Stability, Optimization and Motion Planning for Control-Affine Systems

CANDIDATE

Paolo Mason

SUPERVISORS

Prof. Andrei Agrachev

Dr. Ugo Boscain

Thesis submitted for the degree of
"Doctor Philosophiae"

Academic Year 2005/2006

Contents

Introduction

In this thesis we treat different topics in the framework of control-affine systems i.e. systems of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x)\,, \quad x \in \mathcal{M}\,. \tag{1}$$

where $\mathcal{M}$ is a smooth manifold and the vector fields f_i are assumed to be smooth. These kinds of systems are probably the most studied in non-linear control theory since they are connected to a wide variety of problems that are mathematically very rich, and moreover they are often used in the applications. In this thesis, some of the results achieved relate to general affine systems, while other results focus on nontrivial applications of the general theory to specific systems. In particular we consider problems of optimal control with bounded controls, with applications to the quantum mechanics and mechanical systems. Moreover we consider the problem of the stabilty of control systems in the case of arbitrary controls and the problem of giving a generalized definition of solution for (1) if u is the (distributional) derivative of a non absolutely continuous function.

In the first part of this thesis (Chapters 1–4) we consider the particular case of the single-input control systems with bounded control

$$\dot{x} = f(x) + u(t) g(x)\,, \quad u(t) \in [-1, 1]\,, \quad x \in \mathcal{M}\,. \tag{2}$$

and we limit our study to the case in which $\mathcal{M}$ is a 2-dimensional manifold. For such class of systems we are interested in the following *time optimal problem*: for every pair of points p_0, p_1 in the manifold, we look for the trajectories of (2) connecting p_0 to p_1 in minimum time. These trajectories are called *time optimal trajectories*.

The most important and powerful tool for the study of optimal trajectories is the well known Pontryagin Maximum Principle (in the following PMP, see for instance [4, 34, 56]). It is a first order necessary condition for optimality that generalizes the Weierstraß conditions of Calculus of Variations to problems with non-holonomic constraints. For each optimal trajectory, the PMP provides a lift to the cotangent bundle that is a solution to a suitable pseudo–Hamiltonian system. However, giving a complete solution to an optimization problem remains extremely difficult for several reasons. First, one is faced with the problem of integrating a Hamiltonian system (that generically is not integrable excepted for very special costs). Second, one should manage with some special solutions of the PMP, the so called *abnormal extremals* and *singular trajectories*. Finally, even if one is able to find all the solutions of the PMP, i.e. the *extremal* *trajectories*, it remains the problem of *selecting* among them the optimal trajectories. For these reasons, excepted the case of linear systems with quadratic cost, usually, one can hope to find a complete solution to an optimal control problem (i.e. an *optimal synthesis*) in low dimension only. Indeed even in dimension two, where the techniques developed are very powerful, the problems are often very complicated and a complete solution is difficult to achieve. In dimension three most of the optimal control problems are open (see [5, 23, 37, 59] for results on the structure of the *attainable set* or of the optimal trajectories).

An interesting problem is to determine all the time optimal trajectories of (2) starting from a given point x_0 with $f(x_0) = 0$, i.e. x_0 is a stable point for f. The previous hypothesis is very natural. Indeed, under generic assumptions, it guarantees local controllability. Moreover if we reverse the time, we obtain the problem of stabilizing in minimum time all the points of $\mathcal{M}$ to x_0. For this time-reversed problem, the stability of x_0 also guarantees that once reached the origin, it is possible to stay there with zero control.

To solve this kind of problems, the main techniques have been developed by Sussmann, Bressan, Piccoli and Boscain, see for instance [21, 24, 54, 63] and recently rewritten in [22]. We summarize the main results in Chapter 1. Here we just say that, under generic conditions, the optimal trajectories of these problems can be expressed essentially by means of a piecewise smooth feedback law on $\mathcal{M}$, as it is shown in [53, 22] under generic conditions on the smooth vector fields f, g and in [63] in the analytic case. In particular in this case each optimal trajectory is piecewise $\mathcal{C}^1$, and moreover each $\mathcal{C}^1$ piece can either be a *bang arc* (in this case u is constantly equal to $+1$ or -1) or a *singular arc*, contained inside some special curves.

In Chapter 2, which is based on a joint work with Ugo Boscain, we consider an application of the theory of time optimal synthesis to quantum systems (see [20]).

In the recent past years, people started to approach the design of laser pulses by using geometric control techniques (see for instance [18, 30, 31, 36, 57]). Finite dimensional closed quantum systems are in fact left (or right) invariant control systems on $SU(n)$, or on the corresponding Hilbert sphere $S^{2n-1} \subset \mathbb{C}^n$, where n is the number of atomic or molecular levels.

For these kinds of systems the controllability problem (i.e. proving that for every couple of points p_1, p_2 in the state space one can find controls steering the system from one point to the other) is easy and in particular such systems turn out to be controllable under generic conditions (see for instance [8, 33, 35, 58]).

Concerning optimal control problems, typical costs that are interesting to minimize for applications are the *energy* tranfered by the controlled external fields (controls) to the system and the *time of transfer* among different energy levels.

The problem of minimizing time with no bound on the controls has been deeply investigated in [1, 36] and is now well-understood.

On the other hand the problems of minimizing time or energy with bounded controls are very difficult in general and one can hope to find a complete solution in low dimension only. For instance the minimum energy problem for a two-level system has been studied by D'Alessandro and Dahleh in [31] and the problem of minimizing energy and time in the *rotating wave approximation* ([7]) has been solved only for systems with two and three levels (see [15, 16, 17, 18]).

In this chapter we consider the minimum time population transfer problem for the z-component of the spin of a (spin 1/2) particle, driven by a magnetic field, which is constant along the z-axis and controlled along the x-axis, with bounded amplitude. We let $(-E, E)$ be the two energy levels and M be the bound on the field amplitude. The dynamics of this system is described by the following time dependent Schrödinger equation (in a system of units such that $\hbar = 1$)

$$i\frac{d\psi(t)}{dt} = H(t)\psi(t), \tag{3}$$

where $\psi(.) = (\psi_1(.), \psi_2(.))^T : [0, T] \to \mathbb{C}^2$, $|\psi_1(t)|^2 + |\psi_2(t)|^2 = 1$ (i.e. $\psi(t)$ belongs to the sphere $S^3 \subset \mathbb{C}^2$), and

$$H(t) = \begin{pmatrix} -E & \Omega(t) \\ \Omega(t) & E \end{pmatrix},$$

where $E > 0$ and the control $\Omega(.)$, is assumed to be a real function. Then, the aim is to induce a transition from the first eigenstate of H_0, which, for us, is the *state one* and corresponds to $|\psi_1|^2 = 1$, to any other physical state. In particular we would like to solve the *minimum time population transfer problem*, i.e. the problem of inducing a transition from the first eigenstate of H_0 to the second one.

This problem is reduced to a time optimal control problem on the two dimensional sphere (called Bloch sphere) by means of a Hopf fibration, so that we can apply the general theory discussed in Chapter 1. More precisely we are reduced to study the following control system

$$\begin{aligned} \dot{y} &= F_S(y) + uG_S(y), \quad |u| \le 1, \quad \text{where:} \\ & y \in \mathbf{S}_B := \{(y_1, y_2, y_3) \in \mathbb{R}^3, \quad \sum_{j=1}^{3} y_j^2 = 1\} \\ & F_S(y) := k\cos(\alpha)\begin{pmatrix} -y_2 \\ y_1 \\ 0 \end{pmatrix}, \qquad G_S(y) := k\sin(\alpha)\begin{pmatrix} 0 \\ -y_3 \\ y_2 \end{pmatrix}, \\ & \alpha := \arctan\left(\frac{M}{E}\right) \in]0, \pi/2[, \quad k := 2E/\cos(\alpha) = 2\sqrt{M^2 + E^2}. \end{aligned} \tag{4}$$

The initial point x_0, which in the original problem was the state one, is now represented by the north pole. Then, depending on α, we describe the time optimal synthesis on the sphere. This problem was already partially studied in [19], where the aim was to give a bound on the number of switchings of the optimal trajectories for a single-input control system on $SO(3)$. In particular, for $\alpha \le \pi/4$ the following structure of the extremal trajectories has been obtained. Calling *switching time* a time at which the optimal control switches from -1 to $+1$, or viceversa, every time optimal trajectory starting from the north pole is bang-bang (i.e. it is the concatenation of a finite number of bang arcs) and the difference among two consecutive switching times depends only on the first switching time s and it is equal to the following quantity $v(s)$

$$v(s) = \pi + 2\arctan\left(\frac{\sin(s)}{\cos(s) + \cot^2(\alpha)}\right). \tag{5}$$

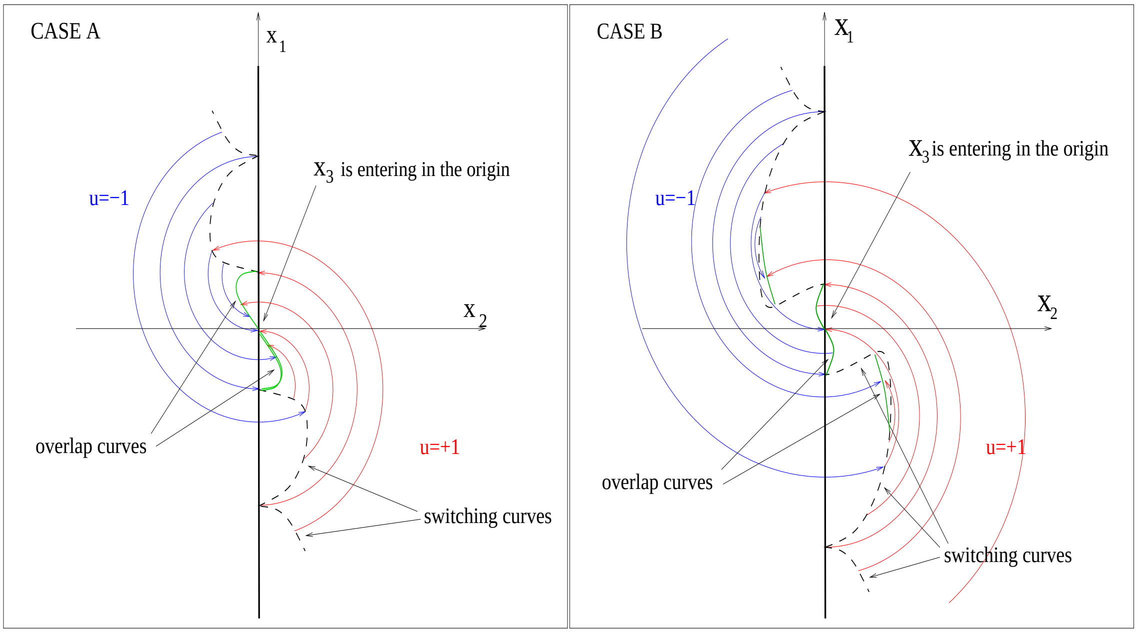

Figure 1: Alternating behaviour of the optimal synthesis in a neighborhood of the south pole for (4)

From this equation one can easily deduce the structure of the *switching curves*, i.e. the curves made by points where the optimal control changes sign. However in [19] the authors were not able to determine where the switching curves *lose optimality* and they conjectured that this can happen only in a neighborhood of the south pole. Moreover they conjectured, with the help of numerical simulations, that the shape of the optimal synthesis in a neighborhood of the south pole strongly depends on the *normalized remainder* $r(\alpha) := \frac{\pi}{2\alpha} - [\frac{\pi}{2\alpha}] \in [0, 1)$, as α goes to 0 and in particular there are two different alternating patterns of the optimal synthesis as shown in Figure 1. In particular the way in which the switching curves lose optimality is different in the two cases. One of the purposes of Chapter 2 is to prove rigorously the existence of this alternating behaviour.

The first results of the chapter completely characterize the time optimal synthesis in the case $\alpha > \pi/4$ and determine the optimal trajectories connecting the north pole to the south pole. In this case the main difficulty is the presence of time optimal trajectories containing singular arcs. However, the trajectories connecting the north pole to the south pole are rather simple to individuate, since they are bang-bang with only one switching. The above results, applied to the two-level quantum system, can be seen as a generalization of those obtained by Brokett, Khaneja and Glaser in [36] where the case of unbounded controls was considered. Indeed if we take M large enough then the optimal trajectories connecting the state one to any other physical state are made by three smooth pieces, where the first one and the third one are bang arcs, and the second piece is a singular arc. This is exactly the structure of the optimal trajectories found in [36]. Moreover the time needed to reach every physical state starting from the state one with unbounded controls can be seen as the limit as M goes to infinity of the time corresponding to the optimal trajectories we find.

The main purpose of the chapter is to provide the explicit expression of the time optimal trajectories steering the system from the north pole to the south pole for $\alpha \leq \pi/4$. This problem is equivalent to the problem of determining the optimal strategy for (3) in order to move from the state 1, corresponding to the energy level $-E$, to the state 2, corresponding to the energy level E. The optimal trajectories reaching the south pole are individuated by using the properties of the extremal trajectories found in [19] and the particular symmetries of the problem, and they are determined up to solving suitable trigonometric equations. Analyzing such equations, we prove that the number of optimal trajectories reaching the south pole changes in an alternating way as α goes to 0 (there are either two or four optimal trajectories reaching the south pole), and therefore, using the definition of local equivalence of optimal synthesis given in [22], it is possible to prove that, in a neighborhood of the south pole, the optimal synthesis presents at least two different patterns that alternate as α goes to 0.

Chapter 3 is based on the paper [46], which is a joint work with Ugo Boscain, Yacine Chitour and Rebecca Salmoni. We continue to study the optimal synthesis for the previous problem, projected on the sphere, focusing on a neighborhood of the south pole, as α tends to zero.

In Chapter 2 we understood that there is an alternating behaviour of the optimal synthesis in a neighborhood of the south pole, depending on $r(\alpha) = \frac{\pi}{2\alpha} - [\frac{\pi}{2\alpha}]$. Therefore it remains to study more precisely the qualitative shape of the optimal synthesis in order to prove that Figure 1 is a good representation of its possible patterns.

We start by considering the extremal front at time $T = [\frac{\pi}{2\alpha}]\pi$, which is the last multiple of π before reaching the south pole if $r(\alpha) \neq 0$, or, in the case $r(\alpha) = 0$, it is the time needed to reach the south pole. We first see that, if we assume $r(\alpha)$ bounded from below by some constant $\bar{r} \in (0, 1)$ and α small enough, such extremal front coincides with the minimum time front at time T. Moreover it is approximately a circle centered at the south pole of radius $2\,r(\alpha)\alpha$. This result gives an answer to one of the questions raised in [19] and recalled above, about the (local) optimality of the switching curves. Indeed we easily see that the switching curves must be locally optimal up to time $[\frac{\pi}{2\alpha}]\pi$ if α is small enough and $r(\alpha) \geq \bar{r}$ for some strictly positive constant $\bar{r}$. We also prove that the switching curves are always locally optimal up to time $([\frac{\pi}{2\alpha}] - 1)\pi$. If $r(\alpha)$ is small with respect to α, the situation is more complicated. However, even in this case, we are able to describe the approximate shape of the extremal front and of the minimum time front.

The aim is then to study the asymptotic behaviour, in a suitable sense, of the time optimal synthesis as α goes to 0 and in the neighborhood of the south pole enclosed by the minimum time front at T. It turns out that this behaviour is strictly connected to the quantity $r(\alpha)$. In particular it is interesting to consider sequences α_k with $\lim_{k\to\infty} \alpha_k = 0$ such that some prescribed relationship among α_k and $r(\alpha_k)$ is satisfied. In this context, we have individuated three possible asymptotic behaviour as α_k goes to 0, corresponding to the following three possibilities:

(1) α_k satisfies $r(\alpha_k) = \bar{r}$ (or $\lim_{k\to\infty} r(\alpha_k) = \bar{r}$) for some $\bar{r} \in (0, 1)$,

(2) α_k satisfies $r(\alpha_k) = K\alpha$ (or $\lim_{k\to\infty} r(\alpha_k)/k = K$) for some $0 < K < \frac{\pi}{4}$,

(3) $r(\alpha_k) = 0$ for every $k > 0$.

In the first two cases equation (4) can be approximated, in the neighborhood of the south pole enclosed by the minimum time front, by simpler control systems. In these cases one can study suitably simplified time optimal control problems on the plane, where the source, instead of being the north pole, is the approximated minimum time front. After having detected the time optimal synthesis for the simplified problem, the next step is to see that it is qualitatively equivalent to the time optimal synthesis of the originary problem in a neighborhood of the south pole and that actually the latter "converges" to the optimal synthesis for the approximating system. In Case **(3)** the neighborhood of the south pole that we consider is the one enclosed by the minimum time front at time $T = ([\frac{\pi}{2\alpha}] - 1)\pi$, which, approximately, is a circle of radius α around the south pole. Then the synthesis for the approximating system contains some singularities that does not allow to apply the methods developed for the first two cases. However the qualitative shape of the synthesis is obtained with the help of the results of Chapter 2 that individuate the time optimal trajectories reaching the south pole.

Chapter 4 is based on [28], which is a joint work with Mireille Broucke and Benedetto Piccoli, and it concerns a particular mechanical system: a planar pendulum on a cart that can move only along one direction. We let the coordinates of such system be the angle with the upright position and the angular velocity and we assume that the control input is the acceleration of the pendulum. Then we look at the time optimal trajectories in order to swing-up the pendulum starting from any possible configuration.

Global stabilization of this model has been studied as a benchmark for nonlinear control by many researchers, for instance, [9, 60, 38], to name a few. Time optimal synthesis has been studied recently in [11] and [65]. These papers are focused on computing exact switching times for an open loop control starting from the down equilibrium.

In contrast, we are interested in computing a globally defined feedback control. A related problem has been studied in [52], with similar techniques.

We first observe that our system assume the form (2) and the starting point for the construction of the synthesis is the equilibrium corresponding to the upright position. Therefore we can try to apply the theory of time optimal syntheses on two dimensional manifolds. However, it has to be observed that, unlike the previous examples, in this case the generic conditions given in [22] are not satisfied (essentially because the vector field that corresponds to g in (2) vanishes if the pendulum is "horizontal"). We recall that such conditions were introduced in order to avoid the so-called Fuller phenomenon and to prove the existence of an optimal synthesis and classification of its singularities in the case of $\mathcal{C}^\infty$ vector fields. Even if such conditions are not satisfied, in the case of the pendulum it is possible to apply the results of Sussmann (see [63]), that hold in the case of analytic vector fields, to see that a *regular* time optimal synthesis exists.

In the chapter we then give a complete qualitative description of the optimal synthesis, and moreover we describe the main frame curves (switching and overlap curves) and points as the solutions of suitable numerical equations involving elliptic integrals.

In Chapter 5, which is based on the paper [47] written in collaboration with U. Boscain and Y. Chitour, we consider a quite different problem, concerning switched systems.

In recent years, the problem of stability and stabilizability of switched systems has attracted increasing attentions since such problems are connected to numerous applications, for instance in control of mechanical systems, aircraft and air traffic control and many other fields (see for instance [39, 40]).

Our work addresses the problem of existence of common polynomial Lyapunov functions for linear switched systems.

By a switched system, we mean a family of continuous–time dynamical systems and a rule that determines at each time which dynamical system is responsible of the time evolution. More precisely, let $\{f_u : \ u \in U\}$ (where U is a subset of $\mathbb{R}^m$, $m \in \mathbb{N}$) be a finite or infinite set of sufficiently regular vector fields on a manifold M, and consider the family of dynamical systems:

$$\dot{x} = f_u(x), \quad x \in M. \tag{6}$$

The rule is given by assigning the so-called switching function, i.e. a function $u(.) : [0, \infty[\rightarrow U \subset \mathbb{R}^m$. Here, we consider the situation in which the switching function is not known a priori and represents some phenomenon (e.g. a disturbance) that is not possible to control.

These kind of systems are sometimes called "n-modal systems", "dynamical polysystems", "polysystems", "input systems". The term "switched system" is often reserved to situations in which the switching function $u(.)$ is piecewise continuous or the set U is finite. For our purposes, we only require $u(.)$ to be a measurable function and U to be a measurable set. Even if these systems are very general, the case of control-affine systems (1) is the most important both from a theoretical point of view and for applications.

A typical problem for switched systems goes as follows. Assume that, for every $u \in U$, the dynamical system $\dot{x} = f_u(x)$ satisfies a given property (P). Then one can investigate conditions under which property (P) still holds for $\dot{x} = f_{u(t)}(x)$, where $u(.)$ is an arbitrary switching function.

In [3, 14, 32], the case of linear switched systems was considered:

$$\dot{x}(t) = A_{u(t)} x(t), \quad x \in \mathbb{R}^n, \quad A_u \in \mathbb{R}^{n \times n}, \tag{7}$$

where n is a positive integer and $u(.) : [0, \infty[\rightarrow U$ is a (measurable) switching function.

Note that, in the case in which the set $\mathbf{A} = \{A_u \ : \ u \in U\}$ is made (or it is the convex hull) of a finite number of matrices, system (7) can be read as a control-affine system. This is indeed the easiest case to study and also the most important for applications.

For systems of the form (7), the problem of asymptotic stability of the origin, uniformly with respect to switching functions, was investigated. A complete solution to this problem has been given only in dimension two, in [14], where necessary and sufficient conditions for stability were found. In dimension larger than two this problems is very difficult and still unsolved. However, in some special cases, the stability of the origin can be proved by studying the properties of the Lie algebra generated by the set of matrices $\mathbf{A}$ (see [3]).

In Chapter 5 we focus on linear switched systems, with the sole assumption that $\mathbf{A}$ is compact. The aim is then to compare the notion of (global) uniform exponential stability (**GUES**) of the origin for a linear switched system with the existence of common polynomial Lyapunov functions.

To understand the importance of this problem, we simply observe that the easiest way of proving stability of a switched system is to look numerically for common Lyapunov functions,

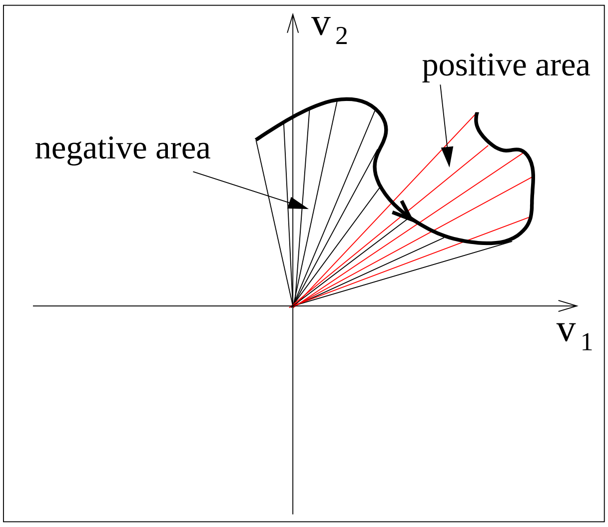

Figure 2: The area of the region enclosed by a curve

possibly inside a family of functions parameterized by a finite number of parameters, as could be a set of polynomials with uniformly bounded degree.

The most used Lyapunov functions in the applications are the quadratic functions. Unfortunately, in [32] Dayawansa and Martin showed an example of linear switched system that does not admit any polynomial quadratic Lyapunov function. They therefore posed the problem of computing a lowest degree for common polynomial Lyapunov functions, for linear switched systems. In Chapter 5 we succeed in solving this open problem.

The first result states that the origin is a **GUES** equilibrium if and only if there exists a polynomial Lyapunov function. We observe in particular that this result and the methods used to prove it, do not give any information on the degree of such polynomial.

Our main result indeed shows that, even for the simplest case of linear switched systems, that is the case of bidimensional systems with single input

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \quad A, B \in \mathbb{R}^{2\times 2}$$

there is not a uniform lower bound on the degree of the common polynomial Lyapunov function. This means that, for any $n \in \mathbb{N}$, there exists a pair of matrices (A, B) such that the origin is a **GUES** equilibrium for the corresponding switched system and there are no polynomial Lyapunov functions of degree $m \leq n$. To prove this fact we use the results of [14], where a complete characterization of the pairs (A, B) ensuring asymptotic stability was given. Then, the proof is essentially based on the fact that the set of “stable” pairs (A, B) is not closed in the usual topology. Our result gives therefore an answer to the problem posed in [32].

In Chapter 6 (see [45]) we discuss the possibility of generalizing the notion of solution of a control-affine system where the control is supposed to belong to a functional space such that the classical results on the local existence and uniqueness do not apply. This problem has been studied extensively by control theorists and also by probabilists for its connections with the field of stochastic processes. The control systems under consideration have the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} \dot{v}_i f_i(x)\,, \tag{8}$$

where $x \in \mathbb{R}^n$ and f_i are smooth vector fields. We observe that, if we define new controls as the derivative of the originary ones, we are again reduced to study control-affine systems.

Equation (8) is important for applications both for mechanical systems (see [26, 27]) and stochastic processes (see [43]).

If we assume that the function v is absolutely continuous then a unique solution exists in Carathéodory sense. This is no more true if we weaken the regularity of v. The case in which the control is still continuous even if it is not absolutely continuous is particularly important, for instance if we look at the control input v as a path of a stochastic process, as it is done for instance in [62]. Indeed a single path of a stochastic process (for instance the Brownian Motion) is often a (Hölder) continuous function, with probability 1.

While in the case $m = 1$ the problem of defining the solution of (8) if v is only continuous, has been solved by Sussmann ([62]), in the case in which $m > 1$ this problem is far from being completely solved (only if the vector fields $f_1, \ldots, f_m$ commute the second case can be easily solved by reducing to the first one). A partial answer has been given by Lyons in [42], where, by means of the Picard iteration method, the author was able to prove the existence and uniqueness of the solution of (8) for a wide class of controls. For instance this result applies to the case in which v is Hölder-α with $\alpha > \frac{1}{2}$.

A further generalization of this result has been still given by Lyons in [43] by lifting the space of controls to an abstract functional space and making the same for the space of trajectories. Then the generalized solution is obtained by means of the Picard iteration method applied to the lifted equation.

In Chapter 6 we discuss these approaches and we propose some alternative ways of generalizing the notion of solution of (8). In particular we focus on the Heisenberg system, the simplest driftless non holonomic system, and we investigate some conjectures by means of examples and counterexamples. For the Heisenberg example our problem has a simple geometric interpretation. Indeed in this case the control v lies in the plane and it is easy to see that every generalized notion of solution is equivalent to a generalized notion of the *area* of the region spanned by $v(t) = (v_1(t), v_2(t))$ on the plane, computed in counterclockwise sense (see Figure 2):

$$\mathcal{A}[v(\cdot)](t) := \int_0^t \frac{1}{2}(v_1(s)\dot{v}_2(s) - \dot{v}_1(s)v_2(s))\; ds\,.$$

We focus in particular on curves belonging to the class of Hölder functions $\mathcal{C}^{0,\alpha}([0,T])$. If $\alpha > 1/2$ a unique generalized notion of area can be given for instance by using the result of [42] described above. In particular the area can be defined as the limit of the areas corresponding to a sequence v_k of smooth functions converging to v in the uniform topology. A simple example shows that an analogous result does not hold in the case $\alpha < 1/2$. Therefore we focus on the case $\alpha = 1/2$, which represents a very particular case, since for instance "almost every" path of

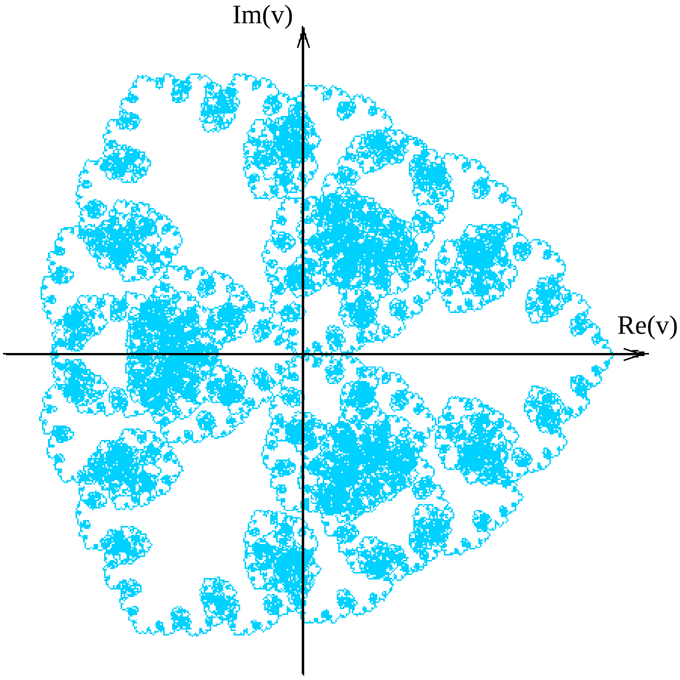

Figure 3: The graph of v in the complex plane

the Brownian motion is Hölder-α, for each fixed $\alpha < \frac{1}{2}$. We consider the function

$$v : [0, 2\pi] \to \mathbb{C} \qquad v(t) := \sum_{k=0}^{+\infty} \frac{1}{2^k} e^{4^k i\, t}.$$

We prove that this function is Hölder-$\frac{1}{2}$ and we see that

- the smooth functions $v^{(k)}(t) = \sum_{k=0}^{N} \frac{1}{2^k} e^{4^k i\, t}$ have uniformly bounded Hölder-$\frac{1}{2}$ constant and converge in the uniform topology to v. The area corresponding to v_k converges to infinity;
- one can construct in a natural way a sequence of piecewise affine functions converging uniformly to v and such that the corresponding area tends to infinity.

It is also possible to find a sequence of smooth functions $w^{(k)}(\cdot)$ with $\mathcal{A}[w^{(k)}(\cdot)](2\pi) = 0$, that converge uniformly to v, so that it does not seem reasonable to associate the value "infinity" to a generalized notion of area for v.
Finally, introducing a suitable modification $\tilde{v}(\cdot)$ of the function $v(\cdot)$, it is possible to construct a sequence of approximating functions converging to $\tilde{v}(\cdot)$ in the $\mathcal{C}^{0,\frac{1}{2}}$ topology (which is stronger than the uniform topology) and such that the corresponding area tends to infinity.

Our conclusion is that the assumption that the control v is Hölder-$\frac{1}{2}$ continuous is not enough to ensure a meaningful definition of generalized solution. Our feeling is that we can expect a meaningful definition of generalized solution only if we assume some additional condition that express the independence of the controls $v_i \quad i = 1, \ldots, m$. Then, the future research on this field will look towards an explicitation of these conditions with the objective of applying them to the case of stochastic processes and, possibly, in the general case in which v is continuous.

Chapter 1

Optimal Control and Time Optimal Synthesis on 2-D Manifolds

In this chapter we want to discuss the basic tools of optimal control theory and time optimal syntesis on two dimensional manifolds that are used throughout this thesis.

1.1 Pontryagin Maximum Principle

In this section we consider single-input control systems of the type

$$\dot{x} = F(x) + u(t)G(x)\,, \tag{1.1}$$

where $x \in \mathcal{M}$ and $u(\cdot) \in [-1, 1]$ is the measurable control. Here $\mathcal{M}$ stands for a smooth manifold and the vector fields f and g are assumed to be smooth. This condition guarantees local existence and uniqueness of the solutions of (1.1), and, in the case in which $\mathcal{M}$ is compact, also global existence. In the general case we assume that the control system is *complete* i.e. for every measurable control function $u(\cdot) : [a, b] \to [-1, 1]$ and every initial state $\bar{x}$, there exists a trajectory $x(\cdot)$ corresponding to $u(\cdot)$, which is defined on the whole interval $[a, b]$ and satisfies $x(a) = \bar{x}$.

To introduce our optimization problem, consider a pair of points (p, q), and assume that there exists a trajectory $\gamma(\cdot) : [0, T_\gamma] \to \mathcal{M}$ of (1.1) with $\gamma(0) = p$, $\gamma(T_\gamma) = q$ and such that T_γ is the minimum time to steer the system from p to q. We call this trajectory a *time-optimal trajectory.*

Clearly the problem of detecting the time-optimal trajectories (also called *time-optimal problem*) is meaningful if and only if the point q belongs to the "attainable set" for the control system (1.1) with initial datum p, i.e. q can be reached in finite time starting from p. If this is the case, then the existence of a time-optimal trajectory is a straightforward consequence of the classical Filippov theorem.

To determine the time-optimal trajectories the key tool is the well-known Pontryagin Maximum Principle (more precisely the particular instance of it that adapts to the time-optimal problem), which is a first order optimality condition.

If we define

$$H(x, \lambda, u) = \lambda \cdot F(x) + u\lambda \cdot G(x)$$

where λ belongs to the cotangent space $\lambda \in T^*\mathcal{M}$, the Pontryagin Maximum Principle for the time-optimal problem associated to a control-affine system of the form (1.1), states the following (see [4, 22, 34]).

Theorem 1.1 (PMP for the time-optimal problem for (1.1)) *For each time-optimal trajectory $x^*(\cdot)$ of (1.1), defined on $[0, T]$ and corresponding to the control $u^*(\cdot)$, there exists a covector $\lambda^*(\cdot)$ and a real number $\lambda_0 \leq 0$ such that the following conditions are satisfied for every $t \in [0, T]$:*

(i) $\dot{\lambda}^*(t) = -\lambda^*(t) \cdot (\nabla F(x(t)) + u\nabla G(x(t)))$,

(ii) $H(x^*(t), \lambda^*(t), u^*(t)) + \lambda_0 = 0$,

(iii) $H(x^*(t), \lambda^*(t), u^*(t)) = \max_{u \in [-1,1]} H(x^*(t), \lambda^*(t), u)$.

In the more general case in which the target and the initial datum are two smooth manifolds $\mathcal{N}_0$ and $\mathcal{N}_1$ the previous statement must be modified by adding the so-called transversality conditions*:*

(iv) $\lambda^*(0) \cdot v = 0 \quad \forall v \in T_{x^*(0)}\mathcal{N}_0\,, \quad \lambda^*(T) \cdot w = 0 \quad \forall w \in T_{x^*(T)}\mathcal{N}_1$.

Remark **1.1** The PMP is just a necessary condition for optimality. A trajectory $x(\cdot)$ (resp. a couple $(x(\cdot), \lambda(\cdot)))$ satisfying the conditions given by the PMP is said to be an *extremal* (resp. an *extremal pair*). An extremal corresponding to $\lambda_0 = 0$ is said to be an *abnormal extremal*, otherwise we call it a *normal extremal*.

We are now interested in determining the extremal trajectories satisfying the conditions given by the PMP. A key role is played by the following:

Definition 1.1 (switching function) *Let $(x(\cdot), \lambda(\cdot))$ be an extremal pair. The corresponding switching function is defined as $\phi(t) := < \lambda(t), G(x(t)) >$.*

Notice that $\phi(\cdot)$ is continuously differentiable (indeed $\dot{\phi}(t) = < \lambda(t), [F, G](x(t)) >$, where $[F, G] = \nabla G \cdot F - \nabla F \cdot G$ is the Lie bracket, is continuous).

Definition 1.2 (bang, singular) *Let $x(\cdot)$, defined in $[a, b]$, be an extremal trajectory and $u(\cdot) : [a, b] \to [-1, 1]$ the corresponding control. We say that $u(\cdot)$ is a* bang *control if $u(t) = +1$ a.e. in $[a, b]$ or $u(t) = -1$ a.e. in $[a, b]$. We say that $u(\cdot)$ is* singular *if the corresponding switching function satisfies $\phi(t) \equiv 0$ in $[a, b]$. A finite concatenation of bang controls is called a* bang-bang *control. A* switching time *of $u(\cdot)$ is a time $\bar{t} \in [a, b]$ such that, for every $\varepsilon > 0$, u is not bang or singular on $(\bar{t} - \varepsilon, \bar{t} + \varepsilon) \cap [a, b]$. An extremal trajectory of the control system (1.1) is said a bang extremal, singular extremal, bang-bang extremal respectively, if it corresponds to a bang control, singular control, bang-bang control respectively. If $\bar{t}$ is a switching time, the corresponding point on the trajectory $x(\bar{t})$ is called a* switching point.

