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PhD Thesis:

On Freed-Witten Anomaly and Charge/Flux Quantization in String/F Theory

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Contents

Preface

String theory, with its richness of dynamical scenarios, represents a microscopic tool of utmost relevance to jointly describe all the fundamental forces of nature and simultaneously it involves a wide and fascinating spectrum of geometrical implications. Because of its nature, in many cases research in string theory cannot be carried through on a purely locally-based analysis, but it should often be supplemented with intrinsically global investigations and consistency checks. The mathematical apparatus needed for such a study, however, turns out to be usually much more sophisticated than the one which is sufficient for field theory-like computations, and, unfortunately, it is not always really familiar to the physics community. The arguments discussed in this thesis constitute manifest examples of this situation.

The central topic is the Freed-Witten anomaly [1], that is a global anomaly of the string path integral measure; thus, any issue regarding it has by definition to be addressed within a global framework, and it turns out that even the elementary tools based on cohomology for some purposes happen to fail to give the correct physical explanations. Hence one unavoidably needs more refined mathematical methods.

The impact of the Freed-Witten anomaly on the string background is *twofold*: either it imposes on the geometry a necessary and sufficient cohomological condition for cancellation or it leads to suitable quantization conditions for the fluxes in order to get rid of ambiguities in the partition function. The former is a condition expressed in terms of classes belonging to the torsion part of integral cohomology of the internal space and it is at one degree in cohomology more with respect to the latter, which regards instead the free part of such cohomology.

Another clear example of the necessity of the global analysis presented in this thesis, is provided by the recent developments on F-theory [2]. Indeed, as opposed to the models based on heterotic string theory, a couple of years ago it has been realized that actually F-theory based models of compactification down to four dimensions can accommodate a limit in which gravity decouples: this means that many computations in the effective gauge theory arising from the compactification can be performed essentially forgetting about the geometrical constraints imposed by the dynamics of the gravitons in the extra-dimensions. This led people to find exciting results and predictions from the phenomenological viewpoint of Grand Unified Theories (GUT).

However, in order to make all this very well-working and elegant machinery really reliable, it must be supported by a correct UV completion, namely by finding a suitable global model in which the already constructed local picture has to be consistently embedded. This requires, among other things, a number of global consistency checks, like cancellation of anomalies and tadpoles, that are clearly missed by the local analysis and for which a more detailed understanding of the geometrical structures underlying such models is compulsory. As a consequence of the global analysis, one can gain a better comprehension of how to treat fluxes, which is crucial in model building, for issues regarding both moduli stabilization and matter chirality. Indeed, for instance, people in F-theory usually introduce by hand fluxes to obtain chiral matter in the double intersections of 7-branes; however, it could very well be that some of these fluxes must be already present due to a strictly topological reason, such as a shift in their quantization condition; and also the opposite could happen, namely that one may want for phenomenological reasons to put to zero some 7-brane gauge flux, which actually turns out to be topologically obstructed to vanish, due to cancellation of anomalies.

In the light of what has been said above, this thesis focuses on the interplay between the intrinsically global phenomenon of the Freed-Witten anomaly and the global properties of the recently constructed F-theory models. For this reason, many of the mathematical methods presented throughout the first part and briefly reviewed in the various appendices are meant as a preparation to their direct physical application in a very adequate context, such as F-theory.

This thesis will be mainly concerned with type II string theories (IIB as long as the F-theory side is concerned) on a ten dimensional target manifold M, which will always be a cartesian product of the form $\mathbb{R}^{1,d} \times \tilde{M}$ made by an "external" part with Minkowskian signature regarded as the space-time where the effective field theory lives, and by an "internal" part being a (9-d)-dimensional compact manifold which, rather, will be further specified when needed.
The thesis is divided into two parts.
The aim of the first is to give a self-contained overview on this topic and to describe the consequences that the cancellation of the Freed-Witten anomaly has on quantization and classification of the various target space fields in the game [3]. Moreover, the closely related topic of the various notions of D-brane charge in type II supergravity theories [4] is introduced and discussed, focusing on the properties of quantization and gauge invariance of each kind of charge. Schematic mathematical introductions (with no pretence of completeness) are provided in the appendices on the topological and geometrical structures and techniques that the Freed-Witten anomaly naturally involves and that are needed to carry out such an analysis. One of them is K-theory whose necessity and virtues in this context are explained in detail. Two separate approaches to the K-theoretical classification of D-branes are then compared, showing their different features, and linked provided a simple choice of background fields is made [5].
The second part, instead, is motivated by the recent progress in local F-theory model building and in particular by the numerous attempt to find global completions of them [2]. A brief introduction on the subject is given, that will focus in particular on the M/F-theory duality and on the mechanism of gauge symmetry enhancement, analyzed either from a geometrical or from a physical (stringy) viewpoint. Then, this part mainly points towards the search of a better control on the many topological constraints that in this context play a major role. Indeed, besides tadpole cancellation, one of the main sources of such global consistency conditions is the Freed-Witten anomaly of the F-theory 7-branes. A direct cohomological generalization of it in the full non-perturbative setup is still lacking and with no doubt constitutes a rather ambitious purpose. The aim here is to make the first steps in this direction [6], working out the right quantization conditions of brane and bulk-type fluxes using the duality between M-theory and F-theory [7] and Witten's quantization condition for the M-theory G-flux [8].

A note for the reader

There are several technical sections that, at a first reading, can be skipped without risk of missing important concepts. These are all contained in the first part and are 1.2, 1.4 and the subsection 3.4.1; they will be indicated with a small asterisk. For a proper understanding of them, the non-familiar reader might need much of the material contained in the appendices and

not just refer to them for the notations.
Finally, the appendix F serves as a schematic toolkit for the reader not familiar with toric techniques to go through the steps of the resolution procedures carried out in the last chapter.

Part I

The Freed-Witten Anomaly and The Road to K-Theory

Introduction

A generic string background, characterized by a B-field and by the presence of D-branes, induces a geometry where traditional mathematical or field theoretical tools have to be updated or upgraded in order to provide an effective description. For instance, it is generally accepted that the appropriate mathematical structure underlying such backgrounds is provided by gerbes, which are generalizations of bundles. In this framework the idea of a gauge bundle associated with the worldvolume of a D-brane is not always adequate and needs to be refined. The Freed-Witten anomaly enters this refinement in a crucial way, because it singles out the allowed field configurations.

These problems have already been analyzed in the literature, starting from the seminal paper of D. Freed and E. Witten [1]. However the analysis has been carried out in a case-by-case basis and a general classifying scheme is still lacking. However, a mathematical tool exists that is capable of encompassing all the particular cases of backgrounds mentioned above: this is Čech hypercohomology of sheaves. Indeed, the first goal of this part is to show that the second hypercohomology group of some specific sheaves (characterizing the target space and the D-brane worldvolumes) furnishes a tool to classify "gerbes with connection" that are Freed-Witten anomaly free. This turns out to be the instrument one needs in order to select the right string backgrounds with D-branes and B-field. To make an example which is more familiar in the physical literature, the first hypercohomology group of the same sheaves classify all the line bundles with connection, so it classifies the (classical) $U(1)$ gauge field theories. To deal with general string backgrounds one needs to go one step further in hypercohomology with respect to this example. Moreover, although the term "hypercohomology" is rarely used in the physical literature, one can find interesting examples of it under a different terminology. For instance, the double BRST complex in local field theory is an example in which the famous descent equations are exactly the cocycle conditions for hypercohomology.

Such mathematical classification will be followed by a case-by-case analysis of the various situations arising from it. Some of them have already been analyzed in the literature, others are new. It is, for instance, well-known that generically one cannot define a canonical gauge theory on a D-brane in the presence of a non-zero B-field, due to the freedom under large gauge transformations. This possibility arises only if some specific conditions are satisfied. A particularly interesting circumstance is the one in which the B-field is flat, where fractional or even irrational charges of sub-branes naturally appear; for a suitable gauge choice, they can be seen as arising from "gauge bundles with non-integral Chern class", whose precise geometrical interpretation is given.

All these concepts, met here from the perspective of the D-brane worldvolume theory, lead to different notions of charge, which is actually a well-known and typical fact arising in the context of gauge theories with Chern-Simons terms [4]. Each notion has only some of the usual features of a physical charge (i.e. conservation, quantization, gauge invariance and localization), but

under particular assumptions, one can construct a new charge matching all of them. Here is where a new mathematical tool, K-theory, enters the game and cures the problems that the old cohomological method has in classifying charges [9]. The necessity of this improvement is stressed from both bulk and worldvolume perspectives.
K-theory is thus introduced in the discussion as a natural generalization of cohomology if one takes into account Freed-Witten anomalies of D-branes. In fact, another approach to it is possible [10], that is more along the spirit of the mathematical meaning of K-theory, being the group of "differences" of vector bundles: it has the advantage of taking into account the gauge bundle on the brane in computing its charge and it is inspired by the physical phenomenon of tachyon condensation of brane/anti-brane pairs. The last goal of this part is to relate these two approaches in the special situation of vanishing B-field, when either there exists a canonical gauge theory on the brane or target space filling D9-branes do not suffer from Freed-Witten anomalies.

This part is organized as follows: in chapter 1 the classification of Freed-Witten anomaly free configurations for the gauge field on the D-brane and the B-field is addressed and simplified by a more concrete analysis of the various cases which arise; chapter 2 contains a discussion on the debated concept of D-brane charge, which starts from the standard cohomological notion and then shows why this notion needs an improvement; finally, in chapter 3 the K-theoretical improvement is described in detail, in both its two main approaches available in the literature.

Chapter 1

Anomaly-free string backgrounds

In this starting chapter the main topic of the thesis is introduced, that is the Freed-Witten anomaly, focusing on the practical mechanism of anomaly cancellation and on the mathematical aspects which will be relevant for the consequent classification of the background fields.

This chapter is organized as follows: in section 1 a general introduction on the Freed-Witten anomaly is given; in section 2 the classification of anomaly free open string backgrounds by means of hypercohomology; in section 3 a case-by-case analysis about the gauge theory on a D-brane arising from the classification is carried out; section 4 contains the geometrical meaning of "gauge bundles with non-integral Chern class", which appeared in the previous section; finally, section 5 contains a brief discussion on the case of stacks of coincident D-branes.
In order to follow the technical details of this chapter, the non-familiar reader may need the appendices A, B and C.

1.1 What is the Freed-Witten anomaly?

Consider type II string theory on a smooth target space M and a single smooth D-brane with world-volume Y. Thus an embedding of the open string world-sheet Σ is given

$$\phi : \Sigma \longrightarrow M \tag{1.1}$$

such that

$$\phi|_{\partial\Sigma} : \partial\Sigma \longrightarrow Y\,. \tag{1.2}$$

In the appropriate units, the Polyakov action for the string contains the following terms, [11]:

$$S \supset \left(\int_{\Sigma} \psi \cdot D_{\phi}\psi \right) + 2\pi \cdot \left(\int_{\Sigma} \phi^{*}B + \int_{\partial\Sigma} \phi^{*}A \right), \tag{1.3}$$

where the Dirac operator D_{ϕ} is coupled to the pull-back of the tangent bundle M to the world-sheet via ϕ and target space fields are redefined in units of 2π in order to formulate their quantization conditions in terms of true integers. After integrating over all the fermionic matter fields in the path integral, one ends up with an expression which is still to be integrated over the ghost fields and over the bosonic part of the configuration space. Such an expression will contain the following problematic terms, directly coming from the integration of the exponential of (1.3):

$$e^{iS} \supset \operatorname{pfaff} D_\phi \cdot \exp\left(2\pi i \cdot \int_\Sigma \phi^* B \right) \cdot \exp\left(2\pi i \cdot \int_{\partial\Sigma} \phi^* A \right), \tag{1.4}$$

where "pfaff" stands for Pfaffian, that is the square root of the determinant. Each term deserves a brief discussion.

- The Pfaffian may be ambiguous. Indeed, evaluated in a point $\phi \in \mathrm{Maps}(\Sigma, M)$, it must satisfy $(\operatorname{pfaff} D_\phi)^2 = \det D_\phi$, so one has a sign ambiguity and a natural definition of it is needed. One can think of it as a section of a line bundle over $\mathrm{Maps}(\partial\Sigma, Y)$ (that is nothing but the loop space of LY of Y), called *pfaffian line bundle*, with natural metric and *flat* connection, [12]: $\mathrm{Pfaff} \to LY$.
- The middle term represents the holonomy of the B-field over the world-sheet. The B-field must be regarded as the connection of a generalization of line bundles, called *1-gerbe* (or simply gerbe): connections on n-gerbes are locally (n+1)-forms, rather than 1-forms and field strengths are represented in de Rham cohomology by (n+2)-forms instead of 2-forms. Precise mathematical definitions and the properties used in the sequel are given in the appendices A and B.
  Being Σ an open surface, this term, which is clearly the analogous of a Wilson loop for a line bundle, is *not* gauge invariant; rather, it is a section of an other line bundle over the loop space of Y: $\mathcal{L}_B \to LY$. For an extended discussion about these holonomies, both in the more familiar line bundle case and in its gerbe generalization, the reader is referred to appendix C.
- The last term, which is a standard Wilson loop over the boundary of the world-sheet of the gauge connection on the brane, is crucial since, provided a suitable quantization condition for the gauge field strength, it cancels the ambiguities of the previous two factors, thus leading to a well-defined function to be integrated over the bosonic configuration space.

Hence the aim is to build up a well defined function out of the section (1.4). This can be clearly achieved by dividing it by another section. Stated in a more precise way, one needs the line bundle $\mathrm{Pfaff} \otimes \mathcal{L}_B$ to be *geometrically trivial.* This means essentially two things:

1. The line bundle is topologically trivial (vanishing first Chern class), i.e. it can be trivialized by means of a *global nowhere-zero section* s that, due to the presence of a metric, one can normalize to have unit norm.
2. s can be chosen to be *parallel*, i.e. killed by the covariant derivative.

Now, since s is going to be the candidate section to divide by the expression (1.4), it is easy to see that the second point above is crucial. Indeed, without that, s would be defined only up to a unit norm function, which would hide completely any physical information contained in (1.4). Thanks to point 2., instead, s will be defined up to a *constant*, which is immaterial for the path integral. The anomaly, then, is nothing but the obstruction to the existence of s.
Thus, the next step is to formulate the necessary and sufficient condition of existence and then, once this is fulfilled, to explicitly construct such an s. Having done that, the above mentioned well-defined function will be:

$$\frac{\operatorname{pfaff} D_\phi \cdot \exp\left(2\pi i \cdot \int_\Sigma \phi^* B \right) \cdot \exp\left(2\pi i \cdot \int_{\partial\Sigma} \phi^* A \right)}{s}. \tag{1.5}$$

To this end, some mathematical properties of the two line bundles in the game are needed. Again, for the technical details the reader is referred to the appendices A, B and C. In particular, as pointed out in appendix C, a line bundle over the loop space of Y is associated with a gerbe on Y and a trivialization of the latter always induces a trivialization of the former. Moreover, consider the following short exact sequences of sheaves on any space X:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] & \underline{\mathbb{R}} \arrow[r, "q"] & \underline{S}^1 \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] \arrow[u, "\Vert"] & \mathbb{R} \arrow[r, "q"] \arrow[u, "\cup"] & S^1 \arrow[r] \arrow[u, "\cup"] & 0\,, \end{tikzcd} \tag{1.6}$$

where $\underline{\mathcal{F}}$ stands for the sheaf of smooth $\mathcal{F}$-valued functions on X, while i and q are respectively an immersion and a quotient map (that is the exponential). As usual, these will induce a long exact sequence in cohomology:

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & H^n(X, \underline{S}^1) \arrow[r, "\varphi", "\simeq"'] & H^{n+1}(X, \mathbb{Z}) \arrow[r] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & H^n(X, \mathbb{R}) \arrow[r, "q_\star"] & H^n(X, S^1) \arrow[r, "\beta"] \arrow[u] & H^{n+1}(X, \mathbb{Z}) \arrow[r, "i_\star"] \arrow[u, "\Vert"] & H^{n+1}(X, \mathbb{R}) \arrow[r] & \cdots\,, \end{tikzcd} \tag{1.7}$$

where $n \geq 1$, $i_\star$, $q_\star$ are respectively the induced immersion and quotient in cohomology and φ, β respectively the Bockstein maps of the first and the second sequence. In particular, φ is an isomorphism because the sheaf $\underline{\mathbb{R}}$ is acyclic and thus all the groups $H^n(X, \underline{\mathbb{R}})$ for $n \geq 1$ are vanishing. All these cohomology groups are finitely generated abelian groups and, in particular, $H^n(X, \mathbb{Z})$ is always of the form

$$H^n(X, \mathbb{Z}) \;=\; \mathbb{Z}^{b_n} \oplus_j \mathbb{Z}_{p(j)^{k(j)}}\,, \tag{1.8}$$

where the first is called the *free* part and the second is called the *torsion* part (b_n, $p(j)$ and $k(j)$ are all integer numbers and $p(j)$ are prime).
Now, the most relevant properties of the two line bundles in question are the following.

- The first Chern class of the Pfaffian line bundle is:

$$c_1(\text{Pfaff}) \;=\; \int_{\phi(\partial\Sigma)} W_3(TY) \quad \in\; H^2(LY, \mathbb{Z})\,, \tag{1.9}$$

  where $W_3(TY) \in H^3(Y, \mathbb{Z})$ is the third integral Stiefel-Whitney class of the brane and actually belongs to the torsion part[1] of the group $H^3(Y, \mathbb{Z})$; in physical terms, it measures the obstruction to have U(1)-charged spinors propagating on the brane. Thus one can say that there is a gerbe associated to the Pfaffian line bundle, whose first Chern class is W_3 itself; this gerbe, and so its associated line bundle on the loop space, is *pure torsion*. Hence, as it is clear from (1.7), if the first Chern class is torsion, i.e. it belongs to $\text{Ker}(i_\star) \subset H^{n+1}(X, \mathbb{Z})$ (n=1 for the bundle, n=2 for the gerbe), then such bundle or gerbe can be realized by means of *constant* transition functions which lift their isomorphism

[1] By exactness of (1.7), $\text{Im}(\beta) = \text{Tors}\, H^{n+1}(X, \mathbb{Z}) \simeq \bar{H}^n(X, S^1)$, where the latter is the group of connected components of $H^n(X, S^1)$, i.e. $H^n(X, S^1)/H^n_0(X, S^1)$, where the denominator is the component connected to the identity. Thus, a class in $H^2(X, S^1)$ with vanishing Bockstein actually lies in $H^n_0(X, S^1) \equiv H^n(X, \mathbb{R})/H^n(X, \mathbb{Z})$ and hence admits a lift to a real class up to integral classes.

classes to $H^{1,2}(X, S^1)$.
Moreover, the constant transition functions of the section pfaff D_ϕ appearing in the path integral are in the class:

$$[\text{pfaff}(D_\phi)_{\alpha\beta}] \;=\; \int_{\phi(\partial\Sigma)} w_2(TY) \quad \in\; H^1(LY, S^1)\,, \tag{1.10}$$

where $w_2(TY) \in H^2(Y, \mathbb{Z}_2)$ is the second Stiefel-Whitney class[2] of the brane and it measures the obstruction to have worldvolume spinors.
Finally, since one can always have locally defined parallel sections, let ρ_α be the *parallel local section* having transition functions $\rho_{\alpha\beta}$ in the class (1.10).

- The first Chern class of the line bundle $\mathcal{L}_B$ is:

$$c_1(\mathcal{L}_B) \;=\; \int_{\phi(\partial\Sigma)} c_1(\mathcal{G}) \quad \in\; H^2(LY, \mathbb{Z})\,, \tag{1.11}$$

where $c_1(\mathcal{G}) \in H^3(Y, \mathbb{Z})$ is the first Chern class of the associated gerbe on Y. Such a gerbe really belongs to the set of *bulk* data and it has a natural field strength, the bulk H-field and a natural connection, the bulk B-field. Here only its *restriction* to Y is present, being the brane embedded as a submanifold in the target space. Although in the sequel this pull-back will not always be written explicitly, notice that every conclusion about this gerbe and its connection, the B-field, will actually concern only their restriction to the brane. In the notations of appendices A and B $\mathcal{G}$ is represented by the following triple:

$$\mathcal{G} \;=\; \{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha\} \qquad \text{such that} \qquad \begin{cases} \check{\delta}^2 g_{\alpha\beta\gamma} = 1 \\ B_\alpha - B_\beta = \mathrm{d}\Lambda_{\alpha\beta} \\ \check{\delta}^1 \Lambda_{\alpha\beta} = (2\pi i)^{-1}\mathrm{d}\,\log g_{\alpha\beta\gamma}\,, \end{cases} \tag{1.12}$$

where $g_{\alpha\beta\gamma}$ are local 0-forms (the transition functions of the gerbe), $\Lambda_{\alpha\beta}$ are local 1-forms (they have no analog for line bundles) and B_α are local 2-forms (the connection).
Moreover, as already said, a trivialization of $\mathcal{G}$ naturally induces a trivialization of $\mathcal{L}_B$: let σ_α be a *local parallel* one induced on $\mathcal{L}_B$.

Within this framework, one can now formulate the necessary and sufficient condition for the anomaly cancellation to be possible, found by Freed and Witten in their seminal paper [1]. This condition is nothing but the statement of *topological triviality* of the line bundle Pfaff $\otimes\, \mathcal{L}_B$. It is written in terms of the Chern classes of the associated gerbes on Y and it reads:

$$W_3(TY) \;=\; c_1(\mathcal{G})\,. \tag{1.13}$$

If (1.13) is fulfilled, Pfaff $\otimes\, \mathcal{L}_B$ can be trivialized by means of a global section. But, as said, this is not enough: in order to achieve geometric triviality, one has to require that also the class in $H^1(LY, S^1)$ of the transition functions of this trivializing global section vanishes, that is the requirement of parallel trivialization. Thus, having $s = \rho_\alpha \cdot \sigma_\alpha$ as the parallel global section to put in the denominator of (1.5) amounts to imposing the following condition:

$$[\sigma_{\alpha\beta}] \;=\; [\rho_{\alpha\beta}]\,. \tag{1.14}$$

[2] In (1.10) the induced map in cohomology of the natural immersion $\mathbb{Z}_2 \hookrightarrow S^1$ is implicitly used.

Therefore, the role of a non-anomalous D-brane is exactly to provide the right reparameterization of the gerbe $\mathcal{G}$ restricted to it, in such a way that the new one induces a trivialization for the associated line bundle whose transitions satisfy the relation (1.14) (thus canceling the ambiguity of the Pfaffian). As is explained in detail in the next section, this reparameterization is represented by a geometrically trivial gerbe $\{g^{-1}_{\alpha\beta\gamma}\eta_{\alpha\beta\gamma}, \Lambda_{\alpha\beta}, \mathrm{d}A_\alpha \equiv F_\alpha\}$: here $\eta_{\alpha\beta\gamma}$ are the transitions of the restricted gerbe $\mathcal{G}$ inducing the transitions $\sigma_{\alpha\beta}$, while F_α is the gauge field strength on the brane that undergoes gauge transformations given by $\mathrm{d}\Lambda_{\alpha\beta}$. These transformations are just the opposite of the B-field ones in (1.12), so that, in the new coordinates, $\mathcal{G}|_Y$ will always be of the form $\{\eta_{\alpha\beta\gamma}, 0, B+F\}$, with $B+F$ being the well-known gauge invariant combination.
It is important to notice at this point that $\mathrm{d}\Lambda_{\alpha\beta}$ is just the local expression of the familiar large gauge transformations $\Phi \in H^2(X,\mathbb{Z})$, which are represented by closed, globally defined and integrally quantized 2-forms. Actually, in the present situation, it is easy to see that they can be locally put to zero, i.e. B and F can be made both globally defined. Indeed, since the gerbe $\mathcal{G}|_Y$ is torsion by (1.13), one can choose the transitions $g_{\alpha\beta\gamma}$ to be constant; thus, using the third relation in (1.12), one finds that $\Lambda_{\alpha\beta}$ are closed under $\check{\delta}^1$, that means also exact because the sheaf of real smooth 1-forms $\Omega^1_{\mathbb{R}}$ which they belong to is acyclic. Hence, nothing forbids to take $\Lambda_{\alpha\beta} = 0$. However, although B and F are now separately globally defined and can be integrated on surfaces of Y, they are *not* immune from large gauge transformations Φ because these are not exact under the de Rham differential and so they do not vanish, even if their local expressions $\Phi_{\alpha\beta} = \mathrm{d}\Lambda_{\alpha\beta}$ do in each double intersection. The reason of such intrinsically global non-triviality of the transition functions is also natural and easy to understand from the exact sequence below in (1.7): this Φ is simply the integral class modulo which B and F, that are the lifts to $H^2(Y,\mathbb{R})$ (when they exist) of their holonomies[3] are defined.

Before ending this section and starting with the actual classification of non-anomalous gauge and B-fields, a comment is in order regarding the necessity of the condition (1.13) for the anomaly cancellation.
At a first sight, the condition (1.13) looks too strong, since for the topological triviality of the tensor product Pfaff $\otimes\, \mathcal{L}_B$ it would be enough to just require the equality of their first Chern classes, $c_1(\text{Pfaff}) = c_1(\mathcal{L}_B)$, while (1.13) is rather the condition of topological triviality of the product of the associated gerbes. However, in general the anomaly could not be detected by genus one Riemann surfaces like $\partial\Sigma \times C$, but only by a map from a surface of higher genus. Thus, Pfaff $\otimes\, \mathcal{L}_B$ could not be any more sufficient; being, instead, (1.13) the only cohomological condition on Y implying triviality of Pfaff $\otimes\, \mathcal{L}_B$, one expects that the properties of factorization of the string measure and of unitarity of string scattering amplitudes will lead to the necessity of (1.13).

1.2 Classification by hypercohomology*

In this section the classification group is described for B-field and A-field (or gauge field) configurations in a string background characterized by the presence of a single D-brane [3]. Such a background is specified in particular by a target space gerbe $\mathcal{G}$ belonging to the following hypercohomology group[4]:

$$\mathcal{G} = [\,\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha\}\,] \in \check{H}^2(M,\, \underline{S}^1 \xrightarrow{\tilde{d}} \Omega^1_{\mathbb{R}} \xrightarrow{d} \Omega^2_{\mathbb{R}}\,)\,, \tag{1.15}$$

[3] $e^{2\pi i B}$, $e^{2\pi i F} \in H^2(Y,S^1)$.

[4] Notations and conventions are found in appendices A and B.

where $\tilde{d} = (2\pi i)^{-1}\, d \circ \log$, $g_{\alpha\beta\gamma}$, as already seen, are functions from triple intersections to S^1, $\Lambda_{\alpha\beta}$ are 1-forms on double intersections and B_α are 2-forms on the open sets of the cover. In (1.15), we denote by $\Omega^p_{\mathbb{R}}$ the sheaf of real p-forms on M. On a single brane $Y \subset M$ we consider the restriction of the target space gerbe, for which we use the same notation $\mathcal{G}\,|_Y = [\,\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha\}\,] \in \check{H}^2(Y, \underline{S}^1 \to \Omega^1_{\mathbb{R}} \to \Omega^2_{\mathbb{R}}\,)$. To give a meaning to the holonomy for open surfaces with boundary on Y, we must fix a specific representative of the class $\mathcal{G}\,|_Y$, i.e., a specific hypercocycle; this operation is analogous to fixing a set of local sections on a line bundle up to pull-back by isomorphism (see appendix C). To compensate for the possible non-definiteness of pfaff D_ϕ, as already stressed in the previous section, this hypercocycle must take the form $\{\eta_{\alpha\beta\gamma}, 0, B+F\}$, with $\eta_{\alpha\beta\gamma}$ representing the class $w_2 \in H^2(Y, S^1)$, denoting by S^1 the *constant* sheaf. The choice of the specific cocycle $\eta_{\alpha\beta\gamma}$ in the class w_2 turns out to be immaterial, as it will be shown later.

In order to obtain the hypercocycle $\{\eta_{\alpha\beta\gamma}, 0, B+F\}$ from any gauge representative $\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha\}$ of the gerbe $\mathcal{G}\,|_Y$, the brane must provide a reparametrization of $\mathcal{G}\,|_Y$, which, by an active point of view, is a hypercoboundary, i.e., a geometrically trivial gerbe. That is, given $\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha\}$, the brane must provide a coordinate change $\{g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}, \Lambda_{\alpha\beta}, dA_\alpha\}$, so that:

$$\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha\} \cdot \{g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}, \Lambda_{\alpha\beta}, dA_\alpha\} = \{\eta_{\alpha\beta\gamma}, 0, B+F\} \tag{1.16}$$

for a globally defined $B+F = B_\alpha + dA_\alpha$. In order for this correction to be geometrically trivial, it should be that:

$$\{g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}, \Lambda_{\alpha\beta}, dA_\alpha\} = \check{\delta}^1\{h_{\alpha\beta}, A_\alpha\}\,, \tag{1.17}$$

i.e. $\{g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}, \Lambda_{\alpha\beta}, dA_\alpha\} = \{\check{\delta}^1 h_{\alpha\beta}, -\tilde{d}h_{\alpha\beta} + A_\beta - A_\alpha, dA_\alpha\}$. For this to hold one must have:

- $\{g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}\} = \{\check{\delta}^1 h_{\alpha\beta}\}$: this is precisely the statement of Freed-Witten anomaly, since, considering the Bockstein homomorphism φ in degree 2 of the sequence above in (1.7), this is equivalent to $\varphi(\,[\,g_{\alpha\beta\gamma}\,]\,) = \varphi(\,[\,\eta_{\alpha\beta\gamma}\,]\,)$, i.e., $c_1(\mathcal{G}|_Y) = W_3(Y)$; only under this condition is $g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}$ trivial in the $\underline{S}^1$-cohomology;

- $A_\beta - A_\alpha = \tilde{d}h_{\alpha\beta} + \Lambda_{\alpha\beta}$: these must be the transition relations for A_α (coherently with [14]); this is always possible since $\check{\delta}^1\{\tilde{d}h_{\alpha\beta}\} = \{\,\tilde{d}(\,\eta_{\alpha\beta\gamma} - g_{\alpha\beta\gamma}\,)\,\} = \{-\tilde{d}g_{\alpha\beta\gamma}\} = -\check{\delta}^1\{\Lambda_{\alpha\beta}\}$ and $\Omega^1_{\mathbb{R}}$ is acyclic.

From the transition relations of A_α we obtain $dA_\beta - dA_\alpha = d\Lambda_{\alpha\beta}$, thus $B+F$ is globally defined. Of course B_α and A_α themselves depends on the gauge choices, while $B+F$ is gauge-invariant.[5]

A comment is worth on the role of the representative $\eta_{\alpha\beta\gamma}$ of the class $w_2(Y) \in \check{H}^2(X, S^1)$. The choice of a different representative corresponds to changing by *constant* local functions the chosen sections of the bundle over the loop space, which define the holonomy for open surfaces. This kind of ambiguity is also present for the Pfaffian, since it also defines a section of a flat bundle with the same holonomy. If $w_2(Y) \neq 0$, there is no possibility to eliminate this non-definiteness. One can only choose the sections for the Pfaffian and for the gerbe, in such a way that on the tensor product one has a *global* flat section, up to an immaterial overall constant. Instead, if $w_2 = 0$, both the pfaffian and the gerbe are geometrically trivial, thus there is a

[5] We remark that, for $W_3(Y) = 0$, from the exact sequences in (1.7), it follows that $w_2(Y)$, having image 0 under the degree-2 Bockstein homomorphism, by exactness can be lifted to a real form G on Y. Therefore, the gerbe $[\,\{\eta_{\alpha\beta\gamma}, 0, B+F\}\,]$ can also be represented by $[\,\{1, 0, B+F+G\}\,]$: however, this is *not* the cocycle one needs, since one needs transition functions realizing the class $w_2(Y)$. These two cocycles are equivalent on closed surfaces, since they represent the same gerbe, but *not* on open ones.

preferred choice, given by a global flat section for both. In this case, one fixes the canonical representative $\eta_{\alpha\beta\gamma} = 1$. In the following the consequences of this fact for the gauge theory of the D-brane will be explained.

The problem is now to jointly characterize B-field and A-field taking into account the gauge transformations contained in the previous description. This unifying role is played by a certain hypercohomology group, which will now be introduced. Since this construction is not very familiar in the literature, it is better for pedagogical reasons to start with the analogous group for line bundles.

1.2.1 Line bundles

Consider an embedding of manifolds $i : Y \to M$: the aim is to describe the group of line bundles on M which are trivial on Y, with a fixed trivialization. Recall that $\underline{S}^1$ is the sheaf of smooth functions on M: it turns out that the sheaf of smooth functions on Y is its pull-back $i^*\underline{S}^1$. One thus obtains a cochain map $(i^*)^p : \check{C}^p(M, \underline{S}^1) \longrightarrow \check{C}^p(Y, \underline{S}^1)$, which can be described as follows: choose a good cover $\mathfrak{U}$ of M restricting to a good cover $\mathfrak{U}|_Y$ of Y, such that every p-intersection $U_{i_0\cdots i_p}|_Y$ comes from a unique p-intersection $U_{i_0\cdots i_p}$ on M. Given a p-cochain $\oplus_{i_0<\cdots<i_p} f_{i_0\cdots i_p}$, restrict $f_{i_0\cdots i_p}$ to $U_{i_0\cdots i_p}|_Y$ whenever the latter is non-empty. In this way one obtains a double complex:

$$\begin{tikzcd}
\check{C}^0(Y,\underline{S}^1) \arrow[r, "\check{\delta}^0"] & \check{C}^1(Y,\underline{S}^1) \arrow[r, "\check{\delta}^1"] & \check{C}^2(Y,\underline{S}^1) \arrow[r, "\check{\delta}^2"] & \cdots \\
\check{C}^0(M,\underline{S}^1) \arrow[r, "\check{\delta}^0"] \arrow[u, "(i^*)^0"] & \check{C}^1(M,\underline{S}^1) \arrow[r, "\check{\delta}^1"] \arrow[u, "(i^*)^1"] & \check{C}^2(M,\underline{S}^1) \arrow[r, "\check{\delta}^2"] \arrow[u, "(i^*)^2"] & \cdots\,.
\end{tikzcd}$$

Denote by $\check{H}^\bullet(M, \underline{S}^1, Y)$ the hypercohomology of this double complex. The claim is that $\check{H}^1(X, \underline{S}^1, Y)$ is the wanted classification group . In fact, the latter can be defined in the following way: choose a line bundle L on M with a fixed set of local sections $\{s_\alpha\}$, so that the transition functions are $\{g_{\alpha\beta}\}$ for $g_{\alpha\beta} = s_\alpha/s_\beta$. Consider $\{s_\alpha|_Y\}$ and express the trivialization by means of local functions $\{f_\alpha\}$ on Y such that $f_\alpha \cdot s_\alpha|_Y$ gives a global section of $L|_Y$. One has that $\check{C}^1(M, \underline{S}^1, Y) = \check{C}^1(M, \underline{S}^1) \oplus \check{C}^0(Y, \underline{S}^1)$, so that one can consider the hypercochain $\{g_{\alpha\beta}, f_\alpha\}$. This turns out to be a hypercocycle: to see this, it is useful to describe the cohomology group $\check{H}^1(M, \underline{S}^1, Y)$.

- *Cocycles:* since $\check{\delta}^1\{g_{\alpha\beta}, f_\alpha\} = \{\check{\delta}^1 g_{\alpha\beta}, ((i^*)^1 g_{\alpha\beta})^{-1} \cdot f_\beta f_\alpha^{-1}\}$, cocycles are characterized by two conditions: $\check{\delta}^1 g_{\alpha\beta} = 0$, i.e., $g_{\alpha\beta}$ is a line bundle L on M, and $(i^*)^1 g_{\alpha\beta} = f_\beta f_\alpha^{-1}$, i.e., f_α trivializes $L|_Y$.

- *Coboundaries:* $\check{\delta}^0\{g_\alpha\} = \{\check{\delta}^0 g_\alpha, (i^*)^0 g_\alpha\}$ thus coboundaries represents line bundles which are trivial on M, with a trivialization on M restricting to the chosen one on Y.

To explain the structure of the coboundaries, it is better to remark that if one chooses different sections $\{s'_\alpha = \varphi_\alpha \cdot s_\alpha\}$, the same trivialization is expressed by $f'_\alpha = \varphi_\alpha|_Y^{-1} \cdot f_\alpha$. Thus the coordinate change is given by $\{\varphi_\alpha^{-1}\varphi_\beta, \varphi_\alpha|_Y\}$, which can be seen, by an active point of view, as a $M \times \mathbb{C}$ with the trivialization $Y \times \{1\}$ on Y, i.e., a trivial bundle with a fixed global section on M restricting to the chosen trivialization on Y. Hence, $\check{H}^1(M, \underline{S}^1, Y)$ is the group that has been looking for.

Line bundles with connection

In this paragraph the analogous group for bundles with connection will be defined. The relevant complex is the following:

$$\begin{tikzcd}
\check{C}^0(M,\Omega^1_{\mathbb{R}}) \oplus \check{C}^0(Y,\underline{S}^1) \arrow[r, "\check{\delta}^0\oplus\check{\delta}^0"] & \check{C}^1(M,\Omega^1_{\mathbb{R}}) \oplus \check{C}^1(Y,\underline{S}^1) \arrow[r, "\check{\delta}^1\oplus\check{\delta}^1"] & \check{C}^2(M,\Omega^1_{\mathbb{R}}) \oplus \check{C}^2(Y,\underline{S}^1) \arrow[r, "\check{\delta}^2\oplus\check{\delta}^2"] & \cdots \\
\check{C}^0(M,\underline{S}^1) \arrow[u, "\tilde{d}\oplus(i^*)^0"] \arrow[r, "\check{\delta}^0"] & \check{C}^1(M,\underline{S}^1) \arrow[u, "\tilde{d}\oplus(i^*)^1"] \arrow[r, "\check{\delta}^1"] & \check{C}^2(M,\underline{S}^1) \arrow[u, "\tilde{d}\oplus(i^*)^2"] \arrow[r, "\check{\delta}^2"] & \cdots .
\end{tikzcd}$$

Denote by $\check{H}^\bullet(M,\underline{S}^1 \to \Omega^1_{\mathbb{R}}, Y)$ the hypercohomology of this double complex. The claim is that the wanted group is $\check{H}^1(M,\underline{S}^1 \to \Omega^1_{\mathbb{R}}, Y)$. The cochains are given by $\check{C}^1(M,\underline{S}^1 \to \Omega^1_{\mathbb{R}}, Y) = \check{C}^1(M,\underline{S}^1) \oplus \check{C}^0(Y,\Omega^1_{\mathbb{R}}) \oplus \check{C}^0(Y,\underline{S}^1)$, so that one is led to consider $\{g_{\alpha\beta}, -A_\alpha, f_\alpha\}$.

- *Cocycles:* since $\check{\delta}^1\{g_{\alpha\beta}, -A_\alpha, f_\alpha\} = \{\check{\delta}^1 g_{\alpha\beta}, -\tilde{d}g_{\alpha\beta} - A_\beta + A_\alpha, ((i^*)^1 g_{\alpha\beta})^{-1} \cdot f_\beta f_\alpha^{-1}\}$, cocycles are characterized by three conditions: $\check{\delta}^1 g_{\alpha\beta} = 1$, i.e., $g_{\alpha\beta}$ is a line bundle L on M, $A_\alpha - A_\beta = \tilde{d}g_{\alpha\beta}$, i.e., A_α is a connection on L, and $(i^*)^1 g_{\alpha\beta} = f_\beta f_\alpha^{-1}$, i.e., f_α trivializes $L|_Y$.

- *Coboundaries:* since $\check{\delta}^0\{g_\alpha\} = \{\check{\delta}^0 g_\alpha, \tilde{d}g_\alpha, (i^*)^0 g_\alpha\}$, coboundaries represents line bundles which are geometrically trivial on M, with a trivialization on M restricting to the chosen one on Y.

1.2.2 Gerbes

In this subsection the analogous group for gerbes with connection is defined. The relevant complex is the following[6]:

$$\begin{tikzcd}
\check{C}^0(M,\Omega^2_{\mathbb{R}}) \oplus \check{C}^0(Y,\Omega^1_{\mathbb{R}}) \arrow[r, "\check{\delta}^0\oplus\check{\delta}^0"] & \check{C}^1(M,\Omega^2_{\mathbb{R}}) \oplus \check{C}^1(Y,\Omega^1_{\mathbb{R}}) \arrow[r, "\check{\delta}^1\oplus\check{\delta}^1"] & \check{C}^2(M,\Omega^2_{\mathbb{R}}) \oplus \check{C}^2(Y,\Omega^1_{\mathbb{R}}) \arrow[r, "\check{\delta}^2\oplus\check{\delta}^2"] & \cdots \\
\check{C}^0(M,\Omega^1_{\mathbb{R}}) \oplus \check{C}^0(Y,\underline{S}^1) \arrow[u, "\begin{bmatrix} d & (i^*)^0 \\ 0 & -\tilde{d} \end{bmatrix}"] \arrow[r, "\check{\delta}^0\oplus\check{\delta}^0"] & \check{C}^1(M,\Omega^1_{\mathbb{R}}) \oplus \check{C}^1(Y,\underline{S}^1) \arrow[u, "\begin{bmatrix} d & (i^*)^1 \\ 0 & -\tilde{d} \end{bmatrix}"] \arrow[r, "\check{\delta}^1\oplus\check{\delta}^1"] & \check{C}^2(M,\Omega^1_{\mathbb{R}}) \oplus \check{C}^2(Y,\underline{S}^1) \arrow[u, "\begin{bmatrix} d & (i^*)^2 \\ 0 & -\tilde{d} \end{bmatrix}"] \arrow[r, "\check{\delta}^2\oplus\check{\delta}^2"] & \cdots \\
\check{C}^0(M,\underline{S}^1) \arrow[u, "\tilde{d}\oplus(i^*)^0"] \arrow[r, "\check{\delta}^0"] & \check{C}^1(M,\underline{S}^1) \arrow[u, "\tilde{d}\oplus(i^*)^1"] \arrow[r, "\check{\delta}^1"] & \check{C}^2(M,\underline{S}^1) \arrow[u, "\tilde{d}\oplus(i^*)^2"] \arrow[r, "\check{\delta}^2"] & \cdots .
\end{tikzcd}$$

Denote by $\check{H}^\bullet(M,\underline{S}^1 \to \Omega^1_{\mathbb{R}} \to \Omega^2_{\mathbb{R}}, Y)$ the hypercohomology of this double complex. The claim is that the wanted group is $\check{H}^2(M,\underline{S}^1 \to \Omega^1_{\mathbb{R}} \to \Omega^2_{\mathbb{R}}, Y)$. The cochains are given by $\check{C}^2(M,\underline{S}^1 \to \Omega^1_{\mathbb{R}} \to \Omega^2_{\mathbb{R}}, Y) = \check{C}^2(M,\underline{S}^1) \oplus \check{C}^1(M,\Omega^1_{\mathbb{R}}) \oplus \check{C}^1(Y,\underline{S}^1) \oplus \check{C}^0(M,\Omega^2_{\mathbb{R}}) \oplus \check{C}^0(Y,\Omega^1_{\mathbb{R}})$, so that we consider $\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, h_{\alpha\beta}, B_\alpha, -A_\alpha\}$.

[6]The maps denoted by matrices are supposed to multiply from the right the row vector in the domain.

- *Cocycles:* since $\check{\delta}^2\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, h_{\alpha\beta}, B_\alpha, -A_\alpha\} = \{\check{\delta}^2 g_{\alpha\beta\gamma}, \tilde{d}g_{\alpha\beta\gamma} + \check{\delta}^1(-\Lambda_{\alpha\beta}), (i^*)^2 g_{\alpha\beta\gamma} \cdot \check{\delta}^1 h_{\alpha\beta}, -d(-\Lambda_{\alpha\beta}) + B_\beta - B_\alpha, -(i^*)^1(-\Lambda_{\alpha\beta}) + \tilde{d}h_{\alpha\beta} + A_\alpha - A_\beta\}$, cocycles are characterized exactly by the condition one needs in order for $\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha\}$ to be a gerbe with connection and $\{h_{\alpha\beta}, A_\alpha\}$ to trivialize it on Y;

- *Coboundaries:* since $\check{\delta}^1\{g_{\alpha\beta}, \Lambda_\alpha, h_\alpha\} = \{\check{\delta}^1 g_{\alpha\beta}, -\tilde{d}g_{\alpha\beta} + \Lambda_\beta - \Lambda_\alpha, ((i^*)^1 g_{\alpha\beta})^{-1} \cdot h_\beta h_\alpha^{-1}, d\Lambda_\alpha, (i^*)^0\Lambda_\alpha - \tilde{d}h_\alpha\}$, coboundaries represent gerbes which are geometrically trivial on M, with a trivialization on M restricting to the chosen one on Y.

There is a last step to obtain the classifying set of B-field and A-field configurations: in general one does not ask for a trivialization of the gerbe on Y, but for a cocycle whose transition functions represent the class $w_2(Y) \in H^2(Y, S^1)$. The transition functions of a coboundary in the previous picture represent the zero class, so they are consistent only for $w_2(Y) = 0$. Hence, one cannot consider the hypercohomology group, but one of its cosets in the group of *cochains* up to coboundaries. In fact, the condition needed is not cocycle condition, but:

$$\check{\delta}^2\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, h_{\alpha\beta}, B_\alpha, -A_\alpha\} = \{0, 0, \eta_{\alpha\beta\gamma}, 0, 0\}\,, \tag{1.18}$$

thus one needs the coset made by cochains satisfying (1.18) up to coboundaries. Actually, anyone of these cosets is needed for $[\,\{\,\eta_{\alpha\beta\gamma}\,\}\,] = w_2(Y) \in \check{H}^2(Y, S^1)$. Their union is denoted by:

$$\check{H}^2_{w_2(Y)}(M, \underline{S}^1 \to \Omega^1_{\mathbb{R}} \to \Omega^2_{\mathbb{R}}, Y) \tag{1.19}$$

and this is the set of configurations that has been looking for.

1.3 Gauge theory on a D-brane

It is now the moment to discuss the possible geometric structures of the gauge theory on the D-brane, arising from the previous picture [3]. The restriction of the H-field to the brane, that is just $c_1(\mathcal{G})$ in (1.13) must be obviously trivial in de Rham cohomology. The main distinction, then, turns out to be whether or not the B-field is flat when restricted to the D-brane, that is whether or not the H-field on Y is zero as representative 3-form.

1.3.1 Generic B-field

Consider the coordinate change given by the D-brane:

$$\begin{gathered} \{g_{\alpha\beta\gamma}, -\,\Lambda_{\alpha\beta}, B_\alpha\} \cdot \{g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}, \Lambda_{\alpha\beta}, dA_\alpha\} = \{\eta_{\alpha\beta\gamma}, 0, B + F\} \\ \{g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}, \Lambda_{\alpha\beta}, dA_\alpha\} = \{\check{\delta}^1 h_{\alpha\beta}, -\tilde{d}h_{\alpha\beta} + A_\beta - A_\alpha, dA_\alpha\}\,. \end{gathered} \tag{1.20}$$

Since, by Freed-Witten anomaly, $[\,\{\,g_{\alpha\beta\gamma}\,\}\,] = [\,\{\,\eta_{\alpha\beta\gamma}\,\}\,] \in \check{H}^2(Y, \underline{S}^1)$ (not the constant sheaf S^1, the sheaf of functions $\underline{S}^1$), one can always choose a gauge $\{\eta_{\alpha\beta\gamma}, 0, B\}$, but one can also consider any gauge $\{\eta_{\alpha\beta\gamma}, 0, B'\}$ with $B' - B$ a closed form representing an integral de Rham class: for a bundle, this corresponds to the free choice of a global automorphism.[7] However,

[7]For gerbes, one directly sees this from the fact that $(1, 0, \Phi)$ is a hypercoboundary for Φ integral. Indeed, one has:

$$\Phi\vert_{U_\alpha} = d\varphi_\alpha \qquad \varphi_\beta - \varphi_\alpha = d\rho_{\alpha\beta} \qquad \rho_{\alpha\beta} + \rho_{\beta\gamma} + \rho_{\gamma\alpha} = c_{\alpha\beta\gamma} \in \mathbb{Z}$$

thus $\varphi_\beta - \varphi_\alpha = \tilde{d}h_{\alpha\beta}$ for $h_{\alpha\beta} = \exp(2\pi i \cdot \rho_{\alpha\beta})$ and $\check{\delta}^1 h_{\alpha\beta} = 1$. Hence, $(1, 0, \Phi) = \check{\delta}^1(h_{\alpha\beta}, \varphi_\alpha)$.

clearly, one can never gauge away the B-field, since it is in general not flat (its holonomy is not even a cohomology class).
Given a certain gauge of the form $\{\eta_{\alpha\beta\gamma}, 0, B\}$, the brane gives a correction $\{1, 0, F\}$ to arrive at the fixed gauge $\{\eta_{\alpha\beta\gamma}, 0, B+F\}$. In fact, (1.20) becomes:

$$\begin{aligned} \{\eta_{\alpha\beta\gamma},&0, B\} \cdot \{1, 0, dA_\alpha\} = \{\eta_{\alpha\beta\gamma}, 0, B+F\} \\ &\{1, 0, dA_\alpha\} = \{\check{\delta}^1 h_{\alpha\beta}, -\tilde{d}h_{\alpha\beta} + A_\beta - A_\alpha, dA_\alpha\}\,. \end{aligned} \tag{1.21}$$

We thus get $\check{\delta}^1 h_{\alpha\beta} = 1$ and $-\tilde{d}h_{\alpha\beta} + A_\beta - A_\alpha = 0$, so $h_{\alpha\beta}$ give a gauge bundle on the brane with connection $-A_\alpha$ and Chern class $[\,-F\,]$. However, since B and F are arbitrary, such bundle is never canonical, because defined up to large gauge transformations $B \to B + \Phi$ and $F \to F - \Phi$ for Φ integral.[8]

Moreover, one still has the freedom to choose a different representative $\eta_{\alpha\beta\gamma} \cdot \check{\delta}^1 \lambda_{\alpha\beta}$ of $w_2(Y) \in \check{H}^2(Y, S^1)$. This is equivalent to consider:

$$\begin{aligned} \{\eta_{\alpha\beta\gamma},&0, B\} \cdot \{\check{\delta}\lambda_{\alpha\beta}, 0, dA_\alpha\} = \{\eta_{\alpha\beta\gamma} \cdot \check{\delta}\lambda_{\alpha\beta}, 0, B+F\} \\ &\{\check{\delta}\lambda_{\alpha\beta}, 0, dA_\alpha\} = \{\check{\delta}h_{\alpha\beta}, -\tilde{d}h_{\alpha\beta} + A_\beta - A_\alpha, dA_\alpha\}\,. \end{aligned} \tag{1.22}$$

One thus obtains that $\check{\delta}h_{\alpha\beta} = \check{\delta}\lambda_{\alpha\beta}$, i.e., $\check{\delta}(h_{\alpha\beta}/\lambda_{\alpha\beta}) = 1$. So one considers the bundle $[\,h_{\alpha\beta}/\lambda_{\alpha\beta}\,]$ instead of $[\,h_{\alpha\beta}\,]$. Since the functions $\lambda_{\alpha\beta}$ are constant, the real image of the Chern class is the same. In fact, by writing $h_{\alpha\beta} = \exp(2\pi i \cdot \tilde{h}_{\alpha\beta})$ and $\lambda_{\alpha\beta} = \exp(2\pi i \cdot \tilde{\lambda}_{\alpha\beta})$, one has that $\tilde{h}_{\alpha\beta} + \tilde{h}_{\beta\gamma} + \tilde{h}_{\gamma\alpha} = \tilde{h}_{\alpha\beta\gamma} \in \mathbb{Z}$ defining the first Chern class, and similarly $\tilde{\lambda}_{\alpha\beta} + \tilde{\lambda}_{\beta\gamma} + \tilde{\lambda}_{\gamma\alpha} = \tilde{\lambda}_{\alpha\beta\gamma} \in \mathbb{Z}$. However, since $\tilde{\lambda}_{\alpha\beta}$ are constant, $\tilde{\lambda}_{\alpha\beta\gamma}$ is a coboundary in the sheaf $\mathbb{R}$ and the real image of the Chern class of $\lambda_{\alpha\beta}$ is 0.
This means that a line bundle *up to the torsion part* has been fixed. Thus, the holonomy of $-A_\alpha$ is defined also up to the torsion part: this ambiguity is compensated for by the one of the Pfaffian, due to the need of obtaining a global section of the tensor product. But if $w_2 = 0$, one can choose the *preferred* representative $\eta_{\alpha\beta\gamma} = 1$, thus completely fixing a line bundle up to large gauge transformation.

1.3.2 Flat B-field

A particularly interesting situation, actually also the most common one, arises when B is flat on Y. its holonomy is now a class $\mathrm{Hol}(B|_Y) \in H^2(Y, S^1)$ (constant sheaf S^1). One can distinguish three cases:

- $w_2(Y) = 0$. This implies, by the way, $W_3(TY) = c_1(\mathcal{G}) = 0$ and it is the simplest case, since in this circumstance the parallel sections ρ_α and σ_α whose product makes the section s of (1.5) are already separately global. Here one can choose, as said, the preferred gauge $\eta_{\alpha\beta\gamma} = 1$, and, via an operation analogous to choosing *parallel* local sections for line bundles, one can obtain $\{1, 0, B\}$, that is $\mathcal{G}$ has been trivialized by means of transition functions equal to 1. Hence, in the notations of the previous section, one gets:

$$\{1, 0, B\} \,\cdot\, \{1, 0, \mathrm{d}A_\alpha \equiv F\} \,=\, \{1, 0, B+F\} \tag{1.23}$$

  and

$$[\mathcal{G}]_{S^1} = [e^{2\pi i\, B}] = \mathrm{Hol}B \in H^2(Y, S^1)\,, \tag{1.24}$$

[8] In particular, one can always choose the gauge $F = 0$, obtaining a flat line bundle.

where, the geometric triviality of the second factor in (1.23) implies:

$$\begin{cases} \check{\delta}^1 h_{\alpha\beta} = 1 \\ \int F \in \mathbb{Z} \\ A_\beta = A_\alpha + (2\pi i)^{-1} \mathrm{d} \log h_{\alpha\beta} \end{cases} \tag{1.25}$$

This means that F is the field strength and represents in de Rham cohomology the *integral* Chern class of a gauge bundle which is *non-canonical* because of large gauge transformations $\Phi \in H^2(Y, \mathbb{Z})$, while A_α is the local expression of its connection, subjected to the usual gauge transformations. Moreover, $B + F \in H^2(Y, \mathbb{R})$ is *not quantized*, but *gauge invariant*.
There is a special sub-case of this case, which occurs when also the quantity in (1.24) is vanishing: in that situation, one can choose the preferred gauge $B = 0$ and the gauge bundle on the brane will be canonically fixed.[9]

- $W_3(TY) = c_1(\mathcal{G}) = 0$. Now one must trivialize the gerbe $\mathcal{G}$ with transition functions in the class $w_2(TY)$, so that one is led essentially to choose among two possibilities. The best known one is:

$$\{1, 0, B\} \cdot \{\eta_{\alpha\beta\gamma}, 0, \mathrm{d}A_\alpha \equiv F\} = \{\eta_{\alpha\beta\gamma}, 0, B + F\} \tag{1.26}$$

and

$$\begin{cases} \check{\delta}^1 h_{\alpha\beta} = \eta_{\alpha\beta\gamma} \\ \int F \in \frac{x}{2} + \mathbb{Z} \\ A_\beta = A_\alpha + (2\pi i)^{-1} \mathrm{d} \log h_{\alpha\beta}\,, \end{cases} \tag{1.27}$$

where $x = 0$ if $w_2(TY) = 0$ and $x = 1$ otherwise. The B-field behaves exactly like in the previous case and satisfies (1.24), since by (1.7) there still exists one that can realize the class $\mathrm{Hol}B$. F instead is "half"-quantized when $w_2 \neq 0$ because it represents the Chern class of a "half"-bundle (in the sense that its square is a "true" bundle). This gauge bundle is still not canonical, because of the large gauge transformations which F is subjected to, unless, as before, $\mathrm{Hol}B = 0$. This is the case of the celebrated half-quantized gauge flux, found for the first time in [1]: the worldvolume spinors must carry a U(1) charge whose electromagnetic field F compensates for the non-existence of the spin bundle allowing the construction of the so called spin^c bundle.
The other possibility is to write:

$$\{\eta_{\alpha\beta\gamma}, 0, B\} \cdot \{1, 0, \mathrm{d}A_\alpha \equiv F\} = \{\eta_{\alpha\beta\gamma}, 0, B + F\} \tag{1.28}$$

and

$$[\mathcal{G}]_{S^1} = w_2(TY) + [e^{2\pi i\, B}] = w_2(TY) + \mathrm{Hol}B \in H^2(Y, S^1)\,, \tag{1.29}$$

where now it is the gauge field F to behave exactly as in the previous case, (1.25); also, the gauge invariant quantity $B + F$ is clearly not quantized, since, upon exponentiation, it realizes the class $[\mathcal{G}]_{S^1} + w_2(TY) \in H^2(Y, S^1)$. A relevant sub-case here occurs when $[\mathcal{G}]_{S^1} = 0$, so that, by (1.29), the B-field gets *quantized* and plays the role the gauge

[9] However, in the end of this subsection it will be shown that, actually, also in this case there is a residual freedom in the choice of the bundle, naturally arising from the hypercohomological analysis.

field had before, namely $\int B \in \frac{x}{2} + \mathbb{Z},$; F, instead, is the integrally quantized but not gauge invariant field strength of a true non-canonical gauge bundle. Therefore, the gauge invariant quantity $B + F$ is in this special sub-case also quantized, although in terms of half integers.
In the second part of this thesis, explicit models will be discussed, in the context of F-theory compactifications, realizing both the possibilities found here within the case of a brane or an orientifold wrapping a non-spin manifold.

- $W_3(TY) = c_1(\mathcal{G}) \neq 0$. This case is the most general one, but also the most unlikely among the various string compactification models available in the literature, due to the quite unusual wrapping by a D-brane of a manifold that is not even spin^c ($W_3 \neq 0$).[10] Anyhow, no global trivialization is now possible for the gerbe $\mathcal{G}$ and there exists no real class realizing $[\mathcal{G}]_{S^1}$. Hence, one has:

$$\{g_{\alpha\beta\gamma}, 0, B\} \cdot \{(g^{-1}\eta)_{\alpha\beta\gamma}, 0, \mathrm{d}A_\alpha \equiv F\} = \{\eta_{\alpha\beta\gamma}, 0, B + F\} \tag{1.30}$$

and

$$\begin{cases} \check{\delta}^1 h_{\alpha\beta} = (g^{-1}\eta)_{\alpha\beta\gamma} \\ \left[e^{2\pi i\, F}\right] = [\mathcal{G}]_{S^1} + w_2(TY) \\ A_\beta = A_\alpha + (2\pi i)^{-1}\mathrm{d}\,\log h_{\alpha\beta}\,, \end{cases} \tag{1.31}$$

where one only has the gauge invariant, not quantized field strength $B + F$, locally given by $\mathrm{d}A_\alpha$. This can be thought of as the *real* (that is in general irrational) Chern class of a canonical generalized line bundle with connection on the brane. In the next section an extensive dissertation is provided about the geometrical nature of such line bundles.

Before ending the section, a comment is in order when $\mathrm{Hol}(B|_Y) = w_2(Y) = 0$: even in this simplest case, the bundle is not completely fixed, but there is a residual gauge freedom. In fact, such configuration is described by $[\,\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, h_{\alpha\beta}, B_\alpha, -A_\alpha\}\,] \in \check{H}^2(M, \underline{S}^1 \to \Omega^1_{\mathbb{R}} \to \Omega^2_{\mathbb{R}}, Y)$ such that $[\,\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha\}\,]$ is geometrically trivial on Y. As said, one can choose on Y the preferred gauge $\{1, 0, h_{\alpha\beta}, 0, -A_\alpha\}$ so that the cocycle condition gives exactly $\{1, 0, \check{\delta}^2 h_{\alpha\beta}, 0, \tilde{d}h_{\alpha\beta} + A_\alpha - A_\beta\} = 0$, i.e. $-A_\alpha$ is a connection on the bundle $[\,h_{\alpha\beta}\,]$. There is still a question: how are the possible representatives $\{1, 0, h_{\alpha\beta}, 0, -A_\alpha\}$ of the same class? Can they all be obtained via a reparameterization of the bundle $[\,h_{\alpha\beta}, A_\alpha\,] \in \check{H}^1(Y, \underline{S}^1 \to \Omega^1_{\mathbb{R}})$? The possible reparameterizations are given by:

$$\begin{aligned} \{1, 0, h_{\alpha\beta}, 0, -A_\alpha\} \cdot \{\check{\delta}^1 g_{\alpha\beta}, -\tilde{d}g_{\alpha\beta} + \Lambda_\beta - \Lambda_\alpha, ((i^*)^1 g_{\alpha\beta})^{-1} \cdot h_\beta h_\alpha^{-1}, d\Lambda_\alpha, (i^*)^0\Lambda_\alpha - \tilde{d}h_\alpha\} \\ = \{1, 0, h'_{\alpha\beta}, 0, -A'_\alpha\} \end{aligned}$$

thus the following conditions hold:

$$\check{\delta}^1 g_{\alpha\beta} = 1 \qquad -\tilde{d}g_{\alpha\beta} + \Lambda_\beta - \Lambda_\alpha = 0 \qquad d\Lambda_\alpha = 0\,. \tag{1.32}$$

Choosing $g_{\alpha\beta} = 1$ and $\Lambda_\alpha = 0$ one simply gets $h'_{\alpha\beta} = h_{\alpha\beta} \cdot h_\beta h_\alpha^{-1}$ and $A'_\alpha = A_\alpha + \tilde{d}h_\alpha$, i.e., a reparameterization of $[\,h_{\alpha\beta}, A_\alpha\,] \in \check{H}^1(Y, \underline{S}^1 \to \Omega^1_{\mathbb{R}})$, and that is what one expects. But, in general, this is not what happens. Indeed, equations (1.32) represent any line bundle $g_{\alpha\beta}$ on the

[10] Every manifold of real dimension less or equal than four and every manifold admitting an almost complex structure are spin^c.

whole target space M with *flat* connection $-\Lambda_\alpha$, thus they represent a residual gauge freedom in the choice of the line bundle over Y: *any flat bundle on Y which is the restriction of a* flat *line bundle over M is immaterial for the gauge theory on the D-brane.* Is there any physical interpretation of this fact?
Consider a line bundle L over Y with connection $-A_\alpha$: its holonomy is in general defined as a function from the loop space of Y to S^1. Actually, what is really relevant in the present classification is not a generic loop, but rather the ones of the form $\partial\Sigma$, with Σ in general not contained in Y: such loops are thus in general not homologically trivial on Y, but they are so on M. Suppose that L extends to $\tilde{L}$ over M: in this case, one can equally consider the holonomy over $\partial\Sigma$ with respect to $\tilde{L}$. If $\tilde{L}$ is flat, such holonomy becomes an S^1-cohomology class evaluated over a contractible loop, thus it is 0. Hence, a bundle extending to a flat one over M gives no contribution to the holonomy over the possible boundaries of the world-sheets. Therefore, also in the case $\mathrm{Hol}(B|_Y) = w_2(Y) = 0$, *a canonically fixed bundle with connection does not exist* on the brane: there exists instead an equivalence class of bundles defined up to flat ones extending to flat target space bundles.
In physical terms this has an important consequence: given a network of separate D-branes, choose two of them and pick two cycles, one for each brane, which are homologous in target space but not necessarily homologically trivial. Since the difference is homologically trivial, one can link them by an open string loop (a cylinder) stretching from one brane to the other. In this way one determines the holonomy only on the difference, i.e. the difference of the holonomies on the two loops, which is not changed by contributions from flat brane-bundles extending to flat target-bundles. Hence, a global uncertainty is left by the hypercohomological classification, represented by flat target space line bundles. One needs than additional information, with respect to the one coming from the anomaly cancellation, for example the holonomy on loops of Y which are not world-sheet boundaries, to completely fix the gauge bundle on each brane.

1.4 Real Chern classes*

In the previous section it has been shown that for B flat one can obtain a gauge theory on a generalized bundle: while bundles are represented by cocycles $\{g_{\alpha\beta}\}$ in Čech cohomology, such generalized bundles are represented by cochains whose coboundary $\check{\delta}^1\{g_{\alpha\beta}\}$ is made by constant functions (not necessarily 1), realizing a class in $\check{H}^2(X, S^1)$ for X any space. In this section it will be shown that even in these cases one can define connections and first Chern class, but the latter turns out to be any closed form, not necessarily integral.

Consider the definition of the Chern class of an ordinary line bundle: $[\,\{g_{\alpha\beta}\}\,] \in \check{H}^1(\mathfrak{U}, \underline{S}^1)$, so that $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$; if $g_{\alpha\beta} = e^{2\pi i \cdot \rho_{\alpha\beta}}$, one has $\rho_{\alpha\beta} + \rho_{\beta\gamma} + \rho_{\gamma\alpha} = \rho_{\alpha\beta\gamma} \in \mathbb{Z}$, so that one obtains a class $[\,\{\rho_{\alpha\beta\gamma}\}\,] \in \check{H}^2(\mathfrak{U}, \mathbb{Z})$ which is the first Chern class. Denoting by Γ_n the subgroup of S^1 given by the n-th root of unity and by $\frac{1}{n}\mathbb{Z}$ the subgroup of $\mathbb{R}$ made by the fractions $\frac{k}{n}$ for $k \in \mathbb{Z}$, then $\Gamma_n = e^{2\pi i \cdot \frac{1}{n}\mathbb{Z}}$. Suppose to have a cochain $\{g_{\alpha\beta}\} \in \check{C}^1(\mathfrak{U}, \underline{S}^1)$ such that $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = g_{\alpha\beta\gamma} \in \Gamma_n$. Then, for $g_{\alpha\beta} = e^{2\pi i \cdot \rho_{\alpha\beta}}$, one has that $\rho_{\alpha\beta} + \rho_{\beta\gamma} + \rho_{\gamma\alpha} = \rho_{\alpha\beta\gamma} \in \frac{1}{n}\mathbb{Z}$, so that one obtains a rational class $c_1 = [\,\{\rho_{\alpha\beta\gamma}\}\,] \in \check{H}^2(\mathfrak{U}, \mathbb{Q})$ such that $n \cdot c_1$ is an integral class. A geometrical interpretation of these classes is now given. A 2-cochain can be thought of as a trivialization of a trivialized gerbe, in the same way as a 1-cochain (i.e. a set of local functions) is a trivialization of a trivialized line bundle; thus a line bundle is a trivialization of a gerbe represented by the coboundary 1, in the same way as a global function is a global section of $X \times \mathbb{C}$. In the following a description is provided first of the easier case of local functions trivializing a line bundle, namely one degree lower in cohomology, and then for the case of gerbes.

1.4.1 Trivializations of line bundles

Definition

As line bundles, which are classes in $\check{H}^1(\mathfrak{U}, \underline{S}^1)$, are trivializations of gerbes represented by the coboundary 1, likewise a section of a line bundle, represented by transition functions equal to 1, is a class in $\check{H}^0(\mathfrak{U}, \underline{S}^1)$, i.e. a function $f : X \to S^1$. A cochain $\{f_\alpha\} \in \check{C}^0(\mathfrak{U}, \underline{S}^1)$ is a section of a trivial bundle represented by transition functions $f_\alpha^{-1} \cdot f_\beta$.

Given a function $f : X \to S^1$, one can naturally define a Chern class $c_1(f) \in H^1(\mathfrak{U}, \mathbb{Z})$, which is the image under the Bockstein map of $f = [\{f_\alpha\}] \in \check{H}^0(\mathfrak{U}, \underline{S}^1)$. One directly computes it as for bundles: since $f_\beta \cdot f_\alpha^{-1} = 1$, for $f_\alpha = e^{2\pi i \cdot \rho_\alpha}$ one has $\rho_\beta - \rho_\alpha = \rho_{\alpha\beta} \in \mathbb{Z}$, so that one can define a class $c_1(f) = [\{\rho_{\alpha\beta}\}] \in \check{H}^1(\mathfrak{U}, \mathbb{Z})$. The geometric interpretation is very simple: $c_1(f)$ is the pull-back under f of the generator of $H^1(S^1, \mathbb{Z}) \simeq \mathbb{Z}$. Suppose now to have a cochain $[\{f_\alpha\}] \in \check{C}^0(\mathfrak{U}, \underline{S}^1)$ such that $f_\alpha^{-1} \cdot f_\beta = f_{\alpha\beta} \in \Gamma_n$. Then $\rho_\beta - \rho_\alpha = \rho_{\alpha\beta} \in \frac{1}{n}\mathbb{Z}$. Therefore one obtains a class $c_1 = [\{\rho_{\alpha\beta}\}] \in \check{H}^1(\mathfrak{U}, \mathbb{Q})$ such that $n \cdot c_1$ is an integral class.

