathcal{U}(z)^n \mathcal{V}(z)^m\right)\ dz = \delta_{n,0}\delta_{n,0} = ⨍\ (\mathfrak{u}^n \mathfrak{v}^m)$$

where $dz = {}^{dk_1 \wedge dk_2}\!/_{(2\pi)^2}$ is the normalized Haar measure. Since the $*$-morphism Π is injective, it follows that $⨍ \equiv \left(\frac{1}{N}\int_{\mathbb{T}^2}\text{Tr}_N\right) \circ \Pi$. Since $\text{Tr}_N(P(z)) = \text{Rk}(P(z)) = c$ is constant, it follows that $⨍\ (\mathfrak{p}) = \frac{c}{N}\int_{\mathbb{T}^2} dz = \frac{c}{N}$.

From the definition of the canonical derivations (equation (2.50)) it follows that

$$\Pi(\partial_j\ (\mathfrak{a})) = (2\pi\partial_{k_j} \otimes \textbf{Id})(\Pi(\mathfrak{a})), \qquad \mathfrak{a} \in \mathfrak{A}^\infty_\theta,$$

where the derivative ∂_{k_j} is defined using the parametrization $z(k) = (\text{e}^{ik_1}, \text{e}^{ik_2})$. Then, using (2.51) one has

$$\mathfrak{C}_1(\mathfrak{p}) = -\frac{1}{N}\left(\frac{i}{2\pi}\int_{\mathbb{T}^2}\text{Tr}_N\left(P(z(k))[\partial_{k_1}P(z(k)); \partial_{k_2}P(z(k))]\right)\ dk_1 \wedge dk_2\right).$$

The quantity in brackets in the right-hand side coincides with the (differential geometric) definition of the first Chern number of the vector bundle (5.2). ■

5.1.2 Standard physical frame for the Hofstadter representation

This section aims to show that the Hofstadter representation admits a natural (or *standard*) physical frame (Definition 4.1.2) with a $\mathbb{Z}^2$-algebra of symmetries. We start with an analysis of the structure of the Hofstadter representation.

Faithfulness of the representation and the GNS construction

Every state on a C^*-algebra induces a cyclic representation on a suitable Hilbert space. If the state is faithful then the representation in injective. This is the content of the *GNS Theorem* (Bratteli and Robinson 1987, Theorem 2.3.16). We want to build the GNS representation of $\mathfrak{A}_\theta$ relative to the faithful state $⨍$ defined by (2.31). We can use $⨍$ to convert the Banach space structure of the algebra $\mathfrak{A}_\theta$ into a pre-Hilbert space structure by means of the positive defined scalar product

$$\langle \mathfrak{a}; \mathfrak{b} \rangle := ⨍ \, (\mathfrak{a}^* \mathfrak{b}). \tag{5.3}$$

Obviously $\langle \mathfrak{a}; \mathfrak{a} \rangle = 0$ if and only if $\mathfrak{a} = 0$, since $⨍$ is faithful. Each $\mathfrak{a} \in \mathfrak{A}_\theta$ can be identified with a vector $\psi_\mathfrak{a}$ of a suitable pre-Hilbert space endowed with the scalar product induced by (5.3). The completion of the space with respect to the norm induced by (5.3) defines a Hilbert space denoted by $\mathcal{H}_{\text{GNS}}$. Any $\mathfrak{a} \in \mathfrak{A}_\theta$ defines a linear operator $\pi_{\text{GNS}}(\mathfrak{a})$ acting on on the dense subset $\{\psi_\mathfrak{b} \ : \ \mathfrak{b} \in \mathfrak{A}_\theta\} \subset \mathcal{H}_{\text{GNS}}$ as $\pi_{\text{GNS}}(\mathfrak{a})\psi_\mathfrak{b} := \psi_{\mathfrak{a}\mathfrak{b}}$. A simple computation shows that $\|\pi_{\text{GNS}}(\mathfrak{a})\psi_\mathfrak{b}\|_{\text{GNS}} \leqslant \|\mathfrak{a}\| \ \|\psi_\mathfrak{b}\|_{\text{GNS}}$, hence $\pi_{\text{GNS}}(\mathfrak{a})$ has a bounded extension to whole $\mathcal{H}_{\text{GNS}}$. By construction the vector $\varpi := \psi_{\mathbb{1}}$ is *cyclic*, indeed

$$[\pi_{\text{GNS}}(\mathfrak{A}_\theta)\varpi] := \{\psi_\mathfrak{a} = \pi_{\text{GNS}}(\mathfrak{a})\varpi \ : \ \mathfrak{a} \in \mathfrak{A}_\theta\}$$

is dense in $\mathcal{H}_{\text{GNS}}$ by construction. Moreover

$$(\varpi; \pi_{\text{GNS}}(\mathfrak{a})\varpi)_{\mathcal{H}_{\text{GNS}}} = ⨍ \, (\mathfrak{a}) \qquad\qquad \mathfrak{a} \in \mathfrak{A}_\theta. \tag{5.4}$$

Equation (5.4) shows that π_{GNS} is an *faithful representation* since $⨍$ is faithful.

Let $\xi_{n,m} \in \mathcal{H}_{\text{GNS}}$ be the vector associated with the element $\mathrm{e}^{-i\pi nm\theta} \ \mathfrak{u}^n \mathfrak{v}^m \in \mathfrak{A}_\theta$, i.e. $\xi_{n,m} := \psi_{\mathrm{e}^{-i\pi nm\theta} \ \mathfrak{u}^n \mathfrak{v}^m}$. A straightforward computation shows that $\{\xi_{n,m}\}_{n,m\in\mathbb{Z}}$ provides an orthonormal basis for $\mathcal{H}_{\text{GNS}}$ with respect to the inner product (5.3). The action of $\pi_{\text{GNS}}(\mathfrak{u})$ and $\pi_{\text{GNS}}(\mathfrak{v})$ on this basis is given by

$$\begin{cases} \pi_{\text{GNS}}(\mathfrak{u}) \ \xi_{n,m} = \mathrm{e}^{i\pi m\theta} \ \xi_{n+1,m} \\ \pi_{\text{GNS}}(\mathfrak{v}) \ \xi_{n,m} = \mathrm{e}^{-i\pi n\theta} \ \xi_{n,m+1}. \end{cases} \tag{5.5}$$

Let $\{\varphi_{n,m}\}_{n,m\in\mathbb{Z}}$ be the Fourier basis of $\mathcal{H}_0 = L^2(\mathbb{T}^2, d^2k)$. From (2.7), with $\theta = -\iota_q\varepsilon_0$, it follows

$$\begin{cases} \mathcal{U}_0 \ \varphi_{n,m} = \mathrm{e}^{i\pi m\theta} \ \varphi_{n+1,m} \\ \mathcal{V}_0 \ \varphi_{n,m} = \mathrm{e}^{-i\pi n\theta} \ \varphi_{n,m+1} \end{cases} \qquad \varphi_{n,m}(k_1, k_2) := \frac{1}{2\pi}\mathrm{e}^{i(nk_1 + mk_2)}. \tag{5.6}$$

Then the unitary map $W : \mathcal{H}_{\text{GNS}} \to \mathcal{H}_0$ defined by $W\xi_{n,m} = \varphi_{n,m}$, intertwines the representations π_{GNS} and π_0, indeed

$$W\pi_{\text{GNS}}(\mathfrak{u})W^{-1} = \mathcal{U}_0 = \pi_0(\mathfrak{u}), \qquad\qquad W\pi_{\text{GNS}}(\mathfrak{v})W^{-1} = \mathcal{V}_0 = \pi_0(\mathfrak{v}).$$

Since π_{GNS} and π_0 are unitarily equivalent and π_{GNS} is faithful it follows that π_0 is faithful as claimed in Lemma 2.3.2.

Hofstadter von Neumann algebra

We refer to Appendix B.1 for basic notions about von Neumann algebras. Let $\mathfrak{M}_0(\mathfrak{A}_\theta) := \pi_0(\mathfrak{A}_\theta)''$ be the von Neumann algebra associated to $\mathfrak{A}_\theta$ in the Hofstadter representation (named *Hofstadter von Neumann algebra*). The *von Neumann density Theorem* (Bratteli and Robinson 1987, Theorem 2.4.11) states that $\mathfrak{M}_0(\mathfrak{A}_\theta) \subset \mathscr{B}(\mathcal{H}_0)$ coincides with the weak (equiv. strong) closure of $\pi_0(\mathfrak{A}_\theta)$. The vector $\varphi_{0,0} = 1/2\pi$ is *cyclic* and *separating* for $\mathfrak{M}_0(\mathfrak{A}_\theta)$ and the vector state induced by $\varphi_{0,0}$ extends to a *faithful normal tracial state* on $\mathfrak{M}_0(\mathfrak{A}_\theta)$ (Boca 2001, Lemma 1.13). According to the usual nomenclature, $\mathfrak{M}_0(\mathfrak{A}_\theta)$ is a *standard* and *finite* von Neumann algebra. Moreover if θ is irrational then $\mathfrak{M}_0(\mathfrak{A}_\theta)$ is a *hyperfinite factor* of type II_1 according to the classification of Murray-von Neumann (Boca 2001, Lemma 1.14 and Corollary 1.16).

The commutant of the Hofstadter representation

The commutant of $\pi_0(\mathfrak{A}_\theta)$, i.e. the von Neumann algebra $\pi_0(\mathfrak{A}_\theta)' = \mathfrak{M}_0(\mathfrak{A}_\theta)'$, plays a special rôle in this work. A straightforward computation shows that the unitary operators $\widehat{\mathcal{U}}_0$ and $\widehat{\mathcal{V}}_0$ defined on the Fourier basis of $\mathcal{H}_0$ by

$$\begin{cases} \widehat{\mathcal{U}}_0\ \varphi_{n,m} = \mathrm{e}^{i\pi m\theta}\ \varphi_{n-1,m} \\ \widehat{\mathcal{V}}_0\ \varphi_{n,m} = \mathrm{e}^{i\pi n\theta}\ \varphi_{n,m+1} \end{cases} \tag{5.7}$$

are element of the commutant since they commute with the generators $\mathcal{U}_0$ and $\mathcal{V}_0$ of $\pi_0(\mathfrak{A}_\theta)$. From (5.7) it follows that

$$\widehat{\mathcal{U}}_0\widehat{\mathcal{V}}_0 = \mathrm{e}^{i2\pi\theta}\ \widehat{\mathcal{V}}_0\widehat{\mathcal{U}}_0, \tag{5.8}$$

namely $\widehat{\mathcal{U}}_0$ and $\widehat{\mathcal{V}}_0$ define an alternative representation of $\mathfrak{A}_\theta$ on the Hilbert space $\mathcal{H}_0$.

PROPOSITION 5.1.2. *The commutant $\pi_0(\mathfrak{A}_\theta)'$ of the Hofstadter representation $\pi_0(\mathfrak{A}_\theta)$ is generated in $\mathscr{B}(\mathcal{H}_0)$ as the weak (equiv. strong) closure of the polynomial algebra spanned by the unitaries $\widehat{\mathcal{U}}_0$ and $\widehat{\mathcal{V}}_0$.*

Proof. Let $A \in \pi_0(\mathfrak{A}_\theta)'$. For any vector $\varphi_{n,m}$ of the Fourier basis one has $A\varphi_{n,m} = \sum_{j,k\in\mathbb{Z}} a_{j,k}^{(n,m)}\varphi_{j,k}$ with $\{a_{j,k}^{(n,m)}\}_{j,k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$. The conditions $[A;\mathcal{U}_0] = 0 = [A;\mathcal{V}_0]$ imply that

$$a_{j-1,k}^{(n-1,m)} = \mathrm{e}^{-i\pi(k-m)\theta}\ a_{j,k}^{(n,m)}, \qquad a_{j,k-1}^{(n,m-1)} = \mathrm{e}^{-i\pi(n-j)\theta}\ a_{j,k}^{(n,m)}. \tag{5.9}$$

Using the definitory equations (5.7), a straightforward computation shows

$$\widehat{\mathcal{U}}_0^{(n-j)}\ \widehat{\mathcal{V}}_0^{(k-m)}\varphi_{n,m} = \widehat{\mathcal{U}}_0^{(n-j)}\ (\mathrm{e}^{i\pi n(k-m)\theta}\varphi_{n,k}) = \mathrm{e}^{i\pi n(k-m)\theta}\ \mathrm{e}^{i\pi k(n-j)\theta}\ \varphi_{j,k}. \tag{5.10}$$

Solving (5.10) with respect $\varphi_{j,k}$ and using (5.9), one has

$$\begin{aligned} A\psi_{n,m} &= \sum_{j,k\in\mathbb{Z}} a_{j,k}^{(n,m)} \left(\mathrm{e}^{-i\pi n(k-m)\theta}\, \mathrm{e}^{-i\pi k(n-j)\theta}\, \widehat{\mathfrak{U}}_0^{(n-j)}\, \widehat{\mathfrak{V}}_0^{(k-m)} \varphi_{n,m} \right) \\ &= \left(\sum_{j,k\in\mathbb{Z}} a_{j,k}^{(n,m)} \left(\mathrm{e}^{-i\pi(k-m)\theta} \right)^n \left(\mathrm{e}^{-i\pi(n-j)\theta} \right)^k \widehat{\mathfrak{U}}_0^{(n-j)}\, \widehat{\mathfrak{V}}_0^{(k-m)} \right) \varphi_{n,m} \\ &= \left(\sum_{j,k\in\mathbb{Z}} a_{j-n,0}^{(0,m-k)}\, \widehat{\mathfrak{U}}_0^{(n-j)}\, \widehat{\mathfrak{V}}_0^{(k-m)} \right) \varphi_{n,m} = \left(\sum_{j,k\in\mathbb{Z}} a_{-j,0}^{(0,-k)}\, \widehat{\mathfrak{U}}_0^{j}\, \widehat{\mathfrak{V}}_0^{k} \right) \varphi_{n,m}. \end{aligned}$$

Since the previous equation holds true for any vector $\varphi_{n,m}$ of the Fourier basis, it follows that $A = \sum_{j,k\in\mathbb{Z}} \alpha_{j,k}\, \widehat{\mathfrak{U}}_0^j\, \widehat{\mathfrak{V}}_0^k$ where $\alpha_{j,k} := a_{-j,0}^{(0,-k)} = (\varphi_{-j,0}; A\varphi_{0,-k})_{\mathcal{H}_0}$. ∎

The Hofstadter physical frame

The separable Hilbert space $\mathcal{H}_0$ and the C^*-algebra $\pi_0(\mathfrak{A}_\theta) \subset \mathscr{B}(\mathcal{H}_0)$ are two ingredients for a physical frame (Definition 4.1.2). To complete the physical frame structure we need to select a maximal commutative subalgebra inside the commutant $\pi_0(\mathfrak{A}_\theta)'$. Let $\theta = {}^{M}/_{N}$ (as in Convention 2.4.3). In view of the commutation relation (5.8) it is easy to check that $\mathfrak{S}_0 := C^*(\widehat{\mathfrak{U}}_0, \widehat{\mathfrak{V}}_0^N)$ defines a commutative C^*-subalgebra of $\pi_0(\mathfrak{A}_{M/N})'$.

LEMMA 5.1.3. *The C^*-algebra $\mathfrak{S}_0$ is maximal commutative inside the commutant $\pi_0(\mathfrak{A}_{M/N})'$.*

Proof. Let $A = \sum_{j,k\in\mathbb{Z}} \alpha_{j,k} \widehat{\mathfrak{U}}_0^j\, \widehat{\mathfrak{V}}_0^k$ be an element in $\pi_0(\mathfrak{A}_\theta)'$ such that $[\widehat{\mathfrak{U}}_0; A] = 0 = [\widehat{\mathfrak{V}}_0^N; A]$. A direct computation shows that $\alpha_{j,k} \neq 0$ only if $k \in N\mathbb{Z}$, which means $A \in \mathfrak{S}_0$. ∎

The triple $\{\mathcal{H}_0, \pi_0(\mathfrak{A}_{M/N}), \mathfrak{S}_0\}$ is a physical frame. We refer to it as the *standard physical frame* of the Hofstadter representation (or simply *Hofstadter physical frame*). It is immediate to recognize that the set $\{\varphi_{0,j}\}_{j=0,\dots,N-1} \subset \mathcal{H}_0$, where $\varphi_{n,m}$ is defined by (5.6), is a wandering system of cardinality N for $\mathfrak{S}_0$. It follows that $\mathfrak{S}_0$ is $\mathbb{Z}^2$-algebra (claim (iii) in Proposition 4.5.2). We refer to $\{\varphi_{0,j}\}_{j=0,\dots,N-1}$ as the *standard wandering system* for the Hofstadter representation.

Obviously $\mathfrak{S}_0$ is not the only maximal commutative C^*-subalgebra of $\pi_0(\mathfrak{A}_{M/N})'$. Indeed let p, q be two integers such that $pq = N$, then $C^*(\widehat{\mathfrak{U}}_0^p, \widehat{\mathfrak{V}}_0^q)$ is another maximal commutative C^*-subalgebra of $\pi_0(\mathfrak{A}_{M/N})'$ different from $\mathfrak{S}_0$, with a wandering system of cardinality N given by $\{\varphi_{i,j}\}_{i=0,\dots,p-1, j=0,\dots,q-1}$. The ambiguity in the choice of the algebra of symmetry is solved by Proposition 4.6.11. Since $\mathfrak{A}_{M/N}$ is a N-homogeneous C^*-algebra and π_0 is a faithful representation it follows that there exists a unique equivalence class of *irreducible* physical frames with wandering property for the Hofstadter representation. Therefore there is no loss of generality fixing the Hofstadter physical frame $\{\mathcal{H}_0, \pi_0(\mathfrak{A}_{M/N}), \mathfrak{S}_0\}$ as the *standard* representative of this class.

5.1.3 Standard physical frame for the Harper representation

In this section we derive the (*standard*) physical frame for the Harper representation. As in the previous section, we start with an analysis of the structure of the Harper representation.

Faithfulness of the representation and generalized eigenvectors

We start this section giving the proof of Lemma 2.3.4 which states that the Harper representation $\pi_\infty : \mathfrak{A}_\theta \to C^*(\mathcal{U}_\infty, \mathcal{V}_\infty) \subset \mathscr{B}(\mathcal{H}_\infty)$ is faithful.

The claim is certainly true when $\theta \in \mathbb{R} \setminus \mathbb{Q}$ since in this case any representation of $\mathfrak{A}_\theta$ is faithful, as explained at the beginning of Section 5.1.1. Then, we can assume that $\theta = {}^M/_N$ (as in Convention 2.4.3). Under this hypothesis the faithfulness of π_∞ follows if one proves that $C^*(\mathcal{U}_\infty^N, \mathcal{V}_\infty^N) \simeq C(\mathbb{T}^2)$ (c.f. Section 5.1.1).

The *Gel'fand-Naĭmark Theorem* states that $C^*(\mathcal{U}_\infty^N, \mathcal{V}_\infty^N) \simeq C(X)$ where the *Gel'fand spectrum* X is (homeomorphic to) the *joint spectrum* of $\mathcal{U}_\infty^N$ and $\mathcal{V}_\infty^N$ (Hörmander 1990, Theorem 3.1.15), the latter being the set of the *common generalized eigenvalues* of $\mathcal{U}_\infty^N$ and $\mathcal{V}_\infty^N$ (Samoilenko 1991, Proposition 2). In view of that, we have to prove that for any $k = (k_1, k_2) \in [0, 2\pi) \times [0, 2\pi)$ there exists a sequence $\{\psi_j^{(k)}\}_{j\in\mathbb{N}} \subset \mathcal{H}_\infty$ of normalized vectors such that

$$\|(\mathcal{U}_\infty^N - \mathrm{e}^{ik_1}\mathbb{1})\psi_j^{(k)}\|_{\mathcal{H}_\infty} \to 0, \qquad \|(\mathcal{V}_\infty^N - \mathrm{e}^{ik_2}\mathbb{1})\psi_j^{(k)}\|_{\mathcal{H}_\infty} \to 0, \qquad \text{if } j \to +\infty.$$

This would mean that the joint spectrum of $\mathcal{U}_\infty^N$ and $\mathcal{V}_\infty^N$ is the full torus $\mathbb{T}^2$, thus it yields $C^*(\mathcal{U}_\infty^N, \mathcal{V}_\infty^N) \simeq C(\mathbb{T}^2)$. To complete the proof we exhibit the generalized eigenvectors, namely

$$\psi_j^{(k)}(x) := \sqrt{\frac{M}{N}} \sum_{n=-j}^{j} \mathrm{e}^{ink_2}\ \varrho_j\left(x + nM - \frac{1}{N}\frac{k_1}{2\pi}\right)$$

where $\varrho_j(x) := \sqrt{\frac{Nj^2}{M(2j+1)}}\,\varrho(xj^2)$ and ϱ is any smooth positive function supported in $(-1, 1)$ which satisfies the normalization condition $\|\varrho\|^2_{\mathcal{H}_\infty} = \int_{\mathbb{R}} \varrho(x)^2\, dx = 1$. The function ϱ_j is supported in $(-{}^1/_{j^2}, {}^1/_{j^2})$ with square-norm $\frac{N}{M(2j+1)}$ and this is enough to show that $\|\psi_j^{(k)}\|^2_{\mathcal{H}_\infty} = \frac{M}{N}\sum_{n=-j}^{j} \|\varrho_j\|^2_{\mathcal{H}_\infty} = 1$.

Harper von Neumann algebra

Let $\mathfrak{M}_\infty(\mathfrak{A}_\theta) := \pi_\infty(\mathfrak{A}_\theta)''$ be the *von Neumann algebra* associated to $\mathfrak{A}_\theta$ in the Harper representation (called *Harper von Neumann algebra*). As usual $\mathfrak{M}_\infty(\mathfrak{A}_\theta)$ coincides with the the weak (equiv. strong) closure of $\pi_\infty(\mathfrak{A}_\theta)$ in $\mathscr{B}(\mathcal{H}_\infty)$. Contrary to the Hofstadter case, the von Neumann algebra $\mathfrak{M}_\infty(\mathfrak{A}_\theta)$ is not standard (whenever $\theta \neq \pm 1$), indeed if $|\theta| < 1$ then $\mathfrak{M}_\infty(\mathfrak{A}_\theta)$ has a cyclic but not separating vector, while if $|\theta| > 1$ then $\mathfrak{M}_\infty(\mathfrak{A}_\theta)$ has a separating but not cyclic vector. The obstruction for $\mathfrak{M}_\infty(\mathfrak{A}_\theta)$ to be a standard von Neumann algebra is discussed by G. G. Emch in (Emch 1996) and follows as a corollary of two general results, one by M. A. Rieffel (Rieffel 1981, Theorem 3.2) and the second by M. Takesaki (Takesaki 1969, Theorem 3).

