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Dirac Operators on Quantum Principal G-Bundles

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Contents

CHAPTER 1

Introduction

In the last decades extensive work has been devoted to the definition, construction and characterization of noncommutative spaces. Motivations for this interest came both from mathematics [C94] and from physics [Be88, C94, DFR94, DFR95, Pad85, Pad87, CMa07]. A noncommutative space can be studied on several levels: algebraic, topological, differential, metric and others. In this thesis we shall be interested, in particular, in the differential and metric structure of (compact) noncommutative spin manifolds and of noncommutative principal bundles. Noncommutative spin manifolds are the central objects of Connes' noncommutative geometry [C94]. A structure of noncommutative spin manifold on a noncommutative topological compact space (which is described by a C^*-algebra A, representing the algebra of continuous functions over the noncommutative space) is codified as *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$. Here $\mathcal{A} \subseteq A$ is a pre-C^*-algebra representing the algebra of smooth functions, $\mathcal{H}$ a Hilbert space representing the space of L^2-spinors and D the Dirac operator. Often one can equip a spectral triple with a real structure J and a certain $\mathbb{Z}_2$-grading γ. Connes introduced a number of axioms for such a real or even spectral triple, in order to describe a noncommutative spin manifold [C95, C96, C00, GBFV, CMa07]. We shall review these axioms in chapter 3. Moreover, we will discuss some additional properties of spectral triples and introduce some tools we shall use elsewhere in this thesis.

Noncommutative principal bundles, instead, were mostly studied on the algebraic level till recently. Their construction relies on the identification of Hopf algebras as noncommutative generalization of groups [Dri87, Wor87, FRT90, Wor91, Kassel]. T. Brzezinski and S. Majid [BM93] introduced the notion of quantum principal bundles, defining them to be H-comodule algebras (where H is a Hopf algebra replacing the structure group of the bundle). Moreover, they introduced also a notion of connection and discussed the role of the differential calculus. These notions were substantially developed recently (see, e.g., [B96, Haj96, BM98b, B99, BH99, BM00, DGH01, BH09, HKMZ11]) and we shall review them in the first sections of chapter 4.

The main goal of this thesis is to study the noncommutative (spin) geometry of some classes of quantum principal bundles. Namely we shall consider noncommutative bundles mainly with classical structure (Lie) group; that is, H-comodule algebras with H a Hopf algebra of smooth functions over a (compact, connected, semisimple) Lie group G. In particular, chapters 4, 5, 6, 7 will be devoted to the cases $G = U(1)$ and $G = \mathbb{T}^n$, while a partial extension of our constructions and of our results to the general case will be discussed in chapter 8. Our aim will be to understand the relation of the spin structure on the total space of a quantum principal bundle with the spin structure on the base space and with the metric structure of the fibres.

Let us briefly recall the classical picture. Consider, first, a principal G-bundle $\pi : P \to M$ (with G a compact, connected, semisimple Lie group), and assume that (M, g) is a Riemannian (spin) manifold. Then, given a bi-invariant metric on G and a connection on P, there is a metric on P such that π is a Riemannian submersion with totally geodesic fibres (see, e.g., [Mor96], lemma 1.1.1). Moreover, under suitable hypotheses, the Dirac operator on the total space of the bundle can be related to that one on the base space. So, in the classical case it is quite straightforward to put a structure of Riemannian (spin) manifold on the total space of a principal G-bundle over a Riemannian (spin) manifold. Moreover, the metric structure obtained in this way depends on the choice of a connection on the bundle and of a bi-invariant metric on the group. In this thesis we shall obtain a (partial) extension of this construction to noncommutative G-bundles. Next, consider the opposite situation. That is, assume to be given a principal G-bundle $\pi : P \to M$ with P and M two Riemannian spin manifolds and assume that π is a Riemannian submersion with totally geodesic fibres. Then we can look for a relation between the spin structure and the Dirac operator on P and those on M. In the $U(1)$ case such a relation admits a quite simple description [Amm98, AmmB98]: the Dirac operator on the total space can be expressed as the sum of three differential operators: a vertical Dirac operator, which acts on the fibres, an horizontal Dirac operator which encodes the metric structure of the base space and a zero order term whose existence is connected with the vanishing of the torsion of the Levi-Civita connection. For a generalization to noncommutative bundles usually the concept of the metric is not available in general, if not indeed codified in terms of a spectral triple. Thus one has to work directly with spectral triples and, if necessary, to introduce other suitable notions. First steps in this direction appeared in [DS13a, DSZ13], for quantum principal $U(1)$-bundles and we shall recall these results. Moreover, we shall introduce a generalization for the case of $\mathbb{T}^n$-bundles.

The thesis is organized as follows. In chapter 2, after a brief introduction to noncommutative topological spaces, we shall quickly review basic notions on noncommutative geometry: the construction of first order differential calculi over noncommutative algebras [C85], the definition and the main properties of Hopf algebras [Sw69, Abe80, Maj95], the K-theory of C^*-algebras and the algebraic K-theory of associative algebras [AtiHir59, AtiHir61, Blck98, Lan03], the Hochschild homology and the Hochschild cohomology [CE, GM], the cyclic cohomology [C85] and, finally, Kasparov's bivariant KK-theory [Kas80].

In chapter 3 we shall review Connes' noncommutative geometry [C94, C95, C96, C00, GBFV, CMa07]. The first part will be dedicated to the definition of spectral triple and real spectral triples and to the discussion of Connes' axioms. Next, we shall recall some properties of spectral triples: the Dirac calculus (that is, the first order differential calculus defined by the Dirac operator), the inner fluctuations of the Dirac operator (which are related to the construction of noncommutative gauge theories [CC96, CC06b, CMa07, CCM07, CC08]), the relation between Connes' noncommutative geometry and Riemannian geometry. Moreover, we shall recall the notion of equivariant spectral triples (with respect to an action or to a coaction of a Hopf algebra): this is the noncommutative formulation of the invariance of the metric and of the spin structure of a Riemannian spin manifold under a group of the transformations. Finally, we shall briefly discuss the relation between spectral triples and Kasparov's KK-theory.

In chapter 4 we shall discuss the algebraic properties of quantum principal bundles. First we shall review the definitions and the general structure of quantum principal bundles [BM93, Haj96, BM98b, B99, BH99, BM00, DGH01, HKMZ11, BH09] discussing both the definition of bundles with universal differential calculus and that of bundles with general calculus. We shall pay attention to the different definitions of (strong) connections, recalling the reasons for which they are equivalent, and to their behaviour under gauge transformations. Then we shall discuss the cleft bundles. Cleftness is a notion close to that of triviality for a quantum principal bundle [BM93]. We shall focus our attention to the case when all the algebras involved are *-algebras. As a new result shall show how the known isomorphism of a cleft extension with a crossed product algebra [BlCM86, DT86, BlM89, Ch98] becomes a *-isomorphism once a suitable structure of *-algebra on the crossed product is introduced. At the end of this part we shall also recall a possible definition of quantum associated bundles [BF12]. Moreover, we will consider the special case of quantum principal $\mathbb{T}^n$-bundles. Requiring the compatibility of the calculus on the total space of the bundle with the de Rham calculus on $\mathbb{T}^n$, we shall show how these bundles admit a characterization which makes easier to see that they share many properties with ordinary principal $\mathbb{T}^n$-bundles. In particular, we shall show how any strong connection, compatible with the de Rham calculus, can be described in terms of a family of n 1-forms; this picture, of course, is closely related to the classical one, when a connection can be described by a $\mathfrak{t}_n$-valued 1-form (here $\mathfrak{t}_n$ is the Lie algebra of $\mathbb{T}^n$), whose components with respect to a suitable basis of $\mathfrak{t}_n$ are exactly n 1-forms on the total space of the bundle.

In chapter 5 we study projectable spectral triples over noncommutative principal torus bundles. The notion of projectability for a $U(1)$-equivariant real spectral triple over a quantum principal $U(1)$-bundle was introduced in [DS13a], for triples of KR-dimension 3. In the first part of this chapter we extend it to triples of any dimension (both even and odd); most of the results discussed here can be found also in the recent paper [DSZ13]. Next we shall consider the more general case of noncommutative principal $\mathbb{T}^n$-bundles, extending the notion of projectability to $\mathbb{T}^n$-equivariant spectral triples (some of these results will appear in [DZ13]). Both in the $U(1)$ and in the more general $\mathbb{T}^n$ case we shall construct twisted Dirac operators: given a projectable spectral triple and a strong connection over the bundle, we use the latter to twist the Dirac operator of the original triple, obtaining in this way a new spectral triple. This gives a way to produce new Dirac operators, encoding, possibly, different geometries. As an example of this issue we discuss, in appendix D, the noncommutative 3-torus (as $U(1)$-bundle over a noncommutative 2-torus), showing that starting from the canonical flat Dirac operator we can produce twisted Dirac operators that could describe geometries with non-trivial curvature. Moreover we shall relate our construction to recent results in KK-theory [Mes11, BMS13].

Chapter 6 is the first of three chapters dedicated to the construction of real spectral triples over cleft quantum principal G-bundles. Of course, we start from the simplest case: we consider $U(1)$-bundles. Given a cleft Hopf-Galois $\mathcal{O}(U(1))$-extension (where $\mathcal{O}(U(1))$ is a suitable Hopf algebra of functions over the circle) $\mathcal{B} \hookrightarrow \mathcal{A}$ together with an $\mathcal{O}(U(1))$-equivariant real spectral triple over $\mathcal{B}$, we construct a $U(1)$-equivariant real spectral triple over $\mathcal{A}$, exploiting the fact that $\mathcal{A}$ is isomorphic to a crossed product algebra $\mathcal{B} \rtimes_\alpha \mathcal{A}$ (in particular, we shall extend some of the results in [BMR10]). We shall see that the triple obtained in this way is projectable, so that one can produce twisted Dirac operators from it. Moreover, we shall discuss some properties of the triple obtained in this way; in particular, we shall see that if the triple over $\mathcal{B}$ satisfied some of Connes' axioms, this will still be true for the triple over $\mathcal{A}$. Finally, we shall give a brief account of the behaviour of the construction under gauge transformations and discuss, as a simple application, the noncommutative 2-torus.

Chapter 7 contains several new results which extend those of chapter 6 to cleft $\mathbb{T}^n$-bundles. In this case we shall exploit the isomorphism of a cleft Hopf-Galois H-extension $\mathcal{B} \hookrightarrow \mathcal{A}$ with a crossed product algebra $\mathcal{B}\#_\sigma H$ [BlCM86, DT86, BlM89, Ch98]. As we shall see in chapter 8, the construction introduced here could be a good candidate for extending our construction to cleft extensions with general Hopf algebra. Also in this case we shall see that the triples we construct are projectable and we shall construct twisted Dirac operators.

In chapter 8 we shall discuss noncommutative principal G-bundles, where G is a compact connected semisimple Lie group. We shall identify them with principal $C^\infty(G)$-comodule algebras. In the first part of the chapter we shall study some general properties of these objects. In particular we shall extend the results discussed in chapter 4 for torus bundles, showing how demanding compatibility of the differential calculus over the total space of the bundle with the de Rham calculus on $C^\infty(G)$ allows to introduce a different characterization of quantum principal G-bundles and, overall, of strong connections: these ones, indeed, can be described by families $\{\omega_a\}$ of 1-forms, with $a = 1, \ldots, \dim(G)$, each 1-form being associated to an element T_a of a basis $\{T_a\}$ of the Lie algebra of G. Next we shall discuss the construction of spectral triple over cleft Hopf-Galois $C^\infty(G)$-extensions. These extensions can be identified with (almost trivial) noncommutative principal G-bundles. We shall see that, under suitable hypotheses, the results of chapter 6 and chapter 7 extend to this more general situation. In particular we shall produce new spectral triples and new (twisted and non-twisted) Dirac operators. In this chapter, moreover, we shall discuss in some detail the action of a gauge transformation on non-universal strong connections and on the objects we introduced in the construction of the new spectral triples.

Finally, in chapter 9 we shall review the main results obtained in this thesis, adding some general comments. Appendix A contains a review of the structure and the properties of noncommutative tori, which we shall use many times as examples. Appendices B and C contain brief accounts of locally convex topological spaces and noncommutative line modules, respectively. In appendix D we shall discuss some further geometric properties of twisted Dirac operators on the noncommutative 3-torus. In particular, we shall recover a (modified) Lichnerowicz formula, obtaining in this way a possible expression for the scalar curvature of the geometry associated to these operators.

CHAPTER 2

Preliminaries

2.1 Introduction to noncommutative topological spaces

In this thesis we study noncommutative spin manifolds and noncommutative (principal or associated) bundles. Instead of 'noncommutative' we shall often use the adjective 'quantum', though in physics, from which this term originates, it usually concerns phase spaces (e.g. cotangent spaces). Roughly, coordinates on such 'virtual' spaces 'fail to commute', and indeed they are described in terms of certain noncommuative algebras of would be 'functions'.

There are two pillars which underly our comprehension of such algebras. On the topological level the first one is the commutative Gelfand-Naimark theorem, which establishes a bijective correspondence between isomorphism classes of commutative (unital) C^*-algebras and homeomorphism classes of (compact) Hausdorff topological spaces. In one direction one associates to a compact Hausdorff topological space X the C^*-algebra $C(X)$ of continuous functions with the sup norm. In the other direction to a C^*-algebra A one associates its spectrum ΣA, with the weak-* topology.

The other one, valid on the topological level and as well on the smooth level, is Serre-Swan theorem [Ser55, Swa62]. Given a vector bundle $E \to X$ over a topological space M, we can consider the space $\Gamma(E) = \Gamma(E, X)$ of continuous sections of E. $\Gamma(E)$ is a $C(X)$-module and, moreover, one can prove that the assignment $E \to \Gamma(E)$ is functorial, so that Γ can be seen as a functor from the category of vector bundles over M to the category of $C(X)$-modules. It is a faithful, full and exact functor. Then Serre-Swan theorem gives a one-to-one correspondence between isomorphism classes of vector bundles over X and isomorphism classes of finitely generated projective $C(M)$-modules. Hence, given a compact noncommutative space A (that is, a unital C^*-algebra), a vector bundle over A can be defined as a finitely generated projective A-module. Notice that, in the noncommutative case, we have to distinguish between left and right modules.

Thus it is fully justified to work with C^*-algebras as algebras of continuous functions over "virtual" noncommutative spaces. Moreover it is well-founded to consider, among all C^*-algebras, those which admit additional structures corresponding to the usual ones of topological and smooth spaces, such as metric structure and well-behaving differential calculus.

2.2 First order differential calculus

To a compact smooth manifold M we can associate the space of smooth 1-forms $\Omega^1(M)$. This is a bimodule over the algebra $C^\infty(M)$ of smooth functions over M, and determines the first order differential calculus over M. In noncommutative geometry one replaces the algebra of smooth functions with a (quite) general (possibly) noncommutative algebra $\mathcal{A}$. Of course, in such a case we do not have anymore canonically given $\Omega^1(M)$, but it is convenient to have some differential calculus over the algebra A. The problem is that there are different candidates and there is, in general, no reason to prefer one or another. Nevertheless it is possible to study some of the properties of all this calculi, and this is the aim of this section. Of course, here we will give only the main results, omitting many details. We refer to e.g. [GBFV, Wor89] for a more detailed discussion.

In this section A is taken to be a unital associative algebra over the field of complex numbers. The symbol $\otimes$ shall denote the algebraic tensor product over $\mathbb{C}$.

Definition 2.2.1. *A* first order differential calculus *for an algebra A is an A-bimodule Γ together with a linear map $d : A \to \Gamma$, obeying the Leibniz rule $d(ab) = (da)b + adb$, such that any element of Γ can be written as $\sum_k a_k db_k$ for some $a_k, b_k \in A$. The map d is called* differential.

Let now A be a unital algebra and consider the following differential calculus.

Definition 2.2.2. *The* universal differential calculus $\Omega^1 A = (\Gamma_u, d_u)$ *over A is the differential calculus defined by the bimodule $\Gamma_u = \ker(m) \subset A \otimes A$, where $m : A \otimes A \to A$ is the multiplication map, and the differential $d(a) = 1 \otimes a - a \otimes 1$. The bimodule structure is simply given by $a(b \otimes c)a' = ab \otimes ca'$.*

$\Omega^1 A$ (in the following we shall omit the differential d_u) is called the *universal calculus*. The reason for this terminology is the following one.

Proposition 2.2.3. *Let N be a sub-bimodule of $\Omega^1 A = \ker(m)$. Consider the bimodule $\Gamma = \Omega^1 A/N$ and the quotient map π. Let $d = \pi \circ d_u$. Then (Γ, d) is a first order differential calculus. Moreover, any first order differential calculus over A can be obtained in this way.*

Proof. See [Wor89], proposition 1.1. □

We shall use the notation $\Omega^1(A)$ for a general first order differential calculus, like that of proposition 2.2.3.

2.3 Hopf algebras

In this section we recall the definition and the main properties of Hopf algebras. For a more detailed treatment we refer to literature (see e.g. [Abe80, Maj95, Sw69]). We will always work on the field of complex numbers, although Hopf algebras can be defined over any field.

Definition 2.3.1. *A* Hopf algebra *is a unital associative algebra H equipped with two algebra maps, $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathbb{C}$, and a linear map $S : H \to H$ obeying the following relations:*
(i) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
(ii) $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$,
(iii) $m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon$,
where $m : H \otimes H \to H$ denotes the multiplication in H and $\eta : \mathbb{C} \to H$ is the unit map. Δ is called the coproduct*, ε the* counit *and S the* antipode *of H.*

We shall adopt Sweedler's notation [Sw69]: $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ for any $a \in H$. Also, the summation symbol will usually be understood, so that we shall simply write

$$\Delta(a) = a_{(1)} \otimes a_{(2)}.$$

Using Sweedler's notation, we can rewrite the properties (i)-(iii) of the maps Δ, ε and S as follows:
(i) $(a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)} \otimes a_{(2)} = a_{(1)} \otimes (a_{(2)})_{(1)} \otimes (a_{(2)})_{(2)}$,
(ii) $\varepsilon(a_{(1)})a_{(2)} = a_{(1)}\varepsilon(a_{(2)}) = a$,
(iii) $S(a_{(1)})a_{(2)} = a_{(1)}S(a_{(2)}) = \varepsilon(a)1_H$.
In particular, from property (i) (called coassociativity) we see that the following notation, and its natural iterated version, are well defined:

$$(\Delta \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \Delta)\Delta(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.$$

Now let $\tau : H \otimes H \to H \otimes H$ be the switch map, i.e. $\tau(x \otimes y) = y \otimes x$.

Definition 2.3.2. *A Hopf algebra H is called* commutative *if it is commutative as associative algebra. It is called* cocommutative *if $\tau \circ \Delta = \Delta$.*

Theorem 2.3.3. *Let H be a Hopf algebra and S its antipode. Then the following properties hold.*
(i) *$S(hl) = S(l)S(h)$ and $S(1) = 1$, i.e. S is an anti-homomorphism of algebras;*
(ii) *$\varepsilon \circ S = \varepsilon$;*
(iii) *$\tau \circ (S \otimes S) \circ \Delta = \Delta \circ S$. That is,*

$$\Delta(S(h)) = Sh_{(2)} \otimes Sh_{(1)};$$

(iv) *the following conditions are equivalent:*
1. *for any $h \in H$, $S(h_{(2)})h_{(1)} = \eta(\varepsilon(h))$,*
2. *for any $h \in H$, $h_{(2)}S(h_{(1)}) = \eta(\varepsilon(h))$,*
3. *$S^2 = 1$;*

(iv) *if H is both commutative and cocommutative, then $S^2 = 1$.*

Proof. See [Abe80], theorem 2.1.4. □

Lemma 2.3.4. *The antipode of a Hopf algebra is unique.*

Proof. See [Maj95], proposition 1.3.1. □

Lemma 2.3.5. *Let H be a Hopf algebra with invertible antipode S. Then, for any $h \in H$, $\Delta(S^{-1}h) = S^{-1}h_{(2)} \otimes S^{-1}h_{(1)}$ and $h_{(2)}S^{-1}h_{(1)} = (S^{-1}h_{(2)})h_{(1)} = \varepsilon(h)$.*

Proof. See [Maj95], section 1.3 (in particular the solution of exercise 1.3.3). □

Before going on we give some examples of Hopf algebras.

Example 2.3.6. **Group algebra**. Let G be a group and let $\mathbb{C}[G]$ be its group algebra, that is the associative algebra whose elements are (finite) complex linear combinations of elements of G, and whose product is the one induced by the product of G. We can make it into a Hopf algebra by defining the maps Δ, ε and S on the elements of G and extending them by linearity to the whole $\mathbb{C}[G]$. In particular, we set, for any $x \in G$,

$$\Delta(x) = x \otimes x, \qquad \varepsilon(x) = 1, \qquad S(x) = x^{-1}.$$

$\mathbb{C}[G]$ is a cocommutative Hopf algebra, with invertible antipode; moreover $S^{-1} = S$.

Example 2.3.7. **Universal enveloping algebra**. Let $\mathfrak{g}$ be a Lie algebra and let $U\mathfrak{g}$ be its universal enveloping algebra. Then it has a natural Hopf algebra structure, with coproduct, counit and antipode defined as follows: for $X \in \mathfrak{g} \subset U\mathfrak{g}$,

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \varepsilon(X) = 0, \qquad S(X) = -X.$$

Let us come back to the general theory of Hopf algebras. Since, as one would expect, we will usually deal with *-algebras, we recall here the definition and the main properties of a Hopf *-algebra. For the details we refer to section 1.7 of Majid's book [Maj95].

Definition 2.3.8. *A Hopf *-algebra H is a Hopf algebra H equipped with an antilinear involution $*$ which makes it into an associative *-algebra, and such that:*

$$\Delta(h^*) = (h_{(1)})^* \otimes (h_{(2)})^*, \qquad \varepsilon(h^*) = \overline{\varepsilon(h)}, \qquad (S \circ *)^2 = \mathrm{id}.$$

Lemma 2.3.9. *If H is a Hopf *-algebra then the antipode S is invertible, and we have: $(Sh)^* = S^{-1}(h^*)$ for any $h \in H$.*

Proof. See [Pas01], 5.1.20. □

Remark 2.3.10. In this thesis we will always assume the Hopf algebras we work with to have an invertible antipode. There are at least two reasons for making this assumption: first, in general we will deal with Hopf *-algebras, and in this case, due to the previous lemma, this will be an empty requirement; in the second place, the lack of invertibility of the antipode carries some technical difficulties: in particular it is an obstruction to the definition of a Hopf algebra structure on the opposite and the co-opposite algebra of a Hopf algebra (see below). Also, some results about the first order differential calculus (see e.g. [Wor87, Wor89]) and the structure of Hopf-Galois extensions [Sch90a, Sch90b] have been proved under the assumption that the antipode is invertible.

As we have anticipated in the remark above, if the antipode of a Hopf algebra is invertible we can define a Hopf algebra structure over its opposite and coopposite algebras [Maj95]. Indeed, let H be a Hopf algebra with invertible antipode S. Then the opposite algebra H^{op} is still a Hopf algebra, with antipode given by S^{-1}. Moreover we can define the co-opposite algebra H^{cop} as follows: it is isomorphic to H as an associative algebra, but the coproduct is given by $\Delta^{cop} = \tau \circ \Delta$ (that is, $\Delta^{cop}(h) = h_{(2)} \otimes h_{(1)}$) and the antipode is S^{-1}. If we perform both "operations", that is we take the co-opposite algebra of the opposite algebra of H, we get a Hopf algebra isomorphic to H itself.

Let now H be any Hopf algebra, A be an associative algebra, and consider the space of linear maps $H \to A$. We can define a product on this space.

Definition 2.3.11. *Let $f, g : H \to A$ be two linear maps. Their* convolution product *is the linear map $f * g : H \to A$ given by:*

$$(f * g)(h) = f(h_{(1)})g(h_{(2)}).$$

Due to the coassociative property of the coproduct of a Hopf algebra, the convolution product is associative. Assume now that A has a unit, which we can see as a map $\eta_A : \mathbb{C} \to A$. Then $f * (\eta_A \circ \varepsilon) = (\eta_A \circ \varepsilon) * f = f$, so η_A is the identity w.r.t. the convolution product. Moreover:

Definition 2.3.12. *We say that a linear map $f : H \to A$ is* convolution invertible *if there exists a linear map $f^{-1} : H \to A$ such that $f^{-1} * f = f * f^{-1} = \eta_A \circ \varepsilon$.*

2.3.1 Actions and equivariant modules

Let H be a Hopf algebra over $\mathbb{C}$. Then we can consider modules over H.

Definition 2.3.13. *A (left) H-*module *is a pair (V, ρ) where V is a complex vector space and ρ is a representation of H on V (as an associative algebra). We shall usually write $h \triangleright v$ ($h \in H$, $v \in V$) for $\rho(h)v$.*

In the same way one defines right H-modules; we will denote by $v \triangleleft h$ (for $v \in V$, $h \in H$) the right action of H on a right H-module V. We mention here only the simplest example of action of H on a vector space V: the *trivial action* $h \triangleright v = v \triangleleft h = \varepsilon(h)v$.

Now let A be an associative algebra (not necessarily unital). consider the following definitions.

Definition 2.3.14. *The algebra A is a* left H-module algebra *if A is a left H-module and the representation of H respects the algebra structure:*

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b),$$

for any $h \in H$, $a, b \in A$. If A is unital we require moreover that

$$h \triangleright 1_A = \varepsilon(h)1_A, \qquad \forall h \in H.$$

Definition 2.3.15. *The algebra A is a* right H-module algebra *if A is a right H-module and the representation of H respects the algebra structure:*

$$(ab) \triangleleft h = (a \triangleleft h_{(1)})(b \triangleleft h_{(2)}),$$

for any $h \in H$, $a, b \in A$. If A is unital we require moreover that

$$1_A \triangleleft h = \varepsilon(h) 1_A, \qquad \forall h \in H.$$

Definition 2.3.16. *The algebra A is an H-*bimodule algebra *if it is both a left and a right H-module algebra, and the left and the right actions commute with each other: $(h \triangleright a) \triangleleft l = h \triangleright (a \triangleleft l)$, for any $h, l \in H$ and any $a \in A$.*

We notice that the trivial action of H on an algebra A makes it into both a left and a right H-module algebra (and also into a H-bimodule algebra, of course).

Now let A be a left H-module algebra and M be a left A-module; we denote simply by am the action of $a \in A$ on $m \in M$. Then we consider the following definition.

Definition 2.3.17. *M is a (left) H-*equivariant A-module *if M is itself a left H-module and*

$$h \triangleright (am) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright m)$$

for any $h \in H$, $a \in A$ and $m \in M$.

In the same way can be defined right H-equivariant A-modules. We conclude this paragraph introducing an additional condition on the action of H on an algebra A in case of *-algebras. Let H be a Hopf *-algebra, A be a *-algebra and assume that A is a (left) H-module algebra, in the sense of definition 2.3.14. Then:

Definition 2.3.18. *The action of H on A is said to be* compatible with the star structure *if*

$$(h \triangleright a)^* = (Sh)^* \triangleright a^*$$

for any $h \in H$, $a \in A$.

When we will consider H-module *-algebras w.r.t. a Hopf *-algebra we will always assume, unless otherwise indicated, that the action of H on A is compatible with the star structure.

2.3.2 Coactions and comodules

Again, let H be a Hopf algebra over $\mathbb{C}$. The coalgebra structure of H allows us to give the following definitions.

Definition 2.3.19. *Let V be a complex vector space. A linear map $\rho_R : V \to V \otimes H$ is said to be a* right coaction *of H on V if:*

$$(\rho_R \otimes \mathrm{id}) \circ \rho_R = (\mathrm{id} \otimes \Delta) \circ \rho_R,$$

$$(\mathrm{id} \otimes \varepsilon) \circ \rho_R = \mathrm{id}.$$

*If ρ_R is a right coaction, V is called a right H-*comodule*.*

Definition 2.3.20. *Let V be a complex vector space. A linear map $\rho_L : V \to H \otimes V$ is said to be a* left coaction *of H on V if:*

$$(\mathrm{id} \otimes \rho_L) \circ \rho_L = (\Delta \otimes \mathrm{id}) \circ \rho_L,$$

$$(\varepsilon \otimes \mathrm{id}) \circ \rho_L = \mathrm{id}.$$

*If ρ_L is a left coaction, V is called a left H-*comodule*.*

For H-comodules we introduce the analogue of Sweedler's notation:

$$\rho_R(v) = v_{(0)} \otimes v_{(1)},$$

$$\rho_L(v) = v_{(-1)} \otimes v_{(0)}.$$

If V is both a left and a right H-comodule we can consider the following definition.

Definition 2.3.21. *V is a H-*bi-comodule* if it is both a left and a right H-comodule, with coactions ρ_L and ρ_R respectively, and*

$$(\rho_L \otimes \mathrm{id}) \circ \rho_R = (\mathrm{id} \otimes \rho_R) \circ \rho_L.$$

As in the case of actions of a Hopf algebra, we can consider coactions on associative algebras. So, let A be an algebra (non necessarily unital); then:

Definition 2.3.22. *The algebra A is a left (right) H-*comodule algebra* if it is an H-comodule and the coaction ρ_L (ρ_R) is an algebra map. If A has a unit we also require that $\rho_{L,R}(1) = 1 \otimes 1$.*

In case A is a *-algebra and H is a Hopf *-algebra, we require also the compatibility between the star structure and the comodule structure:

$$\rho_{L,R} \circ {}^* = ({}^* \otimes {}^*) \circ \rho_{L,R}. \tag{2.3.1}$$

We give here also the following definition, which we shall use in the definition of quantum principal bundles.

Definition 2.3.23. *Let A be a left H-comodule algebra, with left coaction ρ_L. The* invariant subalgebra *of A is the subalgebra*

$$A^{coH} = \{a \in A \mid \rho_L(a) = 1 \otimes a\}.$$

If instead A is a right H-comodule algebra, with coaction ρ_R, then we define:

$$A^{coH} = \{a \in A \mid \rho_R(a) = a \otimes 1\}.$$

Let us mention here one important example of H-comodule algebras: to any Hopf algebra H can be given a structure of (right) H-comodule algebra via the *right adjoint coaction*,

$$\mathrm{ad}_R(h) = h_{(2)} \otimes S(h_{(1)})h_{(3)}. \tag{2.3.2}$$

Let now A be a (right) H-comodule algebra, with coaction ρ_R, and let B be another associative algebra. Then, generalizing (2.3.11), we can define the *right convolution product* of a map $f : A \to B$ with a map $g : H \to B$ as the map $f *_R g : A \to B$ defined by:

$$(f *_R g)(a) = f(a_{(0)}g(a_{(1)}). \tag{2.3.3}$$

In a similar way one defines the *left convolution product* $*_L$ for left H-comodule algebras.

Finally, as for H-module algebras, we can define H-equivariant A-modules for H-comodule algebras. So let A be a left H-comodule algebra and let M be a left A-module; we denote by am, $a \in A$ $m \in M$, the action of A on M.

Definition 2.3.24. *M is a* left H-equivariant (left) A-module *if it is itself a left H-comodule, with coaction ρ'_L, and*

$$\rho'_L(am) = a_{(-1)}m_{(-1)} \otimes a_{(0)}m_{(0)}.$$

A similar definition can of course be given for right A-modules and/or for right H-comodule algebras.

2.3.3 First order differential calculus over Hopf algebras

As for any associative algebra, we can endow a Hopf algebra with a first order differential calculus (see section 2.2). The coalgebra structure of a Hopf algebra allows us to consider a special class of first order differential calculi, which satisfy suitable properties of covariance with respect to the coproduct of the Hopf algebra.

So, let us consider a (general) first order differential calculus $\Omega^1(H)$ on a Hopf algebra H. We know that each element of $\Omega^1(H)$ can be written as a sum $\sum_k a_k db_k$ with $a_k, b_k \in H$. Then we can give the following definitions.

Definition 2.3.25. *The first order differential calculus $\Omega^1(H)$ is* left-covariant *if, for any $a_k, b_k \in H$,*

$$\sum_k a_k db_k = 0 \quad \Rightarrow \quad \sum_k \Delta(a_k)(\mathrm{id} \otimes d)\Delta(b_k) = 0.$$

Definition 2.3.26. *The first order differential calculus $\Omega^1(H)$ is* right-covariant *if, for any $a_k, b_k \in H$,*

$$\sum_k a_k db_k = 0 \quad \Rightarrow \quad \sum_k \Delta(a_k)(d \otimes \mathrm{id})\Delta(b_k) = 0.$$

Definition 2.3.27. *$\Omega^1(H)$ is a* bicovariant *first order differential calculus if it is both left-covariant and right-covariant.*

One of the most relevant properties of left/right-covariant calculi is that they can be characterized by a right ideal $R \subseteq \ker(\varepsilon)$ of H. In order to state this result we need to introduce two maps $r, s : H \otimes H \to H \otimes H$ [Wor87, Wor89]:

$$r(a \otimes b) = (a \otimes 1)\Delta(b) = ab_{(1)} \otimes b_{(2)}, \tag{2.3.4}$$

$$s(a \otimes b) = (1 \otimes a)\Delta(b) = b_{(1)} \otimes ab_{(2)}. \tag{2.3.5}$$

Both r and s are bijections, and their inverses are given by [Wor87, Wor89]:

$$r^{-1}(a \otimes b) = (a \otimes 1)(S \otimes \mathrm{id})\Delta(b) = aS(b_{(1)}) \otimes b_{(2)}, \tag{2.3.6}$$

$$s^{-1}(a \otimes b) = (b \otimes 1)\tau(\mathrm{id} \otimes S^{-1})\Delta(a) = bS^{-1}a_{(2)} \otimes a_{(1)}. \tag{2.3.7}$$

Using these maps we can state the following theorems.

Theorem 2.3.28. *Let $R \subset \ker(\varepsilon)$ be a right ideal of H and let $N = r^{-1}(H \otimes R)$. Then N is a sub-bimodule of* $\ker(m)$. *Moreover, let $\Omega^1(H) = \ker(m)/N$, let $\pi : \ker(m) \to \Omega^1(H)$ be the canonical projection and let $d = \pi \circ d_u$. Then the first order differential calculus $(\Omega^1(H), d)$ is left-covariant. Moreover, any left-covariant first order differential calculus on H can be obtained in this way.*

Proof. See [Wor89], theorem 1.5. □

Theorem 2.3.29. *Let $R \subset \ker(\varepsilon)$ be a right ideal of H and let $N = s^{-1}(H \otimes R)$. Then N is a sub-bimodule of* $\ker(m)$. *Moreover, let $\Omega^1(H) = \ker(m)/N$, let $\pi : \ker(m) \to \Omega^1(H)$ be the canonical projection and let $d = \pi \circ d_u$. Then the first order differential calculus $(\Omega^1(H), d)$ is right-covariant. Moreover, any right-covariant first order differential calculus on H can be obtained in this way.*

Proof. See [Wor89], theorem 1.6. □

Even more interesting is the structure of bicovariant calculi. And this is the class we are interested in (at least, it is the relevant class of differential calculi we need in order to define quantum principal bundles). Before stating the main result we need a definition. We recall that ad_R is the right adjoint coaction, see equation (2.3.2).

Definition 2.3.30. *A linear subset $Q \subseteq H$ is* ad_R-invariant *if $\mathrm{ad}_R(Q) \subseteq Q \otimes H$.*

Theorem 2.3.31. *Let $Q \subset \ker(\varepsilon)$ be a right ideal of H, let $N = r^{-1}(H \otimes Q)$ and let $\Omega^1(H)$ be the associated differential calculus, like in theorem 2.3.28. Then $\Omega^1(H)$ is bicovariant if and only if Q is ad_R-invariant*

Proof. See [Wor89], theorem 1.8. □

In particular any bicovariant differential calculus can be realized in this way. It is clear that the universal calculus $\Omega^1 H$ is a bicovariant calculus (it corresponds to the choice $Q = \{0\}$). Also, we see that any Hopf algebra admits a non-trivial bicovariant differential calculus. Indeed, take $Q = \ker(\varepsilon)$: it is ad_R-invariant,

$$(\varepsilon \otimes \mathrm{id})\mathrm{ad}_R(h) = \varepsilon(h_{(2)})S(h_{(1)})h_{(3)} = S(h_{(1)})h_{(2)} = \varepsilon(h) = 0$$

for any $h \in \ker(\varepsilon)$, and therefore it determines a bicovariant first order differential calculus.

We conclude this paragraph by noticing the fact that a bicovariant differential calculus $\Omega^1(H)$ admits a natural structure of H-bi-comodule, with coactions $\Delta_L^\Omega : \Omega^1(H) \to H \otimes \Omega^1(H)$ and $\Delta_R^\Omega : \Omega^1(H) \to \Omega^1(H) \otimes H$ defined as follows:

$$\Delta_L^\Omega(adb) = a_{(1)}b_{(1)} \otimes a_{(2)}db_{(2)}, \tag{2.3.8}$$

$$\Delta_R^\Omega(adb) = a_{(1)}db_{(1)} \otimes a_{(2)}b_{(2)}. \tag{2.3.9}$$

Both coactions are well defined. Moreover, $\Omega^1(H)$ is both a left and a right H-equivariant H-module. We refer to [Wor89], sections 1 and 2, for the proof of these results.

2.4 K-theory

K-theory is a generalized cohomological theory. Its topological version was first introduced by Sir M. F. Atiyah and F. Hirzebruch in 1959 [AtiHir59, AtiHir61]; they defined the K group $K(X)$ of a topological space X to be the Grothendieck group of stable isomorphism classes of topological vector bundles over X. K-theory proved to be a very useful tool and to have applications in many fields of mathematics (see, e.g., the discussion in [Bak87, C94]). Moreover, Serre-Swan theorem [Ser55, Swa62] allows for a reformulation of K-theory, where topological vector bundles over X are replaced by finitely generated projective modules over $C(X)$. This was generalized to K-theory of C^*-algebras (see [Blck98] and references therein) and it is a main tool in noncommutative topology and noncommutative geometry (see [C94] and references therein).

In this section we will briefly recall the definition and the properties of K-theory of C^*-algebras. Moreover, in the last part, we will discuss the relation between the algebraic K-theory of a pre-C^*-algebra with that of its C^*-completion.

2.4.1 The K_0 group of a C^*-algebra

First of all we present the construction of the K_0 group of a C^*-algebra and we study its main properties. We begin by considering unital C^*-algebras. In the next section we will extend the construction to non-unital algebras using the functoriality properties of K_0. For the details we refer to [Lan03].

Let us begin by discussing some properties of projections in a C^*-algebra. We recall that a projection in a C^*-algebra B is a selfadjoint element p of B such that $p^2 = \mathrm{id}$.

Definition 2.4.1. *Let B be a unital C^*-algebra. Two projections $p, q \in B$ are called:*
- homotopy equivalent*, written $p \sim_h q$, if there is a path $e(t)$ of projections in B such that $e(0) = p$, $e(1) = q$;*

- unitarily equivalent, *written* $p \sim_u q$, *if there is a unitary* $u \in B$ *such that* $q = upu^*$.

Lemma 2.4.2. *Let* p, q *be projections in a unital* C^**-algebra* B*. Then if* p *and* q *are homotopy equivalent then they are unitarily equivalent.*

In general the converse is not true. Now let A be a unital C^*-algebra and let $M_n(A)$ be the space of $n \times n$ matrices with coefficients in A. Consider the disjoint union $\mathcal{M}(A) = \coprod_{n=1}^{\infty} M_n(A)$. We define the *direct sum* of an element $a \in M_k(A)$ with an element $b \in M_l(A)$ as the element

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

in $M_{k+l}(A)$. Next we introduce an equivalence relation on $\mathcal{M}(A)$ as follows: for $p \in M_n(A)$ and $q \in M_m(A)$ we say that $p \sim q$ if $q = p \oplus 0_{m-n}$ (for $m > n$) or $p = q \oplus 0_{n-m}$ (for $n < m$), where 0_k is the zero $k \times k$ matrix. Then we give the following definition.

Definition 2.4.3. *For a* C^**-algebra* A *we define the space* $M_\infty(A)$ *to be the quotient* $\mathcal{M}(A)/\sim$.

We make $M_\infty(A)$ into a *-algebra in the following way: any two classes $[a], [b]$ in $M_\infty(A)$ come from two elements $a, b \in M_n(A)$ for some n, so we can define

$$[a] + [b] = [a + b], \qquad [a] \cdot [b] = [ab];$$

the star structure is simply the one induced by that of $M_n(A)$: $[a]^* = [a^*]$. In the same way we can put a norm on $M_\infty(A)$: $\|[a]\| = \|a\|$. Taking the closure w.r.t. this norm we get a C^*-algebra.

Now we focus our attention on projections. In particular, we define $P_\infty(A)$ as the set of projections in $M_\infty(A)$. We have seen above that in general homotopic equivalence is stronger than unitarily equivalence. A relevant fact is that this is no longer true for P_∞:

Lemma 2.4.4. *The equivalence relations* $\sim_h$ *and* $\sim_u$ *on the set* $P_\infty(A)$ *coincide.*

Now we can define the K_0 group of a unital C^*-algebra A.

Definition 2.4.5. *Let* A *be a unital* C^**-algebra. Then we define* $K_0(A)$ *as the abelian group with one generator for each equivalence class* p *of projections* $p \in P_\infty(A)$ *under the equivalence relation* $\sim_h = \sim_u$*, and addition* $[p] + [q] = [p \oplus q]$ *between these generators.*

The geometric meaning of this definition can be deduced from the result below, recalling that spaces of continuous sections of vector bundles over a topological space X are in correspondence with finitely generated projective modules over $C(X)$.

Proposition 2.4.6. *Let* A *be a unital* C^**-algebra. Let* $p \in M_n(A)$ *and* $q \in M_m(A)$*. Then* $p \sim_h q$ *in* $P_\infty(A)$ *iff* pA^n *and* qA^m *are isomorphic as right* A*-modules.*

Therefore $K_0(A)$ can also be seen as the (Grothendieck) group of isomorphism classes of finitely generated projective modules over A.

2.4.2 K-theory functors and functoriality of K_0

Now that we have defined K_0, we can show that it is actually a functor, and this allows us to introduce the general notion of K-theory functor. We begin with some preliminary definitions.

Definition 2.4.7. *Let A be a C^*-algebra and let $\mathcal{K}$ be the C^*-algebra of compact operators on a separable Hilbert space. Then the tensor product $A_S = \mathcal{K} \otimes A$ is the completion of the algebraic tensor product $\mathcal{K} \odot A$ in the unique*[1] *C^*-norm and it is called the* stabilization *of $\mathcal{A}$. A C^*-algebra B is called* stable *whenever $B_S \simeq B$. Two C^*-algebras B and C are said* stably equivalent *if $B_S \simeq C_S$.*

Definition 2.4.8. *Let A and B be C^*-algebras. Two morphisms of C^*-algebras $\varphi, \psi : A \to B$ are called* homotopic*, written $\varphi \sim_h \psi$, if there is a path $\phi_t : A \to B$ of morphisms of C^*-algebras for which the function $\phi_t(a)$ is continuous in $t \in [0,1]$ for each $a \in A$ and $\phi_0 = \varphi$, $\phi_1 = \psi$.*

Definition 2.4.9. *Two C^*-algebras A and B are said to be* homotopy equivalent*, written $A \sim_h B$, if there are morphisms $\alpha : A \to B$ and $\beta : B \to A$ such that $\alpha \circ \beta \sim_h \mathrm{id}_B$ and $\beta \circ \alpha \sim_h \mathrm{id}_A$.*

Definition 2.4.10. *A C^*-algebra A is called* contractible *if $A \sim_h 0$.*

Definition 2.4.11. *The* cone *of a C^*-algebra A is the C^*-algebra $CA = C_0((0,1], A)$, that is the algebra of continuous functions from $(0,1]$ to A vanishing at zero.*

The suspension *of a C^*-algebra A is the C^*-algebra $SA = C_0((0,1), A)$, that is the algebra of continuous functions from $(0,1)$ to A vanishing at 0 and at 1.*

Lemma 2.4.12. *The cone of any C^*-algebra is contractible.*

Definition 2.4.13. *A functor H from C^*-algebras to abelian groups is called* half-exact *if, given the short exact sequence of C^*-algebras*

$$0 \to J \to A \to A/J \to 0,$$

the corresponding sequence of abelian groups is exact at $H(A)$:

$$H(J) \to H(A) \to H(A/J).$$

Definition 2.4.14. *A functor H from C^*-algebras to abelian groups is called a* K-theory functor *if it has the following properties*

(a) *it is* normalized*: either $H(\mathbb{C}) = \mathbb{Z}$ or $H(\mathbb{C}) = 0$;*

(b) *it is* homotopy-invariant*: if A and B are homotopy equivalent, then $H(A) \simeq H(B)$;*

(c) *it is* stable*: $H(A_S) = H(A)$;*

(d) *it is* continuous*: it commutes with inductive limits;*

(e) *it is* half-exact.

Theorem 2.4.15. *The functor K_0 is a K-theory functor.*

[1] See [GBFV], section 1.A.

The property of half-exactness of the functor K_0 allows us to define in a satisfactory way the K_0 group of a non-unital C^*-algebra A. And, although we shall always deal with unital algebras in this thesis, we need to define the K_0 group of a non-unital algebra since we shall use suspension algebras, which are non-unital, to define higher-rank K-theory groups.

So, let us consider a non-unital C^*-algebra A, and let us denote by A^+ the unitization of A obtained adjoining a unit: $A^+ = A \times \mathbb{C}$, with product $(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$ and sum $(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu)$ [GBFV]. Then we have the short exact sequence

$$0 \to A \to A^+ \to \mathbb{C} \to 0.$$

Hence, if $\pi : A^+ \to A^+/A \simeq \mathbb{C}$ denotes the canonical surjection, we define $K_0(A)$ as the kernel of $\pi_* : K_0(A^+) \to K_0(\mathbb{C})$. In this way we extend K_0 to a (covariant) functor from the category of C^*-algebras, unital and non-unital, to that of abelian groups, this extension still being a K-theory functor.

2.4.3 The K_1 group of a C^*-algebra

There are several different (but equivalent) ways to define the K_1 group of a C^*-algebra. Here we present one of them, while in the next section we will introduce a general construction for higher-rank K-theory groups K_n, and we will show that in the $n = 1$ case it reduces to the definition presented here. We refer to [Blck98, GBFV, C85, C94, Kar78, W-O93] for the details.

Let A be a C^*-algebra and let A^+ denote its unitization. Let $GL_n(A^+)$ be the set of invertible $n \times n$ matrices with values in A^+; we define $GL_n(A)$ to be the group

$$GL_n(A) = \{x \in GL_n(A^+) \mid x \equiv \mathrm{id}_n \bmod M_n(A)\}.$$

It can be shown that $GL_n(A)$ is a normal closed subgroup of $GL_n(A^+)$. Next we embed $GL_n(A)$ into $GL_{n+1}(A)$ in the following way:

$$x \in GL_n(A) \quad \mapsto \quad \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally we define $GL_\infty(A)$ to be the direct limit of the groups $GL_n(A)$. Let $GL_\infty(A)_0$ denote the connected component of the identity in $GL_\infty(A)$. Then we can give the following definition.

Definition 2.4.16. *The K_1 group of the C^*-algebra A is the quotient $K_1(A) = GL_\infty(A)/GL_\infty(A)_0$.*

There is an equivalent characterization of $K_1(A)$. Let $U_\infty(A)$ be the group of unitary matrices defined in the same way as $GL_\infty(A)$ and let $U_\infty(A)_0$ be the connected component of the identity of $U_\infty(A)$. Then one can prove that [W-O93]:

Lemma 2.4.17. *$K_1(A)$ is isomorphic to $U_\infty(A)/U_\infty(A)_0$.*

In particular an element of K_1 can be seen as an equivalence class $[u]$, with $u \in U_n(A^+)$ for some $n > 0$, and the product in $K_1(A)$ can be written in the following way:

$$[u][v] = [uv] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right].$$

This allows us to see that $K_1(A)$ is an abelian group. One can also prove the following result [W-O93].

Proposition 2.4.18. *K_1 is a K-theory functor.*

2.4.4 Higher K-theory groups and Bott periodicity

In the previous section we introduced the concept of suspension of a C^*-algebra (see definition 2.4.11). It can be proved that it gives rise to a covariant functor [W-O93], the *suspension functor*, which we shall denote by S. Also, one can prove that S is an exact functor[2]. Now we will use this functor to introduce higher dimensional K-theory groups and to state one of the main results of algebraic K-theory: the Bott periodicity.

Let us begin by noticing the following fact: the K_1 group of a C^*-algebra A is nothing else than the K_0 group of its suspension. More precisely,

Theorem 2.4.19. *For every C^*-algebra A there is an isomorphism $\theta_A : K_1(A) \to K_0(SA)$ which, whenever $\alpha : A \to B$ is a C^*-algebras morphism, makes the following diagram commutative:*

$$\begin{array}{ccc} K_1(A) & \xrightarrow{\alpha_*} & K_1(B) \\ \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_B} \\ K_0(SA) & \xrightarrow[S\alpha_*]{} & K_0(SB) \end{array}$$

Proof. See [W-O93], theorem 7.2.5. □

This result allows us to give the following definition.

Definition 2.4.20. *The* n-th K-theory group *of a C^*-algebra A is the abelian group $K_n(A) = K_0(S^n A)$.*

Proposition 2.4.21. *For any $n \geq 0$, K_n is a K-theory functor.*

The following results show that the only relevant groups are K_0 and K_1:

Theorem 2.4.22. *For every C^*-algebra A there is an isomorphism $\beta_A : K_0(A) \to K_1(SA)$ such that, for every morphism $\alpha : A \to B$, the following diagram is commutative:*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\alpha_*} & K_0(B) \\ \downarrow{\scriptstyle \beta_A} & & \downarrow{\scriptstyle \beta_B} \\ K_1(SA) & \xrightarrow[S\alpha_*]{} & K_1(SB) \end{array}$$

[2]A functor $F : \mathcal{C} \to \mathcal{D}$, between two abelian categories [McL] $\mathcal{C}$ and $\mathcal{D}$, is said an *exact functor* if it carries exact sequences into exact sequences. More precisely, if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, $0 \to F(A) \to F(B) \to F(C) \to 0$ is a short exact sequence in $\mathcal{D}$ (see [McL], VIII.3).

Proof. See [W-O93], chapter 9. □

Corollary 2.4.23 (Bott periodicity). *For every C^*-algebra A, $K_0(A) \simeq K_0(S^2A)$ and $K_1(A) \simeq K_1(S^2A)$.*

2.4.5 Algebraic K-theory

The definition of the K-theory groups introduced in the previous sections is valid only for C^*-algebras. We introduce now a more general formulation, entirely algebraic, which is applicable to any algebra. We refer to [GBFV, Lo, Kar78, Ros97] for the details of the construction.

Let A be a unital algebra. Consider the following definition.

Definition 2.4.24. *The* algebraic K-theory group $K_0^{alg}(A)$ *of a unital algebra A is the Grothendieck group of isomorphism classes of finitely generated projective right A-modules.*

In the case of C^*-algebras there is no difference with the previous definition. Indeed (see [GBFV], theorem 3.14),

Theorem 2.4.25. *There is an isomorphism $K_0^{alg}(A) \simeq K_0(A)$ for any unital C^*-algebra A.*

Now we introduce the definition of the algebraic K_1 group. As we shall see, it does not coincide with the K_1 group defined above.

Definition 2.4.26. *The* algebraic K-theory *group $K_1^{alg}(A)$ of a unital algebra A is the abelianization $GL_\infty(A)_{ab}$ of the group $GL_\infty(A)$, that is, the quotient of $GL_\infty(A)$ by the subgroup of the commutators.*

We can compare it to the group $K_1(A)$. Let A be a unital C^*-algebra. Then $GL_\infty(A)$ is a topological group (see discussion above). If we endow $GL_\infty(A)$ with the discrete topology we get another topological group, which we denote by $GL_\infty^{disc}(A)$. One can show that the identity map $GL_\infty^{disc}(A) \to GL_\infty(A)$ induces an homomorphism $K_1^{alg}(A) \to K_1(A)$. More precisely (see [GBFV], section 3.7; see also [Ros97]),

Proposition 2.4.27. *Let A be a unital C^*-algebra. Then the identity map $GL_\infty^{disc}(A) \to GL_\infty(A)$ induces an homomorphism $K_1^{alg}(A) \to K_1(A)$ which is surjective and functorial in A.*

Under some conditions on the algebra A this morphism is also injective.

Proposition 2.4.28. *Let A be a* stable *unital C^*-algebra, that is a C^*-algebra such that $A \simeq A \otimes \mathcal{K}$. Then the map of proposition 2.4.27 is an isomorphism of groups.*

Proof. See [Ros97], theorem 1.4. □

It is possible to define also higher dimensional algebraic K-theory groups, but this is far beyond the aim of this thesis. So we shall not discuss them here.

2.4.6 K-theory of pre-C^*-algebras

In this thesis we shall often deal with pre-C^*-algebras: indeed, if we take a look to the commutative case, we see that the algebra of continuous functions over a compact smooth manifold M, $C(M)$, is a C^*-algebra, while the subalgebra of smooth functions, $C^\infty(M)$, is only a pre-C^*-algebra. Since in Connes' noncommutative geometry it is the latter to be involved, we need to define also K-theory of pre-C^*-algebras. In this section we discuss the main properties of this version of K-theory. We refer to [GBFV] for the details. First of all we recall the definition of a pre-C^*-algebra [GBFV].

Definition 2.4.29. *A* pre-C^*-algebra *is a subalgebra of a C^*-algebra that is stable under holomorphic functional calculus.*

Now, let $\mathcal{A}$ be a pre-C^*-algebra. We define its K_0 group to be its algebraic K-theory group K_0^{alg}. That is,

Definition 2.4.30. *The K_0 group of a pre-C^*-algebra $\mathcal{A}$ is the Grothendieck group of isomorphism classes of finitely generated projective right $\mathcal{A}$-modules.*

Now we shall study the relation between the K_0 group of a pre-C^*-algebra and that of its C^*-completion. We begin by stating the following result [Schw92].

Proposition 2.4.31. *Let $\mathcal{A}$ be a pre-C^*-algebra. Then $M_n(\mathcal{A})$ is a pre-C^*-algebra for all n.*

This result implies (see [GBFV], section 3.8) that, if A is the completion of $\mathcal{A}$, the inclusion $\iota : \mathcal{A} \to A$ extends to a morphism $K_0\iota : K_0(\mathcal{A}) \to K_0(A)$. Moreover, one can prove the following fact (see [GBFV], theorem 3.44).

Theorem 2.4.32. *If $\mathcal{A}$ is a Fréchet pre-C^*-algebra with C^*-completion A, the inclusion $\iota : \mathcal{A} \to A$ induces an isomorphism $K_0\iota : K_0(\mathcal{A}) \to K_0(A)$.*

Therefore, the K-theory of Fréchet pre-C^*-algebras is the same as that of the corresponding C^*-completions.

2.5 Hochschild (co)homology

In this section we introduce Hochschild homology and Hochschild cohomology. We will give only a brief discussion, referring to classical literature for the details (see e.g. [CE, GM, GBFV]).

Let A be a unital associative algebra over $\mathbb{C}$ and let A° be its opposite algebra. We define the *enveloping algebra* of A to be the unital associative algebra $A^e = A \otimes A^\circ$. Now let M be an A-bimodule. We can regard it as a right A^e-module, where the right action of A^e on M is given by

$$m(a \otimes b^\circ) = bma,$$

for $m \in M$, $a, b \in A$. Then we can give the following definition

Definition 2.5.1. *Given an A-bimodule M, we define the n-*th Hochschild homology group *of A with values in M to be the group $H_n(A, M) = \mathrm{Tor}_n^{A^e}(M, A)$.*

That is, $H_n(A,-)$ is the n-th left derived functor [HS] of the functor $M \otimes_{A^e} -$. Using the bar resolution to compute $\mathrm{Tor}_n^{A^e}$ we can give an alternative description of Hochschild homology. Skipping some details, we say that the Hochschild homology can be defined to be the homology of the following complex: let $C_n(M,A) = M \otimes A^{\otimes n}$ and let $b_n : C_n \to C_{n-1}$ be the map

$$\begin{aligned} b(m \otimes a_1 \otimes \ldots \otimes a_n) =& m a_1 \otimes \ldots \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\ &+ (-1)^n a_n m \otimes a_1 \otimes \ldots \otimes a_{n-1}. \end{aligned} \tag{2.5.1}$$

Then we can see, by direct computation, that $b_n b_{n-1} = 0$. We call b the collection of the maps b_n. We put a structure of A-bimodule on $C_n(M,A)$ in the following way:

$$a(m \otimes a_1 \otimes \ldots a_n)b = am \otimes a_1 \otimes \ldots \otimes a_n b.$$

We call $C_n(M,A)$ the space of *Hochschild n-chains* with values in M, and we define the Hochschild homology to be the homology of the complex $(C_\bullet(M,A), b)$. The elements ξ of $C_n(M,A)$ such that $b\xi = 0$ will be called *Hochschild n-cycles* and the elements ξ of $C_n(M,A)$ such that there is $\eta \in C_{n+1}(M,A)$ with $b\eta = \xi$ will be called *Hochschild n-borders.*

Now we turn to Hochschild cohomology. Consider, again, an A-bimodule M. Now we see it as a left A^e-module, with action of A^e on M given by

$$(a \otimes b^\circ)m = amb.$$

Then we give the following definition.

Definition 2.5.2. *Given an A-bimodule M, we define the n*-th Hochschild cohomology group *of A with values in M to be the group $H^n(A,M) = \mathrm{Ext}^n_{A^e}(A,M)$*

This means that the n-th cohomology group is the left n-derived functor of the functor $\mathrm{Hom}(-,M)$. Also in this case, using the bar resolution, we can obtain an equivalent definition of Hochschild cohomology. Consider, indeed, the following complex. Let $C^n(A,M)$ be the space of n-linear maps $\varphi : A^n \to M$. Put on $C^n(A,M)$ the following structure of A-bimodule:

$$(a\varphi b)(a_1, \ldots, a_n) = a\varphi(a_1, \ldots, a_n)b.$$

Then define a map $b : C^n(A,M) \to C^{n+1}(A,M)$ by

$$\begin{aligned} b_n\varphi(a_1, \ldots, a_{n+1}) =& a_1\varphi(a_2, \ldots, a_{n+1}) \\ &+ \sum_{j=1}^{n} (-1)^j \varphi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1}\varphi(a_1, \ldots, a_n)a_{n+1}. \end{aligned} \tag{2.5.2}$$

Let now b denote the collection of the operators b_n. Then one can see that $b^2 = 0$. Hence

Hochschild cohomology can be defined as the cohomology of the complex $(C^\bullet(A, M), b)$. The elements of $C_n(A, M)$ will be called *Hochschild n-cochains*, the elements $\varphi \in C^n(A, M)$ such that $b\varphi = 0$ *Hochschild n-cocycles* and those such that there exists $\psi \in C^{n-1}(A, M)$ with $b\psi = \varphi$ *Hochschild n-coborders*. The space of Hochschild n-cocycles will be denoted by $Z^n(A, M)$, that one of n-coborders by $B^n(A, M)$.

2.6 Cyclic cohomology

In this section we introduce an analogue for the de Rham cohomology for noncommutative spaces [C85], the cyclic cohomology, and we discuss briefly its relation with Hochschild homology and K-theory. We begin with the basic definitions.

Let A be a unital associative algebra and let A^* be the space of all linear functionals from A to $\mathbb{C}$. A^* is a bimodule over A, and therefore one can consider Hochschild n-cochains with values in A^*. Then we can notice [GBFV] that a Hochschild n-cochain $\varphi \in C^n(A, A^*)$ can be seen as a linear map $\varphi : A^{\otimes^{n+1}} \to \mathbb{C}$. Under this identification the Hochschild coboundary map b is given by:

$$\begin{aligned} b\varphi(a_0, \dots, a_{n+1}) =& \varphi(a_0a_1, a_2, \dots, a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \varphi(a_0, \dots, a_ia_{i+1}, \dots, a_{n+1}) \\ &+ (-1)^{n+1}\varphi(a_{n+1}a_0, \dots, a_n). \end{aligned}$$

Now let $\lambda : C^\bullet(A, A^*) \to C^\bullet(A, A^*)$ be the operator, of degree 0, defined as follows:

$$\lambda\varphi(a_0, \dots, a_n) = (-1)^n\varphi(a_n, a_0, \dots, a_{n-1}). \tag{2.6.1}$$

Definition 2.6.1. *A Hochschild n-cochain $\varphi \in C^n(A, A^*)$ is called* cyclic *if $\lambda\varphi = \varphi$. A Hochschild n-cocycle $\varphi \in Z^n(A, A^*)$ is called* cyclic *if $\lambda\varphi = \varphi$.*

Let us denote by $C^n_\lambda(A)$ the space of cyclic n-cochains and by $Z^n_\lambda(A)$ the space of cyclic n-cocycles. One can prove the following result (see [C85], part II, corollary 4.).

Proposition 2.6.2. *$(C^\bullet_\lambda(A), b)$ is a subcomplex of the Hochschild complex.*

Therefore we can take the cohomology $HC^\bullet(A) = H^\bullet_\lambda(A)$ of $(C^\bullet_\lambda(A), b)$, and we call it the *cyclic cohomology* of A. Notice that $HC^0(A) = Z^0_\lambda(A)$ is the linear space of traces on A.

2.6.1 Cycles and Chern characters

Definition 2.6.3. *Let $\Omega = \bigoplus_{i=0}^{n} \Omega^i$ be a graded differential algebra, with differential d of degree 1. An* integral *on Ω is a linear map $\int : \Omega \to \mathbb{C}$ such that:*

(i) *$\int \omega_k = 0$ for $\omega_k \in \Omega^k$, $k < n$;*

(ii) *if $\omega_k \in \Omega_k$ and $\omega_l \in \Omega_l$ then $\int \omega_k\omega_l = (-1)^{kl} \int \omega_l\omega k$;*

(iii) *if* $\omega_{n-1} \in \Omega^{n-1}$ *then* $\int d\omega_{n-1} = 0$.

Definition 2.6.4. *A* cycle of dimension n *is a complex graded algebra differential algebra* (Ω, d), $\Omega = \bigoplus_{i=0}^{n} \Omega^i$, *together with an integral* $\int$.

Given two cycles it is straightforward to define their direct sum cycle and their tensor product cycle [C85].

Next we define cycles over an algebra A.

Definition 2.6.5. *Let* A *be an associative algebra. A* cycle *over* A *is an* n*-dimensional cycle* $(\Omega, d, \int)$ *together with a homomorphism* $\rho : A \to \Omega^0$.

Notice that if $(\Omega, d, \int)$ is a cycle over A, for any $a_0, \ldots, a_k \in A$ the object $a_0 da_1 \cdots da_k$ defines an element of Ω^k (here the map ρ is understood).

Now, given a cycle over A of dimension n we can canonically associate to it a cyclic n-cocycle:

Definition 2.6.6. *The* Chern character *of a cycle* $(\Omega, d, \int)$ *of dimension* n *over a unital associative algebra* A *is the* $(n+1)$*-linear functional defined by*

$$\mathrm{ch}_\Omega(a_0, \ldots, a_n) = \int a_0 da_1 \cdots da_n,$$

for any $a_0, \ldots, a_n \in A$.

The fact that ch_Ω is cyclic and that it is a cocycle follows directly from the property of the integral of a cycle. Moreover one can prove the following result (see [GBFV], proposition 8.12; see also [C85], part II, proposition 1 and proposition 8).

Proposition 2.6.7. *An* $(n+1)$*-linear functional* $\tau : A^{n+1} \to \mathbb{C}$ *that vanishes on* $\mathbb{C} \oplus A^n$ *is a cyclic* n*-cocycle if and only if it is the Chern character of a cycle over* A.

2.6.2 Cup product and periodicity of cyclic homology

Now we discuss some properties of cyclic homology. We begin by introducing the cup product. We will use the following characterization of cyclic cocycles (cfr. also proposition 2.6.7).

Definition 2.6.8. *Let* A *be a unital associative algebra. Then the* universal graded differential algebra $\Omega^\bullet A$ *is the graded algebra* $\Omega^\bullet A = \bigoplus_{n=0}^{\infty} \Omega^n A$, *where* $\Omega^0 A = A$ *and* $\Omega^n A$ *is the span of the elements* $a_0 da_1 \cdots da_n$, *with* $a_0, \ldots, a_n \in A$. *The differential* d *is simply defined as follows:*

$$d(a_0 da_1 \cdots da_n) = da_0 \cdots da_n.$$

Proposition 2.6.9. *Let* φ *be an* $(n+1)$*-linear functional on a unital associative algebra* A. *Then the following conditions are equivalent:*

(i) *there exists an n-dimensional cycle $(\Omega, d, \int)$ and a homomorphism $\rho : A \to \Omega^0$ such that*

$$\varphi(a_0, \ldots, a_n) = \int \rho(a_0) d(\rho(a_1)) \cdots d(\rho(a_n)),$$

for any $a_0, \ldots, a_n \in A$;

(ii) *there exists a closed graded trace $\hat{\varphi}$ of dimension n on $\Omega^\bullet A$ such that*

$$\varphi(a_0, \ldots, a_n) = \hat{\varphi}(a_0 da_1 \cdots da_n),$$

for any $a_0, \ldots, a_n \in A$;

(iii) *one has $\varphi(a_0, \ldots, a_n) = (-1)^n \varphi(a_n, a_0, \ldots, a_{n-1})$ and*

$$\sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^{n+1} \varphi(a_{n+1} a_0, \ldots, a_n) = 0,$$

for any $a_0, \ldots, a_n \in A$.

Proof. See [C85], part II, proposition 1. □

Consider now two algebras A and B. In general the equality $\Omega^\bullet(A \otimes B) = \Omega^\bullet(A) \otimes \Omega^\bullet(B)$ does not hold; nevertheless, from the universal property of $\Omega^\bullet(A \otimes B)$ we get a natural homomorphism $\pi : \Omega^\bullet(A \otimes B) \to \Omega^\bullet(A) \otimes \Omega^\bullet(B)$. So we can give the following definition.

Definition 2.6.10. *Consider two arbitrary cocycles $\varphi \in Z^n_\lambda(A)$ and $\psi \in Z^m_\lambda(B)$. Then we define $\varphi \# \psi$ as the $(n + m + 1)$-linear functional associated to the graded trace*

$$\widehat{\varphi \# \psi} = (\hat{\varphi} \otimes \hat{\psi}) \circ \pi.$$

$\varphi \# \psi$ is called the cup product *of φ and ψ.*

Proposition 2.6.11. *The cup product defines a homomorphism $HC^n(A) \otimes HC^m(B) \to HC^{n+m}(A \otimes B)$. Moreover the character of the tensor product of two cycles is the cup product of their characters.*

Proof. See [C85], part II, theorem 9. See also [C94], III.1.α, theorem 12. □

Now we can use the cup product to deduce some properties of cyclic cohomology.

Lemma 2.6.12. *$HC^\bullet(\mathbb{C})$ is a polynomial ring with one generator σ of degree 2.*

Proof. The 2-cocycle σ is defined by $\sigma(1, 1, 1) = 2\pi i$. For the details see [C85], part II, proof of corollary 10. □

Proposition 2.6.13. *For any unital algebra A, $HC^\bullet(A)$ is a module over $HC^\bullet(\mathbb{C})$.*

Proof. Let $\varphi \in Z^n_\lambda(A)$ and let σ be the generator of $HC^\bullet(C)$. Define a map $S : Z^n_\lambda(A) \to Z^{n+2}_\lambda(A)$ by $S\varphi = \varphi \# \sigma = \sigma \# \varphi$. Due to proposition 2.6.11 this defines a map $S : HC^m(A) \to H^{n+2}(A)$. This makes $HC^\bullet(A)$ into a $HC^\bullet(\mathbb{C})$-module. For the details see [C85], part II, corollary 10 and lemma 11. □

2.6.3 Pairing with K-theory

In [C85, C94] Connes introduced a pairing between the cyclic cohomology and the algebraic K-theory of an algebra. In this section we briefly recall the construction. For a different looking, but equivalent, approach see [Kar87]. See also [Lo], chapter 8.

First of all let us define the even and the odd part of the cyclic cohomology of a unital algebra A. We set $HC^{ev}(A) = \bigoplus_n HC^{2n}(A)$ and $HC^{odd}(A) = \bigoplus_n HC^{2n+1}(A)$.

We begin by defining a pairing between $HC^{ev}(A)$ and $K_0^{alg}(A)$. K_0^{alg} was defined in terms of isomorphism classes of finitely generated projective modules: since any finitely generated projective A-module is of the form eA^n for some n and for some idempotent $e \in M_n(A)$, we can equivalently define K_0^{alg} as the Grothendieck group generated by equivalence classes $[e]$ of idempotents e. Let now Tr be the trace on $M_\infty(A)$ (which is well defined on each $M_n(A)$). Then the following result holds.

Proposition 2.6.14. *The following equality defines a bilinear pairing between $K_0(A)$ and $HC^{ev}(A)$:*

$$\langle [e], [\varphi] \rangle = \frac{1}{(2\pi i)^m m!} (\varphi \# Tr)(e, \ldots, e) \tag{2.6.2}$$

for $e \in M_k(A)$ and $\varphi \in Z_\lambda^{2m}(A)$. Moreover one has $\langle [e], [S\varphi] \rangle = \langle [e], [\varphi] \rangle$.

Proof. See [C85], part II, proposition 14. □

Now we pass to the odd case. We recall that any element of $K_1(A)$ is an equivalence class $[u]$ of unitaries in $M_\infty(A)$. Therefore we can define a pairing between $HC^{odd}(A)$ and $K_1(A)$ as follows.

Proposition 2.6.15. *The following equality defines a bilinear pairing between $K_1(A)$ and $HC^{odd}(A)$:*

$$\langle [u], [\varphi] \rangle = \frac{1}{(2\pi i)^m 2^{-(2m+1)}} \frac{1}{(m-1/2)\cdots 1/2} (\varphi \# Tr)(u^{-1}-1, u-1, u^{-1}-1, \ldots, u-1) \tag{2.6.3}$$

for $u \in M_k(A)$ and $\varphi \in Z_\lambda^{2m-1}(A)$. Moreover one has $\langle [u], [S\varphi] \rangle = \langle [u], [\varphi] \rangle$.

Proof. See [C85], part II, proposition 15. □

2.7 Kasparov's KK-theory

Kasparov's bivariant KK-theory [Kas80] is a generalization of the K-theory of C^*-algebras. The basic idea is to see the K-theory functor no more as a functor of a single variable, but as a functor $KK(A, B)$ of two variables, both C^*-algebras. As a functor of the first variable it represents the K-homology (cfr. [BDF77]), while as a functor of the second one it represents ordinary K-theory of C^*-algebras. In this section we will briefly recall the main aspects of KK-theory. For a detailed discussion we refer to literature (see, in particular, [Kas80, BJ83, CS84, Hig87, Cun87, C94, Kuc97]).

2.7.1 C^*-modules

Kasparov's construction relies on Hilbert C^*-modules [GBFV, C94, La].

Definition 2.7.1. *Let B a C^*-algebra with C^*-norm $\|\cdot\|$. A* (Hilbert) right C^*-B-module *is a complex vector space $\mathcal{E}$ which is also a right B-module, together with a bilinear pairing $\langle\cdot,\cdot\rangle : \mathcal{E}\times\mathcal{E}\to B$ such that*
(i) $\langle e,f\rangle = \langle f,e\rangle^*$,
(ii) $\langle e,fb\rangle = \langle e,f\rangle\, b$,
(iii) $\langle e,e\rangle \geq 0$ *and* $\langle e,e\rangle = 0$ *iff* $e=0$,
(iv) *$\mathcal{E}$ is complete in the norm* $\|e\|^2_{\mathcal{E}} = \|\langle e,e\rangle\|$,
for any $e,f\in\mathcal{E}$ and any $b\in B$.

Let $\mathcal{E},\mathcal{F}$ be C^*-B-modules. The vector space of continuous B-module homomorphisms from $\mathcal{E}$ to $\mathcal{F}$ is denoted by $\mathrm{Hom}_B(\mathcal{E},\mathcal{F})$. We can introduce the notion of adjointable morphism:

Definition 2.7.2. *A homomorphism $T:\mathcal{E}\to\mathcal{F}$ is an* adjointable operator *if there exists a B-module homomorphism $T^*:\mathcal{F}\to\mathcal{E}$ such that, for any $e\in\mathcal{E}$, $f\in\mathcal{F}$, $\langle Te,f\rangle = \langle e,T^*f\rangle$. The space of adjointable operators will be denoted by* $\mathrm{Hom}^*_B(\mathcal{E},\mathcal{F})$.

If $\mathcal{F}=\mathcal{E}$, an adjointable operator will also be called an adjointable endomorphism. We will set $\mathrm{End}_B(\mathcal{E}) = \mathrm{Hom}_B(\mathcal{E},\mathcal{E})$ and $\mathrm{End}^*_B(\mathcal{E}) = \mathrm{Hom}^*_B(\mathcal{E},\mathcal{E})$. $\mathrm{End}_B(\mathcal{E})$, with the operator norm, is a Banach algebra. Moreover,

Proposition 2.7.3. $\mathrm{End}^*_B(\mathcal{E})$ *is a closed subalgebra of* $\mathrm{End}_B(\mathcal{E})$, *and it is a C^*-algebra w.r.t. the operator norm and the involution $T\mapsto T^*$.*

Definition 2.7.4. *Two C^*-B-modules $\mathcal{E}$ and $\mathcal{F}$ are* unitarily isomorphic *if there exists a unitary $u\in\mathrm{Hom}^*_B(\mathcal{E},\mathcal{F})$. They are* topologically isomorphic *if there exists an invertible $g\in\mathrm{Hom}^*_B(\mathcal{E},\mathcal{F})$ (with inverse $g^{-1}\in\mathrm{Hom}^*_B(\mathcal{F},\mathcal{E})$).*

Let $\mathcal{E}$ be a C^*-B-module. The B-valued inner product of $\mathcal{E}$ can be used to define a structure of *-algebra on the algebraic tensor product[3] $\mathcal{E}\otimes_B\mathcal{E}$:

$$(e_1\otimes e_2)(f_1\otimes f_2) = e_1\langle e_2,f_1\rangle\otimes f_2, \qquad (e_1\otimes e_2)^* = e_2\otimes e_1.$$

The algebra obtained in this way will be denoted by $\mathrm{Fin}_B(\mathcal{E})$; moreover there is an injective *-homomorphism from $\mathrm{Fin}_B(\mathcal{E})$ into $\mathrm{End}^*_B(\mathcal{E})$, described by the following assignment:

$$(e_1\otimes e_2)(e) = e_1\langle e_2,e\rangle\,.$$

Definition 2.7.5. *The closure of* $\mathrm{Fin}_B(\mathcal{E})$ *in* $\mathrm{End}^*_B(\mathcal{E})$ *w.r.t. the operator norm is the C^*-algebra of B-*compact operators *on $\mathcal{E}$. It will be denoted by $\mathcal{K}_B(\mathcal{E})$.*

Now, since we need it in order to define the KK functor, we introduce the notion of ($\mathbb{Z}_2$-) *graded* C^*-module. Let B be a $\mathbb{Z}_2$-graded C^*-algebra, with grading $\hat\gamma$. Then B decomposes as

[3] $\mathcal{E}$ can be seen also as a left B-module: for $e\in\mathcal{E}$ and $b\in B$ we set $be\equiv eb^*$.

$B^0 \oplus B^1$ (we admit also the trivial case: $B^0 = B$, $B^1 = \{0\}$). We will denote the degree of an element $b \in B$ by $\partial b \in \{0, 1\}$.

Definition 2.7.6. *A C^*-B-module $\mathcal{E}$ is* graded *if there is an element $\gamma \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{E})$ of order 2 (i.e. $\gamma^2 = 1$) such that:*

$$\gamma(eb) = \gamma(e)\hat{\gamma}(b), \qquad \langle \gamma(e_1), \gamma(e_2) \rangle = \hat{\gamma}(\langle e_1, e_2 \rangle),$$

for all $e, e_1, e_2 \in \mathcal{E}$, $b \in B$.

From now on we will assume all C^*-modules to be graded, possibly trivially. Given now two (graded) C^*-modules we define their tensor product. Let A, B be two $\mathbb{Z}_2$-graded C^*-algebras, let $\mathcal{E}$ be a C^*-A-module and $\mathcal{F}$ be a C^*-B-module. Let $C = A \overline{\otimes} B$ be the minimal C^*-tensor product of A and B; that is, the closure of $A \otimes B$ in $\mathcal{L}(\mathcal{H} \otimes \mathcal{H}')$, where $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces carrying a faithful representation of A and B, respectively. In order to make C into a graded algebra we define its multiplication as follows:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\partial b_1 \partial a_2} a_1 a_2 \otimes b_1 b_2.$$

Definition 2.7.7. *The completion $\mathcal{E} \overline{\otimes} \mathcal{F}$ of $\mathcal{E} \otimes \mathcal{F}$ in the inner product*

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle e_1, e_2 \rangle \otimes \langle f_1, f_2 \rangle$$

is a graded C^*-module over the algebra $C = A \overline{\otimes} B$, with grading $\gamma = \gamma_{\mathcal{E}} \otimes \gamma_{\mathcal{F}}$. It is called the exterior tensor product *of $\mathcal{E}$ and $\mathcal{F}$.*

Notice that the *graded tensor product* of maps $\phi \in \mathrm{End}^*_A(\mathcal{E})$ and $\psi \in \mathrm{End}^*_B(\mathcal{F})$,

$$(\phi \otimes \psi)(e \otimes f) = (-1)^{\partial e \partial \psi} \phi(e) \otimes \psi(f),$$

gives a graded inclusion

$$\mathrm{End}^*_A(\mathcal{E}) \overline{\otimes} \mathrm{End}^*_B(\mathcal{F}) \to \mathrm{End}^*_C(\mathcal{E} \overline{\otimes} \mathcal{F})$$

which restricts to an isomorphism $\mathcal{K}_A(\mathcal{E}) \overline{\otimes} \mathcal{K}_B(\mathcal{F}) \simeq \mathcal{K}_C(\mathcal{E} \overline{\otimes} \mathcal{F})$.

We have defined C^*-modules and bounded operators on them. There is also a notion of unbounded operators on C^*-modules (see, e.g., [La, Wor91]), which allows to give an alternative description of the KK functor (cfr. [BJ83]).

Definition 2.7.8. *Let $\mathcal{E}$ be a C^*-B-module. A densely defined closed operator $D : \mathrm{Dom}(D) \to \mathcal{E}$, $\mathrm{Dom}(D) \subset \mathcal{E}$, is called* regular *if D^* is densely defined in $\mathcal{E}$ and $(1 + D^* D)$ has dense range.*

Definition 2.7.9. *A regular operator D is* symmetric *if $\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D^*)$ and $D = D^*$ on $\mathrm{Dom}(D)$. If moreover $\mathrm{Dom}(D) = \mathrm{Dom}(D^*)$ it is called* selfadjoint.

Lemma 2.7.10. *A regular operator $D : \mathrm{Dom}(D) \to \mathcal{E}$ is B-linear and its domain $\mathrm{Dom}(D)$ is a B-submodule of $\mathcal{E}$.*

To a regular operator $D : \mathrm{Dom}(D) \to \mathcal{E}$ we can associate two operators, called respectively the *resolvent* and the *bounded transform* of D, as follows:

$$\mathfrak{r}(D) = (1 + D^*D)^{-1/2}, \tag{2.7.1}$$

$$\mathfrak{b}(D) = D(1 + D^*D)^{-1/2}. \tag{2.7.2}$$

Proposition 2.7.11. *If $D : \mathrm{Dom}(D) \to \mathcal{E}$ is a regular operator, then D^*D is selfadjoint and regular. Moreover,* $\mathrm{Dom}(D)$ *is a core for D^*D and* $\mathrm{Im}(\mathfrak{r}(D)) = \mathrm{Dom}(D)$*.*

Moreover, since $\mathfrak{r}(D) = 1 - \mathfrak{b}(D)^*\mathfrak{b}(D)$, the following holds.

Corollary 2.7.12. *A regular operator D is completely defined by $\mathfrak{b}(D)$.*

There is a useful characterization of regular operators [Wor91]. Let $\mathcal{G}(D)$ denote the graph of D. Let $v \in \mathrm{End}^*_B(\mathcal{E} \oplus \mathcal{E})$ be the unitary defined by $v(e, f) = (-f, e)$. Then $\mathcal{G}(D)$ and $v\mathcal{G}(D^*)$ are orthogonal submodules of $\mathcal{E} \oplus \mathcal{E}$ and, moreover,

Proposition 2.7.13. *A densely defined closed operator $D : \mathrm{Dom}(D) \to \mathcal{E}$, with densely defined adjoint D^*, is regular if and only if $\mathcal{G}(D) \oplus v\mathcal{G}(D^*) \simeq \mathcal{E} \oplus \mathcal{E}$.*

Remark 2.7.14. Whenever $\mathcal{E}$ is a graded module, with grading γ, we will always consider $\mathcal{E} \oplus \mathcal{E}$ as a graded module, with grading $\gamma' = \gamma \oplus (-\gamma)$.

2.7.2 KK-theory: bounded picture

We can now give a brief description of Kasparov's KK-theory [Kas80]. An equivalent characterization, due to S. Baaj and P. Julg [BJ83], will be discussed in the next section. Let A, B be two (eventually trivially) $\mathbb{Z}_2$-graded C^*-algebras.

Definition 2.7.15. *An A-B-bimodule $\mathcal{E}$ is called a C^*-A-B-bimodule if it is a C^*-B-module. It is called a graded bimodule if it admits a $\mathbb{Z}_2$-grading γ which is compatible with the grading of A and which makes it into a graded C^*-B-bimodule.*

Definition 2.7.16. *Let $\mathcal{E}$ be a countably generated graded C^*-A-B-module, with grading operator γ, and let $F \in \mathrm{End}^*_B(\mathcal{E})$ be an odd operator (that is $\gamma F = -F\gamma$). Then $(\mathcal{E}, F)$ is a* Kasparov (A, B)-bimodule *if, for all $a \in A$, $[F, a]$, $a(F^2 - 1)$, $a(F - F^*)$ are B-compact operators; that is, they belong to $\mathcal{K}_B(\mathcal{E})$. We will denote the set of Kasparov (A, B)-bimodules by $\mathcal{E}(A, B)$.*

Definition 2.7.17. *A Kasparov (A, B)-bimodule $(\mathcal{E}, F)$ will be called* degenerate *if, for any $a \in A$,*

$$[F, a] = a(F^2 - 1) = a(F^* - F) = 0.$$

The set of degenerate Kasparov (A, B)-bimodules will be denoted by $\mathcal{D}(A, B)$.

Now we can introduce two equivalence relations on the set of Kasparov bimodules.

Definition 2.7.18. *Two Kasparov (A, B)-bimodules $(\mathcal{E}, F_0)$ and $(\mathcal{E}, F_1)$ are* operatorial homotopic *if there exists a family $(\mathcal{E}, F'_t)$ of Kasparov (A, B)-bimodules with $F'_0 = F_0$, $F'_1 = F_1$ and such that $t \mapsto F_t$ is norm continuous.*

The notion of operatorial homotopy gives rise to an equivalence relation in $\mathcal{E}(A,B)$, which we shall denote by $\sim_{oh}$. We can define also a homotopy transformation between Kasparov bimodules.

Definition 2.7.19. *Two Kasparov (A,B)-bimodules $(\mathcal{E}_0,F_0)$ and $(\mathcal{E}_1,F_1)$ are* homotopic *if there exists a Kasparov bimodule $(\mathcal{E},F)\in\mathcal{E}(A,B\overline{\otimes}C([0,1]))$ such that, if we denote by $\varepsilon_i : B\overline{\otimes}C([0,1])$ the evaluation map at $i\in[0,1]$, $(\mathcal{E}\otimes_{\varepsilon_j}B, F\otimes 1)$ is unitarily equivalent to $(\mathcal{E}_j,F_j)$, for $j=0,1$.*

The notion of homotopy between Kasparov bimodules gives rise to an equivalence relation in $\mathcal{E}(A,B)$, which we shall denote by $\sim_h$.

We can define the sum of two Kasparov bimodules as follows: for $(\mathcal{E}_1,F_1),(\mathcal{E}_2,F_2)\in\mathcal{E}(A,B)$ we set

$$(\mathcal{E}_1,F_1)\oplus(\mathcal{E}_2,F_2)=(\mathcal{E}_1\oplus\mathcal{E}_2,F_1\oplus F_2). \tag{2.7.3}$$

This allows us to take the quotient of $\mathcal{E}(A,B)$ with respect to $\mathcal{D}(A,B)$. So the following definitions of KK groups make sense.

Definition 2.7.20. *The set $KK(A,B)$ is defined as the quotient of the set of Kasparov (A,B)-bimodules by the equivalence relation of homotopy: $KK(A,B)=\mathcal{E}(A,B)/\sim_h$.*

Definition 2.7.21. *The set $\widetilde{KK}(A,B)$ is defined as the quotient of the classes of Kasparov (A,B)-bimodules, up to sum with a degenerate bimodule, by the equivalence relation of operatorial homotopy: $\widetilde{KK}(A,B)=(\mathcal{E}(A,B)/\mathcal{D}(A,B))/\sim_{oh}$.*

Proposition 2.7.22. *Both $KK(A,B)$ and $\widetilde{KK}(A,B)$ are abelian groups w.r.t.* (2.7.3).

Proposition 2.7.23. *$KK(A,B)$ is a quotient of $\widetilde{KK}(A,B)$. If A and B are separable C^*-algebras then $KK(A,B)\simeq\widetilde{KK}(A,B)$.*

Now let $\mathbb{C}\mathrm{l}_j$ denote the j-th complex Clifford algebra. Then, for $j\geq 0$, we define:

$$\begin{aligned} KK_j(A,B)&=KK(A\otimes\mathbb{C}\mathrm{l}_j,B),\\ KK^j(A,B)&=KK(A,B\otimes\mathbb{C}\mathrm{l}_j). \end{aligned} \tag{2.7.4}$$

For $j<0$, instead, we set:

$$KK_j(A,B)=KK^{-j}(A,B),\qquad KK^j(A,B)=KK_{-j}(A,B).$$

In this way we unify the two cases, obtaining a unified KK-theory $KK^\bullet(A,B)$.

Theorem 2.7.24. *For any $j\in\mathbb{Z}$, $KK^j(-,-)$ is a bifunctor, contravariant in the first variable and covariant in the second.*

Moreover one can prove that $KK^\bullet(\mathbb{C},B)$ and $KK^\bullet(A,\mathbb{C})$ are naturally isomorphic to the K-theory of B and to the K-homology of A, respectively. In particular, KK-theory groups share with ordinary K-theory a property of periodicity. Indeed,

Theorem 2.7.25. *The KK groups are periodic modulo 2: $KK^j(A,B)=KK^{j+2}(A,B)$, for any $j\in\mathbb{Z}$.*

Proof. It comes from the periodicity of Clifford algebras, see [Kas80], section 5, theorem 5. □

Remark 2.7.26. If instead of complex algebras and complex Clifford algebras we had considered real ones, the periodicity would have been modulo 8 instead of modulo 2 [Kas80].

2.7.3 KK-theory: unbounded picture

Now we give a different characterization of KK-theory, which relies on unbounded operators on C^*-modules [BJ83].

Definition 2.7.27. *An* unbounded Kasparov (A, B)-bimodule *is a pair* $(\mathcal{E}, D)$ *where* $\mathcal{E}$ *is a graded* C^**-*A*-*B*-bimodule and* D *is an odd selfadjoint regular operator on* $\mathcal{E}$ *such that:*

(i) *all the commutators* $[D, a]$*, for* a *in a dense subalgebra* $\mathcal{A}$ *of* A*, extend to adjointable operators in* $\mathrm{End}^*_B(\mathcal{E})$*,*

(ii) *for any* $a \in \mathcal{A}$*,* $a\mathfrak{r}(D) \in \mathcal{K}_B(\mathcal{E})$*.*

We will denote the set of unbounded Kasparov $(A, B \otimes \mathbb{C}\mathrm{l}_j)$-bimodules, up to unitary equivalence, by $\Psi_j(A, B)$.

Proposition 2.7.28. *Let* $(\mathcal{E}, D) \in \Psi_1(A, B)$*. Then* $(\mathcal{E}, \mathfrak{b}(D))$ *is a Kasparov* (A, B)*-bimodule.*

This implies that there exists a map $\beta : \Psi_1(A, B) \to KK(A, B)$.

Proposition 2.7.29. *The map* $\beta : \Psi_1(A, B) \to KK(A, B)$ *is surjective.*

2.7.4 Kasparov products

Kasparov introduced[4] two product operations in KK-theory. The first one is a bilinear pairing between $KK(A, D)$ and $KK(D, B)$ (where, of course, A, B, D are graded C^*-algebras):

Proposition 2.7.30. *There exists a bilinear associative pairing*

$$KK^i(A, D) \otimes_D KK^j(D, B) \to KK^{i+j}(A, B).$$

We will usually refer to the product of proposition 2.7.30 as to the *Kasparov product.* The second one is the *external Kasparov product*:

Proposition 2.7.31. *For any graded* C^**-algebras* A_1, A_2, B_1, B_2 *there exists an associative bilinear pairing*

$$KK^i(A_1, B_1) \otimes KK^j(A_2, B_2) \to KK^{i+j}(A_1 \overline{\otimes} A_2, B_1 \overline{\otimes} B_2).$$

In both the previous propositions the tensor product between KK groups comes from the graded tensor product of modules.

Remark 2.7.32. The bilinear associative pairings of proposition 2.7.30 and 2.7.31 can also be written in the following way:

$$KK_i(A, D) \otimes_D KK_j(D, B) \to KK_{i+j}(A, B),$$

[4] See [Kas80], section 4, theorem 4.

$$KK_i(A_1, B_1) \otimes KK_j(A_2, B_2) \to KK_{i+j}(A_1\overline{\otimes}A_2, B_1\overline{\otimes}B_2).$$

respectively.

In [BJ83] S. Baaj and P. Julg showed that the external Kasparov product can be recovered from the tensor product of unbounded Kasparov modules. Indeed,

Proposition 2.7.33. *For $i = 1, 2$ let $(\mathcal{E}_i, D_i)$ be unbounded Kasparov (A_i, B_i)-bimodules. Then the operator $D = D_1 \otimes 1 + 1 \otimes D_2$ extends to a selfadjoint regular operator, with compact resolvent, on $\mathcal{E}_1\overline{\otimes}\mathcal{E}_2$. Moreover, the assignment $(\mathcal{E}_1, D_1) \times (\mathcal{E}_2, D_2) \mapsto (\mathcal{E}_1\overline{\otimes}\mathcal{E}_2, D)$ determines a map $\psi_1(A_1, B_1) \times \psi_1(A_2, B_2) \to \psi_1(A_1\overline{\otimes}A_2, B_1\overline{\otimes}B_2)$ which makes the following diagram commutative:*

$$\begin{array}{ccc} \psi_1(A_1, B_1) \times \psi_1(A_2, B_2) & \longrightarrow & \psi_1(A_1\overline{\otimes}A_2, B_1\overline{\otimes}B_2) \\ {\scriptstyle \beta\times\beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ KK(A_1, B_1) \otimes KK(A_2, B_2) & \longrightarrow & KK(A_1\overline{\otimes}A_2, B_1\overline{\otimes}A_2) \end{array}$$

(the bottom line is the external Kasparov product).

So the external Kasparov product can be recovered from the product of unbounded Kasparov bimodules in a quite simple way. It took, instead, many years to come to an analogous result for the Kasparov product of proposition 2.7.30. The first result was that by D. Kucerovsky [Kuc97], who gave a sufficient condition for an unbounded Kasparov (A, B)-bimodule to describe the same class as the Kasparov product of the classes of an (A, D)-bimodule with that of a (D, B)-bimodule. A full characterization of the Kasparov product in terms of unbounded modules has been achieved later by B. Mesland [Mes11]; this characterization of Kasparov product was recently linked to the formulation of gauge theories over noncommutative spaces [BMS13].

We conclude this short discussion of KK-theory noticing that the two Kasparov products of proposition 2.7.30 and proposition 2.7.31 can be written as a unique bilinear coupling (also called *intersection product*, cfr. [Kas80], section 4, theorem 3):

$$KK(A_1, B_1 \otimes D) \otimes_D KK(D \otimes A_2, B_2) \to KK(A_1 \otimes A_2, B_1 \otimes B_2), \qquad (2.7.5)$$

where, of course, the algebras A_i, B_i, D are graded C^*-algebras, all the tensor products are graded tensor products and the completions with respect to the minimal C^*-norms are understood.

CHAPTER 3

Spectral triples and spectral metric spaces

In the previous chapter of this thesis we reviewed some approaches to the study of topological/differential properties of noncommutative spaces. In the last twenty years great attention has been paid to the metric structure of noncommutative spaces. The main steps in this direction, of course, are the one made by A. Connes and his collaborators [C88, C94, C95, C96, CC96, CMa07]. Connes introduced a formulation of spin geometry for noncommutative manifolds, based on spectral triples $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ represents the algebra of smooth functions over a noncommutative space, $\mathcal{H}$ the Hilbert space of L^2-sections of the spinor bundle and D the Dirac operator (for the geometry of spin manifolds see, e.g., [LM]). In this chapter we shall review the main aspects of Connes' formulation of noncommutative geometry. We shall discuss, furthermore, possible definitions of equivariance of a noncommutative geometry under a suitable set of transformations (see, e.g., [PS00, S03]).

3.1 Connes' axioms

We discuss, first of all, Connes' definition of real spectral triples [C94, C95, C96, C00]. We consider here only the more relevant aspects, and we refer to literature for a deeper treatment [C94, CMa07, GBFV]. In what follows $\mathcal{A}$ will always denote a unital complex *-algebra, and it will represent the algebra of "smooth" functions over a noncommutative space. We recall that the geometrical meaning of the hypothesis of the existence of the unit for the algebra $\mathcal{A}$ is that of compactness of the underlying space.

Definition 3.1.1. *A* spectral triple *for an algebra $\mathcal{A}$ is a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{H}$ is a Hilbert space carrying a representation of $\mathcal{A}$ by bounded operators (which we shall simply denote by $\psi \mapsto a\psi$, for any $a \in \mathcal{A}$, $\psi \in \mathcal{H}$) and D is a selfadjoint operator on $\mathcal{H}$ with compact resolvent, such that for any $a \in \mathcal{A}$ the commutator $[D, a]$ is a bounded operator.*

In the commutative case, when $\mathcal{A}$ is the algebra of smooth functions over a Riemannian spin manifold M, $\mathcal{H}$ corresponds to the Hilbert space of L^2-sections of the spinor bundle, and D is the Dirac operator, on the spinor bundle, associated to the Levi-Civita connection [LM]. So, given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we will call $\mathcal{H}$ the *space of spinors* and D the *Dirac operator*.

One can introduce a notion of reality, and, together, an (algebraic) notion of dimension for a spectral triple.

Definition 3.1.2. *A* real spectral triple *of* KR-dimension *j, where $j \in \mathbb{Z}_8$, consists of a package $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ when j is even and of a package $(\mathcal{A}, \mathcal{H}, D, J)$ when j is odd, where $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, J is antiunitary operator and γ is a $\mathbb{Z}_2$-grading on $\mathcal{H}$ such that:*
(i) *for any $a, b \in \mathcal{A}$, $[a, Jb^*J^{-1}] = 0$;*
(ii) *J, D and γ satisfy the following commutation relations:*

$$J^2 = \varepsilon \mathrm{id}, \qquad JD = \varepsilon' DJ$$

and, for j even,

$$J\gamma = \varepsilon'' \gamma J, \qquad \gamma D = -D\gamma,$$

where $\varepsilon, \varepsilon', \varepsilon''$ depend on the KR-dimension and are given in the table below[1].

Table 3.1: Connes' convention is marked by •

n	0	2	4	6	0	2	4	6	1	3	5	7
ϵ	+	−	−	+	+	+	−	−	+	−	−	+
ϵ'	+	+	+	+	−	−	−	−	−	+	−	+
ϵ''	+	−	+	−	+	−	+	−				
	•	•	•	•					•	•	•	•

The operator J will be usually called the *real structure* of the spectral triple. We will often treat together the even and the odd dimensional case. So in general we will write $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ for a real spectral triple, and we will assume $\gamma = \mathrm{id}_{\mathcal{H}}$ when j is odd.

Remark 3.1.3. Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ be a real spectral triple. Then the antiunitary operator J determines a left action of the opposite algebra $\mathcal{A}^\circ$ (or, equivalently, a right action of the algebra $\mathcal{A}$) on the Hilbert space $\mathcal{H}$, given by

$$\pi^\circ(b)\psi = \psi b = Jb^*J^{-1}\psi \tag{3.1.1}$$

for any $b \in \mathcal{A}$ and any $\psi \in \mathcal{H}$. Now we can observe that condition (i) of definition 3.1.2 is the requirement that the action of $\mathcal{A}^\circ$ commutes with the representation of $\mathcal{A}$ on $\mathcal{H}$; that is, J maps $\mathcal{A}$ into its commutant on $\mathcal{H}$. Notice that this recalls the properties of Tomita-Takesaki involution [Tak70].

Now that we have defined real spectral triples we discuss Connes' requirements for a spectral triple being a noncommutative geometry [C94, C96, GBFV]. They consist of six[2] axioms, which corresponds to different properties of ordinary smooth spin manifolds:

[1] See also [DD11].
[2] Actually one usually considers seven conditions, but we incorporated the reality condition into the definition of real spectral triples.

(1) Classical dimension
(2) Regularity
(3) Finiteness, projectivity and absolute continuity
(4) First order condition
(5) Orientation
(6) Poincaré duality

3.1.1 Axiom 1: classical dimension

The first condition we discuss is an analytic condition on the behaviour of the eigenvalues of the Dirac operator D. Before stating the condition we recall briefly some tools of functional analysis.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K}$ be the set of compact operators on $\mathcal{H}$. Let also $\mathcal{L}^1 = \mathcal{L}^1(\mathcal{H}) \subset \mathcal{K}$ be the ideal of trace-class operators. We introduce the so-called *interpolation ideals*[3] between $\mathcal{K}$ and $\mathcal{L}^1$. Let T be a compact operator, and let $|T| = \sqrt{T^*T}$ be its absolute value. Let $\mu_0(T) \geq \mu_1(T) \geq \dots \mu_n(T) \geq \dots$ be the sequence of eigenvalues of $|T|$, counted with multiplicity and arranged in decreasing order. Define the partial sums of eigenvalues as follows: for any $N \in \mathbb{N}$,

$$\sigma_N(T) = \sum_{n=0}^{N-1} \mu_n(T).$$

Notice that the functions σ_N are subadditive: for any $T_1, T_2 \in \mathcal{K}$, $\sigma_N(T_1+T_2) \leq \sigma_N(T_1)+\sigma_N(T_2)$.

Definition 3.1.4. *For any $p \in (1,\infty)$, the interpolation space $\mathcal{L}^{p+} = \mathcal{L}^{(p,\infty)}$ is the space of compact operators T such that $N^{(p-1)/p}\sigma_N(T)$ is a bounded sequence.*

Lemma 3.1.5. *Let T be a compact operator on $\mathcal{H}$. Then T belongs to $\mathcal{L}^{p+}$ if and only if $\mu_n(T) = O(n^{-\frac{1}{p}})$.*

Lemma 3.1.6. *Each $\mathcal{L}^{p+}$ is a two-sided ideal in $\mathcal{L}(\mathcal{H})$. Moreover, for any $1 < p_1 < p_2 < \infty$ we have the inclusion*

$$\mathcal{L}^{p_1+} \subset \mathcal{L}^{p_2+}.$$

We can put a norm on $\mathcal{L}^{p+}$:

$$\|T\|_p = \sup_{N\geq 1} \frac{1}{N^{(p-1)/p}} \sigma_N(T).$$

Such a norm allows us to define the ideal $\mathcal{L}_0^{p+} = \mathcal{L}_0^{p,\infty}$ as the norm-closure of the finite rank operators in $\mathcal{L}^{p,\infty}$. It turns out that a compact operator T belongs to $\mathcal{L}_0^{p,\infty}$ if and only if $\mu_n(T) = o(n^{-1/p})$.

Next, we extend the definition of $\mathcal{L}^{p+}$ to the boundary point $p = 1$. We set:

$$\mathcal{L}^{1+} = \mathcal{L}^{1,\infty} = \{T \in \mathcal{K} \mid \sigma_N(T) = O(\log N)\}.$$

[3] See [C94], IV.2.α. See also [V79, V81].

We can put a norm also on $\mathcal{L}^{1+}$:

$$\|T\|_{1,\infty} = \sup_{N\geq 2} \frac{1}{\log N}\sigma_N(T).$$

We can also define the norm-closed ideal $\mathcal{L}_0^{1+} = \mathcal{L}_0^{1,\infty}$: one can see that $T \in \mathcal{L}_0^{1,\infty}$ if and only if $\mu_n(T) = o(\log N)$.

Now we can state the first of Connes' axiom.

Definition 3.1.7. *A real spectral triple* $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ *fulfils the* classical dimension property *if there is an integer* p *such that*[4] D^{-1} *belongs to* $\mathcal{L}^{p+}$ *but not to* $\mathcal{L}_0^{p+}$. *If such a* p *exists, it is called the* classical dimension *of the triple. If both* $\mathcal{A}$ *and* $\mathcal{H}$ *are finite-dimensional, the classical dimension of the triple is taken to be zero.*

Usually one requires that, if j is the KR-dimension of the triple and p its classical dimension, then $j \equiv p \pmod 8$. But we admit also the case in which such condition is not fulfilled.

Let us conclude this section by mentioning one of the most relevant consequences of the classical dimension property; namely, the possibility of defining a functional on $\mathcal{A}$ using the operator D^{-1}. In order to discuss the construction of such a functional we need to recall briefly what is a Dixmier trace. For the details see, e.g., [Dix66] and [C94], IV.2.β. Let $\ell^\infty(\mathbb{N})$ be the space of bounded sequences. Let $\omega : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ be a linear form satisfying the following conditions:

(a) $\omega(\{\alpha_n\}) \geq 0$ if $\alpha_n \geq 0$,

(b) $\omega(\{\alpha_n\}) = \lim \alpha_n$ if α_n is convergent,

(c) $\omega(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \ldots, \ldots) = \omega(\{\alpha_n\})$.

Then we can give the following definition.

Definition 3.1.8. *Let* $T \in \mathcal{L}^{1+}$. *We define*

$$Tr_\omega(T) = \omega\left(\frac{1}{\log N}\sum_{n=0}^{N-1} \mu_n(T)\right). \tag{3.1.2}$$

Tr_ω *is called* Dixmier trace *relative to the form* ω.

Proposition 3.1.9. *Consider* $T \in \mathcal{L}^{1+}$ *and let* Tr_ω *be a Dixmier trace. Then:*

(i) *if* $T \geq 0$ *then* $Tr_\omega(T) \geq 0$;

(ii) *if* S *is any bounded operator on* $\mathcal{H}$, *then* $Tr_\omega(TS) = Tr_\omega(ST)$;

(iii) $Tr_\omega(T)$ *is independent on the choice of the scalar product on* $\mathcal{H}$, *i.e. it depends only on the Hilbert space* $\mathcal{H}$ *as a topological vector space;*

(iv) $Tr_\omega(T) = 0$ *if* $T \in \mathcal{L}_0^{1+}$.

Proof. See [C94], IV.2.β, proposition 3. □

[4] In all this thesis when we speak of D^{-1} we mean the inverse of the Dirac operator on the orthogonal complement of its kernel, in the case 0 belongs to the spectrum of D.

In order to discuss the dependence of Tr_ω on the choice of ω, we introduce the so-called Cesaro mean. Given a function $f : \mathbb{R}^+ \to \mathbb{C}$ we define its Cesaro mean to be the function

$$M_f(\lambda) = \frac{1}{\log \lambda} \int_1^\lambda f(u) \frac{du}{u}.$$

Proposition 3.1.10. *The following facts hold for Dixmier traces of compact operators on an Hilbert space $\mathcal{H}$.*

a. *Let $T \in \mathcal{L}^{1+}$, $T \geq 0$. Let $f : \mathbb{R}^+ \to \mathbb{C}$ be the step function whose value in $(N-1, N)$ is $\frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T)$. Then $Tr_\omega(T)$ is independent of ω iff $M_f(\lambda)$ converges for $\lambda \to \infty$.*

b. *Let $\mathcal{M} = \{T \in \mathcal{L}^{1+} \mid Tr_\omega(T) \text{ is independent of } \omega\}$. Then $\mathcal{M}$ is a linear space, invariant under conjugation by invertible operators on $\mathcal{H}$.*

c. *$\mathcal{M}$ contains $\mathcal{L}_0^{1+}$ and it is closed w.r.t. $\|\cdot\|_{1,\infty}$.*

Proof. See [C94], IV.2.β, proposition 6. □

A geometric interpretation of Dixmier trace can be obtained by noticing that it is related to the notion of residue of pseudo-differential operators [Man79, Wod84, Gui85]. Indeed we have [C88, C94]:

Theorem 3.1.11. *Let M be an n-dimensional compact manifold and let T be a pseudo-differential operator of order $-n$ acting on sections of a complex vector bundle E on M. Then:*

(i) *the corresponding operator T on $\mathcal{H} = L^2(M, E)$ belongs to the ideal $\mathcal{L}^{1+}(\mathcal{H})$,*

(ii) *the Dixmier trace $Tr_\omega(T)$ is independent of ω and it is equal to the residue[5] $\mathrm{Res}(T)$.*

Now, given $T \in \mathcal{L}^{1+}$, $T \geq 0$, and a Dixmier trace Tr_ω, we can define a positive linear functional $\varphi_\omega : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ by:

$$\varphi_\omega(a) = \mathrm{Tr}_\omega(aT).$$

In particular, if we have a Dirac operator D s.t. D^{-1} belongs to $\mathcal{L}^{p+}$, then we can define a linear positive functional φ_ω^D by setting:

$$\varphi_\omega^D(a) = \mathrm{Tr}_\omega(a|D|^{-p}).$$

Then one can prove the following result (see [GBFV], theorem 10.20):

Theorem 3.1.12. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple of classical dimension p and let Tr_ω be any Dixmier trace on $\mathcal{H}$. Then the functional*

$$a \mapsto Tr_\omega(a|D|^{-p})$$

is a hypertrace on $\mathcal{A}$; that is,

$$Tr_\omega(aT|D|^{-p}) = Tr_\omega(Ta|D|^{-p}),$$

for any bounded operator T on $\mathcal{H}$.