From the exact sequences point of view, the Chern class is the image of the Bockstein map of the sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \underline{\mathbb{R}} \xrightarrow{e^{2\pi i \cdot}} \underline{S}^1 \longrightarrow 0 \; .$$

In the fractional case, since $\check{\delta}^0 f_\alpha$ takes values in Γ_n, the cochain $\{f_\alpha\}$ is a cocycle in $\underline{S}^1 / \Gamma_n$. Thus, one is led to consider the sequence:

$$0 \longrightarrow \tfrac{1}{n}\mathbb{Z} \longrightarrow \underline{\mathbb{R}} \xrightarrow{\pi_{\Gamma_n} \circ e^{2\pi i \cdot}} \underline{S}^1 / \Gamma_n \longrightarrow 0$$

and the image of the Bockstein map is exactly the fractional Chern class. In this way rational Chern classes are constructed, but this is generalizable to any real Chern class. In fact, it is sufficient that $\rho_{\alpha\beta}$ be constant for every α, β to apply the previous construction, using the constant sheaf S^1 instead of Γ_n. The corresponding sequence, which contains all the previous ones by inclusion, is:

$$0 \longrightarrow \mathbb{R} \longrightarrow \underline{\mathbb{R}} \xrightarrow{\pi_{S^1} \circ e^{2\pi i \cdot}} \underline{S}^1 / S^1 \longrightarrow 0 \; .$$

In other words, if the cochain is a cocycle up to constant functions, one obtains a real Chern class. If these constant functions belong to Γ_n, one obtains a rational Chern class in $\frac{1}{n}\mathbb{Z}$.

Geometric interpretation

Thinking of the cochain as a trivialization of $X \times \mathbb{C}$, different trivializations will have different Chern classes, depending on the realization of the trivial bundle as Čech coboundary. This seems quite unnatural from a topological point of view, since the particular trivialization should not play any role. However, by fixing a flat connection, one can distinguish a particular class of trivializations, namely the ones which are parallel with respect to such a connection.

Consider a trivial line bundle with a global section and a flat connection ∇, i.e. $X \times \mathbb{C}$ with a globally defined form A, expressing ∇ with respect to the global section $X \times \{1\}$. The following facts are known:

- by choosing parallel sections $\{f_\alpha\}$, one obtains a trivialization with a real Chern class $c_1(\{f_\alpha\}) \in \check{H}^1(X, \mathbb{R})$, and the local expression of the connection becomes $\{0\}$;

- the globally defined connection A, expressed with respect to 1, is closed by flatness, thus it determines a de Rham cohomology class $[\,A\,] \in H^1_{dR}(X)$.

These two classes actually coincide under the standard isomorphism between Čech and de Rham cohomology. This is the geometric interpretation of real Chern classes: *the real Chern class of a trivialization of $X \times \mathbb{C}$ is the cohomology class of a globally-defined flat connection, expressed with respect to $X \times \{1\}$, for which the trivialization is parallel.*
If the trivial bundle has holonomy 1 (i.e. *geometrically trivial*), one can find a global parallel section: thus there exists a function $f \in \check{H}^1(X, \underline{S}^1)$ trivializing the bundle, and the Chern class of a function is integral. Writing the connection with respect to 1, one gets an integral class $[\,A\,] = [\,f^{-1}df\,]$, while writing it with respect to the global section $f \cdot 1$, one gets 0.

Here is the proof of the statement. Given $\{f_\alpha\} \in \check{C}^0(\mathfrak{U}, \underline{S}^1)$ such that $\check{\delta}^0\{f_\alpha\} \in \check{C}^1(\mathfrak{U}, S^1)$, consider the connection ∇ on $X \times \mathbb{C}$ which is represented by 0 with respect to $\{f_\alpha^{-1}\}$. Representing ∇ with respect to $X \times \{1\}$ one obtains $A_\alpha = \tilde{d}f_\alpha$, and $A_\alpha - A_\beta = \tilde{d}(f_\beta \cdot f_\alpha^{-1}) = 0$. One thus realizes the 1-form A as a Čech cocycle: one has that $A_\alpha = (2\pi i)^{-1} d \log f_\alpha$ and $(2\pi i)^{-1} \log f_\beta - (2\pi i)^{-1} \log f_\alpha = (2\pi i)^{-1} \log g_{\alpha\beta} = \rho_{\alpha\beta}$ which is constant, so that $[\,A\,]_{H^1_{dR}(X)} \simeq [\,\{\rho_{\alpha\beta}\}\,]_{\check{H}^1(X,\mathbb{R})}$. By definition $c_1(\{f_\alpha\}) = [\,\{\rho_{\alpha\beta}\}\,]$, thus $[\,A\,]_{H^1_{dR}(X)} \simeq c_1(\{f_\alpha\})_{\check{H}^1(X,\mathbb{R})}$.
Moreover, for $q_\star : H^1(X, \mathbb{R}) \to H^1(X, S^1)$ the induced quotient map of (1.7), one has that $q_\star\, c_1(\{f_\alpha\}) = q_\star\, [\,\rho_{\alpha\beta}\,] = [\,f_\beta f_\alpha^{-1}\,]_{S^1}$. Thus, for $\check{\delta}^0\{f_\alpha\} \in \check{C}^1(X, S^1)$ (hence, obviously, $\check{\delta}^0\{f_\alpha\} \in \check{Z}^1(X, S^1)$), the first Chern class is one of the possible real lifts of $[\,\check{\delta}^0\{f_\alpha\}\,]_{S^1}$. Therefore, $q_\star\, c_1(\{f_\alpha\})$ is the holonomy of the trivial line bundle on which the connection A, previously considered, is defined.

Hypercohomological description

The trivialized bundle $X \times \mathbb{C}$ with global connection A corresponds to the hypercocycle $\{1, -A\} \in \check{Z}^1(X, \underline{S}^1 \to \Omega^1_{\mathbb{R}})$. For A flat and $\{f_\alpha\}$ parallel sections, one has $[\,\{1, -A\}\,] = [\,\{\check{\delta}^0 f_\alpha, 0\}\,]$, thus the difference is a coboundary:

$$\{1, -A\} \cdot \{\check{\delta}^0 f_\alpha, \tilde{d}f_\alpha\} = \{\check{\delta}^0 f_\alpha, 0\}\,,$$

hence $\tilde{d}f_\alpha = A_\alpha$, so that, as proven before, $[\,A\,] \simeq c_1(\{f_\alpha\})$.
If f is globally defined, one gets $\{1, -A\} \cdot \{1, \tilde{d}f\} = \{1, 0\}$, so that $[\,A\,] = [\,\tilde{d}f\,]$ which is integral: this corresponds to the choice of a global parallel section $f \cdot 1$ in $X \times \mathbb{C}$.

1.4.2 Trivializations of gerbes

Consider now a trivialization of a gerbe $\{h_{\alpha\beta}\} \in \check{C}^1(X, \underline{S}^1)$ such that $\check{\delta}^1\{h_{\alpha\beta}\} \in \check{C}^2(X, S^1)$ and consider a connection $\{-A_\alpha\}$ such that $A_\beta - A_\alpha = \tilde{d}h_{\alpha\beta}$, as for an ordinary bundle. One has $dA_\alpha = dA_\beta$ so that $-F = -dA_\alpha$ is a global closed form whose de Rham class $[\,-F\,]$ is exactly the fractional Chern class of $[\,\{h_{\alpha\beta}\}\,] \in (\check{\delta}^1)^{-1}(\check{C}^2(X, S^1))\,/\,\check{B}^1(X, \underline{S}^1)$. Such a trivialization with connection is an element of the hypercohomology group:

$$\check{H}^1\big(\,X, \underline{S}^1/S^1 \xrightarrow{\tilde{d}} \Omega^1_{\mathbb{R}}\,\big)\,.$$

The interpretation of the Chern class of such trivializations is as before: consider the flat trivial gerbe $[\,\{\check{\delta}^1 h_{\alpha\beta}, 0, 0\}\,]$, and represent it as $[\,\{1, 0, -F\}\,]$:

$$\{1, 0, -F\} \cdot \{\check{\delta}^1 h_{\alpha\beta}, -\tilde{d}h_{\alpha\beta} + A_\beta - A_\alpha, dA_\alpha\} = \{\check{\delta}^1 h_{\alpha\beta}, 0, 0\}$$

from which one obtains:

$$A_\beta - A_\alpha = \tilde{d} h_{\alpha\beta} \qquad dA_\alpha = F|_{U_\alpha} \; .$$

From these data one can now realize F as a Čech class: $\check{\delta}^1 \tilde{d} h_{\alpha\beta} = 0$, thus $(2\pi i)^{-1} \check{\delta}^1 \log h_{\alpha\beta}$ is constant and expresses $[\,F\,]$ as Čech class, which is exactly $c_1(\{h_{\alpha\beta}\})$.

1.5 Stacks of coincident branes

So far the situation in which more than one brane is present in a stack of several coincident ones, thus generating a non-abelian gauge group, has not been discussed. Formally speaking, in such a case one would need the concept of non-abelian cohomology (see [13]). While an analogous technical discussion about this topic is avoided here, the main differences with respect to the abelian case are described and the Kapustin's modification of the Freed-Witten anomaly condition found in [14] revisited in the present framework.

Consider again the fundamental equation (1.20):

$$\begin{gathered}
\{g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha\} \cdot \{g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}, \Lambda_{\alpha\beta}, dA_\alpha\} = \{\eta_{\alpha\beta\gamma}, 0, B + F\} \\
\{g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}, \Lambda_{\alpha\beta}, dA_\alpha\} = \{\check{\delta}^1 h_{\alpha\beta}, -\tilde{d} h_{\alpha\beta} + A_\beta - A_\alpha, dA_\alpha\} \; .
\end{gathered}$$

As already stressed, since $\check{\delta}^1 h_{\alpha\beta} = g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}$, the class $[\,g^{-1}\eta\,] \in H^1(Y, \underline{S}^1)$ must be trivial: this means that $c_1(\mathcal{G}) = W_3(TY)$, which is the Freed-Witten anomaly equation.
In the case of a stack of coincident branes, however, $h_{\alpha\beta} \in U(n)$. Then, thinking of $g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}$ as a multiple of the identity $I_{n\times n}$, the relation $\check{\delta}^1 h_{\alpha\beta} = g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}$ is not a trivialization of $[\,g^{-1}\eta\,] \in H^1(Y, \underline{S}^1)$ any more and it does not imply that $c_1(\mathcal{G}) = W_3(Y)$. Rather, a rank-$n$ bundle $\{h_{\alpha\beta}\}$ such that $\check{\delta}^1\{h_{\alpha\beta}\}$ realizes a class in $H^2(X, \underline{S}^1)$,[11] is called a *twisted bundle* or *non-commutative bundle* [15]. For φ the Bockstein homomorphism in degree 2 of the first sequence in (1.7), define $\zeta \equiv \varphi[\,\check{\delta}^1\{h_{\alpha\beta}\}\,] \in H^3(X, \mathbb{Z})$. Thus, for the relation $\check{\delta}^1 h_{\alpha\beta} = g^{-1}_{\alpha\beta\gamma} \cdot \eta_{\alpha\beta\gamma}$ to hold, one must have:

$$\zeta = W_3(Y) - c_1(\mathcal{G}) \; . \tag{1.33}$$

This is the Kapustin's version of the Freed-Witten anomaly equation for stacks of branes. Hence, it is worth to remark that, while in the abelian case the A-field corresponds to a reparameterization of the gerbe, in the non-abelian case it provides *another non-trivial gerbe*, which tensor-multiplies the gerbe of the B-field. This gerbe, whose first Chern class has been called ζ, is still pure torsion. Indeed, by definition, ζ measures the obstruction for a bundle having transition functions in the quotient $U(n)/U(1) = SU(n)/\mathbb{Z}_n$ to have $U(n)$-structure group.[12]
Now, consider the short exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\text{mod } n} \mathbb{Z}_n \longrightarrow 0 \, , \tag{1.34}$$

which leads to the following long exact sequence in the cohomology of any space, say in this case of the brane Y:

$$\cdots \longrightarrow H^2(Y, \mathbb{Z}) \longrightarrow H^2(Y, \mathbb{Z}_n) \xrightarrow{\beta'} H^3(Y, \mathbb{Z}) \longrightarrow H^3(Y, \mathbb{Z}) \longrightarrow \cdots \, . \tag{1.35}$$

[11] If one realizes the two gerbes involved with constant transition functions (possible, since they are still torsion), this is just the non-abelian generalization of the generalized line bundles encountered in the previous section.

[12] Such bundles without vector structure clearly have adjoint structure, because $U(1)$ is the center of $U(n)$

One can define an n-torsion class $y \in H^2(Y, \mathbb{Z}_n)$ which measures the obstruction to lifting the $SU(n)/\mathbb{Z}_n$-bundle to an $SU(n)$ one. This is the so called *t'Hooft magnetic flux*. The $SU(n)$-valued transition functions $\tilde{h}_{ij}$ of this bundle will satisfy the following cocycle condition on triple overlaps:

$$\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} = y_{ijk}\,, \tag{1.36}$$

y_{ijk} being any 2-cocycle representing the class y. Taking the determinant of both sides of (1.36) shows that y is n-torsion. The claim is:

$$\zeta = \beta'(y)\,, \tag{1.37}$$

obviously implying ζ to be n-torsion too. In fact, suppose $\beta'(y) = 0$, i.e. there exists a $\mathbb{Z}$-valued Čech cocycle z_{ijk} such that $y_{ijk} = \exp(2\pi i z_{ijk}/n)$, and also, by the isomorphism φ in degree 1 of (1.7), a line bundle over Y with transitions t_{ij} whose first Chern class is $-z$. Then, the $U(n)$-valued functions $h_{ij} = \tilde{h}_{ij}t_{ij}^{1/n}$ satisfy the right cocycle condition, $h_{ij}h_{jk}h_{ki} = 1$, namely they define a vector bundle with structure group $U(n)$, hence $\zeta = 0$. The vice-versa is perfectly analogous.

An explicit and familiar example of these gauge bundles without vector structure is provided by Type I strings with an H-flux turned on. Indeed, since the 32 D9 branes that cancel the tadpole are wrapping the entire target space, which is of course spin^c, the modified Freed-Witten anomaly (1.33) will read (see [16]): $\zeta = [H]$, H being a representative of $c_1(\mathcal{G})$. Hence, ζ must be different from zero and it is given by the equation: $\zeta = \beta'(\tilde{w}_2)$. $\tilde{w}_2$ is known as the *generalized second Stiefel Whitney class*: it is just the t'Hooft magnetic coupling of this configuration of space-filling branes which therefore will carry a $Spin(32)/\mathbb{Z}_2$ bundle without vector structure.

Chapter 2

The concept of D-brane charge

The debated concept of D-brane charge in theories with Chern-Simons term is introduced and discussed in this chapter, keeping the parallelism with the properties of quantization and gauge invariance that the cancellation of the Freed-Witten anomaly implies on the brane-induced fluxes. The attention will be focused in particular on the different notions of charge that come out and on how all these concepts open the way to the K-theoretical improvement of the D-brane charge classification problem.

This chapter is organized as follows: in section 1 the standard definition of D-brane charge in cohomology is reviewed, taking into account the Dirac quantization condition; section 2, instead, is an overview of the different notions of charge that it is useful to introduce when the H-flux is turned on in type II supergravities; in section 3 the reasons why cohomology fails to classify D-brane charge are discussed; finally, in section 4 an analogous analysis of the notion of charge is carried out from the perspective of the worldvolume effective theory of a D-brane.

2.1 The standard cohomological definition

A Dirichlet p-brane, or Dp-branes for short, [11], is an extended object lying in the middle between a fundamental quantum (that is a small oscillation around the trivial solution of the equations of motion, i.e. the vacuum) and a soliton (that is a non trivial solution and has its own independent quantum evolution, but it is strongly coupled in the perturbative regime of the fundamental theory). The brane is a classical soliton, in the sense that its trajectory satisfies the Einstein equations of supergravity, but its quantum fluctuations and interactions can be described in the perturbative regime of string theory by means of the fundamental quanta of open and closed strings ($C_{(p+1)}$, A_μ, $G_{\mu\nu}$, $B_{\mu\nu}$, X^i), that are small oscillations around the string vacuum (not around the brane solution, unless one considers the backreaction of the brane). Hence, rather than attempting to construct the quantum mechanics of the brane itself as a strongly coupled soliton, summing over all its possible trajectories in the path integral, one usually treats it as a classical object with small quantum fluctuation around its classical trajectory due to the open strings ending on it (whose low energy spectrum contains the fields representing the transverse position of the brane).

One of the most important features of Dp-branes is that they are electric sources for the Ramond-Ramond (RR) fields of type II string spectrum, which they minimally couple to, in analogy with the theory of electromagnetism, via the so called Wess-Zumino action:

$$S_{WZ} \supset \int_{W_Y} C_{p+1}\,, \tag{2.1}$$

where $W_Y \subset M$ is a timelike manifold representing the (p+1)-dimensional worldvolume swept by the brane as time goes on. Invariance under the gauge transformation $C \to C + d\Lambda$ imposes that the brane worldvolumes are cycles of the target space.

Classical mechanics would suggest to allow charge-preserving homotopic movements of the D-brane but, due to purely stringy interactions, one may want a brane to undergo even processes in which it splits into two disconnected components which then rejoin again after a while. Thus, essentially one has to admit charge-preserving *homological* deformations as well, in order to accommodate also this kind of phenomena.

To obtain a suitable definition of the charge of a D-brane, assume for now that the H-flux vanishes: this hypothesis will lead here to a unique notion of charge and will be relaxed in section 2.2. Then, one should remember that D-branes can also be thought of as sources for violation of the Bianchi identity of the RR field strengths (which thus see the brane as a magnetic source). Namely, at a fixed instant of time, the Dp-brane called Y satisfies the following equation which defines its charge q:

$$\mathrm{d}F_{8-p} \,=\, \mathrm{d}\star F_{p+2} \,=\, \delta^{9-p}(qY) \,=\, \widetilde{\mathrm{PD}}_{\mathbb{R}^d\times\tilde{M}}(q[Y])\,, \tag{2.2}$$

where $F_{p+2} = \mathrm{d}C_{p+1}$ are the so called *unimproved* RR field strengths[1], $\star$ is the Hodge star in the spatial target manifold $\mathbb{R}^d \times \tilde{M}$ and $\widetilde{\mathrm{PD}}$ just means[2] a representative of the class defined by the Poincaré duality. It is important to notice here, that eq. (2.2) clearly implies that the p-cycle wrapped by Y is *trivial* in ordinary homology, being a representative of its Poincaré dual an exact (9-p)-form in $\mathbb{R}^d \times \tilde{M}$, thus trivial in the ordinary cohomology. This would imply that every brane has zero charge, and so unstable, that is of course false. However, what really happens, [17], is that the charge of the brane is taking values in the kernel of the following embedding:

$$i\,:\,H^{9-p}_{cpt}(\mathbb{R}^d\times\tilde{M}) \longrightarrow H^{9-p}(\mathbb{R}^d\times\tilde{M}) \qquad \text{so that} \qquad \mathrm{PD}_{\mathbb{R}^d\times\tilde{M}}(q[Y]) \,\in\, \mathrm{Ker}(i)\,, \tag{2.3}$$

where $H^\bullet_{cpt}$ is the cohomology ring with compact support and the map is defined by "forgetting" that a cohomology class has compact support.

An easy corollary of this statement is that if $d = 0$ then the net charge of every Dp-brane should vanish, because in this case the spatial target manifold coincides with $\tilde{M}$ which is already compact, so i is the identity map. This fact is perfectly reasonable from a physical point of view, because if the spatial target space is compact the flux lines originating from the source brane have nowhere to go and must be sucked up by an other identical source with the opposite charge, which therefore makes the net charge of Dp-brane in the universe to vanish.

The above discussion does not assume Y to be a compact p-cycle of $\tilde{M}$, but is equally well applicable in the case a Dp-brane is also extending along some non-compact directions, [18]. Indeed, one has to use in this case the homology with non-compact support (indicated in the sequel by $H^{ncpt}_\bullet$), which, precisely like compactly supported cohomology with respect to ordinary

[1]The reason of the name "unimproved" will be clear in section 2.2, where (2.2) will be just one among different notions of charge.

[2]The square brackets in its argument denote the homology class.

one, looses its invariance under homotopy. For instance, if $Y \subset \mathbb{R}^d \times \tilde{M}$ is any space-like slice of a Dp-brane, filling $p^{nc} \leq d$ non-compact directions in $\mathbb{R}^d$ and wrapping a p^c-cycle in $\tilde{M}$, such that $p^{nc} + p^c = p$ it will be represented in homology by the following group:

$$[Y] \in H_p^{ncpt}(\mathbb{R}^d \times \tilde{M}) \simeq H_{p-p^{nc}}(\mathbb{R}^{d-p^{nc}} \times \tilde{M}) \simeq H_{p^c}(\tilde{M}), \tag{2.4}$$

where the last equality holds because in the remaining non compact directions the brane is compact (it is a point). For the Poincaré dual, instead, using (2.3) one has:

$$\mathrm{PD}([Y]) \in H_{cpt}^{9-p}(\mathbb{R}^d \times \tilde{M}) \simeq H^{9-p-(d-p^{nc})}(\mathbb{R}^{p^{nc}} \times \tilde{M}) \simeq H^{9-d-p^c}(\tilde{M}). \tag{2.5}$$

Therefore, as the two expressions above suggest and as one naturally expects, one can simply forget about the external space-time, because all the relevant topological properties of the Dp-brane are encoded in the internal space $\tilde{M}$.

Computation of q It should be clear at this point that the value of the charge of a D-brane is in some way controlled by the value at infinity of the corresponding magnetic field strength, like in ordinary gauge theories. If a solution of eq. (2.2) exists for F_{8-p} in $H_{cpt}^{8-p}(\mathbb{R}^d \times \tilde{M})$ such that the flux vanishes at infinity, then the corresponding Dp-brane charge is equal to zero. The following exact sequence will formalize this concept:

$$\cdots \longrightarrow H^{8-p}(M') \xrightarrow{r} H^{8-p}(N) \xrightarrow{\beta} H^{9-p}(M', N) \xrightarrow{i} H^{9-p}(M') \longrightarrow \cdots, \tag{2.6}$$

where $M' \equiv \mathbb{R}^d \times \tilde{M}$, N is the 8-dimensional manifold at infinity of M' (the one linking a point of M'), r restricts classes on M' to classes on N, β is the Bockstein homomorphism of the sequence and the relative cohomology group $H^{9-p}(M', N)$ is the same as the group $H_{cpt}^{9-p}(M')$. Thus (2.3) says that:

$$\mathrm{PD}_{\mathbb{R}^d \times \tilde{M}}(qY) \in \frac{H^{8-p}(N)}{r(H^{8-p}(M'))}, \tag{2.7}$$

which is equivalent to say that the non-triviality of the Dp-brane is measured by the flux lines at infinity of the magnetic (8-p)-forms which cannot be regarded as the restriction of closed forms defined on the whole M'.[3]

In order to practically compute the charge, one has to extract it from (2.2) by integrating both sides and applying Gauss theorem:

$$\int_{\mathcal{B}^{9-p}} \mathrm{PD}_{M'}(qY) = \int_{\mathcal{B}^{9-p}} \mathrm{d}F_{8-p} = \int_{L^{8-p}} F_{8-p}, \tag{2.8}$$

where $\mathcal{B}^{9-p} \subset M'$ is a chain intersecting Y in M' and $L^{8-p} \subset N$ a cycle linking it, such that $L^{8-p} = \partial \mathcal{B}^{9-p}$. By changing representative for Y in its homology class one in general changes class for L^{8-p}; but these classes are indexed by the linking number l, which in turn is equal to the number of intersections of Y with $\mathcal{B}^{9-p}$. Thus:

$$q = \frac{1}{l} \int_{L^{8-p}} F_{8-p}. \tag{2.9}$$

[3]The fields on N that extend on closed forms on the whole M' have clearly no brane source creating them.

2.1.1 The simplest quantum effect

As far as supergravity is concerned, every cohomology group mentioned above is a de Rham cohomology group, and this is the end of the story: D-brane charges lie in the ring $H^\bullet(M',\mathbb{R}) \simeq H^\bullet_{dR}(M')$. However, quantum mechanics induces dramatic effects, the first one of which is the celebrated *Dirac quantization condition* and prescribes to replace the de Rham cohomology with the integral one, so that (2.2) just becomes a differential form approximation to an equation that holds in the ring $H^\bullet(M',\mathbb{Z})$. Indeed this follows from the standard Dirac argument [9], as will be now reviewed. Nevertheless, as it will become clear later on, RR field strengths are very subtle objects and the treatment of their Dirac quantization condition is much more sophisticated (see [19]).
Like the B-field, the RR forms C_{p+1} can be viewed as connections on *p-gerbes* (see section 1.1): as such, as already explained in the previous chapter, they undergo large gauge transformations, $C_{p+1} \to C_{p+1} + \Phi_{p+1}$, with Φ_{p+1} being closed integrally quantized (p+1)-forms. Large gauge transformations may be transition functions between different patches:[4] take for simplicity Y an instantonic brane (with purely space-like worldvolume) lying in the intersection of two of them; one has:

$$\int_{W_Y} \Phi_{p+1} = \int_{W_Y} C^{(+)}_{p+1} - \int_{W_Y} C^{(-)}_{p+1} = \int_{U^+} F_{p+2} - (-)\int_{U^-} F_{p+2} = \int_U F_{p+2}\,, \tag{2.10}$$

where $C^{(\pm)}_{p+1}$ are two local potentials for F_{p+2} and $U = U^+ \cup U^-$, such that $\partial U^+ = -\partial U^- = Y$. The term (2.1) in the classical effective action of the worldvolume theory is of course not invariant under these transformations but, in the quantum theory, one should only require the invariance of the path integral measure e^{iS}. So, apart from the 2π factor, this amounts to require $F_{p+2} \in H^{p+2}(M',\mathbb{Z})$. In mathematical terms this is just equivalent to say that F_{p+2} represents the first Chern class of the p-gerbe mentioned before. Notice that this p-gerbe is defined on $M'\backslash W_Z$, where Z is the electromagnetic dual D(6-p)-brane of Y (Dirac string for C_{p+1}), because only there F_{p+2} is closed:[5] the jump in the connection is exactly measured by the units of dual sources present in the background.
Actually, it should be said that the previous argument for the quantization of the charge of Dp-branes is not really convincing, because Chern Simons terms in the effective action, like the Wess-Zumino term of (2.1) are rather deduced from properties of the supergravity bulk fields, than known a priori. Thus the previous conclusion should be seen as a check of consistency of the unimproved flux quantization with the invariance of the *worldvolume* quantum effective action, rather than as a proof of the quantization itself. In the next chapter a more convincing argument is given for the quantization of these charges, purely based on bulk considerations. The new argument will be formulated in the most general situation of non-vanishing H-flux, thus also automatically justifying the quantization of these charges that will no longer be gauge invariant.

2.2 Different notions of charge

It is common knowledge [4] that in field theories with Chern-Simons terms or modified Bianchi identities, like type II supergravities in 10 dimensions with the H-flux turned on, the concept

[4]For example, if a Dirac string for C_{p+1} is present, C_{p+1} is not globally defined.

[5]This fact is at the origin of the well-known problem of defining a local Lorentz invariant theory for the self-dual D3-branes.

of charge associated to a given gauge field split into three natural notions, each of which shows different properties among the typical ones that should belong to a charge, namely gauge invariance, localization, conservation and quantization. For a more detailed treatment of the topic the reader is referred to [4]. Here, for later purposes, the attention is focused in particular on two of the three notions, namely the Maxwell charge and the Page charge. The third one, i.e. the so called "brane source charge", is associated directly with external sources coupled to supergravity; therefore it is always localized and gauge invariant, even though neither conserved nor quantized, due to the Hanany-Witten effect [20].

When $H \neq 0$, the RR field strengths encountered in the previous section have an *improved* partner defined as follows:

$$G_{p+2} \,=\, F_{p+2} + H \wedge C_{p-1}\,. \tag{2.11}$$

Due to the presence of Chern-Simons terms in these supergravities, the (2.11) are actually the *gauge invariant* field strengths. To see this, it is worth to introduce some notations.[6] First of all, the indices of degree of the forms will be dropped and the language of polyfoms will be adopted, more suitable for the presence of the H-flux, which induces a shift of three degrees; then, the RR potential are locally defined polyforms which are subjected to the large gauge transformations

$$C \to C + \Phi \qquad \text{such that} \qquad (\mathrm{d} + H\wedge)\Phi = 0\,, \tag{2.12}$$

i.e. it is closed under the so called *twisted* de Rham differential.[7] Hence the improved RR field strengths (2.11) are defined as

$$G \,=\, (\mathrm{d} + H\wedge)C\,, \tag{2.13}$$

thus being manifestly gauge invariant, although not closed under the ordinary de Rham differential (they are twisted-closed). However, in this twisted theory, the unimproved field strengths $F = \mathrm{d}C$ are *not* gauge invariant:

$$F \,\longrightarrow\, F - H \wedge \Phi\,, \tag{2.14}$$

but their gauge transformations respect their closure under the ordinary de Rham differential, in the absence of external sources, being closed themselves.
As it is clear from (2.13), even when explicit brane sources are not present, gauge fields themselves naturally carry a gauge invariant charge, being (non-localized) sources for the improved field strengths G:

$$\mathrm{d}G \,=\, -H \wedge F\,. \tag{2.15}$$

This is the so called *Maxwell* charge. It is manifestly conserved, but it is not quantized due to the continuous changes of the integrals of gerbe connections (like the B and the C fields) appearing in the local definition of $H \wedge F$. This is clearly now the charge belonging to the kernel

[6] Another kind of twisting can also be defined which instead keeps the de Rham differential and changes connections and improved field strengths in their twisted versions: $C^{tw} \equiv e^B \wedge C$ and $G^{tw} \equiv e^B \wedge G$, so that $G^{tw} = \mathrm{d}C^{tw}$. This formalism will not be used here, but it is worth to say that it has the advantage of allowing a simple formulation of the generalization of the Freed-Witten anomaly of D(p-2)-branes ending on Dp-branes [21]: the polyform $G^{tw}e^F$ must be topologically trivial when restricted to the Dp-brane worldvolume.

[7] Notice that this operator squares to zero when NS5-branes are not present, thus defining a twisted version of the de Rham cohomology.

of the map $H^\bullet_{cpt}(M') \to H^\bullet(M')$ analogous to (2.3).
As opposed to (2.15), the (localized) charge defined by the current violating the Bianchi identity (2.2), called *Page* charge, is not any more gauge invariant because of (2.14), but it still remains conserved. The aim is now to prove that the Page charge is also quantized, by giving an argument which, based on bulk gerbe trivializations, uses the methods developed in the previous chapter and does not refer at all to any worldvolume theory.[8]
Equation (2.13) looks pretty much like the definition of the gauge invariant connection $B + F$ on the gerbe $\mathcal{G}|_Y$ on the brane, encountered in the previous chapter. G is the gauge invariant connection on a topologically trivial[9], non-flat $(\deg(G) - 1)$-bulk gerbe with curvature (2.15); $H \wedge C$ is the global connection up to large gauge transformation playing the role of the B-field, while $F = \mathrm{d}C$ is the connection of the geometrically trivial gerbe which provides the right trivialization, so that it plays the role of F before and as such it must be integrally quantized in order not to change the S^1 holonomy class of the reparameterized gerbe. The case at hand actually falls in the generic situation analogous to the one presented in subsection 1.3.1. This demonstrates the integrality of the unimproved field-strengths, which have to be regarded as *non-localized* Page charges, because they are carried by the gauge fields, rather than put in as external sources.
Things are a bit more complicated when also *localized* Page charges are present, associated with insertions of monopoles violating the modified Bianchi identities (2.15):

$$\mathrm{d}G = -H \wedge F + \delta_{\mathrm{Page}}(Y)\,, \tag{2.16}$$

where $\delta_{\mathrm{Page}}(Y) = \mathrm{d}F$. So now the integral units of F are measured on the linking manifolds of the monopoles and by definition they coincide with the units of added monopoles: this number is actually fixed by the requirement of the Maxwell charge being in the kernel of the compact support-forgetting map. Hence, one has the relation:

$$[\delta_{\mathrm{Page}}(Y)] = [H \wedge F] \in H^\bullet(M', \mathbb{Z})\,. \tag{2.17}$$

Formula (2.17) shows an important peculiarity of this localized Page charge: it is non-trivial in the ordinary cohomology and thus its net amount could be non-vanishing even if M' is compact, because it is compensated by the contribution of the bulk H-flux. An analog of this circumstance from the perspective of the worldvolume theory of a D-brane will be described in the next section and it will be crucial for the K-theoretical improvement of the charge classification problem.

2.3 The failure of cohomology

In this section, a further crucial quantum effect in the D-brane dynamics will be discussed, that will make the entire cohomological apparatus no longer a correct mathematical device for the classification of charges: this is nothing but the Freed-Witten anomaly.

In the previous chapter it has been said that D-branes wrapping cycles Y such that $W_3(TY) + [H]|_Y \neq 0$ are anomalous. Thus they have to be removed from the classification because they do not represent consistent states.

[8] For an alternative more physical proof, which instead uses Kaluza-Klein reduction from M-theory and T-duality, see [4].

[9] Actually, (2.15) is weaker, since it just assures that this gerbe is torsion. However, a non-vanishing Chern class of this gerbe is not affecting the argument, because the reparameterization performed on it will anyway imply integrality of the Page charge.

Moreover, it can be seen [9] that the Freed-Witten anomaly is also responsible of the appearance of unstable D-branes, which cohomology instead classifies as stable, since they wrap in general non-trivial cycles. They are the so called Maldacena-Moore-Seiberg instanton configurations (MMS) [22] and they have to be classified with the zero charge. Before embarking in a general analysis, it is worth to first look at them from the supergravity point of view, i.e. forgetting about the torsion.

In this case, a Dp-brane is non-anomalous if the H-flux restricted to it is topologically trivial (i.e. an exact 3-form). However, one can still allow for Dp-branes wrapping cycles with non-trivial restriction of the H-flux, provided there is some other lower-dimensional D-brane compensating for the so generated Freed-Witten anomaly. This situation is exactly the worldvolume analog of the Page charge insertions in the bulk, described at the end of the previous section. Indeed, when Freed-Witten anomalies are absent, F, which now is the gauge field strength on the Dp-brane Y, is closed and so $[H] = [\mathrm{d}B] = 0 \in H^3(Y, \mathbb{Z})$, in order for the gerbe with Chern class represented by $\mathrm{d}(B + F)$ to be topologically trivial in de Rham cohomology. When instead $[H]|_Y \neq 0$, a compensating monopole must be present on the worldvolume of the anomalous Dp-brane, such that [23]

$$[\mathrm{d}F] = -[H] \in H^3(Y, \mathbb{Z})\,. \tag{2.18}$$

This monopole is physically represented by a D(p-2)-brane ending on the Dp-brane on a codimension 3 locus given by the Poincaré dual in Y of $\mathrm{d}F$ and its target homology class is indicated by $-[H] \cap [Y]$, where $\cap : H^n \times H_m \to H_{m-n}$ is an operation called cap product. The name MMS-instanton of such configuration is due to the fact that the Dp-brane Y is actually an euclidean instanton, so that W_Y is a purely space-like manifold; then, $\mathrm{d}F = -[H] \in H^3(W_Y, \mathbb{Z})$ and the additional leg of the D(p-2)-brane is time-like[10]. Therefore, the latter brane is unstable, because its charge is eaten up at a given instant of time by a Freed-Witten anomalous spark-like brane.

This new quantum effect, namely the appearance of inconsistent configurations and of additional unstable ones, is beautifully kept into account in the classification procedure by an higher nilpotent differential, known in the mathematical literature as $\mathrm{d}_3 : H^n(M', \mathbb{Z}) \to H^{n+3}(M', \mathbb{Z})$. The idea is then to take a further cohomology, now with respect to this differential, in order to rule out the anomalous branes and to quotient out the unstable ones. To bring the analysis to the full generality, torsion contributions are now reintroduced.

Let $x^{(9-p)} = \mathrm{PD}_{M'}[Y] \in H^{9-p}(M', \mathbb{Z})$ the (9-p)-cohomology class of the Dp-brane Y. As seen many times, Y is free of Freed-Witten anomalies if and only if $W_3(N_{M'}Y) + [H]|_Y \neq 0$, where it has been used the Whitney sum formula for Stiefel-Whitney classes and the fact that M' is spin^c to deduce that $W_3(TY) = W_3(N_{M'}Y)$ Thus, if $Y \overset{i}{\hookrightarrow} M'$ is non-anomalous, the following relation must hold:

$$0 = i_{\#}\left(W_3(N_{M'}Y) + [H]|_Y\right) = \left(\mathrm{Sq}^3 + [H] \cup\right) x^{(9-p)} \equiv \mathrm{d}_3\, x^{(9-p)}\,, \tag{2.19}$$

where $i_{\#}$ is the Gysin map in cohomology (a kind of push-forward, that for cohomology class is non-natural), i.e. $i_\sharp \equiv \mathrm{PD}_{M'} \circ i_\star \circ \mathrm{PD}_Y$, $i_\star$ being the natural push forward in homology; sq^n are operations called Steenrod squares and defined as $\mathrm{sq}^n x^{(9-p)} = i_{\#} w_n(N_{M'}Y)$, while Sq^n are their integral lifts; $[H] \cup x^{(9-p)}$, with $\cup$ being the cup product in integral cohomology, just denotes the

[10] Configurations in which such additional leg is space-like are also possible [9] and are referred to as *baryons* [23]. However, when they are consistent, they lead to high-mass states and are not taken into account in the classification.

Poincaré dual in the target M' of the magnetic monopole $[H] \cap [Y]$ and, finally, the last equality is a definition of d_3. Therefore, equation (2.19) says that if Y is free of Freed-Witten anomalies, its cohomology class must lie in the kernel of d_3. The vice-versa is not true, since the Gysin map can have a non-trivial kernel. Moreover, Dp-branes whose cohomology classes fall in the image of d_3 are unstable, since they are the monopoles of MMS instantons, $x^{(9-p)} = \mathrm{d}_3\,[\delta_s^{6-p}(\mathrm{MMS})]$, where the subscript 's' means that the delta function has only space-like legs.
Hence, taking the cohomology with respect to d_3 is the second step, after ordinary singular cohomology, towards a correct classification of D-brane charges. This is not enough because, as said, some anomalous branes can anyhow fall in the kernel of d_3: thus the idea is to proceed inductively with this series of cohomologies with respect to higher and higher order differentials, until the sequence stops for dimensional reasons; in some situations, it was shown [9, 24] that the above anomalous configurations passing through the second step are ruled out by the next non-trivial differential, namely d_5. This procedure is known in the mathematical literature as a particular kind of spectral sequence, called *Atiyah-Hirzebruch spectral sequence* (AHSS) [25] and, as described in the next chapter, by carrying out it, one ends up with a classification closely related to the K-theoretical one. Actually this constitutes one of the two main approaches available in the literature on the relation between K-theory and D-branes, which will be discussed and compared in the next chapter.

2.4 Charges of branes within branes

Before ending this chapter, it is worth to investigate on the analogous emergence of different notions of charge on the worldvolume theory of a given Dp-brane. Such an analysis is also going to open the way to K-theory, but from a different perspective.

In the next chapter it will be explained in detail why a given D-brane contains lower dimensional D-brane charges [26]; here the attention is focused only on the first one induced in this way, namely a real codimension 2 D-brane, which turns out to be generated by the B-field and the gauge field on the D-brane one starts with.
The Wess-Zumino action, whose first term is written in (2.1), contains also the following term:

$$S_{WZ} \supset \int_{W_Y} (B+F) \wedge C_{p-1}\,, \tag{2.20}$$

which shows that D(p-2)-brane charges naturally appear. From the analysis of the Freed-Witten anomaly cancellation made in the previous chapter, it follows that the quantity $B + F$ is always gauge invariant, but in general non-quantized even in the flat B-field case, when it should represent in $H^2(Y, \mathbb{R})$ the generic class $\mathrm{Hol}B + w_2(TY) \in H^2(Y, S^1)$. By integrating such a quantity over a 2-dimensional surface $S_2 \subset Y$, one obtains a non-localized but conserved charge of D(p-2)-brane that, clearly from its properties, exactly corresponds to the Maxwell charge discussed from a bulk perspective around the formula (2.15):

$$Q_M = \int_{S_2} B + F \in \mathbb{R}\,. \tag{2.21}$$

The absolute value of this charge, by the way, is equal, in the BPS limit, to the tension of the D(p-2)-brane, which is naturally non-quantized.

Notice that fractional brane charges from ADE-orbifolds, as discussed in [27, 28], are automatically taken into account by the analysis of the previous chapter: in that case, it suffices to take:

$$w_2(TY) = 0 \qquad \text{and} \qquad (\mathrm{Hol}B, C_I) = e^{2\pi i \frac{d_I}{|\Gamma|}}\,, \tag{2.22}$$

where d_I is the dimension of the I-th of the N irreducible representations of the discrete orbifold group Γ in which the regular one splits and $|\Gamma|$ is the order of this group, such that $\sum_{I=0}^{N-1} d_I^2 = |\Gamma|$. The second formula of (2.22) is the evaluation of the class $\mathrm{Hol}B \in H^2(Y, S^1)$ on the I-th vanishing 2-cycle of the ADE resolution of the orbifold singularity. Then, the Maxwell charge of the N D(p-2)-branes, obtained by wrapping Dp-branes on each vanishing 2-cycle, is expressed in terms of the gauge fields on the Dp's and of the various restrictions of the B-field, which is taken to be flat on the whole internal space $\tilde{M}$. Therefore, one gets:

$$Q_M^I = \int_{C_I} B + F = \frac{d_I}{|\Gamma|} > 0 \qquad\qquad I = 0, \ldots, N-1\,, \tag{2.23}$$

where the pull-back of the B-field is implicit and $d_0 = 1$ being the dimension of the trivial representation. The Maxwell charges are also the untwisted charges of the D(p-2)-branes, and thus (2.23) guarantees that all the D(p-2)'s are branes and not anti-branes. Usually in this context, people work in a gauge in which

$$\int_{C_I} B = \frac{d_I}{|\Gamma|} \qquad \text{and} \qquad \int_{C_I} F = 0 \qquad\qquad I \neq 0\,. \tag{2.24}$$

On C_0, however, such a frame is not allowed because there is no gauge freedom left. Indeed, the affine 2-cycle C_0 of the extended Dynkin diagram is related to the other 2-cycles by the relation: $C_0 = -\sum_{I\neq 0} d_I C_I$. Since the B-field on it is the restriction of a field coming from the bulk, one is obliged to choose there the gauge:

$$\int_{C_0} B = \frac{1}{|\Gamma|} - 1 = -\sum_{I\neq 0} \frac{d_I^2}{|\Gamma|} \qquad \text{and} \qquad \int_{C_I} F = 1\,. \tag{2.25}$$

In perfect analogy with the bulk analysis, also here there exists another notion of lower-dimensional brane charge: it is non-localized and conserved as well but, unlike the Maxwell one, it is quantized even if the B-field is curved. This is the Page charge, given by the non-gauge invariant quantity:

$$Q_P = \int_{S_2} F \in \frac{x}{2} + \mathbb{Z}\,, \tag{2.26}$$

which is taken from formula (1.27). It is gauge invariant "only" in the very special situation in which the B-field vanishes identically (in fact, there is another one; see below) and it can be quantized in terms of half-integers if the brane only admits $U(1)$-*charged* worldvolume spinors.

Actually, there is an other very special circumstance in IIA string theory in which the Page charge is also gauge invariant [29]: it is the case of D2-branes that are homologically trivial in the target space M', and hence unstable.

Take for simplicity a D2-brane with spatial slices wrapping 2-spheres S^2 in $\mathbb{R}^9$, that are the boundary of a 3-balls $\mathcal{B}^3$, and take a non-vanishing background H-field $H = \mathrm{d}B$. On the D2 an integrally quantized gauge field $F \in H^2(S^2, \mathbb{Z}) = \mathbb{Z}$ lives, which is also gauge invariant, because

$$\int_{S^2} B = \int_{\mathcal{B}^3} H\,, \tag{2.27}$$

is clearly gauge invariant. Then, the Maxwell charge of this worldvolume theory, should correspond to the (observable) number of D-particles (D0-branes) left after the collapse of the unstable D2-brane. However, as said, this charge is not quantized: indeed, here it is obvious from the curvature of the B-field, creating for it a non-trivial, continuously varying, local holonomy on S^2. This leads to the celebrated paradox found by Bachas-Douglas-Schweigert [30].
The paradox was solved by Taylor in the paper quoted above, by noticing that there is a counterterm, coming from the bulk IIA supergravity action, exactly canceling the B-field contribution to the number of particles: this is the Chern-Simons term

$$S_{IIA} \supset \int_M C_1 \wedge H \wedge F_6 = -\int_M C_1 \wedge B \wedge \delta^7(D2) = -\int_{W_{D2}} C_1 \wedge B\,, \tag{2.28}$$

where an integration by part has been performed and suitable fall-off conditions for the fields at the infinity of M chosen. This assures that the number of particles is given by the quantized Page charge, which fortunately in this case turns out to be gauge invariant too, thus leaving no more puzzles.

In general, it seems that, at least when large gauge transformations are transition functions between patches, like in subsection 2.1.1, only the K-theory class of the Page charge is really gauge invariant. Hence this provides another independent reason of the approximate validity of the cohomology-based classification and of the need of the K-theoretical improvement. In the following, a rough and qualitative argument is given to support this conclusion.
Consider the concrete example of a background with a non-vanishing H-flux and made by the following configurations of D-branes:

Brane	0	1	2	3	4	5	6	7	8	9
D5	X	X	X	X	X	X				
D3	X	X	X	X						
MMS5		X	X	X	X	X	X			
NS5	X	X	X	X					X	X
S^2					X	X				
S^3					X	X	X			

where the first row indicates the target dimensions along which the three branes showed below stretch; MMS5, in particular, is an MMS instanton (like the ones introduced in the previous section) having worldvolume along the first 6 spatial directions; S^3 is a 3-sphere linking the NS5-brane, with equator the S^2 above.
At the level of representative forms, a large gauge transformation for the D3-brane Page charge $\int_{S^2} F$, where F is the gauge field on the D5-brane, leads itself to an effective quantized D3 charge:

$$\int_M C_4 \wedge \Phi \wedge \delta^4(D5) \qquad \Longrightarrow \qquad Q_{gauge} = \int_{S^2} \Phi\,. \tag{2.29}$$

One should remember, however, that the corresponding transformation of the B-field is just the opposite of the one of F; moreover, when Dirac strings for the B-field are present, namely NS5-branes, this gauge transformation is related, by the analog of the standard Dirac argument presented in the expression (2.10), to the de Rham class of the bulk gerbe $\mathcal{G}$, represented by $H \in H^3(M', \mathbb{Z})$:

$$\int_{S^2} \Phi = \int_{S^3} H \in \mathbb{Z}\,. \tag{2.30}$$

Therefore, the cohomology class in M' of such gauge-induced D3-brane is of the form:

$$x^{(6)} = [H] \wedge [\delta_s^3(\mathrm{MMS5})] \,. \tag{2.31}$$

The above formula is just the de Rham approximation of an equation that holds in the integral cohomology group $H^6(M', \mathbb{Z})$, with $\wedge$ substituted by $\cup$ and possible torsion contributions included. Recalling now the discussion of the previous section, one realizes that an $x^{(6)}$ of the form (2.31) belongs to the image of the higher differential d_3. Hence, the gauge-induced D3-brane, dual to $x^{(6)}$, will be quotiented out by the second step of AHSS and will lift to a vanishing K-theory charge. This concludes the argument.

Chapter 3

K-theory classification of D-branes

In the previous chapter, a couple of independent reasons have been discussed in order to introduce K-theory as a good tool to classify D-brane charges (see also [9, 31]). Therefore, this chapter is entirely dedicated to this important topic and its aim is to present two different and in some sense complementary approaches to the K-theory classification and to analyze and compare the amount of information one can extract from each of them.

One of the two methods consists of applying the so called Gysin map of K-theory to the gauge bundle of the D-brane, obtaining a K-theory class in space-time [32]. This approach is motivated by the Sen conjecture, which states that a generic configuration of branes and anti-branes with gauge bundle is equivalent, via tachyon condensation, to a stack of coincident space-filling brane-antibrane pairs equipped with an appropriate K-theory class [33].
The second approach, instead, based on the Atiyah-Hirzebruch spectral sequence (AHSS) [25], has already been briefly introduced in the previous chapter, and will be further analyzed here from a technical point of view. It consists of applying the AHSS to the Poincaré dual of the homology cycle of the D-brane: such a sequence rules out some cycles affected by global world-sheet anomalies, e.g. Freed-Witten anomaly, and quotients out some cycles which are actually unstable, e.g. MMS-instantons.

This chapter is organized as follows: in section 1 some assumptions are made and justified on the string background, that will be crucial for the following analysis; section 2 contains an extended review on the first approach to K-theory, namely the one inspired by Sen's conjecture; section 3, instead, contains some technical details concerning the AHSS approach, already introduced in the previous chapter; finally, in section 4 the explicit link between the two methods is first stated and commented, and afterwards proven in full mathematical rigor.
In order to understand the technical parts of this chapter, the reader not familiar with the basics of K-theory and spectral sequences may need the appendices D and E.

3.1 Working hypotheses

Even if the "Gysin map approach" shows many advantages with respect to the former, nevertheless it works properly only in the case of vanishing H-field. This is essentially due to the Freed-Witten anomaly. Indeed, this approach, when applied to IIB string theory, uses stacks of D9-branes filling a spin^c-target space, which, as explained in chapter 1.1 are anomalous when an H-field is turned on. Strictly speaking, recalling Kapustin's modification to the Freed-Witten

anomaly equation discussed in section 1.5, one can still allow a *purely torsion*[1] bulk H-field, provided the gauge bundle on the D9's is turned in a non-commutative one. However, even in this case, if the restriction of such H-field to the D-brane one is willing to classify is non-zero, there are technical problems in constructing the Gysin map in K-theory, which is responsible for the computation of the charge in this approach. Finally, as this approach has the advantage to take into account the gauge bundle on the brane in the definition of the charge, it will be clearly most useful in the case of a canonical gauge bundle; however, as seen in detail in section 1.3, even though the H-field is zero, a non trivial B-field could ruin the canonicity of the gauge field strength because of large gauge transformations.
Therefore, for all these reasons, throughout this chapter it will be assumed the B-field itself to vanish identically. In that special case, one can appreciate the "Gysin map approach" in all its power and compare the information it gives with the one extracted from AHSS.[2]

According to the analysis of section 2.1, no relevant information is lost if one forgets about the non-compact part of the D-branes, since it has a trivial topology. Thus, the attention could be restricted, without loss of generality, to the compact part of the target space, namely $\tilde{M}$, and D-branes will be wrapping cycles in it. For the purposes of this chapter, nothing changes if one takes for simplicity the whole spatial target M' to be compact, i.e. $d = 0$, provided the existence, for every Dp-brane source, of another source of opposite charge somewhere else in which the flux lines can sink (see section 2.1). Moreover, a Wick rotation of the time direction is performed, ending up with a *compact 10-dimensional target space* M, in which the dynamics of D-branes will be analyzed in an euclidean setting. Hence, one looses the physical interpretation of the D-brane worldvolume as a manifold moving in time and of the charge q as a charge conserved in time: rather than considering the homology class of the D-brane volume Y at every instant of time, one considers the homology class of the entire worldvolume W_Y in M, using standard homology with compact support. Then, the physical processes under which one requires conservation of the charge will be deformations of the trajectory, like RG-flows, rather than time-evolutions.

3.2 K-theory from the Sen conjecture

3.2.1 What is K-theory?

Complex vector bundles up to isomorphisms on any topological space, like the D-brane worldvolume W_Y, with the direct sum form a semigroup called $\text{Vect}(W_Y, \oplus)$. One wants to extend this to a group by adding inverses: the K-theory [35] of W_Y, $K(W_Y)$, is just the Grothendieck group associated to this semigroup, like the group of integer numbers $\mathbb{Z}$ under the sum is associated to the semigroup of natural numbers $\mathbb{N}$. However, there is a subtlety. In $\mathbb{N}$ cancellation rules hold: namely, for a, b, c natural numbers,

$$a = b \iff a + c = b + c\,. \tag{3.1}$$

[1]Actually, a generalization of this to allow any kind of H-field, without requiring the presence of an infinite number of D9's, has been proposed by the Australian group, which is based on the theory of *bundle gerbes* [34].

[2]In the second part of this thesis, an instance of this approach in a special case of non-vanishing B-field is presented. It will be a special situation among the ones classified in the second point of section 1.3.2, namely $\text{Hol}B = w_2$ and F integral, so that $B + F$ is still quantized: the role of the canonical gauge bundle will be played by the "half"-bundle whose first Chern class is represented by $B + F$.

In Vect($W_Y, \oplus$), instead, this is not true in general. Indeed, if E, F and G are complex vector bundles on W_Y, then

$$E \simeq F \implies E \oplus G \simeq F \oplus G\,, \tag{3.2}$$

i.e. two different bundles can be isomorphic by adding to them the same bundle. Therefore, in the group extension of Vect($W_Y, \oplus$), that is $K(W_Y)$, the requirement of (3.2) to be an equivalence is non-trivial. Denoting by square brackets the K-theory equivalence classes, one has:

$$[E] = [F] \qquad \Longleftrightarrow \qquad \exists\, G \quad \text{such that} \quad E \oplus G \simeq F \oplus G\,, \tag{3.3}$$

which leads to the so called *stable equivalence relation* in K-theory:

$$[E] = [F] \qquad \Longleftrightarrow \qquad [E] + [G] = [F] + [G]\,. \tag{3.4}$$

Moreover, every vector bundle E is direct summand of a trivial one, i.e. there exists another vector bundle F such that $E \oplus F \simeq n$, for n the trivial vector bundle of some rank n. Hence, the most general K-theory class is of the form $[E] = [F] - [n]$, and the conditions above can more simply be rephrased by substituting G with the trivial bundle n.

In the next subsection, it will be reviewed why the physics of D-branes justifies the need of this kind of mathematical structure [10].

3.2.2 Gauge and gravitational couplings

D-branes have two kinds of interactions:

1. gauge interactions (gauge field configurations from open strings attached to the brane determine the geometry of the gauge bundle on the brane);

2. gravitational interactions (graviton configurations from closed strings fluctuating in the bulk determine the geometry of the tangent bundle of M restricted to the brane, $TM|_{W_Y}$).

Gauge bundles are complex vector bundles, like the ones discussed in the previous subsection, of rank the number of D-branes in the stack and whose topology is partially encoded in their Chern characters, $\text{ch}_k \in H^{2k}(W_Y, \mathbb{Q})$.
The restricted tangent bundle, instead, satisfies the relation $TM|_{W_Y} = TW_Y \oplus N_M W_Y$; they are all real vector bundles, with special orthogonal structure groups, whose topology is partially encoded in degree 4k rational characteristic classes called Pontryagin classes, $p_k \in H^{4k}(W_Y, \mathbb{Q})$.

Minasian and Moore [32] have computed, by means of anomaly-inflow arguments, the unique non-anomalous form of these couplings (namely the one that renders the path integral measure of chiral fermions at the intersection of two D-branes well-defined). In other words, they managed to find the complete Wess-Zumino action for a Dp-brane, of which (2.1) and (2.20) are only the first two parts:

$$S_{WZ} = \int_{W_{Y_p}} i^* C \wedge \text{ch}(E) \wedge e^{\frac{d}{2}} \wedge \frac{\sqrt{\hat{A}(TW_{Y_p})}}{\sqrt{\hat{A}(N_M W_{Y_p})}}\,, \tag{3.5}$$

where $E \to W_{Y_p}$ is the gauge bundle on a Dp-brane worldvolume, $i : W_{Y_p} \to M$ is the embedding of the Dp-brane in the target space, $\hat{A} = \hat{A}(p_k)$ is the so called A-roof genus and d is a class on W_{Y_p} uniquely determined by its modulo 2 reduction $w_2(N_M W_{Y_p})$ (i.e. it can be regarded as the spin^c-class of the normal bundle of the brane)[3]. For the subtle issues related to the proper mathematical definition of the "generalized" holonomy given by the exponentiation of (3.5), the reader is referred to [19]. Thus, as it is clear from (3.5), a D-brane contains lower dimensional brane charges (even below codimension 2) and the old cohomological charge of the initial Dp-brane that contains all the other branes is recovered from the rank of the gauge bundle:

$$x^{9-p} \,=\, i_{\#}(\mathrm{rk}\, E) \,=\, i_{\#}(\mathrm{ch}_0(E))\,, \tag{3.6}$$

$i_{\#}$ being the Gysin map in cohomology, $H^{\bullet}(W_{Y_p}, \mathbb{Q}) \to H^{\bullet}(M, \mathbb{Q})$ (see (2.19)).
In the case of anti-branes, one has to allow for negative charges, hence the gauge bundle is actually a *K-theory class*: a generic class $[E] - [F]$ can be interpreted as a stack of pairs of a brane Y and an anti-brane $\overline{Y}$ with gauge bundle E and F respectively.
Hence a proper definition of the total charge density of a Dp-brane as a class in the target space is the following:

$$Q_p \,=\, i_{\#}\left(\mathrm{ch}(E) \wedge e^{\frac{d}{2}} \wedge \frac{\sqrt{\hat{A}(TW_{Y_p})}}{\sqrt{\hat{A}(N_M W_{Y_p})}}\right)\,. \tag{3.7}$$

3.2.3 The Sen conjecture

Splitting principle

For notational convenience, the following definition will be adopted:

$$G(W_{Y_p}) \,\equiv\, e^{\frac{d}{2}} \wedge \frac{\sqrt{\hat{A}(TW_{Y_p})}}{\sqrt{\hat{A}(N_M W_{Y_p})}}\,. \tag{3.8}$$

So the complete Wess-Zumino action (3.5) becomes:

$$S_{WZ} \,=\, \int_{\mathrm{PD}_{W_{Y_p}}(\mathrm{ch}(E))} i^{*}C \wedge G(W_{Y_p})\,. \tag{3.9}$$

Let $\{q_k \cdot W_{Y_k}\}$ be the set of branes appearing in the Poincaré dual of $\mathrm{ch}(E)$ in W_{Y_p}: the first one is $\mathrm{PD}_{W_{Y_p}}(\mathrm{ch}_0(E)) = q \cdot W_{Y_p}$, so it gives rise to the action without gauge coupling. The other ones are lower dimensional branes. Consider the first one, i.e. $W_{Y_{(1)}} = \mathrm{PD}_{W_{Y_p}}(\mathrm{ch}_1(E))$. Then, the correponding term in the action is $\int_{W_{Y_{(1)}}} i^{*}C \wedge G(W_{Y_p})$, which can be written as $\int_{W_{Y_{(1)}}} i^{*}C \wedge G(W_{Y_{(1)}}) + \int_{W_{Y_{(1)}}} i^{*}C \wedge (G(W_{Y_p}) - G(W_{Y_{(1)}}))$. Since the second term of the sum, $G(W_{Y_p}) - G(W_{Y_{(1)}})$ has vanishing degree 0-part, then $\mathrm{PD}_{W_{Y_{(1)}}}(G(W_{Y_p}) - G(W_{Y_1}))$ is made only by lower-dimensional sub-branes. Let $W_{Y_{(1,1)}}$ be the first one: one gets $\int_{W_{Y_{(1,1)}}} i^{*}C$, which is equal to $\int_{W_{Y_{(1,1)}}} i^{*}C \wedge G(W_{Y_{(1,1)}}) + \int_{W_{Y_{(1)}}} i^{*}C \wedge (1 - G(W_{Y_{(1,1)}}))$. The second term gives again

[3] In the case of almost complex manifolds $X \subset M$, there is a nice expression for this class: $d = -c_1(N_M X)$.

rise only to lower dimensional sub-branes. Proceeding inductively until one arrives at D0-branes, whose G-term is 1, one can write:

$$\int_{W_{Y_{(1)}}} i^*C \wedge G(W_{Y_p}) = \sum_{h=0}^{m} \int_{W_{Y_{(1,h)}}} i^*C \wedge G(W_{Y_{(1,h)}})\,,$$

where, for $h = 0$, $W_{Y_{(1,0)}} = W_{Y_{(1)}}$. Proceeding in the same way for every $W_{Y_{(k)}}$, one obtains a set of sub-branes $\{q_{k,h} \cdot W_{Y_{(k,h)}}\}$, which, using only one index, will still be denoted by $\{q_k \cdot W_{Y_{(k)}}\}$. Therefore one gets:

$$S_{WZ} = \sum_k \int_{W_{Y_{(k)}}} i^*C \wedge G(W_{Y_{(k)}})\,. \tag{3.10}$$

From this expression it is clear that *the Dp-brane Y_p with gauge and gravitational couplings is equivalent to the set of sub-branes $Y_{(k)}$ with trivial gauge bundle.* Moreover, one can show that:

$$i_\#\big(\mathrm{ch}(E) \wedge G(W_{Y_p})\big) = \sum_k (i_k)_\#\, G(W_{Y_{(k)}})\,, \tag{3.11}$$

i.e. the charge densities of the two configurations are the same. In order to prove this, recall the formulae:

$$\begin{aligned} &i_\#(\alpha \wedge i^*\beta) = i_\#(\alpha) \wedge \beta\,, \\ &\int_{W_{Y_p}} \alpha = \int_M i_\#(\alpha) \end{aligned} \tag{3.12}$$

for $\alpha \in H^\bullet(W_{Y_p}, \mathbb{Q})$ and $\beta \in H^\bullet(M, \mathbb{Q})$. Thus:

$$\begin{aligned} \int_{W_{Y_p}} i^*C \wedge \mathrm{ch}(E) \wedge G(W_{Y_p}) &= \int_M i_\#\big[i^*C \wedge \mathrm{ch}(E) \wedge G(W_{Y_p})\big] \\ &= \int_M C \wedge i_\#\big(\mathrm{ch}(E) \wedge G(W_{Y_p})\big) \\ \sum_k \int_{WY_p} i_k^*C \wedge G(W_{Y_{(k)}}) &= \sum_k \int_S (i_k)_\#\big[i_k^*C \wedge G(W_{Y_{(k)}})\big] \\ &= \int_M C \wedge \sum_k (i_k)_\#\big(G(W_{Y_{(k)}})\big)\,. \end{aligned}$$

Since the two terms are equal for every form C, one gets formula (3.11). Hence one can write:

> **Splitting principle:** *a Dp-brane Y_p with gauge bundle is equivalent to a set of sub-branes $Y_{(k)}$ with trivial gauge bundle, such that the total charge density of the two configurations is the same.*

The physical interpretation of this conjecture is the phenomenon of tachyon condensation [33, 10, 9]: the quantization of strings extending from a brane to an anti-brane leads to a tachyonic mode, which represents an instability and generates a process of annihilation of brane and anti-brane worldvolumes via an RG-flow [36], leaving lower dimensional branes. In particular, given a Dp-brane Y_p with gauge bundle $E \to W_{Y_p}$, one can write $[E] = ([E] - [\mathrm{rk}\, E]) + [\mathrm{rk}\, E]$, so that

$[E] - [\mathrm{rk}\, E] \in \tilde{K}(W_{Y_p})$:[4] thus one looks at this configuration as a triple made by a Dp-brane Z_p with gauge bundle $\mathrm{rk}\, E$, a Dp-brane Y_p with gauge bundle E and an anti-Dp-brane $\overline{Z}_p$ with gauge bundle $\mathrm{rk}\, E$. Thus, by tachyon condensation, only Z_p remains (with trivial bundle, i.e., only with its own charge), while Y_p and $\overline{Z}_p$ annihilate, giving rise to lower dimensional branes with trivial bundle, as stated in the splitting principle. Moreover, if one considers a stack of pairs $(Y_p, \overline{Y}_p)$ with gauge bundles E and F respectively, this is equivalent to consider gauge bundles $E \oplus G$ and $F \oplus G$ respectively, since, viewing the factor G as a stack of pairs $(Z_p, \overline{Z}_p)$ with the *same* gauge bundle, one has that by tachyon condensation Z_p and $\overline{Z}_p$ disappear, leaving no other sub-branes. This is exactly the physical interpretation of the stable equivalence relation of K-theory, stated in equation (3.4). This principle is in some sense an inverse of Sen conjecture. However, one should bear in mind that it holds only at the rational level, since it involves Chern characters and the A-roof genus. At the integral level it does not hold in general.

Simplest example

Before getting to the original Sen conjecture, in order to have a flavor of the physics behind this brane-anti-brane annihilation, it is worth to describe in detail here the simplest example [10], namely an equivalent description of a D(p-2)-brane with trivial gauge bundle. This simple configuration is equivalent to a pair made by coincident Dp-brane and anti-Dp-brane equipped with a holomorphic line bundle $\mathcal{L}$ and with a trivial line bundle 1 respectively.
Now, an analysis of the GSO-projected open string spectrum of this system says that $p - p$ and $\bar{p} - \bar{p}$ strings are responsible for the appearance of the usual $U(1) \times U(1)$ Maxwell supermultiplet, while $p - \bar{p}$ and $\bar{p} - p$ strings undergo a reversed GSO projection, so that they contain a complex scalar field T of charge $(1, -1)$ in their spectrum: This is a tachyon. The presence of such a state determines an instability of the system because the configuration with $T = 0$ represents a maximum of the scalar potential computed after integration of all other massive modes. In order to be gauge invariant, this potential must depend only on $|T|$ and it has a one-parameter continuum of minima at $|T| = T_0$, parameterized by the phase of the tachyon field. Suppose there is a locus of real codimension 2 inside the (p+1)-dimensional manifold wrapped by the $Dp - \overline{Dp}$ system on which T vanishes. Then, by requiring the tachyon to approach very quickly at infinity its vev T_0, the $Dp - \overline{Dp}$ system will look like the vacuum everywhere except very close to such a locus. Being T a complex field, it can still have a non-trivial winding number around this locus, which depends on the topology of the gauge group left unbroken: in this case the initial gauge group $U(1) \times U(1)$ gets broken by the vev in its diagonal part $U(1)_{\mathrm{diag}}$,

$$T \longrightarrow T\, e^{i(\theta_1 - \theta_2)} \qquad\qquad \theta_1 \,=\, \theta_2 \quad \text{residual gauge}\,.$$

Thus there can be a non-trivial winding number, which is just the degree of the winding map $\vartheta : S^1 \to U(1)_{\mathrm{diag}}$, where S^1 is the circle which links the vanishing locus inside the ambient manifold.
It is not hard to guess what is the relation of such a topological invariant with the non-trivial complex line bundle on the Dp-brane: it is just its integrated first Chern class, that characterizes completely its topology. Indeed, since T belongs to the $p - \bar{p}$ sector, it is a section of $\mathcal{L} \otimes 1^\vee = \mathcal{L} \otimes 1 = \mathcal{L}$; it can be regarded also as a map $T : 1 \to \mathcal{L}$, which is invertible, and therefore an isomorphism, outside $\{T = 0\}$. The zero locus of T is then precisely the divisor of $\mathcal{L}$ inside the

[4] $\tilde{K}$ is the subgroup of K made by elements $[E] - [F]$ with E and F having the same rank (see appendix D for details).

Dp worldvolume, which, after $Dp - \overline{Dp}$ have annihilated, behaves exactly like a D(p-2)-brane with charge given by:

$$Q_{p-2} \,=\, i_{\#}[(c_1(\mathcal{L})] \,. \tag{3.13}$$

In more mathematical terms, one has the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{W_{Y_p}}(n) \xrightarrow{T} \mathcal{O}_{W_{Y_p}}(m) \xrightarrow{|_{T=0}} \mathcal{O}_{W_{Y_{p-2}}} \longrightarrow 0\,, \tag{3.14}$$

where $\mathcal{O}_X(n)$ is an holomorphic line bundle over X with integrated first Chern class equal to n and, by definition of exact sequence, the last term is defined as follows:

$$\mathcal{O}_{W_{Y_{p-2}}} \equiv \frac{\mathcal{O}_{W_{Y_p}}(m)}{\mathrm{Im}(T)} \simeq \mathrm{Coker}(T)\,, \tag{3.15}$$

and

$$\mathrm{ch}\,(\mathcal{O}_{W_{Y_{p-2}}}) \;=\; \mathrm{ch}\,(\mathcal{O}_{W_{Y_p}}(m)) - \mathrm{ch}\,(\mathcal{O}_{W_{Y_p}}(n))\,. \tag{3.16}$$

$\mathrm{ch}_0(\mathcal{O}_{W_{Y_{p-2}}}) = 0$ is the formalization of the $Dp - \overline{Dp}$ annihilation, while $\mathrm{ch}_1(\mathcal{O}_{W_{Y_{p-2}}})$ represents the density charge of D(p-2)-branes.

Provided one uses a non-abelian gauge theory on the brane-anti-brane system, the generalization to higher codimension is straightforward, taking into account that now the tachyon becomes a *matrix*. Examples of this circumstance will be presented in chapter 6 as an useful and powerful tool for computing certain topological charges.