The commutant of the Harper representation

The latter paper by Takesaki is of particular interest in this work since it provides an explicit description for the commutant $\pi_\infty(\mathfrak{A}_\theta)' = \mathfrak{M}_\infty(\mathfrak{A}_\theta)'$. A simple computation shows that the unitaries

$$\begin{cases} \widehat{\mathcal{U}}_\infty\ \psi(x) = \mathrm{e}^{i\frac{2\pi}{\theta}x}\ \psi(x) \\ \\ \widehat{\mathcal{V}}_\infty\ \psi(x) = \psi(x-1) \end{cases} \qquad \psi \in \mathcal{H}_\infty \tag{5.11}$$

are elements of the commutant $\pi_\infty(\mathfrak{A}_\theta)'$ since they commute with the generators $\mathcal{U}_\infty$ and $\mathcal{V}_\infty$ of the algebra $\pi_\infty(\mathfrak{A}_\theta)$. Moreover,

$$\widehat{\mathcal{U}}_\infty\widehat{\mathcal{V}}_\infty = \mathrm{e}^{i\frac{2\pi}{\theta}}\ \widehat{\mathcal{V}}_\infty\widehat{\mathcal{U}}_\infty, \tag{5.12}$$

namely the unitaries $\widehat{\mathcal{U}}_\infty$ and $\widehat{\mathcal{V}}_\infty$ define a representation of $\mathfrak{A}_{1/\theta}$ on the Hilbert space $\mathcal{H}_\infty$. Since Takesaki proves that $\mathfrak{M}_\infty(\mathfrak{A}_\theta)' = \mathfrak{M}_\infty(\mathfrak{A}_{1/\theta})$ (Takesaki 1969, Theorem 3), one has

PROPOSITION 5.1.4. *The commutant $\pi_\infty(\mathfrak{A}_\theta)'$ of the Harper representation of $\mathfrak{A}_\theta$ is generated in $\mathscr{B}(\mathcal{H}_\infty)$ as the weak (equiv. strong) closure of the polynomial algebra spanned by the unitaries $\widehat{\mathcal{U}}_\infty$ and $\widehat{\mathcal{V}}_\infty$.*

The Harper physical frame

Similarly to the Hofstadter case, the separable Hilbert space $\mathcal{H}_\infty$ and the C^*-algebra $\pi_\infty(\mathfrak{A}_\theta) \subset \mathscr{B}(\mathcal{H}_\infty)$ are two ingredients for a physical frame. To complete the structure, we need to select a maximal commutative subalgebra inside the commutant $\pi_\infty(\mathfrak{A}_\theta)'$ which is generated by the unitaries $\widehat{\mathcal{U}}_\infty$ and $\widehat{\mathcal{V}}_\infty$. Let $\theta = M/N$ (according to Convention 2.4.3). In view of the commutation relation (5.12), it is easy to check that $\mathfrak{S}_\infty := C^*(\widehat{\mathcal{U}}_\infty, \widehat{\mathcal{V}}_\infty^M)$ defines a commutative C^*-subalgebra of $\pi_\infty(\mathfrak{A}_{M/N})'$. With the same proof of Lemma 5.1.3 we have

LEMMA 5.1.5. *The C^*-algebra $\mathfrak{S}_\infty$ is maximal commutative inside the commutant $\pi_\infty(\mathfrak{A}_{M/N})'$.*

The triple $\{\mathcal{H}_\infty, \pi_\infty(\mathfrak{A}_{M/N}), \mathfrak{S}_\infty\}$ is a physical frame and we refer to it as the *standard physical frame* of the Harper representation (or simply *Harper physical frame*). In order to select a wandering system for $\mathfrak{S}_\infty$, we introduce the following family of vectors

$$\chi_{j,n}(x) := \begin{cases} \sqrt{\dfrac{N}{|M|}} & \text{if } \ x \in I_{j,0} \\ \\ 0 & \text{otherwise,} \end{cases} \tag{5.13}$$

where, for any $j, n \in \mathbb{Z}$, the intervals $I_{j,n}$ are defined by

$$I_{j,n} := \begin{cases} \left[j\frac{M}{N} + nM; (j+1)\frac{M}{N} + nM\right) & \text{if} \quad M > 0 \\ \\ \left((j+1)\frac{M}{N} + nM; j\frac{M}{N} + nM\right] & \text{if} \quad M < 0. \end{cases} \tag{5.14}$$

The family $\{\chi_j\}_{j=0,\dots,N-1} \subset \mathcal{H}_\infty$, with $\chi_j \equiv \chi_{j,0}$, is a wandering system of cardinality N for $\mathfrak{S}_\infty$. Indeed, a simple check shows that

$$\mathbb{R} = \bigcup_{j=0}^{N-1} \bigcup_{n \in \mathbb{Z}} I_{j,n}, \qquad \Rightarrow \qquad \mathcal{H}_\infty \simeq \bigoplus_{j=0}^{N-1} \bigoplus_{n \in \mathbb{Z}} L^2(I_{j,n}). \tag{5.15}$$

The characteristic function of each interval $I_{j,n}$ is obtained as $\chi_{j,n} = \widehat{\mathfrak{V}}_\infty^{nM} \chi_j$ and the family of vectors $\mathrm{e}^{i2\pi \frac{N}{M} mx} \chi_{j,n} = \widehat{\mathfrak{U}}_\infty^n \chi_{j,n}$, with $m \in \mathbb{Z}$, provides an orthonormal basis for $L^2(I_{j,n})$. We refer to $\{\chi_j\}_{j=0,\dots,N-1} \subset \mathcal{H}_\infty$ as the *standard wandering system* for the Harper representation. It follows that $\mathfrak{S}_\infty$ is a $\mathbb{Z}^2$-algebra (claim (iii) in Proposition 4.5.2).

Obviously, as in the case of the Hofstadter representation, $\mathfrak{S}_\infty$ is not the only commutative C^*-subalgebra of $\pi_\infty(\mathfrak{A}_{M/N})'$. However, as in the previous case, Proposition 4.6.11, the N-homogeneity of $\mathfrak{A}_{M/N}$ and the faithfulness of π_∞ assure that there is no loss of generality fixing the standard physical frame $\{\mathcal{H}_\infty, \pi_\infty(\mathfrak{A}_\theta), \mathfrak{S}_\infty\}$ as the representative of the equivalence class of the irreducible physical frames with wandering property.

5.2 Bundle decomposition of representations

5.2.1 Bundle decomposition of the Hofstadter representation

In this section we derive the bundle decomposition of the Hofstadter representation according to Definition 2.7.2. We use the technology developped in Chapter 4 to define a generalized Bloch-Floquet transform $\mathcal{F}_0$ which provides a direct integral decomposition of the algebra $\pi_0(\mathfrak{A}_{M/N})$. A relevant and essentially unique vector bundle structure emerges from such a decomposition in virtue of properties of $\mathcal{F}_0$. The content of this section provides the proof of claim (i) in Theorem 2.7.4.

The Bloch-Floquet decomposition in the Hofstadter representation

The standard physical frame of the Hofstadter representation $\{\mathcal{H}_0, \pi_0(\mathfrak{A}_{M/N}), \mathfrak{S}_0\}$, endowed with the standard wandering system $\{\varphi_{0,j}\}_{j=0,\dots,N-1}$ (c.f. Section 5.1.2) determines the *Hofstadter nuclear space* $\Phi_0 \subset \mathcal{H}_0$, as explained at the beginning of Section 4.6. Explicitly, the space Φ_0 consists of finite linear combinations of vectors $\varphi_{n,m}$ (defined by (5.6)) of the Fourier basis of $\mathcal{H}_0$.

Equation (4.22), specialized to the standard wandering system $\{\varphi_{0,j}\}_{j=0,\ldots,N-1}$ of the C^*-algebra $\mathfrak{S}_0$, reads

$$\zeta_0^j(k) := (\mathcal{F}_0\varphi_{0,j})(k) := \sum_{n,m\in\mathbb{Z}} \mathrm{e}^{-ink_1}\mathrm{e}^{-imk_2}\ \widehat{\mathcal{U}}_0^n\ \widehat{\mathcal{V}}_0^{mN}\ \varphi_{0,j} \qquad j = 0,\ldots,N-1, \tag{5.16}$$

where $\mathcal{F}_0$ denotes the generalized Bloch-Floquet transform[1] associated to $\mathfrak{S}_0$ as in (4.17). Obviously, (5.16) has meaning in "distributional" sense. Any vector $\varphi_{i,j}$ of the Fourier basis defines a distribution $\langle\varphi_{i,j}|$ (Dirac notation) such that $\langle\varphi_{i,j}|\varphi_{r,s}\rangle := (\varphi_{i,j};\varphi_{r,s})_{\mathcal{H}_0} = \delta_{i,r}\ \delta_{j,s}$. With this notation, equation (5.16) reads

$$\zeta_0^j(k) = \sum_{n,m\in\mathbb{Z}} \mathrm{e}^{ink_1}\mathrm{e}^{-imk_2}\mathrm{e}^{i\pi(j+mN)n\frac{M}{N}}\ \langle\varphi_{n,j+mN}| \qquad j = 0,\ldots,N-1. \tag{5.17}$$

For any $k\in\mathbb{R}^2$, equation (5.17) defines a frame of N independent distributions $\boldsymbol{\zeta}_0(k) := \{\zeta_0^j(k)\}_{j=0,\ldots,N-1}$ which span, inside Φ_0^* (distibution space) the Hilbert space $\mathcal{H}_0(k)\simeq\mathbb{C}^N$. We can equip $\mathcal{H}_0(k)$ with the Hermitian structure given by the "orthonormality" of the frame $\boldsymbol{\zeta}_0(k)$, i.e. by posing $(\zeta_0^i(k);\zeta_0^j(k))_k = \delta_{i,j}$. It follows from Theorem 4.6.4 that $\mathcal{F}_0$ extends to a unitary map

$$\mathcal{F}_0 : \mathcal{H}_0 \longrightarrow \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}_0(k)\ dz(k). \tag{5.18}$$

We are now in position to exhibit the fiber representation of the algebra $\pi_0(\mathfrak{A}_{M/N})$ subordinate to the direct integral decomposition (5.18). Obviously (surjective representation property, Section 2.3), it is enough to compute the generalized Bloch-Floquet transform of the generators $\mathcal{U}_0$ and $\mathcal{V}_0$. Firstly, one observe that

$$\mathcal{U}_0\ \varphi_{0,j} = \mathrm{e}^{i2\pi j\frac{M}{N}}\ \widehat{\mathcal{U}}_0^{-1}\ \varphi_{0,j} \qquad \mathcal{V}_0\ \varphi_{0,j} = \varphi_{0,j+1} \qquad j = 0,\ldots,N-1. \tag{5.19}$$

Equipped with the notation

$$\mathcal{U}_0(k) := \mathcal{F}_0\ \mathcal{U}_0\ \mathcal{F}_0^{-1}\big|_k, \qquad \mathcal{V}_0(k) := \mathcal{F}_0\ \mathcal{V}_0\ \mathcal{F}_0^{-1}\big|_k,$$

a simple computation shows that

$$\mathcal{U}_0(k)\zeta_0^j(k) = \mathrm{e}^{-ik_1}\ \mathrm{e}^{i2\pi j\frac{N}{M}}\ \zeta_0^j(k), \qquad \mathcal{V}_0(k)\zeta_0^j(k) = \begin{cases} \zeta_0^{j+1}(k) & \text{if} \quad j = 0,\ldots,N-2 \\ \mathrm{e}^{ik_2}\ \zeta_0^0(k) & \text{if} \quad j = N-1. \end{cases} \tag{5.20}$$

Thus, the matrices which describe the action of $\mathcal{U}_0(k)$ and $\mathcal{V}_0(k)$ on the space $\mathcal{H}_0(k)$ with respect to the canonical basis fixed by the frame $\boldsymbol{\zeta}_0(k)$ are

$$\mathcal{U}_0(k) \leftrightarrow \mathbb{U}(\mathrm{e}^{-ik_1}) = \mathrm{e}^{-ik_1}\underline{\mathbb{U}}, \qquad \mathcal{V}_0(k) \leftrightarrow \mathbb{V}(\mathrm{e}^{ik_2}) \tag{5.21}$$

[1]To simplify the notation, we use the symbol $\mathcal{F}_0$ instead $\mathcal{U}_{\mathfrak{S}_0}$. The change is justified by noticing that there is a unique equivalence class of irreducible Hofstadter physical frames (c.f. Section 5.1.2), which allows us to replace $\mathfrak{S}_0$ with 0, and by Remark 4.6.5, which allows us to replace $\mathcal{U}$ with $\mathcal{F}$.

where $\mathbb{U}(\cdot)$ and $\mathbb{V}(\cdot)$ are defined by (2.40) and $\underline{\mathbb{U}} := \mathbb{U}(1)$. Matrices (5.21) provide a frame for a representation of $\mathfrak{A}_\theta$ on the Hilbert space $\mathcal{H}_0(k)$

$$\pi_0^{(k)} : \mathfrak{A}_{M/N} \to C^*(\mathcal{U}_0(k), \mathcal{V}_0(k)) = \mathbf{End}(\mathcal{H}_0(k)) \simeq \mathbf{Mat}_N(\mathbb{C}).$$

The map $\mathcal{F}_0$ induces a fiber representation of $\mathfrak{A}_{M/N}$ which is unitarily equivalent to π_0, namely

$$\mathfrak{A}_{M/N} \xrightarrow{\pi_0} \pi_0(\mathfrak{A}_{M/N}) \xrightarrow{\mathcal{F}_0 \ldots \mathcal{F}_0^{-1}} \int_{\mathbb{T}^2}^{\oplus} \pi_0^{(k)}(\mathfrak{A}_{M/N})\, dz(k).$$

REMARK 5.2.1 (Irreducibility of the fiber representations)**.** Let $\mathbb{L}(t)$ be the $N \times N$ unitary matrix defined by

$$\mathbb{L}(t) := \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \mathrm{e}^{-i\frac{1}{N}t} & 0 & \ldots & 0 \\ 0 & 0 & \mathrm{e}^{-i\frac{2}{N}t} & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & \ldots & \mathrm{e}^{-i\frac{N-1}{N}t} \end{pmatrix}, \qquad t \in \mathbb{R}.$$

A straightforward computation shows that

$$\mathbb{L}(t)\ \mathbb{U}(\lambda)\ \mathbb{L}(t)^{-1} = \mathbb{U}(\lambda), \qquad \mathbb{L}(t)\ \mathbb{V}(\lambda)\ \mathbb{L}(t)^{-1} = \mathrm{e}^{-i\frac{t}{N}}\mathbb{V}(\lambda \mathrm{e}^{it}). \tag{5.22}$$

where $\mathbb{U}(\lambda)$ and $\mathbb{V}(\lambda)$ are defined by (2.40). In particular

$$\mathbb{L}(-k_2)\ \mathbb{V}(\mathrm{e}^{ik_2})\ \mathbb{L}(-k_2)^{-1} = \mathrm{e}^{i\frac{k_2}{N}}\ \mathbb{V}(1) = \mathrm{e}^{i\frac{k_2}{N}}\ \underline{\mathbb{V}}. \tag{5.23}$$

It follows that the representation $\pi_0^{(k)}$, defined by means of the generators (5.21), is unitarily equivalent (via $\mathbb{L}(-k_2)$) to the irreducible representation generated by $\mathrm{e}^{-ik_1}\ \underline{\mathbb{U}}$ and $\mathrm{e}^{i\frac{k_2}{N}}\ \underline{\mathbb{V}}$ (c.f. equation (5.1), Section 5.1.1). The unitary equivalence implies also the irreducibility of $\pi_0^{(k)}$ and this is in accordance with point (iii) of Theorem 4.3.1 in view of the irreducibility of the Hofstadter physical frame $\{\mathcal{H}_0, \pi_0(\mathfrak{A}_{M/N}), \mathfrak{S}_0\}$. ♦◊

Hofstadter vector bundle

According to Theorem 4.7.9, the standard physical frame $\{\mathcal{H}_0, \pi_0(\mathfrak{A}_{M/N}), \mathfrak{S}_0\}$ (c.f. Section 5.1.2) defines a rank N Hermitian vector bundle $\iota : \mathscr{E}_0 \to \mathbb{T}^2$ called *Hofstadter vector bundle*. Moreover, the topology of $\mathscr{E}_0$ depends only on the equivalence class of the physical frame $\{\mathcal{H}_0, \pi_0(\mathfrak{A}_{M/N}), \mathfrak{S}_0\}$.

Bundle decomposition. To complete the proof of (i) in Theorem 2.7.4 we need to show that the Bloch-Floquet transform $\mathcal{F}_0$ induces a bundle decomposition of the C^*-algebra $\pi_0(\mathfrak{A}_{M/N})$ over the vector bundle $\iota : \mathscr{E}_0 \to \mathbb{T}^2$, namely $\mathcal{F}_0\ \pi_0(\mathfrak{A}_{M/N})\ \mathcal{F}_0^{-1} \subset \Gamma(\mathrm{End}(\mathscr{E}_0))$. In view of Theorem 4.7.15 it is enough to show that any element in $\pi_0(\mathfrak{A}_{M/N})$ preserves the Hilbert module Ω_0, the latter being the closure of the nuclear space Φ_0 with respect the module norm $|||\cdot|||$ (4.32). Equation (5.19) assures that

$$\mathcal{U}_0 : \Phi_0 \to \Phi_0, \qquad \mathcal{V}_0 : \Phi_0 \to \Phi_0$$

are continuous linear operators, which implies that the dense subalgebra $\pi_0(\mathfrak{L}_{M/N})$ (finite linear combinations of monomials $\mathcal{U}_0^n\mathcal{V}_0^m$) preserves Φ_0. As for equations (4.36), one can prove that

$$|||O\varphi||| \leqslant \|O\|_{\mathscr{B}(\mathcal{H}_0)}\ |||\varphi|||, \qquad \varphi \in \Phi_0, \quad O \in \pi_0(\mathfrak{L}_{M/N}). \tag{5.24}$$

The density of Φ_0 implies that any $O \in \pi_0(\mathfrak{L}_{M/N})$ extends to a bounded operator $O : \Omega_0 \to \Omega_0$. Let $\{O_n\}_{n\in\mathbb{N}} \subset \pi_0(\mathfrak{L}_{M/N})$ such that $O_n \to O \in \pi_0(\mathfrak{A}_{M/N})$ with respect to the operator norm $\|\cdot\|_{\mathscr{B}(\mathcal{H}_0)}$. For any $\varphi \in \Omega_0$, inequality (5.24) assures that the sequence $\{O_n\varphi\}_{n\in\mathbb{N}} \subset \Omega_0$ converges in norm $|||\cdot|||$ to an element $\tilde{\varphi} \in \Omega_0$. Inequality $\|O\varphi\|_{\mathcal{H}_0} \leqslant |||O\varphi|||$, which follows from equation (4.32), assures that $\tilde{\varphi} = O\varphi$. This proves that any $O \in \pi_0(\mathfrak{A}_{M/N})$ defines a bounded operator $O : \Omega_0 \to \Omega_0$.

Topology. The geometric structure of the vector bundle $\iota : \mathscr{E}_0 \to \mathbb{T}^2$ can be deduced from the frame of sections $\boldsymbol{\zeta}_0(\cdot) := \{\zeta_0^j(\cdot)\}_{j=0,\ldots,N-1}$. It follows immediately from (5.17), that the system of orthonormal sections satisfies the periodic conditions

$$\boldsymbol{\zeta}_0(k_1, k_2) = \boldsymbol{\zeta}_0(k_1 + 2\pi n, k_2 + 2\pi m), \qquad (n, m) \in \mathbb{Z}^2, \ \ (k_1, k_2) \in \mathbb{R}^2.$$

Let $\mathcal{H}_0(k) \subset \Phi_0^*$ be the Hilbert space generated by the orthonormal frame $\boldsymbol{\zeta}_0(k)$. Clearly $\mathcal{H}_0(k + 2\pi\gamma) = \mathcal{H}_0(k) \simeq \mathbb{C}^N$ for any $k \in \mathbb{R}^2$ and $\gamma \in \mathbb{Z}^2$, i.e. the Hilbert space $\mathcal{H}_0(k)$ depend only on the equivalence class $[k] \in \mathbb{R}^2/2\pi\mathbb{Z}^2$. We use the usual identification $\mathbb{R}^2/2\pi\mathbb{Z}^2 \ni [k] \mapsto z(k) = \mathrm{e}^{ik} \in \mathbb{T}^2$ to denote $\mathcal{H}_0(z(k)) := \mathcal{H}_0(k)$. The periodic conditions for the frame $\boldsymbol{\zeta}_0(\cdot)$ imply that the total space $\mathscr{E}_0 := \bigsqcup_{z\in\mathbb{T}^2} \mathcal{H}_0(z)$ is isomorphic to the product space $\mathbb{T}^2 \times \mathbb{C}^N$. In other words, $\boldsymbol{\zeta}_0(\cdot)$ provides a global trivialization for $\iota : \mathscr{E}_0 \to \mathbb{T}^2$, which turns out to be trivial. The triviality of the vector bundle implies the vanishing of the Chern classes and related Chern numbers (Husemoller 1994, Proposition 4.1). In particular $C_1(\mathscr{E}_0) = 0$. The bundle decomposition induced by $\mathcal{F}_0$ and the triviality of the Hofstadter vector bundle imply that

$$\mathcal{F}_0\ \pi_0(\mathfrak{A}_{M/N})\ \mathcal{F}_0^{-1} \subset \Gamma(\mathrm{End}(\mathscr{E}_0)) \simeq \Gamma(\mathrm{End}(\mathbb{T}^2 \times \mathbb{C}^N)) \simeq C(\mathbb{T}^2; \mathrm{Mat}_N(\mathbb{C})),$$

namely the elements of $\mathcal{F}_0\ \pi_0(\mathfrak{A}_{M/N})\ \mathcal{F}_0^{-1}$ are globally defined continuous functions over $\mathbb{T}^2$ with values in the algebra $\mathrm{Mat}_N(\mathbb{C})$. This is in agreement with equation (5.21).

Uniqueness. The bundle decomposition induced by $\mathcal{F}_0$ and the topology of $\iota : \mathscr{E}_0 \to \mathbb{T}^2$ depend uniquely on the equivalence class of the physical frame $\{\mathcal{H}_0, \pi_0(\mathfrak{A}_{M/N}), \mathfrak{S}_0\}$ (Theorem 4.7.9). In other words, equivalent physical frames defines unitarily equivalent bundle decompositions over isomorphic vector bundles. As discussed at the end of Section 5.1.2, for the Hofstadter representation there exists a unique equivalence class of irreducible physical frames with wandering property. This results implies the uniqueness of the irreducible bundle decomposition induced by $\mathcal{F}_0$ as claimed in point (i) of Theorem 2.7.4.

The bundle representation $\widetilde{\pi}_0 := (\mathscr{F}_0 \ldots \mathscr{F}_0^{-1}) \circ \pi_0$ has some features in common with the canonical bundle representation Π defined in Section 5.1.1. The mapping $\widetilde{\pi}_0$ allows us to associate to any projection $\mathfrak{p} \in \text{Proj}(\mathfrak{A}_{M/N})$ a vector subbundle of $\iota : \mathscr{E}_0 \to \mathbb{T}^2$. According to Lemma 2.7.3, $\widetilde{\pi}_0(\mathfrak{p}) := P_0(\cdot) \in \Gamma(\text{End}(\mathscr{E}_0)) \simeq C(\mathbb{T}^2; \mathfrak{Mat}_N(\mathbb{C}))$ defines a vector subbundle of $\mathscr{E}_0$ with total space

$$\mathscr{L}_0(\mathfrak{p}) := \{(z, \text{v}) \in \mathbb{T}^2 \times \mathbb{C}^N \ : \ P_0(z)\text{v} = \text{v}\}, \tag{5.25}$$

according to the notation introduced in (2.44), and projection $\iota : \mathscr{L}_0(\mathfrak{p}) \to \mathbb{T}^2$ defined by $\iota(z, \text{v}) = z$.

Proposition 5.2.2. *Let $\theta = M/N$ as in Convention 2.4.3. For any $\mathfrak{p} \in \text{Proj}(\mathfrak{A}_\theta^\infty)$, let $\iota : \mathscr{L}_0(\mathfrak{p}) \to \mathbb{T}^2$ be the Hermitian vector bundle defined above. The rank of $\mathscr{L}_0(\mathfrak{p})$ is given by $\text{Rk}(\mathfrak{p}) := N \fint (\mathfrak{p})$. Moreover, denoting by $C_0(\mathfrak{p}) := C_1(\mathscr{L}_0(\mathfrak{p}))$ the first Chern number of the vector bundle $\mathscr{L}_0$, one has that $C_0(\mathfrak{p}) = \mathfrak{C}_1(\mathfrak{p})$.*

Proof. Let $f_{(-1,N)} : \mathbb{T}^2 \to \mathbb{T}^2$ be the continuous map defined by

$$f_{(-1,N)}(z_1, z_2) := (z_1^{-1}, z_2^N), \qquad\qquad (z_1, z_2) \in \mathbb{T}^2.$$

The pullback vector bundle of $\mathscr{L}_0(\mathfrak{p})$ induced by $f_{(-1,N)}$ (c.f. Appendix C) is the vector bundle over $\mathbb{T}^2$ which has total space

$$f^*_{(-1,N)} \mathscr{L}_0(\mathfrak{p}) := \{(z, \text{p}) \in \mathbb{T}^2 \times \mathscr{L}_0(\mathfrak{p}) \ : \ f_{(-1,N)}(z) = \iota(\text{p})\} \tag{5.26}$$

where p is any point in $\mathscr{L}_0(\mathfrak{p})$ and ι is the canonical projection of $\mathscr{L}_0(\mathfrak{p})$ over $\mathbb{T}^2$. The canonical projection of the pullback vector bundle is defined by $\iota'(z, \text{p}) = z$, hence $\iota'^{-1}(z) = \{z\} \times \iota^{-1}(f_{(-1,N)}(z))$ and the map $\widehat{f} : f^*_{(-1,N)}\mathscr{L}_0(\mathfrak{p}) \to \mathscr{L}_0(\mathfrak{p})$ defined by $\widehat{f}(z, \text{p}) = \text{p}$ is a linear isomorphism between the fibers $\iota'^{-1}(z)$ and $\iota^{-1}(f_{-1,N}(z))$ (and it is also a vector bundle map). The identification $\iota'^{-1}(z) \simeq \iota^{-1}(f_{-1,N}(z))$ and the observation that

$$\iota^{-1}(f_{(-1,N)}(z)) = \{\text{v} \in \mathbb{C}^N \ : \ P_0(f_{(-1,N)}(z))\text{v} = \text{v}\}$$

show, according to the notation in (5.25), that

$$f^*_{(-1,N)} \mathscr{L}_0(\mathfrak{p}) \simeq \{(z, \text{v}) \in \mathbb{C}^N \ : \ P_0(z_1^{-1}, z_2^N)\text{v} = \text{v}\}. \tag{5.27}$$

Let $\mathfrak{p} = F_{\mathfrak{p}}(\mathfrak{u}, \mathfrak{v})$ be the expression of the abstract projection in terms of the abstract generators of $\mathfrak{A}_\theta$. Using equation (5.21), one has that

$$P_0(z_1^{-1}(k), z_2^N(k)) = F_{\mathfrak{p}}(\mathbb{U}(\mathrm{e}^{ik_1}), \mathbb{V}(\mathrm{e}^{iNk_2})).$$

By using the results of Remark 5.2.1, we can write

$$P_0(z_1^{-1}(k), z_2^N(k)) = \mathbb{L}(-Nk_2)\ F_{\mathfrak{p}}(z_1(k)\underline{\mathbb{U}}, z_2(k)\underline{\mathbb{V}})\ \mathbb{L}(-Nk_2)^{-1}$$

$$= \mathbb{L}(-Nk_2)\ P(z_1(k), z_2(k))\ \mathbb{L}(-Nk_2)^{-1}$$

where $P(\cdot) := \Pi(\mathfrak{p})$ is the projection-valued section in $C(\mathbb{T}^2) \otimes \mathrm{Mat}_N(\mathbb{C})$ associated to the abstract projection $\mathfrak{p}$ by means of the mapping Π. Since the unitary matrix $\mathbb{L}(-Nk_2)$ is globally defined, and so define a global change of the orthonormal frame in the fibers, it follows that

$$f^*_{(-1,N)}\ \mathscr{L}_0(\mathfrak{p}) \simeq \mathscr{L}(\mathfrak{p}) \tag{5.28}$$

where the vector bundle $\mathscr{L}(\mathfrak{p})$ is defined by equation (5.2).