[5] For the definition of residue see, e.g., [GBFV], chapter 7 (especially theorem 7.12).

3.1.2 Axiom 2: regularity

We require the real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ to be regular:

Definition 3.1.13. *Let δ be the derivation on $\mathcal{L}(\mathcal{H})$ defined by:*

$$\delta(T) = [|D|, T].$$

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be regular *if, for any $a \in \mathcal{A}$ and $k \in \mathbb{N}$, both a and $[D, a]$ belong to the domain of δ^k.*

Let us consider now the space of *smooth vectors* $\mathcal{H}^\infty = \bigcap_k \mathrm{Dom}(D^k)$: if the triple is regular one can prove[6] that $\mathcal{H}^\infty$ is a left $\mathcal{A}$-module. Moreover[7], if a triple $(\mathcal{A}, \mathcal{H}, D)$ of classical dimension p is regular, then the functional

$$T \mapsto \mathrm{Tr}_\omega(T|D|^{-p})$$

defines a hypertrace on the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$.

The regularity of a spectral triple is associated to the existence of a so called algebra of generalized differential operators [CM95, Hig04, Hig06, Otg11]. Since we will use this fact later in this thesis, we give a sketch of these notions. Let $\mathcal{H}$ be a Hilbert space and let Δ be an invertible, selfadjoint (usually unbounded) operator on $\mathcal{H}$. Then we can introduce the following definitions.

Definition 3.1.14. *The* Δ-Sobolev space *of order $s \in \mathbb{R}$, denoted $W^s = W^s(\Delta) = W^s(\Delta, \mathcal{H})$, is the Hilbert completion of* $\mathrm{Dom}(\Delta^{\frac{s}{2}})$ *with respect to the inner product given by*

$$\langle \xi, \eta \rangle_{W^s} = \left\langle \Delta^{\frac{s}{2}} \xi, \Delta^{\frac{s}{2}} \eta \right\rangle$$

for any $\xi, \eta \in \mathcal{H}$, where $\langle \cdot, \cdot \rangle$ simply denotes the inner product of $\mathcal{H}$.

Definition 3.1.15. *The space of* Δ-smooth vectors *of $\mathcal{H}$ is*

$$W^\infty = \bigcap_{s \in \mathbb{R}} W^s = \bigcap_{n=0}^{\infty} W^{2n} = \bigcap_{n=0}^{\infty} \mathrm{Dom}(\Delta^n).$$

Now we consider linear operators $P : W^\infty \to W^\infty$. The algebra of these operators will be denoted by $\mathrm{End}(W^\infty)$.

Definition 3.1.16. *A linear operator $W^\infty \to W^\infty$ has* analytic order *at most $t \in \mathbb{R}$ if it extends by continuity to a bounded linear operator $W^{s+t} \to W^s$ for any $s \in \mathbb{R}$.*

We write $\mathrm{Op}^t = \mathrm{Op}^t(\Delta) = \mathrm{Op}^t(\Delta, \mathcal{H})$ *for the class of operators of analytic order at most t. We define then*

$$\mathrm{Op} = \mathrm{Op}^\infty = \bigcup_{t \in \mathbb{R}} \mathrm{Op}^t$$

[6]See [GBFV], lemma 10.22 and section 10.5.
[7]See theorem 3.1.12 and [GBFV], corollary 10.21.

and

$$\mathrm{Op}^{-\infty} = \bigcap_{t \in \mathbb{R}} \mathrm{Op}^t$$

Lemma 3.1.17. *The operators with finite analytic order form a filtered algebra:*
(i) $\mathrm{Op}^s \subseteq \mathrm{Op}^t$ *for* $s \leq t$*,*
(ii) $\mathrm{Op}^s \cdot \mathrm{Op}^t \subseteq \mathrm{Op}^{s+t}$*.*
In particular, Op^0 *is a subalgebra of* Op*, and* $\mathrm{Op}^{-\infty} \subset \mathrm{Op}$ *and* $\mathrm{Op}^t \subset \mathrm{Op}^0$*, for* $t < 0$*, are two-sided ideals.*

Following Higson, we give the following definition.

Definition 3.1.18. *An* $\mathbb{N}$*-filtered subalgebra* $\mathcal{D} \subseteq \mathrm{Op}(\Delta)$ *is called an* algebra of generalized differential operators *if it is closed under the derivation* $[\Delta, \cdot]$ *and satisfies*

$$[\Delta, \mathcal{D}^k] \subseteq \mathcal{D}^{k+1}$$

for any $k \in \mathbb{N}$*.*

Let now $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Let $\Delta = D^2 + 1$, so that $D \in \mathrm{Op}^1(\Delta)$, and let W^∞ be the space of Δ-smooth vectors. Assume that W^∞ is stable under the left action of $\mathcal{A}$. Then define inductively an $\mathbb{N}$-filtered algebra $\mathcal{D} \subset \mathrm{End}(W^\infty)$ as follows:
(1) $\mathcal{D}^0$ is the subalgebra generated by $\mathcal{A} + [D, \mathcal{A}]$,
(2) $\mathcal{D}^1 = \mathcal{D}^0 + [\Delta, \mathcal{D}^0] + \mathcal{D}^0[\Delta, \mathcal{D}^0]$,
(3) $\mathcal{D}^k = \mathcal{D}^{k-1} + \sum_{j=1}^{k-1} \mathcal{D}^j \cdot \mathcal{D}^{k-j} + [\Delta, \mathcal{D}^{k-1}] + \mathcal{D}^0[\Delta, \mathcal{D}^{k-1}]$, for $k \geq 2$.
Then we have the following result.

Theorem 3.1.19. *[Higson] The spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is regular if and only if* $\mathcal{D}^k \subseteq \mathrm{Op}^k$ *for any* $k \in \mathbb{N}$*.*

Proof. See [Hig06], theorem 4.26. For a different proof see [Otg11], theorem 2.4 and section 4. □

There is also a more general criterion of regularity for a spectral triple (actually the result is even stronger than the one we discuss here, but the full version of the theorem would require some more stuff to be introduced, and this is out of the purposes of this thesis):

Theorem 3.1.20. *Let* $(\mathcal{A}, \mathcal{H}, D)$ *be a spectral triple. Let* $\Delta = D^2 + 1$ *and let* δ *be the derivation* $[\Delta^{\frac{1}{2}}, \cdot]$*. Then the following conditions are equivalent:*
(a) *the spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is regular;*
(b) *the set* $\mathcal{A} + [D, \mathcal{A}]$ *is contained in* $\mathrm{Dom}^\infty(\delta)$*;*
(c) *there exists an algebra of generalized differential operators containing* $\mathcal{A} + [D, \mathcal{A}]$ *in degree 0.*

Proof. See [CM95], appendix B, [Hig04], theorem 3.25 and [Hig06], theorem 4.26. □

3.1.3 Axiom 3: finiteness, projectivity and absolute continuity

The regularity condition implies that $\mathcal{H}^\infty$ is a left $\mathcal{A}$-module. We require something more. The additional requirements are the following ones: the first is the so-called *finiteness (and projectivity) property*: the algebra $\mathcal{A}$ is required to be a pre-C^*-algebra[8] and the space of smooth vectors $\mathcal{H}^\infty$ a finitely generated projective left $\mathcal{A}$-module. One immediate consequence of this property is that $\mathcal{A}$ is a Fréchet pre-C^*-algebra [GBFV]. This implies that the K-theory of $\mathcal{A}$ is the same as that of its C^*-completion (see theorem 2.4.32). The second one is the *absolute continuity property*: we require that the following equality,

$$\langle \xi, a\eta \rangle = \mathrm{Tr}_\omega \left(a(\xi|\eta)|D|^{-p} \right), \qquad \forall a \in \mathcal{A}\ \forall \xi, \eta \in \mathcal{H}_\infty,$$

defines a hermitian structure $(\cdot|\cdot)$ on the module $\mathcal{H}_\infty$ [C96, C13].

3.1.4 Axiom 4: first order condition

The fourth requirement is that the real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ satisfies the so-called *first order condition*, that is the requirement that $\mathcal{A}^\circ$ commutes not only with $\mathcal{A}$ but also with $[D, \mathcal{A}]$ so that, for any $a, b \in \mathcal{A}$, we require the following:

$$[[D, a], Jb^* J^{-1}] = 0. \tag{3.1.3}$$

Notice that, using Jacoby identity, one can show (3.1.3) to be equivalent to

$$[[D, Jb^* J^{-1}], a] = 0;$$

we see, therefore, that the first order condition is "symmetric" in $\mathcal{A}$ and $\mathcal{A}^\circ$.

If a spectral triple fulfils the first order condition, we can define a representation of $\mathcal{A} \otimes \mathcal{A}^\circ$-valued Hochschild chains by bounded operators on the Hilbert space $\mathcal{H}$:

$$\pi_D((a_0 \otimes b^\circ) \otimes a_1 \otimes \ldots \otimes a_k) = a_0 Jb^* J^{-1}[D, a_1] \cdots [D, a_k]. \tag{3.1.4}$$

This fact allows us to introduce the orientation axiom.

3.1.5 Axiom 5: orientation

Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ be a real spectral triple of KR-dimension j. We recall that if j is odd we set $\gamma = \mathrm{id}$. In the case of spin geometry of a smooth manifold, the operator γ corresponds to the Clifford representation of an orientation form. So, in the noncommutative case, we require a real spectral triple to fulfil the following property:

Definition 3.1.21. *A real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ of classical dimension n fulfils the* ori-

[8] See definition 2.4.29.

entation condition *if there exists a Hochschild n-cycle* $\boldsymbol{c} \in Z_k(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ *such that*

$$\pi_D(\boldsymbol{c}) = \gamma.$$

3.1.6 Axiom 6: Poincaré duality

Let $(\mathcal{A}, \mathcal{H}^F, F, \gamma)$ be an even Fredholm module [C85, C94, GBFV] over $\mathcal{A}$. Let $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}^F)$ be the representation of $\mathcal{A}$. Then the Hilbert space $\mathcal{H}^F$ splits as $\mathcal{H}^F = \mathcal{H}^F_+ \oplus \mathcal{H}^F_-$ and we can write the representation π and the operator F as

$$\pi(a) = \begin{pmatrix} \pi^+(a) & 0 \\ 0 & \pi^-(a) \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix},$$

for suitable operators P, Q. Now, for any $n > 0$ we can extend the Fredholm module $(\mathcal{A}, \mathcal{H}^F, F, \gamma)$ to a Fredholm module $(\mathcal{A}, \mathcal{H}^F_n, F_n, \gamma_n)$, where

$$\mathcal{H}^F_n = \mathcal{H}^F \otimes \mathbb{C}^n, \qquad F_n = F \otimes \mathrm{id}, \qquad \gamma_n = \gamma \otimes \mathrm{id}.$$

The representation of $\mathcal{A}$ on $\mathcal{H}^F_n$ is simply $\pi_n = \pi \otimes 1$, and π_n can be written as $\pi^+_n \oplus \pi^-_n$. Let now $p \in M_n(\mathcal{A})$ be a projection, so that it defines an element of $K_0(\mathcal{A})$, and consider the following operator:

$$\pi^+_n(p) P \pi^-_n(p) \ : \ \mathcal{H}^F_- \to \mathcal{H}^F_+.$$

It is a Fredholm operator [GBFV, Pas01], so its index is well-defined. Moreover one can prove that it depends only on the class of p in $K_0(\mathcal{A})$, therefore we can use it to define a map $K_0(\mathcal{A}) \to \mathbb{Z}$ as follows:

$$\big\langle [p], (\mathcal{H}^F, F, \gamma) \big\rangle = \mathrm{Index}(\pi^+_n(p) P \pi^-_n(p)). \tag{3.1.5}$$

Now consider an even real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ satisfying the previous axioms. Then $\mathcal{H}$ is a module over $\mathcal{A} \otimes \mathcal{A}^\circ$. Moreover, if we define an operator $F : \mathcal{H} \to \mathcal{H}$ simply taking $F = D|D|^{-1}$, then $(\mathcal{A} \otimes \mathcal{A}^\circ, \mathcal{H}, F, \gamma)$ is an even Fredholm module. Therefore the analogue of (3.1.5) defines a map $K_0(\mathcal{A} \otimes \mathcal{A}^\circ) \to \mathbb{Z}$, and we can see it as a $\mathbb{Z}$-valued pairing between $K_0(\mathcal{A})$ and $K_0(\mathcal{A}^\circ)$. And so it allows us to define an additive form $\langle \cdot, \cdot \rangle : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{Z}$. Such a pairing can be written [GBFV, Pas01] in the following form:

$$\langle [p], [q] \rangle = \mathrm{Index}(\pi^+(p \otimes q^\circ) D_+ \pi^-(p \otimes q^\circ)) \tag{3.1.6}$$

where $D = D^+ \oplus D^-$ accordingly to $\gamma^2 = 1$.

Consider instead an odd real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ satisfying the previous axioms. Let $u \in M_k(\mathcal{A})$ and $v \in M_l(\mathcal{A})$ be two unitaries, defining classes in $K_1(\mathcal{A})$. Then $U = u \otimes v^\circ$ is a unitary operator on $\mathcal{H} \otimes \mathbb{C}^{kl}$. Let now Q be the operator defined as $Q = \frac{1}{2}(1 + D|D|^{-1}) \otimes \mathrm{id}$. Then QUQ is a Fredholm operator on $Q(\mathcal{H} \otimes \mathbb{C}^{kl})$ [GBFV]. This allows us to define an additive form on $K_1(\mathcal{A})$ by

$$\langle [u], [v] \rangle = \mathrm{Index}(QUQ). \tag{3.1.7}$$

Now we can formulate the last of Connes' axioms, the so-called *Poincaré duality condition*: we require that the pairing on $K_i(\mathcal{A})$, defined either by (3.1.6) or (3.1.7), is non-degenerate. Poincaré duality can also be formulated using KK-theory, see section 3.7.

3.2 Differential calculus

In Connes' noncommutative geometry a real spectral triple encodes the (Riemannian) geometry of a noncommutative space. In this section we discuss the first geometric aspect associated to a spectral triple: the differential calculus. While in differential geometry there is a unique reasonable choice for the differential calculus over the algebra of smooth functions on a smooth manifold, i.e. the ordinary de Rham calculus of differential forms, there is no such privileged choice in the noncommutative setup: a priori, the only canonical choice would be to consider the universal differential calculus. But it is clear that this choice is not consistent with the classical case, and so it can not be considered a good candidate for a differential calculus over noncommutative spin manifolds. The situation changes if we are given a spectral triple over a noncommutative algebra $\mathcal{A}$: in this case there is a way to associate to it a first order[9] differential calculus.

Let us begin by taking a look at the commutative case. So, consider a compact smooth manifold M and let $\mathcal{A} = C^\infty(M)$ be the algebra of smooth functions over it. Let us denote by $\Omega^1(\mathcal{A})$ the set of smooth 1-forms over M. Now consider a Clifford module $\mathcal{E}$ over M [BGV, GBFV]. Then there is a map $c : \Omega^1(\mathcal{A}) \to \mathrm{End}(\mathcal{A})$, called Clifford map, and we have:

Proposition 3.2.1. *If D is a generalized Dirac operator on a selfadjoint Clifford module $\mathcal{E}$ then*

$$[D, a] = -ic(da).$$

This result suggests that we can use the Dirac operator to define a differential calculus. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. We recall that any first order differential calculus over $\mathcal{A}$ is determined by a sub-bimodule N of $\Omega^1\mathcal{A}$, where $\Omega^1\mathcal{A}$ is the first order universal calculus over $\mathcal{A}$. We set

$$N_D = \left\{ \sum_j a_j db_j \in \Omega^1\mathcal{A} \;\middle|\; \sum_j a_j[D, b_j] = 0 \right\} \tag{3.2.1}$$

where, of course, we regard $\sum_j a_j[D, b_j]$ as an operator on $\mathcal{H}$. This is the sub-bimodule which determines the calculus we were looking for:

Definition 3.2.2. *The* Dirac operator based *differential calculus over $\mathcal{A}$ is $\Omega^1_D(\mathcal{A}) = \Omega^1\mathcal{A}/N_D$.*

Then we can find an analogue for the Clifford map: it can be identified with the map $\pi_D : \Omega^1_D(\mathcal{A}) \to \mathcal{L}(\mathcal{H})$ defined by

$$\pi_D(adb)\psi = a[D, b]\psi. \tag{3.2.2}$$

This determines a left action of $\Omega^1_D(\mathcal{A})$ on $\mathcal{H}$. Moreover, if $(\mathcal{A}, \mathcal{H}, D, J)$ is a real spectral triple, then we can define also a right action of $\Omega^1_D(\mathcal{A})$:

$$\pi^\circ_D(adb)\psi = J(a[D, b])^* J^{-1}\psi. \tag{3.2.3}$$

[9] Actually this construction extends to higher order differential forms, see e.g. [C94], VI.1.1.

We notice that if the spectral triple fulfils the first order condition then $\pi_D(\eta)$ commutes with the action of $\mathcal{A}^\circ$ and $\pi^\circ_D(\eta)$ with that of $\mathcal{A}$ for every 1-form $\eta \in \Omega^1_D(\mathcal{A})$.

3.3 Inner fluctuations of the Dirac operator

In this section we briefly discuss the so-called inner fluctuations of a Dirac operator. We shall not enter here into the details of the motivations which yield to the construction below, we simply mention that it is connected with Morita equivalence[10]; in particular, the fact that each algebra is Morita equivalent to itself shows that it is geometrically significant to consider, given a Dirac operator D for an algebra $\mathcal{A}$, acting on a Hilbert space $\mathcal{H}$, operators of the form

$$D' = D + A,$$

where $A = A^* \in \Omega^1_D(\mathcal{A})$ is a selfadjoint operator acting on $\mathcal{H}$ on the left[11]. The presence of a real structure allows different modifications of the Dirac operator.

Let be given a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$. Then consider the following definition [CCM07, CMa07].

Definition 3.3.1. *The* inner fluctuations *of the Dirac operator D are given by*

$$D \;\mapsto\; D_A = D + A + \varepsilon' JAJ^{-1},$$

where $A = \sum_j a_j[D, b_j]$ is a selfadjoint one-form $A = A^* \in \Omega^1_D(\mathcal{A})$ and ε' is defined by $JD = \varepsilon' DJ$.

Proposition 3.3.2. *The data $(\mathcal{A}, \mathcal{H}, D_A, J, \gamma)$ define a real spectral triple with the same KR-dimension of the triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$.*

Proof. The one-form A, which is seen as an operator on $\mathcal{H}$, is a bounded operator. Hence D_A is a bounded (selfadjoint) perturbation of D; in particular its resolvent is compact, since so is, by hypothesis, the resolvent of D. Next, the commutation relations involving only J and γ are unchanged. Moreover, in the even dimensional case, so is also the commutation relation between γ and the Dirac operator. Hence we have only to check that the commutation relation between J and D_A is the right one; but this follows by direct computation:

$$D_A J = DJ + AJ + \varepsilon' JA = \varepsilon' J(D + \varepsilon' J^{-1}AJ + A) = \varepsilon' J(D + \varepsilon' JAJ^{-1} + A) = \varepsilon' JD_A.$$

□

Remark 3.3.3. Here we are not requiring the triples to fulfil the first order condition. Indeed, even if the original triple fulfilled it, this would be, in general, no longer true for the triple with the fluctuated Dirac operator D_A. Moreover, the existence of the orientation cycle for the triple with operator D_A is not guaranteed by the existence of the orientation cycle for the original triple,

[10] See [CMa07], chapter 1, section 10.8. See also [CC06b], section 2.

[11] If $A = \sum_j a_j db_j$, then it corresponds to the operator $\sum_j a_j[D, b_j]$, see the discussion in the previous section.

since the fluctuation could not commute both with the elements from $\mathcal{A}$ and from $\mathcal{A}^\circ$. Hence also the orientation condition is, in general, not preserved by inner fluctuations.

Consider now the unitary group $U(\mathcal{A})$ (i.e. the set of the unitary elements of the algebra $\mathcal{A}$). We can define the adjoint action of a unitary $u \in U(\mathcal{A})$ on $\mathcal{H}$ by:

$$\mathrm{Ad}(u)(\psi) = u\psi u^* = JuJ^{-1}u\psi.$$

By direct computation, it can be proved (see [CMa07], chapter 1, section 10.8, proposition 1.141) that:

Proposition 3.3.4. *For any inner fluctuation $D \mapsto D_A$, $A = A^* \in \Omega^1_D$, of the Dirac operator D and for any unitary $u \in U(\mathcal{A})$ we have:*

$$\mathrm{Ad}(u)D_A\mathrm{Ad}(u^*) = D_{\gamma_u(A)},$$

where $\gamma_u(A) = u[D, u^*] + uAu^*$.

This last proposition suggests that, if we view a unitary $u \in U(\mathcal{A})$ as a gauge transformation of some kind, we can identify the one-form A with a gauge potential. This is, indeed, the assumption made by Connes, Chamseddine, Marcolli et al. (see, e.g., [C96, CC96, CC97, CCM07, CMa07, CC08]) in their attempt to recover the Standard Model of elementary particles as a pure gravity theory on a noncommutative space: in their model the (unimodular[12]) inner fluctuations of the Dirac operator correspond to the gauge bosons of the SM.

Remark 3.3.5. Inner fluctuations do not compose properly; that is, inner fluctuations of inner fluctuations of a Dirac operator D can no longer be inner fluctuations of D themselves. Indeed, the commutators of D and D_A, respectively, with elements from $\mathcal{A}$ in general, will differ, so the space of differential 1-forms Ω^1_D and $\Omega^1_{D_A}(\mathcal{A})$ will not coincide. A possible way to overcome this issue has recently been proposed in [CCS13], where are considered fluctuations with also a quadratic term, which violates the first order conditions but allows to obtain a set of transformations closed under composition and invariant with respect to conjugations by a unitary element of $\mathcal{A}$.

3.4 Distance between states

Now we show how, given a spectral triple over a noncommutative space, it is possible to get some information on the metric structure defined by the spectral triple. Let us begin considering the commutative case. Let M be a compact connected Riemannian spin manifold and let D be the Dirac operator on its spinor bundle [C94, BGV]. Let $\mathcal{H}$ be the Hilbert space of L^2-spinors on M; then $C^\infty(M)$ acts on $\mathcal{H}$ by multiplication and D is a selfadjoint operator on $\mathcal{H}$ such that each commutator $[D, f]$, for $f \in C^\infty(M)$, is a bounded operator (equal to $c(df)$, where $c(\cdot)$ is the Clifford map). Then we have the following result.

[12]This is a technical assumption which corresponds to the fact that the gauge group $SU(3) \times SU(2) \times U(1)$ of the SM is the direct product of special unitary groups and not of unitary groups.

Proposition 3.4.1. *Let $d(\cdot,\cdot)$ be the geodesic distance function on M. Then, for any two points $p,q \in M$, we have*

$$d(p,q) = \sup\{|f(p) - f(q)| \mid f \in C^\infty(M),\ \|[D,f]\| \leq 1\}.$$

The points of M can be identified with the pure states on $C(M)$ and d can be seen as a distance on the space of states of the C^*-algebra $C(M)$. Consider now a spectral triple $(\mathcal{A},\mathcal{H},D)$ over a (noncommutative) pre-C^*-algebra $\mathcal{A}$. Let A denote the C^*-completion of $\mathcal{A}$. Then we can define a distance on the space of states on A in the following way.

Definition 3.4.2. *For any two states φ,ψ on A we define their* geodesic distance *to be*

$$d(\varphi,\psi) = \sup\{|\varphi(a) - \psi(a)| \mid a \in \mathcal{A},\ \|[D,a]\| \leq 1\}.$$

3.5 Equivariant spectral triples

In this section we discuss the notion of symmetries of real spectral triples. Making a comparison with the commutative case, we can identify the group of diffeomorphisms of a differentiable manifold with that of automorphisms of the algebra of smooth functions. When we consider a noncommutative space we consider the more general case of Hopf algebra symmetries. This yields to the definition of H-equivariant real spectral triple [PS98, PS00, S01, S03].

When dealing with Hopf algebra symmetries of a real spectral triple $(\mathcal{A},\mathcal{H},D,J,\gamma)$ there are two possibilities: the Hopf algebra H can either act (conventionally on the left) or coact (conventionally on the right) on the algebra $\mathcal{A}$, therefore we shall consider both cases.

We begin from the former. Let H be a Hopf *-algebra acting on the left on the algebra $\mathcal{A}$.

Definition 3.5.1. *A spectral triple $(\mathcal{A},\mathcal{H},D,\gamma)$ is H-*equivariant *if there is a dense subspace $V \subset \mathcal{H}$ with dense intersection with the domain of D such that V is an H-equivariant $\mathcal{A}$-module and, for every $h \in H$, the Dirac operator and (in the even case) the $\mathbb{Z}_2$ grading γ are equivariant: $[\gamma,h] = 0$ and $[D,h] = 0$ on the intersection of V with the domain of D.*

In the case of real spectral triples we add the following requirement.

Definition 3.5.2. *A real spectral triple $(\mathcal{A},\mathcal{H},D,J,\gamma)$ is H-*equivariant *if $(\mathcal{A},\mathcal{H},D,\gamma)$ is an H-equivariant spectral triple and, moreover, for any $h \in H$,*

$$JhJ^{-1} = (Sh)^*$$

on the dense subspace $V \subset \mathcal{H}$.

One of the direct consequences of the equivariance of a spectral triple is that also the space of differential forms is an equivariant module. We point out that the notion of equivariance which we are talking about is different from that introduced by Woronowicz [Wor89] for the differential calculus of an H-comodule algebra; indeed, here we are considering an action of the Hopf algebra H on an algebra $\mathcal{A}$, and not a coaction.

Definition 3.5.3. *A bimodule of differential 1-forms* $\Omega^1(\mathcal{A})$ *is an* H-equivariant differential bimodule *if it is an* H*-equivariant bimodule and the action of* d *intertwines with that of* H*:*

$$d(h \triangleright a) = h \triangleright da$$

for any $h \in H$ *and any* $a \in \mathcal{A}$*.*

Lemma 3.5.4. *Let* N *be the sub-bimodule defining a general differential calculus* $\Omega^1(\mathcal{A})$*. Then* $\Omega^1(\mathcal{A})$ *is an* H*-equivariant differential bimodule if and only if* N *is* H*-invariant; that is,* $H \triangleright N \subseteq N$*.*

Proof. See [S03], corollary 2.21. □

Proposition 3.5.5. *Let* $(\mathcal{A}, \mathcal{H}, D, \gamma)$ *be an* H*-equivariant spectral triple. Then* $\Omega^1_D(\mathcal{A})$ *is an* H*-equivariant differential bimodule.*

Proof. Let us define the action of H on $\Omega^1\mathcal{A}$ simply by

$$h \triangleright adb = (h_{(1)} \triangleright a)d(h_{(2)} \triangleright b).$$

Then the thesis follows directly from the previous lemma and the equivariance of the Dirac operator D. □

Now we consider the second case. Let $\mathcal{A}$ be a left H-comodule algebra, H being a Hopf *-algebra, and denote by Δ_L the coaction. Consider a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ ($\gamma = \mathrm{id}$ in the odd case) and assume that there is a dense subspace $V \subseteq \mathrm{Dom}(D)$ of $\mathcal{H}$, stable under the action of $\mathcal{A}$ and D, which is a left H-equivariant $\mathcal{A}$-module; denote then by ρ_L the left coaction of H on V. Then we give the following definition[13].

Definition 3.5.6. *The real spectral triple* $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ *is* H*-equivariant if:*
(i) $(\rho_L \circ D)v = (\mathrm{id} \otimes D) \circ \rho_L(v)$*. That is,* $(Dv)_{(-1)} \otimes (Dv)_{(0)} = v_{(-1)} \otimes Dv_{(0)}$*;*
(ii) $(\rho_L \circ J)v = (^*\otimes J) \circ \rho_L(v)$*. That is,* $(Jv)_{(-1)} \otimes (Jv)_{(0)} = v^*_{(-1)} \otimes Jv_{(0)}$*;*
And, in the even dimensional case,
(iii) $(\rho_L \circ \gamma)v = (\mathrm{id} \otimes \gamma) \circ \rho_L(v)$*. That is,* $(\gamma v)_{(-1)} \otimes (\gamma v)_{(0)} = v_{(-1)} \otimes \gamma v_{(0)}$*.*

3.6 Spectral metric spaces

In section 3.4 we saw that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over a pre-C^*-algebra $\mathcal{A}$ defines a distance d_D on the states space of the C^*-completion A of $\mathcal{A}$ (see definition 3.4.2). Such a distance induces a tolopogy on the space $\mathcal{S}(A)$ of states over the C^*-algebra A. The space $\mathcal{S}(A)$ is naturally endowed with the weak-* topology [Ru], and if we consider the canonical spectral triple $(C^\infty(M), L^2(S), \not{D})$ over a Riemannian manifold M we can notice that the Dirac operator $\not{D}$ induces exactly the weak-* topology on $\mathcal{S}(C(M))$ [Kan42, KanRub57]. So one can find natural to consider noncommutative

[13] Definition 3.5.6 is a slightly weaker version of the notion of equivariance, with respect to a coaction, adopted, e.g., in [Gos10, BDD11].

spaces, endowed with suitable spectral triples, which fulfil this property: they are usually called *(compact) spectral metric spaces* or *quantum metric spaces* (compact because we consider unital algebras). In this section we recall briefly the main results about these spaces; we refer to literature (especially to Mark Rieffel's works) for a complete discussion [C89, Ri98, Ri99, Ri02, Ri04].

In this section, and only in this section unless explicitly specified, a *spectral triple* (even or odd, it does not matter here) is a triple $(A, \mathcal{H}, D)$ where A is a unital C^*-algebra, $\pi : A \to \mathcal{L}(\mathcal{H})$ is a *faithful* representation of A on a Hilbert space $\mathcal{H}$ and D is a selfadjoint operator on $\mathcal{H}$ such that $(1+D^2)^{-1/2}$ is a compact operator and there is a dense unital *-subalgebra $\mathcal{A} \subset A$ such that the domain of D is invariant under the multiplication by $\pi(a)$ and such that $[D, \pi(a)]$ extends to a bounded operator for any $a \in \mathcal{A}$. We will often omit the representation π, so that $\pi(a)$ will simply be denoted by a.

Let us now give the following definition (see [Ri98] and references therein).

Definition 3.6.1. *A* Lipschitz seminorm *over a unital C^*-algebra A is a seminorm L defined on a dense subalgebra $\mathcal{A}$ of A such that it satisfies the Leibniz property:*

$$L(ab) \leq L(a)\|b\| + \|a\|L(b).$$

Given a Lipschitz norm over A we can define a pseudometric over $\mathcal{S}(A)$ in the following way:

$$d_L(\omega_1, \omega_2) = \sup\{|\omega_1(a) - \omega_2(a)| \mid a \in \mathcal{A},\ L(a) \leq 1\}. \tag{3.6.1}$$

This allows us to give the following definition.

Definition 3.6.2. *A Lipschitz seminorm over a unital C^*-algebra A is called a* Lip-norm *if $L(1) = 0$ and if d_L induces the weak-* topology on $\mathcal{S}(A)$.*

If L is a Lip-norm over A then (A, L) is called a *compact quantum metric space.*

Now, given a spectral triple $(A, \mathcal{H}, D)$, we can use the Dirac operator to define a Lipschitz seminorm on A as follows:

$$L_D(a) = \|[D, \pi(a)]\|, \tag{3.6.2}$$

for any $a \in \mathcal{A}$. The associated pseudometric d_{L_D} is nothing else than the distance d_D. So, if L_D is a Lip-norm then d_D induces the weak-* topology on $\mathcal{S}(A)$ and (A, L_D) is a quantum metric space. In this case we will call $(A, \mathcal{H}, D)$ a *compact spectral metric space.*

We conclude this section by giving a characterization of Lip-norms [Ri98, Ri99, HSWZ11].

Theorem 3.6.3. *Let L be a Lipschitz seminorm over a unital C^*-algebra A and let d_L be the associated pseudometric on $\mathcal{S}(A)$. Then d_L induces the weak-* topology (i.e. L is a Lip-norm, if $L(1) = 0$) if and only if:*

(i) *d_L is bounded;*

(ii) *the set $\mathcal{L}_1 = \{a \in \mathcal{A} \mid L(a) \leq 1,\ \|a\| \leq 1\}$ is totally bounded in $\mathcal{A}$ (w.r.t. $\|\cdot\|$).*

Proof. See [Ri98], theorem 1.9. □

Remark 3.6.4. Condition (ii) of theorem 3.6.3 is equivalent to require that the image of $\mathcal{L}_1$ is totally bounded in $\mathcal{A}/\mathbb{C}1$.

3.7 Spectral triples, KK-theory and Poincaré duality

To a Riemannian spin manifold (M, g) we can associate two classes in Kasparov KK-theory [LRV12]. The first one is the so-called fundamental class $\lambda \in KK(C(M) \otimes \mathbb{C}l(M), \mathbb{C})$; it is represented[14] by the unbounded Kasparov bimodule $(\mathcal{H}_\Lambda, d + d^*)$, where $\mathcal{H}_\Lambda = L^2(\Lambda^\bullet T^*_{\mathbb{C}}M, g)$ and $d + d^*$ is the Hodge–de Rham operator. The second one is a class $\mu \in KK(C(M) \otimes C(M), \mathbb{C})$ and it is represented by the unbounded Kasparov module $(\mathcal{H}, D)$, where $\mathcal{H} = L^2(S)$ is the space of L^2-spinors and D is the Dirac operator associated to the Levi-Civita connection.

In noncommutative geometry a similar result holds: it is possible to associate to a real spectral triple two KK-theory classes as above. Since in this thesis we shall not deal with *Riemannian noncommutative manifolds* (in the sense of [Lord04, LRV12]), we consider here only the class μ, which is associated to the spin structure of the manifold.

Hence, consider an even real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$, fulfilling the regularity and the finiteness condition, so that $\mathcal{A}$ is a Fréchet pre-C^*-algebra. Denote by A its C^*-completion. Then the real structure J determines an action of $A \otimes A^\circ$ on $\mathcal{H}$. Notice now that the operator $\mathfrak{b}(D)$ is an odd regular operator on the graded (with grading γ) Hilbert space $\mathcal{H}$ and, moreover, the analytic properties of the Dirac operator imply that, for any $a \in \mathcal{A}$, the commutator $[D, a]$ is bounded and the operator[15] $a\mathfrak{r}(D)$ is compact. It follows that $(\mathcal{H}, D)$ is an unbounded Kasparov $(A \otimes A^\circ, \mathbb{C})$-bimodule and so we can define $\mu \in KK(A \otimes A^\circ, \mathbb{C})$ to be its equivalence class. Notice that this means that μ is the class of the bounded Kasparov bimodule $(\mathcal{H}, \mathfrak{b}(D))$.

The construction of the fundamental class μ associated to an odd real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ requires some more work [LRV12]. First of all we "double" the triple as follows. Let $\mathbb{C}l_1$ be the Clifford algebra generated by the 2×2 matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Then we consider the spectral triple $(\mathcal{A}' = \mathcal{A} \otimes \mathcal{A}^\circ \otimes \mathbb{C}l_1, \mathcal{H}' = \mathcal{H} \otimes \mathbb{C}^2, D', \gamma')$, where D' and γ are the operators

$$D' = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \qquad \gamma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This triple then defines a class μ in[16] $KK^1(A \otimes A^\circ, \mathbb{C}) \simeq KK^0(A \otimes A^\circ \otimes \mathbb{C}l_1, \mathbb{C})$, which we assume to be the fundamental class associated to the real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$.

The existence of the fundamental class allows to give a different characterization of Poincaré duality [LRV12]. Indeed, given an (even or odd) real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$, defining a class $\mu \in KK^j(A \otimes A^\circ, \mathbb{C})$ ($j = 0$ if the triple is even, $j = 1$ if it is odd) we can reformulate the Poincaré duality condition in the following way: the fundamental class μ determines, for each $i = 0, 1$, an isomorphism[17]

$$- \otimes_A \mu : KK^i(\mathbb{C}, A) \to KK^{i+j}(A^\circ, \mathbb{C}) \simeq KK^{i+j}(A, \mathbb{C}).$$

[14] See [Kas88], 4.2.

[15] $\mathfrak{r}(D)$ and $\mathfrak{b}(D)$ denote, respectively, the resolvent and the bounded transform of D, see equations (2.7.1), (2.7.2).

[16] Here we are using the periodicity properties of KK-theory, see theorem 2.7.25.

[17] With $\otimes_A$ we mean, in this section, the minimal completion of the graded tensor product over the algebra A, see section 2.7.

CHAPTER 4

Quantum principal bundles

The main objects we shall deal with in this thesis are quantum principal bundles (with a Lie group G as structure group), i.e. noncommutative spaces which can be seen as principal fibrations over a noncommutative space, with fibres described by Hopf algebras (of smooth functions over G). Quantum principal bundles were first introduced by T. Brzezinski and S. Majid [BM93] as first steps towards a development of gauge theories over noncommutative spaces based upon quantum groups and principal fibrations. Since then many developments have been made [Haj96, BH99, DGH01, BH04, HKMZ11, BH09, BZ12].

Since one can consider quantum groups as noncommutative generalizations of groups, quantum principal bundles can be defined to be H-comodule algebras, with H a Hopf algebra, fulfilling a set of conditions which correspond to some of the usual properties of principal G-bundles [KN]. It is possible, actually, to consider a more general definition, identifying quantum principal bundles with principal coalgebra extensions, but since in this thesis we shall work with quantum principal bundles with classical structure group, we shall not discuss this part. Since the earliest works [BM93], it was noticed that, as one would like to introduce also a notion of connection on quantum principal bundles, it is necessary to take into consideration the first order differential calculi of the algebras involved. So we can distinguish two different situations: in the first, one considers each of the algebras involved to be endowed with its universal differential calculus; in the second, instead, one considers more general calculi and then introduces a notion of compatibility of the calculus on the total space of the bundle with the calculus on the Hopf algebra.

In the first part of this chapter we shall recall the different formulations and the evolution of the notion of quantum principal bundle. We shall also give the definition of quantum principal bundles with general calculus. The second part, instead, is dedicated to the study of two classes of quantum principal bundles: the cleft bundles, which are the noncommutative counterpart of trivial bundles, and the $\mathbb{T}^n$-bundles. We shall recall, for the former, that they are related with crossed product algebras; for the latter, instead, we shall show how requiring the compatibility of the differential calculus over the total space of the bundle with the de Rham calculus on $\mathbb{T}^n$

yields to an alternative description of noncommutative principal toral fibrations and of strong connections. In particular, we shall see how a strong $\mathbb{T}^n$-connection can be equivalently described by a family of n 1-forms on the total space of the bundle, recovering in this way a more "classical" description of strong connections. Part of these results will be extended, in chapter 8, to quantum principal G-bundle, G being a compact, connected, semisimple Lie group.

4.1 Historical overview

We begin the part of this thesis dedicated to the general theory of noncommutative bundles with a historical overview on the evolution of the definition and the concept itself of quantum principal bundles. Our discussion will not be exhaustive; we refer to literature [BM93, Dur93, Dur96a, Haj96, Dur97a, BH99, DGH01, BH04, HKMZ11, BH09, BZ12] for a more comprehensive treatment. Since Woronowicz's works [Wor87, Wor87b, Wor89], (compact) quantum groups are considered the natural noncommutative generalizations of (compact) Lie groups. On the level of algebras, quantum groups are described by Hopf algebras and hence a candidate for noncommutative bundles are algebra extensions by Hopf algebras. This was the approach adopted by T. Brzezinski and S. Majid, in their seminal paper [BM93].

Definition 4.1.1. *Let H be a Hopf algebra and A a (unital) right H-comodule algebra, with coaction Δ_R. Let $T_R : A \otimes A \to A \otimes H$ be the map*

$$T_R = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R). \tag{4.1.1}$$

T_R will be called the canonical map. *Then A is a* quantum principal bundle *(with universal differential calculus) over the invariant subalgebra $B = A^{coH}$ if the following conditions hold:*
(i) *T_R is surjective;*
(ii) *denote by $\boldsymbol{T}$ the restriction of T_R to $\Omega^1 A = \ker(m) \subset A \otimes A$; then $\ker \boldsymbol{T} \subseteq A(\Omega^1 B)A$.*

In [BM93] it was given also a definition of quantum principal bundle with general calculus. We shall discuss this part of the theory later in this thesis (see section 4.3).

If A is a quantum principal bundle over B, then A will be called the *total space* of the bundle, B the *base space* and H the *structure group.* Now, given a quantum principal bundle (A, H, Δ_R), with invariant subalgebra $B = A^{coH}$, we can define the space of horizontal 1-forms, as one can do for a principal G-bundle over a smooth manifold.

Definition 4.1.2. *The space of* horizontal 1-forms *is the subspace $\Omega^1_{hor} A = A(\Omega^1 B)A$ of $\Omega^1 A$.*

The space of horizontal forms is a sub-A-bimodule of $\Omega^1 A$. We point out here, also, that there is a relevant sub-B-bimodule of the space of horizontal form, which has an important role in the definition of connections:

Definition 4.1.3. *The space of* strongly horizontal 1-forms *is the subspace $\Omega^1_{shor} A = (\Omega^1 B)A$ of $\Omega^1 A$.*

The notion of quantum principal bundle, as given by definition 4.1.1, is equivalent to that of Hopf-Galois extension [ChSw69, Sch90a, Sch90b, BM93, Sch94].

Definition 4.1.4. *A right H-comodule algebra A, with right coaction Δ_R, is a* Hopf-Galois extension *over its invariant subalgebra $B = A^{coH}$ if the canonical map T_R, seen as a map $T_R : A \otimes_B A \to A \otimes H$, is bijective.*

We will usually denote by $B \hookrightarrow A$ a Hopf-Galois extension, omitting the Hopf algebra when there will not be any possible misunderstanding.

Proposition 4.1.5. *An H-comodule algebra A is a Hopf-Galois extension if and only if it is a quantum principal bundle with the universal calculus.*

Proof. See [Haj96], proposition 1.6. See also [B96], lemma 3.2. □

4.1.1 Connections and strong connections

One of the most important notion in the study of principal bundles, in differential geometry, is that of connection. An analogous concept can be introduced in the framework of quantum principal bundles. The study of (strong) connections over Hopf-Galois extensions leads to a refinement of the definition of quantum principal bundle, i.e. to the introduction of the definition of principal comodule algebra. We shall present first the approach of Brzezinski and Majid [BM93], and then we shall discuss the different characterizations of strong connections introduced afterwards [Haj96, DGH01, BH04, HKMZ11, BZ12].

Let $B \hookrightarrow A$ be a Hopf-Galois extension. The coaction Δ_R of H on A induces a right H-coaction $\Delta_R^{A\otimes A}$ on the (algebraic) tensor product $A \otimes A$ in the following way:

$$\Delta_R^{A\otimes A}(a \otimes b) = a_{(0)} \otimes b_{(0)} \otimes a_{(1)} b_{(1)}, \tag{4.1.2}$$

for any $a, b \in A$. $\Delta_R^{A\otimes A}$ restricts to a right coaction on $\Omega^1 A = \ker(m : A \otimes A \to A)$, and the restriction will be denoted by $\Delta_R^{\Omega} : \Omega^1 A \to \Omega^1 A \otimes H$. Now we can introduce a notion of connection for Hopf-Galois extensions [BM93].

Definition 4.1.6. *A* connection *over a Hopf-Galois extension $B \hookrightarrow A$, or, equivalently, over a quantum principal bundle (A, H, Δ_R) with universal differential calculus, is a left A-module projection Π on $\Omega^1 A$ such that:*
(i) $\ker \Pi = \Omega^1_{hor} A$;
(ii) $\Delta_R^{\Omega} \circ \Pi = (\Pi \otimes \mathrm{id}) \circ \Delta_R^{\Omega}$.

If Π is a connection, then $\mathrm{Im}(\Pi)$ is called the space of *vertical forms.* A relevant feature that connections over a quantum principal bundle share with connections over smooth principal G-bundles is that they admit a connection form.

Definition 4.1.7. *A* connection form *over a Hopf-Galois extension $B \hookrightarrow A$ is a linear map $\omega : H \to \Omega^1 A$ such that:*
(i) $\omega(1) = 0$;
(ii) $\boldsymbol{T} \circ \omega = 1 \otimes (\mathrm{id} - \varepsilon)$ *(fundamental vector field condition);*
(iii) $\Delta_R^{\Omega} \circ \omega = (\omega \otimes \mathrm{id}) \circ \mathrm{ad}_R$ *(right adjoint covariance).*

Proposition 4.1.8. *There is a one-to-one correspondence between connections* Π *and connection forms* ω *over a Hopf-Galois extension, given, on exact forms, by*

$$\Pi^{\omega} \circ d = \mathrm{id} *_R \omega, \tag{4.1.3}$$

where $*_R$ *is the right convolution product* (2.3.3).

Proof. See [BM93], proposition 4.4. See also [Haj96], sections 1 and 2. □

Using definition 4.1.3, we can enforce the notion of horizontality for a 1-form, obtaining the space $\Omega^1_{shor}A$ of strongly horizontal 1-forms. There are corresponding notions of strong connections and strong connection forms [Haj96].

Definition 4.1.9. *A connection* Π *over a Hopf-Galois extension* $B \hookrightarrow A$ *is called a* strong connection *iff*

$$(\mathrm{id} - \Pi)(dA) \subseteq \Omega^1_{shor}A.$$

Definition 4.1.10. *A connection form* ω *over a Hopf-Galois extension* $B \hookrightarrow A$ *is called a* strong connection form *iff*

$$da - a_{(0)}\omega(a_{(1)}) \in \Omega^1_{shor}A \qquad \forall a \in A.$$

Proposition 4.1.11. *Equation* (4.1.3) *restricts to a one-to-one correspondence between strong connections and strong connections form.*

Proof. See [Haj96], sections 1 and 2. □

As mentioned before there are different characterizations of strong connections. We recall the following result from [DGH01]. We give directly the main result.

Theorem 4.1.12. *Let* $B \hookrightarrow A$ *be a Hopf-Galois extension. Then the following are equivalent descriptions of a strong connection.*

(1) *a unital left* B*-linear right* H*-colinear splitting* s *of the multiplication map* $B \otimes A \to A$*;*

(2) *a right* H*-colinear homomorphism* $D : A \to \Omega^1_{shor}A$ *with* $D(1) = 0$ *and satisfying the Leibniz rule:* $D(ba) = (db)a + bD(a)$ *for any* $b \in B$, $a \in A$*;*

(3) *a left* A*-linear right* H*-colinear projection* $\Pi : \Omega^1 A \to \Omega^1 A$ *such that* $(\mathrm{id} - \Pi)(dA) \subseteq \Omega^1_{shor}A$*;*

(4) *a homomorphism* $\omega : H \to \Omega^1 A$ *satisfying*

- **a.** $\omega(1) = 0$,
- **b.** $\boldsymbol{T} \circ \omega = 1 \otimes (\mathrm{id} - \varepsilon)$,
- **c.** $\Delta^{\Omega}_R \circ \omega = (\omega \otimes \mathrm{id}) \circ \mathrm{ad}_R$,
- **d.** $da - a_{(0)}\omega(a_{(1)}) \in \Omega^1_{shor}A$ *for any* $a \in A$.

Proof. See [DGH01], theorem 2.3. □

Here by H-colinear map is meant a linear map between two H-comodules which respects the comodule structure. This theorem introduces two new descriptions of a strong connection in addition to (3) and (4) already discussed before. The theorem above allows also to prove some properties of Hopf-Galois extensions. In particular (see [DGH01], corollary 2.4),

Corollary 4.1.13. *If $B \hookrightarrow A$ is a Hopf-Galois extension admitting a strong connection, then*

(i) *A is projective as a left B-module;*

(ii) *B is a direct summand of A as a left B-module;*

(iii) *A is faithfully flat*[1] *over B.*

Definition 4.1.14. *An H-comodule algebra A is called a* principal extension *if it is a Hopf-Galois extension and, moreover, it admits a strong connection.*

Recently another characterization of strong connections over an H-comodule algebra A was introduced in [HKMZ11]. It allows one to change the description of quantum principal bundles from extension to principal comodule algebras [BM98b, BH99, B99, BH09]. We remark that this construction requires that the Hopf algebra H has invertible antipode, which as mentioned in the Introduction is also our assumption (though all the results discussed in this chapter till now hold also for Hopf algebras with non invertible antipode). Denote by Δ_R the coaction of H on A. Then we can define also a left coaction $\Delta_L : A \to H \otimes A$ using the antipode:

$$\Delta_L(a) = S^{-1}(a_{(1)}) \otimes a_{(0)}.$$

This makes A into a left H-comodule algebra. Then we can consider the following definition[2].

Definition 4.1.15. *Let H be a Hopf algebra with invertible antipode. An* HKMZ-connection *on a right H-comodule algebra A is a unital linear map $\ell : H \to A \otimes A$ satisfying:*

(i) $(\mathrm{id} \otimes \Delta_R) \circ \ell = (\ell \otimes \mathrm{id}) \circ \Delta$,

(ii) $(\Delta_L \otimes \mathrm{id}) \circ \ell = (\mathrm{id} \otimes \ell) \circ \Delta$,

(iii) $T_R \circ \ell = 1 \otimes \mathrm{id}$.

Before showing the relation between HKMZ-connections and principal extensions, we recall the definition of the translation map [B96]. Consider a Hopf-Galois extension $B \hookrightarrow A$. By definition the canonical map $T_R : A \otimes_B A \to A \otimes H$ is invertible. Hence we can define a map $\tau : H \to A \otimes_B A$ by

$$\tau(h) = T_R^{-1}(1 \otimes h), \tag{4.1.4}$$

for any $h \in H$. τ is called the *translation map* associated to the extension $B \hookrightarrow A$. We introduce an abbreviated notation for the translation map:

$$\tau(h) \equiv h^{[1]} \otimes h^{[2]},$$

[1] We recall that a right B-module E is *faithfully flat* if the following holds: given any sequence of left B-modules $F' \to F \to F''$, it is exact if and only if the sequence $E \otimes_B F' \to E \otimes_B F \to E \otimes F''$ is exact. For the details see [BourCA], I.3.1.

[2] See [HKMZ11], definition 2.3. Notice that the authors call this object simply a strong connection, or a strong-connection lifting (cfr. [BH09]). For the moment we prefer instead to use a different terminology, until we prove that this is another equivalent description of strong connections.

with summation understood. We list now some properties of τ (see [Sch90b, B96]):

$$\begin{aligned}
(\mathrm{id}\otimes_B \Delta_R)\circ\tau &= (\tau\otimes\mathrm{id})\circ\Delta,\\
((\sigma\circ\Delta_R)\otimes_B \mathrm{id})\circ\tau &= (S\otimes\tau)\circ\Delta,\\
\Delta_{A\otimes_B A}\circ\tau &= (\tau\otimes\mathrm{id})\circ\mathrm{ad}_R,\\
m_A\circ\tau &= \varepsilon,\\
\tau(h\widetilde{h}) &= \widetilde{h}^{[1]}h^{[1]}\otimes_B h^{[2]}\widetilde{h}^{[2]},
\end{aligned} \tag{4.1.5}$$

where $\Delta_{A\otimes_B A}$ is the right coaction of H on $A\otimes_B A$ induced by the usual coaction on $A\otimes A$,

$$\Delta_{A\otimes A}(a\otimes b) = a_{(0)}\otimes b_{(0)}\otimes a_{(1)}b_{(1)},$$

and $\sigma : A\otimes H\to H\otimes A$ is the switch $\sigma(a\otimes h) = h\otimes a$.

Let now A be an H-comodule algebra, with coaction Δ_R, and consider an HKMZ-connection $\ell : H\to A\otimes A$. Let us introduce We introduce an abbreviated notation also for ℓ: for any $h\in H$ we write

$$\ell(h)\equiv \ell(h)^{\langle 1\rangle}\otimes\ell(h)^{\langle 2\rangle}.$$

with summation understood. Now we can rewrite the properties (i)-(iii) in the following way [HKMZ11]:

$$\begin{aligned}
\ell(h)^{\langle 1\rangle}\otimes\left(\ell(h)^{\langle 2\rangle}\right)_{(0)}\otimes\left(\ell(h)^{\langle 2\rangle}\right)_{(0)} &= \ell(h_{(1)})^{\langle 1\rangle}\otimes\ell(h_{(1)})^{\langle 2\rangle}\otimes h_{(2)},\\
\left(\ell(h)^{\langle 1\rangle}\right)_{(0)}\otimes\left(\ell(h)^{\langle 1\rangle}\right)_{(1)}\otimes\ell(h)^{\langle 2\rangle} &= \ell(h_{(2)})^{\langle 1\rangle}\otimes S(h_{(1)})\otimes\ell(h_{(2)})^{\langle 2\rangle},\\
\ell(h)^{\langle 1\rangle}\left(\ell(h)^{\langle 2\rangle}\right)_{(0)}\otimes\left(\ell(h)^{\langle 2\rangle}\right)_{(1)} &= 1\otimes h.
\end{aligned} \tag{4.1.6}$$

Lemma 4.1.16. *Let $\ell : H\to A\otimes A$ be an HKMZ-connection. Then, for any $h\in H$,*

$$\ell(h)^{\langle 1\rangle}\ell(h)^{\langle 2\rangle} = \varepsilon(h).$$

Proof. Apply $\mathrm{id}\otimes\varepsilon$ to the last of (4.1.6). □

Proposition 4.1.17. *A right H-comodule algebra A is a principal extension if and only if it admits an HKMZ-connection.*

Proof. Suppose that $\ell : H\to A\otimes A$ is an HKMZ-connection. We can use it to define a map $\chi : A\otimes H\to A\otimes_B A$ which is an inverse for the canonical map $T_R : A\otimes_B A\to A\otimes H$. We take it to be the composition of the map $\widetilde{\chi} : A\otimes H\to A\otimes A$, defined by

$$\widetilde{\chi}(a\otimes h) = a\ell(h)^{\langle 1\rangle}\otimes\ell(h)^{\langle 2\rangle},$$

with the projection $\pi : A\otimes A\to A\otimes_B A$. If now we take $p\otimes q\in A\otimes A$ we have:

$$\widetilde{\chi}(T_R(p\otimes q)) = pq_{(0)}\ell(q_{(1)})^{\langle 1\rangle}\otimes\ell(q_{(1)})^{\langle 2\rangle}. \tag{4.1.7}$$

Applying $\Delta_R \otimes \mathrm{id}$ to (4.1.7) we obtain:

$$\begin{aligned}(\Delta_R \otimes \mathrm{id})\widetilde{\chi}(T_R(p\otimes q)) &= p_{(0)}q_{(0)}(\ell(q_{(2)})^{\langle 1\rangle})_{(0)} \otimes p_{(1)}q_{(1)}(\ell(q_{(2)})^{\langle 1\rangle})_{(1)} \otimes \ell(q_{(2)})^{\langle 2\rangle} \\ &= \left(p_{(0)}q_{(0)} \otimes p_{(1)}q_{(1)} \otimes 1\right)\left((\ell(q_{(2)})^{\langle 1\rangle})_{(0)} \otimes (\ell(q_{(2)})^{\langle 1\rangle})_{(1)} \otimes \ell(q_{(2)})^{\langle 2\rangle}\right) \\ &= \left(p_{(0)}q_{(0)} \otimes p_{(1)}q_{(1)} \otimes 1\right)\left(\ell(q_{(3)})^{\langle 1\rangle} \otimes S(q_{(2)}) \otimes \ell(q_{(3)})^{\langle 2\rangle}\right) \\ &= p_{(0)}q_{(0)}\ell(q_{(1)})^{\langle 1\rangle} \otimes p_{(1)} \otimes \ell(q_{(1)})^{\langle 2\rangle},\end{aligned} \tag{4.1.8}$$

where we used the second of (4.1.6). Therefore $\widetilde{\chi}(T_R(1\otimes q))$ belongs to $B \otimes A$. Hence, using (4.1.7), we obtain:

$$\begin{aligned}\chi(T_R(p\otimes_B q)) &= p \otimes_B q_{(0)}\ell(q_{(1)})^{\langle 1\rangle}\ell(q_{(1)})^{\langle 2\rangle} \\ &= p\otimes_B q_{(0)}\varepsilon(q_{(1)}) = p\otimes_B q.\end{aligned}$$

where we used lemma 4.1.16. Hence we have proved that $\chi \circ T_R = \mathrm{id}_{A\otimes_B A}$. Next, take $p\otimes h \in A\otimes H$. We have:

$$T_R(\chi(p\otimes h)) = p\ell(h)^{\langle 1\rangle} \otimes_B \ell(h)^{\langle 2\rangle} = p\ell(h)^{\langle 1\rangle}(\ell(h)^{\langle 2\rangle})_{(0)} \otimes (\ell(h)^{\langle 2\rangle})_{(1)} = p\otimes h, \tag{4.1.9}$$

where we used the third of (4.1.6). So we have also $T_R \circ \chi = \mathrm{id}_{A\otimes H}$, which implies that χ is a two-side inverse for the canonical map. In order to show that A is a principal extension we need now only to build a splitting of the multiplication map $B\otimes A \to A$. Consider the map $s : A \to A\otimes A$ defined by

$$s(p) = p_{(0)}\ell(p_{(1)})^{\langle 1\rangle} \otimes \ell(p_{(1)})^{\langle 2\rangle}.$$

With a computation similar to that in (4.1.8), we can see that s takes values in $B\otimes A$. Moreover it is clearly left B-linear, and it is H-colinear due to the first of (4.1.6). Finally, using the last of (4.1.6) we see that $m_A \circ s = \mathrm{id}_A$, and therefore s is the splitting we were looking for. We conclude that A is a principal extension.

Conversely, assume that A is a principal extension. Then we have both a splitting $s : A \to B\otimes A$ of the multiplication map $B\otimes A \to A$ and the translation map τ, defined by equation (4.1.4). Following [BH04], we define a map $\ell : H \to A\otimes A$ as follows:

$$\ell(h) = h^{[1]}s(h^{[2]}) \tag{4.1.10}$$

for any $h\in H$. Since s is left B-linear, and τ takes values in $A\otimes_B A$, ℓ is well-defined. We prove that ℓ fulfils properties (i)-(iii) of definition 4.1.15. Let us begin with the first one. Using the (right) H-colinearity of s we see that

$$((\mathrm{id}\otimes\Delta_R)\circ\ell)(h) = h^{[1]}s((h^{[2]})_{(0)}) \otimes (h^{[2]})_{(1)}. \tag{4.1.11}$$

Now we use the first of (4.1.5) to rewrite (4.1.11) as

$$((\mathrm{id}\otimes\Delta_R)\circ\ell)(h) = (h_{(1)})^{[1]}s((h_{(1)})^{[2]}) \otimes h_{(2)},$$

which is exactly $((\ell\otimes\mathrm{id})\circ\Delta)(h)$, and so (i) of definition 4.1.15 holds. In order to see that also (ii) holds, we use the H-colinearity of s and the second of (4.1.5) to compute the following expression:

$$\begin{aligned}((\Delta_L\otimes\mathrm{id})\circ\ell)(h) &= (S^{-1}\otimes\mathrm{id})\circ((\sigma_{A\otimes H}\circ\Delta_R)\otimes\mathrm{id})(h^{[1]}s(h^{[2]}))\\ &= (S^{-1}\otimes\mathrm{id})\left(S(h_{(1)})\otimes(h_{(2)})^{[1]}s((h_{(2)})^{[2]})\right)\\ &= h_{(1)}\otimes(h_{(2)})^{[1]}s((h_{(2)})^{[2]}) = ((\mathrm{id}\otimes\ell)\circ\Delta)(h),\end{aligned}$$

which is exactly condition (ii). Finally, using property (i), which now we know to hold, we see that

$$(T_R\circ\ell)(h) = (m_A\circ\ell)(h_{(1)})\otimes h_{(2)} = (h_{(1)})^{[1]}m_A(s((h_{(1)})^{[2]}))\otimes h_{(2)}. \tag{4.1.12}$$

But the fact that s is a splitting for the multiplication map, together with the third of (4.1.5), implies that (4.1.12) is actually equal to

$$(h_{(1)})^{[1]}(h_{(1)})^{[2]}\otimes h_{(2)} = \varepsilon(h_{(1)})\otimes h_{(2)} = 1\otimes h,$$

and therefore ℓ fulfils also property (iii). Hence it is an HKMZ-connection. □

Corollary 4.1.18. *There is a one-to-one correspondence between strong connections and HKMZ-connections.*

Proof. The correspondence can be seen as a correspondence $s\leftrightarrow\ell$, where $s\mapsto\ell$ is given by equation (4.1.10), while $\ell\mapsto s$ is given by:

$$s(a) = a_{(0)}\ell(a_{(1)}). \tag{4.1.13}$$

□

Therefore from now on we will refer to an HKMZ-connection simply as to a strong connection, or as to a strong-connection lifting (cfr. [BH09]). So, till now, we have five different characterizations of strong connections (see also theorem 4.1.12). Moreover, we proved that an H-comodule algebra is a principal extension (and therefore a Hopf-Galois extension) if and only if it admits a strong connection.

We conclude this section by noticing that definition 4.1.15 can be equivalently reformulated in the following way [BZ12].

Definition 4.1.19. *Let H be a Hopf algebra with bijective antipode, and let A be a right H-comodule algebra, with coaction Δ_R. Let also m denote the multiplication map of A. Then a linear map $\ell : H\to A\otimes A$ is called a* strong connection *(or a* strong-connection lifting*) if the following hold:*

(i) $\ell(1) = 1\otimes 1$,
(ii) $m\circ\ell = \varepsilon$,
(iii) $(\ell\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta_R)\circ\ell$,
(iv) $(S\otimes\ell)\circ\Delta = (\sigma_{A\otimes H}\otimes\mathrm{id})\circ(\Delta_R\otimes\mathrm{id})\circ\ell$.

The most direct way to see the equivalence between definition 4.1.15 and definition 4.1.19 is to compare conditions (i)-(iv) above to equations (4.1.6).

4.2 Principal comodule algebras

The results discussed in the previous section of this chapter yield to the following definition, which is the one adopted in the more recent works (see, e.g., [BZ12]).

Definition 4.2.1. *Let H be a Hopf algebra with invertible antipode. Then a right H-comodule algebra A is called a* principal comodule algebra *if it admits a strong connection*[3] $\ell : H \to A \otimes A$.

In particular a principal comodule algebra A is a Hopf-Galois extension admitting a strong connection, as it follows from proposition 4.1.17.

4.2.1 Gauge transformations

Since principal comodule algebras are a particular class of principal coalgebra-Galois extensions, we can define their gauge transformations [BM93, B96, Haj96, Dur96b, Dur97a, Dur97b, DGH01].

Definition 4.2.2. *Let A be a principal comodule algebra, and let B denote its invariant subalgebra. A* vertical automorphism *(or* gauge transformation*) is a left B-module automorphism $\mathcal{F} : A \to A$ such that $\mathcal{F}(1) = 1$ and $\Delta_R \circ \mathcal{F} = (\mathcal{F} \otimes \mathrm{id}) \circ \Delta_R$. The set of vertical automorphisms is the group $\mathrm{Aut}_B(A)$.*

Now we study the main properties of vertical automorphisms of principal comodule algebras. As will be clear from the results below, they share many properties with vertical automorphisms of a principal G-bundle.

Proposition 4.2.3. *Vertical automorphisms of a principal comodule algebra A are in bijective correspondence with convolution invertible linear maps $f : H \to A$ such that $f(1) = 1$ and $\Delta_R \circ f = (f \otimes \mathrm{id}) \circ \mathrm{ad}_R$. We shall call these maps* gauge transformations *of the bundle.*

Proof. The correspondence is given by

$$\mathcal{F} \ \mapsto \ f = m_A \circ (\mathrm{id} \otimes_B \mathcal{F}) \circ \tau \tag{4.2.1}$$

(where τ is the translation map, see [B96]), whose inverse is

$$f \ \mapsto \ \mathcal{F} = \mathrm{id} * f. \tag{4.2.2}$$

For the details see [B96], proposition 5.2. □

The gauge transformations $f : H \to A$ form a group, say $\mathcal{G}(A)$, under the convolution product. One can prove that:

Corollary 4.2.4. $\mathrm{Aut}_B(A) \simeq \mathcal{G}(A)$ *as multiplicative groups.*

[3] Or strong connection lifting; see definition 4.1.19.

Proof. See [B96], corollary 5.3. □

Theorem 4.2.5. *Let A be a principal comodule algebra. Let $f \in \mathcal{G}(A)$ be a gauge transformation, and let f^{-1} denote its convolution inverse. Then the following describe left actions of f on strong connections which are compatible with the identifications of theorem 4.1.12 and corollary 4.1.18:*

(i) $(f \triangleright s)(a) = s(a_{(0)} f(a_{(1)})) f^{-1}(a_{(2)})$,

(ii) $(f \triangleright D)(a) = D(a_{(0)} f(a_{(1)})) f^{-1}(a_{(2)})$,

(iii) $(f \triangleright \Pi)(adb) = a\Pi\left(d(b_{(0)} f(b_{(1)}))\right) f^{-1}(b_{(2)}) + ab_{(0)} f(b_{(1)}) df^{-1}(b_{(2)})$,

(iv) $(f \triangleright \omega)(h) = f(h_{(1)})\omega(h_{(2)}) f^{-1}(h_{(3)}) + f(h_{(1)}) df^{-1}(h_{(2)})$,

(v) $(f \triangleright \ell)(h) = f(h_{(1)})\ell(h_{(2)}) f^{-1}(h_{(3)})$.

Proof. The consistency between (i), (ii), (iii), (iv) has already been proved in [DGH01] (see theorem 4.1). We show here that, if ℓ corresponds to s, then the HKMZ-connection corresponding to $(f \triangleright s)$ is the one given in (v) (the proof in the more general case of coalgebra-Galois extension can be found in [BH09]; here we present our direct proof for the case of Hopf algebra extensions). We recall that the correspondence $s \leftrightarrow \ell$ is given by

$$\begin{aligned} s(a) &= a_{(0)}\ell(a_{(1)}), \\ \ell(h) &= h^{[1]} s(h^{[2]}). \end{aligned} \tag{4.2.3}$$

So, if we denote $f \triangleright s$ by s', then the corresponding HKMZ-connection is given by:

$$\begin{aligned} \ell'(h) &= h^{[1]} s'(h^{[2]}) = h^{[1]} s\left({h^{[2]}}_{(0)} f({h^{[2]}}_{(1)})\right) f^{-1}({h^{[2]}}_{(2)}) \\ &= {h_{(1)}}^{[1]} s\left({h_{(1)}}^{[2]} f(h_{(2)})\right) f^{-1}(h_{(3)}), \end{aligned} \tag{4.2.4}$$

where we used (4.1.6). Now we use again (4.2.3), together with (4.1.6), to write, in (4.2.4), s in terms of ℓ, obtaining the following expression:

$$\begin{aligned} \ell'(h) &= {h_{(1)}}^{[1]} ({h_{(1)}}^{[2]} f(h_{(2)}))_{(0)} \ell\left(({h_{(1)}}^{[2]} f(h_{(2)}))_{(1)}\right) f^{-1}(h_{(3)}) \\ &= {h_{(1)}}^{[1]} {h_{(1)}}^{[2]} f(h_{(3)})_{(0)} \ell\left(h_{(2)} f(h_{(3)})_{(1)}\right) f^{-1}(h_{(4)}) \\ &= f(h_{(2)})_{(0)} \ell\left(h_{(1)} f(h_{(2)})_{(1)}\right) f^{-1}(h_{(3)}), \end{aligned} \tag{4.2.5}$$

where, in the last equality, we used the properties of the translation map (see (4.1.5)). Now we use the fact that f is ad_R-equivariant to rewrite (4.2.5) as

$$\begin{aligned} \ell'(h) &= f(h_{(3)})\ell(h_{(1)} S(h_{(2)}) h_{(4)}) f^{-1}(h_{(5)}) \\ &= f(h_{(1)})\ell(h_{(2)}) f^{-1}(h_{(3)}), \end{aligned}$$

which is exactly $(f \triangleright \ell)(h)$, as given by (v). □

Notice that we can write the gauge transformed of a strong connection ℓ under a transformation $f \in \mathcal{G}(A)$ as $f * \ell * f^{-1}$.

4.3 QPBs with general differential calculus

In sections 4.1 and 4.2 we discussed quantum principal bundles, (strong) connections and gauge transformations assuming that all the algebras involved (the Hopf algebra H, the H-comodule algebra A and the invariant subalgebra B) were endowed with the respective universal differential calculus. It is natural, now, to extend this definition and to consider the case of general calculi. Such an extension is unavoidable if we want our theory to cover many relevant cases: indeed, even a complete description of a smooth principal G-bundle can not be achieved using only the universal calculus-based theory, since many of its properties come from the fact that all the spaces involved are endowed with the de Rham calculus, which is far from universal. So, in this section, we discuss quantum principal bundles with general calculus. The main definitions were already given in [BM93]. Due to the results discussed in the previous sections, we restrict ourself to H-comodule algebras A, where H is a Hopf algebra with invertible antipode, which are principal comodule algebras (see definition 4.2.1), so that we already know that they admit a strong connection with respect to the universal calculus.

4.3.1 Definition

Let H be a Hopf algebra with invertible antipode and A a (unital) right H-comodule algebra, with coaction Δ_R. Let $T_R : A \otimes A \to A \otimes H$ be the canonical map. Let N_A be a sub-bimodule of $\ker(m) \subset A \otimes A$ defining[4] a first order differential calculus $\Omega^1(A)$ on A and $Q \subset \ker(\varepsilon)$ be an ad_R-invariant right ideal of H defining[5] a bicovariant first order differential calculus on H. We consider the following H-coaction on $\ker(m) \subset A \otimes A$:

$$\Delta_R^{\Omega}(a \otimes b) = a_{(0)} \otimes b_{(0)} \otimes a_{(1)}b_{(1)}. \tag{4.3.1}$$

Let B be the invariant subalgebra of A and assume that B is endowed with the differential calculus induced by the inclusion $B \hookrightarrow A$: $\Omega^1(B) = \Omega^1 B/(N_A \cap B \otimes B)$. Then we can give the following definition (cf. [BM93, Haj96]).

Definition 4.3.1. *(A, H, Δ_R, N_A, Q) is called a* quantum principal bundle *if:*
(i) *A is a principal comodule algebra,*
(ii) *$\Delta_R^{\Omega}(N_A) \subseteq N_A \otimes H$ (right covariance of the differential calculus),*
(iii) *$T_R(N_A) \subset A \otimes Q$ (fundamental vector field compatibility condition),*
(iv) *$\ker(T) \subset A\Omega^1(A^{coH})A$ (exactness condition), where $\Omega^1(A^{coH}) = \Omega^1 A^{coH}/(N_A \cap \Omega^1 A^{coH})$ and $T : \Omega^1(A) \to A \otimes (\ker(\varepsilon)/Q)$ is the map:*

$$T : [\alpha]_{N_A} \;\mapsto\; ((\mathrm{id} \otimes \pi_Q) \circ T_R)(\alpha),$$

π_Q being the canonical projection $\ker(\varepsilon) \to \ker(\varepsilon)/Q$.

The algebra A will be called *total space* of the bundle, the Hopf algebra H will be called the *structure group* and the invariant subalgebra $B = A^{coH}$ the *base space*. As in the case of QPBs

[4] See proposition 2.2.3.
[5] See theorem 2.3.31.

with universal calculus (i.e. for principal comodule algebras), we can define the sets of horizontal and strongly horizontal 1-forms.

Definition 4.3.2. *The space of* horizontal 1-forms *is the subspace* $\Omega^1_{hor}(A) = A\Omega^1(B)A$ *of* $\Omega^1(A)$.

Definition 4.3.3. *The space of* strongly horizontal 1-forms *is the subspace* $\Omega^1_{shor}(A) = \Omega^1(B)A$ *of* $\Omega^1(A)$.

4.3.2 Connections and strong connections

We have already discussed the theory of (strong) connections over a quantum principal bundle with universal calculus. Now we give the analogous definitions in the case of general first order differential calculus. We refer to [BM93, Haj96] for the details.

Definition 4.3.4. *A left A-module projection* Π *on* $\Omega^1(A)$ *is called a* connection *for the quantum principal bundle* (A, H, Δ_R, N_A, Q) *iff:*
(i) $\ker(\Pi) = \Omega^1_{hor}(A)$,
(ii) $\Delta^{\Omega}_R \circ \Pi = (\Pi \otimes \mathrm{id}) \circ \Delta^{\Omega}_R$ *(right covariance).*

The image of Π will be called the space of *vertical forms* and denoted by $\Omega^1_{ver}(A)$.

Definition 4.3.5. *A connection* Π *is called* strong *iff* $(\mathrm{id} - \Pi)(dA) \subseteq \Omega^1_{shor}(A)$.

Also in this setting we can show that we can associate a (strong) connection 1-form ω to any (strong) connection Π (and viceversa).

Definition 4.3.6. *A* strong connection form *on a quantum principal bundle* (A, H, Δ_R, N_A, Q) *is a homomorphism* $\omega : H \to \Omega^1(A)$ *which satisfies:*
(i) $\omega(\mathbb{C} \oplus Q) = 0$ *(compatibility with the differential structure);*
(ii) $\Delta^{\Omega}_R \circ \omega = (\omega \otimes \mathrm{id}) \circ \mathrm{ad}_R$ *(right adjoint covariance);*
(iii) $T \circ \omega = (\mathrm{id} \otimes \pi_H) \circ (1 \otimes (\mathrm{id} - \varepsilon))$ *(fundamental vector field condition);*
(iv) $da - a_{(0)}\omega(a_{(1)}) \in \Omega^1(B)A$ *for any* $a \in A$ *(strongness).*

A map which fulfils (i)-(iii) but not (iv) will be simply called a *connection form.*

Proposition 4.3.7. *There is a one to one correspondence between strong connections* Π *and strong connection forms* ω*, given, on exact forms, by*

$$\Pi^{\omega} \circ d = \mathrm{id} *_R \omega, \tag{4.3.2}$$

where $*_R$ *is the right convolution product* (2.3.3).

Proof. See formula (47) in [BM93]; see also [Haj96], sections 1 and 2. □

In the case of principal comodule algebras there are five different characterizations of strong connections (see theorem 4.1.12 and corollary 4.1.18). A similar results for QPBs with general calculus is, at the best of our knowledge, not (yet) available. Nevertheless, we can notice the following fact:

Proposition 4.3.8. *Let (A, H, Δ_R, N_A, Q) be a quantum principal bundle and let $B = A^{coH}$. Let $\ell : H \to A \otimes A$ be a strong connection for principal comodule algebra A (see definition 4.1.19). Then, if $\ell(Q) \subseteq N_A$, there is a strong connection form $\omega : H \to \Omega^1(A)$, associated to ℓ, for the quantum principal bundle (A, H, Δ_R, N_A, Q).*

Proof. Let $\omega : H \to \Omega^1 A$ be the universal strong connection form associated to ℓ. Then it descends to a strong connection form $\omega : H \to \Omega^1(A)$ if and only if $\omega(Q) = 0$. Since $\ell = \omega + \varepsilon$, and $Q \subseteq \ker \varepsilon$, this holds if and only if $\ell(Q) = 0$.

□

4.4 Cleft principal comodule algebras

There is an important class of principal comodule algebras: that of cleft Hopf-Galois extensions. As proposed in [BM93], they can be identified with "trivial" quantum principal bundles.

Definition 4.4.1. *A Hopf-Galois extension $B \hookrightarrow A$ is called a* cleft *extension if there is a convolution invertible linear map $\phi : H \to A$ such that $\phi(1) = 1$ and*

$$\Delta_R \circ \phi = (\phi \otimes \mathrm{id}) \circ \Delta.$$

Proposition 4.4.2. *Any cleft Hopf-Galois extension admits a strong connection. In particular, any cleft Hopf-Galois extension is a principal comodule algebra.*

Proof. Let ϕ be a trivialization of a cleft extension $B \hookrightarrow A$ and let $\ell : H \to A \otimes A$ be the map defined by:

$$\ell(h) = \phi^{-1}(h_{(1)}) \otimes \phi(h_{(2)}).$$

Then ℓ is a strong-connection lifting (see definition 4.1.19) [BZ12]. □

In particular we can speak of cleft principal comodule algebras, meaning that they are cleft extensions over their invariant subalgebra. The next sections will be dedicated to the study of their structure and their properties.

4.4.1 Gauge transformations of cleft extensions

Let us consider a cleft Hopf-Galois extension $B \hookrightarrow A$ with trivialization $\phi : H \to A$. Then we can rewrite (4.2.1) as:

$$\mathcal{F} \mapsto f = \phi^{-1} * (\mathcal{F} \circ \phi). \tag{4.4.1}$$

In [BM93], gauge transformations of a trivial quantum principal bundle were defined as convolution invertible linear maps $\Lambda : H \to B$ with $\Lambda(1) = 1$; these maps form a group, say $\mathcal{G}(\mathcal{B})$, under the convolution product. Then we can give the following definition, which can be seen to be consistent with those given above (see the proposition below) [B96].

Definition 4.4.3. *The group of* gauge transformations *of a cleft Hopf-Galois extension $\mathcal{B} \hookrightarrow \mathcal{A}$ is the group $\mathcal{G}(\mathcal{B})$ of unital convolution invertible linear maps $\Lambda : H \to \mathcal{B}$.*

Proposition 4.4.4. *Let $B \hookrightarrow A$ be a cleft Hopf-Galois extension. Then the groups* $\mathrm{Aut}_B(A)$, *$\mathcal{G}(A)$ and $\mathcal{G}(B)$ are isomorphic one to each other.*

Proof. The first isomorphism is given by corollary 4.2.4. The second one is obtained using the following correspondence between maps $f \in \mathcal{G}(A)$ and maps $\Lambda \in \mathcal{G}(B)$:

$$f \;\mapsto\; \Lambda = \phi * f * \phi^{-1}, \tag{4.4.2}$$

with inverse given by

$$\Lambda \;\mapsto\; f = \phi^{-1} * \Lambda * \phi. \tag{4.4.3}$$

For the details see [B96], theorem 5.4. □

From these results we can see that a gauge transformation of a cleft extension can be seen simply as a change of trivialization:

Proposition 4.4.5. *Consider a cleft Hopf-Galois extension $B \hookrightarrow A$ together with a trivialization ϕ. Let $\mathcal{F}$ be a gauge transformation and $\Lambda : H \to B$ the convolution invertible map associated to it by proposition 4.4.4. Then $\phi_\Lambda \equiv \Lambda^{-1} * \phi$ is another trivialization map for the extension $B \hookrightarrow A$, with convolution inverse $\phi_\Lambda^{-1} = \phi^{-1} * \Lambda$.*

Proof. ϕ_Λ is linear and unital. Moreover, $\phi_\Lambda * \phi_\Lambda^{-1} = \Lambda^{-1} * \phi * \phi^{-1} * \Lambda = \varepsilon$ and ϕ_Λ is H-equivariant since ϕ is, and Λ takes values in $\mathcal{B}$ (which is the invariant subalgebra). Thus ϕ_Λ is a trivialization for the extension $\mathcal{B} \hookrightarrow \mathcal{A}$. □

4.4.2 Cleft extensions and crossed products

Let $B \hookrightarrow A$ be a cleft Hopf-Galois extension, with Hopf algebra H. We will show that A is isomorphic to a crossed product $B\#_\sigma H$, where σ is a suitable 2-cocyle on H with values in B [BlCM86, DT86, BlM89, Ch98]. Before proving this result we recall the definition of crossed product of an algebra with a Hopf algebra.

Definition 4.4.6. *Let B be an associative algebra (with unit) and H be a Hopf algebra. A (left)* weak action *of H on B is a bilinear map $H \times B \to B$ (we will use the notation $h \triangleright a$ for the action of $h \in H$ on $a \in B$) such that, for any $h \in H$, $a, b \in B$,*
(i) $h \triangleright ab = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$;
(ii) $h \triangleright 1_B = \varepsilon(h) 1_B$;
(iii) $1 \triangleright a = a$.

Definition 4.4.7. *Let H be a Hopf algebra weakly acting on an algebra B, and let $\sigma : H \times H \to B$ be a $\mathbb{C}$-bilinear map. Let $B\#_\sigma H$ be the algebra whose underlying space is $B \otimes H$ and whose multiplication is given by*

$$(a \otimes h)(b \otimes l) = a(h_{(1)} \triangleright b)\sigma(h_{(2)}, l_{(1)}) \otimes h_{(3)} l_{(2)}.$$

for all $a, b \in B$ and $h, l \in H$. The algebra $B\#_\sigma H$ is called a crossed product *if it is associative with $1 \otimes 1$ as identity element.*

We will denote by $a\#h$ the element $a \otimes h \in B\#_\sigma H$. Notice that we can put on $B\#_\sigma H$ a structure of right H-comodule algebra, simply defining the coaction as $\Delta_R = \mathrm{id} \otimes \Delta$. It is possible to give conditions on σ for which $B\#_\sigma H$ is a crossed product. We begin with the following definition.

Definition 4.4.8. *A bilinear map $\sigma : H \times H \to B$ is called* normal *if $\sigma(h,1) = \sigma(1,h) = \varepsilon(h)1$ for any $h \in H$.*

Lemma 4.4.9. $1\#1$ *is the identity of $B\#_\sigma H$ if and only if σ is normal.*

Proof. See [BlCM86], lemma 4.4. □

Proposition 4.4.10. *Assume σ normal. Then $B\#_\sigma H$ is associative if and only if the following conditions hold:*
(i) (cocycle condition) *for all $h, l, m \in H$:*

$$\left[h_{(1)} \triangleright \sigma(l_{(1)}, m_{(1)})\right] \sigma(h_{(2)}, l_{(2)}m_{(2)}) = \sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, m); \tag{4.4.4}$$

(ii) (twisted module condition) *for all $h, l, m \in H$ and all $a \in B$:*

$$\left(h_{(1)} \triangleright (l_{(1)} \triangleright a)\right) \sigma(h_{(2)}, l_{(2)}) = \sigma(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \triangleright a). \tag{4.4.5}$$

Proof. See [BlCM86], lemma 4.5. □

Hence proposition 4.4.10 gives a complete characterization of crossed products. We can identify a class of "simple" (non-trivial, i.e. non isomorphic to the tensor product algebra $H \otimes B$) crossed-products:

Definition 4.4.11. *A crossed product $B\#_\sigma H$ is called a* smash product *if the cocycle σ is trivial. We will denote a smash product simply by $B\#H$.*

Now consider a cleft Hopf-Galois extension $B \hookrightarrow A$, with trivialization $\phi : H \to A$. The map ϕ induces [BlCM86, Ch98, DT86] a weak action of H on the invariant subalgebra B:

$$h \triangleright b = \phi(h_{(1)}) b \phi^{-1}(h_{(2)}). \tag{4.4.6}$$

Moreover, we can define a cocycle $\sigma : H \times H \to B$ by

$$\sigma(h, l) = \phi(h_{(1)})\phi(l_{(1)})\phi^{-1}(h_{(2)}l_{(2)}). \tag{4.4.7}$$

Proposition 4.4.12. *σ, as defined by 4.4.7, is normal and fulfils conditions (i) and (ii) of proposition 4.4.10. Thus $B\#_\sigma H$ is a crossed product.*

Proof. It follows by direct computation, using the fact that ϕ^{-1} is the convolution inverse of ϕ. □

Now one can prove that any cleft Hopf-Galois extension is isomorphic to a crossed product algebra.

Proposition 4.4.13. *There is a right H-comodule algebra isomorphism $F : B\#_\sigma H \to A$ given by: $F(a\#h) = a\phi(h)$.*

Proof. See [DT86], theorem 11. □

Remark 4.4.14. If the cocycle σ, defined in equation (4.4.7), is trivial we get that A is isomorphic to the smash product $B\#H$. Notice that σ is trivial iff ϕ is an algebra homomorphism.

Remark 4.4.15. If the algebra B is commutative any weak action of the form (4.4.6) is trivial; hence any smash product $B\#H$, with B commutative, is isomorphic, as an algebra, to the tensor product algebra $B\otimes H$. It follows that the usual notion of trivial principal G-bundle in differential geometry corresponds to the case of cleft extensions with trivialization which is an algebra map, that is to smash products.

4.4.3 *-structure of cleft extensions and crossed products

In this thesis we will usually deal with Hopf *-algebras and associative *-algebras. We have already given a definition of H-comodule algebra which takes care of the star structures of both algebras; now we are interested in studying the *-structure of cleft extensions and crossed products. Consider therefore a Hopf-Galois extension $B \hookrightarrow A$, w.r.t. a Hopf *-algebra H, where also A is a *-algebra (and, consequently, so is B). Assume that it is a cleft extension. We know then that A, as an associative algebra, is isomorphic to a crossed product $B\#_\sigma H$, as an associative algebra. So we can look for the conditions under which it is isomorphic to $B\#_\sigma H$ as a *-algebra. Of course, the first thing we need is a way to define an involution on a crossed product algebra. We begin with the following definition[6].

Definition 4.4.16. *Let B be an associative *-algebra and H be a Hopf *-algebra. Then a (weak) action of H on B is called a* (weak) *-action *if the following conditions hold:*
(i) *for every $b \in B$ and every $h \in H$,*

$$(h \triangleright a)^* = (Sh)^* \triangleright a^*;$$

(ii) *the cocycle σ is convolution invertible. In particular, for each $h, l \in H$,*

$$\sigma(h_{(1)}, l_{(1)})\sigma(S^{-1}h^*_{(2)}, S^{-1}l^*_{(2)})^* = \varepsilon(h)\varepsilon(l),$$

$$\sigma(S^{-1}h^*_{(1)}, S^{-1}l^*_{(1)})^*\sigma(h_{(2)}, l_{(2)}) = \varepsilon(h)\varepsilon(l).$$

Consider now a *-algebra B together with a weak *-action of H on it, with cocycle σ, such that $B\#_\sigma H$ is a crossed product (see proposition 4.4.10). Then we can define an involution $*$ on $B\#_\sigma H$ as follows:

$$(b\#h)^* = \sigma(S^{-1}h_{(2)}, h_{(1)})^*(h^*_{(3)} \triangleright b^*)\#h^*_{(4)}, \tag{4.4.8}$$

for any $b\#h \in B\#_\sigma H$. With this operation, $B\#_\sigma H$ is an associative (unital) *-algebra. Indeed:

[6] For the cocycle-free case see definition 2.3.18. See also [S03], section 2.2, and [Maj95], p. 31.

Proposition 4.4.17. *The $*$ operation defined by* (4.4.8) *is an involution, that is* $(b\#h)^{**} = b\#h$ *for every* $b\#h \in B\#_\sigma H$*. Moreover, it makes* $B\#_\sigma H$ *into a* **-algebra: for any* $a, b \in B$ *and any* $h, l \in H$,

$$((a\#h)(b\#l))^* = (b\#l)^*(a\#h)^*.$$

Proof. First of all we show that $^{**} = \mathrm{id}$. We do it by direct computation; given $b\#h \in B\#H$ we have:

$$\begin{aligned}
(b\#h)^{**} &= \Big(\sigma(S^{-1}h_{(2)}, h_{(1)})^*(h^*_{(3)} \triangleright b^*)\#h^*_{(4)}\Big)^* \\
&= \sigma(S^{-1}h^*_{(5)}, h^*_{(4)})^*(h_{(6)} \triangleright (h^*_{(3)} \triangleright b^*)^*)(h_{(7)} \triangleright \sigma(S^{-1}h_{(2)}, h_{(1)}))\#h_{(8)} \\
&= \sigma(S^{-1}h^*_{(5)}, h^*_{(4)})^*(S^{-1}h^*_{(6)} \triangleright (h^*_{(3)} \triangleright b^*))^*(h_{(7)} \triangleright \sigma(S^{-1}h_{(2)}, h_{(1)}))\#h_{(8)} \\
&= \Big[(S^{-1}h^*_{(6)} \triangleright (h^*_{(3)} \triangleright b^*))\sigma(S^{-1}h^*_{(5)}, h^*_{(4)})\Big]^* (h_{(7)} \triangleright \sigma(S^{-1}h_{(2)}, h_{(1)}))\#h_{(8)} \\
&= \Big[\sigma(S^{-1}h^*_{(6)}, h^*_{(3)})((S^{-1}h^*_{(5)})h^*_{(4)} \triangleright b^*)\Big]^* (h_{(7)} \triangleright \sigma(S^{-1}h_{(2)}, h_{(1)}))\#h_{(8)} \\
&= b \cdot \sigma(S^{-1}h^*_{(4)}, h^*_{(3)})^*(h_{(5)} \triangleright \sigma(S^{-1}h_{(2)}, h_{(1)}))\#h_{(6)} \\
&= b \cdot \sigma(S^{-1}h^*_{(6)}, h^*_{(5)})^*(h_{(7)} \triangleright \sigma(S^{-1}h_{(4)}, h_{(1)}))\sigma(h_{(8)}, (S^{-1}h_{(3)})h_{(2)})\#h_{(9)} \\
&= b \cdot \sigma(S^{-1}h^*_{(5)}, h^*_{(4)})^*\sigma(h_{(6)}, S^{-1}h_{(3)})\sigma(h_{(7)}S^{-1}h_{(2)}, h_{(1)})\#h_{(8)}.
\end{aligned}$$

where we used the cocycle condition and the properties of S^{-1} (see lemma 2.3.5). Using (ii) of definition 4.4.16 we see that the product of the first two cocycles of this expression is equal to $\varepsilon(h_{(4)})\varepsilon(S^{-1}h_{(3)})$. Therefore we obtain:

$$\begin{aligned}
(b\#h)^{**} &= b \cdot \varepsilon(h_{(4)})\varepsilon(S^{-1}h_{(3)})\sigma(h_{(5)}S^{-1}h_{(2)}, h_{(1)})\#h_{(6)} \\
&= b \cdot \sigma(h_{(3)}S^{-1}h_{(2)}, h_{(1)})\#h_{(4)} = b\#h.
\end{aligned}$$

So we are left with the proof that $((a\#h)(b\#l))^* = (b\#l)^*(a\#h)^*$. Let us compute separately the two expressions. For the first one we have:

$$\begin{aligned}
((a\#h)(b\#l))^* &= \big(a(h_{(1)} \triangleright b)\sigma(h_{(2)}, l_{(1)})\#h_{(3)}l_{(2)}\big)^* \\
&= \sigma(S^{-1}(h_{(4)}l_{(3)}), h_{(3)}l_{(2)})^* \Big(l^*_{(4)}h^*_{(5)} \triangleright \sigma(h_{(2)}, l_{(1)})^*(h_{(1)} \triangleright b)^*a^*\Big) \#l^*_{(5)}h^*_{(6)} \\
&= \sigma(S^{-1}(h_{(4)}l_{(3)}), h_{(3)}l_{(2)})^*(l^*_{(4)}h^*_{(5)} \triangleright \sigma(h_{(2)}, l_{(1)})^*) \\
&\qquad \cdot (l^*_{(5)}h^*_{(6)} \triangleright (h_{(1)} \triangleright b)^*)(l^*_{(6)}h^*_{(7)} \triangleright a^*)\#l^*_{(7)}h^*_{(8)} \\
&= \sigma(S^{-1}(h_{(4)}l_{(3)}), h_{(3)}l_{(2)})^*(S^{-1}(h_{(5)}l_{(4)}) \triangleright \sigma(h_{(2)}, l_{(1)}))^* \\
&\qquad \cdot (l^*_{(5)}h^*_{(6)} \triangleright (h_{(1)} \triangleright b)^*)(l^*_{(6)}h^*_{(7)} \triangleright a^*)\#l^*_{(7)}h^*_{(8)} \\
&= \big[(S^{-1}(h_{(5)}l_{(4)}) \triangleright \sigma(h_{(2)}, l_{(1)}))\sigma(S^{-1}(h_{(4)}l_{(3)}), h_{(3)}l_{(2)})\big]^* \\
&\qquad \cdot (l^*_{(5)}h^*_{(6)} \triangleright (h_{(1)} \triangleright b)^*)(l^*_{(6)}h^*_{(7)} \triangleright a^*)\#l^*_{(7)}h^*_{(8)}
\end{aligned} \tag{4.4.9}$$

Using the cocycle condition we can rewrite equation (4.4.9) as:

$$((a\#h)(b\#l))^* = \big[\sigma(S^{-1}(h_{(5)}l_{(3)}), h_{(2)})\sigma(S^{-1}(h_{(4)}l_{(2)})h_{(3)}, l_{(1)})\big]^*$$

$$
\begin{aligned}
&\cdot (l^*_{(4)}h^*_{(6)} \triangleright (h_{(1)} \triangleright b)^*)(l^*_{(5)}h^*_{(7)} \triangleright a^*)\#l^*_{(6)}h^*_{(8)} \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^*\sigma(S^{-1}(h_{(3)}l_{(3)}), h_{(2)})^*(l^*_{(4)}h^*_{(4)} \triangleright (h_{(1)} \triangleright b)^*) \\
&\quad \cdot (l^*_{(5)}h^*_{(5)} \triangleright a^*)\#l^*_{(6)}h^*_{(6)} \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^*\sigma(S^{-1}(h_{(3)}l_{(3)}), h_{(2)})^*(S^{-1}(h_{(4)}l_{(4)}) \triangleright h_{(1)} \triangleright b)^* \\
&\quad \cdot (l^*_{(5)}h^*_{(5)} \triangleright a^*)\#l^*_{(6)}h^*_{(6)} \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^* \left[(S^{-1}(h_{(4)}l_{(4)}) \triangleright h_{(1)} \triangleright b)\sigma(S^{-1}(h_{(3)}l_{(3)}), h_{(2)})\right]^* \\
&\quad \cdot (l^*_{(5)}h^*_{(5)} \triangleright a^*)\#l^*_{(6)}h^*_{(6)}
\end{aligned} \tag{4.4.10}
$$

Finally, we use the twisted module condition to rewrite equation (4.4.10) as follows:

$$
\begin{aligned}
((a\#h)(b\#l))^* &= \sigma(S^{-1}l_{(2)}, l_{(1)})^* \left[\sigma(S^{-1}(h_{(4)}l_{(4)}), h_{(1)})(S^{-1}(h_{(3)}l_{(3)})h_{(2)} \triangleright b)\right]^* \\
&\quad \cdot (l^*_{(5)}h^*_{(5)} \triangleright a^*)\#l^*_{(6)}h^*_{(6)} \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^*(S^{-1}l_{(3)} \triangleright b)^*\sigma(S^{-1}(h_{(2)}l_{(4)}), h_{(1)})^*(l^*_{(5)}h^*_{(3)} \triangleright a^*)\#l^*_{(6)}h^*_{(4)} \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^*(l^*_{(3)} \triangleright b^*)\sigma(S^{-1}(h_{(2)}l_{(4)}), h_{(1)})^*(l^*_{(5)}h^*_{(3)} \triangleright a^*)\#l^*_{(6)}h^*_{(4)}
\end{aligned} \tag{4.4.11}
$$

The computation of the other expression, instead, yields:

$$
\begin{aligned}
(b\#l)^*(a\#h)^* &= \left(\sigma(S^{-1}l_{(2)}, l_{(1)})^*(l^*_{(3)} \triangleright b^*)\#l^*_{(4)}\right) \left(\sigma(S^{-1}h_{(2)}, h_{(1)})^*(h^*_{(3)} \triangleright a^*)\#h^*_{(4)}\right) \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^*(l^*_{(3)} \triangleright b^*) \\
&\quad \cdot \left(l^*_{(4)} \triangleright \sigma(S^{-1}h_{(2)}, h_{(1)})^*(h^*_{(3)} \triangleright a^*)\right) \sigma(l^*_{(5)}, h^*_{(4)})\#l^*_{(6)}h^*_{(5)} \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^*(l^*_{(3)} \triangleright b^*) \\
&\quad \cdot (S^{-1}l_{(4)} \triangleright \sigma(S^{-1}h_{(2)}, h_{(1)}))^*(l^*_{(5)} \triangleright h^*_{(3)} \triangleright a^*)\sigma(l^*_{(6)}, h^*_{(4)})\#l^*_{(7)}h^*_{(5)} \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^*(l^*_{(3)} \triangleright b^*) \\
&\quad \cdot \left[(S^{-1}l_{(5)} \triangleright \sigma(S^{-1}h_{(4)}, h_{(1)}))\sigma(S^{-1}l_{(4)}, (S^{-1}h_{(3)})h_{(2)})\right]^* \\
&\quad \cdot (l^*_{(6)} \triangleright h^*_{(5)} \triangleright a^*)\sigma(l^*_{(7)}, h^*_{(6)})\#l^*_{(8)}h^*_{(7)}
\end{aligned} \tag{4.4.12}
$$

Using now both the twisted module condition and the cocycle condition we get, from equation (4.4.12),

$$
\begin{aligned}
(b\#l)^*(a\#h)^* &= \sigma(S^{-1}l_{(2)}, l_{(1)})^*(l^*_{(3)} \triangleright b^*) \\
&\quad \cdot \left[\sigma(S^{-1}l_{(4)}, S^{-1}h_{(3)})\sigma(S^{-1}(h_{(2)}l_{(4)}), h_{(1)})\right]^* \\
&\quad \cdot \sigma(l^*_{(6)}, h^*_{(4)})(l^*_{(7)}h^*_{(5)} \triangleright a^*)\#l^*_{(8)}h^*_{(6)} \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^*(l^*_{(3)} \triangleright b^*)\sigma(S^{-1}(h_{(2)}l_{(4)}), h_{(1)})^* \\
&\quad \cdot \sigma(S^{-1}l_{(4)}, S^{-1}h_{(3)})^*\sigma(l^*_{(6)}, h^*_{(4)})(l^*_{(7)}h^*_{(5)} \triangleright a^*)\#l^*_{(8)}h^*_{(6)} \\
&= \sigma(S^{-1}l_{(2)}, l_{(1)})^*(l^*_{(3)} \triangleright b^*)\sigma(S^{-1}(h_{(2)}l_{(4)}), h_{(1)})^*(l^*_{(5)}h^*_{(3)} \triangleright a^*)\#l^*_{(6)}h^*_{(4)},
\end{aligned} \tag{4.4.13}
$$

where we have used (ii) of definition (4.4.16) to obtain the last equality. Now we see that (4.4.11) and (4.4.13) coincide, and therefore we have proved that $((a\#h)(b\#l))^* = (b\#l)^*(a\#h)^*$. □

Consider now a cleft Hopf-Galois extension $B \hookrightarrow A$, with A a *-algebra, and assume that H is a Hopf *-algebra, with star structure compatible[7] with that of A (this implies that B is a *-algebra). Then we give the following definition.

Definition 4.4.18. *A trivialization $\phi : H \to A$ of a cleft extension $B \hookrightarrow A$ is said to be* unitary *if, for any $h \in H$,*

$$\phi(h)^* = \phi^{-1}((Sh)^*). \tag{4.4.14}$$

Remark 4.4.19. (4.4.14) is equivalent to $(\phi^{-1}(h))^* = \phi((Sh)^*)$.

Proposition 4.4.20. *If ϕ is a unitary trivialization then the weak action* (4.4.6) *is a weak *-action.*

Proof. Both properties (i) and (ii) of definition 4.4.16 can be checked by direct computation using (4.4.14) and (4.4.19). □

Proposition 4.4.21. *If ϕ is a unitary trivialization of a cleft Hopf-Galois extension $B \hookrightarrow A$, and we equip the associated crossed product $B\#_\sigma H$ with the involution* (4.4.8)*, then the isomorphism of proposition 4.4.13 is a *-isomorphism.*

Proof. We show that the map $F : \mathcal{B}\#_\sigma H \to \mathcal{A}$, defined by $F(a\#h) = a\phi(h)$, satisfies $F((a\#h)^*) = (F(a\#h))^* = \phi(h)^* a^*$. We prove this by direct computation. Indeed we have:

$$\begin{aligned} F((a\#h)^*) &= F\left(\sigma(S^{-1}h_{(2)}, h_{(1)})^* (h^*_{(3)} \triangleright a^*)\#h^*_{(4)}\right) \\ &= \sigma(S^{-1}h_{(2)}, h_{(1)})^*(h^*_{(3)} \triangleright a^*)\phi(h^*_{(4)}) = \phi(h_{(1)})^*\phi(S^{-1}h_{(2)})^*\phi(h^*_{(3)})a^*\phi^{-1}(h^*_{(5)}) \\ &= \phi(h^*_{(1)})\phi^{-1}(h^*_{(2)})\phi(h^*_{(3)})a^* = \phi(h)^*a^*. \end{aligned}$$

□

4.5 Quantum associated bundles

In the previous sections of this chapter we have introduced principal comodule algebras as noncommutative analogues of principal bundles. Now we discuss a similar construction for associated bundles [BM93, Haj96, DGH01, BF12]. We begin by recalling the following definition.

Definition 4.5.1. *Let H be a Hopf algebra, A a right H-comodule and V a left H-comodule with coactions, respectively, $\Delta_R : A \to A \otimes H$ and $\rho_L : V \to H \otimes V$. Then the* cotensor product $A\Box_H V$ *is defined as an equalizer:*

$$A\Box_H V \longrightarrow A \otimes V \underset{\mathrm{id}\otimes\rho_L}{\overset{\Delta_R\otimes\mathrm{id}}{\rightrightarrows}} A \otimes H \otimes V\ .$$

That is, $A\Box_H V$ is the subspace of elements $\xi \in A \otimes V$ such that $(\Delta_R \otimes \mathrm{id})\xi = (\mathrm{id} \otimes \rho_L)\xi$.

Now let H be a Hopf algebra with invertible antipode, and A a principal H-comodule algebra.

[7] Cf. equation (2.3.1).

Definition 4.5.2. *If V is a left H-comodule, then the* quantum bundle associated *to A, with fibre V, is the cotensor product $A\Box_H V$.*

Now let B denote the invariant subalgebra of A. By assumption, $B \hookrightarrow A$ is a principal Hopf-Galois extension. This implies the following fact.

Proposition 4.5.3. *Let V be a left H-comodule. Then $A\Box_H V$ is a projective left B-module. Moreover, if V is finite dimensional, $A\Box_H V$ is finitely generated as a left B-module.*

Proof. It follows from corollary 4.1.13. See also [BF12]. □

We recall now that if X is a topological space and E is a vector bundle over X, with finite dimensional fibres, then the space of continuous sections of E is a finitely generated projective $C(X)$-module. Hence, the principality of the extension allows us to interpret quantum associated bundles as (noncommutative) vector bundles, associated to the quantum principal bundle A. Moreover, as noticed, e.g., in [BF12], the following result holds.

Lemma 4.5.4. *Let $B \hookrightarrow A$ be cleft Hopf-Galois extension and let V be a left H-comodule. Then $A\Box_H V$ is a free left B-module.*

4.6 Quantum principal $\mathbb{T}^n$-bundles

Now we restrict our attention to a particular class of quantum principal bundles: the noncommutative analogues of principal $\mathbb{T}^n$-bundles. Although $\mathbb{T}^n$-bundles are probably the simplest examples of noncommutative principal bundles, there are several interesting models which fit into this category (for example quantum Hopf fibrations [BM93, BM98a, HM99, LS05, HMS06], noncommutative tori [DS13a], bundles over quantum lens spaces and quantum projective spaces [Szy03, BZ12, HRZ11]). In this section we discuss and prove some general properties of this kind of bundles, properties that will turn out to be useful in the next chapters.

Our approach will be based on principal comodule algebras; in the literature other proposals for a definition of (noncommutative) principal toric bundle can be found (see e.g. [ENO09, HaMa10, Wag12]). We begin by associating a Hopf algebra to the Lie group $\mathbb{T}^n$. Actually, there are two possible choices, although they are closely related.

Definition 4.6.1. *The algebra $\mathcal{O}(\mathbb{T}^n)$ is the unital complex polynomial *-algebra generated by n commuting unitaries $z_1, \dots, z_n$; that is, it is the commutative algebra whose elements are finite linear combinations*

$$\sum_{k\in\mathbb{Z}^n} \alpha_k z_1^{k_1} \cdots z_n^{k_n},$$

where $\{\alpha_k\} \subset \mathbb{C}$. It is a Hopf *-algebra with respect to the following coproduct, antipode and counit:

$$\Delta(z_i^r) = z_i^r \otimes z_i^r, \qquad S(z_i^r) = z_i^{-r}, \qquad \varepsilon(z_i^r) = 1,$$

for any $i = 1, \dots, n$ and for any $r \in \mathbb{Z}$.

If we introduce the notation $z^k = \prod_{i=1}^{n} z_i^{k_i}$, the maps in definition 4.6.1 can be also characterized by the following relations:

$$\Delta(z^k) = z^k \otimes z^k, \qquad S(z^k) = z^{-k}, \qquad \varepsilon(z^k) = 1.$$

Lemma 4.6.2. *The algebra $\mathcal{O}(\mathbb{T}^n)$ is isomorphic to the group algebra $\mathbb{C}[\mathbb{Z}^n]$.*

Proof. $\mathbb{C}[\mathbb{Z}_n]$ is the algebra of formal (finite) linear combinations $\sum_{m\in\mathbb{Z}^n} \alpha_m \mathbf{m}$. Hence the assignment $z^m \mapsto \mathbf{m}$ determines an isomorphism $\mathcal{O}(\mathbb{T}^n) \simeq \mathbb{C}[\mathbb{Z}^n]$. □

Proposition 4.6.3. *The algebra $\mathcal{O}(\mathbb{T}^n)$ is a dense *-subalgebra of the C^*-algebra of continuous functions over the n-torus $\mathbb{T}^n$. In particular, it inherits a C^*-norm $\|\cdot\|$.*

Proof. Let $\{\varphi_1, \ldots, \varphi_n\}$ be canonical angular coordinates on $\mathbb{T}^n$. For any $k \in \mathbb{Z}^n$ consider the function

$$\psi_k(\varphi) = \exp(ik\cdot\varphi) = \exp\left(\sum_i k_i\varphi_i\right).$$

Then the assignment $z^k \mapsto \psi_k$ determines an injective algebra homomorphism $\Psi : \mathcal{O}(\mathbb{T}^n) \to C(\mathbb{T}^n)$. This means that $\mathcal{O}(\mathbb{T}^n)$ can be seen as a subalgebra of $C(\mathbb{T}^n)$. Finally, since the trigonometric polynomials are dense in $C(\mathbb{T}^n)$, by Stone-Weierstrass theorem, $\mathcal{O}(\mathbb{T}^n)$ is dense in $C(\mathbb{T}^n)$. □

Let now $\mathfrak{t}_n$ denote the (complex) Lie algebra of $\mathbb{T}^n$. It can be described as the Lie algebra generated by n commuting elements $\delta_1, \ldots, \delta_k$. Then we can define an action of $\mathfrak{t}_n$ on $\mathcal{O}(\mathbb{T}^n)$ in the following way:

$$\delta_j(z^k) = k_j z^k, \qquad \text{for } i = 1, \ldots, n.$$

Any δ_j is a derivation of $\mathcal{O}(\mathbb{T}^n)$. We can use them to define a countable family of seminorms[8] on $\mathcal{O}(\mathbb{T}^n)$: for any $k \in \mathbb{N}^n$ and any $a \in \mathcal{O}(\mathbb{T}^n)$, we set

$$\|a\|_k = \|\delta_1^{k_1} \cdots \delta_n^{k_n}(a)\|, \tag{4.6.1}$$

where $\|\cdot\|$ is the C^*-norm on $\mathcal{O}(\mathbb{T}^n)$ induced by the inclusion in $C(\mathbb{T}^n)$ (that is, the sup norm).