The switching function is important because it determines where the controls may switch. In fact, using the PMP, one easily gets:

Proposition 1.1 *A necessary condition for a time t to be a switching is that $\phi(t) = 0$. Therefore, on any interval where ϕ has no zeroes (respectively finitely many zeroes), the corresponding control is bang (respectively bang-bang). In particular, $\phi > 0$ (resp $\phi < 0$) on $[a, b]$ implies $u = 1$ (resp. $u = -1$) a.e. on $[a, b]$. On the other hand, if ϕ has a zero at t and $\dot{\phi}(t)$ is different from zero, then t is an isolated switching.*

1.2 General results about time-optimal trajectories on 2-D manifolds

Let us assume $\mathcal{M}$ to be a two dimensional manifold. For every coordinate chart on the manifold it is possible to introduce the following three functions:

$$\begin{aligned}
\Delta_A(x) &:= Det(F(x), G(x)) = F_1(x)G_2(x) - F_2(x)G_1(x), &(1.2)\\
\Delta_B(x) &:= Det(G(x), [F,G](x)) = G_1(x)[F,G]_2(x) - G_2(x)[F,G]_1(x), &(1.3)\\
f_S(x) &:= -\Delta_B(x)/\Delta_A(x). &(1.4)
\end{aligned}$$

<u>*Remark*</u> **1.2** Notice that, although the functions Δ_A and Δ_B depend on the coordinate chart, the sets $\Delta_A^{-1}(0)$, $\Delta_B^{-1}(0)$ and the function f_S do not, i.e. they are intrinsic objects of the control equation (1.1).

The sets $\Delta_A^{-1}(0), \Delta_B^{-1}(0)$ of zeroes of Δ_A, Δ_B are respectively the set of points where F and G are parallel, and the set of points where G is parallel to $[F, G]$. These loci are fundamental in the construction of the optimal synthesis. Assume indeed that they are smooth embedded one dimensional submanifold of $\mathcal{M}$ and call $\mathcal{M} \setminus (\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0))$ the set of *ordinary points*.
If $x \notin \Delta_A^{-1}(0)$, then it is easy to see that there exists $g(\cdot)$ such that the Lie bracket $[F, G]$ can be decomposed in the following way

$$[F, G](x) = f_S(x)F(x) + g(x)G(x)\,.$$

Assume now that $\bar{t}$ is a switching time for an extremal trajectory $x(\cdot)$, i.e. $\lambda(\bar{t}) \cdot G(x(\bar{t})) = 0$. Then

$$\begin{aligned}
\dot{\phi}(\bar{t}) &= \lambda(\bar{t}) \cdot [F, G](x(\bar{t}))\\
&= \lambda(\bar{t}) \cdot (f_S F + gG)(x(\bar{t}))\\
&= f_S(x(\bar{t}))\left(\lambda(\bar{t}) \cdot F(x(\bar{t}))\right)\,.
\end{aligned}$$

Since, from the PMP, we have that $H(x(\bar{t}), \lambda(\bar{t})) = \lambda(\bar{t}) \cdot F(x(\bar{t})) \geq 0$, we obtain $\dot{\phi} \geq 0$. Note that $\lambda(\bar{t}) \cdot F(x(\bar{t})) \neq 0$ if $x(\bar{t})$ is an ordinary point (otherwise $F(x(\bar{t}))$ and $G(x(\bar{t}))$ would be parallel) and therefore sgn($f_S(x(\bar{t}))$)=sgn($\dot{\phi}(\bar{t})$).

By combining with Proposition 1.1, we have therefore proved the following result.

Lemma 1.1 *Let $\Omega \subset \mathcal{M}$ be an open set such that $\Omega \cap (\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)) = \emptyset$. Then all connected components of $Supp(x(\cdot)) \cap \Omega$, where $x(\cdot)$ is an extremal trajectory of (1.1), are bang-bang with at most one switching. Moreover, if $f_S > 0$ throughout Ω, then $x(\cdot)|_\Omega$ is associated to a constant control equal to $+1$ or -1 or has a switching from -1 to $+1$. If $f_S < 0$ throughout Ω, then $x(\cdot)|_\Omega$ is associated to a constant control equal to $+1$ or -1 or has a switching from $+1$ to -1.*

Since $\dot{\phi}(\bar{t}) = \lambda(\bar{t}) \cdot [F, G](x(\bar{t}))$, we obtain immediately that, for a singular extremal, $\Delta_B(x(t)) \equiv 0$. Therefore, if we assume $x(\cdot)$ to be singular, then it must be

$$0 = \frac{d}{dt}\Delta_B(x(t)) = \nabla\Delta_B \cdot (F(x(t)) + u(t)G(x(t))).$$

so that we obtain the following result.

Lemma 1.2 *Let $x(\cdot)$ be an extremal trajectory that is singular in $[a, b] \subset Dom(x(\cdot))$. Then $Supp(x(\cdot)|_{[a,b]}) \subset \Delta_B^{-1}(0)$ and $x(\cdot)|_{[a,b]}$ corresponds to the so called* singular control $\varphi(x(t))$, *where*

$$\varphi(x) = -\frac{\nabla\Delta_B(x) \cdot F(x)}{\nabla\Delta_B(x) \cdot G(x)}. \tag{1.5}$$

In order to investigate more deeply the properties of the singular extremals it is useful to introduce the following notions

Definition 1.3 • *A* non ordinary arc *is a $\mathcal{C}^2$ one-dimensional connected embedded submanifold S of $\mathcal{M}$ with the property that every $x \in S$ is a non ordinary point.*

• *A non ordinary arc is said* isolated *if there exists a set Ω satisfying the following conditions:*

(C1) *Ω is an open connected subset of $\mathcal{M}$.*

(C2) *S is a relatively closed subset of Ω.*

(C3) *If $x \in \Omega \setminus S$ then x is an ordinary point.*

(C4) *The set $\Omega \setminus S$ has exactly two connected components.*

• *A* turnpike *(resp.* anti-turnpike*) is an isolated non ordinary arc that satisfies the following conditions:*

(S1) *For every $x \in S$ the vectors $F(x) + G(x)$ and $F(x) - G(x)$ are not tangent to S and point to opposite sides of S.*

(S2) *For every $x \in S$ one has $\Delta_B(x) = 0$ and $\Delta_A(x) \neq 0$.*

(S3) *Let Ω be an open set which satisfies* **(C1)–(C4)** *above and $\Delta_A \neq 0$ on Ω. If Ω_+ and Ω_- are the connected components of $\Omega \setminus S$ labeled in such a way $F(x) + G(x)$ points into Ω_+ and $F(x) - G(x)$ points into Ω_-, then the function f_S satisfies*

$$f_S(x) > 0 \text{ (resp. } f_S(x) < 0) \text{ on } \Omega_+$$
$$f_S(x) < 0 \text{ (resp. } f_S(x) > 0) \text{ on } \Omega_-$$

The following two lemmas clarify the relation among turnpikes, antiturnpikes and singular extremals. For a proof of these results, see [22].

Lemma 1.3 *Let* $(x, \lambda) : [0, \bar{t}] \to \mathcal{M}$ *be an extremal pair that verifies* $x(\bar{t}) = \bar{x}$*,* $\bar{x} \in S$ *where* S *is a turnpike or an anti-turnpike, and* $\lambda(\bar{t}) \cdot G(x(\bar{t})) = 0$*. Moreover let* $x' : [0, t'] \to \mathbb{R}^2$ $(t' > \bar{t})$ *be a trajectory such that:*

- $x'|_{[0,\bar{t}]} = \bar{x}$*,*
- $x'([\bar{t}, t']) \subset S$*.*

Then x' *is extremal. Moreover if* ϕ' *is the switching function corresponding to* x' *then* $\phi'|_{[\bar{t},t']} \equiv 0$*.*

Lemma 1.4 *Let* S *be an anti-turnpike and* $x : [c, d] \to \mathcal{M}$ *be an extremal trajectory such that* $x([c, d]) \subset S$*. Then* $x(\cdot)$ *is not optimal.*

Finally, the following lemma shows that the function $\Delta_A^{-1}(0)$ defined above is also useful in order to detect the abnormal extremals

Lemma 1.5 *Let* $x(\cdot)$ *be a bang-bang extremal for the control problem (1.1),* $t_0 \in Dom(x(\cdot))$ *be a time such that* $\phi(t_0) = 0$ *and* $G(x(t_0)) \neq 0$*. Then, the following conditions are equivalent:* **i)** $x(\cdot)$ *is an abnormal extremal;* **ii)** $x(t_0) \in \Delta_A^{-1}(0)$*;* **iii)** $x(t) \in \Delta_A^{-1}(0)$*, for every time* $t \in Dom(x(\cdot))$ *such that* $\phi(t) = 0$*.*

1.3 Exixtence of an optimal synthesis and classification of synthesis singularities

Let $x_0 \in \mathcal{M}$ be a point satisfying $F(x_0) = 0$. Consider the following problem.

(P) For every $x \in \mathcal{M}$ find the time optimal trajectory $\gamma : [0, T] \to \mathcal{M}$ of (1.1) such that $\gamma(0) = x_0$ and $\gamma(T) = x$.

For us, a solution to this problem is the *time optimal synthesis* of the problem. In other words:

Definition 1.4 *The time optimal synthesis for* **(P)** *is the collection of all the trajectories of (1.1) satisfying Problem* **(P)***.*

We start by giving a simple but very useful result that determines the optimal synthesis in a neighborhood of the point x_0 (see [22]).

Proposition 1.2 *Given a system of the form (1.1), with* $\Delta_B(x_0) \neq 0$*, there exists a neighborhood* U *of* x_0 *such that for every* $x \in U$ *the only time optimal trajectory connecting* x_0 *with* x *is bang-bang with at most one switching and it is contained inside* U *.*

Thanks to this result it is then possible to define a local piecewise smooth *feedback law* describing the optimal synthesis in U. In particular the union $\gamma^+ \cup \gamma^-$, where $\gamma^\pm$ are the trajectories starting from x_0 and corresponding to $u = \pm 1$, partitions U in two connected components U^+ and U^-

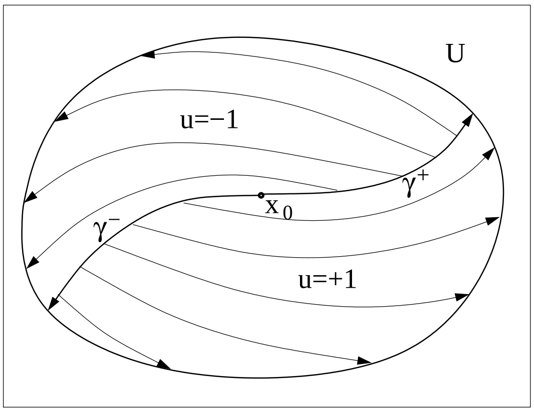

Figure 1.1: Optimal synthesis in a neighborhood of the point x_0

where the control of the corresponding time-optimal trajectories is equal respectively to 1 and -1. See Figure 1.1.

In order to prove the existence of an optimal synthesis we need to introduce the following conditions on the vector fields F and G, which are essential to prove that every optimal trajectory is a finite concatenation of bang and singular arcs (see [22]).

(P1) The vectors $G(x_0)$ and $[F,G](x_0)$ are linearly independent, i.e. $\Delta_B(x_0) \neq 0$.

(P2) Zero is a regular value for Δ_A and Δ_B i.e. $\Delta_A(x) = 0$ implies $\nabla\Delta_A(x) \neq 0$ and similarly for Δ_B.

Define $X(x) = F(x) - G(x)$ and $Y(x) = F(x) + G(x)$ Let Tan_A be the set of points $x \in \Delta_A^{-1}(0)$ such that $X(x)$ or $Y(x)$ is tangent to $\Delta_A^{-1}(0)$. Define Tan_B in the same way using Δ_B rather than Δ_A.

(P3) The set $Bad := (\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)) \cup Tan_A \cup Tan_B$ is locally finite.

(P4) If $x \in Bad,\ \ G(x) = 0$ then $F(x) \cdot \nabla(\Delta_A)(x) \neq 0$.

(P5) If $x \in Bad,\ G(x) \neq 0$ and $x \in (\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)) \cap Tan_A$, then $x \notin Tan_B$, $\partial_y(X \cdot \nabla\Delta_A)|_{y=x} \neq 0$, $X(x) \neq 0$, $Y(x) \neq 0$.

With these assumptions we have the following important result.

Theorem 1.2 *Let $n(x(\cdot))$ denotes the number of bang and singular arcs of $x(\cdot)$. Under generic conditions on $F, G \in \mathcal{C}^3$, for every $\bar{x}$ there exist $\Omega_{\bar{x}}$, neighborhood of $\bar{x}$, and $N_{\bar{x}} \in \mathbb{N}$ such that if γ is optimal and $Supp(\gamma) \subset \Omega_{\bar{x}}$ then:*

$$n(x(\cdot)) \leq N_{\bar{x}}\,.$$

From the above theorem one deduces easily the following fundamental corollary.

Corollary 1.1 *For every $\tau > 0$ and under the above generic conditions on F and G there is an a priori bound on the number of arcs of time optimal trajectories starting from x_0 and defined on $[0, \tau]$.*

We introduce now a more accurate definition of optimal synthesis, in order to describe in a more precise way the qualitative properties of the optimal trajectories. Let $\mathcal{A}_\tau$ be the attainable set for (1.1) at time τ, i.e.

$$\mathcal{A}_\tau := \{\bar{x} \in \mathcal{M} \ : \ \exists x(\cdot) \text{ trajectory of (1.1) with } \ x(0) = x_0\,, \ x(t) = \bar{x} \text{ and } t \leq \tau\,\}.$$

Definition 1.5 (stratification) *A stratification of $\mathcal{A}_\tau$, $\tau > 0$, is a finite collection $\{M_i\}$ of connected embedded $\mathcal{C}^1$ submanifolds of $\mathcal{M}$, called strata, such that the following holds. If $M_j \cap Clos(M_k) \neq \emptyset$ with $j \neq k$ then $M_j \subset Clos(M_k)$ and $dim(M_j) < dim(M_k)$.*

Definition 1.6 (regular optimal synthesis) *A regular optimal synthesis for (1.1) on $\mathcal{A}_\tau$ is a collection of trajectory-control pairs $\{(\gamma_x, u_x) : x \in \mathcal{A}_\tau\}$ satisfying the following properties:*

1. *For every $x \in \mathcal{A}_\tau$, $\gamma_x : [0, t_x] \to \mathcal{M}$ steers the origin to x in minimum time.*
2. *If $y = \gamma_x(t)$ for some $t \in Dom(\gamma_x)$ then γ_y is the restriction to $[0, t]$ of γ_x.*
3. *There exists a stratification of $\mathcal{A}_\tau$ such that $u(x) = u_x(t_x)$ is smooth on each stratum (assuming each u_x left continuous).*

In particular, it is clear from Corollary 1.1 that, under the above generic conditions, the time-optimal synthesis can be built recursively on the number of bang and singular arcs and canceling at each step the non-optimal trajectories.

We want now to give a brief description of the special curves and points that characterizes the optimal synthesis. Let us call u^* the feedback control associated to the optimal synthesis.

Definition 1.7 (Frame Curves and Points) *A one-dimensional connected embedded $\mathcal{C}^2$ submanifold with boundary D of $\mathcal{A}_\tau$ is called a Frame Curve if u^* is discontinuous at each point of D and D is maximal. Frame Points are defined as intersection of Frame Curves.*

Under generic conditions, the algorithm introduced in [22] and briefly described above, constructs only six types of frame curves:

- *the trajectory* γ_{op}^- starting from zero and corresponding to constant control -1. We say that this curve is of kind X;
- *the trajectory* γ_{op}^+ starting from zero and corresponding to constant control $+1$. We say that this curve is of kind Y;
- *singular trajectories* that are trajectories corresponding to the singular control φ. We say that these are FCs of kind S;

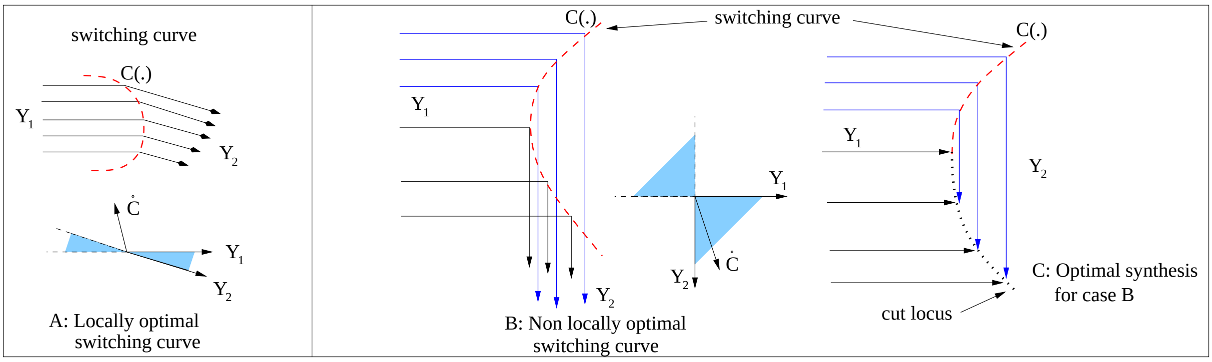

Figure 1.2: Locally optimal switching curves and non locally optimal switching curves with the corresponding synthesis

- *switching curves*, which are also called C frame curves and which are the trajectories formed by switching points. Their peculiarity is that the optimal control changes sign crossing them;
- *overlap curves*, formed by points reached optimally by two distinct trajectories. We call these curves K frame curves;
- the *topological frontier* of the reachable set. We call this curve a F frame curve.

Moreover, in [22] it was also proved that there are exactly 23 qualitatively different frame points that can be constructed by the algorithm of the optimal synthesis, under generic conditions.

Finally, in the next chapters we will make use of the following notions.

Roughly speaking we say that a switching curve is *locally optimal* if it never "reflects" the trajectories (see Figure 1.2 A).

More precisely:

Definition 1.8 *Consider a smooth switching curve C between two smooth vector field Y_1 and Y_2 on a smooth two dimensional manifold. Let $C(s)$ be a smooth parametrization of C. We say that C is* locally optimal *if, for every $s \in Dom(C)$, we have $\dot{C}(s) \neq \alpha_1 Y_1(C(s)) + \alpha_2 Y_2(C(s))$, for every α_1, α_2 s.t. $\alpha_1\alpha_2 \geq 0$. The points of a switching curve on which this relation is not satisfied are called* conjugate points.

The terminology "conjugate points" and "cut locus" comes from Riemannian Geometry. When a family of trajectories is reflected by a switching curve then local optimality is lost and some *cut locus* appear in the optimal synthesis.

Definition 1.9 *A cut locus of the time optimal synthesis is a set of points reached at the same time by two (or more) optimal trajectories. A subset of a cut locus that is a connected $\mathcal{C}^1$ manifold is called* overlap curve.

In particular by the previous results, under generic conditions, a cut locus is a finite union of overlap curves. An example showing how a "reflection" on a switching curves generate a cut locus is portrayed in Figure 1.2 B and C.

Chapter 2

Time Optimal Synthesis for 2-Level Quantum Systems

The dynamics of a n-level quantum system is governed by the time dependent Schrödinger equation (in a system of units such that $\hbar = 1$),

$$i\dot{x}(t) = (H_0 + \sum_{j=1}^{m} \Omega_j(t) H_j) x(t) \tag{2.1}$$

where $x(\cdot)$, defined on $[0, T]$, is a function taking values on the *state space* which is $SU(n)$ (if we formulate the problem for time evolution operator) or the sphere S^{2n-1} (if we formulate the problem for the wave function). The quantity H_0 called the *drift Hamiltonian* is an Hermitian matrix, that is natural to assume diagonalized, i.e., $H_0 = diag(E_1, ..., E_n)$, where $E_1, ..., E_n$ are real numbers representing the *energy levels*. With no loss of generality we can assume $\sum_{j=1}^{n} E_j = 0$. The real valued *controls* $\Omega_1(\cdot), ..., \Omega_m(\cdot)$, represent the *external pulsed field*, while the matrices H_j $(j = 1, ..., m)$ are Hermitian matrices describing the coupling between the external fields and the system. The time dependent Hamiltonian $H(t) := H_0 + \sum_{j=1}^{m} \Omega_j(t) H_j$ is called the *controlled Hamiltonian*.

The first problem that usually one would like to solve is the *controllability problem*, i.e. proving that for every couple of points in the state space one can find controls steering the system from one point to the other. For applications, the most interesting initial and final states are of course the *eigenstates of* H_0.

If $x \in SU(n)$, thanks to the fact that the control system (2.1) is a left invariant control system on the compact Lie group $SU(n)$, this happens if and only if

$$\text{Lie}\{iH_0, iH_1, ...iH_m\} = su(n), \tag{2.2}$$

(see for instance [58]). If the problem is formulated for the wave function, i.e. $x \in S^{2n-1}$, one can have controllability, with less restrictive conditions on the Lie algebra $\text{Lie}\{iH_0, iH_1, ...iH_m\}$, see [6]. The problem of finding easily verifiable conditions under which (2.2) is satisfied has been deeply studied in the literature (see for instance [8, 58]). Here we just recall that the condition (2.2) is generic in the space of Hermitian matrices.

2.1 Optimal control problems for finite dimensional quantum systems

Once that controllability is proved one would like to steer the system, between two fixed points in the state space, in the most efficient way. Typical costs that are interesting to minimize for applications are:

- **Energy transfered by the controls to the system.** $\displaystyle\int_0^T \sum_{j=1}^m \Omega_j^2(t)\ dt$,
- **Time of transfer.** In this case one can attack two different problems one with *bounded* and one with *unbounded* controls.

The problem of minimizing time with unbounded controls has been deeply investigated in [1, 36]. The problems of minimizing time or energy with bounded controls are very difficult in general and one can hope to find a complete solution in low dimension only.

In [18, 16, 17, 15] a special class of systems, for which the analysis can be pushed much further, was studied, namely systems such that the drift term H_0 disappear in the interaction picture (by a unitary change of coordinates and a change of controls). For these systems the controlled Hamiltonian reads

$$H(t) \quad = \quad \begin{pmatrix} E_1 & \mu_1\mathbf{\Omega}_1(t) & 0 & \cdots & 0 \\ \mu_1\mathbf{\Omega}_1^*(t) & E_2 & \mu_2\mathbf{\Omega}_2(t) & \ddots & \vdots \\ 0 & \mu_2\mathbf{\Omega}_2^*(t) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & E_{n-1} & \mu_{n-1}\mathbf{\Omega}_{n-1}(t) \\ 0 & \cdots & 0 & \mu_{n-1}\mathbf{\Omega}_{n-1}^*(t) & E_n \end{pmatrix} \tag{2.3}$$

Here $(^*)$ denotes the complex conjugation involution. The controls $\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_{n-1}$ are complex (they play the role of the real controls $\Omega_1, \ldots, \Omega_m$ in (2.1) with $m = 2(n-1)$) and $\mu_j > 0$, $(j = 1, \ldots, n-1)$ are real constants describing the couplings (intrinsic to the quantum system) that we have restricted to couple only levels j and $j+1$ by pairs.

For $n = 2$ the dynamics (2.3) describes the evolution of the z component of the spin of a (spin 1/2) particle driven by a magnetic field, that is constant along the z-axis and controlled both along the x and y axes, while for $n \geq 2$ it represents the first n levels of the spectrum of a molecule in the *rotating wave approximation* (see for instance [7]), and assuming that each external fields couples only close levels. The complete solution to the optimal control problem between eigenstates of $H_0 = diag(E_1, \ldots, E_n)$, has been constructed for $n = 2$ and $n = 3$, for the minimum time problem with bounded controls (i.e., $|\mathbf{\Omega}_j| \leq M_j$) and for the minimum energy problem $\int_0^T \sum_{j=1}^{n-1} |\mathbf{\Omega}_j(t)|^2\ dt$ (with fixed final time).

<u>*Remark*</u> **2.1** For the simplest case $n = 2$ (studied in [18, 31]), the minimum time problem with bounded control and the minimum energy problem actually coincide. In this case the controlled Hamiltonian is

$$H(t) = \begin{pmatrix} -E & \mathbf{\Omega}(t) \\ \mathbf{\Omega}^*(t) & E \end{pmatrix}, \quad |\mathbf{\Omega}| \leq M, \tag{2.4}$$

and the optimal trajectories, steering the system from the first to the second eigenstate of $H_0 = diag(-E, E)$, correspond to controls in *resonance* with the energy gap $2E$, and with maximal amplitude i.e. $\mathbf{\Omega}(t) = Me^{i[(2E)t+\phi]}$, where $\phi \in [0, 2\pi)$ is an arbitrary phase. The quantity $\omega_R = 2E$ is called the *resonance frequency.* In this case, the time of transfer $T_{\mathbb{C}}$ is proportional to the inverse of the laser amplitude. More precisely (see for instance [18]), $T_{\mathbb{C}} = \pi/(2M)$.

For $n = 3$ the problem has been studied in [16, 15] and it is much more complicated (in particular when the coupling constants μ_1 and μ_2 are different). In the case of minimum time with bounded controls, it requires some nontrivial technical tools of 2-D syntheses theory for distributional systems, that have been developed in [15].

For $n \geq 4$ the problem is very hard and still unsolved, but in [17] it has been proved that the optimal controls steering the system from any couple of eigenstates of H_0 are in *resonance*, i.e. they oscillate with a frequency equal to the difference of energy between the levels that the control is coupling. More precisely

$$\mathbf{\Omega}_j = A_j(t)e^{i[(E_{j+1}-E_j)t+\phi_j]}, \qquad j = 1, ..., n-1 \tag{2.5}$$

where $A_j(\cdot)$ are real functions describing the amplitude of the external fields and ϕ_j are arbitrary phases. Actually, this result holds for more general systems, initial and final conditions, and costs (see [17]).

The problem of minimizing time with bounded controls or energy is even more difficult if it is not possible to eliminate the drift H_0. This happens, for instance, for a system in the form (2.3) with real controls $\mathbf{\Omega}_j(t) = \mathbf{\Omega}_j^*(t)$, $j = 1, ..., n-1$, as we are going to discuss now. (For more details on the elimination of the drift see [18, 16, 17].)

2.2 A spin 1/2 particle in a magnetic field

In this chapter we attack the simplest quantum mechanical model interesting for applications for which it is not possible to eliminate the drift, namely a *two-level quantum system* driven by a *real control.* This system describes the evolution of the z-component of the spin of a (spin 1/2) particle driven by a magnetic field, that is constant along the z-axis and controlled along the x-axis. Equivalently it describes the first two levels of a molecule driven by an external field without the rotating wave approximation. The dynamics is governed by the time dependent Schrödinger equation (in a system of units such that $\hbar = 1$):

$$i\frac{d\psi(t)}{dt} = H(t)\psi(t), \tag{2.6}$$

where $\psi(\cdot) = (\psi_1(\cdot), \psi_2(\cdot))^T : [0, T] \to \mathbb{C}^2$, $\sum_{j=1}^2 |\psi_j(t)|^2 = 1$ (i.e. $\psi(t)$ belongs to the sphere $S^3 \subset \mathbb{C}^2$), and

$$H(t) = \begin{pmatrix} -E & \Omega(t) \\ \Omega(t) & E \end{pmatrix}, \tag{2.7}$$

where $E > 0$ and the control $\Omega(\cdot)$, is assumed to be a real function. With the notation of formula (2.1), the drift Hamiltonian is $H_0 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}$, while $H_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the controllability condition (2.2) is satisfied.

Notice that for a spin 1/2 system, it is equivalent to treat the problem for the wave function or for the time evolution operator since S^3 is diffeomorphic to $SU(2)$. The aim is to induce a transition from the first eigenstate of H_0 (i.e., $|\psi_1|^2 = 1$) to any other *physical state*. We recall that two states $\psi, \psi' \in S^3$ are physically equivalent if they differ by a factor of phase. More precisely by physical state we mean a point of the two dimensional sphere (called the *Bloch sphere*) $\mathbf{S}_B := S^3/\sim$ where the equivalence relation $\sim$ is defined as follows: $\psi \sim \psi'$ (where $\psi, \psi' \in S^3$) if and only if $\psi = \exp(i\Phi)\psi'$, for some $\Phi \in [0, 2\pi)$. The projection from S^3 to $\mathbf{S}_B$ is called *Hopf projection* and it is given explicitly in the next section. A particularly interesting transition is of course from the first to the second eigenstates of H_0 (i.e., from $|\psi_1|^2 = 1$ to $|\psi_2|^2 = 1$).

Due to the presence of the drift, in this case the minimum time problem with bounded control and the minimum energy problem are different. In [31] the authors studied the minimum energy problem (in that case, optimal solutions can be expressed in terms of Elliptic functions), while here we minimize the time of transfer, with bounded field amplitude:

$$|\Omega(t)| \leq M, \quad \text{for every } t \in [0, T], \tag{2.8}$$

where T is the time of the transition and $M > 0$ represents the maximum amplitude available. This problem requires completely different techniques with respect to those used in [31].

Thanks to the reduction to a two dimensional problem (on the Bloch sphere), this problem can be attacked with the techniques of optimal syntheses on 2-D manifolds developed by Sussmann, Bressan, Piccoli and the first author, see for instance [21, 24, 54, 63], recently rewritten in [22], and briefly summarized in this thesis, in Chapter 1.

2.2.1 The control problem on the Bloch sphere $\mathbf{S}_B$

An explicit Hopf projection from S^3 to $\mathbf{S}_B$ is given by:

$$\Pi : \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in S^3 \subset \mathbb{C}^2 \longmapsto y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2\,\mathrm{Re}(\psi_1^* \psi_2) \\ 2\,\mathrm{Im}(\psi_1^* \psi_2) \\ |\psi_1|^2 - |\psi_2|^2 \end{pmatrix} \in \mathbf{S}_B \subset \mathbb{R}^3. \tag{2.9}$$

Notice that Π maps the first eigenstate of H_0 (i.e. $|\psi_1|^2 = 1$) to the *north pole* $P_N := (0, 0, 1)^T$ of $\mathbf{S}_B$, and the second eigenstate (i.e. $|\psi_2|^2 = 1$) to the *south pole* $P_S := (0, 0, -1)^T$.

After setting $u(t) = \Omega(t)/M$, the Schrödinger equation (2.6), (2.7) projects to the following single input affine system (clarified below, after normalizations),

$$\dot{y} \;=\; F_S(y) + uG_S(y), \quad |u| \le 1, \quad \text{where:} \tag{2.10}$$

$$y \in \mathbf{S}_B := \{(y_1, y_2, y_3) \in \mathbb{R}^3, \quad \sum_{j=1}^{3} y_j^2 = 1\} \tag{2.11}$$

$$F_S(y) := k\cos(\alpha)\begin{pmatrix} -y_2 \\ y_1 \\ 0 \end{pmatrix}, \qquad G_S(y) := k\sin(\alpha)\begin{pmatrix} 0 \\ -y_3 \\ y_2 \end{pmatrix}, \tag{2.12}$$

$$\alpha := \arctan\left(\frac{M}{E}\right) \in \, (0, \pi/2), \quad k := 2E/\cos(\alpha) = 2\sqrt{M^2 + E^2}. \tag{2.13}$$

Remark **2.2 (normalizations)** In the following, to simplify the notations, we normalize $k = 1$. This normalization corresponds to a reparametrization of the time. More precisely, if T is the minimum time to steer the state $\tilde{y}$ to the state $\bar{y}$ for the system with $k = 1$, the corresponding minimum time for the original system is $T/(2\sqrt{M^2 + E^2})$. Sometimes we need also the original system (2.6), (2.7) on S^3, with the normalization made in this remark, i.e. the system

$$i\frac{d\psi(t)}{dt} = \tilde{H}(t)\psi(t)\,, \quad \text{where} \quad \tilde{H}(t) = \frac{1}{2}\sin\alpha\begin{pmatrix} -\cot\alpha & u(t) \\ u(t) & \cot\alpha \end{pmatrix}. \tag{2.14}$$

We come back to the original value of k only in Section 2.4.3, where we compare our results with those of other authors.

We refer to Figure 2.1. The vector fields $F_S(y)$ and $G_S(y)$ (that play the role respectively of H_0 and H_1) describe rotations respectively around the axes y_3 and y_1. Let us define the vector fields corresponding to constant control ± 1,

$$X_S^{\pm}(y) := F_S(y) \pm G_S(y). \tag{2.15}$$

The parameter $\alpha \in (0, \pi/2)$ (that is the only parameter of the problem) is the angle between the axes of rotations of F_S and X_S^+. The case $\alpha \ge \pi/4$ (resp. $\alpha < \pi/4$) corresponds to $M \ge E$ (resp. $M < E$).

Definition 2.1 *An admissible control $u(\cdot)$ for the system (2.10)–(2.13) is a measurable function $u(\cdot) : [a, b] \to [-1, 1]$, while an admissible trajectory is a Lipschitz functions $y(\cdot) : [a, b] \to \mathbf{S}_B$ satisfying (2.10) a.e. for some admissible control $u(\cdot)$. If $y(\cdot)$ is an admissible trajectory and $u(\cdot)$ the corresponding control, we say that $(y(\cdot), u(\cdot))$ is an admissible pair.*

For every $\bar{y} \in \mathbf{S}_B$, our minimization problem is then to find the admissible pair steering the north pole to $\bar{y}$ in minimum time. More precisely

Problem (P) *Consider the control system (2.10)-(2.13). For every $\bar{y} \in \mathbf{S}_B$, find an admissible pair $(y(\cdot), u(\cdot))$ defined on $[0, T]$ such that $y(0) = P_N$, $y(T) = \bar{y}$ and $y(\cdot)$ is time optimal.*

For us an optimal synthesis is the collection of all the solutions to the problem **(P)**. More precisely

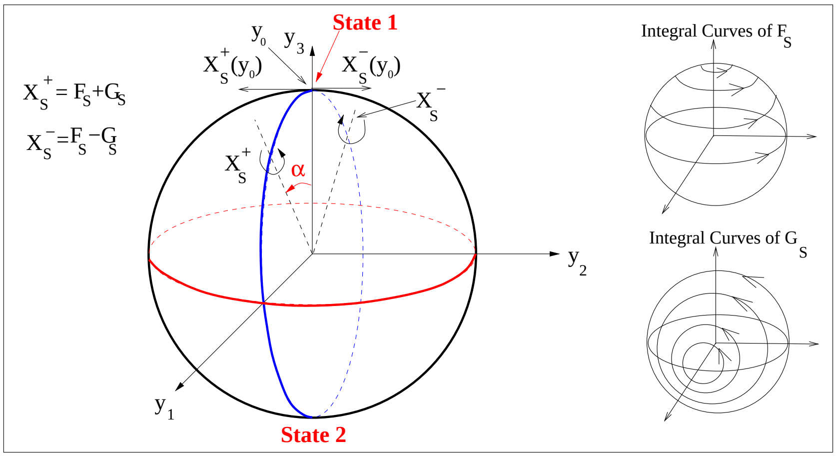

Figure 2.1: The Bloch Sphere

Definition 2.2 (Optimal Synthesis) *An optimal synthesis for the problem* **(P)** *is the collection of all time optimal trajectories* $\Gamma = \{y_{\bar{y}}(\cdot) : [0, b_{\bar{y}}] \mapsto \mathbf{S}_B,\ \bar{y} \in \mathbf{S}_B :\ \ y_{\bar{y}}(0) = P_N,\ y_{\bar{y}}(b_{\bar{y}}) = \bar{y}\}$.

For more elaborated definitions of optimal synthesis see Chapter 1 or [22, 55] and references therein.

Definition 2.3 (bang, singular for the problem (2.10)-(2.13)) *A control* $u(\cdot) : [a, b] \to [-1, 1]$ *is said to be a* bang *control if* $u(t) = +1$ *a.e. in* $[a, b]$ *or* $u(t) = -1$ *a.e. in* $[a, b]$. *A control* $u(\cdot) : [a, b] \to [-1, 1]$ *is said to be a* singular *control if* $u(t) = 0$, *a.e. in* $[a, b]$. *A finite concatenation of bang controls is called a* bang-bang *control. A* switching time *of* $u(\cdot)$ *is a time* $\bar{t} \in [a, b]$ *such that, for every* $\varepsilon > 0$, u *is not bang or singular on* $(\bar{t} - \varepsilon, \bar{t} + \varepsilon) \cap [a, b]$. *A trajectory of the control system (2.10)-(2.13) is said a bang trajectory (or arc), singular trajectory (or arc), bang-bang trajectory, if it corresponds respectively to a bang control, singular control, bang-bang control. If* $\bar{t}$ *is a switching time, the corresponding point on the trajectory* $y(\bar{t})$ *is called a* switching point.