The first consequence of the identification (5.28) is that the rank of the vector bundle $f^*_{(-1,N)}\ \mathscr{L}_0(\mathfrak{p})$ coincides with the rank of the vector bundle $\mathscr{L}(\mathfrak{p})$ which is $\mathrm{Rk}(\mathfrak{p})$ by Proposition 5.1.1. The first part of the claim follows by observing that the operation of pullback does not change the dimension of the fibers.

The second part follows from $C_1(\mathscr{L}(\mathfrak{p})) = -N\mathfrak{C}_1(\mathfrak{p})$ proved in Proposition 5.1.1 and $C_1(\mathscr{L}(\mathfrak{p})) = C_1(f^*_{(-1,N)}\ \mathscr{L}_0(\mathfrak{p})) = -NC_1(\mathscr{L}_0(\mathfrak{p}))$ as a consequence of Lemma C.0.2. ■

5.2.2 Bundle decomposition of the Harper representation

We derive in the present section the bundle decomposition of the Harper representation using the technology developed in Chapter 4. We define a generalized Bloch-Floquet transform $\mathcal{F}_\infty$ which provides a direct integral decomposition of the algebra $\pi_\infty(\mathfrak{A}_{M/N})$ and selects a unique vector bundle structure emerging from such a decomposition. The content of this section provides the proof of claim (ii) in Theorem 2.7.4.

The Bloch-Floquet decomposition in the Harper representation

As explained at the beginning of Section 4.6, we can use the standard physical frame of the Harper representation $\{\mathcal{H}_\infty, \pi_\infty(\mathfrak{A}_{M/N}), \mathfrak{S}_\infty\}$ and the standard wandering system $\{\chi_j\}_{j=0,\ldots,N-1}$ (see Section 5.1.3) to build the *Harper nuclear space* $\Phi_\infty \subset \mathcal{H}_\infty$. Explicitly the space Φ_∞ consists of functions supported on a finite number of intervals $I_{n,j}$, see (5.14), such that inside each $I_{n,j} \subset \mathbb{R}$ they are polynomials in $\mathrm{e}^{i2\pi\frac{N}{M}x}$. Therefore, the elements in Φ_∞ are well defined compactly supported functions on $\mathbb{R}$, which are everywhere smooth except at the points $x_{j,n} := j{M/N} + nM$, $j = 0, \ldots, N-1$, $n \in \mathbb{Z}$.

Equation (4.22), specialized to the standard wandering system $\{\chi_j\}_{j=0,\ldots,N-1}$ of the C^*-algebra $\mathfrak{S}_\infty$, gives

$$\zeta^j_\infty(k) := (\mathcal{F}_\infty\chi_j)(k) = \sum_{n,m\in\mathbb{Z}} \mathrm{e}^{-ink_1}\mathrm{e}^{-imk_2}\ \widehat{\mathbb{U}}^n_\infty\ \widehat{\mathbb{V}}^{mM}_\infty\ \chi_j \qquad j = 1, \ldots, N, \tag{5.29}$$

where $\mathcal{F}_\infty$ denotes the generalized Bloch-Floquet transform associated to $\mathfrak{S}_\infty$ as in (4.17) (for the notation c.f. Note 1 in Section 5.2.1). The distributions (5.29) can be rewritten in a more convenient form.

PROPOSITION 5.2.3. *For any* $(k_1, k_2) \in [0, 2\pi)^2$ *the distributions* $\zeta^j_\infty(k) \in \Phi^*_\infty$ *act on* Φ_∞ *as*

$$\zeta^j_\infty(k; \cdot) = \sqrt{\frac{|M|}{N}} \sum_{m \in \mathbb{Z}} e^{-imk_2}\, \delta\left[\cdot - \frac{M}{N}\left(\frac{k_1}{2\pi} + j\right) - mM\right] \qquad j = 1, \ldots, N \tag{5.30}$$

where $\delta(\cdot - x_0)$ *acts on* $\psi \in \Phi_\infty$ *as the evaluation in* x_0 *(Dirac's delta), i.e.* $\langle \delta(\cdot - x_0); \psi\rangle = \psi(x_0)$.

Proof. For any $\psi \in \Phi_\infty$ one has

$$\begin{aligned}\langle \widehat{\mathfrak{U}}^n_\infty\, \widehat{\mathfrak{V}}^{mM}_\infty\, \chi_j; \psi\rangle : &= \left(\chi_j; \widehat{\mathfrak{U}}^{-n}_\infty\, \widehat{\mathfrak{V}}^{-mM}_\infty\, \psi\right)_{\mathcal{H}_\infty} \\ &= (\text{sign}\theta) \sqrt{\frac{N}{|M|}} \int_0^{\frac{M}{N}} \mathrm{e}^{-i2\pi \frac{N}{M} nx} \psi\left(x + j\frac{M}{N} + mM\right)\, dx. \end{aligned}\tag{5.31}$$

Let $\phi : I_{0,0} \to \mathbb{C}$ be the function defined by

$$\phi(x) := \psi\left(x + j\frac{M}{N} + mM\right),$$

with $I_{0,0}$ defined by (5.14). The definition of Φ_∞ implies that ψ, when restricted to any interval $I_{j,n}$, is a finite linear combination of the exponentials $\mathrm{e}^{i2\pi\frac{N}{|M|}x}$. Then, one has

$$\phi(x) := \sum_{n \in \mathbb{Z}}^{\text{fin}} \hat{\phi}_n\, \mathrm{e}^{i2\pi\frac{N}{|M|}nx}, \qquad \hat{\phi}_n := (\text{sign}\theta)\, \frac{N}{|M|} \int_0^{\frac{M}{N}} \mathrm{e}^{-i2\pi\frac{N}{|M|}nx}\, \phi\,(x)\; dx \tag{5.32}$$

for any $x \in I_{0,0}$. A comparison between (5.31) and (5.32) shows that

$$\sqrt{\frac{N}{|M|}}\, \langle \widehat{\mathfrak{U}}^n_\infty\, \widehat{\mathfrak{V}}^{mM}_\infty\, \chi_j; \psi\rangle = (\text{sign}\theta)\, \frac{N}{|M|} \int_0^{\frac{M}{N}} \mathrm{e}^{-i2\pi\frac{N}{|M|}(\text{sign}\theta)nx} \phi\,(x)\; dx = \hat{\phi}_{(\text{sign}\theta)n}.$$

The above equation implies

$$\begin{aligned}\langle \sum_{n\in\mathbb{Z}} \mathrm{e}^{-ink_1} \widehat{\mathfrak{U}}^n_\infty\, \widehat{\mathfrak{V}}^{mM}_\infty\, \chi_j; \psi\rangle &= \sqrt{\frac{|M|}{N}} \sum_{n\in\mathbb{Z}} \hat{\phi}_{(\text{sign}\theta)n}\, \mathrm{e}^{ink_1} = \sqrt{\frac{|M|}{N}} \sum_{n\in\mathbb{Z}}^{\text{fin}} \hat{\phi}_{(\text{sign}\theta)n}\, \mathrm{e}^{i2\pi\frac{N}{|M|}n\left(\frac{|M|}{N}\frac{k_1}{2\pi}\right)} \\ &= \sqrt{\frac{|M|}{N}} \sum_{n\in\mathbb{Z}}^{\text{fin}} \hat{\phi}_n\, \mathrm{e}^{i2\pi\frac{N}{|M|}n\left(\frac{M}{N}\frac{k_1}{2\pi}\right)} = \phi\left(\frac{M}{N}\frac{k_1}{2\pi}\right)\end{aligned}$$

where we used (5.32) whenever $\frac{M}{N}\frac{k_1}{2\pi} \in I_{0,0}$, i.e. $k_1 \in [0, 2\pi)$. In view of the definition of ϕ, one has

$$\langle \sum_{n\in\mathbb{Z}} \mathrm{e}^{-ink_1} \widehat{\mathfrak{U}}^n_\infty\, \widehat{\mathfrak{V}}^{mM}_\infty\, \chi_j; \psi\rangle = \sqrt{\frac{|M|}{N}}\, \psi\left(\frac{M}{N}\frac{k_1}{2\pi} + j\frac{M}{N} + mM\right).$$

Finally, for any $\psi \in \Phi_\infty$ one has

$$\langle \zeta^j_\infty(k);\psi\rangle = \sqrt{\frac{|M|}{N}}\sum_{m\in\mathbb{Z}} \mathrm{e}^{imk_2}\psi\left[\frac{M}{N}\left(\frac{k_1}{2\pi}+j\right)+mM\right],$$

which implies (5.30). ■

For any $k\in[0,2\pi)^2$, equation (5.30) defines a frame of N independent distributions $\zeta_\infty(k) := \{\zeta^j_\infty(k)\}_{j=0,\dots,N-1}$ which span, inside Φ^*_∞ (distibution space) the Hilbert space $\mathcal{H}_\infty(k)\simeq\mathbb{C}^N$. We can equip $\mathcal{H}_\infty(k)$ with the Hermitian structure given by $(\zeta^i_\infty(k);\zeta^j_\infty(k))_k = \delta_{i,j}$. It follows from Theorem 4.6.4 that $\mathcal{F}_\infty$ extends to a unitary map

$$\mathcal{F}_\infty : \mathcal{H}_\infty \longrightarrow \int^{\oplus}_{\mathbb{T}^2} \mathcal{H}_\infty(k)\, dz(k). \tag{5.33}$$

Analogously to the case of the Hofstadter representation, we can exhibit the fiber representation of the algebra $\pi_\infty(\mathfrak{A}_{M/N})$ subordinate to the direct integral decomposition (5.33) by computing the generalized Bloch-Floquet transform of the generators $\mathcal{U}_\infty$ and $\mathcal{V}_\infty$. Let

$$\mathrm{e}^{i2\pi x}\equiv\sum_{n\in\mathbb{Z}} a_n\,\mathrm{e}^{i2\pi\frac{N}{|M|}nx},\qquad\qquad a_n=\frac{iN(1-\mathrm{e}^{i2\pi\frac{|M|}{N}})}{2\pi(|M|-nN)} \tag{5.34}$$

be the Fourier expansion of $\mathrm{e}^{i2\pi x}$ restricted to the interval $x\in I_{0,0}$. The symbol $\equiv$ means that the series converges in L^2-norm. Moreover, the series converges pointwise in the interior of $I_{0,0}$. Observing that

$$\mathcal{U}_\infty\,\chi_j(x)=\mathrm{e}^{i2\pi x}\chi_j(x)=\mathrm{e}^{i2\pi\frac{M}{N}j}\mathrm{e}^{i2\pi\left(x-j\frac{M}{N}\right)}\chi_j(x)\neq 0 \qquad\qquad \text{if } x-j\frac{M}{N}\in I_{0,0}$$

and using the L^2-expansion (5.34), one has

$$\mathcal{U}_\infty\,\chi_j(x)\equiv\mathrm{e}^{i2\pi\frac{M}{N}j}\sum_{n\in\mathbb{Z}}a_n\,\mathrm{e}^{i2\pi\frac{N}{|M|}n\left(x-j\frac{M}{N}\right)}\chi_j(x)=\mathrm{e}^{i2\pi\frac{M}{N}j}\sum_{n\in\mathbb{Z}}a_n\,\widehat{\mathcal{U}}_\infty^{(\mathrm{sign}\theta)n}\,\chi_j(x). \tag{5.35}$$

It follows that

$$\begin{aligned}
&\langle\sum_{n,m\in\mathbb{Z}}\mathrm{e}^{-ink_1}\mathrm{e}^{-imk_2}\widehat{\mathcal{U}}^n_\infty\,\widehat{\mathcal{V}}^{mM}_\infty\,\mathcal{U}_\infty\chi_j;\psi\rangle\\
&=\mathrm{e}^{-i2\pi\frac{M}{N}j}\sum_{\ell\in\mathbb{Z}}\overline{a}_\ell\langle\sum_{n,m\in\mathbb{Z}}\mathrm{e}^{-ink_1}\mathrm{e}^{-imk_2}\widehat{\mathcal{U}}^{n+(\mathrm{sign}\theta)\ell}_\infty\,\widehat{\mathcal{V}}^{mM}_\infty\chi_j;\psi\rangle\\
&=\mathrm{e}^{-i2\pi\frac{M}{N}j}\,\overline{\left(\sum_{\ell\in\mathbb{Z}}a_\ell\,\mathrm{e}^{i(\mathrm{sign}\theta)k_1\ell}\right)}\langle\sum_{n,m\in\mathbb{Z}}\mathrm{e}^{-ink_1}\mathrm{e}^{-imk_2}\widehat{\mathcal{U}}^n_\infty\,\widehat{\mathcal{V}}^{mM}_\infty\chi_j;\psi\rangle.
\end{aligned} \tag{5.36}$$

In view of (5.34), one has

$$\overline{\left(\sum_{\ell\in\mathbb{Z}}a_\ell\,\mathrm{e}^{i(\mathrm{sign}\theta)k_1\ell}\right)}\psi\equiv\mathrm{e}^{-i\frac{M}{N}k_1}\psi,\qquad\qquad \text{in } \Phi_\infty\subset\mathcal{H}_\infty.$$

The arbitrariness in the choice of the test function $\psi \in \Phi_\infty$ in (5.36) implies

$$\sum_{n,m\in\mathbb{Z}} \mathrm{e}^{-ink_1}\mathrm{e}^{-imk_2}\widehat{\mathfrak{U}}^n_\infty \; \widehat{\mathfrak{V}}^{mM}_\infty \; \mathfrak{U}_\infty\chi_j = \mathrm{e}^{i\frac{M}{N}k_1} \; \mathrm{e}^{i2\pi\frac{M}{N}j} \sum_{n,m\in\mathbb{Z}} \mathrm{e}^{-ink_1}\mathrm{e}^{-imk_2}\widehat{\mathfrak{U}}^n_\infty \; \widehat{\mathfrak{V}}^{mM}_\infty\chi_j$$

where the equality is in distributional sense, i.e. as element of Φ^*_∞. Equipped with the notation

$$\mathfrak{U}_\infty(k) := \mathcal{F}_\infty \; \mathfrak{U}_\infty \; \mathcal{F}^{-1}_\infty\big|_k\,, \qquad\qquad \mathfrak{V}_\infty(k) := \mathcal{F}_\infty \; \mathfrak{V}_\infty \; \mathcal{F}^{-1}_\infty\big|_k\,,$$

the above equation reads

$$\mathfrak{U}_\infty(k)\zeta^j_\infty(k) = \mathrm{e}^{i\frac{M}{N}k_1} \; \mathrm{e}^{i2\pi\frac{M}{N}j} \; \zeta^j_\infty(k), \qquad\qquad j = 0, \ldots, N-1. \tag{5.37}$$

The second generator $\mathfrak{V}_\infty$ acts on the vectors of the wandering system as

$$(\mathfrak{V}_\infty \; \chi_j)(x) = \chi_j\left(x - \frac{M}{N}\right) = \chi_{j+1}(x) \qquad\qquad j = 0, \ldots, N-2$$
$$(5.38)$$
$$(\mathfrak{V}_\infty \; \chi_{N-1})(x) = \chi_0(x - M) = (\widehat{\mathfrak{V}}^M_\infty\chi_0)(x)$$

then

$$\mathfrak{V}_\infty(k)\zeta^j_\infty(k) = \begin{cases} \zeta^{j+1}_\infty(k) & \text{if} \quad j = 0, \ldots, N-2 \\ \mathrm{e}^{ik_2} \; \zeta^0_\infty(k) & \text{if} \quad j = N-1. \end{cases} \tag{5.39}$$

From equations (5.37) and (5.39) one deduces the matrices which describes the action of $\mathfrak{U}_\infty(k)$ and $\mathfrak{V}_\infty(k)$ on the space $\mathcal{H}_\infty(k)$ with respect to the canonical basis fixd by the frame $\boldsymbol{\zeta}_\infty(k)$; explicitly

$$\mathfrak{U}_\infty(k) \leftrightarrow \mathbb{U}(\mathrm{e}^{i\frac{M}{N}k_1}) = \mathrm{e}^{i\frac{M}{N}k_1}\underline{\mathbb{U}}, \qquad\qquad \mathfrak{V}_\infty(k) \leftrightarrow \mathbb{V}(\mathrm{e}^{ik_2}) \tag{5.40}$$

where $\mathbb{U}(\cdot)$ and $\mathbb{V}(\cdot)$ are defined by (2.40) and $\underline{\mathbb{U}} := \mathbb{U}(1)$.

The matrices (5.40) are the frame for a representation of $\mathfrak{A}_{M/N}$ on the Hilbert space $\mathcal{H}_\infty(k)$

$$\pi^{(k)}_\infty : \mathfrak{A}_{M/N} \to C^*(\mathfrak{U}_\infty(k), \mathfrak{V}_\infty(k)) = \mathbf{End}(\mathcal{H}_\infty(k)) \simeq \mathbf{Mat}_N(\mathbb{C}).$$

The map $\mathcal{F}_\infty$ induces a fiber representation of $\mathfrak{A}_{M/N}$ which is unitarily equivalent to π_∞, namely

$$\mathfrak{A}_{M/N} \xrightarrow{\pi_\infty} \pi_\infty(\mathfrak{A}_{M/N}) \xrightarrow{\mathcal{F}_\infty \ldots \mathcal{F}^{-1}_\infty} \int^\oplus_{\mathbb{T}^2} \pi^{(k)}_\infty(\mathfrak{A}_{M/N}) \; dz(k).$$

Analogously to the Hofstadter case, one can prove that the representations $\pi^{(k)}_\infty$ are irreducible (c.f. Remark 5.2.1). This is in agreement with point (iii) of Theorem 4.3.1 in view of the irreducibility of the Harper physical frame $\{\mathcal{H}_\infty, \pi_\infty(\mathfrak{A}_{M/N}), \mathfrak{S}_\infty\}$.

Harper vector bundle

According to Theorem 4.7.9, the standard physical frame $\{\mathcal{H}_\infty, \pi_\infty(\mathfrak{A}_{M/N}), \mathfrak{S}_\infty\}$ (c.f. Section 5.1.3) defines a rank N Hermitian vector bundle $\iota : \mathscr{E}_\infty \to \mathbb{T}^2$ called *Harper vector bundle*. Moreover, the topology of $\mathscr{E}_\infty$ depends only on the equivalence class of the physical frame $\{\mathcal{H}_\infty, \pi_\infty(\mathfrak{A}_{M/N}), \mathfrak{S}_\infty\}$.

Bundle decomposition. To complete the proof of (ii) in Theorem 2.7.4 we need to show that the Bloch-Floquet transform $\mathcal{F}_\infty$ induces a bundle decomposition of the C^*-algebra $\pi_\infty(\mathfrak{A}_{M/N})$ over the vector bundle $\iota : \mathscr{E}_\infty \to \mathbb{T}^2$, namely $\mathcal{F}_\infty\ \pi_\infty(\mathfrak{A}_{M/N})\ \mathcal{F}_\infty^{-1} \subset \Gamma(\mathrm{End}(\mathscr{E}_\infty))$. In view of Theorem 4.7.15 it is enough to show that the any element in $\pi_\infty(\mathfrak{A}_{M/N})$ preserves the Hilbert module Ω_∞ which is the closure of the nuclear space Φ_∞ with respect to the module norm $|||\cdot|||$ (4.32). Analogously to the Hofstadter case, the claim follows from a density argument provided that

$$\mathcal{U}_\infty : \Omega_\infty \to \Omega_\infty, \qquad \mathcal{V}_\infty : \Omega_\infty \to \Omega_\infty. \tag{5.41}$$

The analysis of the generator $\mathcal{V}_\infty$ is simple. Indeed, equations (5.38) imply $\mathcal{V}_\infty : \Phi_\infty \to \Phi_\infty$ which by density proves the second of (5.41).

The analysis of $\mathcal{U}_\infty$ requires more care. Firstly we need to show that $\mathcal{U}_\infty\chi_j \in \Omega_\infty$ for any vector χ_j of the wandering system. This means that we have to prove that $|||\mathcal{U}_\infty\ \chi_j||| < +\infty$. In view of equation (5.35), one has

$$\begin{aligned}
|||\mathcal{U}_\infty\ \chi_j|||^2 &= |||\sum_{n\in\mathbb{Z}}^{\text{fin}} a_n(\widehat{\mathcal{U}}_\infty^{(\mathrm{sign}\theta)n}\ \chi_j)|||^2 \\
&\quad \text{definition (4.32) of } |||\cdot||| \\
&= \sup\nolimits_{k\in[0,2\pi)^2} \|\sum_{n\in\mathbb{Z}} a_n\ \mathrm{e}^{i(\mathrm{sign}\theta)nk_1}\zeta_\infty^j(k)\|_k^2 \\
&= \sup\nolimits_{k_1\in[0,2\pi)} \left(\sum_{n,n'\in\mathbb{Z}} \bar{a}_n\ a_{n'}\ \mathrm{e}^{i(\mathrm{sign}\theta)(n'-n)k_1} \right) \\
&= \sup\nolimits_{k_1\in[0,2\pi)} \left(\sum_{n,n'\in\mathbb{Z}}^{\text{fin}} \bar{a}_n\ a_{n'}\mathrm{e}^{i(\mathrm{sign}\theta)2\pi\frac{N}{|M|}(n'-n)\left(\frac{k_1}{2\pi}\frac{|M|}{N}\right)} \right) < +\infty
\end{aligned}$$

since the series $\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{\pm i2\pi\frac{N}{|M|}nx}$ (5.34) converges pointwise in $I_{0,0}$ and in particular to $\mathrm{e}^{\pm i2\pi x}$ in the interior of $I_{0,0}$. Since $\mathcal{U}_\infty\chi_j \in \Omega_\infty$ for any χ_j, it follows that $\mathcal{U}_\infty\Phi_\infty \to \Omega_\infty$. The density of Φ_∞ and the inequality (5.24) imply the first of (5.41).

Topology. The geometric structure of the vector bundle $\iota : \mathscr{E}_\infty \to \mathbb{T}^2$ can be deduced from the frame of sections $\boldsymbol{\zeta}_\infty(\cdot) := \{\zeta_0^j(\cdot)\}_{j=0,\dots,N-1}$. It follows from (5.30) that the map $\boldsymbol{\zeta}_\infty(\cdot)$, which associates to any $k \in [0,2\pi)^2$ a frame of N independent elements in Φ_∞^*, can be extended to the whole plane $\mathbb{R}^2$ by means of the following *covariance condition*

$$\begin{aligned}
&\zeta_\infty^j(k_1+2\pi,k_2) = \sum_{l=0}^{N-1} G_{j,l}(k_1,k_2)\ \zeta_\infty^l(k_1,k_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad j = 0,\dots,N-1 \\
&\zeta_\infty^j(k_1,k_2+2\pi) = \zeta_\infty^j(k_1,k_2);
\end{aligned} \tag{5.42}$$

here the $N\times N$ unitary matrix $\mathbb{G}(k) := \{G_{j,l}(k)\}_{j,l=0,\dots,N-1}$ is defined for any $k = (k_1,k_2) \in$

$\mathbb{R}^2$ by:

$$\mathbb{G}(k_1,k_2) := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \mathrm{e}^{ik_2} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{5.43}$$

Equations (5.42) can be rewritten in the synthetic form

$$\boldsymbol{\zeta}_0(k_1+2\pi n, k_2+2\pi m) = \mathbb{G}(k)^n \cdot \boldsymbol{\zeta}_0(k_1,k_2), \qquad (n,m)\in\mathbb{Z}^2,\ \ (k_1,k_2)\in\mathbb{R}^2. \tag{5.44}$$

Moreover, a straightforward computation shows that

$$\mathbb{G}(k)^N = \mathrm{e}^{ik_2}\,\mathbb{1}_N = (-1)^{N-1}\,(\mathrm{det}\mathbb{G}(k))\,\mathbb{1}_N. \tag{5.45}$$

Let $\mathcal{H}_\infty(k)\in\Phi^*_\infty$ be the Hilbert space generated by the orthonormal frame $\boldsymbol{\zeta}_\infty(k)$. Clearly $\mathcal{H}_\infty(k+2\pi\gamma)=\mathcal{H}_\infty(k)\simeq\mathbb{C}^N$ for any $k\in\mathbb{R}^2$ and $\gamma\in\mathbb{Z}^2$, i.e. the Hilbert space $\mathcal{H}_\infty(k)$ depends only on the equivalence class $[k]\in\mathbb{R}^2/2\pi\mathbb{Z}^2$. We use the usual identification $\mathbb{R}^2/2\pi\mathbb{Z}^2\ni[k]\mapsto z(k)=\mathrm{e}^{ik}\in\mathbb{T}^2$ to denote $\mathcal{H}_\infty(z(k)):=\mathcal{H}_\infty(k)$.