Lemma 4.6.4. $\Gamma_\delta = \{\|\cdot\|_k \mid k \in \mathbb{N}^n\}$ *is a countable separating family of seminorms on $\mathcal{O}(\mathbb{T}^n)$.*

Proof. The fact that each $\|\cdot\|_k$ is a seminorm follows directly from the fact that $\|\cdot\|$ is a norm. Moreover, since $\|\cdot\| \in \Gamma_\delta$, Γ_δ is a separating family. □

Proposition 4.6.5. $\mathcal{O}(\mathbb{T}^n)$ *is a metrizable locally convex vector space. Moreover, it is a topological algebra, with respect to the same locally convex topology.*

Proof. The first part of the proposition is a direct consequence of lemma 4.6.4 and theorem B.2.5. For the second part it is enough to prove that the topology of $\mathcal{O}(\mathbb{T}^n)$ is determined by a (separable)

[8] See definition B.2.3.

family of sub-multiplicative seminorms. First of all let us notice that $\|\cdot\|$ is sub-multiplicative. Next, for any $k \in \mathbb{N}^n$, $k \neq 0$; due to the linearity of the derivations δ_j and the subadditivity of $\|\cdot\|_k$, let us consider the seminorm $\|\cdot\|'_k = 2^{n+|k|}\|\cdot\|_k$, where $|k| = \sum_i k_i$. The family of the seminorms $\|\cdot\|'_k$, together with the norm $\|\cdot\|$, is still a separating family of seminorms, and it determines the same topology as Γ_δ (indeed, we have simply rescaled some of the seminorms). Now we have to show that any $\|\cdot\|'_k$, $k \neq 0$, is sub-multiplicative. Due to the linearity of the derivations δ_j and the subadditivity of the seminorms $\|\cdot\|'_k$, it is enough to prove that $\|z^r z^s\|'_k \leq \|z^r\|'_k \|z^s\|'_k$ for any $r, s \in \mathbb{Z}^n$. This follows by direct computation; indeed,

$$\|z^r z^s\|'_k = \|z^{r+s}\|'_k = 2^n \prod_{i=1}^{n} |r_i + s_i|^{k_i}, \tag{4.6.2}$$

$$\|z^r z^s\|'_k = 2^{2n} \prod_{i=1}^{n} |r_i|^{k_i} |s_i|^{k_i}. \tag{4.6.3}$$

Hence it is enough to show that $\prod_{i=1}^{n} |r_i + s_i|^{k_i} \leq 2^{n+|k|} \prod_{i=1}^{n} |r_i|^{k_i} |s_i|^{k_i}$ for any $r, s \in \mathbb{Z}^n$. Of course, this is true if the following holds:

$$|m + l|^p \leq 2^{p+1}|m|^p|l|^p \qquad \forall m, l \in \mathbb{Z},\ \forall p \in \mathbb{N}. \tag{4.6.4}$$

If $m \cdot l \leq 0$ (4.6.4) is fulfilled, since either $|m|$ or $|l|$ is greater than $|m + l|$. Without loss of generality, assume now that $m, n > 0$. Then $m \cdot l > m + l$, and so (4.6.4) is satisfied, unless either $m = 1$ or $l = 1$. If $m = l = 1$ (4.6.4) becomes $2^p \leq 2^{p+1}$, and so it is satisfied. If $m = 1$ and $l > 1$, instead, it becomes $(l + 1)^p \leq 2^{p+1} l^p$, which is always true. Hence $\mathcal{O}(\mathbb{T}^n)$ is a topological algebra, with respect to the locally convex topology determined by Γ_δ. □

Proposition 4.6.6. *The maps $S : \mathcal{O}(\mathbb{T}^n) \to \mathcal{O}(\mathbb{T}^n)$ and $\varepsilon : \mathcal{O}(\mathbb{T}^n) \to \mathbb{C}$ are continuous. Moreover, if we endow the algebraic tensor product $\mathcal{O}(\mathbb{T}^n) \otimes \mathcal{O}(\mathbb{T}^n)$ with the projective topology*[9]*, then the coproduct $\Delta : \mathcal{O}(\mathbb{T}^n) \to \mathcal{O}(\mathbb{T}^n) \otimes \mathcal{O}(\mathbb{T}^n)$ is a continuous map.*

Proof. ε is continuous, since it is simply the evaluation at the identity. Next, given $f \in \mathcal{O}(\mathbb{T}^n)$, $Sf(x) = f(x^{-1}) = f(-x)$; hence $\|Sf\|_k = \|f\|_k$ for any $k \in \mathbb{Z}^n$. It follows that also S is continuous. Consider now an element $h \in \mathcal{O}(\mathbb{T}^n)$ given by a finite sum $h = \sum_{r \in \mathbb{Z}^n} \alpha_r z^r$. Then

$$\Delta(h) = \sum_{k \in \mathbb{Z}^n} \alpha_r (z^r \otimes z^r).$$

Due to proposition B.3.9, the projective topology on $\mathcal{O}(\mathbb{T}^n) \otimes \mathcal{O}(\mathbb{T}^n)$ coincides with that induced by the topology of $C^\infty(\mathbb{T}^n \times \mathbb{T}^n)$. In particular, it is defined by the following set of seminorms

$$\Gamma_{\delta\otimes\delta} = \left\{ q_{k,l},\ k, l \in \mathbb{Z}^n \;\middle|\; q_{k,l}(\xi(x,y)) = \|\delta_x^k \delta_y^l \xi(x,y)\| \right\}.$$

[9] See section B.2.

Applying a seminorm $q_{k,l}$ to $\Delta(h)$ we obtain now:

$$\begin{aligned} q_{k,l}(\Delta(h)) &= \|\sum_r \alpha_r r^{k+l} z^r(x) z^r(y)\| = \sup_{x,y\in\mathbb{T}^n} \left|\sum_r \alpha_r r^{k+l} z^r(x) z^r(y)\right| \\ &= \sup_{x,y\in\mathbb{T}^n} \left|\sum_r \alpha_r r^{k+l} z^r(x+y)\right| = \sup_{x\in\mathbb{T}^n} \left|\sum_r \alpha_r r^{k+l} z^r(x)\right| = p_{k+l}(h). \end{aligned} \tag{4.6.5}$$

From theorem B.2.10 it follows then that Δ is continuous. □

A direct consequence of proposition 4.6.5 is that we can consider the metric completion $\overline{\mathcal{O}(\mathbb{T}^n)}$ of the algebra $\mathcal{O}(\mathbb{T}^n)$. By definition this will be a Fréchet algebra, and it is clear that it coincides with the algebra $C^\infty(\mathbb{T}^n)$ of smooth functions on the n-torus. In particular, we shall obtain a structure of Hopf algebra on $C^\infty(\mathbb{T}^n)$, with the coproduct taking values in the completion $C^\infty(\mathbb{T}^n)\overline{\otimes}C^\infty(\mathbb{T}^n) \simeq C^\infty(\mathbb{T}^n \otimes \mathbb{T}^n)$ of the tensor product $C^\infty(\mathbb{T}^n) \otimes C^\infty(\mathbb{T}^n)$.

So, when dealing with quantum principal $\mathbb{T}^n$-bundles, we could either work with $\mathcal{O}(\mathbb{T}^n)$ or with $C^\infty(\mathbb{T}^n)$. In this thesis we shall work with the former, but we have given this short discussion for completeness sake.

4.6.1 The (bicovariant) de Rham differential calculus

In this thesis for quantum principal $\mathbb{T}^n$-bundles we shall mean a suitable class of principal $\mathcal{O}(\mathbb{T}^n)$-comodule (or eventually $C^\infty(\mathbb{T}^n)$-comodule) algebras. This class will be the class of the principal comodule algebras which are quantum principal bundles with respect to the de Rham differential calculus on $\mathcal{O}(\mathbb{T}^n)$ (see definition 4.3.1). Of course, for de Rham calculus on $\mathcal{O}(\mathbb{T}^n)$ we mean the restriction of the ordinary de Rham calculus on $C^\infty(\mathbb{T}^n)$ with respect to the inclusion $\mathcal{O}(\mathbb{T}^n) \subset C^\infty(\mathbb{T}^n)$ (see previous section).

Let us take a look at this calculus. We have seen in section 2.3.3 that any bicovariant first order differential calculus is defined by an ad_R-invariant ideal $Q \subseteq \ker\varepsilon$. The de Rham calculus on $\mathcal{O}(\mathbb{T}^n)$ is a bicovariant calculus and that the ideal which defines it is $Q = (\ker\varepsilon)^2$. This fact actually holds for any Hopf algebra of smooth functions over a compact connected semisimple Lie group G: we shall discuss this in chapter 8. We simply notice here that Q is the ideal of functions vanishing at the identity $e \in G$ with differential vanishing, too.

Let us take a closer look at the ideal Q. In particular let us write down a set of generators.

Proposition 4.6.7. *The ideal $Q = (\ker\varepsilon)^2$ is generated by the elements*

$$q_{k,r} = (z^k - 1)(z^r - 1),$$

for $k, r \in \mathbb{Z}^n$.

Proof. $\ker\varepsilon$ can be identified with the space whose elements are the linear combinations

$$\sum_{k\in\mathbb{Z}^n} \alpha_k z^k \quad \text{s.t.} \quad \sum_{k\in\mathbb{Z}^n} \alpha_k = 0.$$

But any element of this kind can be rewritten as

$$\sum_{k\in\mathbb{Z}^n} \alpha_k(z^k - 1).$$

Therefore $\ker\varepsilon$ is generated by the elements $(z^k - 1)$ for $k \in \mathbb{Z}^n$. It follows directly that the elements $q_{k,r}$ generate Q. □

Corollary 4.6.8. *The ideal Q contains all the elements of the form*

$$q_i^{(r)} \equiv z_i^r - 1 - r(z_i - 1) \qquad i = 1, \dots, n,$$

$$q^{(k)} \equiv z^k - 1 - \sum_{i=1}^{n} \left(z_i^{k_i} - 1\right),$$

for any $r \in \mathbb{Z}$ and any $k \in \mathbb{Z}^n$.

4.6.2 Quantum principal $\mathbb{T}^n$-bundles

We are now ready to introduce a definition of quantum principal $\mathbb{T}^n$-bundles. Given a quantum principal bundle $\mathcal{A}$ with Hopf algebra $H = \mathcal{O}(\mathbb{T}^n)$, we shall provide sufficient conditions on the first order differential calculus of A for it being a quantum principal bundle with general calculus compatible with the de Rham calculus on $\mathcal{O}(\mathbb{T}^n)$. In the rest of this thesis we will adopt the following terminology.

Definition 4.6.9. *A* quantum principal $\mathbb{T}^n$-bundle *is a quantum principal bundle* $(A, \mathcal{O}(\mathbb{T}^n), \Delta_R, N, Q)$ *where $Q = (\ker\varepsilon)^2$ is the ideal which defines the de Rham calculus on $\mathcal{O}(\mathbb{T}^n)$.*

Let us consider now a right $\mathcal{O}(\mathbb{T}^n)$-comodule algebra A. Let Δ_R denote the coaction and B the invariant subalgebra. Assume that $B \hookrightarrow A$ is a Hopf-Galois extension. Then we can split the algebra A as a direct sum,

$$A = \bigoplus_{k\in\mathbb{Z}^n} A^{(k)}, \tag{4.6.6}$$

where the $A^{(k)}$ are the set of elements of homogeneous degree; that is,

$$a \in A^{(k)} \quad \Leftrightarrow \quad \Delta_R(a) = a \otimes z^k.$$

Of course, $A^{(0)} = B$. This allows us to define actions both of the Lie group $\mathbb{T}^n$ and of its Lie algebra $\mathfrak{t}_n$ on A as follows. If $g = (\varphi_1, \dots, \varphi_n)$ is an element of $\mathbb{T}^n$, then we define

$$g \triangleright a = e^{i(k_1\varphi_1 + \dots + k_n\varphi_n)} a \qquad \text{for } a \in A^{(k)}.$$

This corresponds to the following action of the Lie algebra $\mathfrak{t}_n$:

$$\delta_j \triangleright a = \delta_j(a) = k_j a \qquad \text{for } a \in A^{(k)}.$$

Remark 4.6.10. When dealing with pre-C^*-algebras (or C^*-algebras) we will always require the action of $\mathbb{T}^n$ to be norm-continuous (and, therefore, since each element of $\mathbb{T}^n$ acts as an automorphism of A, the action will actually be norm-preserving).

Let now $\Omega^1(A)$ be a (general) first order differential calculus over A. We recall that it is possible to define a $\mathcal{O}(\mathbb{T}^n)$-coaction on $\Omega^1 A$ (see equation (4.1.2)). Then we can give the following definition.

Definition 4.6.11. *A first order differential calculus $\Omega^1(A)$ over a right $\mathcal{O}(\mathbb{T}^n)$-comodule algebra A is $\mathcal{O}(\mathbb{T}^n)$-*covariant *if $\Omega^1(A)$ is an $\mathcal{O}(\mathbb{T}^n)$-equivariant bimodule, with comodule structure inherited from $\Omega^1 A$, and, moreover,*

$$\Delta_R^\Omega(da) = (d \otimes \mathrm{id})\Delta_R(a)$$

in $\Omega^1(A)$.

Remark 4.6.12. When $A = \mathcal{O}(\mathbb{T}^n)$, this definition agrees with definition 2.3.26 (see [Wor89], proposition 1.3).

Remark 4.6.13. A differential calculus $\Omega^1(A)$ defined by a sub-bimodule $N \subseteq A \otimes A$ is H-equivariant if and only if N is equivariant; that is, if and only if $\Delta_R(N) \subseteq N \otimes H$.

Proposition 4.6.14. *Let A be a principal $\mathcal{O}(\mathbb{T}^n)$-comodule algebra, let B be its invariant subalgebra and let $\Omega^1(A)$ be an $\mathcal{O}(\mathbb{T}^n)$-covariant first order differential calculus, defined by a sub-bimodule $N \subset A \otimes A$. Then $(A, \mathcal{O}(\mathbb{T}^n), \Delta_R, N, Q)$, where $Q = (\ker \varepsilon)^2$, is a quantum principal bundle if the following conditions hold:*

(i) *let $a_j, b_j \in A$; then*

$$\sum_j a_j db_j = 0 \text{ in } \Omega^1(A) \qquad \Rightarrow \qquad \sum_j a_j \delta_i(b_j) = 0 \quad \forall i = 1, \dots, n, \tag{4.6.7}$$

(ii) *let $\eta \in \Omega^1 A$, $\eta = \sum_j a_j db_j$; then*

$$\sum_j a_j \delta_i(b_j) = 0 \quad \forall i = 1, \dots, n \qquad \Rightarrow \qquad [\eta]_N \in A\Omega^1(B)A. \tag{4.6.8}$$

Proof. We check (i)-(iv) of definition 4.3.1. (i) is trivially satisfied, since we assumed A to be a principal comodule algebra. Furthermore, (ii) follows directly from the $\mathcal{O}(\mathbb{T}^n)$-covariance of the differential calculus $\Omega^1(A)$.

Let us check condition (iii). Take $\eta \in N$ and write it as $\eta = \sum_j (a_j \otimes b_j - a_j b_j \otimes 1)$. We introduce the following notation: given $a \in A$, we split it as a sum of elements of homogeneous degree:

$$a = \sum_{k \in \mathbb{Z}^n} a^{(k)}.$$

Then we have $\eta = \sum_j \sum_{r\in\mathbb{Z}^n} (a_j \otimes b_j^{(r)} - a_j b_j^{(r)} \otimes 1)$ and

$$T_R(\eta) = \sum_j \sum_{r\in\mathbb{Z}^n} a_j b_j^{(r)} \otimes (z^r - 1). \tag{4.6.9}$$

Let us notice that, for any $i = 1, \dots, n$,

$$\Delta_R \circ \delta_i = (\delta_i \otimes \mathrm{id}) \circ \Delta_R.$$

Using this fact, together with the compatibility with the de Rham calculus (i.e. equation (4.6.7)), we obtain

$$\sum_j \sum_{r\in\mathbb{Z}^n} r_i a_j b_j^{(r)} = 0.$$

Hence,

$$\sum_j \sum_{r\in\mathbb{Z}^n} r_i a_j b_j^{(r)} \otimes (z_i - 1) = 0 \tag{4.6.10}$$

for any $i = 1, \dots, n$. If now we sum, for all i and for each $r \in \mathbb{Z}^n$, (4.6.10) to (4.6.9) we obtain:

$$T_R(\eta) = \sum_j \sum_{r\in\mathbb{Z}^n} a_j b_j^{(r)} \otimes \left((z^r - 1) - \sum_{i=1}^n r_i(z_i - 1) \right). \tag{4.6.11}$$

But from corollary 4.6.8 we know that all the right factors of the terms of the sum (4.6.11) belong to Q, thus $T_R(\eta) \in A \otimes Q$ and (iii) is fulfilled.

We are left with condition (iv). Take $[\eta] \in \Omega^1(A)$, and write the representative η as $\eta = \sum_j (a_j \otimes b_j - a_j b_j \otimes 1)$. Then $T_R(\eta)$ is still given by (4.6.9). We know (see corollary 4.6.8) that, in H/Q, $(z^r - 1)$ is equivalent to $\sum_{i=1}^n r_i(z_i - 1)$, so we can write:

$$T([\eta]) = (\mathrm{id} \otimes \pi_H)(T_R(\eta)) = \sum_j \sum_{r\in\mathbb{Z}^n} a_j b_j^{(r)} \otimes \left[\sum_{i=1}^n r_i(z_i - 1) \right]. \tag{4.6.12}$$

Now assume that at least one of the non-vanishing terms of (4.6.12) has $r_i \neq 0$. Then, imposing $T([\eta]) = 0$, we get

$$\sum_j \sum_{r\in\mathbb{Z}^n} a_j b_j^{(r)} \otimes r_i(z_i - 1) = 0. \tag{4.6.13}$$

But this implies, since the elements z_i are linearly independent also in H/Q, that

$$\sum_j \sum_{r\in\mathbb{Z}^n} r_i a_j b_j^{(r)} = 0.$$

Due to equation (4.6.8), this means that $[\eta]_N \in A\Omega^1(B)A$. Hence $\ker(T) \subseteq A\Omega^1(B)A$, and so condition (iv) of definition 4.3.1 is fulfilled. □

We can also prove the converse: (4.6.7) and (4.6.8) are not only sufficient but also a necessary conditions for a quantum principal bundle to be a $\mathbb{T}^n$ bundle. Indeed,

Proposition 4.6.15. *Let (A, H, Δ_R, N, Q) be a quantum principal bundle, with $H = \mathcal{O}(\mathbb{T}^n)$ and $Q = (\ker \varepsilon)^2$, so that (A, H, Δ_R, N, Q) is a quantum principal $\mathbb{T}^n$ -bundle. Then properties* (4.6.7) *and* (4.6.8) *hold.*

Proof. Take $\eta = \sum_j a_j db_j$ such that it is zero in $\Omega^1(A)$. This means that $\sum_j a_j \otimes b_j - a_j b_j \otimes 1$ belongs to N. Let us introduce the following notation: we write each b_j as a sum of elements of homogeneous degree,

$$b_j = \sum_{k \in \mathbb{Z}^n} b_j^{(k)}.$$

From the definition of quantum principal bundle we know that $T_R(N) \subseteq A \otimes Q$. This means, in particular, that:

$$\sum_{k \in \mathbb{Z}^n} \sum_j a_j b_j^{(k)} \otimes (z^k - 1) \in A \otimes Q.$$

But Q is the ideal which defines the de Rham calculus $\Omega^1_{dR}(H) = \Omega^1 H / N_H$. In particular (see theorem 2.3.31), for any $h \in H$, we have:

$$\sum_{k \in \mathbb{Z}^n} \sum_j a_j b_j^{(k)} \otimes r^{-1}(h \otimes (z^k - 1)) \in A \otimes N_H$$

.

$$\Rightarrow \quad \sum_{k \in \mathbb{Z}^n} \sum_j a_j b_j^{(k)} \otimes h z^{-k} d(z^k) = 0 \in A \otimes \Omega^1_{dR}(H) \tag{4.6.14}$$

We recall that the derivations δ_j form the canonical basis of $\mathfrak{t}_n$, so we can see them also as operators on H. If now we denote by $\{dx^j\}$ the dual basis, then the calculus $\Omega^1_{dR}(H)$ can be described in the following way:

$$h' dh = \sum_i h' \delta_i(h) dx^i.$$

In particular, (4.6.14) is equivalent to

$$\sum_{k \in \mathbb{Z}^n} \sum_j a_j b_j^{(k)} \otimes k_i h dx^i = 0 \qquad \forall i = 1, \dots, n,$$

which implies that $\sum_j a_j \delta_i(b_j) = 0$ for all i, and this concludes the proof that property (4.6.7) holds.

Now we prove that also (4.6.8) holds. Consider $\eta \in \Omega^1 A$, $\eta = \sum_j a_j db_j$, and suppose that $\sum_j a_j \delta_i(b_j) = 0$. Then we can write η, as an element of $A \otimes A$, as $\eta = \sum_j \sum_{r \in \mathbb{Z}^n} (a_j \otimes b_j^{(k)} - a_i b_j^{(k)})$, with $\sum_j \sum_{r \in \mathbb{Z}^n} r_i a_j b_j^{(r)} = 0$. A simple computation now yields to:

$$T([\eta]) = \sum_j \sum_{r \in \mathbb{Z}^n} a_j b_j^{(r)} \otimes [z^r - 1]$$

But, from corollary 4.6.8, we know that $[z^r - 1] = [\sum_i r_i(z_i - 1)]$ in $\ker\varepsilon/Q$. Hence we obtain:

$$T([\eta]) = \sum_i \sum_j \sum_{r\in\mathbb{Z}^n} r_i a_j b_j^{(r)} \otimes [z_i - 1] = 0.$$

That is, $[\eta] \in \ker(T)$. Then condition (iv) of definition 4.3.1 implies that $[\eta] \in A\Omega^1(B)A$. So we have proved that also (4.6.8) is satisfied. □

4.6.3 Strong connections over quantum principal $\mathbb{T}^n$-bundles

In this section we will discuss the main properties of strong connections over quantum principal $\mathbb{T}^n$-bundles. As we shall see, any strong connection can be characterized by a family of n 1-forms, property which reflects the classical case, when a $\mathbb{T}^n$-connection can always be written as an n-dimensional vector of 1-forms, one for each generator of the Lie algebra $\mathfrak{t}_n$.

We begin with the following observation.

Lemma 4.6.16. *Let $(A, \mathcal{O}(\mathbb{T}^n), \Delta_R, N_A, Q)$ be a quantum principal $\mathbb{T}^n$-bundle (that is, $Q = (\ker\varepsilon)^2$ defines the de Rham calculus on $\mathcal{O}(\mathbb{T}^n)$). Then, any strong connection $\omega : \mathcal{O}(\mathbb{T}^n) \to A$ fulfils the following relation:*

$$\omega(z^k) = \sum_{i=1}^n k_i \omega(z_i).$$

Proof. This follows simply by corollary 4.6.8 and condition (i) of definition 4.3.6. Indeed, if we take $q^{(k)} \in Q$, as in corollary 4.6.8, we know that $\omega(q^{(k)}) = 0$. But this means, due to the linearity of ω, that:

$$0 = \omega(z^k) - \sum_{i=1}^n \omega(z_i^{k_i}).$$

We also have to impose that $\omega(q_i^{(r)}) = 0$, which gives $\omega(z_i^{k_i}) = k_i\omega(z_i)$. The two relations together yield to the thesis: $\omega(z^k) = \sum_{i=1}^n k_i\omega(z_i)$ for any $k \in \mathbb{Z}^n$. □

In particular, the connection ω is completely described by a family of n 1-forms $\omega_i \in \Omega^1(A)$. Conversely, is it true that given a suitable family of 1-forms ω_i we can define a strong connection ω simply by $\omega(z^k) = \sum_{i=1}^n k_i\omega_i$? The answer is positive, and the conditions that we have to impose on the family $\{\omega_i\}$ are the n-dimensional analogues of those introduced in [DS13a]:

Definition 4.6.17. *A family of n 1-forms $\{\omega_i\} \subset \Omega^1(A)$ is a* strong $\mathbb{T}^n$-connection *for the quantum principal $\mathbb{T}^n$-bundle $(A, \mathcal{O}(\mathbb{T}^n), \Delta_R, N, Q)$ if the following conditions hold:*
(i) $\delta_j(\omega_i) = 0$ *for any* $i, j = 1, \ldots, n$;
(ii) *if* $\omega_i = \sum_j p_j dq_j$, *with* $p_j, q_j \in A$, *then* $\sum_j p_j\delta_i(q_j) = 1$ *and* $\sum_j p_j\delta_l(q_j) = 0$ *for* $l \neq i$;
(iii) $\forall a \in \mathcal{A},\ (da - \sum_i \delta_i(a)\omega_i) \in \Omega^1(B)A$.

We have to spend here a few words on condition (i) of definition 4.6.17: the action of the generators δ_j of the Lie algebra $\mathfrak{t}_n$ on $\Omega^1(A)$ is defined by

$$\delta_j(adb) = \delta_j(a)db + ad(\delta_j(b)).$$

Proposition 4.6.18. *Let a family of 1-forms $\{\omega_i\} \subset \Omega^1(A)$ be a strong $\mathbb{T}^n$-connection, in the sense of definition 4.6.17, over a quantum principal $\mathbb{T}^n$-bundle $(A, \mathcal{O}(\mathbb{T}^n), \Delta_R, N, Q)$. Then it defines a strong connection ω, in the sense of definition 4.3.6, by:*

$$\omega(z^k) = \sum_{i=1}^{n} k_i \omega_i.$$

Proof. We check properties (i)-(iv) of definition 4.3.6. Let us begin proving (i). By definition we have $\omega(1) = 0$. So we need only to show that $\omega(Q) = 0$. By proposition 4.6.7 we know that Q is generated by the elements $q_{k,r} = (z^k - 1)(z^r - 1)$ for $k, r \in \mathbb{Z}^n$. And we have:

$$\omega(q_{k,r}) = \omega(z^{k+r} - z^k - z^r + 1) = \sum_{i=1}^{n} [(k_i + r_i)\omega_i - k_i\omega_i - r_i\omega_i] = 0.$$

Hence $\omega(Q \oplus \mathbb{C}) = 0$. Next we check (ii). The condition (i) of definition 4.6.17 can be rewritten as

$$\Delta_R^{\Omega}(\omega_i) = \omega_i \otimes 1$$

for any $i = 1, \ldots, n$. Also, the right adjoint coaction ad_R of H is the following one:

$$\mathrm{ad}_R(z^k) = z^k \otimes z^{-k} z^k = z^k \otimes 1.$$

Thus we get:

$$(\omega \otimes \mathrm{id})(\mathrm{ad}_R(z^k)) = \omega(z^k) \otimes 1 = \sum_{i=1}^{n} k_i \omega_i \otimes 1.$$

But we have also

$$\Delta_R^{\Omega}(\omega(z^k)) = \sum_{i=1}^{n} \Delta_R^{\Omega}(k_i\omega_i) = \sum_{i=1}^{n} k_i\omega_i \otimes 1,$$

so (ii) is fulfilled. In order to show that (iii) holds we use the fact that $\Omega^1(A)$ is defined by the sub-bimodule $N \subseteq \ker(m) \subset A \otimes A$. Indeed, taken a representative of the equivalence class of ω_i, it can be written as

$$\hat{\omega}_i = \sum_j p_j \otimes q_j - p_j q_j \otimes 1.$$

If now we split each q_j as a sum of elements of homogeneous degree, w.r.t. the $\mathbb{T}^n$ action, we get

$$\hat{\omega}_i = \sum_j \sum_{k \in \mathbb{Z}^n} p_j \otimes q_j^{(k)} - p_j q_j^{(k)} \otimes 1,$$

so that we can easily compute

$$T_R(\omega_i) = (m \otimes \mathrm{id}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \Delta_R)(\hat{\omega}_0) = \sum_j \sum_{k \in \mathbb{Z}^n} p_j q_j^{(k)} \otimes (z^k - 1).$$

Now, thanks to corollary 4.6.8, we know that, in H/Q, $[z^k - 1] = \sum_j [k_j(z_j - 1)]$. Thus, after applying $\mathrm{id} \otimes \pi_H$ to both sides of previous identity, we get:

$$T(\omega_i) = \sum_j \sum_{k \in \mathbb{Z}^n} \sum_{l=1}^{n} k_l p_j q_j^{(k)} \otimes [z_l - 1].$$

But from condition (ii) of definition 4.6.17 we know that this is equal to $1 \otimes [z_i - 1]$, which is exactly $(\mathrm{id} \otimes \pi_H) \circ (1 \otimes (\mathrm{id} - \varepsilon))(z_i)$. From this we easily deduce that ω fulfils condition (iii).

We are left with the proof that also condition (iv) holds, but it follows immediately using condition (iii) of definition 4.6.17: indeed, if $a \in \mathcal{A}^{(k)}$, then

$$a_{(0)} \omega(a_{(1)}) = a\omega(z_j^k) = \sum_j k_j a \omega_j.$$

□

Proposition 4.6.18 together with lemma 4.6.16 show that, for quantum principal $\mathbb{T}^n$-bundles, definition 4.3.6 and definition 4.6.17 give equivalent characterizations of strong connections.

CHAPTER 5

Projectable spectral triples and twisted Dirac operators

Let M be a (compact) smooth manifold, G a compact semi-simple Lie group and let $\pi : P \to M$ be a principal G-bundle. If M is a Riemannian manifold, with metric tensor g, we can define, for any connection over this bundle, a Riemannian metric on the total space P such that the action of the group is isometric and that the bundle projection π is a Riemannian submersion (a possible generalization of this construction to the noncommutative case will be discussed later; see also the introduction of this thesis for a brief account of the commutative case). Assume, instead, to be given a Riemannian metric on the total space P, such that the bundle projection $\pi : P \to M$ is a Riemannian submersion, the action of G on P is isometric and P is a spin manifold; in this case, what can we say about the metric and the spin structure induced on the base space M? An answer to this question for $U(1)$-bundles was given in [AmmB98, Amm98, GLP96].

Results analogous to those by Ammann and Bär for noncommutative $U(1)$-bundles with spectral geometry of KR-dimension 3 were discussed in [DS13a]. Here[1] we extend them, considering, first, noncommutative $U(1)$-bundles of any dimension and then noncommutative $\mathbb{T}^n$-bundles. We point out that our approach is operatorial and exploits only the algebraic and the spectral properties of the Dirac operator. In particular we can not directly follow [AmmB98, Amm98], where the properties of the spin structure of principal $U(1)$-bundles were proved using the metric tensor and the Christoffel symbols, which, in general, can not be defined in the noncommutative case, but we elaborated another method, which relies on the properties of the Dirac operator.

So, in the first part of this chapter, we shall discuss the $U(1)$ case, both in the even and in the odd dimensional case. Next, we will consider a quantum principal $\mathbb{T}^n$-bundle $(\mathcal{A}, H, \Delta_R, N, Q)$ and we will start from a $\mathbb{T}^n$-equivariant real spectral triple over $\mathcal{A}$ (see below for the definition); this choice corresponds to assume that the action of the structure group is isometric. Then we will discuss some conditions under which it is possible to build a spectral triple for the base space, i.e. for the invariant subalgebra $\mathcal{B} = \mathcal{A}^{coH}$. Finally, we will show how this construction allows us to define spectral triples and Dirac operators *twisted* by a strong connection. This will lead us to

[1] Part of the results discussed here are contained in [DSZ13].

the construction of new Dirac operators: for example, in the case of noncommutative n-tori we get twisted operators that are no more $\mathbb{T}^n$-invariant. In addition, these operators are expected to describe a noncommutative geometry with non zero scalar curvature[2]. We just mention here that a "perturbative" study of the properties of operators of this kind can be found in [DS13b]. Finally, we shall relate our results to recent developments in KK-theory [Mes11, BMS13]: we shall discuss, in particular, the case of $U(1)$ gauge theories on a noncommutative 2-torus.

In this section with *real spectal triple* we mean a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$, in the sense of definition 3.1.2, fulfilling, at least, the classical dimension and the first order condition.

5.1 Twists of spectral triples

We begin by discussing a general way to twist a spectral triple using a (suitable) connection over a module. Let $(\mathcal{B}, \mathcal{H}, D, J)$ be a real spectral triple over a (unital) algebra $\mathcal{B}$. Consider another Hilbert space $\mathcal{H}_M$ together with a representation of $\mathcal{B}$. Let M be a space of $\mathcal{B}$-linear bounded maps $m : \mathcal{H} \to \mathcal{H}_M$. Assume that:

(a) M is a finitely generated projective $\mathcal{B}$-module;

(b) $\mathcal{H}M \equiv M(\mathcal{H})$ is dense in $\mathcal{H}_M$, where $M(\mathcal{H})$ is the linear span of elements $m(h)$, $m \in M$, $h \in \mathcal{H}$;

(c) the multiplication map from $\mathcal{H} \otimes_{\mathcal{B}} M$ to $\mathcal{H}M$ is an isomorphism.

Using the right $\mathcal{B}$-module structure induced on $\mathcal{H}$ by the real structure J, namely

$$hb \equiv Jb^* J^{-1} h \tag{5.1.1}$$

for any $h \in \mathcal{H}$ and any $b \in \mathcal{B}$, we define a left $\mathcal{B}$-module structure on M through:

$$(bm)(h) = m(hb) \qquad \forall m \in M.$$

We introduce a new notation: we write the action of M on the right, that is $m(h) \equiv hm$. Then the $\mathcal{B}$-linearity reads

$$(bh)m = b(hm),$$

while the left $\mathcal{B}$ action on M becomes

$$h(bm) = (hb)m.$$

Also, it follows from the first order condition (see section 3.1.4 or classical textbooks of noncommutative geometry, e.g.[GBFV]) that there is a right action of $\Omega^1_D(\mathcal{B})$ on H, given by:

$$h\omega = -J\omega^* J^{-1} h \qquad \forall \omega \in \Omega^1_D(\mathcal{B}), \tag{5.1.2}$$

[2]Of course, to make this assertion meaningful we need to introduce some well-defined notion of scala curvature. For a possible approach to these problem see appendix D; see also [DS13b] for the construction of "curved" Dirac operators for the noncommutative 2-torus. For a general discussion of scalar curvature for noncommutative manifolds see also [BhMa12, CM11].

where ω^* is the adjoint of ω, s.t. $([D,b])^* = -[D,b^*]$ and

$$h[D,b] = D(hb) - (Dh)b. \tag{5.1.3}$$

Such an action is clearly left $\mathcal{B}$-linear. Also, it induces a left action of $\Omega^1_D(\mathcal{B})$ on M, and $\Omega^1_D(\mathcal{B})M$ is just the space of all compositions $m \circ \omega$ of left $\mathcal{B}$-linear maps. For further details we refer to [DS13a].

Now we pass to connections (or covariant derivatives) and, following [DS13a], we give the following definition.

Definition 5.1.1. *We call a linear map $\nabla : M \to \Omega^1_D(\mathcal{B})M$ a* D-connection *on M if it satisfies:*

$$\nabla(bm) = [D,b]m + b\nabla(m), \ \ \forall b \in \mathcal{B}, \ m \in M.$$

Since we are dealing with maps between Hilbert spaces, we can define their adjoints. In particular, taken $m \in M$, it is clear what is its adjoint $m^\dagger$. Also, it is straightforward to define the adjoint η^* of a 1-form $\eta \in \Omega^1_D(\mathcal{B})$. Thus we can define the adjoint of an element of $\Omega^1_D(\mathcal{B})M$ simply by $(\eta m)^\dagger = m^\dagger \eta^*$. Of course, in general it will not be an element of $\Omega^1_D(\mathcal{B})M$, but we do not need this. Now we can introduce a notion of hermiticity for a D-connection [DS13a].

Definition 5.1.2. *A D-connection ∇ is said to be* hermitian *if, for each $m_1, m_2 \in M$,*
(i) *as an operator on $\mathcal{H}$, $m_1^\dagger \circ m_2 \in J\mathcal{B}J^{-1}$;*
(ii) *writing the action on arbitrary $h \in \mathcal{H}$ on the right, we have:*

$$h\nabla(m_2)m_1^\dagger - hm_2\nabla(m_1)^\dagger = (Dh)m_2m_1^\dagger - D(hm_2m_1^\dagger).$$

Using D-connections we can now define an operator D_M on a the dense subset of $M(\mathcal{H}) \subset \mathcal{H}_M$.

Definition 5.1.3. *We define D_M on $M(\mathrm{Dom}(D)) \subset \mathcal{H}_M$ by:*

$$D_M(hm) = (Dh)m + h\nabla(m) \qquad \forall m \in M.$$

Remark 5.1.4. D_M is well defined. Indeed, for any $b \in \mathcal{B}$ we get, using the Leibniz rule for ∇,

$$(D(hb))m + hb\nabla(m) = (Dh)bm + h\nabla(bm).$$

Proposition 5.1.5. *If ∇ is a hermitian D-connection, the operator D_M is selfadjoint and has compact resolvent. Moreover, all the commutators $[D_M, b]$, for $b \in \mathcal{B}$, are bounded.*

Proof. D_M is a symmetric operator. Indeed, for $h_1, h_2 \in \mathcal{H}$ and $m_1, m_2 \in M$, we have

$$\begin{aligned}
\langle h_1m_1, D_M(h_2m_2)\rangle &= \langle h_1m_1, (Dh_2)m_2\rangle + \langle h_1m_1, h_2\nabla(m_2)\rangle \\
&= \left\langle h_1, (Dh_2)m_2m_1^\dagger \right\rangle + \left\langle h_1, h_2\nabla(m_2)m_1^\dagger \right\rangle \\
&= \left\langle h_1, D(h_2m_2m_1^\dagger) \right\rangle + \left\langle h_1, h_2m_2\nabla(m_1)^\dagger \right\rangle \\
&= \langle (Dh_1)m_1, h_2, m_2\rangle + \langle h_1\nabla(m_1), h_2m_2\rangle = \langle D_M(h_1m_1), h_2m_2\rangle,
\end{aligned}$$

where, in the third equality, we used (ii) of definition 5.1.2. Moreover, since both $m \in M$ and $\nabla(m)$ are bounded operators, and D is selfadjoint, then D_M is selfadjoint.

Next, let us compute the commutator of D_M with $b \in \mathcal{B}$. We get:

$$\begin{aligned}[D_M, b](hm) &= D_M(bhm) - bD_M(hm) \\ &= (Dbh)m + (bh)\nabla(m) - b(Dh)m - b(h\nabla(m)) = ([D,b]h)m.\end{aligned}$$

Hence $\|[D_M, b]\| \leq \|[D,b]\|$. Finally we show that D_M has compact resolvent. If M is a finite free module over $\mathcal{B}$, with basis m_i, we have

$$\sum_j D_M(h_j m_j) = (Dh_j)m_j + h_j\nabla(m_j).$$

Now, $\nabla(m_j)$ can be written as $\omega_{jk}m_k$, and so the second part of the expression above is actually a bounded operator on $\mathcal{H}_M$. Therefore D_M is at most a bounded perturbation of D, which has compact resolvent: it follows that the same has to be true also for D_M. A similar discussion applies to the case when M is a finitely generated projective $\mathcal{B}$-module. □

5.2 Projectable spectral triples for quantum principal $U(1)$-bundles

We begin by considering the simple case of quantum principal $U(1)$-bundles. We extend here the results presented in [DS13a] in order to cover also the even dimensional case[3]. Before entering into the details of the construction we briefly recall the main properties of (projectable) spin structures over principal $U(1)$-bundles [AmmB98, Amm98].

5.2.1 Spin geometry of principal $U(1)$-bundles

Let M be an $(n+1)$-dimensional compact smooth manifold which is also the total space of a principal $U(1)$-bundle over the n-dimensional manifold $N = M/U(1)$. Assume that M, N are Riemannian manifolds, with metric tensors, respectively, $\widetilde{g}$ and g such that:
- the action of $U(1)$ on M is isometric w.r.t to $\widetilde{g}$;
- the bundle projection $\pi : M \to N$ is a Riemannian submersion;
- the fibers are of equal constant length $2\pi l$, for some $l \in \mathbb{R}_+$.

Consider now a (local) orthonormal frame on TM, $e = \{e_0, e_1, \ldots, e_n\}$, such that e is $U(1)$-equivariant and e_0 is the (normalized) Killing vector field of the $U(1)$ action. Then there exists a unique principal connection 1-form $\omega : TM \to \mathbb{R}$ such that $\ker\omega$ is orthogonal to the fibers, for all $m \in M$, w.r.t. the metric $\widetilde{g}$. ω is clearly given by e^0/l, where $\{e^j\}$ is the dual frame of $\{e_j\}$. Conversely, given a principal $U(1)$ connection and a Riemannian metric on the base space N, there exists a unique $U(1)$-invariant metric on M such that the horizontal vectors are orthogonal to the fundamental (Killing) vector field K.

Assume now that M is a spin manifold, and let ΣM be its spinor bundle. The $U(1)$ action either lifts to the spin structure and then to an action $U(1) \times \Sigma M \to \Sigma M$, or to a projective

[3] Part of these new results can be found also in our recent paper [DSZ13].

action. Assuming the former, we have a *projectable* spin structure on M, and it induces a spin structure on N [AmmB98].

Given a structure of this kind, we recall that the Dirac operator $\widetilde{D}$ on sections of ΣM can be written as

$$\widetilde{D} = \sum_{i=0}^{n} \gamma_i \partial_{e_i} + \frac{1}{4} \sum_{i,j,k=0}^{n} \widetilde{\Gamma}_{ij}^{k} \gamma_i \gamma_j \gamma_k,$$

where the γ_i are the gamma matrices, generating the action of the n-dimensional Clifford algebra, associated to the orthonormal frame $\{e_j\}$ and the $\widetilde{\Gamma}_{ij}^{k}$'s are the Christoffel symbols of the Levi-Civita connection on M. In particular [AmmB98] we have:

$$\begin{aligned} -\widetilde{\Gamma}_{ij}^{0} = \widetilde{\Gamma}_{i0}^{j} = \widetilde{\Gamma}_{0i}^{j} &= \frac{l}{2} d\omega(e_i, e_j), \\ \widetilde{\Gamma}_{i0}^{0} = \widetilde{\Gamma}_{0i}^{0} = \widetilde{\Gamma}_{00}^{i} = \widetilde{\Gamma}_{00}^{0} &= 0. \end{aligned} \tag{5.2.1}$$

We notice that the Lie derivative with respect to the Killing field ∂_K, which is equal to $\frac{1}{l}\partial_{e_0}$, differs from the spinor covariant derivative, which is given, according to (5.2.1), by

$$\nabla_{e_0} = \partial_{e_0} + \frac{l}{4} \sum_{j<k} d\omega(e_j, e_k) \gamma_j \gamma_k. \tag{5.2.2}$$

Now, Amman and Bär showed that $\widetilde{D}$ can be expressed as a sum of two first order operators on $L^2(\Sigma M)$ plus a zero order term. The first operator, called the *vertical Dirac operator*, is simply given by

$$D_v = \frac{1}{l} \gamma_0 \partial_K = \gamma_0 \partial_{e_0}.$$

Observe that ∂_{e_0} can be seen as the Dirac operator on the circle $S^1 \simeq U(1)$ with the standard uniform metric that gives S^1 length l. In order to construct the second operator we need to consider separately the case when $(n+1)$ is even and the case when $(n+1)$ is odd.

Consider first the **odd dimensional case**. The space of L^2-spinors on M can be orthogonally decomposed into irreducible representations of $U(1)$,

$$L^2(\Sigma M) = \bigoplus_{k \in \mathbb{Z}} V_k,$$

and this decomposition is preserved by $\widetilde{D}$, since it commutes with the $U(1)$ isometric action. Let now $L = M \times_{U(1)} \mathbb{C}$ be the complex line bundle associated to the principal bundle $\pi : M \to N$. Then one can prove [AmmB98] that there is a natural homothety of Hilbert spaces (which is an isomorphism if the fibres are of length $l = 1$)

$$Q_k : L^2(\Sigma N \otimes L^{-k}) \to V_k,$$

which satisfies

$$Q_k(\gamma_i \psi) = \gamma_i Q_k(\psi) \qquad i = 1, \dots, n$$

and

$$\nabla_{e_i} Q_k(\psi) = Q_k(\nabla_{f_i}\psi) + \frac{1}{4}\sum_{j=1}^{n}\left(\widetilde{\Gamma}_{i0}^{j} - \widetilde{\Gamma}_{ij}^{0}\right)\gamma_0\gamma_j Q_k(\psi),$$

where $f = (f_1, \dots, f_n)$ is the local orthonormal frame on N given by $f_i = \pi_*(e_i)$. Then we can define the *horizontal Dirac operator* D_h as the unique closed linear operator such that, on each V_k, it is given by the composition

$$D_h = Q_k \circ D'_k \circ Q_k^{-1},$$

where $D'_k = \sum_{i=1}^{n}(\gamma_i \otimes \mathrm{id})(\nabla^N_{f_i} \otimes \mathrm{id} + \mathrm{id} \otimes k\nabla^\omega_{f_i})$ is the twisted Dirac operator on $\Sigma N \otimes L^{-k}$. Here ∇^N is the covariant spinor derivative on N coming from the Levi-Civita connection on N, and $k\nabla^\omega$ is the covariant derivative on the bundle L^{-k} associated to the connection $i\omega$. Then we can write the Dirac operator $\widetilde{D}$ in the following way:

$$\widetilde{D} = D_v + D_h + Z,$$

where Z is a zero order operator. Moreover we can give an explicit formula for Z:

$$Z = -\frac{l}{4}\gamma_0 \sum_{j<k} d\omega(e_j, e_k)\gamma_j\gamma_k. \tag{5.2.3}$$

Now we come to the **even dimensional case**. We still have the decomposition

$$L^2(\Sigma M) = \bigoplus_{k\in\mathbb{Z}} V_k,$$

and this decomposition is still preserved by the Dirac operator $\widetilde{D}$. But now the spinor bundle splits as a direct sum $\Sigma M = \Sigma^{(+)} \oplus \Sigma^{(-)}$ accordingly to $\nu^2 = 1$, where

$$\nu = i^{\frac{n+1}{2}}\gamma_1 \cdot \ldots \cdot \gamma_n.$$

Each $\Sigma^{(\pm)}$ corresponds to a spinor bundle $\Sigma^{(\pm)}N$ on the base space. Now, as we have done before, we associate to the $U(1)$ bundle $\pi : M \to N$ the complex line bundle $L = M \times_{U(1)} \mathbb{C}$, with the natural connection $i\omega$. Then [AmmB98] there is a homothety of Hilbert spaces

$$Q_k : L^2((\Sigma^{(+)} \oplus \Sigma^{(-)}) \otimes L_k) \to V_k$$

which satisfies $Q_k(\gamma_i\psi) = \gamma_i Q_k(\psi)$ for each $i = 1, \dots, n$. Now let $f = (f_i, \dots, f_n)$ be the local orthonormal frame on N given by $f_i = \pi_*(e_i)$. Then the *horizontal Dirac operator* D_h is defined, on each V_k, by the composition

$$D_h = Q_k \circ D'_k \circ Q_k^{-1},$$

where

$$D'_k = \sum_{i=1}^{n} (\gamma_i \otimes \mathrm{id}) \left((\nabla^{(+)}_{f_i} \oplus \nabla^{(-)}_{f_i}) \otimes \mathrm{id} + \mathrm{id} \otimes k \nabla^{\omega}_{f_i} \right).$$

Also in this case, then, the Dirac operator $\widetilde{D}$ can be expressed as

$$\widetilde{D} = D_v + D_h + Z,$$

where Z is a zero order term, still given by:

$$Z = -\frac{l}{4} \gamma_0 \sum_{j<k} d\omega(e_j, e_k) \gamma_j \gamma_k. \tag{5.2.4}$$

We conclude this section with the following observation: in both the even and the odd dimensional case, the zero-order term Z is the responsible for the vanishing of the torsion of the metric connection. For the details see [AmmB98, Amm98, DS13a].

5.2.2 $U(1)$-equivariant spectral triples

In section 3.5 we gave the definition of H-equivariant spectral triple. Here we specialize this notion to the $U(1)$ case. Given a coaction of the Hopf algebra $H = \mathcal{O}(U(1))$ on an algebra $\mathcal{A}$, we can define an operator $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(a) = ka$ for $a \in \mathcal{A}^{(k)}$ (see section 4.6.2). This operator can be seen as the (selfadjoint) generator of the enveloping algebra $\mathcal{U} = \mathcal{U}(\mathfrak{u}(1))$ of the Lie algebra of $U(1)$. $\mathcal{U}$ is a Hopf algebra, with coproduct, counit and antipode given by:

$$\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta, \qquad \varepsilon(\delta) = 0, \qquad S(\delta) = -\delta.$$

Of course, $\mathcal{U}$ acts on the algebra $\mathcal{A}$. Therefore we can require a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ to be equivariant w.r.t. $\mathcal{U}$. This yields to the following definition.

Definition 5.2.1. *A* $U(1)$-equivariant real spectral triple *over the algebra $\mathcal{A}$ is a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ ($\gamma = \mathrm{id}$ if the triple is odd) together with a selfadjoint operator δ on $\mathcal{H}$, with domain stable under the action of $\mathcal{A}$, which extends the operator $\delta : \mathcal{A} \to \mathcal{A}$,*

$$\delta(\pi(a)\psi) = \pi(\delta(a))\psi + \pi(a)\delta(\psi),$$

and such that

$$\delta J + J\delta = 0, \qquad [\delta, \gamma] = 0, \qquad [\delta, D] = 0.$$

Actually we require also that the spectrum of δ is $\mathbb{Z}$ (it could be also $\mathbb{Z} + \frac{1}{2}$): this corresponds to the assumption that the $U(1)$ action on the tangent bundle lifts to an action and not to a projective action on the spinor bundle.

Hence, if $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta)$ is a $U(1)$-equivariant real spectral triple, we can split the Hilbert space $\mathcal{H}$ according to the spectrum of δ:

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k,$$

and this decomposition is preserved by the Dirac operator D. Moreover $\pi(\mathcal{A}^{(k)})\mathcal{H}_l \subseteq \mathcal{H}_{k+l}$ for any $k, l \in \mathbb{Z}$; in particular, $\mathcal{H}_0$ is stable under the action of the invariant subalgebra $\mathcal{B} = \mathcal{A}^{coH} = \mathcal{A}^{(0)}$.

5.2.3 Projectable spectral triples: odd case

Now we are ready to study the projectability of the spin structure in the framework of noncommutative geometry. We have to distinguish the odd dimensional case from the even dimensional one. We begin by considering the former.

Let $\mathcal{B} \hookrightarrow \mathcal{A}$ be a principal $\mathcal{O}(U(1))$-comodule algebra, and consider a $U(1)$-equivariant odd real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \delta)$ over the total space $\mathcal{A}$. We give the following definition [DS13a].

Definition 5.2.2. *An odd $U(1)$-equivariant real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta)$ of KR-dimension j is said to be* projectable *along the fibres if there exists a $\mathbb{Z}_2$ grading Γ on $\mathcal{H}$ which satisfies the following conditions,*

$$\Gamma^2 = 1, \qquad \Gamma^* = \Gamma,$$

$$[\Gamma, \pi(a)] = 0 \quad \forall a \in \mathcal{A},$$

$$[\Gamma, \delta] = 0,$$

$$\Gamma J = \begin{cases} J\Gamma & \text{if } j \equiv 1 \pmod 4 \\ -J\Gamma & \text{if } j \equiv 3 \pmod 4. \end{cases}$$

If such a Γ exists, we define the horizontal Dirac operator *D_h by:*

$$D_h \equiv \frac{1}{2}\Gamma[D, \Gamma]$$

We will be interested in a particular class of projectable spectral triples, which should represent the noncommutative counterpart of smooth principal $U(1)$-bundles which are Riemannian manifolds with fibers of constant length (see the discussion in section 5.2.1).

Definition 5.2.3. *A projectable spectral triple* has fibres of constant length *if there is a positive real number l such that, if we set*

$$D_v = \frac{1}{l}\Gamma\delta,$$

the operator

$$Z = D - D_h - D_v$$

is a bounded operator which commutes[4] with the representation of $\mathcal{A}$: $[Z, \pi(a)] = 0$ for any $a \in \mathcal{A}$.

In such a case, the operator D_v will be called the *vertical Dirac operator*, and the number l should represent, up to a 2π factor, the length of the fibres, as like as in the commutative (smooth) case.

[4] In [DS13a] a different choice was made: Z was asked to commute with $\mathcal{A}^\circ$. As pointed out in [DSZ13], remark 4.5, the choice we make here appears more natural. Moreover, it ensures the compatibility between the differential calculus on the total space and the one induced, projecting the spectral triple, on the base space, see remark 5.2.4.

Remark 5.2.4. If a projectable spectral triple satisfies the conditions of definition 5.2.3, then the Dirac operator D and the horizontal Dirac operator D_h determine the same first order differential calculus on $\mathcal{B}$: $[D, b] = [D_h, b]$ for any $b \in \mathcal{B}$.

Consider now a projectable triple $(\mathcal{A}, \mathcal{H}, D, J, \delta, \Gamma)$ of KR-dimension j, with j odd, and assume that it has fibres of constant length. Since Γ and D commute with δ, also D_h does the same. Therefore D_h preserves each $\mathcal{H}_k$. Instead, the real structure intertwines $\mathcal{H}_k$ and $\mathcal{H}_{-k}$: $J\mathcal{H}_k \subseteq \mathcal{H}_{-k}$. In particular, it preserves $\mathcal{H}_0$. Now, let us denote, for any $k \in \mathbb{Z}$, by D_k, and γ_k the restrictions to $\mathcal{H}_k$ of D_h and Γ, respectively. For each $k \in \mathbb{Z}$ we define also an antiunitary operator $j_k : \mathcal{H}_k \to \mathcal{H}_{-k}$ as follows:

$$j_k = \begin{cases} \gamma_{-k} J & \text{if } j \equiv 1 \pmod 4 \\ J & \text{if } j \equiv 3 \pmod 4 \end{cases} \tag{5.2.5}$$

(where the restriction of J to $\mathcal{H}_0$ is understood). Now we can prove the following.

Proposition 5.2.5. *The operators D_k, γ_k, j_k satisfy the commutation relations of a real spectral triple of KR-dimension $j-1$. In particular, if the differential calculus $\Omega^1_D(\mathcal{B})$ is projectable, $(\mathcal{B}, \mathcal{H}_0, D_0, \gamma_0, j_0)$ is an even real spectral triple of KR-dimension $j-1$. Also, for $k \neq 0$, $(\mathcal{B}, \mathcal{H}_k, D_k, \gamma_k)$ are even spectral triples, which are pairwise real.*

Proof. We check here only the commutation relations and the first order condition. For rest of the proof see [DS13a], proposition 4.4. Also, we check these relation only on the subspace $\mathcal{H}_0$, but the extension of the computations below to the general case $k \in \mathbb{Z}$ is straightforward. In order to simplify the notation, the restriction of the various operators to $\mathcal{H}_0$ will be understood.

For $j = 3$ the result is already proved in [DS13a]. Let us consider now the case $j = 1$: $[\Gamma, J] = 0$, $j_0 = \Gamma J$. We need to check that: $j_0^2 = 1$, $j_0 D_0 = D_0 j_0$, $\gamma_0 j_0 = j_0 \gamma_0$. We have:

$$j_0^2 = \gamma_0 J \gamma_0 J = \Gamma J \Gamma J = \Gamma^2 J^2 = 1,$$

$$\begin{aligned} j_0 D_0 &= \frac{1}{2} \Gamma J \Gamma [D, \Gamma] = \frac{1}{2} (\Gamma J \Gamma D \Gamma - \Gamma J D) \\ &= \frac{1}{2} (J D \Gamma - \Gamma J D) = \frac{1}{2} (-D J \Gamma + \Gamma D J) = \frac{1}{2} (\Gamma D J - D \Gamma J) = D_0 j_0, \end{aligned}$$

$$\gamma_0 j_0 = \Gamma \Gamma J = \Gamma J \Gamma = j_0 \gamma_0.$$

Now the case $j = 5$: $[\Gamma, J] = 0$, $j_0 = \Gamma J$. We need to check that $j_0^2 = -1$, $j_0 D_0 = D_0 j_0$, $\gamma_0 j_0 = j_0 \gamma_0$. Since the only difference with the previous case is that now $J^2 = -1$, the proof of the last two relations is the same as before. For the first one:

$$j_0^2 = \gamma_0 J \gamma_0 J = \Gamma J \Gamma J = \Gamma^2 J^2 = -1.$$

We are left with the proof of the proposition for $j = 7$. In this case we have $j_0 = J$, $J\Gamma = -\Gamma J$, and we have to check that $j_0^2 = 1$, $j_0 D_0 = D_0 j_0$, $\gamma_0 j_0 = -j_0 \gamma_0$. We have:

$$j_0^2 = J^2 = 1,$$

$$j_0 D_0 = \frac{1}{2} J\Gamma[D,\Gamma] = \frac{1}{2}\left(J\Gamma D\Gamma - JD\right)$$
$$= \frac{1}{2}\left(\Gamma D\Gamma J - DJ\right) = D_0 j_0,$$
$$\gamma_0 j_0 = \Gamma J = -J\Gamma = -j_0\gamma_0.$$

The first order condition, instead, follows from remark 5.2.4 and from the fact that the triple over $\mathcal{A}$ fulfils the first order condition. □

5.2.4 Projectable spectral triples: even case

Now we extend the notion of projectable spectral triple to the even dimensional case. We give the following definition, which, as we shall see later, is consistent with the results obtained in the commutative (smooth) case [AmmB98, Amm98].

Definition 5.2.6. *An even $U(1)$-equivariant real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta)$ is said to be* projectable *along the fibres if there exists a $\mathbb{Z}_2$ grading Γ on $\mathcal{H}$, which satisfies the following conditions,*

$$\Gamma^2 = 1, \qquad \Gamma^* = \Gamma,$$
$$[\Gamma, \pi(a)] = 0 \quad \forall a \in \mathcal{A},$$
$$[\Gamma, \delta] = 0,$$
$$\Gamma\gamma = -\gamma\Gamma,$$
$$\Gamma J = -J\Gamma.$$

If such a Γ exists, we define the horizontal Dirac operator *D_h by*

$$D_h \equiv \frac{1}{2}\Gamma[D,\Gamma]$$

Also in this case we can introduce the notion of constant length fibres, see definition 5.2.3. Consider now a projectable triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta, \Gamma)$ of KR-dimension j, with j even, and assume that it has fibres of constant length. Also in this case D_h preserves each $\mathcal{H}_k$, and the real structure intertwines $\mathcal{H}_k$ and $\mathcal{H}_{-k}$, $J\mathcal{H}_k \subseteq \mathcal{H}_{-k}$. In particular, it preserves $\mathcal{H}_0$. Let us denote, for any $k \in \mathbb{Z}$, by D_k, and γ_k the restrictions to $\mathcal{H}_k$ of D_h and Γ, respectively. Define an operator ν by $\nu = i\Gamma\gamma$. Then $\nu^* = \nu$ and $\nu^2 = 1$, and we can use it to split $\mathcal{H}_0$. In particular we obtain the following result.

Proposition 5.2.7. *Decompose $\mathcal{H}_0$ as $\mathcal{H}_0 = \mathcal{H}_0^{(+)} \oplus \mathcal{H}_0^{(-)}$, where $\mathcal{H}_0^{(\pm)}$ are the (± 1)-eigenspaces of ν. Then the horizontal Dirac operator D_h preserves both the subspaces $\mathcal{H}_0^{(\pm)}$. Moreover, if we denote by $D_0^{(\pm)}$ the restrictions of D_0 to $\mathcal{H}_0^{(\pm)}$, respectively, $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)})$ are spectral triples.*

Proof. Clearly D_h preserves $\mathcal{H}_0$, so, for the first part of the proposition, we need only to check that $[D_0, \nu] = 0$. We have:

$$[D_0, \nu] = \frac{1}{2}[\Gamma[D,\Gamma], \Gamma\gamma]$$

$$= \frac{1}{2}(\Gamma D\gamma - \Gamma\gamma\Gamma D\Gamma - D\Gamma\gamma + \Gamma\gamma D)$$
$$= \frac{1}{2}(-\Gamma\gamma D - D\gamma\Gamma + D\gamma\Gamma + \Gamma\gamma D) = 0.$$

Next we check that $[D_0, b]$ is bounded for each $b \in \mathcal{B}$. But Γ and γ commutes with $\mathcal{B}$, and $[D, b]$ is bounded since $\mathcal{B} \subset \mathcal{A}$; thus $[D_0, b]$ is bounded for any $b \in \mathcal{B}$. Of course, both $\mathcal{H}_0^{(\pm)}$ are preserved by $\mathcal{B}$. So $(\mathcal{B}, \mathcal{H}^{(\pm)}, D_0^{(\pm)})$, where $D_0^{(\pm)}$ are the restrictions of D_0, are spectral triples.

In order to conclude the proof we should also discuss the analytic behaviour of the Dirac operators $D_0^{(\pm)}$, but the fact that they have compact resolvent, which is what we need, follows from remark 4 of [DS13a]. □

Remark 5.2.8. Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta, \Gamma)$ be as in the previous proposition. Notice that $\Gamma\nu = -\Gamma\nu$, so that $\Gamma\mathcal{H}_0^{(\pm)} \subset \mathcal{H}_0^{(\mp)}$. Since $\Gamma^2 = 1$, Γ determines an isomorphism $\Gamma : \mathcal{H}_0^{(+)} \to \mathcal{H}_0^{(-)}$. Moreover, one can easily see that $D_h\Gamma = -\Gamma D_h$. So, $D_0^{(+)} = -D_0^{(-)}$ w.r.t. the isomorphism $\mathcal{H}_0^{(+)} \simeq \mathcal{H}_0^{(-)}$ determined by Γ. This is nothing else than the noncommutative counterpart of the fact that, in the smooth case, the two Dirac operators $D_0^{(\pm)}$ are associated to the same metric, but they differ by a different choice of orientation [AmmB98, Amm98]. So we can say that the two triples $(\mathcal{B}, \mathcal{H}^{(\pm)}, D_0^{(\pm)})$ differ only by *the choice of (the sign of) the orientation.*

Now we can check if the spectral triples on $\mathcal{B}$ given by the previous proposition are real. We start with the KR-dimension 2 case.

Proposition 5.2.9. *Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta, \Gamma)$ be a projectable real spectral triple of KR-dimension 2, fulfilling the constant length fibres condition. Then the antiunitary operator γJ preserves both the subspaces $\mathcal{H}_0^{(\pm)}$. Moreover, if we denote by $j_0^{(\pm)}$ its restrictions to $\mathcal{H}_0^{(\pm)}$, respectively, then $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)}, j_0^{(\pm)})$ are real spectral triples of KR-dimension 1, and they differ just by a change in the sign of the orientation (see previous remark).*

Proof. We know that both J and γ preserves $\mathcal{H}_0$. So, let j_0 denote the restriction of γJ to $\mathcal{H}_0$. First of all notice that $[j_0, \nu] = 0$, so that j_0 preserves both $\mathcal{H}_0^{(\pm)}$. Also, we see that j_0 commutes with Γ, since the spectral triple is projectable and has KR-dimension 2. Using the following relations,

$$D_0 j_0 = \frac{1}{2}\Gamma[Dj_0, \Gamma],$$

$$j_0 D_0 = \frac{1}{2}\Gamma[j_0 D, \Gamma] = -\frac{1}{2}\Gamma[Dj_0, \Gamma],$$

where the last equality follows from $JD = DJ$ and $\gamma D = -D\gamma$, we see that $D_0 j_0 = -j_0 D_0$, as it should be in KR-dimension 1.

Next, one sees immediately that, since γ commutes with $\mathcal{A}$, j_0 maps $\mathcal{B}$ into its commutant. And, since $-j^2 = \gamma^2 = 1$, and $J\gamma = -\gamma J$, $j_0^2 = 1$. So j_0, and thus $j_0^{(\pm)}$, fulfil all the commutation relations required for a real structure of a real spectral triple of KR-dimension 1 [GBFV]. The last property that we need to check is the first order condition. But it follows from the property $[D_h, b] = [D, b]$, see remark 5.2.4, and from the first order condition for the spectral triple over $\mathcal{A}$.

The last statement of the proposition follows from the fact that Γ intertwines the two triples, as shown in remark 5.2.8. □

In order to extend the result of proposition 5.2.9 to higher dimensional even spectral triples we give the following definition.

Definition 5.2.10. *Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta, \Gamma)$ be an even dimensional projectable real spectral triple of KR-dimension j. Then we define a real structure j_0 on $\mathcal{H}_0$ by:*

$$\begin{array}{ccc|c|c|c} \mathrm{KR-dim} & & 0 & 2 & 4 & 6 \\ j_0 & \equiv & J & \gamma J & J & \gamma J \end{array} \tag{5.2.6}$$

where the restriction of γ and J to $\mathcal{H}_0$ is understood.

With this definition of j_0 we can prove the following proposition, which is the generalization of proposition 5.2.9.

Proposition 5.2.11. *Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta, \Gamma)$ be a projectable even real spectral triple of KR-dimension j,fulfilling the constant length fibres condition. Let $j_0 : \mathcal{H}_0 \to \mathcal{H}_0$ be given by (5.2.6). Then j_0 preserves both the subspaces $\mathcal{H}_0^{(\pm)}$, and $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)}, j_0^{(\pm)})$ are real spectral triples of KR-dimension $(j-1)$. Moreover they differ just by a change in the sign of the orientation.*

Proof. We have already discussed the case $n = 2$. So we prove the proposition separately in the other three cases. All what we need to check is that $j_0^2 = \pm 1$ accordingly to KR-dimension $(j-1)$, that $[j_0, \nu] = 0$, and that $D_0^{(\pm)}$ and $j_0^{(\pm)}$ satisfy the correct commutation relations (see tables below); the other properties (like the first order condition) are fulfilled for the same reasons of the previous proposition. The first condition is easily checked, as follows from [GBFV]:

KR−dim	0	2	4	6
ϵ	+	−	−	+
ϵ'	+	+	+	+
ϵ''	+	−	+	−

KR − dim	1	3	5	7
ϵ	+	−	−	+.
ϵ'	−	+	−	+

where $J^2 = \epsilon$, $JD = \epsilon' DJ$ and $J\gamma = \epsilon''\gamma J$. Let us check that j_0 commutes with ν (from now on the restrictions of the various operators to $\mathcal{H}_0^{(\pm)}$ will be understood). Let $j = 4$. Then $j_0 = J$ and

$$[j_0, \nu] = [J, i\Gamma\gamma] = [J, i\Gamma]\gamma = 0.$$

Let $j = 6$. Then $j_0 = \gamma J$ and

$$[j_0, \nu] = [\gamma J, i\Gamma\gamma] = \gamma[J, i\Gamma]\gamma = 0.$$

Finally, let $j = 0$. Then $j_0 = J$ and

$$[j_0, \nu] = [J, i\Gamma\gamma] = [J, i\Gamma]\gamma = 0.$$

Now we check the commutation relation between j_0 and D_0. But we notice that the commutation relations between D_0 and j_0 are the same of those between j_0 and D. So, if $j = 0$ or $j = 4$ then

$j_0 = J$ and thus $j_0 D_0 = D_0 j_0$, and it is consistent with the requirements, respectively, of KR-dimension 7 and 3; instead, if $j = 6$ then $j_0 D_0 = -D_0 j_0$, as it should be in KR-dimension 5. □

We conclude this section pointing out that, as like as in the odd dimensional case, we can define pairwise real spectral triples $(\mathcal{B}, \mathcal{H}_k, D_k, j_k)$ simply extending the construction discussed above for the $k = 0$ case.

5.2.5 Twisted Dirac operators

In section 5.1 we have described how to twist a Dirac operator using a left module. Now we want to apply this construction to projectable spectral triples. We shall see that, given a quantum principal $U(1)$-bundle $(\mathcal{A}, \mathcal{O}(U(1)), \Delta_R, N, Q)$ (where, we recall, $Q = (\ker \varepsilon)^2$), over the invariant subalgebra $\mathcal{B}$, and a projectable $U(1)$-equivariant real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta, \Gamma)$, it is possible to use strong connections over $\mathcal{A}$ to construct twisted versions of the horizontal Dirac operator D_h. Again, we shall consider separately the odd dimensional and the even dimensional case. Before entering in the details of the construction of twisted Dirac operators, we notice the following facts, which give an additional insight on the geometrical meaning of our construction.

Proposition 5.2.12. *For any $k \in \mathbb{Z}$, $\mathcal{A}^{(k)}$ is a quantum associated bundle to the principal $\mathcal{O}(U(1)$-comodule algebra $\mathcal{A}$. Moreover, it is a finitely generated projective $\mathcal{B}$-module.*

Proof. Consider the left $\mathcal{O}(U(1))$-comodule (V, ρ_L), where $V = \mathbb{C}$ and $\rho_L(\lambda) = z \otimes \lambda$. For any $k \in \mathbb{Z}$ define the $\mathcal{O}(U(1))$-comodule (V^k, ρ_L^k) by setting $V^k = \mathbb{C}$, $\rho_L^k(\lambda) = z^k \otimes \lambda$. Then it is straightforward to see that $\mathcal{A}^{(k)}$ is isomorphic to $\mathcal{A} \square_{\mathcal{O}(U(1))} V^k$ (see definition 4.5.1). It follows (see definition 4.5.2) that $\mathcal{A}^{(k)}$ is a quantum associated bundle.

Next, it is clear that it is a left $\mathcal{B}$-module. Then the fact that it is finitely generated and projective as left $\mathcal{B}$-module follows directly from the fact that $\mathcal{A}$ is a principal comodule algebra (see proposition 4.5.3). □

Proposition 5.2.13. *For any $k \in \mathbb{Z}$, $k \neq 0$, $\mathcal{A}^{(k)}$ is a line module over $\mathcal{B}$.*

Proof. This is a corollary of theorem C.3.3. Indeed, set $E = \mathcal{A}^{(k)}$ and $F = \mathcal{A}^{(-k)}$; let $\mu_1 : E \otimes_{\mathcal{B}} F \to \mathcal{B}$ and $\mu_2 : F \otimes_{\mathcal{B}} E$ denote the multiplication maps. Then both μ_1 and μ_2 are surjective (see lemma C.3.1). This implies that $(\mathcal{A}, \mathcal{A}, E, F, \mu_1, \mu_2)$ is a strict Morita context, from which it follows that μ_1 and μ_2 are, actually, two isomorphisms (see proposition C.2.2). If now we set $\mathrm{ev} = \mu_1$ and $\mathrm{coev} = \mu_2^{-1}$ (cfr. the proof of theorem C.3.3; see also [BB11], theorem 7.3) then we see that, with this choice of evaluation and coevaluation map, E is a weak left module. Since both ev and coev are isomorphism, $E = \mathcal{A}^{(k)}$ is a left line module over $\mathcal{B}$. □

Odd dimensional case

The results discussed in this paragraph are mostly taken from [DS13a]. Let $(\mathcal{A}, \mathcal{H}, J, D, \delta, \Gamma)$ be a projectable $U(1)$-equivariant odd real spectral triple over a quantum principal $U(1)$-bundle $(\mathcal{A}, H, \Delta_R, N, Q)$, and let $\mathcal{B}$ be the invariant subalgebra of $\mathcal{A}$. Assume that the triples has the constant length fibres property; we recall that, in particular, this means that $\Omega^1_D(\mathcal{B}) = \Omega^1_{D_h}(\mathcal{B})$ (see remark 5.2.4). Let $\omega \in \Omega^1_D(\mathcal{A})$ be a strong connection, in the sense of definition 4.3.5. Let us notice that, for any $k \in \mathbb{Z}$, the set $\mathcal{A}^{(k)}$, acting on the right on $\mathcal{H}$ via the right action induced by the real structure (see eq. (5.1.1)), can be regarded as a set of bounded $\mathcal{B}$-linear maps between $\mathcal{H}_0$ and $\mathcal{H}_k$. Also, it fulfils conditions (a) and (b) which we assumed for M in the previous section. So we can take $M = \mathcal{A}^{(k)}$ and use all the previous results. In order to obtain well-behaving twisted Dirac operators we need to introduce an additional requirement on the triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$: we ask that there exists a bounded selfadjoint operator Z' on $\mathcal{H}$ such that

$$(Zh)a = Z'(ha) \qquad \forall h \in \mathcal{H},\ a \in \mathcal{A}.$$

Remark 5.2.14. As we have seen in the previous sections, the real structure which makes the triple over $\mathcal{B}$ a real spectral triple is not always the simple restriction of J to $\mathcal{H}_0$. Nevertheless both J and the collection of j_k induce the same right action of $\mathcal{A}$ on $\mathcal{H}$. Instead, when we use the real structure to define a right action of $\Omega^1_D(\mathcal{B})$ on $\mathcal{H}_0$ (cfr. section 5.1) we shall use the real structure j_0, to be consistent with the results of section 5.1. And in this case it would be not the same to use J, since, at least for some KR-dimensions, its commutation relation with D_0 is different from that of j_0, and the representation of a differential form involves the Dirac operator. Notice that this problem does not arise in the odd dimensional case, since for odd dimensional triple j_0 is always obtained as the restriction of the real structure J to $\mathcal{H}_0$.

We begin the construction of twisted Dirac operators. The first object we need is a connection on $M = \mathcal{A}^{(k)}$.

Proposition 5.2.15. *For any $k \in \mathbb{Z}$, the map $\nabla_\omega : \mathcal{A}^{(k)} \to \Omega^1_D(\mathcal{A})\mathcal{A}^{(k)}$ defined by*

$$\nabla_\omega(a) = [D, a] - ka\omega,$$

where both $a \in \mathrm{Dom}(D)$ and $\nabla_\omega(a)$ are regarded as operators on $\mathcal{H}_0$ acting from the right, defines a D_0-connection over the left $\mathcal{B}$-module $\mathcal{A}^{(k)}$.

Proof. See [DS13a], proposition 5.4. □

Proposition 5.2.16. *The D_0-connection ∇_ω is hermitian if ω is selfadjoint (as an operator on $\mathcal{H}$).*

Proof. See [DSZ13], proposition 5.2. We just recall that the action of D_ω on $hp \in \mathcal{H}$, where $h \in \mathcal{H}_0$ and $p \in \mathcal{A}^{(k)}$ (such that hp is in the domain of $D^{(\pm)}_\omega$), can be written in the following way:

$$D_\omega(hp) = (D + j_0\omega^* j_0^{-1}\delta - Z')(hp). \tag{5.2.7}$$

□

Using the construction discussed in the previous section of this chapter we obtain a family of twisted Dirac operators $D^{(k)}_\omega$, each one acting on $\mathcal{H}_0 \otimes \mathcal{A}^{(k)}$; since the latter can be identified with $\mathcal{H}_k$, we have a family of spectral triples $(\mathcal{B}, \mathcal{H}_k, D^{(k)}_\omega)$. Taking D_ω to be the closure of the direct sum of the operators $D_\omega^{(k)}$, we finally obtain a twisted Dirac operator D_ω, which is an (unbounded) operator on the full Hilbert space $\mathcal{H}$. It is straightforward [DS13a, DSZ13] to check that:

Proposition 5.2.17. *The twisted Dirac operator D_ω is selfadjoint if ω is a selfadjoint one form. Moreover, it has bounded commutators with all the elements of $\mathcal{A}$.*

Proposition 5.2.18. *Let Z be as in definition 5.2.3. Define*

$$\mathcal{D}_\omega = \Gamma\delta + D_\omega.$$

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D}_\omega)$ is a projectable spectral triple with constant length fibres, and the horizontal part of the operator $\mathcal{D}_\omega$ coincides with D_ω.

Proof. See proof of proposition 5.8 in [DS13a]. □

As in [DS13a] we introduce the following notion of *compatibility.*

Definition 5.2.19. *We say that a strong connection ω is* compatible *with a Dirac operator D if D_ω and D_h coincide on a dense subset of $\mathcal{H}$.*

Even dimensional case

Now let $(\mathcal{A}, \mathcal{H}, J, D, \gamma, \delta, \Gamma)$ be a $U(1)$-equivariant even real spectral triple over a quantum principal $U(1)$-bundle $(\mathcal{A}, \mathcal{O}(U(1)), \Delta_R, N, Q)$, and let $\mathcal{B}$ be the invariant subalgebra of $\mathcal{A}$. Assume that the triple has the constant length fibres property and that there exists a bounded operator Z' on $\mathcal{H}$ such that

$$(Zh)a = Z'(ha) \qquad \forall h \in \mathcal{H},\ a \in \mathcal{A}.$$

Let $\omega \in \Omega^1_D(\mathcal{A})$ be a strong connection, in the sense of definition 4.3.5. Let us notice that, for any $k \in \mathbb{Z}$, the set $\mathcal{A}^{(k)}$, acting on the right on $\mathcal{H}$ via the right action induced by the real structure (see eq. (5.1.1)), can be regarded as a set of bounded $\mathcal{B}$-linear maps between $\mathcal{H}_0^{(\pm)}$ and $\mathcal{H}_k^{(\pm)}$ (where, we recall, the $(\pm)$-decomposition is done accordingly to $\nu^2 = 1$). Also, it fulfils conditions (a) and (b) which we assumed for M in the previous section. So we can take $M = \mathcal{A}^{(k)}$ and use all the previous results. Also in this case we have to take care of the fact that the real structure we shall use when defining the left action of differential forms is j_0 and not the simple restriction of J (cfr. remark 5.2.14). Let begin by defining a connection on $M = \mathcal{A}^{(k)}$.

Proposition 5.2.20. *For any $k \in \mathbb{Z}$, the map $\nabla_\omega : \mathcal{A}^{(k)} \to \Omega^1_D(\mathcal{A})\mathcal{A}^{(k)}$ defined by*

$$\nabla_\omega(a) = [D, a] - ka\omega,$$

where both $a \in \mathrm{Dom}(D)$ and $\nabla_\omega(a)$ are regarded as operators on $\mathcal{H}_0^{(\pm)}$ acting from the right[5]*, defines a $D_0^{(\pm)}$-connection over the left $\mathcal{B}$-module $\mathcal{A}^{(k)}$.*

[5] The right action of $\Omega^1_D(\mathcal{B})$ on $\mathcal{H}_0$ is defined via the real structure j_0, see (5.1.2) and remark 5.2.14.

Proof. Due to condition (iii) of definition 4.3.5, $\nabla_\omega(a)$ belongs to $\Omega^1_D(\mathcal{B})\mathcal{A}$. But the constant length fibres property implies that $\Omega^1_D(\mathcal{B}) = \Omega^1_{D_h}(\mathcal{B})$ (see remark 5.2.4. Moreover, D_h commutes with ν and thus it preserves the splitting $\mathcal{H} = \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)}$ for any $k \in \mathbb{Z}$ and the same holds for $\nabla_\omega(a)$, for any $a \in \mathcal{A}$. Now the fact that ∇_ω is both a $D_0^{(+)}$-connection and a $D_0^{(-)}$-connection follows as in the proof of proposition 5.4 in [DS13a]. □

Proposition 5.2.21. *The $D_0^{(\pm)}$-connections ∇_ω are hermitian if ω is selfadjoint (as an operator on $\mathcal{H}$).*

Proof. It follows by direct computation, see [DS13a], lemma 5.5. □

Now we can use the construction discussed in the previous section to twist both the spectral triples $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)}, j_0^{(\pm)})$. This yields a family of spectral triples $(\mathcal{B}, \mathcal{H}_k^{(\pm)}, D_\omega^{(k,\pm)})$, $k \in \mathbb{Z}$. Taking $D_\omega^{(\pm)}$ to be the respective closures of the direct sums of the two families, we obtain two twisted Dirac operators $D_\omega^{(\pm)}$. Notice that the two families differ only by a different choice of the orientation, as follows from proposition 5.2.8.

Proposition 5.2.22. *The twisted Dirac operators $D_\omega^{(\pm)}$ are selfadjoint if ω is a selfadjoint one form, and they have bounded commutators with all the elements of $\mathcal{A}$.*

Proof. Take $h \in \mathcal{H}_0^{(\pm)}$ and $p \in \mathcal{A}^{(k)}$ such that hp is in the domain of $D_\omega^{(\pm)}$. Then we have:

$$\begin{aligned} D_\omega^{(\pm)}(hp) &= (D_0^{(\pm)}h)p + h[D,p] - khp\omega \\ &= (D_0^{(\pm)}h)p + [D, j_0 p^* j_0^{-1}]h + j_0\omega^* j_0^{-1} khp \\ &= D(hp) + ((D_0^{(\pm)} - D)h)p + j_0\omega^* j_0^{-1}\delta(hp) \\ &= (D + j_0\omega^* j_0^{-1}\delta - Z')(hp). \end{aligned} \tag{5.2.8}$$

From (5.2.8) follows, by standard results of functional analysis, the selfadjointness of D_ω. Next, D has bounded commutator with each $a \in \mathcal{A}$; ω is a one-form and so, due to the first order condition, the commutator of the second term of (5.2.8) with $a \in \mathcal{A}$ is simply $j_0\omega^* j_0^{-1}\delta(a)$ and hence is bounded. The commutator with the first term is bounded simply because it is the commutator of two bounded operators. We conlude then that $[D_\omega, a]$ is bounded $\forall a \in \mathcal{A}$. □

Corollary 5.2.23. *Using $D_\omega^{(\pm)}$ we can construct a "full" Dirac operator D_ω simply taking their direct sum. Then the operator D_ω is selfadjoint if ω is selfadjoint and it has bounded commutator with all the elements of the algebra $\mathcal{A}$.*

Proposition 5.2.24. *Let Z be as in definition 5.2.3. Define*

$$\mathcal{D}_\omega = \Gamma\delta + D_\omega.$$

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D}_\omega)$ is a projectable spectral triple with equal length fibres and the horizontal part of the operator $\mathcal{D}_\omega$ coincides with D_ω.

Proof. See proof of proposition 5.8 in [DS13a]. □

5.3 Projectable spectral triples for quantum principal $\mathbb{T}^n$-bundles

We consider now spectral triples over $\mathbb{T}^n$ quantum principal bundles. In this section we will generalize the previous results, giving a definition of projectability for $\mathbb{T}^n$-equivariant spectral triples (see below) and constructing twisted Dirac operators.

5.3.1 Spin geometry of principal $\mathbb{T}^n$-bundles

Let M be an $(m+n)$-dimensional compact smooth manifold and, together, the total space of a principal $\mathbb{T}^n$-bundle over the m-dimensional manifold $N = M/\mathbb{T}^n$. Assume that M, N are Riemannian manifolds, with metric tensors, respectively, $\widetilde{g}$ and g such that:

- the action of $\mathbb{T}^n$ is isometric w.r.t. $\widetilde{g}$;
- the bundle projection $\pi : M \to N$ is a Riemannian submersion;
- the fibres are isometric one to each other; moreover, the length of each fundamental vector field K_j is constant along M.

The last assumption could be weakened (as like as it is possible to define and study projectable triples over $U(1)$ bundles whose fibers are not of constant length [Amm98, AmmB98]) but the more general situation would be much more difficult to treat in the noncommutative case, so we shall not consider it here.

Under these assumptions, there is a unique principal connection 1-form $\omega : TM \to \mathfrak{t}_n$ such that $\ker\omega$ is orthogonal to the fibres, at any point of M, with respect to the metric $\widetilde{g}$. If $\{T_a\}_{a=1,\dots,n}$ is the canonical basis of the Lie algebra of $\mathbb{T}^n$ (we assume each K_a to be the fundamental vector field associated to T_a), then ω will be of the form

$$\omega = \sum_{a=1}^{n} \omega_a \otimes T_a,$$

where each ω_a is a $\mathbb{C}$-valued 1-form on M. Next, for any vector field X on N we shall denote by $\widetilde{X}$ its horizontal lifting. Consider now a (local) orthonormal frame $f = \{f_1, \dots, f_m\}$ on N. Then, if we set

$$\begin{cases} e_j = \frac{1}{l_j} K_j & j = 1, \dots, n, \\ e_{j+n} = \widetilde{f}_j & j = 1, \dots m, \end{cases}$$

where l_j are real positive constants, then $e = \{e_j\}$ is a (local) orthonormal frame form M. Assume now that M is a spin manifold, and let ΣM be its spinor bundle. As in the $U(1)$ case we shall also assume the $\mathbb{T}^n$ action to lift to an action $\mathbb{T}^n \times \Sigma M \to \Sigma M$. In this case we shall speak of *projectable* spin structure. A projectable spin structure on M induces a spin structure on N (this is a straightforward consequence of the analogue property for the $U(1)$ case [AmmB98]).

Assume now that the spin structure over M is projectable. Then the Dirac operator $\widetilde{D}$, acting on L^2-sections of ΣM, will be the following one:

$$\widetilde{D} = \sum_{i=1}^{n+m} \gamma^i \partial_{e_i} + \frac{1}{4} \sum_{i,j,k=1}^{n+m} \widetilde{\Gamma}_{ij}^k \gamma^i \gamma^j \gamma^k,$$

where the γ^j are the gamma matrices, associated to the orthonormal frame $\{e_j\}$, generating the action of the $(n+m)$-dimensional Clifford algebra and $\widetilde{\Gamma}_{ij}^k$ are the Christoffel symbols of the Levi-Civita connection on TM for the frame $\{e_j\}$. Using the Koszul formula we can compute them. Let us use the letters $a, b, c...$ to denote indices from 1 to n and the letters $i, j, k...$ to denote indices from $n+1$ to $n+m$. Then we have:

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k,$$

$$-\widetilde{\Gamma}_{ij}^a = \widetilde{\Gamma}_{ia}^j = \widetilde{\Gamma}_{ai}^j = \frac{l_a}{2} d\omega_a(e_i, e_j),$$

$$\widetilde{\Gamma}_{ib}^a = \widetilde{\Gamma}_{bi}^a = \widetilde{\Gamma}_{ab}^i = \widetilde{\Gamma}_{bc}^a = 0,$$

where the Γ_{ij}^k are the Christoffel symbols of the Levi-Civita connection on TN, with respect to the frame f. Before going on, we notice that the Lie derivative with respect to each Killing vector field differs from the spinor covariant derivative by:

$$\nabla_{e_a} = \partial_{e_a} + \frac{l_a}{4} \sum_{j<k} d\omega_a(e_j, e_k)\gamma^j \gamma^k.$$

We express the Dirac operator $\widetilde{D}$ as a sum of two first order operators plus a zero order term. The first operator, which we shall call the *vertical Dirac operator*, will be given by:

$$D_v = \sum_{a=1}^{n} \frac{1}{l_a} \gamma^a \partial_{K_a} = \sum_{a=1}^{n} \gamma^a \partial_{e_a}.$$

Now we construct the second operator. First of all, we split the Hilbert space $L^2(\Sigma M)$ into irreducible representations of $\mathbb{T}^n$:

$$L^2(\Sigma M) = \bigoplus_{k \in \mathbb{Z}^n} V_k.$$

Next, we reduce ourself to the case when both m and n are even. The general result can be obtained by direct generalization. For each $k \in \mathbb{Z}^n$ consider the vector space $\mathbb{C}$ carrying the irreducible representation of index k of $\mathbb{T}^n$ and form the associated complex bundle $L_k = M \times_{\mathbb{T}^n} \mathbb{C}$. Moreover, endow it with the connection $i\omega$. Then we can prove the following result (Σ_m denotes the m-dimensional spinor representation).

Proposition 5.3.1. *For each $k \in \mathbb{Z}^n$ there is an isomorphism*

$$Q_k : L^2(\Sigma N \otimes L_k) \otimes \Sigma_m \to V_k$$

such that the horizontal covariant derivatives, with respect to the vector fields $\widetilde{f}_i$, are given by

$$\nabla_{\widetilde{f}_i} Q_k(\psi) = Q_k(\nabla_{f_i}\psi) + \sum_{j=n+1}^{n+m} \sum_{a=1}^{n} \frac{l_a}{4} d\omega_a(e_i, e_j)\gamma(K_a/l_a)\gamma(e_j)Q_k(\psi),$$

where V_X is the vector field on N satisfying $d\omega(\widetilde{X}, \cdot) = \left\langle \widetilde{V}_X, \cdot \right\rangle$. Moreover, Clifford multiplication *is preserved, i.e.*

$$Q_k(\gamma(X)\psi) = \gamma(\widetilde{X})Q_k(\psi).$$

Proof. We can write $\Sigma M = SM \times_{\mathrm{Spin}(n+m)} \Sigma_{n+m}$ and $\Sigma M = SN \times_{\mathrm{Spin}(n)} \Sigma_n$ where SM, SN are, respectively, the principal $\mathrm{Spin}(m+n)$- and $\mathrm{Spin}(m)$-bundles defining the spin structures of the two manifolds and Σ_{n+m}, Σ_n are the canonical spin representations of the spin groups. Then, since we assumed both m and n even, we have: $\Sigma_{n+m} = \Sigma_n \otimes \Sigma_m$. Then the proposition follows by direct computations, cfr. the proof of lemma 4.4 in [AmmB98]. □

Then one can see, by direct computation, that, if we define the *horizontal Dirac operator*, on each V_k, by

$$D_h = Q_k \circ (D \otimes \mathrm{id}) \circ Q_k^{-1},$$

where D is the (twisted) Dirac operator on $\Sigma N \otimes L_k$, then $Z = \widetilde{D} - D_v - D_h$ is a zero order operator, which takes the form

$$Z = -\frac{1}{4}\sum_{a=1}^{n} l_a \gamma(K_a/l_a)\gamma(d\omega_a).$$

5.3.2 $\mathbb{T}^n$-equivariant spectral triples

Now we pass to the noncommutative case. We begin by extending definition 5.2.1. Given a coaction of the Hopf algebra $\mathcal{O}(\mathbb{T}^n)$ on an algebra $\mathcal{A}$ we can define operators δ_j, for $j = 1, \dots, n$, which correspond to the selfadjoint generators of the universal enveloping algebra $\mathcal{U}(\mathfrak{t}_n)$ of the Lie algebra $\mathfrak{t}_n$ of $\mathbb{T}^n$, by setting

$$\delta_j(a) = k_j a, \qquad \text{for } a \in \mathcal{A}^{(k)}.$$

$\mathcal{U}$ is a (commutative) Hopf *-algebra, with coproduct, counit and antipode defined by:

$$\Delta(\delta_j) = \delta_j \otimes 1 + 1 \otimes \delta_j, \qquad \varepsilon(\delta_j) = 0, \qquad S(\delta_j) = -\delta_j.$$

This provides us an action of $\mathcal{U}$ on the algebra $\mathcal{A}$, and so we can require a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ to be equivariant w.r.t. $\mathcal{U}$. This yields to the following definition.

Definition 5.3.2. *A* $\mathbb{T}^n$-equivariant real spectral triple *over the algebra $\mathcal{A}$ is a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ ($\gamma = \mathrm{id}$ in the odd case) together with commuting selfadjoint operators $\delta_j : \mathcal{H} \to \mathcal{H}$, for $j = 1, \dots, n$, with (common) domain of selfadjointness stable under the action of $\mathcal{A}$, which extend the operators $\delta_j : \mathcal{A} \to \mathcal{A}$,*

$$\delta_j(\pi(a)\psi) = \pi(\delta_j(a))\psi + \pi(a)\delta_j(\psi),$$

and such that

$$\delta_j J + J\delta_j = 0, \qquad [\delta_j, \gamma] = 0, \qquad [\delta_j, D] = 0.$$

Remark 5.3.3. As like as for $U(1)$-equivariant spectral triples we require the spectrum of each δ_j to be equal to $\mathbb{Z}$.

Now, if $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \{\delta_j\})$ is a $\mathbb{T}^n$-equivariant real spectral triple, the Hilbert space $\mathcal{H}$ splits according to the spectrum of the operators δ_j,

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}^n} \mathcal{H}_k,$$

and this decomposition is preserved by the Dirac operator D. Moreover, for any $k, l \in \mathbb{Z}^n$, $\pi(\mathcal{A}^{(k)})\mathcal{H}_l \subseteq \mathcal{H}_{k+l}$. In particular $\mathcal{H}_0$ is stable under the action of the invariant subalgebra $\mathcal{B} = \mathcal{A}^{coH} = \mathcal{A}^{(0)}$.

5.3.3 Projectable spectral triples: odd case

Now we can extend the notion of projectability to $\mathbb{T}^n$-equivariant spectral triples. We treat separately the odd dimensional and the even dimensional case. We begin with the former.

Definition 5.3.4. *An odd $\mathbb{T}^n$-equivariant real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \{\delta_j\})$, of KR-dimension $n+m$, is said to be* projectable *along the fibres if there exists a $\mathbb{Z}_2$-grading Γ on $\mathcal{H}$ which satisfies the following conditions,*

$$\Gamma^2 = 1, \qquad \Gamma^* = \Gamma,$$

$$[\Gamma, \pi(a)] = 0 \quad \forall a \in \mathcal{A},$$

$$[\Gamma, \delta_j] = 0 \quad \text{for } j = 1, \dots, n,$$

$$J\Gamma = \begin{cases} \Gamma J & \text{if } m \equiv 0 \ (\text{mod } 4), \\ -\Gamma J & \text{otherwise.} \end{cases}$$

We define the horizontal *Dirac operator D_h by:*

$$D_h = \begin{cases} \frac{1}{2}\Gamma[D, \Gamma]_- & \text{for } n \text{ odd} \\ \\ \frac{1}{2}\Gamma[D, \Gamma]_+ & \text{for } n \text{ even} \end{cases} \tag{5.3.1}$$

where $[a, b]_\pm = ab \pm ba$.

It can be imposed, on a projectable spectral triple, a condition equivalent to the constant length fibres condition introduced in the $n = 1$ case (cfr. definition 5.2.3). We give the following definition:

Definition 5.3.5. *We say the bundle $\mathcal{A}$ to have* isometric *fibres if there exists an operator D_v : $\mathcal{H} \to \mathcal{H}$ such that $D = D_v + D_h + Z$ and:*

(a) *$D_v|_{\mathcal{H}_0} = 0$, where $\mathcal{H}_0$ is the common 0-eigenspace of the derivations δ_i;*

(b) *$[D_v, \Gamma] = 0$ if n is odd, $[D_v, \Gamma]_+ = 0$ if n is even;*

(c) *$[D_v, \delta_i] = 0$ for any $i = 1, \dots, n$;*

(d) *Z is a bounded operator;*

(e) *Z commutes with the elements from $\mathcal{A}$: $[Z, a] = 0$ for any $a \in \mathcal{A}$.*

Remark 5.3.6. As like as in the $U(1)$ case (see remark 5.2.4), condition (e) of previous definition implies that $\Omega^1_D(\mathcal{B}) = \Omega^1_{D_h}(\mathcal{B})$.

Proposition 5.3.7. *Let $(\mathcal{A}, \mathcal{H}, D, J, \{\delta_j\}, \Gamma)$ be an odd dimensional projectable spectral triple with isometric fibres, and let $\mathcal{H}_0$ be the common 0-eigenspace of the derivations δ_j. Then, if we denote by D_0 the restriction of D_h to $\mathcal{H}_0$, $(\mathcal{B}, \mathcal{H}_0, D_0)$ is a (usually reducible) spectral triple.*

Moreover, if we denote by J_0 the restriction of J to $\mathcal{H}_0$, then J_0 determines a right action of $\mathcal{B}$ (or a left action of the opposite algebra $\mathcal{B}^\circ$) on $\mathcal{H}_0$ by

$$hb = b^\circ h = J_0 b^* J_0^{-1} h$$

for any $b \in \mathcal{B}$, $h \in \mathcal{H}$. This action fulfils the following properties:
(a) $[b, J_0 c^* J_0^{-1}] = 0$ *for all $b, c \in \mathcal{B}$; that is, J_0 maps $\mathcal{B}$ into its commutant;*
(b) $[[D_0, b], J_0 c^* J_0^{-1}] = 0$ *for all $b, c \in \mathcal{B}$ (*first order condition*).*

Proof. Clearly D_h is a selfadjoint operator, and it has compact resolvent (see [DS13a]). Also, $\mathcal{B}$ preserves $\mathcal{H}_0$ since it is exactly the invariant subalgebra for the $\mathbb{T}^n$ action. Finally, since $[D, b] = [D_h, b]$ for any $b \in \mathcal{B}$ (see remark 5.3.6), D has bounded commutators with the elements from $\mathcal{B}$. We conclude that $(\mathcal{B}, \mathcal{H}_0, D_0)$ is a spectral triple.

Next we prove (a) and (b). (a) follows simply by the fact that j_0 is nothing else than J, and J maps $\mathcal{A}$, and hence $\mathcal{B}$, into its commutant. For what concerns (b), we recall that the triple over $\mathcal{A}$ satisfies the first order condition; that is,

$$[[D, a], J b^* J^{-1}] = 0 \quad \forall a, b \in \mathcal{A}.$$

Using this fact we can see that:

$$\begin{aligned}
[[D_0, b], J_0 c^* J_0^{-1}] &= \frac{1}{2}[[\Gamma D \Gamma, b] \pm [D, b], J c^* J^{-1}] \\
&= \frac{1}{2}[\Gamma [D, b] \Gamma, J c^* J^{-1}] = \frac{1}{2}\Gamma[[D, b], J c^* J^{-1}]\Gamma = 0,
\end{aligned}$$

where we used also the fact that $J\Gamma = -\Gamma J$, according to definition 5.3.4. So the first order condition (b) is fulfilled. □

Lemma 5.3.8. *Let $(\mathcal{A}, \mathcal{H}, D, J, \{\delta_j\}, \Gamma)$, $(\mathcal{B}, D_0, \mathcal{H}_0)$ as above. Then, if we denote by γ_0 the restriction of Γ to $\mathcal{H}_0$,*

$$D_0 \gamma_0 = -\gamma_0 D_0 \qquad \text{for } n \text{ odd},$$

$$D_0 \gamma_0 = \gamma_0 D_0 \qquad \text{for } n \text{ even}.$$

Proof. It follows by direct computation, using the definition of D_h. □

5.3.4 Projectable spectral triples: even case

Now we consider even dimensional triples over principal $\mathcal{O}(\mathbb{T}^n)$-comodule algebras.

Definition 5.3.9. *An even dimensional $\mathbb{T}^n$-equivariant real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \{\delta_j\})$, of KR-dimension $n+m$, is said to be* projectable *along the fibres if there exists a $\mathbb{Z}_2$ grading Γ on $\mathcal{H}$ which satisfies the following conditions,*

$$\Gamma^2 = 1, \qquad \Gamma^* = \Gamma,$$

$$[\Gamma, \pi(a)] = 0 \quad \forall a \in \mathcal{A},$$

$$[\Gamma, \delta_j] = 0 \quad \text{for } j = 1, \dots, n,$$

$$J\Gamma = -\Gamma J,$$

$$\Gamma\gamma = (-1)^n \gamma\Gamma.$$

We define the horizontal *Dirac operator D_h by:*

$$D_h = \begin{cases} \frac{1}{2}\Gamma[D, \Gamma]_- & \text{for } n \text{ odd} \\ \\ \frac{1}{2}\Gamma[D, \Gamma]_+ & \text{for } n \text{ even} \end{cases} \tag{5.3.2}$$

where $[a, b]_\pm = ab \pm ba$.

Also in this case we can introduce the isometric fibres property, see definition 5.3.5.

Proposition 5.3.10. *Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \{\delta_j\}, \Gamma)$ be an even dimensional projectable spectral triple with isometric fibres, and let $\mathcal{H}_0$ be the common 0-eigenspace of the derivations δ_j. Then, if we denote by D_0 the restriction of D_h to $\mathcal{H}_0$, $(\mathcal{B}, \mathcal{H}_0, D_0)$ is a (usually reducible) spectral triple.*

If we denote by J_0 the restriction of J to $\mathcal{H}_0$, then J_0 determines a right action of $\mathcal{B}$ (or a left action of the opposite algebra $\mathcal{B}^\circ$) on $\mathcal{H}_0$ by

$$hb = b^\circ h = J_0 b^* J_0^{-1} h$$

for any $b \in \mathcal{B}$, $h \in \mathcal{H}$. And such an action fulfils the following properties:
(a) $[b, J_0 c^* J_0^{-1}] = 0$ *for all $b, c \in \mathcal{B}$; that is, J_0 maps $\mathcal{B}$ into its commutant;*
(b) $[[D_0, b], J_0 c^* J_0^{-1}] = 0$ *for all $b, c \in \mathcal{B}$ (*first order condition*).*
Moreover both the operators Γ and $\gamma\Gamma$ preserve $\mathcal{H}_0$, and γ anticommutes with D_0.

Proof. The proof is the same as that of proposition 5.3.7. □

5.3.5 Real structure and real spectral triples

The construction of a real structure for the triples considered in the previous sections requires to discuss separately 4 cases. Indeed, if we denote by m the KR-dimension of the triple over $\mathcal{A}$, and we set $j = m - n$ (so that j should be the dimension of the triple over $\mathcal{B}$) we have four different situations: j even and n even, j even and n odd, j odd and n even, j odd and n odd.

Before beginning the discussion, we recall here the dependence on the KR-dimension of the commutation relations between the real structure, the Dirac operator and the $\mathbb{Z}_2$-grading. We use Connes' selection[6] (see [GBFV, DD11]). Given a real spectral triple $(\mathcal{A}, \mathcal{H}, J, D, \gamma)$ we say that it is of KR-dimension j (which we consider always modulo 8) if:

$$J^2 = \varepsilon \cdot \mathrm{id},$$

$$JD = \varepsilon' DJ,$$

and, for j even,

$$J\gamma = \varepsilon'' \gamma J,$$

$$\gamma D = -D\gamma,$$

where $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ according to the following table [GBFV, DD11]:

j	0	2	4	6	1	3	5	7
ϵ	+	−	−	+	+	−	−	+
ϵ'	+	+	+	+	−	+	−	+
ϵ''	+	−	+	−				

j **even,** n **even.** $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ is an even real spectral triple of KR-dimension $m = j + n$. We extend the triple $(\mathcal{B}, \mathcal{H}_0, D_0)$ to an even dimensional real spectral triple $(\mathcal{B}, \mathcal{H}_0, D_0, j_0, \gamma_0)$ of KR-dimension j, where j_0 and γ_0 are defined in the tables below (the restriction of the operators to $\mathcal{H}_0$ is always understood). We recall that D_0 is the restriction of D_h to $\mathcal{H}_0$, where $D_h = \frac{1}{2}\Gamma[D, \Gamma]_+$, so that $\Gamma D_0 = D_0 \Gamma$. Also, we recall that, since n is even, $\Gamma\gamma = \gamma\Gamma$.

Table 5.1: j_0 and γ_0 for the even-even case

j \ n	0	2	4	6
0	J	ΓJ	ΓJ	J
2	J	J	ΓJ	ΓJ
4	J	ΓJ	ΓJ	J
6	J	J	ΓJ	ΓJ

j \ n	0	2	4	6
0	γ	$\gamma\Gamma$	γ	$\gamma\Gamma$
2	γ	$\gamma\Gamma$	γ	$\gamma\Gamma$
4	γ	$\gamma\Gamma$	γ	$\gamma\Gamma$
6	γ	$\gamma\Gamma$	γ	$\gamma\Gamma$

j **even,** n **odd.** $(\mathcal{A}, \mathcal{H}, D, J)$ is an odd real spectral triple of KR-dimension $m = j + n$. We turn the triple $(\mathcal{B}, \mathcal{H}_0, D_0)$ into an even dimensional real spectral triple $(\mathcal{B}, \mathcal{H}_0, D_0', j_0, \gamma_0)$ of KR-dimension j, where $\gamma_0 = \Gamma|_{\mathcal{H}_0}$ and j_0, D_0' are defined in the tables below[7] (the restriction of the operators to $\mathcal{H}_0$ is always understood). We recall that D_0 is the restriction of D_h to $\mathcal{H}_0$, where $D_h = \frac{1}{2}\Gamma[D, \Gamma]$, so that $\Gamma D_0 = -D_0\Gamma$.

Remark 5.3.11. We spend some words about the cases with (j, n) equal to $(0, 3)$, $(0, 5)$, $(4, 3)$ and $(4, 5)$. In all these situations, indeed, it is not possible to find a set of operators (D_0, j_0, γ_0) constructed only using Γ, D and J and fulfilling all the required commutation relations. And this

[6] There is another possible choice, see table 7.3. See also [DD11].

[7] In the cases with (j, n) equal to $(0, 3)$, $(0, 5)$, $(4, 3)$ and $(4, 5)$, actually, the real structure j_0 does not fulfil the right commutation relations. Indeed, j_0^2 has the wrong sign. For a discussion of this issue see remark 5.3.11 and example 5.3.12.

Table 5.2: D_0' and j_0 for the even-odd case

j \ n	1	3	5	7
0	D_0	D_0	D_0	D_0
2	D_0	ΓD_0	ΓD_0	D_0
4	D_0	D_0	D_0	D_0
6	D_0	ΓD_0	ΓD_0	D_0

j \ n	1	3	5	7
0	ΓJ	J^7	ΓJ^7	J
2	J	J	ΓJ	ΓJ
4	ΓJ	J^7	ΓJ^7	J
6	J	J	ΓJ	ΓJ

issue can not be solved changing the commutation relation between J and Γ. Indeed, there are two possible choices: $J\Gamma = \Gamma J$ and $J\Gamma = -\Gamma J$. In the first case $\gamma_0 = \Gamma$ fulfils all the required commutation relations, but j_0^2 has the wrong sign (that is, $j_0^2 = -1$ for $j \equiv 0 \pmod 8$ and $j_0^2 = 1$ for $j \equiv 4 \pmod 8$). In the second one, instead, it is possible to recover a j_0 with the correct commutation relations, by setting $j_0 = \Gamma J$, but then we can not find a suitable γ_0 commuting with j_0. We have chosen to adopt the first convention, since it allows to define all the three operators, even if with j_0^2 with the wrong sign, and, moreover, it appears as the more natural choice (cfr. example 5.3.12 and remark 5.3.13). We conclude this remark with the following observation: the fact that we are not able to define a j_0 fulfilling all the right commutation relations does not mean that such a j_0 does not exist, but only that it can not be expressed only in terms of J, D and Γ.