K-theory classes

Since the H-flux is assumed to vanish, in order not to be Freed-Witten anomalous the Dp-brane must be spin^c; hence, as the whole target space is spin^c, also the normal bundle of the brane is. Thus one can consider, in the case of an even-codimension[5] D-brane, the K-theory Gysin map $i_! : K(W_{Y_p}) \to \tilde{K}(M)$, which is defined in the following way (see the appendix D for further details). Let U be a tubular neighborhood of W_{Y_p}; then,

$$i_! \,=\, \psi^\star \circ \varphi^\star \circ \mathcal{T}\,, \tag{3.17}$$

where $\mathcal{T} : K(W_{Y_p}) \to K(N_M W_{Y_p}) \equiv \tilde{K}(N_M W^+_{Y_p})$ is the so called Thom isomorphism[6] which exists thanks to the spin^c condition on the normal bundle, $\varphi : U^+ \to N_M W^+_{Y_p}$ is a diffeomorphism and $\psi : M \to U^+$ fixes U and sends $M \backslash U$ to the point at infinity.
Recall now the differentiable Riemann-Roch theorem [37, 31]:

$$\mathrm{ch}\big(i_!(E)\big) \wedge \hat{A}(TM) = i_{\#}\big(\mathrm{ch}(E) \wedge e^{\frac{d}{2}} \wedge \hat{A}(T(W_{Y_p}))\big)\,. \tag{3.18}$$

Using (3.18) and (3.12) one obtains:

$$\int_{W_{Y_p}} i^*C \wedge \mathrm{ch}(E) \wedge e^{\frac{d}{2}} \wedge \frac{\sqrt{\hat{A}(TW_{Y_p})}}{\sqrt{\hat{A}(N_M W_{Y_p})}} = \int_M C \wedge \mathrm{ch}\big(i_!(E)\big) \wedge \sqrt{\hat{A}(TM)}\,.$$

[5]In the case of odd-codimension one ends up with $K^1(M)$.

[6]$N_M W^+_{Y_p}$ is the compactification to a point of the normal bundle.

Thus one gets:

$$S_{WZ} \,=\, \int_M C \wedge \mathrm{ch}\big(i_!(E)\big) \wedge \sqrt{\hat{A}(TM)}\,, \tag{3.19}$$

hence:

$$Q_p \,=\, \mathrm{ch}(i_!E) \wedge \sqrt{\hat{A}(TM)}\,. \tag{3.20}$$

Therefore, (3.20) is another expression for Q_p with respect to (3.7), but with an important difference: the $\hat{A}$-factor does not depend on W_{Y_p}, hence all Q_p is a function only of E. Thus, one can consider $i_!E$ as the K-theory analogue of the charge density, considered as an *integral* K-theory class (i.e. with torsion contributions restored): in this case, one really gets a refinement of the classification. The use of Chern characters, instead, obliges to consider rational classes which cannot contain information about the torsion part.

Consider the two expressions found for the rational charge density:

$$\begin{aligned} Q_p^{(1)} &= i_{\#}\big(\mathrm{ch}(E) \wedge G(W_{Y_p})\big)\,, \\ Q_p^{(2)} &= \mathrm{ch}\big(i_!E\big) \wedge \sqrt{\hat{A}(TM)}\,. \end{aligned}$$

$Q_p^{(2)}$ is exactly the charge density of a D9-brane (whose worldvolume coincides with M), whose gauge bundle is the K-theory class $i_!E$. Hence, expressing the charge in the form $Q_p^{(2)}$ for each D-brane in the background is equivalent to think that there exists only a stack of pairs of $D9 - \overline{D9}$ with a suitable K-theory class encoding all the dynamics. Hence a formulation of the Sen conjecture could be the following [33, 10]:

> **Sen's conjecture:** *every configuration of branes and anti-branes with any gauge bundle is dynamically equivalent to a configuration with only a stack of coincident pairs of* $D9 - \overline{D9}$ *with an appropriate K-theory class on it.*

To see that the dynamics is actually equivalent, one can use the splitting principle stated above: since $Q_p^{(1)} = Q_p^{(2)}$, the brane W_{Y_p} with charge $Q_p^{(1)}$ and the D9-brane with charge $Q_p^{(2)}$ split into the same set of sub-branes (with trivial gauge bundle).

To formulate both the splitting principle and the Sen conjecture, only the action has been considered, hence only *rational* classes given by Chern characters and $\hat{A}$-genus. Thus, one can classify the charge density in the two following ways:

- as a rational cohomology class $i_{\#}(\mathrm{ch}(E) \wedge G(W_{Y_p})) \in H^{\mathrm{ev}}(M, \mathbb{Q})$;
- as a rational K-theory class $i_!E \in K_{\mathbb{Q}}(M) \equiv K(M) \otimes_{\mathbb{Z}} \mathbb{Q}$.

These two classification schemes are completely equivalent due to the fact that the Chern character:

$$\mathrm{ch}(\,\cdot\,) \wedge \sqrt{\hat{A}(TM)}:\; K_{\mathbb{Q}}(M) \longrightarrow H^{\mathrm{ev}}(M, \mathbb{Q}) \tag{3.21}$$

is an *isomorphism*[7]. This equivalence is lost at the integral level, since the torsion part of $K(M)$ and $H^{\mathrm{ev}}(M, \mathbb{Z})$ are in general different. Moreover, since at the integral level one does not apply the splitting principle, one does not really have Sen's conjecture: the classification via Gysin map and cohomology are different, and the use of the Gysin map is just *suggested* by the equivalence of the dynamics at rational level.

[7]Even more, it is an isometry of the metric given by the index of the Dirac operator: $\mathrm{ind}\, D_{E\otimes F} = \int_M \mathrm{ch}(E) \wedge \mathrm{ch}(F) \wedge \hat{A}(TM)$.

3.3 K-theory from AHSS

The second approach to the K-theory classification of D-brane charges is the one inspired and induced by the Freed-Witten anomaly and concretely based on the Atiyah-Hirzebruch spectral sequence. This method was already introduced in the previous chapter, thus here some technical details are added for later purposes. For notations and a brief introduction to the AHSS, the reader is referred to appendix E.

Consider for simplicity the case of worldvolumes of *even* codimension in M. Given an appropriate filtration of the target space manifold $M = M^{10} \supset \cdots \supset M^0$, AHSS is simply a sequence of abelian groups E_r^p and of coboundary nilpotent operators $\mathrm{d}_r : E_r^p \to E_r^{p+r}$ such that $E_1^p = C^p(M,\mathbb{Z})$ (the group of simplicial p-cochains on M), $\mathrm{d}_1 \equiv \mathrm{d}$ (the ordinary coboundary operator) and $\mathrm{d}_r^p = 0$ if r is even. As already extensively seen in section 2.3, the cohomology with respect to d_3 has a very precise physical meaning, intimately related to the Freed-Witten anomaly, while the physical counterpart of the following coboundary operators is still lacking as well as their explicit representation[8]. The sequence goes on according to the equation

$$E_{r+1}^p = \frac{\mathrm{Ker}\,\mathrm{d}_r^p}{\mathrm{Im}\,\mathrm{d}_r^{p-r}}\,, \tag{3.22}$$

until it stabilizes after a finite number of steps, to the graded group

$$E_\infty^{\mathrm{ev},\,0}(M) = \bigoplus_{2k} K_{2k}(M)/K_{2k+1}(M)\,, \tag{3.23}$$

where $K_n(M) = \mathrm{Ker}(K(M) \to K(M^{n-1}))$.

Therefore, one starts from a representative of the Poincaré dual of the brane $\mathrm{PD}_M(q \cdot W_{Y_p})$, which for simplicity is now assumed even-dimensional, and, if it survives until the last step, one ends up with a class $\{\mathrm{PD}_M(q \cdot W_{Y_p})\} \in K_{9-p}(M)/K_{10-p}(M)$.

In the rational case, one can build up the corresponding sequence $\mathrm{AHSS}_{\mathbb{Q}}$ [25], which ends with the groups $Q_\infty^{\mathrm{ev},\,0}(M)$, but it stabilizes already after the first step, i.e. at the level of ordinary cohomology. Hence, the class $\{i_\#(\mathrm{ch}(E) \wedge G(W_{Y_p}))\} \in Q_\infty^{\mathrm{ev},\,0}(M)$ is completely equivalent to the cohomology class $i_\#(\mathrm{ch}(E) \wedge G(W_{Y_p})) \in H^{\mathrm{ev}}(M,\mathbb{Q})$.

3.4 Linking the classifications

To summarize, the aim is to classify the charges of D-branes in a 10-dimensional compact euclidean target space M. To achieve this, one can use cohomology or K-theory, with integral or rational coefficients, obtaining the possibilities showed in table 3.1.

In the rational case, as already said, there is a complete equivalence of the three approaches, since the three groups $\bigoplus_{2k} H^{2k}(M,\mathbb{Q})$, $K_{\mathbb{Q}}(M)$ and $\bigoplus_{2k} Q_\infty^{2k,\,0}(M)$ are all canonically isomorphic. Instead, in the integral case there are not such isomorphisms (the three groups are all different), and there is a strong asymmetry due to the fact that in the homological and AHSS classifications *the gauge bundle and the gravitational coupling are not considered at all*, while they are of course taken into account in the Gysin map approach. Up to now only the case of

[8] Actually, in 10 compact dimensions, like the present situation, a further anomaly can arise, related to the fact that some target space homology classes could have no smooth representative cycle. An attempt to relate the anomaly due to the appearance of non-representable cycles and the cohomology of the operator d_5 has been made [38].

	Integer	Rational
Cohomology	$\mathrm{PD}_M(q \cdot W_{Y_p}) \in H^{9-p}(M, \mathbb{Z})$	$i_\#\big(\mathrm{ch}(E) \wedge G(W_{Y_p})\big) \in H^{\mathrm{ev}}(M, \mathbb{Q})$
K-theory (Gysin map)	$i_!(E) \in K(M)$	$i_!(E) \in K_{\mathbb{Q}}(M)$
K-theory (AHSS)	$\{\mathrm{PD}_M(q \cdot W_{Y_p})\} \in E_\infty^{9-p,\,0}(M)$	$\big\{i_\#(\mathrm{ch}(E) \wedge G(W_{Y_p}))\big\} \in Q_\infty^{\mathrm{ev},\,0}(M)$

Table 3.1: Classifications

even-codimensional D-branes has been considered: that is because the Gysin map requires an even-dimensional normal bundle to be valued in $K(M)$. The odd-dimensional case will also be described by considering the brane embedded in the suspension $S^1 M$ of the target space, and the picture will be similar.

Since the two approaches are not equivalent at the integral level, one can wonder what could be the relations among them: it is clear how to link the cohomology class and the AHSS class, since the second level of AHSS is exactly the cohomology. The real goal is to link the Gysin map approach with the one based on AHSS. For the reader's convenience, the result is anticipated here [5]. A detailed and formal proof of it is given in the next subsection, which is rather technical, so non-interested reader can skip it.

It turns out that, if a Dp-brane with gauge bundle E survives until the last step of AHSS, then $i_!(E) \in \mathrm{Ker}(K^{9-p}(M) \longrightarrow K^{9-p}(M^{8-p}))$ and

$$\{\mathrm{PD}_M(W_{Y_p})\}_{E_\infty^{9-p,\,0}} = [i_!(E)]_{E_\infty^{9-p,\,0}} \,. \tag{3.24}$$

Thus, one must first use AHSS to detect possible anomalies, then one can use the Gysin map to get the charge of a non-anomalous brane: such a charge belongs to the equivalence class reached by AHSS, so that Gysin map gives more detailed information, being a particular representative of that class.

Some comments at this point are in order. One could ask why the additional information provided by the Gysin map has to be considered: in fact, one of the advantages of AHSS is that it quotients out unstable configurations. It seems that such additional information keeps into account only instabilities. However, this is not the case.

The charge of a D-brane reached via AHSS (that in this sense simply generalizes cohomology) does not provide complete information about the worldvolume, exactly as the charge of an electron does not provide information about its trajectory: two homologous worldvolumes (or generalized homologous, in the sense of AHSS) are *not* the same trajectory. It turns out, indeed, that for any two gauge bundles E and F on the same Dp-brane[9], $[i_!(E) - i_!(F)]_{E_\infty^{9-p,\,0}} = 0$, which means that $i_!(E) - i_!(F)$ lies in the image of some boundary of AHSS. This implies that, since $\mathrm{rk}(E-F) = 0$, $i_!(E-F)$ is a representative of the AHSS-class reached starting from $\mathrm{PD}_M(0 \cdot W_{Y_p})$.

[9]The square brackets indicating the K-theory classes in the differences are dropped in order not to clutter the notation.

The additional non-trivial and meaningful information contained in $i_!(E-F)$, then, is contained in the charges of the sub-branes of W_{Y_p}, which are sensible to gauge and gravitational couplings, so to the actual trajectory of the Dp-brane.

3.4.1 Proof of the result*

The reader is referred to appendices D and E for the notations throughout this subsection and for any further mathematical explanation. In the application to physics of this general result, X will play the role of the target space M while Y the one of the D-brane worldvolume W_Y.

Even case

Choose a finite triangulation of X which restricts to a triangulation of Y (see [39]). The following notation will be adopted:

- denote the triangulation of X by $\Delta = \{\Delta^m_i\}$, where m is the dimension of the simplex and i enumerates the m-simplices;
- denote by X^p_Δ the p-skeleton of X with respect to Δ.

In the following theorem the definition of "dual cell decomposition" with respect to a triangulation will be needed: the reader is referred to [40] pp. 53-54.

Theorem 3.4.1 *Let X be an n-dimensional compact manifold and $Y \subset X$ a p-dimensional embedded compact submanifold. Let:*

- *$\Delta = \{\Delta^m_i\}$ be a triangulation of X which restricts to a triangulation $\Delta' = \{\Delta^m_{i'}\}$ of Y;*
- *$D = \{D^{n-m}_j\}$ be the dual decomposition of X with respect to Δ;*
- *$\tilde{D} \subset D$ be the subset of D made by the duals of simplices in Δ'.*

Then:

- *the interior of $|\tilde{D}|$ is a tubular neighborhood of Y in X;*
- *the interior of $|\tilde{D}|$ does not intersect X^{n-p-1}_D, i.e.:*

$$|\tilde{D}| \cap X^{n-p-1}_D \subset \partial|\tilde{D}|.$$

Proof: The n-simplices of $\tilde{D}$ are the dual of the vertices of Δ'. Let $\tau = \{\tau^m_i\}$ be the first baricentric subdivision of Δ. For each vertex $\Delta^0_{i'}$ (regarded as an element of Δ), its dual is:

$$\tilde{D}^n_{i'} = \bigcup_{\Delta^0_{i'} \in \tau^n_k} \tau^n_k. \tag{3.25}$$

Moreover, if $\tau' = \{\tau^m_{i'}\}$ is the first baricentric subdivision of Δ' and D' is the dual of Δ' in Y, then:

$$D'^p_{i'} = \bigcup_{\Delta^0_{i'} \in \tau^n_{k'}} \tau^n_{k'}. \tag{3.26}$$

and:

$$\tilde{D}^n_{i'} \cap Y = D'^{\,p}_{i'}.$$

Moreover, consider the $(n-p)$-simplices in $\tilde{D}$ contained in $\partial\tilde{D}^n_{i'}$ (for the fixed i' of formula (3.25)), i.e. $\tilde{D}^{n-p} \cap \tilde{D}^n_{i'}$: it intersects Y transversally in the baricenters of each p-simplex of Δ' containing $\Delta^0_{i'}$: call such baricenters $\{b_1, \ldots, b_k\}$ and the intersecting $(n-p)$-cells $\{\tilde{D}^{n-p}_j\}_{j=1,\ldots,k}$. Since (for a fixed i') $\tilde{D}^n_{i'}$ retracts on $\Delta^0_{i'}$, one can consider a local chart $(U_{i'}, \varphi_{i'})$, with $U_{i'} \subset \mathbb{R}^n$ neighborhood of 0, such that:

- $\varphi^{-1}_{i'}(U_{i'})$ is a neighborhood of $\tilde{D}^n_{i'}$;
- $\varphi_{i'}(D'^{\,p}_{i'}) \subset U_{i'} \cap \big(\{0\} \times \mathbb{R}^p\big)$, for $0 \in \mathbb{R}^{n-p}$ (see eq. (3.26));
- $\varphi_{i'}(\tilde{D}^{n-p}_j) \subset U_{i'} \cap \big(\mathbb{R}^{n-p} \times \pi_p\big(\varphi_{i'}(b_j)\big)\big)$, for $\pi_p : \mathbb{R}^n \to \{0\} \times \mathbb{R}^p$ the projection.

Consider now the natural foliation of $U_{i'}$ given by the intersection with the hyperplanes $\mathbb{R}^{n-p} \times \{x\}$ and its image via $\varphi^{-1}_{i'}$: in this way, one obtains a foliation of $\tilde{D}^n_{i'}$ transversal to Y. If one does so for any i', by construction the various foliations glue on the intersections, since such intersections are given by the $(n-p)$-cells $\{\tilde{D}^{n-p}_j\}_{j=1,\ldots,k}$, and the interior gives a C^0-tubular neighborhood of Y.

Moreover, a $(n-p-r)$-cell of $\tilde{D}$, for $r > 0$, cannot intersect Y since it is contained in the boundary of a $(n-p)$-cell, and such cells intersect Y, which is done by p-cells, only in their interior points b_j.
□

Consider now triples (X, Y, D) satisfying the following condition:

(#) X is an n-dimensional compact manifold and $Y \subset X$ a p-dimensional embedded compact submanifold, such that $n-p$ is even and $\mathcal{N}(Y)$ is spin^c. Moreover, D and $\tilde{D}$ are defined as in theorem 3.4.1.

Lemma 3.4.2 *Let (X, Y, D) be a triple satisfying (#), $U = \mathrm{Int}|\tilde{D}|$ and $\alpha \in K(Y)$. Then:*

- *there exists a neighborhood V of $X \setminus U$ such that $i_!(\alpha)\big|_V = 0$;*
- *in particular, $i_!(\alpha)\big|_{X^{n-p-1}_D} = 0$.*

Proof: By equation (D.8) at page 155:

$$i_!(\alpha) = \psi^*\beta, \qquad \beta = \big(\varphi^+_U\big)^* \circ T(\alpha) \in \tilde{K}(U^+).$$

Let $\beta = [E] - [n]$, and let $V_\infty \subset U^+$ be a neighborhood of ∞ which trivializes E. Then $(\psi^* E)\big|_{\psi^{-1}(V_\infty)}$ is trivial. Hence, for $V = \psi^{-1}(V_\infty)$:

$$(\psi^*\beta)\big|_V = \big[(\psi^* E)\big|_V\big] - [n] = [n] - [n] = 0.$$

By theorem 3.4.1, X^{n-p-1}_D does not intersect the tubular neighborhood $\mathrm{Int}|\tilde{D}|$ of Y, hence $X^{n-p-1}_D \subset \psi^{-1}(V_\infty) = V$, so that $(\psi^*\beta)\big|_{X^{n-p-1}_D} = 0$.
□

Trivial bundle The case of a trivial bundle is first considered.

Theorem 3.4.3 *Let (X, Y, D) be a triple satisfying (#) and $\Phi_D^{n-p} : C^{n-p}(X, \mathbb{Z}) \longrightarrow K(X_D^{n-p}, X_D^{n-p-1})$ be the isomorphism stated in theorem E.2.2. Let:*

$$\pi^{n-p} : X_D^{n-p} \longrightarrow X_D^{n-p}/X_D^{n-p-1}$$

be the projection and $\widetilde{\mathrm{PD}}(Y_\Delta)$ be the representative of $\mathrm{PD}_X Y$ given by the sum of the cells dual to the p-cells of Δ covering Y. Then:

$$i_! \,(Y \times \mathbb{C})\,\big|_{X_D^{n-p}} = (\pi^{n-p})^* \big(\Phi_D^{n-p}\big(\widetilde{\mathrm{PD}}(Y_\Delta)\big)\big).$$

Proof: Define:

$$(U^+)_D^{n-p} = \frac{\overline{X_D^{n-p}\,|_U}}{X_D^{n-p-1}\,|_{\partial U}}$$

so that $(U^+)_D^{n-p} \subset U^+$ sending the denominator to ∞ (the numerator is exactly $\tilde{D}^{n-p}$ of theorem 3.4.1). Define also:

$$\psi^{n-p} = \psi\,\big|_{X_D^{n-p}} : X_D^{n-p} \longrightarrow (U^+)_D^{n-p}.$$

ψ^{n-p} is well-defined since the $(n-p)$-simplices outside U and all the $(n-p-1)$-simplices are sent to ∞ by ψ.

One has:

$$\pi^{n-p}(X_D^{n-p}) \simeq \dot{\bigcup_{i\in I}} S_i^{n-p}.$$

Denote by $\{S_j^{n-p}\}_{j\in J}$, with $J \subset I$, the set of $(n-p)$-spheres corresponding to $\pi^{n-p}\big(\overline{X_D^{n-p}\,|_U}\big)$. Define:

$$\rho : \dot{\bigcup_{i\in I}} S_i^{n-p} \longrightarrow \dot{\bigcup_{j\in J}} S_j^{n-p}$$

as the projection, i.e., ρ is the identity of S_j^{n-p} for every $j \in J$ and sends all the spheres in $\{S_i^{n-p}\}_{i\in I\setminus J}$ to the attachment point. One has that:

$$\psi^{n-p} = \rho \circ \pi^{n-p}.$$

In fact, the boundary of the $(n-p)$-cells intersecting U is contained in ∂U, hence it is sent to ∞ by ψ^{n-p}, while all the $(n-p)$-cells outside U are sent to ∞: hence, the image of ψ^{n-p} is homeomorphic to $\dot{\bigcup}_{j\in J} S_j^{n-p}$ sending ∞ to the attachment point. Thus:

$$(\psi^{n-p})^* = (\pi^{n-p})^* \circ \rho^*.$$

Put $\mathcal{N} = \mathcal{N}(Y)$ and $\tilde{\lambda}_\mathcal{N} = \big(\varphi_U^+\big)^*\big(\lambda_\mathcal{N}\big)$, where $\lambda_\mathcal{N}$ is the Thom class of theorem D.2.2 for the case of the normal bundle. By lemma D.2.1 and equation (D.8) at page 155, one has $i_!\,(Y \times \mathbb{C}) = \psi^* \circ \big(\varphi_U^+\big)^*\big(\lambda_\mathcal{N}\big)$. Then:

$$i_!\,(Y \times \mathbb{C})\,\big|_{X_D^{n-p}} = \psi^*\big(\tilde{\lambda}_\mathcal{N}\big)\,\big|_{X_D^{n-p}} = (\psi^{n-p})^*\Big(\tilde{\lambda}_\mathcal{N}\,\big|_{(U^+)_D^{n-p}}\Big)$$

and

$$\rho^*\Big(\tilde{\lambda}_\mathcal{N}\,\big|_{(U^+)_D^{n-p}}\Big) = \Phi_D^{n-p}\big(\widetilde{\mathrm{PD}}(Y_\Delta)\big)$$

since:

- $\widetilde{\mathrm{PD}}(Y_\Delta)$ is the sum of the $(n-p)$-cells intersecting U;
- hence $\Phi_D^{n-p}\big(\widetilde{\mathrm{PD}}(Y_\Delta)\big)$ gives a $(-1)^{\frac{n-p}{2}}(\eta-1)^{\boxtimes\frac{n-p}{2}}$ factor to each sphere S_j^{n-p} for $j \in J$ and 0 otherwise;
- but this is exactly $\rho^*\Big(\tilde{\lambda}_{\mathcal{N}}\,\big|_{(U^+)_D^{n-p}}\Big)$ since by equation (D.7) at page 154, one has, for $y \in Y$:
$$(\lambda_{\mathcal{N}})\,\big|_{\mathcal{N}_y^+} = \lambda_{\mathbb{R}^{n-p}} = (-1)^{\frac{n-p}{2}}(\eta-1)^{\boxtimes\frac{n-p}{2}}$$
and for the spheres outside U, that ρ sends to ∞, one has that:
$$\begin{aligned}\rho^*\Big(\tilde{\lambda}_{\mathcal{N}}\,\big|_{(U^+)_D^{n-p}}\Big)\Big|_{\dot{\bigcup}_{i\in I\setminus J} S_i^{n-p}} &= \rho^*\Big(\tilde{\lambda}_{\mathcal{N}}\,\big|_{\rho\left(\dot{\bigcup}_{i\in I\setminus J} S_i^{n-p}\right)}\Big)\\ &= \rho^*\big(\tilde{\lambda}_{\mathcal{N}}\,\big|_{\{\infty\}}\big) = \rho^*(0) = 0.\end{aligned}$$

Hence:
$$\begin{aligned} i_!\,(Y\times\mathbb{C})\,\big|_{X_D^{n-p}} &= (\psi^{n-p})^*\Big(\tilde{\lambda}_{\mathcal{N}}\,\big|_{(U^+)_D^{n-p}}\Big)\\ &= (\pi^{n-p})^*\circ\rho^*\Big(\tilde{\lambda}_{\mathcal{N}}\,\big|_{(U^+)_D^{n-p}}\Big)\\ &= (\pi^{n-p})^*\Phi_D^{n-p}\big(\widetilde{\mathrm{PD}}(Y_\Delta)\big).\end{aligned}$$

□

Corollary 3.4.4 *Let (X,Y,D) be a triple satisfying (#) and $\Xi_D^{n-p} : H^{n-p}\,(X,\mathbb{Z}) \longrightarrow \mathrm{Im}\,\Psi \subset K(X_D^{n-p}, X_D^{n-p-2})$ be the isomorphism* (E.7). *Let:*
$$\tilde{\pi}^{n-p} : X_D^{n-p} \longrightarrow X_D^{n-p}/X_D^{n-p-2}$$
be the projection. Then:
$$i_!\,(Y\times\mathbb{C})\,\big|_{X_D^{n-p}} = (\tilde{\pi}^{n-p})^*\big(\Xi_D^{n-p}\big(\mathrm{PD}(Y)\big)\big).$$

Proof: For $\tau \in Z^{n-p}(X,\mathbb{Z})$ and π^* the map of the diagram (E.8), one has $\Xi_D^{n-p}([\tau]) = \pi^*\Phi_D^{n-p}(\tau)$, and $(\tilde{\pi}^{n-p})^*\circ\pi^* = (\pi^{n-p})^*$ since $\pi\circ\tilde{\pi}^{n-p} = \pi^{n-p}$.
□

The following theorem encodes the link among Gysin map and AHSS: since the groups $E_r^{p,\sigma}$ for $r \geq 2$ and the filtration $\mathrm{Ker}\big(K(X)\longrightarrow K(X^{n-p})\big)$ of $K(X)$ do not depend on the particular simplicial structure chosen [25], one can drop the dependence on D.

Theorem 3.4.5 *Let X be an n-dimensional compact manifold and $Y \subset X$ a p-dimensional embedded compact submanifold, such that $n-p$ is even and $\mathcal{N}(Y)$ is spin^c. Let $\{(E_r^p, d_r^p)\}$ be the Atiyah-Hirzebruch spectral sequence, and let $\Xi^{n-p} : H^{n-p}(X,\mathbb{Z}) \xrightarrow{\simeq} E_2^{n-p,0}$ be the isomorphism induced by Φ^{n-p}. Suppose that $\Xi^{n-p}\,\mathrm{PD}(Y)$ is contained in the kernel of all the boundaries $d_r^{n-p,0}$ for $r \geq 2$.*

With this data, define a class:
$$\big\{\Xi^{n-p}\,\mathrm{PD}(Y)\big\}^{(2)}_{E_\infty^{n-p,0}} \in E_\infty^{n-p,0} \simeq \frac{\mathrm{Ker}\big(K(X)\longrightarrow K(X^{n-p-1})\big)}{\mathrm{Ker}\big(K(X)\longrightarrow K(X^{n-p})\big)}.$$

Then:
$$\big\{\Xi^{n-p}\,\mathrm{PD}(Y)\big\}^{(2)}_{E_\infty^{n-p,0}} = \big[i_!\,(Y\times\mathbb{C})\big]_{E_\infty^{n-p,0}}.$$

Proof: The cellular decomposition D considered in the previous theorems will be used. By equations (E.9) and (E.10) one has:

$$E_\infty^{n-p,0} = \mathrm{Im}\Big(\begin{tikzcd} \tilde{K}(X/X_D^{n-p-1}) \arrow[rr, "\Psi"] \arrow[dr, "\pi^*_{n-p-1}"'] & & \tilde{K}(X_D^{n-p}) \\ & \tilde{K}(X) \arrow[ur, "i^*_{n-p}"'] & \end{tikzcd}\Big) \tag{3.27}$$

and, given a representative $\alpha \in \mathrm{Ker}\big(K(X) \longrightarrow K(X_D^{n-p-1})\big) = \mathrm{Im}\,\pi^*_{n-p-1}$, one has that $[\alpha]_{E_\infty^{n-p,0}} = i^*_{n-p}(\alpha) = \alpha|_{X_D^{n-p}}$. One gets:

- the class $\{\Xi_D^{n-p}\,\mathrm{PD}(Y)\}^{(2)}_{E_\infty^{n-p,0}}$, which by construction is equal to $\{\Phi_D^{n-p}\,\widetilde{\mathrm{PD}}(Y)\}^{(1)}_{E_\infty^{n-p,0}}$, by formula (E.12) is given as an element of $\tilde{K}(X_D^{n-p})$ by $(\pi^{n-p})^*\big(\Phi_D^{n-p}\,\widetilde{\mathrm{PD}}(Y)\big)$, for $\pi^{n-p} : X_D^{n-p} \to X_D^{n-p}/X_D^{n-p-1}$;
- by lemma 3.4.2 one has $i_!\,(Y \times \mathbb{C}) \in \mathrm{Ker}\big(K(X) \longrightarrow K(X_D^{n-p-1})\big)$, hence $\big[i_!\,(Y \times \mathbb{C})\big]_{E_\infty^{n-p,0}}$ is well-defined, and, by exactness, $i_!\,(Y \times \mathbb{C}) \in \mathrm{Im}\,\pi^*_{n-p-1}$;
- by theorem 3.4.3, one has $i^*_{n-p}\big(i_!\,(Y \times \mathbb{C})\big) = (\pi^{n-p})^*\big(\Phi_D^{n-p}\big(\mathrm{PD}(Y)\big)\big)$;
- hence $\{\Phi_D^{n-p}\,\widetilde{\mathrm{PD}}(Y)\}^{(1)}_{E_\infty^{n-p,0}} = \big[i_!\,(Y \times \mathbb{C})\big]_{E_\infty^{n-p,0}}$.

□

Consider a trivial vector bundle of generic rank $[r] = Y \times \mathbb{C}^r$. By lemma D.2.1 at page 152, one has that $[r] \cdot \lambda_{\mathcal{N}} = \lambda_{\mathcal{N}}^{\oplus r}$, hence theorem 3.4.3 becomes:

$$i_!\,(Y \times \mathbb{C}^r)\,\big|_{X_D^{n-p}} = (\pi^{n-p})^*\big(\Phi_D^{n-p}\big(\widetilde{\mathrm{PD}}(r \cdot Y_\Delta)\big)\big)$$

and theorem 3.4.5 becomes:

$$\{\Xi^{n-p}\,\mathrm{PD}(r \cdot Y)\}^{(2)}_{E_\infty^{n-p,0}} = \big[i_!\,(Y \times \mathbb{C}^r)\big]_{E_\infty^{n-p,0}}\,.$$

Generic bundle If one considers a generic bundle E over Y of rank r, one can prove that $i_!(E)$ and $i_!(Y \times \mathbb{C}^r)$ have the same restriction to X_D^{n-p}: in fact, the Thom isomorphism gives $T(E) = E \cdot \lambda_{\mathcal{N}}$ and, if one restricts $E \cdot \lambda_{\mathcal{N}}$ to a finite family of fibers, which are transversal to Y, the contribution of E becomes trivial, so it has the same effect of the trivial bundle $Y \times \mathbb{C}^r$. The proof is as follows.

Lemma 3.4.6 *Let (X, Y, D) be a triple satisfying (#) and $E \xrightarrow{\pi} Y$ a bundle of rank r. Then:*

$$i_!\,(E)\,\big|_{X_D^{n-p}} = i_!\,(Y \times \mathbb{C}^r)\,\big|_{X_D^{n-p}}.$$

Proof: referring to the notations in the proof of lemma D.2.1 at page 152, one has that:

$$E \cdot \lambda_{\mathcal{N}} = i^*(\tilde{\pi}^*)^{-1}(E \boxtimes \lambda_{\mathcal{N}}) = i^*(\tilde{\pi}^*)^{-1}\big(\pi_1^* E \otimes \pi_2^* \lambda_{\mathcal{N}}\big).$$

Since X_D^{n-p} intersects the tubular neighborhood in a finite number of cells corresponding under φ_U^+ to a finite number of fibers of $\mathcal{N}$, it is sufficient to prove that, for any $y \in Y$, $(E \cdot \lambda_{\mathcal{N}})\big|_{\mathcal{N}_y^+} = \lambda_{\mathcal{N}}^{\oplus r}\big|_{\mathcal{N}_y^+}$. First of all:

- $i(\mathcal{N}_y^+) = \big(\{y\} \times \mathcal{N}_y\big)^+ \subset \big(\{y\} \times \mathcal{N}\big)^+$;
- $E \cdot \lambda_{\mathcal{N}}\big|_{\mathcal{N}_y^+} = \big(i\,\big|_{\mathcal{N}_y^+}\big)^* \Big\{ \Big[(\tilde{\pi}^*)^{-1}\big(\pi_1^* E \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\Big]\Big|_{i(\mathcal{N}_y^+)} \Big\}$.

In order to obtain the bundle $\Big[(\tilde{\pi}^*)^{-1}\big(\pi_1^* E \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\Big]\Big|_{i(\mathcal{N}_y^+)}$, one can restrict $\tilde{\pi}$ to:

$$\begin{aligned} A = \tilde{\pi}^{-1}\big[i(\mathcal{N}_y^+)\big] &= \tilde{\pi}^{-1}\big[\big(\{y\} \times \mathcal{N}_y\big)^+\big] \\ &= \big(\{y\} \times \mathcal{N}_y^+\big) \cup \big(Y \times \{\infty\}\big) \cup \big(\{\infty\} \times \mathcal{N}^+\big) \end{aligned}$$

and consider $\big(\tilde{\pi}\,\big|_A{}^*\big)^{-1}\Big[\big(\pi_1^* E \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\big|_A\Big]$. Moreover:

- $\big(\pi_1^* E \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\big|_{\{y\}\times\mathcal{N}_y^+} = \big(\mathbb{C}^r \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\big|_{\{y\}\times\mathcal{N}_y^+} \simeq \lambda_{\mathcal{N}}^{\oplus r}\big|_{\mathcal{N}_y^+}$;
- $\big(\pi_1^* E \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\big|_{Y\times\{\infty\}} = \big(\pi_1^* E \otimes 0\big)\big|_{Y\times\{\infty\}} = 0$;
- $\big(\pi_1^* E \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\big|_{\{\infty\}\times\mathcal{N}^+} = \big(0 \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\big|_{\{\infty\}\times\mathcal{N}^+} = 0$.

Hence, since the three components of A intersect each other at most at one point, by lemma E.2.1 at page 159 one gets:

$$\big(\pi_1^* E \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\big|_A = \big(\pi_1^*(Y \times \mathbb{C}^r) \otimes \pi_2^* \lambda_{\mathcal{N}}\big)\big|_A.$$

□

Odd case

Consider now the case of $n-p$ odd. One thus takes into account the unreduced suspension $\hat{S}^1 X$ and the natural embedding $i^1 : Y \to \hat{S}^1 X$. Let U be the tubular neighborhood of Y in X, and let $U^1 \subset \hat{S}^1 X$ be the tubular neighborhood of Y in $\hat{S}^1 X$ obtained by removing the vertices of the double cone to $\hat{S}^1 U$. Then, since $K^1(X) \simeq \tilde{K}(\hat{S}^1 X)$, one considers the Gysin map:

$$i_!^1 : K(Y) \longrightarrow K^1(X).$$

With the neighborhood U^1 considered, one has that $\overline{\hat{S}^1(X_D^{n-p}|_U)} \subset \overline{U^1}$ and $\hat{S}^1(X_D^{n-p-1}|_{\partial U}) \subset \partial U^1$, where ∂U^1 contains also the vertices of the double cone. In this way, one can reformulate the previous results in the odd case, considering $\hat{S}^1(X_D^{n-p})$ and $\hat{S}^1(X_D^{n-p-1})$ rather than X_D^{n-p} and X_D^{n-p-1}.

Consider triples (X, Y, D) safisfying the following condition:

($\#^1$) X is an n-dimensional compact manifold and $Y \subset X$ a p-dimensional embedded compact submanifold, such that $n-p$ is *odd* and $\mathcal{N}(Y)$ is spin^c. Moreover, D is the dual decomposition of Δ as in theorem 3.4.1.

The same theorems stated for the even case are now reformulated, and they can be proved in the same way. One should remember that $\mathcal{N}_{\hat{S}^1 X} Y$ is spin^c if and only $\mathcal{N}_X Y$ is, since $\mathcal{N}_{\hat{S}^1 X} Y = \mathcal{N}_X Y \oplus 1$ so that, by axioms of characteristic classes [41], W_3 must be the same.

Lemma 3.4.7 *Let (X, Y, D) be a triple satisfying ($\#^1$) and $\alpha \in K(Y)$. Then:*

- *there exists a neighborhood V of $X \setminus U^1$ such that $i^1_!(\alpha)\,\big|_V = 0$;*
- *in particular, $i^1_!(\alpha)\,\big|_{\hat{S}^1(X_D^{n-p-1})} = 0$.*

□

Theorem 3.4.8 *Let (X, Y, D) be a triple satisfying ($\#^1$) and $\Phi_D^{n-p} : C^{n-p}\,(X, \mathbb{Z}) \longrightarrow K\big(\hat{S}^1(X_D^{n-p}), \hat{S}^1(X_D^{n-p-1})\big)$ be the isomorphism stated in theorem E.2.2. Let:*

$$\pi^{n-p} : \hat{S}^1(X_D^{n-p}) \longrightarrow \hat{S}^1(X_D^{n-p})/\hat{S}^1(X_D^{n-p-1})$$

be the projection and $\widetilde{\mathrm{PD}}(Y_\Delta)$ be the representative of $\mathrm{PD}_X Y$ given by the sum of the cells dual to the p-cells of Δ covering Y. Then:

$$i^1_!\,(Y \times \mathbb{C})\,\big|_{\hat{S}^1(X_D^{n-p})} = (\pi^{n-p})^*\big(\Phi_D^{n-p}\big(\widetilde{\mathrm{PD}}(Y_\Delta)\big)\big).$$

□

Corollary 3.4.9 *Let (X, Y, D) be a triple satisfying ($\#^1$) and $\Xi_D^{n-p} : H^{n-p}\,(X, \mathbb{Z}) \longrightarrow \mathrm{Im}\,\Psi \subset K\big(\hat{S}^1(X_D^{n-p}), \hat{S}^1(X_D^{n-p-2})\big)$ be the isomorphism* (E.7)*. Let:*

$$\tilde{\pi}^{n-p} : \hat{S}^1(X_D^{n-p}) \longrightarrow \hat{S}^1(X_D^{n-p})/\hat{S}^1(X_D^{n-p-2})$$

be the projection. Then:

$$i^1_!\,(Y \times \mathbb{C})\,\big|_{\hat{S}^1(X_D^{n-p})} = (\tilde{\pi}^{n-p})^*\big(\Xi_D^{n-p}\big(\mathrm{PD}(Y)\big)\big).$$

□

Theorem 3.4.10 *Let X be an n-dimensional compact manifold and $Y \subset X$ a p-dimensional embedded compact submanifold, such that $n-p$ is odd and $\mathcal{N}(Y)$ is spin^c. Let $\{(E_r^p, d_r^p)\}$ be the Atiyah-Hirzebruch spectral sequence, and let $\Xi^{n-p} : H^{n-p}(X, \mathbb{Z}) \xrightarrow{\simeq} E_2^{n-p,0}$ be the isomorphism induced by Φ^{n-p}. Suppose that $\Xi^{n-p}\,\mathrm{PD}(Y)$ is contained in the kernel of all the boundaries $d_r^{n-p,0}$ for $r \geq 2$.*

With this data, define a class:

$$\big\{\Xi^{n-p}\,\mathrm{PD}(Y)\big\}^{(2)}_{E_\infty^{n-p,0}} \in E_\infty^{n-p,0} \simeq \frac{\mathrm{Ker}\big(K(\hat{S}^1 X) \longrightarrow K(\hat{S}^1(X^{n-p-1}))\big)}{\mathrm{Ker}\big(K(\hat{S}^1 X) \longrightarrow K(\hat{S}^1(X^{n-p}))\big)}.$$

Then:

$$\big\{\Xi^{n-p,0}\,\mathrm{PD}(Y)\big\}^{(2)}_{E_\infty^{n-p,0}} = \big[i^1_!\,(Y \times \mathbb{C})\big]_{E_\infty^{n-p,0}}.$$

□

The rational case

The case of rational coefficients will be now analyzed.

Even case Define:

$$K_{\mathbb{Q}}(X) := K(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

One can thus classify the D-brane charge density at rational level as $i_!(E) \otimes \mathbb{Q}$. The Chern character provides an isomorphism $\mathrm{ch} : K_{\mathbb{Q}}(X) \to H^{\mathrm{ev}}(X, \mathbb{Q})$. Since the square root of $\hat{A}(TX)$ is a polyform starting with 1, it also defines an isomorphism, so that the composition:

$$\begin{gathered} \widehat{\mathrm{ch}} : K_{\mathbb{Q}}(X) \longrightarrow H^{\mathrm{ev}}(X, \mathbb{Q}) \\ \widehat{\mathrm{ch}}(\alpha) = \mathrm{ch}(\alpha) \wedge \sqrt{\hat{A}(TX)} \end{gathered}$$

remains an isomorphism. Thus, the classifications with rational K-theory and rational cohomology are completely equivalent.

One can also define a rational version of the Atiyah-Hirzebruch spectral sequence $Q_r^{2k,\sigma}(X) := E_r^{2k,\sigma}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Such sequence [25] collapses at the second level, i.e., at the cohomology: thus $Q_\infty^{2k,\sigma}(X) \simeq Q_2^{2k,\sigma}(X)$. An explicit isomorphism is given by the appropriate component of Chern character:

$$\mathrm{ch}_{\frac{n-p}{2}} : \frac{\mathrm{Ker}\big(K_{\mathbb{Q}}(X) \longrightarrow K_{\mathbb{Q}}(X^{n-p-1})\big)}{\mathrm{Ker}\big(K_{\mathbb{Q}}(X) \longrightarrow K_{\mathbb{Q}}(X^{n-p})\big)} \longrightarrow H^{n-p}(X, \mathbb{Q}).$$

For a bundle which is trivial on the $(n-p-1)$-skeleton, the lower components of ch are zero [25], hence $\mathrm{ch}_{\frac{n-p}{2}} = \widehat{\mathrm{ch}}_{\frac{n-p}{2}}$ but this is not in general true for the higher components. Moreover, since $Q_\infty^{2k,0}$ has no torsion:

$$K_{\mathbb{Q}}(X) = \bigoplus_{2k} Q_\infty^{2k,0}$$

and an isomorphism can be obtained splitting $\alpha \in K_{\mathbb{Q}}(X)$ as $\alpha = \sum_{2k} \alpha_{2k}$ where $\mathrm{ch}(\alpha_{2k}) = \mathrm{ch}_k(\alpha)$. This isomorphism will now be linked with the splitting principle stated in subsection 3.2.3.

If one considers the brane Y with bundle E and the sub-branes $\{q_k \cdot Y_k\}$ verifying the splitting principle, one has that:

$$i_!(E) \otimes_{\mathbb{Z}} \mathbb{Q} = \sum_k (i_k)_!(Y_k \times \mathbb{C}^{q_k}) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{3.28}$$

since the Chern characters of the two terms above are exactly the two terms of formula (3.11). Hence, if one looks at the correspondence:

$$\begin{gathered} K_{\mathbb{Q}}(X) \longleftrightarrow \bigoplus_{2k} Q_\infty^{2k,0} \\ \alpha \longleftrightarrow \oplus_{2k} \left[\alpha_{2k}\right]_{Q_\infty^{2k,0}} \end{gathered}$$

for α_{2k} such that $\mathrm{ch}(\alpha_{2k}) = \mathrm{ch}_k(\alpha)$, one has in particular that $\left[\alpha_{2k}\right]_{Q_\infty^{2k,0}} = \left[(i_k)_!(Y_k \times \mathbb{C}^{q_k})\right]_{Q_\infty^{2k,0}}$.

However, one can also consider the sub-branes $i_* \mathrm{PD}_Y(\mathrm{ch}(E) \wedge G(Y))$, with trivial bundle. Call such sub-branes $\{q'_k \cdot Y'_k\}$. One has that:

$$i_!(E) \otimes_{\mathbb{Z}} \mathbb{Q} \longleftrightarrow \oplus_{2k} \left[(i_k)_!(Y'_k \times \mathbb{C}^{q'_k})\right]_{Q^{2k,0}_\infty}.$$

In fact, as already explained, $(i_k)_*(q_k \cdot Y_k) = \mathrm{PD}_X \widehat{\mathrm{ch}}_k\big(i_!(E)\big)$. Hence:

$$\begin{aligned}\mathrm{ch}_k\big((i_k)_!(Y_k \times \mathbb{C}^{q_k})\big) &= \widehat{\mathrm{ch}}_k\big((i_k)_!(Y_k \times \mathbb{C}^{q_k})\big) = (i_k)_\#(q_k \cdot 1)\\ &= \mathrm{PD}_X(i_k)_*(q_k \cdot Y_k) = \mathrm{ch}_k\, i_!(E).\end{aligned}$$

However, formula (3.28) does not hold for the branes $\{q'_k \cdot Y'_k\}$.

Odd case In this case, one has the isomorphism $\mathrm{ch} : K^1_{\mathbb{Q}}(X) \to H^{\mathrm{odd}}(X, \mathbb{Q})$. Moreover, $H^{\mathrm{odd}}(X, \mathbb{Q}) \simeq H^{\mathrm{ev}}(\hat{S}^1 X, \mathbb{Q})$. Hence one gets the correspondence among:

- $i^1_!(E) \in K^1_{\mathbb{Q}}(X)$;
- $\widehat{\mathrm{ch}}(i^1_! E) \in H^{\mathrm{ev}}(\hat{S}^1 X, \mathbb{Q}) \simeq H^{\mathrm{odd}}(X, \mathbb{Q})$;
- $\oplus_{2k} \left[(i^1_k)_!(Y_k \times \mathbb{C}^{q_k})\right]_{Q^{2k+1,0}_\infty}$.

As before, for the splitting principle:

$$i^1_!(E) \otimes_{\mathbb{Z}} \mathbb{Q} = \sum_k (i^1_k)_!(Y_k \times \mathbb{C}^{q_k}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Part II

F-Theory Side

Introduction

The recent return of interest in F-theory [42] has originated from the suggestion of possible low energy phenomenological implications, more precisely by the possibility to accommodate in this theory a gravity decoupling at the scale of grand-unification together with low energy effective grand-unified actions which extend the MSSM [2]. While the coexistence of all these conditions is still under scrutiny, one of the characteristics of F-theory that makes such conjectures plausible is certainly the symmetry enhancements that can occur in it, which allow for virtually all types of gauge symmetries, that is all type of gauge Lie algebras (with possible bounds only on their rank). What is most interesting for the above mentioned phenomenological applications is in particular the possibility to accommodate theories characterized by the series of exceptional simply-laced Lie algebras.

Independently of its possible phenomenological applications, F-theory vacua are characterized by peculiar aspects that distinguish them from other superstring vacua. Generally speaking F-theory vacua are more 'constrained' than others. In particular the number of 7-branes, their type and, eventually, the type of enhanced symmetry is a result of the dynamics (geometry) rather than put in by hand, as is the case of other compactifications with branes. The price for it is that, generically, the relevant open strings stretching between the branes are mutually non-perturbative. This is not to say, however, that nothing can be said about, for instance, the dynamics in 4D, as refs. [2] abundantly testify. It is therefore important to analyze and understand the dynamics of F-theory. A lot has already been done in the past, but there are still aspects of the theory where the analysis has not been completed.

The first aim of this part of the thesis is to make some further steps towards a better understanding of the "string counterpart" of some peculiar aspects of the F-theory gauge symmetry enhancement. Indeed, the symmetry enhancement in F-theory can be analyzed with various (complementary) techniques: either with algebraic-geometric techniques (Tate's algorithm) [43, 44], or by studying the BPS strings stretched among 7-branes [45, 46, 47, 48], or by means of the (strictly related) Lie algebra realization via string junctions [49]. The purpose here is to focus on the last method and in particular to apply it to a peculiar aspect of the gauge symmetry enhancement, namely the appearance of monodromies leading to non-simply laced Lie algebras. After a brief introduction on the above mentioned methods for describing the symmetry enhancements, it will be shown how to obtain a description of the root system of the non-simply-laced groups by means of F-theory string junctions, attached to a system of (in general, mutually non-perturbative) 7-branes. The analysis is inspired by the analogous description of simply-laced groups made in [49].

However, by making contact with the central topic of the whole thesis, an important and natural question arises, which is without any doubt of high relevance in global F-theory model building and remains still open. That is what would be the general form of the Freed-Witten anomaly for generic configurations of mutually non-local 7-branes. This is essentially due to the

impossibility of addressing the problem by implementing the original perturbative path integral methods (see chapter 1) in such an intrinsically non-perturbative context.
The second goal of this part is in fact to deduce some global constraints on F-theory compactifications, approaching the problem of Freed-Witten anomalies in an effective way. More precisely, the duality between M and F-theory [7] will be used in order to find out the consequences in F-theory compactifications on Calabi Yau fourfolds of the well known Witten's quantization condition of the M-theory G_4 flux [8]. This is indeed nothing but the required flux-quantization for anomaly cancellation of membranes in M-theory and it will be described how to relate it to the D-brane gauge flux quantization already met in chapter 1. The cases of smooth Calabi-Yau fourfolds and of singular ones will be both treated in detail: a general pattern will be given and a number of clarifying, concrete examples provided, which also match some known global constructions available in the literature. [50, 51].

This part is organized as follows: in chapter 4 a brief introduction on F-theory from the type IIB self-S-dual string theory perspective is provided; the aim is of reviewing some specific aspects which will be central in the following discussion, namely the IIB weak coupling limit, the symmetry enhancement mechanism and the M/F-theory duality; chapter 5 will be concerning with the methods of string junctions which will be adopted to give an alternative (stringy) description of the enhancements to non-simply-laced gauge groups; finally, in chapter 6 the effective path towards a general formulation of the Freed-Witten anomaly in F-theory is started, passing through the analysis of the global constraints on flux quantization.

Chapter 4

Topics in F-theory

This introductory chapter aims to give a concise review on selected topics in F-theory needed in the sequel and it is organized as follows: in section 1 the role of S-duality in type IIB string theory is reviewed, focusing on the ensuing properties of 7-branes; in section 2 F-theory is introduced from the perspective of type IIB strings and two kinds of compactification are presented, along with their perturbative limits; section 3 contains a review on the topic of gauge symmetry enhancement in F-theory both from an algebraic-geometric and a physical (stringy) point of view; finally, in section 4 a proper definition of F-theory is given using M-theory and T-duality, which clarifies the nature and the role of the G_4 flux, needed in the sequel.

4.1 IIB string theory and S-duality

Type IIB string theory is conjectured to be invariant under the quantum version of the manifest $SL(2,\mathbb{R})$ self-duality of the type IIB supergravity action, namely $SL(2,\mathbb{Z})$ [11]. The elements of this group can be represented by 2×2 matrices with integral entries and determinant equal to 1:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \qquad \text{if} \qquad a,b,c,d \in \mathbb{Z} \qquad \text{such that} \qquad ad - bc = 1\,, \tag{4.1}$$

and can all be obtained by taking matrix products of two generators:

$$T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad S \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.2}$$

Under this symmetry the field content of the low energy spectrum and the charges of the sources transform in the following way:

- the so called axiodilaton, namely the complex scalar field made out of the axion C_0 and the dilaton ϕ, $\tau \equiv C_0 + ie^{-\phi}$ undergoes a fractional linear transformation:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}\,, \tag{4.3}$$

  which is a representation of $SL(2,\mathbb{Z})$ whose kernel is the $\mathbb{Z}_2$-center of this group $\mathcal{C} \equiv \{I, -I\}$. Therefore τ is only sensible to the quotient $PSL(2,\mathbb{Z}) \equiv SL(2,\mathbb{Z})/\mathcal{C}$. From this action it is clear why in the literature this symmetry is called S-duality, meaning strong-weak coupling duality: indeed, taking for simplicity $C_0 = 0$, the generator S exactly inverts the string coupling g_s defined as the background value $e^{-\langle\phi\rangle}$;

- the B-field together with the RR field C_2 forms a doublet; so do their field strength H and the unimproved F_3 and their electromagnetic duals too. The transformation, for example, of H and F_3 is

$$\begin{pmatrix} H \\ F_3 \end{pmatrix} \longrightarrow \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H \\ F_3 \end{pmatrix} . \tag{4.4}$$

  Correspondingly, F1 and D1 strings, that couple minimally to B and C_2 with strengths p and q respectively, form a doublet under S-duality, such that the complete Wess-Zumino action which describes these couplings (like (2.1)) remains invariant. This fixes the transformation rule of the charges p and q to be:

$$\begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} . \tag{4.5}$$

  Analogous rules hold for the dual sources NS5 and D5-branes;

- the RR field C_8 that couples to D7 branes is more tricky because it is part of a triplet, along with B_8 which couples to the NS7-branes and with new field called D_8. Thus, in principle, there are 3 independent charges which measure the strengths of these three minimal couplings, called respectively[1] p^2, q^2 and r. However, as it will be shown below, only two of them are in fact independent within each given conjugacy class of the S-duality group. Analogous rules hold for the dual instantonic sources;

- D9-branes and the associated volume form C_{10} have an even more difficult behavior under S-duality, but they will not be discussed here since they are not relevant for the purposes of this thesis. The interested reader is referred to [52] and references threin;

- the metric in the Einstein frame G^E and the unimproved RR 5-form F_5 are S-duality invariant.

The manifold of the scalars of type IIB supergravity is the coset $SL(2,\mathbb{R})/SO(2) \simeq H_\tau$, which is isomorphic to the upper half τ-plane, spanned obviously by the axion and by the exponential of the dilaton. Due to the residual gauge symmetry $PSL(2,\mathbb{Z})$, the quantum moduli space is represented by the fundamental region $F \equiv H_\tau/PSL(2,\mathbb{Z})$, which can be chosen to look pictorially like figure 4.1.

The three marked points in fig. 4.1, namely $i\infty$, i and $\rho \equiv -1/2 + i\sqrt{3}/2$, are the three orbifold points of F, i.e. they are fixed under some subgroup of $PSL(2,\mathbb{Z})$. For later use, the isotropy groups of these points are listed here:

- $\tau = i\infty$ is left unchanged by the subgroup $H_{i\infty} \simeq \mathbb{Z} \subset PSL(2,\mathbb{Z})$ generated by $\pm T$ of formula (4.2);

- $\tau = i$ is left unchanged by the subgroup $H_i \simeq \mathbb{Z}_2 \subset PSL(2,\mathbb{Z})$ generated by $\pm S$ of formula (4.2);

- $\tau = \rho$ is left unchanged by the subgroup $H_\rho \simeq \mathbb{Z}_3 \subset PSL(2,\mathbb{Z})$ generated by $\pm T^{-1}S$.

[1] The reason of the squares will become clear later on.

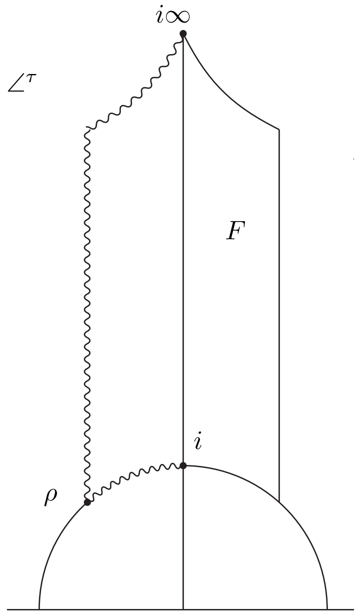

Figure 4.1: Fundamental domain of $PSL(2,\mathbb{Z})$ in a particular S-duality frame.

Every other point of F has trivial isotropy subgroup of $PSL(2,\mathbb{Z})$.
Needless to say, if one changes the S-duality frame by an $SL(2,\mathbb{Z})$ transformation g, the fundamental domain changes and each of the new orbifold points, call it generically τ_0, will be fixed under the new isotropy groups $gH_{\tau_0}g^{-1}$, where H_{τ_0} is the old one. Hence, the elements of the new isotropy groups happen to be simply different representatives in the same conjugacy class as the elements of the old ones.

4.1.1 7-branes

This subsection is entirely dedicated to the 7-branes of type IIB string theory [53] because they will be the most important objects in the sequel.

7-branes are the electric sources of the triplet of RR 8-forms mentioned above, with charges p^2, q^2 and r. They are also 1/2 BPS magnetic sources of the scalars C_0 and ϕ and as such they are essentially defined by two independent data: a $PSL(2,\mathbb{Z})$-matrix determining the change these fields undergo when going counterclockwise along a circle which links the given source in the target space; a sign entering the transformation rule of the local function defining the internal metric and the Killing spinor which preserves half of the supercharges. All together they constitute exactly the amount of information contained in a matrix $M_{(p,q,r)}$ belonging to $SL(2,\mathbb{Z})$. This matrix is called the *monodromy* matrix, for reasons that will become clearer later. The inverse of such a matrix just represents the corresponding anti-7-brane (its charge is the *opposite*) and it is how a 7-brane looks like if one goes *clockwise* along its linking circle.
The most familiar example of these sources is the one of D7-branes, which are magnetic monopoles with respect to $F_1 = \mathrm{d}C_0$: n units of D7-brane, in other words, are defined by the matrix

$$M_{D7} = T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} . \tag{4.6}$$

This element belongs to one of the two representatives of the isotropy group $H_{i\infty}$. Now, if in a local neighborhood of this stack of n D7's, that is just a disk, one defines $z = |z|e^{i\theta}$ to be a local complex coordinate on this disk, then

$$\tau(z) = \tau(|z|e^{i\theta}) = T^n \cdot \tau(|z|e^{i(\theta+2\pi)})\,. \tag{4.7}$$

Let τ_0 be the limit value of τ at the origin. Then, taking the limit for $|z| \to 0$ of both sides in (4.7), one finds[2] that $\tau_0 = T^n \tau_0$. This means $\tau_0 = i\infty$, which is the point fixed by the whole $H_{i\infty}$. In this case it is also particularly easy to find a local explicit representation for τ, different from the trivial one, which is τ constant and equal to $i\infty$. Indeed,

$$\tau(z) \;=\; \frac{n}{2\pi i}\log z + \text{constant} \tag{4.8}$$

fulfills all the requirements.
Had one chosen another fundamental domain, for instance a rotation of fig. 4.1 by some $g \in SL(2,\mathbb{Z})$, this simple example would have been characterized by a stack of another kind of 7-branes with defining monodromy matrix equal to gT^ng^{-1}. These are the so called (p,q)7-branes, namely the ones that can be obtained from the D7's by means of an S-duality transformation.

In general, once a fundamental domain for τ has been chosen, there are as many kind of 7-brane configurations, whose monodromies belong to different $SL(2,\mathbb{Z})$ orbits, as the elements (counted with sign) of all the isotropy groups of the fundamental domain. Although an explicit expression for the axiodilaton, like eq. (4.8), could be not easy to find out in general, the limit value of τ on a given 7-brane configuration must be the one(s) fixed by the monodromy matrix which defines such a configuration.
The most trivial example is the configuration corresponding to the identity, i.e. the positive element of the center $\mathcal{C}$ of $SL(2,\mathbb{Z})$, which is the isotropy group of any $\tau \in F$: this is of course the configuration in which no 7-branes are present.
Less trivially, (4.6) is the family of D7-banes (parametrized by their number), whose elements correspond to the positive elements of the isotropy group $H_{i\infty}$. The negative elements of the same group, i.e. $-T^n$, including $-I$, which belongs to all the isotropy groups, correspond instead to another family of 7-branes that still admit a perturbative picture: this is by means of ordinary D7's on top of an orientifold 7-plane[3] O7.
More difficult and without a perturbative counterpart are the configurations corresponding to the remaining six (torsion) elements of the lifts to $SL(2,\mathbb{Z})$ of the other two isotropy groups, H_i and H_ρ.

All together, one has two families of $SL(2,\mathbb{Z})$ orbits and six other isolated ones. The configurations not corresponding to the family of D7's (or (p,q)7's, in a different S-duality frame) go under the generic name of Q7-branes. The F-theory description of the latter as non-perturbative bound states of ordinary (p,q)7-branes, to be introduced later on, will clarify this scheme and assign to each of these 7-branes a specific Lie algebra, corresponding to the gauge one actually realized in their worldvolume effective theory.

An explicit realization of the monodromy matrices defining all these 7-brane configurations exists, which has the advantage to make visible all the $SL(2,\mathbb{Z})$ orbits discussed above by labeling them by means of a single conjugation invariant. It suffices to write the monodromy as

[2]The monodromy matrix, which in this case is T^n, is independent of $|z|$, since it is independent of the homotopy representative of the local linking circle.

[3]Only orientifold 7-planes of the type $O7^-$ will be considered throughout this thesis.

the exponential of a traceless matrix Q, which is regarded as the charge matrix of the 7-brane configuration:

$$M_{(p,q,r)} = e^Q = \cos(\sqrt{\det Q})I + \frac{\sin(\sqrt{\det Q})}{\sqrt{\det Q}}Q \qquad \text{with} \qquad Q \equiv \begin{pmatrix} r/2 & p^2 \\ -q^2 & -r/2 \end{pmatrix}, \tag{4.9}$$

where the fact that $Q^2 = -\det(Q) \cdot I$ has been used in the second equality. The conjugation invariant used to classify the orbits is $\sqrt{\det Q}$ mod 2π, whose cosine is half the trace of the monodromy matrix, which is manifestly invariant within each $SL(2,\mathbb{Z})$ conjugacy class. One could ask whether the two values of $\sqrt{\det Q}$ mod 2π with the same cosine generate a monodromy matrices in the same orbit or not. The answer is no, because they correspond to different elements of the lifts of the isotropy groups H_i and H_ρ to $SL(2,\mathbb{Z})$.
To determine the allowed values of the above invariant, notice that the condition Tr $M_{(p,q,r)} \in \mathbb{Z}$ must clearly hold. This implies that twice the cosine of $\sqrt{\det Q}$ is an integer. Thus, the allowed values of the invariant will be:

$$\sqrt{\det Q} \;=\; 0\,, \frac{\pi}{3}\,, \frac{\pi}{2}\,, \frac{2\pi}{3}\,, \pi\,, \frac{4\pi}{3}\,, \frac{3\pi}{2}\,, \frac{5\pi}{3} \qquad \text{mod} \qquad 2\pi\,. \tag{4.10}$$

First of all, notice that, as anticipated previously, on any of these 8 classes there is a different relation among the charges p^2, q^2 and r, so that they are not independent. Moreover, the first and the second half of these values are related to each other by the non-trivial element of the center $\mathcal{C}$, which is just the identity of the isotropy subgroups of $PSL(2,\mathbb{Z})$, but it exchanges the two leaves of their lifts to $SL(2,\mathbb{Z})$. This element must then correspond to a configuration of 7-branes which has no backreaction on the axiodilaton. This means that the net D7-brane charge has to vanish not in the trivial way, that is no 7-branes (which would correspond to the identity of $SL(2,\mathbb{Z})$). Hence, perturbative IIB string theory prescribes that an O7-plane should be present, carrying -4 units of physical[4] D7-brane charge, plus 4 D7-branes and 4 D7-images on top of it. This is the perturbative realization of the gauge group $SO(8)$ on the 7-brane worldvolume and the total monodromy around this configuration is, as said, $-I$.
The two families of orbits, which have been found previously, are treated as two single cases, corresponding to the values 0 and π. The former represents, in the chosen S-duality frame, the most familiar configurations of n D7-branes (if n is negative one has anti-D7's). In this case, the values of the charges are: $p^2 = n$, $q^2 = 0$ and $|r| = 2pq = 0$ and the monodromy matrices are just the ones in eq. (4.6). For later use, the monodromy matrix of a generic (p,q)7-brane, which lies in the same conjugacy class of T, is written below. A (p,q)7 is regarded as a single brane if p and q are two relatively prime integers; this is equivalent to the existence of two other integers, s and t, such that the Bezout identity holds: $pt - qs = 1$. Therefore, one has:

$$\begin{pmatrix} p & s \\ q & t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \qquad \text{which implies}$$

$$M_{(p,q)7} = \begin{pmatrix} p & s \\ q & t \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & -s \\ -q & p \end{pmatrix} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}. \tag{4.11}$$

The second family, instead, is the one perturbatively represented in this frame by $4+n$ D7-branes on top of an O7-plane (again negative n gives anti-D7's) and the monodromies are $-T^n$. Notice

[4] Physical just means that the charge of the D7-images are *not* counted as independent: thus a pair of D7-brane/D7-image carries a unit of D7-brane charge. In this convention, an Op-plane has Dp-brane charge equal to -2^{p-5}.

that, except the $n = 0$ case, the realization (4.9) of the monodromy matrix fails to describe the other orbits of this family, since if $n \neq 0$ one between p^2 and q^2 should vanish while the other should diverge.
The remaining six values of $\sqrt{\det Q}$ modulo 2π correspond to the last six orbits (which do not lie in any family). In the same order as in (4.10), the monodromy matrices of the corresponding Q7-branes belong to the conjugacy class of $T^{-1}S$, S, $(T^{-1}S)^2$, $-T^{-1}S$, $-S$, $-(T^{-1}S)^2$.
A summarizing table (table 4.1) will be given in the next section, after having added the information about these configurations arising from the F-theory picture.

4.2 What is F-theory?

The basic flat worldvolume 7-brane solution with a compact transverse space with the topology of a 2-sphere with punctures requires 24 non-coincident and not all mutually perturbative 7-branes, which are the punctures of the sphere [54]. They should not all be mutually perturbative since the total net D7-brane charge must vanish, being the transverse space compact, and no anti-D7-branes can be added in order not to break supersymmetry. This is the most well-known solution of 7-branes in type IIB string theory [53, 48], leading to a worldvolume effective gauge theory on $\mathbb{R}^{1,7}$ with 16 supercharges ($\mathcal{N} = 1, D = 8$) and gauge group depending on the 7-brane configuration.

F-theory [42] can be introduced in this context[5] as a geometric tool to take into account the backreaction of the 7-branes present in the solution. In other words, the S-duality matrices which represent the modification of the axiodilaton background become in F-theory transition functions of an $SL(2, \mathbb{Z})$ fibration over the transverse space and the axiodilaton itself becomes the complex structure modulus of an auxiliary 2-torus fibered over the transverse space (see fig. 4.2). Then, the supersymmetry conditions and the equation of motion for τ enforce the latter to be an elliptic fibration with τ varying holomorphically along the base. Hence, the most important consequence of such a global picture is that the concept of *mutual non-perturbativity* of the 7-branes become now the concept of *mutual non-locality* and the S-duality relating two mutually non-perturbative 7-brane configurations is just the transition function connecting the corresponding two charts of the transverse space in which they lie. Therefore, in each local patch of the transverse space one has a copy IIB string theory and one can always choose the trivialization of the patch in such a way that its axiodilaton τ lies in the fundamental region F of fig. 4.1. F-theory, then, lives on a 12-dimensional manifold; however, there is no metric on the additional 2 spatial direction of the fiber T^2, so they are not physical but are just a trick to get the compact global description of the backreaction. In order to recover the usual 10-dimensional world in which gravity is supposed to live, it suffices to ask the F-theory elliptic fibration to always admit a 0-section reproducing a copy of the base in the total space.
Of course, in the same spirit, F-theory can be generalized to allow more complicated manifolds (provided it be Kähler) as the base space of its compactification, which clearly, by definition, should always occur on elliptic fibrations. In general 7-branes are more complicated divisors of the base and they generically intersects, rather than being globally parallel, as in the case of the 2-sphere. In the remaining chapters of this thesis, the case of F-theory compactified on elliptically fibered Calabi-Yau fourfolds (which give rise to $\mathcal{N} = 1$ supersymmetry in 4 dimensions) will be mainly treated. However, the simplest case will be shortly presented here for illustrative purposes, of F-theory on $K3$, which is the above mentioned 2-torus fibered over

[5]For a more satisfactory definition, see section 4.4.

a 2-sphere (giving rise to $\mathcal{N} = 1, D = 8$).

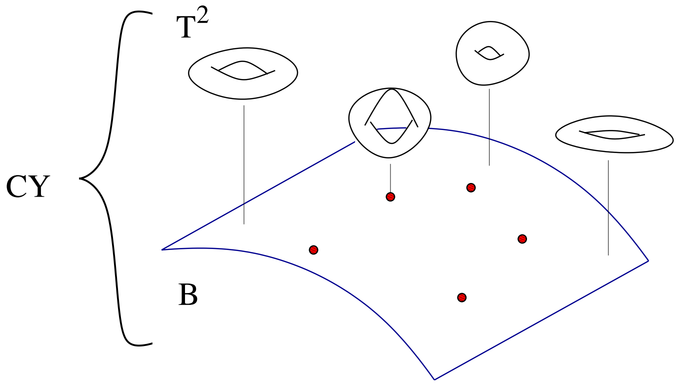

Figure 4.2: The elliptic fibration over a Kähler base manifold B. In case B has more than one complex dimension, the directions drawn in the picture are the two orthogonal ones to the 7-branes.