The covariance condition (5.44) for the frame $\boldsymbol{\zeta}_\infty(\cdot)$ do not provide a global trivialization for $\bigsqcup_{z\in\mathbb{T}^2}\mathcal{H}_\infty(z)$ as in the case of the Hofstadter representation. To build the non trivial Harper vector bundle associated to $\boldsymbol{\zeta}_\infty(\cdot)$ we identify the space $\bigsqcup_{k\in\mathbb{R}^2}\mathcal{H}_\infty(k)$ with a subset of $\mathbb{R}^2\times\Phi^*_\infty$ and we introduce the equivalence relation $\sim_{\mathbb{G}}$, where

$$(k,\mathrm{v})\sim_{\mathbb{G}}(k',\mathrm{v}') \qquad \Leftrightarrow \qquad \begin{cases} k' = k+2\pi\gamma \\ \\ \mathrm{v}' = \mathbb{G}(k)^n\cdot\mathrm{v} \end{cases} \quad \text{for some } \ \gamma:=(n,m)\in\mathbb{Z}^2.$$

The equivalence relation is well posed, indeed $k'=k+2\pi\gamma$ implies that v and v$'$ are in the same space since $\mathcal{H}_\infty(k+2\pi\gamma)=\mathcal{H}_\infty(k)$. This assures that the matrix $\mathbb{G}(k)=\mathbb{G}([k])$ acts as an isomorphism of the space $\mathcal{H}_\infty([k])$. The equivalence class with representative $(k,\mathrm{v})\in\bigsqcup_{k\in\mathbb{R}^2}\mathcal{H}_\infty(k)$ is denoted as $[k,\mathrm{v}]$. Then, the total space $\mathscr{E}_\infty$ of the Harper vector bundle is defined as

$$\mathscr{E}_\infty := \left(\bigsqcup_{k\in\mathbb{R}^2}\mathcal{H}_\infty(k)\right)\Big/\ \sim_{\mathbb{G}}\ .$$

The base space is the flat torus $\mathbb{T}^2=\mathbb{R}^2/2\pi\mathbb{Z}^2$ and the projection to the base space $\iota:\mathscr{E}_\infty\to\mathbb{T}^2$ is $\iota([k,\mathrm{v}])=z(k)$, where z provides the projection $z:\mathbb{R}^2\to\mathbb{T}^2$. One checks that $\iota:\mathscr{E}_\infty\to\mathbb{T}^2$ is a smooth complex vector bundle with typical fiber $\mathbb{C}^N$.

A system of transition functions for $\iota : \mathscr{E}_\infty \to \mathbb{T}^2$ can be described by means of the *standard atlas* of $\mathbb{T}^2$ (c.f. Example C.0.1 and Figure C.1, Appendix C). Let $\{O_a\}_{a=1,\ldots,4}$ the open cover given by (C.2) and $\boldsymbol{g} := \{g_{a,b}\}_{a,b=1,\ldots,4}$, with $g_{a,b} : O_{a,b} \to \mathscr{U}(\mathbb{C}^N)$, the related system of transition functions. Equation (5.44) show that the frame $\boldsymbol{\zeta}_\infty(\cdot)$ is purely periodic in k_2, then it defines a frame of sections for $\mathscr{E}_\infty$ which is globally defined in the k_2-direction. This implies

$$g_{1,3}(z) = \mathbb{1}_N = g_{3,1}(z) \quad \text{if} \quad z \in O_{1,3}, \qquad g_{2,4}(z) = \mathbb{1}_N = g_{4,2}(z) \quad \text{if} \quad z \in O_{2,4}, \tag{5.46}$$

which means that on the overlaps $O_1 \cap O_3$ and $O_2 \cap O_4$ the gluing between the sections is trivial. The other transition functions are non trivial and take into account for the quotient induced by $\sim_{\mathbb{G}}$. The reader can check that (c.f. Appendix C)

$$g_{a,b}(z) = \begin{cases} \mathbb{1}_N & \text{if} \ \ z \in O_{a,b}(k_1 \sim \pi) \\ {}^t\mathbb{G}(z) & \text{if} \ \ z \in O_{a,b}(k_1 \sim 2\pi) \end{cases} \qquad \text{with} \ \ (a,b) = (1,2), \ (1,4), \ (3,2), \ (3,4) \tag{5.47}$$

where ${}^t\mathbb{G}$ denotes the transpose of $\mathbb{G}$. For instance, equation (5.47) specialized to $(a,b) = (1,2)$ assures that the identification between the local frame $\boldsymbol{\zeta}^{(1)}_\infty(k_1,\cdot) = \boldsymbol{\zeta}_\infty(k_1,\cdot)$ in O_1 and the local frame $\boldsymbol{\zeta}^{(2)}_\infty(k_1,\cdot) = \boldsymbol{\zeta}_\infty(k_1 + 2\pi,\cdot)$ in O_2 ($-\epsilon < k_1 < \epsilon$) is given by means of the matrix $\mathbb{G}$, according to (5.44) (c.f. Appendix C). The set of the transition functions is completed by the relations $g_{a,b} = {g_{b,a}}^{-1}$ and $g_{a,a} = \mathbb{1}_N$.

The computation of the first Chern number of the Harper vector bundle needs the choice of a connection $\omega = \{\omega_a\}_{a=1,\ldots,4}$ compatible with the system of transition functions (5.46), (5.47), i.e.

$$\omega_a = dg_{a,b} \ \ {g_{a,b}}^{-1} + g_{a,b} \ \ \omega_b \ \ {g_{a,b}}^{-1}, \qquad\qquad a,b = 1,\ldots,4,$$

(c.f. Appendix C). We can rewrite the above consistency equation in terms of the covariance condition (5.42), namely

$$\omega(k_1,k_2) = d\,{}^t\mathbb{G}(k_1,k_2)\ {}^t\mathbb{G}(k_1,k_2)^{-1} + {}^t\mathbb{G}(k_1,k_2)\ \omega(k_1+2\pi,k_2+2\pi)\ {}^t\mathbb{G}(k_1,k_2)^{-1}. \tag{5.48}$$

A simple computation shows that the matrix valued 1-form

$$\omega_{\text{Har}}(k_1,k_2) := \frac{-i}{N} \begin{pmatrix} \frac{k_1}{2\pi} & 0 & \ldots & 0 \\ 0 & \left(\frac{k_1}{2\pi}+1\right) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \left(\frac{k_1}{2\pi}+(N-1)\right) \end{pmatrix} dk_2 + \mathbb{1}_N dk_1 \tag{5.49}$$

verifies (5.48). The connection ω_{Har}, defined by (5.49), is called *Harper connection*. Since ω_{Har} is diagonal one has that $\omega_{\text{Har}} \wedge \omega_{\text{Har}} = 0$. The *Harper curvature* 2-form K_{Har}, defined according to (C.6), is

$$K_{\text{Har}}(k_1,k_2) = d\omega_{\text{Har}}(k_1,k_2) = \left(\frac{-i}{2\pi N}\ \mathbb{1}_N\right)\ dk_1 \wedge dk_2. \tag{5.50}$$

Notice that K_{Har} is defined globally, as a consequence of the particular choice of the Harper connection. From (C.8), using the curvature K_{Har}, one obtains the (differential) Chern class for the Harper vector bundle

$$c_1(\mathscr{E}_\infty) = \frac{i}{2\pi}\mathrm{Tr}_N\left(K_{\text{Har}}\right) = \frac{1}{(2\pi)^2}\, dk_1 \wedge dk_2. \tag{5.51}$$

The integral of (5.51) over $\mathbb{T}^2$, provides the first Chern number for the Harper vector bundle, namely

$$C_1(\mathscr{E}_\infty) := \int_{\mathbb{T}^2} c_1(\mathscr{E}_\infty) = \frac{1}{(2\pi)^2}\int_0^{2\pi} dk_1 \int_0^{2\pi} dk_2 = 1.$$

Uniqueness. The discussion about the uniqueness of the Harper vector bundle follows as in the case the Hofstadter representation (c.f. Section 5.2).

5.3 Geometric duality between Hofstadter and Harper representations

In this section we provide the proof of Theorem 2.8.1. Firstly, we need to introduce the continuous maps $f_{(n,m)} : \mathbb{T}^2 \to \mathbb{T}^2$, with $n, m \in \mathbb{Z}$, defined by

$$f_{(n,m)}(\mathrm{e}^{ik_1}, \mathrm{e}^{ik_1}) := (\mathrm{e}^{ink_1}, \mathrm{e}^{imk_1}), \qquad (k_1, k_2) \in \mathbb{R}^2. \tag{5.52}$$

Let $\iota : \mathscr{E}_\infty \to \mathbb{T}^2$ be the rank N Harper vector bundle relative to deformation parameter $\theta = {}^M\!/\!_N$. We can use the *untwisting* function $f_{(N,1)}$ to (partially) untwist the vector bundle $\mathscr{E}_\infty$ via pullback. The structure of

$$\iota' : f^*_{(N,1)}\mathscr{E}_\infty \to \mathbb{T}^2 \tag{5.53}$$

can be determined by means of the frame of sections (5.42). By definition of pullback (c.f. Appendix C), the fiber space $(f^*_{(N,1)}\mathscr{E}_\infty)_z := \iota'^{-1}(z)$ coincides with the fiber space $(\mathscr{E}_\infty)_{f_{(N,1)}(z)} := \iota^{-1}(f_{(N,1)}(z))$ which is spanned by the family of sections $\boldsymbol{\zeta}_\infty := \{\zeta^0_\infty, \ldots, \zeta^{N-1}_\infty\}$ evaluated at the point $f_{(N,1)}(z)$. In other words, the structure of $f^*_{(N,1)}\mathscr{E}_\infty$ can be derived from the pullback of the frame $\boldsymbol{\zeta}_\infty$, namely from $f^*_{(N,1)}\boldsymbol{\zeta}_\infty := \boldsymbol{\zeta}_\infty \circ f_{(N,1)}$. From (5.42) it follows that

$$\begin{aligned} f^*_{(N,1)}\boldsymbol{\zeta}_\infty(k_1 + 2\pi, k_2) &= \boldsymbol{\zeta}_\infty(Nk_1 + 2\pi N, k_2) = \mathbb{G}(k_1, k_2)^N \cdot f^*_{(N,1)}\boldsymbol{\zeta}_\infty(k_1, k_2) \\ f^*_{(N,1)}\boldsymbol{\zeta}_\infty(k_1, k_2 + 2\pi) &= \boldsymbol{\zeta}_\infty(Nk_1, k_2 + 2\pi) = f^*_{(N,1)}\boldsymbol{\zeta}_\infty(k_1, k_2). \end{aligned} \tag{5.54}$$

The structure of the vector bundle (5.53) is obtained from (5.54) by means a quotient with respect to the equivalence relation $\sim_{\mathbb{G}^N}$. The procedure is similar to that used to derive the Harper vector bundle $\mathscr{E}_\infty$ from the frame of sections (5.42) and produces a system of transition functions $\tilde{\boldsymbol{g}} := \{\tilde{g}_{a,b}\}_{a,b=1,\ldots,4}$ defined by

$$\tilde{g}_{1,3}(z) = \mathbb{1}_N = \tilde{g}_{3,1}(z) \quad \text{if} \ \ z \in O_{1,3}, \qquad \tilde{g}_{2,4}(z) = \mathbb{1}_N = \tilde{g}_{4,2}(z) \quad \text{if} \ \ z \in O_{2,4}, \tag{5.55}$$

and, in view of equation (5.45),

$$\tilde{g}_{a,b}(z) = \begin{cases} \mathbb{1}_N & \text{if } z \in O_{a,b}(k_1 \sim \pi) \\ {}^t\mathbb{G}(z)^N = (-1)^{N-1}\mathrm{det} g_{a,b}(z)\ \mathbb{1}_N & \text{if } z \in O_{a,b}(k_1 \sim 2\pi) \end{cases} \tag{5.56}$$

whenever $(a,b) = (1,2),\ (1,4),\ (3,2),\ (3,4)$, where $\{g_{a,b}\}_{a,b=1,\dots,4}$ denotes the set of transition functions for the Harper vector bundle $\mathscr{E}_\infty$.

From (5.55) and (5.56), it follows that the vector bundle $f^*_{(N,1)}\mathscr{E}_\infty$ is (isomorphic to) the sum of N copies of a line bundle $\tilde{\iota} : \mathscr{I} \to \mathbb{T}^2$ with system of transition functions $\boldsymbol{\ell} := \{\ell_{a,b}\}_{a,b=1,\dots,4}$ given by

$$\ell_{1,3}(z) = 1 = \ell_{3,1}(z) \quad \text{if } z \in O_{1,3}, \qquad \ell_{2,4}(z) = 1 = \ell_{4,2}(z) \quad \text{if } z \in O_{2,4}, \tag{5.57}$$

and

$$\ell_{a,b}(z) = \begin{cases} 1 & \text{if } z \in O_{a,b}(k_1 \sim \pi) \\ (-1)^{N-1}\mathrm{det} g_{a,b}(z) & \text{if } z \in O_{a,b}(k_1 \sim 2\pi) \end{cases} \tag{5.58}$$

when $(a,b) = (1,2),\ (1,4),\ (3,2),\ (3,4)$. The transition functions (5.57) and (5.58) define the *determinant line bundle* of $\mathscr{E}_\infty$ (c.f. Appendix C), that is $\mathscr{I} \simeq \mathrm{det}(\mathscr{E}_\infty)$ where $\simeq$ denotes the isomorphism of vector bundles over $\mathbb{T}^2$. The latter claim is “tautological” when N is odd. It holds true also for even values of N since any line bundle is uniquely classified by its first Chern class and the transition functions (5.57) and (5.58) determine a unique curvature $K = -\frac{i}{2\pi} dk_1 \wedge dk_2$, independently of the value of N. Summarizing, one has the following isomorphisms of vector bundles

$$f^*_{(N,1)}\mathscr{E}_\infty \ \simeq\ \underbrace{\mathscr{I} \oplus \ldots \oplus \mathscr{I}}_{N-\text{times}} \ \simeq\ (\mathbb{T}^2 \times \mathbb{C}^N) \otimes \mathscr{I}, \qquad \text{with } \mathscr{I} \simeq \mathrm{det}(\mathscr{E}_\infty). \tag{5.59}$$

Since the pullback of a trivial vector bundle is again (equivalent to) a trivial vector bundle, one has $f^*_{(-M,1)}\mathscr{E}_0 \ \simeq\ \mathbb{T}^2 \times \mathbb{C}^N$, where $\mathscr{E}_0$ denotes the Hofstadter vector bundle. Including the latter equivalence in (5.59), one obtains the following isomorphism

$$f^*_{(N,1)}\mathscr{E}_\infty \ \simeq\ f^*_{(-M,1)}\mathscr{E}_0 \ \otimes\ \mathscr{I}, \qquad \text{with } \mathscr{I} \simeq \mathrm{det}(\mathscr{E}_\infty). \tag{5.60}$$

In order to complete the proof of Theorem 2.8.1, we have to extend the duality (5.60) to the vector subbundles of $\mathscr{E}_\infty$ and $\mathscr{E}_0$, associated to a given projection $\mathfrak{p} \in \mathfrak{A}_{M/N}$, according to the “two-fold way” (2.44). The *Serre-Swan Theorem* (Gracia-Bondía et al. 2001, Theorem 2.10 and 3.8) provides an usefull tool to achieve our goal.

Let $\iota : \mathscr{E} \to X$ be a rank N Hermitian vector bundle on the compact and connected base manifold X. The content of the Serre-Swan Theorem is that the vector bundle $\mathscr{E}$ can be obtained as vector subbundle of the trivial vector bundle $X \times \mathbb{C}^{N'} \to X$ (for a suitable $N' > N$) by means of a projection-valued section

$$P_{\mathscr{E}}(\cdot) \in \Gamma(\mathrm{End}(X \times \mathbb{C}^{N'})) \simeq C(N') \otimes \mathrm{Mat}_{N'}(\mathbb{C})$$

as in the proof of Lemma 2.7.3. In other words, the Serre-Swan Theorem claims that $\mathscr{E} \simeq \mathscr{L}(P_{\mathscr{E}}) \subset X \times \mathbb{C}^{N'}$ where the total space is defined by

$$\mathscr{E} \simeq \bigsqcup_{x \in X} \mathrm{Im}(P_{\mathscr{E}})_x \simeq \{(x, \mathrm{v}) \in X \times \mathbb{C}^{N'} \ : \quad P_{\mathscr{E}}(x)\mathrm{v} = \mathrm{v}\}.$$

The Serre-Swan Theorem implies also $\Gamma(\mathbf{End}(\mathscr{E})) = P_{\mathscr{E}}(\cdot)\, \Gamma(\mathbf{End}(X \times \mathbb{C}^M))\, P_{\mathscr{E}}(\cdot)$ (Gracia-Bondía et al. 2001, Lemma 2.18). In particular, any projection-valued section $P(\cdot) \in \Gamma(\mathbf{End}(\mathscr{E}))$ can be written as $P_{\mathscr{E}}(\cdot)\ P(\cdot)\ P_{\mathscr{E}}(\cdot)$ with $\mathrm{Im}(P)_x \subset \mathrm{Im}(P_{\mathscr{E}})_x$ for any $x \in X$. The latter condition is enough to prove that $P_{\mathscr{E}}(\cdot)\ P(\cdot) = P(\cdot)\ P_{\mathscr{E}}(\cdot)$. As usual, the vector subbundle $\mathscr{L}(P) \subset \mathscr{E}$, associated to the projection-valued section $P(\cdot)$, is described by

$$\mathscr{L}(P) \simeq \bigsqcup_{x \in X} \mathrm{Im}(P)_x \simeq \{(x, \mathrm{v}) \in X \times \mathbb{C}^{N'} \ : \ \ P(x)P_{\mathscr{E}}(x)\mathrm{v} = \mathrm{v}\}. \tag{5.61}$$

Equation (5.61) emphasizes the fact that the twisting of the vector subbundle $\mathscr{L}(P)$ is due to the twisting of the "environment" vector bundle $\mathscr{E}$, coded by $P_{\mathscr{E}}(\cdot)$, plus an extra twisting coming from $P(\cdot)$.

Let $\mathfrak{p}$ be an orthogonal projection in the NCT-algebra $\mathfrak{A}_{M/N}$. We can associate to $\mathfrak{p}$ two projection-valued sections

$$\begin{array}{lcll} P_0(\cdot) & := & \mathscr{F}_0\ \pi_0(\mathfrak{p})\mathscr{F}_0^{-1} & \in \Gamma(\mathbf{End}(\mathscr{E}_0)) \\ P_\infty(\cdot) & := & \mathscr{F}_\infty\ \pi_\infty(\mathfrak{p})\mathscr{F}_\infty^{-1} & \in \Gamma(\mathbf{End}(\mathscr{E}_\infty)) \end{array} \tag{5.62}$$

where π_0 (resp. π_∞) is the Hofstadter (resp. Harper) representation and $\mathcal{F}_0$ (resp. $\mathcal{F}_\infty$) is the generalized Bloch-Floquet transform which induces the bundle decomposition (c.f. Sections 5.1.2 and 5.1.3). In view of Lemma 2.7.3, one can associate to projections (5.62) two vector subbundles $\mathscr{L}_0(\mathfrak{p}) \subset \mathscr{E}_0$ and $\mathscr{L}_\infty(\mathfrak{p}) \subset \mathscr{E}_\infty$, with total space defined by

$$\begin{array}{lcll} \mathscr{L}_0(\mathfrak{p}) & = \bigsqcup_{z \in \mathbb{T}^2} \mathrm{Im}(P_0)_z & \simeq & \{(z, \mathrm{v}) \in \mathbb{T}^2 \times \mathbb{C}^N \ : \ P_0(z)\mathrm{v} = \mathrm{v}\} \\ \mathscr{L}_\infty(\mathfrak{p}) & = \bigsqcup_{z \in \mathbb{T}^2} \mathrm{Im}(P_\infty)_z & \simeq & \{(z, \mathrm{v}) \in \mathbb{T}^2 \times \mathbb{C}^{N'} \ : \ P_\infty(z)P_{\mathscr{E}_\infty}(z)\mathrm{v} = \mathrm{v}\} \end{array} \tag{5.63}$$

where $N' > N$ and $P_{\mathscr{E}_\infty}(\cdot)$ is the rank N projection in $\Gamma(\mathbf{End}(\mathbb{T}^2 \times \mathbb{C}^{N'}))$ which defines the non trivial Harper vector bundle according to the Serre-Swan Theorem. Applying the definition of pullback bundle, one has

$$\begin{array}{lcl} f^*_{(-M,1)}\mathscr{L}_0(\mathfrak{p}) & \simeq & \{(z, \mathrm{v}) \in \mathbb{T}^2 \times \mathbb{C}^N \ : \ P_0(f_{(-M,1)}(z))\mathrm{v} = \mathrm{v}\} \\ f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p}) & \simeq & \{(z, \mathrm{v}) \in \mathbb{T}^2 \times \mathbb{C}^{N'} \ : \ P_\infty(f_{(N,1)}(z))P_{\mathscr{E}_\infty}(f_{(N,1)}(z))\mathrm{v} = \mathrm{v}\}. \end{array} \tag{5.64}$$

Any abstract projection $\mathfrak{p} \in \mathfrak{A}_{M/N}$ is realized as a suitable function of the unitaries $\mathfrak{u}$ and $\mathfrak{v}$, namely $\mathfrak{p} = F_{\mathfrak{p}}(\mathfrak{u}, \mathfrak{v})$ where the meaning of $F_{\mathfrak{p}}$ is in the sense of a norm limit of a sequence of polynomials. Using the fact that the bundle representations induced by $\tilde{\pi}_\sharp := (\mathscr{F}_\sharp \ldots \mathscr{F}_\sharp^{-1}) \circ \pi_\sharp$ ($\sharp = 0, \infty$) are faithful, one has

$$\begin{array}{lcl} P_0(\cdot) & = \mathscr{F}_0\ \pi_0(F_{\mathfrak{p}}(\mathfrak{u}, \mathfrak{v}))\mathscr{F}_0^{-1} & = F_{\mathfrak{p}}(\mathcal{U}_0(\cdot), \mathcal{V}_0(\cdot)) \\ P_\infty(\cdot) & = \mathscr{F}_\infty\ \pi_\infty(F_{\mathfrak{p}}(\mathfrak{u}, \mathfrak{v}))\mathscr{F}_\infty^{-1} & = F_{\mathfrak{p}}(\mathcal{U}_\infty(\cdot), \mathcal{V}_\infty(\cdot)) \end{array} \tag{5.65}$$

where the unitary operators $\mathcal{U}_\sharp(\cdot)$ and $\mathcal{V}_\sharp(\cdot)$ are defined by (5.21) and (5.40). From (5.21), one deduces

$$\mathcal{U}_0(-Mk_1,k_2) \leftrightarrow \mathbb{U}(\mathrm{e}^{iMk_1}), \qquad \mathcal{U}_0(-Mk_1,k_2) \leftrightarrow \mathbb{V}(\mathrm{e}^{ik_2}). \tag{5.66}$$

Similarly, from (5.40) it follows that

$$\mathcal{U}_\infty(Nk_1,k_2) \leftrightarrow \mathbb{U}(\mathrm{e}^{iMk_1}), \qquad \mathcal{V}_\infty(Nk_1,k_2) \leftrightarrow \mathbb{V}(\mathrm{e}^{ik_2}) \tag{5.67}$$

and a comparison between (5.66) and (5.67) shows that $\mathcal{U}_\infty(f_{(N,1)}(z)) = \mathcal{U}_0(f_{(-M,1)}(z))$ and $\mathcal{V}_\infty(f_{(N,1)}(z)) = \mathcal{V}_0(f_{(-M,1)}(z))$. In view of (5.65), it follows that

$$P_\infty(f_{(N,1)}(z)) = P_0(f_{(-M,1)}(z)) \qquad \forall\, z \in \mathbb{T}^2. \tag{5.68}$$

The first consequence of equation (5.68) is that the projections $P_\infty(\cdot)$ and $P_0(\cdot)$ have the same rank, which implies that the subbundles $\mathscr{L}_\infty(\mathfrak{p})$ and $\mathscr{L}_0(\mathfrak{p})$ have the same rank too. Point (i) of Theorem 2.8.1 follows observing that $\mathrm{Rank}(P_0(z)) = \mathrm{Rk}(\mathfrak{p})$ for any $z \in \mathbb{T}^2$ (Proposition 5.2.2).