<u>*Remark*</u> **2.3** The definitions of singular trajectory and control, given above are very specific to our problem (2.10)-(2.13). For the definition of singular trajectories for more general systems see Definition 1.2.

In [19] it was proved that, for the same problem (2.10)-(2.13), but in which $y \in \mathbb{R}P^2$, for every couple of points there exists a time optimal trajectory joining them. Moreover it was proved that every time optimal trajectory is a finite concatenation of bang and singular trajectories. Repeating exactly the same arguments and recalling that S^2 is a double covering of $\mathbb{R}P^2$, one easily gets the same result on $\mathbf{S}_B$. More precisely we have:

Proposition 2.1 *For the problem (2.10)-(2.13), for each pair of points* p *and* q *belonging to* $\mathbf{S}_B$, *there exists a time optimal trajectory joining* p *to* q. *Moreover every time optimal trajectory for the problem (2.10)-(2.13) is a finite concatenation of bang and singular trajectories.*

Notice that the previous proposition does not apply if $\alpha = 0$ or $\alpha = \pi/2$, since in these cases the controllability property is lost.

The aim of the chapter is then to study problem **(P)** for every possible value of the parameter α, giving a particular relief to the case in which $\bar{y} = P_S$ (i.e. to the optimal trajectory steering the north to the south pole).

We will not be able to give a complete solution to the problem **(P)**, without the help of numerical simulations. However, thanks to the theory developed in [22] we give a satisfactory description of the optimal trajectories. In the following we describe the main results achieved.

For $\alpha < \pi/4$, every time optimal trajectory is bang-bang and in particular the corresponding control is periodic, in the sense that for every fixed optimal trajectory the time between two consecutive switchings is constant. Moreover it tends to π as α goes to 0. For the original non normalized problem this means that for $M/E << 1$, the optimal control oscillates with frequency of the order of the resonance frequency $\omega_R = 2E$. In this case it is possible to give a satisfactory description of the optimal synthesis excluding a neighborhood of the south pole, in which we are able to find the optimal synthesis only numerically (such results were already given in [19] as we see below).

On the other side, if $\alpha \geq \pi/4$ the computation of the optimal trajectories is simpler since the number of switchings needed to cover the whole sphere is small (less or equal than 2). In this case, for α big enough, we are also able to give the exact value of the time needed to cover the whole sphere. However, there is a new difficulty, namely the presence of singular arcs. Moreover the qualitative shape of the optimal synthesis is rather different if α is close to $\pi/4$ or to $\pi/2$. A relevant fact is that this synthesis contains a singularity (the so called $(S,K)_3$) that is predicted by the general theory (see [22], pp. 61 and 82), and was never observed out from *ad hoc* examples.

The problem of finding explicitly the optimal trajectories from the north pole P_N to the south pole P_S, can be easily solved in the case $\alpha \geq \pi/4$ as a consequence of the construction of the time optimal synthesis. (Coming back to the original non normalized problem we also prove that fixed E, for $M \to \infty$ the time of transfer from P_N to P_S tends to zero.)

For $\alpha < \pi/4$ the problem is more complicated. However, we are able to prove that if $u(t)$ is an optimal control steering the north pole P_N to the south pole P_S in time T, then $u(T-t)$ is as well (see Lemma 2.1 in Section 2.5). Thanks to this fact, we can prove that the optimal trajectories steering the north to the south pole belong to a set Ξ containing at most 8 trajectories (half starting with control $+1$ and half starting with control -1, and switching exactly at the same times). These trajectories are determined in terms of a parameter (the first switching time) that can be easily computed numerically solving suitable equations. Once these trajectories are identified one can check by hands which are the optimal ones.

The analysis can be pushed much forward. We also prove that the cardinality of Ξ depends on the so called *normalized remainder*

$$\mathcal{R} := \frac{\pi}{2\alpha} - \left[\frac{\pi}{2\alpha}\right] \in [0,1), \tag{2.16}$$

where [.] denotes the integer part. In particular, for α small, we prove that if $\mathcal{R}$ is close to zero then Ξ contains exactly 8 trajectories (and in particular there are four optimal trajectories), while if $\mathcal{R}$ is close to 1 then Ξ contains only 4 trajectories (two of them are optimal). The precise description of these facts is contained in Proposition 2.6, Section 2.4.2. As a consequence, the qualitative shape of the time optimal synthesis presents different patterns, that cyclically alternate, in the non controllability limit $\alpha \to 0$, giving a partial proof of a conjecture formulated in a previous paper ([19]), that was supported by numerical simulations, see Remark 2.11. This is probably the most interesting byproduct of this chapter. In the next chapter we will describe more precisely the way in which the shape of the optimal synthesis changes as α goes to zero.

Finally we compare these results with some known results of Khaneja, Brockett and Glaser and with those obtained by controlling the magnetic field both on the x and y directions.

2.3 History of the problem and known facts

The problem **(P)** (although with different purposes) was already partially studied in [19], in the case $\alpha < \pi/4$. In that paper the aim was to give an estimate on the maximum number of switchings for time optimal trajectories on $SO(3)$ (problem first studied by Agrachev and Gamkrelidze in [2], using index theory).

In [19] it has been proved that, for the problem **(P)** in the case $\alpha < \pi/4$, every optimal trajectory is bang-bang. More precisely, it was proved that in the case $\alpha < \pi/4$, if $y(\cdot)$ is a time optimal trajectory starting at the north pole, then it should satisfy the following properties:

i) $y(\cdot)$ is bang bang;

ii) the duration s_i of the first bang arc satisfies $s_i \in [0, \pi]$,

iii) the time duration between two consecutive switchings is the same for all *interior bang arcs* (i.e. excluding the first and the last bang) and it is the following function of s_i defined in the interval $[0, \pi]$,

$$v(s_i) = \pi + 2\arctan\left(\frac{\sin(s_i)}{\cos(s_i) + \cot^2(\alpha)}\right). \tag{2.17}$$

One can immediately check that this function satisfies $v(0) = v(\pi) = \pi$ and $v(s_i) > \pi$ for every $s_i \in]0, \pi)$,

iiii) the time duration of the last arc is $s_f \in [0, v(s_i)]$,

Properties **i)**–**iiii)** are illustrated in Figure 2.2. Moreover, thanks to the analysis given in [19], one easily gets (always in the case $\alpha < \pi/4$):

v) the number of switchings N_y of $y(\cdot)$ satisfies the following inequality

$$N_y \leq N_M := \left[\frac{\pi}{2\alpha}\right] + 1 \tag{2.18}$$

Conditions **i)-v)** define a set of candidate optimal trajectories. Notice that conditions **i)-v)** are just necessary conditions for optimality and one is faced with the problem of selecting, among them, those that are really optimal. In particular given a trajectory satisfying conditions **i)-v)**,

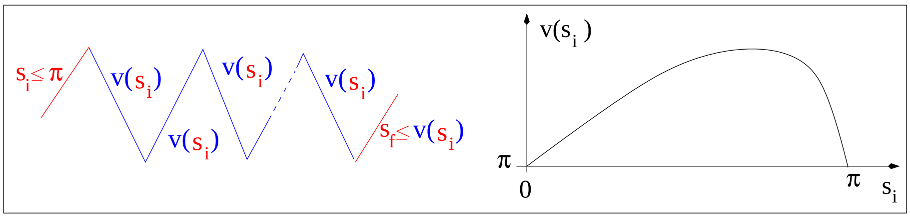

Figure 2.2: Time optimal trajectories for $\alpha < \pi/4$

one would like to find the time after which it is no more optimal. In the following we say that at this time the trajectory loses optimality.

The way in which these candidate optimal trajectories cover the whole sphere is shown in the top of Figure 2.3.

Consider the following curves, made by points where the control switches from $+1$ to -1 or viceversa, called *switching curves*, defined by induction

$$C_1^{\varepsilon}(s) = e^{X_S^{\varepsilon} v(s)} e^{X_S^{-\varepsilon} s} P_N \,, \qquad C_k^{\varepsilon}(s) = e^{X_S^{\varepsilon} v(s)} C_{k-1}^{-\varepsilon}(s)$$
$$(\text{where } \varepsilon = \pm 1 \text{ and } k = 2, ..., N_M - 1). \tag{2.19}$$

See the top of Figure 2.3.

Even if the analysis made in [19] was sufficient to the purpose of giving a bound on the maximum number of switchings for time optimal trajectories on $SO(3)$, some questions remained unsolved. In particular questions about *local optimality* of the switching curves (see Definition 1.8). In [19], the following questions remain unsolved:

Question 1 Are the switching curves C_k^{ε}, $k = 1..., N_M - 1$, locally optimal? More precisely, one would like to understand how the candidate optimal trajectories described above are going to lose optimality.

Question 2 What is the shape of the optimal synthesis in a neighborhood of the south pole?

Numerical simulations suggested some conjectures regarding the above questions. More precisely:

C1 Define $k_{last} = \left[\frac{\pi-\alpha}{2\alpha}\right] - 1$. Then the curves $C_k^{\varepsilon}(s)$, $(k = 1, ..., N_M - 1)$ are locally optimal if and only if $k \le k_{last}$. Notice that $k_{last} \in \{N_M - 3, N_M - 2\}$.

Analyzing the evolution of the minimum time wave front in a neighborhood of the south-pole, it is reasonable to conjecture that:

C2 The shape of the optimal synthesis in a neighborhood of the south pole depends on the so called *remainder*[1] $r := \pi - 2\alpha\left[\frac{\pi}{2\alpha}\right]$. Notice that r belongs to the interval $[0, 2\alpha)$. More precisely, we conjecture that for $\alpha \in (0, \pi/4)$, there exist two positive numbers α_1 and α_2 such that $0 < \alpha_1 < \alpha < \alpha_2 < 2\alpha$ and:

[1]Notice that $r = 2\alpha\mathcal{R}$, where $\mathcal{R}$ has been defined in Formula (2.16). In conjecture **C2**, we use the remainder r, to keep the same notation of [19].

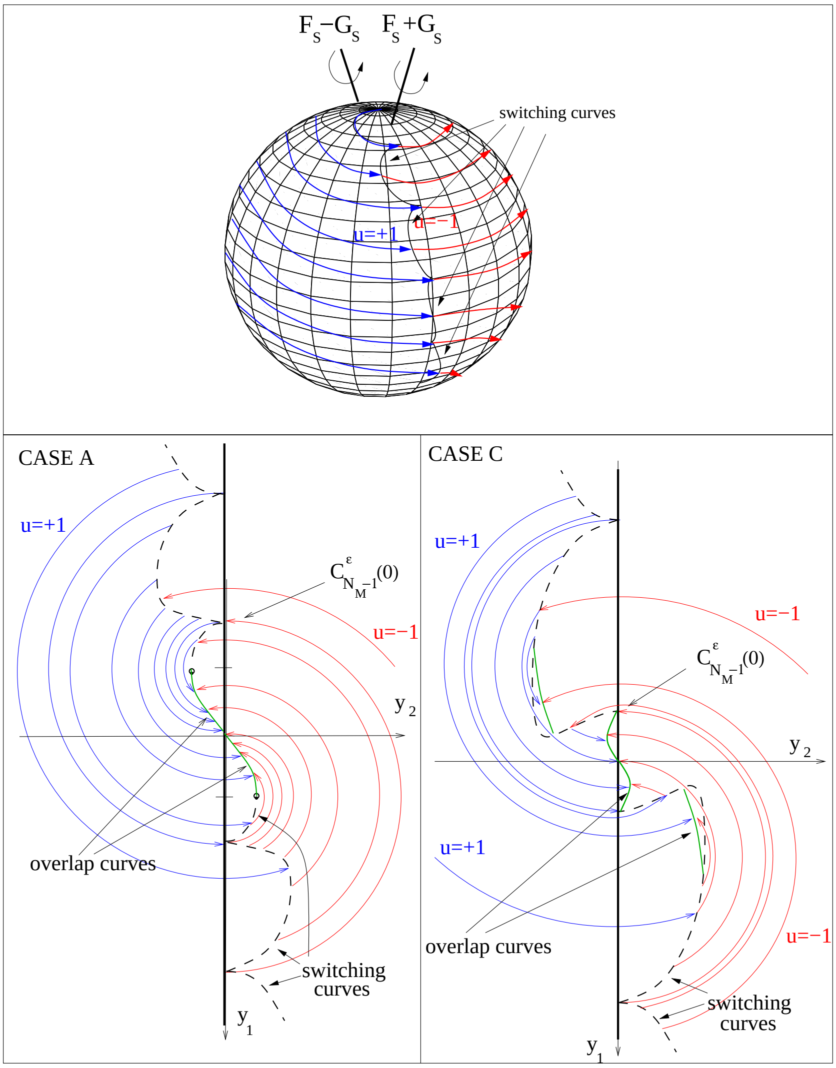

Figure 2.3: Synthesis on the sphere for $\alpha < \pi/4$ and conjectured shape in a neighborhood of the south pole

CASE A: $r \in (\alpha_2, 2\alpha)$. The switching curve $C^{\varepsilon}_{N_M-1}$ glues to an overlap curve that passes through the origin (Fig. 2.3, Case A).

CASE B: $r \in [\alpha_1, \alpha_2]$. The switching curve $C^{\varepsilon}_{N_M-1}$ is not reached by optimal trajectories in the interval $(0, \pi]$. At the point $C^{\varepsilon}_{N_M-1}(0)$ an overlap curve starts and passes through the origin.

CASE C: $r \in (0, \alpha_1)$. The situation is more complicated and it is depicted in the bottom of Fig. 2.3, Case C.

For $r = 0$, the situation is the same as in CASE A, but for the switching curve starting at $C^{\varepsilon}_{N_M-2}(0)$.

2.4 Main results

We give here a brief description of the main results of the chapter. The corresponding proofs are given in Section 2.5. From now on we use the following conventions.

Remark **2.4 (notation)** Recall Definition 2.3. The letter B refers to a bang trajectory and the letter S refers to a singular trajectory. A concatenation of bang and singular trajectories is labeled by the corresponding letter sequence, written in order from left to right. Sometimes, we use a subscript to indicate the time duration of a trajectory so that we use B_t to refer to a bang trajectory defined on an interval of length t and, similarly, S_t for a singular trajectory defined on an interval of length t. Moreover we indicate by γ^+ (resp. γ^-) the trajectory of (2.10)–(2.13) starting at the north pole at time zero and corresponding to control $u \equiv 1$ (resp. $u \equiv -1$). Notice that $\gamma^{\pm}$ are defined for every time, and are periodic. Finally we use the following subsets of $\mathbf{S}_B$: the circle of equation $y_3 = 0$ called *equator*, the set $y_3 > 0$, called *north hemisphere* and the set $y_3 < 0$, called *south hemisphere*.

From Chapter 1, recall the definitions of switching curves, cut loci and overlap curves.

2.4.1 Optimal synthesis for $\alpha \geq \pi/4$

In this section we describe the time optimal synthesis for $\alpha \geq \pi/4$. We divide $\mathbf{S}_B$ in 8 open regions called $\Omega_1^{\pm}, ..., \Omega_3^{\pm}$, $\Omega^{\pm}_{\text{nasty}}$ and in 16 arcs (see Definition 2.4, and Figure 2.4). For every point $\bar{y} \in \mathbf{S}_B \setminus (\Omega^+_{\text{nasty}} \cup \Omega^-_{\text{nasty}})$, Theorem 2.1 gives the optimal trajectories reaching $\bar{y}$.

Unlike the $\alpha < \pi/4$ case, here it is possible to detect the presence of singular trajectories that are optimal, and also of cut loci (even not only in a neighborhood of the south pole).

The region Ω^+_{nasty} (and similarly Ω^-_{nasty}) is more difficult to analyze. It contains a cut locus that should be determined numerically. Even if we are not able to provide an analytic characterization of this locus, we are able to prove the following.

i) $\alpha = \arcsin(1/\sqrt[4]{2})$ is a bifurcation point for the optimal synthesis i.e. the qualitative shape is different if $\alpha \in [\pi/4, \arcsin(1/\sqrt[4]{2}))$ (called **Case 1**) or $\alpha \in [\arcsin(1/\sqrt[4]{2}), \pi/2)$ (called **Case 2**). More precisely, from the point $D^+ := \gamma^+(\pi)$, in **Case 1** it starts an optimal switching curve, while in **Case 2** it starts an overlap curve (see Proposition 2.3). The situation in Ω^-_{nasty} is symmetric.

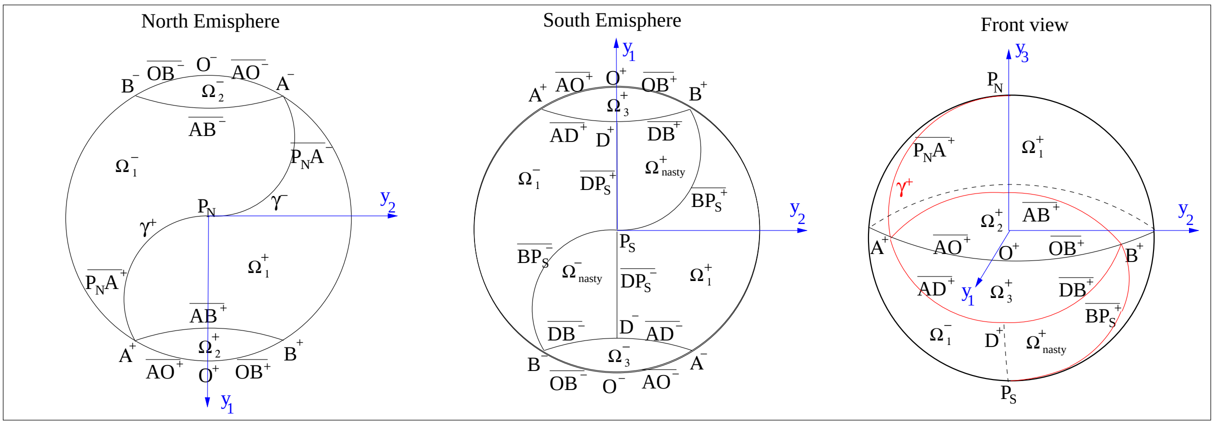

Figure 2.4: Definition 2.4

ii) The south pole belongs to the cut locus and it is reached exactly by four optimal trajectories (see Proposition 2.2).

Numerical computations show that in **Case 2**, the cut locus in Ω^+_{nasty} is an overlap curve connecting D^+ with the south pole, while in **Case 1**, the switching curve starting from D^+ loses local optimality at a point of Ω^+_{nasty} and connects to an overlap curve which reaches the south pole (see Figure 2.5). Remark 2.9 explains that in **Case 2** it is not necessary to compute the cut locus lying in Ω^+_{nasty} to get the expression of the optimal trajectory connecting P_N to a point of Ω^+_{nasty}. The situation in Ω^-_{nasty} is symmetric.

Let us start with the description of the optimal synthesis in $\mathbf{S}_B \setminus (\Omega^+_{\text{nasty}} \cup \Omega^-_{\text{nasty}})$. Even if Definition 2.4 and Theorem 2.1 look complicated, the shape of the optimal synthesis is quite simple as it is shown in Figure 2.5.

Definition 2.4 *According to Figure 2.4, let us define the following curves on* $\mathbf{S}_B$.

- *Let* t_1 *be the first time at which* γ^+ *intersects the equator and let* $A^+ := \gamma^+(t_1)$ *(notice that* $t_1 = \pi - \arccos(\cot^2(\alpha))$*). Define* $\overline{P_N A^+} = Supp\left(\gamma^+|_{[0,t_1]}\right)$.
- *Let* ξ^- *be the trajectory corresponding to control* -1*, starting at time zero from* A^+*. Let* t_2 *be the first positive time at which* ξ^- *intersects the equator (notice that* $t_2 = 2\arccos(\cot^2(\alpha))$*). Define* $B^+ := \xi^-(t_2)$ *and* $\overline{AB^+} = Supp\left(\xi^-|_{[0,t_2]}\right)$.
- *Let* $O^+ = (1,0,0)$*. Define* $\overline{AO^+}$ *(resp.* $\overline{OB^+}$*) as the support of the trajectory corresponding to control zero, starting at* A^+ *(resp.* O^+*) and ending at* O^+ *(resp.* B^+*).*
- *Recall that* $D^+ = \gamma^+(\pi)$*, and define* $\overline{AD^+} = Supp\left(\gamma^+|_{[t_1,\pi]}\right)$, $\overline{DB^+} = Supp\left(\gamma^+|_{[\pi,t_3]}\right)$*, where* t_3 *is the second intersection time of* γ^+ *with the equator (notice that* $t_3 = \pi + \arccos(\cot^2(\alpha)) = t_1 + t_2$*).*

- *Let $\overline{BP_S^+}$ the support of the trajectory corresponding to control -1, starting at B^+ and ending at the south pole.*

- *Let $\overline{DP_S^+}$ the connected subset of the meridian $y_2 = 0$, lying in the south hemisphere and connecting the point D^+ to the south pole.*

Similarly define A^-, B^-, O^-, D^-, $\overline{P_N A^-}$, $\overline{AB^-}$, $\overline{AO^-}$, $\overline{OB^-}$, $\overline{AD^-}$, $\overline{DB^-}$, $\overline{BP_S^-}$, $\overline{DP_S^-}$.

According to Figure 2.4 define $\Omega_1^\pm, \ldots, \Omega_4^\pm, \Omega_{nasty}^\pm$ as the open connected components of the open set obtained subtracting from $\mathbf{S}_B$ all the arcs defined above.

The following theorem holds for every $\alpha \in (\pi/4, \pi/2)$. For the particular value $\alpha = \pi/4$ the claims of the theorem must be modified. Such changes are reported in Remark 2.5.

Theorem 2.1 *Let $\boldsymbol{\gamma}_{\bar{y}}$ be the set of time optimal trajectories steering the north pole to $\bar{y}$. We have the following:*

T1. *If $\bar{y} \in \overline{P_N A^+}$ then $\boldsymbol{\gamma}_{\bar{y}}$ is made by a unique trajectory corresponding to control $+1$ of the form B_t, with $t \leq t_1$.*

T2. *If $\bar{y} \in \overline{AB^+} \setminus B^+$ then $\boldsymbol{\gamma}_{\bar{y}}$ is made by a unique trajectory of the form $B_{t_1} B_t$ (with the first bang corresponding to control $+1$).*

T3. *If $\bar{y} \in \overline{AO^+}$ then $\boldsymbol{\gamma}_{\bar{y}}$ is made by a unique trajectory of the form $B_{t_1} S_s$ (with the first bang corresponding to control $+1$).*

T4. *If $\bar{y} \in \overline{OB^+} \setminus O^+$ then $\boldsymbol{\gamma}_{\bar{y}}$ is made by two trajectories of the form $B_{t_1} S_s B_t$, both starting with control $+1$ and ending respectively with control $+1$ and -1. These two trajectories have the same values of $s \geq 0$ and $t > 0$.*

T5. *If $\bar{y} \in \overline{AD^+}$ then $\boldsymbol{\gamma}_{\bar{y}}$ is made by a unique trajectory corresponding to control $+1$ of the form B_t, with $t \in [t_1, \pi]$.*

T6. *If $\bar{y} \in \overline{DB^+} \setminus B^+$ then $\boldsymbol{\gamma}_{\bar{y}}$ is made by a unique trajectory corresponding to control $+1$ of the form B_t, with $t \in [\pi, t_3)$.*

T7. *If $\bar{y} \in \overline{BP_S^+}$ then $\boldsymbol{\gamma}_{\bar{y}}$ is made by two trajectories respectively of the form $B_{t_1} B_t$ and $B_{t_3} B_{t-t_2}$ and starting with control $+1$.*

T8. *If $\bar{y} \in \Omega_1^+ \cup (\overline{DP_S^-} \setminus P_S)$, then $\boldsymbol{\gamma}_{\bar{y}}$ is made by a unique trajectory of the form $B_t B_{t'}$, with $0 \leq t < t_1$ and the first bang corresponding to control $+1$.*

T9. *If $\bar{y} \in \Omega_2^+$, then $\boldsymbol{\gamma}_{\bar{y}}$ is made by a unique trajectory of the form $B_{t_1} S_s B_t$, with $s > 0$, the first bang arc and the last bang arc corresponding respectively to control $+1$ and -1.*

T10. *If $\bar{y} \in \Omega_3^+$, then $\boldsymbol{\gamma}_{\bar{y}}$ is made by a unique trajectory of the form $B_{t_1} S_s B_t$, with $s > 0$ and both bang arcs corresponding to control $+1$.*

T11. *If $\bar{y} = P_S$ then $\boldsymbol{\gamma}_{\bar{y}}$ is made by the four trajectories of the form $B_{t_1} B_{t_3}$ and $B_{t_3} B_{t_1}$.*

T12. *If $\bar{y} \in \Omega^+_{nasty}$ then every trajectory of $\boldsymbol{\gamma}_{\bar{y}}$ is bang-bang with at most two switchings.*

If $\bar{y}$ belongs to one of the remaining sets defined above, the description of the optimal strategy is analogous, by symmetry.

<u>Remark</u> **2.5** In the case $\alpha = \pi/4$ some changes in the previous statement are required. In particular the points A^+, B^+, O^+ and D^+ coincide (also the points A^-, B^-, O^- and D^- coincide) and, consequently, there are no optimal trajectories containing singular arcs. Another immediate consequence of this fact is that there are only two optimal trajectories reaching the south pole, of the form $B_\pi B_\pi$.

<u>Remark</u> **2.6** Notice that every point of $\overline{OB^+} \setminus O^+$, $\overline{OB^-} \setminus O^-$, $\overline{BP_S^+}$, $\overline{BP_S^-}$ is reached by more than one optimal trajectory, i.e. it belongs to the cut locus. Other points of the cut locus can be identified numerically in Ω^+_{nasty} and Ω^-_{nasty} as explained in the next section.

<u>Remark</u> **2.7** In Theorem 2.1 we do not specify all the durations of the bang arcs. However the missing ones can be obtained simply by following the switching strategy backwards.

<u>Remark</u> **2.8** Note that the region reached by optimal trajectories containing a singular arc $\Omega_2^\pm \cup \Omega_3^\pm \cup \overline{AO^\pm} \cup \overline{OB^\pm}$ become bigger and bigger as α tends to $\pi/2$. Moreover, in this limit, since the modulus of the drift F_S becomes smaller and smaller, the time needed to cover such region tends to infinity. Notice however that the time needed to reach P_S is always 2π. The time needed to reach every point of the sphere for α big enough, and the last point reached by an optimal trajectory containing a singular arc, can be computed explicitly. This is done in Section 2.6.

Since the case $\bar{y} = P_S$ is important also for the determination of the cut locus in $\Omega^+_{\text{nasty}} \cup \Omega^-_{\text{nasty}}$, it is reported in the next section as a separate proposition (see Proposition 2.2).

2.4.1.1 The time optimal synthesis in $\Omega^\pm_{\text{nasty}}$ and optimal trajectories reaching P_S for $\alpha \geq \pi/4$

From next proposition, **T11** of Theorem 2.1 follows. More precisely Proposition 2.2 shows that in the case $\alpha \geq \pi/4$, there are exactly four optimal trajectories steering P_N to P_S, and it characterizes them. As a consequence, the south pole belongs to the cut locus.

Proposition 2.2 *Consider the control system (2.10)–(2.13), and assume $\alpha \geq \pi/4$. Then the optimal trajectories steering the north pole to the south pole are bang-bang with only one switching. More precisely they are the four trajectories corresponding to the four controls*

$$u^{(1)} = \begin{cases} u = 1, \ t \in [0, t_1] \\ u = -1, \ t \in (t_1, T], \end{cases} \qquad u^{(2)} = \begin{cases} 1, \ t \in [0, t_3] \\ -1, \ t \in (t_3, T], \end{cases}$$

$$u^{(3)} = \begin{cases} -1, \ t \in [0, t_1] \\ 1, \ t \in (t_1, T], \end{cases} \qquad u^{(4)} = \begin{cases} -1, \ t \in [0, t_3] \\ 1, \ t \in (t_3, T] \end{cases}$$

where t_1 and t_3 are defined in Definition 2.4, and $T = 2\pi$.

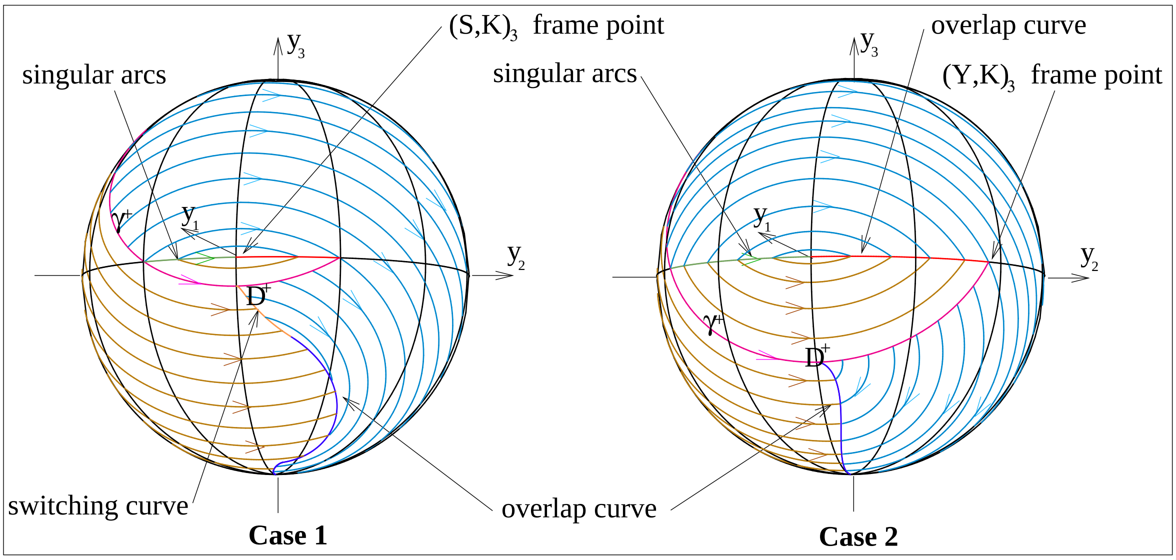

Figure 2.5: Optimal synthesis for $\alpha = \pi/3$ and α slightly larger than $\pi/4$.

One can easily check that the switchings described in Proposition 2.2 occur on the equator ($y_3 = 0$).

The following proposition describes the optimal synthesis in $\Omega^{\pm}_{\text{nasty}}$, in a neighborhood of the points $D^{\pm}$ and the bifurcation occurring at $\alpha = \arcsin(1/\sqrt[4]{2})$.

Proposition 2.3 *Let $\alpha \geq \pi/4$. In a neighborhood of the point D^+ in Ω^+_{nasty}, there exists a switching curve starting at D^+ of the form $e^{v(s)X_S^+} e^{sX_S^-} P_N$. If $\alpha > \pi/4$ this curve is tangent to the equator at D^+. Moreover if $\alpha < \arcsin(1/\sqrt[4]{2})$ (above called* **Case 1***) then the switching curve is optimal near D^+, while if $\alpha \geq \arcsin(1/\sqrt[4]{2})$ (above called* **Case 2***) then the switching curve is not locally optimal near D^+ and an overlap curve starts at the point D^+. A symmetric result holds in a neighborhood of D^- in Ω^-_{nasty}.*

The region Ω^+_{nasty} contains a cut locus that should be determined numerically. In **Case 2**, numerical simulations show that the switching curve starting at D^+ is never optimal, i.e. every point of Ω^+_{nasty} is reached by an optimal trajectory of the form $e^{tX_+}e^{sX_-}P_N$, with $s \in (0, t_1)$ or an optimal trajectory of the form $e^{tX_-}e^{sX_+}P_N$, with $s \in (\pi, t_3)$.

<u>*Remark*</u> **2.9** Notice however that, in **Case 2**, given a point $\bar{y} \in \Omega^+_{\text{nasty}}$, to find the time optimal trajectory reaching $\bar{y}$, it is not necessary to compute the cut locus. Indeed it is sufficient to compare the final times, corresponding to the two switching strategies given above, and to chose the quickest one. The situation in Ω^-_{nasty} is symmetric.

In **Case 1**, the situation is more complicated. The switching curve described by Proposition 2.3 has the expression $C_1^+(s) = e^{X_S^+ v(s)} e^{X_S^- s} P_N$, $s \in (0, t_1)$ where the function $v(\cdot)$ is defined as in the $\alpha < \pi/4$ case, i.e. $v(s) = \pi + 2\arctan\left[(\sin s)/(\cos s + \cot^2\alpha)\right]$. (To verify such formula it is enough to repeat the computations done in [19].) As described by Proposition 2.3, this

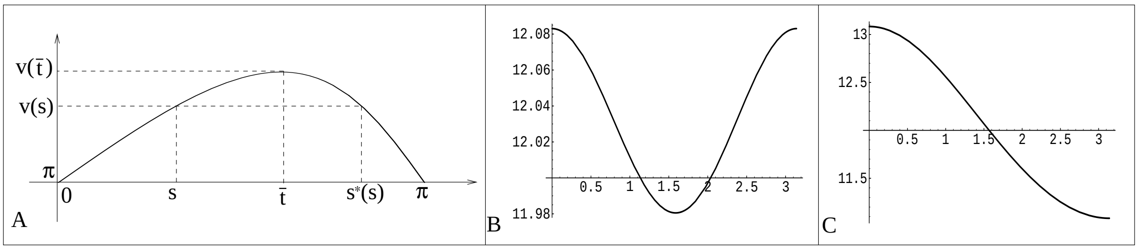

Figure 2.6: Graph of $v(\cdot)$ when $\alpha = \pi/6$ (figure A). Graph of the functions $\mathcal{F}$ and $\mathcal{G}$ when $\alpha = 0.13$ (figure B and C)

switching curve is optimal near D^+ and numerical simulations show that there exists $\bar{s} \in (0, t_1)$ such that there is an optimal trajectory switching on $C_1^+(s)$ if and only if $s \in [0, \bar{s})$, and an overlap curve connecting $C_1^+(\bar{s})$ to the south pole appears. The optimal synthesis for **Case 1** and **Case 2** is depicted in Figure 2.5.

2.4.2 Optimal trajectories reaching the south pole for $\alpha < \pi/4$

In this section we characterize the time optimal trajectories reaching the south pole, in the case $\alpha < \pi/4$. This characterization is more complicated with respect to the case $\alpha \geq \pi/4$, due to the fact that the optimal trajectories have many switchings. The time optimal synthesis for $\alpha < \pi/4$ was already (partially) studied in [19] and it has been described in Section 2.3.

From conditions **i)–iiii)** in Section 2.3, we know that every optimal trajectory starting at the north pole has the form $B_{s_i} B_{v(s_i)} \cdots B_{v(s_i)} B_{s_f}$ where the function $v(s_i)$ is given by formula (2.17). (In the following we do not specify if the first bang corresponds to control $+1$ or -1, since, as a consequence of the symmetries of the problem, if $u(t)$ is an optimal control steering the north pole to the south pole, $-u(t)$ steers the north pole to the south pole as well.) It remains to identify one or more values of s_i, s_f and the corresponding number of switchings n for this trajectory to reach the south pole.

Notice that $\bar{t} = \arccos(-\tan^2(\alpha))$ is the maximum of the function $v(\cdot)$ on the interval $[0, \pi]$, $v(\cdot)$ is increasing on $[0, \bar{t}]$ and decreasing on $[\bar{t}, \pi]$ and $v(0) = v(\pi) = \pi$. Then, given $s \in [0, \pi]$ such that $s \neq \bar{t}$, there is a unique solution $s^*(s) \in [0, \pi]$, $s^*(s) \neq s$, to the equation $v(s^*) = v(s)$. The function $s^*(\cdot)$ is extended to the whole interval $[0, \pi]$ setting $s^*(\bar{t}) = \bar{t}$ (see Figure 2.6 A). Thanks to the symmetries of the problem, we prove that if $\alpha < \pi/4$, s_f is equal either to s_i or to $s^*(s_i)$. This fact is described by Lemma 2.1 stated and proved in Section 2.5.