In order to discuss the issues of the previous remark in a more exhaustive way, we consider an explicit example.

Example 5.3.12. We want to study the behaviour of a projectable spectral triple of KR-dimension 3 over a quantum principal $\mathbb{T}^3$-bundle. In order to avoid the trivial cases without, however, dealing with triples of too high dimension, we shall consider a product geometry between a finite spectral triple [Kra98] and a noncommutative 3-torus. Let us begin by introducing the former. We take it to be the simplest finite real spectral triple of KR-dimension 0 [Kra98, PS08].

Let $\mathcal{A}_F = \mathcal{A}_1 \oplus \mathcal{A}_2 = \mathbb{C} \oplus \mathbb{C}$. Consider the Hilbert space $\mathcal{H}_F = \mathbb{C}^3$. An element $(a_1, a_2) \in \mathcal{A}$ acts on $\mathcal{H} = \mathbb{C}^3$ in the following way:

$$\pi_F(a_1, a_2) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix}.$$

Then we introduce a real structure J_F. We take it to be the composition of the complex conjugation, on each factor $\mathbb{C}$, with the following matrix:

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We see immediately that $J^2 = 1$. Moreover,

$$J_F \pi_F(a_1, a_2)^* J_F^{-1} = J_F \pi_F(\overline{a_1}, \overline{a_2}) J_F^{-1} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_1 \end{pmatrix}.$$

As $\mathbb{Z}_2$-grading orientation operator we can consider the following one:

$$\gamma_F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consistently with the commutation relations characterizing the triples of KR-dimension 0, we have $J_F \gamma_F = \gamma_F J_F$. Finally, as Dirac operator we consider the following one:

$$D_F = \begin{pmatrix} 0 & \overline{a} & a \\ a & 0 & 0 \\ \overline{a} & 0 & 0 \end{pmatrix}, \qquad a \in \mathbb{C}.$$

Then $D_F^* = D_F$, $J_F D_F = D_F J_F$ and $D_F \gamma_F = -\gamma_F D_F$. It follows that $(\mathcal{A}_F, \mathcal{H}_F, D_F, J_F, \gamma_F)$ is a real spectral triple of KR-dimension 0.

Next, let $(\mathcal{A}(\mathbb{T}^3_\theta), \mathcal{H}_T, D_T, J_T)$ be the real spectral triple on the noncommutative 3-torus considered in section 5.4.3 (see also appendix A). That is, $\mathcal{H} = L^2(\mathbb{T}^3) \otimes \mathbb{C}^2$, $J = i\sigma_2 \circ J_0$, where J_0 comes from the Tomita-Takesaki involution, and $D = \sum_{i=1}^{3} \sigma^j \circ \delta_j$. Then the product geometry is obtained in the following way [Van99, DD11]. The Hilbert space is $\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_T$ with the tensor product representation of $\mathcal{A} = \mathcal{A}_F \otimes \mathcal{A}(\mathbb{T}^3_\theta)$; also the real structure is simply $J = J_F \otimes J_T$. For the Dirac operator, instead, we take

$$D = D_F \otimes \mathrm{id} + \gamma_F \otimes D_T.$$

Then $(\mathcal{A}, \mathcal{H}, D, J)$ is a real spectral triple of KR-dimension 3. Moreover, it is $\mathbb{T}^3$-equivariant with respect to the action of $\mathbb{T}^3$ generated by the derivations δ_j, acting on the second factor $\mathcal{H}_T$ of $\mathcal{H}$. The algebra $\mathcal{A}$ is, trivially, a cleft Hopf-Galois $\mathcal{O}(\mathbb{T}^3)$-extension with invariant subalgebra isomorphic to $\mathcal{A}_F$. Also, the common 0-eigenspace $\mathcal{H}_0$ of the derivations δ_j is isomorphic to $\mathcal{H}_F \otimes \mathbb{C}^2$. Is it a projectable triple? The answer is positive. Let us show this fact. According to definition 5.3.4, we need a selfadjoint operator Γ commuting with the derivations δ_j, commuting with the representation of $\mathcal{A}$ and such that: $\Gamma^2 = \mathrm{id}$, $\Gamma J = J\Gamma$. All these requirements are satisfied by the following operator:

$$\Gamma = \gamma_F \otimes \mathrm{id}_{\mathcal{H}}.$$

With this choice of Γ we obtain the following operators:

$$D_h = D_F \otimes \mathrm{id}_{\mathcal{H}_T}$$

$$\Rightarrow\ D_0 = D_F \otimes \mathrm{id}_{\mathbb{C}^2},$$

$$j_0 = J_F \otimes (i\sigma^2 \circ c.c.),$$

$$\gamma_0 = \gamma_F \otimes \mathrm{id}_{\mathbb{C}^2}.$$

Here *c.c.* denotes the complex conjugation. It is straightforward to see that they fulfil the following relations: $D_0 j_0 = j_0 D_0$, $\gamma_0 j_0 = j_0 \gamma_0$ and $\gamma_0 D_0 = -D_0 \gamma_0$. All these relations are consistent with a spectral triple of KR-dimension 0. Instead, for the square of the real structure we obtain $j_0^2 = -1$, hence it has the wrong sign. It is not difficult to see that using only D, J and Γ we can not construct a j_0' fulfilling the previous relations plus $j_0'^2 = 1$. This does not mean, of course, that such an operator does not exists. Indeed, for any 2×2 unitary matrix A anticommuting with $(i\sigma^2 \circ c.c.)$, the operator $A j_0$ satisfies all the required commutation relations. Of course, the most natural choice is $A = i\sigma^2$, so that $j_0' = A j_0 = J_F \otimes \mathrm{id}_{\mathbb{C}^2}$.

Remark 5.3.13. In remark 5.3.11 we pointed out that there are two reasonable choices for the commutation relation between J and Γ: $J\Gamma = -J\Gamma$. If we had considered the second one, the only possible solutions for Γ would have been of this kind: $\Gamma = \mathrm{id}_{\mathcal{H}_F} \otimes \mathrm{id}_{\mathcal{H}_\tau} \otimes \Sigma$, with Σ a suitable selfadjoint matrix acting on the factor $\mathbb{C}^2$ (e.g., $\Sigma = \sigma^1$). Therefore the horizontal Dirac operator would have been, e.g., of this form: $D_h = D_F \otimes \mathrm{id} + \sigma^1 \delta_1$. Even if the restriction of $\mathcal{H}_0$ would have been the same as that of the operator constructed in the previous example, it is clear that we can not ignore the different origin of the two operators, and that the more meaningful choice is the one adopted in the previous example.

j **odd,** n **even.** $(\mathcal{A}, \mathcal{H}, D, J)$ is an odd real spectral triple of KR-dimension $m = j+n$. We turn the triple $(\mathcal{B}, \mathcal{H}_0, D_0)$ into an odd dimensional real spectral triple $(\mathcal{B}, \mathcal{H}_0, D_0', j_0)$ of KR-dimension j, where j_0 and D_0' are defined in the tables below (the restriction of the operators to $\mathcal{H}_0$ is always understood). We recall that D_0 is the restriction of D_h to $\mathcal{H}_0$, where $D_h = \dfrac{1}{2}\Gamma[D, \Gamma]_+$, so that $\Gamma D_0 = D_0 \Gamma$.

Table 5.3: D_0' and j_0 for the odd-even case

j \ n	0	2	4	6
1	D_0	ΓD_0	D_0	ΓD_0
3	D_0	ΓD_0	D_0	ΓD_0
5	D_0	ΓD_0	D_0	ΓD_0
7	D_0	ΓD_0	D_0	ΓD_0

j \ n	0	2	4	6
1	J	ΓJ	ΓJ	J
3	J	J	ΓJ	ΓJ
5	J	ΓJ	ΓJ	J
7	J	J	ΓJ	ΓJ

j **odd,** n **odd.** $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ is an even real spectral triple of KR-dimension $m = j + n$. We turn the triple $(\mathcal{B}, \mathcal{H}_0, D_0)$ into an odd dimensional real spectral triple $(\mathcal{B}, \mathcal{H}_0, D_0', j_0)$ of KR-dimension j, where j_0 and D_0' are defined in the tables below (the restriction of the operators to $\mathcal{H}_0$ is always understood). We recall that D_0 is the restriction of D_h to $\mathcal{H}_0$, where $D_h = \dfrac{1}{2}\Gamma[D, \Gamma]$, so that $\Gamma D_0 = -D_0 \Gamma$.

We conclude this section pointing out that, in all the cases discussed above, the real structure

Table 5.4: D_0' and j_0 for the odd-odd case

j \ n	1	3	5	7
1	D_0	D_0	ΓD_0	ΓD_0
3	D_0	D_0	D_0	D_0
5	D_0	D_0	ΓD_0	ΓD_0
7	D_0	D_0	D_0	D_0

j \ n	1	3	5	7
1	ΓJ	ΓJ	J	J
3	J	ΓJ	ΓJ	J
5	ΓJ	ΓJ	J	J
7	J	ΓJ	ΓJ	J

j_0 maps the algebra $\mathcal{B}$ into its commutant and the triple fulfils the first order condition. Both properties follow from proposition 5.3.7 (and from the analogue result in the even dimensional case, see proposition 5.3.10).

5.3.6 Twisted Dirac operators

Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \{\delta_j\}, \Gamma)$ be a projectable $\mathbb{T}^n$-equivariant real spectral triple over a quantum principal $\mathbb{T}^n$-bundle $(\mathcal{A}, H, \Delta_R, N, Q)$ and let $\mathcal{B}$ be the invariant subalgebra of $\mathcal{A}$. Assume that the differential calculus over $\mathcal{A}$ is projectable and that the triple has isometric fibres. Then we can construct twisted Dirac operators, as like as in section 5.2.5. Actually, in order to get well-behaving operators, we have to add a further requirement to those of definition 5.3.5:
(f) there exists a bounded operator Z' on $\mathcal{H}$ such that

$$(Zh)a = Z'(ha), \qquad \forall h \in \mathcal{H},\ a \in \mathcal{A}.$$

Proposition 5.3.14. *For any $k \in \mathbb{Z}^n$, $\mathcal{A}^{(k)}$ is a quantum bundle associated to the principal $\mathcal{O}(\mathbb{T}^n)$-comodule algebra $\mathcal{A}$. Moreover, it is a finitely generated projective $\mathcal{B}$-module.*

Proof. Let $k \in \mathbb{Z}$. Consider the left $\mathcal{O}(\mathbb{T}^n)$-comodule (V^k, ρ_L^k), where $V = \mathbb{C}$ and $\rho_L(\lambda) = z^k \otimes \lambda$. Then it is straightforward to see that $\mathcal{A}^{(k)}$ is isomorphic to $\mathcal{A}\Box_{\mathcal{O}(\mathbb{T}^n)} V^k$ (see definition 4.5.1). It follows (see definition 4.5.2) that $\mathcal{A}^{(k)}$ is a quantum associated bundle.

Next, it is clear that it is a left $\mathcal{B}$-module. Then the fact that it is finitely generated and projective as left $\mathcal{B}$-module follows directly from the fact that $\mathcal{A}$ is a principal comodule algebra (see proposition 4.5.3). □

Now we can use the results of section 5.1 to build twisted Dirac operators. Indeed, due to the previous results, we can take $M = \mathcal{A}^{(k)}$. We begin defining a connection on $\mathcal{A}^{(k)}$.

Proposition 5.3.15. *Let ω be a $\mathbb{T}^n$ strong connection defined by a family $\omega_i \in \Omega^1_D(\mathcal{A})$, for $i = 1, \dots, n$. Then, for any $k \in \mathbb{Z}^n$, the map $\nabla_\omega : \mathcal{A}^{(k)} \to \Omega^1_D(\mathcal{A})\mathcal{A}^{(k)}$ defined by*

$$\nabla_\omega(a) = [D, a] - \sum_{i=1}^{n} k_i a \omega_i,$$

where both $a \in \mathcal{A}^{(k)}$ and $\nabla_\omega(a)$ are regarded as operators on $\mathcal{H}_0$ acting from the right, defines *a D_0-connection over the left $\mathcal{B}$-module $\mathcal{A}^{(k)}$, where D_0 denotes the restriction of the horizontal Dirac operator D_h to $\mathcal{H}_0$.*

Proof. The proof is the same as that of proposition 5.2.20. □

Proposition 5.3.16. *The D_0-connection ∇_ω is hermitian if all the ω_i are selfadjoint (as operators on $\mathcal{H}$).*

Proof. We check (i) and (ii) of definition 5.1.2. Since we have taken $M = \mathcal{A}^{(k)}$ acting on $\mathcal{H}_0$ on the right via $ha = Ja^*J^{-1}h$, and since J maps $\mathcal{A}$ into its commutant, then (i) is fulfilled. For what concerns (ii), we proceed by direct computation: let $a_1, a_2 \in \mathcal{A}^{(k)}$ and $h \in \mathcal{H}_0$; then, using (5.1.2), we get:

$$\begin{aligned}
&h\left(\nabla_\omega(a_2)a_1^\dagger - a_2\nabla_\omega(a_1)^\dagger - (Dh)a_2a_1^\dagger + D(ha_2a_1^\dagger)\right) \\
&= h\left([D,a_2] - \sum_{i=1}^n k_i a_2\omega_i\right)a_1^\dagger - h\left(a_2\left([D,a_1] - \sum_{i=1}^n k_i a_1\omega_i\right)^\dagger\right) - h[D,a_2a_1^\dagger] \\
&= h\left(\sum_{i=1}^n k_i a_2(\omega_i^\dagger - \omega_i)a_1^\dagger\right),
\end{aligned}$$

which vanishes if $\omega_i^\dagger = \omega_i$. □

Now, we can identify, up to completion, $\mathcal{H}_0\mathcal{A}^{(k)}$ with $\mathcal{H}_k$; hence, we have obtained a family of spectral triples $(\mathcal{B}, \mathcal{H}_k, D_\omega^{(k)})$, $k \in \mathbb{Z}^n$, where each $D_\omega^{(k)}$ is the twisted Dirac operator constructed using the connection ∇_ω on $\mathcal{A}^{(k)}$. Taking D_ω to be the closure of the direct sum of the Dirac operators of this family we obtain a twisted Dirac operator D_ω, acting on (a dense domain of) the whole Hilbert space $\mathcal{H}$.

Proposition 5.3.17. *The twisted Dirac operator D_ω is selfadjoint if all the ω_i are selfadjoint one-forms, and it has bounded commutators with all the elements of $\mathcal{A}$.*

Proof. We compute the action of D_ω on an element hp in its domain, with $h \in \mathcal{H}_0$ and $p \in \mathcal{A}^{(k)}$ (we use (5.1.2) for the right action of one-forms, where J_0 stands either[8] for j_0 or Γj_0):

$$\begin{aligned}
D_\omega(hp) &= (D_0h)p + h[D,p] - \sum_{i=1}^n k_i hp\omega_i \\
&= (D_0h)p + [D, J_0p^*J_0^{-1}]h + \sum_{i=1}^n J_0\omega_i^*J_0^{-1}k_ihp \\
&= D(hp) + ((D_0 - D)h)p + \sum_{i=1}^n J_0\omega_i^*J_0^{-1}h\delta_i(p) \\
&= \left(D + \sum_{i=1}^n J_0\omega_i^*J_0^{-1}\delta_i - Z'\right)(hp).
\end{aligned} \tag{5.3.3}$$

[8]If D_0' - see tables in the previous section - is simply D_0 then we take $J_0 = j_0$; if, instead, $D_0' = \Gamma D_0$, the we take $J_0 = \Gamma j_0$.

Now, the Dirac operator D and the derivations δ_i are selfadjoint, Z' and ω are bounded and selfadjoint; moreover, any δ_i is relatively bounded with respect to D. Then, by Kato-Rellich theorem, D_ω is selfadjoint on $\mathcal{H}$.

Next, D has bounded commutator with each $a \in \mathcal{A}$ and, since any ω_i is a one-form, from the first order condition (which holds also for the triple $(\mathcal{B}, \mathcal{H}_0, D_0)$, see proposition 5.3.7) the commutator of the second term with a is $\sum_i J_0 \omega_i^* J_0^{-1} \delta_i(a)$ and hence is bounded. The third term of (7.3.35) gives commutators between bounded operators, since Z' is bounded, and thus it gives only bounded terms. Therefore $[D_\omega, a]$ is bounded for each $a \in \mathcal{A}$. □

Proposition 5.3.18. *Let D_v be as in definition 5.3.5. Define*

$$\mathcal{D}_\omega = D_v + D_\omega.$$

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D}_\omega)$ is a projectable spectral triple with isometric fibres, and the horizontal part of the operator $\mathcal{D}_\omega$ coincides with D_ω.

Proof. See proof of proposition 5.8 in [DS13a]. □

As in the $U(1)$ we introduce the following notion of *compatibility.*

Definition 5.3.19. *We say that a strong connection ω is* compatible *with a Dirac operator D if D_ω and D_h coincide on a dense subset of $\mathcal{H}$.*

5.4 Projectable spectral triples and twisted Dirac operators for noncommutative tori

Now we apply the results of this chapter to some explicit models: we will show how the canonical flat spectral triples over n-dimensional noncommutative tori are projectable and we will work out explicit formulae for the twisted Dirac operators. An application of our result to a different model (a noncommutative Hopf fibration) can be found in [DSZ13]. We will begin by considering two quantum principal $U(1)$-bundles: the noncommutative 2-torus as a bundle over the circle S^1 and the noncommutative 4-torus as a bundle over a noncommutative 3-torus. The "intermediate" case, that is the 3-dimensional torus over the 2-dimensional one is discussed in [DS13a]. Next, we will consider, again, the noncommutative 3-torus, but now as a $\mathbb{T}^2$-bundle over the circle. Some properties of noncommutative tori will be recalled here, but we refer to appendix A for a more detailed discussion.

5.4.1 T^2_θ as quantum principal $U(1)$-bundle

Let $\mathcal{A} = \mathcal{A}(\mathbb{T}^2_\theta)$ be the unital algebra of a noncommutative 2-torus, that is, the polynomial algebra generated by the two unitaries U, V with the commutation relation $UV = e^{2\pi i\theta} VU$ (θ irrational), and consider the $U(1)$ action on $\mathcal{A}$, associated to the derivation δ_2:

$$\delta_2(U) = 0, \qquad \delta_2(V) = V$$

(we recall that there is also the derivation δ_1, which acts as $\delta_1(U) = U$, $\delta_1(V) = 0$). Then the invariant subalgebra $\mathcal{B}$ is the (commutative) algebra generated by U. In particular, since[9] spectrum$(U) = S^1$, we can identify (Gel'fand-Naimark theorem) $\mathcal{B}$ with a dense subalgebra of $C^\infty(S^1)$.

Let τ be the unique tracial state on $\mathcal{A}$, and let $\mathcal{H}_\tau$ be the associated GNS Hilbert space. Now let $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^2$, and consider the standard flat Dirac operator,

$$D = \sum_{i=1}^{2} \sigma^i \delta_i,$$

where σ_i are the Pauli matrices. Also, let $\mathcal{H}_0$ be the 0-eigenspace[10] of δ_2 in $\mathcal{H}$. We complete this spectral triple with the real structure and the orientation $\mathbb{Z}_2$-grading. They can be taken equal to:

$$\gamma = \mathrm{id} \otimes \sigma_3,$$

$$J = J_0 \otimes (i\sigma^2 \circ c.c.)$$

where $J_0 : \mathcal{H}_\tau \to \mathcal{H}_\tau$ is the Tomita-Takesaki antiunitary involution and *c.c.* denotes the complex conjugation. In this way we obtain a $U(1)$-equivariant even real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta_2)$. Now we can see if there exists an operator Γ such that the spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta_2, \Gamma)$ is projectable.

Proposition 5.4.1. *The unique operators $\Gamma : \mathcal{H} \to, \mathcal{H}$ such that $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta_2, \Gamma)$ is a projectable real spectral triple with equal length fibres, are $\Gamma = \pm \mathrm{id} \otimes \sigma^2$.*

Proof. Since Γ has to commute with $\pi(\mathcal{A})$, and $\mathcal{H}_\tau$ is an irreducible representation of $\mathcal{A}$, we have that the most general form of an admissible Γ is: $\Gamma = \alpha_0 \cdot \mathrm{id} + \sum_{i=1}^{3} \alpha_i \sigma^i$ with $\alpha_i \in \mathbb{C}$. And using $\Gamma\gamma = -\gamma\Gamma$ we see immediately that $\alpha_0 = 0$ and $\alpha_3 = 0$. Next, from $\Gamma^2 = -1$ we obtain:

$$\alpha_1^2 + \alpha_2^2 = 1.$$

There is a last condition to impose; namely: $[D_h, U] = [D, U]$, which comes from the equal length fibres property (see remark 5.2.4). This gives, for any $v \in \mathcal{H}$,

$$\sigma^1 U \delta_1(v) = (\alpha_1\alpha_2\sigma^2 - \alpha_2^2\sigma^1) U \delta_1(v).$$

Which implies that the only solution is $\alpha_2 = \pm 1$. Also, notice that $\Gamma = \pm\sigma^2$ is consistent with the commutation relation $J\Gamma = -\Gamma J$. It follows that $D_h = \sigma^1\delta_1$. Then, if $\Gamma = \pm\sigma^2$ and we define $D_v = \pm\Gamma\delta_2 = \sigma^2\delta_2$, we have $D = D_v + D_h$ and so the equal length fibres condition is fulfilled. □

Now, notice that we can identify both $\mathcal{H}_0^{(\pm)}$ with $L^2(S^1, d\varphi)$; then the restriction of D_h to $\mathcal{H}_0^{(\pm)}$ will be given by:

$$D_0^{\pm} = \pm i \frac{d}{d\varphi}.$$

[9]Here the spectrum of U is taken in the C^*-completion of $\mathcal{A}$.

[10]We are assuming that the spin structure relative to δ_2 is the trivial one [PS06, DS13a]; that is, we assume that the spectrum of δ_2 in $\mathcal{H}$ is $\mathbb{Z}$.

Also, the real structure $j_0^{(\pm)}$ on $\mathcal{H}_0^{(\pm)}$ will simply be the complex conjugation. So, each of the two projected triples is isomorphic to the canonical spectral triple on S^1, corresponding to the uniform metric.

Twisted Dirac operators

Taking $\Gamma = \sigma^2$, so that $D_v = \Gamma\delta_2 = \sigma^2\delta_2$, we see that the spectral triple discussed above has the constant length fibres property with, moreover, $Z = 0$. So we can construct twisted Dirac operators. First of all, we need a strong connection over $\mathcal{A}$. We can prove the following result.

Lemma 5.4.2. *A selfadjoint $U(1)$ strong connection over $\mathcal{A}$ is a one-form*

$$\omega = \sigma^2 + \sigma^1\omega_1,$$

where ω_1 is a selfadjoint element of $\mathcal{B}$.

Proof. Let $\omega \in \Omega^1_D(A)$; then ω can be written as:

$$\omega = \sum_i a_i[D, c_i]$$

with $a_i, c_i \in \mathcal{A}$. This implies that we can generically write ω as:

$$\omega = \sum_i \sigma^i\omega_i$$

with $\omega_i \in \mathcal{A}$. In order to be a strong connection, ω has to fulfil properties (i)-(iii) of definition 4.3.5. In particular we need $[\delta, \omega] = 0$, and this implies $\omega_i \in \mathcal{B}$. Also, ω_3 should be equal to zero, since we cannot obtain an operator such as $\sigma^3\omega^3$ from a commutator $[D, a]$. Thus we are left with a connection of the form:

$$\omega = \sigma^1\omega_1 + \sigma^2\omega_2.$$

Now notice that, for $j = 1, 2$, we can write the Pauli matrices σ^j as: $\sigma^j = U_j^{-1}[D, U_j]$ (where $U_1 = U$, $U_2 = V$). Then ω becomes:

$$\omega = \sum_{j=1}^{2} \omega_j U_j^{-1}[D, U_j].$$

But now we can use condition (ii) of definition 4.3.5; we obtain: $\omega_2 = 1$. Thus the most general $U(1)$ strong connection on $\mathcal{A}$ is:

$$\omega = \sigma^2 + \sigma^1\omega_1, \qquad \omega_1 \in \mathcal{B},$$

which, of course, is selfadjoint if and only if $\omega_1^* = \omega_1$. □

Now we can compute the Dirac operator D_ω obtained twisting D_h by the strong connection ω, in the way described in the previous sections.

Proposition 5.4.3. *For any selfadjoint $U(1)$ strong connection ω, the associated Dirac operator D_ω has the form*

$$D_\omega = D_h - \sigma^1 j_0 \omega_1 j_0^{-1} \delta_2.$$

Proof. From previous results we know that the projectability of the spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \delta)$ implies that there are two spectral triples over the invariant subalgebra $\mathcal{B}$. They are given by $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)}, j_0)$. In order to fix the conventions, we say that on $\mathcal{H}_0^{(+)}$ the Dirac operator D_0 is given by $-\delta_1$, while on $\mathcal{H}_0^{(-)}$ it is given by δ_1 (note that $\nu = -\sigma^1$, and thus σ^1 is diagonal w.r.t. the decomposition $\mathcal{H}_0 = \mathcal{H}_0^{(+)} \oplus \mathcal{H}_0^{(-)}$). Now we can compute the twisted operators $D_{\omega,k}^{(\pm)}$ on $\mathcal{H}_k^{(\pm)}$. Take $h_0 \in \mathcal{H}_0^{(+)}$. Then, for any $a \in \mathcal{A}^{(k)}$, we have:

$$D_{\omega,k}^{(+)}(h_0 a) = (D_0^{(+)} h_0) a - h_0 \nabla_\omega(a) = -\delta_1(h_0 a) + k h_0 a \omega_1.$$

Thus, if we take $h \in \mathcal{H}_k^{(+)}$ we see that the action of the twisted Dirac operator is given by

$$D_{\omega,k}^{(+)}(h) = -\delta_1(h) + j_0 \omega_1 j_0^{-1} \delta_2(h).$$

In the same way, one obtains that, for $h \in \mathcal{H}_k^{(-)}$,

$$D_{\omega,k}^{(-)}(h) = \delta_1(h) - j_0 \omega_1 j_0^{-1} \delta_2(h).$$

If now we put them together, and we consider the collection of all of them for any $k \in \mathbb{Z}$, we get that the full twisted Dirac operator D_ω is given, as an operator on $\mathcal{H}$, by

$$D_\omega = \sigma^1 \delta_1 - \sigma^1 j_0 \omega_1 j_0^{-1} \delta_2,$$

which is equal to $D_h - \sigma^1 j_0 \omega_1 j_0^{-1} \delta_2$. □

Corollary 5.4.4. *The only connection compatible with D, i.e. with the fully $\mathbb{T}^2$-equivariant Dirac operator on the noncommutative 2-torus, is $\omega = \sigma^2$.*

Proof. It follows from previous lemma and definition 5.3.19. □

Now we can compute, given any strong connection ω, the general form of a Dirac operator $D^{(\omega)}$ compatible with such a connection.

Proposition 5.4.5. *Let $\omega = \sigma^2 + \sigma^1 \omega_1$ be a selfadjoint connection. Then the following Dirac operator,*

$$\mathcal{D}_\omega = D - \sigma^1 j_0 \omega_1 j_0^{-1} \delta_2,$$

is compatible with ω.

Proof. It follows from definition 5.3.19 together with the computation of proposition 5.4.3. □

5.4.2 $\mathbb{T}^4_\theta$ as quantum principal $U(1)$-bundle

Let $\mathcal{A}$ be the unital (smooth) algebra of the noncommutative 4-torus, generated by four unitaries $U_1, \ldots, U_4$ with the commutation relations $U_i U_j = e^{2\pi i \theta_{ij}} U_j U_i$, where θ_{ij} is an antisymmetric matrix with no rational entries and no rational relation between them. On $\mathcal{A}$ There is the canonical action of $\mathbb{T}^4$, whose generators are the derivations δ_j,

$$\delta_i(U_j) = \delta_{ij} U_j.$$

As $U(1)$ quantum principal bundle structure we take the one given by the choice $\delta = \delta_4$, and we assume the relative spin structure to be the trivial one. Thus the invariant subalgebra $\mathcal{B}$ is the algebra generated by $U_1, \ldots, U_3$ and is isomorphic to the algebra of a noncommutative 3-torus.

We recall briefly the structure of (one of the) flat $\mathbb{T}^4$-equivariant spectral triples over $\mathcal{A}(\mathbb{T}^4_\theta)$. The commutation relations in KR-dimension 4 are the following ones:

$$J^2 = -1, \qquad JD = DJ, \qquad J\gamma = \gamma J. \tag{5.4.1}$$

In order to work out explicitly the operators, it is useful to recall the structure of the Clifford algebra $\mathbb{C}\mathrm{l}(4)$ (so that we can fix the notation).

The Clifford algebra $\mathbb{C}\mathrm{l}(4)$

The Clifford algebra $\mathbb{C}\mathrm{l}(4)$ is generated by four elements, $\gamma^1, \ldots, \gamma^4$, with the relations

$$\begin{aligned} &{\gamma^i}^2 = 1, \\ &\gamma^i\gamma^j = -\gamma^j\gamma^i \quad \text{for } i \neq j, \\ &{\gamma^i}^* = \gamma^i. \end{aligned} \tag{5.4.2}$$

We can represent the γ^i's as 4×4 matrices, related to the Dirac matrices. In the so-called Dirac representation we can write the matrices γ^i as:

$$\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & i\sigma^j \\ -i\sigma^j & 0 \end{pmatrix}. \tag{5.4.3}$$

Moreover, we can define a matrix $\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4$ which satisfies $\gamma^5\gamma^j = -\gamma^j\gamma^5$, ${\gamma^5}^2 = 1$ and ${\gamma^5}^* = \gamma^5$; in Dirac representation:

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We recall, also, that using the Dirac matrices we can write down a basis for $M_4(\mathbb{C})$. In particular, if we define $\sigma^{ij} \equiv [\gamma^i, \gamma^j]$ (for $i, j = 1, \ldots, 4$), then the basis is given by:

$$\begin{aligned} &\mathrm{id}, \ \gamma^5, && \\ &\gamma^i && i = 1, \ldots, 4, \\ &\gamma^5\gamma^i && i = 1, \ldots, 4, \\ &\sigma^{ij} && i < j. \end{aligned} \tag{5.4.4}$$

A projectable spectral triple

Now, let $\mathcal{H}_\tau$ be the GNS Hilbert space associated to the canonical trace τ on $\mathcal{A}$ [GBFV]. Define $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^4$. We consider the usual flat Dirac operator [GBFV, Ven10]:

$$D = \sum_{j=1}^{4} \gamma^j \delta_j.$$

Then we can take the orientation $\mathbb{Z}_2$-grading to be $\gamma = \gamma^5$. To define J, we recall that it is related to the charge conjugation operator; so we take

$$J = J_0 \otimes (\gamma^4 \gamma^2 \circ c.c.),$$

where $J_0 : \mathcal{H}_\tau \to \mathcal{H}_\tau$ is the Tomita-Takesaki involution and *c.c.* denotes the complex conjugation. Then one can see that the spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ satisfies the relations (5.4.1), and it is also a $U(1)$-equivariant spectral triple. Moreover it is projectable:

Proposition 5.4.6. *The unique operators $\Gamma : \mathcal{H} \to \mathcal{H}$, such that $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta_4, \Gamma)$ is a projectable real spectral triple with equal length fibres, are $\Gamma = \pm \mathrm{id} \otimes \gamma^4$.*

Proof. Since $[\Gamma, \pi(a)] = [\Gamma, \delta] = 0$ for all $a \in \mathcal{A}$, Γ must be of the form $\Gamma = \mathrm{id} \otimes A$ for some matrix $A \in M_4(\mathbb{C})$. Then using the fact that (5.4.4) give a basis of $M_4(\mathbb{C})$, we can write Γ as

$$\Gamma = a + b\gamma^5 + \sum_j c_j \gamma^j + \sum_j d_j \gamma^5 \gamma^j + \sum_{i,j} e_{ij} \sigma^{ij}.$$

From $\Gamma\gamma = -\gamma\Gamma$ we deduce $a = b = e_{ij} = 0$. Thus we are left with

$$\Gamma = \sum_j (\alpha_j \gamma^j + \beta_j \gamma^5 \gamma^j), \qquad \alpha_j, \beta_j \in \mathbb{C},$$

where $\alpha_j \in \mathbb{R}$ and $\beta_j \in i\mathbb{R}$, as follows from the condition $\Gamma = \Gamma^*$. This implies that we can write Γ^2 as:

$$\Gamma^2 = \sum_j \alpha_j^2 + \sum_{i \neq j} 2\alpha_i \beta_j \gamma^5 \gamma^i \gamma^j - \sum_j \beta_j^2.$$

Next, using the condition $\Gamma^2 = -1$, we deduce:

$$\begin{cases} \alpha_i \beta_j = \alpha_j \beta_i & \forall i \neq j \\ \sum_j (\alpha_j^2 - \beta_j^2) = 1 \end{cases} \tag{5.4.5}$$

We have now to impose $[D, b] = [D_h, b]$, for all $b \in \mathcal{B}$ (see remark 5.2.4). Let us compute, first of all, $D_h = \frac{1}{2}\Gamma[D, \Gamma]$ (we use the Einstein convention for the sum over repeated indices):

$$D_h = \frac{1}{2}(\alpha_i \gamma^i + \beta_i \gamma^5 \gamma^i)[\gamma^j \delta_j, \alpha_k \gamma^k + \beta_k \gamma^5 \gamma^k]$$

$$
\begin{aligned}
&= \frac{1}{2}(\alpha_i\gamma^i + \beta_i\gamma^5\gamma^i)(\alpha_k\sigma^{jk}\delta_j - 2\beta_k\gamma^5\delta_k) \\
&= \alpha_j\alpha_k\gamma^k\delta_j - \alpha_j\alpha_j\gamma^k\delta_k + \varepsilon_{ijkl}\alpha_i\alpha_k\gamma^5\gamma^l\delta_j \\
&- \alpha_i\beta_k\gamma^i\gamma^5\delta_k + \varepsilon_{ijkl}\beta_i\alpha_k\gamma^l\delta_j + \beta_i\beta_k\gamma^i\delta_k.
\end{aligned} \tag{5.4.6}
$$

And now, from the condition $[D,b] = [D_h,b]$, using $[\delta_4, \mathcal{B}] = 0$ and the linear independence of the sixteen generators (5.4.4), we get:

$$
\begin{cases}
\varepsilon_{ijkl}\alpha_i\alpha_k\gamma^5\gamma^l\delta_j - \alpha_i\alpha_k\gamma^i\gamma^5\delta_k = 0 \\[2ex]
\sum_{j\neq k}(\alpha_j\alpha_k\gamma^k\delta_j + \beta_i\beta_k\gamma^j\delta_k) + \varepsilon_{ijkl}\beta_i\alpha_k\gamma^l\delta_j = 0 \\[2ex]
\sum_{j\neq k} -\alpha_j^2\gamma^k\delta_k + \sum_i \beta_i^2\gamma^i\delta_i = \sum_{j=1}^{3}\gamma^i\delta_i.
\end{cases} \tag{5.4.7}
$$

The last condition implies:

$$
\begin{cases}
\beta_4^2 = \sum_{j=1}^{3}\alpha_j^2 \\
\text{for } i \neq 4, \quad \sum_{j\neq i} -\alpha_j^2 + \beta_i^2 = 1.
\end{cases} \tag{5.4.8}
$$

If now we use (5.4.8) to compute $\sum_j \beta_j^2$ we get:

$$
\sum_j \beta_j^2 = \sum_{j=1}^{3}\alpha_j^2 + \sum_{i=1}^{3}\left(1 + \sum_{j\neq i}\alpha_j^2\right). \tag{5.4.9}
$$

Comparing (5.4.9) with the second equation of (5.4.5) we obtain the following relation: $\alpha_4^2 + \frac{1}{2}\beta_4^2 = 1$. Now, we know that $\alpha_j \in \mathbb{R}$ and $\beta_j \in i\mathbb{R}$ (therefore $b_j^2 \leq 0$). Thus, from this last relation we obtain $\alpha_4^2 \geq 1$, while from the second equation of (5.4.5) we get $\alpha_4^2 \leq 1$. So the only solutions are $\alpha_4 = \pm 1$, $\alpha_j = \beta_j = \beta_4 = 0$ for $j = 1,2,3$. It is easy to see that such solutions fulfil all the other conditions of (5.4.5), (5.4.7). We conclude that the unique solutions for Γ are $\Gamma = \pm\gamma^4$. Now we take one of the two solutions of the previous proposition, say $\Gamma = \gamma^4$. Then:

$$
D_h = \sum_{i=1}^{3}\gamma^i\delta_i, \qquad D_v = \gamma^4\delta_4,
$$

so that the spectral triple fulfils the constant length fibres condition, with, moreover, $Z = 0$. □

Now we can build the "3-dimensional orientation": $\nu = i\Gamma\gamma = i\gamma^5\gamma^4 = i\gamma^1\gamma^2\gamma^3$. We have $\nu^2 = 1$, $\nu^* = \nu$ as it should be. In 2×2 matrix notation ν is given by:

$$
\nu = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
$$

It is useful to write the down the action of D_h on each of the two eigenspaces of ν. Clearly it is enough to know the action of the matrices γ_j $(j = 1, 2, 3)$. Let us consider the 0-eigenspace $\mathcal{H}_0$ of δ_4, and decompose it accordingly to γ: $\mathcal{H}_0 = \mathcal{H}_0^+ \oplus \mathcal{H}_0^-$. Then any vector can be written as $v = v_+ \oplus v_-$. Moreover, since $\Gamma = \gamma_4$ is an intertwiner between $\mathcal{H}_0^\pm$, these two spaces are isomorphic. If $\mathcal{H}_0 = \mathcal{H}_0^{(+)} \oplus \mathcal{H}_0^{(-)}$ according to ν, then:

$$\begin{aligned} v \in \mathcal{H}_0^{(+)} &\Rightarrow v \text{ is of the form } v = w \oplus (-iw) \\ v \in \mathcal{H}_0^{(-)} &\Rightarrow v \text{ is of the form } v = w \oplus iw \end{aligned} \tag{5.4.10}$$

for some $w \in \mathcal{H}^+$. Using (5.4.3), we see that, for $j = 1, 2, 3$, γ^j acts as $\pm\sigma^j$ on $\mathcal{H}_0^{(\mp)}$. We summarize these results in the following lemma.

Lemma 5.4.7. *Each Hilbert space $\mathcal{H}_0^{(\pm)}$ is isomorphic to $\mathcal{H}_\tau \otimes \mathbb{C}^2$, where $\mathcal{H}_\tau$ is the GNS Hilbert space associated to the canonical tracial state on $\mathcal{B} = \mathcal{A}(\mathbb{T}^3_\theta)$. Moreover the matrices γ^j, $j = 1, 2, 3$, when restricted to $\mathcal{H}_0^{(\pm)}$, act as $\mp\sigma^j$.*

Thus both the spectral triples are isomorphic to the canonical one [DS13a, Ven10] (see also appendix A) on the noncommutative 3-torus, with Dirac operators

$$D_0^{(\pm)} = \mp \sum_{j=1}^{3} \sigma^j \delta_j.$$

We discuss now the real structure. Since J is antiunitary, we see that $[J, i\Gamma] = 0$. And, since $J\gamma = \gamma J$ and $J^2 = -1$ we can take $j_0^{(\pm)} = J$ (restricted to $\mathcal{H}_0^{(\pm)}$) and obtain that $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)}, j_0^{(\pm)})$ are real spectral triples of KR-dimension 3.

Twisted Dirac operators

Now we can proceed to the construction of twisted Dirac operators. First of all we need to characterize the strong $U(1)$ connections over $\mathcal{A}$. We have:

Lemma 5.4.8. *A $U(1)$ selfadjoint strong connection over $\mathcal{A}$ is a one-form*

$$\omega = \gamma^4 + \sum_{j=1}^{3} \gamma^j \omega_j,$$

where ω_j are selfadjoint elements of $\mathcal{B}$.

Proof. Let $\omega \in \Omega^1_D(\mathcal{A})$. Then ω has to be of the form $\omega = \gamma^j \omega_j$, since $[D, a] = \sum_j b_j \gamma^j c_j$ for any $a \in \mathcal{A}$. Moreover, if we impose $[\omega, \delta] = 0$, we get $\omega_j \in \mathcal{B}$ for all $j = 0, \ldots, 4$. Also, since the gamma matrices can be written as $\gamma^j = U_j^{-1}[D, U_j]$, we can write ω as:

$$\omega = \sum_j \omega_j U_j^{-1}[D, U_j].$$

And now, using condition (ii) of definition (4.3.5) we obtain $\omega_0 = 1$. Thus the most general form of a $U(1)$ strong connection over $\mathcal{A}$ is

$$\omega = \gamma^4 + \sum_{j=1}^{3} \gamma^j \omega_j, \qquad \omega_j \in \mathcal{B}.$$

Of course, ω is selfadjoint if and only if each ω_j is selfadjoint. □

As we have done in the two dimensional case, we can now compute the Dirac operator D_ω.

Proposition 5.4.9. *For any selfadjoint $U(1)$ strong connection ω, the associated Dirac operator D_ω has the form*

$$D_\omega = D_h - \sum_{j=1}^{3} \gamma^j J \omega_j J^{-1} \delta_4.$$

Proof. Take $h_0 \in \mathcal{H}_0^{(+)}$ and $a \in \mathcal{A}^{(k)}$ such that $h_0 a$ is in the domain of D_ω. Then, using lemma 5.4.7 we can compute the action of D_ω on $h_0 a$:

$$D_\omega^{(+)}(h_0 a) = (D_0^{(+)} h_0) a - h_0 \nabla_\omega(a) = -\sum_{j=1}^{3} \sigma^j \delta_j(a) + \sum_{j=1}^{3} k(\sigma_j h_0) a \omega_j.$$

Hence, for $h \in \mathcal{H}^{(+)}$ we get:

$$D_\omega^{(+)} = -\sum_{j=1}^{3} \gamma^j \delta_j(h) + \sum_{j=1}^{3} \sigma^j j_0 \omega_j j_0^{-1} \delta_4(h).$$

In the same way, for $h \in \mathcal{H}^{(-)}$ we have:

$$D_\omega^{(-)} = \sum_{j=1}^{3} \gamma^j \delta_j(h) - \sum_{j=1}^{3} \sigma^j j_0 \omega_j j_0^{-1} \delta_4(h).$$

Thus, if now we put all together using the results about gamma matrices' action of lemma 5.4.7, we obtain that, for $h \in \mathcal{H}$,

$$D_\omega(h) = \sum_{j=1}^{3} \gamma^j \delta_j(h) - \sum_{j=1}^{3} \gamma^j J \omega_j J^{-1} \delta_4(h),$$

where we have used the fact that, up to restriction to $\mathcal{H}_0$, $j_0 = J$. □

Corollary 5.4.10. *The only connection compatible with D is $\omega = \gamma^4$.*

Corollary 5.4.11. *Let $\omega = \gamma^4 + \sum_{j=1}^{3} \gamma^j \omega_j$ be a selfadjoint strong connection. Then the following Dirac operator,*

$$\mathcal{D}_\omega = D - \sum_{j=1}^{3} \gamma^j J \omega_j J^{-1} \delta_4,$$

is compatible with ω.

Proof. It follows from definition 5.3.19 together with proposition 5.4.9. □

5.4.3 $\mathbb{T}^3_\theta$ as quantum principal $\mathbb{T}^2$-bundle

Let $\mathcal{A}$ denote the algebra $\mathcal{A}(\mathbb{T}^3_\theta)$ of a noncommutative 3-torus: $\mathcal{A}$ is the polynomial algebra generated by three unitaries U_1, U_2, U_3, with the commutation relations $U_iU_j = e^{2\pi i\theta_{ij}}U_jU_i$. To the canonical action of $\mathbb{T}^3$ on $\mathcal{A}$ corresponds the following action of the generators of its Lie algebra:

$$\delta_i(U_j) = \delta_{ij}U_j.$$

Consider now the $\mathbb{T}^2$ action associated to the derivations δ_1 and δ_2. The invariant subalgebra $\mathcal{B}$ is simply the algebra generated by U_3, and hence it is isomorphic to a dense subalgebra of $C^\infty(S^1)$. As pointed out in appendix A, $\mathcal{B} \hookrightarrow \mathcal{A}$ is a cleft Hopf-Galois extension. In particular, $\mathcal{A}$ is a principal $\mathcal{O}(\mathbb{T}^2)$-comodule algebra.

A projectable spectral triple

Let us consider now the following spectral triple over $\mathcal{A}$. Let $\mathcal{H}_\tau$ denote the GNS Hilbert space associated to the canonical trace τ on $\mathcal{A}$ (cfr [GBFV] and appendix A of this thesis). Set $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^2$. Next, define a Dirac operator by:

$$D = \sum_{j=1}^{3} \sigma^j \delta_j,$$

where the σ^j are the Pauli matrices. The real structure J can be defined in the following way: if J_0 is the Tomita-Takesaki involution on $\mathcal{H}_\tau$ and *c.c.* denotes the complex conjugation[11] on $\mathbb{C}^2$, then we define

$$J = J_0 \otimes (i\sigma^2 \circ c.c.).$$

$(\mathcal{A}, \mathcal{H}, D, J)$ is an odd real spectral triple, of KR-dimension 3, on $\mathcal{A}$. It is straightforward to check that it is $\mathbb{T}^2$-equivariant[12]. Moreover, we know (cfr. appendix A) that the differential calculus $\Omega^1_D(\mathcal{A})$ is a $\mathcal{O}(\mathbb{T}^2)$-covariant calculus, and it makes $\mathcal{A}$ into a quantum principal $\mathbb{T}^2$-bundle. Now we can prove the following result.

Proposition 5.4.12. *$(\mathcal{A}, \mathcal{H}, D, J)$ is a projectable spectral triple, with isometric fibres. Moreover, the operator Γ can be taken equal to $\pm\sigma^3$.*

Proof. Take $\Gamma = \sigma^3$ (the proof is the same for $\Gamma = -\sigma^3$). Then $\Gamma^2 = 1$, $\Gamma^* = \Gamma$ and if commutes both with the representation of $\mathcal{A}$ and with the derivations δ_1, δ_2. Moreover, since $\sigma^2\sigma^3 = -\sigma^3\sigma^2$ and $J_0\sigma^3 = \sigma^3 J_0$, we have: $J\Gamma = -\Gamma J$. Hence Γ satisfies all the requirements of definition 5.3.4. It follows that $(\mathcal{A}, \mathcal{H}, D, J)$ is a projectable $\mathbb{T}^2$-equivariant spectral triple.

Now let us consider the differential calculus. The horizontal Dirac operator is given by:

$$D_h = \frac{1}{2}\Gamma[D, \Gamma]_+ = \sigma^3\delta_3.$$

[11] That is, if $\{e_1, e_2\}$ denotes the canonical basis of $\mathbb{C}^2$ then $c.c(\lambda e_j) = \overline{\lambda} e_j$ for any $\lambda \in \mathbb{C}$.
[12] Indeed, it is $\mathbb{T}^3$-equivariant, see Appendix A.

Take $b \in \mathcal{B}$. Since $\mathcal{B}$ is the invariant subalgebra of $\mathcal{A}$, we have $\delta_1(b) = \delta_2(b) = 0$. Therefore,

$$[D, b] = \sum_j \sigma^j \delta_j(b) = \sigma^3 \delta_3(b).$$

Since $D_h = \sigma^3 \delta_3$, and hence $[D_h, b] = \sigma^3 \delta_3(b)$, it follows that D and D_h generates the same bimodule of forms over $\mathcal{B}$ (cfr. remark 5.3.6). It is straightforward to see, then, that if $\Gamma = \pm\sigma^3$ the triple fulfils the isometric fibres condition. □

Now let $\mathcal{H}_0$ denote the common 0-eigenspace of δ_1 and δ_2. Accordingly to the results of the previous sections, we set $D_0 = D_h|_{\mathcal{H}_0}$ and $D_0' = \Gamma D_0$. Then the real structure j_0 is given by the restriction of ΓJ to $\mathcal{H}_0$. In particular, $j_0 = (J_0 \otimes (\sigma^1 \circ c.c.))|_{\mathcal{H}_0}$. Then $(\mathcal{B}, \mathcal{H}_0, D_0', j_0)$ is a real spectral triple of KR-dimension 1.

Twisted Dirac operators

Let us come back to the triple $(\mathcal{A}, \mathcal{H}, D, J)$. We have seen that it is a projectable $\mathbb{T}^2$-equivariant spectral triple. Now we observe that it has isometric fibres (cfr. definition 5.3.5). Indeed, let D_v be the operator

$$D_v = \sigma^1 \delta_1 + \sigma^2 \delta_2.$$

Then $D = D_v + D_h + Z$, with $Z = 0$, and D_v fulfils (a)-(c) of definition 5.3.5. So we can twist the horizontal Dirac operator D_h. We begin by working out a general form for strong connections over $\mathcal{A}$ (seen as a quantum principal $\mathbb{T}^2$-bundle, i.e. with differential calculus, in this case $\Omega^1_D(\mathcal{A})$, compatible with the de Rham calculus on $\mathcal{O}(\mathbb{T}^2)$).

Lemma 5.4.13. *Any selfadjoint strong connection over $\mathcal{A}$, in the sense of definition 4.6.17, is defined by two selfadjoint 1-forms $\omega^1, \omega^2 \in \Omega^1_D(\mathcal{A})$ such that:*

$$\omega^1 = \sigma^1 + \sigma^3 \omega^1_3,$$

$$\omega^2 = \sigma^2 + \sigma^3 \omega^2_3,$$

with $\omega^i_3 = (\omega^i_3)^* \in \mathcal{B}$.

Proof. Any 1-form $\eta \in \Omega^1_D(\mathcal{A})$ can be written in the following way: $\eta = \sum_{j=1}^{3} \sigma^j \eta_j$, with $\eta_j \in \mathcal{A}$. Hence we write:

$$\omega^i = \sum_{j=1}^{3} \sigma^j \omega^i_j,$$

with $\omega^i_j \in \mathcal{A}$. Imposing condition (i) of definition 4.6.17 we obtain that each ω^i_j has to belong to $\mathcal{B}$. Next we have to impose condition (ii). In order to do this we notice that each σ^j corresponds to the (universal) 1-form $U_j^{-1} dU_j$. Therefore condition (ii) implies that $\omega^i_j = \delta_{ij}$ (for $i, j = 1, 2$). Finally, all the ω^i_j must be selfadjoint, since we are requiring the strong connection to be selfadjoint. □

For $k \in \mathbb{Z}^2$, let now $\mathcal{A}^{(k)}$ denote the subalgebra of $\mathcal{A}$ of homogeneous elements of degree k. Then the connection ω allows us to define a D_0 connection on each $\mathcal{A}^{(k)}$:

$$\nabla_\omega : \mathcal{A}^{(k)} \to \Omega^1_D(\mathcal{A})\mathcal{A}^{(k)},$$

$$\nabla_\omega(a) = [D, a] - \sum_{i=1}^{2} k_i a \omega^i.$$

By direct computation we obtain then, for any $a \in \mathcal{A}^{(k)}$, that:

$$\nabla_\omega(a) = \sigma^3 \delta_3(a) - k_1 \sigma^3 a \omega^1_3 - k_2 \sigma^3 a \omega^2_3.$$

Before computing the twisted Dirac operator D_ω, we recall the following fact: the real structure we shall use here is $\widetilde{j} = \Gamma j_0 = J$ (see the proof of proposition 5.3.17). Then, from equation (7.3.35), we obtain:

$$D_\omega = \sigma^3 \delta_3 - \sigma^3 J \omega^1_3 J^{-1} \delta_1 - \sigma^3 J \omega^2_3 J^{-1} \delta_2.$$

A "full" – three-dimensional – Dirac operator $\mathcal{D}_\omega$ can be obtained simply adding D_v to D_ω.

Remark 5.4.14. The operator D_ω is the twist of the horizontal Dirac operator D_h; that is, of the operator D_0. The twist of the operator D'_0, instead, can be obtained simply multiplying D_ω by Γ.

Remark 5.4.15. The triple $(\mathcal{A}, \mathcal{H}, D_\omega)$ is, actually, a reducible spectral triple (indeed, if we split $\mathcal{H}$ according to σ^3, we obtain two - isomorphic - spectral triples). The reason for this is that the triple $(\mathcal{B}, \mathcal{H}_0, D_0)$ is reducible: indeed, it is the direct sum of two copies (with opposite orientation) of the canonical spectral triple over the circle S^1.

5.5 Projectable spectral triples, KK-theory and gauge theories: the case of the noncommutative 2-torus

In [Mes11] B. Mesland recovered the Kasparov product[13] from the tensor product of unbounded Kasparov bimodules introducing the notion of (smooth) connection over an unbounded Kasparov bimodule and using it to "twist" regular operators. Moreover, in a recent paper [BMS13], these results were used to formulate KK-theory based gauge theories over noncommutative spaces. In this section we shall show how it is possible to interpret our result at the light of Mesland's work. We shall discuss this relation in the concrete case of the noncommutative 2-torus, seen as a $U(1)$-bundle over the circle.

In the first part of this section we shall recall the results in [Mes11, BMS13]. Actually we will skip the main part of that paper, which is about smoothness of modules, operators and connections, and we will discuss here only the "algebraic" part of Mesland's construction. We underline the fact that most of the results that we discuss below actually need additional assumptions on the regularity of the objects involved to be true. As we said, we will not discuss

[13] Cfr. proposition 2.7.30.

these aspects, referring instead to [Mes11].

5.5.1 Lipschitz cycles and unbounded Kasparov products

In what follows all the algebras will be (possibly trivially) $\mathbb{Z}_2$-graded separable C^*-algebras and $\Omega^1 B$ will denote the universal differential calculus over the algebra B. Given a spectral triple $(B, \mathcal{H}, D)$ over a C^*-algebra B, we can define the *Lipschitz algebra* $\mathcal{B}$ to be the subalgebra

$$\mathcal{B} = \{b \in B \mid [D, b] \in \mathcal{L}(\mathcal{H})\}.$$

Then $\mathcal{E}$ is called a *Lipschitz module* if it is a right projective operator $\mathcal{B}$-module[14]. So a Lipschitz module will, in particular, be a pre-C^*-module, and its completion $\overline{\mathcal{E}}$ will be a right C^*-B-module. We give now the following definition [BMS13]:

Definition 5.5.1. *Let $\mathcal{A}, \mathcal{B}$ be Lipschitz algebras. A* Lipschitz $(\mathcal{A}, \mathcal{B})$-bimodule *is a projective operator right $\mathcal{B}$-module $\mathcal{E}$ together with a completely bounded *-homomorphism $\mathcal{A} \to \mathrm{End}^*_{\mathcal{B}}(\mathcal{E})$.*

Consider now an even unbounded (B, C) KK-cycle $(\mathcal{F}, T)$. Then the derivation $b \mapsto [T, \pi(b)]$ defines a first order differential calculus $\Omega^1_T(\mathcal{B})$. We suppose such a derivation to be completely bounded. Assume next to be given a Lipschitz $\mathcal{B}$-module $\mathcal{E}$ together with a connection $\nabla_T : \mathcal{E} \to \mathcal{E}\widetilde{\otimes}_{\mathcal{B}}\Omega^1_T(\mathcal{B})$. Here $\widetilde{\otimes}_{\mathcal{B}}$ denotes the Haagerup tensor product over $\mathcal{B}$, and by connection we mean a map fulfilling the Leibniz rule

$$\nabla(\xi \cdot b) = \xi \otimes db + (\nabla\xi) \cdot b.$$

Notice that ∇_T can be seen as coming from a universal connection; that is, a connection $\nabla : \mathcal{E} \to \mathcal{E}\widetilde{\otimes}_{\mathcal{B}}\Omega^1(B, \mathcal{B})$, where $\Omega^1(B, \mathcal{B})$ is simply the kernel of the multiplication map $m : B\widetilde{\otimes}\mathcal{B} \to B$. Now we can define an operator $\mathrm{id} \otimes_\nabla T : \mathcal{E}\widetilde{\otimes}_{\mathcal{B}}\mathcal{F} \to \mathcal{E}\widetilde{\otimes}_{\mathcal{B}}\mathcal{F}$ by:

$$(\mathrm{id} \otimes_\nabla T)(e \otimes f) = \gamma(e) \otimes Tf + \nabla(\gamma(e))f. \tag{5.5.1}$$

Here it is understood that we are working with $\mathbb{Z}_2$-graded modules, and γ is the grading operator on $\mathcal{E}$. $(\mathrm{id} \otimes_\nabla T)$ is a selfadjoint regular operator on $\mathcal{E}\widetilde{\otimes}_{\mathcal{B}}\mathcal{F}$ (see, e.g., [BMS13], theorem 2.25). We can consider now the following definition.

Definition 5.5.2. *A* Lipschitz cycle *between two spectral triples $(A, \mathcal{H}_1, D_1)$, $(B, \mathcal{H}_2, D_2)$ is a triple $(\mathcal{E}, S, \nabla)$ consisting of:*

(i) *a Lipschitz $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{E}$;*

(ii) *an odd regular selfadjoint operator S on $\mathcal{E}$ with compact resolvent, such that $a \mapsto [S, a] \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{E})$ is a completely bounded derivation;*

(iii) *an even, completely bounded, universal connection $\nabla : \mathcal{E} \to \mathcal{E}\widetilde{\otimes}_{\mathcal{B}}\Omega^1(B, \mathcal{B})$, such that $[\nabla, S] = 0$.*

[14] A projective operator $\mathcal{B}$-module is a right operator $\mathcal{B}$-module equipped with a completely bounded $\mathcal{B}$-valued inner product, such that $\mathcal{E}$ is completely boundedly unitarily isomorphic to $\mathrm{Im}(p)$, for some projection p on $\mathcal{H}_{\mathcal{B}}$, where $\mathcal{H}_{\mathcal{B}} = \ell^2(\mathbb{Z}) \otimes \mathcal{B}$ (Haagerup tensor product) [BMS13].

Given two Lipschitz algebras $\mathcal{A}, \mathcal{B}$, we denote by $\Psi_0^L(\mathcal{A}, \mathcal{B})$ the set of $(\mathcal{A}, \mathcal{B})$ Lipschitz cycles, up to unitary equivalence.

Now we can state the main result [Mes11, BMS13].

Theorem 5.5.3. *Let $(\mathcal{E}, S, \nabla)$ be a Lipschitz cycle for $(\mathcal{A}, \mathcal{B})$ and let $(\mathcal{F}, T)$ be a $(\mathcal{B}, C)$ KK-cycle. Then the pair*

$$(\mathcal{E}\widetilde{\otimes}_{\mathcal{B}}\mathcal{F}, S \otimes \mathrm{id} + \mathrm{id} \otimes_{\nabla} T)$$

is an $(\mathcal{A}, C)$ KK-cycle representing the Kasparov product of $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$.

Till now we have considered even KK-cycles. The extension to the odd case is obtained in the usual way: we define

$$\Psi_i(A, B) = \Psi_0(A, B \otimes \mathbb{C}\mathrm{l}_i),$$

where $\mathbb{C}\mathrm{l}_i$ is Clifford algebra in complex dimension i. The same applies to Lipschitz cycles.

5.5.2 KK-theory and gauge theories

In [BMS13] it was proposed that Lipschitz cycles can be used to define (noncommutative) gauge theories, at least those ones defined over a commutative base space. Given a spectral triple $(A, \mathcal{H}, D)$, defining a spin geometry over the (eventually noncommutative) space A, identified with the total space of a (noncommutative) principal bundle, the idea is to factorize it as a Kasparov product

$$(A, \mathcal{H}, D) \simeq (\mathcal{E}\widetilde{\otimes}_{\mathcal{B}}\mathcal{H}_0, S \otimes \mathrm{id} + \mathrm{id} \otimes_{\nabla} D_0) \in \Psi_0(\mathcal{A}, \mathbb{C}). \tag{5.5.2}$$

In order to do this one shall assume to own a way to "project" the spectral triple to a spectral triple $(B, \mathcal{H}_0, D_0)$ for a "base space" B. In such a picture the connection ∇ will play the role of a gauge field. The authors of [BMS13] proposed also a way to introduce scalar fields in a consistent way, but we shall not deal with them in this thesis, so we don't discuss them here.

The other aspect we are interested in, instead, is that of gauge transformations. Following [BMS13], we give the following definition.

Definition 5.5.4. *The* Lipschitz gauge group *associated to the factorization* (5.5.2) *is*

$$\mathcal{G}(\mathcal{E}) = \{U \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{E}) \mid UU^* = U^*U = \mathrm{id}_{\mathcal{E}},\ U\mathcal{A}U^* = \mathcal{A},\ [S, U] \in \mathrm{End}^*_{B}(\mathcal{E})\}.$$

Dropping the bounded commutator condition we obtain the group of continuous *gauge transformations $\mathcal{G}(\overline{E})$, which can be identified with the C^*-closure of $\mathcal{G}(\mathcal{E})$.*

The action of a Lipschitz gauge transformation $U \in \mathcal{G}(\mathcal{E})$ on a connection ∇ is defined as follows:

$$\nabla \mapsto \nabla^U = U\nabla U^*. \tag{5.5.3}$$

5.5.3 Twisted Dirac operators and gauge transformations for $\mathbb{T}^2_\theta$

Consider the cleft Hopf-Galois $\mathcal{O}(U(1))$-extension $\mathcal{B} \hookrightarrow \mathcal{A}$ where $\mathcal{A} = \mathcal{A}(\mathbb{T}^2_\theta)$ and $\mathcal{B}$, the invariant subalgebra, is isomorphic to (a dense subalgebra of) the algebra of smooth functions over the circle. We have seen, previously in this chapter, that the canonical spectral triple $(\mathcal{A}, \mathcal{H}_\theta, D, J, \gamma)$ is projectable, and so it is possible to twist the Dirac operator D, obtaining new Dirac operators D_ω, $\hat{D}_\omega = D_\omega + D_v$. We want now to study the behaviour of this construction under gauge transformations. In the next section, then, we will relate our results to the KK-theoretical approach introduced above.

Let us consider therefore the space of gauge transformations of the noncommutative 2-torus, seen as a quantum principal $U(1)$-bundle. Since the right adjoint coaction ad_R on $H = \mathcal{O}(U(1))$ is trivial, a gauge transformation $f \in \mathcal{G}(\mathcal{A})$ is simply a convolution invertible linear map $f : H \to \mathcal{B}$, with $f(1) = 1$. Consider now a strong connection form $\omega : H \to \Omega^1_D(\mathcal{A})$, defined by a strong $U(1)$-connection $\omega_0 \in \Omega^1_D(\mathcal{A})$. If we apply a gauge transformation f to ω, we obtain:

$$(f \triangleright \omega)(z^k) = f(z^k)\omega(z^k)f^{-1}(z^k) + f(z^k)df^{-1}(z^k). \tag{5.5.4}$$

Since f, f^{-1} take values in $\mathcal{B}$, and $\omega(z^k) = k\sigma^2 + \sigma^1\omega_1$ (with $\omega_1 \in \mathcal{B}$), then the commutativity of $\mathcal{B}$ allows us to rewrite (5.5.4) as:

$$(f \triangleright \omega)(z^k) = \omega(z^k) + f(z^k)df^{-1}(z^k). \tag{5.5.5}$$

It is now clear that $f \triangleright \omega$ is a strong connection, with respect to the calculus $\Omega^1_D(\mathcal{A})$, if and only if $f(z^k)df^{-1}(z^k) = -k\theta_f$, for some (fixed) 1-form $\theta_f \in \Omega^1_D(\mathcal{B})$. For this reason, we consider the following definition.

Definition 5.5.5. *The space of* differentiable $U(1)$-gauge transformations *is the space* $\mathcal{UG}(\mathcal{A})$ *of gauge transformations* $f \in \mathcal{G}(\mathcal{A})$ *such that:*

(i) $f^{-1}(h) = f(Sh^*)^*$ *for any* $h \in H$;

(ii) *there exists* $\theta_f \in \Omega^1_D(\mathcal{B})$ *such that* $f(z^k)[D, f^{-1}(z^k)] = -k\theta_f$, *for any* $k \in \mathbb{Z}$.

Remark 5.5.6. Condition (i) is equivalent to the requirement that each $f(z^k)$ is a unitary element, with $f^{-1}(z^k) = f(z^k)^*$.

It follows that each gauge transformation $f \in \mathcal{UG}(\mathcal{A})$ preserves the space of strong connections with respect to the calculus $\Omega^1_D(\mathcal{A})$.

Let us consider now a gauge transformation $f \in \mathcal{UG}(\mathcal{A})$. We know that $\omega_f \equiv f \triangleright \omega$ is still a strong connection, and that it can be written in the following way:

$$\omega_f(z^k) = k\omega_0 - k\theta_f.$$

It follows that the D_0-connection associated to ω_f will be:

$$\nabla^{\omega_f}(a) = [D, a] - ka\omega_0 + ka\theta_f.$$

for $a \in \mathcal{A}^{(k)}$. Then the Dirac operator D_{ω_f} shall be given by:

$$D_{\omega_f}(\psi a) = (D_0\psi)a + \psi([D,a] - \delta(a)\omega_0 + \delta(a)\theta_f).$$

That is,

$$D_{\omega_f}(\psi a) = D_\omega(\psi a) + \psi\delta(a)\theta_f. \tag{5.5.6}$$

Now we want to see if it is possible to describe the gauge transformation f in terms of a unitary operator on the Hilbert space $\mathcal{H}_\theta$. The Hilbert space $\mathcal{H}_\theta$ is isomorphic to $L^2(\mathbb{T}^2)\otimes\mathbb{C}^2$, so we can consider the canonical orthonormal basis $\{\psi_{k,l,j} \mid k,l \in \mathbb{Z},\ j=1,2\}$ of eigenvectors of the derivations δ_i (the index j is the "spinor" index associated to the $\mathbb{C}^2$ factor). So we can define an operator $V : \mathcal{H}_\theta \to \mathcal{H}_\theta$ in the following way:

$$V\psi_{k,l,j} = \pi^\circ(f(z^l))\psi_{k,l,j}. \tag{5.5.7}$$

Here $\pi^\circ(b) = J\pi(b)^*J^{-1}$ denotes the representation of $\mathcal{A}^\circ$ induced by the real structure J. Due to (i) of definition 5.5.5 (see also remark 5.5.6), V is a unitary operator, and its inverse is simply given by:

$$V^{-1}\psi_{k,l,j} = V^*\psi_{k,l,j} = \pi^\circ(f^{-1}(z^l))\psi_{k,l,j}.$$

Consider now its action on the twisted Dirac operator D_ω. For $\psi \in \mathcal{H}_0$ and $a \in \mathcal{A}^{(k)}$. we obtain:

$$\begin{aligned}
V^*D_\omega V(\psi a) &= VD_\omega^{(k)}V^*(\psi a) = VD_\omega^{(k)}(\psi a f(z^k)) \\
&= V\Big((D_0\psi)af(z^k) + \psi([D, af(z^k)] - kaf(z^k)\omega_0\Big) \\
&= (D_0\psi)a + \psi\Big([D, af(z^k)]f^{-1}(z^k) - kaf(z^k)\omega_0 f^{-1}(z^k)\Big) \\
&= (D_0\psi)a + \psi\Big([D,a] + a[D, f(z^k)]f^{-1}(z^k) - ka\omega_0\Big) \\
&= D_\omega(\psi a) + k\psi a\eta_f.
\end{aligned}$$

where we used the fact that $\mathcal{B}$ is commutative, that f, f^{-1} take values in $\mathcal{B}$ and that ω_0 is of the form $\sigma^1\omega_1 + \sigma^2$ with $\omega_1 \in \mathcal{B}$. Hence, for a generic vector of the form ψa, $a \in \mathcal{A}$, we have:

$$V^*D_\omega V(\psi a) = D_\omega(\psi a) + \psi\delta(a)\theta_f. \tag{5.5.8}$$

Comparing this result with equation (5.5.6) we see that we have obtained the following relation:

$$V^*D_\omega V = D_{\omega_f}. \tag{5.5.9}$$

This result extends, of course, to the Dirac operator $\hat{D}_\omega = D_v + D_\omega$.

The next step is to study the effect of the (adjoint) action of V on the representation π. Since $L^2(\mathbb{T}^2)$ is also the GNS-representation of $\mathcal{A}$ (which, we recall, is the C^*-completion of $\mathcal{A}$), each basic vector $\psi_{k,l,j}$ can be written as:

$$\psi_{k,l,j} = [U_1^kU_2^l]\otimes e_j,$$

$\{e_1, e_2\}$ being the canonical basis of $\mathbb{C}^2$ and $[a]$ denoting the GNS-equivalence class of $a \in A$. Now, we know that there is a vertical automorphism $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ associated to f, and that it is given by $\mathcal{F} = \mathrm{id} \otimes f$. Hence the gauge transformation can be seen to map the vector $\psi_{kl,l,i} = [U_1^k U_2^l] \otimes e_i$ into the vector $\psi'_{k,l,i} = [U_1^k U_2^l f(z^l)] \otimes e_i$, so that

$$\begin{aligned} V\pi(U_1^m U_2^n)V^*\psi'_{k,l,i} &= V\pi(U_1^m U_2^n)V^*[U_1^k U_2^l f(z^l)] \otimes e_i \\ &= V\pi(U_1^m U_2^n)[U_1^k U_2^l f(z^l) f^{-1}(z^l)] \otimes e_i \\ &= V[U_1^m U_2^n U_1^k U_2^l] \otimes e_i \\ &= [U_1^m U_2^n U_1^k U_2^l f(z^{l+n})] \otimes e_i = \lambda \psi'_{k+m,l+n,i}, \end{aligned}$$

where λ is the coefficient defined by:

$$\pi(U_1^m U_2^n)V^*\psi_{k,l,i} = \lambda \psi_{k+m,l+n,i}.$$

Hence V relates also the representation of $\mathcal{A}$ with its gauge-transformed counterpart. In particular, this means that, as sets of operators on $\mathcal{H}_\theta$, $V\mathcal{A}V^* = \mathcal{A}$.

5.5.4 Twisted Dirac operators, Kasparov products and $U(1)$ gauge theory

We have just introduced a way to implement gauge transformations, at least those belonging to the group $\mathcal{UG}(\mathcal{A})$, in the framework of twisted Dirac operators. Now we want to show that our "model" admits a KK-theoretical interpretation; in particular each twisted Dirac operator can be seen as arising from a Lipschitz cycle, and $\mathcal{UG}(\mathcal{A})$ can be identified with a subgroup of a group $\mathcal{G}(\mathcal{E})$ of Lipschitz gauge transformation. Our results extend the discussion of gauge theory over $\mathcal{A}(\mathbb{T}^2_\theta)$ in [BMS13].

Let A denote the C^*-algebra of functions over a noncommutative 2-torus $\mathbb{T}^2_\theta$, and let $\mathcal{A}$ be the subalgebra of smooth elements (with respect to the canonical $\mathbb{T}^2$ action). Then the canonical real spectral triple $(\mathcal{A}, \mathcal{H}_\theta, D, J, \gamma)$ gives a spectral triple (or K-cycle [C94, GBFV]) $(A, \mathcal{H}_\theta, D)$ for A. This gives us a Lipschitz algebra $\mathrm{Lip}(\mathbb{T}^2_\theta)$, which is the subalgebra of elements of A which have bounded commutator with D. The canonical $\mathbb{T}^2$ action preserves $\mathrm{Lip}(\mathbb{T}^2_\theta)$. Moreover, if we consider the action of one of the two $U(1)$ factors in $\mathbb{T}^2$ (in particular, we shall consider the action generated by δ_2), then fixed-point subalgebra of $\mathrm{Lip}(T^2_\theta)$ is $\mathrm{Lip}(S^1)$ [BMS13], that is the Lipschitz algebra associated to the spectral triple $(C(S^1), L^2(S^1), \partial\!\!\!/)$ (which is simply the algebra of Lipschitz functions on the circle).

In order to build a Lipschitz cycle we have, first of all, to choose a Lipschitz module $\mathcal{E}$. Following [BMS13], we choose it to be a suitable completion[15] of $\mathrm{Lip}(T^2_\theta)^\circ$, that is, of the opposite algebra of $\mathrm{Lip}(\mathbb{T}^2_\theta)$. $\mathcal{E}$ is naturally a left $\mathrm{Lip}(\mathbb{T}^2_\theta)^\circ$-module and a right $\mathrm{Lip}(S^1)^\circ$-module:

$$a^\circ \cdot \xi^\circ \cdot b^\circ = (b\xi a)^\circ,$$

for $a, \xi \in \mathrm{Lip}(\mathbb{T}^2_\theta)$, $b \in \mathrm{Lip}(S^1)$. In particular (see [BMS13], proposition 5.8) $\mathcal{E}$ is a right Lips-

[15] For the details see [BMS13].

chitz module over $\mathrm{Lip}(S^1)^\circ \simeq \mathrm{Lip}(S^1)$, isomorphic to $L^2(S^1)\widetilde{\otimes}\mathrm{Lip}(S^1)$; and, if we denote by $\overline{\mathcal{E}}$ the C^*-completion of $\mathcal{E}$, then the C^*-algebra A° is represented upon $\overline{\mathcal{E}}$ by a *-homomorphism. As shown in [BMS13], then, if we denote by ∇ the canonical Grassmann connection, $\nabla : \mathcal{E} \to \mathcal{E}\widetilde{\otimes}_{\mathrm{Lip}(S^1)}\Omega(C(S^1), \mathrm{Lip}(S^1))$, and $(\mathcal{E}, S = \delta_2, \nabla)$ defines a Lipschitz cycle in $\Psi^l_{-1}(\mathrm{Lip}(\mathbb{T}^2_\theta)^\circ, \mathrm{Lip}(S^1))$. Moreover the Hilbert space $\mathcal{H}_\theta$ is isomorphic to the completion $\mathcal{E}\widetilde{\otimes}_{\mathrm{Lip}(S^1)}L^2(S^1)$ of the tensor product $\overline{\mathcal{E}} \otimes_{C(S^1)} L^2(S^1)$ (see [BMS13], proposition 5.10). On the other hand there is the KK-cycle $(L^2(S_1), \partial\!\!\!/)$, which is an element of $\Psi_0(C(S^1), \mathbb{C})$. In [BMS13] it was shown that $(A^\circ, \mathcal{H}, D)$ is equivalent, as a KK-cycle, to the product of the two cycles above (taking A° instead of A is simply a technical issue). Now we want to consider a more general situation, related to the construction of twisted Dirac operators. That is, we want to consider more general connections on $\mathcal{A}$; in particular, we shall consider those coming from a strong connection ω. Before beginning this task, we notice that $\partial\!\!\!/$ corresponds to the derivation δ_1, and so the calculus $\Omega^1_{\partial\!\!\!/}(\mathrm{Lip}(S^1))$ is the same as the calculus $\Omega^1_{D_0}(\mathrm{Lip}(S^1))$, where D_0 is the restriction to $\mathcal{H}_0$ (i.e. to the 0-eigenspace of δ_2) of the horizontal Dirac operator D_h; indeed, D_h can be written as $\sigma^1\delta_1 \sim \sigma^1\partial\!\!\!/$.

Consider then a strong connection on $\mathcal{A}$ (with respect to the calculus $\Omega^1_D(\mathcal{A})$). We know that it determines a D_0-connection $\nabla^\omega : \mathcal{A} \to \Omega^1_{D_0}(\mathcal{B})\mathcal{A}$. Now we can extend it to a connection on $\mathrm{Lip}(\mathbb{T}^2_\theta)$, since the only requirement of regularity it needs to be well defined is to act on elements of A having bounded commutator with D (which is exactly the Lipschitz condition). Moreover, seeing it as a map acting on the opposite algebra $\mathrm{Lip}(\mathbb{T}^2_\theta)^\circ$, it becomes a map

$$\nabla^\omega : \mathrm{Lip}(\mathbb{T}^2_\theta)^\circ \to \mathrm{Lip}(\mathbb{T}^2_\theta)^\circ\Omega^1_{D_0}(\mathrm{Lip}(S^1)^\circ) \simeq \mathrm{Lip}(\mathbb{T}^2_\theta)^\circ\Omega^1_{\partial\!\!\!/}(\mathrm{Lip}(S^1)).$$

By continuity with respect to the Lipschitz topology (cfr. [BMS13]), then, it can be seen as a map

$$\nabla^\omega : \mathcal{E} \to \mathcal{E}\widetilde{\otimes}_{\mathrm{Lip}(S^1)}\Omega^1_{\partial\!\!\!/}(\mathrm{Lip}(S^1)).$$

Since $\omega(z^k) = k(\sigma^2 + \sigma^1\omega_1)$, with $\omega_1 \in \mathcal{B} \simeq C^\infty(S^1)$ and $S = \delta_2$ is zero on $\mathcal{B}$, then ∇^ω commutes with S. It follows that $(\mathcal{E}, S, \nabla^\omega)$ is a Lipschitz cycle. So we can form the operator

$$S \otimes \mathrm{id} + \mathrm{id} \otimes_{\nabla^\omega} \partial\!\!\!/.$$

Performing the computation as in example 2.35 in [BMS13], we find that this operator coincides with the twisted Dirac operator $\hat{D}_\omega$. So we have shown that the spin geometry of $\mathbb{T}^2_\theta$ defined by the twisted Dirac operator $\hat{D}_\omega$ factorizes as a Kasparov product:

$$(\mathcal{H}_\theta, \hat{D}_\omega) \simeq (\mathcal{E}, S, \nabla^\omega) \otimes_{\mathrm{Lip}(S^1)} (\mathcal{H}, \partial\!\!\!/).$$

Next we look at gauge transformations. We have seen that a gauge transformation $f \in \mathcal{UG}(\mathcal{A})$ is implemented in the spectral triple $(A, \mathcal{H}_\theta, D_\omega)$ by a unitary operator V (see equation 5.5.7). Since V acts trivially on the $\mathbb{C}^2$ factor of $\mathcal{H}_\theta$, and since $\mathcal{H}_\tau$ is a faithful GNS representation of the C^*-algebra A, V can be seen as a map from A to A – and hence from A° to A° – and then restricted to a map $V : \mathcal{E} \to \mathcal{E}$ (requirement (ii) of definition 5.5.5 ensures that V maps Lipschitz elements into Lipschitz elements). Of course, it will still be a unitary operator. Moreover we have noticed that it maps $\mathcal{A}$ into $\mathcal{A}$, and by continuity the same holds for $\mathrm{Lip}(\mathbb{T}^2_\theta)^\circ$. Hence V can be seen as an element of $\mathcal{G}(\mathcal{E})$. To complete the connection between the two approaches we have to see if the connection ∇^ω transforms as a gauge field, in the sense discussed in [BMS13] (see also equation (5.5.3)). By direct computation we obtain, for $a \in \mathcal{A}^{(k)}$,

$$\begin{aligned} V^*\nabla^\omega V(a) &= V^*\nabla^\omega(af(z^k)) = V^*(da \cdot f(z^k) + adf(z^k) - kaf(z^k)\omega_0) \\ &= (da + adf(z^k) cdot f^{-1}(z^k) - ka\omega_0) = da - ka\omega_0 + ka\theta_f = \nabla^{\omega_f} a. \end{aligned}$$

So acting with V on ∇^ω corresponds to transforming the strong connection ω by f.

Let us summarize the results we discussed in this section. We have seen how each strong connection over the noncommutative 2-torus, seen as a quantum principal $U(1)$-bundles, defines a connection over the Lipschitz module associated to the KK-factorization [BMS13]. This allows us, then, to identify the twisted Dirac operator [DS13a] defined by the strong connection with the operator arising from the unbounded Kasparov product construction associated to the KK-factorization. In this way we give a geometrical interpretation, in terms of connections over a (noncommutative) principal $U(1)$-bundle, to the connections appearing in Mesland's construction [Mes11, BMS13].

Next, we considered gauge transformations. We identified a class of $U(1)$ gauge transformations of $\mathbb{T}^2_\theta$ associated to its quantum principal bundle structure. Then we showed how it is possible to implements each of these transformations in the spectral triple defined by the twisted Dirac operator via a unitary operator, and we noticed that this operator defines a gauge transformation in the sense discussed in [BMS13]. Since none of the gauge transformations in $\mathcal{UG}(\mathcal{A})$ is inner (that is, none of them is given by the adjoint action of an element of $\mathcal{A}$) our construction provides a large class of gauge transformations which do not fit into the description of gauge theory in terms of inner fluctuations of the Dirac operator. In particular, we notice that the action of the Pontrjagin dual group $\mathbb{Z}$ of $\mathbb{T}$ through the bounded Dirac operators $e^{2\pi i n\theta\delta_1}$, $n \in \mathbb{Z}$, considered in [BMS13] corresponds to the set of gauge transformations $f_n(z^k) = U^{nk}$.

CHAPTER 6

Spectral triples over cleft principal $\mathcal{O}(U(1))$-extensions

In this chapter we shall discuss a first (simple) example of construction of spectral triples over cleft principal extensions: given a cleft $\mathcal{O}(U(1))$-extension $\mathcal{B} \hookrightarrow \mathcal{A}$ and a real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$, we will exploit the construction by Bellissard, Marcolli and Reihani [BMR10] to build, under suitable hypotheses, a real spectral triple over the algebra $\mathcal{A}$. We will see that the Dirac operator of this triple allows us to define a differential calculus over $\mathcal{A}$ which is compatible with the de Rham calculus on $\mathcal{O}(U(1))$; hence we obtain a structure of quantum principal bundle, with differential calculus compatible with the de Rham calculus on $\mathcal{O}(U(1))$, over $\mathcal{A}$. We will then discuss the properties of such a triple. In particular we will show that it is a projectable spectral triple, and so we will be able to twist it using a strong connection (see chapter 5). Finally, we will study the behaviour of our construction under gauge transformations.

In this chapter $\mathcal{B}$ will denote a unital dense sub-*-algebra of a C^*-algebra B, H will denote the Hopf algebra $\mathcal{O}(U(1))$ (see chapter 2) and we will assume that any real spectral triple we shall consider fulfils at least the first order and the regularity condition.

6.1 Spectral triples over $\mathcal{B} \rtimes_\alpha \mathbb{Z}$

We begin by constructing a real spectral triple over a crossed product algebra $\mathcal{B} \rtimes_\alpha \mathbb{Z}$, extending the results in [BMR10]. Consider a real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ over the pre-C^*-algebra $\mathcal{B}$ such that $(B, \mathcal{H}, D)$ is a compact spectral metric space (see section 3.6). Let $\alpha \in \mathrm{Aut}(\mathcal{B})$ be an automorphism of the algebra $\mathcal{B}$; assume that it is a continuous automorphism, so that it extends to the C^*-completion B. Then α is an isometry of B.

The automorphism α induces an action of $\mathbb{Z}$ both on $\mathcal{B}$ and B, so we can consider the *crossed product algebra* $\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}$ which is the polynomial algebra generated[1] by the elements of $\mathcal{B}$ together with a unitary u, under the relation $ubu^{-1} = \alpha(b)$ for each $b \in \mathcal{B}$, so that u implements the action of α.

[1]Here we mean that the elements of $\mathcal{A}$ are finite sums of monomials bu^k, with $b \in \mathcal{B}$ and $k \in \mathbb{Z}$.

Under the hypotheses above we can construct a spectral triple over $\mathcal{A}$ as follows (see [BMR10] for the details). Let $\mathcal{H}' = \mathcal{H} \otimes \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$; an element $f \in \mathcal{H}'$ can be written as $f = (f_n)_{n\in\mathbb{Z}}$ with $f_n \in \mathcal{H} \otimes \mathbb{C}^2$. The representation of $\mathcal{A}$ on $\mathcal{H}'$ we consider is the *left regular representation* $\hat{\pi} : \mathcal{A} \to \mathcal{L}(\mathcal{H}')$:

$$(\hat{\pi}(a)f)_n = \pi(\alpha^{-n}(a))f_n, \qquad (\hat{u}f)_n \equiv (\hat{\pi}(u)f)_n = f_{n-1}, \tag{6.1.1}$$

where we have extended, in the natural way, the representation π of $\mathcal{B}$ on $\mathcal{H}$ to a representation, still denoted by π, of $\mathcal{B}$ on $\mathcal{H} \otimes \mathbb{C}^2$ (in the following, where there will be no possible misunderstanding, the representation π will be understood, so that the action of $b \in \mathcal{B}$ on $f \in \mathcal{H} \otimes \mathbb{C}^2$ will be simply denoted by bf).

Notice now that $\hat{u}$ is a unitary operator which satisfies $\hat{u}\hat{\pi}(b)\hat{u}^{-1} = (\hat{\pi} \circ \alpha)(b)$, for any $b \in \mathcal{B}$, so that $\hat{\pi}$ is a well-defined representation of the crossed product $\mathcal{B} \rtimes_\alpha \mathbb{Z}$: $\hat{\pi}(\sum_n a_n u^n) = \sum_n \hat{\pi}(a)\hat{u}^n$. The Dirac operator is defined as follows:

$$(\hat{D}f)_n = (D \otimes \sigma^1 + n \cdot \mathrm{id}_{\mathcal{H}} \otimes \sigma^2)f_n. \tag{6.1.2}$$

Proposition 6.1.1. *$(\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}, \mathcal{H}', \hat{D})$ is a spectral triple. Moreover $\hat{u}^{-1}[\hat{D}, \hat{u}]$ commutes with the elements of $\mathcal{A}$.*

Proof. For the first part see [BMR10], section 3.4. The second one follows by direct computation; indeed:

$$\hat{u}^{-1}[\hat{D}, \hat{u}] = \mathrm{id}_{\mathcal{H}} \otimes \mathrm{id}_{\ell^2(\mathbb{Z})} \otimes \sigma^2.$$

□

We want to discuss and extend this result. The first thing we do is to make $(\mathcal{A}, \mathcal{H}', \hat{D})$ into a real spectral triple. In order to achieve this result we need to impose some additional conditions on the triple $(\mathcal{B}, \mathcal{H}, D)$. In particular we need a suitable action of $\mathbb{Z}$ on $\mathcal{H}$, which extends the action generated by α on $\mathcal{B}$.

Definition 6.1.2. *We say that an automorphism $\alpha \in \mathrm{Aut}(\mathcal{B})$ is* implementable *w.r.t. a real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ if there is an invertible bounded operator $\rho : \mathcal{H} \to \mathcal{H}$ such that:*
(i) *$\rho(a\xi) = \alpha(a)\rho(\xi)$ for any $a \in \mathcal{B}$, $\xi \in \mathcal{H}$;*
(ii) *$\langle \rho(\xi), \eta \rangle = \langle \xi, \rho^{-1}(\eta) \rangle$ for any $\xi, \eta \in \mathcal{H}$;*
(iii) *$\rho(J\xi) = J\rho(\xi)$ for any $\xi \in \mathcal{H}$;*
(iv) *$\rho(D\xi) = D\rho(\xi)$ for any $\xi \in \mathcal{H}$;*
(v) *$\rho(\gamma\xi) = \gamma\rho(\xi)$ for any $\xi \in \mathcal{H}$.*

Remark 6.1.3. Given ρ as above, the assignment $k \mapsto \rho^k$ determines an action of $\mathbb{Z}$ on $\mathcal{H}$. Moreover, definition 6.1.2 implies that the spectral triple $(\mathcal{H}, \mathcal{B}, D, J, \gamma)$ is $\mathbb{C}[\mathbb{Z}]$-equivariant, in the sense of definition 3.5.1 and definition 3.5.2 (with the further property that the action of $\mathbb{C}[\mathbb{Z}]$ on $\mathcal{H}$ is a *-action). More precisely, let $\{\underline{k}\}$ be the canonical basis of $\mathbb{C}[\mathbb{Z}]$, for $k \in \mathbb{Z}$; the Hopf *-algebra structure of $\mathbb{C}[\mathbb{Z}]$, then, is described by the following relations:

$$\underline{k} \cdot \underline{l} = \underline{k+l}, \qquad \underline{k}^* = \underline{-k},$$

$$\Delta(\underline{k}) = \underline{k} \otimes \underline{k}, \qquad S(\underline{k}) = \underline{-k}, \qquad \varepsilon(\underline{k}) = 1.$$

We consider the action of $\mathbb{C}[\mathbb{Z}]$ on $\mathcal{B}$ defined by $\underline{k} \triangleright b = \alpha^k(b)$, and the corresponding action on $\mathcal{H}$, $\underline{k} \triangleright \xi = \rho^k(\xi)$. Then condition (i) of definition 6.1.2 implies that $\mathcal{H}$ is a $\mathbb{C}[\mathbb{Z}]$-equivariant left $\mathcal{B}$-module: $h \triangleright (b\xi) = (h_{(1)} \triangleright b)(h_{(2)} \triangleright \xi)$ for any $h \in \mathbb{C}[\mathbb{Z}]$, any $b \in \mathcal{B}$ and any $\xi \iota \mathcal{H}$. Condition (iii) is equivalent to the requirement that $JhJ^{-1} = (Sh)^*$ for any $h \in \mathbb{C}[\mathbb{Z}]$. Conditions (iv) and (v), finally, imply that $[h, D] = [h, \gamma] = 0$ for any $h \in \mathbb{C}[\mathbb{Z}]$.

Remark 6.1.4. Conditions (i) and (ii) above are satisfied for istance if $\mathcal{H}$ is obtained from a GNS construction over (the C^*-completion of) $\mathcal{B}$. The other ones say simply that the metric structure on the noncommutative space associated to $\mathcal{B}$ must be invariant under the action of α. In particular, condition (i) is the usual condition of implementability for an automorphism and (ii) is nothing else than the requirement that ρ is unitary.

Assume now that the automorphism α is implementable w.r.t. the real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$. We have to define a real structure, and, in the even dimensional case (i.e. when the triple over $\mathcal{B}$ is odd dimensional) also a $\mathbb{Z}_2$ grading, for the spectral triple $(\mathcal{A}, \mathcal{H}', \hat{D})$. Before doing this we introduce a "building block" for the construction of the real structure: we define an operator $\widetilde{J} : \mathcal{H}' \to \mathcal{H}'$ by

$$(\widetilde{J} f)_n = J\rho^{-n}(f_{-n}). \tag{6.1.3}$$

Lemma 6.1.5. *The operator $\widetilde{J}$ defined in* (6.1.3) *is an antiunitary operator. Moreover:*
(i) *$\widetilde{J}^2 = \varepsilon \mathrm{id}_{\mathcal{H}'}$, where $J^2 = \varepsilon \mathrm{id}_{\mathcal{H}}$;*
(ii) *$\widetilde{J}$ maps $\hat{\pi}(\mathcal{A}) = \hat{\pi}(\mathcal{B} \rtimes_\alpha \mathbb{Z})$ into its commutant;*
(iii) *$\widetilde{J} \circ (D \otimes \mathrm{id} \otimes \mathrm{id}) = \varepsilon'(D \otimes \mathrm{id} \otimes \mathrm{id}) \circ \widetilde{J}$, where $JD = \varepsilon' DJ$;*
(iv) *if $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ is an even dimensional triple, then $\widetilde{J} \circ (\gamma \otimes \mathrm{id} \otimes \mathrm{id}) = \varepsilon''(\gamma \otimes \mathrm{id} \otimes \mathrm{id}) \circ \widetilde{J}$, where $J\gamma = \varepsilon'' \gamma J$.*

Proof. (i), (iii) and (iv) follow by direct computation using properties (iii), (iv) and (v) of definition 6.1.2. So we are left with the proof of (ii) and of the antiunitarity of $\widetilde{J}$. Let us start from the latter. We have:

$$\begin{aligned}
\left\langle f, \widetilde{J} g \right\rangle_{\mathcal{H}'} &= \sum_n \left\langle f_n, (\widetilde{J} g)_n \right\rangle = \sum_n \left\langle f_n, J\rho^{-n}(g_{-n}) \right\rangle \\
&= \sum_n \left\langle \rho^{-n}(g_{-n}), J f_n \right\rangle = \sum_n \left\langle \rho^n(g_n), J f_{-n} \right\rangle \\
&= \sum_n \left\langle g_n, \rho^{-n}(J f_{-n}) \right\rangle = \sum_n \left\langle g_n, J\rho^{-n}(f_{-n}) \right\rangle = \left\langle g, \widetilde{J} f \right\rangle_{\mathcal{H}'},
\end{aligned}$$

where we used properties (ii) and (iii) of definition 6.1.2 (and where $\langle \cdot, \cdot \rangle$ denotes the natural scalar product in $\mathcal{H} \otimes \mathbb{C}^2$). From this computation it follows easily that J is an antiunitary operator.

Now we show that $\widetilde{J}$ maps $\hat{\pi}(\mathcal{A})$ into its commutant. It is useful to compute the following expressions:

$$\begin{aligned}
(\hat{J}\hat{\pi}(a)\hat{J}^{-1} f)_n &= J\rho^{-n}(\hat{\pi}(a)\hat{J}^{-1} f)_{-n} = J\rho^{-n}(\alpha^n(a)(\hat{J}^{-1} f)_{-n}) \\
&= J\rho^{-n}(\alpha^n(a)\rho^n(J^{-1} f_n)) = JaJ^{-1} f_n,
\end{aligned} \tag{6.1.4}$$

$$\begin{aligned}(\hat{J}\hat{u}\hat{J}^{-1}f)_n &= J\rho^{-n}(\hat{u}\hat{J}^{-1}f)_{-n} = J\rho^{-n}(\hat{J}^{-1}f)_{-n-1} \\ &= J\rho^{-n}\rho^{n+1}(J^{-1}f_{n+1}) = \rho(f_{n+1}).\end{aligned} \tag{6.1.5}$$

From (6.1.4), since J maps $\mathcal{B}$ into its commutant, it is clear that $[\hat{\pi}(a), \widetilde{J}\hat{\pi}(b)\widetilde{J}^{-1}] = 0$ for any $a, b \in \mathcal{B}$. Also, it is easy to see that (6.1.5) implies $[\hat{u}^{\pm 1}, \widetilde{J}\hat{u}\widetilde{J}] = 0$. Thus we need only to check that, for any $a \in \mathcal{B}$, $[\hat{\pi}(a), \widetilde{J}\hat{u}\widetilde{J}^{-1}] = 0$. And we have:

$$\begin{aligned}([\hat{\pi}(a), \widetilde{J}\hat{u}\widetilde{J}^{-1}]f)_n &= \alpha^{-n}(a)(\widetilde{J}\hat{u}\widetilde{J}^{-1}f)_n - (\widetilde{J}\hat{u}\widetilde{J}^{-1}(\hat{\pi}(a)f))_n \\ &= \alpha^{-n}(a)\rho(f_{n+1}) - \rho((\hat{\pi}(a)f)_{n+1}) \\ &= \alpha^{-n}(a)\rho(f_{n+1}) - \rho(\alpha^{-n-1}(a)f_{n+1}) = 0.\end{aligned}$$

□

We proceed then considering the odd and the even dimensional case separately.

Odd dimensional case. Assume that $(\mathcal{B}, \mathcal{H}, D, J)$ is an odd real spectral triple of KR-dimension j and that α is an implementable automorphism. Let $(\mathcal{H}', \hat{\pi})$ denote the left regular representation of $\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}$ introduced above.

Definition 6.1.6. *Let $\widetilde{J}$ be given by (6.1.3). We define an operator $\hat{J} : \mathcal{H}' \to \mathcal{H}'$ by:*

$$\hat{J} = \begin{cases} (\mathrm{id} \otimes \sigma^2) \circ \widetilde{J} & \text{if } j \equiv 1 \pmod 4 \\ \widetilde{J} & \text{if } j \equiv 3 \pmod 4 \end{cases} \tag{6.1.6}$$

where id *denotes the identity operator on $\mathcal{H} \otimes \ell^2(\mathbb{Z})$.*

Proposition 6.1.7. *Let $\hat{\gamma} = \mathrm{id}_{\mathcal{H}} \otimes \mathrm{id}_{\ell^2(\mathbb{Z})} \otimes \sigma^3$. Then $(\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}, \mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma})$ is a real spectral triple of KR-dimension $j+1$.*

Proof. For the analytic properties of the Dirac operator we refer to [BMR10]. Also, the commutation relations between $\hat{D}$, $\hat{J}$, $\hat{\gamma}$ can easily be checked by direct computation (notice that $\sigma^2 J = -J\sigma^2$). The fact that $\hat{J}$ maps $\mathcal{A}$ into its commutant follows from lemma 6.1.5. So, the only thing that we have to check here is the first order condition. First of all we notice that, if $a, b \in \mathcal{B}$, then $[[\hat{D}, \hat{\pi}(a)], \hat{J}\hat{\pi}(b)\hat{J}^{-1}] = 0$ due to (6.1.4); indeed,

$$\begin{aligned}([[\hat{D}, \hat{\pi}(a)], \widetilde{J}\widetilde{b}\widetilde{J}^{-1}]f)_n &= \sigma^1[D, \alpha^{-n}(a)](\widetilde{J}\hat{\pi}(b)\widetilde{J}^{-1}f)_n - JbJ^{-1}([\hat{D}, \hat{\pi}(a)]f)_n \\ &= \sigma^1[[D, \alpha^{-n}(a)], JbJ^{-1}]f_n = 0\end{aligned}$$

since the first order condition holds for the triple $(\mathcal{B}, \mathcal{H}, D, J)$. Also, since $[\hat{D}, \hat{u}] = \hat{u}\sigma^2$, $[\hat{D}, \hat{u}]$ commutes with $\hat{J}\hat{u}^{\pm 1}\hat{J}^{-1}$ (see (6.1.5)). The following two computations conclude the proof of the first order condition. Let $a \in \mathcal{B}$; then:

$$\begin{aligned}([[\hat{D}, \hat{\pi}(a)], \hat{J}\hat{u}\hat{J}^{-1}]f)_n &= ([\hat{D}, \hat{\pi}(a)]\hat{J}\hat{u}\hat{J}^{-1}f)_n - (\hat{J}\hat{u}\hat{J}^{-1}[\hat{D}, \hat{\pi}(a)]f)_n \\ &= \sigma^1[D, \alpha^{-n}(a)](\hat{J}\hat{u}\hat{J}^{-1}f)_n - \rho([\hat{D}, \hat{\pi}(a)]f)_{n+1} \\ &= \sigma^1[D, \alpha^{-n}(a)]\rho(f_{n+1}) - \sigma^1\rho([D, \alpha^{-n-1}(a)]f_{n+1}) = 0,\end{aligned}$$

$$
\begin{aligned}
([[\hat{D}, \hat{u}], \hat{J}\hat{\pi}(a)\hat{J}^{-1}]f)_n &= (\hat{u}\sigma^2 \hat{J}\hat{\pi}(a)\hat{J}^{-1}f)_n - JaJ^{-1}(\hat{u}\sigma^2 f)_n \\
&= \sigma^2(\hat{J}\hat{\pi}(a)\hat{J}^{-1}f)_{n-1} - JaJ^{-1}\sigma^2 f_{n-1} \\
&= \sigma^2 JaJ^{-1} f_{n-1} - JaJ^{-1}\sigma^2 f_{n-1} = 0,
\end{aligned}
$$

(we used repeatedly (6.1.4) and (6.1.5)). □

Even dimensional case. Assume that $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ is an even real spectral triple of KR-dimension j and that α is an implementable automorphism. Let also $(\mathcal{H}', \hat{\pi})$ be the left regular representation of $\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}$ introduced above.

Definition 6.1.8. *Let $\widetilde{J}$ be given by* (6.1.3)*. We define an operator $\hat{J} : \mathcal{H}' \to \mathcal{H}'$ by:*

$$
\hat{J} = \begin{cases} (\gamma \otimes \mathrm{id} \otimes \sigma^1) \circ \widetilde{J} & \text{if } j \equiv 0 \pmod 4 \\ \\ \widetilde{J} & \text{if } j \equiv 2 \pmod 4 \end{cases} \tag{6.1.7}
$$

where id *denotes the identity operator on $\ell^2(\mathbb{Z})$.*

Proposition 6.1.9. *Let $\hat{J}$ be defined by* (6.1.7)*. Then $\hat{D}$ and $\hat{J}$ fulfil the commutation relations of a real spectral triple of KR-dimension $j+1$.*

Proof. It follows by direct computation. □

Now let $\nu = \gamma \otimes \mathrm{id}_{\ell^2(\mathbb{Z})} \otimes \sigma^2$. Then $\nu^* = \nu$, $\nu^2 = 1$ and thus $\mathcal{H}'$ decomposes as $\mathcal{H}' = \mathcal{H}'_+ \oplus \mathcal{H}'_-$. Also, it is easy to see that $[\nu, \hat{D}] = [\nu, \hat{J}] = 0$ (where $\hat{J}$ is defined by (6.1.7)). And, of course, ν commutes with the representation $\hat{\pi}$ of $\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}$. If we denote, respectively, by $\hat{D}_\pm$, $\hat{J}_\pm$ the restrictions of the two operators to $\mathcal{H}'_\pm$, then, using the previous results, we get:

Proposition 6.1.10. *Both $(\mathcal{A}, \mathcal{H}'_\pm, \hat{D}_\pm, \hat{J}_\pm)$ are (odd) real spectral triples of KR-dimension $j+1$. Moreover they differ just by a change of sign of the orientation*[2]*.*

Now let us make some observation about the properties of the spectral triples constructed above, both in the odd and in the even dimensional case.

Definition 6.1.11. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be* irreducible *if there is no closed subspace of $\mathcal{H}$ invariant under the action of the operator algebra generated by $a \in \mathcal{A}$ and D.*

Proposition 6.1.12. *If $(\mathcal{B}, \mathcal{H}, D)$ is an irreducible triple, then the triples of proposition 6.1.7 and proposition 6.1.10 are irreducible, too.*

Now we come to the metric structure of the triples constructed in this section. We assumed that the spectral triple over $\mathcal{B}$ makes the C^*-algebra B into a spectral metric space. In particular, this means that the representation π is faithful. It is then easy to see that this implies that also the representation $\hat{\pi}$ is faithful. Hence we have realized $\mathcal{A}$ as a subalgebra of the C^*-algebra of bounded operators on $\mathcal{H}'$: we can take the completion of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H}')$ obtaining in this way a C^*-algebra A, which, of course, has $\mathcal{A}$ as a dense *-subalgebra. Notice that A is nothing else than the crossed product C^*-algebra $B \rtimes_\alpha \mathbb{Z}$.

[2] See remark 5.2.8. See also the discussion in section 6.2.3.

Proposition 6.1.13. *The Lipschitz seminorms on A induced by the Dirac operators of the spectral triples of proposition 6.1.7 and proposition 6.1.10 are Lip-norms. Hence they give A a structure of compact spectral metric space. Moreover, the two structures associated to the two triples of proposition 6.1.10 are actually the same.*

Proof. For the first part of the proposition see [HSWZ11], theorem 2.11. The second part is a direct consequence of the fact that the two triples differ only by a change of orientation. □

We conclude this section with the following observation. Assume that $\mathcal{B}$ is a locally convex topological algebra[3], with topology defined by a countable separating family $\{p_n\}$ of seminorms. We give a structure of locally convex vector space to $\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}$. We notice, first of all, that as a complex vector space $\mathcal{A}$ is isomorphic to $\mathcal{B} \otimes \mathbb{C}[\mathbb{Z}]$, where $\otimes$ denotes the algebraic tensor product and $\mathbb{C}[\mathbb{Z}]$ is the group algebra[4] of $\mathbb{Z}$. Next, define an operator $\partial : \mathbb{C}[\mathbb{Z}] \to \mathbb{C}[\mathbb{Z}]$ by $\partial(\underline{m}) = m \cdot \underline{m}$. Consider then the following state on $\mathbb{C}[\mathbb{Z}]$: $\varphi : \mathbb{C}[\mathbb{Z}] \to \mathbb{C}$, $\phi(\underline{m}) = \delta_{m,0}$, and define a norm by: $\|\xi\|_{\mathbb{C}[\mathbb{Z}]} = \varphi(\xi^*\xi)^{\frac{1}{2}}$. Then the following maps determines, for $n \geq 0$, a separating family of seminorms on $\mathbb{C}[\mathbb{Z}]$:

$$q_n(\xi) = \|\partial^n(\xi)\|_{\mathbb{C}[\mathbb{Z}]}.$$

It is straightforward to check that the topology associated to this family makes $\mathbb{C}[\mathbb{Z}]$ into a locally convex topological algebra. This allows us to endow $\mathcal{A}$ with the projective topology[5] since, as a vector space, it is isomorphic to the algebraic tensor product of two locally convex spaces. Moreover, since the topology obtained in this way will still be defined by a countable separating family of seminorms, we can give the following definition.

Definition 6.1.14. *We define $\mathcal{A}_\infty$ to be the completion of $\mathcal{A}$ with respect to any translation invariant metric which induces the locally convex projective topology on $\mathcal{A}$. In particular, $\mathcal{A}_\infty$ is a Fréchet algebra.*

Then also $(\mathcal{A}_\infty, \mathcal{H}', \hat{D}, \hat{J})$ and $(\mathcal{A}_\infty, \mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma})$ are real spectral triples.. Although we shall not discuss this issue in this thesis, we point out the following fact: extending from $\mathcal{A}$ to $\mathcal{A}_\infty$ could allow to preserve the finiteness condition; that is, if we assume the triple over $\mathcal{B}$ to fulfil the finiteness axiom, in general we could not expect the triples over $\mathcal{A}$ to do the same. Instead, if we work with $\mathcal{A}_\infty$, we could get triples enjoying the finiteness condition.

6.2 Further properties of spectral triples over $\mathcal{B} \rtimes_\alpha \mathbb{Z}$

When looking to the results of the previous sections a question arises naturally: if the spectral triple over the algebra $\mathcal{B}$ fulfils Connes' axiom, does the same still hold for the spectral triples over $\mathcal{A}$ constructed above? We shall see in this section that in many cases the answer is affirmative (even if to get some axioms to be preserved we will have to impose some additional conditions on the triple over $\mathcal{B}$). We have already seen that the first order condition is preserved; here we shall discuss the classical dimension, the regularity and the orientation condition.

[3]For all the definitions and the results about locally convex spaces we refer to appendix B.
[4]For the Hopf *-algebra structure of $\mathbb{C}[\mathbb{Z}]$ see remark 6.1.3.
[5]See appendix B.