4.2.1 F-theory on $\mathbb{R}^{1,7} \times$ K3

The basic 7-brane solution in IIB supergravity mentioned before is described by means of two local analytic functions on S^2, $\tau(u)$ and $\varphi(u)$, in terms of which the Einstein frame metric and the Killing spinor are given by:

$$\mathrm{d}s^2 = -\mathrm{d}s^2_{1,7} + \mathrm{Im}\,\tau\,|\varphi|^2 \mathrm{d}u\mathrm{d}\bar{u}\,, \tag{4.12}$$

$$\epsilon = \left(\frac{\bar{\varphi}}{\varphi}\right)^{1/4} \epsilon_0\,, \tag{4.13}$$

for some constant spinor ϵ_0 of a given 2-dimensional chirality. The holomorphic functions $\tau(u)$ and $\varphi(u)$ are given implicitly by the following equations:

$$j(\tau) = \frac{4(24f)^3}{4f^3 + 27g^2}\,, \tag{4.14}$$

$$\varphi(u) = c\,\eta^2(\tau)(4f^3 + 27g^2)^{-1/12}\,, \tag{4.15}$$

where f and g are locally defined homogeneous polynomials of degree 8 and 12 respectively in the complex coordinate u, η is the Dedekind eta function, j is the Klein modular invariant function and c is some non-zero complex constant whose absolute value sets the size of the 2-sphere. The Klein function maps the moduli space of elliptic curves F to the 2-sphere, sending the three orbifold points, $i\infty$, i and ρ to ∞, 1 and 0 respectively. Moreover, going counterclockwise around a given puncture of the 2-sphere characterized by the monodromy $M \in SL(2,\mathbb{Z})$ as in eq. (4.1), the function τ continuously changes in its covering space according to eq. (4.3), while φ undergoes the transformation

$$\varphi \longrightarrow (c\tau + d)\varphi\,, \tag{4.16}$$

and thus it is sensible to the full $SL(2,\mathbb{Z})$ group, not just to the quotient of it by its center: this is why a sign must enter, besides the $PSL(2,\mathbb{Z})$ matrix, the definition of the monodromy matrix corresponding to the given puncture.

In order to compute the masses of the states associated to the (p,q)-strings stretching among the 7-branes, one has to take into account the string tension, which for a (p,q)-string reads:

$$T_{p,q} = \frac{1}{\sqrt{\operatorname{Im}\tau}}|p+q\tau|\,. \tag{4.17}$$

One then usually introduces an effective metric for (p,q)-strings, $\mathrm{d}s_{p,q} = T_{p,q}\mathrm{d}s$, which, away from singularities, is just the modulus of an analytic 1-form,

$$\begin{aligned} \mathrm{d}s^2_{p,q} &= |\mathrm{d}w|^2 \qquad \text{where} \\ \mathrm{d}w &= h_{p,q}(u)\,\mathrm{d}u \qquad \text{with} \qquad h_{p,q}(u) \equiv (p+q\tau(u))\varphi(u)\,; \end{aligned} \tag{4.18}$$

hence it is *flat*. However, the metric (4.18) is typically rather complicated, but in the neighborhood of a bunch of punctures very close to one another, say around the point $u = 0$, with all the other punctures very far away, it looks like having a *conical* singularity at that point, with absolute value of the deficit angle equal to the $SL(2,\mathbb{Z})$ invariant characterizing that bunch:

$$h_{p,q} \sim c\,u^{-\frac{\sqrt{\det Q}}{2\pi}} \tag{4.19}$$

at leading order around $u = 0$. For instance, take a stack of n ordinary D7-branes in the vicinity of $u = 0$ and any other brane very far away; around that point the axiodilaton will have the form (4.8) and so, at leading order, the effective metric of an ordinary F1-string will look like

$$h_{1,0} \sim c\,u^{\frac{n}{12}}u^{-\frac{n}{12}} \sim \text{regular}\,, \tag{4.20}$$

where the definition $\eta^2(\tau) = \exp(\pi i\tau/6)\prod_{n=1}^{\infty}(1-\exp(2\pi i n\tau))^2$ has been used and a complete factorization of the polynomial appearing in (4.15) too. Therefore, fundamental strings do not see any metric singularity at the position of the D7-branes, compatibly with the fact that their quantum numbers (1,0) are not changed by the action of the monodromy matrix (4.6). Hence, the deficit angle of this configuration is 0, exactly like the corresponding value of the $SL(2,\mathbb{Z})$ invariant.

An other example is the one of the other family, characterized by the presence of a stack of n D7's on top of an O7. Again here one can uses the same explicit expression as before for the axiodilaton around the configuration, so that one gets:

$$h_{1,0} \sim c\,u^{\frac{n}{12}}\,u^{-\frac{6+n}{12}} \sim c\,u^{-\frac{1}{2}}\,, \tag{4.21}$$

where one uses the fact that the O7 can be seen as a bound state of two mutually non-perturbative (p,q)7-branes. Hence this is a conical singularity of deficit angle equal to π, exactly like the value of the invariant corresponding to this family of configurations.

For the other Q7-branes, a simple explicit expression for τ is not available, but for the present purpose it suffices to take it constant around the given Q7 (with value clearly equal to the fixed point of the corresponding monodromy); by doing so, the deficit angle of the conical singularity, which approximates such Q7 in its neighborhood, says how many constituent (p,q)7-branes it is made of as bound state. Thus, it turns out that one needs, in the same order of the sequence in (4.10), 2, 3, 4, 8, 9, 10 mutually non-local (p,q)7-branes to make the corresponding bound states.

Finally notice that by recombining all the 24 branes in an unique tadpole canceling 7-brane, the conical singularity that is created this way will have deficit angle the whole solid one, i.e. 4π, and the metric at infinity, or else very far away from the recombined brane, will be approximately the leading order expansion of the familiar Fubini-Study metric of the 2-sphere, $h_{p,q} \sim c\, u^{-2}$.

Consider now an elliptic fibration over $\mathbb{P}^1 \simeq S^2$, namely a complex $K3$ surface, described by an hypersurface in an ambient toric threefold defined by the following weight assignments:

$$\begin{array}{ccccc|c} u & v & X & Y & Z & \text{Weierstrass} \\ \hline 1 & 1 & 4 & 6 & 0 & 12 \\ 0 & 0 & 2 & 3 & 1 & 6 \end{array} \quad , \tag{4.22}$$

where X, Y and Z are the homogeneous coordinates of a weighted projective space $W\mathbb{P}^2_{2,3,1}$ fibered[6] over the a $\mathbb{P}^1$ with homogeneous coordinates u and v, while Weierstrass is the following polynomial with variables u and v:

$$Y^2 = X^3 + fXZ^4 + gZ^6\,, \tag{4.23}$$

f and g being the same polynomials met before. As it is clear from the weights, this hypersurface is Calabi-Yau and it is easy to see that it avoids the orbifold points of the $W\mathbb{P}^2_{2,3,1}$ fibration. It defines an elliptic curve fibered over $\mathbb{P}^1$ with complex structure modulus given implicitly by formula (4.14). This fibration has a 0-section, defined by the equation $Z = 0$, regarded as the gravitational brane.
The elliptic curve is not in general a smooth 2-torus everywhere on the base, but degenerates over some points identified with the vanishing locus of the following degree 24 homogeneous polynomial called the discriminant of the elliptic fibration:

$$\Delta \equiv 4f^3 + 27g^2\,. \tag{4.24}$$

In a more geometric language, all these local polynomials become sections of suitable vector bundles. Calling $\mathcal{O}(1)$ the holomorphic line bundle on the ambient threefold whose divisor is the gravitational brane $Z = 0$, and $K(\mathbb{P}^1)$ the canonical line bundle of the 2-sphere, then, from (4.22) one realizes that X, Y, Z, f, g and Δ are sections respectively of [7] $\mathcal{O}(2) \otimes K^{-2}(\mathbb{P}^1)$, $\mathcal{O}(3) \otimes K^{-3}(\mathbb{P}^1)$, $\mathcal{O}(1)$, $K^{-4}(\mathbb{P}^1)$, $K^{-6}(\mathbb{P}^1)$ and $K^{-12}(\mathbb{P}^1)$.

There are generically 24 distinct zeros of Δ and they represent the positions of the 24 (p,q)7-branes of the solution. All together they lie on the Poincaré dual of $12c_1(\mathbb{P}^1)$ and their charges combine in such a way to cancel the 7-brane tadpole. The freedom in choosing the positions of these punctures resides in the various possibilities one has in choosing, modulo $GL(2,\mathbb{C})$ coordinate transformations, the coefficients of the polynomials f and g. They are $9 + 13 - 4 = 18$ independent coefficients, which, along with τ, make the 19 complex structure moduli of $K3$ and, from the viewpoint of the $\mathcal{N} = 1, D = 8$ effective theory, they belong to 18 $U(1)$ vector multiplets[8]. Hence, clearly, not all the positions of the 7-branes can be freely chosen, but there are global obstructions.
On each single zero of Δ a 1-cycle in the fiber is collapsing and the nature of that cycle gives information on the charge of the 7-brane placed at that point[9] (see section 4.4). Actually, there

[6]The fibration is not apparent in the chosen basis of weights, but it is so if one changes basis in (4.22) replacing the first row with itself minus twice the second (see appendix F).
[7]The symbol of the pull-back of the projection map from the ambient threefold to $\mathbb{P}^1$ is dropped.
[8]There are two more vectors in the gravity multiplet.
[9]The precise relation will be evident in the M-theory approach of section 4.4.

is only one singular point on the elliptic fiber[10] (see fig. 4.2). This can be easily seen as follows. Fix a point in the base on which Δ is vanishing and compute the gradient of the Weierstrass equation:

$$\vec{\nabla}(\text{Weierstrass}) = \begin{pmatrix} 3X^2 + fZ^4 \\ 2Y \\ 4fXZ^3 + 6gZ^5 \end{pmatrix} . \tag{4.25}$$

In order for the elliptic curve on that point to be singular, the gradient of its defining equation should be vanishing. Thus, looking at formula (4.25), one realizes that the singular locus should have $Z \neq 0$ because, otherwise, the condition of vanishing gradient would impose also $X = Y = 0$, that cannot be fulfilled in the ambient projective fiber. Hence one can use the gauge freedom to fix $Z = 1$, so that the gradient in (4.25) vanishes if and only if $Y = 0$ and $X = -3g/2f$, that is just one point (notice that the first of the three conditions is just the vanishing of Δ).
These are *only* singularities of the elliptic fiber and *not* of the whole Calabi-Yau, and this is true also in more complicated compactifications. The reason will be evident in section 4.3, where a more powerful technique is discussed to investigate the singularity properties of the Weierstrass hypersurface (4.23) in more difficult settings. The gauge group arising from this fiber singularity is just the abelian U(1) on the (p,q)7-brane, which can always be locally brought into a D7-brane.
However, it can happen that more zeros of Δ coincide: in this case one gets singularities of the entire total space whose nature will depend on the velocity in approaching the singularity of the fiber, i.e. on the order of zero of Δ itself but also of f and g. A systematic way to deal with their classification which is particularly useful in more difficult compactifictions, is briefly presented in section 4.3. Here only an illustrative example will be provided using the techniques showed so far. Suppose $u = 0$ to be a eighth order zero of Δ, a fourth order zero of g and a zero of order at least three of f. Then, according to (4.14) and (4.15), j vanishes at $u = 0$, while $\varphi \sim \eta^2(\tau)u^{-2/3}$. Therefore, a frame can be chosen such that $\tau(0) = \rho$, and the most general transformation it can undergo compatible with this local expression belongs to the conjugacy classes of the matrices $\pm T^{-1}S$. Finally, the transformation rule for φ, eq. (4.16), fixes the sign of the class of monodromies corresponding to this singularity: indeed, $\varphi \to \exp(-4\pi/3)\varphi = \tau\varphi$, which implies that the minus sign must be chosen.

The full classification of the singular fibers of an elliptic surface, due to Kodaira, is given in table 4.1, which also furnishes the list of the A-D-E enhanced gauge symmetries on the worldvolumes of the corresponding bound states.

Sen's weak coupling limit

From what has been said so far, it is clear that it is not obvious how to single out from an F-theory compactification the perturbative regime of type IIB string theory in a frame-invariant manner. Indeed, due to the use of general S-duality transformation to glue the various local charts, it is not possible to set up conventional perturbation theory for fundamental strings in a globally well-defined way. However, Sen [55, 56] proposed a suitable parameterization of the polynomials in the game such that the weak coupling limit of F-theory is performed not on the base space, but rather on the moduli space, thus avoiding the $SL(2,\mathbb{Z})$ ambiguity. In other words, one goes in a region of the $K3$ moduli space in which τ is kept constant with large imaginary part (i.e. small string coupling). Looking at (4.14), the requirement is obviously f^3

[10]The author thanks Andrés Collinucci for having pointed out this aspect.

ord(f)	ord(g)	ord(Δ)	singularity	gauge symmetry	monodromy class
≥ 0	≥ 0	0	I_0 (smooth)	none	$I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
0	0	1	I_1 (dbl. point)	abelian	$[T] = \left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right]$
0	0	n	I_n	A_{n-1}	$[T^n] = \left[\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\right]$
≥ 1	1	2	II (cusp)	abelian	$[T^{-1}S] = \left[\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right]$
1	≥ 2	3	III	A_1	$[S] = \left[\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right]$
≥ 2	2	4	IV	A_2	$[(T^{-1}S)^2] = \left[\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)\right]$
2	3	6	I_0^*	D_4	$-I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$
2	≥ 3	$n+6$	I_n^*	D_{n+4}	$[-T^n] = \left[\left(\begin{smallmatrix} -1 & -n \\ 0 & -1 \end{smallmatrix}\right)\right]$
≥ 2	3				
≥ 3	4	8	IV^*	E_6	$[-T^{-1}S] = \left[\left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right]$
3	≥ 5	9	III^*	E_7	$[-S] = \left[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right]$
≥ 4	5	10	II^*	E_8	$[-(T^{-1}S)^2] = \left[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)\right]$

Table 4.1: The Kodaira classification of singular fibers in elliptic surfaces. The square brackets in the last column denote the conjugacy class in $SL(2,\mathbb{Z})$. The fourth to sixth rows correspond to the so called Argyres-Douglas singularities [57].

being proportional to g^2. Hence, one takes $g = p^3$ and $f = \alpha p^2$ with p a section of $K^{-2}(\mathbb{P}^1)$, so that:

$$\Delta = (4\alpha^3 + 27)p^6 \qquad \text{and} \qquad j = \frac{4(24\alpha)^3}{27 + 4\alpha^3}\,, \tag{4.26}$$

where α can be tuned near $-(27/4)^{1/3}$ to get weak coupling everywhere on the base. Having τ subjected to no monodromies does not mean that the fibration is trivial: indeed, as it is manifest from the form of the discriminant in eq. (4.26), its 24 zeros are collected in 4 groups of six corresponding to the sixth order zeros of the polynomial $p(u)$. There is only one possibility to keep τ constant in such a 7-brane configuration, that has already been discussed in detail in subsection 4.1.1: a frame exists where each of the four groups is made of four D7-branes plus an O7-plane, i.e. the SO(8)-type (D_4) singularity of table 4.1.

Since the monodromy matrix of this non-trivial fibration is $-I$, it is convenient to build up a double cover of the base $\mathbb{P}^1$, called X_1, branched over the loci in which the O7's are placed, namely the four zeros of p. In practice, one adds a new homogeneous coordinate, ξ, to the base ones, whose vanishing locus class is the same as the one of the anti-canonical bundle of the base, and a new equation linking the square of ξ to the polynomial p. Therefore, the new equation and the projective weight assignments will be:

$$\begin{array}{ccc|c} u & v & \xi & \xi^2 = p(u,v) \\ \hline 1 & 1 & 2 & 4 \end{array}\;, \tag{4.27}$$

where it is clear that the so defined X_1 is a Calabi-Yau one-fold, that is a 2-torus. The orientifold loci are now the zeros of ξ and thus the orientifold involution changes the sign of this new coordinate:

$$\sigma : \xi \longrightarrow -\xi \,. \tag{4.28}$$

These two branches of X_1 are identified at the quotient X_1/σ, which thus gives back the original base $\mathbb{P}^1$.
One has arrived this way at the usual perturbative type IIB CY orientifold compactification down to 8 dimensions, that preserves 16 supercharges because of the orientifold involution[11]: the $\mathbb{Z}_2$ monodromy that comes together with σ acts on the fields of the theory exactly as the standard perturbative world-sheet involution $(-1)^{F_L} \circ \Omega$. Hence, this is exactly type IIB string theory on $\mathbb{R}^{1,7} \times T^2/\mathbb{Z}_2$.

4.2.2 F-theory on $\mathbb{R}^{1,3} \times \mathrm{CY}_4$

This much more general F-theory compactification will be the context in which the rest of this thesis will manly develop and is also the one that leads to all the recent striking phenomenological predictions.

The Calabi-Yau fourfold is an elliptic fibration over some Kähler manifold B_3 of complex dimension three, and its geometrical description remains essentially the same as in the $K3$ case, with an important difference. The much larger complex structure moduli space of the Calabi-Yau fourfold divides now in two parts from the IIB perspective: besides the moduli due to the shape of the 7-branes, there is also the contribution from complex structure deformations of the bulk manifold. Typically the number of 7-brane moduli is vastly larger than the number of bulk moduli.
What is really harder, however, is the Sen weak coupling limit of this theory. One first parameterize, without loss of generality, the polynomials f and g as follows:

$$\begin{aligned} f &= -3h^2 + \epsilon\eta \,, \\ g &= -2h^3 + \epsilon h\eta + \frac{\epsilon^2\chi}{12} \,, \end{aligned} \tag{4.29}$$

where ϵ is a complex constant which drives the weak coupling limit, while h, η and χ are sections of $K^{-2}(B_3)$, $K^{-4}(B_3)$ and $K^{-6}(B_3)$ respectively. At leading order in $\epsilon \to 0$ one finds:

$$\Delta \approx -9\epsilon^2 h^2(\eta^2 + h\chi) \qquad \text{and} \qquad j(\tau) \approx \frac{(24)^4}{2}\frac{h^4}{\epsilon^2(\eta^2 + h\chi)} \,. \tag{4.30}$$

Thus, in this limit, g_s goes to 0 everywhere on the base except near $h = 0$. This locus is just one of the components of the vanishing discriminant locus and it is interpreted as the divisor of B_3 wrapped by the O7-plane. For $\epsilon \equiv 0$ there is only the recombined 7-brane wrapping this locus and the situation is essentially the same as in the $K3$ case with $\alpha \equiv -(27/4)^{1/3}$. But when ϵ is not identically zero, a second component of $\Delta = 0$ appears, which should be thought of as the divisor wrapped by a D7-brane compensating the charge of the O7. Thus:

$$\mathrm{O7} \,:\, h(\vec{x}) = 0 \qquad \text{and} \qquad \mathrm{D7} \,:\, \eta^2(\vec{x}) + h(\vec{x})\chi(\vec{x}) = 0 \,, \tag{4.31}$$

where $\vec{x}$ are the coordinates of B_3. All the problems arise from the particular shape of the D7-brane, the so called *Whitney umbrella* shape, which, as it is manifest from eq. (4.31), develops

[11] There is a famous duality relating this string vacuum to heterotic string theory on $\mathbb{R}^{1,7} \times T^2$, but this topic will not be discussed in this thesis.

a double curve singularity at the intersection with the orientifold and additional pinch point singularities on this curve where also $\chi = 0$. Such singularities make many calculations in type IIB, like the D3-brane geometric and gauge tadpoles induced by the D7, no longer reliable. Addressing this issue is the core of the paper [58], in which the authors develop a couple of different techniques to actually calculate the total D3 tadpole in order to match the geometric contribution to it with the prediction deduced from the F-theory lifts of these type IIB orientifold compactifications. One of those methods, based on Sen's tachyon condensation, will be adopted in chapter 6 in order to discuss the weak coupling limits of F-theory compactifications on both smooth and singular Calabi-Yau fourfolds. Before that, fluxes and tadpoles will be briefly reviewed from the M-theory perspective in section 4.4.

Before closing this section, it is important to stress (since it will be crucial for the practical computations) that also in this case it is very natural to think of the Sen limit as a perturbative type IIB string theory on $\mathbb{R}^{1,3} \times X_3$ where, in complete analogy with the previous case, X_3 is the Calabi-Yau threefold that double covers B_3 with branch locus given by the O7-plane (and by the O3-planes, if any, as will be discussed in the paragraph 6.2.1). Hence, one again adds a new coordinate, ξ, odd under the orientifold involution, and a new equation, $\xi^2 = h(\vec{x})$. This leads to an effective $\mathcal{N} = 1$, 4-dimensional gauge theory, which is most desirable from a phenomenological point of view.

4.3 The gauge symmetry enhancement

The purpose of this section is a flash review of the gauge symmetry enhancement in F-theory compactifications, from two different point of view: the former focus on the algebraic mechanism responsible for the appearance of non-simply-laced gauge groups as opposed to simply-laced ones [43, 44], while the latter analyze the phenomenon from the usual, although non-perturbatively generalized, open string perspective [45, 46, 47, 48]. A third method, based on junctions, will be discussed in chapter 5 where its generalization to the non-simply-laced gauge Lie algebras is provided.

4.3.1 Geometric perspective

In order to investigate in a systematic and algorithmic manner the phenomenon of the symmetry enhancement, one has to rewrite the Weierstrass form (4.23) in a more clever way:

$$Y^2 + a_1 XYZ + a_3 YZ^3 = X^3 + a_2 X^2 Z^2 + a_4 XZ^4 + a_6 Z^6 \,, \tag{4.32}$$

where the a_i are locally defined polynomials on the base, or better sections of $K^{-i}(B_3)$.
Then, the appropriate method to study symmetry enhancement is Tate's algorithm. As said, the elliptic fibers degenerate over specific divisors of the base B_3. Let S be one such divisor and let $\sigma = 0$ be its local defining equation. If the discriminant Δ of the curve (4.32) vanishes on it, the corresponding fiber degenerates; if, for instance, Δ is divisible by σ and not by σ^2, one has at $\sigma = 0$ a singularity of Kodaira type I_1, which, as already stressed, does not constitute a singularity of the total space of the fibration and does not give rise to non-abelian enhanced symmetry. In order to come across the latter one needs more severe singularities. The virtue of Tate's algorithm is that it enables one to classify all possible singularities of (4.32) in a systematic tree-like way by analyzing the increasing order of zeros of Δ and of the a_i's.
Suppose, from now on, to be in the local chart where $Z \neq 0$, so that one can fix the gauge by

imposing $Z = 1$. The first non-trivial instance is when the polynomials a_i are such that (4.32) takes the form of the following equation in X, Y, σ:

$$Y^2 + a_1 XY + a_{3,1}\sigma Y = X^3 + a_2 X^2 + a_{4,1}\sigma X + a_{6,2}\sigma^2 \,, \tag{4.33}$$

whose leading order is *quadratic*, and Δ has a double zero at $\sigma = 0$. Thus, (4.33) is manifestly singular at the origin and, in the case its quadratic form is no longer singular, one blows up the origin (that is a codimension 3 submanifold of the ambient fivefold, restricting to a codimension 2 one of the Calabi-Yau fourfold) and resolves the singularity there. Choosing the local patch in which the exceptional divisor of the ambient fivefold is described by the equation $\sigma = 0$, then its intersection with the proper transform of such a resolution is given by the following generically irreducible non-singular quadratic equation:

$$Y_1^2 + a_1 X_1 Y_1 + a_{3,1} Y_1 = a_2 X_1^2 + a_{4,1} X_1 + a_{6,2} \,, \tag{4.34}$$

where $X_1 = X/\sigma$ and $Y_1 = Y/\sigma$ parametrize in the above mentioned patch the lines through the singular point and as such, together with σ, they are the local coordinates of the neighborhood of the exceptional divisor. This is the singularity of Kodaira type I_2, which gives rise to an enhanced symmetry of the $SU(2)$ type: the resolutions of the singular fibers, indeed, are made of just one component, represented by the exceptional divisor (4.34), which is regarded as the unique node of the $SU(2)$ Dynkin diagram.[12]

Continuing, for simplicity, along the same branch of the algorithm, at the next step one encounters two possibilities that differentiate between the non-simply-laced and the simply-laced alternative for the gauge symmetry.[13]

- Require $a_{3,1}, a_{4,1}, a_{6,2}$ to be further divisible by σ, so that (4.33) becomes:

  $$Y^2 + a_1 XY + a_{3,2}\sigma^2 Y = X^3 + a_2 X^2 + a_{4,2}\sigma^2 X + a_{6,3}\sigma^3 \,. \tag{4.35}$$

  Now, by performing the same blow up as before, one ends up with an exceptional divisor made of *two* components represented on each point of the base by the two lines solving the equation

  $$Y_1^2 + a_1 X_1 Y_1 - a_2 X_1^2 = 0 \,. \tag{4.36}$$

  They are not globally defined if the polynomials a_1 and a_2 are generic, and they will experience monodromy as one goes along a closed path on the base. This singularity type is named I_3^{ns} (ns standing for non-split) and gives rise to unconventional gauge symmetry. One can go further along this sub-branch of the algorithm just requiring divisibility by σ of $a_{6,3}$: this induces an $SU(2)$ singularity at the origin ($X_1 = Y_1 = \sigma = 0$) that survives after the blow up, so that a second blow up is necessary to completely resolve the singularity, which leads to a further irreducible exceptional divisor, similar to (4.34), placed at $\sigma = 0$ in the coordinate chart ($X_2 = X_1/\sigma, Y_2 = Y_1/\sigma, \sigma$) This is the type I_4^{ns} singularity, which

[12] As opposed to the "traditional" treatment of this section, in chapter 6 the singularities of the elliptic Calabi-Yau fourfold will be resolved by means of toric methods that, besides the computational power, have the advantage to make manifest and distinct the appearance of the Cartan and the extended nodes in the affine Dynkin diagrams of the corresponding gauge algebras.

[13] Actually, a subtle differentiation arises already at the level I_2: a detailed explanation of this phenomenon as well as of its consequences will be provided, by means of toric methods, in chapter 6.

corresponds to the $Sp(2)$ gauge group.[14]
By induction, one easily constructs in this way the series of $\mathbf{C_n}$ algebras, the resolution of the corresponding singularities being characterized by $n-1$ pairs of non-split exceptional divisors plus an irreducible one; moreover one has also a tower of *unconventional* gauge symmetries for which there are instead n pairs of non-split exceptional divisors.

- Require $a_2, a_{4,1}, a_{6,2}$ to be further divisible by σ, so that (4.33) becomes:

$$Y^2 + a_1 XY + a_{3,1}\sigma Y = X^3 + a_{2,1}\sigma X^2 + a_{4,2}\sigma^2 X + a_{6,3}\sigma^3 \,. \tag{4.37}$$

Blowing up in this case leads to a resolution of the singularity by means of two globally distinct (split) exceptional divisors, described by the equation:

$$Y_1(Y_1 + a_1 X_1 + a_{3,1}) = 0 \,. \tag{4.38}$$

This singularity type is named I_3^s (s standing for split) and gives rise to the familiar $SU(3)$ gauge symmetry. Even though is not necessary, because the singularity is already completely resolved, one can perform a second blow up, now along the codimension two locus in the ambient fivefold described by the equations $Y_1 = \sigma = 0$, which restricts to a codimension one locus in the resolved Calabi-Yau fourfold. This locus is just one component of the exceptional divisor (4.38). Again, going in the patch in which the new exceptional divisor arising from the second blow up is described by $\sigma = 0$, that component of the former exceptional divisor will be substituted by the new generically irreducible and non-singular form:

$$a_1 X_1 Y_2 + a_{3,1} Y_2 = X_1^3 + a_{2,1} X_1^2 + a_{4,2} X_1 + a_{6,3} \,, \tag{4.39}$$

where $Y_2 = Y_1/\sigma$ as before. The other component of the former exceptional divisor (4.38), instead, can be easily shown, by changing local chart, to be a codimension 1 submanifold of the new proper transform. In chapter 6 the toric counterpart of all this will be discussed. One can go on with the algorithm just requiring σ to divide $a_{3,1}$ and $a_{6,3}$. Again this will induce a residual $SU(2)$ singularity at the origin, i.e. the intersection of the two previously found split exceptional divisors: the blow up of such singularity will lead as before to an additional irreducible exceptional divisor, like (4.34) without the a_2 term, placed at $\sigma = 0$ in the coordinate chart (X_2, Y_2, σ). This is the type I_4^s singularity, which corresponds to the $SU(4)$ gauge group.
By induction, one constructs this way the entire series of $\mathbf{A_n}$ algebras, producing, out of the whole resolving procedure, $n/2$ pairs of split exceptional divisors if n is even and $(n-1)/2$ pairs of split exceptional divisors plus an irreducible one if n is odd.

While the full classification of symmetry enhancements can be found in [44], it is worth to write here the polynomials whose factorization distinguishes the split case from the non-split one, also in the other branches of the algorithm, which are relevant for what follows because they contain the orthogonal and the exceptional gauge symmetries. A summarizing table (table 4.2) is anyhow provided below, for later reference.

[14]Throughout this thesis the convention on symplectic groups is adopted such that $Sp(n)$ has rank n, i.e. $Sp(n) \equiv USp(2n)$.

1. For the Kodaira singularity of type $\mathrm{IV}^{\star ns}$ (corresponding to the gauge group F_4), the relevant polynomial is:
$$Y_2^2 + a_{3,2}Y_2 - a_{6,4} = 0\,, \tag{4.40}$$
which factorizes globally if one just requires $a_{6,4} = 0 \bmod \sigma$, thus generating the $\mathrm{IV}^{\star s}$ singularity (namely the E_6 gauge group).

2. For the Kodaira singularities of type $\mathrm{I}^{\star ns}_{2k-3}$, $k \geq 2$ (corresponding to the gauge groups $SO(4k+1)$), the relevant polynomials are:
$$Y_k^2 + a_{3,k}Y_k - a_{6,2k} = 0\,, \tag{4.41}$$
which factorize globally if one just requires $a_{6,2k} = 0 \bmod \sigma$, thus generating the $\mathrm{I}^{\star s}_{2k-3}$ singularities (namely $SO(4k+2)$ gauge groups).

3. For the Kodaira singularities of type $\mathrm{I}^{\star ns}_{2k-2}$, $k \geq 2$ (corresponding to the gauge groups $SO(4k+3)$), the relevant polynomials are:
$$a_{2,1}X_k^2 + a_{4,k+1}X_k + a_{6,2k+1} = 0\,, \tag{4.42}$$
for which one cannot change coordinates in order to formulate their factorization as before in terms of the vanishing mod σ of some polynomial; anyway, if they factor, the associated singularities become $\mathrm{I}^{\star s}_{2k-2}$, $k \geq 2$ (namely $SO(4k+4)$ gauge groups).

4. Finally, the Kodaira singularity $\mathrm{I}^{\star ns}_0$ (corresponding to the gauge group G_2) contains a subtlety. The relevant polynomial is:
$$X_1^3 + a_{2,1}X_1^2 + a_{4,2}X_1 + a_{6,3} = 0\,, \tag{4.43}$$
which describes a triple of non–split exceptional divisors. Clearly (4.43) can either partially or completely split. The former situation is achieved by simply requiring $a_{6,3} = 0 \bmod \sigma$, which leads to the so called type $\mathrm{I}^{\star ss}_0$ (ss standing for semi-split), corresponding to the $SO(7)$ gauge group (a pair of non-split exceptional divisors and a split one). The latter is obtained by further requiring the factorization mod σ of $X_1^2 + a_{2,1}X_1 + a_{4,2}$, which leads to three split exceptional divisors, but, as at point 3., it cannot be formulated in terms of the vanishing mod σ of some polynomial: this is the case of type $\mathrm{I}^{\star s}_0$ (namely the $SO(8)$ gauge group).

In the above algebraic geometric description, what establishes the connection between the specific singularity and the enhanced symmetry is the fact that the intersection matrix of the components of each singular fiber (namely the various exceptional divisors that have been found above) is observed to take the form of the affine Cartan matrix of the corresponding non-Abelian Lie algebra. This gives rise to the A-D-E series of Lie algebras provided that no monodromy acts on the collapsing cycles[15]. On the other hand, when the opposite occurs and, in particular, when, going around the singularity, one picks up an outer automorphism of the Lie algebra, the gauge group gets orbifolded as one shrinks the 2-cycles of the resolution to zero-size, and one ends up with a reduced gauge symmetry. These reductions via outer automorphisms are known, in Lie algebra theory, to be connected to the symmetries of the relevant Dynkin diagrams and to lead to the non-simply-laced algebras; precisely, a $\mathbb{Z}_2$ orbifold leads from $\mathbf{A_{2n-1}}$ to $\mathbf{C_n}$, from $\mathbf{D_n}$ to $\mathbf{B_{n-1}}$ and from $\mathbf{E_6}$ to $\mathbf{F_4}$, while the triality of the $\mathbf{D_4}$ Dynkin diagram leads to $\mathbf{G_2}$.

[15] Actually if monodromies are present but they are all given by elements of the Weyl group (inner automorphisms), they can be undone by a gauge transformation in the fiber and thus one does not break the initial simply-laced gauge group.

type	group	a_1	a_2	a_3	a_4	a_6	Δ
I_0	—	0	0	0	0	0	0
I_1	—	0	0	1	1	1	1
I_2	$SU(2)$	0	0	1	1	2	2
I_3^{ns}	unconven.	0	0	2	2	3	3
I_3^{s}	$SU(3)$	0	1	1	2	3	3
I_{2k}^{ns}	$Sp(k)$	0	0	k	k	$2k$	$2k$
I_{2k}^{s}	$SU(2k)$	0	1	k	k	$2k$	$2k$
I_{2k+1}^{ns}	unconven.	0	0	$k+1$	$k+1$	$2k+1$	$2k+1$
I_{2k+1}^{s}	$SU(2k+1)$	0	1	k	$k+1$	$2k+1$	$2k+1$
II	—	1	1	1	1	1	2
III	$SU(2)$	1	1	1	1	2	3
IV^{ns}	unconven.	1	1	1	2	2	4
IV^{s}	$SU(3)$	1	1	1	2	3	4
$\mathrm{I}_0^{*\,ns}$	G_2	1	1	2	2	3	6
$\mathrm{I}_{2k-3}^{*\,ns}$	$SO(4k+1)$	1	1	k	$k+1$	$2k$	$2k+3$
$\mathrm{I}_{2k-3}^{*\,s}$	$SO(4k+2)$	1	1	k	$k+1$	$2k+1$	$2k+3$
$\mathrm{I}_{2k-2}^{*\,ns}$	$SO(4k+3)$	1	1	$k+1$	$k+1$	$2k+1$	$2k+4$
$\mathrm{I}_{2k-2}^{*\,s}$	$SO(4k+4)^*$	1	1	$k+1$	$k+1$	$2k+1$	$2k+4$
$\mathrm{IV}^{*\,ns}$	F_4	1	2	2	3	4	8
$\mathrm{IV}^{*\,s}$	E_6	1	2	2	3	5	8
III^*	E_7	1	2	3	3	5	9
II^*	E_8	1	2	3	4	5	10
non-min	—	1	2	3	4	6	12

Table 4.2: Tate's algorithm. When the group is not specified it is abelian and the $*$ on the groups $SO(4k+4)$ means that a further factorization condition must be imposed, as explained in point 3. and 4. above.

4.3.2 String perspective

In the framework of algebraic geometry, the above is as much as one can say about the connection between singularity theory and enhancing of gauge symmetry (although some attempts were made in the past to render it more explicit, [59, 60]). The gauge interpretation is supported by the duality with heterotic theory, when the latter exists. But, needless to say, a more direct and physical interpretation is clearly desirable and was indeed put forward in the early stage of F-theory. It was based on the analysis of BPS spectrum of 7-branes. The spectrum of (p,q)7-branes is formed by (p,q)-strings. Since, in general, enhanced symmetry requires, as stressed many times, an assemblage of branes with different p, q charges, it is evident that the strings that enter the game will in general be mutually non-perturbative. The search for BPS string states was carried out in refs. [45, 46, 47, 48] in the case of F-theory compactified on $R_{1,7} \times \mathrm{K3}$, where the BPS condition is fulfilled by computing the geodesics of the effective metric (4.18).

The technique adopted is the one of the so called A-B-C 7-branes, to be introduced in chapter 5 for similar reasons. The idea is to consider an allowed configuration of such mutually non-perturbative (p,q)7-branes, which will eventually collapse and produce the desired enhanced symmetry, and analyze all possible strings stretched between them. These strings, before col-

lapse, will be massive and only a "perturbative" unitary subgroup of the final gauge group is manifest, depending on how many 7-branes typically of the A-type are usually already taken to be coincident. The states with minimal effective length are recognized as BPS states and will identify the massless gauge fields.
The determination of all BPS string states is in principle possible; in practice it is not easy. One reason is that $\tau(z)$ is, in general, a function defined only implicitly, so that only numerical techniques are viable. There are particular regions in the moduli space of the 7-branes positions where τ can be held constant (equal, as seen, to the orbifold points of its fundamental region) [55, 61, 62]. In such instances the search for BPS states can be effectively carried out, but the enhanced symmetries realized in this way are only a limited subset. It is clear that for general τ, things are far more complicated and the control over the BPS states is very hard to realize. Anyhow, the analysis of the constant τ examples, even though it involved simple and far from phenomenologically interesting cases, was important to convince people that this physical intuition of the symmetry enhancement in F-theory is plausible.

Although in this context many things simplify, non-simply-laced gauge groups cannot appear, due to the absence of non-trivial monodromies on $\mathbb{R}^{1,7}$. In chapter 5 yet another method, still based on the use of the A-B-C branes, but improved by the string-junction technique, will be adopted to actually extend such an alternative physical description to non-simply-laced gauge Lie algebras, which from the algebraic-geometric analysis are known to generically arise in the context of Calabi-Yau fourfold compactifications of F-theory.

4.4 The F/M-theory duality

The only rigorous definition of F-theory available so far is the one that starts from M-theory and uses T-duality of string theory [7, 63]. It is worth to briefly present it here because it will clarify some aspects of the previous picture and provide a useful method to treat fluxes and tadpoles, whose integrality properties is the subject of the analysis carried out in the last chapter of this thesis.

The idea is to compactify M-theory on a Calabi-Yau[16] elliptically fibered over some Kähler base space B, with a small T^2 as fiber. Then, one takes one of the two non-trivial 1-cycles of the 2-torus, A_1, as the M-theory circle. So this gives *weakly coupled* type IIA on a fibration of the other non-trivial 1-cycle A_2. Finally, one T-dualizes along A_2, ending up with type IIB on a large circle, which, in the limit of vanishing M-theory T^2 (*F-theory limit*) results in *uncompactified* type IIB string theory. The latter is therefore type IIB on $\mathbb{R}^{1,3} \times B$ with varying axiodilaton τ given by the complex structure modulus of the M-theory T^2. This is exactly the philosophy behind F-theory.
It can be shown that actually, after this chain of dualities and the F-theory limit, Poincaré invariance in the four external dimensions is achieved, which is not obvious given the different nature of the third spatial direction. Moreover, in elliptic fibrations τ varies holomorphically over the base, while the volume of T^2, V remains constant: indeed, the area of a holomorphic curve is given by the integral of the Kähler form, which does not change along the base since the Kähler form is closed. This is not true in the presence of fluxes, when the M-theory internal geometry gets warped and thus becomes the one of a conformal Calabi-Yau [64].

The previous description provides a dictionary to deduce the F-theory equivalents of the

[16]The Calabi-Yau condition must be there to have $\mathcal{N} = 2$ supersymmetry in three dimension, that in the end will turn into $\mathcal{N} = 1$ in four.

other type IIB fields as well as the brane content of the theory. For example, the M-theory 3-form potential locally decomposes as follows:

$$C_3 \;=\; C_3' + B \wedge \sqrt{V}\mathrm{d}x + C_2 \wedge \sqrt{V}\mathrm{d}y + B_1 \wedge \sqrt{V}\mathrm{d}x \wedge \sqrt{V}\mathrm{d}y\,, \tag{4.44}$$

where x and y are periodic coordinates with periodicity 1 along the circles A_1 and A_2 respectively, while C_3', B, C_2 and B_1 do not have legs on T^2. After reduction on A_1, T-duality on A_2 and the limit $V \to 0$, B becomes the B-field of type IIB, C_2 the RR 2-form potential, C_3' half the components of the self-dual RR 4-form potential ($C_4^{(y)} = C_3' \wedge \mathrm{d}y$) and B_1 gives the gauge field that mixes the y direction with the other directions of the IIB target space (this is the result of a T-duality on the B-field along one of its legs).
Similar situation arises for the decomposition of the G_4 flux of M-theory. From now on, the target space of M-theory will be taken to be $\mathbb{R}^{1,2} \times Z_4$, where Z_4 is a Calabi-Yau fourfold with strict SU(4) holonomy elliptically fibered over a Kähler space B_3 of complex dimension three. Thus, G_4 will split in two parts if one wants to keep Lorentz invariance in $\mathbb{R}^{1,2}$. One part is locally of the form:

$$G_4^{\mathrm{warp}} = \frac{1}{3!}\,\epsilon_{\alpha\beta\gamma}\,\partial_i f\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^i\,, \tag{4.45}$$

where $\alpha = 0, 1, 2$ are the external flat directions while $i = 1, \ldots, 6$ are the internal ones, and f is the warp factor of the conformal Calabi-Yau fourfold Z_4. After reduction, T-duality and the F-theory limit, the flux in (4.45), provided i is an index of B_3 in order not to break Lorentz invariance in $\mathbb{R}^{1,3}$, gives rise to the self-dual RR flux of type IIB:

$$F_5 = \frac{1}{4!}\,\epsilon_{\alpha\beta\gamma\delta}\,\partial_i f\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta \wedge \mathrm{d}x^i\,, \tag{4.46}$$

which is a singlet of S-duality. The second part of G_4, instead, is a primitive[17] (and thus self-dual) 4-form of Z_4 and, by the equations of motion, it is harmonic. Hence, it is uniquely determined by its cohomology class which will be indicated with the same symbol and will be essentially the main object of study in chapter 6. Indeed, this flux is quite subtle because it is not really an integral class in general [8]. The self-duality-condition for G_4 is there in order to avoid a runaway behavior of the effective scalar potential. Then supersymmetry requires the vanishing of its (3,1) part (plus complex conjugate) as F-term condition and also the vanishing of its (4,0) part (plus complex conjugate) for compactifications with zero cosmological constant in four dimensions (Minkowski space). Finally, the condition $G_4 \wedge J = 0$ in the remaining (2,2) part is automatic for compactifications on smooth Calabi-Yau fourfolds with strict SU(4) holonomy, as explained below.
This G_4 also contains bulk and brane-type fluxes of type IIB and a brief and propaedeutic discussion on this aspect will occupy the rest of this section.

According to formula (4.44) and the discussion below it about the local potential of G_4, the only fluxes on Z_4 which are not breaking Lorentz invariance in $\mathbb{R}^{1,3}$ are locally of the form:

$$G_4 = H \wedge \sqrt{V}\mathrm{d}x + F_3 \wedge \sqrt{V}\mathrm{d}y\,, \tag{4.47}$$

where the S-duality doublet (H, F_3) of type IIB bulk fluxes appear. The fact that a G_4 of this form satisfies $G_4 \wedge J = 0$ can be seen as follows. For a Z_4 of exactly SU(4) holonomy,

[17] This means that G_4 is of type (2,2) and $G_4 \wedge J = 0$, where J is the Kähler form of Z_4.

$H^2(Z_4) = H^{1,1}(Z_4)$, so all 2-cohomology classes have Poincaré duals represented by linear combinations of holomorphic 6-cycles. Thus, if $\{D_M\}_M$ is a basis of $H^{1,1}(Z_4)$, then

$$\int_{Z_4} G_4 \wedge D_M \wedge D_N = \int_{D_M \cap D_N} G_4 = 0 \qquad \forall\, M, N\,. \tag{4.48}$$

This is because, if Z_4 is smooth, all its divisors, except the gravitational brane (i.e. the 0-section), are elliptic fibration over divisors of the base; thus every intersection of two divisors is either a linear combination of divisors of B_3 or an elliptic fibration over an holomorphic curve of B_3. But, Poincaré-preserving fluxes like (4.47) have one and only one leg along the fiber; so they integrate to zero over all such intersections. This implies that $G_4 \wedge D_M = 0$ as cohomology class for every M. Hence also $G_4 \wedge J = 0$ in cohomology, but, since it is harmonic, pointwise too. This result is as one would have expected from the weak coupling limit of the smooth fourfold case discussed in subsection 4.2.2. In fact, if this condition had not been automatic, it would have appeared as a D-term constraint; however, there are no massless U(1) vectors in the four dimensional effective theory which could generate a D-term, because the gauge group surviving on the Whitney umbrella 7-brane (which is a recombination of brane and image-brane) is $O(1)$, due to the O7. Of course, if the Calabi-Yau fourfold is singular, enhanced gauge groups are supposed to appear and consequently D-term constraints are to be taken into account.

It is crucial to notice here that, due to its monodromy around 7-brane loci, the doublet of IIB bulk fluxes can generate non-trivial excitations localized very close to the 7-branes, which are thought of as worldvolume gauge fluxes. To see this phenomenon in more detail, it is useful to figure out what kind of brane sources one should expect to have in such a context.

1. $\mathbb{R}^{1,2}$-filling M2-brane, which gets mapped into $\mathbb{R}^{1,3}$-filling D3-branes in type IIB.

2. D7-brane, that after T-duality come out of the geometric locus $S_2 \subset B_3$ on which the A_1 cycle of the torus fiber collapses: thus the worldvolume of a D7-brane will be $\mathbb{R}^{1,3} \times S_2$, where S_2 is a divisor of B_3. In general, since (A_1, A_2) is an S-duality doublet, one clearly gets a (p,q)7-brane if the collapsing 1-cycle over S_2 is the linear combination $pA_1 + qA_2$.

3. M5-brane wrapped on a 4-cycle Σ_4 of Z_4 and domain wall in $\mathbb{R}^{1,2}$ (say along the time direction and the first spatial one). Strictly speaking, this source does break Poincaré invariance in $\mathbb{R}^{1,3}$; however, on each side of the wall , away from it, one does have Poincaré invariance and all 4-dimensional field attain their vev's, although they could undergo a discontinuous change across the wall. Since there is no connected path linking a domain wall, no physical wave function can be affected by the generated magnetic flux: therefore one really has two generally different Poincaré vacua on both sides of the wall.
   The magnetic fluxes generated by these extended sources are of the form:

   $$G_4^{M5} = \theta(x^2)\, G_4 \qquad \text{such that} \qquad \mathrm{d}G_4^{M5} = G_4 \wedge \delta(x^2)\mathrm{d}x^2\,, \tag{4.49}$$

   the domain wall being placed at $x^2 = 0$, as the presence of the heaviside function θ in its magnetic flux signals. The 4-cycle Σ_4 wrapped by the M5, instead, is just the Poincaré dual in Z_4 of G_4, and one has for it two possibilities, if again Lorentz invariance in $\mathbb{R}^{1,3}$ is kept:

   (a) Σ_4 is a $pA_1 + qA_2$ fibration over a *3-cycle* $\Sigma_3 \subset B_3$. This maps into a (p,q)5-brane in type IIB wrapping Σ_3 and domain wall in $\mathbb{R}^{1,3}$. The internal magnetic flux of this source maps to a *bulk* flux of the form $pF_3 + qH$.

(b) Σ_4 is a $pA_1 + qA_2$ fibration over a *3-chain* $\Gamma_3 \subset B_3$ with boundary on the collapsing locus of the fiber 1-cycle. This maps into a (p,q)5-brane terminating on (p,q)7-branes. The internal magnetic flux of this source maps to a *brane*-sourced flux F that is the Poincaré dual in S_2 of $\partial\Gamma_3$. The localization of such a gauge flux on the 7-brane worldvolume can be easily described in a local model as follows. Trivialize the given local chart in order the 7-brane at the origin to be a D7 and take the following anti-self-dual, normalized, harmonic 2-form on the elliptic fibration over the disk D_2 surrounding the D7:

$$\omega = \frac{1}{g_s}\,\mathrm{d}\left(\frac{\mathrm{d}x + \mathrm{Re}\,\tau\,\mathrm{d}y}{\mathrm{Im}\,\tau}\right) \qquad \text{such that} \qquad \int_{\partial D_2}\int_{y=0}^{1}\omega = 1\,, \tag{4.50}$$

where τ is given by (4.8) with $n = 1$. Now, the magnetic field-strength of the M5 in question is:

$$G_4 = F \wedge \sqrt{V}\,\omega\,, \tag{4.51}$$

which gives rise to type IIB fluxes of the form:

$$H = \frac{1}{g_s} F \wedge \mathrm{d}\left(\frac{1}{\mathrm{Im}\,\tau}\right)\,, \tag{4.52}$$

$$F_3 = \frac{1}{g_s} F \wedge \mathrm{d}\left(\frac{\mathrm{Re}\,\tau}{\mathrm{Im}\,\tau}\right)\,. \tag{4.53}$$

These expressions are in fact compatible with the monodromy of the doublet (H, F_3) around the locus of a D7-brane, $H \to H$, $F_3 \to F_3 + H$, as it is easy to verify sending $\mathrm{Re}\,\tau \to \mathrm{Re}\,\tau + 1$. Moreover, Freed-Witten anomaly cancellation for the D7-brane (which is surely spin^c, being S_2 a complex surface) assures that H vanishes in the cohomology of S_2, so that F_3 is well defined on the D7-brane locus.

It is important to notice, however, that the distinction between bulk and brane type fluxes outlined above is *not* canonical: it depends on the choice of the 3-chains associated to the brane type fluxes. It is possible, indeed, that, by going around a singularity in the moduli space of the 7-brane configuration, the chosen 3-chain Γ_3 undergoes the transformation $\Gamma_3 \to \Gamma_3 + \Sigma_3$, with Σ_3 some 3-cycle of B_3. Consequently, the brane type flux associated to Γ_3 gets transformed, after the non-contractible loop in the moduli space, to itself plus the bulk type flux dual to Σ_3 in B_3.

4. M5-instanton wrapped on a 6-cycle in Z_4. It can be seen that, in order to have finite action, this source must wrap the entire T^2 fiber, and hence it maps to an instantonic D3-brane in type IIB wrapping a divisor of B_3. The magnetic flux of such M5 is of the form (4.45), which induces a magnetic flux for the D3-instanton of the form (4.46).

A comment is in order at this point about the symmetry enhancement from the M-theory perspective. The various strings stretching between 7-branes that are responsible for the gauge symmetry enhancement, as discussed in subsection 4.3.2, lift in M-theory to membranes wrapping the various "exceptional" components of the singular fiber arising from the complete resolution of the singularity. Thus, the "off-diagonal" gauge bosons, namely the ones responsible for the non-abelianity, are just the massless oscillation of these membranes, while the gauge bosons corresponding to the Cartan generators are given by the integration of the M-theory 3-form C_3 on each exceptional component.

4.4.1 Tadpoles

A few words about tadpole cancellations in M/F-theory, which will turn useful in chapter 6, end the F-theory review.

The net charge of the sources of type 1 above must clearly vanish, because their transverse space is compact. However, since in the 11-dimensional supergravity action there are natural sources of M2 charge, one is obliged to add a suitable number of localized membrane in order to cancel this tadpole. The terms that generate M2-brane charges are essentially two: the Chern-Simons term $\int G_4 \wedge G_4$ and the one-loop correction $I_8(R)$, which is a polynomial of degree four in the curvature [65] and integrates to $\chi(Z_4)/24$ on the Calabi-Yau fourfold. Thus, the number of M2-branes to be added is:

$$N_{M2} \; = \; \frac{\chi(Z_4)}{24} - \frac{1}{2}\int_{Z_4} G_4 \wedge G_4 \,. \tag{4.54}$$

Using eq. (4.47), it is easy to see that the type IIB equivalent of (4.54) is:

$$N_{D3} \; = \; \frac{\chi(Z_4)}{24} - \int_{B_3} F_3 \wedge H \,, \tag{4.55}$$

where all the contributions to the D3-brane tadpole are contained. Indeed, $\chi(Z_4)/24$ takes into account the D3 charge induced by curvature terms, namely by the gravitational interactions of higher dimensional D-branes[18] described in subsection 3.2.2. The second term, instead, contains the D3 charge induced by the gauge couplings of higher dimensional D-branes: if one considers the splitting of F_3 and H in bulk and brane-type fluxes, outlined above, one gets, besides the D3-brane charge induced by the ordinary Chern-Simons term in the type IIB bulk supergravity action, the following brane flux-induced contribution:

$$\text{gauge tadpole} \; = \; -\frac{1}{2}\int_{S_2} F \wedge F \; = \; -\text{ch}_2(F) \,. \tag{4.56}$$

This coincides with the second Chern character of the gauge bundle of the D7-branes, as it should be according to the general theory.[19]

Matching the geometric tadpole $\chi(Z_4)/24$ predicted by F-theory with the curvature induced one in type IIB by D7-branes and O7-planes constitutes a non-trivial global consistency check of the F-theory lift of the considered string compactification. Moreover, a gauge contribution on the type IIB side could be found, which would signal the presence of a Freed-Witten-like gauge flux on the D7-brane(s) that cannot be put to zero (i.e. for example a half-quantized one): then, its presence should be traced back to a non-trivial quantization condition for the M-theory G_4 flux, along the lines of the previous analysis. This is the bottom line of the content of chapter 6.

[18] Actually, also higher dimensional orientifold planes do contribute gravitationally to the D3 tadpole, as will be discussed in chapter 6.

[19] The reader should remember that the orientifold projection of the Sen limit kills the D9-brane and the D5-brane net charges, as it is easy to deduce by performing two T-dualities on type I string theory, which admits D1, D5 and D9-branes.

Chapter 5

Non-simply-laced Lie algebras via F-theory strings

As described in section 4.3, in the case of Calabi-Yau fourfold compactifications of F-theory one should expect non-simply-laced gauge Lie algebra out of the blowing up procedure of non-abelian Kodaira singularities of elliptic fibers. A systematic analysis from the string perspective of the extra massless gauge bosons arising from such enhancing has been carried out for the first time in [66] by means of the technology of string junctions. This chapter contains the results of this paper and is organized as follows: in section 1 a brief introduction on the method of string-junction is provided; in section 2 the analysis of odd orthogonal gauge algebras is carried out after having recalled the structure of the parent even orthogonal gauge algebras; in section 3 the same is done for F_4 arising from the folding of E_6 and in section 4 for G_2 arising from a folding of $\mathbf{D_4}$; finally, section 5 contains some comments about symplectic gauge algebras.

5.1 String junctions

In [47] the importance of string junctions was stressed (for string junctions in F-theory see [49, 67, 68, 69, 70, 71] and also [72]). Indeed, as is well-known, (p,q)-strings may join or split and form string networks. The only condition is that the charges be conserved at the vertices. String junctions is the generic term to indicate any kind of string pattern, from elementary string prongs attached to a 7-brane to complicated networks of strings. String junctions will be basic in the sequel.

In fact, a third technique to analyze symmetry enhancement in F-theory was introduced in [49]. Instead of focusing on BPS states, the idea was to consider the lattice of string junctions related to a given system of 7-branes and define invariant intersection numbers (scalar product) on it. Once this is done the game consists in showing that string junctions of specific composition and length form a realization of the root lattice of a given Lie algebra.

Before discussing how this technology works in the case of Calabi-Yau fourfold compactifications, it is worth to stress again one important difference between the latter and the compactification on $K3$. For F-theory on $K3$, the 7-branes are just points in the internal 2-sphere; hence resolving singularities just amounts to separate some of those points that collapse, ending up with stacks of parallel 7-branes, possibly mutually non-perturbative. On Calabi-Yau fourfolds, instead, 7-branes are regarded as 4-dimensional divisors of the base space B_3, and having stacks of parallel 7-branes after resolution is now a highly non-generic situation. In general 7-branes will intersect in many complicated ways, and, in addition, after the complete resolution of the singularity placed on codimension 1 in the base, nothing guarantees the absence of additional singularities on higher codimension loci. However, rather than attempting to control such global issues, the purpose here is more limited: all the computations will be done strictly locally, in a coordinate patch whose origin will represent the singularity, thus mimicking (locally) the situation of $K3$. Notice, in particular, that in this way the 7-brane type (see also below) is well defined via its monodromy around the local singularity.

In the geometry of an elliptically fibered Calabi-Yau fourfold Z_4, consider the neighborhood of a point where a group of collapsed 7-branes sits and the elliptic fiber degenerates. As has been just explained, one can limit himself to a neighborhood represented by the local coordinates X, Y, σ, in analogy with section 4.3. The singularity is supposed to be located at $\sigma = 0$, where σ represents a coordinate transverse to the bunch of branes. In this sense the geometric environment is locally similar to the compactification on a $K3$ surface, with σ replacing the local coordinate u on $\mathbb{P}^1$. The only difference is that here the picture cannot be made global in a trivial way, because, in general, the space normal to the singularity has not the topology of a 2-sphere with punctures. But this is sufficient for the construction of this chapter.

For practical reasons the notation of [49] will be adopted. As in [47, 49], three types of (p,q)7-branes will be introduced, the so called A-B-C branes already mentioned in subsection 4.3.2. They are chosen in the following way: the A-type brane is an ordinary D7-brane, the B-brane a (1,-1)-brane and finally the C-brane a (1,1)-brane. Thus, choosing the axiodilaton τ to lie in the standard fundamental region F (fig. 4.1) of the quantum moduli space of elliptic curves, in order to take into account the backreaction of such (p,q)7-branes, one is forced to allow discontinuities for τ across their branch cuts, with the following jumps (the cuts being crossed counterclockwise):

$$\begin{aligned} A = [1,0]: &\qquad K_A \equiv M_{1,0}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}; \\ B = [1,-1]: &\qquad K_B \equiv M_{1,-1}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}; \\ C = [1,1]: &\qquad K_C \equiv M_{1,1}^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{5.1}$$

As it is manifest, the discontinuity of τ is just the inverse transformation of the monodromy.

To describe the geometry the brane configuration will be deformed, by separating the branes by a slight amount, so that, afterwards, all the branes will lie at different points of the σ plane near $\sigma = 0$. In order to keep track of the $SL(2,\mathbb{Z})$ transformation properties of the branes and strings, cuts will be drawn in the neighborhood of $\sigma = 0$ in the σ-plane, starting from the branes and going to 'infinity', where 'infinity' is a conventional point where all the cuts end. For definiteness let the cuts go upward. As explained in [47, 49], an (r,s)-string crossing the cut counterclockwise will appear beyond the cut as the string $M_{p,q}^{-1}\begin{pmatrix} r \\ s \end{pmatrix}$, with the monodromy matrix defined in (4.11). By dragging the string down the cut through the point where the brane sits, it will undergo an U-dual version of the Hanany-Witten effect [20]: a third string prong will develop, starting from the brane and joining the string in such a way that at the triple junction the charges are conserved. That is, the finite prong will have charges $(M_{p,q}^{-1} - 1)\begin{pmatrix} r \\ s \end{pmatrix}$.

The above are the basics about junctions. The authors of [49] were able to show that junctions generate a lattice. Consider a junction $\mathbf{J}$, with endpoints on different branes and possibly at infinity. Let b denote a brane index. Then one associates to each brane the charge

$$Q^b(\mathbf{J}) = n_+ - n_- + \sum_{k=1}^{n_b} \begin{vmatrix} r_k & p_b \\ s_k & q_b \end{vmatrix} , \tag{5.2}$$

where n_+ is the number of $\begin{pmatrix} p_b \\ q_b \end{pmatrix}$ prongs departing from the b brane, and n_- is the number of $\begin{pmatrix} p_b \\ q_b \end{pmatrix}$ prongs ending on the b brane. Moreover n_b, a nonnegative integer, is the number of intersections of $\mathbf{J}$ with the cut starting at the b brane and $\begin{pmatrix} r_k \\ s_k \end{pmatrix}$ are the charges of the strings belonging to $\mathbf{J}$ that cross the cut at the k-th intersection in a counterclockwise direction. The charge Q^b can be shown to be invariant under the cut crossing above. Of course there is also a charge associated to the point at infinity. It will be called the *asymptotic* charge.

Now for a brane with label b and type $[p_b, q_b]$, the outgoing $\begin{pmatrix} p_b \\ q_b \end{pmatrix}$ string starting at the brane and going to infinity will be denoted $\mathbf{s}_b$. This is a very simple case of junction whose charges are $Q^a(\mathbf{s}_b) = \delta^a_b$. Moreover, given two junctions $\mathbf{J}_1$ and $\mathbf{J}_2$, their sum is naturally defined as the junction with charges

$$Q^a(\mathbf{J}_1 + \mathbf{J}_2) = Q^a(\mathbf{J}_1) + Q^a(\mathbf{J}_2) .$$

These rules define a lattice in which one can introduce a scalar product as follows: for an $\mathbf{s}$ elementary prong defined above one has

$$< \mathbf{s}, \mathbf{s} >= -1 , \tag{5.3}$$

and for a three strings junction $\mathbf{J}_3$ one has

$$< \mathbf{J}_3, \mathbf{J}_3 >= \begin{vmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{vmatrix} , \tag{5.4}$$

where i is an integer mod 3. It is easy to see that this definition is independent of i. These rules define a (in general degenerate) metric in the junction lattice. For instance if one has n branes of type A, one brane of type B and one of type C, the corresponding elementary prongs $\mathbf{a}_i$ (i=1,. . . ,n), $\mathbf{b}$ and $\mathbf{c}$ departing from them, have the following metric (the missing entries are zero):

$$\begin{aligned} \langle \mathbf{a}_i, \mathbf{a}_j \rangle &= -\delta_{ij} , \\ \langle \mathbf{a}_i, \mathbf{b} \rangle &= -1/2 , \\ \langle \mathbf{a}_i, \mathbf{c} \rangle &= 1/2 , \\ \langle \mathbf{b}, \mathbf{b} \rangle &= -1 , \\ \langle \mathbf{c}, \mathbf{c} \rangle &= -1 , \\ \langle \mathbf{b}, \mathbf{c} \rangle &= 1 . \end{aligned} \tag{5.5}$$

Armed with these tools the authors of [49], by simply selecting the junctions of given length and vanishing asymptotic charge, were able to identify the junctions that correspond to all the roots of the simply-laced Lie algebras. Explicit examples are recalled below, but, especially, the purpose here will be to single out the combinations of these roots which are invariant under the symmetry (if any) of the relevant Dynkin diagram in order to extract the roots of the corresponding non-simply-laced Lie algebras. In this way the root system of the $\mathbf{B_n}$ and $\mathbf{C_n}$ series, and of $\mathbf{F_4}$ and $\mathbf{G_2}$ will be constructed in terms of string junctions. In addition all such roots are shown to be given either in terms of junctions or in terms of Jordan strings (that is, string prongs without three or higher order string mergings). Moreover, the results will be interpreted in a physical perspective in terms of branes and their orientifold images, fractional (involution invariant) branes and string stretching among them.

5.2 Orthogonal Lie algebras

The $\mathbf{D_n} = \mathbf{so(2n)}$ ($n \geq 4$) algebras are constructed out of n A-branes, one B-brane and one C-brane.
The $\mathbf{B_{n-1}} = \mathbf{so(2n-1)}$ ($n \geq 4$) algebras are, instead, $\mathbb{Z}_2$ folding of $\mathbf{D_n}$ (the last two simple roots are identified) and it will be shown how this procedure is physically interpreted by means of a resolution of type $\mathbf{I^*_{n-4}}$ Kodaira singularity.

5.2.1 so(2n) algebras

Let us first review the construction of the $\mathbf{D_n}$ algebras, following the procedure of [49]. The $\mathbf{so(2n)}$ algebras are constructed with n $\mathbf{a}$-type prongs $\mathbf{a}_i$, $i = 1, \ldots, n$, a $\mathbf{b}$ prong and a $\mathbf{c}$ prong. So the relevant vector space in this case is $\mathbb{R}^{n+2}$, spanned by $\{\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}, \mathbf{c}\}$. The roots are the following:

$$\begin{aligned} &\pm(\mathbf{a}_i - \mathbf{a}_j) && 1 \leq i < j \leq n\,, \\ &\pm(\mathbf{a}_i + \mathbf{a}_j - \mathbf{b} - \mathbf{c}) && 1 \leq i < j \leq n\,. \end{aligned} \tag{5.6}$$

They are $2n^2 - 2n$. Counting the n zeros corresponding to Cartan generators, whose open string origin is not manifest, makes $2n^2 - n$, that is the dimension of the $\mathbf{so(2n)}$ algebra. The meaning of the off-diagonal generators is very clear in terms of strings and orientifolds. First of all, as one can see from (5.6), there is no charge left at infinity by these states (no asymptotic charge). Moreover, they all have the same length (the squared norm is equal to -2, computed by means of (5.5)), as it should be for simply-laced algebras. Finally, looking at the coefficients in (5.6), one notes that:

- the root $(\mathbf{a}_i - \mathbf{a}_j)$ just corresponds to the standard string stretching from the i-th A-brane to the j-th one;

- the root $(\mathbf{a}_i + \mathbf{a}_j - \mathbf{b} - \mathbf{c})$ corresponds to a string departing from the i-th A-brane, going across the branch cuts of the B-brane and of the C-brane and eventually “ending” on the j-th A-brane, but with reversed orientation (so it is also departing from the j-th A-brane). In fact the effect of $K_C K_B$ on a fundamental string is to reverse its sign. These are non-orientable strings and, as it is well-known, they have, at the massless level, antisymmetric Chan-Paton factors. Thus the case $(\mathbf{a}_i + \mathbf{a}_i - \mathbf{b} - \mathbf{c})$ is not included because it would correspond to non-orientable strings, which would be massive even in the collapsing limit. Therefore C and B can be thought of as the constituents of a non-perturbative bound state, corresponding to the *orientifold* $O7^-$ of the perturbative theory of the D7’s, and $(\mathbf{a}_i + \mathbf{a}_j - \mathbf{b} - \mathbf{c})$ junctions realize the expected antisymmetric Chan-Paton factors. On the covering space of this $\mathbb{Z}_2$-orbifold, such twisted states simply lift to strings stretching between a brane and the mirror image of another brane. This consideration allows one to write:

$$\bar{\mathbf{a}}_i \equiv \mathbf{b} + \mathbf{c} - \mathbf{a}_i \tag{5.7}$$

defined as the asymptotic string departing from the orientifold image of the i-th A-brane. It has the correct asymptotic charge, the right squared length of a normal $\mathbf{a}$-prong and vanishing scalar product with $\{\mathbf{a}_j\}_{j\neq i}$, as it is easy to verify. In this way the root $(\mathbf{a}_i + \mathbf{a}_j - \mathbf{b} - \mathbf{c})$ becomes $(\mathbf{a}_i - \bar{\mathbf{a}}_j)$, thus representing the familiar string departing from the i-th brane and ending on the image of the j-th one.

All the roots constructed above are represented by string-junctions with vanishing asymptotic charge and all have the same squared length (-2), as it should be for a simply-laced Lie Algebra.

Finally, in order to visualize the folding of the **so(2n)** algebra, its simple roots are written here:

$$\alpha_i = \mathbf{a}_i - \mathbf{a}_{i+1}\,, \qquad i = 1, \ldots, n-1\,, \quad \text{and} \quad \alpha_n = \mathbf{a}_{n-1} - \bar{\mathbf{a}}_n\,. \tag{5.8}$$

5.2.2 so(2n-1) algebras

As already said, these algebras are obtained from the previous ones by identifying the last two simple roots in (5.8), which are exchanged by the $\mathbb{Z}_2$ outer automorphism of the **so(2n)** algebra. From the point of view of the 7-branes, one can achieve this by simply identifying the last A-brane with the fractional one, which lies on the orientifold. So let $\mathbf{a}_0 \equiv \mathbf{a}_n$ for the corresponding outgoing asymptotic string; the identification will thus impose the following relation:

$$2\,\mathbf{a}_0 = \mathbf{b} + \mathbf{c}\,. \tag{5.9}$$

Hence the relevant vector space for the **so(2n-1)** algebra is an $\mathbb{R}^{n+1}$ vector subspace of $\mathbb{R}^{n+2}$, defined by (5.9), which by the way is consistent with the fact that the fractional brane is still a D7. Notice, however, that this prong has now norm equal to 0 and also vanishing scalar product with any other vector. Thus one has to set:

$$\begin{aligned}
\langle \mathbf{a}_0, \mathbf{a}_0 \rangle &= 0\,, \\
\langle \mathbf{a}_0, \mathbf{a}_i \rangle &= 0\,, \\
\langle \mathbf{a}_0, \mathbf{b} \rangle &= 0\,, \\
\langle \mathbf{a}_0, \mathbf{c} \rangle &= 0\,.
\end{aligned} \tag{5.10}$$

Some of the roots of **so(2n-1)** are represented by the junctions

$$\begin{aligned}
&\pm(\mathbf{a}_i - \mathbf{a}_j) \qquad 1 \leq i < j \leq n-1\,, \\
&\pm(\mathbf{a}_i - \bar{\mathbf{a}}_j) \qquad 1 \leq i < j \leq n-1\,,
\end{aligned} \tag{5.11}$$

whose physical meaning is identical to the one described in the previous section, since they just correspond to the (n-1)(2n-4) roots of the maximal **so(2n-2)** subalgebra. The remaining roots are:

$$\pm(\mathbf{a}_i - \mathbf{a}_0) \;\approx\; \pm(\mathbf{a}_i - \bar{\mathbf{a}}_0) \qquad i = 1, \ldots n-1\,. \tag{5.12}$$

These correspond instead to strings stretching from the A-branes to the fractional brane sitting on top of the orientifold (both the orientations are possible). The equivalence is due to the invariance of the fractional brane under the orientifold involution, which means, as stated in (5.9), $\mathbf{a}_0 = \bar{\mathbf{a}}_0$ for the corresponding asymptotic string. A further comment is in order: due to the vanishing norm of the fractional brane, these states have now squared length equal to -1. It is clear then that they correspond to the short roots of the non-simply-laced algebra $\mathbf{B}_{n-1}$. Altogether these are $(n-1)(n-2) + (n-1)(n-2) + 2(n-1) = (n-1)(2n-2)$ non-zero roots. They fill up the root set of **so(2n-1)**. Counting $n-1$ zeros corresponding to the Cartan subalgebra this yields the dimension of **so(2n-1)**.