The following two propositions describe how to identify candidate triples (s_i, s_f, n) for which the corresponding trajectory steers the north pole to the south pole in minimum time. From now on, all along the chapter, we say that a bang-bang trajectory, solution of the system (2.10)–(2.13), is a *candidate optimal trajectory* if it is an extremal trajectory for problem **(P)** reaching the south pole and it has a number n of switchings satisfying $n \leq N_M$ (defined in Formula (2.18)). From Lemma 2.1, there are two kinds of candidate optimal trajectories:

- $s_f = s^*(s_i)$, called TYPE-1-candidate optimal trajectories

- $s_f = s_i$ called TYPE-2-candidate optimal trajectories

Define the following functions, whose geometric meaning is clarified in Section 2.5.2:

$$\theta(s)=2\arccos\left(\sin^2\left(\frac{v(s)}{2}\right)\cos(2\alpha)-\cos^2\left(\frac{v(s)}{2}\right)\right) \tag{2.20}$$

$$\beta(s) = 2\arccos(\sin(\alpha)\cos(\alpha)(1-\cos(s))) \tag{2.21}$$

Proposition 2.4 (TYPE-1-trajectories) *Fixed $\alpha < \pi/4$, the equation for the couple $(s,n) \in [0,\pi] \times \mathbb{N}$:*

$$\mathcal{F}(s) := \frac{2\pi}{\theta(s)} = n, \tag{2.22}$$

has either two or zero solutions. More precisely if (s,n) is a solution to equation (2.22), then $(s^*(s),n)$ is the second one. The TYPE-1-candidate optimal trajectories are then those of the form $B_s \underbrace{B_{v(s)}\cdots B_{v(s)}}_{n-1} B_{s^*(s)}$ and $B_{s^*(s)} \underbrace{B_{v(s)}\cdots B_{v(s)}}_{n-1} B_s$.

Proposition 2.5 (TYPE-2-trajectories) *Fixed $\alpha < \pi/4$, the equation for the couple $(s,n) \in [0,\pi] \times \mathbb{N}$:*

$$\mathcal{G}(s) := \frac{2\beta(s)}{\theta(s)} + 1 = n, \tag{2.23}$$

has exactly two solutions. More precisely these solutions have the form (s_1,n), $(s_2,n+1)$. The trajectories $B_{s_1} \underbrace{B_{v(s_1)}\cdots B_{v(s_1)}}_{n-1} B_{s_1}$ and $B_{s_2} \underbrace{B_{v(s_2)}\cdots B_{v(s_2)}}_{n} B_{s_2}$ are the TYPE-2-candidate optimal trajectories.

In Figure 2.6 B and C the graphs of the functions (2.22) and (2.23) are drawn for a particular value of α, namely $\alpha = 0.13$. Propositions 2.4 and 2.5 select a set of (possibly coinciding) 4 or 8 candidate optimal trajectories (half of them starting with control $+1$ and the other half with control -1) corresponding to triples (s_i, s_f, n). Such triples can be easily computed numerically solving equations (2.22) and (2.23). Then the optimal trajectories can be selected by comparing the times needed to reach the south pole for each of the candidate optimal trajectory. Notice that there are at least two optimal trajectories steering the north to the south pole (one starting with control $+1$ and the other with control -1).

If $\pi/(2\alpha)$ is an integer number $\bar{n}$, then TYPE-1 candidate optimal trajectories coincide with the TYPE-2 candidate optimal trajectories of the form $B_\pi \underbrace{B_\pi ... B_\pi}_{\bar{n}-2} B_\pi$. The remaining trajectories of TYPE-2 are of the form $B_s \underbrace{B_{v(s)}...B_{v(s)}}_{\bar{n}-1} B_s$ for some $s \in (0,\pi)$. Otherwise if $\pi/(2\alpha)$ is not an integer number, define:

$$m := \left[\frac{\pi}{2\alpha}\right], \quad \text{and the normalized remainder} \quad \mathcal{R} := \frac{\pi}{2\alpha} - \left[\frac{\pi}{2\alpha}\right] \in [0,1).$$

where [.] denotes the integer part. The following proposition determines precisely the time optimal trajectories for particular values of the parameter $\mathcal{R}$:

Proposition 2.6 *For m large enough there exist $r_1(m) \leq r_2(m) \in]0,1)$ such that:*

A. *if $\mathcal{R} \in (0, r_1(m)]$ then equation (2.22) admits exactly two solutions that are both optimal, while TYPE-2 candidate optimal trajectories are not.*

B. *if $\mathcal{R} \in (r_1(m), r_2(m))$, then equation (2.22) admits two solutions, that are not optimal.*

C. *if $\mathcal{R} \in (r_2(m), 1)$ then equation (2.22) does not admit any solution.*
Moreover $\lim_{m\to\infty} r_2(m) = 0$.

<u>Remark</u> **2.10** The function $r_2(m)$ can be determined explicitly (see Section 2.5.2.1), while for $r_1(m)$ we are just able to prove the existence, and we conjecture that it can be taken equal to $r_2(m)$.

<u>Remark</u> **2.11** An important consequence of Proposition 2.6 is that for α small, the number of optimal trajectories reaching the south pole is not fixed with respect to α. Indeed such number alternates as $\alpha \to 0$, according to Proposition 2.6: in particular it is equal to 4 if $\mathcal{R} \in (0, r_1(m)]$ and it is equal to 2 if $\mathcal{R} \in (r_2(m), 1[\cup \{0\}$. This is enough to conclude that also the qualitative shape of the optimal synthesis in a neighborhood of the south pole alternates giving a partial proof to the conjecture **C2** of Section 2.3 (originally stated in [19]). In particular it is a proof of the first assertion (on the dependence of the synthesis on the remainder $r = 2\alpha\mathcal{R}$). Moreover notice that the results of Proposition 2.6 perfectly fit with all the other statements of conjecture **C2** with $r_2(m)$ playing the role of $\alpha_1/(2\alpha)$. One can apply the definition of locally equivalent syntheses given in [22] (see Definition 32, pp. 59), to make rigorous the statement that the qualitative shape of the optimal synthesis changes with α. A precise description of the optimal synthesis in a neighborhood of the south pole for α small will be given in the next chapter.

Using the previous analysis one can easily show the following result (of which we skip the proof):

Proposition 2.7 *If N is the number of switchings of an optimal trajectory joining the north to the south pole, then*

$$\frac{\pi}{2\alpha} - 1 \leq N < \frac{\pi}{2\alpha} + 1.$$

Using these inequalities and the fact that, for $\alpha < \pi/6$, the function $2s + \left(\frac{\pi}{2\alpha} - 1\right) v(s)$ is increasing on $[0, \pi]$, one can give a rough estimate of the time needed to reach the south pole:

Proposition 2.8 *The total time T of an optimal trajectory joining the north to the south pole satisfies the inequalities:*

$$\frac{\pi^2}{2\alpha} - 2\pi < T < \frac{\pi^2}{2\alpha} + \pi.$$

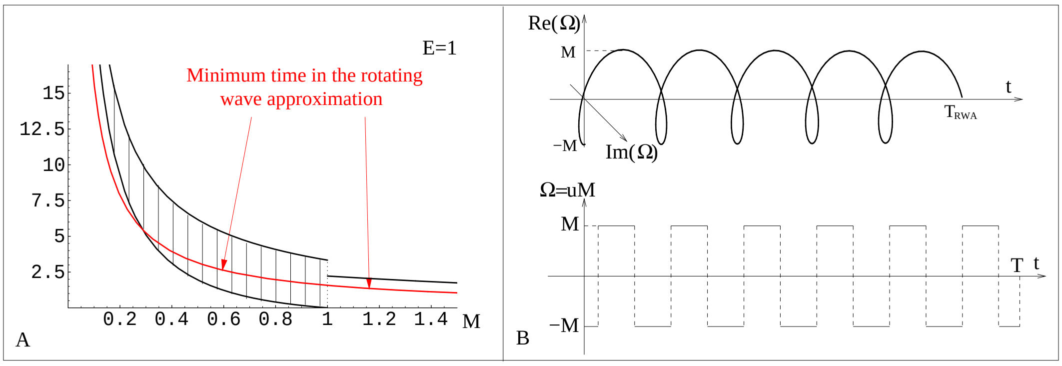

Figure 2.7: A. Estimate of the minimum time to reach the state two and comparison with the time needed with two controls or in the rotating wave approximation B. Comparison between the optimal strategy for our system and in the rotating wave approximation

2.4.3 Comparison with results in the rotating wave approximation and with [36]

In this section we come back to the original value of k i.e. $k = 2E/\cos(\alpha) = 2\sqrt{M^2+E^2}$, and we compare the time necessary to steer the state one to the state two for our model and the model (2.4), described in Remark 2.1, in which we control the magnetic field both along the x and y direction, or we consider a two-level molecule in the rotating wave approximation. We recall that $-E, E$ are the energy levels and M is the bound on the control. For our model, the time of transfer T satisfies:

- for $\alpha \geq \pi/4$ (i.e. for $M \geq E$) then $T = 2\pi/k = \pi/\sqrt{M^2+E^2}$;
- for $\alpha < \pi/4$ (i.e. for $M < E$) then T is estimated by

$$\frac{1}{k}\left(\frac{\pi^2}{2\alpha} - 2\pi\right) < T < \frac{1}{k}\left(\frac{\pi^2}{2\alpha} + \pi\right).$$

On the other hand, for the model (2.4), the time of transfer is $T_{\mathbb{C}} = \pi/(2M)$ (cf. Remark 2.1). Fixed $E = 1$, in Figure 2.7 A the times T and $T_{\mathbb{C}}$ as function of M are compared. Notice that although $T_{\mathbb{C}}$ is bigger than the lower estimate of T in some interval, we always have $T_{\mathbb{C}} \leq T$. This is due to the fact that the admissible velocities of our model are a subset of the admissible velocities of the model (2.4).

Notice that, fixed $E = 1$, for $M \to 0$ we have $T \sim \pi^2/(4M) = (\pi/2)T_{\mathbb{C}}$, while for $M \to \infty$, we have $T \sim \pi/M = 2T_{\mathbb{C}}$.

Remark **2.12** For $M << E$ (i.e. for α small) the difference between two switching times is $v(s)/k \sim \pi/(2E)$. It follows that a time optimal trajectory connecting the north to the south pole (in the interval between the first and the last bang) is periodic with period $P \sim \pi/E$ i.e. with a frequency of the order of the resonance frequency $\omega_R = 2E$ (see Figure 2.7 B). On the other side if $M > E$ then the time optimal trajectory connecting the north with the south pole is the concatenation of two pulses. Notice that if $M >> E$, the time of transfer is of the order of π/M and therefore tends to zero as $M \to \infty$. It is interesting to compare this result with a result of Khaneja, Brockett and Glaser, for a two level system, but with no bound on controls (see [36]). They estimate the infimum time to reach every point of the whole group $SU(2)$ in π/E. On the other side, in Section 2.6 it is proved that the time needed to cover the whole sphere $\mathbf{S}_B = SU(2)/S^1$ goes to $\pi/(4E)$ as M goes to infinity (however this does not contradict the fact that the state two can be reached in an arbitrary small time, as we discussed above).

Notice that our optimal control has the same "qualitative form" of the control computed in [36] i.e. a pulse (bang) followed by an evolution with the drift (singular) followed by a pulse (bang).

2.4.4 Some possible extensions

It is very easy to see that if $\{u_{\bar{y}}\}_{\bar{y}\in\mathbf{S}_B}$ is the collection of all time optimal controls steering the north pole to all the points of $\mathbf{S}_B$, then the same set is also the collection of all time optimal controls starting from the south pole.

Notice that nothing is changing if the controlled magnetic field is in any direction in the x-y plane. If this is not the case, the problem is different. However the same techniques of this chapter could be used to deal with this case, but the solution is probably more complicated.

Another interesting problem could be the variant of **(P)** in which one consider a different initial condition. In this case, generically, one loses the local controllability property (i.e. for small time, the trajectories do not cover a neighborhood of the starting point), but the structure of extremal trajectories (i.e. trajectories satisfying the Pontryagin Maximum Principle) is very similar.

2.5 Proof of the main results

In this section we give the proof of our main results. We start with a lemma, stating a property of optimal trajectories, that is a consequence of the symmetries of the problem. It is used to identify the time optimal trajectories steering the north to the south pole both for $\alpha \geq \pi/4$ and $\alpha < \pi/4$.

Lemma 2.1 *Let $\alpha \in (0, \pi/2)$. Every optimal bang-bang trajectory, connecting the north to the south pole, with more than one switching is such that $v(s_i) = v(s_f)$ where s_i is the first switching time, s_f is the time needed to steer the last switching point to the south pole and $v(s_i)$ is the time between two consecutive switchings.*

Proof of Lemma 2.1. Consider the problem of connecting P_S with P_N in minimum time for the system $\dot{z} = F'_S(z) + uG'_S(z)$ where $z \in S^2$ and $F'_S(z) = -F_S(z)$, $G'_S(z) = -G_S(z)$. The trajectories of this system coincide with those of the system (2.10)–(2.13), but the velocity is reversed. Therefore the optimal trajectories for the new problem coincide with the optimal ones for the system (2.10)–(2.13) connecting P_N to P_S, and the time between two switchings is the same. Since performing the change of coordinates $(z_1, z_2, z_3) \to (y_1, y_2, y_3) = (-z_1, z_2, -z_3)$, the new problem becomes exactly the original problem, we deduce that, if we have more than one switching, it must be $v(s_i) = v(s_f)$. ■

2.5.1 Time optimal synthesis for the two level quantum system for $\alpha \geq \pi/4$

In this section, we apply the theory of optimal syntheses on 2-D manifolds to the system (2.10)–(2.13). Our aim is to describe the time optimal synthesis for $\alpha \geq \pi/4$, i.e. to prove Theorem 2.1 and Propositions 2.2 and 2.3. First we state some general results, holding for $\alpha \in (0, \pi/2)$, regarding time optimal trajectories of the system (2.14), on $S^3 \sim SU(2)$, analogous to those obtained in [19] for $SO(3)$ (in particular the proofs can be repeated using the same arguments).

2.5.1.1 General results on S^3

In this section $\alpha \in (0, \pi/2)$. The first proposition states that singular extremals, defined as extremals for which the switching function vanishes (see Definitions 1.1 and 1.2) correspond to zero control. This fact is very specific for our problem.

Proposition 2.9 *For the normalized minimum time problem on S^3 (2.14), singular extremals are integral curves of the drift, i.e. they must correspond to a control almost everywhere vanishing.*

Since for a fixed $u \in [-1, 1]$ every trajectory of (2.14) is periodic with period $\frac{4\pi}{\sqrt{u^2 \sin^2 \alpha + \cos^2 \alpha}}$ we have that:

Proposition 2.10 *Given an extremal trajectory γ of type B_t (resp S_t), then $t < 4\pi$ (resp. $t < \frac{4\pi}{\cos \alpha}$).*

The following proposition describes the switching behavior of abnormal and bang-bang normal extremals (see Section 1.1 for the definition).

Proposition 2.11 *Let γ be an abnormal extremal of (2.14). Then it is bang-bang and the time duration between two consecutive switchings is always equal to π. In other words, γ is of kind $B_s B_\pi ... B_\pi B_t$ with $s, t \leq \pi$.*
On the other hand, if γ is a bang-bang normal extremal, then the time duration $\mathcal{T}$ along an interior bang arc is the same for all interior bang arcs and verifies $\pi < \mathcal{T} < 2\pi$ (i.e. γ is of kind $B_s B_{\mathcal{T}} ... B_{\mathcal{T}} B_t$ with $s, t \leq \mathcal{T}$).

For the optimal trajectories containing a singular arc we have the following:

Proposition 2.12 *Let γ be a time optimal trajectory containing a singular arc. Then γ is of the type $B_t S_s B_{t'}$, with $s \leq \frac{2\pi}{\cos \alpha}$ if $t > 0$ or $t' > 0$ and $s < \frac{4\pi}{\cos \alpha}$ otherwise.*

These results on $S^3 \sim SU(2)$ are useful to determine the optimal synthesis on $\mathbf{S}_B$, since every optimal trajectory on $\mathbf{S}_B$ is the projection of an optimal trajectory on S^3. This is a simple consequence of the fact that $\mathbf{S}_B$ is an homogeneous space of $SU(2)$:

Proposition 2.13 *A time optimal trajectory γ for the system (2.10)–(2.13) on $\mathbf{S}_B$ starting at P_N is the projection of a time optimal trajectory of (2.14) starting from a point satisfying $|\psi_1|^2 = 1$ (recall that $\psi = (\psi_1, \psi_2)^T \in S^3 \subset \mathbb{C}^2$).*

Remark **2.13** Notice that, since two opposite points on S^3 project on the same point on $\mathbf{S}_B$, it is easy to see from Proposition 2.10, that the projection on $\mathbf{S}_B$ of an optimal trajectory of (2.14) of type B_t (resp S_t), must be such that $t < 2\pi$ (resp. $t < \frac{2\pi}{\cos\alpha}$). More precisely, for a fixed $u \in [-1, 1]$ every trajectory of (2.10)–(2.13) is periodic with period $\frac{2\pi}{\sqrt{u^2\sin^2\alpha+\cos^2\alpha}}$ (the period divides by two after projection).

2.5.1.2 Construction of the synthesis on $\mathbf{S}_B$

In this section we assume $\alpha \geq \pi/4$. We first need to determine the sets $\Delta_A^{-1}(0)$, $\Delta_B^{-1}(0)$, and the function f_S. Checking where F_S is parallel to G_S and where G_S is parallel to $[F_S, G_S]$, one gets $\Delta_A^{-1}(0) = \{y \in \mathbf{S}_B : y_2 = 0\}$ and $\Delta_B^{-1}(0) = \{y \in \mathbf{S}_B : y_3 = 0\}$. To find the function f_S we can choose for instance the coordinate chart defined on each hemisphere by the projection on the plain $\{(y_1, y_2) \in \mathbb{R}^2\}$, obtaining $f_S = (\sin\alpha)y_3/y_2$. Then Lemma 1.1 says that every optimal trajectory belonging to one of the regions $\{y \in \mathbf{S}_B : y_3 > 0, y_2 > 0\}$, $\{y \in \mathbf{S}_B : y_3 < 0, y_2 < 0\}$ is bang-bang with at most one switching. Moreover only the switching from control -1 to control $+1$ is allowed. On the contrary, on the regions $\{y \in \mathbf{S}_B : y_3 > 0, y_2 < 0\}$, $\{y \in \mathbf{S}_B : y_3 < 0, y_2 > 0\}$, the control can switch only from $+1$ to -1. Moreover, thanks to Lemma 1.2, every singular extremal must lie on the equator. The following lemma characterizes the structure of the bang-bang extremals for the problem **(P)**.

Lemma 2.2 *Recall that $t_1 = \pi - \arccos(\cot^2\alpha)$ and $t_3 = \pi + \arccos(\cot^2\alpha)$ and consider a bang-bang extremal for the problem* **(P)***. Then it is of the form $B_sB_{v(s)}B_{v(s)}\ldots$ with $s \in [0, t_1] \cup [\pi, t_3]$, where, on the set $[0, t_1) \cup [\pi, t_3)$, $v(\cdot)$ is defined as follows:*

$$v(s) := \pi + 2\arctan\left(\frac{\sin s}{\cos s + \cot^2\alpha}\right).$$

If $\alpha = \pi/4$ then $t_1 = t_3 = \pi$ and $v(\pi) := \pi$, while if $\alpha > \pi/4$ we set $v(t_1) := v(t_3) := 2\pi$.

Notice that the function $v(\cdot)$ has the same expression (2.17) obtained in the case $\alpha < \pi/4$ (excepted at the points t_1 and t_3). However its interval of definition is different.

Proof of Lemma 2.2. As shown above, the meridian $\Delta_A^{-1}(0)$ and the equator $\Delta_B^{-1}(0)$ divide the sphere in four parts and in each of them the sign of the function f_S is constant and changes when passing through $\Delta_A^{-1}(0)$ or $\Delta_B^{-1}(0)$. In particular, following γ^+ or γ^- (cf. Remark 2.4) in the case in which $\alpha > \pi/4$ this happens at the times t_1 (where the equator is crossed), at time π (where $\Delta_A^{-1}(0)$ is crossed) and at time t_3 (again is the equator to be crossed). Applying Lemma 1.1, we obtain that for an extremal trajectory the first switching may occur only on the intervals $[0, t_1]$ and $[\pi, t_3]$. Exactly as in [19], one shows that the extremal must have the form $B_sB_{v(s)}B_{v(s)}\ldots$ with $s \in [0, t_1] \cup [\pi, t_3]$. The case $\alpha = \pi/4$ is similar. ■

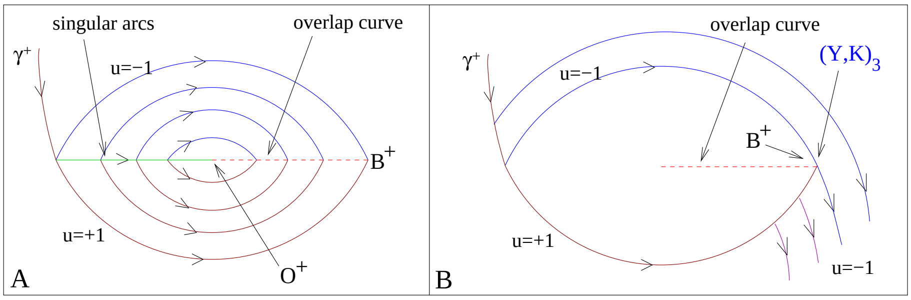

Figure 2.8: The region covered by optimal trajectories with singular arcs and the $(Y, K)_3$ frame point

Remark **2.14** One can also show that every trajectory starting from P_N, of the form $B_s B_{v(s)} B_{v(s)} \dots$ with $s \in [0, t_1] \cup [\pi, t_3]$ is extremal i.e., for every s in such set, there exists an initial value of the covector λ such that the switching function $\phi(\cdot)$ vanishes for the first time at time s.

Unlike the case in which $\alpha < \pi/4$, in the case $\alpha > \pi/4$ it is possible to establish the presence of optimal trajectories containing a singular arc, whose switching behavior is described by the following proposition, illustrated in Figure 2.8 A.

Proposition 2.14 *Let $\alpha \geq \pi/4$. A trajectory γ of (2.10)–(2.13) starting with control $u = 1$ and containing a singular arc is a solution of* **(P)** *if and only if it is of the form $B_t S_s B_{t'}$ and satisfies the following conditions:*

- $t = t_1 = \pi - \arccos(\cot^2 \alpha)$ *i.e. γ coincides with γ^+ until it reaches the equator.*
- $s \leq \arccos(\cot \alpha)/\cos \alpha$ *i.e. the singular arc is optimal until it reaches the point* $O^+ = (1, 0, 0)^T$.
- *If $s = \arccos(\cot \alpha)/\cos \alpha$, then the trajectory is of type $B_t S_s$ (i.e. the time duration of the last bang arc reduces to zero). If $s < \arccos(\cot \alpha)/\cos \alpha$, then γ is optimal until the last bang arc reaches the equator (i.e. it does not exist $\bar{t} \in (0, t')$ such that $\gamma(t + s + \bar{t})$ is contained in the equator).*

An analogous result holds for trajectories starting with control -1.

Remark **2.15** Notice that in the case $\alpha = \pi/4$, Proposition 2.14 provides a singular trajectory degenerated to a point. In other words for $\alpha = \pi/4$ there are no singular trajectories that are optimal.

Remark **2.16** Notice that the previous result completely characterizes the optimal synthesis in some neighborhoods of the points $O^{\pm} = (\pm 1, 0, 0)^T$, namely $\Omega_2^{\pm} \cup \Omega_3^{\pm}$, and moreover it determines the presence of two symmetric overlap curves contained inside the equator. The synthesis around the point O^+ is represented in Figure 2.8 A.

Proof of Proposition 2.14. Consider a trajectory, solution of **(P)**, starting with $u = +1$ and containing a singular arc. Using Propositions 2.12 and 2.13 this trajectory must be of the form $B_t S_s B_{t'}$ and, since the singular arc is contained inside the equator, we have $t = t_1$ (the case $t = t_3$ can be easily excluded). Consider a singular arc containing in its interior the point O^+. This arc contains two points of the form $(y_1^0, -y_2^0, 0)^T$ and $(y_1^0, y_2^0, 0)^T$, with both y_1^0, y_2^0 positive, that can be connected by a bang arc. Using classical comparison theorems for second order ODEs, one can easily compare the time needed to follow such trajectory with the time needed to steer the two points along the singular arc finding that the bang arc is quicker than the singular arc. Therefore a singular arc containing O^+ cannot be optimal. By symmetry, the extremal trajectories that have the same singular arc, but the last bang arc corresponding to opposite control, must meet on a point of the equator. Therefore the arc of the equator which is comprised between the point O^+ (resp O^-) and the second intersection point with γ^+ (resp. γ^-) is an overlap curve. It remains now to verify that the trajectories described above are optimal (until the last bang arc reaches the equator). This is a straightforward consequence of the fact that the quickest bang-bang trajectories that enter the region spanned by such trajectories (i.e. the closure of the regions $\Omega_2^{\pm} \cup \Omega_3^{\pm}$) are not extremal because of Lemma 1.1 (see also Lemma 2.2). ■

Remark **2.17** Notice the trivial fact that, if a trajectory γ defined on the interval $[a, b]$ is optimal between $\gamma(a)$ and $\gamma(b)$, then the restriction of γ in $[c, d]$, $c, d \in [a, b]$, $c < d$, is optimal between $\gamma(c)$ and $\gamma(d)$.

Using Remark 2.17, we have that Proposition 2.14 characterizes completely the time optimal synthesis on $\overline{P_N A^{\pm}}$ and in the closure of $\Omega_2^{\pm} \cup \Omega_3^{\pm}$, i.e. it proves items **T1**–**T6**, **T9** and **T10**, of Theorem 2.1.

Remark **2.18** From Lemma 2.2 we obtain that there are four families of bang-bang trajectories. In particular the families starting with control $+1$ and switching respectively in $[0, t_1]$ and $[\pi, t_3]$ join at the point B^+, generating an amazing $(Y, K)_3$ frame point, in the framework of the classification of [22]. See Figure 2.8 B.

Next we give the proof of Proposition 2.2, from which it follows **T11** of Theorem 2.1, and, using again Remark 2.17, also **T7**.

Proof of Proposition 2.2 By Proposition 2.14, there are no optimal trajectories containing a singular arc joining P_N with P_S. One can easily see that the only possible trajectories steering P_N to P_S with only one switching are those described in the statement of the proposition, that we have to compare with trajectories having more than one switching. Trajectories having two switchings with the first or the last bang longer than π and trajectories with more than two switchings are excluded since from Lemma 2.2 their total time is larger than 2π. Trajectories having two switchings and length of the first arc s_i and the length of the last arc s_f satisfying $s_i, s_f < \pi$ are excluded since by Lemma 2.1 they must satisfy $s_i = s_f$. For these trajectories the total time can be easily computed and it is $2\pi + 2\arcsin\left(\frac{1}{2\sin(\alpha)}\right) > 2\pi$. ■

Item **T8** is proved by the following:

Proposition 2.15 *If* $\bar{y} \in \Omega_1^+ \cup (\overline{DP_S^-} \setminus P_S)$*, then* $\boldsymbol{\gamma}_{\bar{y}}$ *is made by a unique trajectory of the form* $B_t B_{t'}$*, with* $0 \leq t < t_1$ *and the first bang corresponding to control* $+1$*. A similar result holds if* $\bar{y} \in \Omega_1^- \cup (\overline{DP_S^+} \setminus P_S)$*. As a consequence there is not a cut locus in the region* $\Omega_1^+ \cup \Omega_1^-$*. On the other hand* $\Omega_{nasty}^+ \cup \Omega_{nasty}^-$ *contains a cut locus.*

Proof of Proposition 2.15 Define the following three families of extremal trajectories:

$$
\begin{aligned}
\gamma_s^A(t) &:= e^{tX_S^+} e^{sX_S^-} P_N, \quad \text{with} \quad s \in (0, t_1) \quad \text{and} \quad t \leq v(s), \\
\gamma_s^B(t) &:= e^{tX_S^-} e^{sX_S^+} P_N, \quad \text{with} \quad s \in [\pi, t_3) \quad \text{and} \quad t \leq v(s), \\
\gamma_s^C(t) &:= e^{tX_S^-} e^{v(s)X_S^+} e^{sX_S^-} P_N, \quad \text{with} \quad s \in (0, t_1) \quad \text{and} \quad t \leq v(s).
\end{aligned}
$$

First notice that from Proposition 2.2, there are no optimal trajectories of kind γ_s^A reaching the arc $\overline{BP_S^+}$. Now for every point $x \in \overline{DP_S^+}$ the following happens: **i)** there exist $s_A,\ t_A$ such that $x = \gamma_{s_A}^A(t_A)$, and they are unique; **ii)** if there exist $s_B,\ t_B$ (resp. $s_C,\ t_C$) such that $x = \gamma_{s_B}^B(t_B)$, (resp. $x = \gamma_{s_C}^C(t_C)$), then they are unique. By direct computation, one can compare the times the three trajectories need to reach x, i.e. $s_A + t_A, s_B + t_B, s_C + v(s_C) + t_C$, finding that the optimal trajectory is of kind γ^A (these computations are long, not very instructive, and we omit them). From this fact the first part of the claim immediately follows. Moreover it implies that there is not a cut locus in Ω_1^+, since the only trajectories entering such region are those of the form γ^A. The existence of a cut locus in Ω_{nasty}^+ is evident, since no optimal trajectories belonging to the families γ^A, γ^B, γ^C leave Ω_{nasty}^+. The reasoning in Ω_1^- and in Ω_{nasty}^- is similar. ■

End of the proof of Theorem 2.1
To conclude the proof of Theorem 2.1, it remains to prove **T12**. Consider by contradiction an optimal bang-bang trajectory γ defined in $[0, t_\gamma]$ steering P_N to a point of Ω_{nasty}^+, with at least three switchings. Define $\bar{t} = \max\{t \in [0, t_\gamma] : \gamma(t) \notin \Omega_{\text{nasty}}^+\}$. Then, by Remark 2.17, $\gamma|_{[0,\bar{t}]}$ must be optimal between P_N and $\gamma(\bar{t})$. Then, from the results proved above, we deduce that $\gamma|_{[0,\bar{t}]}$ can have at most one switching. Therefore γ switches at least two times in Ω_{nasty}^+ and the arc between them must be completely contained in Ω_{nasty}^+ and this leads to a contradiction since the sign of f_S is constant in Ω_{nasty}^+ (see Lemma 1.1). ■

Before proving Proposition 2.3, notice that the point D^+, which is obtained following the trajectory γ^+ for a time π (see Figure 2.5), belongs to two different families of bang-bang trajectories at time π, one given by trajectories starting with control -1 and switching at time $s \leq t_1$, the other one given by trajectories that start with control 1 and switching at time $s \in [\pi, t_3]$. Moreover, since $v(0) = \pi$, there must be a switching curve starting at D^+ and therefore we deduce that there are two possible behaviors of the optimal synthesis around this point: either this switching curve is optimal or the two fronts continue to intersect generating an overlap curve.
Observe that if $\alpha \geq \pi/3$ the trajectories of the type $B_s B_{v(s)} B_t$ with s small cannot be optimal since the vector fields X_S^+ and X_S^- point to opposite sides on the switching curve (i.e. the

switching curve "reflects the trajectories", see footnote 1.8). In this case the two families of bang-bang trajectories described above must intersect giving rise to an overlap curve. Therefore to prove Proposition 2.3 we assume $\alpha < \pi/3$.

Proof of Proposition 2.3 First we parameterize the switching curve with respect to the first switching time (assuming without loss of generality that this curve starts with $u = -1$):

$$C(s) = e^{v(s)X_S^+} e^{sX_S^-} P_N \,.$$

We consider the functions $\xi_1(s) = \det\left(C(s), C'(s), X_S^+(C(s))\right)$ (here the superscript $'$ denotes the derivative with respect to s) and $\xi_2(s) = \det\left(C(s), C'(s), X_S^-(C(s))\right)$. It is easy to see that the optimality of $C(\cdot)$, for s small, depends on the signs of such functions. Indeed $C(\cdot)$ is locally optimal near the point $D^+ = C(0)$ if and only if for every positive and small enough s, and given a neighborhood of $C(s)$ which is divided in two connected components U_1, U_2 by the trajectory $C(\cdot)$, both $X_S^-(C(s))$ and $X_S^+(C(s))$ point towards U_1 or towards U_2. It is easy to see that this happens if $\xi_1(s)$ and $\xi_2(s)$ have the same sign. Notice that $\xi_1(0) = \xi_2(0) = 0$ and that $\xi_1(s) = \det\left(P_N, X_S^-(P_N), e^{-sX_S^-} X_S^+(e^{sX_S^-} P_N)\right) = 2\cos\alpha\sin^2\alpha\sin s$, which is positive for every $\alpha < \pi/2$ and $s \in (0, \pi)$. To determine the sign of $\xi_2(s)$ near 0 it is enough to look at the sign of the derivative $\xi_2'(0)$ which can be computed directly: $\xi_2'(0) = 4\cos\alpha\sin^2\alpha(1 - 2\sin^4\alpha)$. We deduce that, if $\alpha < \arcsin(1/\sqrt[4]{2})$, the switching curve $C(\cdot)$ is optimal for s small enough. For the particular value $\alpha = \arcsin(1/\sqrt[4]{2})$ one can easily check that the function $\xi_2(\cdot)$ is negative for $s > 0$ small, and then $C(\cdot)$ is no more optimal for $\alpha \geq \arcsin(1/\sqrt[4]{2})$. The tangency of the switching curve starting at D^+ if $\alpha > \pi/4$, is a consequence of the fact that, in this case, the bang-bang trajectory switching at D^+ is an abnormal extremal (see Proposition 1.5 and [22], Proposition 23 pp. 177). ■

2.5.2 Time optimal trajectories that reach the south pole for $\alpha < \pi/4$

The purpose of this section is to characterize the optimal trajectories steering P_N to P_S in the case $\alpha < \pi/4$, i.e. to prove Proposition 2.4 and 2.5. A key tool is Lemma 2.1. Recall the shape of the function $v(s)$, in the case $\alpha < \pi/4$ (see Figure 2.6 A). Given $\alpha < \pi/4$ and $s \in [0, \pi]$ with $s \neq \arccos(-\tan^2\alpha)$, there exists one and only one time $s^*(s) \in [0, \pi]$ different from s, such that $v(s) = v(s^*(s))$. From Section 2.4.2 recall the following definition of candidate optimal trajectories:

- $s_f = s^*(s_i)$ (i.e. TYPE-1-candidate optimal trajectories),
- $s_f = s_i$ (i.e. TYPE-2-candidate optimal trajectories)

A useful relation between s and $s^*(s)$ is given by the following:

Lemma 2.3 *For $\alpha < \pi/4$ and $s \in [0, \pi]$, it holds $s + s^*(s) = v(s)$.*

Proof of Lemma 2.3 Both s and $s^*(s)$ satisfy the following equation in $t \in [0, \pi]$:

$$\cot\left(\frac{1}{2}v(s)\right) = -\frac{\sin(t)}{\cos(t) + \cot^2(\alpha)} \Rightarrow \quad \cos\left(\frac{1}{2}v(s) - t\right) = -\cos\left(\frac{1}{2}v(s)\right)\cot^2(\alpha).$$

Therefore, since $\frac{1}{2}v(s) - t \in [-\pi, \pi] \quad \forall s, t \in [0, \pi]$ and $s^*(s) \neq s$, it must be: $s^*(s) - \frac{1}{2}v(s) = \frac{1}{2}v(s) - s \quad \Rightarrow \quad s + s^*(s) = v(s)$. ■

The description of candidate optimal trajectories is simplified by the following lemma, of which we skip the proof.