6.2.1 Dimension

If the Dirac operator D on $\mathcal{H}$ is p^+-summable, then the Dirac operator $\hat{D}$ (and hence the operators $\hat{D}_\pm$) is $(p+1)^+$-summable. This follows as in the case of product spectral triples; for further details see [DS13a, GBFV].

6.2.2 Regularity

Let the spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ over $\mathcal{B}$ be *regular*. We want to see if this implies that also the spectral triples of proposition 6.1.7 and 6.1.10 are regular. We will use the results of section 3.1.2, in particular theorem 3.1.19 and theorem 3.1.20.

Since $(\mathcal{B}, \mathcal{H}, D)$ is a regular spectral triple, there exists an algebra of generalized differential operators $\mathcal{D}_\mathcal{B} \subset \mathrm{End}(W^\infty)$ such that $\mathcal{B}+[D,\mathcal{B}]$ is dense in $\mathcal{D}^0_\mathcal{B}$. Here W^∞ is the space of Δ-smooth vectors, where $\Delta = D^2+1$. Consider now the Hilbert space $\mathcal{H}' = \mathcal{H} \otimes \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$. We can extend Δ to an operator $\hat{\Delta}$ on $\mathcal{H}'$, simply defined by $\hat{\Delta} = \Delta \otimes \mathrm{id}$. Then the space of $\hat{\Delta}$-smooth vectors is just $\hat{W}^\infty = W^\infty \otimes \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$.

Now, we assumed that the automorphism α is implementable, via an operator $\rho : \mathcal{H} \to \mathcal{H}$ (see definition 6.1.2). In particular ρ commutes with D, and so also with Δ. Furthermore, it is an isometry of $\mathcal{H}$. It follows that the space W^∞ is invariant under the action of ρ (notice that ρ is an isometry also w.r.t. the inner products $\langle\cdot,\cdot\rangle_{W^s}$, and so ρ is an isometry of W^∞, too). This allows us to extend the action of α to $\mathrm{End}(W^\infty)$. Indeed, given $P \in \mathrm{End}(W^\infty)$, we define $\alpha(P)$ by:

$$\alpha(P)\xi = \rho(P\rho^{-1}(\xi)).$$

Let $P \in \mathcal{D}_\mathcal{B}$. Then we define an operator $\hat{P}$ acting on $\hat{W}^\infty \subset \mathcal{H}'$ in the following way:

$$(\hat{P}f)_n = \alpha^{-n}(P) f_n,$$

for any $f \in \hat{W}^\infty$. Then we can consider the filtered algebra $\hat{\mathcal{D}}$ of operators $\hat{P}$, where $P \in \mathcal{D}_\mathcal{B}$. Of course, the filtration is the one induced by the filtration of $\mathcal{D}_\mathcal{B}$.

Lemma 6.2.1. $\hat{\mathcal{D}}_\mathcal{B} \subseteq \mathrm{Op}(\hat{\Delta})$.

Proof. Since ρ is an isometry of W^∞ then $\alpha(P)$ has the same analytic properties of P. Hence if P belongs to $\mathrm{Op}^t(\Delta)$ so will do $\alpha^k(P)$, for any $k \in \mathbb{Z}$. Since $\mathcal{D}_\mathcal{B} \subseteq \mathrm{Op}(\Delta)$ this implies that $\hat{\mathcal{D}}_\mathcal{B} \subseteq \mathrm{Op}(\hat{\Delta})$. □

Now let $\hat{H} \subset \mathcal{A}$ be the (unital) *-algebra generated by u, u^{-1}. Consider the following $\mathbb{N}$-filtered algebra:

$$\mathcal{D}^k_A = \hat{\mathcal{D}}^k_\mathcal{B} \cdot \hat{H} + \hat{H} \cdot \hat{\mathcal{D}}^k_\mathcal{B} + \sum_{j=1}^{3} (\mathrm{id} \otimes \sigma^j)(\hat{\mathcal{D}}^k_\mathcal{B} \cdot \hat{H} + \hat{H} \cdot \hat{\mathcal{D}}^k_\mathcal{B}).$$

By construction $\mathcal{A} + [D, \mathcal{A}]$ is dense in $\mathcal{D}^0_\mathcal{A}$. Let now Δ' be the operator $\hat{D}^2 + 1$ on $\mathcal{H}'$. Notice that it is equal to $\hat{\Delta} + \delta^2$. Let W'^∞ be the space of Δ'-smooth vectors of $\mathcal{H}'$. Then it is easy to see that $W'^\infty \subset \hat{W}^\infty$. Also, since any $P \in \hat{\mathcal{D}}_\mathcal{B}$ acts as the identity on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}$, $\hat{\mathcal{D}}_\mathcal{B}$ can be seen as an ($\mathbb{N}$-filtered) subalgebra of $\mathrm{End}(W'^{\infty})$. And the same holds for $\hat{H}$. Hence $\mathcal{D}_{\mathcal{A}}$ is a subalgebra of $\mathrm{End}(W'^{\infty})$. Moreover, since α commutes with Δ', it satisfies $[\Delta', \mathcal{D}^k_{\mathcal{A}}] \subseteq \mathcal{D}^{k+1}_{\mathcal{A}}$. So, in order to prove that it is an algebra of generalized differential operators, we have only to show that each $\mathcal{D}^k_{\mathcal{A}}$ is contained in $\mathrm{Op}^k(\Delta')$. It is clear that $\hat{\mathcal{D}}^k_{\mathcal{B}}$ is contained in $\mathrm{Op}^k(\Delta')$. Also, $\hat{H}$ and $(\mathrm{id} \otimes \sigma^j)\ \hat{H}$ (for any $j = 1, 2, 3,$) are contained in $\mathrm{Op}^k(\Delta')$; more precisely both of them are contained in $\mathrm{Op}^0(\Delta')$. Hence $\mathcal{D}^k_{\mathcal{A}}$ is contained in $\mathrm{Op}^k(\Delta')$. It follows (see theorem 3.1.19) that the spectral triples of proposition 6.1.7 and proposition 6.1.10 are regular spectral triples.

6.2.3 Orientation

One of the requirements of Connes' noncommutative geometry is the existence, for a real spectral triple of KR-dimension n, of an orientation Hochschild n-cycle (cfr. section 3.1.5). So a natural question is the following one: given a spectral triple over $\mathcal{B}$ with an orientation cycle $\mathbf{c}_{\mathcal{B}} \in Z_n(\mathcal{B}, \mathcal{B} \otimes \mathcal{B}^{\circ})$, can we define orientation $(n+1)$-cycles $\mathbf{c} \in Z_{n+1}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\circ})$ for the triples constructed in proposition 6.1.7 and proposition 6.1.10? In this section we shall see that it is possible if the orientation cycle $\mathbf{c}_{\mathcal{B}}$ is α-invariant (see below).

In the case of tensor product algebras $\mathcal{B} \otimes H$, the cycle can be obtained [DD11], from the orientation cycles on $\mathcal{B}$ and on H, using the shuffle product [Lo]. Since we are dealing with a smash product instead of a tensor product, we need to modify a little the construction. For any $k \in \mathbb{Z}$, let us consider the following Hochschild 1-cycle with values in $H \otimes H^{\circ}$:

$$\mathbf{c}^k_H = (z^{-k} \otimes 1) \otimes z^k.$$

Notice that we can write any Hochschild chain $\mathbf{c} \in C_p(\mathcal{B}, \mathcal{B} \otimes \mathcal{B}^{\circ})$ as

$$\mathbf{c} = \sum (a_0 \otimes b_0^{\circ}) \otimes a_1 \otimes \cdots \otimes a_p. \tag{6.2.1}$$

Now we give the following definition.

Definition 6.2.2. *For any Hochschild p-cycle $\mathbf{c} \in Z_p(\mathcal{B}, \mathcal{B} \otimes \mathcal{B}^{\circ})$, written as in equation (6.2.1), we define its* twisted shuffle product *with $\mathbf{c}^k_H$ as the Hochschild $(p{+}1)$-chain $\mathbf{c} \times_{\alpha} \mathbf{c}^k_H \in C_{p+1}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\circ})$ defined by:*

$$\begin{aligned} \mathbf{c} \times_{\alpha} \mathbf{c}^k_H &= \sum (a_0 u^{-k} \otimes b_0^{\circ}) \otimes u^k \otimes a_1 \otimes \cdots \otimes a_p \\ &+ \sum_{j=2}^{p} \sum (-1)^{j-1} (a_0 u^{-k} \otimes b_0^{\circ}) \otimes \alpha^k(a_1) \otimes \cdots \otimes \alpha^k(a_{j-1}) \otimes u^k \otimes a_j \otimes \cdots \otimes a_p \\ &+ (-1)^p (a_0 u^{-k} \otimes b_0^{\circ}) \otimes \alpha^k(a_1) \otimes \cdots \otimes \alpha^k(a_p) \otimes u^k. \end{aligned} \tag{6.2.2}$$

Remark 6.2.3. The twisted shuffle product $\times_{\alpha}$ can be extended, by linearity in the second variable, to a map $C_p(\mathcal{B}, \mathcal{B} \otimes \mathcal{B}^{\circ}) \otimes C_1(H, H) \to C_{p+1}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\circ})$ defining:

$$\mathbf{c} \times_{\alpha} (z^r \otimes z^k) = \sum (a_0 u^r \otimes b_0^{\circ}) \otimes u^k \otimes a_1 \otimes \cdots \otimes a_p$$

$$
\begin{aligned}
&+ \sum_{j=2}^{p} \sum (-1)^{j-1} (a_0 u^r \otimes b_0^\circ) \otimes \alpha^k(a_1) \otimes \cdots \otimes \alpha^k(a_{j-1}) \otimes u^k \otimes a_j \otimes \cdots \otimes a_p \\
&+ (-1)^p (a_0 u^r \otimes b_0^\circ) \otimes \alpha^k(a_1) \otimes \cdots \otimes \alpha^k(a_p) \otimes u^k.
\end{aligned} \tag{6.2.3}
$$

for any $k, r \in \mathbb{Z}$.

Lemma 6.2.4. *The twisted shuffle product* (6.2.2), (6.2.3), *is linear also in the first variable (i.e. in $\boldsymbol{c}$).*

Next we introduce an action of α on the space of Hochschild p-chains over $\mathcal{B}$: for $\mathbf{c} \in C_p(\mathcal{B} \otimes \mathcal{B}^\circ)$, written as in equation (6.2.1), we define

$$
\alpha(\mathbf{c}) = \sum (\alpha(a_0) \otimes b_0^\circ) \otimes \alpha(a_1) \otimes \cdots \otimes \alpha(a_p). \tag{6.2.4}
$$

Definition 6.2.5. *An Hochschild p-chain $\boldsymbol{c} \in C_p(\mathcal{B} \otimes \mathcal{B}^\circ)$ is* α-invariant *if $\alpha(\boldsymbol{c}) = \boldsymbol{c}$.*

Then we can prove the following result.

Lemma 6.2.6. *If $\boldsymbol{c}$ is an α-invariant Hochschild p-cycle, then the shuffle products $\boldsymbol{c} \times_\alpha \boldsymbol{c}_H^k$ are Hochschild $(p+1)$-cycles.*

Proof. Let $b_\mathcal{A}$ the Hochschild boundary operator on the Hochschild complex $C_\bullet(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$. Then we prove that $b_\mathcal{A}(\mathbf{c} \times_\alpha \mathbf{c}_H^k) = 0$. First of all we introduce the following notation: according to (6.2.2) we can write $\mathbf{c} \times_\alpha \mathbf{c}_H^k$ as

$$
\mathbf{c} \times_\alpha \mathbf{c}_H^k = \sum_{j=1}^{p+1} \mathbf{c}_j.
$$

We compute now each $b_\mathcal{A}\mathbf{c}_k$. For $\mathbf{c}_1$ we have:

$$
\begin{aligned}
b_\mathcal{A}\mathbf{c}_1 = &\sum (a_0 \otimes b_0^\circ) \otimes a_1 \otimes \cdots \otimes a_p \\
&- \sum (a_0 u^{-k} \otimes b_0^\circ) \otimes u^k a_1 \otimes a_2 \otimes \cdots \otimes a_p \\
&+ \sum_{i=1}^{p-1} \sum (-1)^{i+1} (a_0 u^{-k} \otimes b_0^\circ) \otimes u^k \otimes a_1 \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \\
&+ \sum (-1)^{p+1} (a_p a_0 u^{-k} \otimes b_0^\circ) \otimes u^k \otimes a_1 \otimes \cdots \otimes a_{p-1}.
\end{aligned} \tag{6.2.5}
$$

Next, for $j = 2, \ldots, p$ we obtain:

$$
\begin{aligned}
b_\mathcal{A}\mathbf{c}_j = &\sum (-1)^{j-1} (a_0 a_1 u^{-k} \otimes b_0^\circ) \otimes \alpha^k(a_2) \otimes \cdots \otimes \alpha^k(a_{j-1}) \otimes u^k \otimes a_j \otimes \cdots \otimes a_p \\
&+ \sum_{i=1}^{j-2} \sum (-1)^i (-1)^{j-1} (a_0 u^{-k} \otimes b_0^\circ) \otimes \alpha^k(a_1) \otimes \cdots \otimes \alpha^k(a_i a_{i+1}) \otimes \\
&\qquad \cdots \alpha^k(a_{j-1}) \otimes u^k \otimes a_j \otimes \cdots a_p \\
&+ \sum (a_0 u^{-k} \otimes b_0^\circ) \otimes \alpha^k(a_1) \otimes \cdots \otimes u^k a_{j-1} \otimes a_j \otimes \cdots \otimes a_p \\
&- \sum (a_0 u^{-k} \otimes b_0^\circ) \otimes \alpha^k(a_1) \otimes \cdots \otimes \alpha^k(a_{j-1}) \otimes u^k a_j \otimes a_{j+1} \otimes \cdots \otimes a_p
\end{aligned} \tag{6.2.6}
$$

$$
\begin{aligned}
&+\sum_{i=j}^{p-1}\sum(-1)^{i+1}(-1)^{j-1}(a_0u^{-k}\otimes b_0^\circ)\otimes\alpha^k(a_1)\otimes \\
&\qquad\qquad\cdots\otimes\alpha^k(a_{j-1})\otimes u^k\otimes a_j\otimes a_ia_{i+1}\otimes\cdots\otimes a_p \\
&+\sum(-1)^{p+1}(-1)^{j-1}(a_pa_0u^{-k}\otimes b_0^\circ)\otimes\alpha^k(a_1)\otimes\alpha^k(a_{j-1})\otimes u^k\otimes a_j\otimes\cdots\otimes a_{p-1}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
b_{\mathcal{A}}\mathbf{c}_{p+1} &= \sum(-1)^p(a_0a_1u^{-k}\otimes b_0^\circ)\otimes\alpha^k(a_2)\otimes\cdots\otimes\alpha^k(p)\otimes u^k \\
&+\sum_{i=1}^{p-1}\sum(-1)^i(-1)^p(a_0u^{-k}\otimes b_0^\circ)\otimes\alpha^k(a_1)\otimes\cdots\otimes\alpha^k(a_ia_{i+1})\otimes\cdots\otimes\alpha^k(a_p)\otimes u^k \\
&+\sum(a_0u^{-k}\otimes b_0^\circ)\otimes\alpha^k(a_1)\otimes\cdots\otimes\alpha^k(a_{p-1})\otimes u^ka_p \\
&-\sum(\alpha^k(a_0)\otimes b_0^\circ)\otimes\alpha^k(a_1)\otimes\cdots\otimes\alpha^k(a_p).
\end{aligned}
\tag{6.2.7}
$$

Now, the first line of (6.2.5) cancels out with the last of (6.2.7), due to the α-invariance of $\mathbf{c}$. Next, the terms containing a factor u^ka_i in (6.2.6) and (6.2.7) sum up to zero. What remains is nothing else than $b_{\mathcal{B}}\mathbf{c}\times_\alpha\mathbf{c}_H^k$, which is zero since $\mathbf{c}$ is a cycle and the twisted shuffle product is linear (lemma 6.2.4). □

Proposition 6.2.7. *Assume that the spectral triple $(\mathcal{B},\mathcal{H},D,J,\gamma)$ over $\mathcal{B}$ has KR-dimension n (if n is odd then $\gamma=\mathrm{id}$), and let $\mathbf{c}_{\mathcal{B}}$ be an orientation n-cycle for it. Then, if $\mathbf{c}_{\mathcal{B}}$ is α-invariant, the normalized twisted shuffle product $\mathbf{c}_{\mathcal{A}}=i^n(n+1)^{-1}\mathbf{c}_{\mathcal{B}}\times_\alpha\mathbf{c}_H^1$ gives an orientation $(n+1)$-cycle for the triples of proposition 6.1.7 or of proposition 6.1.10, according to the parity of n.*

Proof. For the moment we consider together the odd dimensional and the even dimensional case. First of all we compute $\pi_{\hat{D}}(\mathbf{c}_{\mathcal{A}})$. Write $\mathbf{c}_{\mathcal{B}}$ as in (6.2.1). Then, for any $f=(f_k)\in\mathcal{H}'$, we have:

$$
\begin{aligned}
(\pi_{\hat{D}}(\mathbf{c}_{\mathcal{A}})f)_k &= i^n(n+1)^{-1}\sum(\hat\pi(a_0)\hat u^{-1}\hat J\hat\pi(b_0^*)\hat J^{-1}[\hat D,\hat u][\hat D,\hat\pi(a_1)]\cdots[\hat D,\hat\pi(a_n)]f)_k \\
&+i^n(n+1)^{-1}\sum_{i=1}^{n}\sum(-1)^{j-1}(\hat\pi(a_0)\hat u^{-1}\hat J\hat\pi(b_0^*)\hat J^{-1}[\hat D,\hat\pi(a_1)] \\
&\qquad\cdots[\hat D,\hat\pi(a_{j-1})][\hat D,\hat u][\hat D,\hat\pi(a_j)]\cdots[\hat D,\hat\pi(a_n)]f)_k \\
&+i^n(n+1)^{-1}\sum(-1)^n(\hat\pi(a_0)\hat u^{-1}\hat J\hat\pi(b_0^*)\hat J^{-1}[\hat D,\hat\pi(a_1)]\cdots[\hat D,\hat\pi(a_n)][\hat D,\hat u]f)_k \quad (6.2.8)\\
&= i^n(n+1)^{-1}\left(\sum\pi(a_0)J\pi(b_0^*)J^{-1}[D,\pi(a_1)]\cdot[D,\pi(a_n)]\right) \\
&\qquad\cdot\left(\sigma^2(\sigma^1)^n+\sum_{j=1}^{n}(-1)^{j-1}(\sigma^1)^{j-1}\sigma^2(\sigma^1)^{n-j+1}+(-1)^n(\sigma^1)^n\sigma^2\right)f_k \\
&= i^n(n+1)^{-1}(n+1)\pi_D(\mathbf{c}_{\mathcal{B}})\sigma^2(\sigma^1)^nf_k = i^n\gamma\sigma^2(\sigma^1)^nf_k.
\end{aligned}
$$

If n is odd (and so $n+1$ is even) γ is simply the identity. Therefore $\pi_{\hat{D}}(\mathbf{c}_{\mathcal{A}})$ is equal to $i^n(-i)\sigma^3=\pm\sigma^3=\pm\hat\gamma$. If n is even, instead, it is equal to $\pm\gamma\otimes\sigma^2$. That is, it is equal to $\pm\nu$. Hence it acts as the identity or minus the identity on $\mathcal{H}'_\pm$, which is what we need, since in this case the triples are odd, and so the representation of the orientation cycle has simply to be the identity. □

6.3 Quantum principal bundle structure on $\mathcal{B} \rtimes_\alpha \mathbb{Z}$

Consider, as in the previous section, a real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ over a unital algebra $\mathcal{B}$ and an implementable automorphism $\alpha \in \mathrm{Aut}(\mathcal{B})$. We shall assume in this section that the representation of $\mathcal{B}$ on $\mathcal{H}$ is faithful, and so the same holds for the representations of $\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}$. We shall show, in this section, that we can define a structure of quantum principal $U(1)$-bundle on $\mathcal{A}$; in particular we will see that it is a principal $\mathcal{O}(U(1))$-comodule algebra and that it is a quantum principal $U(1)$-bundle with the first order differential calculus $\Omega^1_{\hat{D}}(\mathcal{A})$. We will also give a quite explicit formula for strong connections over $\mathcal{A}$.

First of all we recall that any element of $\mathcal{A}$ can be written as

$$a = \sum_{n \in \mathbb{Z}} b_n u^n, \qquad b_n \in \mathcal{B},$$

where the sum contains only a finite number of non-zero terms. So we can define a map $\Delta_R : \mathcal{A} \to \mathcal{A}_\infty \otimes H$ as follows. We set

$$\Delta_R(b) = b \otimes 1 \quad b \in \mathcal{B},$$

$$\Delta_R(u) = u \otimes z,$$

and we extend it as an algebra *-homomorphism to the whole $\mathcal{A}$. Then it is straightforward to check that:

Lemma 6.3.1. $(\mathcal{A}, \Delta_R)$ *is a right* $\mathcal{O}(U(1))$*-comodule algebra.*

Actually we can say something more.

Proposition 6.3.2. $\mathcal{B} \hookrightarrow \mathcal{A}$ *is a cleft Hopf-Galois extension. In particular,* $\mathcal{A}$ *is a principal* $\mathcal{O}(U(1))$*-comodule algebra.*

Proof. It is clear that T_R is surjective. We prove injectivity. Notice that any element of $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$ can be written as:

$$A = \sum_{n,m \in \mathbb{Z}} b_{n,m} u^n \otimes u^m.$$

So, for a generic $A \in \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$ we have:

$$T_R(A) = \sum_{n,m \in \mathbb{Z}} b_{n,m} u^{n+m} \otimes z^m.$$

If now we impose $T_R(A) = 0$ we get, due to the mutual linear independence of the elements z^m,

$$\sum_{n \in \mathbb{Z}} b_{n,m} u^{n+m} = \left(\sum_{n \in \mathbb{Z}} b_{n,m} u^n \right) u^m = 0$$

for all $m \in \mathbb{Z}$, which implies $A = 0$. Thus T_R is also injective when restricted to $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$.

Finally, to prove that that $\mathcal{B} \hookrightarrow \mathcal{A}$ is a cleft Hopf-Galois extension, we give its trivialization:

$$\phi(z^k) = u^k, \qquad \phi^{-1}(z^k) = u^{-k}.$$

From proposition 4.4.2 it follows that $\mathcal{A}$ is a principal comodule algebra. □

Now we want to give to $(\mathcal{A}, H, \Delta_R)$ a structure of quantum principal $U(1)$-bundle. In order to achieve this result, we need to specify a differential calculus over $\mathcal{A}$. We recall that when we speak of quantum principal $U(1)$-bundles we assume to consider, on the Hopf algebra $H = \mathcal{O}(U(1))$, the de Rham calculus, $\Omega^1(H) = \Omega^1_{dR}(H)$, which is the bicovariant calculus associated to the ad_R-invariant ideal $Q = (\ker \varepsilon)^2$. On $\mathcal{A}$, instead, we consider the differential calculus induced by the Dirac operator $\hat{D}$. Take $\Omega^1(\mathcal{A}) = \Omega^1_{\hat{D}}(\mathcal{A}) = \Omega^1\mathcal{A}/N$ where N, seen as a sub-bimodule of $\mathcal{A} \otimes \mathcal{A}$, is given by:

$$N = \left\{ \sum_j p_j \otimes q_j \in \mathcal{A} \otimes \mathcal{A} \;\middle|\; \sum_j \hat{\pi}(p_j)[\hat{D}, \hat{\pi}(q_j)] = 0 \right\}.$$

Since $\mathcal{B} \hookrightarrow \mathcal{A}$ is a Hopf-Galois extension, we can define (cfr. section 4.6.2) an action of the Lie algebra $\mathfrak{t}_1 \simeq \mathbb{C}$ on $\mathcal{A}$. If we denote by δ the canonical generator of $\mathfrak{t}_1$, then we can extend it to an unbounded operator on $\ell^2(\mathbb{Z})$, and therefore on $\mathcal{H}'$, in the following way:

$$(\delta f)_n = n f_n.$$

It follows that δ satisfies the Leibniz rule:

$$\delta(\hat{\pi}(a)\xi) = \hat{\pi}(\delta(a))\xi + \hat{\pi}(a)\delta(\xi)$$

for any $a \in \mathcal{A}$ and any $\xi \in \mathrm{Dom}(\delta)$. Now we can observe that the Dirac operator $\hat{D}$ can be written as:

$$\hat{D} = D \otimes \sigma^1 + (\mathrm{id} \otimes \sigma^2) \circ \delta. \tag{6.3.1}$$

Moreover the operator δ is selfadjoint and it defines an action of $U(1)$ on the Hilbert space $\mathcal{H}'$; it is easy to see, since δ anticommutes with $\hat{J}$, that:

Proposition 6.3.3. *The spectral triples constructed in proposition 6.1.7 and proposition 6.1.10 are $U(1)$-equivariant w.r.t. the $U(1)$ action defined by δ.*

Next, we look at the differential calculus defined by the Dirac operator $\hat{D}$.

Proposition 6.3.4. *The differential calculus $\Omega^1_{\hat{D}}(\mathcal{A})$ is a right H-covariant first order differential calculus.*

Proof. We prove that $\Delta_R^{\Omega}(N) \subseteq N \otimes H$. Take $\eta \in N$, $\eta = \sum p dq$. We know that we can write it in the following form:

$$\eta = \sum \sum_{k,l \in \mathbb{Z}} p_l u^l d(q_k u^k),$$

with $p_l, q_k \in \mathcal{B}$. Using the Leibniz rule for the differential d, then, we obtain:

$$\eta = \sum\sum_{k,l\in\mathbb{Z}} \left(p_l u^l (dq_k) u^k + p_l u^l q_k du^k\right).$$

Now, the fact that η belongs to N means that $\pi_{\hat{D}}(\eta) = 0$. That is,

$$\sum\sum_{k,l\in\mathbb{Z}} \left[\left(\hat{\pi}(p_l)\hat{u}^l[D\otimes \mathrm{id}, \hat{\pi}(q_k)]\hat{u}^k\right)\otimes\sigma^1 + k\left(\hat{\pi}(p_l)\hat{u}^l\hat{\pi}(q_k)\hat{u}^k\right)\otimes\sigma^2\right] = 0.$$

Since the Pauli matrices are linearly independent, we get:

$$\begin{aligned}\sum\sum_{k,l\in\mathbb{Z}} \hat{\pi}(p_l)\hat{u}^l[D\otimes \mathrm{id}, \hat{\pi}(q_k)]\hat{u}^k = 0,\\ \sum\sum_{k,l\in\mathbb{Z}} k\hat{\pi}(p_l)\hat{u}^l\hat{\pi}(q_k)\hat{u}^k = 0.\end{aligned} \tag{6.3.2}$$

The second of (6.3.2) can be rewritten using the properties of the representation $\hat{\pi}$:

$$\begin{aligned}0 &= \sum\sum_{k,l\in\mathbb{Z}} k\hat{\pi}(p_l)\hat{u}^l\hat{\pi}(q_k)\hat{u}^k = \sum\sum_{k,l\in\mathbb{Z}} k\hat{\pi}(p_l)\hat{\pi}(\alpha^l(q_k))\hat{u}^l\hat{u}^k\\ &= \sum\sum_{k,l\in\mathbb{Z}} k\hat{\pi}(p_l)\hat{\pi}(\alpha^l(q_k))\hat{u}^{l+k}.\end{aligned} \tag{6.3.3}$$

In the same way the first of (6.3.2) becomes:

$$0 = \sum\sum_{k,l\in\mathbb{Z}} \hat{\pi}(p_l)[D\otimes \mathrm{id}, \hat{\pi}(\alpha^l(q_k))]\hat{u}^{l+k}. \tag{6.3.4}$$

Since the operators $\hat{u}^n$, for different n, are linearly independent, from (6.3.3) and (6.3.4) we obtain that, for any $n \in \mathbb{Z}$,

$$\sum\sum_{k+l=n} k\hat{\pi}(p_l)\hat{\pi}(\alpha^l(q_k))\hat{u}^{l+k} = \sum\sum_{k+l=n} \hat{\pi}(p_l)[D\otimes \mathrm{id}, \hat{\pi}(\alpha^l(q_k))]\hat{u}^{l+k} = 0. \tag{6.3.5}$$

Let us split, now, η in a different way: write $\eta = \sum_{n\in\mathbb{Z}} \eta_n$, with

$$\eta_n = \sum\sum_{k+l=n} p_l u^l d(q_k u^k).$$

Then we can compute, for each n, $\pi_{\hat{D}}(\eta_n)$:

$$\begin{aligned}\pi_{\hat{D}}(\eta_n) &= \sum\sum_{k+l=n} \hat{\pi}(p_l)\hat{u}^l[\hat{D}, \hat{\pi}(q_k)\hat{u}^k]\\ &= \sum\sum_{k+l=n} \left[\left(\hat{\pi}(p_l)\hat{u}^l[D\otimes \mathrm{id}, \hat{\pi}(q_k)]\hat{u}^k\right)\otimes\sigma^1 + k\left(\hat{\pi}(p_l)\hat{u}^l\hat{\pi}(q_k)\hat{u}^k\right)\otimes\sigma^2\right]\\ &= \sum\sum_{k+l=n} \left[\left(\hat{\pi}(p_l)[D\otimes \mathrm{id}, \hat{\pi}(\alpha^l(q_k))]\hat{u}^{l+k}\right)\otimes\sigma^1 + k\left(\hat{\pi}(p_l)\hat{\pi}(\alpha^l(q_k))\hat{u}^{l+k}\right)\otimes\sigma^2\right] = 0,\end{aligned} \tag{6.3.6}$$

due to (6.3.5). This means that each η_n belongs to N. Now let us compute $\Delta_R^\Omega(\eta)$. It is easy to see that:

$$\Delta_R^\Omega(\eta) = \sum \sum_{k+l=n} \Big(p_l u^l d(q_k u^k)\Big) \otimes z^{l+k}.$$

But this means that $\Delta_R^\Omega(\eta) = \sum_n \eta_n \otimes z^n$, which implies, since each η_n belongs to N, that it belongs to $N \otimes H$. This concludes the proof of the covariance of the calculus. □

Now we can prove that, with the first order differential calculus defined by $\hat{D}$, $\mathcal{A}$ is a quantum principal $U(1)$-bundle.

Theorem 6.3.5. *If N is the sub-bimodule of $\mathcal{A} \otimes \mathcal{A}$ which defines the calculus $\Omega^1_{\hat{D}}(\mathcal{A})$, $(\mathcal{A}, H = \mathcal{O}(U(1)), \Delta_R, N, Q)$ is a quantum principal $U(1)$-bundle.*

Proof. We have already proved that $\mathcal{A}$ is a principal comodule algebra and that the calculus $\Omega^1_{\hat{D}}$ is H-covariant. So we have only to prove that (i) and (ii) of proposition 4.6.14 are satisfied.

Let $\sum_j p_j dq_j$ be zero in $\Omega^1_{\hat{D}}(\mathcal{A})$; using expression (6.3.1) for the Dirac operator $\hat{D}$, we get then

$$\sum_j \hat{\pi}(p_j)[\hat{D}, \hat{\pi}(q_j)] = 0$$

$$\Rightarrow \quad \sum_j \hat{\pi}(p_j)[D \otimes \mathrm{id}, \hat{\pi}(q_j)](\mathrm{id} \otimes \sigma^1) + \hat{\pi}(p_j)\hat{\pi}(\delta(q_j))(\mathrm{id} \otimes \sigma^2) = 0.$$

Since σ^1 and σ^2 are linearly independent, this implies that $\sum_j p_j \delta(q_j) = 0$. So condition (i) is fulfilled. Now let us prove condition (ii). Take $\eta \in \Omega^1 \mathcal{A}$, $\eta = \sum p dq$, and assume that $\sum p\delta(q) = 0$. Then rewrite η as $\eta = \sum \sum_{k \in \mathbb{Z}} pd(q_k u^k)$, with $q_k \in \mathcal{B}$. Using the Leibniz rule we obtain:

$$\eta = \sum \sum_{k \in \mathbb{Z}} \Big(p(dq_k)u^k + pq_k du^k\Big).$$

In order to prove that $[\eta]_N$ belongs to $\mathcal{A}\Omega^1_{\hat{D}}(\mathcal{B})\mathcal{A}$ it is then enough to show that $\sum \sum_{k \in \mathbb{Z}} pq_k du^k$ is zero in $\Omega^1_{\hat{D}}(\mathcal{A})$. But this follows by direct computation. Indeed,

$$\begin{aligned} \pi_{\hat{D}}\Big(\sum \sum_{k \in \mathbb{Z}} pq_k du^k\Big) &= \sum \sum_{k \in \mathbb{Z}} \hat{\pi}(pq_k)[\hat{D}, \hat{u}^k] = \sum \sum_{k \in \mathbb{Z}} k\hat{\pi}(pq_k)\hat{u}^k \otimes \sigma^2 \\ &= \sum \hat{\pi}(p\delta(q)) \otimes \sigma^2 = 0. \end{aligned}$$

□

Now that we have seen that $(\mathcal{A}, H, \Delta_R, N, Q)$ is a quantum principal $U(1)$-bundle we can give a characterization of its strong connections. Indeed, the fact that $\hat{D}$ can be written as in equation (6.3.1) leads, by direct computation, to the following result.

Lemma 6.3.6. *Any 1-form $\eta \in \Omega^1_{\hat{D}}(\mathcal{A})$ can be written, as an operator on $\mathcal{H}'$, as follows:*

$$\eta = \sum_{n\in\mathbb{Z}} \hat{\pi}(p_n)[\hat{D}, \hat{u}^n] + \sum_j \hat{\pi}(q_j)[D, \hat{\pi}(b_j)](\mathrm{id}\otimes\sigma^1) \tag{6.3.7}$$

with $p_n, q_j \in \mathcal{A}$, $b_j \in \mathcal{B}$.

We can use this lemma to prove the following fact.

Proposition 6.3.7. *Any strong connection $\omega_1 \in \Omega^1_{\hat{D}}(\mathcal{A})$, in the sense of definition 4.6.17, over the quantum principal $U(1)$-bundle $(\mathcal{A}, H, \Delta_R, N, Q)$ can be written, as an operator on $\mathcal{H}'$, in the following form:*

$$\omega_0 = \sigma^2 + \sum_j \hat{\pi}(p_j)[D\otimes\sigma^1, \hat{\pi}(q_j)] \tag{6.3.8}$$

with $p_j, q_j \in \mathcal{B}$.

Proof. Take ω_0 written as in equation (6.3.7):

$$\begin{aligned}\omega_0 &= \sum_{n\in\mathbb{Z}} \hat{\pi}(r_n)[\hat{D}, \hat{u}^n] + \sum_j \hat{\pi}(p_j)[D\otimes\sigma^1, \hat{\pi}(q_j)] \\ &= \sum_{n\in\mathbb{Z}} n\hat{\pi}(r_n)\hat{u}^n(\mathrm{id}\otimes\sigma^2) + \sum_j \hat{\pi}(p_j)[D\otimes\mathrm{id}, \hat{\pi}(q_j)](\mathrm{id}\otimes\sigma^1).\end{aligned} \tag{6.3.9}$$

Since σ^1 and σ^2 are linearly independent, condition (i) of definition 4.6.17 implies that the first term of (6.3.9) reduces to $\hat{\pi}(r)\sigma^2$ with $r \in \mathcal{B}$. Also, it implies that p_j belongs to the invariant subalgebra $\mathcal{B}$ for every j. Finally, using condition (ii), and writing $\mathrm{id}\otimes\sigma^2$ as $\hat{u}^{-1}[\hat{D}, \hat{u}]$, we get $r = 1$. □

One can also check that condition (iii) is fulfilled for any ω_1 written as in (6.3.8). We recall that the associated strong connection form ω (see definition 4.3.6) is defined by the relation $\omega(z^k) = k\omega_0$, $k \in \mathbb{Z}$.

6.4 Projectability and twisted Dirac operators

We have shown that the spectral triples of propositions 6.1.7 and 6.1.10 are $U(1)$-equivariant triples (see proposition 6.3.3). It is then interesting to answer whether or not they are *projectable* triples.

Proposition 6.4.1. *Let $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma}, \delta)$ be the $U(1)$-equivariant even real spectral triple of proposition 6.1.7. Then it is a projectable triple with constant length fibres. Moreover, we can take $\Gamma = \hat{u}^{-1}[\hat{D}, \hat{u}]$ (where $\hat{u}$ is defined by (6.1.1)).*

Proof. Take $\Gamma = \hat{u}^{-1}[\hat{D}, \hat{u}]$. Then by direct computation one sees that $\Gamma = \mathrm{id}\otimes\sigma^2$ [BMR10]. We immediately get $\Gamma^2 = 1$, $\Gamma^* = \Gamma$, $[\Gamma, \hat{\pi}(a)] = 0$ for any $a \in \mathcal{A}$, $[\Gamma, \delta] = 0$. Moreover, since $\hat{\gamma}$ is nothing else that σ^3, we have also that $\Gamma\hat{\gamma} = -\hat{\gamma}\Gamma$. So, in order to prove that such a triple is projectable, we have only to check that Γ has the right commutation relation with $\hat{J}$. But Γ

commutes with $1 \otimes \sigma^2$ and it anticommutes with $\widetilde{J}$, since $\widetilde{J}$ is an antiunitary operator. Thus $\Gamma \hat{J} = -\hat{J}\Gamma$ independently from the KR-dimension of the triple.

Finally, it is straightforward to see that it has constant length fibres; indeed, $D_h = D \otimes \sigma^1$, so, if we take $D_v = (\mathrm{id} \otimes \sigma^2) \circ \delta$, we have $\hat{D} = D_h + D_v$. □

Proposition 6.4.2. *Let $(\mathcal{A}, \mathcal{H}'_\pm, \hat{D}_\pm, \hat{J}_\pm, \delta)$ be the $U(1)$-equivariant odd real spectral triples of proposition 6.1.10. Then they are projectable spectral triples with constant length fibres. Moreover we can take $\Gamma_\pm$ equal to $\hat{u}^{-1}[\hat{D}, \hat{u}]$, restricted either to $\mathcal{H}'_+$ or to $\mathcal{H}'_-$.*

Proof. Take $\Gamma = \hat{u}^{-1}[\hat{D}, \hat{u}]$. Then, by direct computation, one sees that $\Gamma = \mathrm{id} \otimes \sigma^2$ [BMR10]. Hence we get $\Gamma^2 = 1$, $\Gamma^* = \Gamma$, $[\Gamma, \hat{\pi}(a)] = 0$ for any $a \in \mathcal{A}$, $[\Gamma, \delta] = 0$. Moreover, since $\nu = \gamma \otimes \mathrm{id} \otimes \sigma^2$, $[\Gamma, \nu] = 0$ and so Γ restricts to both $\mathcal{H}'_\pm$. Next, the fact that the commutation relation with $\hat{J}$ is the right one for any even KR-dimension (see definition 5.2.2) follows from the following relations:

$$\Gamma(\mathrm{id} \otimes \sigma^1) = -(\mathrm{id} \otimes \sigma^1)\Gamma, \quad \Gamma\widetilde{J} = -\widetilde{J}\Gamma, \quad [\Gamma, \gamma \otimes \mathrm{id}] = 0.$$

For the proof of the fact that the the triples fulfil the constant length fibres condition, see proposition 6.4.1. □

We can now use the results of section 5.2.5 to define twisted Dirac operators associated to a strong connection ω. The direct application of the results of section 5.2.5 leads, indeed, to the following results.

Proposition 6.4.3. *Let $\omega : H \to \Omega^1_{\hat{D}}(\mathcal{A})$ be a strong connection form given by $\omega(z^k) = k\omega_0$, $k \in \mathbb{Z}$. Then the twisted Dirac operator D_ω has the form:*

$$D_\omega = \hat{D} + j_0 \omega_0^* j_0^{-1} \delta,$$

where j_0 is obtained from $\hat{J}$, accordingly to the discussion in sections 5.2.3 and 5.2.4. Moreover, D_ω is selfadjoint if ω_0 is selfadjoint.

Proposition 6.4.4. *Let $\hat{D}_\omega = D_v + D_\omega$. Then $(\mathcal{A}, \mathcal{H}, \hat{D}_\omega)$ is a projectable triple with equal length fibres, and the horizontal part of $\hat{D}_\omega$ coincides with D_ω.*

Then, using the description of strong connections given in proposition 4.6.17, we obtain:

Corollary 6.4.5. *The unique strong connection ω_0 for which $\hat{D}_{\omega_0} = \hat{D}$ is $\omega_0 = \hat{u}^{-1}[D, \hat{u}]$.*

6.5 Spectral triples over cleft $\mathcal{O}(U(1))$-extensions

In the previous sections, given a (suitable) spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$, we have built a spectral triple for the crossed product algebra $\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}$. As we have seen this algebra is nothing else than a cleft $\mathcal{O}(U(1))$-extension of $\mathcal{B}$. Now we look at this construction from the opposite point of view: we start with a cleft extension and, using the construction above, we define a spectral triple over it. Of course, we will have to impose some conditions over the extension to get a well-behaving real spectral triple.

So, let $\mathcal{B} \hookrightarrow \mathcal{A}$ be a cleft Hopf-Galois extension, with respect to the Hopf algebra $H = \mathcal{O}(U(1))$. Assume that it admits a *unitary trivialization* ϕ. This means, in particular, that (cfr. definition 4.4.18)

$$\phi(z^k)^*\phi(z^k) = 1 \qquad \forall k \in \mathbb{Z}.$$

That is, any element $\phi(z^k)$ is unitary. Moreover, up to a gauge transformation, we can always assume that ϕ is an algebra homomorphism. In particular, we can take $\phi(z^k) = u^k$, where $u = \phi(z)$. Under this hypothesis $\mathcal{A}$ is isomorphic to the smash product $\mathcal{B}\#H$. We also see that the following holds.

Proposition 6.5.1. *$\mathcal{A}$ is generated by $\mathcal{B}$ and by the unitary $u = \phi(z)$. Hence it is isomorphic to the crossed product algebra $\mathcal{B} \rtimes_\alpha \mathbb{Z}$, where the automorphism α is given by $\alpha(b) = ubu^{-1}$ (for $b \in \mathcal{B}$).*

Proof. Take $a \in \mathcal{A}$ s.t. $\Delta_R(a) = a \otimes z^k$ with $k \neq 0$. Then au^{-k} satisfies $\Delta_R(au^{-k}) = au^{-k} \otimes 1$ and thus it belongs to the invariant subalgebra $\mathcal{B}$. But this implies that $a = bu^k$ for some $b \in \mathcal{B}$, and so $\mathcal{A}$ is generated by $\mathcal{B}$, u and u^{-1}. □

Assume now that $\mathcal{B}$ is a pre-C^*-algebra, with C^*-completion B, and let $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ be a real (even or odd) spectral triple over $\mathcal{B}$. Assume also that it defines a spectral metric space structure for the C^*-algebra B. Then the automorphism α is a *-homomorphism of pre-C^*-algebras and so it is norm decreasing [BC91]. Since the same holds for α^{-1}, actually α is norm preserving, and so it extends to an automorphism $\alpha : B \to B$. This means, in particular, that we can apply the construction of the previous sections to get a real spectral triple over $\mathcal{A}$. We work out some more details of this construction.

Let $\phi(z^k) = u^k$ as above. Since it is both a *-homomorphism and a unitary trivialization, it defines a *-action of the Hopf algebra H on $\mathcal{B}$:

$$z^k \triangleright b = \phi(z^k) b \phi^{-1}(z^k) = u^k b u^{-k} = \alpha^k(b)$$

for any $k \in \mathbb{Z}$ and any $b \in \mathcal{B}$. Since α preserves the C^*-norm of $\mathcal{B}$, this action extends to an action on the C^*-algebra B.

Now we consider an H-equivariant (see definition 3.5.2) real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$. We notice that the requirement of H-equivariance corresponds to the implementability of the automorphism α (cfr. definition 6.1.2). So, under this hypothesis, we can repeat the construction discussed in the previous sections. This yields the following results.

Proposition 6.5.2. *Let $(\mathcal{B}, \mathcal{H}, D, J)$ be an odd real spectral triple of KR-dimension j. Then $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma})$, where $\mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma}$ are as in proposition 6.1.7 and the representation of $\mathcal{A}$ on $\mathcal{H}'$ is induced by that of $\mathcal{B} \rtimes_\alpha \mathbb{Z}$ under the isomorphism of proposition 6.5.1, defines an even real spectral triple over $\mathcal{A}$ of KR-dimension $j+1$.*

Proposition 6.5.3. *Let $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ be an even real spectral triple of KR-dimension j. Then $(\mathcal{A}, \mathcal{H}'_\pm, \hat{D}_\pm, \hat{J}_\pm)$, where $\mathcal{H}'_\pm, \hat{D}_\pm, \hat{J}_\pm$ are as in proposition 6.1.10 and the representation of $\mathcal{A}$ on $\mathcal{H}'$ is induced by that of $\mathcal{B} \rtimes_\alpha \mathbb{Z}$ under the isomorphism of proposition 6.5.1, define two odd real spectral triples over $\mathcal{A}$, of KR-dimension $j+1$, which differ just by a change of sign in the orientation.*

Now assume that $(\mathcal{B}, \mathcal{H}, D)$ is a spectral metric space. This means that the representation $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ is faithful, which implies that also the representation $\hat{\pi} : \mathcal{A} \to \mathcal{L}(\mathcal{H}')$ is faithful. Hence we can consider the C^*-completion A of $\hat{\pi}(\mathcal{A})$ in $\mathcal{L}(\mathcal{H}')$. A is a C^*-algebra which has $\mathcal{A}$ as a dense *-subalgebra. Then we can prove the following fact.

Proposition 6.5.4. *The Lipschitz seminorms on A, induced by the Dirac operators of the spectral triples of proposition 6.5.2 and proposition 6.5.3, are Lip-norms. Hence each of them gives A a structure of compact spectral metric space. Moreover, the two structures associated to the two triples of proposition 6.5.3 are actually the same.*

Proof. Let $\mathcal{A}_0$ be the space of finite linear combinations of elements bu^k, $b \in \mathcal{B}$, $k \in \mathbb{Z}$. Then proposition 6.5.1 implies that $\mathcal{A}_0$ is isomorphic to $\mathcal{B} \rtimes_\alpha \mathbb{Z}$. So the thesis follows directly from proposition 6.1.13 □

6.5.1 Behaviour under gauge transformations

Let $\mathcal{A}, \mathcal{B}$ as above and let ϕ be a unitary trivialization. We can study what happens if we make a gauge transformation. From proposition 4.4.4 we know that a gauge transformation is a convolution invertible map $\Lambda : H \to \mathcal{B}$ with $\Lambda(1) = 1$. We also know that a gauge transformation corresponds to a change of trivialization (see proposition 4.4.5). Since we restricted ourself to unitary trivialization, we cannot consider the whole group $\mathcal{G}(\mathcal{B})$. Instead, we consider only *unitary* gauge transformations; that is, maps $\Lambda \in \mathcal{H}(\mathcal{B})$ which take values in the group $U(\mathcal{B})$ of unitaries of the algebra $\mathcal{B}$.

Corollary 6.5.5. *If Λ takes values in $U(\mathcal{B})$, then $\Lambda^{-1}(z^k) = \Lambda(z^k)^*$. Moreover, if ϕ is a unitary trivialization, then so are $\Lambda^{-1} * \phi$ and $\Lambda * \phi$.*

Actually, since the construction discussed in this chapter depends only on $u = \phi(z)$, the only relevant part of Λ is its value on z. So we consider the following situation. Let $u = \phi(z)$; given any unitary u_Λ, such that $u^* u_\Lambda \in \mathcal{B}$, we define $\phi_\Lambda(z^k) = u_\Lambda^k$, for any k. This new trivialization is the one obtained from ϕ with the gauge transformation $\Lambda(z^k) = u_\Lambda^k u^{-k}$: $\phi_\Lambda = \Lambda * \phi$. Let now $(\mathcal{A}, \mathcal{H}'_\Lambda, \hat{D}_\Lambda, \hat{J}_\Lambda, \hat{\gamma}_\Lambda)$ be the spectral triple constructed from u_Λ, and denote by $\hat{\pi}_\Lambda$ the representation of $\mathcal{A}$ on $\mathcal{H}'_\Lambda$.

Proposition 6.5.6. *The representations $(\mathcal{H}', \hat{\pi})$, $(\mathcal{H}'_\Lambda, \hat{\pi}_\Lambda)$ are unitarily equivalent.*

Proof. Let α_Λ be the automorphism of $\mathcal{B}$ given by the adjoint action of u_Λ: $\alpha^n_\Lambda(b) = u^n_\Lambda b u^{-n}_\Lambda$. Then the representation $\hat{\pi}_\Lambda$ on $\mathcal{H}'_\Lambda$ is given by ($b \in \mathcal{B}$):

$$(\hat{\pi}_\Lambda(b) f^\Lambda)_n = \alpha_\Lambda^{-n}(b) f^\Lambda_n, \qquad (\hat{u}_\Lambda f^\Lambda)_n = f^\Lambda_{n-1}.$$

Now, there is a natural isomorphism $\mathcal{H}' \simeq \mathcal{H}'_\Lambda$, which identifies the two $\mathcal{H}$ factors. Hence we can define a unitary operator $U : \mathcal{H}' \to \mathcal{H}'_\Lambda$ by:

$$(Uf)_n = \alpha^{-n}(\Lambda(z^n)^*) f_n.$$

Then we have

$$(U\hat{\pi}(b)U^* f)_n = \alpha^{-n}(\Lambda(z^n)^*)\alpha^{-n}(b)\alpha^{-n}(\Lambda(z^n)) = (\hat{\pi}_\Lambda(b)f)_n,$$

$$\begin{aligned}(U\hat{u}U^* f)_n &= \alpha^{-n}(\Lambda(z^n)^*)\alpha^{-n+1}(\Lambda(z^{n-1}))f_{n-1}\\ &= \alpha^{-n}(\Lambda(z^n)^*)\alpha^{-n}(\Lambda(z)^*\Lambda(z^n))f_{n-1} = (\hat{\pi}_\Lambda(u)f)_n\end{aligned}$$

where we used the following relation:

$$\begin{aligned}\alpha^{-n+1}(\Lambda(z^{n-1})) &= u^{-n+1}\Lambda(z^{n-1})u^{n-1} = u^{-n+1}u_\Lambda^{n-1}\\ &= u^{-n+1}u_\Lambda^{-1}u_\Lambda^{n} = u^{-n+1}u^{-1}\Lambda(z)^*u_\Lambda^{n} =\\ &= u^{-n}\Lambda(z)^*\Lambda(z^n)u^n = \alpha^{-n}(\Lambda(z)^*\Lambda(z^n))\end{aligned}$$

Hence $U\hat{\pi}U^* = \hat{\pi}_\Lambda$. □

Now we can see what happens, under this unitary equivalence, to the Dirac operator $\hat{D}$. Clearly it can be seen as an operator $\hat{D}^\Lambda$ on $\mathcal{H}'_\Lambda$ simply taking $\hat{D}^\Lambda = U\hat{D}U^*$. It acts on $\mathcal{H}'_\Lambda$ in the following way:

$$(\hat{D}^\Lambda f)_n = (D \otimes \sigma^1 + \alpha^{-n}(\Lambda(z^n)^*)[D, \alpha^{-n}(\Lambda(z^n))] + n \cdot \mathrm{id} \otimes \sigma^2) f_n.$$

Hence, in terms of $\hat{D}_\Lambda$, $\hat{D}^\Lambda$ is given by:

$$\hat{D}^\Lambda = \hat{D}_\Lambda + U[\hat{D}_\Lambda, U^*]. \tag{6.5.1}$$

6.6 An example: the noncommutative torus

We conclude this chapter with an application. It is nothing new nor particular, just an example of how it works. We consider to the (smooth) noncommutative 2-torus $\mathcal{A} = \mathcal{A}(\mathbb{T}^2_\theta)$ as a (cleft) Hopf-Galois extension over a dense subalgebra $\mathcal{B}$ of the algebra of smooth functions over the circle. We denote by U, V the unitary generators of $\mathcal{A}$, with the commutation relation $UV = e^{i\theta}VU$. We identify U with the generator of $\mathcal{B}$, and we take as trivialization $\phi(z^k) = V^k$; of course, this trivialization is unitary. Then the automorphism α is given by:

$$\alpha(U^k) = VU^kV^* = e^{-ik\theta}U^k.$$

Now we consider the following spectral triple over $\mathcal{B}$. We take $\mathcal{H} = L^2([0, 2\pi], d\varphi)$, $D = -i\frac{d}{d\varphi}$, $J = c.c.$ (complex conjugation). Of course, $(\mathcal{B}, \mathcal{H}, D, J)$ is a real spectral triple of dimension 1. We write also explicitly a basis for $\mathcal{H}$; we take eigenvectors of D: $\psi_n = \frac{1}{\sqrt{2\pi}}e^{in\varphi}$, for $n \in \mathbb{Z}$.

We construct now an operator $\rho : \mathcal{H} \to \mathcal{H}$, which makes the automorphism α implementable for the triple $(\mathcal{A}, \mathcal{H}, D, J)$, in the following way. Identify $\mathcal{H}$ with the GNS Hilbert space relative to the state associated to the Lebesgue integral on S^1, so that $\psi_n \sim [U^n]$. Hence, since $\alpha(U^n) = e^{-in\theta}U^n$, we define ρ by:

$$\rho\psi_n = e^{-in\theta}\psi_n.$$

One can easily check that, with such a ρ, α is an implementable automorphism.

Now we consider the Hilbert space $\mathcal{H}' = \mathcal{H} \otimes \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$. We have a basis of $\mathcal{H}'$ given by $\psi_{n,m} = (\psi^+_{n,m}, \psi^-_{n,m})$ $(m, n, \in \mathbb{Z})$, where, for any m, $\psi^\pm_{n,m} = \psi_n \in \mathcal{H}$ and $\delta\psi_{n,m} = m\psi_{n,m}$. The representation $\hat{\pi}$ is given by:

$$\hat{U}\psi_{n,m} \equiv \hat{\pi}(U)\psi_{n,m} = e^{in\theta}\psi_{n+1,m}, \qquad \hat{V}\psi_{n,m} \equiv \hat{\pi}(V)\psi_{n,m} = \psi_{n,m+1}.$$

The real structure is given, according to (6.1.3) and to definition (6.1.6), by

$$\hat{J}\psi_{n,m} = e^{-inm\theta}(\mathrm{id} \otimes \sigma_2)\psi_{-n,-m}.$$

The Dirac operator can be written as

$$D = \delta_1(\mathrm{id} \otimes \sigma^1) + \delta_2(\mathrm{id} \otimes \sigma^2),$$

where $\delta_2 = \delta$ and $\delta_1\psi_{n,m} = n\psi_{n,m}$. The $\mathbb{Z}_2$-grading γ is simply $\gamma = \mathrm{id} \otimes \sigma_3$.

As one can easily check, this is nothing else than the canonical $\mathbb{T}^2$-equivariant real spectral triple (see, e.g., [PS06]) on the noncommutative 2-torus $\mathcal{A}(\mathbb{T}^2_\theta)$. Now, using the results of section 6.2.3, we can associate to the spectral triple constructed above a Hochschild orientation 2-cyle. We begin by noticing that $\mathbf{c}_\mathcal{B} = U^{-1} \otimes 1 \otimes U$ is a Hochschild orientation 1-cycle for the spectral triple over $\mathcal{B}$. Then we consider the twisted shuffle product (6.2.2) of $\mathbf{c}_\mathcal{B}$ with the cocycle $\mathbf{c}^1_H = z^{-1} \otimes 1 \otimes z$ over $H = \mathcal{O}(U(1))$. By definition we have:

$$\begin{aligned}
\mathbf{c}_\mathcal{B} \times_\alpha \mathbf{c}^1_H &= U^*V^* \otimes 1 \otimes V \otimes U - U^*V^* \otimes 1 \otimes \alpha(U) \otimes V \\
&= U^*V^* \otimes 1 \otimes V \otimes U - e^{-i\theta}U^*V^* \otimes 1 \otimes U \otimes V \\
&= U^*V^* \otimes 1 \otimes V \otimes U - V^*U^* \otimes 1 \otimes U \otimes V,
\end{aligned}$$

which is, up to a multiplicative constant, nothing else than the usual orientation cycle of the canonical $\mathbb{T}^2$-equivariant spectral triple over the noncommutative 2-torus (see appendix A; see also [GBFV], chapter 12, pp. 546-548).

CHAPTER 7

Spectral triples over cleft principal $\mathcal{O}(\mathbb{T}^n)$-extensions

We generalize the construction of the previous chapter to cleft quantum principal $\mathbb{T}^n$-bundles: given a cleft $\mathcal{O}(\mathbb{T}^n)$-extension $\mathcal{B} \hookrightarrow \mathcal{A}$, admitting unitary trivializations, and a $\mathcal{O}(\mathbb{T}^n)$-equivariant real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$, we will build, under suitable hypotheses, a real spectral triple over the algebra $\mathcal{A}$. The main difficulty here is that we do not consider only cocycle-free trivializations. As in the $U(1)$ case, we will see that the Dirac operator of this triple allows us to define a differential calculus over $\mathcal{A}$ which is compatible with the de Rham calculus on $\mathcal{O}(\mathbb{T}^n)$, obtaining in this way a structure of quantum principal $\mathbb{T}^n$-bundle over $\mathcal{A}$. We will then discuss the properties of such a triple. In particular we will show that it is a projectable spectral triple, and so we can twist it using a strong connection (cfr. chapter 5).

Actually similar results could be obtained by iterating the procedure of chapter 6, but the direct approach adopted in this chapter may be better suited for a generalization to arbitrary Hopf algebra. Partial results in this direction will be discussed in the next chapter of this thesis.

In this chapter $\mathcal{B}$ will denote a pre-C^*-algebra with C^*-completion B, H will denote the Hopf algebra $\mathcal{O}(\mathbb{T}^n)$ and $\phi : H \to \mathcal{A}$ will be a *unitary* trivialization of the principal extension $\mathcal{B} \hookrightarrow \mathcal{A}$. For a spectral triple we shall mean a real spectral triple fulfilling, at least, the classical dimension and the first order condition.

In the $U(1)$ case we used the fact that a cleft extension is isomorphic to a crossed product algebra $\mathcal{B} \rtimes_\alpha \mathbb{Z}$. Here we shall do something similar: we use the isomorphism $\mathcal{A} \simeq \mathcal{B}\#_\sigma H$ (see proposition 4.4.13 and proposition 4.4.21).

7.1 Properties of real cleft extensions $\mathcal{A} \simeq \mathcal{B}\#_\sigma\mathcal{O}(\mathbb{T}^n)$

First of all we discuss some properties of cleft extensions $\mathcal{B} \hookrightarrow \mathcal{A}$ with Hopf algebra $H = \mathcal{O}(\mathbb{T}^n)$ and unitary trivialization ϕ. In particular we will compute some useful relations, using the twisted module condition and the cocycle condition, involving the cocycle σ and the weak action of H.

Lemma 7.1.1. *Let ϕ be a unitary trivialization and let σ be the cocycle associated to it. Then, for every $r, s \in \mathbb{Z}^n$,*

$$\sigma(z^r, z^s)\sigma(z^r, z^s)^* = \sigma(z^r, z^s)^*\sigma(z^r, z^s) = 1.$$

Proof. First of all we notice that, in the case $H = \mathcal{O}(\mathbb{T}^n)$, the condition of unitarity of a trivialization reads:

$$\phi^{-1}(z^k)^* = \phi(z^k) \qquad \forall k \in \mathbb{Z}^n. \tag{7.1.1}$$

Then we have:

$$\begin{aligned}\sigma(z^r, z^s)\sigma(z^r, z^s)^* &= \phi(z^r)\phi(z^s)\phi^{-1}(z^{r+s})\phi^{-1}(z^{r+s})^*\phi(z^s)^*\phi(z^r)^* \\ &= \phi(z^r)\phi(z^s)\phi^{-1}(z^{r+s})\phi(z^{r+s})\phi^{-1}(z^s)\phi^{-1}(z^r) = 1\end{aligned}$$

In the same way one proves that $\sigma(z^r, z^s)^*\sigma(z^r, z^s) = 1$. □

Lemma 7.1.2. *Let ϕ be a unitary trivialization and let σ be the cocycle associated to it. Then, for every $k \in \mathbb{Z}^n$,*

$$z^k \triangleright \sigma(z^{-k}, z^k)^* = \sigma(z^k, z^{-k})^*.$$

Proof. Using (7.1.1) we get:

$$\begin{aligned}z^k \triangleright \sigma(z^{-k}, z^k)^* &= \phi(z^k)\phi(z^k)^*\phi(z^{-k})^*\phi^{-1}(z^k) \\ &= \phi(z^k)\phi^{-1}(z^k)\phi(z^{-k})^*\phi(z^k)^* \\ &= \phi(z^{-k})^*\phi(z^k)^* = \sigma(z^k, z^{-k})^*.\end{aligned}$$

□

7.2 Weak actions and equivariant spectral triples

We shall deal with crossed product algebras $\mathcal{A} = \mathcal{B}\#_\sigma H$, in general with non-trivial cocycle σ. This implies that we shall not have an action but only a weak *-action of H on $\mathcal{B}$. So we need to give the definition of equivariance of a real spectral triple with respect to a weak *-action of a Hopf *-algebra H.

Hence, assume to be given a weak action of H on $\mathcal{B}$, with cocycle σ. Consider a real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$; let π be the representation of $\mathcal{B}$ on $\mathcal{H}$, and let π° the representation of the opposite algebra induced by the real structure J; namely,

$$\pi^\circ(b) = J\pi(b^*)J^{-1}.$$

Then we give the following definition.

Definition 7.2.1. *A real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ is* equivariant *w.r.t. a weak action of the Hopf algebra H, with associated cocycle σ, if:*

(i) *there is a weak *-action of H on a dense subspace $V \subset \mathcal{H}$ which fulfils the following properties:*

1. *$h \triangleright \pi(b)v = \pi(h_{(1)} \triangleright b)(h_{(2)} \triangleright v)$ for any $h \in H$, $b \in \mathcal{B}$, $v \in V$,*

2. twisted module condition: *for any $h, l, m \in H$ and any $v \in V$*

$$\pi^\circ(\sigma(h_{(2)}, l_{(2)}))(h_{(1)} \triangleright l_{(1)} \triangleright v) = \pi(\sigma(h_{(1)}, l_{(1)}))(h_{(2)} l_{(2)} \triangleright v),$$

3. any $h \in H$, seen as an operator on V, can be extended to a bounded operator on $\mathcal{H}$;

(ii) $h \triangleright Dv = D(h \triangleright v)$ *for any $h \in H$, $v \in V$;*

(iii) $[\pi(\sigma(h,l)), D] = [\pi^\circ(\sigma(h,l)), D] = 0$ *for any $h, l \in H$;*

(iv) $h \triangleright \gamma v = \gamma(h \triangleright v)$ *for any $h \in H$, $v \in V$;*

(v) $h \triangleright Jv = J((Sh)^* \triangleright v)$ *for any $h \in H$, $v \in V$;*

(vi) $\langle w, h \triangleright v \rangle = \langle h^* \triangleright w, v \rangle$ *for any $h \in H$, $v, w \in V$;*

(vii) *for any $h \in H$, $v \in V$: $\pi(\sigma(h_{(2)}, S^{-1}h_{(1)}))\pi^\circ(\sigma(S^{-1}h^*_{(3)}, h^*_{(4)})^*)v = \varepsilon(h)v$.*

Remark 7.2.2. The only "unnatural" assumption is condition (vii); as we will see it is a sufficient condition, at least for $\mathbb{T}^n$-bundles, to build a *-representation of $\mathcal{B}\#_\sigma H$ on the tensor product of $\mathcal{H}$ with a suitable Hilbert space of spinors on H.

7.3 Spectral triples over cleft $\mathcal{O}(\mathbb{T}^2)$-extensions

Now we begin the construction of the spectral triples over $\mathcal{A}$, where $\mathcal{A}$ is a cleft $\mathcal{O}(\mathbb{T}^2)$-extension admitting a unitary trivialization ϕ.. We present first a simpler case: we take the dimension of the torus, which is the structure group of our bundle, to be equal to 2. We do it because in this case we have not to take care of some subtleties coming from the dependence on the KR-dimension of the commutation relations between the operators appearing in a real spectral triple (see [DD11] and the discussion in the next sections of this thesis). Nevertheless, we shall see that all the results and demonstrations in this sections, which do not involve commutation relations depending on the KR-dimension, apply with no changes to the general case of T^n bundles, to be discussed later.

7.3.1 Construction of the real spectral triples

Let $\mathcal{A} \simeq \mathcal{B}\#_\sigma H$, the isomorphism being determined by the (unitary) trivialization ϕ, which induces a weak action of H on $\mathcal{B}$ with cocycle σ. Let $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ be an H-equivariant real spectral triple (in the odd dimensional case $\gamma = \mathrm{id}$), in the sense of definition 7.2.1. Consider the Hilbert space $\mathcal{H}' = \mathcal{H} \otimes \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$. Choose an orthonormal basis $\{e_k\}_{k \in \mathbb{Z}^2}$ of $\ell^2(\mathbb{Z}^2)$; we define a representation π_H of H on $\ell^2(\mathbb{Z}^2)$ simply by setting:

$$\pi_H(z^h)e_k = e_{k+h}.$$

π_H is a *-representation, and each $\pi_H(z^k)$ is a unitary operator on $\ell^2(\mathbb{Z}^2)$. We introduce also a left H coaction ρ_L on[1] $\mathcal{H}$, defined on the basis by

$$\rho_L(e_k) = z^k \otimes e_k.$$

[1]Actually ρ_L is a coaction $\rho_L : V \to H \otimes V$, where $V \subseteq \mathcal{H}$ is the subspace of vector $\sum_k \alpha_k e_k$, $\{\alpha_k\} \subset \mathbb{C}$ with only a finite number of elements different from zero.

Now we use these tools to define a representation of $\mathcal{A} \simeq \mathcal{B}\#_\sigma H$ on $\mathcal{H}'$. Let $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ be the representation of $\mathcal{B}$ on $\mathcal{H}$; with a little abuse of notation we denote also by π the natural extension of this representation to $\mathcal{H} \otimes \mathbb{C}^2$ (and by π° the analogous extension of the representation of the opposite algebra induced by the real structure J). Then we define

$$\hat{\pi}(b\#h)(v \otimes w) = \pi(b)\pi^\circ(\sigma(h_{(2)}, w_{(-1)}))(h_{(1)} \triangleright v) \otimes \pi_H(h_{(3)})w_{(0)} \tag{7.3.1}$$

Proposition 7.3.1. *$\hat{\pi} : \mathcal{A} \simeq \mathcal{B}\#H \to \mathcal{L}(\mathcal{H}')$, defined by* (7.3.1)*, is a *-representation of $\mathcal{A}$.*

Proof. First of all we show that it is a representation. For $a, b \in \mathcal{B}$ and $h, l \in H$ we have:

$$\begin{aligned}\hat{\pi}(a\#h)\hat{\pi}(b\#l)v \otimes w &= \hat{\pi}(a\#h)\left[\pi(b)\pi^\circ(\sigma(l_{(2)}, w_{(-1)}))(l_{(1)} \triangleright v) \otimes \pi_H(l_{(3)})w_{(0)}\right] \\ &= \pi(a)\pi(h_{(1)} \triangleright b)\pi^\circ(\sigma(h_{(4)}, l_{(3)}w_{(-1)})) \\ &\quad \cdot \pi^\circ(h_{(3)} \triangleright \sigma(l_{(2)}, w_{(-2)}))(h_{(2)} \triangleright l_{(1)} \triangleright v) \otimes \pi_H(h_{(5)}l_{(4)})w_{(0)}.\end{aligned} \tag{7.3.2}$$

$$\begin{aligned}\hat{\pi}((a\#h)(b\#l))v \otimes w &= \hat{\pi}(a(h_{(1)} \triangleright b)\sigma(h_{(2)}, l_{(1)})\#h_{(3)}l_{(2)})v \otimes w \\ &= \pi(a(h_{(1)} \triangleright b)\sigma(h_{(2)}, l_{(1)}))\pi^\circ(\sigma(h_{(4)}l_{(3)}, w_{(-1)})) \\ &\quad \cdot (h_{(3)}l_{(2)} \triangleright v) \otimes \pi_H(h_{(5)}l_{(4)})w_{(0)}\end{aligned} \tag{7.3.3}$$

Using now the twisted module condition and the cocycle conditions we see that we can perform the following replacements: in equation (7.3.2)

$$\begin{aligned}&\pi^\circ(\sigma(h_{(4)}, l_{(3)}w_{(-1)}))\pi^\circ(h_{(3)} \triangleright \sigma(l_{(2)}, w_{(-2)})) \\ &\qquad \to \pi^\circ(\sigma(h_{(4)}l_{(3)}, w_{(-1)}))\pi^\circ(\sigma(h_{(3)}, l_{(2)})),\end{aligned}$$

and in equation (7.3.3)

$$\pi^\circ(\sigma(h_{(2)}, l_{(1)}))(h_{(3)}l_{(2)} \triangleright v) \quad \to \quad \pi^\circ(\sigma(h_{(3)}, l_{(2)}))(h_{(2)} \triangleright l_{(1)} \triangleright v).$$

After this operation we see that (7.3.2) and (7.3.3) actually coincide, and so $\hat{\pi}$ is a representation of $\mathcal{A}$. Now we show that it is a *-representation. The scalar product on $\mathcal{H}'$ is simply the product of the scalar product on $\mathcal{H} \otimes \mathbb{C}^2$ (which comes directly from that of $\mathcal{H}$) with that one of $\ell^2(\mathbb{Z}^2)$. It is enough to show that $\hat{\pi}((a\#z^k)^*) = \hat{\pi}(a\#z^k)^*$. Recall that $(a\#z^k)^* = \sigma(z^{-k}, z^k)^*(z^{-k} \triangleright a^*)\#z^{-k}$. We have:

$$\begin{aligned}\left\langle v' \otimes e_r, \hat{\pi}((a\#z^k)^*)v \otimes e_s\right\rangle &= \left\langle v' \otimes e_r, \hat{\pi}(\sigma(z^{-k}, z^k)^*(z^{-k} \triangleright a^*)\#z^{-k})v \otimes e_s\right\rangle \\ &= \left\langle v' \otimes e_r, \pi(\sigma(z^{-k}, z^k)^*(z^{-k} \triangleright a^*))\pi^\circ(\sigma(z^{-k}, z^s))(z^{-k} \triangleright v) \otimes e_{s-k}\right\rangle.\end{aligned} \tag{7.3.4}$$

But we know that $\{e_k\}$ is an orthonormal basis, so (7.3.4) is different from zero if and only if $s = r + k$. In this case we get:

$$\begin{aligned}&\left\langle v \otimes e_r, \hat{\pi}((a\#z^k)^*)v' \otimes e_s\right\rangle = \\ &= \left\langle \pi((z^{-k} \triangleright a^*)^*)\pi(\sigma(z^{-k}, z^k))\pi^\circ(\sigma(z^{-k}, z^{k+r})^*)v' \otimes e_{k+r}, (z^{-k} \triangleright v) \otimes e_{k+r}\right\rangle\end{aligned}$$

$$= \left\langle \pi(z^k \triangleright (z^{-k} \triangleright a^*)^*)\pi(z^k \triangleright \sigma(z^{-k}, z^k))\pi^\circ(z^k \triangleright \sigma(z^{-k}, z^{k+r})^*)(z^k \triangleright v') \otimes e_{k+r}, v \otimes e_{k+r} \right\rangle$$
$$= \left\langle \pi(\sigma(z^k, z^{-k}))\pi(a)\pi^\circ(z^k \triangleright \sigma(z^{-k}, z^{k+r})^*)(z^k \triangleright v') \otimes e_{k+r}, v \otimes e_{k+r} \right\rangle \qquad (7.3.5)$$

where, in the last equality, we have used the twisted module condition and the relation of lemma 7.1.2. Now we use the cocycle condition, the twisted module condition, the results of lemma 7.1.1 and lemma 7.1.2 and the fact that $\sigma(z^k, 1) = \sigma(1, z^k) = 1$ to rewrite (7.3.5) as:

$$\begin{aligned} &\left\langle v \otimes e_r, \hat{\pi}((a\#z^k)^*)v' \otimes e_s \right\rangle = \\ &= \left\langle \pi(\sigma(z^k, z^{-k}))\pi^\circ(\sigma(z^k, z^{-k})^*)\pi(a)\pi^\circ(\sigma(z^k, z^r))(z^k \triangleright v') \otimes e_{k+r}, v \otimes e_{k+r} \right\rangle \end{aligned} \qquad (7.3.6)$$

which, due to property (vii) in definition 7.2.1, is equal to $\langle \hat{\pi}(a\#z^k)v' \otimes e_r, v \otimes e_{r+k} \rangle$. Therefore $\hat{\pi}$ is a *-representation of $\mathcal{A}$. Remark: we often used the fact that $[\pi(a), \pi^\circ(b)] = 0$ for any $a, b \in \mathcal{B}$. ☐

Before going on we introduce an action of the Lie algebra of $\mathbb{T}^2$ on $\mathcal{H}'$. The left coaction ρ_L trivially extends to (a dense subspace of) $\mathcal{H}'$; so, if we denote by δ_1 and δ_2 the representative of the two commuting generators of the Lie algebra of $\mathbb{T}^2$, we can define their action in the following way:

$$\rho_L(w) = z^k \otimes w \quad \Rightarrow \quad \delta_j(w) \equiv k_j w.$$

Now we can use $\mathcal{H}'$ to build a spectral triple over $\mathcal{A}$ (for the moment, without real structure): the only thing we need is a Dirac operator. Take:

$$\hat{D} = D \otimes \mathrm{id}_{\ell^2(\mathbb{Z}^2)} \otimes \sigma^1 + \delta_1 \circ (\mathrm{id} \otimes \sigma^2) + \delta_2 \circ (\mathrm{id} \otimes \sigma^3). \qquad (7.3.7)$$

Proposition 7.3.2. *$(\mathcal{A}, \mathcal{H}', \hat{D})$ is a spectral triple. Moreover, $\phi^{-1}(z^k)[\hat{D}, \phi(z^k)]$ commutes with the elements of $\mathcal{A}$.*

Proof. In order to see that $(\mathcal{A}, \mathcal{H}', \hat{D})$ is a spectral triple we need to show that $\hat{D}$ has compact resolvent and that it has bounded commutators with all the elements of $\mathcal{A}$. Now, if λ_j, $j \in \mathbb{Z}$, are the eigenvalues on D (and therefore they goes to infinity as j goes to infinity), then the eigenvalues of $\hat{D}$ are $t_{j,m,n} = \pm\sqrt{\lambda_j^2 + m^2 + n^2}$ for $j, m, n \in \mathbb{Z}$, and so their inverses converge to zero as either j or m or n go to infinity. Hence $\hat{D}$ has compact resolvent.

Next, by direct computation, using the equivariance of the trivialization ϕ and the identification $1\#h = \phi(h)$, we get, for any $k \in \mathbb{Z}^2$,

$$\phi^{-1}(z^{-k})[\hat{D}, \phi(z^k)] = k_1 \mathrm{id} \otimes \sigma^2 + k_2 \mathrm{id} \otimes \sigma^3. \qquad (7.3.8)$$

In particular the commutator between $\hat{D}$ and $\phi(h)$ is bounded for any $h \in H$ and $\phi^{-1}(z^k)[\hat{D}, \phi(z^k)]$ commutes with $\hat{\pi}(\mathcal{A})$.

We are left with the proof that any commutator $[\hat{D}, \hat{\pi}(a\#h)]$ is a bounded operator. Without loss of generality, we can take $h = z^k$ for some $k \in \mathbb{Z}^2$. Then the action of such a commutator on a vector $v \otimes e_s \in \mathcal{H}'$ is given by:

$$\begin{aligned}[\hat{D}, \hat{\pi}(a\#z^k)]v \otimes e_s &= [D \otimes \sigma^1, \pi(a)\pi^\circ(\sigma(z^k, z^s))](z^k \triangleright v) \otimes e_{k+s} \\ &\quad + (k_1\sigma^2 + k_2\sigma^3)\pi(a)\pi^\circ(\sigma(z^k, z^s))(z^k \triangleright v) \otimes e_{k+s} \\ &= [D \otimes \sigma^1, \pi(a)]\pi^\circ(\sigma(z^k, z^s))(z^k \triangleright v) \otimes e_{k+s} \\ &\quad + (k_1\sigma^2 + k_2\sigma^3)\pi(a)\pi^\circ(\sigma(z^k, z^s))(z^k \triangleright v) \otimes e_{k+s}.\end{aligned} \tag{7.3.9}$$

Thus such a commutator can be written as a sum of two operators, and it is enough to show that each of them is a bounded operator. By hypothesis, the commutator of D with $\pi(a)$ is bounded. Moreover, condition (i).3 of definition 7.2.1 implies that z^k acts on $\mathcal{H}$ as a bounded operator. Next, the map $e_s \mapsto e_{s+k}$ extends to a bounded (actually, unitary) operator on $\mathcal{H}$. So we have only to show that $\|\pi^\circ(\sigma(z^k, z^s))\| \leq C$, C being some real positive constant, independently from the value of $s \in \mathbb{Z}^2$. But we know (see lemma 7.1.1) that $\sigma(z^k, z^s)$ is unitary, hence such a constant exists and it can, actually, be taken equal to 1. □

The next step in the construction of a real spectral triple is the definition of a real structure. First of all, let J_H be the antiunitary operator $J_H : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$ defined by:

$$J_H(\lambda e_k) = \overline{\lambda} e_{-k} \qquad \forall k \in \mathbb{Z}^2, \lambda \in \mathbb{C}.$$

Then we define an operator $\widetilde{J} : \mathcal{H}' \to \mathcal{H}'$ by:

$$\widetilde{J}(v \otimes w) = \pi(\sigma(S^{-1}w_{(-2)}, w_{(-3)})^*)(w^*_{(-1)} \triangleright Jv) \otimes J_H w_{(0)} \tag{7.3.10}$$

(we consider $\widetilde{J}$ acting as the identity on the $\mathbb{C}^2$ factor).

Lemma 7.3.3. *The operator $\widetilde{J}$ defined in* (7.3.10) *is an antiunitary operator. Moreover:*
(i) $\widetilde{J}^2 = \varepsilon \mathrm{id}_{\mathcal{H}'}$, *where* $J^2 = \varepsilon \mathrm{id}_{\mathcal{H}}$*;*
(ii) $\widetilde{J}$ *maps* $\hat{\pi}(\mathcal{A})$ *into its commutant;*
(iii) $\widetilde{J} \circ (D \otimes \mathrm{id}) = \varepsilon'(D \otimes \mathrm{id}) \circ \widetilde{J}$, *where* $JD = \varepsilon' DJ$*;*
(iv) *if* $(\mathcal{B}\,H, D, J, \gamma)$ *is an even dimensional triple, then* $\widetilde{J} \circ (\gamma \otimes \mathrm{id}) = \varepsilon''(\gamma \otimes \mathrm{id}) \circ \widetilde{J}$, *where* $J\gamma = \varepsilon''\gamma J$.

Proof. First of all let us show that $\widetilde{J}$ is antiunitary. We have:

$$\begin{aligned}\left\langle v \otimes e_r, \widetilde{J}(v' \otimes e_s)\right\rangle &= \left\langle v \otimes e_r, \pi(\sigma(z^{-s}, z^s)^*)(z^{-s} \triangleright Jv') \otimes J_H e_s\right\rangle \\ &= \left\langle \pi(z^s \triangleright \sigma(z^{-s}, z^s))(z^s \triangleright v) \otimes e_r, Jv' \otimes e_{-s}\right\rangle \\ &= \left\langle \pi(\sigma(z^s, z^{-s}))(z^s \triangleright v) \otimes e_r, Jv' \otimes e_{-s}\right\rangle\end{aligned} \tag{7.3.11}$$

where we used the result of lemma 7.1.2. Now, since $\{e_k\}$ is an orthonormal basis, (7.3.11) is different from zero only if $s = -r$; and in this case we obtain:

$$\begin{aligned}\left\langle v \otimes e_r, \widetilde{J}(v' \otimes e_{-r})\right\rangle &= \left\langle \pi(\sigma(z^{-r}, z^r))(z^{-r} \triangleright v) \otimes e_r, Jv' \otimes e_r\right\rangle \\ &= \left\langle v' \otimes e_{-r}, J\pi(\sigma(z^{-r}, z^r))(z^{-r} \triangleright v) \otimes J_H e_r\right\rangle\end{aligned}$$

$$\begin{aligned}
&= \langle v' \otimes e_{-r}, J\pi(\sigma(z^{-r}, z^r))J^{-1}J(z^{-r} \triangleright v) \otimes J_H e_r \rangle \\
&= \langle v' \otimes e_{-r}, \pi^\circ(\sigma(z^{-r}, z^r)^*)J(z^{-r} \triangleright v) \otimes J_H e_r \rangle \\
&= \langle v' \otimes e_{-r}, \pi(\sigma(z^{-r}, z^r)^*)\pi(\sigma(z^{-r}, z^r))\pi^\circ(\sigma(z^{-r}, z^r)^*)J(z^{-r} \triangleright v) \otimes J_H e_r \rangle \\
&= \langle v' \otimes e_{-r}, \pi(\sigma(z^{-r}, z^r)^*)(z^{-r} \triangleright Jv) \otimes J_H e_r \rangle \\
&= \left\langle v' \otimes e_{-r}, \widetilde{J}(v \otimes e_r) \right\rangle
\end{aligned} \tag{7.3.12}$$

(where we used the results of lemma 7.1.1 and 7.1.2 and property (vii) of definition 7.2.1) which shows that $\widetilde{J}$ is antiunitary. Next we compute $\widetilde{J}^2$. We have

$$\begin{aligned}
\widetilde{J}^2(v \otimes e_k) &= \widetilde{J}\left(\pi(\sigma(z^{-k}, z^k)^*)(z^{-k} \triangleright Jv) \otimes e_{-k}\right) \\
&= \pi(\sigma(z^k, z^{-k})^*)J\pi(z^k\sigma(z^{-k}, z^k)^*)J(z^k \triangleright z^{-k} \triangleright v) \otimes e_k \\
&= \varepsilon\pi(\sigma(z^k, z^{-k})^*)J\pi(\sigma(z^k, z^{-k})^*)J^{-1}(z^k \triangleright z^{-k} \triangleright v) \otimes e_k \\
&= \varepsilon\pi(\sigma(z^k, z^{-k})^*)\pi^\circ(\sigma(z^k, z^{-k}))(z^k \triangleright z^{-k} \triangleright v) \otimes e_k \\
&= \varepsilon\pi(\sigma(z^k, z^{-k})^*)\pi(\sigma(z^k, z^{-k}))v \otimes e_k \\
&= \varepsilon v \otimes e_k,
\end{aligned} \tag{7.3.13}$$

where we used the twisted module condition together with the result of lemma 7.1.1. (i) follows directly from (7.3.13). Next we prove (ii). First of all we notice that:

$$\rho_L(J_H w) = ({}^* \otimes J_H)\rho_L(w). \tag{7.3.14}$$

Before completing the proof of this lemma, we state and prove two lemmas:

Lemma 7.3.4. *For every $k, r \in \mathbb{Z}^n$ we have:*

$$z^{k-r} \triangleright z^r \triangleright \sigma(z^{-k}, z^k)^* = \sigma(z^{k-r}, z^r)\sigma(z^k, z^{-k})^*\sigma(z^{k-r}, z^r)^*$$

Proof. It follows by direct computation. □

Lemma 7.3.5. *The action of the opposite algebra $\mathcal{A}^\circ$ on $\mathcal{H}'$ induced by $\widetilde{J}$ is given by:*

$$\widetilde{J}\hat{\pi}(a\#z^r)\widetilde{J}^{-1}(v \otimes e_k) = \pi^\circ((z^{k-r} \triangleright a)^*)\pi^\circ(\sigma(z^{k-r}, z^r)^*)v \otimes e_{k-r}. \tag{7.3.15}$$

Proof. Let ε be as in (i) of lemma 7.3.3. Then we have

$$\begin{aligned}
&\widetilde{J}\hat{\pi}(a\#h)\widetilde{J}^{-1}(v \otimes e_k) = \varepsilon\widetilde{J}\hat{\pi}(a\#h)\widetilde{J}(v \otimes e_k) \\
&\quad = J\pi(z^{k-r} \triangleright a)J^{-1}\pi(\sigma(z^{k-r}, z^{r-k})^*(z^{k-r} \triangleright \sigma(z^r, z^{-k})^*)) \\
&\qquad \cdot J\pi(z^{k-r} \triangleright z^r \triangleright \sigma(z^{-k}, z^k)^*)J^{-1}(z^{k-r} \triangleright z^r \triangleright z^{-k} \triangleright v) \otimes e_{k-r} \\
&\quad = J\pi(z^{k-r} \triangleright a)J^{-1}\pi(\sigma(z^k, z^{-k})^*\sigma(z^{k-r}, z^r)^*) \\
&\qquad \cdot J\pi(\sigma(z^{k-r}, z^r)\sigma(z^k, z^{-k})^*\sigma(z^{k-r}, z^r)^*)J^{-1}(z^{k-r} \triangleright z^r \triangleright z^{-k} \triangleright v) \otimes e_{k-r} \\
&\quad = J\pi(z^{k-r} \triangleright a)J^{-1}\pi(\sigma(z^k, z^{-k})^*\sigma(z^{k-r}, z^r)^*)
\end{aligned} \tag{7.3.16}$$

$$\cdot \pi^\circ(\sigma(z^{k-r}, z^r)^*)\pi^\circ(\sigma(z^k, z^{-k}))\pi^\circ(\sigma(z^{k-r}, z^r))(z^{k-r} \triangleright z^r \triangleright z^{-k} \triangleright v) \otimes e_{k-r}$$

where we used lemma 7.3.4 for the last equality. Now, using the twisted module condition, we get from equation 7.3.16 the following expression.

$$\begin{aligned}
&\widetilde{J}\hat{\pi}(a\#h)\widetilde{J}^{-1}(v \otimes e_k) = \\
&\quad = J\pi(z^{k-r} \triangleright a)J^{-1}\pi(\sigma(z^k, z^{-k})^*\sigma(z^{k-r}, z^r)^*) \\
&\qquad \cdot \pi^\circ(\sigma(z^{k-r}, z^r)^*)\pi^\circ(\sigma(z^k, z^{-k}))\pi(\sigma(z^{k-r}, z^r))(z^k \triangleright z^{-k} \triangleright v) \otimes e_{k-r} \\
&\quad = J\pi(z^{k-r} \triangleright a)J^{-1}\pi(\sigma(z^k, z^{-k})^*\sigma(z^{k-r}, z^r)^*)\pi(\sigma(z^{k-r}, z^r)) \\
&\qquad \cdot \pi^\circ(\sigma(z^{k-r}, z^r)^*)\pi^\circ(\sigma(z^k, z^{-k}))(z^k \triangleright z^{-k} \triangleright v) \otimes e_{k-r} \\
&\quad = J\pi(z^{k-r} \triangleright a)J^{-1}\pi(\sigma(z^k, z^{-k})^*) \\
&\qquad \cdot \pi^\circ(\sigma(z^{k-r}, z^r)^*)\pi(\sigma(z^k, z^{-k}))v \otimes e_{k-r} \\
&\quad = J\pi(z^{k-r} \triangleright a)J^{-1}\pi^\circ(\sigma(z^{k-r}, z^r)^*)v \otimes e_{k-r}
\end{aligned} \tag{7.3.17}$$

from which (7.3.15) follows directly. □

We come back to the proof of lemma 7.3.3. We compute explicitly the commutator between $\widetilde{J}\hat{\pi}(a\#h)\widetilde{J}^{-1}$ and $\hat{\pi}(b\#l)$ using (7.3.15).

$$\begin{aligned}
&[\widetilde{J}\hat{\pi}(a\#z^r)\widetilde{J}^{-1}, \hat{\pi}(b\#z^s)](v \otimes w) = \\
&\quad = \pi^\circ((z^{k+s-r} \triangleright a)^*)\pi^\circ(\sigma(z^{k+s-r}, z^r)^*)\pi(b)\pi^\circ(\sigma(z^s, z^k))(z^s \triangleright v) \otimes e_{k+s-r} \\
&\qquad - \pi(b)\pi^\circ(\sigma(z^s, z^{k-r}))\pi^\circ(z^s \triangleright (z^{k-r} \triangleright a)^*)\pi^\circ(z^s \triangleright \sigma(z^{k-r}, z^r)^*)(z^s \triangleright v) \otimes e_{k+s-r}.
\end{aligned} \tag{7.3.18}$$

Using the twisted module condition and the following relation,

$$(z^s \triangleright \sigma(z^{k-r}, z^r)^*)\sigma(z^s, z^{k-r}) = \sigma(z^s, z^k)\sigma(z^{k+s-r}, z^r)^*,$$

we can rewrite equation (7.3.18) as follows:

$$\begin{aligned}
&[\widetilde{J}\hat{\pi}(a\#z^r)\widetilde{J}^{-1}, \hat{\pi}(b\#z^s)](v \otimes w) = \\
&\quad = \pi^\circ((z^{k+s-r} \triangleright a)^*)\pi^\circ(\sigma(z^{k+s-r}, z^r)^*)\pi(b)\pi^\circ(\sigma(z^s, z^k))(z^s \triangleright v) \otimes e_{k+s-r} \\
&\qquad - \pi(b)\pi^\circ((z^{k+s-r} \triangleright a)^*)\pi^\circ(\sigma(z^{k+s-r}, z^r)^*)\pi^\circ(\sigma(z^s, z^k))(z^s \triangleright v) \otimes e_{k+s-r} = 0
\end{aligned} \tag{7.3.19}$$

since J maps $\pi(a)$ into its commutant on $\mathcal{H} \otimes \mathbb{C}^2$. The proof of (iii) and (iv) is straightforward: they are direct consequences of the properties of equivariance stated in definition 7.2.1. □

Now, following [DD11], we can define, in terms of $\widetilde{J}$, a real structure $\hat{J}$ with the correct commutation relations, obtaining in this way a real spectral triple over the algebra $\mathcal{A}$ (in the even dimensional case we can define a $\mathbb{Z}^2$-grading $\hat{\gamma}$, too). We consider separately the odd dimensional and the even dimensional case.

Odd dimensional case. Let $(\mathcal{B}, \mathcal{H}, D, J)$ be a odd real spectral triple of KR-dimension j. Then we consider the following operator on $\mathcal{H}'$.

Definition 7.3.6. *Let $\widetilde{J}$ as above. We define an operator $\hat{J}: \mathcal{H}' \to \mathcal{H}'$ by:*

$$\hat{J} = \begin{cases} (\mathrm{id} \otimes \sigma^2) \circ \widetilde{J} & \text{if } j \equiv 1 \pmod 4 \\ \\ (\mathrm{id} \otimes \sigma^3) \circ \widetilde{J} & \text{if } j \equiv 3 \pmod 4 \end{cases} \tag{7.3.20}$$

where id *is the identity operator on $\mathcal{H} \otimes \ell^2(\mathbb{Z}^2)$.*

Proposition 7.3.7. *Let $\hat{J}$ be defined by* (7.3.20)*. Then $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J})$ is a real spectral triple of KR-dimension $j+2$.*

Proof. Due to proposition 7.3.2 and lemma 7.3.3 we have only to prove that $\hat{J}$ and $\hat{D}$ fulfil the right commutation relations and that the first order condition holds. The commutation relations can be checked by direct computation. Here we prove only the first order condition. By direct computation (using the equivariance of the Dirac operator D) we can see that:

$$\begin{aligned} [\hat{D}, \hat{\pi}(a\# z^r)](v \otimes e_k) =& \sigma^1[D, \pi(a)\pi^\circ(\sigma(z^r, z^k))](z^r \triangleright v) \otimes e_{k+r} \\ &+ (r_1\sigma^2 + r_2\sigma^3)\pi(a)\pi^\circ(\sigma(z^r, z^k))(z^r \triangleright v) \otimes e_{k+r}. \end{aligned} \tag{7.3.21}$$

Using (7.3.21) and (7.3.15) we get:

$$\begin{aligned} &[[\hat{D}, \hat{\pi}(a\# z^r)], \hat{J}\hat{\pi}(b\# z^s)\hat{J}^{-1}](v \otimes e_k) \\ &\quad = [\hat{D}, \hat{\pi}(a\# z^r)] \left(\pi^\circ(z^{k-s} \triangleright b^*)\pi^\circ(\sigma(z^{k-s}, z^s)^*)v \otimes e_{k-s}\right) \\ &\qquad - \hat{J}\hat{\pi}(b\# z^s)\hat{J}^{-1}\left(\sigma^1[D, \pi(a)\pi^\circ(\sigma(z^r, z^k))](z^r \triangleright v) \otimes e_{k+r}\right) \\ &\qquad - \hat{J}\hat{\pi}(b\# z^s)\hat{J}^{-1}\left((r_1\sigma^2 + r_2\sigma^3)\pi(a)\pi^\circ(\sigma(z^r, z^k))(z^r \triangleright v) \otimes e_{k+r}\right) \\ &\quad = \sigma^1[D, \pi(a)\pi^\circ(\sigma(z^r, z^{k-s}))]\pi^\circ(z^r \triangleright z^{k-r} \triangleright b^*)\pi^\circ(z^r \triangleright \sigma(z^{k-s}, z^s))(z^r \triangleright v) \otimes e_{k+r-s} \\ &\qquad + (r_1\sigma^2 + r_2\sigma^3)\pi(a)\pi^\circ(\sigma(z^r, z^{k-s}))\pi^\circ(z^r \triangleright z^{k-r} \triangleright b^*)\pi^\circ(z^r \triangleright \sigma(z^{k-s}, z^s))(z^r \triangleright v) \otimes e_{k+r-s} \\ &\qquad - \sigma^1\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)[D, \pi(a)\pi^\circ(\sigma(z^r, z^k))](z^r \triangleright v) \otimes e_{k+r-s} \\ &\qquad - (r_1\sigma^2 + r_2\sigma^3)\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi(a)\pi^\circ(\sigma(z^r, z^k))(z^r \triangleright v) \otimes e_{k+r-s} \\ &\quad = \sigma^1[D, \pi(a)]\pi^\circ(\sigma(z^r, z^{k-s}))\pi^\circ(z^r \triangleright z^{k-r} \triangleright b^*)\pi^\circ(z^r \triangleright \sigma(z^{k-s}, z^s))(z^r \triangleright v) \otimes e_{k+r-s} \\ &\qquad + (r_1\sigma^2 + r_2\sigma^3)\pi(a)\pi^\circ(\sigma(z^r, z^{k-s}))\pi^\circ(z^r \triangleright z^{k-r} \triangleright b^*)\pi^\circ(z^r \triangleright \sigma(z^{k-s}, z^s))(z^r \triangleright v) \otimes e_{k+r-s} \\ &\qquad - \sigma^1\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))[D, \pi(a)](z^r \triangleright v) \otimes e_{k+r-s} \\ &\qquad - (r_1\sigma^2 + r_2\sigma^3)\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))\pi(a)(z^r \triangleright v) \otimes e_{k+r-s} \\ &\quad = \sigma^1[D, \pi(a)]\pi^\circ((z^r \triangleright \sigma(z^{k-s}, z^s))(z^r \triangleright z^{k-r} \triangleright b^*)\sigma(z^r, z^{k-s}))(z^r \triangleright v) \otimes e_{k+r-s} \\ &\qquad + (r_1\sigma^2 + r_2\sigma^3)\pi(a)\pi^\circ((z^r \triangleright \sigma(z^{k-s}, z^s))(z^r \triangleright z^{k-r} \triangleright b^*)\sigma(z^r, z^{k-s}))(z^r \triangleright v) \otimes e_{k+r-s} \\ &\qquad - \sigma^1\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))[D, \pi(a)](z^r \triangleright v) \otimes e_{k+r-s} \\ &\qquad - (r_1\sigma^2 + r_2\sigma^3)\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))\pi(a)(z^r \triangleright v) \otimes e_{k+r-s}, \end{aligned} \tag{7.3.22}$$

where we used property (iii) of definition 7.2.1. Now, using the twisted module condition we get the following relation:

$$(z^r \triangleright z^k \triangleright b^*)\sigma(z^r, z^{k-s}) = \sigma(z^r, z^{k-s})(z^{k+r-s} \triangleright b^*).$$

Also, by direct computation we can see that:

$$z^r \triangleright \sigma(z^{k-s}, z^s)^* = \sigma(z^r, z^k)\sigma(z^{k+r-s}, z^s)^*\sigma(z^r, z^{k-s})^*.$$

Therefore the term in π° in the first two lines of the last expression in (7.3.22) becomes:

$$\begin{aligned}
&\pi^\circ \Big((z^r \triangleright \sigma(z^{k-s}, z^s))(z^r \triangleright z^{k-r} \triangleright b^*)\sigma(z^r, z^{k-s})\Big) \\
&\quad = \pi^\circ \Big(\sigma(z^r, z^k)\sigma(z^{k+r-s}, z^s)^*\sigma(z^r, z^{k-s})^*\sigma(z^r, z^{k-s})(z^{k+r-s} \triangleright b^*)\Big) \\
&\quad = \pi^\circ \Big(\sigma(z^r, z^k)\sigma(z^{k+r-s}, z^s)^*(z^{k+r-s} \triangleright b^*)\Big) \\
&\quad = \pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k)).
\end{aligned}$$

Hence we can rewrite equation (7.3.22) in the following way:

$$\begin{aligned}
&[[\hat{D}, \hat{\pi}(a\#z^r)], \hat{J}\hat{\pi}(b\#z^s)\hat{J}^{-1}](v \otimes e_k) \\
&\quad = \sigma^1[D, \pi(a)]\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))(z^r \triangleright v) \otimes e_{k+r-s} \\
&\qquad + (r_1\sigma^2 + r_2\sigma^3)\pi(a)\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))(z^r \triangleright v) \otimes e_{k+r-s} \\
&\qquad - \sigma^1\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))[D, \pi(a)](z^r \triangleright v) \otimes e_{k+r-s} \\
&\qquad - (r_1\sigma^2 + r_2\sigma^3)\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))\pi(a)(z^r \triangleright v) \otimes e_{k+r-s} \\
&\quad = \sigma^1[D, \pi(a)]\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))(z^r \triangleright v) \otimes e_{k+r-s} \\
&\qquad - \sigma^1\pi^\circ(z^{k+r-s} \triangleright b^*)\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))[D, \pi(a)](z^r \triangleright v) \otimes e_{k+r-s} \\
&\quad = \sigma^1[[D, \pi(a)], \pi^\circ(z^{k+r-s} \triangleright b^*)]\pi^\circ(\sigma(z^{k+r-s}, z^s)^*)\pi^\circ(\sigma(z^r, z^k))[D, \pi(a)](z^r \triangleright v) \otimes e_{k+r-s}.
\end{aligned} \tag{7.3.23}$$

But then we see that it is equal to zero, since the first order condition holds for the spectral triple $(\mathcal{B}, \mathcal{H}, D, J)$. This concludes the proof of the proposition.

Remark: in the computation of equation (7.3.23) we have used the fact that $\pi^\circ(\mathcal{B})$ is in the commutant of $\pi(\mathcal{B})$, together with the commutation property of D with $\pi^\circ(\sigma(\cdot, \cdot))$ (see (iii) of definition 7.2.1). □

Even dimensional case. Let $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ be an even real spectral triple of KR-dimension j. Then we can define a real structure $\hat{J}$ on $\mathcal{H}'$ in the following way.

Definition 7.3.8. *Let $\widetilde{J}$ as above. We define an operator $\hat{J} : \mathcal{H}' \to \mathcal{H}'$ by:*

$$\hat{J} = \begin{cases} (\gamma \otimes \mathrm{id} \otimes \sigma^2) \circ \widetilde{J} & \text{if } j \equiv 0 \pmod 4 \\ \\ (i\gamma \otimes \mathrm{id} \otimes \sigma^3) \circ \widetilde{J} & \text{if } j \equiv 2 \pmod 4 \end{cases} \tag{7.3.24}$$

where id *is the identity operator on* $\ell^2(\mathbb{Z}^2)$.

Proposition 7.3.9. *Let* $\hat{J}$ *be defined by* (7.3.24), *and let* $\hat{\gamma} = \gamma \otimes \mathrm{id} \otimes \sigma^1$. *Then* $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma})$ *is a real spectral triple of KR-dimension* $j + 2$.

Proof. That the commutation relations between $\hat{D}$, $\hat{J}$ and $\hat{\gamma}$ are the right ones follows by direct computation. For the rest, the proof is the same as that of proposition 7.3.7. ☐

We conclude this section with the following observation.

Definition 7.3.10. *A spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is said to be* irreducible *if there is no closed subspace of* $\mathcal{H}$ *invariant under the action of the operator algebra generated by* $a \in \mathcal{A}$ *and* D.

Proposition 7.3.11. *If* $(\mathcal{B}, \mathcal{H}, D)$ *is an irreducible triple, then the triples of proposition 7.3.7 and proposition 7.3.9 are irreducible, too.*

7.3.2 Quantum principal bundle structure

We started from a (cleft) Hopf-Galois extension $\mathcal{B} \hookrightarrow \mathcal{A}$, with Hopf algebra $H = \mathcal{O}(\mathbb{T}^2)$. This is not enough to say that we are working with a quantum principal $\mathbb{T}^2$-bundle: indeed, the definition[2] of quantum principal $\mathbb{T}^n$-bundles involves the differential calculus on the algebra $\mathcal{A}$. In this section we will show how the construction of the spectral triples over $\mathcal{A}$ discussed in the previous sections allows us to define a differential calculus on $\mathcal{A}$ compatible with the de Rham calculus on $H = \mathcal{O}(\mathbb{T}^2)$.

Lemma 7.3.12. *Let* $\Omega^1_{\hat{D}}(\mathcal{A})$ *be the first order differential calculus associated to the Dirac operator* $\hat{D}$. *Then it is a right* $\mathcal{O}(\mathbb{T}^2)$*-covariant calculus.*

Proof. We generalize the proof of proposition 6.3.4. Let N be the sub-bimodule of the universal calculus $\Omega^1\mathcal{A}$ defining the calculus $\Omega^1_{\hat{D}}(\mathcal{A})$. Then $\eta \in N$ iff $\pi_{\hat{D}}(\eta) = 0$. Take $\eta \in N$, $\eta = \sum pdq$. We can write it also in the following way:

$$\begin{aligned}\eta &= \sum \sum_{k,l \in \mathbb{Z}^2} (p_l \# z^l) d(q_k \# z^k) \\ &= \sum \sum_{k,l \in \mathbb{Z}^2} \Big[(p_l \# z^l)(d(q_k \# 1))(1 \# z^k) + (p_l \# z^l)(q_k \# 1) d(1 \# z^k) \Big] .\end{aligned}$$

Then the condition $\pi_{\hat{D}}(\eta) = 0$ implies:

$$\begin{aligned}0 = \pi_{\hat{D}}(\eta) &= \sum \sum_{k,l \in \mathbb{Z}^2} \hat{\pi}(p_l \# z^l) [\hat{D}, \hat{\pi}(q_k \# z^k) \\ &= \sum \sum_{k,l \in \mathbb{Z}^2} \Big[\hat{\pi}(p_l \# z^l) [D \otimes \mathrm{id} \otimes \sigma^1, \hat{\pi}(q_k \# 1)] \hat{\pi}(1 \# z^k) \\ &\qquad + (k_1 (\mathrm{id} \otimes \sigma^2) + k_2 (\mathrm{id} \otimes \sigma^3)) \hat{\pi}(p_l \# z^l) \hat{\pi}(q_k \# z^k) \Big] .\end{aligned}$$

[2] See definition 4.6.9.