The second consequence of (5.68) comes from a comparison of the fiber spaces of the subbundles $f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p})$ and $f^*_{(-M,1)}\mathscr{L}_0(\mathfrak{p})$. From (5.64) it follows that, for any fixed $z \in \mathbb{T}^2$, the two fiber spaces over z of $f^*_{(-M,1)}\mathscr{L}_0(\mathfrak{p})$ and $f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p})$ are deremined by the same projection (5.68) acting on a N-dimensional complex vector space. Nevertheless in the Hofstadter case this vector space does not depend on z and coincides with $\mathbb{C}^N$ as a consequence of the triviality of the Hofstadter vector bundle $\mathscr{E}_0$. Conversely, in the Harper case the vector space in which the projection acts depends on z as a consequence of the non triviality of the Harper vector bundle $\mathscr{E}_\infty$ and it is defined by $P_{\mathscr{E}_\infty}(f_{(N,1)}(z))\mathbb{C}^{N'}$. Then $f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p})$ and $f^*_{(-M,1)}\mathscr{L}_0(\mathfrak{p})$ coincide locally but $f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p})$ has an extra twist induced by the rank N projection $(P_{\mathscr{E}_\infty} \circ f_{(N,1)})(\cdot)$. The latter is the projection associated (via Serre-Swan Theorem) to the vector bundle $f^*_{(N,1)}\mathscr{E}_\infty \simeq (\mathbb{T}^2 \times \mathbb{C}^N) \otimes \mathscr{I}$ (c.f. equation 5.59). This isomorphism suggests that the non triviality due to $(P_{\mathscr{E}_\infty} \circ f_{(N,1)})(\cdot)$ can be factorized out by means of the tensor product with the line bundle $\mathscr{I}$. Let $P_{\mathscr{I}}(\cdot) \in \Gamma(\mathrm{End}(\mathbb{T}^2 \times \mathbb{C}^{N''}))$, with $N'' > 1$, be the rank one projection-valued section which defines the line bundle $\mathscr{I}$. The equivalence between $f^*_{(N,1)}\mathscr{E}_{\mathrm{Har}}$ and $(\mathbb{T}^2 \times \mathbb{C}^N) \otimes \mathscr{I}$ allows us to write $(P_{\mathscr{E}_\infty} \circ f_{(N,1)})(\cdot) = \mathbb{1}_N \otimes P_{\mathscr{I}}(\cdot)$ where $\mathbb{1}_N$ denotes the rank N constant projection. It follows that

$$\begin{aligned}(P_\infty \circ f_{(N,1)})(\cdot)\ (P_{\mathscr{E}_\infty} \circ f_{(N,1)})(\cdot) &= [(P_\infty \circ f_{(N,1)})(\cdot) \otimes \mathbb{1}_{N''}]\ [\mathbb{1}_N \otimes P_{\mathscr{I}}(\cdot)] \\ &= (P_\infty \circ f_{(N,1)})(\cdot) \otimes P_{\mathscr{I}}(\cdot) \\ &= (P_0 \circ f_{(-M,1)})(\cdot) \otimes P_{\mathscr{I}}(\cdot).\end{aligned}$$

The above equation implies the the *geometric duality*

$$f^*_{(N,1)}\mathscr{L}_\infty(\mathfrak{p}) \ \simeq \ f^*_{(-M,1)}\mathscr{L}_0(\mathfrak{p}) \ \otimes \ \mathscr{I}.$$

Appendix A

Technicalities concerning adiabatic results

Since in the analysis of Chapter 3 we include a periodic vector potential A_Γ (as a new ingredient with respect to the standard literature) we include a short discussion of the self-adjointness and the spectral properties of the operators H_{BL} and H_{per}.

A.1 Self-adjointness and domains

A.1.1 Self-adjointness of H_{BL} and H_{per}

The *second Sobolev space* $\mathcal{H}^2(\mathbb{R}^2)$ is defined to be the set of all $\psi \in L^2(\mathbb{R}^2)$ such that $\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}\psi \in L^2(\mathbb{R}^2)$ in the sense of distributions for all $n := (n_1, n_2) \in \mathbb{N}^2$ with $|n| := n_1 + n_2 \leqslant 2$. One can proves that $\mathcal{H}^2(\mathbb{R}^2)$ is the closure of $C_{\mathrm{c}}^\infty(\mathbb{R}^2, \mathbb{C})$ with respect to the *Sobolev norm* $\| \cdot \|_{\mathcal{H}^2} := \|(\mathbb{1} - \Delta_x) \cdot \|_{L^2}$ and has a Hilbert space structure. Similarly the *second magnetic-Sobolev space* $\mathcal{H}^2_{\mathrm{M}}(\mathbb{R}^2)$ is defined to be the set of all $\psi \in L^2(\mathbb{R}^2)$ such that $D_1^{n_1} D_2^{n_2}\psi \in L^2(\mathbb{R}^2)$ in the sense of distributions for all $n \in \mathbb{N}^2$ with $|\alpha| \leqslant 2$, where $D_1 := (\partial_{x_1} + \frac{i}{2}x_2)$ and $D_2 := (\partial_{x_2} - \frac{i}{2}x_1)$. One can prove that $\mathcal{H}^2_{\mathrm{M}}(\mathbb{R}^2)$ is the closure of $C_{\mathrm{c}}^\infty(\mathbb{R}^2, \mathbb{C})$ with respect to the *magnetic-Sobolev norm* $\| \cdot \|_{\mathcal{H}^2_{\mathrm{M}}} := \|(\mathbb{1} - \Delta_M) \cdot \|_{L^2}$, where $\Delta_M := {D_1}^2 + {D_2}^2$ is the *magnetic-Laplacian*. Moreover, $\mathcal{H}^2_{\mathrm{M}}(\mathbb{R}^2)$ has a natural Hilbert space structure. For further details see (Reed and Simon 1975, Section IX.6 and IX.7) or (Lieb and Loss 2001, Chapter 7).

Proof of Proposition 3.2.4.
We prove the claim for the dimensionless operators, namely we fix all the physical constants equal to 1 in (3.1) and (3.7).
- *Step 1.* First of all we prove that H_{per} is essentially self-adjoint on $C_{\mathrm{c}}^\infty(\mathbb{R}^2, \mathbb{C})$ and self-adjoint on $\mathcal{H}^2(\mathbb{R}^2)$. Notice that

$$H_{\mathrm{per}} = \frac{1}{2}\left[-i\nabla_x - A_\Gamma(x)\right]^2 + V_\Gamma(x) = -\frac{1}{2}\Delta_x + T_1 + \frac{1}{2}T_2 \tag{A.1}$$

with $T_1 := iA_\Gamma \cdot \nabla_x$ and $T_2 := i(\nabla_x \cdot A_\Gamma) + |A_\Gamma|^2 + 2V_\Gamma$. The free Hamiltonian $-{}^1\!/\!_2\Delta_x$ is a self-adjoint operator with domain $\mathcal{H}^2(\mathbb{R}^2)$, essentially self-adjoint on $C_{\mathrm{c}}^\infty(\mathbb{R}^2, \mathbb{C})$ and from Assumption 3.2.2 it follows that T_2 is infinitesimally bounded with respect to $-{}^1\!/\!_2\Delta_x$ (notice that $T_2 - 2V_\Gamma$ is bounded). The symmetric operator T_1 is unbounded with domain $\mathcal{D}(T_1) \supset \mathcal{H}^2(\mathbb{R}^2)$. Let $\psi \in \mathcal{H}^2(\mathbb{R}^2)$, then

$$\left\|(A_\Gamma)_j \partial_{x_j}\psi\right\|^2_{L^2} \leqslant \|(A_\Gamma)_j\|^2_\infty \int_{\mathbb{R}^2} \lambda_j^2\, |\widehat{\psi}(\lambda)|^2\, d^2\lambda$$

with $\widehat{\psi}(\lambda)$ the Fourier transform of $\psi(x)$. For every $a > 0$, if $b = \frac{1}{2a}$ then $\lambda_j^2 \leqslant (a|\lambda|^2 + b)^2$. It follows that $\left\|(A_\Gamma)_j \partial_{x_j}\psi\right\|_{L^2}^2 \leqslant C\|(a|\lambda|^2 + b)\ \widehat{\psi}\|_{L^2}^2$ which implies that for all $a' > 0$ (arbitrary small) there exists a b' (depending on a') such that

$$\left\|(A_\Gamma)_j \partial_{x_j}\psi\right\|_{L^2}^2 \leqslant a'\ \||\lambda|^2\ \widehat{\psi}\|_{L^2}^2 + b'\ \|\widehat{\psi}\|_{L^2}^2 = a'\ \|\Delta_x\psi\|_{L^2}^2 + b'\ \|\psi\|_{L^2}^2.$$

This inequality implies that T_1 is infinitesimally bounded with respect to $-1/2\Delta_x$ and the thesis follows form the *Kato-Rellich Theorem* (Reed and Simon 1975, Theorem X.12).
- *Step 2.* The Bloch-Landau Hamiltonian is

$$H_{\mathrm{BL}} := \frac{1}{2}\left[-i\nabla_x - A_\Gamma(x) - A(x)\right]^2 + V_\Gamma(x) + \phi(x). \tag{A.2}$$

Assumptions 3.2.2 and 3.2.3 imply that $(A_\Gamma + A)_j \in C^1(\mathbb{R}^2, \mathbb{R})$, $j = 1, 2$, and $V_\Gamma + \phi \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ and this assures that H_{BL} is essentially self-adjoint on $C^\infty_{\mathrm{c}}(\mathbb{R}^2, \mathbb{C})$ (Reed and Simon 1975, Theorem X.34). Let $A = A_0 + A_B$ be the decomposition of the external vector potential with A_0 smooth and bounded and $A_B = \frac{1}{2}(-x_2, x_1)$. By posing $D := \nabla_x - iA_B$, $\Delta_M := |D|^2$ and $A_{\mathrm{b}} := A_\Gamma + A_0$, the Hamiltonian H_{BL} reads

$$H_{\mathrm{BL}} = -\frac{1}{2}\Delta_M - A_{\mathrm{b}} \cdot D + \frac{1}{2}T.$$

where $T := i(\nabla_x \cdot A_{\mathrm{b}}) + |A_{\mathrm{b}}|^2 + 2(V_\Gamma + \phi)$. The operator $T - 2V_\Gamma$ is bounded and the observation that $A_{\mathrm{b}} \cdot D$ is infinitesimally bounded with respect to $-1/2\Delta_M$ is an immediate consequence of Lemma A.1.2. The assumption $\int_{M_\Gamma} |V_\Gamma(x)|^2\ d^2x < +\infty$ implies that V_Γ is *uniformly locally* L^2 and hence, infinitesimally bounded with respect to $-\Delta_x$ (Reed and Simon 1978, Theorem XIII.96). As proved in (Avron et al. 1978, Theorem 2.4) this is enough to claim that V_Γ is aslo infinitesimally bounded with respect to $-\Delta_M$. Therefore, by the *Kato-Rellich Theorem* it follows that the domain of self-adjointness of H_{BL} coincides with the domain of self-adjointness of the magnetic-Laplacian, which is $\mathcal{H}^2_{\mathrm{M}}(\mathbb{R}^2)$. ■

A.1.2 Band spectrum of H_{per}

We describe the spectral properties of the periodic Hamiltonian. The Bloch-Floquet transform maps unitarily H_{per} in $H^{\mathcal{Z}}_{\mathrm{per}} := \int^{\oplus}_{M^*_\Gamma} H_{\mathrm{per}}(k)\ d^2\underline{k}$. Then to have information about the spectrum of H_{per} we need to study the spectra of the family of Hamiltonians

$$H_{\mathrm{per}}(k) = \frac{1}{2}\left[-i\nabla_\theta + k - A_\Gamma(\theta)\right]^2 + V_\Gamma(\theta) = -\frac{1}{2}\Delta_\theta + T_1(k) + \frac{1}{2}T_2(k)$$

where $T_1(k) := i(A_\Gamma - \frac{k}{2}) \cdot \nabla_\theta$ and $T_2(k) := i(\nabla_\theta \cdot A_\Gamma) + |k|^2 + |A_\Gamma|^2 + 2V_\Gamma$ are operators acting on the Hilbert space $\mathcal{H}_{\mathrm{f}} := L^2(\mathbb{V}, d^2\theta)$ with $\mathbb{V} := \mathbb{R}^2/\Gamma$ (*Voronoi torus*).

Proof of Proposition 3.3.3.
- (i) The operator $-1/2\Delta_\theta$ on the Hilbert space $\mathcal{H}_{\mathrm{f}}$, is essentially selfadjoint on $C^\infty(\mathbb{V})$, has domain of self-adjointness $\mathcal{D} := \mathcal{H}^2(\mathbb{V})$ and its spectrum is pure point with $\{\mathrm{e}^{i\theta\cdot\gamma^*}\}_{\gamma^*\in\Gamma^*}$

a complete orthogonal system of eigenvectors. If Assumption 3.2.2 holds true, then $T_2(k)$ infinitesimally bounded with respect to $-{}^1\!/\!_2\Delta_\theta$, indeed $T - 2V_\Gamma$ is bounded and V_Γ is infinitesimally bounded (Reed and Simon 1978, Theorem XIII.97). With a Fourier estimate similar to those in the proof of Proposition 3.2.4 one can also show that $T_1(k)$ is infinitesimally bounded with respect to $-{}^1\!/\!_2\Delta_\theta$, hence the *Kato-Rellich Theorem* implies that $H_{\text{per}}(k)$ is essentially self-adjoint on $C^\infty(\mathbb{T}^2)$ and self-adjoint on the domain $\mathcal{D}$. Moreover since $-{}^1\!/\!_2\Delta_\theta$ is bounded below then also $H_{\text{per}}(k)$ is bounded below.
- (ii) For all ζ in the resolvent set of $-{}^1\!/\!_2\Delta_\theta$ the *resolvent operator* $r_0(\zeta) := (-{}^1\!/\!_2\Delta_\theta - \zeta\mathbb{1}_{\mathcal{H}_\text{f}})^{-1}$ is compact. Since $T_1(k) + \frac{1}{2}T_2(k)$ is a bounded perturbation of $-{}^1\!/\!_2\Delta_\theta$ it follows that $H_{\text{per}}(k)$ has compact resolvent (Reed and Simon 1978, Theorem XIII.68) and moreover it has a purely discrete spectrum with eigenvalues $\mathcal{E}_n(k) \to +\infty$ as $n \to +\infty$ (Reed and Simon 1978, Theorem XIII.64).
- (iii) The continuity of the function $\mathcal{E}_n(\cdot)$ follows from the *perturbation theory of discrete spectrum* (Reed and Simon 1978, Theorem XII.13). Indeed, as discussed in Remark 3.3.2, $H_{\text{per}}(\cdot)$ is an *analytic family (of type A) in the sense of Kato*. Finally, since $H_{\text{per}}(k - \gamma^*) = \tau(\gamma^*)H_{\text{per}}(k)\tau(\gamma^*)^{-1}$, with $\tau(\gamma^*)$ a unitary operator, then $\mathcal{E}_n(\cdot)$ are Γ^*-periodic. ■

A.1.3 The Landau Hamiltonian H_L

The *Landau Hamiltonian* is the operator

$$H_\text{L} := -\frac{1}{2}\Delta_M = \frac{1}{2}\left(K_1^2 + K_2^2\right) = \frac{1}{2}\left[\left(-i\frac{\partial}{\partial x_1} + \frac{1}{2}x_2\right)^2 + \left(-i\frac{\partial}{\partial x_2} - \frac{1}{2}x_1\right)^2\right] \tag{A.3}$$

where $K_j := -iD_j$, with $j = 1, 2$, are the **kinetic momenta**. The Landau Hamiltonian H_L is essentially self-adjoint on on $C_\text{c}^\infty(\mathbb{R}^2;\mathbb{C}) \subset L^2(\mathbb{R}^2)$ (Reed and Simon 1975, Theorem X.34) and its domain of self-adjointness is exactly the second magnetic-Sobolev space $\mathcal{H}^2_\text{M}(\mathbb{R}^2)$ defined in Section A.1.1. To describe the spectrum of H_L is helpful to introduce another pair of operators: $G_1 := -i\partial_{x_1} - \frac{1}{2}x_2$ and $G_2 := i\partial_{x_2} - \frac{1}{2}x_1$. The operators K_1, K_2, G_1, G_2 are all essentially self-adjoint on $C_\text{c}^\infty(\mathbb{R}^2;\mathbb{C})$ (they have *deficiency indices* equal to zero) and on this domain the following commutation relations hold true

$$[K_1; K_2] = [G_1; G_2] = i\mathbb{1}, \quad [G_j; K_i] = 0. \tag{A.4}$$

The last of (A.3) implies $[G_j; H_\text{L}] = 0$, hence the operators G_1 and G_2 are responsible for the degeneration of the spectral eigenspaces of H_L. It is a common lore to introduce the *annihilation operator* $\mathfrak{a} := {}^i\!/\!\sqrt{2}(K_2 - iK_1)$ (its adjoint $\mathfrak{a}^\dagger$ is called *creation operator*) and the *degeneration operator* $\mathfrak{g} := {}^i\!/\!\sqrt{2}(G_2 - iG_1)$. They fulfill the following (bosonic) commutation relation

$$[\mathfrak{a}; \mathfrak{a}^\dagger] = [\mathfrak{g}; \mathfrak{g}^\dagger] = \mathbb{1}, \quad [\mathfrak{g}; H_\text{L}] = [\mathfrak{g}^\dagger; H_\text{L}] = 0, \quad [\mathfrak{a}; H_\text{L}] = \mathfrak{a}, \quad [\mathfrak{a}^\dagger; H_\text{L}] = -\mathfrak{a}^\dagger. \tag{A.5}$$

The last two relations follow from the equality $H_\text{L} = \mathfrak{a}\mathfrak{a}^\dagger - {}^1\!/\!_2\mathbb{1} = \mathfrak{a}^\dagger\mathfrak{a} + {}^1\!/\!_2\mathbb{1}$. Define the *ground state* $\psi_0 \in L^2(\mathbb{R}^2)$ as the normalized solution of $\mathfrak{g}\psi_0 = 0 = \mathfrak{a}\psi_0$, i.e.

$\psi_0(x) = C\mathrm{e}^{-\frac{1}{4}|x|^2}$. The *generalized Hermite function* of order (n,m) is defined to be $\psi_{n,m} := \frac{1}{\sqrt{n!m!}} \left(\mathfrak{g}^\dagger\right)^m \left(\mathfrak{a}^\dagger\right)^n \psi_0$. We denote by $\mathcal{L} \subset L^2(\mathbb{R}^2)$ the set of the finite linear combinations of the vectors $\psi_{n,m}$ and we will call it the *Hermite domain*. Clearly $\mathcal{L} \subset \mathcal{S}(\mathbb{R}^2)$ (the Schwartz space).

Lemma A.1.1. *With the notation above:*

(i) *the set* $\{\psi_{n,m} \ : \ n,m = 0,1,2,\ldots\}$ *is a complete orthonormal basis for* $L^2(\mathbb{R}^2)$ *and so* $\mathcal{L}$ *is a dense domain;*

(ii) *the spectrum of* H_L *is pure point and is given by* $\{\lambda_n := (n + 1/2) \ : \ n = 0,1,2,\ldots\}$, *moreover* $H_L\ \psi_{n,m} = \lambda_n\ \psi_{n,m}$ *for every* $m = 0,1,2,\ldots$ *(degeneration index);*

(iii) H_L *is essentially self-adjoint on* $\mathcal{L}$ *and the closure of* $\mathcal{L}$ *with respect to the magnetic-Sobolev norm coincides with the magnetic-Sobolev space* $\mathcal{H}^2_{\mathbf{M}}(\mathbb{R}^2)$.

Proof. - (i) Let $\mathcal{W} : L^2(\mathbb{R}^2, d^2x) \to L^2(\mathbb{R}, du) \otimes L^2(\mathbb{R}, dv)$ be the unitary map which transforms the conjugate pairs (K_1, K_2) and (G_1, G_2) into the canonical pairs $(u, -i\partial_u)$ and $(v, -i\partial_v)$. The existence of such a unitary $\mathcal{W}$ is discussed in Appendix A.2. Obviously $\mathfrak{a} \mapsto \widetilde{\mathfrak{a}} = 1/\sqrt{2}(u + \partial_u)$, $\mathfrak{g} \mapsto \widetilde{\mathfrak{g}} = 1/\sqrt{2}(v + \partial_v)$. Moreover, $\widetilde{\psi}_0 := \mathcal{W}\psi_0$ is the solution of $\widetilde{\mathfrak{g}}\widetilde{\psi}_0 = \widetilde{\mathfrak{a}}\widetilde{\psi}_0 = 0$, namely $\widetilde{\psi}_0(u,v) = h_0(u) \otimes h_0(v)$ where $h_0(t) := \pi^{-\frac{1}{4}}\mathrm{e}^{-\frac{1}{2}t^2}$ is the 0-th Hermite function. Then $\widetilde{\psi}_{n,m}(u,v) := (\mathcal{W}\psi_{n,m})(u,v) = h_n(u) \otimes h_m(v)$ which shows that the functions $\widetilde{\psi}_{n,m}$ define an orthonormal basis for $L^2(\mathbb{R}, du) \otimes L^2(\mathbb{R}, dv)$ since the Hermite functions h_n are an orthonormal system for $L^2(\mathbb{R})$. The claim follows since $\mathcal{W}$ is a unitary map.
- (ii) Clearly $H_{\mathrm{L}}\psi_0 = (\mathfrak{a}^\dagger\mathfrak{a} + 1/2\mathbb{1})\psi_0 = 1/2\psi_0$ and from relations (A.5) it follows that $H_{\mathrm{L}}\psi_{n,m} = \frac{1}{\sqrt{n!m!}}\left(\mathfrak{g}^\dagger\right)^m H_{\mathrm{L}} \left(\mathfrak{a}^\dagger\right)^n \psi_0 = 1/2\psi_{n,m} + \frac{1}{\sqrt{n!m!}}\left(\mathfrak{g}^\dagger\right)^m (\mathfrak{a}^\dagger\mathfrak{a}) \left(\mathfrak{a}^\dagger\right)^n \psi_0 = \lambda_n\ \psi_{n,m}$. Then the generalized Hermite functions $\psi_{n,m}$ are a complete set of orthonormal eigenvectors for H_{L}. This proves that the spectrum of H_{L} is pure point.
- (iii) The operator H_{L} is essentially self-adjoint on $\mathcal{L}$ since the deficiency indices are both zero. This implies the last part of the claim. ■

Lemma A.1.2. *The operators* K_1, K_2, $\mathfrak{a}$ *and* $\mathfrak{a}^\dagger$ *are infinitesimally bounded with respect to* H_L.

Proof. Since $K_1 = 1/\sqrt{2}(\mathfrak{a} + \mathfrak{a}^\dagger)$ and $K_2 = 1/i\sqrt{2}(\mathfrak{a} - \mathfrak{a}^\dagger)$ it is enough to prove the claim for $\mathfrak{a}$ and $\mathfrak{a}^\dagger$. Let $\psi := \sum_{n,m=0}^{+\infty} c_{n,m}\ \psi_{n,m} \in \mathcal{H}^2_{\mathbf{M}}(\mathbb{R}^2)$. An easy computation shows that

$$\|\mathfrak{a}\psi\|^2_{L^2} = \sum_{n,m=0}^{+\infty} |c_{n,m}|^2\ n, \qquad \|\mathfrak{a}^\dagger\psi\|^2_{L^2} = \sum_{n,m=0}^{+\infty} |c_{n,m}|^2\ (n+1)\,.$$

Since $n \leqslant n+1 \leqslant 2\left(n + \frac{1}{2}\right) \leqslant a\left(n+\frac{1}{2}\right)^2 + \frac{1}{a}$ holds true for any $a > 0$ (arbitrarily small), then

$$\|\mathfrak{a}^\sharp\psi\|^2_{L^2} \leqslant a \sum_{n,m=0}^{+\infty} |c_{n,m}|^2 \left(n + \frac{1}{2}\right)^2 + b \sum_{n,m=0}^{+\infty} |c_{n,m}|^2 = a\|H_{\mathrm{L}}\psi\|^2_{L^2} + b\|\psi\|^2_{L^2}$$

with $b := \frac{1}{a} + \frac{1}{2}$ where $\mathfrak{a}^\sharp$ denotes either $\mathfrak{a}$ or $\mathfrak{a}^\dagger$. ■

A.2 Canonical transform for fast and slow variables

This section is devoted to the concrete realization of the *von Neumann unitary* $\mathcal{W}$ introduced (in abstract way) in Section 3.4.2. The unitary $\mathcal{W}$ maps the fast and slow variables, which satisfy canonical commutation relation, into a set of canonical Schrödinger operators. In Section A.2.1 we derive a general version of the transform $\mathcal{W}$ "by hand", as a composition of three sequential transforms. In Section A.2.2 we compute the integral kernel of $\mathcal{W}$.