The simple roots of **so(2n-1)** are:

$$\alpha_i = \mathbf{a}_i - \mathbf{a}_{i+1}\,, \qquad i = 1, \ldots, n-2 \quad \text{and} \quad \alpha_{n-1} = \mathbf{a}_{n-1} - \mathbf{a}_0\,. \tag{5.13}$$

Therefore the roots α_i, $i = 1, \ldots, n-2$ are long, while α_{n-1} is short. The Cartan matrix of $\mathbf{B_{n-1}}$ is easily recovered, using the scalar product (5.5)[1]:

$$\begin{aligned}
\langle \alpha_i, \alpha_i \rangle &= -2 \qquad && 1 \le i \le n-2\,, \\
\langle \alpha_i, \alpha_{i+1} \rangle &= 1 \qquad && 1 \le i \le n-2\,, \\
\langle \alpha_{n-1}, \alpha_{n-1} \rangle &= -1\,. &&
\end{aligned} \tag{5.14}$$

All in all, in this Lie Algebra there are 2n-2 short roots, while the remaining ones are long, and all are still represented by string junctions with vanishing charge at infinity.

Actually one can say more. The physical meaning of the roots (5.11) and (5.12) will say how they behave under the breaking of the odd orthogonal gauge algebra to the maximal subalgebra that can be realized perturbatively. Suppose one resolves the non-split $\mathbf{I^*_{n-4}}$ Kodaira singularity, the one relevant for the $\mathbf{B_{n-1}}$ algebra, in two groups of 7-branes, one made of $n-1$ A-branes on top of each other, and the other made by the fractional A-brane on top of the CB orientifold. In this way, the manifest *perturbative* subalgebra of **so(2n-1)** will be $\mathbf{su(n-1)} \times \mathbf{u(1)}$. Hence, for the breaking

$$\mathbf{so(2n-1)} \quad \longrightarrow \quad \mathbf{su(n-1)} \times \mathbf{u(1)}\,, \tag{5.15}$$

the branching rule for the adjoint representation is[2]:

$$(n-1)(2n-1) \quad \longrightarrow \quad (n-1)^2 - 1 + 1 + 2 \times \frac{(n-1)(n-2)}{2} + 2 \times (n-1)\,; \tag{5.16}$$

that is, the adjoint of **so(2n-1)** goes into the adjoint plus two copies of the 2-antisymmetric plus two copies of the fundamental of $\mathbf{su(n-1)} \times \mathbf{u(1)}$.
It is very easy now to match this representation content with the roots (5.11), (5.12).

- The first set of roots in (5.11) (and the $n-1$ zeros corresponding to the Cartan generators) fill up the weights of the $(n-1)^2$-dimensional adjoint representation of $\mathbf{su(n-1)} \times \mathbf{u(1)}$, i.e. they correspond to the gauge vectors of the manifest perturbative subalgebra.

[1] All the entries that are not written are vanishing.
[2] The $\mathbf{u(1)}$ charges are not taken into account, as they cannot be detected by an analysis like the present one based on string junctions.

- The second set of roots in (5.11) fill two copies of the 2-antisymmetric representation[3] of $\mathbf{su(n-1)}$, and are therefore responsible of the enhancing of the perturbative subalgebra $\mathbf{su(n-1)} \times \mathbf{u(1)}$ to the maximal subalgebra **so(2n-2)**.

- The roots in (5.12) fill up two copies of the fundamental[4] of $\mathbf{su(n-1)}$, since, as said, they are just the strings stretched between the fractional brane and one of the $n-1$ A-branes in the stack.

5.3 E_6 and F_4

An analogous construction is now made that will lead from a 7-brane model for E_6 to the one corresponding to F_4, since the latter algebra can be viewed as the folding of the former one under the $\mathbb{Z}_2$ automorphism group of its Dynkin diagram. It is worth to start by reviewing the procedure for E_6, following again [49].

5.3.1 The E_6 algebra

E_6 is constructed out of five A-branes, one B-brane and two C-branes. Hence the string realization of the E_6 algebra is based on five prongs $\mathbf{a}_1, \ldots, \mathbf{a}_5$, one prong $\mathbf{b}$ and two prongs $\mathbf{c}_1, \mathbf{c}_2$. Define the images of the $\mathbf{a}$-prongs as follows:

$$\bar{\mathbf{a}}_i^I \equiv \mathbf{b} + \mathbf{c}_I - \mathbf{a}_i \qquad 1 \leq i \leq 5 \quad \text{and} \quad I = 1, 2\,, \tag{5.17}$$

according to which of the two C-branes is taken to make the orientifold with the B-brane. Using the same scalar product as (5.5) in the $\mathbb{R}^8$ generated by $\{\mathbf{a}_1, \ldots, \mathbf{a}_5, \mathbf{b}, \mathbf{c}_1, \mathbf{c}_2\}$, with in addition $\langle \mathbf{c}_I, \mathbf{c}_J \rangle = -\delta_{ij}$, one can see that the definition (5.17) is still compatible with the metric behavior of the $\mathbf{a}$-prongs of (5.7) Some of the roots of E_6 are then as usual identified with the junctions

$$\begin{array}{ll} \pm(\mathbf{a}_i - \mathbf{a}_j) & 1 \leq i < j \leq 5\,, \\ \pm(\mathbf{a}_i - \bar{\mathbf{a}}_j^I) & 1 \leq i < j \leq 5, \quad I = 1, 2\,, \end{array} \tag{5.18}$$

which represent 20+40=60 states. Other obvious states are the two corresponding to the $C-C$ string with both orientations:

$$\pm(\mathbf{c}_1 - \mathbf{c}_2)\,. \tag{5.19}$$

Finally, to identify the remaining 10 roots which complete the 72 roots of the E_6 algebra one has to remember that, in the non-perturbative set of 8 branes we are considering here, there are 2 mutually non-local orientifolds: one is the CB bound state already encountered (that is an orientifold for the A-branes) and the other is the bound state BA^4 (which instead turns out to change the sign of the charge of the C-strings). So now one has additional states represented by strings stretching from a C-brane to its images under the five possible orientifolds one can

[3] More precisely, the 2-antisymmetric and the (n-2)-antisymmetric, since their string representatives have opposite orientations.

[4] More precisely, the fundamental and the (n-1)-antisymmetric (antifundamental), since their string representatives have opposite orientations.

construct this way (i.e. depending on which A-brane one leaves apart in making the bound state). If one defines such images as follows

$$\bar{\mathbf{c}}_2^i \equiv \sum_{\substack{k=1\\k\neq i}}^{5} \mathbf{a}_k - 2\,\mathbf{b} - \mathbf{c}_2 \qquad\qquad 1 \leq i \leq 5\,, \tag{5.20}$$

the new states will take the usual form:

$$\pm(\mathbf{c}_1 - \bar{\mathbf{c}}_2^i) \qquad\qquad 1 \leq i \leq 5\,. \tag{5.21}$$

Notice that the definition (5.20) has in fact the right asymptotic charges for **c**-prongs, as well as their correct squared length and vanishing scalar product with $\mathbf{c}_1$. The factor of 2 in front of the **b**-prong, needed for charge conservation, looks unexpected since, as said, the new orientifold is made of four A-branes and *one* B-brane. However, it has a very easy explanation in terms of prongs of a string-junction. Indeed, it is not difficult to show[5] that the actual number of prongs of the string-junction created by the already mentioned U-dual version of the Hanany-Witten effect[6] out of a (p,q)-string and a (p',q')-brane is:

$$\text{number of prongs} \;=\; 2 + \left| \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \right| . \tag{5.22}$$

When the determinant is zero, nothing happens and one ends up with a 2-pronged string-junction that is just the standard Jordan string. When instead an asymptotic C-string is crossing an A-brane a 3-pronged junction is created, while when it crosses a B-brane a 4-pronged junction appear. Hence, apart from the sign that simply gives the correct orientation of the prong, the absolute value of each coefficient in the linear combination defining the image prongs coincides with the modulus of the determinant of the corresponding matrix of charges.
Again, as for the **so(2n)** case, all the roots constructed above are string-junctions with vanishing asymptotic charge and all have the same squared length (-2), as it should be for a simply-laced Lie algebra.

In order to visualize the folding of the E_6 algebra, its simple roots are written here:

$$\begin{aligned} &\alpha_1 = \mathbf{a}_1 - \mathbf{a}_2\,, \qquad \alpha_2 = \mathbf{a}_2 - \mathbf{a}_3\,, \qquad \alpha_3 = \mathbf{a}_3 - \mathbf{a}_4\,, \\ &\alpha_4 = \mathbf{a}_4 - \bar{\mathbf{a}}_5^1\,, \qquad \alpha_5 = \mathbf{c}_1 - \mathbf{c}_2\,, \qquad \alpha_6 = \mathbf{a}_4 - \mathbf{a}_5\,, \end{aligned} \tag{5.23}$$

so that it is manifest an **so(10)** subalgebra with simple roots $\{\alpha_i\}_{i\neq 5}$.

5.3.2 The F_4 algebra

F_4 is algebraically generated by folding the E_6 Dynkin diagram under its $\mathbb{Z}_2$ symmetry group. Acting on the simple roots in (5.23), this symmetry maps $\alpha_1 \to \alpha_5$, $\alpha_2 \to \alpha_4$, while leaving α_3 and α_6 unchanged. In terms of F-theory strings, this is generated by the prong correspondences

$$\begin{aligned} \mathbf{a}_1 &\longrightarrow \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_2\,, \\ \mathbf{a}_2 &\longrightarrow \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_1\,, \\ \mathbf{c}_1 &\longrightarrow \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{a}_2\,, \\ \mathbf{c}_2 &\longrightarrow \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{a}_1\,, \end{aligned} \tag{5.24}$$

[5] One uses the fact that fundamental strings can end on D1 branes and a generic S-duality rotation.

[6] This process, occurring when a string crosses a 7-brane, can be thought of as a movement in the region of the moduli space given by the positions of the 7-branes.

while $\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{b}$ remain unchanged.

Now, as proceeded for the odd orthogonal algebras, one has to take the quotient of the $\mathbb{R}^8$ vector space one started with to construct E_6 by the equivalence relations implied by this $\mathbb{Z}_2$ symmetry. In the previous **so(2n-1)** case, there was just one condition to impose on the vector space of the parent **so(2n)** algebra, namely the fractionality of the last A-brane with respect to the only one orientifold present. In this case, instead, the correspondences (5.24) amount to *two* independent identifications[7], which can be expressed either for the first two A-branes or for the two C-branes. Choosing for example the first formulation, according to the given definitions of the images (5.17) and (5.20) one gets:

$$\mathbf{a}_I \;\approx\; \bar{\mathbf{c}}_J^J + \mathbf{b} - \mathbf{a}_I \qquad I = 1,2 \quad I \neq J\,. \tag{5.25}$$

Physically speaking, this means that the A_I-brane becomes fractional with respect to the *non-perturbative* orientifold $\tilde{C}_J^J B$, whose C-type brane is itself in turn the image of the C_J-brane under the relatively non-local orientifold $BA_5A_4A_3A_I$. Equivalently, the same correspondences amount to identify both the C-branes with their images with respect to suitable non-perturbative orientifolds.
Hence the relevant vector space for the F_4 algebra will be:

$$\frac{\mathrm{Span}\,\{\mathbf{a}_1,\ldots,\mathbf{a}_5,\mathbf{b},\mathbf{c}_1,\mathbf{c}_2\}}{\{\mathbf{a}_I \approx \bar{\mathbf{c}}_J^J + \mathbf{b} - \mathbf{a}_I\}_{\substack{I=1,2\\ I\neq J}}} \;\simeq\; \mathbb{R}^6\,. \tag{5.26}$$

In order to write down the string-junctions representing the roots of the non-simply laced algebra just constructed, one can proceed in two steps:

- single out the roots of the parent E_6 algebra that are not touched by the $\mathbb{Z}_2$ symmetry, which, therefore, remain with the same squared length: these will be the analogs of the roots (5.11) of the manifest **so(2n-2)** subalgebra of **so(2n-1)**;
- build up singlets under the $\mathbb{Z}_2$ symmetry, by taking linear combinations of the vectors, to find out the remaining roots, which therefore will have double the length of the previous ones: working on the covering space of this symmetry with the invariant combinations turns out to be for F_4 more straightforward[8] than expressing the roots in coordinates of the quotient, as it has been done for the **so(2n-1)** case.

It is useful to start by looking at the simple roots (5.23) of E_6. The first of the rules above says that α_3 and α_6 become the short simple roots of F_4, since they are left unchanged by the $\mathbb{Z}_2$ symmetry; the second, instead, prescribes to take as long simple roots of F_4 the two invariant combinations out of the remaining four simple roots of E_6 that are pairwise exchanged by $\mathbb{Z}_2$: these are clearly $\alpha_1 + \alpha_5$ and $\alpha_2 + \alpha_4$. Thus, the simple roots of F_4 will be:

$$\begin{aligned} \alpha_1 &= \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{c}_1 - \mathbf{c}_2\,, \qquad & \alpha_2 &= \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4 - \bar{\mathbf{a}}_5^1\,, \\ \alpha_3 &= \mathbf{a}_3 - \mathbf{a}_4\,, & \alpha_4 &= \mathbf{a}_4 - \mathbf{a}_5\,. \end{aligned} \tag{5.27}$$

[7]This, by the way, is consistent with the rank being lowered by two instead of one.

[8]But we will pay the price of missing the physical picture of these states as strings stretching between a regular and a fractional brane.

The first two are long (squared length -4), the last two short (squared length -2). Using the scalar product (5.5) with one more $\mathbf{c}$-prong (with $\langle \mathbf{c}_I, \mathbf{c}_J \rangle = -\delta_{ij}$) and the definition (5.17), one gets for the Cartan matrix[9]:

$$\begin{aligned} \langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle &= -4\,, & \langle \alpha_3, \alpha_3 \rangle = \langle \alpha_4, \alpha_4 \rangle &= -2\,, \\ \langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_3 \rangle &= 2\,, & \langle \alpha_3, \alpha_4 \rangle &= 1\,. \end{aligned} \tag{5.28}$$

This is consistent with [73], vol. II, App.F, ch.8 (changing sign).

One can now write down explicitly all the roots of F_4. Like for the simple roots, of the 72 roots of E_6 one third passes directly to the quotient without any change (the short roots of step one above); these are immediately recognized among the roots of (5.18) and (5.21):

$$\begin{aligned} &\pm(\mathbf{a}_i - \mathbf{a}_j) && 3 \le i < j \le 5\,, \\ &\pm(\mathbf{a}_I - \bar{\mathbf{a}}_i^J) && 3 \le i \le 5 \qquad I = 1,2 \quad I \neq J\,, \\ &\pm(\mathbf{c}_1 - \bar{\mathbf{c}}_2^i) && 3 \le i \le 5\,. \end{aligned} \tag{5.29}$$

They are 24 and explicitly look like:

$$\begin{aligned} &\pm(\mathbf{a}_3 - \mathbf{a}_4)\,, && \pm(\mathbf{a}_1 + \mathbf{a}_3 - \mathbf{b} - \mathbf{c}_2)\,, \\ &\pm(\mathbf{a}_4 - \mathbf{a}_5)\,, && \pm(\mathbf{a}_1 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c}_2)\,, \\ &\pm(\mathbf{a}_3 - \mathbf{a}_5)\,, && \pm(\mathbf{a}_1 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_2)\,, \\ &\pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_4 + \mathbf{a}_5 - 2\,\mathbf{b} - \mathbf{c}_1 - \mathbf{c}_2)\,, && \pm(\mathbf{a}_2 + \mathbf{a}_3 - \mathbf{b} - \mathbf{c}_1)\,, \\ &\pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_5 - 2\,\mathbf{b} - \mathbf{c}_1 - \mathbf{c}_2)\,, && \pm(\mathbf{a}_2 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c}_1)\,, \\ &\pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 - 2\,\mathbf{b} - \mathbf{c}_1 - \mathbf{c}_2)\,, && \pm(\mathbf{a}_2 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_1)\,. \end{aligned} \tag{5.30}$$

Of the remaining two thirds of the E_6 roots, only a half survives to the quotient, namely the 24 singlet combinations (the long roots of step two above), and they are written directly in the explicit form:

$$\begin{aligned} &\pm(\mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_2)\,, \quad \pm(\mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_1)\,, \\ &\pm(\mathbf{a}_1 + \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_2)\,, \quad \pm(\mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_1)\,, \\ &\pm(\mathbf{a}_1 + \mathbf{a}_3 + \mathbf{a}_4 - \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_2)\,, \quad \pm(\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 - \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_1)\,, \\ &\pm(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{c}_1 - \mathbf{c}_2)\,, \quad \pm(\mathbf{a}_1 + \mathbf{a}_2 + 2\,\mathbf{a}_3 - 2\,\mathbf{b} - \mathbf{c}_1 - \mathbf{c}_2)\,, \\ &\qquad \pm(\mathbf{a}_1 + \mathbf{a}_2 + 2\,\mathbf{a}_4 - 2\,\mathbf{b} - \mathbf{c}_1 - \mathbf{c}_2)\,, \\ &\qquad \pm(\mathbf{a}_1 + \mathbf{a}_2 + 2\,\mathbf{a}_5 - 2\,\mathbf{b} - \mathbf{c}_1 - \mathbf{c}_2)\,, \\ &\qquad \pm(2\,\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - 3\,\mathbf{b} - \mathbf{c}_1 - 2\,\mathbf{c}_2)\,, \\ &\qquad \pm(\mathbf{a}_1 + 2\,\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - 3\,\mathbf{b} - 2\,\mathbf{c}_1 - \mathbf{c}_2)\,. \end{aligned} \tag{5.31}$$

All in all we have 24 short + 24 long = 48 roots, still represented by string junctions with vanishing charge at infinity. Adding the four zeros corresponding to the Cartan generators yields a total of 52, the dimension of F_4.

[9] The entries that are not written are vanishing.

With the labeling used for the simple roots (5.27), the roots (5.30) and (5.31) coincide with the roots of [73], vol. II, App.F, ch.8.
There are no problems even concerning the last roots of (5.31). For instance, for the very last one:

$$\mathbf{a}_1 + 2\,\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - 3\,\mathbf{b} - 2\,\mathbf{c}_1 - \mathbf{c}_2 \quad = \quad 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + \alpha_4\,.$$

What has been said so far simply means that the roots of the Lie algebra $\mathbf{F_4}$ can be constructed in terms of junctions, i.e. the folding of $\mathbf{E_6}$ leads again to string junctions. It remains to understand the origin of the $\mathbb{Z}_2$ symmetry of $\mathbf{E_6}$. To this end one has to unravel the meaning of the transformations (5.24). The E_6 singularity (IV^{*s}, in Kodaira classification) will be resolved by arranging the 8 branes, for instance, as follows: take a group formed by $BA_3A_4A_5$ at the center, then A_1A_2 on the left and C_1C_2 on the right, with the relevant cuts going upward. Looking at the first of (5.24), the $\mathbf{a}_1$ on the left is just the usual elementary prong going downward to infinity. The junction on the right ($\mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_2$) is also going to infinity and its asymptotic charge is the same as for $\mathbf{a}_1$. This junction can be easily undone and represented by a Jordan string that ends on C_2 coming from the left, after having crossed the cuts of B, A_5, A_4 and A_3. This is a Jordan string that, after the crossings, has the charge of a fundamental string. Indeed one can easily verify that $K_B K_A^3 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In other words, looking from the left through the screen formed by $BA_3A_4A_5$ at a string ending on C_2, one sees a fundamental string. A similar construction holds for the second transformation in (5.24) with A_1, C_2 exchanged with A_2, C_1 (see fig. 5.1).

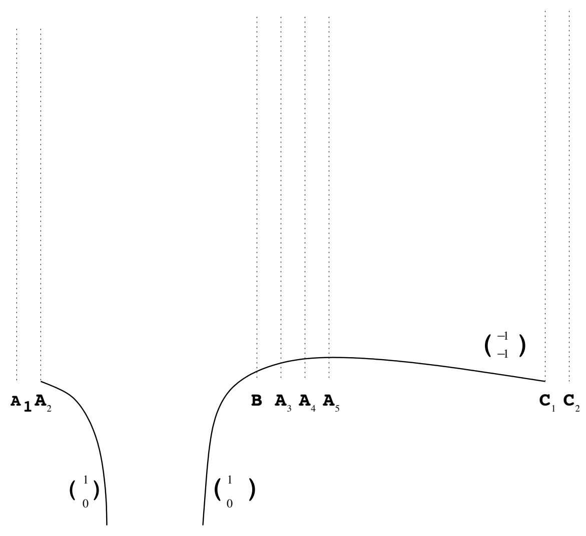

Figure 5.1: The Jordan strings representing the junctions $\mathbf{a}_2$ and $\mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{c}_1$.

Consider next the third transformation in (5.24). In this case the $\mathbf{c}_1$ prong on the left is the usual elementary prong departing from C_1 and going down to infinity. The junction $\mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{a}_2$ on the right can be undone and replaced by a string ending on A_2 and crossing backward successively the cuts of B, A_3, A_4, A_5, and emerging behind the $BA_3A_4A_5$ screen as a $\mathbf{c}$ prong that goes down to infinity. In other words, looking from the right through the $BA_3A_4A_5$ screen, one sees **c**-strings instead of the original (oppositely oriented) fundamental strings (see fig. 5.2). Likewise for the fourth transformation in (5.24) with A_2, C_1 exchanged with A_1, C_2. The conclusion is that the screen formed by $BA_3A_4A_5$ changes fundamental strings to **c**-strings while reversing the orientation, and viceversa. The fact that fundamental strings can be seen as oppositely oriented **c**-strings and viceversa, creates a $\mathbb{Z}_2$ symmetry among the roots of $\mathbf{E}_6$. This symmetry is only evident when B, A_3, A_4, A_5 collapse before the other branes, and A_1, A_2 and C_1, C_2 collapse symmetrically with respect to the $BA_3A_4A_5$ screen. The orbifold with respect to this $\mathbb{Z}_2$ symmetry gives rise, in the collapsing limit, to $\mathbf{F_4}$. This is the F-theory string description of the $\mathbf{E}_6$ folding to $\mathbf{F_4}$.

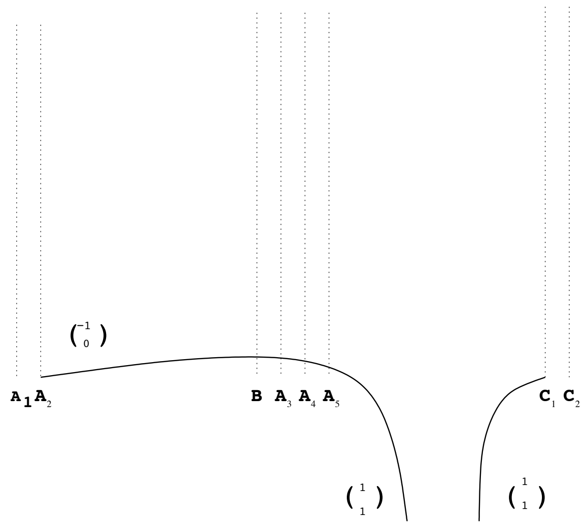

Figure 5.2: The Jordan strings representing the junctions $\mathbf{c}_1$ and $\mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{b} - \mathbf{a}_2$.

5.4 The $\mathbf{G_2}$ algebra

It is possible to carry out the same procedure for the $\mathbf{G_2}$ algebra that comes from the $\mathbf{so(8)}$ one via a triple folding under the extended outer automorphism group of $\mathbf{D_4}$ (due to its triality). It is worth, then, to briefly review the root structure of this parent algebra.

As said in section 5.2, the $\mathbf{so(8)}$ algebra is constructed with four **a**-prongs, one **b**-prong and one **c**-prong. The relevant vector space is an $\mathbb{R}^6$ generated by $\{\mathbf{a}_1, \ldots, \mathbf{a}_4, \mathbf{b}, \mathbf{c}\}$ and the roots are:

$$\begin{aligned} &\pm(\mathbf{a}_i - \mathbf{a}_j) && 1 \leq i < j \leq 4\,, \\ &\pm(\mathbf{a}_i + \mathbf{a}_j - \mathbf{b} - \mathbf{c}) && 1 \leq i < j \leq 4\,. \end{aligned} \tag{5.32}$$

They are 24. Adding the 4 Cartan generators makes 28 dimensions.
The simple roots are:

$$\begin{aligned} \alpha_1 &= \mathbf{a}_1 - \mathbf{a}_2\,, & \alpha_2 &= \mathbf{a}_2 - \mathbf{a}_3\,, \\ \alpha_3 &= \mathbf{a}_3 - \mathbf{a}_4\,, & \alpha_4 &= \mathbf{a}_3 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c}\,. \end{aligned} \tag{5.33}$$

The symmetries of the $\mathbf{D_4}$ Dynkin diagram are the ones of the equilateral triangle, namely they form a group $\mathbf{T^3}$ made of the $\mathbb{Z}_3$ rotations of the roots $\alpha_{1,3,4}$, together with the three reflections:

$$\begin{array}{lll} \tau_1(\alpha_{1,4}) = \alpha_{4,1}\,, & \tau_1(\alpha_i) = \alpha_i\,, & i = 2,3\,, \\ \tau_2(\alpha_{3,4}) = \alpha_{4,3}\,, & \tau_2(\alpha_i) = \alpha_i\,, & i = 1,2\,, \\ \tau_3(\alpha_{1,3}) = \alpha_{3,1}\,, & \tau_3(\alpha_i) = \alpha_i\,, & i = 2,4\,. \end{array} \tag{5.34}$$

For instance, by folding $\mathbf{D_4}$ under τ_2 alone, the algebra $\mathbf{B_3}$, corresponding to $\mathbf{so(7)}$ has been obtained in section 5.2. As far as $\mathbf{G_2}$ is concerned, instead, one needs all the reflections (actually just two of them will be enough it will be shown), but one can disregard the invariance under the rotations, since the latter are simply products of two reflections.

Hence, in terms of string-junctions the reflections (5.34) are generated by

$$\begin{aligned} \tau_1(\mathbf{a}_1) &= \frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c})\,, \\ \tau_1(\mathbf{a}_2) &= \frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{b} + \mathbf{c})\,, \\ \tau_1(\mathbf{a}_3) &= \frac{1}{2}(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{b} + \mathbf{c})\,, \\ \tau_1(\mathbf{a}_4) &= \frac{1}{2}(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{b} + \mathbf{c})\,, \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} \tau_3(\mathbf{a}_1) &= \frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4)\,, \\ \tau_3(\mathbf{a}_2) &= \frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4)\,, \\ \tau_3(\mathbf{a}_3) &= \frac{1}{2}(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4)\,, \\ \tau_3(\mathbf{a}_4) &= \frac{1}{2}(-\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4)\,, \end{aligned} \tag{5.36}$$

while, as already known (compare with (5.9)),

$$\tau_2(\mathbf{a}_4) = \mathbf{b} + \mathbf{c} - \mathbf{a}_4\,, \qquad \tau_2(\mathbf{a}_i) = \mathbf{a}_i\,, \quad i = 1,2,3\,, \tag{5.37}$$

and in any case $\mathbf{b}$ and $\mathbf{c}$ are left unchanged

$$\tau_i(\mathbf{b}) = \mathbf{b}\,, \qquad \tau_i(\mathbf{c}) = \mathbf{c}\,, \qquad i = 1,2,3\,.$$

First of all, notice that only two independent constraints on $\mathbb{R}^6$ are imposed by the joint action of these three reflections, which is consistent with the rank being lowered by two units. Indeed, using the usual definition for the images (5.7), the correspondences (5.35), (5.36) and (5.37) amount to the following identifications:

$$\begin{aligned} \tau_1 &\implies \bar{\mathbf{a}}_4 \approx \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_1\,, \\ \tau_2 &\implies \mathbf{a}_4 \approx \bar{\mathbf{a}}_4\,, \\ \tau_3 &\implies \mathbf{a}_4 \approx \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_1\,. \end{aligned} \tag{5.38}$$

One soon recognizes in the second constraint above the fractional nature of the fourth A-brane and it can be immediately seen that one of the three identifications is not independent of the other two. Thus, the relevant vector space for the $\mathbf{G_2}$ algebra will be given by the following quotient:

$$\frac{\mathrm{Span}\,\{\mathbf{a}_1,\ldots,\mathbf{a}_4,\mathbf{b},\mathbf{c}\}}{\{\mathbf{a}_4 \approx \bar{\mathbf{a}}_4 \approx \mathbf{a}_2+\mathbf{a}_3-\mathbf{a}_1\}} \;\simeq\; \mathbb{R}^4\,. \tag{5.39}$$

As in the previous cases, one can now proceed to the explicit construction of the roots. By looking at the 4 simple roots of $\mathbf{so(8)}$, one readily notices that just one of them, α_2 is not touched at all by any of the elements of the triality group $\mathbf{T^3}$ (as it corresponds to the middle node in the $\mathbf{D_4}$ Dynkin diagram): thus it corresponds to a short root (first step) and it passes to the quotient keeping its squared length equal to -2. The remaining three simple roots of $\mathbf{so(8)}$ are pairwise exchanged by the $\{\tau_i\}_{i=1,2,3}$, so that there exists clearly only one invariant linear combination of them, namely $\alpha_1+\alpha_3+\alpha_4$: this corresponds to a long root (second step) and, as such, it survives to the quotient but it has three times the squared length of the previous one, i.e. -6. Hence, the simple roots of $\mathbf{G_2}$ will be:

$$\beta_1 \equiv \mathbf{a}_1-\mathbf{a}_2+2\mathbf{a}_3-\mathbf{b}-\mathbf{c}\,, \qquad \beta_2 \equiv \mathbf{a}_2-\mathbf{a}_3\,. \tag{5.40}$$

Using the usual scalar product (5.5), it is easy to find out the Cartan matrix of the $\mathbf{G_2}$ algebra:

$$\langle\beta_1,\beta_1\rangle = -6\,, \qquad \langle\beta_2,\beta_2\rangle = -2\,, \qquad \langle\beta_1,\beta_2\rangle = 3\,.$$

It is now possible to write down explicitly all the roots of $\mathbf{G_2}$. As seen for the simple roots, of the 24 roots of the parent $\mathbf{D_4}$ one fourth of them passes directly to the quotient without any change (short roots):

$$\pm(\mathbf{a}_2-\mathbf{a}_3)\,, \qquad \pm(\mathbf{a}_1-\bar{\mathbf{a}}_2)\,, \qquad \pm(\mathbf{a}_1-\bar{\mathbf{a}}_3)\,. \tag{5.41}$$

They are 6 and explicitly look like:

$$\begin{aligned} \pm\beta_2 &= \pm(\mathbf{a}_2-\mathbf{a}_3)\,, \\ \pm(\beta_1+2\beta_2) &= \pm(\mathbf{a}_1+\mathbf{a}_2-\mathbf{b}-\mathbf{c})\,, \\ \pm(\beta_1+\beta_2) &= \pm(\mathbf{a}_1+\mathbf{a}_3-\mathbf{b}-\mathbf{c})\,. \end{aligned} \tag{5.42}$$

Of the remaining three fourths of the $\mathbf{D_4}$ roots, only one third survives the quotient, namely the 6 singlet combinations (long roots), and they will be written directly in the explicit form:

$$\begin{aligned} \pm\beta_1 &= \pm(\mathbf{a}_1-\mathbf{a}_2+2\mathbf{a}_3-\mathbf{b}-\mathbf{c})\,, \\ \pm(\beta_1+3\beta_2) &= \pm(\mathbf{a}_1+2\mathbf{a}_2-\mathbf{a}_3-\mathbf{b}-\mathbf{c})\,, \\ \pm(2\beta_1+3\beta_2) &= \pm(2\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3-2\mathbf{b}-2\mathbf{c})\,. \end{aligned} \tag{5.43}$$

All in all one has 6 short + 6 long = 12 roots, still represented by string junctions with vanishing charge at infinity. Adding the 2 zeros corresponding to the Cartan generators makes a total of 14, the dimension of $\mathbf{G_2}$.

Notice that the $\mathbf{a}_4$-prong has disappeared from the roots of $\mathbf{G_2}$. However, this is only apparent. In fact, in the simple root β_1 the junction $\mathbf{a}_1-\mathbf{a}_2$ is easily interpretable, but the junction $2\mathbf{a}_3 - \mathbf{b} - \mathbf{c}$ would represent an A-string starting from the brane A_3, circling around the CB orientifold and returning to the *same* brane with *opposite* orientation, i.e. it would be a non-orientable string. Such a string would be massive in the collapsing limit. The paradox is explained by correctly interpreting β_1 as $\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{a}_3 - \bar{\mathbf{a}}_4$. In this case, all the involved junctions ($\mathbf{a}_1 - \mathbf{a}_2$, $\mathbf{a}_3 - \mathbf{a}_4$ and $\mathbf{a}_3 - \bar{\mathbf{a}}_4$) are massless Jordan strings, the first two orientable, the last not (as viewed in the quotient of the orientifold involution), and the paradox disappears.

So far it has been shown that the folding of the $\mathbf{D}_4$ algebra down to $\mathbf{G}_2$ can be implemented in terms of string junctions. Actually it is possible to show that the symmetry responsible for such folding can be interpreted in a natural way as a symmetry of F-theory string configurations. The transformation (5.37) has already been understood by interpreting a CB bound state as an orientifold. This reduces the transformations that need to be interpreted to the set (5.35). It is easy to see that these transformations can be replaced by the following ones

$$
\begin{aligned}
\tau_1(\mathbf{a}_1 + \mathbf{a}_2) &= \mathbf{a}_1 + \mathbf{a}_2 \,, \\
\tau_1(\mathbf{a}_1 - \mathbf{a}_2) &= \mathbf{a}_3 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c} \,, \\
\tau_1(\mathbf{a}_1 - \mathbf{a}_3) &= \mathbf{a}_2 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c} \,, \\
\tau_1(\mathbf{a}_3 - \mathbf{a}_4) &= \mathbf{a}_3 - \mathbf{a}_4 \,.
\end{aligned} \tag{5.44}
$$

In fact from these one can derive (5.35) and vice-versa. The interpretation of the first and last equations are of course trivial. As for the others, consider the following brane resolution of the relevant I_0^{*s} singularity: the CB block at the center, A_1, A_2 at the left and A_3, A_4 at the right. Then $\mathbf{a}_1 - \mathbf{a}_2$ represents a fundamental string departing from A_1 and ending on A_2, while $\mathbf{a}_3 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c}$ represents a fundamental string departing from A_3 going around the orbifold CB and returning to A_4 with reversed orientation. We know that the latter is the junction $\mathbf{a}_3 - \bar{\mathbf{a}}_4$ which has been already identified by τ_2 with $\mathbf{a}_3 - \mathbf{a}_4$ (see fig. 5.3). The symmetry of this configuration under reflection with respect to the CB block is evident (in this local representation).

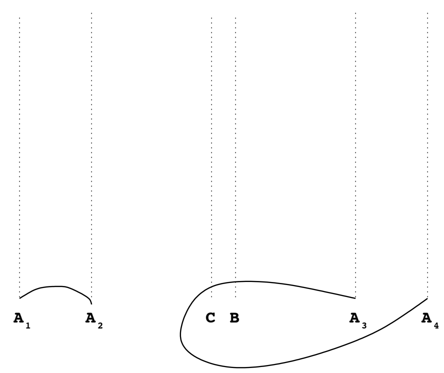

Figure 5.3: The Jordan strings representing the junctions $\mathbf{a}_1 - \mathbf{a}_2$ and $\mathbf{a}_3 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c}$.

Going to the third of the (5.44), the junction $\mathbf{a}_1 - \mathbf{a}_3$ is a fundamental string that departs from A_1 and ends on A_3 without crossing any cut. On the other hand $\mathbf{a}_2 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c}$ is a fundamental string that departs from A_2, crosses the C and B cuts and ends on A_4 with reverse orientation. Again, due to the presence of an orientifold, one identifies A_4 with its mirror image, which means identifying $\mathbf{a}_2 + \mathbf{a}_4 - \mathbf{b} - \mathbf{c}$ with a string stretching from A_2 to A_4 without crossing cuts. Even after these moves, the symmetry of the configuration is not immediately evident. But it is easy to see that using (5.44) and in addition $\tau_1(\mathbf{a}_2 - \mathbf{a}_3) = \mathbf{a}_2 - \mathbf{a}_3$, which also follows from (5.35), one can pass from the second to the third transformation in (5.44). Since the former is a symmetry also the latter is.

5.5 Symplectic algebras

The last non-simply-laced Lie algebras in the classification are the symplectic ones.
The $\mathbf{sp(n)}$ $(n > 1)$ algebra[10] is a $\mathbb{Z}_2$ folding of the $\mathbf{su(2n)}$ algebra under its $\mathbb{Z}_2$ outer automorphism that reflects the nodes of the $\mathbf{A_{2n-1}}$ Dynkin diagram with respect to the central one.

The $\mathbf{A_{2n-1}}$ algebra is obviously realized by means of $\mathbf{a}$-type junctions stretching among $2n$ D7 branes on top of one another (giving rise to $U(2n)$ gauge group). The positive roots of this algebra are:

$$\mathbf{a}_i - \mathbf{a}_j \qquad\qquad 1 \leq i < j \leq 2n\,, \tag{5.45}$$

while the simple ones are:

$$\alpha_i \equiv \mathbf{a}_i - \mathbf{a}_{i+1} \qquad\qquad 1 \leq i \leq 2n-1\,. \tag{5.46}$$

The $\mathbb{Z}_2$ symmetry acts on these simple roots as:

$$\alpha_i \quad \leftrightarrow \quad \alpha_{2n-i}\,, \tag{5.47}$$

so that α_n (corresponding to the central node) remains unchanged. This $\mathbb{Z}_2$ symmetry is realized by the independent correspondences

$$\mathbf{a}_i \quad \leftrightarrow \quad -\mathbf{a}_{2n-i+1}\,. \tag{5.48}$$

As is evident from (5.48), by imposing such constraints one halves the dimension of the $\mathbb{R}^{2n}$ vector space one started with, so that the relevant vector space for the $\mathbf{sp(n)}$ algebra will be the following quotient:

$$\frac{\mathrm{Span}\,\{\mathbf{a}_1, \ldots, \mathbf{a}_{2n}\}}{\{\mathbf{a}_i + \mathbf{a}_{2n-i+1} \approx 0\}_{i=1,\ldots,n}} \quad \simeq \quad \mathbb{R}^n\,. \tag{5.49}$$

Following the example of the previous cases, it is easy to explicitly construct the positive roots. First one has a fraction of $\frac{1}{2n-1}$ that passes directly to the quotient:

$$2\mathbf{a}_i - 2\mathbf{a}_{2n-i+1} \qquad\qquad 1 \leq i \leq n\,, \tag{5.50}$$

which descend straight from the invariant roots of $\mathbf{su(2n)}$. They are long roots. Of the remaining $\frac{2n-2}{2n-1}\cdot n(2n-1) = 2n(n-1)$ positive roots of $\mathbf{A_{2n-1}}$, only a half survives to the quotient, namely the singlet combinations. They are:

$$\mathbf{a}_i - \mathbf{a}_j + \mathbf{a}_{2n-j+1} - \mathbf{a}_{2n-i+1} \qquad\qquad 1 \leq i < j < 2n \quad i+j \leq 2n\,. \tag{5.51}$$

They are $n(n-1)/2 + \sum_{i=1}^{n-1} i = n(n-1)$ long positive roots.
Altogether they are n^2, still represented by string junctions with vanishing charge at infinity.

[10] Also here it is adopted the convention for which n stands for the rank of the algebra. So $sp(2) \sim so(5)$.

To these one must add the n^2 negative roots and the n Cartan generators of the $\mathbf{C_n}$ algebra. In total one gets $n(2n+1)$, which is the dimension of $\mathbf{sp(n)}$.

It is easy to see that a set of simple roots is

$$\begin{aligned}\gamma_i &\equiv \mathbf{a}_i - \mathbf{a}_{i+1} + \mathbf{a}_{2n-i} - \mathbf{a}_{2n-i+1} \qquad 1 \le i \le n-1\,, \\ \gamma_n &\equiv 2\mathbf{a}_n - 2\mathbf{a}_{n+1}\,.\end{aligned} \tag{5.52}$$

These are $n-1$ short and 1 long. The Cartan matrix of $\mathbf{C_n}$ is easily recovered, using the scalar product (5.5)[11]:

$$\begin{aligned}\langle\gamma_i,\gamma_i\rangle &= -4 \qquad 1 \le i \le n-1\,, \\ \langle\gamma_i,\gamma_{i+1}\rangle &= 2 \qquad 1 \le i \le n-1\,, \\ \langle\gamma_n,\gamma_n\rangle &= -2\,.\end{aligned} \tag{5.53}$$

In conclusion, it is easy to realize the folding of $\mathbf{A_{2n-1}}$ and obtain the roots of the Lie algebra $\mathbf{C}_n$ in terms of junctions. However, this is only a formal operation, without any string interpretation behind it. In fact, having at hand only A-branes, it is impossible to construct an orientifold or a screen like in the previous cases, since fundamental strings that cross A-cuts remain fundamental strings. Thus, since there has not been so far any realization of $O7^+$ planes out of F-theory 7-branes[12], one concludes that the $\mathbf{C_n}$ Lie algebras cannot be realized in the geometry considered in this chapter.

[11] All the entries that are not written are vanishing.

[12] Without intersecting branes, Sp(n) gauge symmetry can be realized only in the presence of $O7^+$ planes.

Chapter 6

7-branes and Freed-Witten anomaly

The subject of this final chapter is essentially the content of the paper [6] about the impact of Freed-Witten anomaly cancellation on the gauge theory of the F-theory 7-branes. The central issue is the subtle quantization condition of the G_4 flux in the context of both smooth and singular Calabi-Yau fourfold compactifications of F-theory. While the principal aim is to find a general pattern and to give formulae with range of validity as wide as possible, a number of explicit and concrete examples are provided in order to get a detailed understanding of the proper quantization conditions of fluxes; as an aside, the description of such models will also show several topological features regarding the classification of the B and A-fields and the Sen tachyon condensation outlined and discussed at length throughout the first part of the thesis.

This chapter is organized as follows: in section 1 an introduction to the problem is given focusing on the subtle nature of several topological effects; section 2 contains a detailed treatment of the problem in the context of F-theory on smooth elliptic Calabi-Yau fourfolds; crosschecks of the results are made from the viewpoint of type IIB at weak coupling; finally, in section 3 the same problem is addressed in the context of fourfold compactifications with $Sp(N)$ singularities, both in a specific, illustrative example and in the general case, taking advantage from their nice and intuitive weak coupling picture.

The reader not familiar with basic toric methods for resolving singularities may need the basic material covered in appendix F while reading section 6.3.

6.1 Setting up the problem

In the preface of this thesis it was said that the Freed-Witten anomaly has a double impact on the string background: one is an equation in terms of torsion characteristic classes expressing the obstruction to anomaly cancellation, the other is the right quantization condition to be imposed on fluxes in order to actually get rid of the anomaly, once the background fulfills the former condition. In this chapter, the second aspect is addressed in the context of F-theory, by using its duality with M-theory.

In M-theory a phenomenon pretty similar to the global ambiguity of the string path integral measure arises, but now in the context of the quantum theory of membranes [8]. Indeed, let M_{11} be the 11-dimensional target space of M-theory, which is obviously forced to be a spin manifold. Then, it is well known that its property of being spin implies that its first Pontryagin class

$p_1(M_{11})$ is an even class[1] in the fourth cohomology of M_{11}: let $\lambda(M_{11}) = p_1(M_{11})/2$. Now, an M2-brane has a 3-dimensional worldvolume and M-theory is just the theory of the embeddings of this manifold in M_{11}. Such a non-linear sigma model is taken to be supersymmetric but, unlike the case of strings, there is no chirality for the worldvolume spinors to play with. Hence, while for the closed string path integral, the ambiguity of the Pfaffian of the world-sheet Dirac operator cancels between left and right movers [1], here an ambiguity does survive and it can be shown to be related to topological properties of the target as follows. Bringing the membrane worldvolume along a non trivial circle C, after having completed the loop the Pfaffian of the Dirac operator comes back to itself up to the following sign:

$$(-1)^{(\lambda\,,\,M2\times C)}\,, \tag{6.1}$$

where the round brackets in the exponent just mean evaluation of the class λ on the 4-dimensional manifold $M2 \times C$. The only possibility to compensate for such a global anomaly in order to render well-defined the partition function of the theory of membranes is to require an equal ambiguity for the holonomy of the C_3 potential over the M2 worldvolume that also enters the path integral measure. This is achieved by simply requiring a suitable quantization condition for the G_4 field strength[2]:

$$G_4 - \frac{\lambda}{2} \in H^4(M_{11}, \mathbb{Z})\,. \tag{6.2}$$

Notice, in particular, that this anomaly can always be canceled, namely there are no global obstructions, but only a suitable quantization condition for the G-flux must be chosen to get a consistent quantum theory. Moreover, the fact that closed strings never have this problem for their path integral implies that the H-flux, which in turn is responsible for the well-definiteness of the holonomy of the B-field over the world-sheet, must be always an integral class in the third cohomology of the string target M_{10}.

One particular consequence in type IIB string theory of this quantization condition will be essentially the topic of this chapter, but it is worth mentioning here also the other string implications.

For example, already for the simplest situation of M-theory on $M_{10} \times S^1$, (6.2) implies an analogous quantization condition for the unimproved RR field-strength of type IIA F_4, the electric field of the D2-brane. A beautiful explanation of this phenomenon from a purely string perspective has been given by Witten [76] in the case of vanishing H-flux. The origin of this shift in the quantization of F_4 is traced back to the self-duality of the type IIB RR flux F_5. Indeed, the quantum mechanics of a self-dual field is quite subtle: in order to construct the partition function of such a field one needs to choose a maximal set of commutative periods to sum on and two independent $\mathbb{Z}_2$-data for each such period, i.e. its sign and its shift from integrality. In the case of type IIB on $M_{10} = M_9 \times S^{1'}$ it is particularly easy to separate the two information. In this circumstance, one has $H^5(M_{10}, \mathbb{Z}) = H^5(M_9, \mathbb{Z}) \oplus H^4(M_9, \mathbb{Z})$, and thus:

$$F_5^{(10)} = F_5^{(9)} + F_4^{(9)} \wedge a'\,, \tag{6.3}$$

where a' is the generator of $H^1(S^{1'}, \mathbb{Z})$. Now, choosing for instance to sum over the periods of $F_4^{(9)}$, then, calling $x = [F_5^{(9)}] \in H^5(M_9, \mathbb{Z})$ the periods of $F_5^{(9)}$ which are just the Poincaré duals

[1] Notice that the vice-versa of this statement is not true, because $p_1 \mod 2 = w_2^2$, which can very well be vanishing even though the manifold admits no spin structures (i.e. $w_2 \neq 0$).

[2] Square brackets indicating the cohomology classes are dropped, as well as factors of 2π.

in M_9 of the ones of $F_4^{(9)}$, the set of shifts of the periods one is summing over can be read off from the following expression:

$$\Omega(x) = (-1)^{(\lambda \cup x \,,\, M_9)} \,. \tag{6.4}$$

Hence, the shift from the integral quantization of $F_4^{(9)}$ is the class $\lambda/2$, exactly as predicted by the M-theory computation. Formula (6.4), by the way, gives also the set of signs to be taken into account if one instead chooses to compute the partition function by summing over the periods of $F_5^{(9)}$.

For the other set of data that one needs, namely the signs of the periods of $F_4^{(9)}$, or alternatively, the shift of the periods of $F_5^{(9)}$, i.e. $\Omega(\mathrm{PD}_{M_9}x)$, there is unfortunately no explicit expression like (6.4). One could wonder what would be the M-theory interpretation of the shift of quantization for $F_5^{(9)}$ given by this unknown factor. One reasonable hypothesis could be as follows. Perform a T-duality along $S^{1'}$; then, $F_5^{(9)}$ becomes part of the type IIA flux $F_6^{(10)}$. So this signals a sort of compensation of an ambiguity in the path integral of D4-branes. The latter, in turn, lift to M5-branes wrapping the M-theory S^1 and the above mentioned shift is so traced back to a shift in the quantization of the G_7 flux with one leg along S^1, namely the Hodge-dual in M_{11} of G_4 with all its legs in M_{10}.

Besides this effect on the quantization of type IIA bulk fluxes, (6.2) have another important consequence in the case D6-branes are present, namely M_{11} is a non-trivial S^1 fibration over M_{10}, as explored in [74, 75]. In this situation, the so called Taub-NUT geometry, codimension 3 loci of the base on which the S^1 fiber collapses are interpreted as D6-branes of type IIA string theory, and the shift in the quantization of the M-theory G_4 flux directly induces a shift à la Freed-Witten in the quantization condition of the gauge flux F on the D6-branes. This phenomenon is naturally expected to happen because the open strings responsible for the possible half-quantization of F lift to *closed* M2-branes, since the S^1 is collapsing on the boundaries of the open strings (the D6-branes loci): therefore the ambiguity in the path integral measure of such open strings gets directly related to the above mentioned one in the path integral measure of closed membranes leading to (6.2).

The search of the analogous effect for the gauge flux on the F-theory 7-branes essentially drives the analysis carried out in [6], since in this case the whole powerful machinery of algebraic geometry can be used to greatly simplify the computations. This phenomenon would hence signal the presence of a Freed-Witten anomaly on the F-theory 7-brane that is cancelled by requiring the A-fields to be a connection on a "half"-line bundle (see subsection 1.3.2). The context is exactly the one described in section 4.4, with an M_{11} elliptically fibered over B_3 and the G_4 flux inducing brane-type flux as in (4.51). In other words, a spin^c bundle is supposed to arise on a D7-brane wrapping a non-spin divisor $S_2 \subset B_3$; thus, the gauge flux F appearing on it cannot be chosen to vanish because of its half-quantization, which in turn is generated by an half-quantized G_4-flux living on the F-theory lift of the given type IIB model.

Notice that, on the other hand, the bulk type fluxes in (4.47) do not undergo such a shift in their quantization in order to preserve the well-definiteness of the path integral measure of fundamental strings and their S-dual D1-branes.

Therefore, in the next sections, the attention will be focused on the λ class of the F-theory internal manifold, that decides the right quantization condition for G_4 according to (6.2). F-theory on Calabi-Yau fourfolds will be considered (see section 4.2.2), which leads to the most phenomenologically relevant models of $\mathcal{N} = 1$ gauge theories in four dimensions. For a complex manifold X there is a nice explicit expression for its first Pontryagin class in terms of its Chern classes:

$$p_1(X) \;=\; c_1^2(X) - 2\,c_2(X)\,. \tag{6.5}$$

This implies that, for the Calabi-Yau fourfolds Z_4 relevant or the sequel, $\lambda(Z_4) = -c_2(Z_4)$. Thus one is led to study the possibility of being odd of the second Chern class of the F-theory Calabi-Yau fourfolds in order to deduce the right quantization condition for G_4 according to

$$G_4 + \frac{c_2(Z_4)}{2} \;\in\; H^4(Z_4, \mathbb{Z})\,, \tag{6.6}$$

and consequently the right one for the D7 gauge flux F according to the previous discussion. Both the smooth and the singular cases will be addressed for Z_4, where a number of new results are found about the Freed-Witten anomaly cancellation of the F-theory 7-branes. A general pattern for the form and for the property of being odd of $c_2(Z_4)$ will be also conjectured on the base of several concrete examples which will be constructed in detail and matched with models available in the literature.

6.1.1 A further subtle topological effect

There exists another subtle topological effect suffered by the M-theory G-flux which will not be analyzed any further in the rest of this thesis from an F-theory point of view.

Consider the effective theory on the 6-dimensional worldvolume of an M5-branes. Witten has conjectured [76] that the topological type of the *restricted* G-flux is constrained by the following equation:

$$G_4|_{M5} \;\;=\;\; \beta\,(\theta(M5)) \;\;\in\; \mathrm{Tors}\,H^4(M5, \mathbb{Z})\,, \tag{6.7}$$

where β is the Bockstein map in degree 3 of the second long exact sequence in (1.7) and $\theta(M5) \in H^3(M5, S^1)$ is the image of a 2-torsion class belonging to $H^3(M5, \mathbb{Z}_2)$.
(6.7) is manifestly the natural generalization to M5-branes of the Freed-Witten anomaly cancellation condition for ordinary D-branes in type II string theories (1.13), with open M2-branes ending on the M5 playing the role of the open strings. Indeed, in the special case of an M5-brane worldvolume like $M5 = D4 \times S^1$, where S^1 is the M-theory circle, (6.7) restricts exactly to the standard Freed-Witten equation for a D4-brane:

$$H|_{D4} \;\;=\;\; \beta\,(w_2(D4)) \;\;=\;\; W_3(D4) \;\;\in\; \mathrm{Tors}\,H^3(M5, \mathbb{Z})\,, \tag{6.8}$$

where w_2, meant as a class in $H^2(D4, S^1)$, is the restriction to the D4-brane of the class θ. However, the type IIA interpretation of (6.7) in the case of an M5 not wrapping the M-theory circle is not clear: it should be related to some sort of anomaly of the NS5-branes.

As usual, once the torsion condition (6.7) is fulfilled, a suitable quantization condition for the gauge flux on the M5-brane is required in order to get rid of the anomaly. The type IIA counterpart of this phenomenon has been provided by Witten [76] in the case of topologically trivial H-flux, again using the self-duality of the 3-form field T_3 present in the spectrum of the M5 effective theory. In fact, in the special situation of $M5 = D4 \times S^1$, one has $H^3(M5, \mathbb{Z}) = H^3(D4, \mathbb{Z}) \oplus H^2(D4, \mathbb{Z})$. Then, since the restriction of G_4 to the M5 is assumed to be exact, $\theta(M5)$ must be in the kernel of the Bockstein for (6.7), which is the M5 analog of the D4-brane being spin^c because of the topological triviality of the restricted H-flux. Moreover, the self-dual field T_3, which satisfies the equation $G_4|_{M5} = \mathrm{d}T_3$, can be decomposed according to

$$T_3^{(M5)} = T_3^{(D4)} + F \wedge a\,, \tag{6.9}$$

where F is the gauge field on the D4 and a is the generator of $H^1(S^1,\mathbb{Z})$. Thus, the quantum mechanics of T_3 prescribes a shift in the quantization condition of F which can be deduced by a formula analogous to (6.4):

$$\Omega'(x) = (-1)^{(w_2 \cup x\ ,\ D4)}\,, \tag{6.10}$$

where $x = [T_3^{(D4)}] \in H^3(D4,\mathbb{Z})$ are the periods chosen to be summed on for the construction of the partition function of $T_3^{(M5)}$. Therefore, the shift from the integral quantization of F is the class $w_2'/2$, w_2' being any integral lift of w_2, exactly as required by the cancellation of the Freed-Witten anomaly on a spin^c D4-brane (see subsection 1.3.2). There is more: it is possible to show that in case the worldvolume of such D4-brane fails to be spin^c, the partition function of $T_3^{(M5)}$ vanishes identically, therefore indicating the presence of a global anomaly for the M5-brane, justified by the fact that now the class θ does not belong to the connected component of $H^3(M5,S^1)$ which contains the identity (see the footnote at page 7).
However, the shift in the quantization condition of $T_3^{(D4)}$, given by the unknown sign function $\Omega'(\mathrm{PD}_{D4}x)$, has not yet a clear interpretation in type IIA string theory.

6.2 Smooth Calabi-Yau fourfolds

The detailed study of the second Chern class of smooth elliptically fibered Calabi-Yau fourfold Z_4, admitting a Weierstrass representation (4.32), is carried out in this section. The attention will be focused on Weierstrass models with E_8 fibrations, due to their connection with IIB string theory. However, the analysis can be repeated for E_6 and E_7 fibrations, whose weak coupling limits were constructed in [77], yielding the same conclusion. The strategy for deciding whether it is even or odd is to reduce the problem to a much easier one formulated on the base B_3 of the elliptic fibration, by using its Weierstrass form. The result will be that the second Chern class of such Calabi-Yau manifolds is always *even*. The physical consequences of this fact will be spelt out.
Smooth elliptically fibered Calabi-Yau fourfolds can be constructed in practice (even if sometimes it could be not sufficient) by taking for instance B_3 to be an *almost Fano* threefold, i.e. $c_1(B_3) \geq 0$ when integrated over any 2-cycle of the manifold. This is because what really creates singularities (although it is not necessary) is the absence of non-identically vanishing sections of the canonical bundle of B_3: indeed, if $c_1(B_3)$ becomes negative on some 2-cycle, the canonical bundle will have no non-vanishing section when restricted to that locus, and thus the polynomials f and g of (4.23) vanish as well, being sections of suitable powers of the canonical bundle. But, as it is clear from table 4.1, this surely implies a singularity of the total space of the fibration.

Using adjunction formulae, as in [78], it is easy to compute the total Chern class of a smooth Calabi-Yau fourfold described by the Weierstrass polynomial. Let M_5 be the ambient fivefold of Z_4, which is, as usual, a $W\mathbb{P}^2_{2,3,1}$ fibration over B_3, and $F \in H^2(M_5,\mathbb{Z})$ be the first Chern class of the line bundle[3] on M_5 $\mathcal{O}(1) \otimes K^{-1}(B_3)$. Thus the Poincaré dual of the gravitational brane $Z = 0$ will be, in this notation, $F - c_1(B_3)$. Now, recalling the general construction of an elliptically fibered Calabi-Yau outlined for $K3$ in subsection 4.2.1, it is straightforward to deduce the following adjunction formula expressing the total Chern class of Z_4 in terms of F

[3]As in chapter 4, the pull-back of the projection map from M_5 to B_3 is always implicit in the notation.

and of the one of B_3:

$$c\,(Z_4) \quad = \quad \frac{c\,(B_3)\cdot(1+2F)\cdot(1+3F)\cdot(1+F-c_1(B_3))}{1+6F}\,. \tag{6.11}$$

The fact that the element XYZ always belongs to the Stanley-Reisner ideal of the ambient space M_5 is expressed easily by the following constraint:

$$F^2(F-c_1(B_3))=0\,, \tag{6.12}$$

which just means that those three coordinates cannot all vanish at the same time in M_5. Since the Weierstrass equation (4.32) defining Z_4 represents a divisor of class $6F$, on Z_4 the constraint (6.12) simply reduces to the condition $F^2 = F\cdot c_1(B_3)$. Therefore, the second order in the expansion of (6.11), namely the second Chern class of Z_4, will be:

$$c_2(Z_4) \quad = \quad 12F^2+c_2(B_3)-c_1^2(B_3)\,. \tag{6.13}$$

Before trying to understand if (6.13) is even or odd, some comments on its structure are in order (see [79] for similar observations). The reader should remember from the discussion of section 4.4 that, in order to preserve Poincaré invariance in $\mathbb{R}^{1,3}$, the flux G_4 must be of the form (4.47), namely it must have one and only one leg along the elliptic fiber. Now, since the second and the third term in (6.13) are pull-backs from the base, they cannot have any leg along the fiber. The first term, on the other hand, can be written as $c_1^2(B_3)+c_1(B_3)\cdot(F-c_1(B_3))$, where the first term is again all along the base, while the second has two legs on the base and two on the fiber, $F-c_1(B_3)$ being as said the Poincaré dual of the 0-section. Hence, the structure of $c_2(Z_4)$ is necessarily unacceptable if one wants to keep space-time Poincaré invariance, and thus its possible disparity would induce a Poincaré-breaking G_4 flux which cannot be put to zero because half-quantized. This is just a consequence of the already stressed fact (see page 78) that, on a smooth elliptic Calabi-Yau fourfold of strict SU(4) holonomy, any 4-cycle which is Poincaré dual to a Poincaré-preserving G_4 does not intersect any of the possible intersections of two divisors. Indeed, apart from $c_2(B_3)$, which comes anyhow from a class of the base, (6.13) is manifestly a linear combination of wedge products of two classes belonging to $H^2(Z_4,\mathbb{Z}) = H^{1,1}(Z_4,\mathbb{Z})$, which never have only one leg along the T^2 fiber. To summarize, one can divide the fourth cohomology group of Z_4 in the part given by combinations of wedges of two divisor classes plus its orthogonal complement in the non-degenerate metric given by wedging and integration:

$$H^4(Z_4,\mathbb{Z}) = \left(H^{1,1}\wedge H^{1,1}\right)\oplus\left(H^{1,1}\wedge H^{1,1}\right)^{\perp}\,(Z_4,\mathbb{Z})\,. \tag{6.14}$$

All the acceptable 4-classes lie in the second summand, while the first summand contains only unacceptable ones.
Therefore, the G_4 flux will be always quantized (and so can be chosen to be vanishing), while, in order not to break Poincaré invariance in $\mathbb{R}^{1,3}$, one must be sure that $c_2(Z_4)$ is always even.

In order to address this issue, it is convenient to reduce the problem to another one formulated entirely on the base space of the fibration, which will turn out to be much more tractable.
The term $12F^2$ in (6.13) is twice the Poincaré dual of a true, perfectly decent 4-cycle, which is a divisor of B_3. Indeed, this 4-cycle is described in Z_4 by the two equations $X=0$ and $Y=0$, which have class $2F$ and $3F$ respectively. Therefore, after having gauge fixed $Z=1$, the Weierstrass equation, for example in the easier form (4.23), implies that such 4-cycle is the divisor of the base described by the equation $g(x_i)=0$. Hence, its Poincaré dual is an integrally

quantized 4-class and thus $12F^2$ is always an even class.
In this way, one is led to analyze a pure class of the base, namely $c_2(B_3) - c_1^2(B_3) \in H^4(B_3, \mathbb{Z})$. So the aim would be to study the value of this class on all the divisors of B_3, in order to decide whether or not it is always even. Fortunately, some basic facts about Steenrod squares and Wu classes in algebraic topology help one to find out very quickly a pretty general answer to this problem, which is stated and proven below.

Fact: for any *smooth, complex* manifold X of *complex dimension at most three*, the characteristic class $c_2 - c_1^2$ of its tangent bundle is always *even*.

Proof: it is convenient to reduce the integral 4-class in question to a class modulo 2 and study whether the latter vanishes or not. The modulo 2 reduction of c_1^2 is clearly w_2^2 because $c_1 \bmod 2 = w_2$ and the quotient homomorphism $q : \mathbb{Z} \to \mathbb{Z}_2$ induces an homomorphism of cohomology rings under the cup product $\cup$. For c_2, instead, one notices that, since the manifold is complex, it is equal to the class λ apart from the sign which is killed by the mod 2 reduction. But [8] the mod 2 reduction of λ is the fourth Stiefel-Whitney class w_4, and thus, for complex manifolds[4], $c_2 \bmod 2 = w_4$. Therefore, one is led to analyze the class:

$$c_2 - c_1^2 \mod 2 \quad = \quad w_4 + w_2^2 \quad \in \ H^4(X, \mathbb{Z}_2)\,. \tag{6.15}$$

Here it is where Steenrod squares and Wu classes come about. The Steenrod squares, already introduced in formula (2.19), are operations in the $\mathbb{Z}_2$-cohomology of a given space M of real dimension n, such that $\mathrm{sq}^i : H^k(M, \mathbb{Z}_2) \to H^{k+i}(M, \mathbb{Z}_2)$. Some of their basic properties and their connection with Stiefel-Whitney classes and Wu classes are needed for this proof and they are summarized below [41, 80].

1. sq^0 is the identity map.
2. The total Steenrod square respects the cup product, namely:
$$\mathrm{sq}^k(x \cup y) = \sum_{i+j=k} \mathrm{sq}^i(x) \cup \mathrm{sq}^j(y)\,.$$
3. The Wu classes v_i are defined as the unique representatives (by Riesz theorem) of the functionals (upon integration on M) $\mathrm{sq}^i(x)$ for $x \in H^{n-i}(M, \mathbb{Z}_2)$, namely:
$$\mathrm{sq}^i(x) \equiv v_i \cup x\,, \qquad x \in H^{n-i}(M, \mathbb{Z}_2)\,, \qquad 0 \le i \le n\,.$$
4. The total Stiefel-Whitney class equals the total Steenrod square applied to the total Wu class, namely:
$$w_i = \sum_{j=0}^{i} \mathrm{sq}^{i-j}(v_j)\,.$$
5. They satisfy the so called Wu formula:
$$\mathrm{sq}^i(w_j) = \sum_{t=0}^{i} \begin{pmatrix} j+t-i-1 \\ t \end{pmatrix} w_{i-t}\, w_{j+t}\,.$$

[4]In general, for a spin^c manifold which is not complex, one has [76, 81] $w_4 = \lambda \bmod 2 = \frac{p_1 - \alpha^2}{2} \bmod 2$, where α is the spin^c class. In particular, if the manifold is complex, then $\alpha = c_1$.

6. Since $\mathrm{sq}^i(x) = 0$ if $x \in H^j(M, \mathbb{Z}_2)$ with $i > j$, then, by definition:

$$v_i = 0 \qquad \forall\, i > \left[\frac{n}{2}\right] .$$

Using these properties, one can find very easily the expressions of the Wu classes in terms of the Stiefel-Whitney ones; for example, for the first four one gets:

$$\begin{aligned} v_1 &= w_1 , \\ v_2 &= w_2 + w_1^2 , \\ v_3 &= w_1 w_2 , \\ v_4 &= w_4 + w_2^2 + w_1^4 + w_1 w_3 . \end{aligned} \tag{6.16}$$

Now, since the manifold at hand, X, is complex, it is in particular orientable, so that w_1 of its tangent bundle vanishes. Hence, in the present situation, the fourth Wu class displayed in (6.16) becomes exactly the reduction modulo 2 of the integral class of (6.15):

$$v_4 = c_2 - c_1^2 \mod 2 . \tag{6.17}$$

However, the last of the properties listed above says that if $\dim_{\mathbb{C}} X \leq 3$, then $v_i = 0$ for all $i \geq 4$. This concludes the proof.
□

Notice that, as far as the purposes of F-theory are concerned, the hypotheses of this result constitute no limitation at all, since the smooth base B_3 of the elliptic fibration the result will be applied to, is always a Kähler manifold, and thus in particular complex. Moreover, as it is clear from the proof above, there is no reason, on the other hand, for the fourth Wu class of a generic Calabi-Yau fourfold to vanish. In that case, due to the triviality of the canonical bundle, it is equal to the fourth Stiefel-Whitney class of the Calabi-Yau manifold: $v_4(CY_4) = w_4(CY_4) = \lambda(CY_4) \bmod 2$. Therefore, in order to prove that this quantity is instead vanishing for Z_4, it was crucial for the latter to be elliptically fibered with a *smooth* Weierstrass representation. In section 6.3, in fact, the hypothesis of smoothness will be relaxed, by allowing non-abelian singularities for Z_4 of the Kodaira types listed in table 4.2. In that case, the computation of the second Chern class by means of the adjunction formula (6.11) will no longer be reliable, and thus one is forced to resolve the singularity by a series of blow-up's. Only after the complete resolution one is allowed to compute c_2 using adjunction, but the resolved fourfold, although still elliptic, strict Calabi-Yau (all the resolutions in the series will be crepant), will in general not admit any more a Weierstrass description of the elliptic fibration.