Lemma 2.4 *Set:*

$$Z(s) = \frac{1}{\rho}\begin{pmatrix} 0 & \cot\left(\frac{1}{2}v(s)\right) & -\sin(\alpha) \\ -\cot\left(\frac{1}{2}v(s)\right) & 0 & 0 \\ \sin(\alpha) & 0 & 0 \end{pmatrix}$$

where $\rho = \sqrt{\cot^2\left(\frac{1}{2}v(s)\right) + \sin^2(\alpha)}$ *. Then, if* $\theta(s)$ *is defined as in (2.20), we have* $e^{\theta(s)Z(s)} = e^{v(s)X_S^-} e^{v(s)X_S^+}$*.*

Notice that the matrix $Z(s) \in so(3)$ is normalized in such a way that the map $t \mapsto e^{tZ(s)} \in SO(3)$ represents a rotation around the axes $R(s) = \left(0, \sin(\alpha), \cot(\frac{1}{2}v(s))\right)^T$ with angular velocity equal to one.

To prove the results stated in Section 2.4.2 we study separately the two possible cases listed above:

Proof of Proposition 2.4. In this case we consider TYPE-1-candidates optimal trajectories. Assume that the optimal trajectory starts with $u = -1$ (the case $u = 1$ is symmetric) and has an even number n of switchings. Then it must be

$$P_S = e^{s_f X_S^-} \underbrace{e^{v(s_i)X_S^+} \cdots \cdots e^{v(s_i)X_S^+}}_{n-1 \text{ times}} e^{s_i X_S^-} P_N \tag{2.24}$$

where P_N and P_S denote respectively the north and the south pole, and we have that

$$e^{s_i X_S^-} P_S = e^{v(s_i)X_S^-} e^{v(s_i)X_S^+} \cdots \cdots e^{v(s_i)X_S^+} e^{s_i X_S^-} P_N = e^{\frac{1}{2}n\theta(s_i)Z(s_i)} e^{s_i X_S^-} P_N$$

from which we deduce that s_i must satisfy

$$\frac{1}{2}n\theta(s_i) = \pi + 2p\pi \quad \text{for some integer } p.$$

It is easy to see that a value of s_i which satisfies previous equation with $p > 0$ doesn't give rise to a candidate optimal trajectory since the corresponding number of switchings is larger than N_M. Therefore in previous equation it must be $p = 0$. If n is odd, instead than (2.24) we have

$$P_S = e^{s_f X_S^+} \underbrace{e^{v(s_i)X_S^-} \cdots \cdots e^{v(s_i)X_S^+}}_{n-1 \text{ times}} e^{s_i X_S^-} P_N \tag{2.25}$$

and, moreover, by symmetry:

$$P_N = e^{s_f X_S^-} e^{v(s_i)X_S^+} \cdots \cdots e^{v(s_i)X_S^-} e^{s_i X_S^+} P_S.$$

Then, combining with (2.25) and using the relation Lemma 2.3, we find:

$$P_N = e^{-s_i X_S^-} \underbrace{e^{v(s_i) X_S^-} \ldots \ldots e^{v(s_i) X_S^+}}_{2n\ times} e^{s_i X_S^-} P_N = e^{-s_i X_S^-} e^{n\theta(s_i) Z(s_i)} e^{s_i X_S^-} P_N.$$

Since $e^{s_i X_S^-} P_N$ is orthogonal to the rotation axis $R(s_i)$ corresponding to $Z(s_i)$, previous identity is satisfied if and only if $n\theta(s_i) = 2m\pi$ with m positive integer. As in the previous case, for a candidate optimal trajectory, it must be $m = 1$. ■

Proof of Proposition 2.5. Here we consider TYPE-2-candidate optimal trajectories. For simplicity call $s_i = s_f = s$. Assume, as before, that the optimal trajectory starts with $u = -1$. If this trajectory has $n = 2q + 1$ switchings then it must be

$$P_S = e^{s X_S^+} e^{q\theta(s) Z(s)} e^{s X_S^-} P_N.$$

In particular the points $e^{-s X_S^+} P_S$ and $e^{s X_S^-} P_N$ must belong to a plane invariant with respect to rotations generated by $Z(s)$ and therefore the difference $e^{s X_S^-} P_N - e^{-s X_S^+} P_S$ must be orthogonal to the rotation axis $R(s)$. Actually it is easy to see that this is true for every value $s \in [0, \pi]$, since both $e^{-s X_S^+} P_S$ and $e^{s X_S^-} P_N$ are orthogonal to $R(s)$. Since the integral curve of $Z(s)$ passing through $e^{s X_S^-} P_N$ and $e^{-s X_S^+} P_S$ is a circle of radius 1, it is easy to compute the angle $\beta(s)$ between these points. In particular the distance between $e^{s X_S^-} P_N$ and $e^{-s X_S^+} P_S$ coincides with $2\sin(\frac{\beta(s)}{2})$, and so one easily gets the expression $\beta(s) = 2\arccos(\sin(\alpha)\cos(\alpha)(1 - \cos(s)))$. Then Proposition 2.5 is proved when n is odd.

Assume now that the optimal trajectory has $n = 2q + 2$ switchings, then we can assume without loss of generality that $P_S = e^{s X_S^-} e^{v(s) X_S^+} e^{q\theta(s) Z(s)} e^{s X_S^-} P_N$. First of all it is possible to see that $e^{-v(s) X_S^+} e^{-s X_S^-} P_S$ is orthogonal to $R(s)$. So it remains to compute the angle $\tilde{\beta}(s)$ between the point $e^{s X_S^-} P_N$ and the point $e^{-v(s) X_S^+} e^{-s X_S^-} P_S$ on the plane orthogonal to $R(s)$. As before the distance between these points coincides with $2\sin(\frac{\tilde{\beta}(s)}{2})$. Instead of computing directly $\tilde{\beta}(s)$, we compute the difference between the angle $\tilde{\beta}(s)$ and the angle $\beta(s)$ defined above. We know that

$$\begin{aligned} 2\sin(\frac{\beta(s)}{2} - \tilde{\beta}(s)) &= |e^{-v(s) X_S^+} e^{-s X_S^-} P_S - e^{-s X_S^+} P_S| \\ &= |e^{-s X_S^-} P_S - e^{v(s) X_S^+} e^{-s X_S^+} P_S| \\ &= |e^{-s X_S^-} P_S - e^{s^*(s) X_S^+} P_S|. \end{aligned}$$

Using the fact that s and $s^*(s)$ satisfy the relation $v(s) = v(s^*(s))$ one can easily find that

$$|e^{-s X_S^-} P_S - e^{s^*(s) X_S^+} P_S| = 2\sqrt{1 - \cos^2(\alpha)\sin^2\left(\frac{1}{2}v(s)\right)}.$$

Therefore $\beta(s) = \tilde{\beta}(s) + 2\arccos\left(\cos(\alpha)\sin\left(\frac{1}{2}v(s)\right)\right)$. This leads to $\beta(s) - \tilde{\beta}(s) = \theta(s)/2$ and the proposition is proved also in the case n is even. ■

2.5.2.1 Proof of Proposition 2.6, on the alternating behavior of the optimal synthesis

In this section we need to consider also the dependence on α of the functions $v(s), \theta(s), \beta(s), \mathcal{F}(s), \mathcal{G}(s)$. Therefore we switch to the notation $v(s,\alpha), \theta(s,\alpha), \beta(s,\alpha), \mathcal{F}(s,\alpha), \mathcal{G}(s,\alpha)$.

The claims on existence of solutions of Proposition 2.6 come from the fact that $\mathcal{F}(0) = \mathcal{F}(\pi) = \frac{\pi}{2\alpha}$ and the only minimum point of $\mathcal{F}$ occurs at $\bar{s} = \pi - \arccos(\tan^2(\alpha))$. It turns out that the image of $\mathcal{F}$ is a small interval whose length is of order α and therefore equation (2.22) has a solution only if α is close enough to $\frac{\pi}{2m}$ for some integer number m. This proves **C.** with $r_2(m)$ satisfying $r_2(m) = O(1/m)$.

On the other hand it is possible to estimate the derivative of $\mathcal{G}$ with respect to s showing that it is negative in the open interval $(0,\pi)$. Therefore, since $\mathcal{G}(0) = \frac{\pi}{2\alpha} + 1$ and $\mathcal{G}(\pi) = \frac{\pi}{2\alpha} - 1$, equation (2.23) has always two positive solutions.

For the particular values $\alpha = \frac{\pi}{2m}$, where $m > 1$ is an integer number, the solutions to the equations (2.22) and (2.23) give rise to two candidate optimal trajectories: the first one has exactly m bang arcs, all of length π (TYPE-1 and TYPE-2 candidate optimal trajectory at the same time), while the second one has one more switching and is a TYPE-2 candidate optimal trajectory. We want to see that the optimal trajectory is the first one. For this purpose, we need to estimate the time needed to reach the south pole by the second candidate optimal trajectory showing that it is greater than $m\pi = \frac{\pi^2}{2\alpha}$.

First, using the Taylor expansions with respect to α and centered at 0 of $\beta(\pi/2,\alpha)$ and $\theta(\pi/2,\alpha)$, one obtains

$$\mathcal{G}\Big(\frac{\pi}{2},\alpha\Big) = \frac{\pi}{2\alpha} - \alpha\frac{\pi}{4} + o(\alpha). \tag{2.26}$$

We want now to estimate the solution $s(\alpha)$ of the equation $\mathcal{G}(s,\alpha) = \frac{\pi}{2\alpha}$. This can be done using (2.26) and the following estimate on the derivative of $\mathcal{G}(\cdot)$, with respect to s, near $s = \pi/2$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{G}(s,\alpha) = -1 + o\left(|\alpha| + \left|\frac{\pi}{2} - s\right|\right).$$

Then it is easy to find that $s(\alpha) = \dfrac{\pi}{2} - \alpha\dfrac{\pi}{4} + o(\alpha)$, and, consequently, $v(s(\alpha),\alpha) = \pi + 2\alpha^2 + o(\alpha^2)$. Therefore $2s(\alpha) + \Big(\dfrac{\pi}{2\alpha} - 1\Big)\, v(s(\alpha),\alpha) = \dfrac{\pi^2}{2\alpha} + \alpha\dfrac{\pi}{2} + o(\alpha)$. In particular, for $\alpha = \frac{\pi}{2m}$ this expression gives the time needed to reach the south pole by the candidate optimal trajectory and, since for m large enough it is larger than $m\pi = \frac{\pi^2}{2\alpha}$, we conclude that this trajectory cannot be optimal. Since the solutions to the equations (2.22),(2.23) change continuously with respect to α for each fixed number of switchings n, we easily deduce that, if we slightly decrease α starting from the value $\frac{\pi}{2m}$, the solution of (2.22) for $n = m$ does not give rise to an optimal trajectory.

For α slightly smaller than $\bar{\alpha} := \frac{\pi}{2m}$ there is a TYPE-2 candidate optimal trajectory corresponding to a solution $(s_1(\alpha), m+1)$ of (2.23), where $s_1(\cdot)$ is continuous (on $[\bar{\alpha} - \varepsilon, \bar{\alpha}]$) and $s_1(\bar{\alpha}) = 0$, and there is also a TYPE-1 candidate optimal trajectory corresponding to a solution $(s_2(\alpha), m)$ of (2.22) where $s_2(\cdot)$ is continuous (on $[\bar{\alpha} - \varepsilon, \bar{\alpha}]$) and $s_2(\bar{\alpha}) = 0$. Clearly for $\alpha = \bar{\alpha}$ these trajectories coincide. So we have to compare the time to reach the south pole for such trajectories with α close to $\bar{\alpha}$.

We start with the TYPE-1 candidate optimal trajectory. From equation (2.22) we have that $\frac{\mathrm{d}}{\mathrm{d}\alpha}\theta(s_2(\alpha),\alpha) = 0$. We use a subscript s, α to denote the partial differentiation with respect to such variables. Since $\theta_s(0,\alpha) = 0$ we cannot apply directly the implicit function theorem near $(0,\bar{\alpha})$. However, if we set $\tilde{s}_2(\alpha) = s_2^2(\alpha)$ we find that $\tilde{s}_2'(\alpha) = \frac{2s_2(\alpha)\theta_\alpha(s_2(\alpha),\alpha)}{\theta_s(s_2(\alpha),\alpha)}$ (the superscript $'$ denotes differentiation with respect to α), and then, passing to the limit as $(s_2(\alpha),\alpha)$ tends to $(0,\bar{\alpha})$, one easily finds that $\tilde{s}_2'(\bar{\alpha}) = -\frac{2}{\sin(\bar{\alpha})^3\cos(\bar{\alpha})}$.

Now we want to determine the way in which the total time $T_2(\alpha) = mv(s_2(\alpha),\alpha)$ changes. It is easy to see that $T_2(\alpha)$ is not differentiable at $\bar{\alpha}$, therefore we introduce the function $F(\alpha) = (T_2(\alpha) - T_2(\bar{\alpha}))^2 = m^2(v(s_2(\alpha),\alpha) - \pi)^2$.

Then

$$\begin{aligned} F'(\alpha) &= 2m^2 \frac{\mathrm{d}}{\mathrm{d}\alpha} v\big(s_2(\alpha),\alpha\big)\big(v\big(s_2(\alpha),\alpha\big) - \pi\big) \\ &= 2m^2 \Big(v_s\big(s_2(\alpha),\alpha\big)s_2'(\alpha) + v_\alpha\big(s_2(\alpha),\alpha\big)\Big) \end{aligned}$$

and, after the substitution $s_2'(\alpha) = \frac{\tilde{s}_2'(\alpha)}{2s_2(\alpha)}$ we can pass to the limit as α converges to $\bar{\alpha}$ obtaining

$$F'(\bar{\alpha}) = m^2 v_s^2(0,\bar{\alpha})\tilde{s}_2'(\bar{\alpha}) = -8m^2\tan\bar{\alpha}\,.$$

Now we consider the TYPE-2 candidate optimal trajectory and we want to estimate $s_1(\alpha)$. From equation (2.23) we have that $s_1(\cdot)$ is implicitly defined by the equation $\Phi(s_1(\alpha),\alpha) := 2\beta(s_1(\alpha),\alpha) - m\theta(s_1(\alpha),\alpha) = 0$. As before it is easy to see that $s_1(\cdot)$ is not differentiable at $\bar{\alpha}$ and therefore we introduce the parameter $\tilde{s}_1(\alpha) = s_1^2(\alpha)$. As before, it is possible to compute the derivative $\tilde{s}_1'(\alpha)$:

$$\tilde{s}_1'(\bar{\alpha}) = -\lim_{\alpha\to\bar{\alpha}} \frac{2s_1(\alpha)\Phi_\alpha(s_1(\alpha),\alpha)}{\Phi_s(s_1(\alpha),\alpha)} = -\frac{2m}{\sin\bar{\alpha}\cos\bar{\alpha}(1+m\sin^2\bar{\alpha})}\,.$$

We have now to estimate the total time $T_1(\alpha) = 2s_1(\alpha) + mv\big(s_1(\alpha),\alpha\big)$ for α close to $\bar{\alpha}$. After defining

$$\Lambda(\alpha) = T_1(\alpha) - T_1(\bar{\alpha}) = 2s_1(\alpha) + m\big(v(s_1(\alpha),\alpha) - \pi\big)\,, \qquad G(\alpha) = \Lambda(\alpha)^2\,,$$

we can compute the derivative of $G(\cdot)$ as follows

$$\begin{aligned} G'(\bar{\alpha}) &= 2\lim_{\alpha\to\bar{\alpha}}\big(\Lambda'(\alpha)\Lambda(\alpha)\big) \\ &= 2\lim_{\alpha\to\bar{\alpha}}\big(s_1(\alpha)\Lambda'(\alpha)\big)\lim_{\alpha\to\bar{\alpha}}\big(\Lambda(\alpha)/s_1(\alpha)\big)\,, \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} s_1(\alpha)\Lambda'(\alpha) &= \tilde{s}_1'(\alpha) + m\Big(\frac{1}{2}v_s(s_1(\alpha),\alpha)\tilde{s}_1'(\alpha) + v_\alpha\big(s_1(\alpha),\alpha\big)s_1(\alpha)\Big)\,, \\ \Lambda(\alpha)/s_1(\alpha) &= 2 + m\frac{v\big(s_1(\alpha),\alpha\big) - v\big(0,\alpha\big)}{s_1(\alpha)}\,. \end{aligned}$$

Then, passing to the limit in (2.27) we obtain

$$\begin{aligned} G'(\bar\alpha) &= \big(2+mv_s(0,\bar\alpha)\big)^2\tilde s_1'(\bar\alpha) \\ &= -\big(2+2m\sin^2\bar\alpha\big)^2\frac{2m}{\sin\bar\alpha\cos\bar\alpha(1+m\sin^2\bar\alpha)} \\ &= -\frac{8m(1+m\sin^2\bar\alpha)}{\sin\bar\alpha\cos\bar\alpha}. \end{aligned}$$

Since

$$\frac{8m(1+m\sin^2\bar\alpha)}{\sin\bar\alpha\cos\bar\alpha} > 8m^2\tan\bar\alpha$$

we deduce that $G(\alpha)$ decreases faster than $F(\alpha)$ as α goes to $\bar\alpha$ and, since $T_1(\alpha)$ and $T_2(\alpha)$ are decreasing for α close to $\bar\alpha$, we have that $T_2(\alpha) > T_1(\alpha)$, i.e. the TYPE-1 trajectory is optimal for $\alpha \in [\bar\alpha-\varepsilon,\bar\alpha]$.

2.6 The time needed to reach every point of the Bloch sphere starting from the north pole in the case $\alpha \in [\pi/4,\pi/2)$

In this section we assume $\alpha \in [\pi/4,\pi/2)$. If α is close to $\pi/4$ it is easy to verify that the south pole is not the last point reached by bang-bang trajectories (the last point reached belongs to the cut locus present in the region $\Omega^{\pm}_{\text{nasty}}$) and the time needed to cover the whole sphere is slightly larger than 2π.

On the other hand, if α is large enough then the velocity along a singular arc is small and therefore the time needed to move along trajectories containing singular arcs is larger than 2π. The following proposition gives the asymptotic behavior of the total time needed to reach every point from the north pole and determines the last point reached by the optimal synthesis for α large enough.

Proposition 2.16 *Let $T(\alpha)$ the time needed to cover the whole sphere. Then, if α is large enough*

$$\begin{aligned} T(\alpha) &= \frac{\pi}{2\cos\alpha}+\pi-\frac{2\arcsin(\cot\alpha)}{\cos\alpha}+2\arcsin(\cot^2\alpha) \\ &= \frac{\pi}{2\cos\alpha}+\pi-2+O\Big(\frac{\pi}{2}-\alpha\Big) \end{aligned} \tag{2.28}$$

and the last points reached for a fixed value of α are $\pm(\sqrt{1-\cot^2\alpha},\cot\alpha,0)^T$.

Proof of Proposition 2.16 From Proposition 2.2 the last points reached by optimal trajectories of the form $B_tS_sB_{t'}$ must lie on overlap curves which are subsets of the equator. Therefore it is enough to estimate the maximum time to reach these overlap curves. Assume that the first bang arc corresponds to the control $u=1$ and denote by β the angle corresponding to the arc of the equator between the last point of the singular arc and the point $O^+=(1,0,0)^T$. Notice that $\beta \in (0,\arccos(\cot\alpha))$. Then it is easy to find the expression $T(\alpha,\beta)$ of the time needed to reach the overlap curve along that optimal trajectory:

$$T(\alpha,\beta) = \pi-\arccos(\cot^2\alpha)+\frac{\arccos(\cot\alpha)}{\cos\alpha}-\frac{\beta}{\cos\alpha}+\arccos\left(\frac{\cos^2\alpha-\tan^2\beta}{\cos^2\alpha+\tan^2\beta}\right).$$

The conclusion follows finding the maximum with respect to β of the previous quantity, which corresponds to the value $\bar{\beta} = \arcsin(\cot\alpha)$. Notice that $\bar{\beta}$ belongs to the interval of definition of β only if $\alpha > \operatorname{arccot}(\sqrt{2}/2)$. ■

Remark **2.19** Notice that, if $\alpha > \operatorname{arccot}(\sqrt{2}/2)$, then the set of points of the sphere reached within time t, with t in a left neighborhood of $T(\alpha)$, is not simply connected. More precisely there are two symmetric neighborhoods of the points $\pm(\sqrt{1-\cot^2\alpha}, \cot\alpha, 0)^T$ that are not reached in time less or equal than t.

Remark **2.20** Recall that for system (2.6) the time needed to cover the whole sphere for α close enough to $\pi/2$ is obtained dividing by $k = \frac{2E}{\cos\alpha}$ the expression (2.28). Therefore, if we fix E it turns out that this quantity converges to $\frac{\pi}{4E}$ as M goes to infinity.

Chapter 3

An Example of Limit Time Optimal Synthesis

In this chapter we continue the analysis of the time optimal synthesis relative to the problem **(P)** introduced in the previous chapter (see Section 2.2.1), in the case $\alpha < \pi/4$.

In Chapter 2 we gave a complete description of the time optimal synthesis starting from the north pole in the case $\alpha \geq \pi/4$ and we determined the optimal trajectories connecting the north pole to the south pole of the sphere. However the results obtained did not answer completely to the questions raised in [19] concerning the optimal synthesis in the case $\alpha < \pi/4$ (see Questions 1 and 2 in Section 2.3, and the subsequent conjectures about the local optimality of the switching curves and the shape of the optimal synthesis as α goes to 0).

To attack these problems we will make use of techniques completely different with respect to the techniques used in the previous chapter. Moreover the purposes of this chapter are not strictly connected with the application to the two-level system studied before, so that, in the following, we will use slightly different notations and we will need several additional definitions.

Let $\alpha \in]0, \pi/2[$. On the unit sphere $S^2 \subset \mathbb{R}^3$, consider the control system $(\Sigma)_\alpha$ defined by

$$(\Sigma)_\alpha \quad \dot{x} = (F_\alpha + uG_\alpha)x, \quad x = (x_1, x_2, x_3)^T, \quad \|x\|^2 = 1, \quad |u| \leq 1, \tag{3.1}$$

where F_α and G_α are two 3×3 skew-symmetric matrices representing two orthogonal rotations with axes of length respectively $\cos(\alpha)$ and $\sin(\alpha)$, $\alpha \in]0, \pi/2[$ (for the precise meaning of length, see Section 3.1). With no loss of generality, we assume that

$$F_\alpha := \begin{pmatrix} 0 & -\cos(\alpha) & 0 \\ \cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad G_\alpha := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\alpha) \\ 0 & \sin(\alpha) & 0 \end{pmatrix}. \tag{3.2}$$

and we define the matrices $X_\pm = F_\alpha \pm G_\alpha$.

For a definition of the problem of time optimal synthesis and the related definitions we refer to Chapter 2. Here we briefly recall the main results and properties of the optimal trajectories that will be useful in the following.

From the previous chapter we have that the extremals associated to $(\Sigma)_\alpha$ (i.e. the trajectories candidate for time optimality obtained after using the PMP) and starting from the north pole

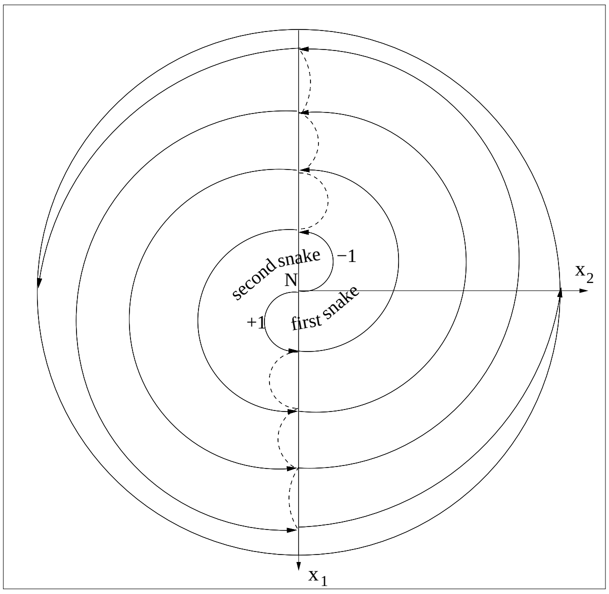

Figure 3.1: The "two-snakes" configuration defined by the extremal flow

are bang-bang trajectories of the type $e^{s_f X_{-\varepsilon'}} e^{v(s_i)X_{\varepsilon'}} \ldots e^{v(s_i)X_{-\varepsilon}} e^{s_i X_\varepsilon} N$ (recall that N and S denote respectively the north pole and the south pole). Here the initial time duration s_i verifies $s_i \in [0, \pi]$, all the time durations of the interior bang arcs are equal to $v(s_i)$, where the function $v(\cdot)$ is defined as

$$v(s) = \pi + 2 \arctan \left(\frac{\sin(s)}{\cos(s) + \cot^2(\alpha)} \right). \tag{3.3}$$

and the final time duration s_f verifies $s_f \leq v(s_i)$. Of particular importance for the construction of the TOS, are the *switching curves*, i.e. the curves made by points where the control switches from $+1$ to -1 or viceversa and defined inductively by

$$C_1^\varepsilon(s) = e^{X_\varepsilon v(s)} e^{X_{-\varepsilon} s} N, \quad C_k^\varepsilon(s) = e^{X_\varepsilon v(s)} C_{k-1}^{-\varepsilon}(s), \text{ (where } \varepsilon = \pm 1 \text{ and } k = 2, ..., k_M), \tag{3.4}$$

where $k_M := \left[\frac{\pi}{2\alpha}\right]$. Since being an extremal is only a necessary condition for time optimality, it is crucial to determine the time after which an extremal is no more optimal. In the following, we say that at this time the trajectory loses optimality. In [19], it was shown that the number of bangs must be lower than or equal to $k_M + 1$ and the extremals cover the sphere S^2 according to the "two-snakes" configuration as depicted in Figure 3.1 The two "snakes" correspond to extremal trajectories starting respectively with control $+1$ and -1.
However in [19], the authors were not able to construct the complete TOS associated to $(\Sigma)_\alpha$. In particular, they did not show the optimality of all the extremals up to $k_M - 1$ bang arcs and they could not complete analytically the construction of the synthesis in a neighborhood of the south pole S. There, the minimum time front develops singularities due to the compactness of S^2. However, as explained in Section 2.3, in [19] numerical simulations describing the evolution of the extremal front suggested the emergence of an interesting phenomenon. In particular the

optimal synthesis in a neighborhood of the south pole presents different patterns that ciclically alternate as α goes to 0, depending on the following quantity (*normalized remainder*)

$$r(\alpha) := \frac{\pi}{2\alpha} - \left[\frac{\pi}{2\alpha}\right]. \tag{3.5}$$

The main purpose of this chapter consists in studying the TOS associated to $(\Sigma)_\alpha$ as α tends to zero, focusing in particular on its behavior inside neighborhoods of the south pole. Roughly speaking, we want to determine, as α tends to zero, what could be a possible limit for the TOS associated to $(\Sigma)_\alpha$ (as suggested for instance by the patterns depicted in bottom of Fig. 2.3) and then to prove the convergence (in some suitable sense) of the TOS associated to $(\Sigma)_\alpha$ to that limit. To proceed, we embark on the study of a geometric object $\mathcal{F}(\alpha, T)$ called the *extremal front at time* T along $(\Sigma)_\alpha$ and defined as the set of points reached at time T by extremal trajectories starting from N (see subsection 3.2.1 for a precise definition). Recall that, according to the PMP, $\mathcal{F}(\alpha, T)$ contains the *minimum time front* $OF(\alpha, T)$, i.e. the set of points reached at time T by time optimal trajectories. When $\mathcal{F}(\alpha, T) = OF(\alpha, T)$, we say that $\mathcal{F}(\alpha, T)$ is *optimal*.

The first result we prove says that $\mathcal{F}(\alpha, T)$ is actually optimal for $T \leq (k_M - 1)\pi$ and α small enough (see Remark 3.5 below). Moreover, we show that $\mathcal{F}(\alpha, (k_M - 1)\pi)$ is essentially a circle of radius $2(1 + r(\alpha))\alpha$ (see Remark 3.4 below). We therefore obtain rigorously the optimal synthesis up to a neighborhood $C(\alpha)$ of the south pole of size proportional to α, i.e. we confirm the "two-snakes" configuration for the TOS associated to $(\Sigma)_\alpha$ outside $C(\alpha)$. That neighborhood $C(\alpha)$ is the connected component delimited by $\mathcal{F}(\alpha, (k_M - 1)\pi)$ and containing the south pole. We next prove $\mathcal{F}(\alpha, k_M\pi)$ is made up of the union of two curves $\mathcal{F}^\varepsilon(\alpha, k_M, \cdot) : [0, \pi] \to S^2$, $\varepsilon = \pm$, such that, for α small enough, $\mathcal{F}^\varepsilon$ admits a convergent power series of the type $\sum_{l \geq 0} f_l^\varepsilon(s, r)\alpha^l$, where the $f_l^\varepsilon(s, r)$ are real-analytic functions of $(s, r) \in [0, 2\pi] \times [0, 1]$, 2π-periodic in s with

$$f_0^+(s, r) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \qquad f_1^+(s, r) = \begin{pmatrix} -2rc_s \\ 2rs_s \\ 0 \end{pmatrix}, \qquad f_2^+(s, r) = \begin{pmatrix} \frac{\pi}{2}(4r + c_s)s_s^2 \\ \frac{\pi}{4}(3 + 8rc_s + c_{2s})s_s \\ 2r^2 \end{pmatrix}, \tag{3.6}$$

and, for every $l \geq 0$,

$$f_l^- = \Pi_{x_3} f_l^+. \tag{3.7}$$

Here r simply stands for $r(\alpha)$ and Π_{x_3} is the orthogonal symmetry with respect to the x_3-axis (see Proposition 3.2 below).

The previous result is the central tool to understand the asymptotic analysis of the TOS associated to $(\Sigma)_\alpha$, as α tends to zero. First of all, it is clear that, in general, there is no constraint between α and $r(\alpha)$ as α tends to zero. As a consequence, the power series $\sum_{l \geq 0} f_l^\varepsilon(s, r)\alpha^l$ must be understood as a function of two *independent* variables, which implies that the asymptotic behavior of the TOS associated to $(\Sigma)_\alpha$, as α tends to zero, depends in fact of two *independent* variables α and $r := r(\alpha)$. It makes therefore no sense to expect some global behavior under the sole condition "α tends to zero". Then, one must impose particular relationships between the two independent variables α and r in order to define any asymtotic behavior. In other words we must let α goes to zero, only along certain subsequences $(\alpha_k)_{k \geq 0}$ where a specific relationship holds between α_k and $r(\alpha_k)$. The analysis of Eq. (3.6) will help us to determine such relationships. Indeed, the first idea consists of looking for a condition so that $\mathcal{F}^\varepsilon(\alpha, k_M, \cdot)$ is well approximated by $f_0^\varepsilon(\cdot, r) + f_1^\varepsilon(\cdot, r)\alpha$. This is the case if and only if $r(\alpha)$ remains bounded away from zero as α tends to zero. To simplify further the discussion, it is therefore reasonable to consider the following.

$$(C1) \quad \text{for } \bar{r} \in (0,1), \text{ let } \alpha \text{ tend to zero so that } r(\alpha) = \bar{r}. \tag{3.8}$$

It is clear that $(C1)$ defines a decreasing sequence $(\alpha_k)_{k\geq 0}$ tending to zero.

The next step in the analysis of Eq. (3.6) consists of understanding when $f_1^\varepsilon(\cdot, r)\alpha$ is comparable to $f_2^\varepsilon(\cdot, r)\alpha^2$ as α tends to zero. This happens if and only if $r(\alpha)/\alpha$ remains bounded above and below by positive constants as α tends to zero. Let us therefore consider the statement below.

$$(C2) \quad \text{for } K > 0, \text{ let } \alpha \text{ tend to zero so that } r(\alpha) = K\alpha. \tag{3.9}$$

Condition $(C2)$ defines a decreasing sequence $(\alpha_k)_{k\geq 0}$ tending to zero and $\mathcal{F}^\varepsilon(\alpha, k_M, \cdot)$ is well approximated by $S + (f_1^\varepsilon(\cdot, K) + f_2^\varepsilon(\cdot, 0))\alpha^2$.

Pushing further the analysis of Eq. (3.6) leads to determine when $f_1^\varepsilon(\cdot, r)\alpha$ is negligible with respect to $f_2^\varepsilon(\cdot, r)\alpha^2$ as α tends to zero. This is exactly assuming that $r(\alpha)/\alpha$ tends to zero as α tends to zero. In that case, $\mathcal{F}^\varepsilon(\alpha, k_M, \cdot)$ is well approximated by $S + f_2^\varepsilon(\cdot, 0)\alpha^2$ and that corresponds to the following.

$$(C3) \quad \text{let } \alpha \text{ tend to zero so that } r(\alpha) = 0. \tag{3.10}$$

Again, Condition $(C3)$ defines a decreasing sequence $(\alpha_k)_{k\geq 0}$ tending to zero.

Finally, we see that the study of the asymptotic analysis of the TOS associated to $(\Sigma)_\alpha$, as α tends to zero, splits up in three distinct cases defined respectively by (Ci), $i = 1, 2, 3$. For instance, let us describe the results obtained in the case $(C1)$. Fix $\bar{r} \in (0,1)$ and assume that $(C1)$ holds true. It is easy to see that the "two-snakes" configuration for the TOS associated to $(\Sigma)_\alpha$ propagates until $\mathcal{F}(\alpha, k_M\pi)$. The latter turns out to be optimal and equal to the boundary of a neighborhood $C(\alpha) \subset S^2$ of the south pole S which is approximated, up to $\mathcal{O}(\alpha^2)$, by a circle of center S and radius $2\bar{r}\alpha$. Moreover, the third coordinate of every point of $C(\alpha)$ is equal to $-1 + \mathcal{O}(\alpha^2)$. To describe the behavior of the TOS inside $C(\alpha)$ (and eventually explain the patterns of bottom of Fig. 2.3), one must rescale the problem since $C(\alpha)$ collapses on S as α tends to zero. Clearly, the rescaling factor is $1/\alpha$. Also notice that only the dynamics of the two first coordinates is not trivial. Consequently, we are now in a position to define a possible limit behavior for the TOS inside $C(\alpha)$. Let M_α be the linear mapping from $\mathbb{R}^3$ onto $\mathbb{R}^2$ defined as the composition of the projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ followed by the dilation by $1/\alpha$. Denote $(\widetilde{\Sigma})_\alpha$ and $\tilde{F}(\alpha, k_M\pi)$ to be respectively the images by M_α of $(\Sigma)_\alpha$ and $\mathcal{F}(\alpha, k_M\pi)$. Then, $(\widetilde{\Sigma})_\alpha$ and $\tilde{F}(\alpha, k_M\pi)$ are respectively perturbations by $\mathcal{O}(\alpha^2)$ of the standard linearized pendulum and $C_{2\bar{r}}$, the planar circle of center $(0,0)$ and radius $2\bar{r}$. Up to a rescaling, the candidate limit TOS, as α tends to zero, for the TOS associated to $(\Sigma)_\alpha$ inside $C(\alpha)$ consists therefore of starting from $C_{2\bar{r}}$ and reaching in minimum time any point inside $C_{2\bar{r}}$ along the dynamics of the standard linearized pendulum. To prove such a result, we first study the above mentioned optimal control problem and show that (see Section 3.3.2) the corresponding TOS is characterized by an overlap curve γ^o_{pen}: "above" γ^o_{pen}, the control u takes the constant value 1 and "below" γ^o_{pen}, it is equal to -1 (see Fig. 3.2). Finally, the asymptotic result we prove in Section 3.3.2 is the following.