Since the Pauli matrices are linearly independent, this means that:

$$\sum\sum_{k,l\in\mathbb{Z}^2} \hat{\pi}(p_l\#z^l)[D\otimes \mathrm{id}\otimes\sigma^1, \hat{\pi}(q_k\#1)]\hat{\pi}(1\#z^k) = 0,$$

$$\sum\sum_{k,l\in\mathbb{Z}^2} (k_1(\mathrm{id}\otimes\sigma^2) + k_2(\mathrm{id}\otimes\sigma^3))\hat{\pi}(p_l\#z^l)\hat{\pi}(q_k\#z^k) = 0.$$

But this implies also that, for any $r\in\mathbb{Z}^2$,

$$\sum\sum_{k+l=r} \hat{\pi}(p_l\#z^l)[D\otimes \mathrm{id}\otimes\sigma^1, \hat{\pi}(q_k\#1)]\hat{\pi}(1\#z^k) = 0, \tag{7.3.25}$$

$$\sum\sum_{k+l=r} (k_1(\mathrm{id}\otimes\sigma^2) + k_2(\mathrm{id}\otimes\sigma^3))\hat{\pi}(p_l\#z^l)\hat{\pi}(q_k\#z^k) = 0. \tag{7.3.26}$$

Now we split η in a different way: we write it as $\eta = \sum_{r\in\mathbb{Z}^2}\eta_r$, where the η_r are defined by:

$$\eta_r = \sum\sum_{k+l=r}(p_l\#z^l)d(q_k\#z^k).$$

Then equation (7.3.25) and equation (7.3.26) imply that $\pi_{\hat{D}}(\eta_r) = 0$ for any $r\in\mathbb{Z}^2$; that is, each η_r belongs to N. But if now we compute $\Delta_R^{\Omega}(\eta)$ we obtain $\Delta_R^{\Omega}(\eta) = \sum_{r\in\mathbb{Z}^2}\eta_r\otimes z^r$, hence it belongs to $N\otimes H$. It follows that $\Delta_R^{\Omega}(N)\subseteq N\otimes H$; that is, $\Omega^1_{\hat{D}}(\mathcal{A})$ is a right H-covariant calculus. □

Proposition 7.3.13. *Let $\Omega^1_{\hat{D}}(\mathcal{A})$ be the first order differential calculus associated to the Dirac operator $\hat{D}$, and let N be the sub-bimodule of $\mathcal{A}\otimes\mathcal{A}$ defining it. Let $Q = (\ker\varepsilon)^2$ be the ideal of $H = \mathcal{O}(\mathbb{T}^2)$ which defines the de Rham calculus $\Omega^1_{dR}(H)$. Then $(\mathcal{A}, H, \Delta_R, N, Q)$ is a quantum principal $\mathbb{T}^2$-bundle.*

Proof. Due to lemma 7.3.12, it is enough to show that $\Omega^1_{\hat{D}}(H)$ satisfies (i) and (ii) of proposition 4.6.14. Let us begin by showing that (i) is fulfilled. We recall that the sub-bimodule defining $\Omega^1_{\hat{D}}(\mathcal{A})$ is:

$$N = \left\{\sum_j a_j\otimes b_j \in \mathcal{A}\otimes\mathcal{A} \;\middle|\; \sum_j \hat{\pi}(a_j)[\hat{D}, \hat{\pi}(b_j)] = 0\right\}. \tag{7.3.27}$$

Take now $\eta\in N$. We can write it as $\eta = \sum_{k,l\in\mathbb{Z}^2}(a_{kl}\#z^k)\otimes(b_{kl}\#z^l)$. Exploiting the fact that η belongs to N, we get, using (7.3.21),

$$\begin{aligned} 0 &= \sum_{k,l\in\mathbb{Z}^2}\hat{\pi}(a_{kl}\#z^k)\Big(\sigma^1[D, \pi(b_{kl})\pi^\circ(\sigma(z^l, w_{(-1)})](z^l\triangleright v)\otimes\pi_H(z^l)w_{(0)} \\ &\qquad + (l_1\sigma^2 + l_2\sigma^3)\pi(b_{kl})\pi^\circ(\sigma(z^l, w_{(-1)}))(z^l\triangleright v)\otimes\pi_H(z^l)w_{(0)}\Big) \\ &= \sum_{k,l\in\mathbb{Z}^2}\hat{\pi}(a_{kl}\#z^k)\Big(\sigma^1[D, \pi(b_{kl})\pi^\circ(\sigma(z^l, w_{(-1)})](z^l\triangleright v)\otimes\pi_H(z^l)w_{(0)} \\ &\qquad + (l_1\sigma^2 + l_2\sigma^3)\hat{\pi}(b_{kl}\#z^l)(v\otimes w)\Big). \end{aligned} \tag{7.3.28}$$

Since the Pauli matrices are linearly independent, from equation (7.3.28) we obtain, for $i = 1, 2$,

$$\sum_{k,l\in\mathbb{Z}^2} l_i(a_{kl}\#z^k)(b_{kl}\#z^l) = \sum_{k,l\in\mathbb{Z}^2}(a_{kl}\#z^k)\delta_i(b_{kl}\#z^l) = 0,$$

and this concludes the proof that (i) is fulfilled. We prove that also (ii) holds. Take $\eta \in \Omega^1\mathcal{A}$, $\eta = \sum p dq$, and assume that $\sum p\delta_i(q) = 0$, for $i = 1, 2$. Then rewrite η as $\eta = \sum\sum_{k,l\in\mathbb{Z}^2}(p_l\#z^l)d(q_k\#z^k)$. Using the Leibniz rule we obtain then:

$$\eta = \sum\sum_{k,l\in\mathbb{Z}^2}\left[(p_l\#z^l)(d(q_k\#1))(1\#z^k) + (p_l\#z^l)(q_k\#1)d(1\#z^k)\right].$$

In order to prove that $[\eta]_N$ belongs to $\mathcal{A}\Omega^1_{\hat{D}}(\mathcal{B})\mathcal{A}$ it is then enough to show that

$$\pi_{\hat{D}}\left(\sum\sum_{k,l\in\mathbb{Z}^2}(p_l\#z^l)(q_k\#1)d(1\#z^k)\right) = 0.$$

Let us compute it. We have:

$$\begin{aligned}
\pi_{\hat{D}}\left(\sum\sum_{k,l\in\mathbb{Z}^2}(p_l\#z^l)(q_k\#1)d(1\#z^k)\right) &= \sum\sum_{k,l\in\mathbb{Z}^2}\hat{\pi}(p_l\#z^l)\hat{\pi}(q_k\#1)[\hat{D},\hat{\pi}(1\#z^k)] \\
&= \sum\sum_{k,l\in\mathbb{Z}^2}(k_1\otimes\sigma^2 + k_2\otimes\sigma^3)\hat{\pi}(p_l\#z^l)\hat{\pi}(q_k\#z^k) \\
&= \sum\sum_{k,l\in\mathbb{Z}^2}\left[(\mathrm{id}\otimes\sigma^2)\hat{\pi}\left((p_l\#z^l)\delta_1(q_k\#z^k)\right) + (\mathrm{id}\otimes\sigma^3)\hat{\pi}\left((p_l\#z^l)\delta_2(q_k\#z^k)\right)\right] = 0.
\end{aligned}$$

□

7.3.3 Projectability and twisted Dirac operators

Now we show that the triples constructed in the previous sections are projectable spectral triples. First of all it is not difficult to see that:

Proposition 7.3.14. *The spectral triples of proposition 7.3.7 and proposition 7.3.9 are equivariant spectral triples with respect to the $\mathbb{T}^2$-action generated by the two commuting derivations δ_1, δ_2.*

We can now prove the following results.

Proposition 7.3.15. *Let $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J}, \{\delta_j\})$ be the $\mathbb{T}^2$-equivariant odd real spectral triple of proposition 5.3.7. Then it is a projectable triple with isometric fibres. Moreover we can take (under the isomorphism $\mathcal{A} \simeq \mathcal{B}\#_\sigma H$)*

$$\Gamma = -i\hat{\pi}(\phi^{-1}(z_1))[\hat{D},\hat{\pi}(\phi(z_1))]\hat{\pi}(\phi^{-1}(z_2))[\hat{D},\hat{\pi}(\phi(z_2))]. \tag{7.3.29}$$

Proof. Let Γ be as in (7.3.29). By direct computation one sees that $\Gamma = \mathrm{id} \otimes \sigma^1$. Thus we have immediately $\Gamma^2 = 1$, $\Gamma^* = \Gamma$, $[\Gamma, \hat{\pi}(a)] = 0$ for any $a \in \mathcal{A}$, $[\Gamma, \delta_j] = 0$. And $\Gamma\hat{J} = -\hat{J}\Gamma$. Hence $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma}, \delta)$ is a projectable triple. Finally, it is straightforward to see that the isometric fibres conditions is fulfilled. Indeed, for $\Gamma = \pm\mathrm{id} \otimes \sigma^1$ we have $D_h = D \otimes \sigma^1$; so if we take $D_v = (\mathrm{id} \otimes \sigma^2)\delta_1 + (\mathrm{id} \otimes \sigma^3)\delta_2$ we obtain $\hat{D} = D_v + D_h$. □

Proposition 7.3.16. *Let $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma}, \{\delta_j\})$ be the $\mathbb{T}^2$-equivariant even real spectral triple of proposition 5.3.10. Then it is a projectable spectral triples with isometric fibres. Moreover we can take ($\mathcal{A} \simeq \mathcal{B}\#_\sigma H$)*

$$\Gamma = -i\hat{\pi}(\phi^{-1}(z_1))[\hat{D}, \hat{\pi}(\phi(z_1))]\hat{\pi}(\phi^{-1}(z_2))[\hat{D}, \hat{\pi}(\phi(z_2))]. \tag{7.3.30}$$

Proof. Let Γ be as in (7.3.30). Also in this case, $\Gamma = \mathrm{id} \otimes \sigma^1$. The only difference with proposition 7.3.15 is that we have to check also the commutation relation between Γ and $\hat{\gamma}$. Since $\hat{\gamma} = \gamma \otimes \mathrm{id} \otimes \sigma^1$, we have $\Gamma\hat{\gamma} = \hat{\gamma}\Gamma$, which is consistent with definition 5.3.9. The fact that the triple has isometric fibres follows as in proposition 7.3.15. □

Now we can compute the twisted Dirac operator D_ω (which will be a "j-dimensional" Dirac operator, where j is the dimension of the triple over $\mathcal{B}$) associated to a strong connection ω, which will be defined by a family of n 1-forms ω_i (see definition 4.6.17). First of all we work out an explicit formula for the admissible ω_i. Any component ω_i of a family defining a strong connection, due to condition (i) of definition 4.6.17, must be of the following form:

$$\omega_i = \omega_{i,1}\sigma^1 + \omega_{i,2}\sigma^2 + \omega_{i,3}\sigma^3 \tag{7.3.31}$$

with $\omega_{i,j} \in \Omega^1_D(\mathcal{B})$. More precisely, we can write ω_i as:

$$\omega_i = \sum_j a_{i,j}[\hat{D}, b_{i,j}] + c_{i,1}\sigma^2 + c_{i,2}\sigma^3, \tag{7.3.32}$$

where $a_{i,j}, b_{i,j}, c_i$ belong to $\mathcal{B}$. Now, from condition (ii), we get, using $\sigma^2 = (1\#z_1^{-1})[\hat{D}, 1\#z_1]$, $\sigma^3 = (1\#z_2^{-1})[\hat{D}, 1\#z_2]$,

$$c_{1,1} = 1, \qquad c_{1,2} = 0,$$
$$c_{2,1} = 0, \qquad c_{2,2} = 1.$$

Therefore we get obtain following expressions[3] for ω_1 and ω_2:

$$\begin{aligned}\omega_1 &= (1\#z_1^{-1})[\hat{D}, 1\#z_1] + \sum_j (a_{1,j}\#1)[D \otimes \mathrm{id}, b_{1,j}\#1]\sigma^1 \\ &= \sigma^2 + \sum_j (a_{1,j}\#1)[D \otimes \mathrm{id}, b_{1,j}\#1]\sigma^1,\end{aligned} \tag{7.3.33}$$

[3]We consider both of them as operators on $\mathcal{H}'$. We omit the representation $\hat{\pi}$ to simplify the notation.

$$\begin{aligned}\omega_2 &= (1\#z_2^{-1})[\hat{D}, 1\#z_2] + \sum_j (a_{2,j}\#1)[D\otimes \mathrm{id}, b_{2,j}\#1]\sigma^1 \\ &= \sigma^3 + \sum_j (a_{2,j}\#1)[D\otimes \mathrm{id}, b_{2,j}\#1]\sigma^1,\end{aligned} \tag{7.3.34}$$

Now we can compute the twisted Dirac operator D_ω. We know that in general, for $\mathbb{T}^n$ bundles, it can be written as

$$D_\omega = D - \sum_{i=1}^{n} j_0 \omega_i^* j_0^{-1} \delta_i - Z,$$

where D is the Dirac operator of the triple over $\mathcal{A}$ and j_0 is defined as follows:

Table 7.1: j_0

n	0	1	2	3	4	5	6	7
j_0	$\Gamma\hat{J}$	$\Gamma\hat{J}$	$\hat{J}$	$\hat{J}$	$\Gamma\hat{J}$	$\Gamma\hat{J}$	$\hat{J}$	$\hat{J}$

Therefore we get, using $\Gamma = \sigma^1$,

Table 7.2: j_0

n	0	1	2	3	4	5	6	7
j_0	$i\gamma\sigma^3\widetilde{J}$	$i\sigma^3\widetilde{J}$	$i\gamma\sigma^3\widetilde{J}$	$i\sigma^3\widetilde{J}$	$i\gamma\sigma^3\widetilde{J}$	$i\sigma^3\widetilde{J}$	$i\gamma\sigma^3\widetilde{J}$	$i\sigma^3\widetilde{J}$

It is now clear that, since σ^2 anticommutes and σ^3 commutes with $\widetilde{J}$, both σ^2 and σ^3 commute with j_0; instead, σ^1 anticommutes with j_0. Therefore, writing the 1-forms ω_i in the short form

$$\omega_1 = \sigma^2 + \sigma^1\omega_1^{\mathcal{B}},$$

$$\omega_2 = \sigma^3 + \sigma^1\omega_2^{\mathcal{B}},$$

we obtain that the twisted Dirac operator D_ω is given by:

$$D_\omega = D\otimes\sigma^1 + \sum_{i=1}^{2} (\mathrm{id}\otimes\sigma^1) j_0 (\omega_i^{\mathcal{B}})^* j_0^{-1}\delta_i. \tag{7.3.35}$$

Adding the vertical Dirac operator we obtain a full, $j+2$ dimensional Dirac operator:

$$\hat{D}_\omega = D\otimes\sigma^1 + (\mathrm{id}\otimes\sigma^2)\delta_1 + (\mathrm{id}\otimes\sigma^3)\delta_2 + \sum_{i=1}^{2} (\mathrm{id}\otimes\sigma^1) j_0 (\omega_i^{\mathcal{B}})^* j_0^{-1}\delta_i. \tag{7.3.36}$$

Corollary 7.3.17. *The only connection compatible with the Dirac operator $\hat{D}$ is the Maurer-Cartan connection $\omega = \phi^{-1} * d\phi$.*

7.4 Spectral triples over cleft $\mathcal{O}(\mathbb{T}^n)$-extensions

Now we generalize the construction of the previous section to cleft $\mathcal{O}(\mathbb{T}^n)$-extensions $\mathcal{B} \hookrightarrow \mathcal{A}$, for $n > 2$, admitting a unitary trivialization which determines an isomorphism $\mathcal{A} \simeq \mathcal{B}\#_\sigma H$, with $H = \mathcal{O}(\mathbb{T}^n)$. Roughly speaking, the only things to be changed with respect to the 2-dimensional case are are in the $\mathbb{C}^2$ factor, where we have to replace the Pauli matrices with suitable γ matrices acting on some representation of an n or $n+1$ dimensional Clifford algebra. More precisely we modify the part of the Dirac operator, the real structure and the $\mathbb{Z}^2$ grading acting on the $\mathbb{C}^2$ factor that have to be chosen accordingly to n and to the KR-dimension of the triple over $\mathcal{B}$. We follow [DD11] to get the right formulae.

Since we want to use the results of [DD11], we adopt a slightly different convention w.r.t. the one we worked with in the previous sections. So, if we specialize the discussion below to the $n = 2$ case we see that there are some minor differences w.r.t. the results obtained previously. But it is only a matter of conventions on the commutation relations between the elements of the triple. We describe shortly, following [DD11], the conventions that we will use.

Given a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ ($\gamma = \mathrm{id}$ if the triple is odd) the commutation relations between D, J and γ are given by

$$J^2 = \varepsilon \mathrm{id}, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon'' \gamma J,$$

where $\varepsilon, \varepsilon', \varepsilon''$ are given by the following table.

Table 7.3: Connes' selection in [GBFV] is marked by •

n	0	2	4	6	0	2	4	6	1	3	5	7
ϵ	+	−	−	+	+	+	−	−	+	−	−	+
ϵ'	+	+	+	+	−	−	−	−	−	+	−	+
ϵ''	+	−	+	−	+	−	+	−				
	•	•	•	•					•	•	•	•

Notice that altogether there are twelve different possibilities, which can be labelled by *KR*-dimension $n \in \mathbb{Z}_8$ with the additional index ϵ' if n is even (so for example the case $(\epsilon, \epsilon', \epsilon'') = (+, -, -)$ is labelled by 2_-). We keep the notation of [DD11], so we place this additional index also in the case of odd n, though it is redundant there. We notice also that, in the even dimensional case, we pass from the n_- to the n_+ case by multiplying the real structure by the $\mathbb{Z}_2$ grading γ.

Given a real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$, we define, on $\mathcal{H}' = \mathcal{H} \otimes \ell^2(\mathbb{Z}^n) \otimes \mathbb{C}^{2^{[n/2]}}$, the representation $\hat{\pi}$ and the antiunitary operator $\widetilde{J}$ exactly as in the 2-dimensional case. We will get $\hat{D}$ and $\hat{J}$ acting on the $\mathbb{C}^{2^{[n/2]}}$ factor in a such a way that we obtain the right commutation relations.

Next we choose a convention for the γ matrices. Let n equal either to $2m$ or to $2m+1$. We denote by $\gamma^1, \ldots, \gamma^{2m}$ the canonical generators of the representation of the n-dimensional complex Clifford algebra on $\mathbb{C}^{2^m}$; they satisfy the relations

$$\gamma^i \gamma^j + \gamma^j \gamma^i = \delta_{ij}.$$

We define also an operator $\gamma^{2m+1} = \lambda_n \gamma^1 \cdots \gamma^{2m}$, where $\lambda_n = 1, -i, i, 1$ accordingly to $n \equiv 0, 1, 2, 3 \pmod 4$. γ^{2m+1} is defined in a such a way that $(\gamma^{2m+1})^2 = 1$. Next, we introduce the derivations δ_j, $j = 1, \dots, n$, acting on $\mathcal{H}$ by:

$$(\delta_j f)_k = k_j f_k, \qquad \forall f \in \mathcal{H}',\ k \in \mathbb{Z}^n.$$

7.4.1 $\mathbb{T}^{2m}$-bundles

We consider first the case of smash products $\mathcal{B}\#_\sigma H$, $H = \mathcal{O}(\mathbb{T}^n)$ with $n = 2m$. Since the construction depends on the parity of the KR-dimension of the triple over $\mathcal{B}$, we have to consider two different situations.

Odd dimensional case. Assume that the triple over $\mathcal{B}$ has dimension j, with j odd. Then we define[4] the Dirac operator $\hat{D}$ as:

$$\hat{D} = D \otimes \mathrm{id} \otimes \gamma^{2m+1} + \sum_{j=1}^{n} (\mathrm{id} \otimes \gamma^j) \circ \delta_j.$$

As real structure we take, instead, $\hat{J}$ defined as:

$$\hat{J} = \begin{cases} \widetilde{J} & \text{for } j + n \equiv 3 \pmod 4 \\ (\mathrm{id} \otimes \gamma^{2m+1}) \circ \widetilde{J} & \text{for } j + n \equiv 1 \pmod 4 \end{cases}$$

Then we get immediately, using the results in [DD11], that $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J})$ is an odd dimensional real spectral triple of KR-dimension $n + j$.

Even dimensional case. Assume that the triple over $\mathcal{B}$ has dimension j, with j even. Then we define[5] the Dirac operator $\hat{D}$ as:

$$\hat{D} = D \otimes \mathrm{id} \otimes \gamma^{2m+1} + \sum_{j=1}^{n} (\mathrm{id} \otimes \gamma^j) \circ \delta_j.$$

As $\mathbb{Z}_2$ grading $\hat{\gamma}$ we take $\hat{\gamma} = \gamma \otimes \mathrm{id} \otimes \gamma^{2m+1}$, while the real structure $\hat{J}$ is defined as follows:

$$\hat{J} = \begin{cases} \widetilde{J} & \text{for } j + n \equiv 2 \pmod 4 \\ (\mathrm{id} \otimes \gamma^{2m+1}) \circ \widetilde{J} & \text{for } j + n \equiv 0 \pmod 4 \end{cases}$$

Then we get immediately, using the results in [DD11], that $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma})$ is an even dimensional real spectral triple of KR-dimension $n + j$ whose "parity" is given by the following table:

7.4.2 $\mathbb{T}^{2m+1}$-bundles

Now we consider the case of smash products $\mathcal{B}\#_\sigma H$, $H = \mathcal{O}(\mathbb{T}^n)$ with $n = 2m + 1$. Still, we have to consider two different situations.

Odd dimensional case. Assume that the triple over $\mathcal{B}$ has dimension j, with j odd. Then we define a new Hilbert space, $\mathcal{H}'' = \mathcal{H}' \otimes \mathbb{C}^2$, and we extend trivially all the operators on $\mathcal{H}'$ to operators on $\mathcal{H}''$. Explicitly, we have $\mathcal{H}'' = \mathcal{H} \otimes \ell^2(\mathbb{Z}^n) \otimes \mathbb{C}^{2^m} \otimes \mathbb{C}^2$. Then we define the Dirac

[4] This is not the only possible choice, see [DD11] for the details.

[5] See previous footnote.

j \ n	0	2	4	6
0_+	0_+	2_-	4_+	6_-
2_+	2_+	4_-	6_+	0_-
4_+	4_+	6_-	0_+	2_-
6_+	6_+	0_-	2_+	4_-
0_-	0_-	2_+	4_-	6_+
2_-	2_-	4_+	6_-	0_+
4_-	4_-	6_+	0_-	2_+
6_-	6_-	0_+	2_-	4_+

operator $\hat{D}$, the real structure $\hat{J}$ and the $\mathbb{Z}_2$ grading $\hat{\gamma}$ in the following way [DD11]:

$$\hat{D} = D \otimes \mathrm{id} \otimes \mathrm{id} \otimes \sigma^1 + \sum_{j=1}^{n} (\mathrm{id} \otimes \mathrm{id} \otimes \gamma^j \otimes \sigma^2) \circ \delta_j,$$

$$\hat{J}^{\pm} = \widetilde{J} \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes M^{\pm} K),$$

$$\hat{\gamma} = \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \sigma^3,$$

where $M^{\pm}$ are two complex matrices specified by the table below and K is the complex conjugation operator defined for the canonical basis of $\mathbb{C}^2$ (i.e., if (e_1, e_2) is the canonical basis, we have $K(\lambda e_i) = \overline{\lambda} e_i$ for every $\lambda \in \mathbb{C}$).

Table 7.4: Matrices M^+, M^-.

j \ n	1	3	5	7
1	σ_2, σ_1	σ_3, σ_0	σ_2, σ_1	σ_3, σ_0
3	σ_0, σ_3	σ_1, σ_2	σ_0, σ_3	σ_1, σ_2
5	σ_2, σ_1	σ_3, σ_0	σ_2, σ_1	σ_3, σ_0
7	σ_0, σ_3	σ_1, σ_2	σ_0, σ_3	σ_1, σ_2

The resulting triple $(\mathcal{A}, \mathcal{H}'', \hat{D}, \hat{J}^{\pm}, \hat{\gamma})$ is an even dimensional real spectral triple of KR-dimension $(j+n)_{\pm}$.

Even dimensional case. Assume that the triple over $\mathcal{B}$ has dimension j, with j even. Then we define[6] the Dirac operator $\hat{D}$ as:

$$\hat{D} = D \otimes \mathrm{id} \otimes \gamma^{2m+1} + \sum_{j=1}^{n} (\mathrm{id} \otimes \gamma^j) \circ \delta_j.$$

[6] This is not the only possible choice, see [DD11] for the details.

The real structure $\hat{J}$ is defined as follows: if $JD = DJ$ we take

$$\hat{J} = \begin{cases} \widetilde{J} & \text{for } j+n \equiv 3 \pmod 4 \\ (\gamma \otimes \mathrm{id}) \circ \widetilde{J} & \text{for } j+n \equiv 1 \pmod 4 \end{cases}$$

instead, if $JD = -DJ$, we take

$$\hat{J} = \begin{cases} \widetilde{J} & \text{for } j+n \equiv 1 \pmod 4 \\ (\gamma \otimes \mathrm{id}) \circ \widetilde{J} & \text{for } j+n \equiv 3 \pmod 4 \end{cases}$$

Then we can see, using the results in [DD11], that $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J})$ is an odd dimensional real spectral triple of KR-dimension $n + j$.

7.5 Further properties

7.5.1 Dimension

If the Dirac operator D on $\mathcal{H}$ is p^+-summable, then the Dirac operator $\hat{D}$ is $(p+n)^+$-summable. This follows as in the case of product spectral triples; for further details see [DS13a, GBFV].

7.5.2 Regularity

Assume that the spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$ over $\mathcal{B}$ be *regular*. Let us see that this implies that also the spectral triples built in the previous sections are regular. We will use the results of section 3.1.2, in particular theorem 3.1.19 and theorem 3.1.20. We discuss here only the case when $n = 2$, but the extension to the general one is straightforward.

Since $(\mathcal{B}, \mathcal{H}, D)$ is a regular spectral triple, there exists an algebra of generalized differential operators $\mathcal{D}_{\mathcal{B}} \subset \mathrm{End}(W^\infty)$ such that $\mathcal{B} + [D, \mathcal{B}]$ is dense in $\mathcal{D}^0_{\mathcal{B}}$. Here W^∞ is the space of Δ-smooth vectors, where $\Delta = D^2 + 1$. Consider now the Hilbert space $\mathcal{H}' = \mathcal{H} \otimes \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$. We can extend Δ to an operator $\hat{\Delta}$ on $\mathcal{H}'$, simply defined by $\hat{\Delta} = \Delta \otimes \mathrm{id}$. Then the space of $\hat{\Delta}$-smooth vectors is just $\hat{W}^\infty = W^\infty \otimes \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$.

Now, due to condition (ii) of definition 7.2.1 W^∞ is stable under the (weak) action of the Hopf algebra $H = \mathcal{O}(\mathbb{T}^2)$. For any $b \in \mathcal{D}_{\mathcal{B}}$ and any $k \in \mathbb{Z}^2$ consider now the operator $\hat{b}^{(k)}$ on $\mathcal{H}'$ defined by: $\hat{b}^{(k)}(v \otimes e_s) = \pi(b)\pi^\circ(\sigma(z^k, z^s))(z^k \triangleright v) \otimes e_s$. All the operators of this kind form an $\mathbb{N}$-graded algebra $\hat{\mathcal{D}}_{\mathcal{B}}$, and the space $\hat{W}^\infty$ is stable under the action of $\hat{\mathcal{D}}_{\mathcal{B}}$. Moreover,

Lemma 7.5.1. $\hat{\mathcal{D}}_{\mathcal{B}} \subseteq \mathrm{Op}(\hat{\Delta})$.

Now let $\hat{H} \subset \mathcal{A}$ be the (unital) *-algebra generated by $(1 \# z^k)$, $k \in \mathbb{Z}^2$. Consider the following $\mathbb{N}$-filtered algebra:

$$\mathcal{D}^n_A = \hat{\mathcal{D}}^n_{\mathcal{B}} \cdot \hat{H} + \hat{H} \cdot \hat{\mathcal{D}}^n_{\mathcal{B}} + \sum_{j=1}^{3} (\mathrm{id} \otimes \sigma^j)(\hat{\mathcal{D}}^n_{\mathcal{B}} \cdot \hat{H} + \hat{H} \cdot \hat{\mathcal{D}}^n_{\mathcal{B}}).$$

By construction $\mathcal{A} + [D, \mathcal{A}]$ is contained in $\mathcal{D}^0_{\mathcal{A}}$. Let now Δ' be the operator $\hat{D}^2 + 1$ on $\mathcal{H}'$. Notice that it is equal to $\hat{\Delta} + \delta_1^2 + \delta_2^2$. Let W'^∞ be the space of Δ'-smooth vectors of $\mathcal{H}'$. Then it is easy to see that $W'^{\infty} \subset \hat{W}^{\infty}$. Also, since any $P \in \hat{\mathcal{D}}_{\mathcal{B}}$ acts as the identity on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}$, $\hat{\mathcal{D}}_{\mathcal{B}}$ can be seen as an ($\mathbb{N}$-filtered) subalgebra of $\mathrm{End}(W'^{\infty})$. And the same holds for $\hat{H}$. Hence $\mathcal{D}_{\mathcal{A}}$ is a subalgebra of $\mathrm{End}(W'^{\infty})$. Moreover, it satisfies $[\Delta', \mathcal{D}^n_{\mathcal{A}}] \subseteq \mathcal{D}^{k+1}_{\mathcal{A}}$. So, in order to prove that it is an algebra of generalized differential operators, we have only to show that each $\mathcal{D}^n_{\mathcal{A}}$ is contained in $\mathrm{Op}^n(\Delta')$. It is clear that $\hat{\mathcal{D}}^n_{\mathcal{B}}$ is contained in $\mathrm{Op}^n(\Delta')$. Also, $\hat{H}$ and $(\mathrm{id} \otimes \sigma^j)\ \hat{H}$ (for any $j = 1, 2, 3,$) are contained in $\mathrm{Op}^n(\Delta')$; more precisely both of them are contained in $\mathrm{Op}^0(\Delta')$. Hence $\mathcal{D}^n_{\mathcal{A}}$ is contained in $\mathrm{Op}^n(\Delta')$. It follows (see theorem 3.1.19) that the spectral triples constructed in the previous section are regular spectral triples.

7.5.3 Orientation

An orientation Hochschild cycle for the spectral triples $(\mathcal{A}, \mathcal{H}', \hat{D}, \hat{J}, \hat{\gamma})$ and $(\mathcal{A}, \mathcal{H}'', \hat{D}, \hat{J}, \hat{\gamma})$ can be obtained extending the construction of section 6.2.3. We begin by showing the construction in the first case. Later we will show how it extends quite trivially to the other one. So, in the first part of this section, we take $n = 2m$.

Let us consider, for any $i = 1, \ldots, n$, the following Hochschild 1-cycle with values in $H \otimes H^\circ$:

$$\mathbf{c}^{(i)}_H = (z_i^{-1} \otimes 1) \otimes z_i \tag{7.5.1}$$

Since the action of H on $\mathcal{B}$ is actually a weak action, with (possibly) non-trivial cocycle σ, the corresponding Hochschild 1-cycle on $\mathcal{A}$ is not simply $(\phi(z_i^{-1}) \otimes 1) \otimes \phi(z_i)) = ((1\# z_i^{-1}) \otimes 1\#1) \otimes (1\# z_i)$ but it is:

$$(\phi^{-1}(z_i) \otimes 1) \otimes \phi(z_i) = ((\sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes 1\#1) \otimes (1\# z_i). \tag{7.5.2}$$

So we introduce the following generalization of the shuffle product [Lo] and of the twisted shuffle product of section 6.2.3. First of all we recall that any Hochschild chain $\mathbf{c} \in C_p(\mathcal{B}, \mathcal{B} \otimes \mathcal{B}^\circ)$ can be written as

$$\mathbf{c} = \sum (a_0 \otimes b_0^\circ) \otimes a_1 \otimes \cdots \otimes a_p, \tag{7.5.3}$$

with $a_0, b_i \in \mathcal{B}$. Then we give the following definition.

Definition 7.5.2. *For any Hochschild p-cycle $\mathbf{c} \in \mathbb{Z}_p(\mathcal{B}, \mathcal{B} \otimes \mathcal{B}^\circ)$, written as in equation (7.5.3), we define its* twisted shuffle product *with the 1-cycle $\mathbf{c}^{(i)}_H$ as the Hochschild $(p{+}1)$-chain $\mathbf{c} \times_\sigma \mathbf{c}^{(i)}_H \in C_{p+1}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ defined by:*

$$\begin{aligned}
\boldsymbol{c} \times_\sigma \boldsymbol{c}^{(i)}_H = & \sum ((a_0 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ \# 1)) \otimes 1\# z_i \otimes a_1\#1 \otimes \cdots \otimes a_p\#1 \\
& + \sum_{j=2}^{p} \sum (-1)^{j-1} ((a_0 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ \# 1)) \otimes (z_i \triangleright a_1)\#1 \otimes \cdots \\
& \qquad \cdots \otimes (z_i \triangleright a_{j-1})\#1 \otimes 1\# z_i \otimes a_j\#1 \otimes \cdots \otimes a_p\#1 \\
& + \sum (-1)^p ((a_0 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ \# 1)) \otimes (z_i \triangleright a_1)\#1 \otimes \cdots \\
& \qquad \cdots \otimes (z_i \triangleright a_p)\#1 \otimes 1\# z_i
\end{aligned} \tag{7.5.4}$$

Lemma 7.5.3. *The twisted shuffle product* (7.5.4) *is linear in the first factor (i.e. in* $\mathbf{c}$*).*

Now we extend the (weak) action of H on $\mathcal{B}$ to a weak action on Hochschild p-chain over $\mathcal{B}$. For $\mathbf{c} \in C_p(\mathcal{B}, \mathcal{B} \otimes \mathcal{B}^\circ)$, written as in equation (7.5.3), and for $h \in H$, we define:

$$h \triangleright \mathbf{c} = \sum ((h_{(1)} \triangleright a_0) \otimes b_0^\circ) \otimes (h_{(2)} \triangleright a_1) \otimes \cdots \otimes (h_{(p+1)} \triangleright a_p). \tag{7.5.5}$$

Definition 7.5.4. *A Hochschild p-chain* $\mathbf{c} \in C_p(\mathcal{B}, \mathcal{B} \otimes \mathcal{B}^\circ)$ *is H-invariant if* $h \triangleright \mathbf{c} = \varepsilon(h)\mathbf{c}$ *for any* $h \in H$.

Proposition 7.5.5. *If* $\mathbf{c}$ *is an H-invariant Hochschild p-cycle, then its shuffle products* $\mathbf{c} \times_\sigma \mathbf{c}_H^{(i)}$, *for* $i = 1, \ldots, n$, *are Hochschild* $(p+1)$*-cycles.*

Proof. We will use the following relations:

$$(a_0\sigma(z_i^{-1}, z_i)^* \# z_i^{-1})(1\# z_i) = a_0\sigma(z_i^{-1}, z_i)^* \# 1,$$

$$(a_j \# 1)(1\# z_i) = a_j \# z_i,$$

$$(1\# z_i)(a_0\sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) = (z_i \triangleright a_0\sigma(z_i^{-1}, z_i)^* \sigma(z_i, z_i^{-1}))\# 1 = (z_i \triangleright a_0)\# 1,$$

$$(1\# z_i)(a_j \# 1) = (z_i \triangleright a_j)\# z_i.$$

Let $b_\mathcal{A}$ the Hochschild boundary operator on the Hochschild complex $C_\bullet(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$. Then we prove that $b_\mathcal{A}(\mathbf{c} \times_\sigma \mathbf{c}_H^{(i)}) = 0$. First of all we introduce the following notation: accordingly to (7.5.4) we can write $\mathbf{c} \times_\sigma \mathbf{c}_H^{(i)}$ as

$$\mathbf{c} \times_\sigma \mathbf{c}_H^{(i)} = \sum_{j=1}^{p+1} \mathbf{c}_j.$$

We compute now each $b_\mathcal{A}\mathbf{c}_j$. For $\mathbf{c}_1$ we have:

$$\begin{aligned} b_\mathcal{A}\mathbf{c}_1 = & \sum ((a_0\#1) \otimes (b_0^\circ \#1)) \otimes a_1\#1 \otimes \cdots \otimes a_p\#1 \\ & - \sum ((a_0\sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ\#1)) \otimes (z_i \triangleright a_1)\# z_i \otimes a_2\#1 \otimes \cdots \otimes a_p\#1 \\ & + \sum_{k=1}^{p-1} \sum (-1)^{k+1} ((a_0\sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ\#1)) \otimes 1\# z_i \otimes \cdots \otimes a_k a_{k+1}\#1 \otimes \cdots \otimes a_p\#1 \\ & + \sum (-1)^{p+1} ((a_p a_0\sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ\#1)) \otimes 1\# z_i \otimes a_1 \otimes \cdots \otimes a_{p-1}. \end{aligned} \tag{7.5.6}$$

Next we compute $b_\mathcal{A}\mathbf{c}_j$ for $j = 2, \ldots, p$. We obtain, using the twisted module condition and the unitarity of $\sigma(z_i^{-1}, z_i)$,

$$\begin{aligned} b_\mathcal{A}\mathbf{c}_j = & \sum (-1)^{j-1} ((a_0 a_1\sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ\#1)) \otimes (z_i \triangleright a_2)\#1 \otimes \cdots \\ & \cdots \otimes (z_i \triangleright a_{j-1})\#1 \otimes 1\# z_i \otimes a_j\#1 \otimes \cdots \otimes a_p \\ & + \sum_{k=1}^{j-2} (-1)^k (-1)^{j-1} ((a_0\sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ\#1)) \otimes (z_i \triangleright a_1) \otimes \cdots \end{aligned}$$

$$
\begin{aligned}
&\cdots \otimes (z_i \triangleright a_k a_{k+1})\#1 \otimes \cdots \otimes (z_i \triangleright a_{j-1})\#1 \otimes 1\#z_i \otimes a_j\#1 \otimes \cdots \otimes a_p\#1 \\
&+ \sum ((a_0 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ \#1)) \otimes (z_i \triangleright a_1)\#1 \otimes \cdots \\
&\cdots \otimes (z_i \triangleright a_{j-1})\#z_i \otimes a_j\#1 \otimes \cdots \otimes a_p\#1 \\
&- \sum ((a_0 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ \#1)) \otimes (z_i \triangleright a_1)\#1 \otimes \cdots \\
&\cdots \otimes (z_i \triangleright a_{j-1})\#1 \otimes (z_i \triangleright a_j)\#z_i \otimes a_{j+1}\#1 \otimes \cdots \otimes a_p\#1 \\
&+ \sum_{k=j}^{p-1} (-1)^{k+1}(-1)^{j-1}((a_0 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ \#1)) \otimes (z_i \triangleright a_1)\#1 \otimes \cdots \\
&\cdots \otimes (z_i \triangleright a_{j-1})\#1 \otimes (z_i \triangleright a_j)\#z_i \otimes a_{j+1}\#1 \otimes \cdots \otimes a_p \\
&+ \sum (-1)^{p+1}(-1)^{j-1}((a_p a_0 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ \#1)) \otimes (z_i \triangleright a_1)\#1 \otimes \cdots \\
&\cdots \otimes (z_i \triangleright a_{j-1})\#1 \otimes 1\#z_i \otimes a_j\#1 \otimes \cdots \otimes a_{p-1}\#1.
\end{aligned} \tag{7.5.7}
$$

Finally,

$$
\begin{aligned}
b_{\mathcal{A}} \mathbf{c}_{p+1} = &\sum (-1)^p ((a_0 a_1 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ \#1)) \otimes (z_i \triangleright a_2)\#1 \otimes \cdots \otimes (z_i \triangleright a_p)\#1 \otimes 1\#z_i \\
&+ \sum_{k=1}^{p-1} \sum (-1)^k (-1)^p ((a_0 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ)\#1)) \otimes (z_i \triangleright a_2)\#1 \otimes \cdots \\
&\quad \cdots \otimes (z_i \triangleright a_j a_{j+1}) \otimes \cdots \otimes (z_i \triangleright a_p) \otimes 1\#z_i \\
&+ \sum ((a_0 \sigma(z_i^{-1}, z_i)^* \# z_i^{-1}) \otimes (b_0^\circ)\#1)) \otimes (z_i \triangleright a_2)\#1 \otimes \cdots \otimes (z_i \triangleright a_p)\#z_i \\
&- \sum (((z_i \triangleright a_0)\#1) \otimes (b_0^\circ \#1)) \otimes \otimes (z_i \triangleright a_2)\#1 \otimes \cdots \otimes (z_i \triangleright a_p)\#1
\end{aligned} \tag{7.5.8}
$$

Now, the first line of (7.5.6) cancels out with the last of (7.5.8), due to the H-invariance of $\mathbf{c}$. Next, the terms containing a factor $(z_i \triangleright u^k a_k)\#z_i$ in (7.5.7) and (7.5.8) sum to zero. What remains is nothing else than $b_{\mathcal{B}}\mathbf{c} \times_\sigma \mathbf{c}_H^{(i)}$, which is zero since $\mathbf{c}$ is a cycle and the twisted shuffle product is linear (lemma 7.5.3). □

Before going on we notice the following fact.

Lemma 7.5.6. *If $\mathbf{c}$ is H-invariant then so is $\mathbf{c} \times_\sigma \mathbf{c}_H^{(i)}$, for any $i = 1, \ldots, n$.*

Now, given an orientation p-cycle $\mathbf{c}_{\mathcal{B}}$ for the spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$, we can iterate the twisted shuffle product to obtain a Hochschild $(p+n)$-chain $\mathbf{c}_{\mathcal{A}}$:

$$
\mathbf{c}_{\mathcal{A}} = \nu_{p,n}^{-1} \left(\left(\left((\mathbf{c}_{\mathcal{B}} \times_\sigma \mathbf{c}_H^{(1)}) \times_\sigma \mathbf{c}_H^{(2)} \right) \times_\sigma \cdots \right) \times_\sigma \mathbf{c}_H^{(n)} \right), \tag{7.5.9}
$$

where $\nu_{p,n}$ is the normalization factor [DS13a] (λ_n is the phase which enters in the definition of γ^{2m+1}, see above)

$$
\nu_{p,n} = \lambda_n^* \frac{(p+n)!}{p!}.
$$

Corollary 7.5.7. *If $\mathbf{c}_{\mathcal{B}}$ is H-invariant then $\mathbf{c}_{\mathcal{A}}$ is a Hochschild $(p+n)$-cycle over $\mathcal{A}$ with values in $\mathcal{A} \otimes \mathcal{A}^\circ$.*

Proof. Due to lemma 7.5.6 we can iterate proposition 7.5.5. □

Now we compute $\pi_{\hat{D}}(\mathbf{c}_{\mathcal{A}})$. We begin proving the following preparatory result.

Lemma 7.5.8. *For $i = 1, \ldots, n$, the representation on $\mathcal{H}'$ of the Hochschild $(p+1)$-cycle is $\mathbf{c}_{\mathcal{B}} \times_\sigma \mathbf{c}_H^{(i)})$ is given by:*

$$\pi_{\hat{D}}(\mathbf{c}_{\mathcal{B}} \times_\sigma \mathbf{c}_H^{(i)}) = (p+1)\gamma^i(\gamma^{2m+1})^p\gamma.$$

Proof. Consider a vector $v \otimes e_k \in \mathcal{H}'$. Using the definition of $\pi_{\hat{D}}$ we get:

$$\begin{aligned}
\pi_{\hat{D}}(\mathbf{c}_{\mathcal{B}} \times_\sigma \mathbf{c}_H^{(i)})(v \otimes e_k) = &\sum \hat{\pi}(a_0\sigma(z_i^{-1}, z_i)^*\#z_i^{-1})\hat{J}\hat{\pi}(b_0^*\#1)\hat{J}^{-1}[\hat{D}, \hat{\pi}(1\#z_i)] \\
&\cdot [\hat{D}, \hat{\pi}(a_1\#1)] \cdots [\hat{D}, \hat{\pi}(a_p\#1)](v \otimes e_k) \\
&+ \sum_{j=2}^{p} \sum (-1)^{j-1}\hat{\pi}(a_0\sigma(z_i^{-1}, z_i)^*\#z_i^{-1})\hat{J}\hat{\pi}(b_0^*\#1)\hat{J}^{-1}[\hat{D}, \hat{\pi}((z_i \triangleright a_1)\#1)] \\
&\cdots [\hat{D}, \hat{\pi}((z_i \triangleright a_{j-1})\#1)][\hat{D}, \hat{\pi}(1\#z_i)][\hat{D}, \hat{\pi}(a_j\#1)] \\
&\cdots [\hat{D}, \hat{\pi}(a_p\#1)](v \otimes e_k) \\
&+ \sum (-1)^p\hat{\pi}(a_0\sigma(z_i^{-1}, z_i)^*\#z_i^{-1})\hat{J}\hat{\pi}(b_0^*\#1)\hat{J}^{-1}[\hat{D}, \hat{\pi}((z_i \triangleright a_1)\#1)] \\
&\cdots [\hat{D}, \hat{\pi}((z_i \triangleright a_p)\#1)][\hat{D}, \hat{\pi}(1\#z_i)](v \otimes e_k)
\end{aligned} \tag{7.5.10}$$

Let us compute the first line of equation (7.5.10). Using the definition of $\hat{\pi}$ and of $\hat{D}$ we get:

$$\begin{aligned}
&\sum \hat{\pi}(a_0\sigma(z_i^{-1}, z_i)^*\#z_i^{-1})\hat{J}\hat{\pi}(b_0^*\#1)\hat{J}^{-1}[\hat{D}, \hat{\pi}(1\#z_i)] \\
&\qquad\qquad \cdot [\hat{D}, \hat{\pi}(a_1\#1)] \cdots [\hat{D}, \hat{\pi}(a_p\#1)](v \otimes e_k) \\
&= \sum \Big[\pi^\circ(b_0)\pi(a_0\sigma(z_i^{-1}, z_i)^*)\pi^\circ(\sigma(z_i^{-1}, z^{k+f_i}))(z_i^{-1} \triangleright \gamma^i\pi^\circ(\sigma(z_i, z^k))(z_i \triangleright (\gamma^{2m+1})^p \\
&\qquad \cdot [D, \pi(a_1)] \cdots [D, \pi(a_p)]v))\Big] \otimes e_k \\
&= \sum \gamma^i(\gamma^{2m+1})^p\Big[\pi^\circ(b_0)\pi(a_0)\pi(\sigma(z_i^{-1}, z_i)^*)\pi^\circ((z_i^{-1} \triangleright \sigma(z_i, z^k))\sigma(z_i^{-1}, z^{k+f_i})) \\
&\qquad \cdot (z_i^{-1} \triangleright z_i \triangleright [D, \pi(a_1)] \cdots [D, \pi(a_p)]v)\Big] \otimes e_k \\
&= \gamma^i(\gamma^{2m+1})^p\Big[\pi(\sigma(z_i^{-1}, z_i)^*)\pi^\circ(\sigma(z_i^{-1}, z_i))(z_i^{-1} \triangleright z_i \triangleright [D, \pi(a_1)] \cdots [D, \pi(a_p)]v)\Big] \otimes e_k \\
&= \gamma^i(\gamma^{2m+1})^p\left(\pi(\sigma(z_i^{-1}, z_i)^*)\sigma(z_i^{-1}, z_i)\gamma v\right) \otimes e_k \\
&= \gamma^i(\gamma^{2m+1})^p\gamma v \otimes e_k
\end{aligned} \tag{7.5.11}$$

Here f_i denotes the element of $\mathbb{Z}^n$ defined by $(f_i)_j = \delta_{ij}$. In the computation we used the twisted module condition, the unitarity of $\sigma(z_i^{-1}, z_i)$ and the fact that $\mathbf{c}_{\mathcal{B}}$ is an orientation cycle for the triple over $\mathcal{B}$, which implies that $\pi_D(\mathbf{c}_{\mathcal{B}}) = \gamma$ (if p is odd, $\gamma = \text{id}$). Next we compute the second line of equation (7.5.10).

$$\begin{aligned}
&\sum_{j=2}^{p}\sum(-1)^{j-1}\hat{\pi}(a_0\sigma(z_i^{-1},z_i)^*\# z_i^{-1})\hat{J}\hat{\pi}(b_0^*\#1)\hat{J}^{-1}[\hat{D},\hat{\pi}((z_i\triangleright a_1)\#1)]\\
&\qquad\cdots[\hat{D},\hat{\pi}((z_i\triangleright a_{j-1})\#1)][\hat{D},\hat{\pi}(1\# z_i)][\hat{D},\hat{\pi}(a_j\#1)]\cdots[\hat{D},\hat{\pi}(a_p\#1)](v\otimes e_k)\\
&\quad=\sum_{j=2}^{p}\sum(-1)^{j-1}\pi^\circ(b_0)\pi(a_0\sigma(z_i^{-1},z_i)^*)(\gamma^{2m+1})^{j-1}\gamma^i(\gamma^{2m+1})^{p-j+1}\\
&\qquad\cdot\pi^\circ(\sigma(z_i^{-1},z^{k+f_i}))\Big[z_i^{-1}\triangleright[D,\pi(z_i\triangleright a_1)]\cdots[D,\pi(z_i\triangleright a_{j-1})]\\
&\qquad\cdot\pi^\circ(\sigma(z_i,z^k))\,(z_i\triangleright[D,\pi(a_j)]\cdots[D,\pi(a_p)]v)\Big]\otimes e_k\\
&\quad=\sum_{j=2}^{p}\sum\gamma^i(\gamma^{2m+1})^p\pi^\circ(b_0)\pi(a_0\sigma(z_i^{-1},z_i)^*)\pi^\circ(\sigma(z_i^{-1},z^{k+f_i}))[D,\pi(z_i^{-1}\triangleright z_i\triangleright a_1)]\cdots\\
&\qquad\cdots[D,\pi(z_i^{-1}\triangleright z_i\triangleright a_{j-1})]\pi^\circ(z_i^{-1}\triangleright\sigma(z_i,z^k))\left(z_i^{-1}\triangleright z_i\triangleright[D,\pi(a_j)]\cdots[D,\pi(a_p)]v\right)\otimes e_k\\
&\quad=\sum_{j=2}^{p}\sum\gamma^i(\gamma^{2m+1})^p\pi^\circ(b_0)\pi(a_0\sigma(z_i^{-1},z_i)^*)\pi^\circ(\sigma(z_i^{-1},z^{k+f_i}))[D,\pi(z_i^{-1}\triangleright z_i\triangleright a_1)]\\
&\qquad\cdots[D,\pi(z_i^{-1}\triangleright z_i\triangleright a_{j-1})]\pi^\circ(\sigma(z_i^{-1},z^{k+f_i})^*)\pi^\circ(\sigma(z_i^{-1},z_i))\\
&\qquad\cdot\left(z_i^{-1}\triangleright z_i\triangleright[D,\pi(a_j)]\cdots[D,\pi(a_p)]v\right)\otimes e_k
\end{aligned}\tag{7.5.12}$$

Using the unitarity of $\sigma(z^r,z^s)$, the twisted module condition and the first order condition, we can rewrite (7.5.12) in the following way.

$$\begin{aligned}
&\sum_{j=2}^{p}\sum\gamma^i(\gamma^{2m+1})^p\pi^\circ(b_0)\pi(a_0\sigma(z_i^{-1},z_i)^*)[D,\pi(z_i^{-1}\triangleright z_i\triangleright a_1)]\\
&\qquad\cdots[D,\pi(z_i^{-1}\triangleright z_i\triangleright a_{j-1})]\pi(\sigma(z_i^{-1},z_i))[D,\pi(a_j)]\cdots[D,\pi(a_p)]v\otimes e_k\\
&\quad=\sum_{j=2}^{p}\sum\gamma^i(\gamma^{2m+1})^p\pi^\circ(b_0)\pi(a_0\sigma(z_i^{-1},z_i)^*)\pi(\sigma(z_i^{-1},z_i))\\
&\qquad\cdot[D,\pi(a_1)]\cdots[D,\pi(a_{j-1})][D,\pi(a_j)]\cdots[D,\pi(a_p)]v\otimes e_k\\
&\quad=\sum_{j=2}^{p}\sum\gamma^i(\gamma^{2m+1})^p\pi^\circ(b_0)\pi(a_0)[D,\pi(a_1)]\cdots\cdots[D,\pi(a_p)]v\otimes e_k\\
&\quad=\gamma^i(\gamma^{2m+1})^p\gamma v\otimes e_k.
\end{aligned}\tag{7.5.13}$$

In the same way one can see that also the last line of equation (7.5.10) is equal to $\gamma^i(\gamma^{2m+1})^p\gamma v\otimes e_k$. This concludes the proof of the proposition. □

Proposition 7.5.9. *For $n=2m$ we have: $\pi_{\hat{D}}(\boldsymbol{c}_{\mathcal{A}})=\hat{\gamma}$.*

Proof. Let p be odd. Then $\hat{\gamma}=\mathrm{id}$, $(\gamma^{2m+1})^p=\gamma^{2m+1}$, $\gamma=\mathrm{id}$. Iterating the result of lemma 7.5.8

we get:

$$\pi_{\hat{D}}(\mathbf{c}_{\mathcal{A}}) = \lambda_n^* \gamma^{2m} \cdots \gamma^1 \gamma^{2m+1} \gamma = \gamma^{2m+1} \gamma^{2m+1} = \mathrm{id} = \hat{\gamma}.$$

For p even, instead, we have $\hat{\gamma} = \gamma^{2m+1}$, $(\gamma^{2m+1})^p = \mathrm{id}$. Which implies that

$$\pi_{\hat{D}}(\mathbf{c}_{\mathcal{A}}) = \lambda_n^* \gamma^{2m} \cdots \gamma^1 \gamma = \gamma^{2m+1} \gamma = \hat{\gamma}.$$

□

We are left with the $n = 2m+1$ case. If p is even than the Dirac operator is the same as in the previous discussion and so lemma 7.5.8 still holds. Hence we can take $\mathbf{c}_{\mathcal{A}}$ as defined by equation (7.5.9) and, with a computation similar to that of proposition 7.5.9, we obtain the following.

Proposition 7.5.10. *Let p be even and let $n = 2m+1$. Let $\boldsymbol{c}_{\mathcal{A}}$ be as in (7.5.9). Then $\pi_{\hat{D}}(\boldsymbol{c}_{\mathcal{A}}) = \hat{\gamma} = \mathrm{id}$.*

Let us consider now the case when both n and p are odd. The Dirac operator is given by

$$\hat{D} = D \otimes \sigma^1 + \sum_{j=1}^{n} \gamma^j \delta_j \otimes \sigma^2,$$

while the $\mathbb{Z}_2$ grading is simply given by $\hat{\gamma} = \mathrm{id} \otimes \sigma^3$. Then, with a computation similar to that of lemma 7.5.8, we can prove the following lemma.

Lemma 7.5.11. *For $i = 1, \ldots, n$, the representation on $\mathcal{H}'' = \mathcal{H}' \otimes \mathbb{C}^2$ of the Hochschild $(p+1)$-cycle $\boldsymbol{c}_{\mathcal{B}} \times_\sigma \boldsymbol{c}_H^{(i)}$ is given by:*

$$\pi_{\hat{D}}(\boldsymbol{c}_{\mathcal{B}} \times_\sigma \boldsymbol{c}_H^{(i)}) = (p+1) \gamma^i \sigma^2 (\sigma^1)^p.$$

Iterating this result we get:

Proposition 7.5.12. *Let $\mathbf{c}'_{\mathcal{A}}$ be defined by*

$$\boldsymbol{c}'_{\mathcal{A}} = \mu_{p,n}^{-1} \left(\left(\left((\boldsymbol{c}_{\mathcal{B}} \times_\sigma \boldsymbol{c}_H^{(1)}) \times_\sigma \boldsymbol{c}_H^{(2)} \right) \times_\sigma \cdots \right) \times_\sigma \boldsymbol{c}_H^{(n)} \right), \tag{7.5.14}$$

where $\mu_{p,n}$ is the normalization factor

$$\nu_{p,n} = -i \lambda_n^* \frac{(p+n)!}{p!}.$$

Then we have $\pi_{\hat{D}}(\boldsymbol{c}'_{\mathcal{A}}) = \hat{\gamma}$.

CHAPTER 8

Quantum principal G-bundles and gauge theories

In the previous chapters we discussed various aspects of noncommutative principal $\mathbb{T}^n$-bundles. In particular, we focused our attention to bundles with differential calculus compatible with the de Rham calculus on $\mathbb{T}^n$, showing how this property allows a description of strong connections more close to the classical one; indeed, a $\mathbb{T}^n$-connection over a bundle of this kind can be described by a family of n 1-forms over the total space of the bundle, which corresponds to the classical picture of a connection as a $\mathfrak{t}_n$-valued 1-form over the total space, where $\mathfrak{t}_n$ is the Lie algebra of $\mathbb{T}^n$. Moreover, we have introduced a way to construct spectral triples over cleft $\mathbb{T}^n$-bundles defining a Dirac calculus compatible with the de Rham calculus on $\mathbb{T}^n$. This provides a way to put a structure of quantum principal $\mathbb{T}^n$-bundle over cleft $\mathcal{O}(\mathbb{T}^n)$-extensions (with suitable properties).

In this chapter we shall extend – partially – these results to noncommutative principal G-bundles, G being a compact connected semisimple Lie group. First of all, we shall discuss the structure of quantum principal G-bundles with calculus compatible with the de Rham calculus on G. Next, we shall consider cleft extensions and, under suitable hypotheses, we shall work out a construction of a real spectral triple whose Dirac operator determines a first order differential calculus compatible with the de Rham calculus. Finally, we shall introduce twisted Dirac operators and study their behaviour under gauge transformations.

8.1 Quantum principal G-bundles: definition, general properties and strong connections

In chapter 4 we gave a characterization of quantum principal $\mathbb{T}^n$-bundles. In this section we will extend this analysis to the more general case of quantum principal bundles whose "structure group" (which, we recall, is actually a Hopf algebra) is the algebra of smooth functions over a (compact, semisimple, connected) Lie group G. In particular we shall relate strong connections to the usual concept of (gauge) connections on a principal G-bundle, i.e. to $\mathfrak{g}$-valued 1-forms (here $\mathfrak{g}$ denotes the Lie algebra of G).

8.1.1 The algebra $H = C^\infty(G)$ and its differential calculus

Let G be a compact connected semisimple Lie group of dimension n, $\mathfrak{g}$ be its Lie algebra, and consider the algebra $C^\infty(G)$ of (complex) smooth functions on G. Let $\{X_j\}$ be any linear basis of $\mathfrak{g}$; each X_j is a (left invariant) vector field on G, so it acts as a derivation on $C^\infty(M)$. In particular, we recall that its action can be written in the following way:

$$X_j(f) = \frac{d}{dt} R_{\exp tX_j}(f)\bigg|_{t=0}, \tag{8.1.1}$$

where R_g is the right regular representation of G on $C^\infty(G)$, $(R_g f)(h) = F(hg)$ and exp is the exponential map of G. Then we can define a family of seminorms on $C^\infty(G)$: for any $r \in \mathbb{N}^n$ and any $f \in C^\infty(G)$ we set

$$p_n(f) = \sup_{g\in G} |X_1^{r_1} \cdots X_n^{r_n}(f)(g)|.$$

Proposition 8.1.1. *$C^\infty(G)$ is a nuclear Fréchet algebra with respect to the locally convex topology defined by the seminorms p_n.*

Proof. It follows from corollary B.3.8. ☐

Corollary 8.1.2. *There is an isomorphism of Fréchet algebras $C^\infty(G)\overline{\otimes}C^\infty(G) \simeq C^\infty(G\times G)$.*

Proof. See proposition B.3.9. ☐

Now let $H = C^\infty(G)$ and consider the maps $\Delta : H \to H\overline{\otimes}H \simeq C^\infty(G\times G)$, $S : H \to H$, $\varepsilon : H \to \mathbb{C}$ defined as follows (for $f \in H$, $g, g' \in G$):

$$\Delta(f)(g, g') = f(gg'),$$

$$S(f)(g) = f(g^{-1}),$$

$$\varepsilon(f) = f(e),$$

where $e \in G$ is the identity element.

Proposition 8.1.3. *The maps Δ, S, ε are continuous maps.*

Proof. The proof that S and ε are continuous is straightforward. Let us prove, instead, that Δ is continuous. The Lie algebra of $G\times G$ is simply $\mathfrak{g}\oplus\mathfrak{g}$. Hence, given a linear basis $\{X_j\}_{j=1,\dots n}$ of $\mathfrak{g}$, we can consider the following basis of $\mathfrak{g}\oplus\mathfrak{g}$: $\{Y_1,\dots,Y_n,Z_1,\dots,Z_n\}$, with $Y_j = X_j$ acting on the first factor and $Z_j = X_j$ acting on the second factor. Then the topology of $C^\infty(G\times G)$ is defined by the seminorms

$$q_{I,J}(f) = \sup_{(g,h)\in G\times G} Y_{i_1}\cdots Y_{i_r} Z_{j_1}\cdots Z_{j_s}(f)(g,h)|,$$

where $I = \{i_1,\dots,i_r\}$, $J = \{j_1,\dots,j_s\}$. Take now $f \in H$; then $\Delta(f)$ can be seen as an element of $C^\infty(G\times G)$ and we have, for any $g,h\in G$,

$$(X_i\Delta(f))(g,h)=\frac{d}{dt}f(ge^{tX_i}h)\Big|_{t=0}=\frac{d}{dt}f(ghh^{-1}e^{tX_i}h)\Big|_{t=0}=\frac{d}{dt}f(ghe^{t\mathrm{Ad}_{h^{-1}}(X_i)})\Big|_{t=0} \tag{8.1.2}$$

where e^X is the exponential map. Hence, if we denote by $R_i^j(g)$ the matrix elements of the adjoint representation of G on $\mathfrak{g}$, we have:

$$(X_i\Delta(f))(g,h)=\frac{d}{dt}f(ghe^{t\sum_j R_i^j(h^{-1})X_j})\Big|_{t=0}=\sum_j(R_i^j(h^{-1})X_jf)(gh). \tag{8.1.3}$$

By iterating this procedure, then, we obtain the following relation (we use here Einstein convention: the sum over repeated indices is understood):

$$\begin{aligned} &Y_{i_1}^{r_1}\cdots Y_{i_r}Z_{j_1}\cdots Z_{j_s}(\Delta(f))(g,h)\\ &\qquad=[R_{i_1}^{k_1}(h^{-1})\cdots R_{i_r}^{k_r}(h^{-1})X_{k_1}\cdots X_{k_r}X_{j_1}\cdots X_{j_s}(f)](gh)\end{aligned} \tag{8.1.4}$$

Using (8.1.4) we can perform the following estimate.

$$\begin{aligned} q_{I,J}(\Delta(f)) &= \sup_{(g,h)\in G\times G}|Y_{i_1}\cdots Y_{i_r}Z_{j_1}\cdots Z_{j_s}(f)(g,h)|\\ &= \sup_{(g,h)\in G\times G}|[R_{i_1}^{k_1}(h^{-1})\cdots R_{i_r}^{k_r}(h^{-1})X_{k_1}\cdots X_{k_r}X_{j_1}\cdots X_{j_s}(f)](gh)|\\ &\leq \sup_{(g,h)\in G\times G}|R_{i_1}^{k_1}(h^{-1})|\cdots|R_{i_r}^{k_r}(h^{-1})|\cdot|[X_{k_1}\cdots X_{k_r}X_{j_1}\cdots X_{j_s}(f)](gh)|\end{aligned} \tag{8.1.5}$$

Each R_i^j is a smooth function on G. So, if we denote by $\|\cdot\|_\infty$ the sup norm on $C^\infty(G)$, we obtain, from (8.1.6),

$$q_{I,J}(\Delta(f))\leq\|R_{i_1}^{k_1}\|_\infty\cdots\|R_{i_r}^{k_r}\|_\infty\cdot\left(\sup_{(g,h)\in G\times G}|X_{k_1}\cdots X_{k_r}X_{j_1}\cdots X_{j_s}(f)(gh)|\right). \tag{8.1.6}$$

In particular we have shown that $q_{I,J}\circ\Delta$ is less or equal than a finite multiple of a continuous seminorm on H. Therefore (see proposition B.2.7, proposition B.2.8 and remark B.2.9) it is a continuous seminorm. It follows (see theorem B.2.10) that Δ is a continuous map. □

The next step toward the definition of quantum principal G-bundles is the choice of a first order differential calculus on the algebra $H=C^\infty(G)$. The most natural choice is to consider the de Rham calculus $\Omega^1(H)=\Omega^1_{dR}(H)$. It is [Wor89] the bicovariant calculus defined by[1] the ad_R-invariant ideal $Q=(\ker\ \varepsilon)^2$. Indeed, the de Rham calculus can be characterized in the following way. Let $\{T_j\}$ be a linear basis of $\mathfrak{g}$ and let $\{\varepsilon^j\}$ be the dual basis of $\mathfrak{g}^*$. Since the space of sections on the cotangent bundle T^*G can be identified with $C^\infty(G)\otimes\mathfrak{g}^*$, the (de Rham)

[1] See theorem 2.3.31.

exterior differential d^G on G can be described as follows:

$$d^G(f) = \sum_j T_j(f)\varepsilon^j.$$

Here T_j is seen as a (left invariant) vector field on G, and so it is clear which is its action on f (in particular, it is given by (8.1.1)). It follows that we can write $T_j(f)$ as:

$$T_j(f) = f_{(1)} \cdot (df_{(2)})_e(T_j).$$

Consider now a universal one form $\eta = \sum adb \in \Omega^1 H$. If we impose $\eta = 0$ in $\Omega^1_{dR}(H)$ we obtain:

$$\sum_j \sum ab_{(1)}(db_{(2)})_e(T_j)\varepsilon^j = 0.$$

Since the ε^j are linearly independent, this means that $\sum ab_{(1)}(db_{(2)})_e(T_j) = 0$ for any j. Now, η can be seen as an element of $H \otimes H$: $\eta = a \otimes b - ab \otimes 1$. Hence[2], $r(\eta) = \sum ab_{(1)} \otimes b_{(2)} - ab \otimes 1$. If now we apply $(\mathrm{id} \otimes \mathrm{ev}_e) \circ (\mathrm{id} \otimes d)$ to $r(\eta)$ we obtain:

$$(\mathrm{id} \otimes \mathrm{ev}_e) \circ (\mathrm{id} \otimes d) \circ r(\eta) = \sum ab_{(1)}(db_{(2)})_e(T_j) = 0.$$

That is, $r(\eta) \in H \otimes Q$. Conversely, it is straightforward to check that $r^{-1}(H \otimes Q)$ is contained in the sub-bimodule N defining the de Rham calculus $\Omega^1_{dR}(H)$.

We conclude this section with the following observation: since Q is defined by two closed conditions ($f(e) = 0$, $(df)_e = 0$), it is a closed ideal of H. In particular, since H is a Fréchet algebra, Q is a Fréchet space.

8.1.2 Quantum principal G-bundles

Now we can give the definition of quantum principal G-bundles. Here and in the following sections H will denote the Fréchet algebra $C^\infty(G)$, with G a compact connected semi-simple Lie group of dimension n.

Definition 8.1.4. *Let $\mathcal{A}$ be a Fréchet algebra and an H-comodule algebra. Assume that the coaction $\Delta_R : \mathcal{A} \to \mathcal{A} \otimes H$ is a continuous*[3] *map. Let $N \subseteq \mathcal{A} \otimes \mathcal{A}$ be a sub-bimodule defining a first order differential calculus $\Omega^1(\mathcal{A})$. Then $\mathcal{A}$ is called a* quantum principal G-bundle *if $(\mathcal{A}, H, \Delta_R, N, Q)$, where $Q = (\ker\varepsilon)^2$, is a quantum principal bundle*[4].

Consider now a principal extension $\mathcal{B} \hookrightarrow \mathcal{A}$ (with continuous coaction) with respect to the Hopf algebra H. We know that it is a quantum principal bundles with the universal calculus (both on $\mathcal{A}$ and on H). A natural question is the following one: which are the conditions on a differential calculus $\Omega^1(\mathcal{A}) = \Omega^1\mathcal{A}/N$ which make $(\mathcal{A}, H, \Delta_R, N, Q)$ a quantum principal G-bundle? The answer is the content of the proposition below. Before stating it we need to introduce an action

[2]For the definition of the map r see equation (2.3.4).
[3]With respect to the projective topology (see appendix B) on $\mathcal{A} \otimes H$.
[4]In the sense of definition 4.3.1.

of the Lie algebra $\mathfrak{g}$ on the algebra $\mathcal{A}$. It is defined as follows: for $X \in \mathfrak{g}$ and $a \in \mathcal{A}$,

$$X(a) = a_{(0)}a_{(1)}(X), \tag{8.1.7}$$

where, for $f \in H$, we set

$$f(X) = X(f)(e) = \frac{d}{dt}f(\exp tX)|_{t=0}. \tag{8.1.8}$$

Proposition 8.1.5. *Let $\mathcal{A}$ be a principal H-comodule algebra, with invariant subalgebra $\mathcal{B}$ and continuous coaction Δ_R, and let $\Omega^1(\mathcal{A})$ be an H-covariant first order differential calculus, defined by a sub-bimodule $N \subset \mathcal{A} \otimes \mathcal{A}$. Then $(\mathcal{A}, H, \Delta_R, N, Q)$ is a quantum principal G-bundle if the following hold:*

(i) *for any $j = 1, \dots, \dim G$,*

$$\sum adb = 0 \ \text{ in } \Omega^1(\mathcal{A}) \quad \Rightarrow \quad \sum aT_j(b) = 0; \tag{8.1.9}$$

(ii) *let $\eta \in \Omega^1\mathcal{A}$, $\eta = \sum adb$. Then;*

$$\sum aT_j(b) = 0 \quad \forall j = 1, \dots, \dim G \quad \Rightarrow \quad [\eta]_N \in \mathcal{A}\Omega^1(\mathcal{B})\mathcal{A}. \tag{8.1.10}$$

Here $\{T_j\}$ is any linear basis of the Lie algebra $\mathfrak{g}$.

Proof. We check properties (i)-(iv) of definition 4.3.1. (i) is trivially satisfied, since we assumed $\mathcal{A}$ to be a principal comodule algebra. Also (ii) is fulfilled: indeed, the covariance of the calculus implies that N is right H-equivariant. Let us check property (iii). Let $\eta \in \Omega^1\mathcal{A}$ be zero in $\Omega^1(\mathcal{A})$. That is, it is an element of N. We can write η as $\eta = \sum p \otimes q - pq \otimes 1$. Then an element $f \in H$ belongs to Q iff $f(e) = 0$, $(df)_e = 0$. Hence, an element ξ of $\mathcal{A} \otimes H$ belongs to $\mathcal{A} \otimes Q$ iff

$$\begin{cases} (\mathrm{id} \otimes \mathrm{ev}_e)T_R(\xi) = (\mathrm{id} \otimes \varepsilon)T_R(\xi) = 0, \\ \\ ((\mathrm{id} \otimes \mathrm{ev}_e) \circ (\mathrm{id} \otimes d) \circ T_R)(\xi) = ((\mathrm{id} \otimes \varepsilon) \circ (\mathrm{id} \otimes d) \circ T_R)(\xi) = 0. \end{cases} \tag{8.1.11}$$

The form η trivially satisfies the first of (8.1.11). Indeed,

$$(\mathrm{id} \otimes \varepsilon)T_R(\eta) = pq_{(0)}\varepsilon(q_{(1)}) - pq = pq - pq = 0.$$

Next, by direct computation we obtain:

$$\begin{aligned} ((\mathrm{id} \otimes \varepsilon) \circ (\mathrm{id} \otimes d) \circ T_R)(\eta) &= (\mathrm{id} \otimes \varepsilon)\left(\sum \sum_j pq_{(0)} \otimes q_{(1)}q_{(2)}(T_j)\varepsilon^j\right) \\ &= \sum \sum_j pq_{(0)} \otimes q_{(1)}(T_j)\varepsilon^j = \sum_j \sum pq_{(0)}q_{(1)}(T_j) \otimes \varepsilon^j \end{aligned} \tag{8.1.12}$$

Here $\{\varepsilon^j\}$ is the dual basis of $\{T_j\}$. But now $\sum pq_{(0)}q_{(1)}(T_j) = \sum pT_j(q) = 0$ by hypothesis; hence also (8.1.12) is equal to zero. It follows that $T_R(\eta)$ belongs to $\mathcal{A} \otimes Q$.

In order to conclude the proof of the proposition we need only to show that also property (iv) is fulfilled. Take $\eta \in \Omega^1\mathcal{A}$, $\eta = \sum adb$, such that $T(\eta) = 0$. Since η can be written, as an element of $\mathcal{A} \otimes \mathcal{A}$, as $\eta = \sum(a \otimes b - ab \otimes 1)$, we obtain:

$$T_R(\eta) = \sum(ab_{(1)} \otimes b_{(2)} - ab \otimes 1).$$

Now, $T(\eta) = 0$ means that $T_R(\eta)$ belongs to $\mathcal{A} \otimes Q$. Imposing this condition we obtain:

$$\begin{cases} (\mathrm{id} \otimes \mathrm{ev}_e) T_R(\eta) = 0, \\ (\mathrm{id} \otimes \mathrm{ev}_e) \circ (\mathrm{id} \otimes d) T_R(\eta) = 0. \end{cases} \tag{8.1.13}$$

From the second of (8.1.13) we get, then,

$$\sum_j \sum ab_{(1)} T_j(b_{(2)}) \otimes \varepsilon^j = 0.$$

But the ε^j, since they form a frame for the cotangent bundle of G, are linearly independent, so we obtain that:

$$\sum ab_{(1)} T_j(b_{(2)}) = 0$$

for any $j = 1, \ldots, \dim G$. Condition (ii) of the thesis of this proposition now implies that $[\eta]_N$ belongs to $\mathcal{A}\Omega^1(\mathcal{B})\mathcal{A}$. Hence $\ker(T) \subseteq \mathcal{A}\Omega^1(\mathcal{B})\mathcal{A}$; that is, also condition (iv) of definition 4.3.1 is fulfilled. □

We can also prove the converse. That is,

Proposition 8.1.6. *Let $(\mathcal{A}, H, \Delta_R, N, Q)$ be a quantum principal G-bundle. Then* (8.1.9) *and* (8.1.10) *hold.*

Proof. Take $\eta \in N$ (that is, $\eta = 0$ in $\Omega^1(\mathcal{A})$) and write it as $\eta = \sum a \otimes b - ab \otimes 1$. Then condition (iii) of definition 4.3.1 implies that $T_R(\eta) \in \mathcal{A} \otimes Q$; that is, $(\mathrm{id} \otimes \mathrm{ev}_e) \circ (\mathrm{id} \otimes d) T_R(\eta) = 0$. Making this condition explicit, we obtain:

$$\begin{aligned} 0 &= (\mathrm{id} \otimes \mathrm{ev}_e) \circ (\mathrm{id} \otimes d) T_R(\eta) = (\mathrm{id} \otimes \mathrm{ev}_e) \circ (\mathrm{id} \otimes d) \Big(\sum ab_{(1)} \otimes b_{(2)} - ab \otimes 1 \Big) \\ &= \sum_j \sum ab_{(1)} b_{(2)}(T_j) \otimes \varepsilon^j = \sum_j \sum a T_j(b) \otimes \varepsilon^j. \end{aligned}$$

Since the ε^j are linearly independent, this implies that $\sum a T_j(b) = 0$ for any $j = 1, \ldots, \dim G$. So we have proved (8.1.9).

The computation above shows also that, for any $\eta \in \Omega^1 \mathcal{A}$, $\eta = \sum a db$, $T_R(\eta)$ belongs to $\mathcal{A} \otimes Q$ if $\sum a T_j(b) = 0$ for any j. But this means that if this condition holds, $T(\eta) = 0$. From condition (iv) of definition 4.3.1, this implies that $[\eta]_N \in \mathcal{A}\Omega^1(\mathcal{B})\mathcal{A}$. So we have proved also (8.1.10). □

8.1.3 Strong connections over quantum principal G-bundles

The next step in the study of quantum principal G-bundles is the characterization of strong connections. We begin introducing the following definition. Here and in the rest of this section

$(\mathcal{A}, H, \Delta_R, N, Q)$ will denote a quantum principal G-bundle. Moreover, we assume $\dim(G) = n$ and we fix a linear basis $T_1, \ldots, T_n$ of the Lie algebra $\mathfrak{g}$.

Definition 8.1.7. *A* strong G-connection *for the G-bundle $\mathcal{A}$ is a family $\omega^1, \ldots, \omega^n$ of 1-forms $\omega^i \in \Omega^1(\mathcal{A})$ such that:*

(i) *for any $j = 1, \ldots, n$ and for any $g \in G$,*

$$\omega^j_{(0)}\omega^j_{(1)}(g) = R^j_k(g)\omega^k,$$

where R is the adjoint representation of G on $\mathfrak{g}$;

(ii) *for any $j, k = 1, \ldots, n$, if $\omega^j = \sum p^j dq^j$, then*

$$\sum p^j T_k(q^j) = \delta_{jk},$$

where $T_k(a) = a_{(0)}a_{(1)}(T_k)$, for any $a \in \mathcal{A}$;

(iii) *$\forall a \in \mathcal{A}$, $da - \sum_j a_{(0)} \cdot a_{(1)}(T_j)\omega^j \in \Omega^1(\mathcal{B})\mathcal{A}$, where $\mathcal{B}$ is the invariant subalgebra of $\mathcal{A}$.*

Proposition 8.1.8. *Let $\{\omega_i\}$ be a strong G-connection over the bundle $\mathcal{A}$. Then the map $\omega : H \to \mathcal{A}$, defined by*

$$\omega(h) = \sum_{j=1}^{n} h(T_j)\omega^j, \tag{8.1.14}$$

is a strong connection (form), in the sense of definition 4.3.6.

Proof. We check (i)-(iv) of definition 4.3.6. Let us begin by proving that (i) is fulfilled by ω. Of course, $\omega(1) = 0$. Next, take $h \in Q$. This means that $h(e) = 0$ and $(dh)_e = 0$. Hence, for any $j = 1, \ldots, n$,

$$h(T_j) = \frac{d}{dt}h(\exp tT_j)\Big|_{t=0} = (dh)_e(T_j) = 0.$$

That is, $\omega(h) = 0$, and so $\omega(Q) = 0$. Now we consider condition (ii), that is the covariance with respect to the right adjoint coaction. Let $h \in H$ and $g \in G$. Then we have:

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{ev}_g) \circ (\omega \otimes \mathrm{id}) \circ \mathrm{ad}_R(h) &= \omega(h_{(2)}) \otimes h_{(1)}(g^{-1})h_{(3)}(g) \\ &= \sum_{j=1}^{n} h_{(1)}(g^{-1})h_{(2)}(T_j)h_{(3)}(g)\omega^j \\ &= \sum_{j=1}^{n} \frac{d}{dt}h(g^{-1} \cdot \exp(tT_j) \cdot g)\omega^j \\ &= f(\mathrm{Ad}_g T_j)\omega^j = \sum_{j,k=1}^{n} h(T_k)R^k_j(g^{-1})\omega^j,\end{aligned} \tag{8.1.15}$$

where R is the adjoint representation of G on $\mathfrak{g}$. From the other side of the equation which defines property (ii), instead, we obtain:

$$(\mathrm{id} \otimes \mathrm{ev}_g) \circ \Delta_R \circ \omega(h) = \sum_{j=1}^{n} h(T_j)\omega^j_{(0)}\omega^j_{(1)}(g) \tag{8.1.16}$$

It is clear that equations (8.1.15) and (8.1.16) are equal if and only if

$$\omega^j_{(0)}\omega^j_{(1)}(g) = \sum_{k=1}^{n} R^{jk}(g)\omega_k,$$

which is exactly condition (i) of definition 8.1.7. Since condition (iv) of definition 4.3.6 follows directly from (iii) of definition 8.1.7, we are left with the proof of condition (iii).

Write each ω_j, seen as an element of $\mathcal{A} \otimes \mathcal{A}$, as $\omega^j = \sum p^j \otimes q^j - p^j q^j \otimes 1$. Hence we obtain, for any $h \in H$,

$$T_R(\omega(h)) = \sum_j \sum h(T_j)\left(p^j q^j_{(0)} \otimes q^j_{(1)} - p^j q^j \otimes 1\right). \tag{8.1.17}$$

On the other side, we have also:

$$(1 \otimes (id - \varepsilon))(h) = 1 \otimes (h - h(e)). \tag{8.1.18}$$

In order to prove condition (iii) we have to show that the difference between (8.1.17) and (8.1.18) belongs to $\mathcal{A} \otimes Q$, and so is zero in $\mathcal{A} \otimes H/Q$. This means that it must vanish at e, and its differential must vanish, too. More precisely, we have to show that:

$$\begin{cases} (\mathrm{id} \otimes \mathrm{ev}_e)\left[\sum_j \sum h(T_j)\left(p^j q^j_{(0)} \otimes q^j_{(1)} - p^j q^j \otimes 1\right) - 1 \otimes (h - h(e))\right] = 0, \\ (\mathrm{id} \otimes (\mathrm{ev}_e \circ d^G))\left[\sum_j \sum h(T_j)\left(p^j q^j_{(0)} \otimes q^j_{(1)} - p^j q^j \otimes 1\right) - 1 \otimes (h - h(e))\right] = 0. \end{cases} \tag{8.1.19}$$

The first of (8.1.19) is trivially fulfilled. Let us look at the second one. We can rewrite it as:

$$(\mathrm{id} \otimes \mathrm{ev}_e)\left[\sum_{j,k} \sum h(T_j) p^j q^j_{(0)} q^j_{(1)}(T_k) \otimes \varepsilon^k - \sum_k h(T_k) \otimes \varepsilon^k\right]$$

where, we recall, $\{\varepsilon^k\}$ is the basis of $\mathfrak{g}^*$ dual to $\{T_k\}$. Relabelling the indices, we obtain:

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{ev}_e)&\left[\sum_k h(T_j)\left(\sum_j \sum p^k q^k_{(0)} q^k_{(1)}(T_j) \otimes \varepsilon^k - 1 \otimes \varepsilon^j\right)\right] \\ &= \sum_k h(T_k)\left(\sum_j \sum p^k T_j(q^k) - 1\right) \otimes \varepsilon^j,\end{aligned}$$

which is zero due to condition (ii) of definition 8.1.7. □

Now we could ask ourself if the converse holds. That is, given a strong connection ω (in the sense of definition 4.3.6), does it come from a family of 1-forms $\{\omega^i\}$ defining a strong G-connection?

Proposition 8.1.9. *Let $\omega : H \to \Omega^1(\mathcal{A})$ be a strong connection (form) over the quantum principal G-bundle $\mathcal{A}$. Then ω is defined by a strong G-connection $\{\omega_i\}$, as in proposition 8.1.8.*

Proof. Let Q be the ideal defining the de Rham differential calculus on H. Then Q can be identified with the ideal of functions on G vanishing at e with differential vanishing at e, too. By hypothesis, $\omega(\mathbb{C}) = 0$, $\omega(Q) = 0$. In particular, given $f, f' \in H$ with $(df)_e = (df')_e$, this means that $\omega(f - f') = 0$. Hence $\omega(f)$ depends only on the behaviour of f at the identity $e \in G$. It follows that ω can be seen as a linear map $\omega : \mathcal{O}_e \to \Omega^1\mathcal{A}$, where $\mathcal{O}_e$ is the set of germs of smooth functions at e (that is, the stalk at e of the sheaf of smooth functions on G). Now, given a germ $[f] \in \mathcal{O}_e$, we can write each of its representatives as $f = f(e) + \sum_j (df)_e(T_j)t_j + \widetilde{f}$, where $\widetilde{f}$ is a smooth function with differential vanishing at e and $\{t_1, \ldots, t_n\}$ is a set of local coordinates at e (in particular, they are the coordinates associated to the basis $\{T_j\}$ of $\mathfrak{g}$ by the exponential map). Then $\omega([\widetilde{f}]) = 0$, and so $\omega(f) = \sum_j (df)_e(T_j)\omega([t_j])$. Now it is enough to define $\omega^j = \omega([t_j])$ to get

$$\omega(f) = \sum_{j=1}^{n} f(T_j)\omega^j.$$

Finally, ω^j has to fulfil (i)-(iii) of proposition 8.1.8: from the proof of proposition 8.1.8, indeed, it is clear that these are not only sufficient but also necessary conditions. □

A nice corollary of the discussion above is the following property of a strong connection over a quantum principal G-bundle (which can be deduced from the proof of the previous proposition).

Corollary 8.1.10. *Let $\omega : H \to \Omega^1(\mathcal{A})$ be a strong connection for the quantum principal G-bundle $\mathcal{A}$. Then ω satisfies the following* Leibniz rule*:*

$$\omega(fg) = \varepsilon(g)\omega(f) + \varepsilon(f)\omega(g) = g(e)\omega(f) + f(e)\omega(g),$$

for any $f, g \in G$.

We conclude this section discussing some regularity properties of strong connections over quantum principal G-bundles. In the definition of a quantum principal G-bundles we required $\mathcal{A}$ to be a locally convex topological algebra. Then we can put the projective topology on the tensor product $\mathcal{A} \otimes \mathcal{A}$. Since the multiplication map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is continuous, its kernel $\Omega^1\mathcal{A} = \ker(m)$ is a locally convex vector space and then so is the quotient space $\Omega^1(\mathcal{A})$ (with the quotient topology).

Lemma 8.1.11. *Each T_j, seen as a linear map $\mathcal{A} \to \mathcal{A}$, is continuous.*

Proof. T_j is continuous as a map $H \to H$ (it is a vector field on G). Then, since its action on $\mathcal{A}$ is given by the composition (all the tensor product are endowed with the projective topology)

$$\mathcal{A} \xrightarrow{\Delta_R} \mathcal{A} \otimes H \xrightarrow{\mathrm{id}\otimes T_j} \mathcal{A} \otimes H \xrightarrow{\mathrm{id}\otimes\varepsilon} \mathcal{A},$$

and since the coaction Δ_R and the counit ε are continuous maps, it is continuous as a map $T_j : \mathcal{A} \to \mathcal{A}$. □

Lemma 8.1.12. *Each strong connection $\omega : H \to \Omega^1(\mathcal{A})$ on a quantum principal G-bundle $\mathcal{A}$ is a continuous map.*

Proof. It is a direct consequence of the continuity of $T_j : H \to H$. □

8.2 Spectral triples over cleft Hopf-Galois $C^\infty(G)$-extension

Consider a cleft Hopf-Galois $C^\infty(G)$-extension $\mathcal{B} \hookrightarrow \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are unital *-algebras, and assume that it admits a unitary trivialization ϕ. Then we know[5] that $\mathcal{A}$ is isomorphic to the crossed product $\mathcal{B}\#_\sigma C^\infty(G)$, where the cocyle $\sigma : C^\infty(G) \otimes C^\infty(G) \to \mathcal{B}$ is given by

$$\sigma(h,l) = \phi(h_{(1)})\phi(l_{(1)})\phi^{-1}(h_{(2)}l_{(2)})$$

and the weak action of $C^\infty(G)$ on $\mathcal{B}$ by

$$h \triangleright b = \phi(h_{(1)})b\phi^{-1}(h_{(2)}).$$

Consider now a real spectral triple $(\mathcal{B}, \mathcal{H}_\mathcal{B}, D, J, \gamma)$ (with $\gamma = \mathrm{id}$ in the odd dimensional case). Assume also that it is equivariant with respect to the weak action of $H = \mathbb{C}^\infty(G)$ on $\mathcal{B}$ (see definition 7.2.1). Our aim is, starting from this triple, to construct a spectral triple over the crossed product algebra $\mathcal{A} \simeq \mathcal{B}\#_\sigma H$, extending the results of the previous chapter. The first thing we need is a spectral triple over the Hopf algebra H.

8.2.1 A spectral triple over $C^\infty(G)$

Let G be a compact connected Lie group, with Lie algebra $\mathfrak{g}$. Assume[6] that G admits a bi-invariant metric which is G-spin; that is, the adjoint representation of G on $\mathfrak{g}$ lifts to $\mathrm{Spin}(\mathfrak{g})$. The tangent bundle TG is trivial; in particular, it is isomorphic to $G \times \mathfrak{g}$. Moreover, any linear map $\gamma : \mathfrak{g} \to \mathfrak{so}(\mathfrak{g})$ determines a left G-invariant metric connection on TG [Sle85, Sle87]. In particular, we can define a real family of connections by

$$\gamma_a(\xi) = a \cdot \mathrm{ad}_\xi, \qquad \xi \in \mathfrak{g},\ a \in \mathbb{R}.$$

In the same way, the spin bundle is isomorphic to $G \times S$, S being a suitable [Sle85] left $\mathfrak{g}$-module. Then each γ_a lifts to a left G-invariant metric connection on $G \times S$, determined by

$$\gamma_a^S : \mathfrak{g} \to \mathfrak{u}(S),$$

$$\gamma_a^S(\xi) = a \cdot d\rho(\xi),$$

where $\rho : G \to GL(S)$ is the representation of G on S determined by the lift of the adjoint action [Sle85, Sle87], and $\mathfrak{u}(S)$ is the Lie algebra of $U(S)$, the group of unitary transformations of S. The connection obtained in this way is flat if and only if $a = 0$ or $a = 1$. Instead, the case $a = \frac{1}{2}$ corresponds to the Levi-Civita connection.

Now, let $\{T_j\}$ be an orthonormal basis of $\mathfrak{g}$ (with respect to the hermitian scalar product which

[5] See proposition 4.4.13 and proposition 4.4.21.

[6] This is true, e.g., for the groups $SU(n)$, $SO(n)$, $Sp(n)$ [Adams].

induces the bi-invariant metric on G) and denote by $c : \mathfrak{g} \otimes S \to S$ the Clifford multiplication map. Moreover, let $r : G \otimes L^2(G, dg) \otimes S \to L^2(G, dg) \otimes S$ (where dg is the Haar measure on G) be the map $r_g(f)(h) = f(gh)$, for any $g, h \in G$. Then we can write the Dirac operator $\not{D}$ on $G \otimes S$ in the following way:

$$\not{D} = \sum_j c(T_j)\left(dr(T_j) + \gamma_a^S(T_j)\right). \tag{8.2.1}$$

We recall that, for any (differentiable) S-valued function f, we can write $dr(T_j)f(g)$ as:

$$dr(T_j)f(g) = \frac{d}{dt} f(ge^{tT_j})\bigg|_{t=0}.$$

If now we put on S the scalar product induced by the metric tensor associated to the bi-invariant metric of G, we can consider the Hilbert space $\mathcal{H}_G = L^2(G, S)$. Then $\not{D}$ extends to a selfadjoint operator on $\mathcal{H}_G$ and $(H = C^\infty(G), \mathcal{H}_G, \not{D})$ is a spectral triple. Moreover, since we assumed G to be a spin manifold, there exists an antiunitary operator J_G and, if G is even dimensional, a $\mathbb{Z}_2$-grading γ_G on $\mathcal{H}_G$ such that $(H, \mathcal{H}_G, \not{D}, J_G, \gamma_G)$ is a real spectral triple (if G is odd dimensional, of course, $\gamma_G = \mathrm{id}$). We shall not work out the explicit form of the real structure J_G; we simply notice that, given a vector $f \otimes s$ ($f \in C^\infty(G)$, $s \in S$) of $\mathcal{H}_G$, it must act in the following way:

$$J_G(f \otimes s) = f^* \otimes J_S s,$$

for some antiunitary operator J_S. Similarly, the grading γ_G acts only on S: $\gamma_G(f \otimes s) = f \otimes \gamma_S s$.

Consider now the (dense) subspace $\mathcal{H}^\infty = H \otimes S$ of $\mathcal{H}_G$. Since $H = C^\infty(G)$ is a Hopf algebra, we can define in a very natural way both a left and a right coaction of H on $\mathcal{H}^\infty$. Let us consider the former, $\rho_L : \mathcal{H}^\infty \to H \otimes \mathcal{H}^\infty$, $\rho_L(f \otimes s) = f_{(1)} \otimes (f_{(2)} \otimes s)$. Then we can prove the following result.

Proposition 8.2.1. *The real spectral triple $(H, \mathcal{H}_G, \not{D}, J_G, \gamma_G)$ is equivariant with respect to the left H-comodule structure of H (and $\mathcal{H}_G$).*

Proof. $\mathcal{H}^\infty$ is clearly a left H-equivariant $\mathcal{A}$-module, stable under the action of $\not{D}$. We check (i)-(iii) of definition 3.5.6. The fact that (ii) and (iii) are satisfied follows from the discussion above. In the same way, we see that the second part of $\not{D}$ is equivariant. So it is enough to prove the proposition in the $a = 0$ case. But in this case the equivariance follows from the fact that the left coaction of H on itself associated to the coproduct is the coaction associated to the left regular representation of G on $C^\infty(G)$, and the Dirac operator is clearly invariant with respect to it, since it is defined in terms of the right regular representation. □

We conclude this section noticing that, of course, the Dirac operator $\not{D}$ induces the (bi-covariant) de Rham differential calculus on G (even if it is only left equivariant and not right equivariant).

8.2.2 Spectral triples over $\mathcal{B}\#_\sigma C^\infty(G)$

Now we are ready to construct a spectral triple for the crossed product algebra $\mathcal{A} = \mathcal{B}\#_\sigma H$, where we have set $H = C^\infty(G)$. First of all, let us consider the Hilbert space $\hat{\mathcal{H}} = \mathcal{H}_\mathcal{B} \otimes \mathcal{H}_G$ and let us define a representation of $\mathcal{A}$ on $\hat{\mathcal{H}}$ in the following way:

$$\hat{\pi}(a\#h)(v \otimes w) = a\pi^\circ(\sigma(h_{(2)}, w_{(-1)}))(h_{(1)} \triangleright v) \otimes h_{(3)}w_{(0)} \tag{8.2.2}$$

for any $a \in \mathcal{B}$, $h \in H$, $v \in \mathcal{H}_\mathcal{B}$, $w \in V$, where, we recall, V is the dense subspace of $\mathcal{H}_G$ introduced in the previous section.

Proposition 8.2.2. *The map $\hat{\pi}$ is a *-homomorphism from $\mathcal{A}$ to the algebra of bounded operators on $\hat{\mathcal{H}}$. Moroever, assume that the cocycle σ satisfies the following condition of boundedness: there exists $C \in \mathbb{R}^+$ such that*[7]

$$\|\sigma(h,l)\|_\mathcal{B} \leq C\|h\|_2\|l\|_2$$

for any $h, l \in H$. Then $\hat{\pi}$ gives a representation of $\mathcal{A}$ on $\hat{\mathcal{H}}$ by bounded operators.

Proof. The fact that $\hat{\pi}$ is an algebra map follows as in proposition 7.3.1. We prove that it respects the *-structure. That is, we have to show that, for any $x, v \in \mathcal{H}_\mathcal{B}$, and any $y, w \in \mathcal{H}_G$

$$\langle x \otimes y, \hat{\pi}(a\#h)v \otimes w\rangle = \langle \hat{\pi}((a\#h)^*)x \otimes y, v \otimes w\rangle \,.$$

Let us compute the two sides separately. The first computation is quite trivial:

$$\langle x \otimes y, \hat{\pi}(a\#h)v \otimes w\rangle = \left\langle x, a\pi^\circ(\sigma(h_{(2)}, w_{(-1)}))(h_{(1)} \triangleright v)\right\rangle_{\mathcal{H}_\mathcal{B}} \left\langle y, h_{(3)}w_{(0)}\right\rangle_{\mathcal{H}_G} . \tag{8.2.3}$$

Before computing the r.h.s. we need a lemma.

Lemma 8.2.3. *For any $y, w \in V \subseteq \mathcal{H}_G$ we have:*

$$y^*_{(-1)} \left\langle y_{(0)}, w\right\rangle_{\mathcal{H}_G} = (Sw_{(-1)}) \left\langle y, w_{(0)}\right\rangle_{\mathcal{H}_G} .$$

Proof. $y^*_{(-1)} \left\langle y_{(0)}, w\right\rangle_{\mathcal{H}_G}$ is a smooth section of the spinor bundle of G, hence it can be seen as a function from G to S. Moreover, we recall that the scalar product on $\mathcal{H}_G$ is obtained from the L^2 scalar product on $C^\infty(G)$ associated to the Haar measure together with a suitable inner product $\langle\cdot,\cdot\rangle_S$ on S. Hence if we evaluate $y^*_{(-1)} \left\langle y_{(0)}, w\right\rangle_{\mathcal{H}_G}$ at a generic point $h \in G$ we obtain:

$$\begin{aligned} y^*_{(-1)} \left\langle y_{(0)}, w\right\rangle_{\mathcal{H}_G}(h) &= \int_G \langle y^*(hg), w(g)\rangle_S \, dg = \int \left\langle y^*(g), w(h^{-1}g)\right\rangle_S dg \\ &= w_{(-1)} \left\langle y, w_{(0)}\right\rangle_{\mathcal{H}_G}(h^{-1}) = (Sw_{(-1)}) \left\langle y, w_{(0)}\right\rangle_{\mathcal{H}_G}(h). \end{aligned}$$

□

[7] $\mathcal{B}$ is a dense subset, by hypothesis, of a C^*-algebra $\mathcal{B}$, and $\|\cdot\|_\mathcal{B}$ denotes the induced C^*-norm. $\|\cdot\|_2$, instead, denotes the L^2-norm on $H = C^\infty(G) \subseteq L^2(G, dg)$.

Now we come back to the proof of the proposition. Using the lemma above and the properties of equivariance of the spectral triple over $\mathcal{B}$ (see definition 7.2.1) we get:

$$\begin{aligned}
&\langle \hat{\pi}((a\#h)^*)x\otimes y, v\otimes w\rangle = \left\langle \hat{\pi}(\sigma(S^{-1}h_{(2)}, h_{(1)})^*(h^*_{(3)}\triangleright a^*)\#h^*_{(4)})x\otimes y, v\otimes w\right\rangle \\
&= \left\langle \sigma(S^{-1}h_{(2)}, h_{(1)})^*(h^*_{(3)}\triangleright a^*)\pi^\circ(\sigma(h^*_{(5)}, y_{(-1)}))(h^*_{(4)}\triangleright x)\otimes h^*_{(6)}y_{(0)}, v\otimes w\right\rangle \\
&= \left\langle \sigma(S^{-1}h_{(2)}, h_{(1)})^*(h^*_{(3)}\triangleright a^*)\pi^\circ(\sigma(h^*_{(5)}, (S^{-1}h^*_{(6)})(S^{-1}w^*_{(-1)})))(h^*_{(4)}\triangleright x)\otimes y, v\otimes h_{(7)}w_{(0)}\right\rangle \\
&= \left\langle (h^*_{(4)}\triangleright x)\otimes y, \pi^\circ(\sigma(h^*_{(5)}, (S^{-1}h^*_{(6)})(S^{-1}w^*_{(-1)}))^*)(h^*_{(3)}\triangleright a^*)^*\sigma(S^{-1}h_{(2)}, h_{(1)})v\otimes h_{(7)}w_{(0)}\right\rangle \\
&= \left\langle (h^*_{(4)}\triangleright x)\otimes y, \pi^\circ(\sigma(h^*_{(5)}, (S^{-1}h^*_{(6)})(S^{-1}w^*_{(-1)}))^*)(S^{-1}h_{(3)}\triangleright a)\sigma(S^{-1}h_{(2)}, h_{(1)})v\otimes h_{(7)}w_{(0)}\right\rangle \\
&= \Big\langle x\otimes y, \pi^\circ(h_{(7)}\triangleright\sigma(h^*_{(8)}, (S^{-1}h^*_{(9)})(S^{-1}w^*_{(-1)}))^*) \\
&\qquad \cdot (h_{(4)}\triangleright(S^{-1}h_{(3)}\triangleright a))(h_{(5)}\triangleright\sigma(S^{-1}h_{(2)}, h_{(1)}))(h_{(6)}\triangleright v)\otimes h_{(10)}w_{(0)}\Big\rangle.
\end{aligned} \tag{8.2.4}$$

Using again the properties of equivariance (definition 7.2.1) we see that we can rewrite (8.2.4) in the following way:

$$\begin{aligned}
&\langle \hat{\pi}((a\#h)^*)x\otimes y, v\otimes w\rangle = \\
&\quad = \Big\langle x\otimes y, \pi^\circ(\sigma(S^{-1}h^*_{(5)}, h^*_{(6)})^*)\pi^\circ(\phi(h_{(2)})\phi(w_{(-2)})\phi^{-1}(h_{(7)}w_{(-1)})) \\
&\qquad \cdot\sigma(h_{(4)}, S^{-1}h_{(5)})a(h_{(1)}\triangleright v)\otimes h_{(8)}w_{(0)}\Big\rangle.
\end{aligned} \tag{8.2.5}$$

But now from condition (vii) of definition 7.2.1 we see that (8.2.5) reduces to

$$\left\langle x\otimes y, \pi^\circ(\sigma(h_{(2)}, w_{(-1)}))a(h_{(1)}\triangleright v)\otimes w_{(0)}\right\rangle$$

which is the same as equation (8.2.3). Hence $\hat{\pi}$ is a *-homomorphism. Finally, we show that each $\hat{\pi}(a\#h)$ is a bounded operator. We have already pointed out that $\mathcal{H}_H$ is isomorphic to $L^2(G)\otimes S$. From Peter-Weyl theory [Bump, Waw] we know that the matrix elements of unitary irreducible representations of G are dense in $L^2(G)$; moreover, the functions

$$\sqrt{\dim(u)}u_{ij},$$

with u a unitary irreducible representation and u_{ij} its matrix elements with respect to some orthonormal basis, form an orthonormal basis of $L^2(G)$. Since $\Delta_(u_{ij}) = \sum_k u_{ik}\otimes u_{kj}$, we have:

$$\begin{aligned}
\|\Delta(u_{ij})\|_2^2 = \|\sum_k u_{ik}\otimes u_{kj}\|_2^2 &= \sum_{k,l}\int_{G\otimes G} u^*_{ik}(g)u^*_{kj}(g')u_{il}(g)u_{lj}(g')dgdg' \\
&= \int_G u^*_{ij}(g)u_{ij}(g)dg = \|u_{ij}\|_2^2.
\end{aligned}$$

It follows that Δ is bounded with respect to the L^2-norm (both on H and on $H\otimes H$). This implies then that also Δ_L is bounded. Now, by hypothesis, for any $h \in H$, $\sigma(h,\cdot) : H \to \mathcal{B}$ is bounded; it follows that $\sigma(h,\cdot) \otimes \mathrm{id} : H \otimes H \to \mathcal{B} \otimes H$ is bounded, with the same norm as $\sigma(h,\cdot)$ [Ryan]. So $(\sigma(h,\cdot) \otimes \mathrm{id}) \circ \Delta_L$ is bounded. From this, and from the fact that each $h \in H$ acts on $\mathcal{H}_{\mathcal{B}}$ as a bounded operator (see definition 7.2.1), it follows that each $\hat{\pi}(a\#h)$ is a bounded operator. □

From now on we shall always assume the cocycle σ to be a bounded map, in the sense of proposition 8.2.2. The next step is the construction of a Dirac operator. Since the definition depends on the KR-dimension of the triples involved [DD11], we consider separately the different cases. In the following we shall denote by j the KR-dimension of the triple over $\mathcal{B}$ and by n the dimension, as smooth manifold, of the Lie group G.

Even-even case

Consider first the case when both j and n are even. This means, in particular, that both the two $\mathbb{Z}_2$-gradings γ and γ_G are non-trivial. According to [DD11], there are two possible choices for the Dirac operator on $\hat{\mathcal{H}}$:

$$\begin{aligned} \hat{D} &= D \otimes \mathrm{id}_{\mathcal{H}_G} + \gamma \otimes \not{D}, \\ \hat{D}' &= D \otimes \gamma_G + \mathrm{id}_{\mathcal{H}_{\mathcal{B}}} \otimes \not{D}. \end{aligned} \tag{8.2.6}$$

We shall adopt the first choice, but all the results presented in this chapter can be proved also under the second one. Indeed, we recall, the two choices are unitarily equivalent, the unitary transformation being given by

$$U = \frac{1}{2}(\mathrm{id} \otimes \mathrm{id} + \gamma \otimes \mathrm{id} + \mathrm{id} \otimes \gamma_G - \gamma \otimes \gamma_G).$$

Proposition 8.2.4. *The triple $(\mathcal{A}, \hat{\mathcal{H}}, \hat{D})$ is a spectral triple. That is, the Dirac operator $\hat{D}$ is selfadjoint, it has compact resolvent and all the commutators $[\hat{\pi}(a), \hat{D}]$, $a \in \mathcal{A}$, are bounded.*

Proof. The first part of the proof follows from the results in [DD11]. We have only to prove that $\hat{D}$ has bounded commutators with all the elements of $\mathcal{A}$. Consider therefore a commutator $[\hat{D}, \hat{\pi}(a\#h)]$. It can be written as the sum of two terms. The first one is $[D \otimes \mathrm{id}\hat{\pi}(a\#1)]\hat{\pi}(1\#h)$ which is equal to $([D,a] \otimes \mathrm{id})\hat{\pi}(1\#h)$, and so is bounded since it is equivalent to the product of two bounded operators. The action of the second term on a vector $v \otimes w$, instead, is given by:

$$\gamma a \pi^{\circ}(\sigma(h_{(2)}, w_{(-1)}))(h_{(1)} \triangleright v) \otimes [\not{D}, h_{(3)}] w_{(0)}.$$

This shows that also the second term is a bounded operator. Indeed, the commutator of $\not{D}$ with any element of H is a bounded operator and so the boundedness of $[\hat{D}, \hat{\pi}(a\#h)]$ follows by arguments similar to those used in the proof of proposition 8.2.2. □

Finally, since the triple so obtained should be even, we must define a $\mathbb{Z}_2$-grading. We take it to be:

$$\hat{\gamma} = \gamma \otimes \gamma_G.$$

Even-odd case

Suppose now, instead, that j is even while n is odd. Now we have only one non-trivial $\mathbb{Z}_2$-grading, so the unique choice for $\hat{D}$ is:

$$\hat{D} = D \otimes \mathrm{id}_{\mathcal{H}_G} + \gamma \otimes \not{D} \tag{8.2.7}$$

Obviously, the results of proposition 8.2.4 apply also to this case.

Odd-even case

It is the same situation as in the even odd case: the Dirac operator is taken to be

$$\hat{D} = D \otimes \gamma_G + \mathrm{id}_{\mathcal{H}_\mathcal{B}} \otimes \not{D} \tag{8.2.8}$$

and the results of proposition 8.2.4 still hold.

Odd-odd case

When both j and n are odd we have to enlarge the Hilbert space: indeed, we take $\hat{\mathcal{H}} = \mathcal{H}_\mathcal{B} \otimes \mathcal{H}_G \otimes \mathbb{C}^2$. The representation $\hat{\pi}$ will be, of course, defined in the same way as above, and $\mathcal{A}$ will act trivially on the $\mathbb{C}^2$ factor. Finally, the Dirac operator $\hat{D}$ will be taken equal to:

$$\hat{D} = D \otimes \mathrm{id} \otimes \sigma^1 + \mathrm{id} \otimes \not{D} \otimes \sigma^2. \tag{8.2.9}$$

It is straightforward to check that the proof of proposition 8.2.4 can be easily adapted to this case, so we obtain a spectral triple also in this case. Moreover, it should be an even triple and, indeed, we can define a $\mathbb{Z}_2$-grading in the following way:

$$\hat{\gamma} = \mathrm{id} \otimes \mathrm{id} \otimes \sigma^3.$$

8.2.3 Real structure and real spectral triples

The construction of a real structure for the triples discussed in the previous sections deserves a distinguished paragraph. We recall that in the case of quantum $\mathbb{T}^n$ bundles we were able to find a general formula for the real structure. In the more general case of quantum principal G-bundles, we are able to construct a real structure only when the crossed product algebra A is actually a smash product; that is, the trivialization ϕ is an algebra map, so that the cocycle σ is trivial. We present it only in the even-even case, but the extension to the other three situations is straightforward.

Consider therefore a cleft extension $\mathcal{B} \hookrightarrow \mathcal{A}$ with unitary trivialization ϕ and assume the latter to be an algebra map. Consider also, as like as above, an H-equivariant even real spectral triple $(\mathcal{B}, \mathcal{H}_\mathcal{B}, D, J, \gamma)$ of KR-dimension j. We define the following map on $\hat{\mathcal{H}}$:

$$\hat{J}(v \otimes w) = (w^*_{(-1)} \triangleright Jv) \otimes J_H w_{(0)}. \tag{8.2.10}$$

Proposition 8.2.5. *$(\mathcal{A}, \hat{\mathcal{H}}, \hat{D}, \hat{J}, \hat{\gamma})$ is an even real spectral triple of KR-dimension $j+n$, where*[8] *n is the dimension of the Lie group G.*

Proof. The proof is a straightforward generalization of the proof of lemma 7.3.3, obtained using lemma 8.2.3. □

Even if we have still not been able to work out a real structure for the general case, we have proved a partial result, which, after all, is what we need in the rest of this thesis: it is possible to define, without introducing a real structure, a right action of $\mathcal{A}$ on $\hat{\mathcal{H}}$ (or, equivalently, a representation of the opposite algebra $\mathcal{A}^\circ$) commuting with $\hat{\pi}(\mathcal{A})$. Again, we discuss here only the even-even case, the extension to the other situations being straightforward. For $v \otimes w \in \hat{\mathcal{H}}$ and $a\#h \in \mathcal{A} \simeq \mathcal{B}\#_\sigma H$ consider the following map:

$$\hat{\pi}^\circ(a\#h)v \otimes w = \pi^\circ((w_{(-2)} \triangleright a)\sigma(w_{(-1)}, h_{(1)}))v \otimes w_{(0)}h_{(2)}, \tag{8.2.11}$$

where the right action of H on $\mathcal{H}_H$ is the same as the left one, since H is commutative and $\mathcal{H}_H$ is a space of L^2-sections.