A.2.1 The transform $\mathcal{W}$ built "by hand"

Let $\mathcal{H}_{\text{phy}} := L^2(\mathbb{R}^2, d^2r)$ be the initial Hilbert space, with $r := (r_1, r_2)$. Let $Q_r := (Q_{r_1}, Q_{r_2})$ where Q_{r_j} is the multiplication operator by r_j and $P_r := (P_{r_1}, P_{r_2})$ where $P_{r_1} := -i\hbar\partial_{r_j}$, with $j = 1, 2$. Consider the *fast* and *slow* operators

$$\text{(fast)} \begin{cases} K_1 := -\dfrac{\alpha}{2\beta}\, v \cdot Q_r - \dfrac{\alpha\beta}{\hbar}\, w^* \cdot P_r \\ K_2 := \quad \dfrac{\alpha}{2\beta}\, w \cdot Q_r - \dfrac{\alpha\beta}{\hbar}\, v^* \cdot P_r \end{cases} \qquad \text{(slow)} \begin{cases} G_1 := \dfrac{1}{2}\, v \cdot Q_r - \dfrac{\beta^2}{\hbar}\, w^* \cdot P_r \\ G_2 := \dfrac{1}{2}\, w \cdot Q_r + \dfrac{\beta^2}{\hbar}\, v^* \cdot P_r \end{cases} \tag{A.6}$$

with $\alpha, \beta \in \mathbb{C}$ and $v, w, v^*, w^* \in \mathbb{R}^2$ such that $v \cdot v^* = w \cdot w^* = 1$, $v^* \cdot w = v \cdot w^* = 0$ and $|v \wedge w| = \ell^2 > 0$.

Remark A.2.1. The choice $v = b^*$, $w = a^*$, $\alpha = \sqrt{\iota_q}$ and $\beta = \sqrt{\iota_q}\,\delta$ defines the operators (3.48), while the choice $v = v^* = (0, -1)$, $w = w^* = (-1, 0)$, $\alpha = \beta = 1$ defines the *kinetic momenta* and the related conjugate operators introduced in Section A.1.3. ♦◊

Observing that $[a \cdot Q_r + b \cdot P_r; c \cdot Q_r + d \cdot P_r] = i\hbar(a \cdot d - b \cdot c)\mathbb{1}_{\mathcal{H}}$ one deduce that the operators (A.6) verify the following *canonical commutation relations* (CCR)

$$[K_1, K_2] = i\alpha^2 \mathbb{1}_{\mathcal{H}}, \qquad [G_1, G_2] = i\beta^2 \mathbb{1}_{\mathcal{H}}, \qquad [K_i, G_j] = 0, \;\; i, j = 1, 2. \tag{A.7}$$

The *Stone-von Neumann uniqueness theorem* (Bratteli and Robinson 1997, Corollary 5.2.15) assures the existence of a unitary map $\mathcal{W}$ (*von Neumann unitary*)

$$\mathcal{W} : \mathcal{H}_{\text{phy}} \longrightarrow \mathcal{H}_{\text{w}} := \mathcal{H}_{\text{s}} \otimes \mathcal{H}_{\text{f}} := L^2(\mathbb{R}, dx_{\text{s}}) \otimes L^2(\mathbb{R}, dx_{\text{f}}) \tag{A.8}$$

such that

$$\mathcal{W} G_1 \mathcal{W}^{-1} := Q_{\text{s}} = \text{multiplication by } x_{\text{s}}, \qquad \mathcal{W} G_2 \mathcal{W}^{-1} := P_{\text{s}} = -i\beta^2 \frac{\partial}{\partial x_{\text{s}}} \tag{A.9}$$

$$\mathcal{W} K_1 \mathcal{W}^{-1} := Q_{\text{f}} = \text{multiplication by } x_{\text{f}}, \qquad \mathcal{W} K_2 \mathcal{W}^{-1} := P_{\text{f}} = -i\alpha^2 \frac{\partial}{\partial x_{\text{f}}}. \tag{A.10}$$

In other words, $(Q_{\text{s}}, P_{\text{s}})$ is a pair of operators which defines a Schrödinger representation on the Hilbert space $\mathcal{H}_{\text{s}} := L^2(\mathbb{R}, dx_{\text{s}})$ while the pair $(Q_{\text{f}}, P_{\text{f}})$ defines a Schrödinger representation on the Hilbert space $\mathcal{H}_{\rm f} := L^2(\mathbb{R}, dx_{\rm f})$. Our purpose is to give an explicit construction for $\mathcal{W}$. Firstly one considers the change of coordinates $(r_1, r_2) \mapsto (k_1 := \frac{v\cdot r}{\ell}, k_2 := \frac{w\cdot r}{\ell})$. The inverse transforms are defined by $r_1(k) = \frac{1}{\ell}(w_2 k_1 - v_2 k_2)$ and $r_2(k) = \frac{1}{\ell}(v_1 k_2 - w_1 k_1)$. The map $\mathcal{J} : \mathcal{H}_{\rm phy} \to L^2(\mathbb{R}^2, d^2k)$ defined by $(\mathcal{J}\psi)(k) := \psi(r(k))$ for any $\psi \in \mathcal{H}_{\rm phy}$ is unitary since the change of coordinates is invertible and isometric. Moreover $\mathcal{J} Q_{r_j} \mathcal{J}^{-1}$ acts on $L^2(\mathbb{R}^2, d^2k)$ as the multiplication by $r_j(k)$, while $\mathcal{J} P_{r_j} \mathcal{J}^{-1} = \frac{v_j}{\ell} P_{k_1} + \frac{w_j}{\ell} P_{k_2}$ where $P_{k_j} := -i\hbar\partial_{k_j}$, with $j = 1, 2$. Then

$$\mathcal{J} G_1 \mathcal{J}^{-1} := \frac{\ell}{2} Q_{k_1} - \frac{\beta^2}{\hbar}\frac{1}{\ell} P_{k_2}, \qquad \mathcal{J} G_2 \mathcal{J}^{-1} := \frac{\ell}{2} Q_{k_2} + \frac{\beta^2}{\hbar}\frac{1}{\ell} P_{k_1} \tag{A.11}$$

$$\mathcal{J} K_1 \mathcal{J}^{-1} := -\frac{\alpha}{\beta}\frac{\ell}{2} Q_{k_1} - \frac{\alpha\beta}{\hbar}\frac{1}{\ell} P_{k_2}, \qquad \mathcal{J} K_2 \mathcal{J}^{-1} := \frac{\alpha}{\beta}\frac{\ell}{2} Q_{k_2} - \frac{\alpha\beta}{\hbar}\frac{1}{\ell} P_{k_1}. \tag{A.12}$$

Let $\mathcal{F}_{2,\mu} : L^2(\mathbb{R}, dk_2) \to L^2(\mathbb{R}, d\zeta_2)$ be the k_2-*Fourier transfor* of weight μ, defined by $(\mathcal{F}_{2,\mu}\psi)(\zeta_2) := \sqrt{\frac{|\mu|}{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-i\mu\zeta_2 k_2} \psi(k_2)\, dk_2$ and let $\Pi_1 : L^2(\mathbb{R}, dk_1) \to L^2(\mathbb{R}, d\zeta_1)$ be the k_1-*parity operator* defined by $(\Pi_2\psi)(\zeta_1) := \psi(-\zeta_1)$ (namely by the change of coordinates $k_1 \mapsto \zeta_1$). Let $\mathbb{I}$ the unitary map which identifies $L^2(\mathbb{R}^2, d^2k)$ with $L^2(\mathbb{R}, dk_1) \otimes L^2(\mathbb{R}, dk_2)$. Let $Q_\zeta := (Q_{\zeta_1}, Q_{\zeta_2})$ whit Q_{ζ_j} the multiplication operator by ζ_j and $P_\zeta := (P_{\zeta_1}, P_{\zeta_2})$ where $P_{\zeta_1} := -i\hbar\partial_{\zeta_j}$, with $j = 1, 2$. One can check that

$$\mathcal{F}_{2,\mu} Q_{k_2} \mathcal{F}_{2,\mu}^{-1} = -\frac{1}{\mu\hbar} P_{\zeta_2}, \quad \mathcal{F}_{2,\mu} P_{k_2} \mathcal{F}_{2,\mu}^{-1} = \mu\hbar Q_{\zeta_2}, \quad \Pi_1 Q_{k_1} \Pi_1^{-1} = -Q_{\zeta_1}, \quad \Pi_1 P_{k_1} \Pi_1^{-1} = -P_{\zeta_1}. \tag{A.13}$$

Fix $\mu := -\frac{\ell^2}{2\beta^2}$, then the unitary map $\mathcal{T} := (\Pi_1 \otimes \mathcal{F}_{2,\mu}) \circ \mathbb{I} \circ \mathcal{J} : \mathcal{H} \to L^2(\mathbb{R}, d\zeta_1) \otimes L^2(\mathbb{R}, d\zeta_2)$ acts on the operators (A.6) in the following way

$$\mathcal{T} G_1 \mathcal{T}^{-1} := -\frac{\ell}{2}\left(Q_{\zeta_1} - Q_{\zeta_2}\right), \qquad \mathcal{T} G_2 \mathcal{T}^{-1} := -\frac{1}{\ell}\frac{\beta^2}{\hbar}\left(P_{\zeta_1} - P_{\zeta_2}\right) \tag{A.14}$$

$$\mathcal{T} K_1 \mathcal{T}^{-1} := \frac{\alpha}{\beta}\frac{\ell}{2}\left(Q_{\zeta_1} + Q_{\zeta_2}\right), \qquad \mathcal{T} K_2 \mathcal{T}^{-1} := \frac{1}{\ell}\frac{\alpha\beta}{\hbar}\left(P_{\zeta_1} + P_{\zeta_2}\right). \tag{A.15}$$

Now we can consider the change of coordinates $(\zeta_1, \zeta_2) \mapsto (x_{\rm s}, x_{\rm f})$ defined by

$$\begin{cases} x_{\rm s} = -\dfrac{\ell}{2}(\zeta_1 - \zeta_2) \\ x_{\rm f} = \dfrac{\alpha}{\beta}\dfrac{\ell}{2}(\zeta_1 + \zeta_2) \end{cases} \qquad \begin{cases} \zeta_1 = -\dfrac{1}{\ell}\left(x_{\rm s} - \dfrac{\beta}{\alpha}\, x_{\rm f}\right) \\ \zeta_2 = \dfrac{1}{\ell}\left(x_{\rm s} + \dfrac{\beta}{\alpha}\, x_{\rm f}\right) \end{cases}$$

The jacobian of this transformation is $|\partial(\zeta_1, \zeta_2)/\partial(x_{\rm s}, x_{\rm f})| = \frac{2}{\ell^2}\left|\frac{\beta}{\alpha}\right| =: C$, then the map $(\mathcal{R}\psi)(x_{\rm s}, x_{\rm f}) := \sqrt{C}\, \psi(\zeta(x_{\rm s}, x_{\rm f}))$ defines a unitary map $\mathcal{R} : L^2(\mathbb{R}^2, d^2\zeta) \to L^2(\mathbb{R}^2, dx_{\rm s}\, dx_{\rm f})$. With a direct computation one can check that $\mathcal{R} Q_{\zeta_j} \mathcal{R}^{-1}$ acts on $L^2(\mathbb{R}^2, dx_{\rm s}\, dx_{\rm f})$ as the multiplication by $\zeta_j(x_{\rm s}, x_{\rm f})$, while $\mathcal{R} P_{\zeta_j} \mathcal{R}^{-1} = \frac{\ell\hbar}{2\alpha\beta}\left(\frac{(-1)^j\alpha}{\beta} P_{\rm s} + P_{\rm f}\right)$, with $j = 1, 2$. This shows that the unitary map $\mathcal{W} := \mathbb{I} \circ \mathcal{R} \circ \mathbb{I}^{-1} \circ \mathcal{L} := \mathbb{I} \circ \mathcal{R} \circ \mathbb{I}^{-1} \circ (\Pi_1 \otimes \mathcal{F}_{2,\mu}) \circ \mathbb{I} \circ \mathcal{J}$ is the von Neumann unitary that verifies the relations (A.9) and (A.10).

A.2.2 The integral kernel of $\mathcal{W}$

The unitary operators $\mathcal{J} : \mathcal{H}_{\text{phy}} \to L^2(\mathbb{R}^2, d^2k)$ and $\mathcal{R} : L^2(\mathbb{R}^2, d^2\zeta) \to L^2(\mathbb{R}^2, dx_{\text{s}}\ dx_{\text{f}})$ related to the change of coordinates $(r_1, r_2) \mapsto (k_1, k_2)$ and $(\zeta_1, \zeta_2) \mapsto (x_{\text{s}}, x_{\text{f}})$ can be written as an integral operators

$$(\mathcal{J}\psi)(k) = \int_{\mathbb{R}^2} J(r;k)\ \psi(r)\ d^2r, \qquad (\mathcal{R}\psi)(x_{\text{s}}, x_{\text{f}}) = \int_{\mathbb{R}^2} R(\zeta; x_{\text{s}}, x_{\text{f}})\ \psi(\zeta)\ d^2\zeta$$

with distributional integral kernels

$$J(r_1, r_2; k_1, k_2) := \delta\left(r_1 - \frac{1}{\ell}(w_2 k_1 - v_2 k_2)\right)\ \delta\left(r_2 + \frac{1}{\ell}(w_1 k_1 - v_1 k_2)\right)$$
$$R(\zeta_1, \zeta_2; x_{\text{s}}, x_{\text{f}}) := \sqrt{C}\ \delta\left(\zeta_1 + \frac{1}{\ell}\left(x_{\text{s}} - \frac{\beta}{\alpha}\ x_{\text{f}}\right)\right)\ \delta\left(\zeta_2 - \frac{1}{\ell}\left(x_{\text{s}} + \frac{\beta}{\alpha}\ x_{\text{f}}\right)\right)$$

with $C = \frac{2}{\ell^2}\left|\frac{\beta}{\alpha}\right|$. The k_1-parity operator Π_1 can be written as an integral operator with the distributional kernel $\delta(r_1 + k_1)$ while the integral kernel of the k_2-Fourier transform $\mathcal{F}_{2,\mu}$, with $\mu := -\frac{\ell^2}{2\beta^2}$, is $\frac{\ell}{2|\beta|\sqrt{\pi}} \mathrm{e}^{i\frac{\ell^2}{2\beta^2}\zeta_2 k_2}$. Then the unitary map $\mathbb{I}^{-1} \circ (\Pi_1 \otimes \mathcal{F}_{2,\mu}) \circ \mathbb{I} : L^2(\mathbb{R}^2, d^2k) \to L^2(\mathbb{R}^2, d^2\zeta)$ as the integral distributional kernel

$$L(k_1, k_2; \zeta_1, \zeta_2) := \delta(\zeta_1 + k_1)\ \frac{\ell}{2|\beta|\sqrt{\pi}} \mathrm{e}^{i\frac{\ell^2}{2\beta^2}\zeta_2 k_2}.$$

Summarizing the total transform $\mathcal{W} : L^2(\mathbb{R}^2, d^2r) \to L^2(\mathbb{R}^2, dx_{\text{s}}\ dx_{\text{f}})$ (up to the obvious identification $\mathbb{I}$) can be expressed as an integral operator

$$(\mathcal{W}\psi)(x_{\text{s}}, x_{\text{f}}) = \int_{\mathbb{R}^2} W(r; x_{\text{s}}, x_{\text{f}})\ \psi(r)\ d^2r$$

with a (total) integral distributional kernel

$$W(r_1, r_2; x_{\text{s}}, x_{\text{f}}) := \frac{\ell}{\sqrt{2\pi|\alpha\beta|}}\ \delta\left(v \cdot r - \left(x_{\text{s}} - \frac{\beta}{\alpha}\ x_{\text{f}}\right)\right)\ \mathrm{e}^{i\frac{w \cdot r}{2\beta^2}\left(x_{\text{s}} + \frac{\beta}{\alpha}\ x_{\text{f}}\right)}.$$

Appendix B

Basic notions on operator algebras theory

For a comprehensive exposition of the theory of C^*-algebras, von Neumann algebras and related topics, we refer to (Bratteli and Robinson 1987, Dixmier 1982, Dixmier 1981).

B.1 C^*-algebras, von Neumann algebras, states

A *C^*-algebra* $\mathfrak{A}$ is a complex algebra, closed with respect to a norm $\|\cdot\|$ and endowed with an *involution* (or *adjoint*) $*$. The topological structure and the $*$-structure are related by the *C^*-property* $\|AA^*\| = \|A\|^2$ for any $A \in \mathfrak{A}$. If $\mathfrak{A}$ as a *unit* $\mathbb{1}$ (i.e. $\mathbb{1}A = A = A\mathbb{1}$ for any $A \in \mathfrak{A}$), then $\mathfrak{A}$ is called *unital*. If $AB = BA$ for any $A, B \in \mathfrak{A}$, then $\mathfrak{A}$ is said *commutative*.

A *representation* π of the C^*-algebra $\mathfrak{A}$ is a $*$-morphism $\pi : \mathfrak{A} \to \mathscr{B}(\mathcal{H})$, with $\mathscr{B}(\mathcal{H})$ the C^*-algebra of bounded operators on the Hilbert space $\mathcal{H}$ (endowed with the operator norm and the adjoint operation). The representation is called *faithful* if the map π is injective.

A *state* ω for a unital C^*-algebra $\mathfrak{A}$ is a linear map $\omega : \mathfrak{A} \to \mathbb{C}$ which is *normalized* $\omega(\mathbb{1}) = 1$ and *positive* $\omega(AA^*) \geqslant 0$ for any $A \in \mathfrak{A}$. The state ω is *faithful* if $\omega(AA^*) = 0$ if and only if $A = 0$. A *trace* is a state such that $\omega(AB) = \omega(BA)$ for any $A, B \in \mathfrak{A}$.

Let $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ be a C^*-algebra of bounded operators on the Hilbert space $\mathcal{H}$. The *commutant* of $\mathfrak{A}$ is defined by

$$\mathfrak{A}' := \{T \in \mathscr{B}(\mathcal{H}) \ : \ [T; A] = 0, \quad \forall\, A \in \mathfrak{A}\}.$$

The *von Neuamma* algebra $\mathfrak{M}(\mathfrak{A})$ associated to $\mathfrak{A}$ is the *double commutant* (i.e. the commutant of the commutant) of $\mathfrak{A}$, namely $\mathfrak{M}(\mathfrak{A}) = \mathfrak{A}''$ where $\mathfrak{A}'' := (\mathfrak{A}')'$. The algebra $\mathfrak{M}(\mathfrak{A})$ coincides with the strong (equiv. weak) closure of $\mathfrak{A}$ (Bratteli and Robinson 1987, Bicommutant Theorem 2.4.11). A vector $\psi \in \mathcal{H}$ is *cyclic* for $\mathfrak{M}(\mathfrak{A})$ if

$$[\mathfrak{M}(\mathfrak{A})\psi] := \{\phi \in \mathcal{H} \ : \ \phi = A\psi, \quad A \in \mathfrak{M}(\mathfrak{A})\}$$

is dense in $\mathcal{H}$. A vector $\psi \in \mathcal{H}$ is *separating* for $\mathfrak{M}(\mathfrak{A})$ if $A\psi = 0$ with $A \in \mathfrak{M}(\mathfrak{A})$ implies $A = 0$. A von Neumann algebra with a cyclic and separating vector is called *standard*.

A commutative C^*-algebra $\mathfrak{S} \subseteq \mathscr{B}(\mathcal{H})$ is said *maximal commutative* if there is no other commutative C^*-algebra in $\mathscr{B}(\mathcal{H})$ which contains properly $\mathfrak{S}$. Clearly the condition of maximal commutativity implies the existence of a unit.

B.2 Gel'fand theory, joint spectrum and basic measures

Let $\mathfrak{A}$ be a unital C^*-algebra and $\mathfrak{A}^\times$ the *group of the invertible elements* of $\mathfrak{A}$. The *algebraic spectrum* of $A \in \mathfrak{A}$ is defined to be $\sigma_{\mathfrak{A}}(A) := \{\lambda \in \mathbb{C} \ : \ (A - \lambda \mathbb{1}) \notin \mathfrak{A}^\times\}$. If $\mathfrak{A}_0$ is a non unital C^*-algebra and $\imath : \mathfrak{A}_0 \hookrightarrow \mathfrak{A}$ is the canonical embedding of $\mathfrak{A}_0$ in the unital C^*-algebra $\mathfrak{A}$ (Bratteli and Robinson 1987, Proposition 2.1.5) then one defines $\sigma_{\mathfrak{A}_0}(A) := \sigma_{\mathfrak{A}}(\imath(A))$ for all $A \in \mathfrak{A}_0$. This shows that the notion of spectrum is strongly linked to the existence of the unit. If $\mathfrak{A}$ is unital and $C^*(A) \subset \mathfrak{A}$ is the unital C^*-subalgebra generated algebraically by A, its adjoint $A^\dagger$ and $\mathbb{1}$ (=: A^0 for definition) then $\sigma_{\mathfrak{A}}(A) = \sigma_{C^*(A)}(A)$ (Bratteli and Robinson 1987, Proposition 2.2.7). As a consequence we have that if $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ is a concrete C^*-algebra of operators on the Hilbert space $\mathcal{H}$ and $A \in \mathfrak{A}$ then the algebraic spectrum $\sigma_{\mathfrak{A}}(A)$ agrees with the *Hilbertian spectrum* $\sigma(A) := \{\lambda \in \mathbb{C} \ : \ (A - \lambda \mathbb{1}) \notin \mathrm{GL}(\mathcal{H})\}$ where $\mathrm{GL}(\mathcal{H}) := \mathscr{B}(\mathcal{H})^\times$ is the group of the invertible bounded linear operators on the Hilbert space $\mathcal{H}$.

Let $\mathfrak{S}$ by a commutative C^*-algebra. A *character* of $\mathfrak{S}$ is a nonzero homomorphism $x : \mathfrak{S} \to \mathbb{C}$ (also called *pure state*). The *Gel'fand spectrum* of $\mathfrak{S}$, denoted by $X(\mathfrak{S})$ or simply by X, is the set of all characters of $\mathfrak{S}$. The space X, endowed with the $*$-weak topology (topology of the pointwise convergence on $\mathfrak{S}$) becomes a topological Hausdorff space, which is compact if $\mathfrak{S}$ is unital and only locally compact otherwise (Bratteli and Robinson 1987, Theorem 2.1.11A). If $\mathfrak{S}$ is *separable* (namely it is generated algebraically by a countable family of commuting elements) then the $*$-weak topology in X is metrizable (Brézis 1987, Theorem III.25) and if, in addition, $\mathfrak{S}$ is also unital then X is compact and metrizable which implies (Choquet 1966, Proposition 18.3 and Theorem 20.9) that X is second-countable (has a countable basis of open sets), separable (has a countable everywhere dense subset) and complete. Summarizing, the Gel'fand spectrum of a commutative separable unital C^*-algebra has the structure of a *Polish space* (separable complete metric space).