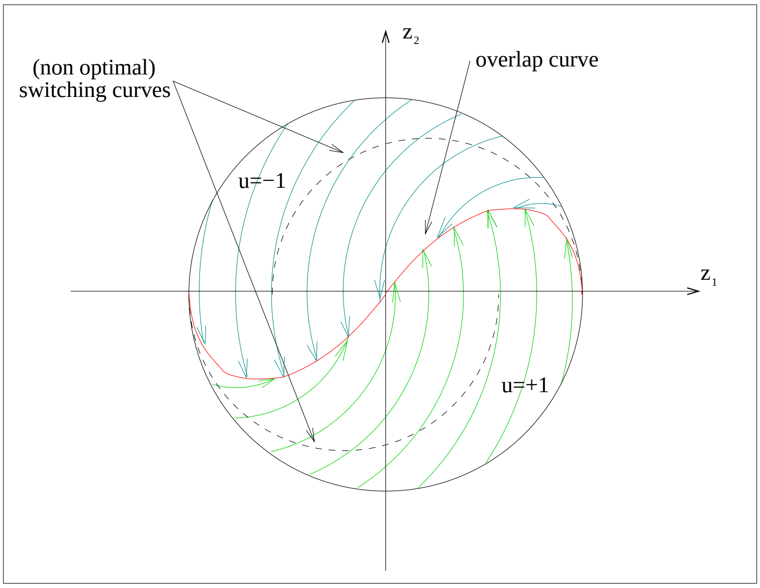

Figure 3.2: Optimal synthesis for the linear pendulum

Theorem 3.1 *Let $\mathcal{F}(\alpha, k_M\pi)$ be the extremal front at time $k_M\pi$ of the TOS associated to $(\Sigma)_\alpha$. For $\alpha \in (0, \pi/4)$, define respectively $(\widetilde{\Sigma})_\alpha$ and $\tilde{F}(\alpha, k_M\pi)$ as the images of $(\Sigma)_\alpha$ and $\mathcal{F}(\alpha, k_M\pi)$ by M_α. Fix $\bar{r} \in (0,1)$ and let α tend to zero with $r(\alpha)$ fixed in $[\varepsilon, 1-\varepsilon]$. Then, the TOS associated to $(\widetilde{\Sigma})_\alpha$ inside $\tilde{F}(\alpha, k_M\pi)$ is characterized by an overlap curve γ^o_α, converging to γ^o_{pen} in the C^0 topology, and, "above" γ^o_α, the control u takes the constant value -1 and "below" γ^o_α, it is equal to 1. Here γ^o_{pen} is the overlap curve of the TOS for the optimal control problem consisting of starting from $C_{2r(\alpha)}$, the planar circle of center $(0,0)$ and radius $2r(\alpha)$, and reaching in minimum time any point inside $C_{2r(\alpha)}$ along the following control system*

$$(Pen): \quad \begin{cases} \dot{z}_1 = -z_2, \\ \dot{z}_2 = z_1 + u, \qquad (z_1, z_2) \in \mathbb{R}^2, \quad |u| \le 1. \end{cases} \tag{3.11}$$

Moreover, the convergence of γ^o_α to γ^o_{pen} is uniform with respect to $r(\alpha) \in [\varepsilon, 1-\varepsilon]$.

3.1 Notations

All along the paper we use the notation $\varepsilon = \pm 1$. The set $so(3)$ of 3×3 skew-symmetric matrices is a three-dimensional vector space on which the following bilinear map

$$< A, B >= -Tr(AB), \quad A, B \in so(3),$$

is an inner product. For $A \in so(3)$, $\|A\| := \sqrt{< A, A >}$ is the norm (or length) of A. With the above notations, F and G are perpendicular and normalized so that $\|F\| = \cos(\alpha)$ and $\|G\| = \sin(\alpha)$.

Let Id be the 3×3 identity matrix. We recall that $N = (0,0,1)^T$ and denote the south pole as $S = (0,0,-1)^T$. Set $c_t := \cos(t)$ and $s_t := \sin(t)$ for $t \in [0, 2\pi)$. Recall that $X_+ := F_\alpha + G_\alpha$

and $X_- := F_\alpha - G_\alpha$ and we have

$$X_+ = \begin{pmatrix} 0 & -c_\alpha & 0 \\ c_\alpha & 0 & -s_\alpha \\ 0 & s_\alpha & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & -c_\alpha & 0 \\ c_\alpha & 0 & s_\alpha \\ 0 & -s_\alpha & 0 \end{pmatrix}.$$

Let Π_{x_3} be the orthogonal symmetry with respect to the x_3-axis, i.e. Π_{x_3} is represented in the canonical basis by $Diag(-1,-1,1)$. Then, we have the trivial but following useful property.

$$\Pi_{x_3} X_\varepsilon = X_{-\varepsilon}\Pi_{x_3}. \tag{3.12}$$

We next recall standard formulas for a rotation e^{tX} of $SO(3)$ in terms of its axis X (whose length is equal to one) and its angle t. We have

$$e^{tX} = Id + s_t X + (1-c_t)X^2. \tag{3.13}$$

Moreover, for $t \in [0, 2\pi)$, we have

$$e^{\Theta(t)Z_-(t)} := e^{tX_+}e^{tX_-} \quad e^{\Theta(t)Z_+(t)} := e^{tX_-}e^{tX_+},$$

where the unit vectors $Z_+(t), Z_-(t)$ are defined by

$$Z_+(t) = \begin{pmatrix} 0 & -C(t) & -B(t) \\ C(t) & 0 & 0 \\ B(t) & 0 & 0 \end{pmatrix}, \quad Z_-(t) = \begin{pmatrix} 0 & -C(t) & B(t) \\ C(t) & 0 & 0 \\ -B(t) & 0 & 0 \end{pmatrix}, \tag{3.14}$$

with $B(t) := \frac{s_\alpha s_{t/2}}{\sqrt{s_{t/2}^2 s_\alpha^2 + c_{t/2}^2}}$, $C(t) := -\frac{c_{t/2}}{\sqrt{s_{t/2}^2 s_\alpha^2 + c_{t/2}^2}}$ and the angle $\Theta(t)$ by

$$\cos(\Theta(t)/2) = s_{t/2}^2 c_{2\alpha} - c_{t/2}^2, \tag{3.15}$$

for $t > 0$ and $\Theta(0) = 2\pi$.

3.2 The Extremal Front

3.2.1 Definition and description

As said in the introduction, $\mathcal{F}(\alpha, T)$ the extremal front along $(\Sigma)_\alpha$ at time T is the set of points reached at time T by extremal trajectories starting from N, i.e.

$$\mathcal{F}(\alpha, T) := \{\bar{x} \in S^2 : \ \exists \text{ an extremal pair } (x(\cdot), \lambda(\cdot)) \text{ such that } x(0) = N, \quad x(T) = \bar{x}\}. \tag{3.16}$$

<u>*Remark*</u> **3.1** The extremal front $\mathcal{F}(\alpha, T)$ is actually made up of two families of extremal trajectories, which are parametrized by the length of the first bang arc, the one of the last bang arc and the number of arcs:

$$\begin{aligned} \Xi^+(s,t) &= \overbrace{e^{X_\varepsilon t}e^{X_{-\varepsilon}v(s)}\cdots e^{X_-v(s)}e^{X_+s}}^{n \ \text{terms}} N, & (3.17)\\ \Xi^-(s,t) &= \underbrace{e^{X_{\varepsilon'} t}e^{X_{-\varepsilon'}v(s)}\cdots e^{X_+v(s)}e^{X_-s}}_{n' \ \text{terms}} N, & (3.18)\end{aligned}$$

where $s \in [0, \pi]$, $t \in [0, v(s)]$, the number of bang arcs (n and n' respectively) is an integer and

(-) $\varepsilon = +1$ (resp. $\varepsilon = -1$), if n is odd (resp. even),

(-) $\varepsilon' = +1$ (resp. $\varepsilon' = -1$), if n' is even (resp. odd).

Roughly speaking, we would like to compute the limit, as $\alpha \to 0$, of $\mathcal{F}(\alpha, T)$, when T is such that the extremal front reaches a neighborhood of the south pole.

The idea is that, once one knows the extremal front $\mathcal{F}(\alpha, T)$ and if it is optimal, then one can continue to build the synthesis for times bigger than T using $\mathcal{F}(\alpha, T)$ as a source for the minimization problem.

The computation of the front $\mathcal{F}(\alpha, T)$ is not easy since it is the product of several exponentials of matrices. Moreover, if $\mathcal{F}(\alpha, T)$ crosses some switching curve, then the number of exponentials in general depend on the point.

This problem is overcome by considering $\mathcal{F}(\alpha, T)$ only at times equal to multiples of π. Indeed, first notice that, for $T = \pi\left[\frac{\pi}{2\alpha}\right]$, the extremal front reaches the points $C^{\pm}_{k_M}(0)$, i.e. the points where the last switching curves $C^{\pm}_{k_M}$ start. Thanks to Proposition 3.1 below, at these times, every extremal trajectory has exactly the same number of switchings. The extremal front at times that are not multiple of π can be obtained *a posteriori*, continuing the extremal front, as explained above.

Since both the extremal trajectories and the switching curves $C^{\pm}_k(s)$ ($k = 1, ..., k_M$) are parameterized by the first switching time $s \in [0, \pi]$, then the time at which the point $C^{\pm}_k(s)$ is reached is $T_k(s) = s + kv(s)$.

Lemma 3.1 *Let k be an integer satisfying* $1 \le k \le \mathcal{N}_{mon} := \left[\dfrac{(\cot(\alpha)^2 - 1)^2}{2\cot(\alpha)^2 - 1)}\right]$, *then $T_k(s)$ is a strictly increasing function of s.*

Proof. It holds

$$\frac{d}{ds}T_k(s) = \frac{1 + 2\,c_s\cot(\alpha)^2 + \cot(\alpha)^4 + k\left(2 + 2\,c_s\cot(\alpha)^2\right)}{1 + 2\,c_s\cot(\alpha)^2 + \cot(\alpha)^4}. \tag{3.19}$$

Let $\mathcal{N}$ and $\mathcal{D}$ be respectively the numerator and the denominator of the above fraction. It is clear that $\mathcal{D}$ is never vanishing on $[0, \pi]$. On the other hand, $\mathcal{N}$, as a function of s, reaches its minimum at $s = \pi$, where it is equal to $(\cot(\alpha)^2 - 1)^2 - k(2\cot(\alpha)^2 - 1)$, and then the conclusion follows easily. ■

As a consequence, we obtain the following important corollary.

Corollary 3.1 *Let k be an integer satisfying $1 \le k \le \mathcal{N}_{mon}$. If an extremal trajectory is switching at time $T = k\pi$, then the length s of the first bang arc satisfies $s = 0$ or $s = \pi$.*

Since for α small $k_M \le \mathcal{N}_{\text{mon}}$, then for $T = k\pi$, k positive integer with $k \le [\pi/(2\alpha)]$, all extremal trajectories switch exactly k times (except the trajectories with length of the first switching equal to 0 or π that switch $k - 1$ times). Therefore, the extremal front $\mathcal{F}(\alpha, k\pi)$ is described by the next proposition.

Proposition 3.1 *Let k be a positive integer such that $1 \leq k \leq [\pi/(2\alpha)]$. Then, we have*

$$\mathcal{F}(\alpha, k\pi) = \left\{\mathcal{F}^{+}(\alpha, k, s), \quad s \in [0,\pi]\right\} \bigcup \left\{\mathcal{F}^{-}(\alpha, k, s), \quad s \in [0,\pi]\right\}, \quad \textit{where}: \tag{3.20}$$

$$\mathcal{F}^{+}(\alpha, k, s) := \begin{cases} e^{(k\pi-(k-1)v(s)-s)X_{-}}\, e^{\frac{k-1}{2}\Theta(v(s))Z_{-}}\, e^{sX_{+}}N, & \textit{for } k \textit{ odd}, \\ e^{(k(\pi-v(s))-s)X_{+}}\, e^{\frac{k}{2}\Theta(v(s))Z_{-}}\, e^{sX_{+}}N, & \textit{for } k \textit{ even}. \end{cases} \tag{3.21}$$

The expression for $\mathcal{F}^{-}$ is the same as the expression for $\mathcal{F}^{+}$ after exchanging the subscripts $+$ and $-$. As a consequence, $\mathcal{F}^{-\varepsilon} = \Pi_{x_3}\mathcal{F}^{\varepsilon}$, where Π_{x_3} is the orthogonal symmetry with respect to the x_3-axis.

<u>Remark</u> **3.2** Since $\mathcal{F}$ is a continuous closed curve, one has that

$$\mathcal{F}^{\varepsilon}(\alpha, k, t) = \mathcal{F}^{-\varepsilon}(\alpha, k, t+\pi), \quad t \in [0,\pi] \ (\text{mod } 2\pi).$$

3.2.2 Description of the extremal front $\mathcal{F}(\alpha, k_M\pi)$ and consequences

As sketched in the introduction, we must describe the optimal synthesis on S^2 deprived of a neighborhood of the south pole. For that purpose, we will provide the precise asymptotics of $\mathcal{F}(\alpha, k_M\pi)$, as α tends to zero, and derive, from its topological nature, the minimum time front at time $k_M\pi$. The key point consists of noticing that $\mathcal{F}^{+}(\alpha, k, s)$ and $\mathcal{F}^{-}(\alpha, k, s)$ are analytic functions with respect to some parameters, for α small enough and thus admit power series expansions. In the following statement, r stands for $r(\alpha)$.

Proposition 3.2 *For α small enough, $\mathcal{F}^{\varepsilon}$ with $\varepsilon = \pm$ and defined in Eq. (3.20), admit convergent power series of the type*

$$\sum_{l \geq 0} f_l^{\varepsilon}(s, r)\alpha^l, \tag{3.22}$$

where the $f_l^{\varepsilon}(s,r)$ are real-analytic functions of $(s,r) \in [0,2\pi] \times [0,1]$, 2π-periodic in s and verifying Eqs. (3.6) and (3.7). As a consequence, the extremal front $\mathcal{F}(\alpha, k_M\pi)$ is continuous closed curve, piecewise analytic with discontinuities at $s = 0, \pi$ for derivatives of order greater than or equal to one.

<u>Remark</u> **3.3** The remainder r and the parameter α enter as independent variables in the previous expressions since the singular part $1/\alpha$ actually only appears (up to a constant scalar) through its integer part k_M.

Proof of Proposition 3.2.

In the sequel, several functions of the type $f(\alpha, s, r)$ will be considered, with (s,r) belonging to a compact subset I of $[0,2\pi] \times [0,1]$. For such functions $f(\alpha, s, r)$, an asymptotic development in powers of α will be evaluated as α tends to zero. We will use Landau notations to express the remainder and sometimes, we want that remainder to do not depend on the parameter

$(s,r) \in I$. For that purpose, we use $\mathcal{O}_I(\alpha^m)$ to say that the remainder is of order α^m uniformly with respect to $(s,r) \in I$. In most of the cases, we have $I := [0,\pi] \times [0,1]$.

For $\alpha < \pi/4$, we set

$$k_M = \left[\frac{\pi}{2\alpha}\right] \qquad r(\alpha) := \frac{\pi}{2\alpha} - k_M.$$

Note that $r(\alpha) \in [0,1)$. In the sequel, we will sometimes drop the dependence of $r(\alpha)$ with respect to α in order to take advantage of the independence between the variables α and r in particular expressions. When necessary we replace k_M by $\frac{\pi}{2\alpha} - r$.

From Eqs. (3.14), we first define, for $\alpha < \pi/4$, $s \in [0,\pi]$ and $\varepsilon = \pm$,

$$Z_\varepsilon(s) := Z_\varepsilon(v(s)), \quad b(s) := B(v(s)), \quad c(s) := C(v(s)), \quad \xi(s) := \cot(v(s)/2),$$

where $v(s)$ was introduced in Eq. (3.3). For simplicity, we did not consider an explicit dependence with respect to α and r and will also drop sometimes the dependence with respect to s. Note that ξ is an even function of α.

We start the proof of the proposition for $\mathcal{F}^+$. The following lemma will be instrumental for the rest of the paper.

Lemma 3.2 *For α small enough, $\mathcal{F}^+(\alpha, k_M, s)$ is equal to a convergent power series*

$$\mathcal{F}^+(\alpha, k_M, s) = \sum_{l \geq 0} f_l^+(s,r)\alpha^l, \tag{3.23}$$

where the $f_l(s,r)$ are real-analytic functions of $(s,r) \in [0,\pi] \times [0,1]$, 2π-periodic in s and f_l^+ are given in Eq. (3.6), for $l = 0,1,2$.

Proof of Lemma 3.2. We assume that k_M is odd and define

$$\theta(s) := (k_M - 1)\Theta(v(s))/2.$$

We first prove the existence of the convergent power series in Eq. (3.23). It consists in showing that the quantities ψ, Z_ε and θ admit such power series. Since $\xi(s) = \pi/2 + \mathcal{O}_I(\alpha^2)$, the claim holds for Z_ε. For ψ, the claim reduces to prove a similar conclusion for $k_M(v(s) - \pi)$ and thus for $\frac{v(s)-\pi}{\alpha}$. The last expression is equal to

$$\frac{2}{\alpha}\arctan(s_\alpha^2 \mu(s)),$$

where $\mu(s) := \frac{s_s}{c_\alpha^2 + s_\alpha^2 c_s}$. The conclusion follows easily.

As for θ, we first set $\beta(s) := \Theta(v(s))$ and rewrite Eq. (3.15) as

$$\cos(\beta(s)) = 1 - F_\alpha(s),$$

with

$$F_\alpha(s) := 2s_\alpha^2 \left[1 + \frac{c_\alpha^2 s_\alpha^2 \mu^2(s)}{1 + s_\alpha^4 \mu^2(s)} + 2s_\alpha^2 (1 + \frac{c_\alpha^2 s_\alpha^2 \mu^2(s)}{1 + s_\alpha^4 \mu^2(s)})^2\right]. \tag{3.24}$$

We first need to determine a convergent power series for β from the expression $\beta(s) = \arccos(1 - F_\alpha(s))$. Note that $|F_\alpha(s)| \leq 5\alpha^2$. We first expand $\arccos(1 - F_\alpha)$ in a power series in F_α. For that purpose, consider the power series expansion of $(1-t)^{-1/2}$ given by

$$(1-t)^{-1/2} = 1 + \sum_{m\geq 1} s_m t^m,$$

with radius of convergence equal to 1. Since

$$\frac{d}{dF_\alpha}(\arccos(1-F_\alpha)) = \frac{1}{\sqrt{2F_\alpha}} \frac{1}{\sqrt{1-F_\alpha/2}},$$

we get, after simple integration that

$$\arccos(1-F_\alpha) = \sqrt{F_\alpha}(1 + \sum_{m\geq 1} \frac{s_m}{2^{m+1/2}(m+1/2)} F_\alpha^m),$$

Finally, from Eq. (3.24), F_α can be written as $2s_\alpha^2(1 + s_\alpha^2 H_\alpha(s))$ with H_α uniformly bounded by 3. Then,

$$\sqrt{F_\alpha(s)} = \sqrt{2} s_\alpha (1 + s_\alpha^2 H_\alpha(s))^{1/2}. \tag{3.25}$$

Since $F_\alpha(s)$ and $\mu(s)$ can be expanded in power series, as well as $\frac{s_\alpha}{\alpha}$, we deduce that it is also the case for $\beta(s)$ for α small enough. To finish the proof of the claim, it suffices to notice that the existence of a convergent power series for $\theta(s)$ is equivalent to that of $\sqrt{F_\alpha(s)}/\alpha$. The latter is obviously true from Eq. (3.25).

■

We next compute f_l^+ for $l = 0, 1, 2$. We start from the formula

$$\mathcal{F}^+(\alpha, k_M, s) = e^{\psi(s)X_-} e^{\theta(s)Z_-} e^{sX_+} N. \tag{3.26}$$

Replacing k_M by $\frac{\pi}{2\alpha} - r$, $\mathcal{F}^+(\alpha, k_M, s)$ can be seen as a function of (s, α, r) on $[0, 2\pi] \times [-a, a] \times [0, 1]$, $a > 0$ small enough. Recall that

$$\psi(s) := k_M \pi - (k_M - 1)v(s) - s.$$

We deduce that

$$\psi(s) = -\frac{\pi}{\alpha} \arctan(\frac{s_s s_\alpha^2}{c_\alpha^2 + c_s s_\alpha^2}) + r(v(s) - \pi) + v(s) - s,$$

and then, for a small enough, ψ can be expanded as a power series in α with coefficients real-analytic in (s, r). To proceed, we list the Taylor expansions for several scalar quantities, obtained after elementary computations.

$$
\begin{aligned}
b(s) &= 1-\frac{s_s^2}{2}\alpha^2+\mathcal{O}_I(\alpha^3) && (3.27)\\
c(s) &= s_s\alpha+\mathcal{O}_I(\alpha^3) && (3.28)\\
\xi(s) &= -s_s\alpha^2+\mathcal{O}_I(\alpha^4) && (3.29)\\
\psi(s) &= \pi-s-\pi s_s\alpha+2(1-r)s_s\alpha^2+\mathcal{O}_I(\alpha^3) && (3.30)\\
\theta(s) &= \pi-2\alpha(1+r)+\frac{\pi s_s^2}{2}\alpha^2+\mathcal{O}_I(\alpha^3) && (3.31)\\
Z_- &= \begin{pmatrix} 0 & -\alpha s_s & 1-\frac{s_s^2}{2}\alpha^2 \\ \alpha s_s & 0 & 0 \\ -1+\frac{s_s^2}{2}\alpha^2 & 0 & 0 \end{pmatrix}+\mathcal{O}_I(\alpha^3). && (3.32)
\end{aligned}
$$

Using Eq. (3.31), we get that

$$
\begin{aligned}
\sin(\theta(s)) &= 2\alpha(1+r)-\frac{\pi s_s^2}{2}\alpha^2+\mathcal{O}_I(\alpha^3) && (3.33)\\
\cos(\theta(s)) &= -1+2\alpha^2(1+r)^2+\mathcal{O}_I(\alpha^3). && (3.34)
\end{aligned}
$$

Using Eqs. (3.13) and (3.30), we obtain

$$
e^{\psi(s)X_-}=\begin{pmatrix} -c_s+\pi s_s^2\alpha & -s_s-\pi c_s s_s\alpha & -(1+c_s)\alpha \\ s_s+\pi s_s c_s\alpha & c_s+\pi s_s^2\alpha & s_s\alpha \\ -(1+c_s)\alpha & s_s\alpha & 1 \end{pmatrix}, \tag{3.35}
$$

and using Eq. (3.13), we have

$$
e^{sX_+}N=\begin{pmatrix} s_\alpha c_\alpha(1-c_s) \\ -s_\alpha s_s \\ 1-s_\alpha^2(1-c_s) \end{pmatrix}=\begin{pmatrix} \alpha(1-c_s) \\ -\alpha s_s \\ 1-\alpha^2(1-c_s) \end{pmatrix}+\mathcal{O}_I(\alpha^3). \tag{3.36}
$$

An easy computation yields

$$
Z_-^2=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -c^2(s) & b(s)c(s) \\ 0 & b(s)c(s) & -b^2(s) \end{pmatrix}=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\alpha^2 s_s^2 & \alpha s_s \\ 0 & \alpha s_s & -1+\alpha^2 s_s^2 \end{pmatrix}+\mathcal{O}_I(\alpha^3). \tag{3.37}
$$

Using Eqs. (3.13), (3.36) and the previous equation, we get

$$
e^{\theta(s)Z_-}e^{sX_+}N=\begin{pmatrix} \alpha(1+2\,r+c_s)-\alpha^2\frac{\pi}{2}\,s_s^2 \\ \alpha\,s_s \\ -1+\alpha^2\,\left(1+2\,r+2\,r^2+c_s+2\,r\,c_s\right) \end{pmatrix}+\mathcal{O}_I(\alpha^3). \tag{3.38}
$$

Applying $e^{\psi(s)X_-}$ to the previous equation and using Eqs. (3.13),(3.35), we finally get Eq. (3.6).

As for the case k_M even, similar computations lead to identical formulas for f_l with $l=0,1,2$. Finally, the results for $\mathcal{F}^-$ are obtained similarly together with Eq. (3.12).

■

Since $\mathcal{F}$ is piecewise analytic, we can deduce a power series expansion for the tangent vector to $\mathcal{F}$.

Corollary 3.2 *For α small enough, $\mathcal{F}^\varepsilon$ with $\varepsilon = \pm$ and defined in Eq. (3.20), are analytic maps and their tangent vectors verify, for $s \in [0, \pi]$,*

$$\frac{d}{ds}\mathcal{F}^\varepsilon(\alpha, k_M, s) = \sum_{l \geq 1} \frac{\partial}{\partial s} f_l^\varepsilon(s, r)\alpha^l. \tag{3.39}$$

In particular, $\frac{\partial}{\partial s} f_l^- = -\frac{\partial}{\partial s} f_l^+$ for $l = 1, 2$.

Proof of Corollary 3.2: This is an immediate consequence of the analyticity of $\mathcal{F}^\varepsilon$, $\varepsilon = \pm$.
■

<u>Remark</u> **3.4** From the previous computations, we also have a power series expansion for $\mathcal{F}^\varepsilon(s, k_M - 1, \alpha)$. Indeed, we just have to replace r by $1 + r$. In that case, the leading term is equal to $N + f_1^\varepsilon(s, 1 + r)\alpha$, meaning that $\mathcal{F}(\alpha, (k_M - 1)\pi$ is a circle a radius $2(1 + r)\alpha$ with an error term in $\mathcal{O}_I(\alpha^2)$.

3.3 Case $r(\alpha) = \bar{r} \in (0, 1)$

In that section, we assume that α tends to zero so that $r(\alpha) = \bar{r}$ for a constant $\bar{r} \in (0, 1)$, i.e. α takes the values of the decreasing sequence $(\frac{\pi}{k+\bar{r}})_{k \geq 0}$. We first describe the minimum time front at $t = k_M\pi$, then identify and study the candidate for the limiting behavior and finally prove Theorem 3.1.

3.3.1 Description of the minimum time front at $t = k_M\pi$

The purpose of the paragraph is to prove the following proposition.

Proposition 3.3 *For α small enough with $r(\alpha) \in [\varepsilon, 1 - \varepsilon]$, the extremal front $\mathcal{F}(\alpha, k_M\pi)$ is homeomorphic to the circle e^{is}, $s \in [0, 2\pi]$. In particular, every point of $\mathcal{F}(\alpha, k_M\pi)$ is reached by a unique extremal trajectory starting at the north pole. As a consequence, the switching curves defined inductively in Eq. (3.4) are optimal up to $k = k_M$ and $OF(\alpha, k_M\pi)$, the minimum time front in time $k_M\pi$ coincides with $\mathcal{F}(\alpha, k_M\pi)$.*

Proof of Proposition 3.3: From Proposition 3.2 and Corollary 3.2, we get that the extremal front $\mathcal{F}(\alpha, k_M\pi)$ is the union of two arcs, $\mathcal{F}^+(\alpha, k_M, s)$, $s \in [0, \pi]$ and $\mathcal{F}^-(\alpha, k_M, s)$, $s \in [0, \pi]$ so that, for $\varepsilon = \pm$ and $s \in [0, \pi]$,

$$\mathcal{F}^\varepsilon(\alpha, k_M, s) = \begin{pmatrix} -2r\varepsilon\alpha c_s \\ 2r\varepsilon\alpha s_s \\ -1 \end{pmatrix} + \mathcal{O}_I(\alpha^2), \tag{3.40}$$

and

$$\frac{d}{ds}\mathcal{F}^+(\alpha, k_M, s) = 2r\varepsilon\alpha \begin{pmatrix} s_s \\ c_s \\ 0 \end{pmatrix} + \mathcal{O}_I(\alpha^2). \tag{3.41}$$

Moreover, at $s = 0$ and $s = \pi$, the derivatives of $\mathcal{F}^{\varepsilon}(\alpha, k_M, s)$ are only one-sided, i.e. as $s > 0$ tends to zero and $s < \pi$ tends to π. By a trivial continuity argument, one can parameterize $\mathcal{F}(\alpha, k_M\pi)$ as a closed continuous curve γ defined on $[0, 2\pi]$ so that $\gamma(s) = \mathcal{F}^{+}(\alpha, k_M, s)$ for $s \in [0, \pi]$ and $\gamma(s) = \mathcal{F}^{-}(\alpha, k_M, s-\pi)$ for $s \in [\pi, 2\pi]$. Moreover, with the previous computations, it is immediate that γ is in fact piecewise $\mathcal{C}^1$ with possible discontinuity jumps for $\frac{d}{ds}\gamma$ at $s = 0$ and $s = \pi$.

Since the curve γ is in an neighborhood of the south pole of size proportional to α (thanks to Eq. (3.40)), it is enough to prove that the orthogonal projection γ_1 of γ on the (x, y)-plane is homeomorphic to the circle e^{is}, $s \in [0, 2\pi]$. Using Eq. (3.40), we see that $\|\gamma_1(s)\| = 2r\alpha + \mathcal{O}_I(\alpha^2)$ on $[0, 2\pi]$, which implies that the continuous function $\|\gamma_1(s)\|$ is always strictly positive for α small enough. We can therefore parameterize γ_1 using polar coordinates (ρ, β), i.e., for $s \in [0, 2\pi]$,

$$\gamma_1(s) = \rho(s)e^{i\beta(s)},$$

where $\rho(s) := \|\gamma_1(s)\|$ and the function β are defined on $[0, 2\pi]$, continuous and piecewise $\mathcal{C}^1$, with possible jumps of discontinuity for their derivatives at $s = 0$ and $s = \pi$. In addition $\rho(0) = \rho(2\pi)$ and $|\beta(2\pi) - \beta(0)|/2\pi$ is an integer. To show Proposition 3.3, it suffices now to prove that β is a monotone bijection from $[0, 2\pi]$ to the interval bounded by $\beta(0)$ and $\beta(2\pi)$ and $|\beta(2\pi) - \beta(0)| = 2\pi$. The latter simply results from Eq. (3.41). Indeed, from that equation, we get that $\frac{d}{ds}\beta(s) = -1 + \mathcal{O}(\alpha)$ where β is differentiable and the one-sided derivatives at $s = 0$ and $s = \pi$ verify the same equation. We deduce that β is strictly decreasing and $|\beta(2\pi)-\beta(0)|/2\pi = 1$ for α small enough.

We next show that $OF(\alpha, k_M\pi)$, the minimum time front in time $k_M\pi$ coincides with $\mathcal{F}(\alpha, k_M\pi)$. By the results of [20], we first notice that any time minimal trajectory starting at the north pole reaches the south pole in time $T > k_M\pi$. Therefore $OF(\alpha, k_M\pi)$ is not empty and is included in $\mathcal{F}(\alpha, k_M\pi)$ according to the PMP. According to Theorem 27 of [22], $OF(\alpha, k_M\pi)$ is a one-dimensional piecewise C^1 compact embedded submanifold of S^2. By an easy topological argument, we deduce from the above that $OF(\alpha, k_M\pi)$ coincides with $\mathcal{F}(\alpha, k_M\pi)$. Since every point Q of $\mathcal{F}(\alpha, k_M\pi)$ belongs to $OF(\alpha, k_M\pi)$ and is reached in time $k_M\pi$ by a unique extremal trajectory γ_Q, then γ_Q must be optimal. We immediately deduce the last statement of Proposition 3.3.

Remark **3.5** In Remark 3.4, we showed that $\mathcal{F}(\alpha, (k_M - 1)\pi$ is essentially a circle a radius $2(1 + r)\alpha$, without assuming any particular relation between r and α. With the same reasoning as above, we also deduce first that $\mathcal{F}(\alpha, (k_M - 1)\pi$ is homeomorphic to the circle e^{is}, $s \in [0, 2\pi]$ and finally that $OF(\alpha, (k_M - 1)\pi)$, the minimum time front in time $(k_M - 1)\pi$ coincides with $\mathcal{F}(\alpha, (k_M - 1)\pi)$, i.e. $\mathcal{F}(\alpha, T)$ is optimal for $T \leq (k_M - 1)\pi$.

3.3.2 Optimal synthesis for the linear pendulum control problem

Recall that $M_\alpha : \mathbb{R}^3 \to \mathbb{R}^2$ is the composition of the projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ followed by the dilation by $1/\alpha$. With the results of the previous subsection, it is clear that the original control problem on S^2 can be reduced, in a neighborhood of the south pole, to a planar control problem inside $\tilde{F}(\alpha, k_M\pi) := M_\alpha(F(\alpha, k_M\pi))$ along $(\widetilde{\Sigma})_\alpha$, the control system obtained as the image of $(\Sigma)_\alpha$ by M_α and defined by

$$(\widetilde{\Sigma})_\alpha : \quad \begin{cases} \dot{z_1} = -\cos(\alpha) z_2, \\ \dot{z_2} = \cos(\alpha) z_1 + u\frac{\sin(\alpha)}{\alpha}\sqrt{1-(\alpha z_1)^2-(\alpha z_2)^2}, \quad (z_1,z_2)\in\mathbb{R}^2, \quad |u|\le 1. \end{cases} \tag{3.42}$$

It is therefore natural to conjecture (simply set $\alpha = 0$ in $\tilde{F}(\alpha, k_M\pi)$ and $(\Sigma)_\alpha$) that the limit synthesis should be that of connecting $C(0, 2r(\alpha))$ to any point inside $B(0, 2r(\alpha))$ along the control system (Pen) given by Eq. (3.11), which we rewrite as

$$(Pen) \quad \dot{z} = A_0 z + u b_0, \quad \text{with} \quad A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.43}$$

where $z \in \mathbb{R}^2$ and $u \in [-1, 1]$. The control system (Pen) corresponds to a linear pendulum with a forcing term. Set $X^p_\varepsilon z := A_0 z + \varepsilon b_0$.

Theorem 3.1 simply states that the conjecture is correct and, as a first step for an argument, we describe, in more details in this subsection, the conjectured limiting synthesis. More generally, we investigate the following time optimal synthesis problem

(P) *Fixed* $\rho \in]0, 2]$, *for any given* $\bar{y} \in B(0, \rho)$ *find a time optimal trajectory connecting the circle of radius* ρ *centered at the origin to* $\bar{y}$ *along the control system* (Pen).

Remark **3.6** Notice that the circle of radius ρ centered at the origin is a trajectory of the system (3.43) with $u = 0$, i.e. it corresponds to a constant energy level for the uncontrolled system.

It is not difficult to see that the solutions of problem **(P)** must be bang-bang trajectories. Indeed $\Delta_B = \det(b_0, -A_0 b_0) = -1$ everywhere and therefore singular arcs, that are subsets of $\Delta_B^{-1}(0)$, cannot appear. To determine the TOS, we first look for the switching curves, i.e. the curves made by switching points. First, we know that every extremal trajectory for the problem **(P)** must satisfy the transversality condition

$$< \lambda(0), T_{z(0)}M >= 0 \quad (\lambda(0) \text{ and } z(0) \text{ are the initial data for the state and adjoint variables}),$$

where the manifold M describes the constraint on the initial point. In our case, the transversality condition essentially translates into the property that the unit vector $\lambda(0)$ is parallel to $z(0) \in M$ (identifying the cotangent space with the plane $\mathbb{R}^2$). To determine completely $\lambda(0)$, it is enough to observe that a necessary condition for $z(\cdot)$ to be optimal is that $\dot{z}(0)$ points inside the disk $B(0, \rho)$, i.e., if we denote by u_{opt} the corresponding control, then

$$< z(0), \dot{z}(0) >\le 0 \iff < z(0), A_0 z(0) + u_{opt} b_0 >\le 0 \iff < z(0), u_{opt} b_0 >\le 0\,.$$

Therefore, $u_{opt} = -\text{sgn}< z(0), b_0 >$. On the other hand, from the maximality condition of the PMP, it must hold $u_{opt} =\text{sgn}< \lambda(0), b_0 >$ and, therefore, one can define $\lambda(0) := -z(0)/\rho$. Finally $u_{opt} = -\text{sgn}(z_2(0))$ (except at the points $\pm(\rho, 0)$), while the switching time t_{sw} must satisfy the condition $< \lambda(t_{sw}), b_0 >= \lambda_2(t_{sw}) = 0$.
Consider now the adjoint system

$$\begin{cases} \dot{\lambda}_1 = -\lambda_2, \\ \dot{\lambda}_2 = \lambda_1. \end{cases} \tag{3.44}$$

If we identify $\mathbb{R}^2$ with the complex plane, so that $z = z_1 + iz_2$ and $\lambda = \lambda_1 + i\lambda_2$, then the equations (3.43), (3.44) become

$$\dot{z} = i(z + u) \qquad \text{and} \qquad \dot{\lambda} = i\lambda.$$

Moreover we can set $z(0) = -\rho e^{-i\theta}$ and $\lambda(0) = e^{-i\theta}$ for some $\theta \in [0, 2\pi[$ and the corresponding solutions are:

$$\begin{cases} z(t) = (z(0) + u_{opt})e^{it} - u_{opt} = -\rho e^{i(t-\theta)} + u_{opt}(e^{it} - 1), \\ \lambda(t) = \lambda(0)e^{it} = e^{i(t-\theta)}. \end{cases}$$

The switching curves are determined by the relation $\theta + t_{sw} \equiv 0 \pmod{\pi}$ and this allows to conclude that the switching curves are the following two semicircles of radius 1:

$$\begin{cases} z(\theta) = 1 - \rho - e^{i\theta} & \theta \in [0, \pi[, \\ z(\theta) = \rho - 1 - e^{i\theta} & \theta \in [\pi, 2\pi[. \end{cases}$$

These switching curves cannot be optimal for $\rho < 2$ since they are not locally optimal. Indeed, if we consider a neighborhood of $z(\theta)$ (with $\theta \neq 0, \pi$), which is divided by $z(\cdot)$ in two regions, then X_+^p and X_-^p do not point in the same region. We conclude that the optimal trajectories are bang arcs and the corresponding control depends on the sign of the component $z_2(0)$ of the starting point.