Proposition 8.2.6. *$\hat{\pi}^\circ : \mathcal{A}^\circ \to \mathcal{L}(\mathcal{H})$ is a *-algebra map. Moreover, $[\hat{\pi}^\circ(\mathcal{A}^\circ), \hat{\pi}(\mathcal{A})] = 0$.*

Proof. First of all we prove that $\hat{\pi}^\circ$ is an algebra map. That is, we show that:

$$\hat{\pi}^\circ((a\#h)(b\#l))v \otimes w = \hat{\pi}^\circ(b\#l)\hat{\pi}^\circ(a\#h)v \otimes w. \tag{8.2.12}$$

The r.h.s. of equation (8.2.12) is given by:

$$\begin{aligned}&\hat{\pi}^\circ(b\#l)\hat{\pi}^\circ(a\#h)v \otimes w \\ &\qquad = \pi^\circ((w_{(-4)} \triangleright a)\sigma(w_{(-3)}, h_{(1)})(w_{(-2)}h_{(2)} \triangleright b)\sigma(w_{(-1)}h_{(3)}, l_{(1)}))v \otimes w_{(0)}h_{(4)}l_{(2)}.\end{aligned} \tag{8.2.13}$$

Let us compute the l.h.s. Using the cocycle condition (4.4.4) and the twisted module condition (4.4.5), we obtain:

$$\begin{aligned}\hat{\pi}^\circ((a\#h)(b\#l))v \otimes w &= \hat{\pi}^\circ(a(h_{(1)} \triangleright b)\sigma(h_{(2)}, l_{(1)})\#h_{(3)}l_{(2)})v \otimes w \\ &= \pi^\circ((w_{(-4)} \triangleright a)(w_{(-3)} \triangleright h_{(1)} \triangleright b)(w_{(-2)} \triangleright \sigma(h_{(2)}, l_{(1)}))\sigma(w_{(-1)}, h_{(3)}l_{(2)}))v \otimes w_{(0)}h_{(4)}l_{(3)} \\ &= \pi^\circ((w_{(-4)} \triangleright a)(w_{(-3)} \triangleright h_{(1)} \triangleright b)\sigma(w_{(-2)}, h_{(1)})\sigma(w_{(-1)}h_{(3)}, l_{(1)}))v \otimes w_{(0)}h_{(4)}l_{(2)} \\ &= \pi^\circ((w_{(-4)} \triangleright a)\sigma(w_{(-3)}, h_{(1)})(w_{(-2)}h_{(2)} \triangleright b)\sigma(w_{(-1)}h_{(3)}, l_{(1)}))v \otimes w_{(0)}h_{(4)}l_{(2)}\end{aligned} \tag{8.2.14}$$

which is equal to (8.2.13). Hence we have proved (8.2.12). Next we have to show that it is a *-algebra map. In particular, it is enough to show that the following holds:

$$\langle x \otimes y, \hat{\pi}^\circ(a\#h)v \otimes w\rangle = \langle \hat{\pi}^\circ((a\#h)^*)x \otimes y, v \otimes w\rangle. \tag{8.2.15}$$

[8] We recall that we are considering the case when n is even.

Let us compute the r.h.s. of the equation above. Again using the cocycle condition, the twisted module condition and the fact that ϕ is a unitary trivialization, we can show it to be equal to:

$$\begin{aligned}
&\langle \hat{\pi}^\circ((a\#h)^*)x\otimes y, v\otimes w\rangle \\
&\quad = \left\langle \pi^\circ(\sigma(S^{-1}(h_{(2)}y^*_{(-2)}), h_{(1)})^*(y_{(-1)}h^*_{(3)} \triangleright a^*)^* x \otimes y_{(0)}h^*_{(4)}, v\otimes w\right\rangle \\
&\quad = \left\langle x\otimes y_{(0)}h^*_{(4)}, \pi^\circ((S^{-1}(h_{(3)}y^*_{(-1)})\triangleright a)\sigma(S^{-1}(h_{(2)}y^*_{(-2)}), h_{(1)}))v\otimes w\right\rangle .
\end{aligned} \tag{8.2.16}$$

Now, using the result of lemma 8.2.3, we can rewrite (8.2.17) in the following way.

$$\begin{aligned}
&\langle \hat{\pi}^\circ((a\#h)^*)x\otimes y, v\otimes w\rangle \\
&\quad = \left\langle x\otimes y, \pi^\circ((S^{-1}(Sw_{(-2)})\triangleright a)\sigma(S^{-1}(Sw_{(1)}), h_{(1)}))v\otimes w_{(0)}h_{(2)}\right\rangle \\
&\quad = \left\langle x\otimes y, \pi^\circ((w_{(-2)}\triangleright a)\sigma(w_{(-1)}, h_{(1)}))v\otimes w_{(0)}h_{(2)}\right\rangle = \langle x\otimes y, \hat{\pi}(a\#h)v\otimes w\rangle .
\end{aligned} \tag{8.2.17}$$

Finally, we show that $\hat{\pi}$ maps $\mathcal{A}$ into its commutant over $\hat{\mathcal{H}}$. We have:

$$\begin{aligned}
&\hat{\pi^\circ}(a\#h)\hat{\pi}(b\#l)v\otimes w \\
&\quad = \pi^\circ((l_{(3)}w_{(-2)}\triangleright a)\sigma(l_{(4)}w_{(-1)}, h_{(1)}))b\pi^\circ(\sigma(l_{(2)}, w_{(-3)}))(l_{(1)}\triangleright v)\otimes l_{(5)}w_{(0)}h_{(2)};
\end{aligned} \tag{8.2.18}$$

$$\begin{aligned}
&\hat{\pi}(b\#l)\hat{\pi^\circ}(a\#h)v\otimes w \\
&\quad = b\pi^\circ(\sigma(l_{(2)}, w_{(-3)})(l_{(3)}w_{(-2)}\triangleright a)\sigma(l_{(4)}w_{(-1)}, h_{(1)}))(l_{(1)}\triangleright v)\otimes l_{(5)}w_{(0)}h_{(2)}
\end{aligned} \tag{8.2.19}$$

(in the computation of the second equation we used the twisted module condition and the cocycle condition). Now, since π° maps $\mathcal{B}$ into its commutant, equation (8.2.19) is equal to (8.2.18), and so $\hat{\pi}$ maps $\mathcal{A}^\circ$ (or, equivalently, $\mathcal{A}$) into the commutant of $\hat{\pi}(\mathcal{A})$. □

8.2.4 Quantum principal G-bundles structure

In the case of T^n-bundles we have seen that our construction of a spectral triple over a cleft Hopf-Galois extension $\mathcal{B} \hookrightarrow \mathcal{A}$ induces a structure of quantum principal bundle with respect to the de Rham calculus, the calculus over $\mathcal{A}$ being the one determined by the new Dirac operator. Now we want to see if an analogous result hold for cleft $C^\infty(G)$-extensions. We shall discuss only the even-even case (i.e., we assume the triple over $\mathcal{B}$ to be even, and the dimension n of the Lie group G to be even, too), the extension to the other situations being straightforward.

The Dirac operator $\hat{D}$ determines a first order differential calculus $\Omega^1_{\hat{D}}(\mathcal{A}) = \Omega^1\mathcal{A}/N_{\hat{D}}$. Since $\hat{D}$ acts on $\mathcal{H}_H$ as the Dirac operator $\not{D}$, and since the de Rham calculus on G is bi-covariant, if the crossed product structure of $\mathcal{A}$ is trivial this calculus will be right $C^\infty(G)$-equivariant. In the general case is not so straightforward to see that this still holds; it is instead not so difficult to obtain a weaker result: from the results in [BM98a] it follows that we can construct a right equivariant calculus $\Omega^1(\mathcal{A}) = \Omega^1\mathcal{A}/N$ still compatible with the Dirac operator $\hat{D}$ (that is, $N \subseteq N_{\hat{D}}$) and, moreover, with the de Rham calculus on G: indeed, the Dirac operator $\hat{D}$ can be seen[9] to be the operator associated to the Maurer-Cartan connection $\theta = \phi^{-1} * d\phi$, and so

[9] We shall discuss this aspects later in this chapter.

applying the construction in [BM98a] with θ as strong connection we obtain a calculus compatible both with $\hat{D}$ and with the de Rham calculus on G.

However, for simplicity in the rest of this chapter we shall work under the assumption that the calculus $\Omega^1_{\hat{D}}(\mathcal{A})$ is right $C^\infty(G)$-equivariant.

Lemma 8.2.7. *If the representation $\hat{\pi}$ is faithful, the differential calculus $\Omega^1_{\hat{D}}(\mathcal{A})$ enjoys properties (i) and (ii) of proposition 8.1.5.*

Proof. Let us begin by showing that (i) is satisfied. Take $\eta = \sum(a\#h)d(b\#l)$ such that $\pi_{\hat{D}}(\eta) = 0$. This means, in particular, that

$$\sum \hat{\pi}(a\#h)\hat{\pi}(b\#1)[\gamma \otimes \not{D}, \hat{\pi}(1\#l)] = 0.$$

By direct computation then we obtain:

$$\sum\sum_j \hat{\pi}(a\#h)\hat{\pi}(b\#1)\left(\pi^\circ(\sigma(h_{(2)}, w_{(-1)}))(l_{(1)} \triangleright \gamma v) \otimes c(T_j)l_{(3)}l_{(4)}(T_j)\right) = 0. \tag{8.2.20}$$

Let us consider now the elements $\alpha_j = \sum(a\#h)T_j(b\#l)$. Each of them is zero if and only if the following expression is zero:

$$\sum\sum_j c(T_j)\hat{\pi}((a\#h)T_j(b\#l)); \tag{8.2.21}$$

this follows from the linear independence of the $c(T_j)$ and from the fact that $\hat{\pi}$ is faithful. But now we see that (8.2.21) is equal to the left term of equation (8.2.20), and so it is equal to zero.

Next we have to show that also (ii) is satisfied. So, take $\eta \in \Omega^1\mathcal{A}$, $\eta = \sum(a\#h)d(b\#l)$, such that

$$\sum(a\#h)T_j(b\#l) = 0 \tag{8.2.22}$$

for any $j = 1, \ldots, \dim G$. We have to show that $[\eta]$ belongs to $\mathcal{A}\Omega^1(\mathcal{B})\mathcal{A}$. But this follows from the fact that (8.2.22) implies that

$$\pi_{\mathrm{id}\otimes \not{D}}(\eta) = 0.$$

□

Due to proposition 8.1.5, then, we get the following.

Theorem 8.2.8. *Let N be the sub-bimodule of $\mathcal{A} \otimes \mathcal{A}$ defining the differential calculus $\Omega^1_{\hat{D}}(\mathcal{A})$ and let $Q = (\ker\varepsilon)^2$. Assume that the representation $\hat{\pi}$ is faithful. Then $(\mathcal{A}, H, \Delta_R, N, Q)$ is a quantum principal G-bundle.*

8.3 Strong connections and twisted Dirac operators

In the previous section we have shown that, under suitable conditions, the differential calculus associated to the Dirac operator $\hat{D}$ is compatible with the de Rham calculus on $C^\infty(G)$, so that the cleft Hopf-Galois extension $\mathcal{B} \hookrightarrow \mathcal{A}$ admits a structure of quantum principal G-bundle. In particular we can consider strong G-connections on it, and we can use them to build twisted Dirac operators.

We have proved in proposition 8.2.6 that it is possible to define a structure of $\mathcal{A}$-bimodule (and, hence, in particular, of right $\mathcal{A}$-module) on $\hat{\mathcal{H}}$. Going a little further, we can see that $\hat{\mathcal{H}}$ can be identified with (the closure of) $\mathcal{H}_0\mathcal{A}$, where $\mathcal{H}_0 = \mathcal{H}_\mathcal{B} \otimes S$, the right action of $\mathcal{A}$ being the one defined by $\hat{\pi}^\circ$. So, according with the discussion in section 5.1, we can begin by defining a D-connection on $\mathcal{A}$. Here we denote by D the Dirac operator $D \otimes \mathrm{id}_S$ on $\mathcal{H}_0$; we shall do the same for the real structure: J will actually denote the operator $J \otimes c.c.$.

So, let us be given a strong connection $\omega : H \to \Omega^1_{\hat{D}}(\mathcal{A})$, and consider the following map (the representation $\hat{\pi}$ here is understood):

$$\begin{gathered}\nabla^\omega : \mathcal{A} \to \Omega^1_{\hat{D}}(\mathcal{A})\mathcal{A}, \\ \nabla^\omega(a) = [\hat{D}, a] - a_{(0)}\omega(a_{(1)}),\end{gathered} \tag{8.3.1}$$

where we see $\Omega^1_{\hat{D}}(\mathcal{A})\mathcal{A}$ as a space of operators on $\mathcal{H}$. Since ω is strong, ∇^ω takes values, actually, in $\Omega^1_D(\mathcal{B})\mathcal{A}$. Moreover,

Proposition 8.3.1. *$\nabla^\omega : \mathcal{A} \to \Omega^1_D(\mathcal{B})\mathcal{A}$ is a D-connection.*

Proof. We have to show[10] that, for any $b \in \mathcal{B}$ and any $a \in \mathcal{A}$,

$$\nabla^\omega(ba) = [D, b]a + b\nabla^\omega a. \tag{8.3.2}$$

But this follows by direct computation. Indeed, the l.h.s. of (8.3.2) is equal to:

$$\begin{aligned}\nabla(ba) &= [\hat{D}, ba] - (ba)_{(0)}\omega((ba)_{(1)}) = [\hat{D}, b]a + b[\hat{D}, a] - ba_{(0)}\omega(a_{(1)}) \\ &= [D, b]a + b\nabla^\omega a,\end{aligned}$$

where we used the fact that $\mathcal{B}$ is the invariant subalgebra of $\mathcal{A}$. □

Now we can use ∇^ω to build a twisted Dirac operator D_ω. According with the discussion in section 5.1, we set

$$D_\omega(\xi a) = (D\xi)a + \xi\nabla^\omega a, \tag{8.3.3}$$

for any $\xi \in \mathcal{H}_0$, $a \in \mathcal{A}$. The right action of $\Omega^1_D(\mathcal{B})$ on $\mathcal{H}_0$ is the one defined by the real structure J. Although the D-connection ∇^ω will in general not be hermitian, in the sense of definition 5.1.2, we can easily find sufficient conditions on a strong connections ω for which D_ω is a selfadjoint operator.

Proposition 8.3.2. *Let ω be a strong connection, defined*[11] *by a family $\{\omega^j\} \subset \Omega^1_{\hat{D}}(\mathcal{A})$ of 1-forms. Then if each ω^j is selfadjoint, as an operator on $\hat{\mathcal{H}}$, D_ω is a selfadjoint operator. Moreover, it has has compact resolvent and bounded commutators with the elements $\mathcal{B}$.*

[10] See definition 5.1.1.
[11] See definition 8.1.7 and proposition 8.1.8.

Proof. The second part of the proposition is a direct consequence of proposition 4.7 in [DS13a]. So we have only to prove that D_ω is symmetric. First of all consider two vectors $\xi, \eta \in \mathcal{H}_0$ and two elements $b\phi(h), b'\phi(l) \in \mathcal{A}$, with $b, b' \in \mathcal{B}$; then the scalar product $\langle \eta b'\phi(l), \xi b\phi(h)\rangle$, which is nothing else than the scalar product in $\hat{\mathcal{H}}$, can be written as follows:

$$\left\langle \eta b'\phi(l), \xi b\phi(h)\right\rangle = \int_G \left\langle \eta b', \xi b\right\rangle_{\mathcal{H}_0} \overline{l(g)} h(g) dg.$$

Hence we have (here we use the fact that each ω^j is selfadjoint):

$$\begin{aligned}
&\left\langle \eta b'\phi(l), D_\omega(\xi b\phi(h))\right\rangle = \\
&= \left\langle \eta b'\phi(l), D(\xi b)\phi(h) + \xi b\left[\sum_j \phi(h_{(1)})h_{(2)}(T_j)(c(T_j) - \omega^j)\right]\right\rangle \\
&= \left\langle D(\eta b')\phi(l), \xi b\phi(h)\right\rangle + \sum_j \left\langle \eta b'\phi(l)c(T_j)^*, \xi b\phi(h_{(1)})h_{(2)}(T_j)\right\rangle \\
&\quad - \sum_j \left\langle \eta b'\phi(l)\omega^j, \xi b\phi(h_{(1)})h_{(2)}(T_j)\right\rangle .
\end{aligned} \tag{8.3.4}$$

Now let us consider the second term of the last expression in (8.3.4). For what said above, we have:

$$\begin{aligned}
\left\langle \eta b'\phi(l)c(T_j)^*, \xi b\phi(h_{(1)})h_{(2)}(T_j)\right\rangle &= \int_G \left\langle \eta b'c(T_j)^*, \xi b\right\rangle_{\mathcal{H}_0} \overline{l(g)} h_{(1)}(g)h_{(2)}(T_j) dg \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_G \left\langle \eta b'c(T_j)^*, \xi b\right\rangle_{\mathcal{H}_0} \overline{l(g)} h(ge^{tT_j}) dg \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_G \left\langle \eta b'c(T_j)^*, \xi b\right\rangle_{\mathcal{H}_0} \overline{l(ge^{-tT_j})} h(g) dg \\
&= \left\langle \eta b'\phi(l_{(1)})l_{(2)}(-T_j)c(T_j)^*, \xi b\phi(h)\right\rangle \\
&= \left\langle \eta b'\phi(l_{(1)})l_{(2)}(T_j)c(T_j), \xi b\phi(h)\right\rangle
\end{aligned} \tag{8.3.5}$$

where the last equality follows from the fact that $c(T_j)^* = -c(T_j)$ (which comes from the selfadjointness of $\not{D}$). Now let us consider the third term of the last expression in (8.3.4). With a computation similar to the one above we obtain the following:

$$\begin{aligned}
\left\langle \eta b'\phi(l)\omega^j, \xi b\phi(h_{(1)})h_{(2)}(T_j)\right\rangle &= \left\langle \eta b'\phi(l_{(1)})\omega^j_{(0)} l_{(2)}(-T_j)\omega^j_{(1)}(-T_j), \xi b\phi(h)\right\rangle \\
&= \left\langle \eta b'\phi(l_{(1)})\omega^j_{(0)} l_{(2)}(T_j)\omega^j_{(1)}(T_j), \xi b\phi(h)\right\rangle .
\end{aligned} \tag{8.3.6}$$

Due to the covariance properties of the strong G-connection ω (see (i) of definition 8.1.7), we obtain from (8.3.6) the following relation.

$$\begin{aligned}
\left\langle \eta b'\phi(l)\omega^j, \xi b\phi(h_{(1)})h_{(2)}(T_j)\right\rangle &= \sum_k \left\langle \eta b'\phi(l_{(1)})l_{(2)}(T_j)R^j_k(T_j)\omega^k, \xi b\phi(h)\right\rangle \\
&= \sum_k \left\langle \eta b'\phi(l_{(1)})l_{(2)}(T_j)\delta^j_k\omega^k, \xi b\phi(h)\right\rangle
\end{aligned}$$

$$= \sum_k \left\langle \eta b'\phi(l_{(1)})l_{(2)}(T_j)\omega^j, \xi b\phi(h)\right\rangle . \qquad (8.3.7)$$

Now we see that, using (8.3.5) and (8.3.7) to rewrite (8.3.4), we have:

$$\left\langle \eta b'\phi(l), D_\omega(\xi b\phi(h))\right\rangle = \left\langle D_\omega(\eta b'\phi(l)), \xi b\phi(h)\right\rangle .$$

That is, D_ω is a symmetric operator. □

We conclude this section showing that, actually, the "horizontal part" of the Dirac operator $\hat{D}$, which is simply D, is nothing else that the twisted Dirac operator associated to the *Maurer-Cartan connection* $\theta = \phi^{-1} * d\phi$. First of all we have to show that θ is compatible with the de Rham calculus.

Proposition 8.3.3. *The Maurer-Cartan connection $\theta = \phi^{-1} * d\phi$ is a strong connection with respect to the de Rham differential calculus on $H = C^\infty(G)$. In particular, it is defined by a strong G-connection $\{\theta^j\}$.*

Proof. We have to prove that θ fulfils property (i)-(iv) of definition 4.3.6, Q being the ideal $Q = (\ker\varepsilon)^2$. In order to show that (i) is satisfied, we have to prove that $\pi_{\hat{D}}(\theta(Q)) = 0$. So, take $q \in Q$; then $\theta(q) = \phi(q_{(1)})^{-1}d\phi(q_{(2)})$. Since, for any $h \in H$, $\phi(h)$ corresponds to $1\#h$ under the isomorphism $\mathcal{A} \simeq \mathcal{B}\#_\sigma H$, it follows that:

$$\pi_{\hat{D}}(\theta(q)) = \hat{\pi}(\phi^{-1}(q_{(1)})[\gamma \otimes \not{D}, \phi(q_{(2)})].$$

But then we have:

$$\begin{aligned}
\pi_{\hat{D}}(\theta(q))v \otimes w &= \hat{\pi}(\phi^{-1}(q_{(1)}))\left(\pi^\circ(\sigma(q_{(3)}, w_{(-1)}))(q_{(2)} \triangleright v) \otimes [\not{D}, q_{(4)}]w_{(0)}\right) \\
&= \sum_j \hat{\pi}(\phi^{-1}(q_{(1)}))\left(\sigma(q_{(3)}, w_{(-1)}))(q_{(2)} \triangleright v) \otimes q_{(4)}c(T_j)q_{(5)}(T_j)w_{(0)}\right) \\
&= \sum_j \hat{\pi}(\phi^{-1}(q_{(1)}))\hat{\pi}(\phi(q_{(2)}))\left(v \otimes c(T_j)q_{(3)}(T_j)w\right) \\
&= \sum_j v \otimes c(T_j)q(T_j)w = \sum_j v \otimes (dq)_e(T_j)w = 0,
\end{aligned}$$

since $q \in Q$ and so $(dq)_e = 0$. Next, property (ii) follows from the well-known fact that θ is a strong connection w.r.t. the universal calculus. We prove that also (iii) holds: we show that $T \circ \theta = (\mathrm{id} \otimes \pi_H) \circ (1\otimes)(\mathrm{id} - \varepsilon)$. Since $T = (\mathrm{id} \otimes \pi_H) \circ T_R$, for any $h \in H$ we have:

$$\begin{aligned}
T \circ \theta(h) &= T(\phi^{-1}(h_{(1)}) \otimes \phi(h_{(2)}) - \varepsilon(h) \otimes 1) \\
&= (\mathrm{id} \otimes \pi_H)(1 \otimes h - \varepsilon(h) \otimes 1) = (\mathrm{id} \otimes \pi_H) \circ (1 \otimes (\mathrm{id} - \varepsilon))(h).
\end{aligned}$$

Finally, we check property (iv).Take $a \in \mathcal{A}$, $a = b\phi(h)$ with $b \in \mathcal{B}$. Then

$$\begin{aligned}
da - a_{(0)}\theta(a_{(1)}) &= d(b\phi(h)) - b\phi(h_{(1)})\phi^{-1}(h_{(2)})d\phi(h_{(3)}) \\
&= db \cdot \phi(h) + bd\phi(h) - bd\phi(h) = db \cdot \phi(h) \in \Omega^1(\mathcal{B})\mathcal{A},
\end{aligned}$$

hence θ satisfies also the strongness condition. Hence it is a strong connection w.r.t. the de Rham calculus on H. In particular, due to proposition 8.1.9, θ can be defined in terms of a strong G-connection $\{\theta^j\}$. □

So, since θ is a strong connection, we can consider the twisted Dirac operator D_θ: it is straightforward to see that it is simply given by $D \otimes \mathrm{id}$, so θ is *compatible* with the Dirac operator D. As we shall see in the next section, if we consider a different trivialization things can change completely: not only the corresponding twisted Dirac operator will no longer be equal to D, or, in general, to an inner fluctuation of D, but the associated Maurer-Cartan connection could not be a strong connection with respect to the calculus associated to the "old" trivialization ϕ: this means that our construction depends strongly on the choice of the trivialization, and so of the isomorphism $\mathcal{A} \simeq \mathcal{B} \#_\sigma H$. And two different trivializations can yield to two different Dirac operators, $\hat{D}$, $\hat{D}'$, determining two first order differential calculi so different one from the other that a universal strong connection compatible with the first will in general not be compatible with the second one. So the choice of the trivialization is also the choice of the Dirac operator and the selection of a distinguished class of strong connections. All these aspects will be discussed in more detail in the next section, when we shall consider the behaviour of our construction under gauge transformations.

8.4 Gauge transformations

Let $\mathcal{B} \hookrightarrow \mathcal{A}$ be a cleft principal $C^\infty(G)$-comodule algebra and let ϕ be a unitary trivialization. We know that it determines a weak action of $H = C^\infty(G)$ on $\mathcal{B}$, which we shall denote by $\triangleright$; moreover, we shall denote by σ the corresponding cocycle. Consider then a real spectral triple $(\mathcal{B}, \mathcal{H}_B, J, D, \gamma)$ and assume than it is H-equivariant w.r.t. the weak action of H determined by ϕ (assume also the representation of $\mathcal{B}$ on $\mathcal{H}$ to be faithful). Then, under these hypotheses, we can use the results of the previous sections to build a spectral triple $(\mathcal{A}, \hat{\mathcal{H}}, \hat{D})$ for the algebra $\mathcal{A}$, $\mathcal{A}$ being identified with the algebra $\mathcal{A} = \mathcal{B} \#_\sigma H$; we shall denote by $\hat{\pi}$ the representation of $\mathcal{A}$ on $\hat{\mathcal{H}}$.

Now let $\Lambda : H \to \mathcal{B}$ be a gauge transformation. We know[12] that Λ can be seen as a change of trivialization, from ϕ to $\phi_\Lambda = \Lambda^{-1} * \phi$; and ϕ_Λ will induce an identification $\mathcal{A} \simeq \mathcal{B} \#_{\sigma_\Lambda} H$. Assume the triple over $\mathcal{B}$ to be H-equivariant also w.r.t. the weak action associated to the trivialization ϕ_Λ. Then we can construct a spectral triple $(\mathcal{A}, \hat{\mathcal{H}}_\Lambda, \hat{D}_\Lambda)$. We shall denote the representation of $\mathcal{A}$ on $\hat{\mathcal{H}}_\Lambda$ by $\hat{\pi}_\Lambda$.

Due to the results of the previous section, we know that each of the two spectral triples above induces a structure of quantum principal G-bundle on $\mathcal{A}$. A natural question is the following one: which is the relation between the two spectral triples, and which is the relation between the two quantum principal bundle structures? To give some answer to this question we begin by considering the behaviour of strong connections under gauge transformations. In this section we shall use the results discussed in section 4.2.1 and in section 4.4.1.

[12] See proposition 4.4.5.

Let $\omega : H \to \Omega^1_{\hat{D}}(\mathcal{A})$ be a strong connection. We know (see proposition 8.1.8) that it is defined in terms of a strong G-connection $\{\omega^j\} \subseteq \Omega^1_{\hat{D}}(\mathcal{A})$. Now let us compute the gauge transformed of ω. We recall that to Λ is associated the gauge transformation $f \in \mathcal{G}(\mathcal{A})$, $f = \phi^{-1} * \Lambda * \phi$, whose convolution inverse is simply $f^{-1} = \phi^{-1} * \Lambda^{-1} * \phi$. We know that ω transforms according to the following rule:

$$f \triangleright \omega = f * \omega * f^{-1} + f * df^{-1}.$$

Using the fact that $\omega(h)$, for $h \in H$, can be written as $\omega(h) = \sum_j h(T_j)\omega^j$, we obtain:

$$\begin{aligned}(f \triangleright \omega)(h) =& \sum_j f(h_{(1)})h_{(2)}(T_j)\omega^j f(h_{(3)}) + (\phi^{-1} * \Lambda * \phi * d\phi^{-1} * \Lambda^{-1} * \phi)(h) \\ &+ (\phi^{-1} * \Lambda * d\Lambda^{-1} * \phi)(h) + (\phi^{-1} * d\phi)(h).\end{aligned} \tag{8.4.1}$$

Due to the arbitrariness of the ω^j, it is clear that, in general, it won't be possible to rewrite equation (8.4.1) as $(f \triangleright \omega)(h) = \sum_j h(T_j)\eta^j$, $\{\eta^j\} \subseteq \Omega^1_{\hat{D}}(\mathcal{A})$ being a strong G-connection. Hence we expect, in general, that the gauge transformed of a strong connection w.r.t. a certain first order differential calculus $\Omega^1_{\hat{D}}(\mathcal{A})$ will no longer be a strong connection w.r.t. the same calculus. In particular, if we wish to consider the whole space of (unitary) gauge transformations, it is not possible to deal with it in the framework given by a single spectral triple, but we would have to introduce a more general setup. Such an aspect is with no doubt interesting, but it appears quite difficult to study it in a generic situation.

Instead, here we choose to select a space of gauge transformations which leave the space of strong connections w.r.t. to a given calculus unchanged. Also, we restrict ourself to cleft extensions admitting unitary trivializations which are also algebra homomorphisms. Hence, let us consider a cleft Hopf-Galois extension $\mathcal{B} \hookrightarrow \mathcal{A}$ with a unitary trivialization $\phi : H \to \mathcal{A}$ which is also an algebra homomorphism (here, of course, $H = C^\infty(G)$, G being an even-dimensional compact connected semisimple Lie group). It follows from the uniqueness of the antipode that $\phi^{-1} = \phi \circ S$. Since ϕ is a homomorphism, $\mathcal{A}$ is isomorphic to the smash product $\mathcal{B}\#H$. Given an H-equivariant even real spectral triple $(\mathcal{B}, \mathcal{H}, D, J, \gamma)$, using the results of the first part of this chapter, we can construct now a real spectral triple $(\mathcal{A}, \hat{\mathcal{H}}, \hat{D}, \hat{J}, \hat{\gamma})$ for the algebra $\mathcal{A}$. As usual, the Dirac operator $\hat{D}$ defines a differential calculus $\Omega^1_{\hat{D}}(\mathcal{A})$, and we know that $\mathcal{A}$ is a quantum principal bundle w.r.t. this calculus. Now let us consider a gauge transformation $\Lambda : H \to \mathcal{B}$ with the following properties:

(a) Λ is an algebra homomorphism, so that $\Lambda^{-1} = \Lambda \circ S$;
(b) $\phi_\Lambda = \Lambda^{-1} \circ \phi$ is still a unitary trivialization;
(c) Λ (and hence Λ^{-1}) takes values[13] in $\mathcal{B} \cap (\mathcal{A} \oplus \Omega^1_{\hat{D}}(\mathcal{A}))'$.

Lemma 8.4.1. *If Λ fulfils (a)-(b)-(c) then, for $f = \phi * \Lambda * \phi^{-1}$, the following hold:*
(i) *f is an algebra homomorphism;*
(ii) *$f^{-1} = f \circ S$;*
(iii) *f takes values in $\mathcal{A} \cap (\mathcal{A} \oplus \Omega^1_{\hat{D}})'$*

[13]The commutant is taken, of course, in $\hat{\mathcal{H}}$.

Let now ω be a strong G-connection, defined by a family $\{\omega^j\} \subset \Omega^1_{\hat{D}}$. The action of f on ω is given by equation (8.4.1). Now we can prove the following result.

Proposition 8.4.2. *Let Λ be a gauge transformation satisfying (a), (b) and (c). If there exists a family of 1-forms $\{\lambda^j\} \in \Omega^1_D(\mathcal{B})$ such that*

$$\Lambda(h_{(1)})[D, \Lambda^{-1}(h_{(2)})] = \sum_j h(T_j)\lambda^j \tag{8.4.2}$$

for any $h \in H$, then $f \triangleright \omega$, where $f = \phi * \Lambda * \phi^{-1}$, is a strong connection w.r.t. the differential calculus $\Omega^1_{\hat{D}}(\mathcal{A})$. In particular, it is defined by a family $\{\omega^j_\Lambda\} \subset \Omega^1_{\hat{D}}(\mathcal{A})$.

Proof. We have to check that $f \triangleright \omega$ fulfils (i)-(iv) of definition 4.3.6. The proof that (ii),(iii) and (iv) hold is straightforward. So we have only to prove that $(f \triangleright \omega)(Q) = 0$ in $\Omega^1_{\hat{D}}(\mathcal{A})$, where $Q = (\ker \varepsilon)^2$. Let us begin by considering the first term in (8.4.1). Using (i)-(iii) of lemma 8.4.1 we obtain:

$$\begin{aligned}(f * \omega * f^{-1})(h) &= \sum_j f(h_{(1)})h_{(2)}(T_j)\omega^j f^{-1}(h_{(3)}) = \sum_j f(h_{(1)})h_{(2)}(T_j)f(Sh_{(3)})\omega^j \\ &= \sum_j f(h_{(1)} \cdot h_{(2)}(T_j) \cdot Sh_{(3)})\omega^j\end{aligned} \tag{8.4.3}$$

Now, as a function on G, $h_{(1)} \cdot h_{(2)}(T_j) \cdot Sh_{(3)}$ is equal to

$$(h_{(1)}h_{(2)}(T_j)Sh_{(3)})(g) = \left.\frac{d}{dt}\right|_{t=0} h(ge^{tT_j}g^{-1}),$$

which is zero if $(dh)_e = 0$. It follows that (8.4.3) is equal to zero for any $h \in Q$. Consider then the second term of (8.4.1):

$$(\phi^{-1} * \Lambda * \phi * d\phi^{-1} * \Lambda^{-1} * \phi)(h) = \phi^{-1}(h_{(1)})\Lambda(h_{(2)})\phi(h_{(3)})[\hat{D}, \phi^{-1}(h_{(4)})]\Lambda^{-1}(h_{(5)})\phi(h_{(6)}). \tag{8.4.4}$$

By direct computation we can see that, for any $l \in H$, $\phi(l_{(1)})[\hat{D}, \phi^{-1}(h_{(2)})] = -\varepsilon(h)(\gamma \otimes \mathrm{id})$. Hence (8.4.5) becomes:

$$(\phi^{-1}*\Lambda*\phi*d\phi^{-1}*\Lambda^{-1}*\phi)(h) = -\phi^{-1}(h_{(1)})\Lambda(h_{(2)})\varepsilon(h_{(3)})(\gamma\otimes\mathrm{id})\Lambda^{-1}(h_{(4)})\phi(h_{(5)}) = -\varepsilon(h)(\gamma\otimes\mathrm{id}).$$

In particular, it is zero for $h \in Q$. Now the third term of (8.4.1). Before computing it we notice that from (c) and from the equivariance of D it follows that each λ^j commutes with $\phi(H)$. So we have[14]:

$$\begin{aligned}(\phi^{-1} * \Lambda * d\Lambda^{-1} * \phi)(h) &= \phi^{-1}(h_{(1)})\Lambda(h_{(2)})[\hat{D}, \Lambda^{-1}(h_{(3)})]\phi(h_{(4)}) \\ &= \phi^{-1}(h_{(1)})\Lambda(h_{(2)})[D \otimes \mathrm{id}, \Lambda^{-1}(h_{(3)})]\phi(h_{(4)}) \\ &= \sum_j \phi^{-1}(h_{(1)})h_{(2)}(T_j)\lambda^j\phi(h_{(3)})\end{aligned}$$

[14]We use also the fact that D and $\hat{D}$ induces the same first order differential calculus on $\mathcal{B}$.

$$
\begin{aligned}
&= \sum_j \phi^{-1}(h_{(1)})h_{(2)}(T_j)\phi(h_{(3)})\lambda^j \\
&= \sum_j \phi(Sh_{(1)})h_{(2)}(T_j)\phi(h_{(3)})\lambda^j \qquad (8.4.5)\\
&= \sum_j \phi(Sh_{(1)} \cdot h_{(2)}(T_j) \cdot h_{(3)})\lambda^j.
\end{aligned}
$$

But now $(Sh_{(1)}h_{(2)}(T_j)h_{(3)})(g) = \frac{d}{dt}\Big|_{t=0} h(g^{-1}e^{tT_j}g)$, and so it is zero if $(dh)_e = 0$. It follows that (8.4.5) is equal to zero for any $h \in Q$. Finally, the fourth term of (8.4.1) is nothing else but the Maurer-Cartan connection associated to the trivialization ϕ, so it is itself a strong G-connection w.r.t. the calculus $\Omega^1_{\hat{D}}(\mathcal{A})$ (see proposition 8.3.3). □

With proposition 8.4.2 we have identified a class of gauge transformations which transform strong G-connections into strong G-connections with respect to the same differential calculus. Observe that, in the classical case (i.e. for a trivial principal G-bundle $P \to M$) this space coincides more or less with the ordinary space of differentiable gauge transformations $\varphi : M \to G$.

Now we consider a related but different aspect: given a gauge transformation Λ, with suitable properties (possibly, different from (a)-(c)) is there any relation between the spectral triple $(\mathcal{A}, \hat{\mathcal{H}}, \hat{D})$, associated to a trivialization ϕ, and the spectral triple $(\mathcal{A}, \hat{\mathcal{H}}_\Lambda, \hat{D}_\Lambda)$, associated to the trivialization $\phi_\Lambda = \Lambda^{-1} * \phi$?

First of all we have to relate the representation of $\mathcal{A}$ on $\hat{\mathcal{H}}$ with that on $\hat{\mathcal{H}}_\Lambda$ (we shall denote the former by $\hat{\pi}$ and the latter by $\hat{\pi}_\Lambda$). Consider the (unitary) map $V : \hat{\mathcal{H}} \to \hat{\mathcal{H}}_\Lambda$, defined by

$$
V(v \otimes w) = \pi^\circ(\Lambda(w_{(-1)}))v \otimes w_{(0)}. \qquad (8.4.6)
$$

We notice immediately that V^* acts in the following way:

$$
V^*(v \otimes w) = \pi^\circ(\Lambda^{-1}(w_{(-1)}))v \otimes w_{(0)}.
$$

In order to show that V is an intertwining between the two representations we need to introduce a further requirement on Λ:

(d) for any $h \in H$ and any $v \in \mathcal{H}$, $\pi^\circ(\Lambda(h_{(2)}))(h_{(1)} \triangleright v) = \Lambda(h_{(1)})(h_{(2)} \triangleright_\Lambda v)$.

Here $\triangleright$ and $\triangleright_\Lambda$ denote, respectively, the weak actions of H on $\mathcal{H}$ associated to ϕ and to ϕ_Λ.

Remark 8.4.3. Requirement (d), despite perhaps its appearance, is quite natural: indeed it comes from the following relation between $\triangleright$ and $\triangleright_\Lambda$, seen as weak actions on $\mathcal{B}$, $(h_{(1)} \triangleright b)\Lambda(h_{(2)}) = \Lambda(h_{(1)})(h_{(2)} \triangleright_\Lambda b)$, which can be checked by direct computation.

Proposition 8.4.4. *Let Λ be a gauge transformation*[15] *fulfilling (d). Then, for any $a \in \mathcal{A}$,*

$$
V\hat{\pi}(a)V^* = \hat{\pi}_\Lambda.
$$

Proof. We know that ϕ and ϕ_Λ determine two isomorphisms between $\mathcal{A}$ and the crossed product algebras associated to them. In particular, we shall denote by $\mathcal{B}\#_\sigma H$ the crossed product algebra

[15] Such that ϕ_Λ is still a unitary trivialization.

defined by ϕ, and by $\mathcal{B}\#_{\sigma_\Lambda}H$ the one defined by ϕ_Λ. Moreover, we shall denote the elements of the former by $b\#h$ and the elements of the latter by $b\#_\Lambda h$. Now, consider an element $a = b\#h$ of $\mathcal{A}$ under the usual isomorphism; that is, $a = b\phi(h)$. In order to compute $\hat{\pi}_\Lambda(a)$ we have to know which is the corresponding element in $\mathcal{B}\#_{\sigma_\Lambda}H$. An easy computation shows that a corresponds to $b\Lambda(h_{(1)})\#_\Lambda h_{(2)}$. Hence, in order to prove the proposition we have to show that

$$V\hat{\pi}(b\#h)V^* = \hat{\pi}_\Lambda(b\Lambda(h_{(1)})\#_\Lambda h_{(2)}). \tag{8.4.7}$$

The computation of the r.h.s. of (8.4.7) is straightforward:

$$\hat{\pi}_\Lambda(b\Lambda(h_{(1)})\#_\Lambda h_{(2)})v \otimes w = \pi^\circ(\sigma_\Lambda(h_{(3)}, w_{(-1)}))b\Lambda(h_{(1)})(h_{(2)} \triangleright_\Lambda v)\#h_{(4)}w_{(0)}. \tag{8.4.8}$$

But if now we consider the l.h.s. of (8.4.7) we get:

$$\begin{aligned}
&V\hat{\pi}(b\#h)V^*v \otimes w = \\
&\quad = \pi^\circ(\Lambda(h_{(3)}w_{(-1)}))\pi^\circ(\sigma(h_{(2)}, w_{(-2)}))b(h_{(1)} \triangleright \pi^\circ(\Lambda^{-1}(w_{(-3)}))v)\#h_{(4)}w_{(0)} \\
&\quad = \pi^\circ(\Lambda(h_{(4)}w_{(-1)}))\pi^\circ(\sigma(h_{(3)}, w_{(-2)}))b\pi^\circ(h_{(2)} \triangleright \Lambda^{-1}(w_{(-3)}))(h_{(1)} \triangleright v)\#h_{(5)}w_{(0)} \\
&\quad = \pi^\circ(\Lambda(h_{(4)}w_{(-1)}))\pi^\circ(\sigma(h_{(3)}, w_{(-2)}))\pi^\circ(h_{(2)} \triangleright \Lambda^{-1}(w_{(-3)}))b(h_{(1)} \triangleright v)\#h_{(5)}w_{(0)} \\
&\quad = \pi^\circ\left((h_{(2)} \triangleright \Lambda^{-1}(w_{(-3)}))\sigma(h_{(3)}, w_{(-2)})\Lambda(h_{(4)}w_{(-1)})\right) b(h_{(1)} \triangleright v)\#h_{(5)}w_{(0)} \\
&\quad = \pi^\circ\left(\Lambda(h_{(2)})\sigma_\Lambda(h_{(3)}, w_{(-1)})\right) b(h_{(1)} \triangleright v)\#h_{(4)}w_{(0)} \\
&\quad = \pi^\circ(\sigma_\Lambda(h_{(3)}, w_{(-1)}))b\pi^\circ(\Lambda(h_{(2)}))(h_{(1)} \triangleright v)\#h_{(4)}w_{(0)} \\
&\quad = \pi^\circ(\sigma_\Lambda(h_{(3)}, w_{(-1)}))b\Lambda(h_{(1)})(h_{(2)} \triangleright_\Lambda v)\#h_{(4)}w_{(0)}
\end{aligned} \tag{8.4.9}$$

where, in the last equality, we have used property (d). □

Next we look at the Dirac operators: we can compare $\hat{D}$ with $V^*\hat{D}_\Lambda V$, since they are both operators on $\hat{\mathcal{H}}$. By direct computation we obtain the following result.

$$V^*\hat{D}_\Lambda V(v \otimes w) = \hat{D}(v \otimes w) + \pi^\circ(\Lambda^{-1}(w_{(-1)}))[D, \pi^\circ(\Lambda(w_{(-2)}))]v \otimes w_{(0)}. \tag{8.4.10}$$

Now let us consider the Maurer-Cartan connection associated to the trivialization ϕ_Λ:

$$\theta_\Lambda = \phi_\Lambda^{-1} * d\phi_\Lambda.$$

Since θ_Λ is nothing else than the gauge transformed of $\theta = \phi^{-1} * d\phi$, from proposition 8.4.2 we know that if Λ satisfies (a), (b), (c) then θ_Λ is a strong G-connection w.r.t. the calculus $\Omega^1_{\hat{D}}(\mathcal{A})$. So it defines a D-connection on $\mathcal{A}$:

$$\nabla_{\theta_\Lambda} a = [\hat{D}, a] - a_{(0)}\pi_{\hat{D}}\theta_\Lambda(a_{(1)}).$$

In particular, a simple computation yields to the following:

$$\nabla_{\theta_\Lambda}\phi(h) = -\Lambda(h_{(1)})[D, \Lambda^{-1}(h_{(2)})]\phi(h_{(3)}).$$

According to the results of the previous section, then, if we take a vector $\xi \in \mathcal{H}_0$ we have[16]:

$$\begin{aligned}\xi(-\Lambda(h_{(1)})[D,\Lambda^{-1}(h_{(2)})]\phi(h_{(3)})) &= -(\xi\Lambda(h_{(1)}))[D,\Lambda^{-1}(h_{(2)})]\phi(h_{(3)})\\ &= -\big(D(\xi\Lambda(h_{(1)})\Lambda^{-1}(h_{(2)}))\phi(h_{(3)}) - (D(\xi\Lambda(h_{(1)})))\Lambda^{-1}(h_{(2)})\phi(h_{(3)})\big)\\ &= -\pi^\circ(\Lambda^{-1}(h_{(2)}))[\pi^\circ(\Lambda(h_{(1)})),D]\xi\phi(h_{(3)}) = \pi^\circ(\Lambda^{-1}(h_{(2)}))[D,\pi^\circ(\Lambda(h_{(1)}))]\xi\phi(h_{(3)}),\end{aligned}$$

from which follows that $V^*\hat{D}_\Lambda V$ coincides (as an operator on on $\hat{\mathcal{H}}$) with the twisted Dirac operator D_{θ_Λ}. Hence V implements the gauge transformation Λ.

[16] We use here (5.1.3).

CHAPTER 9

Conclusions

In this conclusive chapter we review the results obtained, adding some general considerations and spending some words on possible applications and/or developments of the results discussed in this thesis.

The noncommutative geometry of noncommutative torus bundles. We have studied the noncommutative geometry of quantum principal $U(1)$- and $\mathbb{T}^n$-bundles, focusing on the relation between the geometry of the total space and the geometry of the base space. We have considered two different situations. In the first case, we assumed given a $\mathbb{T}^n$-bundle together with a $\mathbb{T}^n$-equivariant spectral triple (defining therefore a noncommutative spin geometry invariant under the $\mathbb{T}^n$-action), and we discussed the conditions under which this triple is *projectable*; that is, it can be projected to a triple on the base space in a way such that the bundle projection respects the metric structure (in the commutative case, more precisely, the bundle projection is a Riemannian submersion). Next we have shown that, under some additional assumptions[1], the Dirac operator on the total space of the bundle can be written as a sum of three operators: a first order operator D_v, called the vertical Dirac operator, acting along the fibres; the horizontal Dirac operator D_h, which contains the informations on the metric structure of the base space; a zero order term Z, which, at least in the commutative case, is related to the vanishing of the torsion of the Levi-Civita connection. On the other side, we have considered cleft (that is, almost trivial) quantum principal $\mathbb{T}^n$-bundles $\mathcal{A}$ over a base space $\mathcal{B}$ endowed with a noncommutative spin geometry, described, as usual, by a real spectral triple on the algebra $\mathcal{B}$. Then, assuming some equivariance conditions on the triple over $\mathcal{B}$ with respect to the (weak) action of the Hopf algebra $\mathcal{O}(\mathbb{T}^n)$, we have constructed a real spectral triple for the algebra $\mathcal{A}$. This spectral triple was easily proved to be $\mathbb{T}^n$-equivariant and projectable, so that our construction preserves the geometry of the base space. We can give a geometric interpretation of our construction. As we have discussed in chapter 7, the representation of the algebra $\mathcal{A}$, the Dirac operator and the real

[1] We refer here to the constant length fibres condition (see definition 5.2.3), for the $U(1)$ case, and to the isometric fibres condition (see definition 5.3.5), for the $\mathbb{T}^n$ case.

structure depend on the choice of the trivialization of the cleft extension $\mathcal{B} \hookrightarrow \mathcal{A}$. So, the first step is to choose a trivialization ϕ. In the classical case, this amounts to "fix the gauge"; that is, to fix a trivialization of a principal G-bundle; and to a fixed trivialization it corresponds a Maurer-Cartan connection: the same, as we have seen, holds in the noncommutative case, the Maurer-Cartan connection being the strong connection form $\theta = \phi^{-1} * d\phi$. Then we have seen that the Dirac operator of the triple over $\mathcal{A}$ can be interpreted as the Dirac operator obtained twisting the Dirac operator of the triple over $\mathcal{B}$ with the Maurer-Cartan connection. Hence, the spectral triple over $\mathcal{A}$ that we have constructed defines a geometry which should correspond to the one obtained by identifying the space of horizontal vector fields with the kernel of the Maurer-Cartan connection, endowing it with the metric coming from that of the base space and then putting on the space of vertical vector fields the scalar product associated to the Killing-Cartan form of $\mathfrak{t}_n = \mathrm{Lie}(\mathbb{T}^n)$.

The twisted Dirac operators. A distinguished feature of projectable triples is the possibility to construct twisted Dirac operators. In chapter 5 we exploited the construction introduced in [DS13a] to define twisted Dirac operators on projectable $\mathbb{T}^n$-equivariant spectral triple (fulfilling the isometric fibres condition[2]). So, given a projectable $\mathbb{T}^n$-equivariant real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ on a quantum principal $\mathbb{T}^n$-bundle $\mathcal{B} \hookrightarrow \mathcal{A}$ and a strong connection form $\omega : \mathcal{O}(\mathbb{T}^n) \to \Omega^1_D(\mathcal{A})$ we can construct the relative twisted Dirac operators D_ω, $\mathcal{D}_\omega = D_\omega + D_v$. Making a comparison with the classical case [Amm98, AmmB98, Mor96] we can see which is the geometrical meaning of the latter: it is the Dirac operator associated to the metric obtained "gluing" the metric on the base space with the canonical metric on $\mathbb{T}^n$ via the connection ω. From [Amm98, AmmB98] and from the discussion in appendix D we see that it is not exactly such an operator. Indeed, it could be seen as the operator obtained from a spin connection compatible with the metric described above but with non-zero torsion: to recover the operator associated to the torsionless Levi-Civita connection, at least in the commutative case, one should add to $\mathcal{D}_\omega$ a suitable zero order term. Hence, leaving apart the issue about the torsion, the construction of twisted Dirac operators is a way to find new Dirac operators defining, possibly, non-trivial geometries: indeed, even starting from a flat Dirac operator, one can obtain Dirac operators describing geometries with, e.g., non trivial scalar curvature (of course, in order to make this assertion precise one need a definition of scalar curvature for noncommutative spaces[3]). Moreover, looking at the behaviour under (a suitable class of) gauge transformations, we have shown, in chapter 8, that we can define gauge transformations of twisted Dirac operators in a way consistent with the transformation law of strong connections.

Quantum principal G-bundles: the role of the differential calculus. Since the earliest works on quantum principal bundles [BM93], the differential calculi of the algebras involved (the algebra defining the total space of the bundle and the Hopf algebra representing the structure group) were taken into consideration, and one of the requirements for a comodule algebra being a quantum principal bundle was a compatibility conditions between the two calculi (see definition 4.3.1). Bicovariant calculi [Wor89] on Hopf algebras and bundles with non-universal calculus

[2]See definitions 5.2.3 and 5.3.5.

[3]For some results in this direction see [CT11, FK12, CM11, FK11, BhMa12, DS13b]; see also appendix D.

have been extensively studied in many works (see, e.g., [BM92, BM93, SS95, Haj96, BM98a, Maj98, MajOe99, Maj02]). In this thesis we considered only principal bundles with a classical (compact, connected, semisimple) Lie group G as structure group[4]. For this class of bundles there is a natural[5] choice of (bicovariant) differential calculus on the Hopf algebra: the de Rham calculus of differential forms over G. We have chosen to restrict ourself to this class. There are several reasons for making this choice, connected with the properties of the calculus on the total space of the bundle arising from the assumed compatibility with the de Rham calculus on the Hopf algebra. The first interesting property we have underlined is the possibility to introduce equivalent characterizations both of the bundles and of the strong connections. In particular, any strong connection with respect to a differential calculus compatible with the de Rham calculus on a quantum principal G-bundle can be described[6] by a family of 1-forms, one for each element of a fixed linear basis of the Lie algebra of G: we obtain in this way a picture of strong connections very close to the usual one of the classical case, when a connection of a principal G-bundle P can be described as a $\mathfrak{g}$-valued 1-form on P (here $\mathfrak{g}$ is the Lie algebra of G). The second aspect concerns the construction of twisted Dirac operators. From a pure algebraic point of view, D-connections (see chapter 5) could be defined also without assuming compatibility with the de Rham calculus. But in such a general case it would be much more difficult to control the regularity properties of the operators obtained in this way. In particular, the selfadjointness of the twisted Dirac operators constructed in this thesis is deeply connected with the assumption of compatibility with the de Rham calculus of the calculus on the total space of the bundle; more precisely, it is connected with the particular form that strong connections assume as a consequence of this property. So the choice of the calculus, besides adding further algebraic structure on a quantum principal bundles, can be a way to select a class of strong connections with some desirable regularity properties. In chapter 6 and chapter 7, moreover, we have shown that it is possible to construct spectral triples on a class of cleft Hopf-Galois extensions whose Dirac operators define first order differential calculi compatible with the de Rham calculus. This provides a way to put a structure of quantum principal G-bundle (in the sense of definition 4.6.9 and definition 8.1.4) on a given cleft Hopf-Galois extension, under the assumption that the base space admits a suitable equivariant spectral triple.

Gauge transformations and gauge theory. Since the first works on the geometry of noncommutative spaces attention was paid to the definition and the construction of gauge theories [CR87, D-VKM90, CL91, BM93, C94]. Among the various approaches, two of them are of special interest for us: the first is the one based on quantum principal bundles [BM93], the second is the one based on Connes' noncommutative geometry [C94, C96, CMa07]. One of the fundamental bricks in the construction of a gauge theory is the identification of a group of (local) gauge transformations. In the case of quantum principal bundles (with universal differential calculus), as we discussed in chapter 4, this group can be taken to be the group of vertical automorphisms[7]. In the case of gauge theories based upon Connes' noncommutative geometry, instead, gauge

[4] That is, we considered principal H-comodule algebras, with H a suitable algebra of smooth functions over G.
[5] This is not the only possible choice, of course.
[6] See the discussion in chapter 4 and chapter 8.
[7] See definition 4.2.2.

transformations are connected with inner fluctuations of the Dirac operator, and therefore the gauge group can be identified with (a subgroup of) the group of inner automorphisms of the algebra defining the noncommutative space. In particular, in the first case the group of gauge transformations is defined in an entirely algebraic way; there is therefore no canonical way to select a group of transformations with some desired degree of regularity (looking at the classical case, e.g., one could wish to select a group of continuous or derivable transformations). Moreover, when working with quantum principal bundles with general calculus, it could happen that a certain gauge transformation of this kind does not preserve strong connections with respect to the given calculus. So we think that it is possible to make the hypothesis that, at least in some situations, it could be useful to have a way to select a smaller group of transformations. On the other side, as pointed out recently in [BMS13], the group of inner automorphisms could be a too small group, and it could be necessary to enlarge it, including transformations which are not inner. In this thesis we have discussed some, very partial, results in this direction. First, in section 5.5, working on the noncommutative 2-torus, seen as a $U(1)$-bundle over the circle, we have selected a class of gauge transformations and we interpreted it as a subgroup of the set of gauge transformations introduced in [BMS13]. A similar space of gauge transformations, for quantum principal G-bundles, is then described in chapter 8. Both these sets of gauge transformations have the following properties: first of all, they preserve strong connections compatible with the de Rham calculus; next, each gauge transformation belonging to one of them can be implemented by a unitary operator in a suitable spectral triple, and the transformation law of the Dirac operator of this triple is consistent with the construction of twisted Dirac operators discussed in this thesis.

Further developments. We conclude this thesis discussing which could be possible extensions and/or applications of our results. Looking to what we have done in chapters 6, 7, 7 it is quite natural to consider, as a possible extension of our work, the construction of spectral triples over (some suitable class) of cleft Hopf-Galois H-extensions, with H a Hopf algebra more general than an algebra of smooth functions over a classical Lie group. Our construction relies on some properties of the Hopf algebra involved, apart from the equivariance of the spectral triple on the base space: we used the fact that there is a (real) spectral triple also on the Hopf algebra and that the Dirac operator of this triple defines the bicovariant de Rham calculus. This property could fail in the general case. Indeed, consider for example the case $H = SU_q(2)$; we know that it is possible to define spectral triples on the quantum $SU(2)$ [ChaP03a, ChaP03b, ChaP06] and, even, real spectral triples [DLSSV05], but the Dirac operators of these triples do not define a bicovariant calculus on $SU_q(2)$. The problem, actually, is deeper: on $SU_q(2)$ (but the same applies also to other quantum groups [Schm02]) it is not possible to define a bicovariant first order differential calculus in terms of commutators with a given operator which has, at the same time, bounded commutators with all the elements from $SU_q(2)$ [Schm99]; that is, it is not possible to define a bicovariant calculus on $SU_q(2)$ via the Dirac operator of a spectral triple. Although this appears to be really a severe obstruction, some recent results [KMT05, KW11, KS12] seem to suggest that a way to overcome this issue could be to consider twisted spectral triples [CM06, Mos10].

Another possibility to extend our result is to introduce a suitable definition of projectable spectral triples for quantum principal G-bundles, with G any (compact, connected, semisimple) Lie group. This would amount to extend the notion of projectable spinors [Mor96, GLP98] to the noncommutative case, and then work out a suitable decomposition of the Dirac operator in a vertical and a horizontal part (plus, possibly, a zero order term). Possible indications in this direction could come from the structure of the spectral triples we constructed in chapter 8, which have to turn out to be projectable, whatever definition of projectable spectral triples for quantum principal G-bundles one considers.

Finally, in order to be able to deal with more general spaces, it could be useful to extend our construction of real spectral triples to non-cleft bundles. We do not expect that an extension to the very general case is possible. However, there is a class of bundles, larger than that one of cleft bundles, which contains a number of interesting examples and which could be a candidate for an extension of this kind: the class of locally trivial quantum principal bundles [BuK96, CaMa00, CaMa02, Zie05, HMS06, HKMZ11, HRuZ11]. The main difficulty in a task of this kind would probably be to give a local description of the noncommutative spin geometries, both of the base space and of the total space; that is, to describe them in terms of collections of spectral triples, with well-behaving gluing maps (compatible with the structure of locally trivial bundle, of course).

APPENDIX A

Noncommutative tori

Noncommutative tori are probably the best known and most widely discussed examples of noncommutative spaces. For a review of the literature and the applications of noncommutative tori the reader can see [GBFV], chapter 12. In this appendix we will review the basic properties of noncommutative tori, with the unique some results which are used elsewhere in this thesis.

A.1 The C^*-algebra $C(\mathbb{T}^n_\theta)$

The C^*-algebra $C(\mathbb{T}^n_\theta)$ defining an n-dimensional noncommutative torus is the universal C^*-algebra generated by n unitaries $U_1, \ldots, U_n$ under the relations

$$U_i U_j = e^{2\pi\theta_{ij}} U_j U_i,$$

where $\theta = (\theta_{ij})$ is an $n \times n$ skewsymmetric real matrix. Of course, for $\theta = 0$, we recover the algebra $C(\mathbb{T}^n)$ of continuous functions over an n-torus.

As in the case of the classical n-torus, we can define an action of the torus $\mathbb{T}^n$ on the algebra $C(\mathbb{T}^n_\theta)$ in the following way. Let $z = (\varphi_1, \ldots, \varphi_n)$ be the set of canonical angular coordinates on $\mathbb{T}^n$. Then we define:

$$z \triangleright U_j = \begin{cases} e^{i\varphi_i} U_i & \text{if } \ i = j, \\ U_j & \text{if } \ i \neq j \end{cases} \tag{A.1.1}$$

Averaging on $\mathbb{T}^n$ this action yields a linear operator $E : C(\mathbb{T}^n_\theta) \to C(\mathbb{T}^n_\theta)$,

$$E(a) = \int_{\mathbb{T}^n} (z \triangleright a) dz.$$

One can show (see [GBFV], section 12.2) that the image of E is just $\mathbb{C} \cdot \mathrm{id}$, so we are allowed to define a functional τ on $C(\mathbb{T}^n_\theta)$ by $E(a) \equiv \tau(a)\mathrm{id}$. It turns out that τ *is a faithful tracial state*. Consider now the following definition.

Definition A.1.1. *We say that a $n \times n$ skewsymmetric matrix θ is* quite irrational *if the lattice Λ_θ generated by its columns is such that $\Lambda_\theta + \mathbb{Z}^n$ is dense in $\mathbb{R}^n$.*

Then we can prove the following fact.

Proposition A.1.2. *If θ is quite irrational, the tracial state τ on $C(\mathbb{T}^n_\theta)$ is unique.*

Proof. Let τ' any tracial state on $C(\mathbb{T}^n_\theta)$. Then $\tau'(U_s U_r U_{-s}) = \tau'(U_r)$. Since

$$U_s U_r U_{-s} = z^s \triangleright U_r, \quad \text{with } z_j^s = e^{2\pi i \sum_k \theta_{jk} s_k},$$

then $U_s A U_{-s} = z^s \triangleright A$ for all $A \in C(\mathbb{T}^n_\theta)$. Hence $\tau'(z^s \triangleright A) = \tau'(A)$ for every $s \in \mathbb{Z}^n$. Now, for each fixed $A \in C(\mathbb{T}^n_\theta)$, the set $\{z \in \mathbb{T}^n \mid \tau'(z \triangleright A) = \tau'(A)\}$ is closed in $\mathbb{T}^n$. Moreover it is dense, since in contains every z^s and θ is quite irrational. Therefore,

$$\tau'(A) = \int_{\mathbb{T}^n} \tau'(z \triangleright A) dz = \tau'(E(A)) = \tau'(\tau(A)\mathrm{id}) = \tau(A).$$

Since A is arbitrary, we conclude that $\tau' = \tau$. □

Corollary A.1.3. *If θ is quite irrational, the C^*-algebra $C(\mathbb{T}^n_\theta)$ is simple.*

Proof. Let J be a closed two-sided ideal of $\mathbb{T}^n_\theta$. Suppose that there is a nonzero element $a \in J$. Then $a^* a \in J$ is positive and nonzero. Moreover, from the previous proof, $z(s) \triangleright a^* a = U_s a^* a U_{-s} \in J$ for $s \in \mathbb{Z}^n$, so $\{z \in \mathbb{T}^n \mid z \triangleright a^* a \in J\}$ is dense in $\mathbb{T}^n$ whenever θ is quite irrational. Since J is closed, this set is the whole n-torus $\mathbb{T}^n$, and so

$$\tau(a^* a)\mathrm{id} = E(a^* a) = \int_{\mathbb{T}^n} z \triangleright (a^* a) dz$$

lies in J, too. But $\tau(a^* a) > 0$ since τ is faithful. Therefore $\mathrm{id} \in J$, and thus $J = \mathbb{T}^n_\theta$. □

A.2 The algebras $\mathcal{A}(\mathbb{T}^n_\theta)$ and $C^\infty(\mathbb{T}^n_\theta)$

The algebra $C(\mathbb{T}^n_\theta)$ determines the noncommutative n-torus as a topological noncommutative space. We want now to specify a smooth structure and a differential structure. We begin by considering the following (dense) subalgebra of $C(\mathbb{T}^n)$.

Definition A.2.1. *The algebra $\mathcal{A}(\mathbb{T}^n_\theta)$ is the complex polynomial *-algebra generated by the n unitaries $U_1, \ldots, U_n$. That is, its elements are linear combinations*

$$\sum_{k \in \mathbb{Z}^n} \alpha_k U_1^{k_1} \cdots U_n^{k_n},$$

with only a finite number of α_k different from zero.

Since $\mathcal{A}(\mathbb{T}^n_\theta)$ is a stable subalgebra of $C(\mathbb{T}^n)$, it inherits the action of $\mathbb{T}^n$ introduced in the previous section. Moreover, this action corresponds to an action of the Lie algebra $\mathfrak{t}_n$. Indeed, if we denote by $\delta_1, \dots, \delta_n$ the canonical generators of $\mathfrak{t}_n$, then we can define

$$\delta_i(U_j) = \delta_{ij} U_j. \tag{A.2.1}$$

Each δ_j acts as a derivation on $\mathcal{A}(\mathbb{T}^n_\theta)$; that is, it satisfies the Leibniz rule. Using the derivations δ_j, we can define a family of seminorms on $\mathcal{A}(\mathbb{T}^n_\theta)$: for any $k \in \mathbb{Z}$ and any $a \in \mathbb{T}^n_\theta$ we set

$$p_k(a) = \|\delta_1^{k_1} \cdots \delta_n^{k_n}(a)\|,$$

where $\|\cdot\|$ is the C^*-norm of $C(\mathbb{T}^n_\theta)$. The family of the seminorms p_k is a separating family of seminorms, hence it induces a locally convex topology on $\mathcal{A}(\mathbb{T}^n_\theta)$ (cfr. appendix B), which makes it a locally convex topological algebra. Moreover, since it is a countable family, $\mathcal{A}(\mathbb{T}^n_\theta)$ is metrizable; in particular, it admits a translation invariant norm which induces the same topology as the seminorms p_k. Hence we can consider the following definition.

Definition A.2.2. *The algebra $C^\infty(\mathbb{T}^n_\theta)$ is the completion of $\mathcal{A}(\mathbb{T}^n_\theta)$ as a locally convex space, with topology induced by the family of seminorms $\{p_k\}_{k\in\mathbb{Z}}$.*

$C^\infty(\mathbb{T}^n_\theta)$ is, therefore, a Fréchet algebra. Moreover it can be seen that it is nothing else than the algebra of smooth elements of $C(\mathbb{T}^n_\theta)$ under the action of $\mathbb{T}^n$; since the latter is a strongly continuous action, it follows[1] that $C^\infty(\mathbb{T}^n_\theta)$ is a Frëchet pre-C^*-algebra.

A.3 The differential calculus

Any noncommutative torus $\mathcal{A}(\mathbb{T}^n_\theta)$ admits an $n-$dimensional first order differential calculus. It is defined in the following way: if $\{e^j\}_{j=1,\dots,n}$ denotes the dual basis of $\{\delta_j\}$, then for any $a \in \mathcal{A}(\mathbb{T}^n_\theta)$ we define:

$$da = \sum_j \delta_j(a) \otimes e^j.$$

The differential calculus $\Omega^1(\mathcal{A}(\mathbb{T}^n_\theta))$ then is the quotient of $\Omega^1\mathcal{A}(\mathbb{T}^n_\theta)$ by the sub-bimodule N,

$$N = \left\{ \sum a d_u b \in \Omega^1\mathcal{A}(\mathbb{T}^n_\theta) \;\middle|\; \sum a db = 0 \right\},$$

where d_u denotes the universal differential. In the same way one can define an n-dimensional calculus $\Omega^1(C^\infty(\mathbb{T}^n_\theta))$ over $C^\infty(\mathbb{T}^n_\theta)$.

Notice that each e^j can be also written in the following way:

$$e^j = U_j^* dU_j.$$

[1]See e.g. [GBFV], proposition 3.45.

A.4 Equivariant spectral triples over noncommutative tori

Any noncommutative n-torus admits[2] a family of $\mathbb{T}^n$-equivariant real spectral triples. We discuss here only those we need for our examples. For an exhaustive discussion we refer to literature [GBFV, PS06, Ven10].

Let $\mathcal{H}_\tau$ be the GNS Hilbert space associated to the tracial state τ. It is easy to see that $\mathcal{H}_\tau$ is isomorphic to $L^2(\mathbb{T}^n)$, which, moreover, can be identified with $\ell^2(\mathbb{Z}^n)$. Therefore it is straightforward to define an orthonormal basis $\{\psi_k \mid k \in \mathbb{Z}^n\}$ of $\mathcal{H}_\tau$. Consider now the Hilbert space $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^{2^{[n/2]}}$ (here $[t]$ denotes the integer part of $t \in \mathbb{R}^+$). Then tensoring the basis of $\mathcal{H}_\tau$ with the canonical basis of $\mathbb{C}^{2^{[n/2]}}$ yields an orthonormal basis of $\mathcal{H}$: $\{\psi_{k,i} \mid k \in \mathbb{Z}^n,\ i = 1, \dots, 2^{[n/2]}\}$.

Now, since θ is an antisymmetric matrix, we can write it as $\theta = A - A^t$, for some matrix A with real entries such that the representation of $\mathcal{A}(\mathbb{T}^n_\theta)$ on $\mathcal{H}$, which comes from the GNS representation of $\mathcal{A}(\mathbb{T}^n_\theta)$ on $\mathcal{H}_\tau$, can be written in the following way [Ven10]: for any $k, l \in \mathbb{Z}^n$ and any $i \in \{1, \dots, 2^{[n/2]}\}$,

$$U^k \psi_{l,i} = e^{\frac{1}{2}(k \cdot Ak + k \cdot Al)} \psi_{k+l,i},$$

where we introduced the notation

$$U^k = \prod_{i=1}^{n} U_i^{k_i}.$$

Now we define the Dirac operator. First of all we extend the derivations δ_j to selfadjoint operators on $\mathcal{H}$. This is straightforward: it is enough to set $\delta_j \psi_{k,i} = k_j \psi_{k,i}$. Next, let $\gamma^1, \dots, \gamma^n$ denotes the gamma matrices which generates the Clifford algebra $\mathbb{C}\mathrm{l}_n$. They are $2^{[n/2]} \times 2^{[n/2]}$ matrices, acting then on $\mathbb{C}^{2^{[n/2]}}$ (and hence on $\mathcal{H}$). We define the Dirac operator D simply by:

$$D = \sum_j \gamma^j \delta_j.$$

Next, one can find a bounded linear operator Λ on $\mathcal{H}$ which commutes with each δ_j and such that $\Lambda\Lambda^* = \mathrm{id}$ and $D\Lambda = -\varepsilon' \Lambda D$ (where $\varepsilon' = \pm 1$ accordingly to KR-dimension $n \pmod 8$). Then the real structure will be defined by [Ven10]:

$$J\psi_{k,i} = e^{k \cdot Ak} \Lambda \psi_{-k,i}.$$

Finally, if n is even we set $\gamma = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} \gamma^{\sigma(i)}$. Then $(\mathcal{A}(\mathbb{T}^n_\theta), \mathcal{H}, D, J, \gamma)$ (with $\gamma = \mathrm{id}$ in the odd dimensional case) is a real spectral triple. Moreover, it is a $\mathbb{T}^n$-equivariant spectral triples, with respect to the $\mathbb{T}^n$-action generated by the derivations δ_j.

This spectral triple can be shown to fulfil Connes' axioms [GBFV, C96]. In particular, it admits an orientation cocycle, which can be written (see [GBFV], exercise 12.13 and lemmas

[2] The complete classification in the $n = 2$ case was done in [PS06]. An extension of this result to the general case can be found in [Ven10].

12.15, 12.16) in the following way (for $n = 2m$ or $n = 2m+1$):

$$\mathbf{c} = \frac{(-i)^m}{n!(2\pi i)^n} \sum_{\sigma \in S_n} (-1)^\sigma U_n^* \cdots U_1^* \otimes U_{\sigma(1)} \otimes U_{\sigma(2)} \otimes \cdots \otimes U_{\sigma(n)}.$$

A.5 Quantum principal bundle structure of noncommutative tori

Any $(n+m)$-dimensional torus can be seen as a principal $\mathbb{T}^m$-bundle over an n-dimensional torus. The same holds for noncommutative tori [DS13a, DZ13]. Indeed, consider a noncommutative torus $\mathcal{A}(\mathbb{T}^{n+m}_\theta)$ and write the matrix θ as

$$\theta = \left(\begin{array}{c|c} \theta' & \vdots \\ \hline \cdots & \ddots \end{array} \right)$$

so that θ' is a skewsymmetric $n \times n$ matrix. Then the first n generators, $U_1, \ldots, U_n$, of $\mathcal{A}(\mathbb{T}^{n+m}_\theta)$ can be seen as the generators of the noncommutative torus $\mathcal{A}(\mathbb{T}^n_{\theta'})$. We shall now see that $\mathcal{A}(\mathbb{T}^n_\theta)$ is a quantum principal $\mathbb{T}^m$-bundle over $\mathcal{A}(\mathbb{T}^n_{\theta'})$.

First of all we have to introduce a right coaction Δ_R of $H = \mathcal{O}(\mathbb{T}^m)$ on $\mathcal{A}(\mathbb{T}^{n+m}_\theta)$. We define it as follows:

$$\begin{aligned} \Delta_R(U_j) &= U_j \otimes 1 \qquad\qquad j \leq n, \\ \Delta_R(U_{n+j}) &= U_{n+j} \otimes z^j. \end{aligned}$$

In particular, the invariant subalgebra $(\mathcal{A}(\mathbb{T}^{n+m}_\theta))^{coH}$ coincides with the algebra $\mathcal{A}(\mathbb{T}^n_{\theta'})$.

Proposition A.5.1. *$\mathcal{A}(\mathbb{T}^n_{\theta'}) \hookrightarrow \mathcal{A}(\mathbb{T}^{n+m}_\theta)$ is a cleft Hopf-Galois $\mathcal{O}(\mathbb{T}^n)$-extension.*

Proof. First of all let us prove that it is a Hopf-Galois extension. We have to show that the canonical map $T_R : \mathcal{A}(\mathbb{T}^{n+m}_\theta) \otimes_{\mathcal{A}(\mathbb{T}^n_{\theta'})} \mathcal{A}(\mathbb{T}^{n+m}_\theta) \to \mathcal{A}(\mathbb{T}^{n+m}_\theta) \otimes H$ is an isomorphism. Here $H = \mathcal{O}(\mathbb{T}^m)$. We begin by observing that a generic element of $\mathcal{A}(\mathbb{T}^{n+m}_\theta) \otimes H$ can be written as a sum of monomials of the form

$$U_1^{k_1} \cdots U_{n+m}^{k_{n+m}} \otimes z_1^{r_1} \cdots z_n^{r_m}.$$

If now we set $a = \lambda U_1^{k_1} \cdots U_n^{k_n} U_{n+1}^{k_{n+1}-r_1} \cdots U_{n+m}^{k_{n+m}-r_m}$ and $b = U_{n+1}^{r_1} \cdots U_{n+m}^{r_m}$, where λ is a suitable phase, then

$$T_R(a \otimes b) = U_1^{k_1} \cdots U_{n+m}^{k_{n+m}} \otimes z_1^{r_1} \cdots z_n^{r_m}.$$

Since T_R is linear, this implies that it is a surjection. Before proving the injectivity, we introduce the following notation: for any $\alpha \in \mathbb{Z}^{n+m}$, we define

$$U_\alpha = U_1^{\alpha_1} \cdots U_{n+m}^{\alpha_{n+m}}. \tag{A.5.1}$$

The monomials U_α fulfil the following commutation relation:

$$U_\alpha U_\beta = \varphi(\alpha, \beta) U_{\alpha+\beta},$$

where $\varphi(\alpha, \beta)$ is a phase coming from the commutation relations of $\mathcal{A}(\mathbb{T}^{n+m}_\theta)$. Now observe that any element of $\mathcal{A}(\mathbb{T}^{n+m}_\theta)\otimes_{\mathcal{A}(\mathbb{T}^n_{\theta'})}\mathcal{A}(\mathbb{T}^{n+m}_\theta)$ can be written as a sum of monomials like $U_\beta\otimes U_{\overline{\alpha}}$, where $\overline{\alpha}\in\mathbb{Z}^m$ is seen as an element of $\mathbb{Z}^{n+m}$ with the first n entries equal to zero. Hence, if we take $A\in\ker(T_R)$, we can write it as:

$$A=\sum_{\overline{\alpha}\in\mathbb{Z}^m}\sum_{\beta\in\mathbb{Z}^{n+m}}c_{\overline{\alpha}\beta}U_\beta\otimes U_{\overline{\alpha}}.$$

Applying T_R to this equation yields to the following expression:

$$T_R(A)=\sum_{\gamma\in\mathbb{Z}^{n+m}}\sum_{\overline{\alpha}+\beta=\gamma}c_{\overline{\alpha}\beta}\varphi(\beta,\overline{\alpha})U_\gamma\otimes\prod_{j=1}^m z_j^{\overline{\alpha}_j}.$$

Then, since the elements U_γ are linearly independent, $T_R(A)=0$ implies that, for any $\gamma\in\mathbb{Z}^{n+m}$,

$$\sum_{\overline{\alpha}+\beta=\gamma}c_{\overline{\alpha}\beta}\varphi(\beta,\overline{\alpha})\prod_{j=1}^m z_j^{\overline{\alpha}_j}=0. \tag{A.5.2}$$

But also the monomials $\prod_{j=1}^m z_j^{\overline{\alpha}_j}=0$ are linearly independent (for different $\overline{\alpha}$). Hence (A.5.2) implies that all the coefficients $c_{\overline{\alpha}\beta}$ must be equal to zero, and so $A=0$. This shows that $\mathcal{A}(\mathbb{T}^n_{\theta'})\hookrightarrow\mathcal{A}(\mathbb{T}^{n+m}_\theta)$ is a Hopf-Galois extension. Moreover it is cleft since the map $\phi: H\to\mathcal{A}(\mathbb{T}^{n+m}_\theta)$, defined by $\phi(1)=1$, $\phi(z^{\overline{\alpha}})=U_{\overline{\alpha}}$, is a unitary trivialization. □

Corollary A.5.2. *$\mathcal{A}(\mathbb{T}^n_{\theta'})\hookrightarrow\mathcal{A}(\mathbb{T}^{n+m}_\theta)$ is a principal comodule algebra. In particular, it admits a strong connection.*

Proof. It follows directly from the fact that it is a cleft extension, see proposition 4.4.2. □

We have shown that $\mathcal{A}(\mathbb{T}^{n+m}_\theta)$ is a quantum principal bundle with respect to the universal calculus. Now we consider the first order differential calculus $\Omega^1(\mathcal{A}(\mathbb{T}^{n+m}_\theta))$ of section A.3, and we show that it makes $\mathcal{A}(\mathbb{T}^{n+m}_\theta)$ into a quantum principal $\mathbb{T}^m$-bundle[3] (see definition (4.6.9)). Due to proposition 4.6.14, it is enough to show that the calculus $\Omega^1(\mathcal{A}(\mathbb{T}^{n+m}_\theta))$ is *compatible with the de Rham calculus* on $\mathbb{T}^m$. Hence, take $\sum pdq\in\Omega^1\mathcal{A}(\mathbb{T}^{n+m}_\theta)$ such that it is equal to zero in $\Omega^1(\mathcal{A}(\mathbb{T}^{n+m}_\theta))$. By definition of the calculus, this means that

$$\sum_{j=1}^{n+m}\sum p\delta_j(q)\otimes e^j=0,$$

which implies, in particular, that $\sum p\delta_j(q)=0$ for $j=n+1,\ldots,n+m$. So we have shown that (4.6.7) for the first order differential calculus $\Omega^1(\mathcal{A}(\mathbb{T}^{n+m}))$. In order to prove that $\mathbb{T}^{n+m}$ is a quantum principal $\mathbb{T}^m$-bundle, we have to show that also (4.6.8) does hold. So, consider $\eta\in\Omega^1\mathcal{A}(\mathbb{T}^{n+m}_\theta)$, and assume that $\eta=\sum pdq$, with $\sum p\delta_j(q)=0$ for any $j=n+1,\ldots,n+m$. This means, in particular that $[\eta]$, as an element of $\Omega^1(\mathcal{A}(\mathbb{T}^{n+m}_\theta))$, can be written in the following

[3]Notice that $\Omega^1(\mathcal{A}(\mathbb{T}^{n+m}_\theta))$ coincides with the Dirac operator based differential calculus $\Omega^1_D(\mathcal{A}(\mathbb{T}^{n+m}_\theta))$, where D is the Dirac operator of the canonical spectral triple discussed in this appendix.

way:

$$[\eta] = \sum \sum_{j=1}^{n} p\delta_j(q) \otimes e^j.$$

But this means, exactly, that $[\eta]$ belongs to $\mathcal{A}(\mathbb{T}^{n+m})\Omega^1(\mathcal{A}(\mathbb{T}^n))\mathcal{A}(\mathbb{T}^{n+m})$. Therefore also (4.6.8) holds. It follows that $\mathcal{A}(\mathbb{T}_\theta^{n+m})$ is a quantum principal $\mathbb{T}^m$-bundle.

We conclude this section by noticing that it is possible to give an explicit characterization of strong connections over the quantum principal $\mathbb{T}^m$-bundle $\mathbb{T}_\theta^{n+m}$.

Proposition A.5.3. *Let $\omega : H \to \Omega^1(\mathcal{A}(\mathbb{T}_\theta^{n+m}))$ be a strong connection form. Then it can be written in the following way:*

$$\omega(z^k) = \sum_{i=1}^{m} \sum_{j=1}^{n} k_i b_{ij} \otimes e^j + \sum_{i=1}^{m} k_i \otimes e^{n+i}, \tag{A.5.3}$$

with $b_{ij} \in \mathcal{A}(\mathbb{T}_{\theta'}^n)$, for any $k \in \mathbb{Z}^m$.

Proof. From proposition 4.6.18, we know that ω is defined by $\omega(z^k) = \sum_{i=1}^{m} k_i\omega_i$, where each ω_i is a 1-form over $\mathcal{A}(\mathbb{T}_\theta^{n+m})$ fulfilling properties (i)-(iii) of definition 4.6.17. Now, any ω_i can be expressed as

$$\omega_i = \sum_{j=1}^{n+m} b_{ij} \otimes e^j.$$

Condition (i), that is invariance under the $\mathbb{T}^m$-action, implies that each b_{ij} has to belong to the invariant subalgebra $\mathcal{A}(\mathbb{T}_{\theta'}^n)$. Next, since each 1-form $1 \otimes e^j$ can also be written as $U_j^{-1}dU_j$, condition (ii) implies that, for $j > n$, $b_{ij} = \delta_{ij}$. □

Corollary A.5.4. *Under the isomorphism $\Omega^1(\mathcal{A}(\mathbb{T}_\theta^{n+m})) \simeq \Omega_D^1(\mathcal{A}(\mathbb{T}_\theta^{n+m}))$, a strong connection ω will be of the following form:*

$$\omega(z^k) = \sum_{i=1}^{m} \sum_{j=1}^{n} k_i b_{ij} \otimes \gamma^j + \sum_{i=1}^{m} k_i \otimes \gamma^{n+i}.$$

APPENDIX B

Locally convex vector spaces

In this appendix we shall give a brief account of definitions and properties of topological vector spaces, locally convex vector spaces, Fréchet spaces and Fréchet algebras. We shall work over the field of complex numbers. Part of the results, however, can be worked out for modules over a topological division ring [BourGT, BourTVS].

B.1 Topological vector spaces

Definition B.1.1. *A* topological vector space *is a vector space V endowed with a topology $\mathcal{T}$ such that:*

(a) *every point of V is closed with respect to $\mathcal{T}$;*

(b) *the vector space operations are continuous with respect to $\mathcal{T}$.*

In particular, the topology $\mathcal{T}$ of a topological vector space is translation invariant; that is, for each $v \in V$, the translation operator $T_v : V \to V$, $T_v(w) = v + w$, is a homeomorphism. It follows that $\mathcal{T}$ is completely determined by any local basis[1] of neighborhoods [Ru]. Moreover,

Theorem B.1.2. *Every topological vector space is a Hausdorff space.*

Proof. See [Ru], theorem 1.12. □

Consider now a topological vector space V, with topology $\mathcal{T}$, and suppose that it is metrizable. That is, there is a metric d on V which is compatible with the topology $\mathcal{T}$. Then the balls with radius $1/n$, for $n \in \mathbb{N}^+$, centered in 0 form a local basis for $\mathcal{T}$. In addition, one can also show that:

Theorem B.1.3. *If V is a topological vector space with a countable local basis, then there is a metric d on V such that:*

[1] In the case of topological vector spaces, in general, for local basis we shall always mean a local basis of neighborhoods at 0.

(i) *d is compatible with the topology of V;*
(ii) *the open balls centered at 0 are balanced (that is, if B is an open ball centered at 0, $\alpha B \subseteq B$ for any $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$);*
(iii) *d is translation-invariant: for any $v, w, z \in V$, $d(v+z, w+z) = d(v, w)$.*

Proof. See [Ru], theorem 1.24. □

B.2 Locally convex vector spaces

Let V be a topological vector space. Then an open set $C \subset V$ is *convex* if, for any $t \in [0,1]$ and for any $v, w \in C$, $tv + (1-t)w$ belongs to C.

Definition B.2.1. *A topological vector space V is called a* locally convex (vector) space *if its topology has a local basis Γ whose members are convex open sets.*

In the case of locally convex spaces we can enforce theorem B.1.3:

Theorem B.2.2. *If V is a locally convex vector space with a countable local basis then the metric of theorem B.1.3 can be chosen so that all the open balls are convex.*

Proof. See [Ru], theorem 1.24. □

Local convexity of a topological vector space is strictly linked to the existence of suitable families of seminorms.

Definition B.2.3. *A* seminorm *on a vector space V is a function $p : V \to \mathbb{R}^+$ such that*
(i) *$p(v+w) \leq p(v) + p(w)$,*
(ii) *$p(\lambda v) = |\lambda| p(v)$,*
for any $v, w \in V$ and any $\lambda \in \mathbb{C}$.

Clearly, a seminorm is a *norm* if $p(v) = 0$ iff $v = 0$.

Definition B.2.4. *A family Γ of seminorms on a vector space X is called a* separating *family if for each $v \neq 0$ in V there exists at least a seminorm $p \in \Gamma$ such that $p(v) \neq 0$.*

Locally convex vector spaces can be equivalently characterized in terms of seminorms. Indeed, a family Γ of seminorms defines a topology $\mathcal{T}_\Gamma$, which can be identified with the translation invariant topology defined by the following local basis,

$$\mathcal{B} = \bigcup_{p \in \Gamma} \mathcal{B}_p,$$

where $\mathcal{B}_p$ is the set of balls of radius $1/n$, for $n \in \mathbb{N}^+$, w.r.t. the seminorm p. Then one can show the following fact.

Theorem B.2.5. *A topological vector space X is a locally convex space if and only if its topology is defined by a separating family of seminorms. X is metrizable if and only if we can choose a countable family of seminorms.*

Proof. See [Ru], theorems 1.36, 1.37. See also [Con], IV.1; [Schaef], II.4; [BourTVS], chapter II, section 4. □

In the case of metrizable locally convex spaces, actually we can say something more. Indeed,

Proposition B.2.6. *Let $\Gamma = \{p_i\}_{i\in\mathbb{N}^+}$ be a (separating) countable family of seminorms defining a locally convex topology on V. Then, if we define, for any $v, w \in V$,*

$$d(v,w) = \sum_{n\in\mathbb{N}^+} \frac{1}{2^n} \cdot \frac{p_n(v-w)}{1+p_n(v-w)},$$

d is a translation-invariant metric on V, and the topology defined by Γ is the same as the topology defined by d.

Proof. See [Con], chapter IV, proposition 2.1. □

Let us recall some properties of seminorms.

Proposition B.2.7. *Let V be a topological vector space and let p be a seminorm on V. Then the following are equivalent:*
(i) *p is continuous;*
(ii) *$\{x \in V \mid p(x) < 1\}$ is open;*
(iii) *0 belongs to the interior of $\{x \in V \mid p(x) < 1\}$;*
(iv) *0 belongs to the interior of $\{x \in V \mid p(x) \leq 1\}$;*
(v) *p is continuous at 0;*
(vi) *there is a continuous seminorm q on V such that $p \leq q$.*

Proof. See [Con], chapter IV, proposition 1.3. □

Proposition B.2.8. *Let V be a topological vector space. If $p_1, \ldots, p_n$ are continuous seminorms, then $p_1 + \ldots + p_n$ and $\max_i(p_i)$ are continuous seminorms.*

Proof. See [Con], chapter IV, proposition 1.4. □

Remark B.2.9. Assume that the topology of a topological space V is determined by a family Γ of seminorms on V. Then it is often convenient to enlarge Γ and assume that it is closed under finite sums. Moreover, one could also assume that Γ consists of all continuous seminorms. Indeed, in either case the resulting topology on V remains unchanged.

Now we consider linear maps $T : V \to W$ between two locally convex spaces and we look for continuity conditions.

Theorem B.2.10. *Let V, W be locally convex space, with topologies induced by families Γ, Γ' of seminorms, respectively. Let $T : V \to W$ be a linear map. Then the following are equivalent:*
(i) *T is continuous;*
(ii) *for any seminorm $q \in \Gamma'$, there exists $c \in \mathbb{R}^+$ and a seminorm $p \in \Gamma$ such that, for any $v \in V$, $q(T(v)) \leq c \cdot p(v)$;*
(iii) *for every continuous seminorm p on W, $p \circ T$ is a continuous seminorm on V.*

Proof. See [BourTVS], chapter II, proposition 4 and [Con], chapter IV. □

Take now two locally convex vector spaces E and F. Consider then the algebraic tensor product $E \otimes F$. We want to turn it into a locally convex vector space. In particular we look for a topology with respect to which the bilinear map $\varphi : E \times F \to E \otimes F$, $\varphi(e, f) = e \otimes f$, is continuous ($E \times F$ is a topological space with respect to the product topology). Such a topology exists, but, in general, it is not unique. Nevertheless, if we consider the family of locally convex topologies for which the map φ is continuous, its upper bound is the so called *projective topology* [Schaef]. It is the finest topology which makes the map φ a continuous map. It is possible to give a description of the projective topology on $E \otimes F$ in terms of seminorms (see [Schaef], chapter III, section 6.3).

Definition B.2.11. *Let p be a seminorm on E and q a seminorm on F. Then the* tensor product seminorm *$p \otimes q$ on $E \otimes F$ is defined as:*

$$(p \otimes q)(\xi) = \inf\left\{ \sum_i p(e_i)q(f_i) \mid \xi = \sum_i e_i \otimes f_i \right\}.$$

Lemma B.2.12. *For any $e \in E$, $f \in F$, $(p \otimes q)(e \otimes f) = p(e)q(f)$.*

Definition B.2.13. *A family Γ of seminorms is called a* directed family *if for each pair of seminorms $p_1, p_2 \in \Gamma$ there exists a seminorm $p_3 \in \Gamma$ such that* $\sup\{p_1, p_2\} \leq p_3$.

Proposition B.2.14. *Let the topologies on E and F be defined by two directed families of seminorms Γ_E and Γ_F, respectively. Then the projective topology on $E \otimes F$ is defined by the (directed) family of tensor product seminorms*

$$\{p \otimes q \mid p \in \Gamma_E,\ q \in \Gamma_F\}.$$

There is another relevant topology on the algebraic tensor product $E \otimes F$, the so called *inductive topology* [Schaef]: without entering into details, it is the finest locally convex topology for which the bilinear map φ defined above is separately continuous. In this thesis we shall denote simply by $E \otimes F$ the tensor product with the projective topology and with $E \otimes^i F$ that one with the inductive topology. However, since we shall usually deal with nuclear spaces[2], the two tensor product spaces will be homeomorphic, and so there will be no distinction to make:

Theorem B.2.15. *A locally convex space E is nuclear if and only if, for any other locally convex space F, the identity map between the spaces $E \otimes^p F$ and $E \otimes^i F$ is a homeomorphism.*

Proof. See [Piet], 7.3.3. □

In particular, if E is a nuclear locally convex space, we shall denote its topological (i.e. endowed with the projective, or equivalently with the injective, topology) tensor product with another locally convex space simply by $E \otimes F$.

[2]For the definition of nuclear space see [Schaef, Piet], chapter III, section 7.

B.3 Fréchet spaces and Fréchet algebras

We have seen that a locally convex space is metrizable if its topology is defined by a countable separating family of seminorms. So it is natural to consider the class of complete locally convex spaces.

Definition B.3.1. *A complete metrizable, with translation invariant metric, locally convex vector space is called a* Fréchet space*.*

Hence, given a locally convex vector space V, with topology defined by a countable separating family of seminorms, its metric completion $\overline{V}$ is a Fréchet space. In particular, given two locally convex vector spaces E and F we define $E\overline{\otimes}F$ to be the completion of the tensor product $E \otimes^p F$.

Now consider associative algebras over the field of complex numbers. We give the following definitions [Mall].

Definition B.3.2. *A* topological algebra *is an algebra A which is a topological vector space in such a way that the multiplication map $A \times A \to A$ is separately continuous.*

If it is locally convex as a vector space, we shall speak of locally convex topological algebra*.*

In the previous section we saw that any locally convex topology is defined by a separating family of seminorms. The same holds for locally convex algebras, but in this case we have to require the seminorms to be sub-multiplicative; that is:

$$p(ab) \leq p(a)p(b)$$

for any elements a, b of the algebra.

Definition B.3.3. *A topological algebra A is called a* Fréchet algebra *if it is a Fréchet space.*

Proposition B.3.4. *If an algebra A is a Fréchet space and the multiplication $A \times A \to A$ is separately continuous (so that A is a Fréchet algebra), the multiplication is jointly continuous.*

Proof. See [Wael], chapter VII, proposition 1. □

Now consider two locally convex algebras A and B and form the (algebraic) tensor product $A \otimes B$. Then:

Proposition B.3.5. *$A \otimes^p B$, that is the algebraic tensor product of A and B endowed with the projective topology, is a locally convex topological algebra. In particular, if the multiplication in A and B is (jointly) continuous, then the same holds for the multiplication in $A \otimes^p B$.*

Proof. See [Mall], chapter X, lemma 3.1. □

Corollary B.3.6. *Let A and B be two Fréchet algebras. Then the completion $A\overline{\otimes}B$ of the tensor product $A \otimes^p B$ is a Fréchet algebra.*

Proof. See [Gr66], proposition I.5. □

Consider now a smooth manifold M of dimension n. Let $\{K\}$ be a countable compact covering of M such that each compact set K is contained in an open chart $(U, \{x_1, \dots, x_n\})$. For each compact set K of the covering and for each $r \in \mathbb{N}^n$, consider the following seminorm on the algebra $C^\infty(M)$ of smooth functions over M:

$$p_{K,r}(f) = \sup_{x \in K} |\partial_{x_1}^{r_1} \cdots \partial_{x_n}^{r_n}(f)(x)|.$$

These seminorms determine a structure of locally convex space on $C^\infty(M)$. Moreover,

Proposition B.3.7. *$C^\infty(M)$, with the topology defined above, is a nuclear Fréchet algebra.*

Proof. See [Schw], page 88. See also [Gr66], chapter II, page 54. □

Corollary B.3.8. *If M is a compact smooth manifold, then the algebra $C^\infty(M)$ is a nuclear Fréchet algebra with respect to the topology defined by the following seminorms:*

$$p_D(f) = \sup_{x \in M} |D(f)(x)|,$$

where D varies on a basis of the space of the algebra of differential operators on M.

Proof. It follows directly from the previous proposition. See also [GBFV], section 3.8. □

Proposition B.3.9. *Let M be a compact smooth manifold. Then there is an isomorphism of Fréchet algebras $C^\infty(M)\overline{\otimes}C^\infty(M) \simeq C^\infty(M \times M)$.*

Proof. See [Gr66], theorem II.13. □

APPENDIX C

Line modules, Morita contexts and Hopf-Galois $\mathbb{C}[\mathbb{Z}]$-extensions

In this appendix we shall briefly recall the main aspects of the theory of noncommutative line modules, which are the generalization of line bundles, and its relation with Morita contexts and Hopf-Galois $\mathbb{C}[\mathbb{Z}]$-extensions [BB11]. As usual, we shall work over the field of complex number. The symbol $\otimes$ will denote the algebraic tensor product over $\mathbb{C}$, and the symbol $\otimes_A$ the algebraic tensor product over the algebra A of a right A-module with a left A-module.

C.1 Line modules

Let A be a unital associative algebra and E a left A-module. Then the *left dual* E' of E is the right module of left A-linear maps from E to A. The right A-module structure is the following one: if $\alpha \in E'$ and $a \in A$, then $\alpha \cdot a$ is defined by

$$(\alpha \cdot a)(e) = \alpha(e)a \qquad \forall e \in E.$$

If E is a finitely generated projective A-module then [BB11] there exist, for $i = 1, \ldots, n$, elements $e^i \in E$, $e_i \in E'$ such that any $f \in E$ can be written as $f = \sum_i e_i(f) \cdot e^i$. Also, any functional $\alpha \in E'$ satisfies $\alpha = \sum_i e_i \cdot \alpha(e^i)$.

Assume now that E is an A-bimodule, which is finitely generated and projective as left A-module. In this case E' is a bimodule[1], too, and we can define two bimodule maps, $\mathrm{ev} : E \otimes_A E' \to A$ and $\mathrm{coev} : A \to E' \otimes_A E$ by:

$$\mathrm{ev}(e \otimes \alpha) = \alpha(e), \qquad \mathrm{coev}(1_A) = \sum_i e_i \otimes e^i, \tag{C.1.1}$$

for $e \in E$, $\alpha \in E'$. Here 1_A denotes the unit of A. ev is called *evaluation map*, coev *coevaluation*

[1]With the following left module structure: $(a \cdot \alpha)(e) = \alpha(e \cdot a)$ for any $a \in A$, $e \in E$, $\alpha \in E'$.

map. They satisfy the following relations:

$$(\mathrm{ev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}(1_A)) = \mathrm{id}_E,$$

$$(\mathrm{id} \otimes \mathrm{ev}) \circ (\mathrm{coev}(1_A) \otimes \mathrm{id}) = \mathrm{id}_{E'}.$$

Definition C.1.1. *An A-bimodule E, which is finitely generated and projective as left A-module, is called a* weak (left) line module *if the coevaluation map is an isomorphism. If, in addition, also the evaluation map is an isomorphism, it is called a* (left) line module.

Proposition C.1.2. *Let E be an A-bimodule, which is finitely generated and projective as left A-module. Then the following are equivalent:*

(i) *E is a weak left line module;*

(ii) *every left module map from E to E is given by the right action of some element of A, and the only $a \in A$ for which $E \cdot a = 0$ is $a = 0$.*

Proof. See [BB11], proposition 3.2. □

Proposition C.1.3. *Let E be a weak left line module. Then if* ev *is surjective, it is an isomorphism.*

Proof. See [BB11], proposition 3.6. □

C.2 Morita contexts

Let A and B be two unital algebras. Denote by ${}_A\mathcal{M}$ the category of left A-modules, by ${}_B\mathcal{M}$ the category of left B-modules and by ${}_A\mathcal{M}_B$, ${}_B\mathcal{M}_A$ the categories of $A-B-$ and $B-A-$bimodules, respectively. Then [BB11, Bass, BeKe],

Definition C.2.1. *A* Morita context *for the algebras A and B consists of two bimodules $E \in {}_A\mathcal{M}_B$ and $F \in {}_B\mathcal{M}_A$, together with two bimodule maps $\mu_1 : E \otimes_B F \to A$ and $\mu_2 : F \otimes_A E \to B$ such that*

$$\begin{aligned} \mu_1 \otimes \mathrm{id} = \mathrm{id} \otimes \mu_2 \quad &: \quad E \otimes_B F \otimes_A E \to E, \\ \mu_2 \otimes \mathrm{id} = \mathrm{id} \otimes \mu_1 \quad &: \quad F \otimes_A E \otimes_B F \to F. \end{aligned} \tag{C.2.1}$$

A Morita context is strict *if μ_1 and μ_2 are surjective.*

Proposition C.2.2. *If $(A, B, E, F, \mu_1, \mu_2)$ is a strict Morita context, then:*

(i) *μ_1 and μ_2 are isomorphisms;*

(ii) *E and F are finitely generated projective left $A-$ and $B-$modules, respectively;*

(iii) *E and F are finitely generated projective right $B-$ and $A-$modules, respectively.*

Proof. See [Bass], chapter II, theorem 3.5. □

Proposition C.2.3. *There is a one-to-one correspondence between equivalences*[2] *between the categories $E \in {}_A\mathcal{M}$ and $E \in {}_B\mathcal{M}$ and strict Morita contexts $(A, B, E, F, \mu_1, \mu_2)$. The functors*

[2]For the notion of equivalence between two categories see, e.g., [Bass, McL].

associated to a strict Morita context are:

$$E \otimes_B - \; : \; {}_B\mathcal{M} \to {}_A\mathcal{M},$$

$$F \otimes_A - \; : \; {}_A\mathcal{M} \to {}_B\mathcal{M}.$$

Proof. See [Bass], chapter II, theorem 3.5. □

In particular, any *autoequivalence* of the category ${}_A\mathcal{M}$ corresponds to a strict Morita context $(A, A, E, F, \mu_1, \mu_2)$, where both E and F are A-bimodules.

C.3 Line module and Hopf-Galois $\mathbb{C}[\mathbb{Z}]$-extensions

Now we consider Hopf-Galois extensions by the Hopf algebra $\mathbb{C}[\mathbb{Z}]$. We recall that it can be described as the commutative polynomial complex *-algebra generated by a unitary element z, with Hopf algebra structure defined by the following relations:

$$\Delta(z^k) = z^k \otimes z^k, \qquad S(z^k) = z^{-k}, \qquad \varepsilon(z^k) = 1,$$

for any $k \in \mathbb{Z}$. Any Hopf-Galois $\mathbb{C}[\mathbb{Z}]$-extension $B \hookrightarrow A$ has the following structure: A is a graded algebra, $A = \bigoplus_{k\in\mathbb{Z}} A^{(k)}$, with $A^{(0)} = B$. Moreover,

Lemma C.3.1. *A $\mathbb{Z}$-graded algebra $A = \bigoplus_{k\in\mathbb{Z}} A^{(k)}$ is a Hopf-Galois $\mathbb{C}[\mathbb{Z}]$-extension over $A^{(0)}$ if and only if every product $A^{(k)} \otimes A^{(l)} \to A^{(k+l)}$ is surjective.*

Proof. See [BB11], proposition 7.1. □

Now let A be a unital algebra and E an A-bimodule. Then we can define the *$\mathbb{Z}$-graded tensor algebra* $T_{\mathbb{Z}}(A)$ in the following way [BB11]:

$$T_{\mathbb{Z}}(E)^{(k)} = \begin{cases} A & n = 0 \\ E^{\otimes_A^n} & n > 0 \\ E'^{\otimes_A^{-n}} & n < 0 \end{cases}$$

Here E' is the dual bimodule of E. If L is a weak line module then it is possible to put a structure of associative algebra on $T_{\mathbb{Z}}(L)$ (see [BB11], proposition 6.1). Moreover,

Lemma C.3.2. *Let L be a weak left line module over the algebra A. Then L is a left line module if and only if $T_{\mathbb{Z}}(L)$ is a Hopf-Galois $\mathbb{C}[\mathbb{Z}]$-extension.*

Proof. See [BB11], proposition 7.2. □

Now we can state and prove the result we are interested in[3].

[3] This result is taken from [BB11] (see theorem 7.3). We give here a sketch of the proof because we shall use it elsewhere in this thesis.

Theorem C.3.3. *Let A be a unital associative algebra. Then there is a one-to-one correspondence between:*

(i) *autoequivalences of the category ${}_A\mathcal{M}$;*

(ii) *left line modules over A;*

(iii) *Hopf-Galois $\mathbb{C}[\mathbb{Z}]$-extensions of A.*

Proof. (i) ⇒ (ii). An autoequivalence of ${}_A\mathcal{M}$ corresponds (see above) to a strict Morita context $(A, A, E, F, \mu_1, \mu_2)$. Moreover, due to proposition C.2.2, E and F are finitely generated and projective both as left and right A-modules. Now, by definition of strict Morita context, the maps $\mu_1 : E \otimes_A F \to A$ and $\mu_2 : F \otimes_A E \to A$ are isomorphisms of A-bimodules. If now we set $\mathrm{ev} = \mu_1$ and $\mathrm{coev} = \mu_2^{-1}$ we see that, due to (C.2.1), they behave like the evaluation and the coevaluation maps of a weak left line module. Moreover, since we started from a strict Morita context, both ev and coev are isomorphisms; hence E is a left line module over A.

(ii) ⇒ (iii). It L is a left line module then $T_{\mathbb{Z}}$ is a Hopf-Galois $\mathbb{C}[\mathbb{Z}]$-extension (see lemma C.3.2).

(iii) ⇒ (i). Let B be a Hopf-Galois $\mathbb{C}[\mathbb{Z}]$ extension over A. Then B can be split into a direct sum of subspaces of homogeneous degree: $B = \bigoplus_{k \in \mathbb{Z}} B^{(k)}$. In particular, $B^{(0)} = A$. Now set $E = C^{(1)}$ and $F = C^{(-1)}$. Clearly E and F are A-bimodules. Moreover, the multiplication maps $\mu_1 : E \otimes_A F \to A$ and $\mu_2 : F \otimes_A E \to A$ are surjective, due to lemma C.3.1. Also, μ_1, μ_2 fulfils (C.2.1), as follows from the fact that B is an associative ring. Hence $(A, A, E, F, \mu_1, \mu_2)$ is a strict Morita context, and so it determines an autoequivalence of the category ${}_A\mathcal{M}$ of left A-modules. □

APPENDIX D

Twisted Dirac operators, curvature and torsion of noncommutative tori

In chapter 5 we discussed, among other things, the construction of twisted Dirac operators. In particular, we worked out explicitly the twisted Dirac operators for (low dimensional) noncommutative tori. In this section we shall study some "geometric" properties of these operators. The reason is the following one: we know very well that the canonical spectral triple over a noncommutative n-torus is flat, in the sense that it corresponds exactly to the flat, $\mathbb{T}^n$-invariant, geometry of the smooth n-torus. In some recent works [CT11, FK12, FK11, CM11, BhMa12, DS13b] it was pointed out that various modifications of the flat Dirac operator of a noncommutative torus can lead to noncommutative geometries in which it is possible to associate a nontrivial curvature or a nontrivial torsion to the new Dirac operator. For this reason, we found interesting to discuss if and how it is possible to associate (possibly) nontrivial curvature and/or torsion to the twisted Dirac operators built in this thesis (and in [DS13a]).