The *Gel'fand-Naĭmark Theorem* (Bratteli and Robinson 1987, Section 2.3.5), (Gracia-Bondía et al. 2001, Section 1.2) states that there is a canonical isomorphism between any commutative unital C^*-algebra $\mathfrak{S}$ and the commutative C^*-algebra $C(X)$ of the continuous complex valued functions on its spectrum endowed with the norm of the uniform convergence. The *Gel'fand isomorphism* $C(X) \ni f \stackrel{\mathscr{G}}{\mapsto} A_f \in \mathfrak{S}$ maps any continuous f in the unique element A_f which satisfies the relation $f(x) = x(A_f)$ for all $x \in X$. Then we can use the continuous functions on X to "label" the elements of $\mathfrak{S}$. If $\mathfrak{S}_0$ is a non-unital commutative C^*-algebra then the Gel'fand-Naĭmark Theorem proves the isomorphism between $\mathfrak{S}_0$ and the commutative C^*-algebra $C_0(X_0)$ of the continuous complex valued functions vanishing at infinity on the locally compact space X_0 which is the spectrum of $\mathfrak{S}_0$. If $\mathfrak{S}_0 \subset \mathscr{B}(\mathcal{H})$ we define the *multiplier algebra* (or *idealizer*) of $\mathfrak{S}_0$ to be $\mathfrak{S} := \{B \in \mathscr{B}(\mathcal{H}) \ : \ BA, AB \in \mathfrak{S}_0 \quad \forall A \in \mathfrak{S}_0\}$ (Gracia-Bondía et al. 2001, Definition 1.8 and Lemma 1.9). Obviously $\mathfrak{S}$ is a unital C^*-algebra and the commutativity of $\mathfrak{S}_0$ implies the commutativity of $\mathfrak{S}$. Moreover $\mathfrak{S}$ contains $\mathfrak{S}_0$ as an *essential ideal*. The Gel'fand spectrum X of $\mathfrak{S}$ corresponds to the *Stone-Čech compactification* of the spectrum X_0. Since $C(X) \simeq C_{\rm b}(X_0)$, the Gel'fand isomorphism asserts that the multiplier algebra $\mathfrak{S}$ can be described as the unital commutative C^*-algebra of bounded continuous functions on the locally compact space X_0 (Gracia-Bondía et al. 2001, Proposition 1.10). For every $A_f \in \mathfrak{S}$ one has that $\sigma_{\mathfrak{S}}(A_f) = \{f(x) \ : \ x \in X\}$ (Hörmander 1990, Theorem 3.1.6) then A_f is invertible if and only if $0 < |f(x)| \leqslant \|A_f\|_{\mathfrak{S}}$ for all $x \in X$.

We often consider the relevant case of a *finitely generated* unital commutative C^*-algebra, i.e. of a $\mathfrak{S}$ algebraically generated by a finite family $\{A_1, \dots, A_N\}$ of commuting normal elements, their adjoints and $\mathbb{1}$ (=: A_j^0 by definition).

Let $f_1, \dots, f_N$ be the continuous functions which label the elements of the generating system. The map $X \ni x \overset{\varpi}{\mapsto} (f_1(x), \dots, f_N(x)) \in \mathbb{C}^N$ is a homeomorphism of the Gel'fand spectrum X on a compact subset of $\mathbb{C}^N$ called the *joint spectrum* of the generating system $\{A_1, \dots, A_N\}$ (Hörmander 1990, Theorem 3.1.15). Then, when $\mathfrak{S}$ is finitely generated, we can identify the Gel'fand spectrum X with its homeomorphic image $\varpi(X)$ (the joint spectrum) which is a compact, generally proper, subset of $\sigma_{\mathfrak{S}}(A_1) \times \dots \times \sigma_{\mathfrak{S}}(A_N)$. When $\{A_1, \dots, A_N\} \subset \mathscr{B}(\mathcal{H})$ a necessary and sufficient condition for $\lambda := (\lambda_1, \dots, \lambda_N)$ to be in $\varpi(X)$ is that there exists a sequence of normalized vectors $\{\psi_n\}_{n\in\mathbb{N}}$ such that $\|(A_j - \lambda_j)\psi_n\| \to 0$ if $n \to \infty$ for all $j = 1, \dots, N$ (Samoilenko 1991, Proposition 2).

REMARK B.2.1 (*Dual group*)**.** The Gel'fand theory has an interesting application to abelian locally compact groups $\mathbb{G}$. Usually the *dual group* (or character group) $\widehat{\mathbb{G}}$ is defined to be the set of all continuous characters of $\mathbb{G}$, namely the set of all the continuous homomorphism of $\mathbb{G}$ into the group $\mathbb{S}^1 := \{z \in \mathbb{C} \ : \ |z| = 1\}$. However, to endow $\widehat{\mathbb{G}}$ with a natural topology it is useful to give an equivalent definition of dual group. Since $\mathbb{G}$ is locally compact and abelian there exists a unique (up to a multiplicative constant) invariant *Haar measure* on $\mathbb{G}$ denoted by dg. The space $L^1(\mathbb{G})$ becomes a commutative Banach $*$-algebra, if multiplication is defined by convolution; it is called the *group algebra* of $\mathbb{G}$. If $\mathbb{G}$ is discrete then $L^1(\mathbb{G})$ is unital otherwise $L^1(\mathbb{G})$ has always an *approximate unit* (Rudin 1962, Theorems 1.1.7 and 1.1.8). Every $\chi \in \widehat{\mathbb{G}}$ defines a linear multiplicative functional $\widehat{\chi}$ on $L^1(\mathbb{G})$ by $\widehat{\chi}(f) := \int_{\mathbb{G}} f(g)\chi(-g)\, d\mu(g)$ for all $f \in L^1(\mathbb{G})$ (the Fourier transform). This map defines a one to one correspondence between $\widehat{\mathbb{G}}$ and the Gel'fand spectrum of the algebra $L^1(\mathbb{G})$ (Rudin 1962, Theorem 1.2.2). This enables us to consider $\widehat{\mathbb{G}}$ as the Gel'fand spectrum of $L^1(\mathbb{G})$. When $\widehat{\mathbb{G}}$ is endowed with the $*$-weak topology with respect to $L^1(\mathbb{G})$ then it becomes a Hausdorff locally compact space. Moreover $\widehat{\mathbb{G}}$ is compact (resp. discrete) if $\mathbb{G}$ is discrete (resp. compact) (Rudin 1962, Theorem 1.2.5). ♦◊

Let X be a compact Polish space and $\mathcal{B}(X)$ the Borel σ-algebra generated by the topology of X. The pair $\{X, \mathcal{B}(X)\}$ is called *standard Borel space*. A mapping $\mu : \mathcal{B}(X) \to [0, +\infty]$ such that: $\mu(\emptyset) = 0$, $\mu(X) \leqslant \infty$ which is additive with respect to the union of countable families of pairwise disjoint subsets of X is called a *(finite) Borel measure*. If $\mu(X) = 1$ then we will said that μ is a *probability* Borel measure. Any Borel measure on a standard Borel space $\{X, \mathcal{B}(X)\}$ is *regular*, i.e. for all $Y \in \mathcal{B}(X)$ one has that $\mu(Y) = \sup\{\mu(K) \ : \ K \subset Y,\ K \text{ compact}\} = \inf\{\mu(O) \ : \ Y \subset O,\ O \text{ open}\}$.

Let N be the union of all the open sets $O_\alpha \subset X$ such that $\mu(O_\alpha) = 0$. The closed set $X \setminus N$ is called the *support* of μ. If μ is a regular Borel measure then $\mu(N) = 0$ and μ is concentrated on its support.

Let $\mathfrak{S}$ be a unital commutative C^*-algebra acting on the separable Hilbert space $\mathcal{H}$ with Gel'fand spectrum X. For all pairs $\psi, \varphi \in \mathcal{H}$ the mapping $C(X) \ni f \mapsto (\psi; A_f\varphi)_{\mathcal{H}} \in \mathbb{C}$ is a continuous linear functional on $C(X)$; hence the *Riesz-Markov Theorem* (Rudin 1987, Theorem 2.14) implies the existence of a unique regular (complex) Borel measure $\mu_{\psi,\varphi}$, with finite total variation, such that $(\psi; A_f\varphi)_{\mathcal{H}} = \int_X f(x)\, d\mu_{\psi,\varphi}(x)$ for all $f \in C(X)$. We will refer to $\mu_{\psi,\varphi}$ as a *spectral measure*. The union of the supports of the (positive) spectral measures $\mu_{\psi,\psi}$ is dense, namely for every open set $O \subset X$ there exists a $\psi \in \mathcal{H}$ such that $\mu_{\psi,\psi}(O) > 0$. A positive measure μ on X is said to be *basic* for the C^*-algebra $\mathfrak{S}$ if: for every $Y \subset X$, Y is locally μ-negligible if and only if Y is locally $\mu_{\psi,\psi}$-negligible for any $\psi \in \mathcal{H}$. From the definition it follows that: (i) if there exists a basic measure μ on X, then every other basic measure is *equivalent* (has the same null sets) to μ; (ii) for all $\psi, \varphi \in \mathcal{H}$ the spectral measure $\mu_{\psi,\varphi}$ is *absolutely continuous* with respect to μ, and there exists a unique element $h_{\psi,\varphi} \in L^1(X)$ (the *Radon-Nikodym derivative*) such that $\mu_{\psi,\varphi} = h_{\psi,\varphi}\mu$; (iii) since the union of the supports of the measures $\mu_{\psi,\psi}$ is dense in X, then the support of a basic measure μ is the whole X (Dixmier 1981, Chapter 7, Part I). The existence of a basic measure for a commutative C^*-algebra $\mathfrak{S} \subset \mathscr{B}(\mathcal{H})$ is a general fact. Indeed the existence of a basic measure is equivalent to the existence of a cyclic vector ϕ for the commutant $\mathfrak{S}'$ and the basic measure can be chosen to be the spectral measure $\mu_{\phi,\phi}$ (Dixmier 1981, Chapter 7, Proposition 3, Part I). Since a vector ϕ is cyclic for $\mathfrak{S}'$ if and only if it is separating for the commutative von Neumann algebra $\mathfrak{S}'' \supset \mathfrak{S}$, and since any commutative von Neumann algebra of operators on a separable Hilbert space has a separating vector, it follows that any commutative unital C^*-algebra $\mathfrak{S}$ of operators which acts on a separable Hilbert space has a basic measure carried on its spectrum (Dixmier 1981, Chapter 7, Propositions 4, Part I).

B.3 Direct integral of Hilbert spaces

General references about the notion of a direct integral of Hilbert spaces can be found in (Dixmier 1981, Part II, Chapters 1-5) or in (Maurin 1968, Chapter I, Section 6). In the following we assume that the pair $\{X, \mathcal{B}(X)\}$ is a standard Borel space and μ a (regular) Borel measure on X. For every $x \in X$ let $\mathcal{H}(x)$ be a Hilbert space with scalar product $(\ ;\ )_x$. The set $\mathfrak{F} := \prod_{x\in X} \mathcal{H}(x)$ (Cartesian product) is called a *field of Hilbert spaces* over X. A *vector field* $\varphi(\cdot)$ is an element of $\mathfrak{F}$, namely a map $X \ni x \mapsto \varphi(x) \in \mathcal{H}(x)$. A countable family $\{\xi_j(\cdot)\ :\ j \in \mathbb{N}\}$ of vector fields is called a *fundamental family of measurable vector fields* if:

a) for all $i, j \in \mathbb{N}$ the functions $X \ni x \mapsto (\xi_i(x); \xi_j(x))_x \in \mathbb{C}$ are measurable;

b) for each $x \in X$ the set $\{\xi_j(x)\ :\ j \in \mathbb{N}\}$ spans the space $\mathcal{H}(x)$.

The field $\mathfrak{F}$ has a *measurable structure* if it has a fundamental family of measurable vector fields. A vector field $\varphi(\cdot) \in \mathfrak{F}$ is said to be *measurable* if all the functions $X \ni x \mapsto$

$(\xi_j(x); \varphi(x))_x \in \mathbb{C}$ are measurable for all $j \in \mathbb{N}$. The set of all measurable vector fields is a linear subspace of $\mathfrak{F}$. By the Gram-Schmidt orthonormalization we can always build a fundamental family of orthonormal measurable fields (see (Dixmier 1981, Propositions 1 and 4, Part II, Chapter 1). Such a family is called a *measurable field of orthonormal frames*. Two fields are said to be equivalent if they are equal μ-almost everywhere on X. The *direct integral* $\mathfrak{H}$ of the Hilbert spaces $\mathcal{H}(x)$ (subordinate to the measurable structure of $\mathfrak{F}$), is the Hilbert space of the equivalence classes of measurable vector fields $\varphi(\cdot) \in \mathfrak{F}$ satisfying

$$\|\varphi(\cdot)\|^2_{\mathfrak{H}} := \int_X \|\varphi(x)\|^2_x \, d\mu(x) < \infty. \tag{B.1}$$

The scalar product on $\mathfrak{H}$ is defined by

$$\langle \varphi_1(\cdot); \varphi_2(\cdot)\rangle_{\mathfrak{H}} := \int_X (\varphi_1(x); \varphi_2(x))_x \, d\mu(x) < \infty. \tag{B.2}$$

The Hilbert space $\mathfrak{H}$ is often denoted by the symbol $\int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$. It is separable if X is separable.

Let ν be a positive measure equivalent to μ. The Radon-Nikodym theorem ensures the existence of a positive $\rho \in L^1(X, \mu)$ with $\frac{1}{\rho} \in L^1(X, \nu)$ such that $\nu = \rho\mu$. Let $\mathfrak{H}$ be the direct integral with respect to μ, $\mathfrak{K}$ the direct integral with respect to ν and $\varphi(\cdot) \in \mathfrak{H}$. The mapping $\mathfrak{H} \in \varphi(\cdot) \mapsto \varphi'(\cdot) \in \mathfrak{K}$ defined by $\varphi'(x) = \frac{1}{\sqrt{\rho(x)}}\varphi(x)$ for all $x \in X$ is an unitary map of $\mathfrak{H}$ onto $\mathfrak{K}$ and for fixed μ and ν. This isomorphism does not depend on the choice of the representative for ρ and it is called the *canonical rescaling isomorphism*.

A *(bounded) operator field* $A(\cdot)$ is a map $X \ni x \mapsto A(x) \in \mathscr{B}(\mathcal{H}(x))$. It is called measurable if the function $X \ni x \mapsto (\xi_i(x); A(x)\xi_j(x))_x \in \mathbb{C}$ is measurable for all $i, j \in \mathbb{N}$. A measurable operator field is called a *decomposable operator* in the Hilbert space $\mathfrak{H}$. Let $f \in L^\infty(X)$ (with respect to the measure μ); then the map $X \ni x \mapsto M_f(x) := f(x)\mathbb{1}_x \in \mathscr{B}(\mathcal{H}(x))$ (with $\mathbb{1}_x$ the identity in $\mathcal{H}(x)$) defines a simple example of decomposable operator called *diagonal operator*. When $f \in C(X)$, the diagonal operator $M_f(\cdot)$ is called a *continuously diagonal operator*. Denote by $C(\mathfrak{H})$ (resp. $L^\infty(\mathfrak{H})$) the set of the continuously diagonal operators (resp. the set of diagonal operators) on $\mathfrak{H}$. Suppose that $\mathcal{H}(x) \neq 0$ μ-almost everywhere on X, then the following facts hold true (Dixmier 1981, Part II, Chapter 2, Section 4): (i) $L^\infty(\mathfrak{H})$ is a commutative von Neumann algebra and the mapping $L^\infty(X) \ni f \mapsto M_f(\cdot) \in L^\infty(\mathfrak{H})$ is a (canonical) isomorphism of von Neumann algebras; (ii) the commutant $L^\infty(\mathfrak{H})'$ is the von Neumann algebra of decomposable operators on $\mathfrak{H}$; (iii) the mapping $C(X) \ni f \mapsto M_f(\cdot) \in C(\mathfrak{H})$ is a (canonical) homomorphism of C^*-algebras which becomes an isomorphism if the support of μ coincides with X; in this case X is the Gel'fand spectrum of $C(\mathfrak{H})$ and μ is a basic measure.

B.4 Periodic cyclic cohomology for the NCT-algebra

Let $(\mathfrak{A}^\infty_\theta)^{\otimes k} := \mathfrak{A}^\infty_\theta \otimes \ldots \otimes \mathfrak{A}^\infty_\theta$ (k-times) and denote by $C^k(\mathfrak{A}^\infty_\theta) := \mathrm{Hom}((\mathfrak{A}^\infty_\theta)^{\otimes(k+1)}; \mathbb{C})$ the space of $(k+1)$-linear functionals on $\mathfrak{A}^\infty_\theta$ with value in $\mathbb{C}$. The elements of $C^k(\mathfrak{A}^\infty_\theta)$

are called *Hochschild k-cochains*. The *Hochschild differential* $b : C^k(\mathfrak{A}_\theta^\infty) \to C^{k+1}(\mathfrak{A}_\theta^\infty)$ is defined by

$$(b\varphi)(\mathfrak{a}_0, \dots, \mathfrak{a}_k, \mathfrak{a}_{k+1}) := \sum_{j=0}^{k} (-1)^j \varphi(\mathfrak{a}_0, \dots, \mathfrak{a}_j\mathfrak{a}_{j+1}, \dots \mathfrak{a}_{k+1}) + (-1)^{k+1}\varphi(\mathfrak{a}_{k+1}\mathfrak{a}_0, \dots \mathfrak{a}_k).$$

One can check that $b^2 = 0$. The cohomology of the complex $(C^\bullet(\mathfrak{A}_\theta^\infty), b)$ is called the *Hochschild cohomology* of $\mathfrak{A}_\theta^\infty$ and will be denoted by $H^\bullet(\mathfrak{A}_\theta^\infty)$. A *Hochschild k-cocycle* is a k-cochain $\varphi \in C^k(\mathfrak{A}_\theta^\infty)$ such that $b\varphi = 0$. A *Hochschild k-coboundary* is a k-cochain $\varphi \in C^k(\mathfrak{A}_\theta^\infty)$ such that $\varphi = b\varphi'$ for some $\varphi' \in C^{k-1}(\mathfrak{A}_\theta^\infty)$.

A k-cochain $\varphi \in C^k(\mathfrak{A}_\theta^\infty)$ is called *cyclic* if $\varphi(\mathfrak{a}_k, \mathfrak{a}_0, \dots, \mathfrak{a}_{k-1}) = (-1)^k\varphi(\mathfrak{a}_0, \dots, \mathfrak{a}_k)$. Cyclic cochain form a subcomplex $(C_\lambda^\bullet(\mathfrak{A}_\theta^\infty), b) \subset (C^\bullet(\mathfrak{A}_\theta^\infty), b)$ of the Hochschild compex called the *Connes complex*. Its cohomology is called *cyclic cohomology* of $\mathfrak{A}_\theta^\infty$ and will be denoted by $CH^\bullet(\mathfrak{A}_\theta^\infty)$. A cocycle for the cyclic cohomology is called a *cyclic cocycle*.

An important tool in the cyclic cohomology is the *periodic operator* which is a map of degree $+2$ between cyclic cocycles which is defined for all $\varphi \in C^k(\mathfrak{A}_\theta^\infty)$ by

$$\begin{aligned}(S\varphi)(\mathfrak{a}_0, \dots, \mathfrak{a}_{k+2}) := &- \frac{1}{(k+2)(k+1)} \sum_{j=1}^{k+1} \varphi(\mathfrak{a}_0, \dots, \mathfrak{a}_{j-1}\mathfrak{a}_j\mathfrak{a}_{j+1}, \dots \mathfrak{a}_{k+2}) \\ &- \frac{1}{(k+2)(k+1)} \sum_{1 \leqslant i < j \leqslant n+1}^{k+1} \varphi(\mathfrak{a}_0, \dots, \mathfrak{a}_{i-1}\mathfrak{a}_i, \dots, \mathfrak{a}_j\mathfrak{a}_{j+1}, \dots \mathfrak{a}_{k+2}).\end{aligned}$$

The map S defines the *periodicity homomorphism* $S : CH^k(\mathfrak{A}_\theta^\infty) \to CH^{k+2}(\mathfrak{A}_\theta^\infty)$ (Gracia-Bondía et al. 2001, Lemma 10.4).

The periodicity map S defines two direct systems of abelian groups. Their inductive limits

$$PH^{\mathrm{ev}}(\mathfrak{A}_\theta^\infty) := \lim_{\to} CH^{2k}(\mathfrak{A}_\theta^\infty), \qquad PH^{\mathrm{od}}(\mathfrak{A}_\theta^\infty) := \lim_{\to} CH^{2k+1}(\mathfrak{A}_\theta^\infty)$$

form a $\mathbb{Z}_2$ graded group $PH^\bullet(\mathfrak{A}_\theta^\infty) := PH^{\mathrm{ev}}(\mathfrak{A}_\theta^\infty) \oplus PH^{\mathrm{od}}(\mathfrak{A}_\theta^\infty)$ called the *periodic cyclic cohomology* of the algebra $\mathfrak{A}_\theta^\infty$. It turns out that $PH^{\mathrm{ev}}(\mathfrak{A}_\theta^\infty) \simeq PH^{\mathrm{od}}(\mathfrak{A}_\theta^\infty) \simeq \mathbb{C}^2$ independently of θ.

To describe the cohomology it is enough to exhibit a cyclic cocycle in each cohomology class. Since a cyclic 0-cocycle is a trace, $CH^0(\mathfrak{A}_\theta^\infty)$ is one-dimensional with generator the canonical trace; i.e. $CH^0(\mathfrak{A}_\theta^\infty) = \mathbb{C}[⨍]$. With the two standard derivation ∂_1 and ∂_2 one can define two cyclic 1-cocycles ψ_1 and ψ_2 by $\psi_j(\mathfrak{a}_0, \mathfrak{a}_1) := ⨍ (\mathfrak{a}_0 \;\partial_j\; \mathfrak{a}_1)$. It turns out that $CH^1(\mathfrak{A}_\theta^\infty) = \mathbb{C}[\psi_1] \oplus \mathbb{C}[\psi_2]$. Next there is a 2-cocycle obtained by promoting the trace $⨍$ to a cyclic trilinear form $(S\;⨍)(\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2) := ⨍ (\mathfrak{a}_0\mathfrak{a}_1\mathfrak{a}_2)$. However, there is another cyclic 2-cocycle that is not in the range of S and it is defined by

$$\mathfrak{C}_1(\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2) := -\frac{i}{2\pi} ⨍ \left(\mathfrak{a}_0(\partial_1\, \mathfrak{a}_1)(\partial_2\, \mathfrak{a}_2) - \mathfrak{a}_0(\partial_2\, \mathfrak{a}_1)(\partial_1\, \mathfrak{a}_2)\right).$$

It turns out that $CH^3(\mathfrak{A}_\theta^\infty) = \mathbb{C}[S\;⨍] \oplus \mathbb{C}[\mathfrak{C}_1]$. For $k \geqslant 3$ the cohomology groups are stable under repeated application of S, i.e. $CH^k(\mathfrak{A}_\theta^\infty) = S(CH^{k-2}(\mathfrak{A}_\theta^\infty)) \simeq \mathbb{C} \oplus \mathbb{C}$. It follows that $PH^{\mathrm{ev}}(\mathfrak{A}_\theta^\infty)$ is generated by $[⨍]$ and $[\mathfrak{C}_1]$, while $PH^{\mathrm{od}}(\mathfrak{A}_\theta^\infty)$ is generated by $[\psi_1]$ and $[\psi_2]$.

Appendix C
Basic notions on vector bundles theory

In this appendix we introduce some notations concerning the theory of *vector bundles*. For a detailed and complete exposition about this subject, we refer to (Lang 1985) or (Gracia-Bondía et al. 2001) or (Greub et al. 1972). The reader expert in this field can skip this appendix.

Vector bundle and local frames

A rank N (complex) vector bundle $\iota : \mathscr{E} \to X$ is the datum of two topological spaces $\mathscr{E}$ (*total space*) and X (*base space*, connected and compact for our purposes) and a continuous map ι (canonical projection) such that for any $x \in X$, the set $\mathscr{E}_x := \iota^{-1}(x)$ is a complex vector space isomorphic to $\mathbb{C}^N$ (*fiber space*) and there exists an open neighborhood O_x of x and N continuous maps $\zeta^j : O_x \to \mathscr{E}$ such that $\iota \circ \zeta^j = \mathrm{id}$ on O_x and such that the map

$$O_x \times \mathbb{C}^N \ni (y, \mathrm{v}) \overset{\Phi_{O_x}}{\longmapsto} v_0\, \zeta^0(y) + \ldots + v_{N-1}\, \zeta^{N-1}(y) \in \mathscr{E}_y, \qquad \mathrm{v} := (v_0, \ldots, v_{N-1}) \tag{C.1}$$

is a homeomorphism of $O_x \times \mathbb{C}^N$ onto $\pi^{-1}(O_x)$. The maps ζ^j are called *(canonical) local sections* and the collection $\boldsymbol{\zeta} := \{\zeta^0, \ldots, \zeta^{N-1}\}$ is called a *(canonical) local frame* over O_x.

Local triviality and triviality

Let $\iota_j : \mathscr{E}_j \to X$, $j = 1, 2$, be two vector bundles over the same base space X. Morphisms in the category of vector bundles over a fixed X are continuous *bundle maps* $\Theta : \mathscr{E}_1 \to \mathscr{E}_2$ satisfying $\iota_2 \circ \Theta = \iota_1$ such that the associated *fiber maps* $\Theta_x : \iota_1^{-1}(x) \to \iota_2^{-1}(x)$ are linear for any $x \in X$. If the morphism Θ is an homeomorphism between $\mathscr{E}_1$ and $\mathscr{E}_2$ such that Θ_x is a linear isomorphism for every $x \in X$, then the vector bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ are said to be *isomorphic* or *equivalent*. A rank N vector bundle is trivial if it is equivalent to the *product bundle* $\mathrm{pr}_1 : X \times \mathbb{C}^N \to X$. A vector bundle is trivial if and only if it admits a *global frame* $\boldsymbol{\zeta}$ defined on the whole X.