As a last comment, it is important to stress that when λ of the fourfold is an even class as in this case, the geometric tadpole of F-theory, namely the $\chi(Z_4)/24$ mentioned in subsection 4.4.1, is always an integer, as Witten proved using index theorems [8]. Thus, since in this case one is allowed to choose a vanishing G-flux (because it is integrally quantized), one has not to worry about a possible non-integrality of the number of D3-branes. Moreover, even the minimal choice for a non-vanishing G-flux, namely $G_4 = c_2(Z_4)/2$, does not constitute a problem in this smooth case. Indeed, by computing the holomorphic Euler number of Z_4 by means of the Hirzebruch-Riemann-Roch theorem, one has:

$$2 = \chi_0(Z_4) = \int_{Z_4} \mathrm{Td}(Z_4) = \frac{1}{720} \int_{Z_4} 3c_2^2(Z_4) - c_4(Z_4) , \tag{6.18}$$

where Td is the total Todd-class and, thanks to the strict SU(4) holonomy, it has been used that $h^{0,1} = h^{0,2} = h^{0,3} = 0$ for Z_4. Therefore, one gets:

$$\frac{1}{2}\int_{Z_4} G_4 \wedge G_4 = \frac{1}{8}\int_{Z_4} c_2^2(Z_4) = \frac{\chi(Z_4)}{24} + 60\,, \tag{6.19}$$

which is integral. This argument also shows that the total M2-tadpole (4.54) is always an integer at least if one adopts the minimal choice for G_4, and it is equal to -60. In this smooth case, the two terms are separately integer; in section 6.3 it will be shown that in general, in the singular case, they are separately non-integral (when computed of course on the blown-up Calabi-Yau), but the total tadpole remains the same as here because the previous argument is not affected if a Weierstrass representation is missing.

6.2.1 The perturbative type IIB perspective

It has just been found that F-theory on an elliptic Calabi-Yau fourfold with smooth Weierstrass representation always has a quantized G-flux which preserves 4-dimensional Poincaré invariance and no other obligatorily present flux that breaks it.
The aim of this subsection is simply to check these F-theory expectations from the weak coupling limit viewpoint, reached via the Sen procedure outlined in subsection 4.2.2. However, as already stressed, the computations there turn out to be problematic due to the singular shape of the recombined D7-brane (Whitney umbrella). In order to avoid this complication, instead of directly working out the resolution of the singular space, a much easier technique will be adopted that is based on the Sen tachyon condensation and used for similar purposes in [58, 50, 51].

Let X_3 be the Calabi-Yau threefold that double covers B_3, branched over the locus invariant under the involution $\sigma : X_3 \to X_3$ displayed in (4.28). The complete expression of the involution, which takes into account also the $\mathbb{Z}_2$-transformation that acts on the fields of the theory, is $\tilde{\sigma} \equiv \sigma \circ (-1)^{F_L} \circ \Omega$ where $(-1)^{F_L}$ changes the sign of the Ramond states of the left-moving sector, while Ω denotes world-sheet orientation reversal.
The strategy now is to consider a pair of D9-branes and a pair of the corresponding anti-D9-branes with suitable vector bundles on them, which will eventually tachyon-condense according to the theory sketched in subsection 3.2.3 and leave the right configuration of D7 and D3's one had before. In general, as explained in [58], it is crucial to have an even number of D9-branes (and consequently of anti-D9's) in order to avoid a discrete anomaly. Indeed, the worldvolume theory of a D3-brane probe placed with its image on the $O7^-$ in the presence of r D9-branes is an $SU(2)$ $\mathcal{N} = 1$ gauge theory coupled to r chiral multiplets in the fundamental of $SU(2)$ coming from open strings stretching from the D3 (and its image) to the r D9's (open strings stretching to the image anti-D9's are simply the image strings and thus should not be counted separately as independent). Therefore, if r is odd, one has an odd number of Weyl fermions in the fundamental of $SU(2)$ and this results in a $\mathbb{Z}_2$-anomaly [82]. In the present situation one has $r = 2$; in section 6.3, cases with bigger r will be also discussed because they will be needed in analyzing the Sen limit of singular F-theory compactifications.

The orientifold 7-plane $O7^-$ is placed on the perfectly regular divisor $\xi = 0$ of X_3. The study made in [50] reveals that if the first Chern class of the normal (line) bundle of the O7 inside X_3 is odd, then the D7-brane seems to be forced to assume a non-generic shape, which in turn would signal the presence of a gauge flux à la Freed-Witten which cannot be put to zero: indeed, this flux would create a superpotential that would constrain the transverse moduli of the D7-brane.

After having reviewed this argument, it will be actually proven here that the necessity of such a gauge field is only apparent because it vanishes as one turns on a discrete B-field, consistent with the orientifold projection, whose presence is anyway required for cancellation of the Freed-Witten anomaly of bulk open strings. This is as expected from F-theory, which, as said, does not predict the appearance of any D7-brane gauge flux with shifted quantization condition.
Since, by adjunction, $c_1(N_{X_3}O7) = -c_1(O7) = c_1(B_3)|_{O7}$, the disparity of the class of $\xi = 0$ just means that the divisor wrapped by the O7 is not spin, which also implies that B_3 itself is not spin. In this circumstance, let $G = c_1(B_3)$ such a class, odd on some 2-cycles of O7; the class of the normal bundle of O7 in B_3 will thus be $2G|_{O7}$. In order to cancel the 7-brane tadpole, as explained in subsection 4.2.1, all the 7-branes present should wrap a divisor of class $12\,G$: hence, because the O7 is a bound state of two mutually non-local 7-branes and it enters to the fourth power the leading order discriminant (4.30), the recombined D7-image-D7-brane of Whitney umbrella shape should wrap a (singular) divisor of class $8G$.[5]
The configuration of X_3-filling D9 and anti-D9-branes with gauge bundles, which will be considered, is the following:

$$\begin{array}{cccc} \overline{D9_1} & \overline{D9_2} & \qquad D9_1 & D9_2 \\ \mathcal{O}(-aG) & \mathcal{O}((a-4)G) & \qquad \mathcal{O}(aG) & \mathcal{O}((4-a)G)\,, \end{array}$$

where a is some integer. This is compatible with the O7-projection because the anti-D9's are exactly the orientifold images of the D9's. The tachyon T of this configuration is a 2×2 matrix-valued section of $E\otimes E$, where E is the rank-two holomorphic vector bundle on the D9's, namely $\mathcal{O}(aG)\oplus\mathcal{O}((4-a)G)$. The orientifold involution acts on the tachyon as follows:[6]

$$\tilde{\sigma}^*(T(\vec{x},\xi)) \;=\; -{}^tT(\vec{x},-\xi)\,, \tag{6.20}$$

where $\vec{x}$ are the local coordinates on B_3. To survive such an orientifold projection, therefore, the tachyon is constrained to take the following form:

$$T(\vec{x},\xi) \;=\; \begin{pmatrix} 0 & \eta(\vec{x}) \\ -\eta(\vec{x}) & 0 \end{pmatrix} + \xi \begin{pmatrix} \rho(\vec{x}) & \psi(\vec{x}) \\ \psi(\vec{x}) & \tau(\vec{x}) \end{pmatrix}\,, \tag{6.21}$$

where η is the same polynomial, of degree $4G$, as in (4.31), while ρ,ψ,τ are other locally defined polynomials such that $\rho\tau-\psi^2\equiv\chi$ of formula (4.31). Now, matching the degrees of these three new polynomials should be greater or equal than zero in order to be well-defined sections: this leads the free parameter a to be actually bounded in an interval with semi-integer extrema. Indeed, by looking at the chosen configuration of bundles on the $\overline{D9}-D9$ system, one readily realizes that ρ,ψ,τ have degrees $(2a-1,3,7-2a)G$ respectively. Hence, a should lie in the interval

$$\frac{1}{2} < a < \frac{7}{2}\,, \tag{6.22}$$

but it cannot take the values of the extrema because they are not integral. This affects the shape of the D7-brane locus, which is by definition

$$S_2\,:\,\det(T) \;=\; \eta^2+\xi^2(\rho\tau-\psi^2) \;=\; 0\,, \tag{6.23}$$

[5]Throughout this chapter the charge of D-branes will be computed from the point of view of the double cover X_3 and therefore it will be twice the physical D-brane charge, since the charge of the images is counted separately.

[6]The minus sign corresponds to the choice of the $O7^-$; the $O7^+$ involution would have had the plus sign. The transpose, instead, comes from the fact that Ω exchanges the open string endpoints.

because the form of χ is non-generic unless one of the two bounds for a is saturated (i.e. one between ρ and τ becomes a constant). This means, as anticipated, that this system is apparently constrained to accommodate a Freed-Witten-like gauge flux on the D7, which freezes its shape.

This phenomenon can also be deduced from the contribution to the D3-tadpole of this system, whose practical computation will be now sketched for later use.
As explained in subsection 3.2.2, gauge and gravitational couplings of a Dp-brane induce lower dimensional D-brane charges according to the non-anomalous coupling (3.5). Here one has D9-branes wrapping the whole Calabi-Yau threefold, which means that their normal bundle is trivial and the $\hat{A}$-genus becomes equal to the Todd class of X_3. Thus the contribution to lower dimensional D-brane charge densities due to the $\overline{D9} - D9$ system is:

$$\Gamma_{D9} = \mathrm{ch}\left([E] - [\bar{E}]\right) \cdot \sqrt{\mathrm{Td}(X_3)} = \left(e^{aG} + e^{(4-a)G} - e^{-aG} - e^{(a-4)G}\right) \cdot \left(1 + \frac{c_2(X_3)}{24}\right) \quad (6.24)$$

where $\bar{E}$ is the gauge bundle on the anti-D9's and the square brackets denote the K-theory class, as in subsection 3.2.1. Also an Op-plane contributes to lower dimensional D-brane charges via gravitational coupling only (since there is no gauge bundle on it), according to the formula [83]:

$$S_{WZ}^{Op^{\pm}} = \pm\, 2^{p-4} \int_{Op} i^* C \wedge \frac{\sqrt{L\left(\frac{1}{4} T Op\right)}}{\sqrt{L\left(\frac{1}{4} N_{X_3} Op\right)}} \,, \quad (6.25)$$

where i is the embedding of the Op-plane worldvolume in the target space, while L is the Hirzebruch genus. Hence, for the case of the $O7^-$, the first non-trivial induced charge density is the D3-brane one and it is easy to see that it reduces to:

$$\Gamma_{O7} = \frac{\chi(O7)}{6} \,. \quad (6.26)$$

First of all, the zeroth order term in (6.24) vanishes, indicating that no net charge of D9-brane is left after the tachyon condensation (stated in the mathematical terms of subsection 3.2.1, this means that on the $\overline{D9} - D9$ system there is a class in $\tilde{K}(X_3)$). Then, the first order term in (6.24) gives the right D7-brane charge, namely $8G$, while the second order term vanishes, compatibly with the fact that D5-branes are projected out by the orientifold involution.
The third order term in (6.24) and the contribution (6.26) constitute the total induced D3-brane charge, which must be compensated by an equal number of explicitly added D3-sources in order to cancel the D3-brane tadpole. It is possible to single out two different contribution to this charge density:

$$Q_{D3} = Q_{gauge} + Q_{grav} \,. \quad (6.27)$$

The gauge contribution reads:

$$Q_{gauge} = 4\left(a - \frac{1}{2}\right)\left(a - \frac{7}{2}\right) G^3 \,, \quad (6.28)$$

which manifestly disappears, as it should, when the bound (6.22) is artificially saturated (i.e. absence of D7-brane gauge flux). The gravitational contribution, instead, reads:

$$Q_{grav} = \frac{29}{2} G^3 + \frac{c_2(X_3)}{2} G \,. \quad (6.29)$$

As stressed in subsection 4.4.1, the curvature-induced contribution (6.29) should exactly match the geometric tadpole (first term in (4.54)) predicted by the F-theory lift of the system (or better twice it, because the image-D3's are counted separately); and indeed it does:

$$\frac{\chi(Z_4)}{12} = \int_{B_3} 30G^3 + c_2(B_3)G\,, \tag{6.30}$$

where, analogously to [78], the fourth order term of (6.11) has been used together with the fact that the class F integrates to 1 on the generic T^2 fiber (i.e. the 0-section is not a multi-section). Using the adjunction formula $c_2(B_3)|_{X_3} = c_2(X_3) - G^2|_{X_3}$, this can be turned into an integral which involves only Calabi-Yau threefold data[7]:

$$\frac{\chi(Z_4)}{12} = \frac{1}{2}\int_{X_3} 29G^3 + c_2(X_3)G\,, \tag{6.31}$$

where the factor 1/2 takes into account that X_3 is a double cover of B_3. This exactly matches the integration on X_3 of (6.29).

Therefore, if things stay like this, it seems impossible to eliminate the gauge contribution (6.28), with the unexpected consequence of a compulsory presence of D7-brane gauge flux. However, one can turn on a 2-torsion B-field (with topological type related to $w_2(B_3)$, see below) to cure this problem. The details of how such a discrete B-field modifies the interval (6.22) making its extrema integral will be now described.

If a B-field is turned on in this type IIB orientifold compactification, this should be done compatibly with the orientifold projection, which acts on it as:

$$\tilde{\sigma}^*(B(\vec{x},\xi)) \;=\; -\sigma^* B(\vec{x},-\xi)\,, \tag{6.32}$$

σ^* being the pull-back map of the target space involution, acting on its cohomology ring. Hence, in order for the B-field to survive to the projection, one would have to require $B + \sigma^* B = 0$. However, since, as seen many times, what really matters at the quantum level is the holonomy of the B-field, one is led to impose the weaker condition $B + \sigma^* B \in H^2(X_3, \mathbb{Z})$, namely $\mathrm{Hol}(B + \sigma^* B) = 1$. One could now wonder which representative in the second de Rham cohomology of X_3 should be chosen for the B-field. This, of course, does not matter for closed strings which do not feel large gauge transformations; but for open strings different choices are in general not equivalent, because integral shifts for the B may change their path integral measure. Again, as stressed many times in chapter 1, the right trivialization is provided by the gauge fields on the D9-branes, in such a way that the combination $B + F$ remains gauge invariant. Actually there is the freedom to choose only one overall trivialization, on one of the four D9's, as the choice for the other gauge fields is constrained by the orientifold involution. Indeed, for example, one can conventionally choose to trivialize the B-field on $D9_1$ by means of the gauge field F_1, namely one chooses the couple (F_1, B) on $D9_1$. Then, the involution would imply the couple $(-\sigma^* F_1, -\sigma^* B)$ for the $\overline{D9}_1$. But, since the bulk B-field is the same and these are branes filling the whole target space, the right trivialization on the $\overline{D9}_1$ turns out to be $F_1' = -\sigma^* F_1 - B - \sigma^* B$, so that on the $\overline{D9}_1$ one has the couple (F_1', B) (so one has just applied the large gauge transformation $B + \sigma^* B$ to the previous trivialization). Analogously, choose the couple (F_2', B) on the $\overline{D9}_2$, namely the trivialization F_2' on it, which is anyhow fixed by the 7-brane tadpole cancellation

[7] The computation on X_3 is actually the only reliable one, since B_3 is singular on the O7 locus in the weak coupling limit and sometimes also because of the presence of O3-planes.

$-(F_1' + F_2') = 4G$. Then, again applying the large gauge transformation $B + \sigma^* B$, the right trivialization on the $D9_2$ will be $F_2 = -\sigma^* F_2' - B - \sigma^* B$, so that on the $D9_2$ one has the couple (F_2, B).
Now, as opposed to the target space orientifold involution defined in [51], in the present case it just reverses the sign of a coordinate: therefore, its action on cohomology is trivial, namely σ^* is the identity map. With the minimal choice $2B = G$, thought of as a class of X_3 by pull-back[8], one has the following new configuration:

$$\begin{array}{cccc} \overline{D9_1} & \overline{D9_2} & D9_1 & D9_2 \\ \mathcal{O}(-(a+1)G) & \mathcal{O}((a-4)G) & \mathcal{O}(aG) & \mathcal{O}((3-a)G)\ . \end{array}$$

It is important to stress that such gauge freedom was absent in the previous configuration, because $\mathrm{Hol}B = 0$ there, and thus the choice $B = 0$ was *canonical* (see subsection 1.3.2).
If one computes the interval for a in this new configuration, one easily finds:

$$0 \le a \le 3\,, \tag{6.33}$$

where the extrema are now integral and the bound can be honestly saturated.

It is useful to verify, as a crosscheck, that the D3 charge computation, based as before on the rational K-theory formula (3.20), gives, upon saturation of the bound (i.e. by killing the gauge contribution), precisely the same number as the F-theory geometric tadpole. But now it is crucial to remember that the Chern character of the K-theory class on the $\overline{D9} - D9$ system must be represented by the exponential of the gauge invariant combination $B + F$, that in this particular case turns out to be also quantized (although only in terms of half-integers), because of the constraint of the orientifold projection on the B-field. So in this case $\exp(B + F)$ is the only available concept of gauge-induced lower-dimensional D-brane charges and it should enter the Wess-Zumino action of the $\overline{D9} - D9$ system.
Therefore, the new contribution due to the D9's and image-D9's will be:

$$\Gamma^B_{D9} = \left(e^{(a+\frac{1}{2})G} + e^{(\frac{7}{2}-a)G} - e^{-(a+\frac{1}{2})G} - e^{(a-\frac{7}{2})G}\right) \cdot \left(1 + \frac{c_2(X_3)}{24}\right)\,, \tag{6.34}$$

while the contribution of the O7-plane remains obviously the same as in (6.26).
Putting everything together, one finds again vanishing D9's and D5's net charges, the right D7-brane charge ($8G$) and finally Q_{grav} as in (6.29), while for the gauge contribution to the D3-tadpole:

$$Q^B_{gauge} \;=\; 4\,a\,(a-3)\,G^3\,, \tag{6.35}$$

which can manifestly be killed by saturating the new bound (6.33) and there is no real topological obstruction to choose a vanishing gauge flux on the D7-brane. Notice that in the case the class G is *even* the bound (6.22) can be saturated. Consequently, the B-field just introduced becomes integrally quantized and hence one can more easily choose the canonical gauge $B = 0$.

Finally note that that the M-theory origin of this B-field is *not* G_4, because the associated H-field vanishes identically for Freed-Witten anomaly cancellation (for B_3, $W_3 = \zeta = 0$, in the notations of section 1.5). It is rather the holonomy of the C_3-field (connection on the membrane 2-gerbe) that generates this B via the F-theory limit.

[8] On B_3, instead, this is exactly what has been said above, namely $2B \bmod 2 = w_2(B_3)$. Hence, $\mathrm{Hol}B \in H^2(X_3, \Gamma_2)$, where Γ_2, as in section 1.4, represents the subgroup of S^1 given by the square roots of unity.

O3-planes

This paragraph contains some comments about the possible presence of O3-planes in F-theory compactifications, resulting in codimension four singularities of different nature with respect to the non-abelian one discussed in the next section.

First of all, notice that the $\mathbb{Z}_2$-orbifold singularity of B_3 represented by the O7-plane is only an artifact of the perturbative limit. Indeed, as seen in subsection 4.2.1, the O7 should really be thought of as a bound state of two mutually non-local 7-branes placed at a non-perturbatively small distance of the order of e^{-1/g_s}. Thus the F-theory background, which resolves this distance, does not feel any singularity at all.

A different story, instead, is the one related to other possible fixed loci of the orientifold involution: there can be, in fact, codimension three such loci (points) on the Calabi-Yau threefold, which become singularities at the quotient. These are interpreted as O3-planes and, since they are blind to the S-duality, they contain no charge for the axio-dilaton and thus the elliptic fiber does not degenerate on them. For this reason, F-theory does not see them and they generate point-like orbifold singularities also on the elliptic Calabi-Yau fourfold. This can be rigorously shown as follows.

An O3-plane is present, for instance, whenever the point described[9] by $\xi = 1$ and all the coordinates of X_3 set to zero is fixed under the orientifold involution, due to some projective rescaling[10] $\xi \to \lambda\xi$. Let $h \equiv \xi^2$, according to the notations of subsection 4.2.2, the coordinate that substitutes ξ in the quotient B_3: due to the fact that h scales with λ^2, B_3 has a $\mathbb{Z}_2$-orbifold singularity in $h = 1$. The question is how this singularity is seen by the elliptic fibration via its Weierstrass representation (4.23). In this situation, (4.23) simplifies to:

$$Y^2 = X^3 + ah^2XZ^4 + bh^3Z^6\,, \tag{6.36}$$

where the Sen parameterization of f and g, (4.29), has been used and a, b are constants which take into account the now present dependence on h also of all the other terms in (4.29), apart from the first. The two projective rescalings of the four homogeneous coordinates appearing in (6.36) are:

$$\begin{aligned} (h,X,Y,Z) \sim (\lambda^2 h, \lambda^2 X, \lambda^3 Y, Z) \quad &\Longleftrightarrow \quad (h,X,Y,Z) \sim (\lambda^2 h, X, Y, \lambda^{-1} Z)\,, && (6.37)\\ (h,X,Y,Z) \quad &\sim \quad (h, \mu^2 X, \mu^3 Y, \mu Z)\,, && (6.38)\end{aligned}$$

where the first line displays two different and equivalent projective weight assignments for the rescaling of the base coordinates (the second choice is obtained by subtracting the μ-weights from the λ-weights of the first choice). It is easy to see, then, that as long as $Y \neq 0$ and $Z \neq 0$, one does not get any orbifold singularity in $h = 1$, because any of the two choices of basis for the first rescaling fixes completely the sign freedom of λ.

However, if one among Y and Z vanishes, one immediately gets singularities. Indeed, take for example $Y = 0$. Thus, one can fix the μ-rescaling (6.38) by choosing $Z = 1$, since, (6.36) with $Y = 0$ imposes $Z \neq 0$; then, by adopting the first choice of basis in (6.37), it is evident that the points $(1, X_i, 0, 1)$, where X_i $(i = 1, 2, 3)$ solve the cubic equation $X^3 + aX + b = 0$, are fixed under the gauge transformation $\lambda = -1$: Hence these are generically three points with a $\mathbb{Z}_2$-orbifold singularity.

[9]The circumstance is a bit different from the general setting outlined in subsection 4.2.2, since here ξ is treated as a *toric* coordinate of X_3, and thus not as a polynomial function of the others.

[10]This is what happens, for example, in the case of the quintic, discussed in [50].

Similarly, if $Z = 0$, eq. (6.36) imposes $X \neq 0$ and one can fix it to 1 using part of the μ-gauge (6.38); then, by choosing the second basis in (6.37), it is evident that the points $(1, 1, \pm 1, 0)$ are fixed under the gauge transformation $\lambda = -1$; but this is actually just *one point*, because the residual freedom in the μ-rescaling exchanges $(1, 1, 1, 0)$ with $(1, 1, -1, 0)$: Hence this is one point with a $\mathbb{Z}_2$-orbifold singularity.

The presence of O3-planes enters crucially the above discussion as far as the integrality of the gravitational tadpole (6.29) is concerned. Indeed, while $c_2(X_3)$ is always even for the argument of page 105, the first term is not always integral. However, if O3-planes are absent, B_3 is a perfectly regular manifold from the F-theory point of view, and thus one can compute its holomorphic Euler number by means of the Hirzebruch-Riemann-Roch theorem. First, notice that, thanks to the strict SU(4) holonomy, the elliptic Calabi-Yau fourfold Z_4 has no holomorphic forms apart from the constant and the unique (4,0)-form; hence, $h^{0,1}(B_3) = h^{0,2}(B_3) = h^{0,3}(B_3) = 0$, because, otherwise holomorphic forms 1,2 or 3-forms would be generated on Z_4 by pull-back. Therefore, one has:

$$1 = \chi_0(B_3) = \int_{B_3} \mathrm{Td}(B_3) = \frac{1}{24} \int_{B_3} c_1(B_3) c_2(B_3) \,. \tag{6.39}$$

But since $c_1(B_3) = G$ and, by adjunction, $c_2(X_3) = c_2(B_3)|_{X_3} + G^2|_{X_3}$, one gets:

$$\frac{1}{2} \int_{X_3} c_2(X_3) G = 24 + \frac{1}{2} \int_{X_3} G^3 \,, \tag{6.40}$$

and thus, using (6.29):

$$\int_{X_3} Q_{grav} \quad = \quad 24 + 15 \int_{X_3} G^3 \,, \tag{6.41}$$

which is manifestly integral. Note that also its half, namely the physical D3-brane charge, is integral. Indeed, if B_3 is smooth, $G^3 \bmod 2 = w_2^3(B_3) = 0$, since the top cohomology group of B_3 does not contain torsion. Taking into account formula (6.31), this independently confirms that in the smooth case $\chi(Z_4)$ is a multiple of 24.

This is just a particular manifestation of a more general fact. Indeed, as will be proven below,

$$\int_{X_3} G^3 \bmod 2 \quad = \quad \# O3 \bmod 2 \,. \tag{6.42}$$

Therefore, in the absence of O3-planes one recovers in (6.29) the integrality of the tadpole proved before, while if they are present in odd number (like in the quintic example), then the gravitational tadpole is half-integral. However, in the latter case, one should add to it the contribution of the O3's themselves to the D3-brane tadpole, which, as a consequence of eq. (6.31), are not seen by the naïve F-theory computation of the geometric tadpole [50]. This is maybe explained by the high codimension (four, as previously seen) of the elliptic fibration singularities generated by the O3-planes. The contribution of $O3^{\pm}$-planes to the D3-brane charge, as measured from the covering space X_3, is (see formula (6.25)):

$$\int_{X_3} Q_{O3} \quad = \quad \pm \frac{\# O3}{2} \,, \tag{6.43}$$

which exactly compensates the half-integrality of the first term in (6.29), giving a perfectly integral gravitational tadpole.
Eq. (6.42) can be proven as follows. Independently of the presence of O3-planes, the Calabi-Yau threefold X_3 and the O7-plane are perfectly regular varieties. Thus, applying adjunction, $c_2(O7) = c_2(X_3)|_{O7} + G^2|_{O7}$, and then:

$$\chi(O7) = \int_{O7} c_2(O7) = \int_{X_3} c_2(X_3)G + G^3 \,, \tag{6.44}$$

which implies that $\chi(O7) \mod 2 = \int_{X_3} G^3 \mod 2$. If one now calculates the index on X_3 equivariant under the $\mathbb{Z}_2$ orientifold involution, one gets:

$$L_\sigma = \sum_{i=0}^{6}(-1)^i(b^i_+ - b^i_-) = 2h^{0,0} + 2\,(h^{1,1}_+ - h^{1,1}_-) + 2\,(h^{2,1}_- - h^{2,1}_+ + h^{3,0})\,, \tag{6.45}$$

where b^i are the Betti numbers of X_3 and the subscript $\pm$ refers to the behavior under σ of the corresponding forms. In the second equality of (6.45) the following information have been used: $h^{1,0} = h^{2,0} = 0$, due to the strict SU(3) holonomy of X_3; the unique holomorphic three form of X_3 is odd under the involution; the volume form of X_3 is σ-invariant, thus the scalar product between $H^{1,1} = H^{1,1}_+ \oplus H^{1,1}_-$ and $H^{2,2} = H^{2,2}_+ \oplus H^{2,2}_-$ respect the $\pm$ direct decomposition, and hence $h^{1,1}_\pm = h^{2,2}_\pm$; similar situation holds for the group $H^{2,1}$. But, on the other hand L_σ is equal to the Euler number of the loci of X_3 fixed under σ:

$$L_\sigma = \chi(O7) + \#\, O3\,. \tag{6.46}$$

Therefore, since (6.45) is manifestly an even number, (6.46) and (6.44) together imply (6.42).

Remark To conclude this section, it is important to mention that the integrality of the physical D3-brane charge (i.e. as computed on the base manifold B_3) remains an open issue when O3-planes are present. Indeed, it has been proven above that independently of the presence of O3-planes, the D3-brane charge measured from the Calabi-Yau double cover is always integer, but *not* that it is always an even number! It turns out that actually, if the absolute value of the net O3-plane charge, $|n_{O3^+} - n_{O3^-}|$, is odd, then only one of its signs leads to an integral D3-brane charge as measured on B_3 (the other choice gives a semi-integer value, which is unacceptable). For example, in the quintic case considered in [50], one has only one O3-plane, but only if the latter is an $O3^+$ one gets an *integer* number of D3-branes to be added for tadpole cancellation. Therefore the type of the most abundant O3-planes present seems to be a crucial ingredient as far as the integrality of the D3 tadpole is concerned. Along the lines of [84], there should be a connection between the appearance of such $O3^+$ and of a B-field in the bulk, which anyway is necessarily present, according to formula (6.42), if the number of O3-planes is odd. Nevertheless, such a conjecture will not be analyzed any further in this thesis.

6.3 Singular Calabi-Yau fourfolds

As anticipated in the previous section, the case of singular elliptic fourfolds, which is obviously more appealing for GUT model building, will be now treated in detail. The results found for the smooth case will not hold any more essentially because of two reasons: first, after blow-up of the singularity, one looses in general the Weierstrass representation; second, the appearance

of exceptional divisors makes the separation between Poincaré-preserving and breaking G-fluxes no longer as clear as in the previous case. Moreover, one should expect any possible odd value of the second Chern class of the blown-up fourfold to be detected on 4-cycles of the exceptional divisors[11]: indeed, outside of them, the geometrical description of the elliptic fibration remains the same as in the smooth case, and also c_2, being quantized, stays constant during the continuous blow-down process. Such odd values can then be related to the possible absence of spin structure on the stack of D7-branes, which now wraps a perfectly regular divisor of B_3. This would turn into a Freed-Witten-like gauge flux on the D7's whose elimination is topologically obstructed, due to its shifted quantization condition. This effect could finally be verified in the Sen limit, as done before for the smooth case.
All these facts will be actually shown to happen in the next subsection, in a simple toy model in which an $SU(2)$ singularity is forced by hand on a generic (non-singular) divisor of the base $B_3 = \mathbb{P}^3$. A general pattern will be conjectured too, based on Fulton's formula of the total Chern class of blown-up manifolds [85, 86].
In subsection 6.3.4 the case of general $Sp(N)$ singularities (which of course contains the previous $SU(2)$ case) will be addressed. Again, a final answer for the quantization of the gauge flux and formulae for gauge and gravitational contributions to the D3 tadpole will be given, starting from the $\mathbb{P}^3$ toy model and then generalizing the results to any B_3 and any degree of the O7. Moreover, it will be made clear which element of the Cartan torus of a given non-abelian theory the gauge flux shifted in its quantization corresponds to. In any case, as will be described in the next sections, it is undoubted that the gauge flux corresponding to the affine node of the extended Dynkin diagram of a given singularity remains always integrally quantized: this node, in fact, is always present (see appendix F), *also* in the smooth situation, and, at least in unitary gauge theories, it represent the U(1)-trace factor whose gauge boson decouples in the non-abelian case.

The general formulae derived throughout this section are checked in the specific working example considered by calculations performed on SAGE [87] for $Sp(N)$ with $N = 1, \ldots, 4$.

6.3.1 SU(2) singularities

In this subsection, an explicit and concrete toy model will be constructed in detail, in which the F-theory elliptic Calabi-Yau fourfold suffers from a non-abelian singularity of the simplest possible Kodaira type, namely I_2 (see table 4.2). For simplicity a toric base will be chosen, the projective threefold $\mathbb{P}^3$, and, using the Tate classification of table 4.2, the transverse moduli of a generic holomorphic, non-singular divisor $S_2 \subset B_3$ will be constrained by hand in order to generate on it an $SU(2)$ singularity of the total space of the fibration. Then, by means of toric methods, a single blow-up of such fourfold Z_4 will be performed, which is sufficient to completely resolve the singularity. Finally, the second Chern class of the blown-up fourfold $\tilde{Z}_4$ will be computed and analyzed as compared to the first Chern class modulo 2 of S_2 (i.e. $w_2(S_2)$). The reader is referred to appendix F for notations and some mathematical details about toric blow-up's that will be needed in the sequel.

Suppose for the moment one constrains the Calabi-Yau fourfold moduli in order to force a type I_2 Kodaira singularity on the simple toric locus $X = Y = x_1 = 0$: Tate's classification, then, prescribes to drop from the Weierstrass equation (4.32) every monomial which is not at least quadratic in those three coordinates (otherwise one would not get any singularity of the total space; see eq. (4.33), with σ playing the role of x_1). The analysis of this situation is pretty

[11] In particular on the ones that collapse after blow-down, which are $\mathbb{P}^1$-fibrations over a divisor of the brane worldvolume S_2.

much like the one presented in appendix F, except that here there are two more base coordinates, x_3 and x_4, which anyhow, at any time, will simply play the role of spectators. Therefore, the fan of the resolved ambient toric fivefold $\tilde{M}_5$, which is a $W\mathbb{P}^2_{2,3,1}$-fibration over $\mathbb{P}^3$, will be made by the following lattice vectors:

$$
\begin{array}{lccccccccccc}
 & \left\{\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array}\right. & (1 & , & 0 & , & 0 & , & 0 & , & 0) & x_1 \\
 & & (0 & , & 1 & , & 0 & , & 0 & , & 0) & X \\
 & & (0 & , & 0 & , & 1 & , & 0 & , & 0) & Y \\
\text{canonical} & & (0 & , & 0 & , & 0 & , & 1 & , & 0) & x_3 \\
 & & (0 & , & 0 & , & 0 & , & 0 & , & 1) & x_4 \\
 & & (0 & , & -2 & , & -3 & , & 0 & , & 0) & Z \\
 & & (-1 & , & -8 & , & -12 & , & -1 & , & -1) & x_2 \\
\hline
\text{additional} & & (1 & , & 1 & , & 1 & , & 0 & , & 0) & v \quad .
\end{array} \tag{6.47}
$$

One can realize from (6.47) that the last two entries of the lattice vectors play no essential role in the blow-up procedure and hence one can disregard the last two coordinates of the base, x_3 and x_4, which span the 4-5 plane in the five dimensional lattice. The variety represented by this fan is also readily recognized as a fibration on a $\mathbb{P}^3$, with the following projection map:

$$\pi \,:\, (x_1, x_2, x_3, x_4, X, Y, Z, v) \longrightarrow (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) \,=\, (vx_1, x_2, x_3, x_4)\,. \tag{6.48}$$

Thus, the fan of the fiber is generated by the coordinates whose non-zero entries are only the second and the third, namely X, Y and Z. From (6.47) one can easily deduce the table of projective weights for the ambient $\tilde{M}_5$ and for the proper transform which describes the blown-up Calabi-Yau fourfold as an hypersurface of $\tilde{M}_5$:

x_1	x_2	x_3	x_4	X	Y	Z	v	proper transform
1	1	1	1	0	0	−4	0	0
0	0	0	0	2	3	1	0	6
1	0	0	0	1	1	0	−1	2

(6.49)

The relation of this toric blow-up with the one performed in subsection 4.3.1 by means of "traditional" methods [40] is exactly as described in appendix F. Finally, the Stanley-Reisner ideal of such a variety is:

$$\text{SR ideal} : \{x_1x_2x_3x_4\,;\, XYZ\,;\, x_1XY\,;\, x_2x_3x_4v\,;\, vZ\}\,. \tag{6.50}$$

It is now the moment to write down explicitly the equation of the proper transform

$$
\begin{aligned}
Y^2 + a_1(x_1v, x_i)XYZ + a_{3,1}(x_1v, x_i)x_1YZ^3 \;&=\; vX^3 + a_2(x_1v, x_i)X^2Z^2 + a_{4,1}(x_1v, x_i)x_1XZ^4 \\
&\quad +a_{6,2}(x_1v, x_i)x_1^2Z^6 \qquad\qquad i = 2, 3, 4\,,
\end{aligned} \tag{6.51}
$$

which has manifestly lost the Weierstrass representation characteristic of the pre-blow-up elliptic fibration (4.33) and where the polynomials depend on the coordinates of the new $\mathbb{P}^3$ base.
As explained in appendix F, the fiber is everywhere elliptic on the base except where the former singularity was, namely, in the new coordinates, $x_1v = 0$. Here, the fiber splits in two components: one, on $v = 0$, is the fiber of the exceptional divisor of the blow-up, and it represents

the Cartan node of the $SU(2)$ Dynkin diagram, while the other, on $x_1 = 0$, is the affine node of the extended diagram, present also in the non-singular case, and it is fibered over the divisor $S_2 \simeq \mathbb{P}^2$ with coordinates x_2, x_3 and x_4. The exceptional divisor itself is also fibered on S_2, and it is given by simply substituting $v = 0$ in eq. (6.51):

$$Y^2 + a_1(x_i)XYZ + a_{3,1}(x_i)x_1YZ^3 = a_2(x_i)X^2Z^2 + a_{4,1}(x_i)x_1XZ^4 + a_{6,2}(x_i)x_1^2Z^6 \,, \quad (6.52)$$

where $i = 2, 3, 4$. Since Z must be different from zero when $v = 0$ (see (6.109)), one can gauge fix $Z = 1$ in the above equation, which, afterwards, appears as an irreducible non-singular quadratic equation describing a $\mathbb{P}^1$ of degree *two* in the $\mathbb{P}^2$ of coordinates (x_1, X, Y). The other component, namely the affine curve fibered over S_2 is obtained by substituting $x_1 = 0$ in eq. (6.51):

$$Y^2 + a_1(x_i)XYZ = vX^3 + a_2(x_i)X^2Z^2 \,, \quad (6.53)$$

where again $i = 2, 3, 4$. Notice that X must be different from zero in the above equation, otherwise Y would be forced to be also vanishing, which is not allowed by (6.109). Therefore, one can gauge fix $X = 1$ and realize that the above equation will become linear and describe a degree *one* $\mathbb{P}^1$ with coordinates (Y, Z) by eliminating the coordinate v in favor of them.

Before entering the discussion of the second Chern class, it is worth to first probe a bit more this illustrative example, in order to see the properties of the intersections of these two components and describe their relation with D7-branes. The actual computation of second Chern class and tadpoles and the investigation of the integrality property of fluxes will be performed afterwards, directly in the general case of a non-toric S_2 of degree n. Of course, the results one would obtain in the present toric example are simply recovered by putting $n = 1$ in the general expressions which will be found later.

What one is really dealing with here is the type I_2^{ns} Kodaira singularity of table 4.2, although from the general treatment of subsection 4.3.1 it turns out that the separation between split and non-split case starts from the next step of Tate's algorithm, that is I_3 for this branch. Indeed, as it appears from the table, already for I_2 the two cases differ for the order of zero of a_2: if $a_2 = 0 \bmod x_1$, then one has the true $SU(2)$ gauge group, otherwise, in case a_2 is completely generic, one has the group $Sp(1)$ which is anyhow isomorphic to $SU(2)$.
Nevertheless, the explicit difference between the two cases shows up upon analyzing the intersection between the Cartan node and the affine node of the blown-up fiber. Such intersection is a codimension three locus in $\tilde{M}_5$ (made by two points fibered on S_2, as shown later) and, in the non-split case, it reads:

$$\begin{cases} v = 0 \\ x_1 = 0 \\ Y^2 + a_1(x_i)XYZ - a_2(x_i)X^2Z^2 = 0 \qquad\qquad i = 2, 3, 4 \,. \end{cases} \quad (6.54)$$

One can see here that the difference between split and non-split just resides in the possibility of factorizing the above polynomial: if one is in this more generic non-split case, in fact, such polynomial cannot be factorized, and thus monodromies can occur along the brane worldvolume S_2. Notice, however, that here what are undergoing monodromies are actually the two intersection points between the two components of the blown-up fiber, rather than the two components themselves (as in the next non-split cases of the algorithm; see subsection 4.3.1). So there is no folding of the Dynkin diagram, being the latter made only of one Cartan node ($\mathbf{A_1}$). On the other hand, in the less generic split case, the last monomial in (6.54) becomes $a_{2,1}(x_i)x_1vX^2Z^2$ which vanishes because of any of the above equations, and thus the polynomial factorizes. Therefore, no monodromy will exchange the two intersection points, as they are perfectly globally defined over S_2.

To see more geometrically this codimension three manifold described by (6.54), first go to a basis in which the blown-up ambient fivefold and the proper transform have the following weights:

$$\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & X & Y & Z & v & \text{proper transform} \\ \hline 1 & 1 & 1 & 1 & 8 & 12 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 6 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 2 \end{array} \quad . \tag{6.55}$$

Since, by (6.109), $Z \neq 0$ in (6.54), one can fix the second gauge by taking $Z = 1$. Then, taking into account $v = x_1 = 0$, the residual SR-ideal will be made by XY and $x_2x_3x_4$; thus, one obtains an ambient toric threefold which is a $\mathbb{P}^1_{X,Y}$-fibration over $S_2 \simeq \mathbb{P}^2_{x_2,x_3,x_4}$, as it is evident from the weights below:

$$\begin{array}{ccccc|c} x_2 & x_3 & x_4 & X & Y & \text{proper transform} \\ \hline 1 & 1 & 1 & 8 & 12 & 24 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array} \quad \sim \quad \begin{array}{ccccc|c} x_2 & x_3 & x_4 & X & Y & \text{proper transform} \\ \hline 1 & 1 & 1 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array} \quad . \tag{6.56}$$

On this ambient threefold one is imposing what remains of the proper transform, namely the equation $Y^2 + a_1XY - a_2X^2 = 0$, where X must be different from zero, otherwise Y would vanish as well. Hence, one can also fix the last gauge (second row in (6.56)) by taking $X = 1$. Finally, one arrives at the following degree eight equation in the previous ambient threefold, seen now as $W\mathbb{P}^3_{1114}$:

$$Y^2 + a_1(x_i)Y - a_2(x_i) = 0 \qquad i = 2, 3, 4\,. \tag{6.57}$$

It is clear at this point that the intersection between the exceptional divisor and the affine one is a $\mathbb{Z}_2$-fibration over S_2, namely, on each point of the brane worldvolume there are two points $Y_\pm$, which are the solutions of (6.57). The fiber of course degenerates in one point on the curve given by the vanishing locus of the discriminant of (6.57), i.e. $a_1^2 + 4a_2 = 0$.

Now, what is the physics behind all that? As said, these two points are not globally defined on S_2, but they can flip when they do not coincide. They are locally described by the two different functions $Y_\pm(x_i)$, and they represent the positions of the two D7-branes making together the stack on which the $SU(2)$ gauge theory lives. On the curve $a_1^2(x_i) + 4a_2(x_i) = 0$, the two points degenerate in only one, $Y_+(x_i) = Y_-(x_i) = -a_1(x_i)/2$, which is globally defined on S_2. In the Sen weak coupling limit, the two D7's in question are involution-invariant because one is the image of the other and the monodromy which swaps them is represented by the orientifold involution itself. Therefore, the O7 will be placed where the D7-brane intersects its image, namely where Y_+ and Y_- coincide. Hence, the O7 will wrap the degree eight divisor $h \equiv a_1^2 + 4a_2$ inside B_3. In type IIB, however, the two D7-branes are on top of each other, on the non-singular divisor $x_1 = 0$, because their separation in the F-theory lift is only on the fiber. Moreover, since on the curve $x_1 = h = 0$ the proper transform is $(Y + a_1/2)^2$, the point $Y = -a_1/2$ is apparently a singularity of the whole F-theory fibration, but actually it is not; it is a singularity only of the $\mathbb{Z}_2$-fibration over S_2. Indeed, to see this one has to take the complete proper transform (6.51) and calculate its gradient *before* restricting it to the locus in question, because otherwise

one misses the transversal directions which actually make the locus smooth. The intersection between the O7 and the D7's stack is in fact non-singular because for example the derivative of (6.51) with respect to v evaluated on this locus is X^3, which as seen before, must be different from zero.
A more systematic proof that, after the blow-up induced by the lattice vector v of the SU(2)-singular Calabi-Yau fourfold, no further singularity (even in lower codimension) is going to remain can be found in the appendix G. The argument given there may be generalized to worse singularities, obtaining the result that any singularity will disappear once the series of blow-up's prescribed by the list of lattice vectors corresponding to the given singularity [44] has been completely performed.

It is now the right moment to finally introduce numbers into the discussion. In order to compute the second Chern class of the blown-up Calabi-Yau fourfold, the hypothesis of the $SU(2)$ singularity being on a toric divisor of $\mathbb{P}^3$ will be relaxed, allowing a generic divisor described by the equation $P_n(x_1, \dots, x_4) = 0$ to be the worldvolume of the D7-brane stack, where P_n is a polynomial of degree n in $\mathbb{P}^3$. Since such S_2 is not in general toric, a trick is necessary in order to treat it in the same way as in the previous case. It is sufficient to add a new coordinate to the ambient fivefold, σ (with exactly the same role as in subsection 4.3.1), and a new equation, $\sigma = P_n(x_1, \dots, x_4)$. The projective weight assignments of the ambient sixfold, of the two equations defining the proper transform will then be:

$$\begin{array}{ccccccccc|c|c} x_1 & x_2 & x_3 & x_4 & \sigma & X & Y & Z & v & \text{eq.(6.59)} & v\sigma = P_n \\ \hline 1 & 1 & 1 & 1 & n & 8 & 12 & 0 & 0 & 24 & n \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & 2 & 0 \end{array} \quad , \tag{6.58}$$

$$Y^2 + a_1 XYZ + a_{3,1}\sigma Y Z^3 = vX^3 + a_2 X^2 Z^2 + a_{4,1}\sigma X Z^4 + a_{6,2}\sigma^2 Z^6 \,, \tag{6.59}$$

with a_i being polynomial functions of $x_1 \dots x_4$. The SR-ideal of the ambient toric sixfold is:

$$\text{SR ideal} : \{x_1x_2x_3x_4\sigma \,;\, x_1x_2x_3x_4 v \,;\, XYZ \,;\, \sigma XY \,;\, vZ\} \,. \tag{6.60}$$

Notice that on the blown-up Calabi-Yau fourfold $x_1 \dots x_4$ cannot vanish all at the same time because of the first two element in (6.60) and of the equation $v\sigma = P_n$. Finally, as before, the relation with the traditional blow-up gives exactly the expected weights for the s, t, u coordinates (see appendix F).
The second Chern class of the blown-up elliptic Calabi-Yau fourfold $\tilde{Z}_4$, defined by (6.58-6.59) is, by adjunction:

$$c_2(\tilde{Z}_4) \;=\; \underbrace{11F^2 + 92FH + 182H^2}_{c_2(Z_4)} \;+\; \underbrace{(n-28)EH}_{\Delta c_2} \,, \tag{6.61}$$

where H is the hyperplane class of $\mathbb{P}^3$ (with weights $(1,0,0)$), E is the class of the exceptional divisor $v = 0$ (with weights $(0,0,-1)$) and F, differently from the notation of section 6.2, represents the class of the 0-section (gravitational brane) $Z = 0$ (with weights $(0,1,0)$). Notice that, with this choice of basis, $EF = 0$ because vZ is an element of the SR-ideal of the ambient toric sixfold. The first piece in (6.61), $c_2(Z_4)$, is exactly the second Chern class of the pre-blow-up Calabi-Yau fourfold as it would turn out by applying adjunction in case it was non-singular; the second piece, Δc_2, instead, is the addition due to the blow-up, which of course depends on

the exceptional divisor class.
The intersection numbers of $\tilde{Z}_4$ have been calculated with the help of the toric package in SAGE [87]. The results are as follows. The geometric tadpole of F-theory is:

$$\frac{\chi(\tilde{Z}_4)}{24} = 972 - \frac{1}{4}n(n-28)^2 \,, \tag{6.62}$$

which is manifestly integral when n is even and a quarter of integer if n is odd. In the latter case, therefore, by Witten's argument mentioned in section 6.2, the second Chern class of $\tilde{Z}_4$ must be *odd* and hence should give rise to a Freed-Witten-like gauge flux on the D7's. This is as expected because, when n is odd, the stack of D7-branes is wrapping a *non-spin* divisor of $\mathbb{P}^3$, and consequently a half-quantized worldvolume flux must be added in order to cancel the Freed-Witten anomaly of open strings. Nevertheless, the integral of $c_2(\tilde{Z}_4)$ on every 4-cycle whose Poincaré dual is the product of two among the three classes H, F and E, turns out to be even. This is a consequence of the quadratic nature of the exceptional fiber described by equation (6.52). Roughly speaking, in order to find an odd number, one should integrate c_2 on a 4-cycle whose Poincaré dual is $HE/2$ where $E/2$ restricts to one leaf of the exceptional $\mathbb{P}^1$-fibration over S_2 and H further restricts to a curve of S_2. This odd number comes of course from the integration of the additional piece Δc_2 in (6.61), and it is $-n(n-28)$, which has the same parity as n, as it should be. This argument also agrees with the expectation that the flux corresponding to the affine node of the resolved fiber is never going to get a shifted quantization condition: indeed, as said below formula (6.53), the affine node is a linear $\mathbb{P}^1$ and therefore no division by 2 must be performed in integrating c_2.
Hence, one naturally recognizes the half-quantized flux arising this way to be the one associated to the Cartan generator of $SU(2)$ and not to the $U(1) \subset U(2)$ which eventually gets decoupled (affine node). Indeed, the latter is present also in the non-singular case, where it was proven in subsection 6.2.1 that the half-quantization of the gauge flux is *fake*, because compensated by turning on a (necessary) discrete B-field. As a confirmation of this fact, if one integrates $c_2(\tilde{Z}_4)$ on the Poincaré dual of $H(nH-E)$ one gets an even integer for all n.
Moreover, the two pieces of (6.61) are orthogonal, i.e. $c_2(Z_4) \cdot \Delta c_2 = 0$ when integrated on $\tilde{Z}_4$, and thus there is no mixed contribution to the minimal G-flux induced tadpole of F-theory, $-c_2(\tilde{Z}_4)^2/8$. Hence, the non-integrality of the geometric tadpole of $\tilde{Z}_4$ (6.62) should be compensated by the compulsory addition of $-(\Delta c_2)^2/8$, which is due to the blow-up. Indeed:

$$\frac{\chi(\tilde{Z}_4)}{24} - \frac{(\Delta c_2)^2}{8} = 972 = \frac{\chi(Z_4)}{24} \,. \tag{6.63}$$

This number is *equal* to the geometric tadpole *alone* computed on the pre-blow-up Calabi-Yau fourfold Z_4 as it was non-singular. Such a nice result is physically explained as follows.
From the type IIB point of view, such blow-up transition is perfectly smooth: One is not changing the Calabi-Yau threefold, but rather there is a recombination/separation of branes via tachyon-condensation processes (see subsection 6.3.2); these processes are allowed if all the charges are conserved (kinematics) *and* if the D-terms admit the splittings (dynamics). From a 4-dimensional point of view, nevertheless, they are two different vacua of the theory; in M-theory they are also different vacua, because the transition here is much more complicated and one is really moving in the enlarged (because of the brane deformation moduli) Kähler moduli space of Calabi-Yau fourfolds. However, the duality between F and M-theory (see section 4.4) assures that the number of membrane to be added before the transition to cancel the tadpole is *equal* to the one after the transition. This constraint, indeed, although not required a priori, comes

from the IIB kinematic constraint of conservation of the D3-brane charge, which is necessary for the above mentioned recombination/separation processes. Therefore, having found the same D3-tadpole after the blow-up procedure constitutes a non-trivial consistency check of the correctness of the whole reasoning[12].

It is worth making one last observation. The 4d vacuum arising after the smooth transition mentioned above cannot be supersymmetric because the minimal G_4-flux chosen cannot fulfill the self-duality condition. Indeed, $G_4 = \Delta c_2/2$ has a negative square, as can be deduced from eq. (6.62). Therefore in the orbit of vacua spanned by brane recombination/separation processes, which keep the D3-brane charge to the value 972, the only supersymmetric point is the vacuum corresponding to the smooth F-theory configuration, where a vanishing G_4-flux has been chosen.

Sen's limit of the SU(2) configuration

To conclude the treatment of the $SU(2)$ singularity over a generic divisor of $\mathbb{P}^3$, the Sen limit of such configurations will be discussed here focusing on some interesting features it shows.

First of all, the configuration with type I_2^s singularity is intrinsically non-perturbative [88], although the gauge group $SU(2)$ would not suggest it. In this split case, the equation $h = a_1^2 + 4a_{2,1}x_1 = 0$ defining the orientifold is non-generic and the O7 intersects itself on the locus where also the D7-branes sit, $x_1 = 0$ (like the Whitney umbrella D7-brane). Moreover, the equation $\xi^2 = h(\vec{x})$ of degree 8, which defines the double cover Calabi-Yau threefold in the ambient $W\mathbb{P}^4_{1,1,1,1,4}$, is singular in the codimension four locus in the ambient (which restricts to a codimension three one in the Calabi-Yau) $\xi = a_1 = a_2 = x_1 = 0$: this is a point-like singularity of a conifold type. By deforming the Calabi-Yau threefold with a term in a_2, Δa_2, without a factor of x_1 in front, i.e. $\xi^2 - a_1^2 - 4a_{2,1}x_1 = \Delta a_2$, one ends up with the Calabi-Yau threefold corresponding to the more generic non-split case, I_2^{ns}. Here everything is fine and a natural Sen limit exists, since such $Sp(1)$ configuration can be perturbatively realized.
In the perturbative $SU(2)$ configuration one has to deal, as seen, with a D7-brane with coinciding D7-image, which intersects transversally the O7. The D7-brane and its image (counted separately) carry a total charge of $2nH$, where $n < 16$ is the degree of the polynomial defining their worldvolume and H is the hyperplane class of $\mathbb{P}^3$. But, since a total D7-brane charge of $32H$ is needed in order to cancel the tadpole (the O7 has charge $8H$ in $\mathbb{P}^3$), there must be an orientifold invariant Whitney umbrella D7-brane, with the usual $O(1)$ gauge group, carrying charge $(32 - 2n)H$. As opposed to the Sen limit of the smooth case, discussed in subsection 6.2.1, here one needs 4 D9-branes and 4 anti-D9-branes for the K-theory description of the configuration. The tachyon of the system is given by:

$$T = \begin{pmatrix} \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} + \xi \begin{pmatrix} \rho & \psi \\ \psi & \tau \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & P_n \\ -P_n & 0 \end{pmatrix} \end{pmatrix}, \tag{6.64}$$

so that the total tadpole-canceling D7-brane wraps the manifold:

$$\det T = P_n^2(\eta^2 + \xi^2(\rho\tau - \psi^2)) = 0\,. \tag{6.65}$$

[12] It is not at all clear a priori why one should take the c_2 of the blown-up Calabi-Yau to be the characteristic class which really contains the physical information about flux quantization, despite the fact that it is the most natural candidate.

Following the prescription given in [58] to realize on $P_n = 0$ the $Sp(1)$ stack of D7-image-D7 with gauge group-breaking flux in the adjoint, the right configuration of D9 and anti-D9-branes is:

$$
\begin{array}{cccc}
\overline{D9_1} & \overline{D9_2} & \overline{D9_3} & \overline{D9_4} \\
\mathcal{O}((n-14)H) & \mathcal{O}(-2H) & \mathcal{O}\left((14-n)H\right) & \mathcal{O}\left(-14H\right) \\
 & & & \\
D9_1 & D9_2 & D9_3 & D9_4 \\
\mathcal{O}((14-n)H) & \mathcal{O}(2H) & \mathcal{O}\left((n-14)H\right) & \mathcal{O}\left(14H\right) .
\end{array}
$$

In order to find the above assignments for the Chern classes of the various bundles in the Whitney sum, four independent conditions have been imposed[13]:

1. right total D7-brane charge, i.e. $32H$;
2. degree n for the polynomial P_n;
3. singular D7 of generic Whitney umbrella shape, i.e., for example, $\deg \tau = 0$;
4. gauge flux on the transverse D7 stack as expected, from the F-theory computation, to give rise to the right gauge contribution to the D3 tadpole (see eq (6.62)).

Therefore, the first couple of D9's and anti-D9's is responsible for the Whitney umbrella D7-brane, while the second one for the $Sp(1)$ stack. Notice that for consistency one must require $14 - n > 2$, i.e. $n < 12$, otherwise one looses the generic shape of the Whitney umbrella.

Out of the above $D9 - \overline{D9}$ system, one can compute all the induced lower dimensional D-brane charges.

The D9-brane and the D5-brane net charges vanish as they should because of the orientifold projection. The D7-brane charge, instead, by construction is the right one to cancel the 7-brane tadpole, namely $32H$.

For the D3-brane charge, as usual (see subsection 6.2.1), there is the contribution from the $D9 - \overline{D9}$ system and the one from the O7-plane. Using that the triple intersection number of the hyperplane class H in $W\mathbb{P}^4_{1,1,1,1,4}[8]$ is equal to 2, a simple calculation leads to the following expected result:

$$
Q_{D3} \;=\; 1944 \;=\; 2 \times \frac{\chi(Z_4)}{24}\,, \tag{6.66}
$$

which coincides with the total tadpole predicted by the F-theory lift (6.63) as computed in the covering space. Now, in order to separate in (6.66) the gauge contribution from the gravitational one, recall [7] that the effective D3-brane charge induced by the D7-brane-type fluxes ensuing upon weak coupling limit from G_4 is:[14]

$$
-\frac{1}{2}\int_{Z_4} G_4 \wedge G_4 \;=\; \frac{1}{2}\int_{S_2} \mathrm{Tr}\, F \wedge F \;=\; \mathrm{ch}_2(F)\,. \tag{6.67}
$$

By construction, the gauge flux on the D7 stack is of the form [58]:

$$
F \;=\; \frac{1}{2}(28-n)\, H \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.68}
$$

[13] The B field here is absent since the degree of the O7 in X_3 is even. Therefore the line bundles on the anti-D9's are just the inverses. The general situation is presented in subsection 6.3.4.

[14] The bulk fluxes part is put to zero here because, as said, no shifted quantization rules arise for them.

where the two by two matrix is thought of as taking into account the opposite contributions of the D7-brane and of its image. This is a flux in the adjoint of $SU(2)$, aligned with its Cartan generator. The D3-tadpole contribution, as seen by the Calabi-Yau threefold, induced by the gauge flux in (6.68) is thus:

$$\int_{S_2} \mathrm{ch}_2(F) = \frac{1}{2}\int_{X_3} \frac{1}{4}\, 2\,(n-28)^2 H^2 \cdot nH = \frac{n}{2}\,(n-28)^2 \,. \tag{6.69}$$

This is exactly the amount one expects from the F-theory calculation, namely $-2 \times (\Delta c_2)^2/8$. Consequently, by subtracting it to the total tadpole (6.66), one obtains twice the geometric tadpole (6.62). Notice that in (6.69), the integration on S_2 is perfectly well-defined, since the worldvolume of the D7-stack is non-singular. Moreover, the flux (6.68) does not induce any D5-brane charge, since it is traceless and so the first Chern character vanishes. This is again consistent with the orientifold projection[15].
It is important to remark that, for n odd the half-quantized D7-brane gauge flux (6.68), required for Freed-Witten anomaly cancellation of open strings attached to the non-spin D7-stack, cannot be put to zero. Therefore it necessarily breaks $SU(2)$ to its Cartan torus $U(1)$.

6.3.2 Bound states via tachyon condensation

As already mentioned in the previous subsection, one should be able to connect the $SU(2)$ configuration just described to an other description of the same type IIB supersymmetric vacuum with a single, recombined Whitney umbrella brane carrying all the D7-charge. This is suggested at the kinematic level by the equality of the total D3-brane tadpole (6.63). Indeed, this is possible by applying to the tachyon field (6.64) a double change of base of the D9's and the anti-D9's (namely two independent automorphisms of the domain and of the codomain). In this subsection, the details of such a procedure are described. The process of brane recombination is analyzed, first in the unorientifolded case as a warm-up, and then, in the orientifolded case of interest. This discussion is based on [89].

Unorientifolded case

Before directly addressing the case of interest, it is worth to begin by postponing tadpole cancellation and orientifolding. To describe a single D7-brane as a condensation of a D9/anti-D9 brane[16], recall the procedure of subsection 3.2.3 and use an exact sequence of line bundles as follows:

$$\begin{array}{ccccc} \mathcal{O}(-D+F) & \xrightarrow{T} & \mathcal{O}(F) & \rightarrow & \mathcal{O}(F)|_D \,, \\ \overline{\mathrm{D9}} & & \mathrm{D9} & & \mathrm{D7} \end{array} \tag{6.70}$$

whereby the notation $\mathcal{O}(\omega_2)$ is the line-bundle of first Chern class ω_2, and D denotes the divisor class on which the D7-brane is defined. This sequence describes a D7-brane on D with a line-bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = F - \frac{1}{2}D$. The third term in the sequence is a skyscraper-like sheaf localized on D (the cokernel of the tachyon map T). The tachyon field is then a section of the line bundle:

$$T \in \Gamma\big((\mathcal{O}(-D+F))^* \otimes \mathcal{O}(F)\big) = \Gamma\big(\mathcal{O}(D)\big)\,. \tag{6.71}$$

[15]Strictly speaking, it is $i_\# \mathrm{ch}_1(F)$, where i is the D7 embedding in B_3, that is forced to vanish. But in this case, being F restriction of a class of the target, this is equivalent to the requirement $\mathrm{ch}_1(F) = 0$.

[16]Alternatively, one can think in terms of D4-branes as D6/anti-D6 condensates.

Suppose now that two D7-branes with divisor and flux data (D_1, F_1) and (D_2, F_2) intersect along a curve. Then, the following brane recombination process can be triggered by a FI term, provided one chooses the right values for the complexified Kähler modulus:

$$(D_1, F_1) + (D_2, F_2) \quad \mapsto \quad (D_1 + D_2, \tilde{F}) \,, \tag{6.72}$$

where the recombined brane resides on a divisor class equivalent to the sum of the two constituent divisor classes. The charge vector (polyform) for a brane (D, F) is in general:

$$\begin{aligned} \Gamma_1 &= Q_{D7} + Q_{D5} + Q_{D3} \\ &= D + \left(F - \tfrac{1}{2}\,D\right) D + \left(\tfrac{1}{24}\,(c_2(X)\,D + 4\,D^3) + \tfrac{1}{2}\,F\,D\,(F - D)\right) \,, \end{aligned} \tag{6.73}$$

whereby the D7-, D5-, and D3-charges are represented by two-, four-, and six-forms, respectively. If one imposes conservation of all three types of charges during the recombination process, one arrives at the following unique constraints (modulo redefinitions):

$$F_1 - F_2 = D_1 \,, \quad \tilde{F} = F_1 , . \tag{6.74}$$

The aim is to describe this process in terms of D9/anti-D9 condensation. To this end, combine two exact sequences into one as follows:

$$\mathcal{O}(F_1 - D_1) \oplus \mathcal{O}(F_2 - D_2) \xrightarrow{T} \mathcal{O}(F_1) \oplus \mathcal{O}(F_2) \to \mathcal{O}(F_1)|_{D_1} \oplus \mathcal{O}(F_2)|_{D_2} \,. \tag{6.75}$$

Then, the entries of T are sections of the following line-bundles:

$$T \in \begin{pmatrix} \mathcal{O}(D_1) & \mathcal{O}(D_2 + F_1 - F_2) \\ \mathcal{O}(D_1 + F_2 - F_1) & \mathcal{O}(D_2) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(D_1) & \mathcal{O}(D_2 + D_1) \\ \mathcal{O} & \mathcal{O}(D_2) \end{pmatrix} \tag{6.76}$$

where in the last equality the constraints (6.74) have been used. A general Ansatz for the tachyon matrix is then:

$$T \in \begin{pmatrix} T_1 & \psi_{1,2} \\ \lambda & T_2 \end{pmatrix} \,. \tag{6.77}$$

When the D9's and anti-D9's annihilate, a D7-brane will be left along the locus where T fails to be invertible, i.e. along

$$|T| = T_1\,T_2 - \lambda\,\psi_{1,2} = 0 \,. \tag{6.78}$$

Here, one realizes that if $\psi_{1,2} = 0$, then the system corresponds to the union of two different branes on D_1 and D_2, respectively. However, switching on a vev for $\psi_{1,2}$ corresponds to recombining these two intersecting branes into a single, smooth divisor of class $D_1 + D_2$.

In order to see this more directly, one can perform basis transformations on the rank two D9-stack, and the rank two anti-D9-stack, respectively:

$$R : \mathcal{O}(F_1 - D_1) \oplus \mathcal{O}(F_2 - D_2) \quad \to \quad \mathcal{O}(F_1 - D_1) \oplus \mathcal{O}(F_2 - D_2) \quad , \tag{6.79}$$

$$L : \mathcal{O}(F_1) \oplus \mathcal{O}(F_2) \quad \leftarrow \quad \mathcal{O}(F_1) \oplus \mathcal{O}(F_2) \tag{6.80}$$

such that the tachyon transforms as

$$T \mapsto T' = L.T.R \,. \tag{6.81}$$

Choosing

$$L = \begin{pmatrix} \frac{1}{a} & -\frac{T_1}{a\,\lambda} \\ \\ 0 & -\frac{1}{b\,\lambda} \end{pmatrix}\,, \quad R = \begin{pmatrix} b & a\,T_2 \\ \\ 0 & -\lambda\,a \end{pmatrix}\,, \quad a, b \in \mathbb{C}^* \tag{6.82}$$

we find

$$T' = \begin{pmatrix} 0 & T_1\,T_2 - \lambda\,\psi_{1,2} \\ -1 & 0 \end{pmatrix}\,. \tag{6.83}$$

The $(2,1)$ entry is saying that the first anti-D9 in (6.75) with flux $\mathcal{O}(F_1 - D_1)$ is annihilating the second D9 in (6.75) with flux $\mathcal{O}(F_2 = F_1 - D_1)$, leaving behind just one non-trivial sequence:

$$\mathcal{O}(F_1 - D_1 - D_2) \xrightarrow{T_1\,T_2 - \lambda\,\psi_{1,2}} \mathcal{O}(F_1)\,, \tag{6.84}$$

corresponding to a D7 on $D_1 + D_2$ with flux F_1. Note, that one could have predicted the rule (6.74) simply by inspecting the off-diagonal elements of T, and requiring that at least one of those be a section of the bundle $\mathcal{O}(D_1 + D_2)$.

Orientifolded case

It is now time to move on to the orientifolded, D7-tadpole canceling case. For simplicity of notation, focus on the working example, the $W\mathbb{P}^4_{1,1,1,1,4}[8]$ geometry. The following 2×2 tachyon profile

$$T_2 \;=\; \begin{pmatrix} 0 & \eta_{16} \\ -\eta_{16} & 0 \end{pmatrix} + \xi \begin{pmatrix} \rho_{24} & \psi_{12} \\ \psi_{12} & \tau_0 \end{pmatrix}\,, \tag{6.85}$$

where the subscripts indicate degrees, corresponds to the Sen limit of a smooth F-theory fourfold. However, one realizes that by tuning the entries as follows:

$$\eta_{16} = \tilde{\eta}_{16-n}\,P_n\,, \quad \rho_{24} = \tilde{\rho}_{24-2n}\,{P_n}^2\,, \quad \psi_{12} = \tilde{\psi}_{12-n}\,P_n\,, \tag{6.86}$$

one arrives at a D7-configuration of the following form (dropping the degrees):

$$\det T = {P_n}^2\,\left(\tilde{\eta}^2 + \xi^2\,(\tilde{\rho}\,\tau_0 - \tilde{\psi}^2)\right)\,. \tag{6.87}$$

This corresponds to a configuration with one Whitney umbrella D7-brane of degree $32 - 2\,n$, and one $SU(2)$-stack on $P_n = 0$. On the other hand, from the techniques in [58], it is known that this configuration can be described by a rank four D9/anti-D9 pair, with tachyon of the following form:

$$T_4 \;=\; \begin{pmatrix} \xi\,\tilde{\rho}_{24-2\,n} & \tilde{\eta}_{16-n} + \xi\,\tilde{\psi}_{12-n} & \frac{\xi\,P_n\,\tilde{\rho}_{24-2\,n}}{\lambda_0} & \lambda_0 \\ -\tilde{\eta}_{16-n} + \xi\,\tilde{\psi}_{12-n} & -\xi\,\tau_0 & 0 & 0 \\ \\ \frac{\xi\,P_n\,\tilde{\rho}_{24-2\,n}}{\lambda_0} & 0 & 0 & P_n \\ -\lambda_0 & 0 & -P_n & 0 \end{pmatrix}\,. \tag{6.88}$$

This describes a Whitney umbrella brane on the upper-left 2×2 block, an $SU(2)$ stack on the lower-right block, and two off-diagonal 'binding blocks' whose degrees will be justified shortly. T_4 has the same determinant as (6.87). The aim now is to use the technology described in unorientifolded case above to connect T_4 back to T_2, in order to prove that they are physically equivalent.