To conclude the description of the synthesis, it is enough to determine the cut locus, i.e. the set of points that are reached by two or more optimal trajectories at the same time. Assume that $z \in \mathbb{C}$ belongs to the cut locus. Then, there exist $s \in [0, \pi)$, $s' \in [\pi, 2\pi)$ and t such that

$$\begin{cases} z = -\rho e^{i(t-s)} + 1 - e^{it}, \\ z = -\rho e^{i(t-s')} - 1 + e^{it}. \end{cases} \tag{3.45}$$

Therefore $|z - 1 + e^{it}| = |z + 1 - e^{it}| = \rho$. In particular, denoting by $\bar{z}$ the complex conjugate to z, we have

$$(z - 1 + e^{it})(\bar{z} - 1 + e^{-it}) - (z + 1 - e^{it})(\bar{z} + 1 - e^{-it}) = -4z_1 + 4z_1 \cos t + 4z_2 \sin t = 0, \tag{3.46}$$
$$(z - 1 + e^{it})(\bar{z} - 1 + e^{-it}) + (z + 1 - e^{it})(\bar{z} + 1 - e^{-it}) = 2z_1^2 + 2z_2^2 + 4 - 4\cos t = 2\rho^2. \tag{3.47}$$

From (3.46) we have that $\cos t = \frac{z_1^2 - z_2^2}{z_1^2 + z_2^2}$, and, substituting in (3.47), we find that z must satisfy the equation

$$z_1^4 + z_2^4 + 2z_1^2 z_2^2 - \rho^2 z_1^2 + (4 - \rho^2) z_2^2 = 0.$$

It is not difficult to see that the locus defined by this equation does not coincide exactly with the cut locus, which is a proper subset consisting in a $\mathcal{C}^1$ curve.
Moreover from the previous computations we have $\rho e^{is'} = \rho e^{is} + 2 - 2e^{it}$ and, since $\rho e^{is'} + \rho e^{is} = 2\rho e^{is'} - 2 + 2e^{it}$ and $\rho e^{is'} - \rho e^{is} = 2 - 2e^{it}$ are orthogonal in the complex plane, we find easily the following equation:

$$(2 - \rho \cos s')(\cos t - 1) - \rho \sin s' \sin t = 0.$$

Consequently, for $t \in [0, 2\pi[$ and $s' \in [\pi, 2\pi[$, one has, along the overlap curve

$$t = -2 \arctan \frac{\rho \sin s'}{2 - \rho \cos s'}. \tag{3.48}$$

This expression will be useful in the following. Notice that combining (3.45) and (3.48) one could easily find a parametrization of the overlap curve in terms of s'. We also remark that, in the case $\rho = 2$, the overlap curve coincides exactly with the switching curves and with the trajectories, corresponding to $u = \pm 1$, that connect the points $(\pm\rho, 0)$ to the origin. The precise shape of the optimal synthesis is portrayed in Fig. 3.2 for a particular value of $\rho < 2$.

3.3.3 Proof of Theorem 3.1

The complete argument is actually divided in two parts, i.e. we must decompose $\tilde{F}(\alpha, k_M\pi)$ as the disjoint union of two sets and prove that the statement of the theorem holds true in each of these sets. For that purpose, let $P_\varepsilon^\alpha := C^+\varepsilon_{k_M}(0)$ for $\varepsilon = \pm$ and, for $\xi > 0$, $B_\varepsilon^\alpha(\xi)$, the ball of center P_ε^α and radius ξ. Similarly, set $P_\varepsilon := (2\varepsilon r(\alpha), 0)$ for $\varepsilon = \pm$ and, for $\xi > 0$, $B_{pen}(\xi)$, the ball of center P_ε and radius ξ. For $\xi > 0$, define

$$F(\alpha, \xi) := \tilde{N}(\alpha, k_M\pi) \cap \Big(B_+^\alpha(\xi) \cup B_-^\alpha(\xi)\Big), \quad G(\alpha, \xi) := \tilde{N}(\alpha, k_M\pi) \setminus F(\alpha, \xi),$$

where $\tilde{N}(\alpha, k_M\pi)$ is the open neighborhood of 0 enclosed by $\tilde{F}(\alpha, k_M\pi)$.

As said previously, Theorem 3.1 is easily deduced from the following two propositions.

Proposition 3.4 *Consider the notations defined above. There exist ξ_0, α_0 and C_0 such that, for every $\xi < \xi_0$ and $\alpha < \alpha_0$ with $r(\alpha)$ fixed in $[\varepsilon, 1-\varepsilon]$, the time optimal synthesis associated to $(\Sigma)_\alpha$ inside $F(\alpha, \xi)$ is characterized by an overlap curve γ_α^o defined on $I_\xi := [C_0\xi, \pi - C_0\xi]$. Moreover, γ_α^o converges to γ_{pen}^o in the C^0 topology of I_ξ, and, "above" γ_α^o, the control u takes the constant value -1 and "below" γ_α^o, it is equal to 1.*

Proposition 3.5 *Consider the notations defined above. There exist ξ_0, α_0 and C_0 such that, for every $\xi < \xi_0$ and $\alpha < \alpha_0$ with $r(\alpha)$ fixed in $[\varepsilon, 1-\varepsilon]$, the time optimal synthesis associated to $(\Sigma)_\alpha$ inside $G(\alpha, \xi)$ is characterized by an overlap curve γ_α^o defined on $[0, C_0\xi] \cup [\pi - C_0\xi, \pi]$. Moreover, γ_α^o converges to γ_{pen}^o in the C^0 topology of $[0, C_0\xi] \cup [\pi - C_0\xi, \pi]$, and, "above" γ_α^o, the control u takes the constant value -1 and "below" γ_α^o, it is equal to 1.*

We must consider separately $F(\alpha, \xi)$ and $G(\alpha, \xi)$ because, in each of these subsets, the implicit function arguments we rely on are different. It is clear that an appropriate choice of ξ allows one to obtain Theorem 3.1 from Propositions 3.4 and 3.5. We will therefore only provide the complete proofs of the propositions.

At that point of the discussion, one must check whether the switching curves $C_{k_M}^\varepsilon$, $\varepsilon = \pm 1$ are optimal or not. In that regard and similarly to the case of the linear pendulum, we have the following result:

Lemma 3.3 *Let $\bar{r} \in (0, 1)$. Then, if α is small enough and $r(\alpha) = \bar{r}$, the switching curve $C_{k_M}^\varepsilon$ is never locally optimal.*

Proof of Lemma 3.3. We only treat the case where $S(s) = C^+_{k_M}(s)$ and k_M is odd. As in the proof of Lemma 3.4, we get the following asymptotic expansions (in the normalized coordinates):

$$S(s) = \begin{pmatrix} 2r-1+c_s \\ s_s \end{pmatrix} + \mathcal{O}_I(\alpha), \qquad S(0) = \begin{pmatrix} 2r+\mathcal{O}_I(\alpha) \\ 0 \end{pmatrix}, \tag{3.49}$$

$$S'(s) = \begin{pmatrix} -s_s \\ c_s \end{pmatrix} + \mathcal{O}_I(\alpha), \qquad S'(0) = \begin{pmatrix} 0 \\ 1+\mathcal{O}_I(\alpha) \end{pmatrix}, \tag{3.50}$$

$$S''(s) = \begin{pmatrix} -c_s \\ -s_s \end{pmatrix} + \mathcal{O}_I(\alpha). \tag{3.51}$$

Integrating the above equation, we have

$$S'(s) = S'(0) + \int_0^s S''(\tau)d\tau = \begin{pmatrix} -s_s + \mathcal{O}_I(s\alpha) \\ c_s + \mathcal{O}_I(\alpha) \end{pmatrix}, \tag{3.52}$$

$$S(s) = S(0) + \int_0^s S'(\tau)d\tau = \begin{pmatrix} 2r-1+c_s+\mathcal{O}_I(\alpha) \\ s_s + \mathcal{O}_I(s\alpha), \end{pmatrix} \tag{3.53}$$

and therefore

$$\begin{aligned} \frac{1}{c_\alpha}X_\pm(S(s)) &= \begin{pmatrix} -S_2(s) \\ S_1(s) \pm \frac{\tan\alpha}{\alpha}\sqrt{1-\alpha^2 S_1(s)^2 - \alpha^2 S_2(s)^2} \end{pmatrix} \\ &= \begin{pmatrix} -s_s + \mathcal{O}_I(s\alpha) \\ 2r-1+c_s+\mathcal{O}_I(\alpha) \pm (1+\mathcal{O}_I(\alpha^2)) \end{pmatrix}. \end{aligned}$$

Here S_i, $i = 1, 2$, denote the component of S. Dividing the above equation by $1 + \mathcal{O}_I(\alpha)$, we can assume that the first component is identically equal to $-s_s$. The same can be done with the expression (3.52), so that it is possible to compare the three vectors obtained in this way simply by looking at the second components, which are equal respectively to $2r - 1 + c_s \pm 1 + \mathcal{O}_I(\alpha)$ and $c_s + \mathcal{O}_I(\alpha)$. In particular, the fact that $S(\cdot)$ is nowhere locally optimal if α is small enough follows from the inequalities $2r - 2 + c_s + \mathcal{O}_I(\alpha) < c_s + \mathcal{O}_I(\alpha) < 2r + c_s + \mathcal{O}_I(\alpha)$. ■

A straightforward consequence of the previous result is the presence of a non trivial cut locus in the neighborhood of the south pole enclosed by $F(\alpha, k_M\pi)$. It remains to clearly define that cut-locus, which is the purpose of Propositions 3.4 and 3.5.

3.3.3.1 Proof of Proposition 3.4

As usual, we only provide an argument in the case k_M odd and we fix the remainder equal to $r \in [\varepsilon, 1-\varepsilon]$. We also restrict the discussion, when necessary, to the sequence (α_k), $k \geq 0$ tending to zero so that $r(\alpha_k) = r$.

According to Proposition 3.3 and its proof, $\mathcal{F}(\alpha, k_M\pi)$ is homeomorphic to a circle. To describe the synthesis inside the neighborhood of the south pole enclosed by $\mathcal{F}(\alpha, k_M\pi)$, it is more convenient to use the two dimensional control system $(\widetilde{\Sigma})_\alpha$, which is rewritten as follows by using Eq. (3.42),

$$\dot{z} = c_\alpha A_0 z + u\frac{s_\alpha}{\alpha}\sqrt{1-\alpha^2\|z\|^2}b_0.$$

Set $X^p_\varepsilon(\alpha)z := c_\alpha A_0 z + \varepsilon\frac{s_\alpha}{\alpha}\sqrt{1-\alpha^2\|z\|^2}b_0$. Recall that $\tilde{N}(\alpha, k_M\pi)$ is the open neighborhood of 0 enclosed by $\tilde{F}(\alpha, k_M\pi)$, which is itself the image by M_α of $\mathcal{F}(\alpha, k_M\pi, \cdot)$. We derive that $\tilde{F}(\alpha, k_M\pi)$ is also homeomorphic to a circle and there exists a piecewise smooth parameterization $\gamma_\alpha : [0, 2\pi] \to \tilde{F}(\alpha, k_M\pi)$ so that $\gamma_\alpha(0) = P^\alpha_-$, $\gamma_\alpha(\pi) = P^\alpha_+$ with a loss of regularity only occuring at $s = 0, \pi$ (with two-sided differentials at any order).

Taking into account Lemma 3.3, the cut-locus inside $\tilde{N}(\alpha, k_M\pi)$ is the set of points $M \in \mathbb{R}^2$, besides P^α_ε, such that there exists $(s, s', t) \in (0, \pi) \times (\pi, 2\pi) \times (0, 2\pi)$ for which $M = e^{tX^p_+(\alpha)}\gamma(s') = e^{tX^p_-(\alpha)}\gamma(s)$.

In view of applying an inverse function result for characterizing the overlap curve, we consider the map Φ defined on $[0, \pi] \times [\pi, 2\pi] \times [0, 2\pi]$ by

$$\Phi(s, s', t) := (s, e^{tX^p_+(\alpha)}\gamma(s') - e^{tX^p_-(\alpha)}\gamma(s)),$$

which takes values in $\mathbb{R}^3$. Similarly, for $k \geq 0$, we consider the map Φ_k defined on $[0, \pi] \times [\pi, 2\pi] \times [0, 2\pi]$ by

$$\Phi_k(s, s', t) := (s, e^{tX^p_+(\alpha_k)}\gamma_{\alpha_k}(s') - e^{tX^p_-(\alpha_k)}\gamma_{\alpha_k}(s)).$$

For (Pen), the overlap curve is defined as the set of points M, besides P^α_ε, such that there exists $(s, s', t) \in (0, \pi) \times (\pi, 2\pi) \times (0, 2\pi)$ with $\Phi(s, s', t) = (s, 0, 0)$ and then, the overlap curve can be seen as the map on $w : [0, \pi] \to \mathbb{R}^3$ defined implicitly by $\Phi(w(s)) = (s, 0, 0)$.

Similarly, we would like to define the overlap curve corresponding to $(\Sigma)_{\alpha_k}$, for k large enough, as the function w_k defined by $\Phi_k(w_k(s)) = (s, 0, 0)$. To proceed, we will apply Theorem 3.3. The first task consists of computing $\det D\Phi$ along the overlap curve.

Lemma 3.4 *Along the set of triples* $(s, s', t) \in (0, \pi) \times (\pi, 2\pi) \times (0, 2\pi)$ *for which* $e^{tX^p_+(\alpha)}\gamma(s') = e^{tX^p_-(\alpha)}\gamma(s)$, *we have*

$$\det \mathcal{D}\Phi(s, s', t) = \frac{2\rho(4-\rho^2)\sin s'}{(2-\rho\cos s')^2 + (\rho \sin s')^2}.$$

Proof of Lemma 3.4: One has

$$\det \mathcal{D}\Phi(s, s', t) = \det((e^{tX^p_+(\alpha)})_*\frac{d\gamma}{ds'}, X^p_+(\alpha)e^{tX^p_+(\alpha)}\gamma(s') - X^p_-(\alpha)e^{tX^p_-(\alpha)}\gamma(s)).$$

By taking into account that $\Phi(s, s', t) = 0$, the previous determinant is equal to twice the first component of $(e^{tX^p_+(\alpha)})_*\frac{d\gamma}{ds'}$, i.e., $\det \mathcal{D}\Phi(s, s', t) = 2\rho\sin(s' - t)$. Using Eq. (3.48), one concludes. ■

Set D_{2r} to be equal to the disk centered at 0 of radius $2r$ and $B_\varepsilon(\xi)$, $\varepsilon = \pm$ and $\xi > 0$, the closed balls centered at $P_\varepsilon := (\varepsilon, 0)$ of radius ξ. Finally define $G(\xi)$ as the complement in D_{2r} of (D_{2r} intersected with $B_+(\xi) \cup B_-(\xi)$.

Fix now $\xi > 0$ and choose $0 < s_1(\xi) < s_2(\xi) < \pi$ with $s_1(\xi)$ and $s_2(\xi)$ tending to zero and π respectively as ξ tends to zero, in such a way that $\|P_- - \gamma(s_1(\xi))\| = \|P_+ - \gamma(s_2(\xi))\| := \bar{\xi}$. Finally, choose the compact K to be equal to $G(\bar{\xi})$. Thanks to Lemma 3.4, it is now easy to see that the hypotheses of Theorem 3.3 are verified, and thus Proposition 3.4 is proved. ■

3.3.3.2 Proof of Proposition 3.5

Fix $\xi_0 > 0$ small and the remainder r in $[\varepsilon, 1-\varepsilon]$. Consider the sequence (α_k), $k \geq 0$ tending to zero so that $r(\alpha_k) = r$.

Let φ_k be the map defined on $[0,\pi] \times [\pi, 2\pi] \times [0, 2\pi]$ by

$$\varphi_k(s, s', t) = e^{tX_+^p(\alpha_k)}\gamma_k(s') - e^{tX_-^p(\alpha_k)}\gamma_k(s).$$

For the rest of this paragraph, we drop the index k to get lighter notations.

From the Taylor expansion of φ around the points $(0, 2\pi, 0)$ and $(\pi, \pi, 0)$, we derive the asymptotic behaviors of the cut locus close to the points P_ε^α, $\varepsilon = \pm$, since that cut locus belongs to the level set $\varphi = 0$. We will only perform computations at $(0, 2\pi, 0)$ since they are entirely similar at $(\pi, \pi, 0)$.

Let us call $\varphi^{(1)}$ $\varphi^{(2)}$ the two components of φ. We use $\varphi_s^{(1)}$ to denote the partial derivative of the component $\varphi^{(1)}$ with respect to s evaluated in $(0, 2\pi, 0)$ and we define in an analogous way all the (multiple) partial derivatives evaluated in $(0, 2\pi, 0)$. After computations, we have $\varphi_s^{(1)} = \varphi_{\tilde{s}}^{(1)} = \varphi_t^{(1)} = 0$. In the sequel, we set $\tilde{s} := s' - 2\pi$.

$$\varphi_{ss}^{(1)} = -2r + \mathcal{O}(\alpha)\,, \qquad \varphi_{\tilde{s}\tilde{s}}^{(1)} = 2r + \mathcal{O}(\alpha)\,, \qquad \varphi_{tt}^{(1)} = 2 + \mathcal{O}(\alpha)\,, \qquad \varphi_{s\tilde{s}}^{(1)} = 0\,,$$

$$\varphi_{st}^{(1)} = -2r + \mathcal{O}(\alpha)\,, \qquad \varphi_{\tilde{s}t}^{(1)} = 2r + \mathcal{O}(\alpha)\,,$$

and

$$\varphi_s^{(2)} = 2r + \mathcal{O}(\alpha)\,, \qquad \varphi_{\tilde{s}}^{(2)} = -2r + \mathcal{O}(\alpha)\,, \qquad \varphi_t^{(2)} = 2r + \mathcal{O}(\alpha)\,.$$

We thus get

$$\begin{aligned} \varphi^{(1)}(s, \tilde{s}, t) &= \varphi_{ss}^{(1)} s^2 + \varphi_{\tilde{s}\tilde{s}}^{(1)} \tilde{s}^2 + \varphi_{tt}^{(1)} t^2 + 2\varphi_{st}^{(1)} st + 2\varphi_{\tilde{s}t}^{(1)} \tilde{s}t + \mathcal{O}(|(s, \tilde{s}, t)|^3) \\ &= -2rs^2 + 2r\tilde{s}^2 + 2t^2 - 4rst + 4r\tilde{s}t + \mathcal{O}(\alpha|(s, \tilde{s}, t)|^2) + \mathcal{O}(|(s, \tilde{s}, t)|^3)\,, \end{aligned} \tag{3.54}$$

and

$$\begin{aligned} \varphi^{(2)}(s, \tilde{s}, t) &= \varphi_s^{(2)} s + \varphi_{\tilde{s}}^{(2)} \tilde{s} + \varphi_t^{(2)} t + \mathcal{O}(|(s, \tilde{s}, t)|^2) \\ &= 2rs - 2r\tilde{s} - 2t + \mathcal{O}(\alpha|(s, \tilde{s}, t)|) + \mathcal{O}(|(s, \tilde{s}, t)|^2)\,, \end{aligned} \tag{3.55}$$

where, here, $\mathcal{O}(\cdot)$ is uniform with respect to α. It is important to notice that the maps $\varphi^{(i)}$, $i = 1, 2$, are in fact smooth where they are defined for $s > 0$ and $\tilde{s} < 0$.

Fix $\xi_0 > 0$ small. We are looking at the cut locus in a neighborhood of P_ε^α, and thus, we can assume $|(s, \tilde{s}, t)| < \xi_0$ for some $\xi_0 > 0$. The purpose of subsequent computations consists of expressing $\tilde{s} < 0$ and $t > 0$ as functions of s, for $0 \leq s \leq \xi_0$, by using the equations $\varphi^{(1)} = 0$ and $\varphi^{(2)} = 0$.

From $\varphi^{(2)} = 0$, we obtain that $t = \mathcal{O}(|(s, \tilde{s})|)$ and then

$$t = rs - r\tilde{s} + \mathcal{O}(\alpha|(s, \tilde{s})|) + \mathcal{O}(|(s, \tilde{s})|^2). \tag{3.56}$$

Substituting this expression inside (3.54) the equation $\varphi^{(1)} = 0$ becomes

$$-2r(s - \tilde{s})[s + \tilde{s} + r(s - \tilde{s})] + \mathcal{O}(\alpha|(s, \tilde{s})|^2) + \mathcal{O}(|(s, \tilde{s})|^3) = 0.$$

Since $s\tilde{s} < 0$ for $s > 0$, we can divide the previous equation by $s - \tilde{s}$ obtaining

$$s(1+r) + \tilde{s}(1-r) + \mathcal{O}(\alpha|(s,\tilde{s})|) + \mathcal{O}(|(s,\tilde{s})|^2) = 0, \tag{3.57}$$

and then, $|\tilde{s}| = \mathcal{O}(|s|)$. Therefore, from this estimate and the above ones, we immediately obtain that

$$\tilde{s} = -\Big(\frac{1+r}{1-r} + \mathcal{O}(\alpha)\Big)s + \mathcal{O}(s^2), \qquad t = \Big(\frac{2r}{1-r} + \mathcal{O}(\alpha)\Big)s + \mathcal{O}(s^2). \tag{3.58}$$

We next use the previous Taylor expansions to prove the existence of the overlap curve in a ξ_0-neighborhood of P_-^α, for α small enough. We first apply the implicit function theorem to $\varphi^{(2)}(s,s',t) = 0$ to get that $t = h(s,s')$ defined in $V(0,2\pi)$, a ξ_0-neighborhood of $(0,2\pi)$ in $[0,\pi]\times[\pi,2\pi]$. By Eq. (3.56) (and by remembering that $\varphi^{(2)}$ is smooth on $V(0,2\pi)$), the function h is $\mathcal{C}^1$ in $V(0,2\pi)$, for α small enough.

In order to finally define the overlap curve for $(\Sigma)_\alpha$, it remains to express s' as a $\mathcal{C}^1$ function of s, for s in a ξ_0-neighborhood of 0 in $[0,\pi]$. For doing so, we apply the implicit function theorem to $\phi(s,s') = 0$, where ϕ is the C^1-map defined on $V(0,2\pi)$ by

$$\phi(s,s') := \frac{\varphi^{(1)}(s,s',h(s,s'))}{s - s' + 2\pi}.$$

This is possible thanks to Eq. (3.57) and we deduce that $s' = j(s)$ in $V_{s'}$, a ξ_0-neighborhood of 0 in $[0,\pi]$. The map j is $\mathcal{C}^1$ in $V_{s'}$. Similarly, $t = h(s,j(s)$. The proof of Proposition 3.5 is finished. ■

Remark **3.7** In the proof of Proposition 3.4, instead of applying the inverse function theorem to Φ_k, we could have conducted computations using the map φ_k, as explicit as in the previous argument together with the expansions of Proposition 3.2. In that case, it is possible to obtain immediately from the equations $\varphi_k^{(1)} = 0$ and $\varphi_k^{(2)} = 0$, a invertible linear part with respect to the variables (s',t) and thus conclude with a simple implicit function theorem.

3.4 Case $r = K\alpha$

3.4.1 Description of the minimum time front at $t = k_M\pi$

Fix $K > 0$ and consider the sequence (α_k) such that $r(\alpha_k) = K\alpha_k$, $k \geq 0$. As before, we drop the index k when possible. For α_k small enough, one deduces, from the analysis of [20], that the south pole is not reached at time $k_m\pi = [\frac{\pi}{2\alpha}]\pi$. The next result provides a description of the extremal front at time $k_m\pi$.

Lemma 3.5 *Define the planar curve* $\mathcal{L} : [0,2\pi] \to \mathbb{R}^2$ *by*

$$\mathcal{L}(s) = \begin{pmatrix} c_s(-2K + \pi s_s^2/2) \\ s_s(\pi + 2K - \pi s_s^2/2) \end{pmatrix}. \tag{3.59}$$

Then, for $s \in [0,\pi]$, *we have*

$$\mathcal{F}^+(\alpha, k_M, s) = (\alpha^2\mathcal{L}(s), -1)^T + \mathcal{O}_I(\alpha^3), \tag{3.60}$$

and

$$\frac{d}{ds}\mathcal{F}^+(\alpha, k_M, s) = (\alpha^2 \frac{d}{ds}\mathcal{L}(s), 0)^T + \mathcal{O}_I(\alpha^3). \tag{3.61}$$

At $s = 0$ and $s = \pi$, the derivatives are only one-sided, i.e. as $s > 0$ tends to zero and $s < \pi$ tends to π.

Similarly, we have, for $s \in [0, \pi]$,

$$\mathcal{F}^-(\alpha, k_M, s) = (\alpha^2 \mathcal{L}(s + \pi), -1)^T + \mathcal{O}_I(\alpha^3), \tag{3.62}$$

and

$$\frac{d}{ds}\mathcal{F}^-(\alpha, k_M, s) = (\alpha^2 \frac{d}{ds}\mathcal{L}(s + \pi), 0)^T + \mathcal{O}_I(\alpha^3), \tag{3.63}$$

with one-sided derivatives at $s = 0$ and $s = \pi$.

Proof of Lemma 3.5. This is immediate from Proposition 3.2 applied in the case $r(\alpha) = K\alpha$. ■

For $K < \pi/4$, consider $\theta_d \in (0, \pi/2)$ with $\sin(\theta_d) = 2\sqrt{K/\pi}$. The curve $\mathcal{L}(s)$ has two double points $D^+ = \mathcal{L}(s_1^+) = \mathcal{L}(s_2^+)$, with $s_1^+ = \theta_d$ and $s_2^+ = \pi - \theta_d$, and $D^- = \mathcal{L}(s_1^-) = \mathcal{L}(s_2^-)$, with $s_1^- = \pi + \theta_d$ and $s_2^- = 2\pi - \theta_d$. It also has four cuspidal points Cp_i^ε, $i = 1, 2$ and $\varepsilon = \pm$, corresponding to the values $s = s_{cusp,i}^\varepsilon$, where $s_s^2 = \frac{2+4K/\pi}{3}$.

Finally, let σ be the closed Jordan curve defined as the restriction of $\mathcal{L}(s)$ to $[0, s_1^+] \cup [s_2^+, s_1^-] \cup [s_2^-, 2\pi]$. If $K > \pi/4$, we simply define σ to be $\mathcal{L}$.

At the light of the previous result, we get that $\mathcal{F}(\alpha, k_M\pi, \cdot)$, the complete extremal front at time $k_M\pi$, $\mathcal{F}(\alpha, k_M\pi, \cdot)$, is contained in V_α, a $\mathcal{O}(\alpha^2)$ neighborhood of the south pole. In order to understand the shape of the optimal synthesis inside V_α, we must rescale the whole problem by N_α, the linear mapping from to $\mathbb{R}^3$ onto $\mathbb{R}^2$ defined as the composition of the orthogonal projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ followed by the dilation by $1/\alpha^2$.

For $x \in V_\alpha$, we first consider $(\Lambda)_\alpha$, the image of (Σ) by N_α, i.e. $(\Lambda)_\alpha$ is the planar control system given by

$$(\Lambda)_\alpha : \quad \begin{cases} \dot{z_1} = -c_\alpha z_2, \\ \dot{z_2} = c_\alpha z_1 + u\frac{s_\alpha}{\alpha^2}\sqrt{1 - \alpha^4\|z\|^2}. \end{cases} \tag{3.64}$$

Let $\mathcal{L}_\alpha$ be the image of $\mathcal{F}(\alpha, k_M\pi, \cdot)$ by N_α. From Lemma 3.5, $\mathcal{L}_\alpha$ converges to $\mathcal{L}$ in the C^1 topology. It is clear that, for $K > \pi/4$, $\mathcal{L}_\alpha : [0, 2\pi] \to \mathbb{R}^2$ is homeomorphic to e^{is}, $s \in [0, 2\pi]$. In the case where $K < \pi/4$, the next lemma shows that, for α small enough, $\mathcal{L}_\alpha$ has the same shape as $\mathcal{L}$.

Lemma 3.6 *If $K < \pi/4$, then $\mathcal{L}_\alpha$ is described by the following picture, where $Cp_i^\varepsilon(\alpha) = \mathcal{L}_\alpha(s_{cusp,i}^+(\alpha))$, $i = 1, 2$ and $\varepsilon = \pm$, are cuspidal points and $D^\varepsilon(\alpha)$ are double points with*

$$D^+(\alpha) = \mathcal{L}_\alpha(s_1^+(\alpha)) = \mathcal{L}_\alpha(s_2^+(\alpha)), \qquad D^-(\alpha) = \mathcal{L}_\alpha(s_1^-(\alpha)) = \mathcal{L}_\alpha(s_2^-(\alpha)), \tag{3.65}$$

where $s_{cusp,i}^+(\alpha)$ and $s_i^\varepsilon(\alpha)$ tend respectively to $s_{cusp,i}^+$ and s_i^ε as α tends to zero, for $i = 1, 2$ and $\varepsilon = \pm$. For α small enough, set σ_α, the closed curve defined as the restriction of $\mathcal{L}_\alpha(s)$ to $[0, s_{1,\alpha}^+] \cup [s_{2,\alpha}^+, s_{1,\alpha}^-] \cup [s_{2,\alpha}^-, 2\pi]$. Then, it is a Jordan curve.

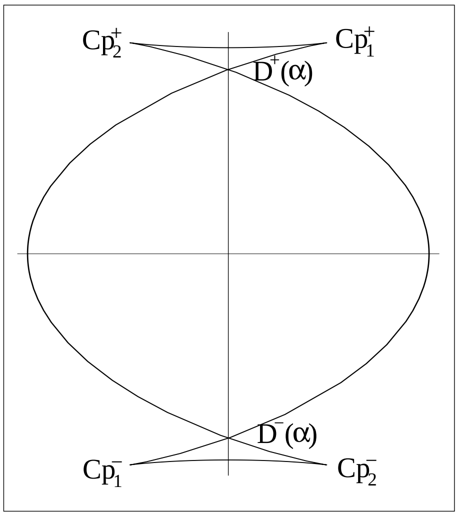

Figure 3.3: Graph of the function $\mathcal{L}_\alpha$ for $K < \pi/4$

Proof of Lemma 3.6: For $i = 1, 2$ and $\varepsilon = \pm$, the existence of the cuspidal points $Cp_i^\varepsilon(\alpha)$ is obtained by applying the implicit function theorem to the equation $DL(s, \alpha) = 0$ where

$$DL(s, \alpha) := \frac{d}{ds}\mathcal{L}_\alpha(s),$$

in the neighborhood of each $(s^\varepsilon_{cusp,i}, 0)$. We have

$$\partial_s DL(s^\varepsilon_{cusp,i}, 0) = \frac{d^2}{ds^2}\mathcal{L}(s^\varepsilon_{cusp,i}) \neq 0$$

and we conclude. The uniqueness of these four points, on $[0, 2\pi]$, is trivial since $DL(s, \alpha) = \frac{d}{ds}\mathcal{L}(s) + \mathcal{O}_I(\alpha)$.

Similarly, for $\varepsilon = \pm$, the existence of the double points $D\varepsilon(\alpha)$ follows after applying the implicit function theorem to

$$DP(s, s', \alpha) = \mathcal{L}_\alpha(s) - \mathcal{L}_\alpha(s'),$$

in the neighborhood of each $(s_1^\varepsilon, (s_2^\varepsilon, 0)$. For the uniqueness, we proceed as before. ■

In the case $K > \pi/4$, we also define σ_α to be equal to $\mathcal{L}_\alpha$.

As a consequence, we are able to characterize $OF(\alpha, k_M\pi)$, the minimum time front at time $k_M\pi$ when $K \neq \pi/4$.

Proposition 3.6 *For α small enough and $K \neq \pi/4$, the minimum time front at time $k_M\pi$, $OF(\alpha_k, k_M\pi)$ is equal to $\tilde{\sigma}_\alpha$, the inverse image on S^2, by N_α, of σ_α.*

Remark **3.8** As a consequence, we deduce that, for $K > \pi/4$, the optimal synthesis between $\mathcal{F}(\alpha, (k_M - 1)\pi)$ and $\mathcal{F}(\alpha, k_M\pi)$ is simply given by the extremal flow whereas, for $K < \pi/4$, there is a loss of optimality along certain extremal curves starting at $\mathcal{F}(\alpha, (k_M - 1)\pi)$ before reaching $\mathcal{F}(\alpha, k_M\pi)$. The values of s corresponding to such curves can be deduced from the previous characterizations of $\mathcal{F}(\alpha, k_M\pi)$ and $OF(\alpha_k, k_M\pi)$.

Proof of Proposition 3.6: recall that $OF(\alpha_k, k_M\pi)$ is a piecewise $\mathcal{C}^1$ submanifold of $\mathcal{F}(\alpha, k_M)$. As in the proof of Proposition 3.3, the result to establish in the case $K > \pi/4$ is a consequence of the fact that $\sigma_\alpha = \mathcal{L}_\alpha : [0, 2\pi] \to \mathbb{R}^2$ is homeomorphic to e^{is}, $s \in [0, 2\pi]$.

In the case $K < \pi/4$, σ_α is a piecewise $\mathcal{C}^1$ Jordan curve homeomorphic to e^{is}, $s \in [0, 2\pi]$. A simple topological argument yields the conclusion. ■

3.4.2 Limit of the synthesis

It remains to describe the limiting dynamics close to the south pole. In order to take the limit, as α tends to zero, in $(\Lambda)_\alpha$, one must reparameterize by the time αt. The limit is then given by the control system

$$(\Lambda): \qquad \begin{cases} \dot{z_1} = 0, \\ \dot{z_2} = u. \end{cases}$$

We now describe the optimal synthesis for the limit problem, i.e. for the problem of reaching in minimum time every point inside σ along (Λ) and starting from σ. Because of the symmetries of σ and because the tangent vector to σ is vertical only at $s = 0$ and $s = \pi$, there exists a unique overlap curve $(Seg)_K$, defined as the segment of the z_1-axis between the points $(-2K, 0)$ and $(2K, 0)$. Above it, the input u takes the constant value -1 and, below that overlap curve, the constant value 1. Integral curves are clearly vertical lines.

We next intend to prove that the optimal synthesis consisting of reaching in minimum time every point inside σ_α along $(\Lambda)_\alpha$ and starting from σ_α converges to the previous synthesis in the following sense.