Trivializing covering and transition functions

A *trivializing covering* $\{(O_a, \boldsymbol{\zeta}_a)\}_{a \in \mathcal{I}}$ for the vector bundle $\iota : \mathscr{E} \to X$ is the datum of an open covering $\{O_a\}_{a \in \mathcal{I}}$ for X such that each O_a carries a local frame $\boldsymbol{\zeta}_a := \{\zeta_a^0, \ldots, \zeta_a^N\}$. If X is compact the trivializing covering can be chosen in such a way that $\mathcal{I}$ is finite. If $O_a \cap O_b \neq \emptyset$, then the continuous (matrix-valued) function $g_{a,b} : O_a \cap O_b \to \mathrm{GL}(\mathbb{C}^N)$ defined by $g_{a,b}(x) := \Phi_{O_a}^{-1} \circ \Phi_{O_b}\big|_x$ according to (C.1), is called *transition function* between the local trivializations in O_a and O_b. The family $\boldsymbol{g} := \{g_{a,b}\}_{a,b\in\mathcal{I}}$ satisfies the consistency conditions

$$g_{a,a}(x) = \mathbb{1}_N \quad \forall\, x \in O_a \qquad\qquad g_{a,b}(x)\cdot g_{b,c}(x) = g_{c,a}(x) \quad \forall\, x \in O_a \cap O_b \cap O_c.$$

Suppose that $\boldsymbol{\zeta}_a(x) = \mathbb{G}_{a,b}(x)\cdot\boldsymbol{\zeta}_b(x)$ for any $x \in O_a \cap O_b$ with $\mathbb{G}_{a,b} : O_a \cap O_b \to \mathrm{GL}(\mathbb{C}^N)$. A simple computation shows

$$\mathbb{C}^N \ni \mathrm{v}^{(a)} \overset{\Phi_{O_a}|_x}{\longrightarrow} \sum_{j=0}^{N-1} \zeta_a^j(x) v_j^{(a)} = \sum_{k=0}^{N-1} \zeta_a^k(x) \left(\sum_{j=0}^{N-1} \mathbb{G}_{a,b}^{j,k}\, v_j^{(a)}\right) \overset{\Phi_{O_b}^{-1}|_x}{\longrightarrow} \mathrm{v}^{(b)} = {}^t\,\mathbb{G}_{a,b}\cdot \mathrm{v}^{(a)} \in \mathbb{C}^N$$

which implies that $g_{a,b}(x) = {}^t\mathbb{G}_{a,b}{}^{-1}(x)$.

Example C.0.1 (Standard atlas for the two dimensional torus)**.** Since we are mainly interested in vector bundles with $\mathbb{T}^2$ as base manifold, we provide the description of an atlas for the two-dimensional torus. The compact space $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ can be covered by four neighborhoods $\{O_a\}_{a=1,\ldots,4}$, where each O_a is homeomorphic to a subset of $\mathbb{R}^2$. Let

$$\begin{aligned}
O_1 &:= \{z(k) = \mathrm{e}^{ik} \in \mathbb{T}^2 \;:\; -\epsilon < k_1 < \pi + \epsilon,\ -\epsilon < k_2 < \pi + \epsilon\}\\
O_2 &:= \{z(k) = \mathrm{e}^{ik} \in \mathbb{T}^2 \;:\; \pi - \epsilon < k_1 < 2\pi + \epsilon,\ -\epsilon < k_2 < \pi + \epsilon\}\\
O_3 &:= \{z(k) = \mathrm{e}^{ik} \in \mathbb{T}^2 \;:\; -\epsilon < k_1 < \pi + \epsilon,\ \pi - \epsilon < k_2 < 2\pi + \epsilon\}\\
O_4 &:= \{z(k) = \mathrm{e}^{ik} \in \mathbb{T}^2 \;:\; \pi - \epsilon < k_1 < 2\pi + \epsilon,\ \pi - \epsilon < k_2 < 2\pi + \epsilon\}
\end{aligned} \tag{C.2}$$

with $\epsilon > 0$ and small enough. For every *chart* O_a we define the *local coordinate system* ϕ_a which maps O_a in the fixed open set $V \subset [-\epsilon, \pi+\epsilon] \times [-\epsilon, \pi+\epsilon]$ of the plane $\mathbb{R}^2$; explicitly

$$\phi_1(\mathrm{e}^{ik}) := (k_1, k_2), \quad \phi_2(\mathrm{e}^{ik}) := (k_1 - \pi, k_2), \quad \phi_3(\mathrm{e}^{ik}) := (k_1, k_2 - \pi), \quad \phi_4(\mathrm{e}^{ik}) := (k_1 - \pi, k_2 - \pi).$$

Any intersection $O_{a,b} := O_a \cap O_b$ is non empty, and is made by the union of two disjoint set. For instance (c.f. Figure C.1) $O_{1,2} = O_{1,2}(k_1 \sim \pi) \cup O_{1,2}(k_1 \sim 2\pi)$, where $O_{1,2}(k_1 \sim \pi)$ is the intersection around $k_1 = \pi$ (mod. 2π) and $O_{1,2}(k_1 \sim 2\pi)$ is the intersection around $k_1 = 0$ (mod. 2π). The *standard atlas* $\{O_a, \phi_a\}_{a=1,\ldots,4}$ is smooth and endows $\mathbb{T}^2$ with the structure of a smooth manifold. ◀▷

Tensor product of vector bundles

The structure of a vector bundle (up to isomorphisms) can be entirely recovered by the set $\boldsymbol{g}$ of its transition functions (Lang 1985, Proposition 1.2). One can use this result to define the *tensor product* of two vector bundles $\iota_1 : \mathscr{E}_1 \to X$ and $\iota_2 : \mathscr{E}_2 \to X$. The latter is the vector bundle $\iota : \mathscr{E}_1 \otimes \mathscr{E}_2 \to X$ with fiber spaces $\iota^{-1}(x) := (\mathscr{E}_1 \otimes \mathscr{E}_2)_x \simeq \iota_1^{-1}(x) \otimes \iota_2^{-1}(x)$ and transition functions given by the tensor product of the transition functions of $\mathscr{E}$ and $\mathscr{E}'$, relative to the same trivializing cover.

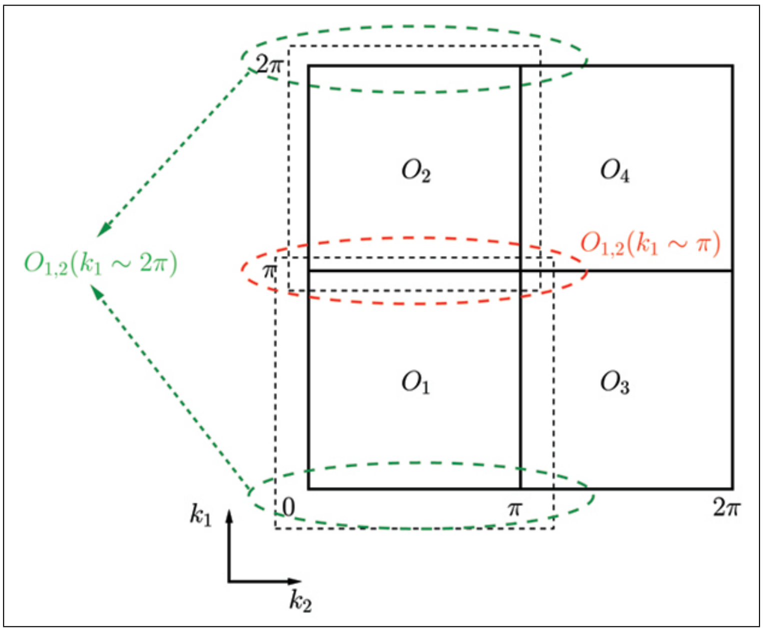


Figure C.1: *Standard atlas for the two dimensional torus* $\mathbb{T}^2 \simeq [0, 2\pi)^2$.

Determinant line bundle

Let $\iota : \mathscr{E} \to X$ be a rank N vector bundle. The q*-th exterior power* of $\mathscr{E}$ is a vector bundle

$$\tilde{\iota} : \wedge^q \mathscr{E} \to X$$

with fiber over $x \in X$ given by

$$\tilde{\iota}^{-1}(x) =: (\wedge^q \mathscr{E})_x = \wedge^q \mathscr{E}_x$$

where the symbol $\wedge^q$ denotes the q-th exterior power of the N dimensional vector space $\mathscr{E}_x := \iota^{-1}(x)$ (Greub et al. 1972, Chapter II, Section 2.11). The rank of $\wedge^q \mathscr{E}$ is $\frac{N!}{q!(N-q)!}$ for $1 \leqslant q \leqslant N$. One can extend the definition of the exterior power to any integer q by means of the following convetions

$$\wedge^q \mathscr{E} \simeq X \times \{0\} \quad \text{if} \;\; q > N, \qquad\qquad \wedge^q \mathscr{E} \simeq X \times \mathbb{C} \quad \text{if} \;\; q = 0.$$

Let $\boldsymbol{g} := \{g_{a,b}\}_{a,b \in \mathcal{I}}$ be the system of transition functions of the vector bundle $\mathscr{E}$ subordinate to the open covering $\{O_a\}_{a \in \mathcal{I}}$. The system of transition functions of $\wedge^q \mathscr{E}$, subordinate to the same cover, is given by $\tilde{\boldsymbol{g}} := \{\wedge^q g_{a,b}\}_{a,b \in \mathcal{I}}$, where for any $x \in O_a \cap O_b$, $\wedge^q g_{a,b}(x) \in \wedge^q \text{GL}(\mathbb{C}^N)$ denotes the q-th exterior power of the matrix $g_{a,b}(x)$.

When $q = N$ the vector bundle $\wedge^N \mathscr{E}$ has rank 1; it is called *determinant line bundle* of $\mathscr{E}$ and denoted by

$$\tilde{\iota} : \det(\mathscr{E}) \to X$$

The system of transition functions of $\det(\mathscr{E})$, subordinate to the open covering $\{O_a\}_{a \in \mathcal{I}}$, is given by $\tilde{\boldsymbol{g}} := \{\det(g_{a,b})\}_{a,b \in \mathcal{I}}$, where for any $x \in O_a \cap O_b$, $\det(g_{a,b})(x) \in \mathbb{C}$ denotes the determinant of the matrix $g_{a,b}(x)$.

Pullback vector bundle

Let $\iota : \mathscr{E} \to X$ be a Hermitian vector bundle and $f : X \to X$ a continuous function. The *pullback bundle* $\iota' : f^*\mathscr{E} \to X$ is the vector bundle with total space

$$f^*\mathscr{E} := \{(x, \mathrm{v}) \in X \times \mathscr{E} \ : \ \iota(\mathrm{v}) = f(x)\} \tag{C.3}$$

and canonical projection $\iota'(x, v) = x$. In other words we have the following isomorphism between the fiber spaces: $(f^*\mathscr{E})_x \simeq \mathscr{E}_{f(x)}$ for all $x \in X$. Let $\{(O_a, \boldsymbol{\zeta}_a)\}_{a\in\mathcal{I}}$ be a trivializing covering for $\iota : \mathscr{E} \to X$ with transition functions $\boldsymbol{g} := \{g_{a,b}\}_{a,b\in\mathcal{I}}$. It follows from the definition that a trivializing covering for $\iota' : f^*\mathscr{E} \to X$ is given by $\{(f^{-1}(O_a), f^*\boldsymbol{\zeta}_a)\}_{a\in\mathcal{I}}$ where $f^*\boldsymbol{\zeta}_a := \boldsymbol{\zeta}_a \circ f$. In other words, the local frames of the pullback vector bundle are given by the pullback of the local frames of the original vector bundle. We remark that a (local) frame transforms by pullback and not by pushforward as for vector fields. The transition functions of a pullback vector bundle are given by $f^*\boldsymbol{g} := \{f^*g_{a,b}\}_{a,b\in\mathcal{I}}$, where $f^*g_{a,b} : f^{-1}(O_a) \cap f^{-1}(O_b) \to \mathrm{GL}(\mathbb{C}^N)$ is defined by $f^*g_{a,b} := g_{a,b} \circ f$. Namely, the transition functions of a pullback vector bundle are given by the pullback of the transition functions of the original vector bundle. In particular the pullback of a trivial vector bundle is again a trivial vector bundle.

Module of continuous sections and endomorphism bundle

Let $\iota : \mathscr{E} \to X$ be a rank N vector bundle with trivializing covering $\{(O_a, \boldsymbol{\zeta}_a)\}_{a\in\mathcal{I}}$ and transition functions $\boldsymbol{g} := \{g_{a,b}\}_{a,b\in\mathcal{I}}$. The set of continuous sections is denoted by $\Gamma(\mathscr{E})$. Notice that $\Gamma(\mathscr{E})$ is a $C(X)$-module and the action of $C(X)$ is just the (left) multiplication in each fiber, i.e. $(f \cdot s)(x) := f(x)s(x)$ for any $s \in \Gamma(\mathscr{E})$ and $f \in C(X)$.

The *endomorphism bundle* of the vector bundle $\iota : \mathscr{E} \to X$ is the rank N^2 vector bundle $\iota : \mathrm{End}(\mathscr{E}) \to X$ which has as typical fiber the vector space $\mathrm{End}(\mathbb{C}^N) = \mathrm{Mat}_N(\mathbb{C})$ and transition functions $\{\mathrm{Ad}g_{a,b}\}_{a,b\in\mathcal{I}}$ given by

$$A_a(x) = \mathrm{Ad}g_{a,b}(x) \cdot A_b(x) := g_{a,b}(x) \cdot A_b(x) \cdot g_{a,b}^{-1}(x), \qquad\qquad x \in O_a \cap O_b,$$

where $A_a := A|_{O_a} : O_a \to \mathrm{Mat}_N(\mathbb{C})$ and $A_b := A|_{O_b} : O_b \to \mathrm{Mat}_N(\mathbb{C})$ denote the local expressions of the continuous section $A \in \Gamma(\mathrm{End}(\mathscr{E}))$. The space $\Gamma(\mathrm{End}(\mathscr{E}))$ has the structure of a unital C^*-algebra as showed by the *(localization) isomorphism* $\Gamma(\mathrm{End}(\mathscr{E})) \simeq \mathrm{End}_{C(X)}(\Gamma(\mathscr{E}))$, the latter being the C^*-algebra of the adjontable operators on the $C(X)$-module $\Gamma(\mathscr{E})$ (Proposition 4.7.13). In the case of a trivial vector bundle $\mathscr{E} = X \times \mathbb{C}^N$, then $\Gamma(\mathrm{End}(\mathscr{E})) = C(X, \mathrm{Mat}_N(\mathbb{C})) \simeq C(X) \otimes \mathrm{Mat}_N(\mathbb{C})$.

Hermitian structure and L^2-sections

Each complex vector bundle $\iota : \mathscr{E} \to X$ can be endowed (in many ways) with a *Hermitian structure*. For that, one can define a positive definite sesquilinear form $(\cdot;\cdot)_x$ on each $\iota^{-1}(O_a) \simeq O_a \times \mathbb{C}^N$, for O_a in a trivializing open cover of X, and then glue the metric together by using a partition of unity. Thereupon, we get a *pairing* $\{\cdot;\cdot\} : \Gamma(\mathscr{E}) \times \Gamma(\mathscr{E}) \to$

$C(X)$, defined by $\{s;r\}(x) := (s(x);r(x))_x$ for any $x \in X$, $s,r \in \Gamma(\mathscr{E})$, which is $C(X)$-sesquilinear, conjugate -symmetric and positive definite.

Assume that the base space X is a finite dimensional oriented manifold (with real dimension n) with a Riemannian metric g. Denote with $\Omega^r(X)$ the *exterior form bundle* of the r-forms over X. Let $\mu_g \in \Omega^n(X)$ be the *Riemannian volume form* associated to the metric g, i.e. (in local coordinates)

$$\mu_g = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$$

where dx^i are the 1-forms providing an oriented basis for the cotangent bundle T^*X and $|g|$ is the absolute value of the determinant of the metric tensor g. The Hermitian structure of $\iota : \mathscr{E} \to X$ and the Riemannian strucure of the oriented manifold X allow to define the notion of scalar product on the space of sections $\Gamma(\mathscr{E})$:

$$\langle s;r \rangle := \int_X \{s;r\}(x)\ \mu_g(x) \qquad s,r \in \Gamma(\mathscr{E}). \tag{C.4}$$

The completion of $\Gamma(\mathscr{E})$ with respect to the norm $\|s\|_{L^2} := \sqrt{\langle s;s \rangle}$ defines the Hilbert space $L^2(\mathscr{E})$ called the space of L^2*-sections* of $\mathscr{E}$.

Connections, curvatures, Chern classes

According to the general (axiomatic) definition (Husemoller 1994, definition 3.2, Chapter 17), the *Chern classes* of a vector bundle $\iota : \mathscr{E} \to X$ are a sequence of elements of the *integer cohomology* group $H^\bullet(X;\mathbb{Z}) := \bigoplus_j H^j(X;\mathbb{Z})$. The j-th Chern class of $\mathscr{E}$, denoted by $\tilde{c}_j(\mathscr{E})$, is an element of $H^{2j}(\mathbb{T}^2;\mathbb{Z})$. The *total Chern class* is defined as

$$\tilde{c}(\mathscr{E}) = \tilde{c}_0(\mathscr{E}) + \tilde{c}_1(\mathscr{E}) + \tilde{c}_2(\mathscr{E}) + \cdots$$

with $\tilde{c}_0(\mathscr{E}) = 1$. The definition of Chern class is *functorial*, namely

$$\tilde{c}(f^*\mathscr{E}) = f^*\tilde{c}(\mathscr{E})$$

for any continuous map $f : X \to X$.

There are various equivalent ways to construct the Chern classes. Assuming that X is a (Riemannian) manifold we can define the Chern classes by means of differential geometric tools. A *connection* ω for the Hermitian vector bundle $\iota : \mathscr{E} \to X$, subordinate to the trivializing cover $\{(O_a, \boldsymbol{\zeta}_a)\}_{a\in\mathcal{I}}$, is a collection $\{\omega_a\}_{a\in\mathcal{I}} \subset \Omega^1(O_a;\mathfrak{u}(\mathbb{C}^N))$ of local differential 1-forms (called *gauge potentials*) with value in $\mathfrak{u}(\mathbb{C}^N) := i\mathrm{Her}_N(\mathbb{C})$ (the algebra of anti-Hermitian matrices) which satisfies the transformation rule

$$\omega_a = dg_{a,b}\ \ {g_{a,b}}^{-1} + g_{a,b}\ \ \omega_b\ \ {g_{a,b}}^{-1}, \tag{C.5}$$

where $\boldsymbol{g} := \{g_{a,b}\}_{a,b\in\mathcal{I}}$ are the transition functions of $\mathscr{E}$. $\mathfrak{u}(\mathbb{C}^N)$ is the Lie algebra of the group $\mathscr{U}(\mathbb{C}^N)$ (unitary group on $\mathbb{C}^N$), which is the group in which the transition functions $\{g_{a,b}\}_{a,b\in\mathcal{I}}$ take values. The *curvature* K^ω associated to the connection $\omega = \{\omega_a\}_{a\in\mathcal{I}}$ is the collection $\{K_a^\omega\}_{a\in\mathcal{I}} \subset \Omega^2(O_a;\mathfrak{u}(\mathbb{C}^N))$ of local differential 2-forms with value in $\mathfrak{u}(\mathbb{C}^N)$ defined by the *local structural equation*

$$K_a^\omega := d\omega_a + \frac{1}{2}[\omega_a;\omega_a] = d\omega_a + \omega_a \wedge \omega_a. \tag{C.6}$$

The *de Rham-Chern classes* are defined by means of the *Chern-Weil homomorphism* as

$$c(\mathscr{E}) := \det\left(\mathbb{1}_N + \frac{i}{2\pi}\,K^\omega\right) = 1 + c_1(\mathscr{E}) + \ldots + c_N(\mathscr{E}) \tag{C.7}$$

where $c_j(\mathscr{E}) \in H^{2j}_{\text{d.R.}}(X)$ (the $(2j)^{\text{th}}$ de Rham complex cohomology group). In particular from equation (C.7) one deduces

$$c_1(\mathscr{E}) = \frac{i}{2\pi}\mathrm{Tr}_N\left(K^\omega\right). \tag{C.8}$$

If the base manifold X has dimension n, then $c_j(\mathscr{E}) = 0$ if $j > {}^n\!/\!_2$. Moreover, the definition (C.7) does not depend on a particular choice of the curvature K^ω (Husemoller 1994, Chapter 19). The equivalence $\tilde{c}_j(\mathscr{E}) \simeq c_j(\mathscr{E})$ between the axiomatic definition and the differential geometric definition comes from the *de Rham Theorem* $H^j(X,\mathbb{C}) \simeq H^j_{\text{d.R.}}(X)$ and the *coefficient morphism* $H^j(X,\mathbb{Z}) \to H^j(X,\mathbb{C})$.

Given a vector bundle $\iota : \mathscr{E} \to X$, denote with $\det(\mathscr{E})$ the determinant line bundle. One can prove that (Greub et al. 1973, Chapter IX, section 9.18)

$$c_j(\det(\mathscr{E})) = c_j(\mathscr{E}). \tag{C.9}$$

Let $\iota : \mathscr{E} \to X$ be a rank N Hermitian vector bundle with base space of dimension 2 (as for $\mathbb{T}^2$). The *(total) Chern character* is defined by

$$\mathrm{ch}(\mathscr{E}) = N + c_1(\mathscr{E})$$

where $c_1(\mathscr{E})$ denotes first Chern class of $\mathscr{E}$. This definition agrees with the general one since the base manifold has dimension two and $c_1(\mathscr{E})^{\wedge n} = 0$ whenever $n > 1$. Let $\iota' : \mathscr{E}' \to X$ be a rank N' Hermitian vector bundle. From the general formula $\mathrm{ch}(\mathscr{E} \otimes \mathscr{E}') = \mathrm{ch}(\mathscr{E}) \wedge \mathrm{ch}(\mathscr{E}')$ it follows the useful relation

$$c_1(\mathscr{E} \otimes \mathscr{E}') = N'c_1(\mathscr{E}) + Nc_1(\mathscr{E}'). \tag{C.10}$$

Chern numbers for vector bundles over $\mathbb{T}^2$

Let $\iota : \mathscr{E} \to \mathbb{T}^2$ be a rank N Hermitian vector bundle. The first Chern class $c_1(\mathscr{E})$ is an equivalence class of closed 2-forms and any representative can be integrated over $\mathbb{T}^2$ (the only 2-cycle of the homology) producing the same integer number. The number

$$C_1(\mathscr{E}) := \int_{\mathbb{T}^2} c_1(\mathscr{E}),$$

called the *first Chern number* of the vector bundle $\mathscr{E}$, is a quantity depending only on the topology of the vector bundle.

The simple lemma below is useful to compute the first Chern numbers of pullback vector bundles. Indeed the functoriality of the Chern classes implies

$$C_1(f^*\mathscr{E}) := \int_{\mathbb{T}^2} c_1(f^*\mathscr{E}) = \int_{\mathbb{T}^2} f^* c_1(\mathscr{E})$$

where $f^* c_1(\mathscr{E})$ denotes the pullback of a 2-form over $\mathbb{T}^2$.

Lemma C.0.2. *Let* $\omega \in \Omega^2(\mathbb{T}^2)$ *be a 2-form and* $f : \mathbb{T}^2 \to \mathbb{T}^2$ *the continuous function defined by* $f(z_1, z_2) = (z_1^n, z_2^m)$ *with* $n, m \in \mathbb{Z}$*. Then*

$$\int_{\mathbb{T}^2} f^*\omega = nm \int_{\mathbb{T}^2} \omega.$$

Proof. The local expression of the two form ω is $\omega(z(k)) = h(k)\, dk_1 \wedge dk_2$. The pullback $f^*\omega$, defined by $f^*\omega|_z(X_1, X_2) = \omega|_{f(z)}(f_*(X_1), f_*(X_2))$ for any $X_2, X_2 \in T_z\mathbb{T}^2$ has local expression $(f^*\omega)(z(k)) = nm\, h(nk_1, mk_2) dk_1 \wedge dk_2$. An integration shows that

$$\int_{\mathbb{T}^2} f^*\omega = nm \int_0^{2\pi} dk_1 \int_0^{2\pi} dk_2\; h(nk_1, mk_2) = nm \int_{\mathbb{T}^2} \omega$$

since h is a periodic function defined on $\mathbb{T}^2$. ■

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