To keep the total D7-charge equal to $32\,H$, one imposes that T_4 be a map $T_4 : \bar{E} \mapsto E$, where

$$E = \mathcal{O}(14-n) \oplus \mathcal{O}(2) \oplus \mathcal{O}(a) \oplus \mathcal{O}(-a+n)\,, \quad a \in \mathbb{Z}\,. \tag{6.89}$$

In order for this system to connect to the original tachyon

$$T_2 : \bar{F} \mapsto F = \mathcal{O}(14) \oplus \mathcal{O}(2)\,, \tag{6.90}$$

one is led to impose $a = 14$. Therefore, one finds here exactly the same vector bundle as the one used in the Sen limit of the perturbative $SU(2)$ configuration in subsection 6.3.1. By reshuffling the line-bundles, one can view the rank four bundle E as

$$E = F \oplus \mathcal{O}(14-n) \oplus \mathcal{O}(n-14)\,. \tag{6.91}$$

If one rewrites T_4 in the basis:

$$\begin{aligned} T_4 : &\mathcal{O}(-2) \oplus \mathcal{O}(14-n) \oplus \mathcal{O}(n-14) \oplus \mathcal{O}(-14) \\ &\mapsto \mathcal{O}(14-n) \oplus \mathcal{O}(14) \oplus \mathcal{O}(2) \oplus \mathcal{O}(n-14)\,, \end{aligned} \tag{6.92}$$

then it takes the following form:

$$T_4 \;=\; \begin{pmatrix} \tilde{\eta} + \xi\,\tilde{\psi} & \lambda & \xi\,\tilde{\rho} & \frac{\xi\,P_n\,\tilde{\rho}}{\lambda} \\ 0 & P_n & \frac{\xi\,P_n\,\tilde{\rho}}{\lambda} & 0 \\ -\xi\,\tau & 0 & -\tilde{\eta} + \xi\,\tilde{\psi} & 0 \\ 0 & 0 & -\lambda & -P_n \end{pmatrix}, \tag{6.93}$$

where the degrees have been dropped for compactness. If one inspects the 2×2 in the upper-left and lower-right corners, one recognizes the pattern of (6.77). One can now apply transformations of the form (6.82) to act on the lower and upper block separately:

$$L_l = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{-\tilde{\eta}+\xi\,\tilde{\psi}}{\lambda} \\ 0 & 0 & 0 & -\frac{1}{\lambda} \end{pmatrix}, \qquad R_l = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & P_n \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \tag{6.94}$$

$$L_u = \begin{pmatrix} -1 & 0 & 0 & 0 \\ P_n & -\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad R_u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\tilde{\eta}+\xi\,\tilde{\psi}}{\lambda} & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.95}$$

After performing these transformations and bringing everything back to the original basis:

$$T_4 : \mathcal{O}(n-14) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-14) \oplus \mathcal{O}(14-n) \mapsto \mathcal{O}(14-n) \oplus \mathcal{O}(2) \oplus \mathcal{O}(14) \oplus \mathcal{O}(n-14)\,, \tag{6.96}$$

one has:

$$T_4 = \begin{pmatrix} -\xi\,\tilde{\rho} & 0 & 0 & -1 \\ 0 & -\xi\,\tau & P_n\,(-\tilde{\eta} + \xi\,\tilde{\psi}) & 0 \\ 0 & P_n\,(\tilde{\eta} + \xi\,\tilde{\psi}) & -\xi\,{P_n}^2\,\tilde{\rho} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6.97}$$

To make this result more appealing, one can perform one more worthwile basis transformation such that

$$T_4 : \mathcal{O}(-14)\oplus\mathcal{O}(-2)\oplus\mathcal{O}(n-14)\oplus\mathcal{O}(14-n)\mapsto\mathcal{O}(14)\oplus\mathcal{O}(2)\oplus\mathcal{O}(14-n)\oplus\mathcal{O}(n-14)\,, \quad (6.98)$$

Now the tachyon profile takes the following form:

$$T_4 = \begin{pmatrix} -\xi\,\tilde\rho\,{P_n}^2 & P_n\,(\tilde\eta+\xi\,\tilde\psi) & 0 & 0 \\ P_n\,(-\tilde\eta+\xi\,\tilde\psi) & -\xi\,\tau & 0 & 0 \\ 0 & 0 & -\xi\,\tilde\rho & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (6.99)$$

One can immediately identify the upper-left block as a Whitney brane of degree 32, with its entries tuned as stated in (6.86). The lower-right block has constant determinant one, which means that it is an isomorphism between the D9-stack with $\mathcal{O}(14-n)\oplus\mathcal{O}(n-14)$ and its image anti-D9-stack. Hence, the lower-right block physically annihilates, leaving behind one recombined Whitney umbrella. This transformation goes one step further in showing that T_2 and T_4 are physically connected than just checking charge conservation.

This can be generalized to construct a rank N D7-stack on $P_n=0$ with a Whitney brane of degree $32-2\,N$. Starting with the Whitney brane of degree 32 made of a D9-stack with $\mathcal{O}(2)\oplus\mathcal{O}(14)$, and its image anti-D9-stack, one simply tags along pairs as follows:

$$\begin{aligned}&\underbrace{\mathcal{O}(2)\oplus\mathcal{O}(14)}_{\text{Whitney of deg 32}}\quad\oplus\underbrace{\mathcal{O}(14-n)\oplus\mathcal{O}(n-14)}_{\text{trivial}}\\ &\qquad\oplus\underbrace{\mathcal{O}(14-2\,n)\oplus\mathcal{O}(2\,n-14)}_{\text{trivial}}\oplus\ldots\oplus\underbrace{\mathcal{O}(14-N\,n)\oplus\mathcal{O}(N\,n-14)}_{\text{trivial}}\end{aligned} \quad (6.100)$$

and do the same for the anti-D9's. Adding these pairs is a trivial operation since their respective tachyon block will be of the form

$$\begin{pmatrix} \cdot & -1 \\ 1 & 0 \end{pmatrix}, \quad (6.101)$$

which is an isomorphism. Then, the next step is to perform a basis transformation analogous to the one performed for the $Sp(1)$ case. Instead of doing this explicitly, one can understand qualitatively what such transformations do. Before the transformation, one looks at the branes as coming in pairs as indicated in (6.100). After the transformation, the branes will be effectively paired as follows:

$$\begin{aligned}&\underbrace{\mathcal{O}(2)\oplus\mathcal{O}(14-N\,n)}_{\text{Whitney of deg }32-2n}\quad\oplus\underbrace{\mathcal{O}(14)\oplus\mathcal{O}(n-14)}_{\text{deg n brane}}\\ &\qquad\oplus\underbrace{\mathcal{O}(14-n)\oplus\mathcal{O}(2\,n-14)}_{\text{deg n brane}}\oplus\ldots\oplus\underbrace{\mathcal{O}(14-(N-1)\,n)\oplus\mathcal{O}(N\,n-14)}_{\text{deg n brane}}.\end{aligned} \quad (6.102)$$

The very first pair (with its image anti-D9 pair) corresponds to the remaining Whitney brane of degree $32-n$, and each of the following N pairs (with their respective anti-D9 pairs) will give rise to a 2×2 block in the tachyon profile, all in all corresponding to N D7-branes (with N images) on $P_n=0$.

This pattern of line bundles will be adopted in dealing with the general structure of Sp(N) stacks later on in (6.119) and (6.127).

6.3.3 General pattern for the SU(2) configuration

This subsection only contains a conjecture on how gauge flux quantization in $SU(2)$ singular F-theory compactifications would be described in the case of general base B_3. The conjecture is essentially based on Fulton's formula for the total Chern class of a blown-up manifold [85, 86].

Suppose one characterizes the original Calabi-Yau fourfold Z_4 as a (singular) hypersurface inside a five-dimensional ambient space M_5. In any case, the singularity is placed on a codimension three locus, $S_2 \in M_5$, which, for the ambient space, is a perfectly regular surface. Now, one can perform the blow-up of S_2 as submanifold of M_5, and one has the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "g"] & \tilde{M}_5 \arrow[d, "f"] \\ S_2 \arrow[r, "i"] & M_5 \end{tikzcd} \tag{6.103}$$

where E is the exceptional divisor, fibered over S_2 and $\tilde{M}_5$ is the blown-up ambient fivefold. The singularity is of multiplicity 2 in Z_4. Therefore:

$$c_1(N_{\tilde{M}_5}\tilde{Z}_4) = f^*c_1(N_{M_5}Z_4) - \mathrm{mult}_{Z_4}(S_2)\cdot E|_{\tilde{Z}_4} = f^*c_1(N_{M_5}Z_4) - 2\,E|_{\tilde{Z}_4}\,, \tag{6.104}$$

where the blow-down map restricted to $\tilde{Z}_4$, i.e. $f|_{\tilde{Z}_4} : \tilde{Z}_4 \to Z_4$, has been implicitly used. Now, in this case, Fulton's formula reads [86]:

$$c_2(\tilde{M}_5) = f^*c_2(M) - j_{\#}g^*(2c_1(S_2) + c_1(N_{M_5}S_2))\,,$$

where $j_{\#}$ is the Gysin map in cohomology induced by the embedding j. By adjunction and commutativity of diagram (6.103) one gets:

$$c_2(\tilde{M}_5) = f^*c_2(M) - E\,f^*c_1(M_5) - j_{\#}g^*c_1(S_2)\,. \tag{6.105}$$

The second Chern class of the blown-up Calabi-Yau fourfold $\tilde{Z}_4$ is easily recovered, again by adjunction, as the restriction of (6.105).

The first term in the right hand side of (6.105) represents the second Chern class of the pre-blow-up ambient fivefold and gives rise to the always present second Chern class of the original Calabi-Yau fourfold treated as it was smooth. All the rest, instead, gives rise to what has been called Δc_2 in subsection 6.3.1, i.e. the crucial additional term due to the blow-up process. Using naïvely adjunction for $S_2 \subset Z_4 \subset M_5$, one has, in the same basis as at page 119, $c_1(M_5) = 6(c_1(B_3)+F)$ and $c_1(S_2) = F + c_1(B_3) - \mathrm{PD}_{B_3}(S_2)$. Therefore, formula (6.105), after restriction to $\tilde{Z}_4$, can be rewritten as follows:

$$c_2(\tilde{Z}_4) = \underbrace{11F^2 + 23c_1(B_3)F + 11c_1^2(B_3) + c_2(B_3)}_{c_2(Z_4)} + \underbrace{E\,\mathrm{PD}_{B_3}(S_2) - 7Ec_1(B_3)}_{\Delta c_2}\,, \tag{6.106}$$

where the general fact $EF = 0$ has been used; $c_1(B_3)$ must be understood as $f^*\pi^*c_1(B_3)$ with $\pi : Z_4 \to B_3$ and $E\,\mathrm{PD}_{B_3}(S_2)$ as $j_{\#}g^*\mathrm{PD}_{B_3}(S_2)$, while F is meant pulled back by f to $\tilde{Z}_4$.

Formula (6.106) gives of course the right result (6.61) in the case discussed in subsection 6.3.1, namely $B_3 = \mathbb{P}^3$, but also gives the general structure of the second Chern class of the blown-up fourfold for any B_3 in terms only of the Chern classes of the base and of the class of

the brane worldvolume.
To extend this formula to worse singularities, in principle, one should iterate the toric blow-up procedure introducing as many new lattice vectors as the complete resolution of the given singularity requires (see [44] for a list of such vectors for the resolution of several Kodaira singularities). Consequently, also formula (6.106) gets replaced by another one whose form will depend on the particular singularity, being the result of a number of iterations.
Rather than attempting to perform such iteration, in the next subsection the general expression for the gauge field and the two tadpole contributions will be provided purely in the type IIB context for the entire series of $\mathbf{C_N}$ gauge algebras (i.e. symplectic groups $Sp(N)$). The results will be found by analyzing the Sen weak coupling limit of such F-theory configurations; from them one will recognize the $SU(2)$ results of this subsection by simply putting $N = 1$ ($Sp(1) = SU(2)$).

6.3.4 Sp(N) singularities

Before embarking in the full generalization of the D7-brane gauge flux quantization rules for the Sp(N) family of F-theory configurations, it is instructive to first present in detail the case of $Sp(2) \sim SO(5)$ singularity on a toric degree one divisor of $\mathbb{P}^3$ (for example, $x_1 = 0$). In this case (and actually for $Sp(3)$ and $Sp(4)$ too) a crosscheck with the available computer results of SAGE [87] has been made, finding complete agreement.

In order to completely resolve the $Sp(2)$ singularity (Kodaira type I_4^{ns}), one is required to add to the fan of the ambient fivefold M_5 the *two* additional vectors [44] $v_1 = (1, 1, 1, 0, 0)$ and $v_3 = (1, 2, 2, 0, 0)$, where the first one is exactly the only one needed for $SU(2)$.
It is easy to deduce the following projective weights for the various homogeneous coordinate in the game:

$$\begin{array}{ccccccccc|c} x_1 & x_2 & x_3 & x_4 & X & Y & Z & v_1 & v_3 & \text{proper transform} \\ \hline 1 & 1 & 1 & 1 & 8 & 12 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 0 & 6 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 2 \end{array} \quad . \tag{6.107}$$

Therefore, the blow-up generated by v_3 is along the codimension three locus in $\tilde{M}_5$ given by $X = Y = v_1 = 0$, which has multiplicity 2 in the proper transform after the first blow-up. The final proper transform, whose wight assignments are displayed in (6.107), has the following form:

$$\begin{aligned} Y^2 + a_1(x_1v_1v_3, x_i)XYZ + a_{3,2}(x_1v_1v_3, x_i)x_1^2v_1YZ^3 \; &= v_1v_3^2X^3 + a_2(x_1v_1v_3, x_i)X^2Z^2 + \\ +a_{4,2}(x_1v_1v_3, x_i)x_1^2v_1XZ^4 + a_{6,4}(x_1v_1v_3, x_i)x_1^4v_1^2Z^6 & \qquad\qquad i = 2, 3, 4\,, \end{aligned} \tag{6.108}$$

The Stanley-Reisner ideal of the blown-up ambient toric variety is:

$$\text{SR ideal} : \{x_1x_2x_3x_4\,;\, XYZ\,;\, x_1XY\,;\, x_2x_3x_4v_1\,;\, v_1Z\,;\, v_1XY\,;\, x_2x_3x_4v_3\,;\, v_3Z\,;\, x_1v_3\}\,. \tag{6.109}$$

From the last element one readily deduce that the exceptional divisor ensuing from the second blow-up, E_3 does not intersect the affine node $x_1 = 0$ of the resolved fiber. This means that its corresponding Cartan node must be the one of the Dynkin diagram of $Sp(2)$ which is invariant under the non-split monodromy; in other words it is the $\mathbf{A_3}$ invariant node (the middle one) under the symmetry of the $SU(4)$ Dynkin diagram, whose folding gives rise to the $Sp(2)$ one.

After having gauge fixed all the non-zero coordinates, the exceptional divisor E_1 is given by the following genus zero *quadratic* curve fibered over $S_2 \simeq \mathbb{P}^2_{x_i}$:

$$Y^2 + a_1(x_i)Y - a_2 = 0\,. \tag{6.110}$$

Analogously, the exceptional divisor E_3 is given by a fibration of the irreducible *quadratic* $\mathbb{P}^1$:

$$Y^2 + a_1(x_i)XY + a_{3,2}(x_i)v_1Y = a_2(x_i)X^2 + a_{4,2}(x_i)v_1X + a_{6,4}(x_i)v_1^2\,. \tag{6.111}$$

Finally, the affine curve is linear, as expected (see subsection 6.3.1), and it is given again by formula (6.53) with $v = v_1$. By intersecting E_1 with E_3 one obtains the usual $\mathbb{Z}_2$-fibration over S_2, with fiber given by the two points:

$$Y_\pm = \frac{-a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}\,. \tag{6.112}$$

The $\mathbb{Z}_2$-fiber degenerates over the O7, $h \equiv a_1^2 + 4a_2 = 0$. Those two points can swap over along closed paths on the base. The same of course happens to the intersection between E_1 and the affine node. All this clearly corresponds to the exchange of the two external nodes of $\mathbf{A_3}$, or in other words to a pairwise exchange of the four D7's realizing the $SU(4)$ gauge group.

The second Chern class of the blown-up fourfold $\tilde{Z}_4$ is, by adjunction:

$$c_2(\tilde{Z}_4) \;=\; c_2(Z_4) + \Delta c_2\,, \tag{6.113}$$

where the first term is the usual one of the smooth case (first piece in (6.61)), while the additional term reads:

$$\Delta c_2 \;=\; -H(27E_1 + 52E_3)\,. \tag{6.114}$$

The intersection numbers have been computed by means of SAGE [87] and the results are the following:

$$\begin{aligned} \frac{\chi(\tilde{Z}_4)}{24} &= \frac{1267}{2} \\ \frac{(\Delta c_2)^2}{8} &= -\frac{677}{2}\,. \end{aligned} \tag{6.115}$$

By subtracting these two contributions, one obtains the usual D3-brane total tadpole $n_{D3} = 972$: This constitutes a non-trivial check of the validity of the results.

It is now the moment to reinterpret these quantities from the type IIB perspective at weak coupling, by means of the usual Sen limit procedure, which will lead to the correct expression of the Freed-Witten gauge flux on the D7 stack. From (6.115), one obviously deduces that $c_2(\tilde{Z}_4)$ is odd. Therefore a gauge flux on the stack of four D7-branes is expected, which is quantized in terms of half integers.

In order to realize a supersymmetric configuration of two D7-branes and two anti-D7-branes on top of one another, wrapping the toric divisor $x_1 = 0 \subset \mathbb{P}^3$, transverse to the O7-plane, one needs six D9 branes and six anti-D9-branes. By requiring conditions analogous to the ones of the $SU(2)$ case, and following the procedure explained in 6.3.2, one ends up easily with the following Whitney sum of line bundles on the D9-branes (on the anti-D9's there are just the

$$\begin{array}{cccccc} D9_1 & D9_2 & D9_3 & D9_4 & D9_5 & D9_6 \\ \mathcal{O}(12H) & \mathcal{O}(2H) & \mathcal{O}(-13H) & \mathcal{O}(14H) & \mathcal{O}(-12H) & \mathcal{O}(13H)\,. \end{array}$$

inverse bundles):
The tachyon field will be a 6×6 matrix with determinant:

$$\det T \;=\; x_1^4(\eta^2 + \xi^2(\rho - \psi^2))\,, \tag{6.116}$$

where it is evident the separation of the two D7-brane components, namely the $Sp(2)$-stack and the singular Whitney umbrella D7-brane.
Out of the above $D9 - \overline{D9}$ system one readily extrapolates the various lower dimensional D-brane charges: D9 and D5-brane charge vanish as the should; D7 charge is the right one for the tadpole, namely $32H$. Finally, the total D3-brane charge is again equal to 1944, which confirms that this configuration is connected to the smooth elliptic fourfold on $\mathbb{P}^3$ by brane recombination/separation processes.
The gauge flux induced by tachyon condensation on the D7 stack turns out to be of the following form:

$$F \;=\; \frac{1}{2}H \begin{pmatrix} 27 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & -27 & 0 \\ 0 & 0 & 0 & -25 \end{pmatrix} \;=\; \frac{1}{2}H(27C_1 + 25C_3)\,, \tag{6.117}$$

where $C_1 = \mathrm{diag}(1,0,-1,0)$ and $C_3 = \mathrm{diag}(0,1,0,-1)$ are the two Cartan matrices of $Sp(2)$. In particular C_1 corresponds to the Cartan node already existent in the $SU(2)$ case, as it has the same coefficient in front (see (6.68)). Moreover, it is clear that such a gauge flux in the adjoint of $Sp(2)$ is quantized in terms of half integers along both the chosen Cartan directions and it necessarily breaks $Sp(2)$ to its Cartan torus[17] $U(1)^2$. Finally this gauge field generates exactly the gauge contribution to the D3-brane tadpole expected by the F-theory computation:

$$\int_{S_2} \mathrm{ch}_2(F) \;=\; 677\,, \tag{6.118}$$

as one can see from formula (6.115).

It is now very easy to guess the behavior of higher rank symplectic singularities. The general formulae for any $Sp(N)$ will be given in the following directly for a D7 stack wrapping a divisor of $\mathbb{P}^3$ of any degree n. Notice, however, that N cannot be arbitrarily large: indeed, for $N \geq 6$, the order of zero of the discriminant at that singularity is greater than 10, resulting in the loss of triviality of the canonical bundle of the fourfold.
Let $Sp(0) = 1$ by convention, which corresponds to the smooth case of section 6.2. The Sen limit of the F-theory $Sp(N)$ configuration on $P_n = 0 \subset \mathbb{P}^3$ works by means of $2N+2$ D9-branes and $2N+2$ anti-D9-branes. Following the procedure in 6.3.2, one deduces that the gauge bundle on the D9's is the following Whitney sum of line bundles (on the anti-D9's there are just the inverse bundles):

$$\mathcal{O}((14 - nN)H) \oplus \mathcal{O}(2H) \bigoplus_{i=1}^{N} [\mathcal{O}((in - 14)H) \oplus \mathcal{O}((14 - (i-1)n)H)]\,. \tag{6.119}$$

[17] More precisely, this is the Cartan torus generated by C_1, C_3. Strictly speaking, the gauge flux (6.117) still preserves the $Sp(1)$ factor with orthogonal Cartan generator in the Killing metric.

The tachyon field is a $(2N+2) \times (2N+2)$ with determinant:

$$\det T \;=\; P_n^{2N}(\eta^2 + \xi^2(\rho - \psi^2))\,. \tag{6.120}$$

As for the $SU(2)$, consistency requires $nN < 12$. One can verify that the ansatz (6.119) fulfills, upon tachyon condensation, the requirements of vanishing of D9 and D5-brane charge and of right tadpole-canceling D7-brane charge. Moreover, a straightforward computation shows that it also predicts the right total D3-brane tadpole, namely 1944, as found from the F-theory perspective. This family of configuration is therefore entirely connected to the smooth case by means of a change of basis performed on the tachyon, as expected from 6.3.2.
The gauge flux induced on the stack of N D7's plus N image-D7's turns out to be of the following form:

$$F = \frac{1}{2}H \sum_{i=1}^{N} (28 - n(2i-1))\, C_{2i-1} \qquad \text{with} \qquad (C_{2i-1})_{jk} = \delta_{ij}\,\delta_{ik} - \delta_{i+N,j}\,\delta_{i+N,k}\ , \tag{6.121}$$

where C_{2i-1} are the N, $2N \times 2N$ Cartan matrices of $Sp(N)$. The contribution of such a gauge field to the D3-brane tadpole then reads:

$$\text{Tadpole}|_{\text{gauge}} \;=\; \frac{nN}{2}\left[(28 - nN)^2 + n^2\,\frac{N^2-1}{3}\right]\,, \tag{6.122}$$

while the contribution due to gravitational interactions obviously reads:

$$\text{Tadpole}|_{\text{grav}} \;=\; 1944 - \frac{nN}{2}\left[(28 - nN)^2 + n^2\,\frac{N^2-1}{3}\right]\,. \tag{6.123}$$

It is clear that, when n is even, namely the D7 stack is wrapping a *spin* manifold inside $\mathbb{P}^3$, regardless of the value of N, (6.121), (6.122) and (6.123) are all integral quantities. In particular the gauge flux on the D7-branes is integrally quantized and there is no topological obstruction in putting it to zero.[18] On the other hand, if the stack is not spin, namely n is odd, then (6.121) implies that a gauge flux on it is generated to compensate for its Freed-Witten anomaly: this flux is quantized in terms of *half*-integers and such a shift in its quantization rule arises along all the chosen Cartan directions of the gauge group $Sp(N)$. This unavoidable flux breaks $Sp(N)$ to the Cartan torus $U(1)^N$ parametrized by C_{2i-1}. However, as already stressed for the $Sp(2)$ case (see the footnote at page 131), the $SU(N)$ gauge group orthogonal to the Cartan direction (6.121) survives, so that the group $SU(N) \times U(1)$ is left unbroken by the half-quantized flux. This matches with the expectations from many models available in the literature (see for example [90])

To summarize, in the physical transition from a single D7-brane to an $Sp(N)$-stack plus a 'remainder' brane, a flux is created on the former stack. If the D7 stack wraps a non-spin manifold, this gauge field is already sufficient to make the path integral measure of open strings attached to the D7 stack well-defined [1], so that no further topological obstruction is allowed: this argument rules out the possibility of having gauge bundles without vector structure, which would imply [14] the presence of a non-trivial 't Hooft magnetic flux and, consequently, a new ambiguity of the measure. Finally, the gauge contribution to the D3 tadpole (6.122) (and

[18] Notice however that by doing so one unavoidably changes the total induced D3-brane charge, thus violating the kinematical constraint necessary to connect by physical processes this family of singular configurations to the smooth one.

clearly also the gravity one, (6.123)) is an integer even in the non-spin case, as long as the rank of the original gauge group, N, is even, in agreement with the results found from the F-theory perspective.[19]

General pattern for Sp(N) configurations

The only thing left to discuss as far as $Sp(N)$ singularities are concerned, is the generalization of the above formulae to any base B_3 of the elliptic fibration and to any degree of the O7-plane as a divisor of the base (in the toy model example it was of degree 4). An additional complication arises here because, as discussed in subsection 6.2.1, when the O7-plane has an odd degree, a non-trivial B-field must be taken into account in the bulk. Recalling the construction of the double cover Calabi-Yau threefold over the base B_3, the Chern class of the normal bundle of the O7-plane in B_3 is twice the one of the anti-canonical bundle of B_3 itself. Therefore, it is useful to define:

$$\begin{aligned} G &\equiv c_1(B_3) = \frac{1}{2}\,\mathrm{PD}_{B_3}(\{h=0\}) \in H^2(B_3,\mathbb{Z})\,, \\ \pi^*(G) &= \mathrm{PD}_{X_3}(\{\xi=0\}) \in H^2(X_3,\mathbb{Z})\,, \end{aligned} \tag{6.124}$$

where ξ is the X_3 homogeneous coordinate transverse to the O7 and $\pi : X_3 \to B_3$ is the orientifold projection. Let also p be a $\mathbb{Z}_2$-parameter measuring the obstruction for B_3 to admit spin structure, namely:

$$p = \begin{cases} 0 & w_2(B_3) = 0 \\ 1 & w_2(B_3) \neq 0\,. \end{cases} \tag{6.125}$$

Let finally D be the Chern class of the normal bundle of the D7-branes stack inside B_3:

$$D = \mathrm{PD}_{B_3}(S_2) \in H^2(B_3,\mathbb{Z})\,, \tag{6.126}$$

where S_2 is the divisor of B_3 wrapped by the stack. In the sequel, all the cohomology classes defined so far will be implicitly meant pulled-back to the Calabi-Yau threefold.

It is easy to find that the Sen weak coupling limit of the F-theory configuration with $Sp(N)$ singularity ($N \geq 0$) on the divisor $S_2 \subset B_3$ can be obtained upon tachyon condensation of a system of $2N+2$ D9-branes and $2N+2$ anti-D9-branes, with gauge bundles as follows:

$$\begin{aligned} D9 \qquad & \mathcal{O}\left(\frac{1}{2}(7-p)\,G - N\,D\right) \oplus \mathcal{O}\left(\frac{1}{2}(1-p)\,G\right) \oplus \\ & \bigoplus_{i=1}^{N}\left[\mathcal{O}\left(i\,D - \frac{1}{2}(7+p)\,G\right) \oplus \mathcal{O}\left(\frac{1}{2}(7-p)\,G - (i-1)\,D\right)\right]\,, \\ \overline{D9} \qquad & \mathcal{O}\left(N\,D - \frac{1}{2}(7+p)\,G\right) \oplus \mathcal{O}\left(-\frac{1}{2}(1+p)\,G\right) \oplus \\ & \bigoplus_{i=1}^{N}\left[\mathcal{O}\left(\frac{1}{2}(7-p)\,G - i\,D\right) \oplus \mathcal{O}\left((i-1)\,D - \frac{1}{2}(7+p)\,G\right)\right]\,, \end{aligned} \tag{6.127}$$

[19] Recall that here one is counting the D3-brane charge contributions from the point of you of the Calabi-Yau threefold double cover of $\mathbb{P}^3$, so its value is *twice* the one found in F-theory.

and with a B-field of the form:

$$B = \frac{p}{2}\, G\,. \tag{6.128}$$

It is a matter of easy algebraic manipulations to show that indeed the above $D9 - \overline{D9}$ system realizes the right charge densities of lower dimensional D-branes, provided one computes them taking into account the B-field (6.128), as in subsection 6.2.1. The D9-brane and D5-brane charges vanish; the D7-brane charge density is $8G$ distributed among the Whitney and the $Sp(N)$-stack; finally the total D3-brane charge density is independent of the values of p, N and it reads:

$$Q_{D3} \;=\; \frac{29}{2}\, G^3 + \frac{1}{2}\, c_2(X_3)\, G\,, \tag{6.129}$$

which agrees with the smooth case formula (6.27), remembering that the Whitney umbrella has generic shape (saturation of the bound). The total D-brane charge (as seen by the double cover) is of course obtained by integrating (6.129) over the Calabi-Yau threefold X_3.
The induced gauge flux on the stack of D7 and anti-D7-branes is the following:

$$F = \frac{1}{2} \sum_{i=1}^{N} ((7-p)\, G - (2i-1)\, D)\, C_{2i-1} \quad , \quad (C_{2i-1})_{jk} = \delta_{ij}\, \delta_{ik} - \delta_{i+N,j}\, \delta_{i+N,k}\;, \tag{6.130}$$

where G and D are meant restricted to S_2. Again this flux is breaking the gauge group $Sp(N)$ down to $SU(N) \times U(1)$. Note that the quantization condition of the gauge flux (6.130) is still regulated only by the even/odd-ness of the class D of the stack, i.e. by the first Chern class of the normal bundle of S_2 in X_3. Indeed, the first term in the expression (6.130) is always an even class, due to (6.125). Hence, the reduction modulo 2 of the cohomology class $(7G - (2i-1)D)\,|_{S_2}$ is $w_2\,(N_{X_3} S_2) = w_2(S_2)$. Therefore, the gauge flux (6.130) is the right one to cancel the Freed-Witten anomaly of the D7 stack wrapping S_2, having holonomy in the class $w_2(S_2)$ [1, 3].

The gauge contribution to the D3-brane charge density induced by the flux (6.130) is easily computed:

$$Q_{gauge} \;=\; \frac{N}{4}\, D \left[(7\, G - N\, D)^2 + \frac{N^2-1}{3}\, D^2 \right]. \tag{6.131}$$

Then, clearly, the gravitational contribution reads:

$$Q_{grav} \;=\; Q_{D3} - Q_{gauge}\,, \tag{6.132}$$

with Q_{D3} as in (6.129). Putting $N = 1$ in the square brackets of eq. (6.131), one easily recognizes the same structure of $(\Delta c_2)^2$, with Δc_2 as in the general Fulton's formula (6.106). It would be interesting to find an iteration of Fulton's formula in the case of more than one blow-up ($N > 1$) and verify that its structure agrees with eq. (6.131).

To summarize, for an $Sp(N)$ singularity on a divisor S_2 of a general Kähler base manifold B_3, and a double cover CY threefold X_3, the induced physical D3-brane charge (i.e. as measured on B_3) is:

$$n_{D3} \;=\; \frac{1}{4} \int_{X_3} 29\, \pi^* c_1(B_3)^3 + c_2(X_3)\, \pi^* c_1(B_3)\,,$$

where $\pi : X_3 \to B_3$ is the double cover projection. The gauge contribution to this D3 tadpole is:

$$n_{D3}^{gauge} \;=\; \frac{N}{8}\int_{X_3}\pi^*\left\{\mathrm{PD}_{B_3}S_2\,\left[(7\,c_1(B_3)-N\,\mathrm{PD}_{B_3}S_2)^2+\frac{N^2-1}{3}\,(\mathrm{PD}_{B_3}S_2)^2\right]\right\}\,,$$

where PD denotes Poincaré duality. The gravitational contribution is simply $n_{D3}^{grav} = n_{D3} - n_{D3}^{gauge}$.

The gauge flux on the D7-brane stack wrapping the divisor S_2 is:

$$F=\frac{1}{2}\sum_{i=1}^{N}\left[(7-p)\,c_1(B_3)-(2i-1)\,\mathrm{PD}_{B_3}S_2\right]|_{S_2}\,C_{2i-1}\quad ,\quad (C_{2i-1})_{jk}=\delta_{ij}\,\delta_{ik}-\delta_{i+N,j}\,\delta_{i+N,k}\;.$$

Such a flux breaks the gauge group $Sp(N)$ down to $SU(N)\times U(1)$. The bulk B-field reads:

$$B \;=\; \frac{p}{2}\,c_1(B_3)\,,$$

where $p=0,1$ according to whether B_3 is spin or not respectively.

Results for the smooth configuration are recovered by just putting $N=0$.

Conclusions and outlook

Out of the topics discussed in this thesis, one can extract many interesting ideas for future investigations.

K-theory As seen in chapter 2, the simplest method to classify D-brane charges is *cohomology* (de Rham in supergravity, integral in quantum theory) but, due to subtle quantum effects (e.g. Freed-Witten anomaly), a refined structure is needed. K-theory seems to be a good candidate. In the case of vanishing H-flux, it is possible to precisely relate the two approaches to K-theory emerging from the literature (see chapter 3): one motivated by Freed-Witten anomaly and based on the Atiyah-Hirzebruch spectral sequence and the other motivated by Sen's *tachyon condensation* and based on the push-forward to space-time (Gysin map) of the gauge bundle of the brane. It turns out that the Gysin map at a fixed instant of time provides a representative element of the equivalence class obtained via the spectral sequence.
Since these methods lead to two different kinds of charge (one conserved only in time, the other also under RG-flow), in order to understand what K-theory really classifies, it would be very important in the future to complete this link with the *twisted* case ($H \neq 0$), in which a generalization of the Gysin map is needed (maybe on the lines of [91]). Moreover, it would be worth to also think about a precise physical interpretation of the higher differentials of the sequence, which is still lacking.

Gerbes In chapter 1 it has been explained how the Freed-Witten anomaly naturally contains the mathematical concept of (abelian) *gerbe*, which is a generalization of a (line) bundle in which the connection is a two form (B-field).
This setting, along with the practice of trivializing gerbes with connection, actually allows to jointly describe non anomalous A and B-field configurations by means of a coset of a certain hypercohomology group. All of this is quite interesting as either it successfully accounts for quantized, not gauge invariant Page charges, interpreted as possibly fractional sub-branes, or it opens the way to the K-theoretical point of view (being instead, in particular situations, their K-theory classes gauge invariant). However, there are still some missing points at the foundational level, in the most general case of non-vanishing H-flux. In particular, one could ask whether there exists or not a twisted version of abelian high-rank gerbe, whose connection would be the RR potential with the corresponding holonomy, in complete analogy with the untwisted case.
Hence a possible plan would be to give a precise mathematical meaning to the Wess-Zumino action in a generic background and, in connection with K-theory, try to clarify what should be the argument of the Gysin map, starting from the case in which a canonical gauge "bundle" exists but has a non-integral first Chern class. Filling in those gaps could help a lot in understanding the geometry behind the debated concept of charge in supergravities with Chern-Simons terms.

7-branes F-theory GUT models seem to provide a promising starting point for making detailed contact between string theory and the Standard Model. In this context, various non-perturbative bound states of 7-branes, giving rise to enhanced gauge groups, play a crucial role. Such configurations are referred to as *Q7 branes* of type IIB string theory (see chapter 4).
Bearing in mind that also orientifolds can be viewed as bound states, but with a monodromy in a "perturbative" conjugacy class, it would be very interesting and really important for the consistency of these models, to find the general extension of Freed-Witten anomaly to the Q7's. With this purpose, a possible future plan would be to look for a world-sheet theory of such objects by carefully analyzing the BPS open string states that are responsible for the symmetry enhancement; by the way, an analogous study for the case of non-simply laced gauge groups has led to the content of chapter 5.
In chapter 6 a detour towards this project has been taken by analyzing the quantization conditions for D7-brane gauge fluxes which arise from the one of the M-theory G-flux and which are related to Freed-Witten anomaly cancellation of open strings attached to the D7's. For the smooth Calabi-Yau fourfold case a complete answer has been found (no Freed-Witten-like gauge field arises), while for the singular case only the $Sp(N)$-type singularities have been spelt out in detail. Clearly an analogous analysis for more complicated singularities (for example for the other branches of Tate's algorithm) allows one to get information about the mechanism of Freed-Witten anomaly cancellation for general Q7-branes (for instance the ones responsible for exceptional gauge groups). Such an analysis is thus not necessarily restricted to ordinary stacks of D7-branes, but rather it allows for general gauge group enhancing.
It would be also interesting to study in depth the topics sketched in subsection 6.1.1, related to the general conjecture of the Freed-Witten anomaly for M5-branes, and to find the consequences of such conjecture in the F-theory context, including its effects on flux quantization rules.

Orientifolds The purpose of addressing the same issues for type I superstrings is very intriguing; but first a systematic study of the Polyakov approach to them is needed, since it looks the most appropriate one to detect anomalies of the path integral measure. A work on the mathematical side of this area has been done [92], which tries to correctly define spinors on the *orientable* double covering of the world-sheet, with suitable boundary conditions so to reproduce the standard choices of spin structures [93]. Then, the equivalence of these spinors with *pinors* of the unorientable manifold is shown in full generality, and the example of surfaces spelt out.
It would be useful to put the content of such a paper in a more physical language, in order to make contact with the Polyakov formalism used to do actual one-loop computations [94]. Moreover, according to the original motivations, it would be worth to understand what is the correct number of local degrees of freedom to sum on in the path integral and what are the repercussions on anomalies (everything in connection with the recent papers about this topic by Distler, Freed and Moore [95, 96]). Having done this, the idea is to carry through in this orientifold scenario the classification of brane and bulk type fields and eventually gain new insights about charge/flux quantization conditions and about the KO-theory, which classifies type I D-brane charges.

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Appendix A

Čech Hypercohomology

The reader is referred to [13] for a comprehensive treatment of hypercohomology.

Given a sheaf $\mathcal{F}$ on a topological space X with a good cover $\mathfrak{U} = \{U_i\}_{i\in I}$, one can construct the complex of Čech cochains:

$$\check{C}^0(\mathfrak{U},\mathcal{F}) \xrightarrow{\check{\delta}^0} \check{C}^1(\mathfrak{U},\mathcal{F}) \xrightarrow{\check{\delta}^1} \check{C}^2(\mathfrak{U},\mathcal{F}) \xrightarrow{\check{\delta}^2} \cdots$$

whose cohomology is by definition Čech cohomology of $\mathcal{F}$. Recall, in particular, that $\check{\delta}^p : \check{C}^p(\mathfrak{U},\mathcal{F}) \to \check{C}^{p+1}(\mathfrak{U},\mathcal{F})$ is defined by $(\check{\delta}^p g)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1}(-1)^i g_{\alpha_0\cdots\check{\alpha}_i\cdots\alpha_{p+1}}$. If, instead of a single sheaf, one has a complex of sheaves:

$$\cdots \xrightarrow{d^{i-2}} \mathcal{F}^{i-1} \xrightarrow{d^{i-1}} \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \xrightarrow{d^{i+1}} \cdots$$

one can still associate to it a cohomology, called *hypercohomology* of the complex. To define it, one considers the double complex made by the Čech complexes of each sheaf:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\check{C}^0(\mathfrak{U},\mathcal{F}^{q+1}) \arrow[u, "d^{q+1}"] \arrow[r, "\check{\delta}^0"] & \check{C}^1(\mathfrak{U},\mathcal{F}^{q+1}) \arrow[u, "d^{q+1}"] \arrow[r, "\check{\delta}^1"] & \check{C}^2(\mathfrak{U},\mathcal{F}^{q+1}) \arrow[u, "d^{q+1}"] \arrow[r, "\check{\delta}^2"] & \cdots \\
\check{C}^0(\mathfrak{U},\mathcal{F}^{q}) \arrow[u, "d^{q}"] \arrow[r, "\check{\delta}^0"] & \check{C}^1(\mathfrak{U},\mathcal{F}^{q}) \arrow[u, "d^{q}"] \arrow[r, "\check{\delta}^1"] & \check{C}^2(\mathfrak{U},\mathcal{F}^{q}) \arrow[u, "d^{q}"] \arrow[r, "\check{\delta}^2"] & \cdots \\
\check{C}^0(\mathfrak{U},\mathcal{F}^{q-1}) \arrow[u, "d^{q-1}"] \arrow[r, "\check{\delta}^0"] & \check{C}^1(\mathfrak{U},\mathcal{F}^{q-1}) \arrow[u, "d^{q-1}"] \arrow[r, "\check{\delta}^1"] & \check{C}^2(\mathfrak{U},\mathcal{F}^{q-1}) \arrow[u, "d^{q-1}"] \arrow[r, "\check{\delta}^2"] & \cdots \\
\vdots \arrow[u, "d^{q-2}"] & \vdots \arrow[u, "d^{q-2}"] & \vdots \arrow[u, "d^{q-2}"] &
\end{tikzcd}$$

One then considers the associated total complex[1]:

$$T^n = \bigoplus_{p+q=n} \check{C}^p(\mathfrak{U},\mathcal{F}^q)\,, \qquad d^n = \bigoplus_{p+q=n} \left(\check{\delta}^p + (-1)^p\, d^q\right).$$

[1]The notations of [13] are adopted, in which the two boundaries of the double complex commute, so that the boundary of the total complex has a factor $(-1)^p$. In the most common notations, instead, the two boundaries anticommute.

By definition, the Čech *hypercohomology* of the complex of sheaves is the cohomology of the total complex $H^\bullet(T^n, d^n)$. It is denoted by:

$$\check{H}^\bullet\big(\mathfrak{U}, \cdots \xrightarrow{d^{i-1}} \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \xrightarrow{d^{i+1}} \cdots\big) .$$

Using hypercohomology one can describe the group of line bundles with connection, up to isomorphism and pull-back of the connection, on a space X. Recall that a bundle with connection is specified by a couple $(\{h_{\alpha\beta}\}, \{A_\alpha\})$ where $\check{\delta}\{h_{\alpha\beta}\} = 1$ and $A_\alpha - A_\beta = (2\pi i)^{-1} d \log h_{\alpha\beta}$. The bundle is trivial if there exists a 0-cochain $\{f_\alpha\}$ such that $\check{\delta}^0\{f_\alpha\} = \{h_{\alpha\beta}\}$. Consider the complex of sheaves on X:

$$\underline{S}^1 \xrightarrow{\tilde{d}} \Omega^1_{\mathbb{R}} ,$$

where $\underline{S}^1$ is the sheaf of smooth S^1-valued functions, $\Omega^1_{\mathbb{R}}$ the sheaf of 1-forms and $\tilde{d} = (2\pi i)^{-1}\, d \circ \log$. (The complex is trivially extended to left and right by 0.) The associated Čech double complex is given by:

$$\begin{tikzcd}
\check{C}^0(\mathfrak{U}, \Omega^1_{\mathbb{R}}) \arrow[r, "\check{\delta}^0"] & \check{C}^1(\mathfrak{U}, \Omega^1_{\mathbb{R}}) \arrow[r, "\check{\delta}^1"] & \check{C}^2(\mathfrak{U}, \Omega^1_{\mathbb{R}}) \arrow[r, "\check{\delta}^2"] & \cdots \\
\check{C}^0(\mathfrak{U}, \underline{S}^1) \arrow[r, "\check{\delta}^0"] \arrow[u, "\tilde{d}"] & \check{C}^1(\mathfrak{U}, \underline{S}^1) \arrow[r, "\check{\delta}^1"] \arrow[u, "\tilde{d}"] & \check{C}^2(\mathfrak{U}, \underline{S}^1) \arrow[r, "\check{\delta}^2"] \arrow[u, "\tilde{d}"] & \cdots
\end{tikzcd}$$

Thus one has that $\check{C}^1(\mathfrak{U}, \underline{S}^1 \to \Omega^1_{\mathbb{R}}) = \check{C}^1(\mathfrak{U}, \underline{S}^1) \oplus \check{C}^0(\mathfrak{U}, \Omega^1_{\mathbb{R}})$. Given a line bundle $L \to X$ one fixes a set of local sections, with respect to $\mathfrak{U}$, determining transition functions $\{g_{\alpha\beta}\}$ and local representation of the connection $\{A_\alpha\}$. The claim is that $(g_{\alpha\beta}, -A_\alpha) \in \check{C}^1(\mathfrak{U}, \underline{S}^1 \to \Omega^1_{\mathbb{R}})$ is a cocycle. In fact, by definition, $\check{\delta}^1(g_{\alpha\beta}, -A_\alpha) = (\check{\delta}^1 g_{\alpha\beta}, -\tilde{d}g_{\alpha\beta} + \check{\delta}^0(-A_\alpha))$, thus the cocycle condition gives $\check{\delta}^1 g_{\alpha\beta} = 0$, i.e., $g_{\alpha\beta}$ must be transition functions of a line bundle, and $A_\alpha - A_\beta = (2\pi i)^{-1} d \log g_{\alpha\beta}$, the latter being exactly the gauge transformation of a connection. Moreover, coboundaries are of the form $\check{\delta}^0(g_\alpha) = (\check{\delta}^0 g_\alpha, \tilde{d}g_\alpha)$ and it is easy to prove that these are exactly the possible local representations of the trivial connection ∂_X on the trivial bundle $X \times \mathbb{C}$, i.e., the unit element of the group of line bundles with connection. Thus, such group is isomorphic to:

$$\check{H}^1(\mathfrak{U}, \underline{S}^1 \xrightarrow{\tilde{d}} \Omega^1_{\mathbb{R}}) .$$

Appendix B

Gerbes

The reader is referred to [97] for a clear introduction to gerbes. The approach of [13] will be adopted here.

A gerbe with connection is defined by a triple $(\{g_{\alpha\beta\gamma}\}, \{\Lambda_{\alpha\beta}\}, \{B_\alpha\})$ where $\check{\delta}\{g_{\alpha\beta\gamma}\} = 1$, $\check{\delta}^1\{\Lambda_{\alpha\beta}\} = \{(2\pi i)^{-1} d \log g_{\alpha\beta\gamma}\}$ and $B_\alpha - B_\beta = d\Lambda_{\alpha\beta}$. The gerbe is trivial if there exists a 1-cochain $\{f_{\alpha\beta}\}$ such that $\check{\delta}\{f_{\alpha\beta}\} = \{g_{\alpha\beta\gamma}\}$.
As the group of isomorphism classes of line bundles on X is isomorphic to $\check{H}^1(X, \underline{S}^1)$, the group of gerbes on X up to isomorphism can be identified with $\check{H}^2(X, \underline{S}^1)$. Throughout the present thesis, this is regarded as the definition of a gerbe.

Consider the complex of sheaves:

$$\underline{S}^1 \xrightarrow{\tilde{d}} \Omega^1_{\mathbb{R}} \xrightarrow{d} \Omega^2_{\mathbb{R}}$$

where $\underline{S}^1$ is the sheaf of smooth S^1-valued functions, $\Omega^p_{\mathbb{R}}$ the sheaf of p-forms and $\tilde{d} = (2\pi i)^{-1}\, d \circ \log$. (The complex is trivially extended to left and right by 0.) In analogy with the case of line bundles, define the equivalence classes of gerbes with connection as the elements of the group:

$$\check{H}^2(X, \underline{S}^1 \to \Omega^1_{\mathbb{R}} \to \Omega^2_{\mathbb{R}})\ .$$

The Čech double complex is given by:

$$\begin{array}{ccccccc}
\check{C}^0(\mathfrak{U}, \Omega^2_{\mathbb{R}}) & \xrightarrow{\check{\delta}^0} & \check{C}^1(\mathfrak{U}, \Omega^2_{\mathbb{R}}) & \xrightarrow{\check{\delta}^1} & \check{C}^2(\mathfrak{U}, \Omega^2_{\mathbb{R}}) & \xrightarrow{\check{\delta}^2} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\check{C}^0(\mathfrak{U}, \Omega^1_{\mathbb{R}}) & \xrightarrow{\check{\delta}^0} & \check{C}^1(\mathfrak{U}, \Omega^1_{\mathbb{R}}) & \xrightarrow{\check{\delta}^1} & \check{C}^2(\mathfrak{U}, \Omega^1_{\mathbb{R}}) & \xrightarrow{\check{\delta}^2} & \cdots \\
\uparrow{\scriptstyle \tilde{d}} & & \uparrow{\scriptstyle \tilde{d}} & & \uparrow{\scriptstyle \tilde{d}} & & \\
\check{C}^0(\mathfrak{U}, \underline{S}^1) & \xrightarrow{\check{\delta}^0} & \check{C}^1(\mathfrak{U}, \underline{S}^1) & \xrightarrow{\check{\delta}^1} & \check{C}^2(\mathfrak{U}, \underline{S}^1) & \xrightarrow{\check{\delta}^2} & \cdots
\end{array}$$

Thus one has that $\check{C}^2(\mathfrak{U}, \underline{S}^1 \to \Omega^1_{\mathbb{R}} \to \Omega^2_{\mathbb{R}}) = \check{C}^2(\mathfrak{U}, \underline{S}^1) \oplus \check{C}^1(\mathfrak{U}, \Omega^1_{\mathbb{R}}) \oplus \check{C}^0(\mathfrak{U}, \Omega^2_{\mathbb{R}})$. By definition, $\check{\delta}^1(g_{\alpha\beta\gamma}, -\Lambda_{\alpha\beta}, B_\alpha) = (\check{\delta}^2 g_{\alpha\beta\gamma}, \tilde{d} g_{\alpha\beta\gamma} + \check{\delta}^1(-\Lambda_{\alpha\beta}), -d(-\Lambda_{\alpha\beta}) + \check{\delta}^0 B_\alpha)$. Thus the cocycle condition gives $\check{\delta}^2 g_{\alpha\beta\gamma} = 0$, i.e., $g_{\alpha\beta\gamma}$ must be transition functions of a gerbe, and:

$$\begin{aligned}
&B_\alpha - B_\beta = d\Lambda_{\alpha\beta} \\
&\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = (2\pi i)^{-1} d \log g_{\alpha\beta\gamma}\ .
\end{aligned}$$

Coboundaries are of the form $\check{\delta}^1(h_{\alpha\beta}, -A_\alpha) = (\check{\delta}^1 h_{\alpha\beta}, -\tilde{d} h_{\alpha\beta} + \check{\delta}^0(-A_\alpha), d(-A_\alpha))$, thus the gerbes of this form are geometrically trivial.

Appendix C

Holonomies and Wilson loops

The purpose of this appendix is to give a precise definition of the holonomy integrals that appear in (1.3). Given the complexity of the definition for gerbes, a description of the more familiar subject of the holonomy for line bundles is given to start with.

C.1 Line bundles

C.1.1 Global description

Consider a line bundle with connection (L, ∇) on X and a *closed* curve $\gamma : S^1 \to X$ with a fixed point $x = \gamma(e^{2\pi i \cdot t})$: parallel transport along γ gives a linear map $t_x : L_x \to L_x$, which can be thought of as a number $\mathrm{Hol}_\nabla(\gamma) \in S^1$ thanks to the canonical isomorphism $L_x^\checkmark \otimes L_x \simeq \mathbb{C}$ given by $\varphi \otimes v \simeq \varphi(v)$ (such a number is independent of the chosen point x). Thus, denoting by LX the loop space of X, parallel transport defines a function $\mathrm{Hol}_\nabla : LX \to S^1$ called *holonomy* of ∇.

What can be said about open curves? Given a curve $\gamma : [0, 1] \to X$, put $x = \gamma(0)$ and $y = \gamma(1)$: parallel transport defines a linear map $t_{x,y} : L_x \to L_y$, which is no longer canonically a number, since $L_x^\checkmark \otimes L_y$ is not canonically isomorphic to $\mathbb{C}$. Thus, given a curve $\gamma \in CX$, CX being the space of open curves on X, holonomy is an element of a 1-dimensional vector space $CL_\gamma = L_x^\checkmark \otimes L_y$: this vector space will be described as the fiber over γ of a line bundle $CL \to CX$, so that holonomy defines a section of CL. In fact, consider the bundle $L^\checkmark \boxtimes L \to X \times X$, i.e., $L^\checkmark \boxtimes L = \pi_1^* L^\checkmark \otimes \pi_2^* L$ for $\pi_1, \pi_2 : X \times X \to X$ the projections to the first and second factor, respectively. One has a natural map $\pi : CX \to X \times X$ given by $\pi(\gamma) = (\gamma(0), \gamma(1))$, so that one can define $CL = \pi^*(L^\checkmark \boxtimes L)$. By construction $CL_\gamma = (L^\checkmark \boxtimes L)_{\pi(\gamma)} = (L^\checkmark \boxtimes L)_{(\gamma(0),\gamma(1))} = L^\checkmark_{\gamma(0)} \otimes L_{\gamma(1)}$, so one obtains exactly the desired fiber. Thus the holonomy defines a section $\mathrm{Hol}_\nabla : CX \to CL$. Moreover $c_1(CL) = \pi^*(\pi_2^*\, c_1(L) - \pi_1^*\, c_1(L))$.

As one can see from the expression of $c_1(CL)$, if L is trivial so is CL. There is more: *a trivialization of L determines a trivialization of CL*. In fact, if $s : X \to L$ is a global section, it determines canonically a global section $s^\checkmark : X \to L^\checkmark$ given by $s^\checkmark(s) = X \times \{1\}$, thus a section $s^\checkmark \boxtimes s : X \times X \to L^\checkmark \boxtimes L$, thus, by pull-back, a global section $\pi^*(s^\checkmark \boxtimes s) : CX \to CL$. What is happening geometrically? A global section $s : X \to L$ provides a way to identify the fibers of L, hence a linear map $L_x \to L_y$ becomes the number λ such that $s_x \to \lambda \cdot s_y$. Thus, *for a trivial bundle with a fixed global section, the holonomy is a well-defined function also over the space of open curves.*

Similarly, *a system of local sections of* L*, with respect to a good cover* $\mathfrak{U} = \{U_i\}_{i\in I}$*, determines a system of local sections of* CL, with respect to the cover $\mathfrak{V}$ defined in the following way:

- fix a triangulation τ of S^1, i.e. a set of vertices $\sigma^0_1, \ldots, \sigma^0_l \in S^1$ and of edges $\sigma^1_1, \ldots, \sigma^1_l \subset S^1$ such that $\partial\sigma^1_i = \sigma^0_{i+1} - \sigma^0_i$ for $1 \leq i < l$ and $\partial\sigma^1_l = \sigma^0_1 - \sigma^0_l$;
- consider the following set of indices:

$$J = \left\{ (\tau, \varphi) : \quad \begin{array}{l} \bullet\ \tau = \{\sigma^0_1, \ldots, \sigma^0_{l(\tau)}; \sigma^1_1, \ldots, \sigma^1_{l(\tau)}\} \text{ is a triangulation of } S^1 \\ \bullet\ \varphi : \{1, \ldots, l(\tau)\} \longrightarrow I \text{ is a function} \end{array} \right\} ;$$

- one obtains the covering $\mathfrak{V} = \{V_{(\tau,\sigma)}\}_{(\tau,\sigma)\in J}$ of LX by:

$$V_{(\tau,\varphi)} = \{\gamma \in LX : \ \gamma(\sigma^1_i) \subset U_{\varphi(i)}\} \, .$$

Consider $\gamma \in V_{(\tau,\varphi)}$: then $L^{\checkmark}_{\gamma(0)} \otimes L_{\gamma(1)}$ is isomorphic to $\mathbb{C}$ via $s_{\varphi(1)}$ and $s_{\varphi(l(\tau))}$, so that one has a local trivialization $V_{(\tau,\varphi)} \times \mathbb{C}$, giving a local section $V_{(\tau,\varphi)} \times \{1\}$. Thus, one can describe the transition functions of CL for $\mathfrak{V}$ in terms of the ones of L for $\mathfrak{U}$. In particular, the local expression of parallel transport along γ with respect to the fixed local sections is given by $\{\rho_{(\tau,\varphi)}\}$ such that $t_{\gamma(0),\gamma(1)}(x, z)_{\varphi(1)} = (x, \rho_{(\tau,\varphi)} \cdot z)_{\varphi(l)}$. Then, if $\gamma \in V_{(\tau,\varphi)} \cap V_{(\tau',\varphi')}$, one has, with respect to the second chart, $t_{\gamma(0),\gamma(1)}(x, z)_{\varphi'(1)} = (x, \rho_{(\tau',\varphi')} \cdot z)_{\varphi'(l')}$. Then, since $(x, z)_{\varphi(1)} = (x, g_{\varphi(1),\varphi'(1)} \cdot z)_{\varphi'(1)}$, one gets:

$$\begin{aligned} t_{\gamma(0),\gamma(1)}(x, z)_{\varphi(1)} &= (x, \rho_{(\tau,\varphi)} \cdot z)_{\varphi(l)} = (x, g_{\varphi(l),\varphi'(l')} \cdot \rho_{(\tau,\varphi)} \cdot z)_{\varphi'(l')} \\ t_{\gamma(0),\gamma(1)}(x, g_{\varphi(1),\varphi'(1)} \cdot z)_{\varphi'(1)} &= (x, \rho_{(\tau',\varphi')} \cdot g_{\varphi(1),\varphi'(1)} \cdot z)_{\varphi'(l')} \end{aligned}$$

so that $g_{\varphi(l),\varphi'(l')} \cdot \rho_{(\tau,\varphi)} = \rho_{(\tau',\varphi')} \cdot g_{\varphi(1),\varphi'(1)}$, thus, $\rho_{(\tau,\varphi)} = \rho_{(\tau',\varphi')} \cdot (g^{-1}_{\varphi(l),\varphi'(l')} \cdot g_{\varphi(1),\varphi'(1)})$. Hence the transition functions of CL are exactly $g_{(\tau,\varphi),(\tau',\varphi')}(\gamma) := g^{-1}_{\varphi(l),\varphi'(l')}\gamma(1) \cdot g_{\varphi(1),\varphi'(1)}\gamma(0)$. In particular, a trivialization $g_{ij} = g_i^{-1} g_j$ of L determines a trivialization of CL given by $g_{(\tau,\varphi),(\tau',\varphi')} = g^{-1}_{(\tau,\varphi)} g_{(\tau',\varphi')}$ for $g_{(\tau,\varphi)}(\gamma) = g_{\varphi(1)}\gamma(0) \cdot g_{\varphi(l)}\gamma(1)^{-1}$, as it is easy to verify.

It is possible to generalize a little bit this construction: consider a line bundle $L \to X$ and a subset $Y \subset X$ and consider the space $C_Y X$ of open curves in X with boundary in Y, i.e. such that $\gamma(0), \gamma(1) \in Y$. In this case, one has $\pi : C_Y X \to Y \times Y$ and the holonomy is a section of the bundle $C_Y L = \pi^*(L|_Y{}^{\checkmark} \boxtimes L|_Y)$. Thus, in order to have a function one only needs the triviality of $L|_Y$ and a global section of its; it is not necessary the whole L be trivial; similarly, in order to have a set of local sections of $C_Y L$ one just needs a set of local sections of $L|_Y$.

C.1.2 Local description

One can now express the holonomy using local expression of the connection, so that one can generalize it to gerbes. Considering the covering $\mathfrak{V}$ of LX previously defined, for a closed curve $\gamma \in V_{(\tau,\varphi)}$ define[1]:

$$\int_\gamma A \;:=\; \sum_{i=1}^{l(\tau)} \left[\left(\int_{\gamma(\sigma^1_i)} A_{\varphi(i)} \right) + \tfrac{1}{2\pi i} \log g_{\varphi(i),\varphi(i+1)}\big(\gamma(\sigma^0_{i+1})\big) \right] \tag{C.1}$$

[1]The index i of the triangulation is regarded as a cyclic index, thus $l + 1 = 1$.

and one can prove that this is a well-defined function in $\mathbb{R}/\mathbb{Z}$. It is worth to stress that the definition of the holonomy depends not only on the local connetion $\{A_\alpha\}$ but also on the cocycle $\{g_{\alpha\beta}\}$.

For γ open one must skip the last transition function. First of all an analogous open cover for the space of open curves CX will be described:

- fix a triangulation τ of $[0,1]$, i.e., a set of vertices $\sigma^0_1, \dots, \sigma^0_l, \sigma^0_{l+1} \in [0,1]$ and of edges $\sigma^1_1, \dots, \sigma^1_l \subset [0,1]$ such that:
  - $\partial\sigma^1_i = \sigma^0_{i+1} - \sigma^0_i$ for $1 \leq i \leq l$;
  - $\sigma^0_1 = 0$ and $\sigma^0_{l+1} = 1$; these are called *boundary vertices*;
- consider the following set of indices:

$$J = \left\{ (\tau, \varphi) : \begin{array}{l} \bullet\ \tau = \{\sigma^0_1, \dots, \sigma^0_{l(\tau)}, \sigma^0_{l(\tau)+1}; \sigma^1_1, \dots, \sigma^1_{l(\tau)}\} \text{ is a triangulation of } [0,1] \\ \bullet\ \varphi : \{1, \dots, l(\tau)\} \longrightarrow I \text{ is a function} \end{array} \right\} ;$$

- one obtains a covering $\{V_{(\tau,\sigma)}\}_{(\tau,\sigma)\in J}$ of CX by:

$$V_{(\tau,\varphi)} = \{\gamma \in CX : \ \gamma(\sigma^1_i) \subset U_{\varphi(i)}\} .$$

Thus, define:

$$\int_\gamma A := \left(\sum_{i=1}^{l(\tau)-1} \int_{\gamma(\sigma^1_i)} A_{\varphi(i)} + \log g_{\varphi(i),\varphi(i+1)}\big(\gamma(\sigma^0_{i+1})\big) \right) + \int_{\gamma(\sigma^1_l)} A_{\varphi(l)} . \tag{C.2}$$

In this case the integral is not well-defined as a function, but, as seen, it is *a section of a line bundle* $CL \to CX$ with transition functions $\tilde{g}_{(\tau,\varphi),(\tau',\varphi')}(\gamma) = g_{\varphi(l),\varphi'(l)}\gamma(1)^{-1} \cdot g_{\varphi(1),\varphi'(1)}\gamma(0)$. If, for a submanifold $Y \subset X$, one asks that $\partial\gamma \subset Y$ and one chooses a trivialization of $L|_Y$ given by $g_{\alpha\beta}(y) = g^{-1}_\alpha(y) \cdot g_\beta(y)$, one can express the transition functions of CL as $\tilde{g}_{(\tau,\varphi),(\tau',\varphi')}(\gamma) = (g_{\varphi(l)}\gamma(1) \cdot g_{\varphi(1)}\gamma(0)^{-1}) \cdot (g_{\varphi'(l)}\gamma(1) \cdot g_{\varphi'(1)}\gamma(0)^{-1})^{-1}$, thus one obtains a trivialization of CL given by $\tilde{g}_{(\tau,\varphi)}(\gamma) = g_{\varphi(l)}\gamma(1)^{-1} \cdot g_{\varphi(1)}\gamma(0)$. With respect to this trivialization, the holonomy becomes a function given by:

$$\begin{aligned} \int_\gamma A := & \left(\sum_{i=1}^{l(\tau)-1} \int_{\gamma(\sigma^1_i)} A_{\varphi(i)} + \log g_{\varphi(i),\varphi(i+1)}\big(\gamma(\sigma^0_{i+1})\big) \right) + \int_{\gamma(\sigma^1_l)} A_{\varphi(l)} \\ & + \tfrac{1}{2\pi i} \left(\log g_{\varphi(l)}(\gamma(1)) - \log g_{\varphi(1)}(\gamma(0)) \right) . \end{aligned} \tag{C.3}$$

C.1.3 Cohomology classes and cocycles

It is worth to remark here the following facts, which will be useful later to better figure out by analogy the case of gerbes. Fix a good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ on a space X:

- when one specifies a *cohomology class* $\alpha = [\{g_{\alpha\beta}\}] \in \check{H}^1(\mathfrak{U}, C^\infty(\,\cdot\,, \mathbb{C}^*))$, one associates to it an *equivalence class up to isomorphism* of line bundles, represented by[2]:

$$\Big(\bigsqcup (U_\alpha \times \mathbb{C}) \Big) \Big/ \sim , \qquad (x, z)_\alpha \sim (x, g_{\alpha\beta}(x) \cdot z)_\beta, \text{ for } x \in U_{\alpha\beta} ; \tag{C.4}$$

[2]This equivalence class is much larger than the class made by the bundles of the form (C.4) for the various representatives $\{g_{\alpha\beta}\}$ of α, since there are all the bundles which are not of the form (C.4) but only isomorphic to one of them.

- when one specifies a *cocycle* $\{g_{\alpha\beta}\} \in \check{Z}^1(\mathfrak{U}, C^\infty(\,\cdot\,, \mathbb{C}^*))$, one associates to it the *equivalence class* of a line bundle *with a fixed set of local sections* $\{s_\alpha : X \to L\}$ *up to isomorphism with relative pull-back of the sections*, such that $g_{\alpha\beta} = s_\alpha / s_\beta$. In this case one has dependence on the covering $\mathfrak{U}$; but this is obvious since the local sections themselves determine the covering by their domains. One has a canonical representative for each of these classes given by (C.4).

If a line bundle L is given with a fixed set of local sections $\{s_\alpha : U_\alpha \to L\}$, it is canonically isomorphic to a line bundle of the form (C.4) for $g_{\alpha\beta} = s_\alpha / s_\beta$ (of course the sections $\{s_\alpha\}$ do not make $\{g_{\alpha\beta}\}$ a coboundary since they are not functions, they are sections of a bundle). The isomorphism is simply given by $\varphi(s_\alpha)_x = (x, 1)_\alpha$, and it can be applied to any bundle isomorphic to L with the pull-back of the sections $\{s_\alpha\}$.

C.2 Gerbes

The situation of gerbes is analogous to the one of bundles. In particular, the holonomy of a gerbe over a closed surface is a well defined function, while the holonomy for a surface with boundary Σ is a section of a bundle over the space $\mathrm{Maps}(\Sigma, X)$. If one considers the maps such that $\phi(\partial\Sigma) \subset Y$, then a trivialization of the gerbe on Y, if it exists, determines a trivialization of the bundle, so that the holonomy becomes a well-defined function.

C.2.1 Closed surfaces

Definition C.2.1 *Given a topological space X and a closed compact surface Σ, the* space of maps from Σ to X*, called ΣX, is the set of continuous maps:*

$$\Gamma : \Sigma \longrightarrow X$$

equipped with the compact-open topology.

A natural open covering for the space of maps is now described. In particular:

- fix a triangulation τ of Σ, i.e.:
  - a set of vertices $\sigma^0_1, \ldots, \sigma^0_l \in \Sigma$;
  - a subset $E \subset \{1, \ldots, l\}^2$, determining a set of oriented edges $\{\sigma^1_{(a,b)} \subset \Sigma\}_{(a,b)\in E}$ such that $\partial\sigma^1_{(a,b)} = \sigma^0_b - \sigma^0_a$; if $(a,b) \in E$ then $(b,a) \notin E$ and declare $\sigma^1_{(b,a)} := -\sigma^1_{(a,b)}$;
  - a subset $T \subset \{1, \ldots, l\}^3$, determining a set of oriented triangles $\{\sigma^2_{(a,b,c)} \subset \Sigma\}_{(a,b,c)\in T}$ such that $\partial\sigma^2_{(a,b,c)} = \sigma^1_{(a,b)} + \sigma^1_{(b,c)} + \sigma^1_{(c,a)}$; given a, b, c only one permutation of them belongs to T and for a permutation ρ declare $\sigma^2_{\rho(a),\rho(b),\rho(c)} := (-1)^\rho \sigma^2_{(a,b,c)}$;

  satisfying the following conditions:

  - every point $P \in \Sigma$ belongs to at least one triangle, and if it belongs to more than one triangle then it belongs to the boundary of each of them;
  - every edge $\sigma^1_{(a,b)}$ lies in the boundary of exactly two triangles $\sigma^2_{(a,b,c)}$ and $\sigma^2_{(b,a,d)}$, inducing on it opposite orientations, and $\sigma^2_{(a,b,c)} \cap \sigma^2_{(b,a,d)} = \sigma^1_{(a,b)}$; if a point $p \in \Sigma$ belongs to an edge $\sigma^1_{(a,b)}$ and it's not a vertex, than the only two triangles containing

it are the ones having $\sigma^1_{(a,b)}$ as common boundary; thus, there exists a function $b: E \to T^2$ such that $\sigma^1_{(a,b)} \subset \partial\sigma^2_{b^1(a,b)}$ and $-\sigma^1_{(a,b)} \subset \partial\sigma^2_{b^2(a,b)}$;

– for every vertex σ^0_i there exists a finite set of triangles $\{\sigma^2_{(i,a_1,a_2)}, \dots, \sigma^2_{(i,a_{k_i},a_1)}\}$ having σ^0_i as vertex, such that $\sigma^2_{(i,a_j,a_{j+1})} \cap \sigma^2_{(i,a_{j+1},a_{j+2})} = \sigma^1_{(i,a_{j+1})}$ (the notation is such that $k_i + 1 = 1$), these triangles are the only one containing σ^0_i and their union is a neighborhood of it; thus, there exists a function $B: \{1, \dots, l\} \to \coprod_{i=1}^{l} T^{k_i}$, such that $B(i) \in T^{k_i}$ and $B(i) = \{\sigma^2_{(i,a_1,a_2)}, \dots, \sigma^2_{(i,a_{k_i},a_1)}\}$;

- consider the following set of indices:

$$J = \left\{ (\tau, \varphi) : \begin{array}{l} \bullet\ \tau = \left\{\sigma^0_1, \dots, \sigma^0_{l(\tau)}, E, T\right\} \text{ is a triangulation of } \Sigma \\ \bullet\ \varphi : T \longrightarrow I \text{ is a function} \end{array} \right\}$$

and a covering $\{V_{(\tau,\sigma)}\}_{(\tau,\sigma)\in J}$ of ΣX is given by:

$$V_{(\tau,\varphi)} = \{\Gamma \in \Sigma X : \ \Gamma(\sigma^2_{(a,b,c)}) \subset U_{\varphi(a,b,c)}\}.$$

One can prove that these sets are open in the compact-open topology and that they cover ΣX.