D.1 The commutative case

Our discussion will be mainly focused on the noncommutative 3-torus[1]. We begin by considering the commutative case, reviewing the results obtained by Ammann and Bär [Amm98, AmmB98] and applying them to a 3-torus, seen as a principal $U(1)$-bundle over a flat 2-torus. Let M denote a smooth 3-torus $\mathbb{T}^3$ and N a smooth 2-torus $\mathbb{T}^2$, so that $M \to N$ is a principal $U(1)$-bundle. Put on N a flat, $\mathbb{T}^2$-equivariant, metric g. Then, any connection form ω induces a metric $\widetilde{g}$ on M such that the bundle projection $\pi : (M, \widetilde{g}) \to (N, g)$ is a Riemannian submersion and all the fibres have equal length. Up to rescaling the metric, we can assume this length to be equal to 1. The metric $\widetilde{g}$ can be characterized in the following way. Let K denote the Killing vector field associated to the $U(1)$ action and let $f_1 = \partial_1$, $f_2 = \partial_2$ be the canonical (local) orthonormal frame on N. Then $\{e_1 = \widetilde{f}_1, e_2 = \widetilde{f}_2, e_3 = K\}$, where $\widetilde{X}$ denotes the horizontal lift of a vector field X

[1] For the twisted Dirac operator on a noncommutative 3-torus see [DS13a]. Here we will just recall its explicit expression.

with respect to the connection ω, is a local orthonormal frame for $\widetilde{g}$. Using the Koszul formula, we can easily work out the Christoffel symbols $\widetilde{\Gamma}^k_{ij}$ of the Levi-Civita connection, with respect to the frame $\{e_j\}$:

$$\begin{aligned} &\widetilde{\Gamma}^k_{ij} = 0 && \text{for } i,j,k = 1,2, \\ &\widetilde{\Gamma}^3_{ij} = -\widetilde{\Gamma}^j_{i3} = -\widetilde{\Gamma}^j_{3i} = -\frac{1}{2} d\omega(e_i, e_j) && \text{for } i,j = 1,2, \\ &\widetilde{\Gamma}^3_{i3} = \widetilde{\Gamma}^3_{3i} = \widetilde{\Gamma}^i_{33} = 0 && \text{for } i = 1,2,3. \end{aligned} \tag{D.1.1}$$

It follows that the spinor connection $\nabla^{\Sigma M}$ on the spinor bundle ΣM can be written in the following way:

$$\nabla^{\Sigma M}_{e_i} \psi = \partial_{e_i} \psi + \frac{1}{4} \widetilde{\Gamma}^k_{ij} \gamma(e_j) \gamma(e_k) \psi, \tag{D.1.2}$$

where $\gamma : TM \to \mathrm{End}(\Sigma M)$ is the Clifford multiplication map. We recall that the Clifford multiplication extends to a map $\gamma : \Omega^\bullet(M) \to \mathrm{End}(\Sigma M)$ defined by:

$$\gamma(\alpha)\psi = \sum_{i_1 < \ldots < i_p} \alpha(e_{i_1}, \ldots, e_{i_p}) \gamma(e_{i_1}) \cdots \gamma(e_{i_p}) \psi$$

for any p-form α. Then the Dirac operator is given by

$$\widetilde{D} = -\gamma \circ \nabla^{\Sigma M} = \sum_j -\gamma(e_j) \nabla^{\Sigma M}_{e_j} \tag{D.1.3}$$

and it is a selfadjoint operator on the Hilbert space $L^2(\Sigma M)$. Now, the $U(1)$ action allows us to split $L^2(\Sigma M)$ as a direct sum of eigenspaces of the Killing vector field K. More precisely,

$$L^2(\Sigma M) = \bigoplus_{k \in \mathbb{Z}} V_k,$$

where V_k is the eigenspace of eigenvalue ik of the Lie derivative $\mathcal{L}_K$. Moreover one can prove the following result.

Lemma D.1.1. *For any $\psi \in \Gamma^\infty(\Sigma M)$,*

$$\nabla_K \psi = \mathcal{L}_K \psi + \frac{1}{4} \gamma(d\omega) \psi. \tag{D.1.4}$$

Proof. See [AmmB98], lemma 4.3. □

Consider now the complex line bundle $L = M \times_{U(1)} \mathbb{C}$ associated to the principal bundle $M \to N$, together with the connection given by $i\omega$. Then, since the dimension of N is even and we have taken the fibres to have length equal to 1, the following holds.

Proposition D.1.2. *For any $k \in \mathbb{Z}$, there is an isomorphism of Hilbert spaces*

$$Q_k : L^2(\Sigma N \otimes L^{-k}) \to V_k$$

such that the horizontal covariant derivatives are given by

$$\nabla^{\Sigma M}_{\widetilde{X}} Q_k(\psi) = Q_k(\nabla^{\omega}_X \psi) + \frac{1}{4}\gamma(K)\gamma(\widetilde{V}_X) Q_k(\psi),$$

where V_X is the vector field on N defined by $d\omega(\widetilde{X}, \cdot) = \widetilde{g}(\widetilde{V}_X, \cdot)$ and ∇^{ω} is the twisted spin connection on $\Sigma N \otimes L^{-k}$. Moreover, the Clifford multiplication is preserved; that is,

$$Q_k(\gamma(X)\psi) = \gamma(\widetilde{X}) Q_k(\psi),$$

where we used $\gamma(\cdot)$ to denote both the Clifford multiplication on ΣM and on ΣN.

Proof. See [AmmB98], lemma 4.4. □

Then[2] the Dirac operator $\widetilde{D}$ can be written, on each V_k, as sum of three operators,

$$\widetilde{D} = D_h + D_v + Z,$$

where $D_h = Q_k \circ D^N \circ Q_k^{-1}$, $D_v = \gamma(K)\mathcal{L}_K$ and $Z = -\frac{1}{4}\gamma(K)\gamma(d\omega)$. Now we compute explicitly the spin covariant derivatives and the Dirac operator for our 3-torus $(M, \widetilde{g})$. We begin by writing the connection form ω as:

$$\omega = \omega_1 dx^1 + \omega_2 dx^2 + dx^3.$$

Then $d\omega = -\omega_{12} dx^1 \wedge\ dx^2$, where we set $\omega_{12} = \partial_2\omega_1 - \partial_1\omega_2$. Next, we consider the following Clifford multiplication map on the spinor bundle ΣN (and, hence, on each $\Sigma N \otimes L^{-k}$)[3]:

$$\gamma(f_1) = \gamma(\partial_1) = i\sigma^1, \qquad \gamma(f_2) = \gamma(\partial_2) = i\sigma^2.$$

It follows that $Q_k^{-1}\gamma(d\omega)Q_k = -\omega_{12}\sigma^1\sigma^2 = -i\omega_{12}\sigma^3$. Moreover, by direct computation one can see that

$$V_{f_1} = V_{\partial_1} = -\omega_{12} f_2 = -\omega_{12}\partial_2, \qquad V_{f_2} = V_{\partial_2} = \omega_{12} f_1 = \omega_{12}\partial_1.$$

Therefore, $Q_k^{-1}\gamma(V_{f_1})Q_k = -\omega_{12}\sigma^1$ and $Q_k^{-1}\gamma(V_{f_2})Q_k = \omega_{12}\sigma^2$. The last thing we need is $\gamma(K)$. Using the fact that Clifford multiplication is preserved by Q_k and the properties of Clifford algebras, we deduce that the only possible choices are $Q_k^{-1}\gamma(K)Q_k = \pm i\sigma^3$. We fix this convention assuming $Q_k^{-1}\gamma(K)Q_k = i\sigma^3$.

Using these results, together with lemma D.1.1 and proposition D.1.2, we can obtain the explicit expressions of the spinor covariant derivatives $\nabla^{\Sigma M}_{e_i}$. In what follows Q will denote the collection of the maps Q_k, and ∂_3 the operator corresponding to the Lie derivative $\mathcal{L}_K$ (or, equivalently, to the operator ∂_K [AmmB98]); in particular, ∂_3 corresponds to the multiplication by ik on each Hilbert space $L^2(\Sigma N \otimes L^{-k})$. For $\psi \in \bigoplus_k L^2(\Sigma N \otimes L^{-k})$,

$$Q^{-1}\nabla_{e_1} Q(\psi) = \partial_1\psi - \omega_1\partial_3\psi - \frac{i}{4}\omega_{12}\sigma^1, \tag{D.1.5a}$$

[2]See [AmmB98], theorem 4.1.

[3]We recall here that the Hilbert space $L^2(\Sigma N)$ can be identified with $C(\mathbb{T}^2) \otimes \mathbb{C}^2$ with the scalar product given by the integral on the torus, since we are considering the flat (Haar) metric on $N = \mathbb{T}^2$.

$$Q^{-1}\nabla_{e_2}Q(\psi) = \partial_2\psi - \omega_2\partial_3\psi - \frac{i}{4}\omega_{12}\sigma^2, \tag{D.1.5b}$$

$$Q^{-1}\nabla_{e_3}Q(\psi) = \partial_3\psi + \frac{i}{4}\omega_{12}\sigma^3. \tag{D.1.5c}$$

Now we see that the Dirac operator $\widetilde{D}$ is given by:

$$Q^{-1}\widetilde{D}Q = \sum_{j=1}^{3} -i\sigma^j\partial_j + i\sigma^1\omega_1\partial_3 + i\sigma^2\omega_2\partial_3 - \frac{1}{4}\omega_{12}.$$

In particular we notice that it differs from the twisted Dirac operator $\hat{D}_\omega$ only for the scalar term $Z = -\frac{1}{4}\omega_{12}$. We take this as a suggestion for the noncommutative case: in the next sections we shall look for a suitable modification of the noncommutative Dirac operator $\hat{D}_\omega$ which should correspond to the Dirac operator obtained from a metric compatible, torsionless connection. Moreover, we shall see that the construction discussed below allows to recover a notion of curvature.

D.2 Tangent bundle, horizontal lifts and Levi-Civita connection

As a first attempt one could try to mimic the commutative case in the most straightforward way. Hence we look for a noncommutative analogue of an orthonormal frame, from which we would derive the analogue of Christoffel symbols. In general this would be a very difficult task. Indeed, in noncommutative geometry the definition of the tangent bundle of a noncommutative space is far from straightforward. But for noncommutative tori, the $\mathbb{T}^n$ action (or better, the associated action of the Lie algebra $\mathfrak{t}_n$) allows to give a reasonable definition of tangent bundle: let $\delta_1, \ldots, \delta_n$ denote the generators of the $\mathfrak{t}_n$-action on a noncommutative torus $\mathcal{A}(\mathbb{T}^n_\theta)$ and consider the following definition.

Definition D.2.1. *The space of (complex)* smooth vector fields *of a noncommutative n-torus $\mathbb{T}^n_\theta$ is the $\mathbb{C}$-linear space $\mathfrak{X}(\mathbb{T}^n_\theta) = (\mathcal{A}(\mathbb{T}^n_\theta))^\circ \otimes \mathbb{C}^n$, where the $\mathbb{C}^n$ factor is the linear space of the derivations $\delta_1, \ldots, \delta_n$.*

Hence a smooth vector field over $\mathbb{T}^n$ is a linear combination $X = \sum_{j=1}^n (a^j)^\circ\delta_j$, where each a_j belongs to $\mathcal{A}(\mathbb{T}^n_\theta)$. We can define a $(\mathcal{A}(\mathbb{T}^n_\theta))^\circ \otimes \mathcal{A}(\mathbb{T}^n_\theta)$-valued action of $\mathfrak{X}(\mathbb{T}^n_\theta)$ on $\mathcal{A}(\mathbb{T}^n_\theta)$ in the following way: for any $X = \sum(a_j^\circ)\delta_j \in \mathfrak{X}(\mathbb{T}^n_\theta)$ and any $f \in \mathcal{A}(\mathbb{T}^n_\theta)$, we set

$$X(f) = \sum_j (a_j)^\circ \otimes \delta_j(f).$$

Moreover, we can put a structure of left $\mathcal{A}(\mathbb{T}^n_\theta)$-module on $\mathfrak{X}(\mathbb{T}^n_\theta)$: $f \cdot X = \sum_j (fa_j)^\circ\delta_j$. In the following two lemmas we point out two (trivial) properties of $\mathfrak{X}(\mathbb{T}^n_\theta)$, which show that it behaves as the space of sections of the tangent bundle of a smooth manifold.

Lemma D.2.2. *Each element of $\mathfrak{X}(\mathbb{T}^n_\theta)$ is a derivation of $\mathcal{A}(\mathbb{T}^n_\theta)$; that is, it satisfies the Leibniz rule*

$$X(ab) = X(a)b + aX(b) \qquad \forall a, b \in \mathbb{T}^n_\theta.$$

Lemma D.2.3. $X(\mathbb{T}^n_\theta)$ *is a finitely generated projective left* $\mathcal{A}(\mathbb{T}^n_\theta)$*-module.*

Now let us consider, in full generality, a noncommutative torus $\mathcal{A}(\mathbb{T}^{n+m}_\theta)$ as a $\mathbb{T}^m$ bundle over a noncommutative torus $\mathcal{A}(\mathbb{T}^n_{\theta'})$. Identifying the latter with the invariant subalgebra of the former, and assuming the $\mathbb{T}^m$-action to be the one generated by the derivations $\delta_{n+1}, \dots, \delta_{n+m}$, we can identify the space $\mathfrak{X}(\mathbb{T}^{n+m}_\theta)$ with the span, over $(\mathcal{A}(\mathbb{T}^{n+m}_\theta))^\circ$, of the derivations $\delta_1, \dots, \delta_{n+m}$ and the space $\mathfrak{X}(\mathbb{T}^n_{\theta'})$ with the span, over $(\mathcal{A}(\mathbb{T}^n_{\theta'}))^\circ$, of the derivations $\delta_1, \dots, \delta_n$.

The inclusion $\mathcal{A}(\mathbb{T}^n_{\theta'}) \hookrightarrow \mathcal{A}(\mathbb{T}^{n+m}_\theta)$ corresponds, from a geometrical point of view, to a $\mathbb{T}^n$-equivariant submersion $\pi : \mathcal{A}(\mathbb{T}^{n+m}_\theta) \to \mathcal{A}(\mathbb{T}^n_{\theta'})$. Following this idea, we define the push-forward of π, as a linear map $\pi_* : \mathfrak{X}(\mathbb{T}^{n+m}_\theta) \to \mathfrak{X}(\mathbb{T}^n_{\theta'})$. We could give simply the definition and check its properties, but first we want to give some motivations for our choice.

Let us consider a principal G-bundle $\pi : P \to M$, where G is a compact Lie group. If we take a vector field $X \in \Gamma(TP)$ and a function $f \in \mathbb{C}^\infty(M)$, then the push-forward $\pi_* X$ is the element of $\Gamma(TM)$ defined by:

$$(\pi_* X)(f)_x(x) = X_p(f \circ \pi)(p), \tag{D.2.1}$$

for any $x \in M$, where p is any point of P such that $\pi(p) = x$. In particular, the value of (D.2.1) does not depend on the choice of p on the fibre over x. This means that for any $g \in G$ we have:

$$(\pi_* X)_x(f)(x) = X_{g \cdot p}(f \circ \pi)(g \cdot p). \tag{D.2.2}$$

Nevertheless, we see that the definition of the push-forward is a pointwise definition. In noncommutative geometry such a pointwise description is usually non available, hence we have to pay attention and check if we are defining something meaningful. It is quite clear that it is not possible to define, in a global way, the push-forward of all of the vector fields of the total space of a bundle; instead, it is possible if one restricts itself to projectable vector fields.

Now we come back to noncommutative tori. We begin by introducing a definition of projectable vector field, motivated from the fact that invariance under the $\mathbb{T}^m$-action corresponds to invariance under the coaction of the Hopf algebra $H = \mathcal{O}(\mathbb{T}^m)$. Let us introduce the following right coaction of H on $\mathfrak{X}(\mathbb{T}^{m+n}_\theta)$:

$$\rho^{\mathfrak{X}}_R\left(\sum_j (a^j)^\circ \delta_j \right) = \sum_j (a^j_{(0)})^\circ \delta_j \otimes a^j_{(1)}.$$

Definition D.2.4. *A vector field* $X \in \mathfrak{X}(\mathbb{T}^{n+m}_\theta)$ *is said to be* projectable *if* $\rho^{\mathfrak{X}}_R(X) = X \otimes 1$. *Hence, the space of projectable vector fields is the space* $\mathfrak{X}(\mathbb{T}^{n+m}_\theta)^{coH}$.

Then we can define, on projectable vector fields, the push-forward π_* as the $\mathcal{A}(\mathbb{T}^n_{\theta'})$-bimodule map $\pi_* : \mathfrak{X}(\mathbb{T}^{n+m}_\theta)^{coH} \to \mathfrak{X}(\mathbb{T}^n_{\theta'})$ defined by

$$\pi_*\left(\sum_{j=1}^{n+m} X^j \delta_j \right) = \sum_{j=1}^{n} X^j \delta_j.$$

Before going on and introducing a notion of horizontal lift of vector fields, we spend some words about the differential calculus and the relation between differential forms and vector fields.

In this whole chapter we will assume any noncommutative n-torus to be endowed with the first order differential calculus $\Omega^1(\mathcal{A}(\mathbb{T}^n_\theta))$ discussed in appendix A: it is described by the bimodule $\Omega^1(\mathcal{A}(\mathbb{T}^n_\theta)) = \mathcal{A}(\mathbb{T}^n_\theta)\otimes\mathbb{C}^n$, where the canonical basis $\{e^1,\dots,e^n\}$ of the second factor can be identified with the dual of the basis $\{\delta_1,\dots,\delta_n\}$ (see the discussion in appendix A; see also [C80]). Consequently we have a pairing $\langle e^i,\delta_j\rangle = e^i(\delta_j)=\delta_{ij}$. This can be extended to a bilinear pairing $\langle\cdot,\cdot\rangle : \Omega^1(\mathcal{A}(\mathbb{T}^n_\theta))\times\mathfrak{X}(\mathbb{T}^n_\theta)\to(\mathbb{T}^n_\theta)^\circ$ in the following way:

$$\left\langle \sum_i \eta_i\otimes e^i, \sum_j (a^j)^\circ\delta_j\right\rangle = \sum_{i,j}(\eta_i a^j)^\circ\left\langle e^i,\delta_j\right\rangle = \sum_i(\eta_i a^i)^\circ. \tag{D.2.3}$$

Let now $\omega:\mathcal{O}(\mathbb{T}^m)\to\Omega^1(\mathcal{A}(\mathbb{T}^{n+m}_\theta))$ be a $\mathbb{T}^m$ strong connection form. From proposition 4.3.7, we know that ω corresponds to a projection Π^ω on $\Omega^1(\mathcal{A}(\mathbb{T}^{n+m}_\theta))$, which identifies the space of vertical forms:

$$\Omega^1_{ver}(\mathcal{A}(\mathbb{T}^{n+m}_\theta)) = \{\eta\in\Omega^1(\mathcal{A}(\mathbb{T}^{n+m}_\theta))\mid \Pi^\omega(\eta)=\eta\}.$$

We can use this fact to introduce a notion of horizontal vector field and, then, to define the horizontal lift of vector fields.

Definition D.2.5. *A* horizontal vector field *for the $\mathbb{T}^m$-bundle $\mathbb{T}^{n+m}_\theta$ is a vector field $X\in\mathfrak{X}(\mathbb{T}^{n+m}_\theta)$ such that*

$$\langle\eta,X\rangle = 0$$

for any vertical form $\eta\in\Omega^1_{ver}(\mathcal{A}(\mathbb{T}^{n+m}_\theta))$, where $\langle\cdot,\cdot\rangle$ is the pairing defined in equation (D.2.3). *The linear space of horizontal vector fields will be denoted by $\mathfrak{X}_{hor}(\mathbb{T}^{n+m}_\theta)$.*

Definition D.2.6. *A* horizontal lift *of a vector field $X\in\mathfrak{X}(\mathbb{T}^n_{\theta'})$, with respect to a strong connection ω, is a horizontal projectable vector field $\widetilde{X}\in\mathfrak{X}_{hor}(\mathbb{T}^{n+m}_\theta)^{coH}$ such that $\pi_*\widetilde{X}=X$.*

Theorem D.2.7. *The horizontal lift of a generic vector $X\in\mathfrak{X}(\mathbb{T}^n_{\theta'})$ exists and is unique.*

Proof. First of all we work out the general expression of a vertical form $\eta\in\Omega^1_{ver}(\mathbb{T}^{n+m}_\theta)$. Let η be a 1-form, which we write as $\eta=\sum_{j=1}^{n+m}\eta_j\otimes e^j$. Then, for ω written as in proposition A.5.3, we can see, using the fact that $1\otimes e^j = U_j^* dU_j$, that

$$\Pi^\omega(\eta) = \sum_{i=1}^n\sum_{j=1}^m \eta_{j+n}b_{ji}\otimes e^i + \sum_{i=n+1}^{m+n}\eta_i\otimes e^i. \tag{D.2.4}$$

Imposing $\Pi^\omega(\eta)=\eta$ we obtain: $\eta_i=\sum_{j=1}^m\eta_{n+j}b_{ji}$ for $i=1,\dots,n$. Hence any vertical 1-form η can be written as in equation (D.2.4), and so it is completely determined by $\eta_{n+1},\dots,\eta_{n+m}$. Consider now a vector field $X\in\mathfrak{X}(\mathbb{T}^{n+m}_\theta)$. Imposing the horizontality condition implies that, for any η written as in equation (D.2.4), X must satisfy the following relation:

$$\sum_{i=1}^n\sum_{j=1}^m\eta_{n+j}b_{ji}X^i + \sum_{j=1}^m\eta_{n+j}X^{j+n} = 0$$

$$\Rightarrow \quad \sum_{j=1}^{m} \eta_{n+j} \left(\sum_{i=1}^{n} b_{ji} X^i + X^{j+n} \right) = 0.$$

This must hold for any choice of the η_j, and so X is horizontal if and only if

$$X^{j+n} = -\sum_{i=1}^{n} b_{ji} X^i \qquad \text{for } j = 1, \dots, m. \tag{D.2.5}$$

In particular, any horizontal vector field is completely determined by $X^1, \dots, X^n$. Now, take a vector field $Y \in \mathfrak{X}(\mathbb{T}^n_{\theta'})$, and write it as $Y = \sum_{j=1}^{n} (Y^j)^\circ \delta_j$ and assume that $\widetilde{Y} = (\widetilde{Y}^j)^\circ \delta_j$ is a horizontal lift of Y. We get immediately that $\widetilde{Y}^j = Y^j$ for $j = 1, \dots, n$, by definition of π_*. Furthermore, the elements $\widetilde{Y}^{n+j}$, for $j = 1, \dots, n$ are easily computed using equation (D.2.5):

$$\widetilde{Y}^{n+j} = -\sum_{i=1}^{n} b_{ji} Y^i.$$

Therefore the horizontal lift of Y exists and, since we have computed an explicit expression for it, it is also unique. □

Corollary D.2.8. *The horizontal lifts of the vector fields $\delta_1, \dots, \delta_n \in \mathfrak{X}(\mathbb{T}^n_{\theta'})$ are given by:*

$$\widetilde{\delta_j} = \delta_j - \sum_{i=1}^{m} b^\circ_{ij} \delta_{n+i} \qquad j = 1, \dots, n. \tag{D.2.6}$$

Let us consider now a noncommutative 3-torus as a $U(1)$-bundle over a noncommutative 2-torus. In this section we shall use the notation $\mathcal{A} = \mathcal{A}(\mathbb{T}^3_\theta)$, $\mathcal{B} = \mathcal{A}(\mathbb{T}^2_{\theta'})$, $H = \mathcal{O}(\mathbb{T}^3)$. The canonical spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ (cfr. appendix A) on $\mathcal{A}$ is a projectable triple, and the twisted Dirac operator associated to a hermitian strong connection ω is [DS13a]

$$D_\omega = \sum_{j=1}^{3} \sigma^j \delta_j - \sigma^1 J \omega_1 J^{-1} \delta_3 - \sigma^2 J \omega_2 J^{-1} \delta_3, \tag{D.2.7}$$

where ω is given by $\omega(z^k) = k\left(1 \otimes e^3 + \omega_1 \otimes e^1 + \omega_2 \otimes e^2\right)$, with $\omega_i = \omega_i^* \in \mathcal{B}$. Now, as we have seen above, the connection ω allows us to define the horizontal lift of any vector field over the base space $\mathbb{T}^2_{\theta'}$. In particular, we can consider the basis $\{\partial_1 = i\delta_1, \partial_2 = i\delta_2\}$ of $\mathfrak{X}(\mathcal{B})$ and take the horizontal lifts of ∂_1 and ∂_2. We obtain, using equation (D.2.5), the vectors

$$E_1 = \partial_1 - \omega_1^\circ \partial_3, \qquad E_2 = \partial_2 - \omega_2^\circ \partial_3.$$

We complete them to a basis $\{E_1, E_2, E_3\}$ of $\mathfrak{X}(\mathcal{A})$ by taking $E_3 = \partial_3 = i\delta_3$. In the commutative case [Amm98, AmmB98], one consider then the metric for which (the analogue of) $\{E_1, E_2, E_3\}$ is an orthonormal frame and compute the associated Dirac operator. We look for a similar result in the noncommutative case. The first step is the introduction of the analogue of the Levi-Civita connection. We begin with some recall of Riemannan geometry. It is well known (see, e.g., [Lee], theorem 5.4) that the Levi-Civita connection ∇ on a Riemannian manifold (M, g) can be defined through the Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) =& \partial_X(g(Y,Z)) + \partial_Y(g(X,Z)) - \partial_Z(g(X,Y)) \\ &+ g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X). \end{aligned} \tag{D.2.8}$$

Hence, if $\{E_i\}$ is a (local) orthonormal frame, then we obtain:

$$2g(\nabla_{E_i} E_j) = g([E_i,E_j],E_k) - g([E_i,E_k],E_j) - g([E_j,E_k],E_i). \tag{D.2.9}$$

Then, still in the commutative case, if we write (using Einstein convention) $[E_i, E_j] = c_{ij}^k E_k$, we can use equation (D.4.2) to compute the symbols Γ_{ij}^k, obtaining

$$\Gamma_{ij}^k = \frac{1}{2}\left(c_{ij}^k - c_{ik}^j - c_{jk}^i\right). \tag{D.2.10}$$

We would like to produce noncommutative Christoffel symbols in a similar way. But a problem arises: in the noncommutative case the commutator $[X, Y]$ of two vector fields is (in general) no longer a vector field. In particular, in our specific case we see that:

$$[E_1, E_2] = (\delta_2\omega_1^\circ - \delta_1\omega_2^\circ)\delta_3 + [\omega_1^\circ, \omega_2^\circ]\delta_3^2.$$

Hence, in order to go on along this way, we need to overcome this issue. There could be many different ways to achieve this scope. Here we choose to use the so-called $\star$-product formalism [ABDMSW05, A06, ADMW06, A07, A09], which allows to define a "twisted" commutator $[\cdot,\cdot]_\star$ which preserves the space of vector fields. In the next section we shall briefly recall the main aspects of this formalism and we will apply them to noncommutative tori. Later, we will use these results to work out Levi-Civita and spin connections and to discuss torsion and curvature.

D.3 $\star$-geometries and noncommutative tori

By $\star$-geometries we mean a wide class of noncommutative manifolds, obtained as deformations of the algebras of functions over smooth manifolds via $\star$-products. Following Aschieri et al. [ABDMSW05, A06, ADMW06], we consider here $\star$-products associated with a deformation of the algebra of smooth functions over a manifold M obtained using a twist $\mathcal{F}$ of the Lie algebra of infinitesimal diffeomorphisms of M. In particular, the twists we consider are elements $\mathcal{F}$ of $U\mathfrak{X}[[\lambda]]\otimes U\mathfrak{X}[[\lambda]]$, where $\mathfrak{X}$ is the Lie algebra of vector fields over M, $U\mathfrak{X}$ is its universal enveloping algebra, and $U\mathfrak{X}[[\lambda]]$ denotes the algebra of formal power series in λ. We will not discuss most of the algebraic aspects of Lie algebras' twists; for all the details we refer to classical literature [Dri83, Dri90, Res90, Kassel]. Here we will briefly recall the construction of the $\star$-product and the main properties of $\star$-geometries. Then, we will apply these results to the specific case of noncommutative tori.

Consider a smooth manifold M, and let $\mathfrak{X} = \mathfrak{X}(M)$ be the space of smooth vector fields; that is, the space of smooth sections of the tangent bundle TM. $\mathfrak{X}$ is a Lie algebra, with the commutator given by the Lie derivative: $[X, Y] = \mathcal{L}_X Y$. Hence we can consider its universal enveloping algebra $U\mathfrak{X}$. We know [Sw69, Maj95, Kassel] that $U\mathfrak{X}$ admits a Hopf algebra structure, where coproduct, counit and antipode are defined by:

$$\begin{aligned} \delta(X) &= X \otimes 1 + 1 \otimes X, & \Delta(1) &= 1 \otimes 1, \\ S(X) &= -X, & S(1) &= 1, \\ \varepsilon(X) &= 0, & \varepsilon(1) &= 1, \end{aligned}$$

for any $X \in \mathfrak{X}$. From $U\mathfrak{X}$ we can obtain the algebra $U\mathfrak{X}[[\lambda]]$ of formal power series in λ, which is still a Hopf algebra. Then we can give the following definition.

Definition D.3.1. *A* twist *$\mathcal{F}$ for the Lie algebra $\mathfrak{X}$ is an invertible element $\mathcal{F} \in U\mathfrak{X}[[\lambda]] \otimes U\mathfrak{X}[[\lambda]]$ which satisfies the following relations:*

$$\begin{aligned} (\mathcal{F} \otimes 1) \cdot (\Delta \otimes \mathrm{id})\mathcal{F} &= (1 \otimes \mathcal{F}) \cdot (\mathrm{id} \otimes \mathcal{F})\mathcal{F}, \\ (\varepsilon \otimes \mathrm{id})\mathcal{F} &= (\mathrm{id} \otimes \varepsilon)\mathcal{F} = 1. \end{aligned} \tag{D.3.1}$$

Moreover[4]*, we require $\mathcal{F} = 1 \otimes 1 + O(\lambda)$.*

In what follows we shall denote, with an abuse of notation, the algebra $U\mathfrak{X}[[\lambda]]$ simply by $U\mathfrak{X}$. Now let A denote the algebra $C^\infty(M)[[\lambda]]$. Then a twist $\mathcal{F}$ can be seen as a map $\mathcal{F} : A \otimes A \to A \otimes A$. Hence we can consider the following operation on A.

Definition D.3.2. *The* $\star$-product *of two functions $g, h \in A$ is the function $g \star h = m_A(\mathcal{F}^{-1}(g \otimes h))$, where $m_A : A \otimes A \to A$ is the multiplication map.*

We introduce now the following notation: we write

$$\mathcal{F} = f^\alpha \otimes f_\alpha, \qquad \mathcal{F}^{-1} = \overline{f}^\alpha \otimes \overline{f}_\alpha,$$

where the sum over α is understood and the elements $f^\alpha, f_\alpha, \overline{f}^\alpha, \overline{f}_\alpha$ belong to $U\mathfrak{X}$. Then the $\star$-product can be written in the following form:

$$g \star h = \overline{f}^\alpha(g)\overline{f}_\alpha(h). \tag{D.3.2}$$

Lemma D.3.3. *The $\star$-product is $\mathbb{C}$-linear, associative and, if A is unital, $1 \in A$ is the unit element for $\star$.*

We shall denote by $A_\star$ the algebra which has A as underlying space and $\star$ as product. Also the algebra $U\mathfrak{X}$ can be twisted, and there are different ways to do this: we can twist the Hopf algebra structure, the associative algebra structure or both of them. We shall present here only a part of this possibilities, for a complete discussion see, e.g., [ADMW06, Dri90, Maj95]. Consider the element $\mathcal{F}_{21} \in U\mathfrak{X} \otimes U\mathfrak{X}$ defined by $\mathcal{F}_{21} = f_\alpha \otimes f^\alpha$. Then we can introduce the so called *universal $\mathcal{R}$-matrix*

$$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}.$$

[4] Actually this condition can be recovered from (D.3.1), see [ADMW06].

$\mathcal{R}$ is an invertible element of $U\mathfrak{X} \otimes U\mathfrak{X}$. For the other properties of $\mathcal{R}$ we refer to [ADMW06]. We introduce also the following notation:

$$\mathcal{R} = R^\alpha \otimes R_\alpha, \qquad \mathcal{R}^{-1} = \overline{R}^\alpha \otimes \overline{R}_\alpha.$$

Lemma D.3.4. *For any $g, h \in A$, $g \star h = \overline{R}^\alpha(h) \star \overline{R}_\alpha(g)$.*

Now let us define a $\star$-product also for the (Hopf) algebra $U\mathfrak{X}$. We take it to be the map $\star : U\mathfrak{X} \otimes U\mathfrak{X} \to U\mathfrak{X}$ defined by

$$X \star Y \equiv \mathcal{L}_{\overline{f}^\alpha}(X)\mathcal{L}_{\overline{f}_\alpha}(Y) \equiv \overline{f}^\alpha(X)\overline{f}_\alpha(Y). \tag{D.3.3}$$

We define $U\mathfrak{X}_\star$ to be the associative algebra which has $U\mathfrak{X}$ as underlying vector space and $\star$, defined by equation (D.3.3), as multiplication map. We shall see in a few that $U\mathfrak{X}_\star$ can be made into a Hopf algebra. Before doing this, we introduce also the deformed commutator of any two generators of $U\mathfrak{X}$: for $X, Y \in \mathfrak{X}$ we set

$$[X, Y]_\star = X \star Y - \overline{R}^\alpha(Y) \star \overline{R}_\alpha(X). \tag{D.3.4}$$

Proposition D.3.5. *$[\cdot,\cdot]_\star$ is a (bilinear) map $\mathfrak{X} \otimes \mathfrak{X} \to \mathfrak{X}$. That is, the twisted commutator (or $\star$-commutator) of two vector fields is again a vector field.*

The space $\mathfrak{X}_\star$, endowed with the commutator $[\cdot,\cdot]_\star$, is a deformed Lie algebra. It is a left $A_\star$-module in the natural way: $g \star X = \overline{f}_\alpha(g)\overline{f}_\alpha(X)$. Also, we can state the twisted analogues of the antisymmetry property and of the Jacoby identity for the $\star$-commutator.

Lemma D.3.6. *Let $X, Y, Z \in \mathfrak{X}$. Then:*
(i) $[X, Y]_\star = -[\overline{R}^\alpha(Y), \overline{R}_\alpha(X)]_\star$,
(ii) $[X, [Y, Z]_\star]_\star = [[X, Y]_\star, Z]_\star + [\overline{R}^\alpha(Y), [\overline{R}_\alpha(X), Z]_\star]_\star$.

Proof. See, e.g., [ADMW06], section 3.2 and appendix A.2. □

We can also define a deformed version of the Lie derivative of a function $f \in A$ with respect to a vector fields $X \in \mathfrak{X}$; we take it to be [A07]:

$$\mathcal{L}^\star_X(g) = \overline{f}^\alpha(X)\overline{f}_\alpha(g). \tag{D.3.5}$$

$\mathcal{L}^\star$ fulfils the twisted versions of the usual properties of a Lie derivative:

Lemma D.3.7. *Let $g, h \in A$ and let $X \in \mathfrak{X}$. Then:*
(i) $\mathcal{L}^\star_{g\star X}(h) = g \star \mathcal{L}^\star_X(h)$,
(ii) $\mathcal{L}^\star_X(g \star h) = \mathcal{L}^\star_X(g) \star h + \overline{R}^\alpha(g)\mathcal{L}^\star_{\overline{R}_\alpha(X)}(h)$ (deformed Leibniz rule).

Proof. See [A07], section 4. □

It is possible to introduce also the deformed space of differential forms [ADMW06, A07], but we shall not use it anywhere in this thesis, so we skip this part. Instead, now we consider a noncommutative n-torus, we show that it is possible to see it as a ⋆-deformation of the smooth torus $\mathbb{T}^n$ and we use this fact to build the deformed Lie algebra of vector fields over $\mathbb{T}^n_\theta$ we were looking for. So, let $C^\infty(\mathbb{T}^n_\theta)$ denote the smooth algebra of a noncommutative n-torus, which we assume to be generated by n unitaries $U_1, \ldots, U_n$, with the commmutation relations $U_iU_j = e^{-i\theta_{ij}}U_jU_i$. We can define a bijective $\mathbb{C}$-linear map $\varphi : C^\infty(\mathbb{T}^n_\theta) \to C^\infty(\mathbb{T}^n)$. First of all we introduce the *Weyl symbols* $W(k)$, for $k \in \mathbb{Z}^n$:

$$W(k) = e^{\frac{i}{2}\sum_{i<j} k_i\theta_{ij}k_j} U_1^{k_1} \cdots U_n^{k_n}.$$

Then we set $\varphi(W(k)) = (2\pi)^{-n/2} e^{i\sum_j k_jx_j}$, where $x_1, \ldots, x_n$ are the canonical angular coordinates on $\mathbb{T}^n \simeq S^1 \times \ldots S^1$. Since the product rule of $C^\infty(\mathbb{T}^n_\theta)$ reads

$$W(k)W(h) = e^{\frac{i}{2}\sum_{ij} k_i\theta_{ij}h_j} W(k+h),$$

the following relation holds:

$$\varphi(W(k)W(h)) = e^{\frac{i}{2}\sum_{ij}\theta_{ij}\delta^x_i\delta^y_j}\varphi(W(k))(x)\varphi(W(h))(y)\Big|_{x=y},$$

where $\delta^x_i = -i\frac{\partial}{\partial x_i}$. This means that $C^\infty(\mathbb{T}^n_\theta)$ can be identified with the twisted algebra $C^\infty(\mathbb{T}^n)_\star$, where the ⋆-product is the following one:

$$(g \star h)(x) = e^{\frac{i}{2}\sum_{ij}\theta_{ij}\delta^x_i\delta^y_j} g(x)h(y)\Big|_{x=y}. \tag{D.3.6}$$

This corresponds to the ⋆-product induced by the following twist:

$$\mathcal{F} = e^{-\frac{i}{2}\sum_{ij}\theta_{ij}\delta^x_i\otimes\delta^y_j} \tag{D.3.7}$$

whose inverse is simply given by:

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\sum_{ij}\theta_{ij}\delta^x_i\otimes\delta^y_j}. \tag{D.3.8}$$

We shall use the notation introduced above, $\mathcal{F} = f^\alpha \otimes f_\alpha$, etc., but only as a formal tool: we are not considering $\mathcal{F}$ as a power series in θ. We just find that writing expressions in this way makes the computations more clear.

Now we come to vector fields. First of all, we notice that the derivations δ_j fulfil the following relation: $\varphi(\delta_j(A))(x) = \delta^x_j\varphi(A)(x)$, for any $A \in C^\infty(\mathbb{T}^n_\theta)$. Then we can identify δ_j with δ^x_j. Hence the space of ⋆-vector fields $\mathfrak{X}_\star$ can be identified[5] with the space $\mathfrak{X}(C^\infty(\mathbb{T}^n_\theta))$ introduced in the previous section. But now it is endowed with the ⋆-commutator $[\cdot,\cdot]_\star$, which, therefore, can be seen as a bilinear map $\mathfrak{X}(C^\infty(\mathbb{T}^n_\theta)) \otimes \mathfrak{X}(C^\infty(\mathbb{T}^n_\theta)) \to \mathfrak{X}(C^\infty(\mathbb{T}^n_\theta))$.

Remark D.3.8. If we restrict all the maps to $\mathbb{T}^n_\theta \subset C^\infty(T^n_\theta)$, we obtain a ⋆-commutator $[\cdot,\cdot]_\star$ on $\mathfrak{X}(\mathbb{T}^n_\theta)$. Moreover, all the results in this section hold also for the opposite algebra $(\mathbb{T}^n_\theta)^\circ$.

[5]Notice that they are isomorphic as left $C^\infty(\mathbb{T}^n_\theta) \simeq C^\infty(\mathbb{T}^n)_\star$-modules.

D.4 Connections, torsion and curvature on $\mathbb{T}^3_\theta$

In the previous section we have introduced the notion of deformed space $\mathfrak{X}_\star$ of vector fields over a smooth manifold M. Now, following [ADMW06, A06, A07], we give the definition of covariant derivative of deformed vector fields.

Definition D.4.1. *A* $\star$-covariant derivative *in the space* $\mathfrak{X}_\star$ *of vector fields over* $A_\star$*, along the vector fields* $X \in \mathfrak{X}_\star$*, is a linear map* $\nabla^\star_X : \mathfrak{X}_\star \to \mathfrak{X}_\star$ *such that:*
(i) $\nabla^\star_{X+Y} Z = \nabla^\star_X Z + \nabla^\star_Y Z$,
(ii) $\nabla^\star_{f\star X} Y = f \star \nabla^\star_X Y$,
(iii) $\nabla^\star_X (f \star Y) = \mathcal{L}^\star_X(f) \star Y + \overline{R}^\alpha(f) \star \nabla^\star_{\overline{R}_\alpha(X)} Y$,
for any $Y, Z \in \mathfrak{X}_\star$ *and any* $f \in A_\star$*. A map* $\nabla^\star : \mathfrak{X}_\star \otimes \mathfrak{X}_\star \to \mathfrak{X}_\star$ *defined by* $(X,Y) \mapsto \nabla^\star_X Y$*, where each* $\nabla^\star_X$ *is a* $\star$*-covariant derivative, will be called a* $\star$-connection*.*

Given a $\star$-connection $\nabla^\star$ on $\mathfrak{X}_\star$ we can define its curvature and its torsion.

Definition D.4.2. *The* curvature R *of a* $\star$*-connection* $\nabla^\star$ *is the* $\mathbb{C}$*-linear map* $R : \mathfrak{X}_\star \otimes \mathfrak{X}_\star \otimes \mathfrak{X}_\star \to \mathfrak{X}_\star$ *defined by:*

$$R(X,Y,Z) = \nabla^\star_X \nabla^\star_Y Z - \nabla^\star_{\overline{R}^\alpha(Y)} \nabla^\star_{\overline{R}_\alpha(X)} Z - \nabla^\star_{[X,Y]_\star} Z$$

for any $X, Y, Z \in \mathfrak{X}_\star$*.*

Definition D.4.3. *The* curvature T *of a* $\star$*-connection* $\nabla^\star$ *is the* $\mathbb{C}$*-linear map* $T : \mathfrak{X}_\star \otimes \mathfrak{X}_\star \to \mathfrak{X}_\star$ *defined by:*

$$T(X,Y) = \nabla^\star_X Y - \nabla^\star_{\overline{R}^\alpha(Y)} \overline{R}_\alpha(X) - [X,Y]_\star$$

for any $X, Y \in \mathfrak{X}_\star$*.*

Both the torsion and the curvature of a $\star$-connection are $\star$-antisymmetric. More precisely [ADMW06],

Lemma D.4.4. *Let* $\nabla^\star$ *be a* $\star$*-connection and let* T *and* R *be, respectively, its torsion and its curvature. Then,*

$$T(X,Y) = -T(\overline{R}^\alpha(Y), \overline{R}_\alpha(X)),$$

$$R(X,Y,Z) = -R(\overline{R}^\alpha(Y), \overline{R}_\alpha(X), Z),$$

for any $X, Y, Z \in \mathfrak{X}_\star$*.*

Moreover, both of them fulfil the following properties of $A_\star$-linearity [ADMW06].

Lemma D.4.5. *Let* $\nabla^\star$ *be a* $\star$*-connection and let* T *and* R *be, respectively, its torsion and its curvature. Then,*

$$T(f \star X, Y) = f \star T(X,Y),$$

$$T(X \star f, Y) = \overline{R}^\alpha(f) \star T(\overline{R}_\alpha(X), Y),$$

$$R(f \star X, Y, Z) = f \star R(X,Y,Z),$$

$$R(X \star f, Y, Z) = \overline{R}^\alpha(f) \star R(\overline{R}_\alpha(X), Y, Z),$$

for any $X, Y, Z \in \mathfrak{X}_\star$ *and any* $f \in A_\star$.

Now assume that the space $\mathfrak{X}_\star$ admits global frames. That is, there are $E_1, \ldots, E_n \in X_\star$ such that any other vector field $X \in X_\star$ can be written as $X = \sum_j x^j \star E_j$, with $x^j \in A_\star$. Then we can introduce the following description of a $\star$-connection and of its torsion and its curvature. We set (using Einstein convention):

$$\nabla^\star_{E_i} E_j = \Gamma^k_{ij} \star E_k,$$

$$T(E_i, E_j) = T^k_{ij} \star E_k,$$

$$R(E_i, E_j, E_k) = R^l_{ijk} \star E_l,$$

with $\Gamma^k_{ij}, T^k_{ij}, R^l_{ijk} \in A_\star$ for any $i, j, k = 1, \ldots, n$. This notation is relevant, in particular, when we assume (or we interpret) $E_1, \ldots, E_n$ as an orthonormal frame.

We consider now the noncommutative 3-torus $\mathbb{T}^3_\theta$, seen as a quantum principal $U(1)$-bundle over $\mathbb{T}^2_\theta$, together with the reference frame $\{E_1, E_2, E_3\}$ discussed in the first part of this chapter. We recall that E_3 is nothing else than the Killing vector field associated to the $U(1)$ action and E_1, E_2 are the horizontal lifts of the canonical orthonormal frame $f_i = \partial_i$ on the flat noncommutative 2-torus. We can work out a set of Christoffel symbols associated to this frame. Indeed, on a smooth Riemannian manifold (M, g), the Koszul formula reads:

$$\begin{aligned} 2g(\nabla_X Y, Z) =& \partial_X(g(Y, Z)) + \partial_Y(g(X, Z)) - \partial_Z(g(X, Y)) \\ &+ g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X). \end{aligned} \tag{D.4.1}$$

Hence, if $\{E_i\}$ is a (local) orthonormal frame, then we obtain:

$$2g(\nabla_{E_i} E_j) = g([E_i, E_j], E_k) - g([E_i, E_k], E_j) - g([E_j, E_k], E_i). \tag{D.4.2}$$

If now we set (using the Einstein convention)

$$\nabla_{E_i} E_j = \Gamma^k_{ij}, \qquad [E_i, E_j] = c^k_{ij} E_k,$$

we can use equation (D.4.2) to compute the symbols Γ^k_{ij}, obtaining

$$\Gamma^k_{ij} = \frac{1}{2}\left(c^k_{ij} - c^j_{ik} - c^i_{jk}\right). \tag{D.4.3}$$

In the noncommutative case, therefore, we can use (D.4.3) to define the Christoffel symbols associated to the frame $\{E_1, E_2, E_3\}$. Indeed, we can define the coefficients c^k_{ij} using the $\star$-commutator:

$$[e_i, e_j]_\star = c^k_{ij} \star e_k.$$

Then, by direct computation, we can see that the only nonzero Γ_{ij}^k are the following ones:

$$\begin{aligned} \Gamma_{12}^3 &= \frac{1}{2}\omega_{12}^\circ, & \Gamma_{21}^3 &= -\frac{1}{2}\omega_{12}^\circ, \\ \Gamma_{13}^2 &= -\frac{1}{2}\omega_{12}^\circ, & \Gamma_{31}^2 &= -\frac{1}{2}\omega_{12}^\circ, \\ \Gamma_{23}^1 &= \frac{1}{2}\omega_{12}^\circ, & \Gamma_{23}^1 &= \frac{1}{2}\omega_{12}^\circ, \end{aligned} \tag{D.4.4}$$

where $\omega_{12}^\circ = \partial_2\omega_1^\circ - \partial_1\omega_2^\circ$. These symbols define a $\star$-connection $\nabla^\star$, which we may see as a map $\nabla^\star : \mathfrak{X}(\mathbb{T}_\theta^3) \otimes \mathfrak{X}(\mathbb{T}_\theta^3) \to \mathfrak{X}(\mathbb{T}_\theta^3)$. We can compute both the torsion and the curvature of $\nabla^\star$, using the formulae of definition D.4.3 and D.4.2. Before this, it is useful to compute the expressions of $\nabla_{\partial_i}^\star \partial_j$. We will use the fact that the vector fields ∂_i can be expressed as follows,

$$\partial_1 = E_1 + \omega_1^\circ \star E_3, \qquad \partial_2 = E_2 + \omega_2^\circ \star E_3, \qquad \partial_3 = E_3,$$

and the properties of a $\star$-covariant derivative (see definition D.4.1). We obtain:

$$\nabla_{\partial_1}^\star \partial_3 = \nabla_{E_1+\omega_1^\circ\star E_3}^\star E_3 = \nabla_{E_1}^\star E_3 + \omega_1^\circ \star \nabla_{E_3}^\star E_3 = \Gamma_{13}^2 \star E_2, \tag{D.4.5a}$$

$$\nabla_{\partial_2}^\star \partial_3 = \nabla_{E_2+\omega_2^\circ\star E_3}^\star E_3 = \nabla_{E_2}^\star E_3 + \omega_2^\circ \star \nabla_{E_3}^\star E_3 = \Gamma_{23}^1 \star E_1, \tag{D.4.5b}$$

$$\nabla_{\partial_3}^\star \partial_1 = \nabla_{E_3}^\star (E_1 + \omega_1^\circ \star E_3) = \nabla_{E_3}^\star E_1 = \Gamma_{31}^2 \star E_2, \tag{D.4.5c}$$

$$\nabla_{\partial_3}^\star \partial_2 = \nabla_{E_3}^\star (E_2 + \omega_2^\circ \star E_3) = \nabla_{E_3}^\star E_2 = \Gamma_{32}^1 \star E_1, \tag{D.4.5d}$$

$$\nabla_{\partial_3}^\star \partial_3 = \nabla_{E_3}^\star E_3 = 0, \tag{D.4.5e}$$

$$\begin{aligned} \nabla_{\partial_1}^\star \partial_1 &= \nabla_{E_1+\omega_1^\circ\star E_3}^\star (E_1 + \omega_1^\circ \star E_3) = \\ &= \nabla_{E_1}^\star E_1 + \nabla_{E_1}^\star (\omega_1^\circ \star E_3) + \omega_1^\circ \star \nabla_{E_3}^\star E_1 + \omega_1^\circ \star \nabla_{E_3}^\star (\omega_1^\circ \star E_3) \\ &= \mathcal{L}_{E_1}^\star (\omega_1^\circ) \star E_3 + \overline{R}^\alpha (\omega_1^\circ) \star \nabla_{\overline{R}_\alpha(E_1)}^\star E_3 + \omega_1^\circ \star \Gamma_{31}^2 \star E_2 \\ &= \partial_1(\omega_1^\circ) \star E_3 + \omega_1^\circ \star \Gamma_{13}^2 \star E_2 + \omega_1^\circ \star \Gamma_{31}^2 \star E_2, \end{aligned} \tag{D.4.5f}$$

$$\begin{aligned} \nabla_{\partial_1}^\star \partial_2 &= \nabla_{E_1+\omega_1^\circ\star E_3}^\star (E_2 + \omega_2^\circ \star E_3) = \\ &= \nabla_{E_1}^\star E_2 + \nabla_{E_1}^\star (\omega_2^\circ \star E_3) + \omega_1^\circ \star \nabla_{E_3}^\star E_2 + \omega_1^\circ \star \nabla_{E_3}^\star (\omega_2^\circ \star E_3) \\ &= \Gamma_{12}^3 \star E_3 + \mathcal{L}_{E_1}^\star (\omega_2^\circ) \star E_3 + \overline{R}^\alpha (\omega_2^\circ) \star \nabla_{\overline{R}_\alpha(E_1)}^\star E_3 + \omega_1^\circ \star \Gamma_{32}^1 \star E_1 \\ &= \partial_1(\omega_2^\circ) \star E_3 + \Gamma_{12}^3 \star E_3 + \omega_2^\circ \star \Gamma_{13}^2 \star E_2 + \omega_1^\circ \star \Gamma_{32}^1 \star E_1, \end{aligned} \tag{D.4.5g}$$

$$\begin{aligned} \nabla_{\partial_2}^\star \partial_1 &= \nabla_{E_2+\omega_2^\circ\star E_3}^\star (E_1 + \omega_1^\circ \star E_3) = \\ &= \nabla_{E_2}^\star E_1 + \nabla_{E_2}^\star (\omega_1^\circ \star E_3) + \omega_2^\circ \star \nabla_{E_3}^\star E_1 + \omega_2^\circ \star \nabla_{E_3}^\star (\omega_1^\circ \star E_3) \\ &= \Gamma_{21}^3 \star E_3 + \mathcal{L}_{E_2}^\star (\omega_1^\circ) \star E_3 + \overline{R}^\alpha (\omega_1^\circ) \star \nabla_{\overline{R}_\alpha(E_2)}^\star E_3 + \omega_2^\circ \star \Gamma_{31}^2 \star E_2 \\ &= \partial_2(\omega_1^\circ) \star E_3 + \Gamma_{21}^3 \star E_3 + \omega_1^\circ \star \Gamma_{23}^1 \star E_1 + \omega_2^\circ \star \Gamma_{31}^2 \star E_2, \end{aligned} \tag{D.4.5h}$$

$$\begin{aligned}\nabla^\star_{\partial_2}\partial_2 &= \nabla^\star_{E_2+\omega_2^\circ\star E_3}(E_2+\omega_2^\circ\star E_3) = \\ &= \nabla^\star_{E_2}E_2 + \nabla^\star_{E_2}(\omega_2^\circ\star E_3) + \omega_2^\circ\star\nabla^\star_{E_3}E_2 + \omega_1^\circ\star\nabla^\star_{E_3}(\omega_2^\circ\star E_3) \\ &= \mathcal{L}^\star_{E_2}(\omega_2^\circ)\star E_3 + \overline{R}^\alpha(\omega_2^\circ)\star\nabla^\star_{\overline{R}_\alpha(E_2)}E_3 + \omega_2^\circ\star\Gamma^1_{32}\star E_1 \\ &= \partial_2(\omega_2^\circ)\star E_3 + \omega_2^\circ\star\Gamma^1_{23}\star E_2 + \omega_2^\circ\star\Gamma^1_{32}\star E_1.\end{aligned} \tag{D.4.5i}$$

Now we are ready to compute the torsion of $\nabla^\star$. From lemma D.4.4 we know that $T(E_i,E_i)=0$ for any $i=1,2,3$. Next, by direct computation we get the following expressions:

$$\begin{aligned}T(E_1,E_2) &= \nabla^\star_{E_1}E_2 - \nabla^\star_{\overline{R}_\alpha(E_2)}E_1 - [E_1,E_2]_\star = \\ &= \Gamma^3_{12}\star E_3 - \nabla^\star_{\partial_2}\partial_1 + \nabla^\star_{\partial_2}(\omega_1\star\partial_3) + \omega_2\star\nabla^\star_{\partial_3}\partial_1 \\ &\quad - \nabla^\star_{\overline{R}^\alpha(\omega_2)\star\partial_3}(\overline{R}_\alpha(\omega_1)\star\partial_3) - 2\Gamma^3_{12}\star E_3 \\ &= -\Gamma^3_{12}\star E_3 - \Gamma^3_{21}\star E_3 = -\frac{1}{2}\omega_{12}^\circ\star E_3 + \frac{1}{2}\omega_{12}^\circ\star E_3 = 0\end{aligned} \tag{D.4.6a}$$

$$\begin{aligned}T(E_1,E_3) &= \nabla^\star_{E_1}E_3 - \nabla^\star_{\overline{R}_\alpha(E_3)}E_1 - [E_1,E_3]_\star = \nabla^\star_{E_1}E_3 - \nabla^\star_{E_3}E_1 \\ &= \Gamma^2_{13}\star E_2 - \Gamma^2_{31}\star E_2 = -\frac{1}{2}\omega_{12}^\circ\star E_2 + \frac{1}{2}\omega_{12}^\circ\star E_2 = 0\end{aligned} \tag{D.4.6b}$$

$$\begin{aligned}T(E_2,E_3) &= \nabla^\star_{E_2}E_3 - \nabla^\star_{\overline{R}_\alpha(E_3)}E_2 - [E_2,E_3]_\star = \nabla^\star_{E_2}E_3 - \nabla^\star_{E_3}E_2 \\ &= \Gamma^1_{23}\star E_1 - \Gamma^1_{32}\star E_1 = \frac{1}{2}\omega_{12}^\circ\star E_1 - \frac{1}{2}\omega_{12}^\circ\star E_1 = 0\end{aligned} \tag{D.4.6c}$$

In the same way (or using the properties of T, see lemma D.4.4) one can see that also $T(E_2,E_1)$, $T(E_3,E_1)$ and $T(E_3,E_2)$ are zero. This, together with the linearity properties of the torsion (see lemma D.4.5), implies that $T=0$. Hence $\nabla^\star$ is a *torsionless* $\star$-connection. The next step is the computation of the curvature of $\nabla^\star$. Performing it in the same way we did for the torsion, we obtain the following results.

$$R(E_i,E_i,E_j) = 0 \qquad \forall i,j=1,2,3, \tag{D.4.7a}$$

$$R(E_1,E_2,E_1) = -\frac{1}{2}\partial_1(\omega_{12}^\circ)\star E_3 + \frac{3}{4}(\omega_{12}^\circ)^2\star E_2, \tag{D.4.7b}$$

$$R(E_1,E_2,E_2) = -\frac{1}{2}\partial_2(\omega_{12}^\circ)\star E_3 - \frac{3}{4}(\omega_{12}^\circ)^2\star E_1, \tag{D.4.7c}$$

$$\begin{aligned}R(E_1,E_2,E_3) &= \frac{1}{2}\partial_1(\omega_{12}^\circ)\star E_1 + \frac{1}{2}\partial_2(\omega_{12}^\circ)\star E_2 \\ &\quad - \frac{1}{4}\omega_{12}^\circ\star\omega_1^\circ\star\omega_{12}^\circ\star E_2 + \frac{1}{4}\omega_1^\circ\star(\omega_{12}^\circ)^2\star E_2 \\ &\quad + \frac{1}{4}\omega_{12}^\circ\star\omega_2^\circ\star\omega_{12}^\circ\star E_1 - \frac{1}{4}\omega_2^\circ\star(\omega_{12}^\circ)^2\star E_1,\end{aligned} \tag{D.4.7d}$$

$$\begin{aligned}R(E_1,E_3,E_1) &= -\frac{1}{2}\partial_1(\omega_{12}^\circ)\star E_2 - \frac{1}{4}(\omega_{12}^\circ)^2\star E_3 \\ &\quad + \frac{1}{4}\omega_1^\circ\star(\omega_{12}^\circ)^2\star E_1 - \frac{1}{4}\omega_{12}^\circ\star\omega_1\star\omega_{12}^\circ\star E_1,\end{aligned} \tag{D.4.7e}$$

$$R(E_1,E_3,E_2) = \frac{1}{2}\partial_1(\omega_{12}^\circ)\star E_1 + \frac{1}{4}\omega_1^\circ\star(\omega_{12}^\circ)^2\star E_2 - \frac{1}{4}\omega_{12}^\circ\star\omega_1^\circ\star\omega_{12}^\circ\star E_2, \tag{D.4.7f}$$

$$R(E_1, E_3, E_3) = \frac{1}{4}(\omega^\circ_{12})^2 \star E_1, \tag{D.4.7g}$$

$$R(E_2, E_1, E_1) = \frac{1}{2}\partial_1(\omega^\circ_{12}) \star E_3 - \frac{3}{4}(\omega^\circ_{12})^2 \star E_2, \tag{D.4.7h}$$

$$R(E_2, E_1, E_2) = \frac{1}{2}\partial_2(\omega^\circ_{12}) \star E_3 + \frac{3}{4}(\omega^\circ_{12})^2 \star E_1, \tag{D.4.7i}$$

$$\begin{aligned} R(E_2, E_1, E_3) = &-\frac{1}{2}\partial_2(\omega^\circ_{12}) \star E_2 - \frac{1}{2}\partial_1(\omega^\circ_{12}) \star E_1 \\ &- \frac{1}{4}\omega^\circ_{12} \star \omega^\circ_2 \star \omega^\circ_{12} \star E_1 + \frac{1}{4}\omega^\circ_2 \star (\omega^\circ_{12})^2 \star E_1 \\ &+ \frac{1}{4}\omega^\circ_{12} \star \omega^\circ_1 \star \omega^\circ_{12} \star E_2 - \frac{1}{4}\omega^\circ_1 \star (\omega^\circ_{12})^2 \star E_2, \end{aligned} \tag{D.4.7j}$$

$$R(E_2, E_3, E_1) = -\frac{1}{2}\partial_2(\omega^\circ_{12}) \star E_2 + \frac{1}{4}\omega^\circ_2 \star (\omega^\circ_{12})^2 \star E_1 - \frac{1}{4}\omega^\circ_{12} \star \omega^\circ_2 \star \omega^\circ_{12} \star E_1, \tag{D.4.7k}$$

$$\begin{aligned} R(E_2, E_3, E_2) = &\frac{1}{2}\partial_2(\omega^\circ_{12}) \star E_1 - \frac{1}{4}(\omega^\circ_{12})^2 \star E_3 \\ &+ \frac{1}{4}\omega^\circ_2 \star (\omega^\circ_{12})^2 \star E_2 - \frac{1}{4}\omega^\circ_{12} \star \omega_2 \star \omega^\circ_{12} \star E_2, \end{aligned} \tag{D.4.7l}$$

$$R(E_2, E_3, E_3) = \frac{1}{4}(\omega^\circ_{12})^2 \star E_2, \tag{D.4.7m}$$

$$R(E_3, E_i, E_j) = -R(E_i, E_3, E_j) \qquad \forall i, j, \tag{D.4.7n}$$

where the last relation was obtained using the $\star$-antisymmetry property of the curvature of a $\star$-connection (see lemma D.4.4). Now we write the curvature of $\nabla^\star$ in the following way:

$$R(E_i, E_j, E_k) = \sum_l R_{ijkl} \star E_l.$$

The symbols R_{ijkl} can be easily read directly from equations (D.4.7a) - (D.4.7n). We can make a step further, introducing the analogue of the Ricci tensor. We set:

$$R_{ij} = \sum_k R_{kijk}.$$

We can then take the "trace" of R_{ij}: we define the *Ricci $\star$-curvature* to be $R = \sum_i R_{ii}$. We find:

$$R = R_{11} + R_{22} + R_{33} = -\frac{1}{2}(\omega^\circ_{12})^2 - \frac{1}{2}(\omega^\circ_{12})^2 + \frac{1}{2}(\omega^\circ_{12})^2 = -\frac{1}{2}(\omega^\circ_{12})^2. \tag{D.4.8}$$

Hence we can give the following interpretation of our construction: the $\star$-connection $\nabla^\star$ can be seen as a noncommutative analogue of the Levi-Civita connection, and it corresponds to a metric with scalar curvature equal to $-\frac{1}{2}(\omega^\circ_{12})^2$. In particular, if $\omega_{12} \neq 0$ we obtain a curved Riemannian structure over the noncommutative 3-torus.

D.5 Spin Laplacian, twisted Dirac operator and Lichnerowicz formula

The next step is to study the construction discussed in the previous sections in the framework of Connes' noncommutative geometry. That is, we need to associate a spectral triple ($\mathcal{A} = \mathcal{A}(\mathbb{T}^3_\theta), \mathcal{H}, \mathcal{D}$) to the $\star$-connection $\nabla^\star$. This is what we shall do in this section. As we will see, the Dirac operator $\mathcal{D}$ will simply be a bounded perturbation of the twisted Dirac operator $\hat{D}_\omega$ [DS13a] (see also chapter 5 of this thesis). Moreover, we will define an analogue for the spin Laplacian and prove that it is related to the square of the Dirac operator by a modified Lichnerowicz formula (which reduces to the classical one in the limit $\theta \to 0$).

The first object we have to define is the spin connection associated to the $\star$-connection $\nabla^\star$. We define it giving the spinor covariant derivatives ∇_{E_i} as operators on the Hilbert space of L^2 spinors. We begin by considering the canonical flat real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ over $\mathcal{A} = \mathcal{A}(\mathbb{T}^3_\theta)$, where[6] $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^2$, $D = \sum_j \sigma^j \delta_j$ and $J = J_0 \otimes (i\sigma^2 \circ c.c.)$. Due to what we have seen in the first part of this chapter, in the commutative case we can identify $\mathcal{H}$ with $\bigoplus_k L^2(\Sigma\mathbb{T}^2 \otimes L^{-k})$, and so the Dirac operator associated to the metric induced by the connection ω can be seen as an operator on $\mathcal{H}$. Hence, in the noncommutative case we define the Dirac operator $\mathcal{D}$ already as an operator on $\mathcal{H}$. For the same reason, moreover, we can define a Clifford map $\gamma : \mathfrak{X}_\star \to \mathcal{L}(\mathcal{H})$ by setting $\gamma(E_j) = -i\sigma^j$ and extending it by left $\mathcal{A}$-linearity (but, actually, we shall not use this extension).

According to classical results, the spinor covariant derivatives, with respect to an orthonormal frame $\{E_i\}$, can be written in the following way:

$$\nabla_{E_i} = \mathcal{L}_{E_i} + \frac{1}{4}\sum_{jk} \gamma(E_j)\gamma(E_k)\Gamma^k_{ij}, \tag{D.5.1}$$

where the Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection. Taking (D.5.1) to be the definition of the spin connection also in the noncommutative case, we obtain, for any $\psi \in \mathcal{H}$,

$$\nabla_{E_1}\psi = \partial_1\psi - \omega_1^\circ\partial_3\psi - \frac{i}{4}\omega_{12}^\circ\sigma^1\psi, \tag{D.5.2a}$$

$$\nabla_{E_2}\psi = \partial_2\psi - \omega_2^\circ\partial_3\psi - \frac{i}{4}\omega_{12}^\circ\sigma^2\psi, \tag{D.5.2b}$$

$$\nabla_{E_3}\psi = \partial_3\psi + \frac{i}{4}\omega_{12}^\circ\sigma^3\psi. \tag{D.5.2c}$$

We recall that $\mathcal{A}^\circ$ acts on $\mathcal{H}$ via the representation induced by the real structure J. Defining the Dirac operator simply by $\mathcal{D} = \sum_j \gamma(E_j)\nabla_{E_j}$, we arrive to the following expression:

$$\mathcal{D} = \sigma^1\delta_1 + \sigma^2\delta_2 + \sigma^3\delta_3 - \sigma^1\omega_1^\circ\delta_3 - \sigma^2\omega_2^\circ\delta_3 - \frac{1}{4}\omega_{12}^\circ. \tag{D.5.3}$$

In particular we see that $\mathcal{D}$ differs from the twisted Dirac operator $\hat{D}_\omega$ only for the bounded

[6] $\mathcal{H}_\tau$ is the GNS Hilbert space associated to the tracial state τ, cfr. appendix A; J_0 is the Tomita-Takesaki antiunitary involution on $\mathcal{H}_\tau$.

"scalar" term $Z = -\frac{1}{4}\omega^\circ_{12}$. Next we introduce the spinor Laplacian. We know [Fri00] that the spinor Laplacian on a Riemannian manifold can be written as

$$\Delta^S = -\sum_j \nabla_{E_j}\nabla_{E_j} + \sum_j \mathrm{div}(E_j)\nabla_{E_j},$$

where ∇ is the spin connection and $\{E_j\}$ is a (local) orthonormal frame. In our case the divergence of each E_j is zero, so we are left with:

$$\Delta^S\psi = -\sum_{j=1}^{3} \nabla_{E_j}\nabla_{E_j}\psi \tag{D.5.4}$$

for any $\psi \in \mathcal{H}$. We compute separately the three terms of equation (D.5.4):

$$\begin{aligned}\nabla_{E_1}\nabla_{E_1}\psi =&\partial_1^2\psi - \partial_1(\omega^\circ_1)\partial_3\psi - 2\omega^\circ_1\partial_1\partial_3\psi - \frac{i}{4}\partial_1(\omega^\circ_{12})\sigma^1\psi - \frac{i}{2}\omega^\circ_{12}\sigma^1\partial_1\psi \\ &+ (\omega^\circ_1)^2\partial_3^2\psi + \frac{i}{4}\omega^\circ_1\omega^\circ_{12}\sigma^1\partial_3\psi + \frac{i}{4}\omega^\circ_{12}\omega^\circ_1\sigma^1\partial_3\psi - \frac{1}{16}(\omega^\circ_{12})^2\psi,\end{aligned} \tag{D.5.5a}$$

$$\begin{aligned}\nabla_{E_2}\nabla_{E_2}\psi =&\partial_2^2\psi - \partial_2(\omega^\circ_2)\partial_3\psi - 2\omega^\circ_2\partial_2\partial_3\psi - \frac{i}{4}\partial_2(\omega^\circ_{12})\sigma^2\psi - \frac{i}{2}\omega^\circ_{12}\sigma^2\partial_2\psi \\ &+ (\omega^\circ_2)^2\partial_3^2\psi + \frac{i}{4}\omega^\circ_2\omega^\circ_{12}\sigma^2\partial_3\psi + \frac{i}{4}\omega^\circ_{12}\omega^\circ_2\sigma^2\partial_3\psi - \frac{1}{16}(\omega^\circ_{12})^2\psi,\end{aligned} \tag{D.5.5b}$$

$$\nabla_{E_3}\nabla_{E_3}\psi = \partial_3^2\psi + \frac{i}{2}\omega^\circ_{12}\sigma^3\partial_3\psi - \frac{1}{16}(\omega^\circ_{12})^2\psi. \tag{D.5.5c}$$

As we anticipated, we compare now Δ^S with the square of the Dirac operator $\mathcal{D}$. First of all, let us compute $\mathcal{D}^2$. Writing $\mathcal{D} = D + X_\omega + Z$, where D is the flat Dirac operator on the noncommutative 3-torus, $D = \sum_{j=1}^{3}\sigma^j\delta_j$, Z is the scalar term $-\frac{i}{4}\omega^\circ_{12} = -\frac{i}{4}J\omega_{12}J^{-1}$, and X_ω is the operator $X_\omega = -\sigma^1\omega^\circ_1\delta_3 - \sigma^2\omega^\circ_2\delta_3$, we obtain the following expressions:

$$D^2\psi = -\partial_1^2\psi - \partial_2^2\psi - \partial_3^2\psi, \tag{D.5.6a}$$

$$X_\omega^2\psi = -(\omega^\circ_1)^2\partial_3^2\psi - (\omega^\circ_2)^2\partial_3^2\psi - i(\omega^\circ_1\omega^\circ_2 - \omega^\circ_2\omega^\circ_1)\sigma^3\partial_3^2\psi, \tag{D.5.6b}$$

$$Z^2\psi = \frac{1}{16}(\omega^\circ_{12})^2\psi, \tag{D.5.6c}$$

$$(DX_\omega + X_\omega D)\psi = \partial_1(\omega_1)^\circ\partial_3\psi + \partial_2(\omega^\circ_2)\partial_3\psi + 2\omega^\circ_1\partial_1\partial_3\psi + 2\omega^\circ_2\partial_2\partial_3\psi - i\omega^\circ_{12}\sigma^3\partial_3\psi, \tag{D.5.6d}$$

$$\begin{aligned}(DZ + ZD)\psi =&\frac{i}{2}\omega^\circ_{12}\sigma^1\partial_1\psi + \frac{i}{4}\partial_1(\omega^\circ_{12})\sigma^1\partial_1\psi + \frac{i}{2}\omega^\circ_{12}\sigma^2\partial_2\psi \\ &+ \frac{i}{4}\partial_2(\omega^\circ_{12})\sigma^1\partial_2\psi + \frac{i}{2}\omega^\circ_{12}\sigma^3\partial_3\psi,\end{aligned} \tag{D.5.6e}$$

$$(X_\omega Z + ZX_\omega)\psi = -\frac{i}{4}\omega^\circ_1\omega^\circ_{12}\sigma^1\partial_3 - \frac{i}{4}\omega^\circ_{12}\omega^\circ_1\sigma^1\partial_3 - \frac{i}{4}\omega^\circ_2\omega^\circ_{12}\sigma^2\partial_3 - \frac{i}{4}\omega^\circ_{12}\omega^\circ_2\sigma^2\partial_3 \tag{D.5.6f}$$

In order to compute $\mathcal{D}_\omega^2 - \Delta^S$, we compare equations (D.5.5a) - (D.5.5c) with equations (D.5.6a) - (D.5.6f). We see the following facts.

- All the terms of $D^2\psi$ (equation (D.5.6a)) cancel out with the leading terms of equations (D.5.5a), (D.5.5b), (D.5.5c).
- All the terms proportional to $(\omega_i^\circ)^2\partial_3^2\psi$ in equation (D.5.6b) are cancelled by the corresponding ones in equation (D.5.5a) and equation (D.5.5c).
- The terms proportional to $i\omega_{12}^\circ\sigma^3\partial_3\psi$ (equations (D.5.5c), (D.5.6d), (D.5.6e)) sum up to zero.
- The sum of all the terms proportional to $(\omega_{12}^\circ)^2\psi$ (equations (D.5.5a), (D.5.5b), (D.5.5c), (D.5.6c)) is equal to $-\frac{1}{8}(\omega_{12}^\circ)^2\psi$.
- The terms proportional, respectively, to $\partial_i(\omega_i^\circ)\partial_3\psi$ and to $\omega_i^\circ\partial_i\partial_3\psi$ in equation (D.5.6d) cancel out with the corresponding ones in equation (D.5.5a) and equation (D.5.5b).
- The terms proportional, respectively, to $\omega_{12}^\circ\sigma^i\partial_i\psi$ and to $\partial_i(\omega_{12}^\circ)\sigma^i\psi$ in equation (D.5.6e) cancel out with the corresponding ones in equation (D.5.5a) and equation (D.5.5b).
- The terms proportional, respectively, to $\omega_{12}^\circ\omega_i^\circ\sigma^i\partial_3\psi$ and to $\omega_i^\circ\omega_{12}^\circ\sigma^i\partial_3\psi$ in equation (D.5.6f) cancel out with the corresponding ones in equation (D.5.5a) and equation (D.5.5b).

Hence we are left with

$$(\mathcal{D}_\omega^2 - \Delta)\psi = -\frac{1}{8}(\omega_{12}^\circ)^2\psi + \Xi\psi, \tag{D.5.7}$$

where

$$\Xi\psi = i(\omega_2^\circ\omega_1^\circ - \omega_1^\circ\omega_2^\circ)\sigma^3\delta_3^2(\psi). \tag{D.5.8}$$

We have therefore obtained a modified version of the Licherowicz formula, which reduces, as expected, to the classical one in the case $\theta = 0$. Indeed, Ξ is different from zero if and only if ω_1 and ω_2 do not commute. Moreover, from (D.5.7) we obtain that the scalar curvature should be equal to $-\frac{1}{2}(\omega_{12}^\circ)^2$, and this is consistent with the computation of the Ricci curvature performed in the previous section (cfr. equation (D.4.8)). It is interesting to notice that the new term appearing in this Licherowicz-like formula is a second order pseudodifferential operator and it is proportional to σ^3, so that it distinguishes the two polarizations of a spinor field.

Bibliography

[Abe80] E. Abe, *Hopf algebras*, Cambridge University Press, Cambridge (1980)

[Adams] J. F. Adams, *Lectures on Lie groups*, Benjamin, New York (1969)

[Amm98] B. Ammann, *The Dirac Operator on Collapsing S^1-bundles*, arXiv:math/9812116v1 (1998)

[AmmB98] B. Ammann and C. Bär, *The Dirac Operator on Nilmanifolds and Collapsing Circle Bundles*, Ann. Global Anal. and Geom. **16**, 221 (1998)

[A06] P. Aschieri, *Noncommutative Symmetries and Gravity*, J. Phys. Conf. Ser. **53**, 799 (2006)

[A07] P. Aschieri, *Noncommutative Gravity and the *-Lie algebra of diffeomorphisms*, Fortsch. Phys. **55**, 649 (2007)

[A09] P. Aschieri, *Star Product Geometries*, Russ. J. Math. Phys. **16**, 371 (2009)

[ABDMSW05] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Mayer, P. Schupp and J. Wess, *A Gravity Theory on Noncommutative Spaces*, Class. Quant. Grav. **22**, 3511 (2005)

[AC93] P. Aschieri and L. Castellani, *An introduction to noncommutative differential geometry on quantum groups*, Int. J. Mod. Phys. A **8**, 1667 (1993)

[ADMW06] P. Aschieri, M. Dimitrijevic, F. Mayer and J. Wess, *Noncommutative Geometry and Gravity*, Class. Quant. Grav. **23**, 1883 (2006)

[AtiHir59] M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. AMS **65**, 276 (1959)

[AtiHir61] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, in *Differential geometry*, ed. by C. B. Allendoerfer, Proc. Symp. Pure Math. vol. III, AMS, Providence, pp. 7-38 (1961)

[BJ83] S. Baaj and P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans le C^*-modules hilbertiens*, C. R. Acad. Sc. Paris **296**, 875 (1983)

[Bak87] A. Bak, *Editorial*, K-theory **1**, 1 (1987)

[Bass] H. Bass, *Algebraic K-theory*, W. A. Benjamin Inc., New York (1968)

[BB11] E. J. Beggs and T. Brzezinski, *Line bundles and the Thom construction in noncommutative geometry*, arXiv:1012.1475v2 (2011)

[Be88] J. V. Bellissard, *Ordinary quantum Hall effect and noncommutative cohomology*, in *Proc. on localization in disordered systems (Bad Schandau, 1986)*, Teubner, Leipzig (1988)

[BMR10] J. V. Bellissard, M. Marcolli and K. Reihani, *Dynamical Systems on Spectral Metric Spaces*, arXiv:1008.4617v1 (2010)

[BGV] N. Berline, E.Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag, Berlin (1991)

[BeKe] A. J. Berrick and M. E. Keating, *Categories and Modules, with K-theory in view*, Cambridge University Press, Cambridge (2000)

[BDD11] J. Bhowmick, F. D'Andrea and L. Dabrowski, *Quantum Isometries of the finite noncommutative geometry of the Standard Model*, Comm. Math. Phys. **307**, 101 (2011)

[BhMa12] T. A. Bhuyain and M. Marcolli, *The Ricci flow on non noncommutative two-tori*, Lett. Math. Phys. **101**, 173 (2012)

[Blck98] B. Blackadar, *K-Theory for Operator Algebras*, Cambridge University Press (1998)

[BC91] B. Blackadar and J. Cuntz, *Differential Banach algebra norms and smooth subalgebras of C^*-algebras*, J. Operator Theory **26**, 255 (1991)

[BlCM86] R. J. Blattner, M. Cohen and S. Montgomery, *Crossed products and inner actions of Hopf algebras*, Trans. Amer. Math. Soc. **298**, 671 (1986)

[BlM89] R. J. Blattner and S. Montgomery, *Crossed products and Galois extensions of Hopf algebras*, Pacific J. Math. **137**, 37 (1989)

[BvS11] J. Boeijink and W. D. van Suijlekom, *The noncommutative geometry of Yang-Mills fields*, J. Geom. Phys. **61**, 1122 (2011)

[BourCA] N. Bourbaki, *Commutative Algebra*, Hermann, Paris (1972)

[BourGT] N. Bourbaki, *General Topology, part I and part II*, Hermann, Paris (1966)

[BourTVS] N. Bourbaki, *Topological Vector Spaces*, Springer-Verlag, Heidelberg (2003)

[BMS13] S. Brain, B. Mesland and W. D. van Suijlekom, *Gauge theory for spectral triples and the unbounded Kasparov product*, arXiv:1306.1951v1 (2013)

[BDF77] L. Brown, R. Douglas and P. Fillmore, *Extensions of C^*-algebras and K-homology*, Ann. Math. **105**, 265 (1977)

[B96] T. Brzezinski, *Translation Map in Quantum Principal Bundles*, J. Geom. Phys. **20**, 349 (1996)

[B96b] T. Brzezinski, *Quantum homogeneous spaces as quotient spaces*, J. Math. Phys. **37**, 2388 (1996)

[B99] T. Brzezinski, *Modules Associated to Coalgebra Galois Extensions*, J. Algebra **215**, 290 (1999)

[BF12] T. Brzezinski and S. A. Fairfax, *Bundles over Quantum Real Weighted Projective Spaces*, Axioms **1**, 201 (2012)

[BH99] T. Brzezinski and P. M. Hajac, *Coalgebra extensions and algebra coextensions of Galois type*, Comm. Algebra **27**, 1347 (1999)

[BH04] T. Brzezinski and P. M. Hajac, *The Chern-Galois character*, C. R. Acad. Sci. Paris **338**, 113 (2004)

[BH09] T. Brzezinski and P. M. Hajac, *Galois-Type Extensions and Equivariant Projectivity*, arXiv:0901.0141 (2009)

[BM92] T. Brzezinski and S. Majid, *A class of bicovariant differential calculi on Hopf algebras*, Lett. Math. Phys. **26**, 67 (1992)

[BM93] T. Brzezinski and S. Majid, *Quantum Group Gauge Theory on Quantum Spaces*, Comm. Math. Phys. **157**, 591 (1993). Erratum: Comm. Math. Phys. **167**, 235 (1995)

[BM98a] T. Brzezinski and S. Majid, *Quantum differentials and the q-monopole revisited*, Acta Appl. Math. **54**, 185 (1998)

[BM98b] T. Brzezinski and S. Majid, *Coalgebra Bundles*, Comm. Math. Phys. **191**, 467 (1998)

[BM00] T. Brzezinski and S. Majid, *Quantum geometry of algebra factorisations and coalgebra bundles*, Comm. Math. Phys. **213**, 491 (2000)

[BZ12] T. Brzezinski and B. Zielinski, *Quantum principal bundles over quantum real projective spaces*, J. Geom. Phys. **62**, 1097 (2012)

[BuK96] R. J. Budzynski and W. Kondracki, *Quantum principal fiber bundles: Topological aspects*, Rep. Math. Phys. **37**, 365 (1996)

[Bump] D. Bump, *Lie Groups*, Springer, New York (2004)

[CaMa00] D. Calow and R. Matthes, *Covering and gluing of algebras and differential algebras*, J. Geom. Phys. **32**, 364 (2000)

[CaMa02] D. Calow and R. Matthes, *Connections on locally trivial quantum principal fibre bundles*, J. Geom. Phys. **41**, 114 (2002)

[CE] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press (1956)

[ChaP03a] P. S. Chakraborty and A. Pal, *Equivariant spectral triples on the quantum* $SU(2)$ *group*, K-theory **28**, 107 (2003)

[ChaP03b] P. S. Chakraborty and A. Pal, *Spectral triples and associated Connes-de Rham complex for the quantum SU(2) and the quantum sphere*, Comm. Math. Phys. **240**, 447 (2003)

[ChaP06] P. S. Chakraborty and A. Pal, *On equivariant Dirac operators for* $SU_q(2)$, Proc. Indian Acad. Sci. **116**, 531 (2006)

[CC96] A. H. Chamseddine and A. Connes, *Universal Formula for Noncommutative Geometry Actions: Unification of Gravity and the Standard Model*, Phys. Rev. Lett. **77**, 4868 (1996)

[CC97] A. H. Chamseddine and A. Connes, *The Spectral Action Principle*, Comm. Math. Phys. **186**, 731 (1997)

[CC06a] A. H. Chamseddine and A. Connes, *Scale Invariance in the Spectral Action*, J. Math. Phys. **47** (2006): 063504

[CC06b] A. H. Chamseddine and A. Connes, *Inner fluctuations of the spectral action*, J. Geom. Phys. **57**, 1 (2006)

[CC08] A. H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. **58**, 38 (2008)

[CCM07] A. H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the Standard Model with neutrino mixing*, Adv. Theor. Math. Phys. **11**, 991 (2007)

[CCS13] A. H. Chamseddine, A. Connes and W. van Suijlekom, *Inner Fluctuations in Noncommutative Geometry without the first order condition*, arXiv:1304.7583 (2013)

[ChSw69] S. U. Chase and M. E. Sweedler, *Hopf algebras and Galois theory*, Lecture Notes in Math. **97**, Springer-Verlag, Berlin and New York (1969)

[Ch98] X.-H. Chen, *Cleft extensions for a Hopf algebra* $k_q[X, X^{-1}, Y]$, Glasgow Math. J. **40**, 147 (1998)

[C80] A. Connes, *C^*-algèbres et géométrie différentielle*, C. R. Acad. Sci. Paris **290A**, 599 (1980)

[C85] A. Connes, *Noncommutative differential geometry*, Publ. Math. IHES **39**, 257 (1985)

[C88] A. Connes, *The action functional in noncommutative geometry*, Comm. Math. Phys. **117**, 673 (1988)

[C89] A. Connes, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic Theory Dynam. Systems **9**, 207 (1989)

[C94] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego (1994)

[C95] A. Connes, *Geometry from the Spectral Point of View*, Lett. Math. Phys. **34**, 203 (1995)

[C96] A. Connes, *Gravity Coupled with Matter and the Foundation of Non-commutative Geometry*, Comm. Math. Phys. **182**, 155 (1996)

[C00] A. Connes, *A short survey of noncommutative geometry*, J. Math. Phys. **41** (2000): 3832

[C13] A. Connes, *On the spectral characterization of manifolds*, J. Noncomm. Geom. **7**, 1 (2013)

[CL91] A. Connes and J. Lott, *Particle models and noncommutative geometry*, Nucl. Phys. B - Proc. Suppl. **18**, 29 (1991)

[CMa07] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, Colloquium Publications, vol. 55, AMS and Hindustan Book Agency (2007)

[CM95] A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5**, 174 (1995)

[CM06] A. Connes and H. Moscovici, *Type III and spectral triples*, arXiv:math/0609703 (2006)

[CM11] A. Connes and H. Moscovici, *Modular Curvature for Noncommutative Two-Tori*, arXiv:1110.3500v1 (2011)

[CR87] A. Connes and M. Rieffel, *Yang-Mills theory over quantum tori*, Contemp. Math. **62**, 237 (1987)

[CS84] A. Connes and G. Skandalis, *The longitudinal index theorem for foliations*, Publ. RIMS Kyoto **20**, 1139 (1984)

[CT11] A. Connes and P. Tretkoff, *The Gauss-Bonnet Theorem for the noncommutative two torus*, in *Noncommutative Geometry, Arithmetic, and Related Topics*, ed. by C. Consani and A. Connes, Johns Hopkins University Press (2011), pp. 141-158

[Con] J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York (1990)

[Cun87] J. Cuntz, *A New Look at KK-Theory*, K-Theory **1**, 31 (1987)

[DD11] L. Dabrowski and G. Dossena, *Product of real spectral triples*, Int. J. Mod. Phys. **8**, 1833 (2011)

[DGH01] L. Dabrowski, H. Grosse and P. M. Hajac, *Strong Connections and Chern-Connes Pairing in the Hopf-Galois Theory*, Comm. Math. Phys. **220**, 301 (2001)

[DLSSV05] L. Dabrowski, G. Landi, A. Sitarz, W. van Suijlekom and J. C. Varilly, *The Dirac operator on $SU_q(2)$*, Comm. Math. Phys. **259**, 729 (2005)

[DS13a] L. Dabrowski and A. Sitarz, *Noncommutative circle bundles and new Dirac operators*, Comm. Math. Phys. **318**, 111 (2013)

[DS13b] L. Dabrowski and A. Sitarz, *Curved noncommutative torus and Gauss–Bonnet*, J. Math. Phys. **54** (2013): 013518

[DSZ13] L. Dabrowski, A. Sitarz and A. Zucca, *Dirac operators on noncommutative principal circle bundles*, Int. J. Geom. Methods in Mod. Phys. (in press); preprint version, arXiv:1305.6185 (2013)

[DZ13] L. Dabrowski and A. Zucca, *Dirac operators on noncommutative principal torus bundles*, arXiv:1308.4738 (2013)

[Dix66] J. Dixmier, *Existence de traces non normales*, C. R. Acad. Sci. Paris Ser A-B **262** (1966)

[DT86] Y. Doi and M. Takeuchi, *Cleft comodule algebras for a bialgebra*, Comm. in Algebra **14**, 801 (1986)

[DFR94] S. Doplicher, K. Fredenhagen and J. E. Roberts, *Space-time quantization induced by classical gravity*, Phys. Lett. B **331**, 39 (1994)

[DFR95] S. Doplicher, K. Fredenhagen and J. E. Roberts, *The quantum structure of space-time at the Planck scale and quantum fields*, Comm. Math. Phys. **172**, 187 (1995)

[Dri83] V. G. Drinfeld, *On constant quasiclassical solutions of the Yang-Baxter equations*, Soviet Math. Dokl. **28**, 667 (1983)

[Dri87] V. G. Drinfeld, *Quantum Groups*, in *Proceedings of the ICM*, ed. by A. Gleason, AMS, Rhode Island (1987), pp. 798-820

[Dri90] V. G. Drinfeld, *Quasi-Hopf Algebras*, Leningrad Math. J. **1**, 1419 (1990)

[D-VKM90] M. Dubois-Violette, R. Kerner and J. Madore, *Noncommutative differential geometry and new models of gauge theory*, J. Math. Phys. **31**, 323 (1990)

[Dur93] M. Durdevic, *Quantum Principal Bundles*, arXiv:hep-th/9311029v1 (1993)

[Dur96a] M. Durdevic, *Geometry of quantum principal bundles I*, Comm. Math. Phys. **175**, 457 (1996)

[Dur96b] M. Durdevic, *Quantum Gauge Transformations and Braided Structure on Quantum Principal Bundles*, arXiv:q-alg/9605010 (1996)

[Dur97a] M. Durdevic, *Geometry of Quantum Principal Bundles II*, Rev. Math. Phys. **9**, 531 (1997)

[Dur97b] M. Durdevic, *Quantum Principal Bundles and Corresponding Gauge Theories*, J. Phys. A **30**, 2027 (1997)

[ENO09] S. Echterhoff, R. Nest and H. Oyono-Oyono, *Principal noncommutative torus bundles*, Proc. Lond. Math. Soc. **99**, 1 (2009)

[FRT90] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1**, 193 (1990)

[FK11] F. Fathizadeh and M. Khalkhali, *Scalar Curvature for the Noncommutative Two Torus*, arXiv:1110.3511v1 (2011)

[FK12] F. Fathizadeh and M. Khalkhali, *The Gauss-Bonnet Theorem for Noncommutative Two Tori With a General Conformal Structure*, J. Noncomm. Geom. **6**, 457 (2012)

[Fri00] T. Friedrich, *Dirac Operators in Riemannian Geometry*, AMS, Providence, Rhode Island (2000)

[GM] S. I. Gelfand and Y. I. Manin, *Methods of Homological Algebra*, Springer-Verlag, Heidelberg (2003)

[Gil] P. B. Gilkey, *Invariance theory, the heat equation and the Atiyah-Singer index theorem*, Publish or Perish Inc., Wilmington (1984). Electronic reprint (1996)

[GLP96] P. Gilkey, J. Leahy and J. H. Park, *The spectral geometry of the Hopf fibration*, J. Phys. A **29**, 5645 (1996)

[GLP98] P. Gilkey, J. Leahy and J. H. Park, *Eigenforms of the spin Laplacian and projectable spinors for principal bundles*, Nucl. Phys. B **514**, 740 (1998)

[Gos10] D. Goswami, *Quantum isometry group for spectral triples with real structure*, SIGMA **6**, 007-7 (2010)

[GBFV] J. M. Gracia-Bondia, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, Boston (2001)

[Gr66] A. Grothendieck, *Produits Tensoriels Topologiques et Espaces Nucléaires*, Memoirs of the AMS **16**, Providence (1966)

[Gui85] V. W. Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv. in Math. **55**, 131 (1985)

[Haj96] P. M. Hajac, *Strong Connections on Quantum Principal Bundles*, Comm. Math. Phys. **182**, 579 (1996)

[HKMZ11] P. M. Hajac, U. Kraehmer, R. Matthes and B. Zielinski, *Piecewise principal comodule algebras*, J. Noncomm. Geom. **5**, 591 (2011)

[HM99] P. M. Hajac and S. Majid, *Projective module description of the q-monopole*, Comm. Math. Phys. **206**, 247 (1999)

[HMS06] P. M. Hajac, R. Matthes and W. Szymanski, *A Locally Trivial Quantum Hopf Fibration*, Algebras and Representation Theory **9**, 121 (2006)

[HRZ11] P. M. Hajac, A. Rennie and B. Zielinski, *The K-theory of Heegaard quantum lens spaces*, arXiv:1110.5897v1 (2011)

[HRuZ11] P. M. Hajac, J. Rudnik and B. Zielinski, *Reductions of piecewise trivial principal comodule algebras*, arXiv:1101.0201 (2011)

[HaMa10] K. Hannabuss and V. Mathai, *Noncommutative principal torus bundles via parametrised strict deformation quantization*, in *Superstrings, Geometry, Topology, and C^*-algebras*, Proc. Symp. Pure Math. vol. 81, ed. by R. S. Doran, G. Friedman and J. Rosenberg, AMS, Providence, RI (2010), pp. 133-148

[HSWZ11] A. Hawkins, A. Skalski, S. White and J. Zacharias, *Spectral triples on crossed products arising from equicontinuous actions*, arXiv:1103.6199v3 (2011)

[Hig87] N. Higson, *A characterization of KK-theory*, Pacific J. Math. **126**, 253 (1987)

[Hig04] N. Higson, *The local index formula in noncommutative geometry*, Contemporary developments in algebraic K-theory, ICTP, Lect. Notes, XV, Trieste, pp. 443-536 (2004)

[Hig06] N. Higson, *The residue index theorem of Connes and Moscovici*, Surveys in noncommutative geometry, Clay Math. Proc., vol. 6, AMS, Providence, RI (2006), pp. 71-126

[HS] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer-Verlag, New York (1971)

[KL13] J. Kaad and M. Lesch, *Spectral flow and the unbounded Kasparov product*, Adv. in Math (in press, DOI:10.1016/j.aim.2013.08.015), 495 (2013)

[KS12] J. Kaad and R. Senior, *A twisted spectral triple for quantum SU(2)*, J. Geom. Phys. **62**, 731 (2012)

[Kan42] L. V. Kantorovic, *On the translocation of masses*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 199 (1942)

[KanRub57] L. V. Kantorovic and G. S. Rubinstein, *On a functional space and certain extremum problems*, Dokl. Akad. Nauk SSSR (N.S.) **115**, 1058 (1957)

[Kar78] M. Karoubi, *K-Theory. An Introduction*, Springer-Verlag, Berlin (1978)

[Kar87] M. Karoubi, *Homologie cyclique et K-théorie*. Astérisque 149 (1987)

[Kas80] G. G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk. SSSR Ser. Mat. **44**, 571 (1980). English transl., Math. USSR-Izv. **16**, 513 (1981)

[Kas88] G. G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91**, 147 (1988)

[Kassel] C. Kassel, *Quantum groups*, Springer-Verlag, New York (1995)

[KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, vol.1*, Interscience (Wiley), New York (1963)

[Kra98] T. Krajewski, *Classification of finite spectral triples*, J. Geom. Phys. **28**, 1 (1998)

[KW11] U. Krähmer and E. Wagner, *Twisted spectral triples and covariant differential calculi*, Banach Center Publ. **93**, 177 (2011)

[KT81] H. F. Kreimer and M. Takeuchi, *Hopf algebras and Galois extensions of an algebra*, Indiana Univ. Math. J. **30**, 675 (1981)

[Kuc97] D. Kucerovsky, *The KK-Product of Unbounded Modules*, K-Theory **11**, 17 (1997)

[KMT05] J. Kustermans, G. Murphy and L. Tuset, *Quantum groups, differential calculi and the eigenvalues of the Laplacian*, Trans. Am. Math. Soc. **357**, 4681 (2005)

[La] E. C. Lance, *Hilbert C^*-modules*, London Math. Soc. Lect. Notes Series 210, Cambridge University Press (1995)

[LPR06] G. Landi, C. Pagani and C. Reina, *A Hopf bundle over a quantum four-sphere from the symplectic group*, Comm. Math. Phys. **263**, 65 (2006)

[LRZ09] G. Landi, C. Reina and A. Zampini, *Gauged Laplacians on quantum Hopf bundles*, Comm. Math. Phys. **287**, 179 (2009)

[LS05] G. Landi and W. van Suijlekom, *Principal fibrations from noncommutative spheres*, Comm. Math. Phys. **260**, 203 (2005)

[LZ11] G. Landi and A. Zampini, *Calculi, Hodge operators and Laplacians on a quantum Hopf fibration*, Rev. Math. Phys. **23**, 575 (2011)

[Lan03] N. P. Landsman, *Lecture Notes on C^*-Algebras and K-theory*, available at `http://www.science.uva.nl/~npl/CK.pdf` (2003)

[Lee] J. M. Lee, *Riemannian manifolds. An introduction to curvature*, Springer-Verlag, New York (1997)

[LM] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton University Press (1994)

[Lich63] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris Sér A **257**, 7 (1963)

[Lo] J.-L. Loday, *Cyclic Homology*, Springer-Verlag, Berlin (1998)

[Lord04] S. Lord, *Riemannian noncommutative geometry*, PhD Thesis, University of Adelaide (2004)

[LRV12] S. Lord, A. Rennie and J. C. Várilly, *Riemannian Manifolds in Noncommutative Geometry*, J. Geom. Phys. **62**, 1611 (2012)

[Maj95] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge (1995)

[Maj98] S. Majid, *Classification of bicovariant differential calculi*, J. Geom. Phys. **25**, 199 (1998)

[Maj02] S. Majid, *Riemannian Geometry of Quantum Groups and Finite Groups with Nonuniversal Differentials*, Comm. Math. Phys. **225**, 131 (2002)

[MajOe99] S. Majid and R. Oeckl, *Twisting of Quantum Differentials and the Planck Scale Hopf Algebra*, Comm. Math. Phys. **205**, 617 (1999)

[Mall] A. Mallios, *Topological Algebras. Selected Topics*, North-Holland, Amsterdam (1986)

[Man79] Yu. I. Manin, *Algebraic aspects of non-linear differential equations*, J. Soviet Math. **11**, 1 (1979)

[McL] S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, New York (1998)

[Mes11] B. Mesland, *Unbounded bivariant K-theory and correspondences in noncommutative geometry*, arXiv:0904.4383v4 (2011)

[Mes11b] B. Mesland, *Lecture notes on KK-theory and spectral triples*, notes for the course *KK-theory and spectral triples* given at CIMPA-UNESCO school, Bangkok, June 2011

[MS] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton Univ. Press, Princeton (1974)

[Mor96] A. Moroianu, *Opérateur de Dirac et submersions riemanniennes*, PhD thesis, Ècole Polytechnique (1996)

[Mos10] H. Moscovici, *Local index formula and twisted spectral triples*, Quanta of maths **11**, 465 (2010)

[Otg11] U. Otgonbayar, *Pseudo-differential operators and regularity of spectral triples*, arXiv:0911.08164v4 (2011)

[OR05] N. Ozawa and M. A. Rieffel, *Hyperbolic group C^*-algebras and free-product C^*-algebras as compact quantum metric spaces*, Canad. J. Math. **57**, 1056 (2005)

[Pad85] T. Padmanabhan, *Physical significance of Planck length*, Ann. Phys. **165**, 38 (1985)

[Pad87] T. Padmanabhan, *Limitations on the operatorial definition of spacetime events and quantum gravity*, Class. Quant. Grav. **4**, 107 (1987)

[Pas01] M. Paschke, *Über nichtkommutative Geometrien, ihre Symmetrien und etwas Hochenergiephysik*, PhD thesis, Mainz (2001)

[PS98] M. Paschke and A. Sitarz, *Discrete spectral triples and their symmetries*, J. Math. Phys. **39**, 6191 (1998)

[PS00] M. Paschke and A. Sitarz, *The geometry of noncommutative symmetries*, Acta Phys. Pol. B **31**, 1897 (2000)

[PS06] M. Paschke and A. Sitarz, *On Spin Structures and Dirac Operators on the Noncommutative Torus*, Lett. in Math. Phys. **77**, 317 (2006)

[PS08] M. Paschke and A. Sitarz, *Discrete spectral triples and their symmetries*, J. Math. Phys. **39**, 6191 (1998)

[Piet] A. Pietsch, *Nuclear Locally Convex Spaces*, Springer-Verlag, Berlin (1972)

[Res90] N. Reshetikhin, *Multiparameter Quantum Groups And Twisted Quasitriangular Hopf Algebras*, Lett. Math. Phys. **20**, 3 31 (1990)

[Ri98] M. A. Rieffel, *Metrics on states from actions of compact groups*, Doc. Math. **3**, 215 (1998)

[Ri99] M. A. Rieffel, *Metrics on state spaces*, Doc. Math. **4**, 559 (1999)

[Ri02] M. A. Rieffel, *Group C^*-algebras as compact quantum metric spaces*, Doc. Math. **7**, 605 (2002)

[Ri04] M. A. Rieffel, *Compact quantum metric spaces* in *Operator algebras, quantization, and noncommutative geometry*, Contemp. Math. **365**, 315 (2004)

[RLL] M. Rordam, F. Larsen and M. Laustsen, *An Introduction to K-Theory for C^*-Algebras*, Cambridge University Press (2000)

[Ros97] J. Rosenberg, *The algebraic K-theory of operator algebras*, K-theory **12**, 75 (1997)

[Ru] W. Rudin, *Functional analysis*, 2nd ed., McGraw-Hill, New York (1991)

[Ryan] R. A. Ryan, *Introduction to tensor product of Banach spaces*, Springer-Verlag, London (2002)

[Schaef] H. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, New York (1971)

[Schm99] K. Schmüdgen, *Commutator representations of differential calculi on the quantum group $SU_q(2)$*, J. Geom. Phys. **31**, 241 (1999)

[Schm02] K. Schmüdgen, *Commutator representations of covariant differential calculi on quantum groups*, Lett. Math. Phys. **59**, 95 (2002)

[SS95] K. Schmüdgen and A. Schüler, *Classification of bicovariant differential calculi on quantum groups of type A, B, C and D*, Comm. Math. Phys. **167**, 635 (1995)

[Sch90a] H.-J. Schneider, *Principal homogeneous spaces for arbitrary Hopf algebras*, Isr. J. Math. **72**, 167 (1990)

[Sch90b] H.-J. Schneider, *Representation theory of Hopf-Galois extensions*, Isr. J. Math. **72**, 196 (1990)

[Sch94] H.-J. Schneider, *Hopf Galois Extensions, Crossed Products and Clifford Theory*, in *Advances in Hopf Algebras*, ed. by J. Bergen and S. Montgomery, Lecture Notes in Pure and Applied Mathematics **158**, Marcel Dekker, Inc., pp. 267-298 (1994)

[Schw] L. Schwartz, *Théorie des distributions*, Hermann, Paris (1966)

[SWZ93] P. Schupp, P. Watts and B. Zumino, *Bicovariant quantum algebras and quantum Lie algebras*, Comm. Math. Phys. **157**, 305 (1993)

[Schw92] L. B. Schweitzer, *A short proof that $M_n(A)$ is local if A is local and Fréchet*, Int. J. Math. **3**, 581 (1992)

[Ser55] J.-P. Serre, *Faisceaux Algébriques Cohérents*, Ann. Math. **61**, 197 (1955)

[S01] A. Sitarz, *Twists and Spectral Triples for Isospectral Deformations*, Lett. Math. Phys. **58**, 69 (2001)

[S03] A. Sitarz, *Equivariant Spectral Triples.* In: P. M. Hajac and W. Pusz (eds.), *Noncommutative Geometry and Quantum Groups*, vol. 61. Banach Centre Publications, IMPAN, Warszawa, pp. 231-263 (2003)

[Ska84] G. Skandalis, *Some Remarks on Kasparov Theory*, J. Funct. Analysis **56**, 337 (1984)

[Sle85] S. Slebarski, *Dirac operators on a compact Lie group*, Bull. London Math. Soc. **17**, 579 (1985)

[Sle87] S. Slebarski, *The Dirac Operator on Homogeneous Spaces and Representations of Reductive Lie Groups I-II*, Amer. J. Math. **109**, pp. 283-301 and pp. 499-520 (1987)

[Sui05] W. D. v. Suijlekom, *The Geometry of Noncommutative Spheres and Their Symmetries*, PhD Thesis (2005)

[Swa62] R. G. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105**, 264 (1962)

[Sw69] M. E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York (1969)

[Szy03] W. Szymanski, *Quantum lens spaces and principal actions on graph C^*-algebras*, Pacific J. Math. **211**, 249 (2003)

[Tak70] M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras*, Lect. Notes in Math. **128**, Springer, Berlin (1970)

[Van99] F. J. Vanhecke, *On the product of real spectral triples*, Lett. Math. Phys. **50**, 157 (1999)

[Ven10] J. J. Venselaar, *Classification of spin structure on the noncommutative n-torus with Clifford structure*, arXiv:1003.5156v2 (2010)

[V79] D. Voiculescu, *Some results on norm-ideal perturbations of Hilbert space operators I*, J. Operator Theory **2**, 3 (1979)

[V81] D. Voiculescu, *Some results on norm-ideal perturbations of Hilbert space operators II*, J. Operator Theory **5**, 77 (1981)

[Wael] L. Waelbroeck, *Topological Vector Spaces and Algebras*, Springer-Verlag, Berlin (1971)

[Wag12] S. Wagner, *A Geometric Approach to Noncommutative Principal Torus Bundles*, arXiv:1108.4294v2 (2012)

[Waw] A. Wawrzynczyk, *Group Representations and Special Functions*, Polish Scientific Publishers, Warszawa (1984)

[W-O93] N. E. Wegge-Olsen, *K-Theory and C^*-Algebras. A Friendly Approach*, Oxford University Press (1993)

[Wod84] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent. Math. **75**, 143 (1984)

[Wor87] S. L. Woronowicz, *Compact Matrix Pseudogroups*, Comm. Math. Phys. **111**, 613 (1987)

[Wor87b] S. L. Woronowicz, *Twisted $SU(2)$ Group. An Example of a Non-Commutative Differential Calculus*, Publ. RIMS, Kyoto University, vol. **28**, 117 (1987)

[Wor89] S. L. Woronowicz, *Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups)*, Comm. Math. Phys. **122**, 125 (1989)

[Wor91] S. L. Woronowicz, *Unbounded elements affiliated with C^*-algebras and non-compact quantum groups*, Comm. Math. Phys. **136**, 399 (1991)

[Zie05] B. Zielinski, *Locally coalgebra-Galois extensions*, arXiv:math/0512150 (2005)