For a fixed $\Gamma \in \Sigma X$, there exists $(\tau, \varphi) \in J$ such that $\Gamma \in V_{(\tau,\varphi)}$. The function $\varphi : T \to I$ induces two functions:

- $\varphi^E : E \to I^2$, given by $\varphi^E(a,b) = \big(\varphi(b^1(a,b)), \varphi(b^2(a,b)\big)$;
- $\varphi^V : \{1, \dots, l\} \to \coprod_{i=1}^{l} (I^3)^{k_i - 2}$, such that $\varphi^V(i) \in (I^3)^{k_i-2}$ and $\big(\varphi^V(i)\big)^j = \big(\varphi(B^1(i)), \varphi(B^j(i)), \varphi(B^{j+1}(i))\big)$.

Define:

$$\begin{aligned} \int_\Gamma B := \sum_{(a,b,c)\in T_\tau} \int_{\Gamma(\sigma^2_{(a,b,c)})} B_{\varphi(a,b,c)} + \sum_{(a,b)\in E_\tau} \int_{\Gamma(\sigma^1_{(a,b)})} \Lambda_{\varphi^E(a,b)} \\ + \sum_{i=1}^{l} \sum_{j=1}^{k_i} \log g_{(\varphi^V(i))^j}\big(\Gamma(\sigma^0_i)\big) \ . \end{aligned} \tag{C.5}$$

The last term needs some clarifications. The logarithm can be taken since a good covering has been chosen, so the intersections are contractible. Of course, it's defined up to $2\pi i\,\mathbb{Z}$, so the quantity that can be well-defined as a number is $\exp\big(\int_\Gamma B\big)$. The sum is taken in the following way: consider the star of triangles having σ^0_i as common vertex (each of them associated to a chart via φ) and, since one is considering 0-simplices, that corresponds to 2-cochains, one considers the possible triads with first triangle fixed $\big(\varphi^V(i)\big)^j = \big(\varphi(B^1(i)), \varphi(B^j(i)), \varphi(B^{j+1}(i))\big)$ and sums over them. The fact that one fixed $B^1(i)$ as first triangle has no effect, since one could consider any other possibility $\big(\varphi^V_\alpha(i)\big)^j = \big(\varphi(B^\alpha(i)), \varphi(B^j(i)), \varphi(B^{j+1}(i))\big)$. In fact, by cocycle condition with indices $(1, i, i+1, \alpha)$ one has that $g_{1,i+1,\alpha} \cdot g_{1,i,i+1} = g_{i,i+1,\alpha} \cdot g_{1,i,\alpha}$, thus $g_{\alpha,i,i+1} = g^{-1}_{1,i,\alpha} \cdot g_{1,i,i+1} \cdot g_{1,i+1,\alpha}$, but in the cyclic sum the extern terms simplify, hence the sum involving $g_{\alpha,i,i+1}$ is equal to the sum involving $g_{1,i,i+1}$. Finally, one summed over $j = 1, \dots, k_i$, but for $j = 1$ and $j = k_i$ one obtains trivial terms, hence the real sum is for $j = 2, \dots, k_i - 1$.

C.2.2 Surfaces with boundary

Definition C.2.2 *Given a topological space X and a compact surface* with boundary Σ, *the* space of maps from Σ to X, *called ΣX, is the set of continuous maps:*

$$\Gamma : \Sigma \longrightarrow X$$

equipped with the compact-open topology.

As before:

- fix a triangulation τ of Σ, i.e.:
  - a set of vertices $\sigma_1^0, \ldots, \sigma_l^0 \in \Sigma$;
  - a subset $E \subset \{1, \ldots, l\}^2$, determining a set of oriented edges $\{\sigma^1_{(a,b)} \subset \Sigma\}_{(a,b)\in E}$ such that $\partial\sigma^1_{(a,b)} = \sigma^0_b - \sigma^0_a$; if $(a, b) \in E$ then $(b, a) \notin E$ and declare $\sigma^1_{(b,a)} := -\sigma^1_{(a,b)}$;
  - a subset $T \subset \{1, \ldots, l\}^3$, determining a set of oriented triangles $\{\sigma^2_{(a,b,c)} \subset \Sigma\}_{(a,b,c)\in T}$ such that $\partial\sigma^2_{(a,b,c)} = \sigma^1_{(a,b)} + \sigma^1_{(b,c)} + \sigma^1_{(c,a)}$; given a, b, c only one permutation of them belongs to T and for a permutation ρ declare $\sigma^2_{\rho(a),\rho(b),\rho(c)} := (-1)^\rho\sigma^2_{(a,b,c)}$;

  satisfying the usual conditions for triangulations; there exists a partition $E = BE \,\dot{\cup}\, IE$ in *boundary edges* and *internal edges*, and two functions:

  - $b : IE \to T^2$ such that $\sigma^1_{(a,b)} \subset \partial\sigma^2_{b^1(a,b)}$ and $-\sigma^1_{(a,b)} \subset \partial\sigma^2_{b^2(a,b)}$;
  - $b : BE \to T$ such that $\sigma^1_{(a,b)} \subset \partial\sigma^2_{b(a,b)}$;

  moreover, there exists a partition $\{0, \ldots, l\} = BV \,\dot{\cup}\, IV$ in *boundary vertices* and *internal vertices*, such that:

  - for $i \in IV$, there exists a finite set of triangles $\{\sigma^2_{(i,a_1,a_2)}, \ldots, \sigma^2_{(i,a_{k_i},a_1)}\}$ having σ^0_i as vertex; $\sigma^2_{(i,a_j,a_{j+1})} \cap \sigma^2_{(i,a_{j+1},a_{j+2})} = \sigma^1_{(i,a_{j+1})}$ with a cyclic order (the notation is such that $k_i + 1 = 1$); these triangles are the only ones containing σ^0_i and their union is a neighborhood of it; thus, there exists a function $B : IV \to \coprod_{i\in IV} T^{k_i}$, such that $B(i) \in T^{k_i}$ and $B(i) = \{\sigma^2_{(i,a_1,a_2)}, \ldots, \sigma^2_{(i,a_{k_i},a_1)}\}$.
  - for $i \in BV$, there exists a finite set of triangles $\{\sigma^2_{(i,a_1,a_2)}, \ldots, \sigma^2_{(i,a_{k_i-1},a_{k_i})}\}$ (without $\sigma^2_{(i,a_{k_i},a_1)}$) having σ^0_i as vertex; $\sigma^2_{(i,a_j,a_{j+1})} \cap \sigma^2_{(i,a_{j+1},a_{j+2})} = \sigma^1_{(i,a_{j+1})}$ for $1 < i < k_i$, these triangles are the only ones containing σ^0_i and their union is a neighborhood of it; thus, there exists a function $B : BV \to \coprod_{i=\in BV} T^{k_i-1}$, such that $B(i) \in T^{k_i-1}$ and $B(i) = \{\sigma^2_{(i,a_1,a_2)}, \ldots, \sigma^2_{(i,a_{k_i-1},a_{k_i})}\}$;

- consider the following set of indices:

$$J = \left\{ (\tau, \varphi) : \begin{array}{l} \bullet\ \tau = \left\{\sigma^0_1, \ldots, \sigma^0_{l(\tau)}, E, T\right\} \text{ is a triangulation of } \Sigma \\ \bullet\ \varphi : T \longrightarrow I \text{ is a function} \end{array} \right\}$$

  and a covering $\{V_{(\tau,\sigma)}\}_{(\tau,\sigma)\in J}$ of ΣX is given by:

$$V_{(\tau,\varphi)} = \{\Gamma \in \Sigma X : \ \Gamma(\sigma^2_{(a,b,c)}) \subset U_{\varphi(a,b,c)}\}.$$

One can prove that these sets are open in the compact-open topology and that they cover ΣX.

For the holonomy in this case, the only possibility is to use the same definition as for closed surfaces, omitting the boundary edges and vertices in the integration. This forbids the well-definedness of the integral as a function. One obtains a *line bundle* $\tilde{L}$ over the space of maps $\mathrm{Maps}(\partial\Sigma, Y)$ with the following properties (we call $\mathcal{G}$ the gerbe):

- $c_1(\tilde{L})$ depends on $c_1(\mathcal{G})$, thus, if $\mathcal{G}$ is trivial then $\tilde{L}$ is trivial too;
- a particular realization of $\mathcal{G}$ as a Čech hypercocycle (see appendices A and B for notations) determines a realization of $\tilde{L}$ as Čech cocycle; in particular, if $\mathcal{G}$ is of the form $\{g_{\alpha\beta\gamma}, 0, B\}$ with $g_{\alpha\beta\gamma}$ constant, one obtains a realization of $\tilde{L}$ with constant transition functions whose class in $H^1(\mathrm{Maps}(\partial\Sigma, Y), S^1)$ depends on $[\,\{g_{\alpha\beta\gamma}\}\,] \in H^2(Y, S^1)$. In particular, for a realization of the form $\{\eta_{\alpha\beta\gamma}, 0, B\}$ with $[\,\eta_{\alpha\beta\gamma}\,] = w_2(Y)$, one gets a realization of the $\tilde{L}$ with the same class as the parallel sections of the Pfaffian line bundle.

One can prove that the function for a specific trivialization can be obtained in the following way. Consider a trivial gerbe $\{g_{\alpha\beta\gamma}\} \in \check{B}^2(X, \underline{S}^1)$, and let $g_{\alpha\beta\gamma} = g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha}$. One has:

$$\begin{aligned}
&B_\alpha - B_\beta = d\Lambda_{\alpha\beta} \\
&\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = \frac{1}{2\pi i}(d\log g_{\alpha\beta} + d\log g_{\beta\gamma} + d\log g_{\gamma\alpha}) \\
&\big(\Lambda_{\alpha\beta} - d\log g_{\alpha\beta}\big) + \big(\Lambda_{\beta\gamma} - d\log g_{\beta\gamma}\big) + \big(\Lambda_{\gamma\alpha} - d\log g_{\gamma\alpha}\big) = 0 \\
&\delta\left\{\Lambda_{\alpha\beta} - d\log g_{\alpha\beta}\right\} = 0
\end{aligned}$$

and, since the sheaf of 1-forms is fine, hence acyclic, one obtains:

$$\Lambda_{\alpha\beta} - d\log g_{\alpha\beta} = A_\alpha - A_\beta \;.$$

The integral of the connection will be now defined. For a fixed $\Gamma \in \Sigma X$, there exists $(\tau, \varphi) \in J$ such that $\gamma \in V_{(\tau,\varphi)}$. Define:

$$\int_\Gamma B \;:=\; \sum_{(a,b,c)\in T_\tau} \left(\int_{\Gamma(\sigma^2_{(a,b,c)})} B_{\varphi(a,b,c)} + \int_{\Gamma(\partial\sigma^2_{(a,b,c)})} A_{\varphi(a,b,c)} \right) . \tag{C.6}$$

As before, the logarithm can be taken since a good cover has been chosen, and it is defined up to $2\pi i\,\mathbb{Z}$, so that $\exp\big(2\pi i \cdot \int_\Gamma B\big)$ is a well-defined number. The contribution of A to the internal edges cancel in pairs, so only the integral of A on boundary terms remains. That is why this expression is usually denoted by:

$$\int_\Gamma B + \oint_{\partial\Gamma} A \;.$$

This expression is equivalent to the one obtained by changing B choosing transition functions on the boundary corresponding to the fixed realization.

Appendix D

Useful K-theory notions

In this appendix the main K-theoretical constructions, which are used throughout the thesis, are briefly recalled.

D.1 Products in K-theory

$K(X)$ has a natural ring structure given by the tensor product: $[E] \otimes [F] := [E \otimes F]$. Such product restricts to $\tilde{K}(X)$. In general, one can define a product:

$$K(X) \otimes K(Y) \xrightarrow{\boxtimes} K(X \times Y) \tag{D.1}$$

where, if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projections, $E \boxtimes F = \pi_1^* E \otimes \pi_2^* F$. The fiber of $E \boxtimes F$ at (x, y) is $E_x \otimes E_y$[1]. By fixing a marked point for X and Y, this restricts to [98]:

$$\tilde{K}(X) \otimes \tilde{K}(Y) \xrightarrow{\boxtimes} \tilde{K}(X \wedge Y)\,, \tag{D.2}$$

where $X \wedge Y = X \times Y \setminus (\{pt\} \times Y \cup X \times \{pt\})$. Indeed, first of all[2]:

$$\tilde{K}(X \times Y) \simeq \tilde{K}(X \wedge Y) \oplus \tilde{K}(Y) \oplus \tilde{K}(X). \tag{D.3}$$

In fact:

- since X is a retract of $X \times Y$ via the projection, one has that $\tilde{K}(X \times Y) = K(X \times Y, X) \oplus \tilde{K}(X) = \tilde{K}(X \times Y/X) \oplus \tilde{K}(X)$;
- since Y is a retract of $X \times Y/X$ via the projection, one also has $\tilde{K}(X \times Y/X) = K(X \times Y/X, Y) \oplus \tilde{K}(Y) = \tilde{K}(X \wedge Y) \oplus \tilde{K}(Y)$.

Combining, one obtains (D.3). The explicit isomorphism in (D.3) is given, for $\alpha = [E] - [F] \in \tilde{K}(X \times Y)$, by:

$$\alpha \longrightarrow \big(\alpha - \pi_1^* \,\alpha|_X - \pi_2^* \,\alpha|_Y\big) \oplus \pi_2^* \,\alpha|_Y \oplus \pi_1^* \,\alpha|_X.$$

Let $\alpha \in \tilde{K}(X)$ and $\beta \in \tilde{K}(Y)$: then $\alpha \boxtimes \beta|_X = 0$ and $\alpha \boxtimes \beta|_Y = 0$. In fact:

$$\alpha \boxtimes \beta|_X = \alpha \otimes (\pi_2^* \,\beta)|_X = \alpha \otimes i_1^* \pi_2^* \,\beta = \alpha \otimes (\pi_2 i_1)^* \,\beta\,,$$

where $i_1 : X \to X \times Y$. But $\pi_2 i_1 : X \to Y$ is the constant map with value y_0, and the pull-back of a bundle by a constant map is trivial. Hence $(\pi_2 i_1)^* \,\beta = 0$. Similarly for Y. Hence, by (D.3), one obtains $\alpha \boxtimes \beta \in \tilde{K}(X \wedge Y)$.

[1] If $X = Y$ and $\Delta : X \to X \times X$ is the diagonal embedding, then $E \otimes F = \Delta^*(E \boxtimes F)$.

[2] (D.3) is actually true for $\tilde{K}^{-n}(X \times Y)$ for any n, with the same proof.

D.1.1 Non-compact case

For a generic (also non-compact) space X, one uses K-theory with compact support, i.e., $K(X) = \tilde{K}(X^+)$ where X^+ is the compactification of X to a point. One can easily prove that $X^+ \wedge Y^+ = (X \times Y)^+$. Hence, the product (D.2) exactly becomes:

$$K(X) \otimes K(Y) \xrightarrow{\boxtimes} K(X \times Y) \tag{D.4}$$

also for the non-compact case.

D.2 Thom isomorphism

Let X be a compact topological space and $E \xrightarrow{\pi} X$ a fiber bundle (not necessarily complex). Then, $K(E)$ has a natural structure of $K(X)$-module. The proof is as follows.

One does not have a natural pull-back $\pi^* : K(X) \to K(E)$ since one considers the compactification E^+, and there are no possibilities to extend continuously π to E^+. Hence, one makes use of the product (D.4): considering the embedding $i : E \to X \times E$ given by $i(e) = \big(\pi(e), e\big)$,[3] which trivially extends to $i : E^+ \to (X \times E)^+$ by $i(\infty) = \infty$, one can define a product:

$$\begin{gathered} K(X) \otimes K(E) \longrightarrow K(E) \\ \alpha \otimes \beta \longrightarrow i^*(\alpha \boxtimes \beta). \end{gathered} \tag{D.5}$$

This product defines a structure of $K(X)$-module on $K(E)$.

Lemma D.2.1 *$K(E)$ is* unitary *as $K(X)$-module.*

Proof: Consider the following maps:

$$\begin{aligned} &\pi_1 : X^+ \times E^+ \longrightarrow X^+ \\ &\pi_2 : X^+ \times E^+ \longrightarrow E^+ \\ &i : E^+ \longrightarrow (X \times E)^+ \\ &\tilde{\pi} : X^+ \times E^+ \longrightarrow X^+ \wedge E^+ = (X \times E)^+ \\ &\tilde{\pi}_2 : (X \times E)^+ \longrightarrow E^+ \end{aligned}$$

where $i(e) = \big(\pi(e), e\big)$ and the others are defined in the obvious way. Since the map:

$$r : X^+ \times E^+ \longrightarrow \big(X^+ \times \{\infty\}\big) \cup \big(\{\infty\} \times E^+\big)$$

given by $r(x, e) = (x, \infty)$ and $r(\infty, e) = (\infty, e)$ [4] is a retraction, $\tilde{\pi}^* : \tilde{K}\big((X \times E)^+\big) \longrightarrow \tilde{K}(X^+ \times E^+)$ is injective [35]. Then, by the definition of the module structure, for $\alpha \in K(X) = \tilde{K}(X^+)$ and $\beta \in K(E) = \tilde{K}(E^+)$ one can reformulate (D.5) as[5]:

$$\alpha \cdot \beta = i^*(\tilde{\pi}^*)^{-1}(\alpha \boxtimes \beta) = i^*(\tilde{\pi}^*)^{-1}\big(\pi_1^*\alpha \otimes \pi_2^*\beta\big).$$

[3] For such an embedding it is not necessary to have a marked point on X.

[4] The map r is continuous because X is compact, so that its ∞-point is disjoint from it.

[5] With respect to (D.5), one thinks $\alpha \boxtimes \beta \in \tilde{K}(X^+ \times E^+)$ and writes explicitly $(\tilde{\pi}^*)^{-1}$.

For $\alpha = 1$, one has $\alpha|_X = X \times \mathbb{C}$ and $\alpha|_{\{\infty\}} = 0$. Hence one has:

$$(1 \boxtimes \beta)\,\big|_{X\times E^+} = \pi_2^*\beta\,\big|_{X\times E^+}$$
$$(1 \boxtimes \beta)\,\big|_{\{\infty\}\times E^+} = 0.$$

But:

- since $\pi_2\,\big|_{X\times E^+} = (\tilde{\pi}_2 \circ \tilde{\pi})\,\big|_{X\times E^+}$, one has $\pi_2^*\beta\,\big|_{X\times E^+} = \tilde{\pi}^*\tilde{\pi}_2^*\beta\,\big|_{X\times E^+}$;
- since $\tilde{\pi}_2 \circ \tilde{\pi}\,(\{\infty\} \times E^+) = \{\infty\}$ and $\beta \in \tilde{K}(E^+)$, one has $(\tilde{\pi}^*\tilde{\pi}_2^*\beta)\,\big|_{\{\infty\}\times E^+} = 0$.

Hence $1 \boxtimes \beta = \tilde{\pi}^*\tilde{\pi}_2^*\beta$, so that:

$$1 \cdot \beta = i^*(\tilde{\pi}^*)^{-1}\tilde{\pi}^*\tilde{\pi}_2^*\beta = i^*\tilde{\pi}_2^*\beta = (\tilde{\pi}_2 \circ i)^*\beta = \mathrm{id}^*\beta = \beta.$$

□

Consider a vector space $\mathbb{R}^{2n}$ as a fiber bundle on a point $\{x\}$. Then we have:

- $K(x) = \mathbb{Z}$;
- $K(\mathbb{R}^{2n}) = \tilde{K}\big((\mathbb{R}^{2n})^+\big) = \tilde{K}(S^{2n}) = \mathbb{Z}$.

Hence $K(x) \simeq K(\mathbb{R}^{2n})$. The idea of the Thom isomorphism is to extend this to a generic bundle $E \to X$ with fiber $\mathbb{R}^{2n}$. To achieve this, one tries to write such isomorphism in a way that extends to a generic bundle. Actually, this generalization works for E a spin^c-bundle of *even* dimension.

Consider the spin group $\mathrm{Spin}(2n)$ (see [99]). The spin representation acts on $\mathbb{C}^{2^n}$, and it splits in the two irreducible representations of positive and negative chirality, acting on the subspaces S^+ and S^- of $\mathbb{C}^{2^n}$ of dimension 2^{n-1}. Also the group $\mathrm{Spin}^c(2n)$, defined as $\mathrm{Spin}(2n) \otimes_{\mathbb{Z}_2} U(1)$, acts on $\mathbb{C}^{2^n}$ via the standard spin^c representation, and the same splitting in chirality holds: let $S^+_{\mathbb{C}}$ and $S^-_{\mathbb{C}}$ be the two corresponding subspaces, thinking of them as $\mathrm{Spin}^c(2n)$-modules instead of $\mathrm{Spin}(2n)$-modules. For $\mathbb{C}\mathrm{l}(2n)$ the complex Clifford algebra of dimension $2n$, $\mathbb{C}^{2^n}$ is also a $\mathbb{C}\mathrm{l}(2n)$-module, and, for $v \in \mathbb{R}^{2n} \subset \mathbb{C}\mathrm{l}(2n)$, one has $v \cdot S^+_{\mathbb{C}}(\mathbb{R}^{2n}) = S^-_{\mathbb{C}}(\mathbb{R}^{2n})$. One thus considers the following complex:

$$0 \longrightarrow \mathbb{R}^{2n} \times S^+_{\mathbb{C}}(\mathbb{R}^{2n}) \xrightarrow{\;c\;} \mathbb{R}^{2n} \times S^-_{\mathbb{C}}(\mathbb{R}^{2n}) \longrightarrow 0$$

where c is the Clifford multiplication by the first component: $c\,(v, z) = (v, v \cdot z)$. Such sequence of trivial bundles on $\mathbb{R}^{2n}$ is exact when restricted to $\mathbb{R}^{2n} \setminus \{0\}$, hence the alternated sum:

$$\lambda_{\mathbb{R}^{2n}} = \big[\mathbb{R}^{2n} \times S^-_{\mathbb{C}}(\mathbb{R}^{2n})\big] - \big[\mathbb{R}^{2n} \times S^+_{\mathbb{C}}(\mathbb{R}^{2n})\big]$$

naturally gives a class in $K\big(\mathbb{R}^{2n}, \mathbb{R}^{2n}\setminus\{0\}\big)$ [35]. The sequence is exact in particular in $\mathbb{R}^{2n}\setminus B_1(0)$, $B_1(0)$ being the 2n-dimensional open ball of radius 1 centered in the origin with boundary $S^1(0)$; hence it defines a class:

$$\lambda_{\mathbb{R}^{2n}} \in K\big(\mathbb{R}^{2n}, \mathbb{R}^{2n} \setminus B_1(0)\big) = \tilde{K}\big(\,\overline{B_1(0)}\,/\,S^1(0)\big) = \tilde{K}(S^{2n}).$$

One can prove that, for η the dual of the tautological line bundle on $\mathbb{CP}^1$, whose sheaf of sections is usually denoted by $\mathcal{O}_{\mathbb{CP}^1}(1)$, if one identifies S^2 with $\mathbb{CP}^1$ topologically, one has:

$$\lambda_{\mathbb{R}^{2n}} = (-1)^n \cdot (\eta - 1)^{\boxtimes n} \tag{D.6}$$

i.e., it is a generator of $\tilde{K}(S^{2n}) \simeq \mathbb{Z}$.

For a generic spin^c-bundle $\pi : E \to X$ of dimension $2n$, let $S^{\pm}_{\mathbb{C}}(E)$ be the bundles of complex chiral spinors associated to E, i.e. one considers the spin^c-lift of the orthogonal frame bundle $SO(E)$ to $\mathrm{Spin}^c(E)$; call $S_{\mathbb{C}}(E)$ the vector bundle with fiber $\mathbb{C}^{2^n}$ associated to the representation $\mathrm{Spin}^c(2n) \subset \mathbb{C}\mathrm{l}(2n) \hookrightarrow \mathbb{C}^{2^n}$: this bundle splits into $S_{\mathbb{C}}(E) = S^+_{\mathbb{C}}(E) \oplus S^-_{\mathbb{C}}(E)$. Such bundle is naturally a $\mathbb{C}\mathrm{l}(E)$-module.

One can lift $S^{\pm}_{\mathbb{C}}(E)$ to E by π^*. Then one considers the complex:

$$0 \longrightarrow \pi^* S^+_{\mathbb{C}}(E) \xrightarrow{c} \pi^* S^-_{\mathbb{C}}(E) \longrightarrow 0$$

where c is the Clifford multiplication given by the structure of $\mathbb{C}\mathrm{l}(E)$-module: for $e \in E$ and $s_e \in \big(\pi^* S^+_{\mathbb{C}}(E)\big)_e$, define $c(s_e) = e \cdot s_e$. Such sequence is exact when restricted to $E \setminus B_1(E)$, where, for any fixed metric on E, $B_1(E)$ is the union of open balls of radius 1 on each fiber. Hence one naturally obtains:

$$\lambda_E = \big[\pi^* S^-_{\mathbb{C}}(E)\big] - \big[\pi^* S^+_{\mathbb{C}}(E)\big]$$

as a class in $K\big(E, E \setminus B_1(E)\big) = \tilde{K}\big(\overline{B_1(E)} \,/\, S_1(E)\big) = \tilde{K}(E^+) = K(E)$. The following fundamental theorem holds (see [99, 100] and, only for the complex case, [35, 98]):

Theorem D.2.2 (Thom isomorphism) *Let X be a compact topological space and $\pi : E \to X$ an* even *dimensional spin^c-bundle. For*

$$\lambda_E = \big[\pi^* S^-_{\mathbb{C}}(E)\big] - \big[\pi^* S^+_{\mathbb{C}}(E)\big] \in K(E)$$

the map:

$$\begin{aligned} T : K(X) &\longrightarrow K(E) \\ \alpha &\longrightarrow \alpha \cdot \lambda_E \end{aligned}$$

is a group isomorphism.

One can now see that the construction for a generic $2n$-dimensional spin^c-bundle $E \to X$ is a generalization of the construction of $\mathbb{R}^{2n}$. In fact, for $x \in X$:

- $\big(\pi^* S^{\pm}_{\mathbb{C}}(E)\big)\big|_{E_x} = E_x \times \big(S^{\pm}_{\mathbb{C}}(E)\big)_x \simeq \mathbb{R}^{2n} \times S^{\pm}_{\mathbb{C}}(\mathbb{R}^{2n})$;
- Clifford multiplication restricts on each fiber E_x to Clifford multiplication in $\mathbb{R}^{2n} \times S_{\mathbb{C}}(\mathbb{R}^{2n})$.

Hence:

$$\lambda_E\big|_{E_x} \simeq \lambda_{\mathbb{R}^{2n}}. \tag{D.7}$$

In particular, one sees that, for $i : E^+_x \to E^+$, the restriction $i^* : K(E) \to K(E_x) \simeq \mathbb{Z}$ is surjective.

D.3 Gysin map

Let X be a compact smooth n-manifold equipped with a metric and $Y \subset X$ a compact embedded p-submanifold such that $n - p$ is even and the normal bundle $\mathcal{N}(Y) = (TX\,|_Y)/\,TY$ is spin^c. Then, since Y is compact, there exists a tubular neighborhood U of Y in X, i.e. there exists an homeomorphism $\varphi_U : U \to \mathcal{N}(Y)$.

If $i : Y \to X$ is the embedding, from this data one can naturally define an homomorphism, called **Gysin map**:

$$i_! : K(Y) \longrightarrow \tilde{K}(X).$$

In fact:

- one first applies Thom isomorphism $T : K(Y) \longrightarrow K\big(\mathcal{N}(Y)\big) = \tilde{K}\big(\mathcal{N}(Y)^+\big)$;
- then one naturally extends φ_U to $\varphi_U^+ : U^+ \longrightarrow \mathcal{N}(Y)^+$ and applies $\big(\varphi_U^+\big)^* : K\big(\mathcal{N}(Y)\big) \longrightarrow K(U)$;
- there is a natural map $\psi : X \to U^+$ given by:

$$\psi(x) = \begin{cases} x & \text{if } x \in U \\ \infty & \text{if } x \in X \setminus U \end{cases}$$

  hence one applies $\psi^* : K(U) \longrightarrow \tilde{K}(X)$.

Summarizing:

$$i_!\,(\alpha) = \psi^* \circ \big(\varphi_U^+\big)^* \circ T\,(\alpha). \tag{D.8}$$

Remark: One could try to use the immersion $i : U^+ \to X^+$ and the retraction $r : X^+ \to U^+$ to have a splitting $K(X) = K(U) \oplus K(X,U) = K(Y) \oplus K(X,U)$. But this is false, since the immersion $i : U^+ \to X^+$ is not continuous: *since X is compact*, $\{\infty\} \subset X^+$ is open, but $i^{-1}(\{\infty\}) = \{\infty\}$, and $\{\infty\}$ is not open in U^+ since U is non-compact.

Appendix E

The Atiyah-Hirzebruch spectral sequence

E.1 Spectral sequences for cohomological theories

As explained in [101], given the following assignments, for $p, q, r \in \mathbb{Z} \cup \{-\infty, +\infty\}$:

- for $-\infty \leq p \leq q \leq \infty$, an abelian group $H(p, q)$, such that $H(p, q) = H(0, q)$ for $p \leq 0$ and there exists $l \in \mathbb{N}$ such that $H(p, q) = H(p, +\infty)$ for $q > l$;
- for $p \leq q \leq r$, $a, b \geq 0$, $p + a \leq q + b$, two maps:

$$\begin{aligned} \Psi &: H(p + a, q + b) \to H(p, q) \\ \Delta &: H(p, q) \to H(q, r) \end{aligned}$$

satisfying appropriate axioms (see [101] chap. XV par. 7), one can define:

$$\begin{aligned} E_r^p &= \mathrm{Im}\big(H(p, p + r) \xrightarrow{\Psi} H(p - r + 1, p + 1)\big) \\ d_r^p \;&=\; \Delta^{p-r+1, p+1, p+r+1}\big|_{\mathrm{Im}(\Psi^{p,p+r}_{p-r+1,p+1})} \;:\; E_r^p \longrightarrow E_r^{p+r} \\ F^p H &= \mathrm{Im}\big(H(p, +\infty) \xrightarrow{\Psi} H(0, +\infty)\big). \end{aligned} \tag{E.1}$$

Then:

- the groups $F^p H$ are a filtration of $H \equiv H(0, +\infty)$;
- $E_{r+1}^p = H(E_r^p, d_r^p)$;
- the sequence $\{E_r^p\}_{r \in \mathbb{N}}$ stabilizes to $F^p H / F^{p+1} H$.

In particular, one has a commutative diagram:

$$\begin{tikzcd} H(p, p + r) \arrow[r, "\Psi_1"] \arrow[d, "\Delta_1"] & H(p - r + 1, p + 1) \arrow[d, "\Delta_2"'] \\ H(p + r, p + 2r) \arrow[r, "\Psi_2"] & H(p + 1, p + r + 1). \end{tikzcd} \tag{E.2}$$

The reader can verify that (see [101] chap. XV):

- $\mathrm{Im}(\Psi_1) = E_r^p$ and $\mathrm{Im}(\Psi_2) = E_r^{p+r}$;
- $d_r^p = \Delta_2 \big|_{\mathrm{Im}(\Psi_1)} : E_r^p \longrightarrow E_r^{p+r}$.

In this language, the limit of the sequence is:

$$E_0^p H := E_\infty^p = \mathrm{Im}\big(H(p, +\infty) \xrightarrow{\Psi} H(0, p+1)\big). \tag{E.3}$$

Such a limit is the associated graded group of the filtration of $H \equiv H(0, +\infty)$ given by:

$$F^p H = \mathrm{Im}\big(H(p, +\infty) \xrightarrow{\Psi} H(0, +\infty)\big)$$

i.e., $E_0^p H = F^p H / F^{p+1} H$.

Given a topological space X with a finite filtration:

$$\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^m = X$$

one can consider a generic cohomological theory $H^\bullet$ and define:

- $H(p, q) = \bigoplus\limits_n H^n(X^{q-1}, X^{p-1})$;
- $\Psi : H(p + a, q + b) \longrightarrow H(p, q)$ is induced (by the axioms of cohomology) by the map of couples $i : (X^{q-1}, X^{p-1}) \longrightarrow (X^{q+b-1}, X^{p+a-1})$;
- $\Delta : H(p, q) \to H(q, r)$ is the composition of the map $\pi^* : H^\bullet(X^{q-1}, X^{p-1}) \longrightarrow H^\bullet(X^{q-1})$ induced by $\pi : (X^{q-1}, \emptyset) \to (X^{q-1}, X^{p-1})$, and the Bockstein map $\beta : H^\bullet(X^{q-1}) \longrightarrow H^{\bullet+1}(X^{r-1}, X^{q-1})$.

Remark: the shift by -1 in the definition of $H(p, q)$ is necessary to have $H(0, +\infty) = \bigoplus_n H^n(X)$. It would not be necessary if one declared $X^0 = \emptyset$, but this is not coherent with the case of finite simplicial complexes, since, in that case, X^0 denotes the 0-skeleton.

Since K-theory is a cohomological theory, it is natural to consider the spectral sequence associated to it for a given filtration $\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^m = X$: such sequence is called **Atiyah-Hirzebruch spectral sequence** (AHSS). In particular, groups and maps are defined in the following way (for $p \leq q \leq r; a, b \geq 0; p + a \leq q + b$):

- $H(p, q) = \bigoplus\limits_n K^n(X^{q-1}, X^{p-1})$;
- $\Psi : K^\bullet(X^{q+b-1}, X^{p+a-1}) \longrightarrow K^\bullet(X^{q-1}, X^{p-1})$ is induced by pull-back of the map $i : X^{q-1}/X^{p-1} \longrightarrow X^{q+b-1}/X^{p+a-1}$;
- $\Delta : K^\bullet(X^{q-1}, X^{p-1}) \longrightarrow K(X^{r-1}, X^{q-1})$ is the composition of the map $\pi^* : K^\bullet(X^{q-1}, X^{p-1}) \longrightarrow K^\bullet(X^{q-1})$ induced by $\pi : X^{q-1} \to X^{q-1}/X^{p-1}$, and the K-theory Bockstein map $\delta : K^\bullet(X^{q-1}) \longrightarrow K^{\bullet+1}(X^{r-1}, X^{q-1})$.

E.2 K-theory and simplicial cohomology

Lemma E.2.1 *For $k \in \mathbb{N}$ and $0 \leq i \leq k$, let:*

$$X = \dot{\bigcup_{i=0,\dots,k}} X_i$$

be the one-point union of k topological spaces. Then:

$$\tilde{K}^n(X) = \bigoplus_{i=0}^{k} \tilde{K}^n(X_i)$$

Proof: For $n = 0$, one can construct the isomorphism $\varphi : \tilde{K}(X) \to \bigoplus \tilde{K}(X_i)$: it is simply given by $\varphi(\alpha)_i = \alpha|_{X_i}$. To build φ^{-1}, consider $\big\{[E_i] - [n_i]\big\} \in \bigoplus \tilde{K}(X_i)$, with n_i the trivial vector bundle of rank n_i. By adding and subtracting a trivial bundle one can suppose $n_i = n_j$ for every i, j, so that one considers $\big\{[E_i] - [n]\big\}$. Since the intersection of the X_i is a point and the bundles E_i have the same rank, one can glue them to a bundle E on X (see [35] pp. 20-21): then one declares $\varphi^{-1}\big(\,\{[E_i] - [n]\}\,\big) = \big(\,[E] - [n]\,\big)$.

For $n = 1$, one first notes that, for $\hat{S}^1 X$ the *unreduced* suspension of X,[1] $\tilde{K}\big(\hat{S}^1(X_1 \dot{\cup} X_2)\big) = \tilde{K}(\hat{S}^1 X_1 \dot{\cup} \hat{S}^1 X_2)$, since quotienting by a contractible space (the linking between vertices of the cones and the joining point) one obtains the same space. Hence $\tilde{K}^1(X_1 \dot{\cup} X_2) = \tilde{K}^1(X_1) \oplus \tilde{K}^1(X_2)$. Then, by induction, the thesis extends to finite families. $\square$

Remark: the previous lemma holds only for the one-point union of a *finite* number of spaces.

In the following theorem the group of simplicial cochains $C^p(X, \mathbb{Z})$ of a finite simplicial complex is supposed to coincide with the group of chains $C_p(X, \mathbb{Z})$: that is because, being the dimension finite, one can define the coboundary operator δ^p directly on chains, asking that the coboundary of a simplicial p-simplex σ^p is the alternated sum of the $(p+1)$-simplices whose boundary contains σ^p (while the boundary operator ∂^p is the alternated sum of the $(p-1)$-simplices contained in the boundary of σ^p). One can use this definition since the group of p-cochains as usually defined, i.e. $\mathrm{Hom}(C_p(X, \mathbb{Z}), \mathbb{Z})$, is canonically isomorphic to $C_p(X, \mathbb{Z})$ in the case of finite simplicial complexes, and the usual coboundary operator corresponds under this isomorphism to the one defined above.

Theorem E.2.2 *Let X be a n-dimensional simplicial complex, X^p be the p-skeleton of X for $0 \leq p \leq n$ and $C^p(X, \mathbb{Z})$ be the group of simplicial p-cochains. Then, for any p such that $0 \leq 2p \leq n$ or $0 \leq 2p+1 \leq n$, there are isomorphisms:*

$$\begin{aligned} \Phi^{2p} &: C^{2p}(X, \mathbb{Z}) \longrightarrow K(X^{2p}, X^{2p-1}) \\ \Phi^{2p+1} &: C^{2p+1}(X, \mathbb{Z}) \longrightarrow K^1(X^{2p+1}, X^{2p}) \end{aligned}$$

which can be summarized by:

$$\Psi^p : C^p(X, \mathbb{Z}) \longrightarrow K^p(X^p, X^{p-1}).$$

Moreover:

$$K^1(X^{2p}, X^{2p-1}) = K(X^{2p+1}, X^{2p}) = 0.$$

[1] The reader should remember that $K(\hat{S}^1 X) = K(S^1 X)$.

Proof: Denote the simplicial structure of X by $\Delta = \{\Delta_i^m\}$, where m is the dimension of the simplex and i enumerates the m-simplices, so that $X^{2p} = \bigcup_{i=0}^{k} \Delta_i^{2p}$. Then the quotient by X^{2p-1} is given by k spheres of dimension $2p$ attached to a point:

$$X^{2p}/X^{2p-1} = \dot{\bigcup_i} S_i^{2p}.$$

By lemma (E.2.1) one obtains $\tilde{K}(X^{2p}/X^{2p-1}) = \bigoplus_i \tilde{K}(S^{2p})$, and, by Bott periodicity, $\tilde{K}(S^{2p}) = \tilde{K}(S^0) = \mathbb{Z}$. Hence:

$$K(X^{2p}, X^{2p-1}) = \bigoplus_i \mathbb{Z} = C^{2p}(X, \mathbb{Z}).$$

For the odd case, let $X^{2p+1} = \bigcup_{j=0}^{h} \Delta_j^{2p+1}$. One has by lemma E.2.1:

$$\begin{aligned} K^1(X^{2p+1}, X^{2p}) &= \tilde{K}^1\Big(\dot{\bigcup_j} S_j^{2p+1}\Big) = \bigoplus_j \tilde{K}^1\big(S_j^{2p+1}\big) \\ &= \bigoplus_j \tilde{K}(S_j^{2p+2}) = \bigoplus_j \mathbb{Z} = C^{2p+1}(X, \mathbb{Z}). \end{aligned}$$

In the same way, $K^1(X^{2p}, X^{2p-1}) = \bigoplus_j \tilde{K}^1(S_j^{2p}) = \bigoplus_j \tilde{K}(S_j^{2p+1}) = 0$, and similarly for $K(X^{2p+1}, X^{2p})$.
□

The explicit isomorphisms Φ^{2p} and Φ^{2p+1} are given by:

$$\Phi^{2p}\big(\Delta_i^{2p}\big) = \begin{cases} (-1)^p(\eta - 1)^{\boxtimes p} & \in \tilde{K}\big(S_i^{2p}\big) \\ \\ 0 & \in \tilde{K}\big(S_j^{2p}\big) \quad \text{for } j \neq i \end{cases}$$

and:

$$\Phi^{2p+1}\big(\Delta_i^{2p+1}\big) = \begin{cases} (-1)^{p+1}(\eta - 1)^{\boxtimes (p+1)} & \in \tilde{K}^1\big(S_i^{2p+1}\big) \\ \\ 0 & \in \tilde{K}^1\big(S_j^{2p+1}\big) \quad \text{for } j \neq i \end{cases}$$

where the overall factors $(-1)^p$ and $(-1)^{p+1}$ have been put for coherence with (D.6).

Remark: such isomorphisms are canonical, since every simplex is supposed to be oriented and $\eta - 1$ is distinguishable from $1 - \eta$ also up to automorphisms of X (in the first case the trivial bundle has negative coefficient, in the second case the non-trivial one, so that, for example, they have opposite first Chern classes).

E.3 The spectral sequence

The spectral sequence will now be built, by using the groups:

$$H(p, q) = \bigoplus_n K^n(X^{q-1}, X^{p-1}).$$

E.3.1 The first step

The first step is:

$$E_1^p = H(p, p+1) = \bigoplus_n K^n(X^p, X^{p-1}).$$

The presence of the grading in the spectral sequence is now considered [101]. Since K^n is determined by the parity of n, one can use the $\mathbb{Z}_2$-index σ:

$$E_1^{p,\,\sigma} = K^{p+\sigma}(X^p, X^{p-1}).$$

By theorem E.2.2, one has isomorphisms:

$$\begin{array}{rclrcl} E_1^{2p,\,0} & \simeq & C^{2p}(X,\mathbb{Z}) & E_1^{2p,\,1} & = & 0 \\ E_1^{2p+1,\,0} & \simeq & C^{2p+1}(X,\mathbb{Z}) & E_1^{2p+1,\,1} & = & 0. \end{array}$$

Since $K(x_0) = \mathbb{Z}$ and $K^1(x_0) = 0$, one can write in a compact form:

$$E_1^{p,\,\sigma} \simeq C^p\big(X, K^\sigma(x_0)\big). \tag{E.4}$$

For $r = 1$, in the diagram (E.2) at page 157 one has $\Psi_1 = \Psi_2 = \mathrm{id}$, hence $d_1^p = \Delta_2$, i.e., $d_1^p = \Delta^{p,p+1,p+2}$. In particular:

$$d_1^p : \bigoplus_n K^n(X^p, X^{p-1}) \longrightarrow \bigoplus_n K^n(X^{p+1}, X^p)$$

is the composition:

$$\begin{tikzcd} \tilde{K}^{p+\sigma}(X^p/X^{p-1}) \arrow[rr, "d_1^{p,\,\sigma}"] \arrow[dr, "\pi^*"'] & & \tilde{K}^{p+\sigma+1}(X^{p+1}/X^p) \\ & \tilde{K}^{p+\sigma}(X^p) \arrow[ur, "\delta"'] & \end{tikzcd}$$

Another way to describe d_1^p can be obtained considering the exact sequence induced by $X^p/X^{p-1} \xrightarrow{i} X^{p+1}/X^{p-1} \xrightarrow{\pi} X^{p+1}/X^p$: then $d_1^{p,\,\sigma}$ is the corresponding map δ:

$$d_1^{p,\,\sigma} : \tilde{K}^{p+\sigma}(X^p/X^{p-1}) \longrightarrow \tilde{K}^{p+\sigma+1}(X^{p+1}/X^p). \tag{E.5}$$

E.3.2 The second step

As $E_1^{p,\,0} \simeq C^p\big(X,\mathbb{Z}\big)$, one has that [25] $E_2^{p,\,0} \simeq H^p\big(X,\mathbb{Z}\big)$, i.e. that $d_1^{p,\,0}$ is the simplicial coboundary operator under the isomorphism (E.4). By the first formula of (E.1) one has $E_2^p = \mathrm{Im}\big(H(p,p+2) \xrightarrow{\Psi} H(p-1,p+1)\big)$, i.e.:

$$E_2^{p,\,\sigma} = \mathrm{Im}\big(K^{p+\sigma}(X^{p+1}, X^{p-1}) \xrightarrow{\Psi} K^{p+\sigma}(X^p, X^{p-2})\big). \tag{E.6}$$

Thus, for $\sigma = 0$, one obtains the isomorphism:

$$\Xi^p : H^p(X,\mathbb{Z}) \longrightarrow \mathrm{Im}\,\Psi \subset K^p(X^p, X^{p-2}). \tag{E.7}$$

Cocycles and coboundaries

Consider now the maps:

$$\begin{aligned} j &: X^p/X^{p-1} \longrightarrow X^{p+1}/X^{p-1} \\ \pi &: X^p/X^{p-2} \longrightarrow X^p/X^{p-1} = \frac{X^p/X^{p-2}}{X^{p-1}/X^{p-2}} \\ i &: X^p/X^{p-2} \longrightarrow X^{p+1}/X^{p-1} \end{aligned}$$

These maps induce a commutative diagram:

$$\begin{tikzcd} E_1^{p,\sigma} = \tilde{K}^{p+\sigma}(X^p/X^{p-1}) \arrow[dr, "\pi^*"] & \\ \tilde{K}^{p+\sigma}(X^{p+1}/X^{p-1}) \arrow[u, "j^*"] \arrow[r, "i^*"] & \tilde{K}^{p+\sigma}(X^p/X^{p-2}) \end{tikzcd} \tag{E.8}$$

where i^*, j^*, π^* are maps of the Ψ-type. One has that $E_2^{p,\sigma} = \operatorname{Im} i^*$ by (E.6).

It turns out that:

1. $\operatorname{Ker} d_1^{p,\sigma} = \operatorname{Im} j^*$;
2. $\operatorname{Im} d_1^{p-1,\sigma} = \operatorname{Ker} \pi^*$.

The first statement follows directly from (E.5) using the exact sequence:

$$\cdots \longrightarrow \tilde{K}^{p+\sigma}(X^{p+1}/X^{p-1}) \xrightarrow{j^*} \tilde{K}^{p+\sigma}(X^p/X^{p-1}) \xrightarrow{d_1^{p,\sigma}} \tilde{K}^{p+\sigma+1}(X^{p+1}/X^p) \longrightarrow \cdots$$

and the second from the exact sequence:

$$\cdots \longrightarrow \tilde{K}^{p+\sigma-1}(X^{p-1}/X^{p-2}) \xrightarrow{d_1^{p-1,\sigma}} \tilde{K}^{p+\sigma}(X^p/X^{p-1}) \xrightarrow{\pi^*} \tilde{K}^{p+\sigma}(X^p/X^{p-2}) \longrightarrow \cdots .$$

Since $\operatorname{Im} i^* \simeq H^p(X, \mathbb{Z})$ and $d_1^{p,0}$ corresponds to the simplicial coboundary under this isomorphism, it follows that:

- cocycles in $C^p(X, K^\sigma(x_0))$ correspond to classes in $\operatorname{Im} j^*$, i.e. to classes in $\tilde{K}^{p+\sigma}(X^p/X^{p-1})$ that are restriction of classes in $\tilde{K}^{p+\sigma}(X^{p+1}/X^{p-1})$;
- coboundaries in $C^p(X, K^\sigma(x_0))$ correspond to classes in $\operatorname{Ker} \pi^*$, i.e. to classes in $\tilde{K}^{p+\sigma}(X^p/X^{p-1})$ that are 0 when lifted to $\tilde{K}^{p+\sigma}(X^p/X^{p-2})$;
- $\operatorname{Im} \pi^*$ corresponds to cochains (not only cocycles) up to coboundaries and its subset $\operatorname{Im} i^*$ corresponds to cohomology classes;
- given $\alpha \in \operatorname{Im} i^*$, one can lift it to a class in $\tilde{K}^{p+\sigma}(X^p/X^{p-1})$ choosing different trivializations on X^{p-1}/X^{p-2}, and the different homotopy classes of such trivializations determine the different respresentative cocycles of the class.

E.3.3 The last step

Notation: denote $i_p : X^p \to X$ and $\pi_p : X \to X/X^p$ for any p.

Recall equation (E.3):

$$E_\infty^p = \mathrm{Im}\big(H(p,+\infty) \xrightarrow{\Psi} H(0,p+1)\big)$$

which, in the present case, becomes:

$$E_\infty^{p,\sigma} = \mathrm{Im}\big(K^{p+\sigma}(X,X^{p-1}) \xrightarrow{\Psi} K^{p+\sigma}(X^p)\big) \tag{E.9}$$

where Ψ is obtained by the pull-back of $i : X^p \to X/X^{p-1}$. Since $i = \pi_{p-1} \circ i_p$, the following diagram commutes:

$$\begin{tikzcd} \tilde{K}^{p+\sigma}(X/X^{p-1}) \arrow[rr, "\Psi"] \arrow[dr, "\pi_{p-1}^*"'] & & \tilde{K}^{p+\sigma}(X^p) \\ & \tilde{K}^{p+\sigma}(X). \arrow[ur, "i_p^*"'] & \end{tikzcd} \tag{E.10}$$

Remark: in the previous triangle one cannot say that $i_p^* \circ \pi_{p-1}^* = 0$ by exactness, since by exactness $i_p^* \circ \pi_p^* = 0$ at the same level p, as follows by $X^p \to X \to X/X^p$.

By exactness of $K^{p+\sigma}(X,X^{p-1}) \xrightarrow{\pi_{p-1}^*} K^{p+\sigma}(X) \xrightarrow{i_{p-1}^*} K^{p+\sigma}(X^{p-1})$, one has that:

$$\mathrm{Im}\,\pi_{p-1}^* = \mathrm{Ker}\,i_{p-1}^*.$$

Since trivially $\mathrm{Ker}\,i_p^* \subset \mathrm{Ker}\,i_{p-1}^*$, one obtains that $\mathrm{Ker}\,i_p^* \subset \mathrm{Im}\,\pi_{p-1}^*$. Moreover:

$$\mathrm{Im}\,\Psi = \mathrm{Im}\left(i_p^* \circ \pi_{p-1}^*\right) = \mathrm{Im}\left(i_p^* \,|_{\mathrm{Im}\,\pi_{p-1}^*}\right) \simeq \frac{\mathrm{Im}\,\pi_{p-1}^*}{\mathrm{Ker}\,i_p^*} = \frac{\mathrm{Ker}\,i_{p-1}^*}{\mathrm{Ker}\,i_p^*}$$

hence, finally:

$$\begin{aligned} E_\infty^{p,0} &= \frac{\mathrm{Ker}\big(K^p(X) \longrightarrow K^p(X^{p-1})\big)}{\mathrm{Ker}\big(K^p(X) \longrightarrow K^p(X^p)\big)} \\ E_\infty^{p,1} &= 0 \end{aligned} \tag{E.11}$$

i.e., $E_\infty^{p,0}$ is made by p-classes on X which are 0 on X^{p-1}, up to classes which are 0 on X^p.

E.3.4 From the first to the last step

Now it will be shown how to link the first and the last step of the sequence. In general, as already seen, one has:

$$E_1^p = H(p,p+1) \qquad E_\infty^p = \mathrm{Im}\big(H(p,+\infty) \xrightarrow{\Psi} H(0,p+1)\big).$$

There is a natural Ψ-map:

$$\iota : H(p,+\infty) \longrightarrow H(p,p+1)$$

so that an element $\alpha \in E_1^p$ survives up to the last step if and only if $\alpha \in \mathrm{Im}\,\iota$ and its class in E_∞^p is $\Psi \circ (\iota^{-1})(\alpha)$, which is well-defined since $\mathrm{Ker}\,\iota \subset \mathrm{Ker}\,\Psi$. One thus defines, for $\alpha \in \mathrm{Im}\,\iota \subset E_1^p$:

$$\{\alpha\}^{(1)}_{E_\infty^p} := \Psi \circ (\iota^{-1})(\alpha).$$

where the upper 1 means that one is starting from the first step.

For AHSS, considering p even and $\sigma = 0$, this becomes:

$$E_1^{p,\,0} = K(X^p, X^{p-1}) \qquad E_\infty^{p,\,0} = \mathrm{Im}\big(K(X, X^{p-1}) \xrightarrow{\Psi} K(X^p)\big)$$

and:

$$\iota : K(X, X^{p-1}) \longrightarrow K(X^p, X^{p-1}).$$

In this case, $\iota = i^*$ for $i : X^p/X^{p-1} \to X/X^{p-1}$. Thus, the classes in $E_1^{p,\,0}$ surviving up to the last step are the ones which are restrictions of a class defined on all X/X^{p-1}. Moreover, $\Psi = j^*$ for $j : X^p \to X/X^{p-1}$, and $j = i \circ \pi^p$ for $\pi^p : X^p \to X^p/X^{p-1}$. Hence $\Psi = (\pi^p)^* \circ \iota$, so that, for $\alpha \in \mathrm{Im}\,\iota \subset E_1^{p,\,0}$:

$$\{\alpha\}^{(1)}_{E_\infty^{p,\,0}} := (\pi^p)^*(\alpha). \tag{E.12}$$

Since in this thesis it is needed to start from an element $\beta \in E_2^{p,\,0}$, which survives up to the last step, it is also defined

$$\{\beta\}^{(2)}_{E_\infty^{p,\,0}}$$

as the class in $E_\infty^{p,\,0}$ corresponding to β.

E.4 Rational K-theory and cohomology

Consider, before concluding, the Atiyah-Hirzebruch spectral sequence in the rational case. In particular, consider the groups:

$$H(p,q) = \bigoplus_n K^n_{\mathbb{Q}}(X^{q-1}, X^{p-1})$$

where $K^n_{\mathbb{Q}}(X,Y) := K^n(X,Y) \otimes \mathbb{Q}$. In this case the sequence is made by the groups $Q_r^{p,\,\sigma} = E_r^{p,\,\sigma} \otimes \mathbb{Q}$. In particular:

$$\begin{aligned} &Q_2^{p,\,0} \simeq H^p\big(X, \mathbb{Q}\big) && Q_2^{p,\,1} = 0 \\ &Q_\infty^{p,\,0} = \frac{\mathrm{Ker}\big(K^p_{\mathbb{Q}}(X) \longrightarrow K^p_{\mathbb{Q}}(X^{p-1})\big)}{\mathrm{Ker}\big(K^p_{\mathbb{Q}}(X) \longrightarrow K^p_{\mathbb{Q}}(X^p)\big)} && Q_\infty^{p,\,1} = 0. \end{aligned} \tag{E.13}$$

Such sequence collapses at the second step [25], hence $Q_\infty^{p,\,0} \simeq Q_2^{p,\,0}$. Since:

- $\bigoplus_p Q_\infty^{p,\,0}$ is the graded group associated to the chosen filtration of $K_{\mathbb{Q}}(X) \oplus K^1_{\mathbb{Q}}(X)$;
- in particular, by (E.13), $\bigoplus_{2p} Q_\infty^{2p,\,0}$ is the graded group of $K_{\mathbb{Q}}(X)$ and $\bigoplus_{2p+1} Q_\infty^{2p+1,\,0}$ is the graded group of $K^1_{\mathbb{Q}}(X)$;
- $Q_\infty^{p,\,0} \simeq H^p\big(X, \mathbb{Q}\big)$, thus it has no torsion;

it follows that:

$$K_{\mathbb{Q}}(X) = \bigoplus_{2p} Q_\infty^{2p,\,0} \qquad K^1_{\mathbb{Q}}(X) = \bigoplus_{2p+1} Q_\infty^{2p+1,\,0}$$

hence:

$$K_{\mathbb{Q}}(X) \simeq H^{\mathrm{ev}}(X, \mathbb{Q}) \qquad K^1_{\mathbb{Q}}(X) \simeq H^{\mathrm{odd}}(X, \mathbb{Q}).$$

In particular, the isomorphisms of the last equation are given by Chern character:

$$\begin{aligned}
\mathrm{ch} : K_{\mathbb{Q}}(X) &\longrightarrow H^{\mathrm{ev}}(X, \mathbb{Q}) \\
\mathrm{ch} : K^1_{\mathbb{Q}}(X) &\longrightarrow H^{\mathrm{ev}}(S^1 X, \mathbb{Q}) \simeq H^{\mathrm{odd}}(X, \mathbb{Q}) \,,
\end{aligned}$$

and they are isomorphisms of rings.

Appendix F

Toric resolutions

In this appendix some basic techniques of toric geometry will be presented for illustrative purpose: they should be sufficient to understand all the steps of the blow-up processes performed in section 6.3 to resolve non-abelian singularities. For a more complete treatment, the reader is referred to [102, 103].
In particular, the fan of the ambient variety of $K3$, which can be easily visualized, will be constructed in detail, showing in this context its property of being a $W\mathbb{P}^2_{2,3,1}$-fibration over the 2-sphere. Then the Kodaira singularity of type I_2 will be forced on a toric divisor of the base and the weight assignments for the blown-up $K3$ will be deduced. Finally, most important, the resolved $K3$ will be shown to be still an elliptic fibration over the 2-sphere, but with a *two*-component fiber, regarded as the affine node and the Cartan node of the extended Dynkin diagram of $SU(2)$.

Consider the following parameterization of the $K3$ surface (just a change of basis with respect to the one in (4.22)):

$$\begin{array}{ccccc|c} x_1 & x_2 & X & Y & Z & \text{Weierstrass} \\ \hline 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 3 & 1 & 6 \end{array} \quad , \tag{F.1}$$

where now the homogeneous coordinate of the base $\mathbb{P}^1$ are called x_1, x_2. The Weierstrass polynomial describing $K3$ as a divisor of the ambient threefold, will generically be a sum of monomials like the following

$$x_1^{a+1}\, X^{b+1}\, Y^{c+1}\, Z^{d+1}\, x_2^{e+1}\,, \tag{F.2}$$

with a, b, c, d, e integer numbers. First of all, these numbers are not all independent, due to the two projective rescalings. Thus compatibility with (F.1) clearly requires:

$$\begin{cases} a + e - 2d &= 0 \\ 2b + 3c + d &= 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} d &= -2b - 3c \\ e &= -a - 4b - 6c \end{cases} . \tag{F.3}$$

Moreover, they are subjected to several conditions, which can be represented by means of vectors belonging to an integral three dimensional[1] lattice in the following way. Write the generic condition as:

$$w_1 a + w_2 b + w_3 c \geq -1\,; \tag{F.4}$$

[1]The dimension of the lattice is always equal to the dimension of the ambient space.

then, the lattice vector corresponding to a given condition will have coordinates equal to the coefficient w_i used to express that condition in the form of (F.4).
There are conditions which are always present: these are the ones that assure well-definiteness of the Weierstrass section, namely the positivity of every exponent in (F.2). Thus, one has:

$$\begin{array}{lccccccl} a \geq -1 & (1 & , & 0 & , & 0) & & x_1 \\ b \geq -1 & (0 & , & 1 & , & 0) & & X \\ c \geq -1 & (0 & , & 0 & , & 1) & & Y \\ d \geq -1 & (0 & , & -2 & , & -3) & & Z \\ e \geq -1 & (-1 & , & -4 & , & -6) & & x_2\ , \end{array} \tag{F.5}$$

where the third column assigns to each vector the coordinate whose exponent is displayed in the first column. As fare as $K3$ is smooth, this is the end of the story, and the vectors in (F.5) make up the so called *fan* of the ambient toric variety of $K3$. Indeed, it is easy to verify that the linear relations between them exactly coincide with the vanishing weighted sums of the corresponding coordinates (with weights given by (F.1)). For each further condition imposed, one gets a new vector with a new associated coordinate: the new condition represents some constraint on the coefficient of the Weierstrass polynomial, which may generate a singularity of the elliptic Calabi-Yau, and the additional vector, together with the previous ones, makes up the fan of the blown-up ambient threefold.
From the coordinates of the vectors in (F.5), one realizes that the vector associated to Z comes between the ones associated to x_1 and x_2, thus breaking the convex cone made by the last two; this implies that x_1x_2 is an element of the Stanley-Reisner ideal of this toric variety, and thus these two coordinates make up a 2-sphere. Moreover, it can be shown that the whole toric threefold is a fibration on such a $\mathbb{P}^1$. This can be quickly seen by verifying that the projection along the line generated by x_1 does not destroy any cone. The fan of the fiber is finally singled out as made by the vectors in the kernel of such a projection: these are obviously X, Y and Z (they all have zero in the first entry), and consequently XYZ constitutes the other element of the Stanley-Reisner ideal.

Now it is the moment to force the type I_2 singularity for example on the toric divisor $x_1 = 0$ of the base. In analogy to (4.32), the singularity of $K3$ will be located on the codimension three place in the ambient threefold (i.e. a point) described by $X = Y = x_1 = 0$. According to table 4.2, the only new condition to impose is that each monomial in the Weierstrass equation should be *at least quadratic* in those three coordinates (see eq. (4.33), where σ plays the role of x_1). Therefore, looking at (F.2) and (F.4), the right lattice vector to be added is $(1, 1, 1)$ and the associated coordinate will be called v. It is evident that one is blowing up the locus $X = Y = x_1 = 0$ in the original ambient threefold because v destroys the cone given by those three coordinates and the element XYx_1 must be added to the Stanley-Reisner ideal. Therefore, knowing all the lattice vectors, one can easily construct the table of projective weights for the blown-up ambient threefold and for the proper transform equation, which will no longer have the Weierstrass representation:

$$\begin{array}{cccccc|c} x_1 & x_2 & X & Y & Z & v & \text{proper transform} \\ \hline 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 & 6 \\ 1 & 0 & 1 & 1 & 0 & -1 & 2 \end{array} \quad . \tag{F.6}$$

Again, this toric manifold can be seen to be a fibration over a 2-sphere, whose coordinate now are (vx_1, x_2), as they both have projective weights $(1, 0, 0)$ in (F.6). It is clear, then, that also

x_2 and v cannot vanish at the same time, and thus also vx_2 belongs to the Stanley-Reisner ideal of the variety. This can be verified by changing basis in (F.6), substituting the first row with itself minus the third and adding a positive Fayet-Iliopulos parameter for the first U(1) gauge charge (see [7]). There is actually one last element of the ideal, namely vZ, that is not manifest, but can again be recognized by choosing yet an other basis of weights.
While the fan of the fiber is still made by the vectors corresponding to X, Y and Z, the fiber of the blown-up $K3$ over a generic point of the base is everywhere a T^2, given by the proper transform on that point, except for one point, that is when $vx_1 = 0$. Indeed, as it is manifest, on this base point the fiber splits into two parts given by the proper transform in which one puts $v = 0$ and $x_1 = 0$ respectively. It can be shown that the $x_1 = 0$ component has the topology of a $\mathbb{P}^1$ with degree *one*, i.e. a linear equation in the $W\mathbb{P}^2_{1,1,2}$ defined by the coordinates Y, Z, v expressing v in terms of the other two; the $v = 0$ component, instead, has the topology of a $\mathbb{P}^1$ with degree *two*, i.e. a complete irreducible quadratic equation in the $\mathbb{P}^2$ defined by the coordinates x_1, X, Y. The two components intersect in two points, as seen by imposing $v = x_1 = 0$ in the proper transform equation. The $v = 0$ component is the true exceptional divisor and thus represents the Cartan node of the $SU(2)$ Dynkin diagram, while the $x_1 = 0$ component is always there since it is the former base locus of singularity and it represents the affine node of the extended Dynkin diagram of $SU(2)$; the latter is the only one present when the singularity is the abelian I_1, which is only a fiber singularity.

As last useful information, the relation between this toric blow-up and the one performed in subsection 4.3.1 with "traditional" methods [40] is as follows. If s, t and u are the homogeneous coordinates of $\mathbb{P}^2$ which have to be introduced to make the blow-up of a codimension three locus by means of the traditional method, then the map which links the toric method to the latter is the following:

$$\varphi \,:\, (x_1, x_2, X, Y, Z, v) \longrightarrow (\tilde{x}_1, \tilde{x}_2, \tilde{X}, \tilde{Y}, \tilde{Z}, s, t, u) \;=\; (vx_1, x_2, vX, vY, Z, X, Y, x_1)\,. \tag{F.7}$$

Indeed, the table of projective weights of the new homogeneous coordinates is:

$$\begin{array}{cccccccc} \tilde{x}_1 & \tilde{x}_2 & \tilde{X} & \tilde{Y} & \tilde{Z} & s & t & u \\ \hline 1 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 3 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\,, \tag{F.8}$$

which are exactly the right weight assignments for the homogeneous coordinates of a toric threefold blown-up with traditional techniques. Notice that they are compatible with the (two independent) new equations which have to be added and which express the linear dependence of the two vectors (s, t, u) and (X, Y, x_1), namely

$$\mathrm{rank}\begin{pmatrix} s & X \\ t & Y \\ u & x_1 \end{pmatrix} \;=\; 1\,. \tag{F.9}$$

Appendix G

Smoothness of the blown-up fourfold

In this appendix an easy calculation is presented to prove that, in the case of I_2^{ns} singularities, the only blow-up induced by the lattice vector $v = (1, 1, 1, 0, 0)$ is sufficient to *completely* resolve the elliptic Calabi-Yau fourfold. It is expected that the same conclusion holds for worse singularities, provided all the blow-up's induced by the relevant lattice vectors are performed. A list of such vectors corresponding to any given Kodaira singularity can be found in [44].

Suppose to place the $SU(2)$ singularity on $x_1 = 0 \subset \mathbb{P}^3$ and consider the proper transform of the elliptic fibration after the blow-up induced by v:

$$\begin{aligned} Y^2 + a_1(x_1v, x_i)XYZ + a_{3,1}(x_1v, x_i)x_1YZ^3 &= vX^3 + a_2(x_1v, x_i)X^2Z^2 + \\ +a_{4,1}(x_1v, x_i)x_1XZ^4 + a_{6,2}(x_1v, x_i)x_1^2Z^6 & \qquad\qquad i = 2, 3, 4\,, \end{aligned} \tag{G.1}$$

and compute its gradient $\vec{\nabla}$. Since any possible residual singularity in (G.1) must lie on the exceptional divisor, it suffices to restrict the gradient to the submanifold $v = 0$. Gauge-fixing $Z = 1$, one obtains:

$$\vec{\nabla}|_E = \begin{pmatrix} a_1Y - 2a_2X - a_{4,1}x_1 \\ 2Y + a_1X + a_{3,1}x_1 \\ a_1XY + 3a_{3,1}x_1Y - 2a_2X^2 - 4a_{4,1}x_1X - 6a_{6,2}x_1^2 \\ a_{3,1}Y - a_{4,1}X - 2a_{6,2}x_1 \\ \partial_ia_1XY + \partial_ia_{3,1}x_1Y - \partial_ia_2X^2 - \partial_ia_{4,1}x_1X - \partial_ia_{6,2}x_1^2 \\ \partial_1a_1x_1XY + \partial_1a_{3,1}x_1^2Y - X^3 - \partial_1a_2x_1X^2 - \partial_1a_{4,1}x_1^2X - \partial_1a_{6,2}x_1^3 \end{pmatrix}, \tag{G.2}$$

where the eight rows are the derivatives of (G.1) with respect to X, Y, Z, x_1, x_i, v respectively and ∂_1 means the derivative of the polynomials with respect to the first argument. In analogy with the notations of [44], define:

$$\begin{aligned} b_2 &\equiv a_1^2 + 4a_2\,, \\ b_{4,1} &\equiv a_1a_{3,1} + 2a_{4,1}\,, \\ b_{6,2} &\equiv a_{3,1}^2 + 4a_{6,2}\,, \end{aligned} \tag{G.3}$$

where $b_2 = h$ is the O7-plane in the weak coupling limit. Then, after some algebraic manipulations, the conditions of vanishing of the gradient read:

$$\left\{\begin{array}{rcl} Y & = & -\frac{a_1 X + a_{3,1} x_1}{2} \\ b_2 X + b_{4,1} x_1 & = & 0 \\ b_{4,1} X + b_{6,2} x_1 & = & 0 \\ b_2 X^2 + 4 b_{4,1} x_1 X + 3 b_{6,2} x_1^2 & = & 0 \\ 4X^3 + (\partial_1 b_2 X^2 + 2\partial_1 b_{4,1} x_1 X + \partial_1 b_{6,2} x_1^2) x_1 & = & 0 \\ \partial_i b_2 X^2 + 2\partial_i b_{4,1} x_1 X + \partial_i b_{6,2} x_1^2 & = & 0\,. \end{array}\right. \tag{G.4}$$

It is evident from the above equations that residual singularities of the blown-up fourfold (if any) cannot lie on the intersection between the exceptional divisor and the affine component of the resolved fiber, $x_1 = 0$: indeed, the fourth equation would imply $X = 0$ and so, by the first equation, Y should vanish too, that is impossible in the ambient variety.
Therefore, one can gauge-fix in (G.4) $x_1 = 1$. Now a case-by-case analysis is needed.

- If $X = 0$, then $b_{4,1} = b_{6,2} = \partial_I b_{6,2} = 0$ $(I = 1, i)$, which are too many conditions to be imposed on S_2, and thus generically no solution is found.
- If $X \neq 0$, but $b_2 = 0$, then again there are too many conditions on S_2, namely $b_2 = b_{4,1} = b_{4,1} = 0$, plus others coming from the derivatives.
- If $X \neq 0$ and also $b_2 \neq 0$, then $X = -b_{4,1}/b_2$, with $b_{4,1} \neq 0$. Both the third and the fourth equation above implie $b_{8,2} \equiv b_2 b_{6,2} - b_{4,1}^2 = 0$, which restricts to a curve in S_2; but afterwards four further equations on the derivatives have to be imposed, which generically do not intersect.

This concludes the proof. No residual singularities survive after this toric blow-up of the original elliptic, I_2^{ns}-singular Calabi-Yau fourfold.

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