Theorem 3.2 *Assume that $K \neq \pi/4$. As α tends to zero, the time optimal synthesis associated to $(\Lambda)_\alpha$ inside σ_α is characterized by an overlap curve $(Seg)^\alpha_K$, converging to $(Seg)_K$ in the C^0 topology, and, above $(Seg)^\alpha_K$, the control u takes the constant value -1 and below $(Seg)^\alpha_K$, it is equal to* 1*. Moreover, there exist only two time optimal trajectories reaching the origin and, in the case $K < \pi/4$, these trajectories start from D^ε_α, $\varepsilon = \pm$, the double points of $\mathcal{L}_\alpha$.*

Proof of Theorem 3.2. Fix $K \neq \pi/4$. We first notice that, for α small enough, there are not switching curves inside σ_α. Therefore, the cut-locus may only occur as images by N_α of points $M \in S^2$ such that $M = e^{\frac{t}{\alpha}X_-}\tilde{\sigma}(s) = e^{\frac{t}{\alpha}X_+}\tilde{\sigma}(s')$ for $t \in [0, \frac{2\pi}{\alpha}]$, $s \in [0, \pi]$ and $s' \in [\pi, 2\pi]$. Proceeding exactly as in the proof of Theorem 3.1, we apply inverse function arguments first in neighborhoods of $\sigma_\alpha(0)$ and $\sigma_\alpha(\pi)$ and second in a region enclosed by σ_α excluding such neighborhoods. It is then easy to determine the values of the input u in each connected component of the region enclosed by σ_α minus $(Seg)^\alpha_K$.

By a continuity argument, it is clear that there exist only two time optimal trajectories reaching the origin, one above $(Seg)^\alpha_K$ and one below. Finally, suppose that $K < \pi/4$. In that case, it was proved in [20] that the only extremals starting at a point $\mathcal{L}_\alpha(s)$ and reaching the origin from above the overlap curve $(Seg)^\alpha_K$ correspond to values of s verifying one of the following three possibilities as α tends to zero: (a) s tends to zero, (b) s tends to $\pi/2$, (c) $\mathcal{L}_\alpha(s)$ is a double point also associated to $s' = v(s) - s$. In view of what precedes, only possibility (c) is allowed for optimality. Theorem 3.2 is proved. ■

Remark **3.9** As a consequence of the previous argument, we get that, for α small enough and $K < \pi/4$,

$$s_2^+(\alpha) = v(s_1^+(\alpha)) - s_1^+(\alpha), \ \ s_2^-(\alpha) = 2\pi + v(s_1^-(\alpha) - \pi) - s_1^+(\alpha),$$

where $s_i^\varepsilon(\alpha)$, $i = 1, 2$ $\varepsilon = \pm$, were defined in (3.65).

3.5 Case $r(\alpha) = 0$

We assume here that $r(\alpha) = 0$, i.e. $\alpha_k = \frac{\pi}{2k}$ for $k \geq 1$. From Proposition 3.2, we know that the extremal front at time $([\frac{\pi}{2\alpha}] - 1)\pi = \frac{\pi}{2\alpha} - \pi$, encloses the south pole, is optimal and is approximately (in the $\mathcal{C}^1$ sense) a circle of radius 2α around the south pole. Moreover, at time $[\frac{\pi}{2\alpha}]\pi$, we know that the extremal front must contain the south pole and is equal, up to $\mathcal{O}(\alpha^3)$, to $(\alpha^2\mathcal{L}, -1)^T$ given in (3.60) and (3.62) with $K = 0$. In that case, the minimum time front reduces to the south pole.

It is then easy to see that the only candidate for a limit for the synthesis question in the case

$$r(\alpha) = 0$$

is the synthesis of the linear pendulum studied in Section 3.3.2 and corresponding to $\rho = 2$. Let us first describe briefly that synthesis. Let D_2 and C_2 be the disc and the circle centered at the origin and of radius 2 respectively. The overlap curve inside D_2 coincides with the switching curves and with the trajectories, corresponding to $u = \pm 1$, connecting the points $(\pm\rho, 0)$ to the origin. In particular, it means that an optimal trajectory of the synthesis starting at any point $P \in C_2$ reaches the origin, and thus, there exist an infinite number of optimal trajectories from C_2 to the origin.

For $\alpha > 0$ and $r(\alpha) = 0$, the situation is rather different. Let us first define $\tilde{F}(\alpha, (k_M - 1)\pi)$ to be the image of $\mathcal{F}(\alpha, (k_M - 1)\pi)$ by M_α. Then, for α small enough, it was shown in [20], that the only optimal trajectories starting from $\tilde{F}(\alpha, (k_M - 1)\pi)$ and reaching the origin are those starting at P_+^α and P_-^α. Let us refer to them as γ^+ and γ^-. Therefore, in the case $r(\alpha) = 0$, the synthesis for $\alpha > 0$ is rather different than the synthesis of the limit candidate when α tends to zero. It is a clear indication that the case $r(\alpha) = 0$ is more delicate than the cases $r(\alpha)$ positive constant or $r(\alpha) = K\alpha$. The next proposition collects the few remarks we are able to show for the case $r(\alpha) = 0$.

Proposition 3.7 *Assume that $r(\alpha) = 0$ and α is small enough. Then the switching curve $C_{k_M}^+$ (resp. $C_{k_M}^-$) is optimal for some interval $[0, s(\alpha)]$, $s(\alpha) < \pi$, and it is above (resp. below) γ^+ (resp. γ^-) as long as it is optimal. Moreover, we have*

$$\lim_{\alpha \to 0, r(\alpha)=0} s(\alpha) = \bar{s} := \arccos\sqrt{1/3}. \qquad (3.66)$$

Proof of Proposition 3.7. We only provide an argument for $C_{k_M}^+$. To prove the first statement of the proposition, we reason by contradiction. We thus get the existence of an optimal trajectory starting at $\mathcal{F}(\alpha, (k_M - 1)\pi)$ above P_α^+ and reaching the origin, which is equal to the concatenation of an integral curve of X_- and a piece of γ^+. Therefore, an optimal integral curve of X_-, starting

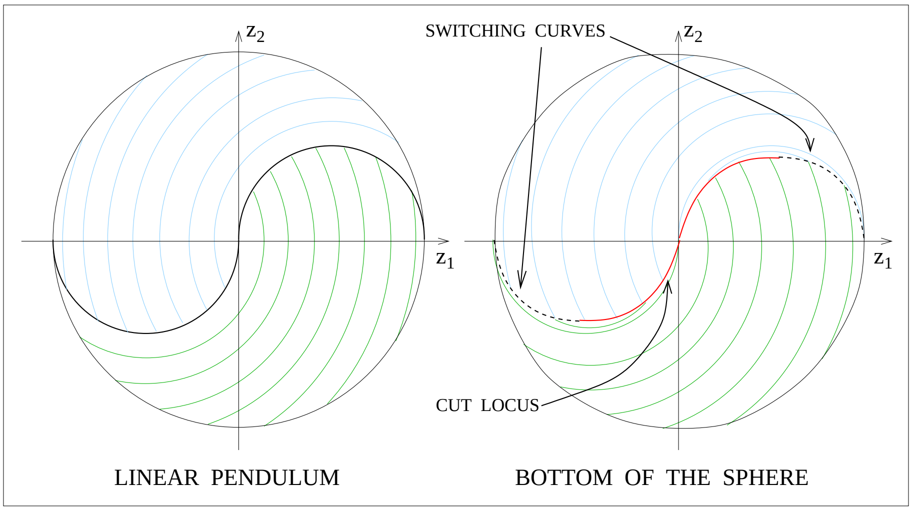

Figure 3.4: Comparison between the optimal synthesis for the linear pendulum and the optimal synthesis on the bottom of the sphere in the case $r(\alpha) = 0$

above γ^+, must either switch or loose optimality before reaching γ^+. If the second possibility occurs, we must have an overlap i.e., at that point an optimal integral curve of X_+ arrives also at that point. Close to P_α^+, the latter would imply that the optimal integral curve of X_+ starts at $\mathcal{F}(\alpha, (k_M - 1)\pi)$ above P_α^+. This is impossible because, from every point of $\mathcal{F}^+(\alpha, (k_M - 1)\pi)$, the value of the optimal control is -1. Let $s(\alpha) \leq \pi$ be the first value of s for which $C_{k_M}^+$ ceases to be optimal. Define

$$H(s) := \det\left(X_+(C_{k_M}^+(s)), \frac{dC_{k_M}^+}{ds}(s), C_{k_M}^+(s)\right),$$

for $s \in [0, \pi]$. Then, $s(\alpha)$ is the smallest solution in $(0, \pi]$ of $H(s) = 0$. It is easy to see that H must be take the value zero before π. We deduce that $s(\alpha) < \pi$. By taking the asymptotic expansion of the previous expression as α tends to zero, we get

$$H(s) = \frac{\pi}{4} s_s \alpha^3 (1 + 3\cos(2s) + \mathcal{O}(\alpha)).$$

Then $s(\alpha)$ must converge to $\bar{s}$ as α tends to zero, the smallest solution in $[0, \pi]$ of $1+3\cos(2s) = 0$. ■

We conclude that there exists a (piecewise) $\mathcal{C}^1$ cut locus passing through the origin (corresponding to the south pole S of the sphere) and connecting $C_{k_M}^+(s(\alpha))$ with $C_{k_M}^-(s(\alpha))$. The shape of the synthesis is portraied in Figure 3.4

3.6 A generalization of the inverse function theorem

The following version of the inverse function theorem is used in the argument of Proposition 3.4.

Theorem 3.3 *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ map and $\mathcal{K} \subset \mathbb{R}^n$ a compact set such that $\Phi_{|\mathcal{K}} : \mathcal{K} \to \Phi(\mathcal{K})$ is bijective and the differential $D\Phi(x)$ is invertible for $x \in K$. Then, there exists an open neighborhood $U \supset \mathcal{K}$ such that $\Phi_{|U}$ is a $\mathcal{C}^1$ diffeomorphism.*

Let now $(\Phi_k)_{k\geq 1}$ be a sequence of $\mathcal{C}^1$ maps converging in the $\mathcal{C}^1_{loc}$ sense to Φ. Then, for every open set $\tilde{U}$ with closure included in U, there exists $\bar{k}$ such that, for every $k \geq \bar{k}$, ${\Phi_k}_{|\tilde{U}}$ is a $\mathcal{C}^1$ diffeomorphism and, for every compact subset $\tilde{\mathcal{K}}$ of $\tilde{U}$, $\Phi(\tilde{\mathcal{K}}) \subset \Phi_k(\tilde{U})$ and $\lim\limits_{k\to\infty} \Phi_k^{-1}(v) = \Phi^{-1}(v)$ uniformly with respect to $v \in \Phi(\tilde{\mathcal{K}})$.

Proof. Let us define, for $k \geq 0$, the following open neighborhoods of $\mathcal{K}$

$$A := \{x \in \mathbb{R}^n \, : \, \det D\Phi(x) \neq 0\} \qquad A_k := \cup_{x\in\mathcal{K}} B\Big(x, \frac{1}{k}\Big) \cap A\,.$$

In view of the inverse function theorem, in order to conclude the proof of the first part, it is enough to show that for k large enough the restriction $\Phi_{|A_k}$ is one-to-one.

We argue by contradiction. Let $x_k \neq y_k \in A_k$ such that $\Phi(x_k) = \Phi(y_k) \quad \forall k$. Then, up to extractions of subsequences, we can assume that the two sequences converge to $\bar{x}$ and $\bar{y}$ respectively. Since $\bar{x}, \bar{y} \in \cap_k A_k = \mathcal{K}$ and $\Phi(\bar{x}) = \Phi(\bar{y})$, we deduce that $\bar{x} = \bar{y}$. However, since $\det D\Phi(\bar{x}) \neq 0$, we have that Φ is bijective in a neighborhood of $\bar{x}$, which contradicts the assumption $\Phi(x_k) = \Phi(y_k)$ for k large enough.

The proof of the second part is similar. First, fix a subset $\tilde{U}$ of U. By the uniform convergence of $D\phi_k$ to $D\phi$ on every compact subset of U, we get $\det D\Phi_k(x) \neq 0$ for every $x \in \tilde{U}$ and k large enough. We also obtain that Φ_k is one-to-one with the same argument as above. For the remaining results to establish, they simply follow from the uniform convergence of Φ_k to Φ on every compact subset of U. ∎

Chapter 4

Time Optimal Swing-Up of the Planar Pendulum

In this chapter we discuss the global structure of the time optimal trajectories to swing up a planar pendulum on a cart. Specifically, we consider only the dynamics of the pendulum and take the acceleration w of the cart as the control input.

Let x_1 be the angle among the pendulum and the upright position, increasing in the clockwise direction. Then the equations of motion are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{mgl}{I}\sin x_1 - \frac{mgl}{I} u \cos x_1 \,, \end{aligned} \tag{4.1}$$

where m is the pendulum's mass, I its moment of inertia, l the distance from the pivot to the centre of mass, $g \approx 9.81$ the gravitational field strength, $u = \frac{w}{g}$ is the control input. Notice that the domain of our system is the cylinder $S \times \mathbb{R}$.

4.1 Qualitative Analysis of the Pendulum

Our aim is to stabilize in minimum time the system (4.1) to the origin, which is an equilibrium for the uncontrolled system, starting from any arbitrary point of $S \times \mathbb{R}$. For this purpose we use the techniques of time optimal synthesis on two dimensionsional manifolds described in Chapter 1.

We consider the time optimal control of the pendulum with equations of motion corresponding to $\frac{mgl}{I} = 1$ and with the bound $|u| \leq 1$, so that (4.1) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - u \cos x_1 \,, \end{aligned} \tag{4.2}$$

i.e. it takes the form (1.1) with $F(x) = \begin{pmatrix} x_2 \\ \sin x_1 \end{pmatrix}$ and $G(x) = \begin{pmatrix} 0 \\ -\cos x_1 \end{pmatrix}$.

We apply the Pontryagin Maximum Theorem to the time optimal problem, and we use the notations and the results described in Chapter 1 to describe its solutions.

The Hamiltonian is $H = \lambda_1 x_2 + \lambda_2(\sin x_1 - u\cos x_1)$, and the adjoint variables satisfy the differential equation

$$\begin{aligned}\dot{\lambda}_1 &= -\lambda_2(\cos x_1 + u\sin x_1)\\ \dot{\lambda}_2 &= -\lambda_1\,.\end{aligned} \tag{4.3}$$

The switching function is $\phi = -\lambda_2 \cos x_1$ and, by Proposition 1.1, the optimal control is $u^* = \mathrm{sgn}(\phi)$. We compute the Lie bracket

$$[F,G](x) = \cos x_1 \frac{\partial}{\partial x_1} + x_2 \sin x_1 \frac{\partial}{\partial x_2}$$

and the functions

$$\begin{aligned}\Delta_A(x) &= -x_2 \cos x_1\\ \Delta_B(x) &= \cos^2 x_1\,.\end{aligned}$$

The set of ordinary points, i.e. the set of points x with $\Delta_A(x) \neq 0$ and $\Delta_B(x) \neq 0$ is

$$\Omega = \{x : x_2 \neq 0, x_1 \neq \pm\frac{\pi}{2}\}\,.$$

Also

$$f_S(x) = \frac{\cos x_1}{x_2}\,.$$

Ω is split into four regions where f_S has a constant sign. In the region

$$\Big\{x \mid x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2}), x_2 > 0\Big\} \;\cup\; \Big\{x \mid x_1 \in (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi), x_2 < 0\Big\}$$

$f_S(x) > 0$ so by Lemma 1.1, the optimal control can switch at most once from $u = -1$ to $u = +1$. In the region

$$\Big\{x \mid x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2}), x_2 < 0\Big\} \;\cup\; \Big\{x \mid x_1 \in (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi), x_2 > 0\Big\}$$

$f_S(x) < 0$, and the optimal control can switch at most once from $u = +1$ to $u = -1$. See Figure 4.1. Note that system (4.2) does not satisfy all the generic conditions in [22] recalled in Chapter 1, since both the sets $\Delta_A^{-1}(0)$ and $\Delta_B^{-1}(0)$ contain the line $x_2 = 0$ and therefore $\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)$ is not locally finite. Although the generic conditions are not satisfied, the existence of a regular time optimal synthesis can be deduced from the analiticity of the vector fields F and G, according to the results of Sussmann ([63]). In the following sections we will be able to give a satisfactory description of the optimal synthesis.

4.1.1 Extremal trajectories

First, we note that there are no optimal trajectories containing singular arcs. Indeed a singular arc must be contained inside $\Delta_B^{-1}(0)$ i.e. it must be a vertical segment, which is not allowed by equation (4.2).
Since the switching function is $\phi = -\lambda_2 \cos x_1$ and for an extremal trajectory the control is defined a.e. as $u^* = \mathrm{sgn}(\phi)$, we deduce that the switchings occur either if $\lambda_2 = 0$ or if $x_1 = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$.

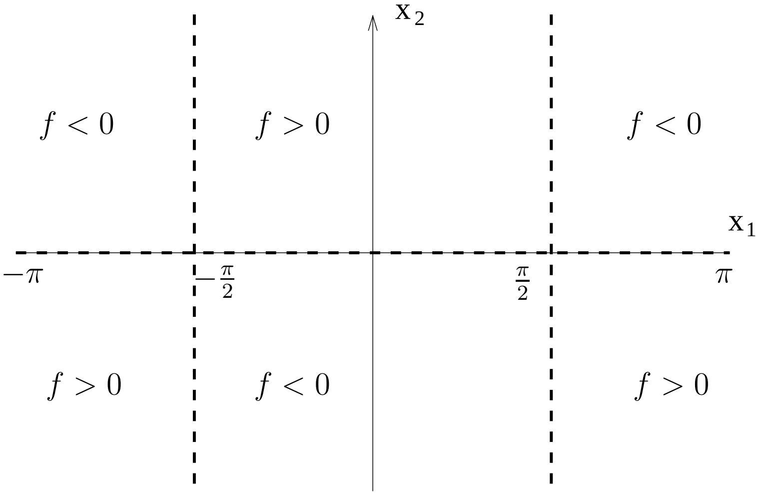

Figure 4.1: Regions where $f > 0$ and $f < 0$ for the pendulum system.

Definition 4.1 *We define γ^+ (resp. γ^-) as the trajectory of (4.2) defined on $(-\infty, 0]$ that reaches the origin with $u = 1$ (resp. $u = -1$) at time 0 and such that the control switches occur exactly at $x_1 = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$.*

Remark **4.1** Notice that the controls corresponding to the trajectories γ^+ and γ^- are respectively given (almost everywhere) by $u(t) = \text{sgn}(\cos x_1(t))$ and $u(t) = -\text{sgn}(\cos x_1(t))$.

Proposition 4.1 *The trajectory γ^+ (resp. γ^-) is extremal on any interval $[\bar{t}, 0]$ and for every point p of γ^+ (resp. γ^-) there exists an extremal trajectory that reaches for the first time γ^+ (resp γ^-) at p and then follows γ^+ (resp. γ^-) until it touches the origin.*

Proof. It is enough to prove the second part of the claim. Let $p = \gamma^+(t_0)$ and consider the extremal trajectory $x(\cdot)$ such that $x(t_0) = p$ the adjoint variables $\lambda(\cdot)$ satisfy $\lambda(t_0) = (1, 0)$. We want to see that the trajectory $x(\cdot)$ differs from γ^+ before reaching the point p and then coincides with γ^+ until it reaches the origin.
We know that the Hamiltonian coincides with $x_2(t_0)$ and therefore it is positive, moreover we deduce from equation (4.3) that $\lambda_2 > 0$ on $(t_0 - \varepsilon, t_0)$ and $\lambda_2 < 0$ on $(t_0, t_0 + \varepsilon)$ for ε small enough, and so the maximality condition given by the PMP is satisfied if the control is $u(t) = -\text{sgn}(\cos x_1(t))$ on $(t_0 - \varepsilon, t_0)$ and $u(t) = \text{sgn}(\cos x_1(t))$ on $(t_0, t_0 + \varepsilon)$. From Remark 4.1 we deduce that our extremal trajectory $x(t)$ is such that $x(t) \neq \gamma^+(t)$ on the interval $(t_0 - \varepsilon, t_0)$ and then coincides with γ^+ on $[t_0, t_0 + \varepsilon)$. By contradiction, assume that there exists $t_1 \in (t_0, 0)$ such that $x(t) = \gamma^+(t)$ on $[t_0, t_1]$ and $x(t) \neq \gamma^+(t)$ on $(t_1, t_1 + \varepsilon)$ for ε small enough. Therefore it must be $u(t) = \text{sgn}(\cos x_1(t))$ on $(t_1 - \varepsilon, t_1)$ and $u(t) = -\text{sgn}(\cos x_1(t))$ on $(t_1, t_1 + \varepsilon)$ and, since we have also that $u(t) = \text{sgn}(\phi(t)) = -\text{sgn}(\lambda_2(t))\text{sgn}(\cos x_1(t))$ a.e. on such intervals, it turns out that $\lambda_2(t_1) = 0$. Since $x_2(t_1) > 0$ and $H(\lambda(t_1), x(t_1)) > 0$ it must be $\lambda_1(t_1) > 0$. Therefore $\dot{\lambda}_2(t_1) = -\lambda_1(t_1) < 0$ which gives a contradiction since from our assumptions $\lambda_2(t) < 0$ on (t_0, t_1).
The proof is analogous for γ^-. ■

Remark **4.2** If we choose as initial condition $x(0) = 0$ and $\lambda(0) = (1, 0)$ we have $H = 0$ that implies $\lambda_2 \neq 0$ for every point of the extremal trajectory such that $x_2 \neq 0$ (otherwise $\lambda_2 = 0$ implies $\lambda_1 = 0$, which is impossible from the PMP). We deduce immediately that γ^- is the corresponding extremal trajectory and it is an abnormal extremal.
Analogously, taking $\lambda(0) = (-1, 0)$, we find that γ^+ is an abnormal extremal.

Remark **4.3** Notice that, if the point $p = (x_1(t_0), x_2(t_0))$ defined in the previous proof is such that $\cos(x_1(t_0)) = 0$, then the corresponding trajectory, unlike the trajectory γ^+, does not have a switching at p.

The following result, which generalizes Proposition 4.1 and Lemma 1.1 in this particular case, is useful to determine the optimal synthesis and in particular the switching curves.

Proposition 4.2 *Consider a normal extremal $x(\cdot) = (x_1(\cdot), x_2(\cdot)) : [t_1, t_2] \to \mathbb{R}$ with $x_2(t) \neq 0$ on (t_1, t_2) and let S be the set of switching times of $x(\cdot)$ and $K = \{t \in (t_1, t_2) : \; x_1(t) = \frac{\pi}{2} + k\pi\,,\ \exists k \in \mathbb{Z}\}$. Then we have the following three possibilities:*

(i) $K = S$,

(ii) *there exists* $\bar{t} \in (t_1, t_2) \setminus K$ *such that* $S = K \cup \{\bar{t}\}$,

(iii) *there exists* $\bar{t} \in K$ *such that* $S = K \setminus \{\bar{t}\}$.

Moreover the corresponding control $u(\cdot)$ must be such that $u(t) = \mathrm{sgn}(x_2)\,\mathrm{sgn}(\cos x_1(t))$ a.e. on $(t_1, \bar{t})$ and $u(t) = -\mathrm{sgn}(x_2)\,\mathrm{sgn}(\cos x_1(t))$ a.e. on $(\bar{t}, t_2)$.

Proof. We have $\mathrm{sgn}(\phi(t)) = -\mathrm{sgn}(\lambda_2(t))\,\mathrm{sgn}(\cos x_1(t))$. Therefore (i) is equivalent to $\mathrm{sgn}(\lambda_2(t)) = \varepsilon$ *a.e* (where $\varepsilon \in \{-1, 1\}$), while (ii) and (iii) hold if and only if $\mathrm{sgn}(\lambda_2(t)) = \varepsilon$ a.e on $(t_1, \bar{t})$ and $\mathrm{sgn}(\lambda_2(t)) = -\varepsilon$ a.e on $(\bar{t}, t_2)$.
Assume that $x(\cdot)$ does not satisfy (i), then in particular there exists $\bar{t}$ such that $\lambda_2(\bar{t}) = 0$. Then, using the fact that $H = \lambda_1(\bar{t})x_2(\bar{t}) > 0$, we obtain $\mathrm{sgn}(\lambda_1(\bar{t})) = \mathrm{sgn}(x_2(\bar{t}))$. From this equality and since $\dot{\lambda}_2 = -\lambda_1$, we find that $\mathrm{sgn}(\lambda_2(t)) = \mathrm{sgn}(x_2(\bar{t}))$ on $(\bar{t} - \varepsilon, \bar{t})$ and $\mathrm{sgn}(\lambda_2(t)) = -\mathrm{sgn}(x_2(\bar{t}))$ on $(\bar{t}, \bar{t} + \varepsilon)$. To conclude the proof it is enough to see that, if the sign of x_2 is fixed, then there is only one time $\bar{t}$ with $\lambda_2(\bar{t}) = 0$. Assume by contradiction that $\bar{t}_1 < \bar{t}_2$ are such that $\lambda_2(\bar{t}_1) = \lambda_2(\bar{t}_2) = 0$ and $\lambda_2(\cdot) \neq 0$ on $(\bar{t}_1, \bar{t}_2)$. Then, since $\mathrm{sgn}(\lambda_2(t)) = -\mathrm{sgn}(x_2)$ on $(\bar{t}_1, \bar{t}_1 + \varepsilon)$ and $\mathrm{sgn}(\lambda_2(t)) = \mathrm{sgn}(x_2)$ on $(\bar{t}_2 - \varepsilon, \bar{t}_2)$ the continuous function $\lambda_2(\cdot)$ must annihilate somewhere on $(\bar{t}_1, \bar{t}_2)$ and we find a contradiction. ■

Remark **4.4** Note that, if we put (4.2) in the form (4.1) with $G(x) = (0, |\cos x_1|)^T$, then Proposition 4.2 simply states that every extremal trajectory in the upper or lower half-plane can switch at most once. This statement is therefore analogous to Lemma 1.1.

The results above permits to partition the cylinder in such a way that in each region the behaviour of the extremal trajectories is essentially different. In particular we are able to individuate four kinds of regions, as it is shown in Figure 4.2. These regions are delimited by $\gamma^{\pm}$, by segments of the axes x_1, x_2 and by (arcs of) special trajectories of (4.2). Such trajectories

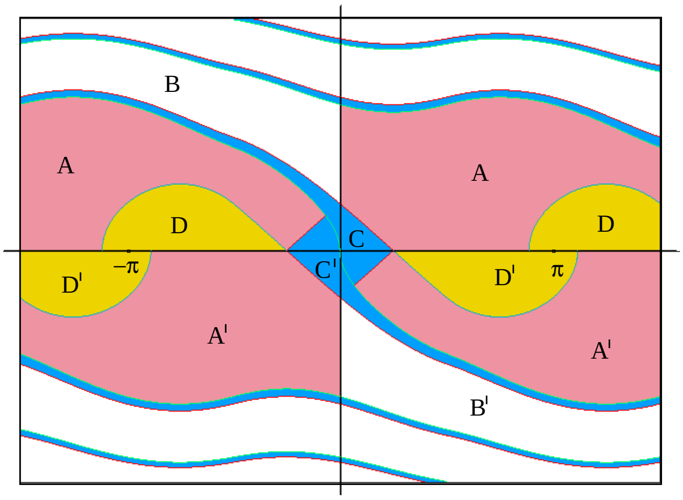

Figure 4.2: Qualitative shape of the synthesis

correspond to arcs of the separatrices for $u = \pm 1$ (that start from the points $(\pm\frac{\pi}{4}, 0)$) and their extensions with $u = \pm\,\mathrm{sgn}(\cos x_1(t))$.

We give now a brief description of the trajectories of (4.2) satisfying the conditions given by Proposition 4.2 inside each region in Figure 4.2, until they reach the corresponding boundary.

- If the initial condition for the minimization problem is inside **A** then it is easy to see that the trajectories corresponding to $u = -\,\mathrm{sgn}(\cos x_1(t))$ must reach the boundary of **A** at some $\gamma^+(t_0)$. On the other hand all the trajectories corresponding to $u = \mathrm{sgn}(\cos x_1(t))$ reach the boundary of **D**, **D′** or **C′**.
- If the initial condition is inside **B** then all the trajectories reach the boundary in a point of γ^+ or at the boundary of **A**.
- If the initial condition is inside **C** then the trajectories corresponding to $u = \mathrm{sgn}(\cos x_1(t))$ stay in **C** until they reach γ^-, while every trajectory corresponding to $u = -\,\mathrm{sgn}(\cos x_1(t))$ must reach the boudary of **A** or **B**.
- All the trajectories that start inside **D** must cross the x_1 axis.

The descriptions of the trajectories starting from the regions **A′**, **B′**, **C′** and **D′** are analogous. In the next sections we will describe the optimal synthesis inside these regions.

4.1.2 Switching curves

Note that, by the previous results, a switching curve (different from $x_1 = \frac{\pi}{2} + k\pi$) can be generated only by a front of extremal trajectories that cross the x_1 axes before reaching the origin. In particular, as we see below, there exists a switching curve C corresponding to extremal trajectories that first switch on C from $u = 1$ to $u = -1$, then (after passing the x_1 axes) switch on $x_1 = \frac{\pi}{2}$ from $u = -1$ to $u = 1$ and finally follow the trajectory γ^- until it reaches the origin

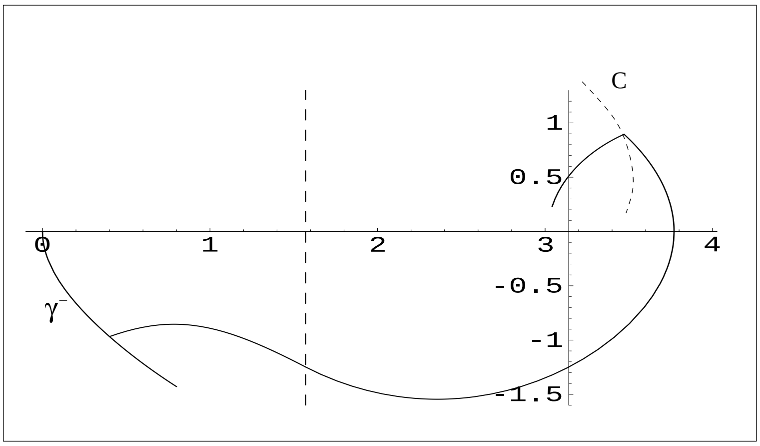

Figure 4.3: The switching curve C

(see Figure 4.3). Notice that, after passing the x_1 axes and before reaching γ^-, these extremal trajectories must "stay below" region $\mathbf{D}'$. We want to determine an equation that describes C. In particular this curve can be found describing the locus in which λ_2 annihilates. To do this we follow backwards the extremal trajectories that reach the origin, integrating by quadratures the equation corresponding to the adjoint variable λ_2.
First we observe that there is a first integral of the system (for a fixed value of u):

$$h(x) = \frac{1}{2}x_2^2 + \cos x_1 + u \sin x_1 \,.$$

Moreover we know that along an extremal trajectory the value of the Hamiltonian is a constant, that we call H. Then:

$$-\dot{\lambda}_2 x_2 + \lambda_2(\sin x_1 - u\cos x_1) = H \quad \Longrightarrow \quad \dot{\lambda}_2 = \frac{\lambda_2}{x_2}(\sin x_1 - u\cos x_1) - \frac{H}{x_2}$$

If $x_2 \neq 0$ and $\dot{x}_2 \neq 0$ we can locally look at λ_2 as a function of x_2 and it holds:

$$\frac{\dot{\lambda}_2}{\dot{x}_2} = \frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \frac{\lambda_2}{x_2} - \frac{H}{x_2(\sin x_1 - u\cos x_1)} \,.$$

We can write the right-hand side of the equation in terms of x_2 using the first integral defined above.
If $u = \pm 1$ we have

$$\sin x_1 - u\cos x_1 = \sqrt{2}\sin(x_1 - u\frac{\pi}{4})\,,$$

while

$$h(x) = \frac{1}{2}x_2^2 + \cos x_1 + u\sin x_1 = \frac{1}{2}x_2^2 + \sqrt{2}\cos(x_1 - u\frac{\pi}{4})\,.$$

Then

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \frac{\lambda_2}{x_2} - \varepsilon\frac{H}{x_2\sqrt{2 - (h - \frac{1}{2}x_2^2)^2}}\,,$$

where $\varepsilon = \text{sgn}(\sin x_1 - u\cos x_1)$ is the only term depending on x_1. The explicit solution to this equation, with initial condition $\lambda_2(x_2^0) = \lambda_2^0$ can be easily obtained with the method of variation of parameters:

$$\lambda_2(x_2) = x_2\left(\frac{\lambda_2^0}{x_2^0} - \varepsilon H \int_{x_2^0}^{x_2} \frac{\mathrm{d}y}{y^2\sqrt{2-(h-\frac{1}{2}y^2)^2}}\right). \tag{4.4}$$

The previous integral cannot be solved exactly, but it can be written in terms of elliptic integrals and it can be easily computed numerically.

Take an extremal trajectory $\tilde{\gamma}$ as described above, and assume that the last switching occurs at $p_0 = (x_1^0, x_2^0) \in \gamma^-$, with $x_1^0 < \pi/4$ and with $\lambda_2^0 = 0$. We want to find an equation for the switching point on C associated to this trajectory, using the continuity of the adjoint variables and the fact that $\lambda_2 = 0$ at this point.

Observe that the value of H is a fixed positive number (and in particular we could rescale $\lambda_1(x_2^0)$ in such a way that $H = 1$), while $h(\cdot)$ is fixed between two consecutive switching points.

It's easy to see that $h(\cdot) = 1$ along γ^- and $h(\cdot) = h_1 = 1 + 2\sin x_1^0$ for the bang arc of $\tilde{\gamma}$ before the last switching point p_0. Note that this arc must be below the separatrix corresponding to $u = 1$ and so we need to assume $h_1 > \sqrt{2} \quad\Rightarrow\quad x_1^0 > \arcsin((\sqrt{2}-1)/2)$.

Consider now a point $\bar{p} = (\bar{x}_1, \bar{x}_2)$ of $\tilde{\gamma}$ with $\bar{x}_1 > 3\pi/4$ and $\bar{x}_2 < 0$. Before reaching the point p_0 from $\bar{p}$ the trajectory $\tilde{\gamma}$ switches at $x_1 = \pi/2$, and, since after the switching the value of $h(\cdot)$ becomes $h_1 = 1 + 2\sin x_1^0$, before the switching it must be $h(\cdot) = h_2 = -1 + 2\sin x_1^0$. Using the expression (4.4) we are able to compute the value of $\lambda_2(\bar{x}_2)$:

$$\begin{aligned}\lambda_2(\bar{x}_2) \;=\; & H\bar{x}_2\Bigg(\int_{x_2^0}^{x_2(x_1=\frac{\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} - \int_{x_2(x_1=\frac{\pi}{4})}^{x_2(x_1=\frac{\pi}{2})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} \\ & - \int_{x_2(x_1=\frac{\pi}{2})}^{x_2(x_1=\frac{3\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} + \int_{x_2(x_1=\frac{3\pi}{4})}^{\bar{x}_2} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}\Bigg)\end{aligned} \tag{4.5}$$

and

$$\begin{aligned} x_2^0 &= -\sqrt{2(1-\cos x_1^0 + \sin x_1^0)} \\ x_2(x_1 = \frac{\pi}{4}) &= -\sqrt{2(1+2\sin x_1^0 - \sqrt{2})} \\ x_2(x_1 = \frac{\pi}{2}) &= -2\sqrt{\sin x_1^0} \\ x_2(x_1 = \frac{3\pi}{4}) &= -\sqrt{2(-1+2\sin x_1^0 + \sqrt{2})}. \end{aligned}$$

Note that the integrals in (4.5) are generalized integrals (the integrands are not well-defined at the extremes).

We compute now $\lim_{\bar{x}_2 \to 0} \lambda_2(\bar{x}_2)$. Applying the equality

$$\int_{z_1}^{z_2} \frac{\mathrm{d}y}{y^2\sqrt{2-(h-\frac{1}{2}y^2)^2}} = -\frac{\sqrt{2-(h-\frac{1}{2}y^2)^2}}{y(2-h^2)}\Bigg|_{z_1}^{z_2} - \frac{1}{4(2-h^2)}\int_{z_1}^{z_2} \frac{y^2\mathrm{d}y}{\sqrt{2-(h-\frac{1}{2}y^2)^2}} \tag{4.6}$$

to the last integral in (4.5) we find immediately that $\lim_{\bar{x}_2\to 0}\lambda_2(\bar{x}_2) = -\frac{H}{\sqrt{2-h_2^2}}$ and it's easy to see that this limit is invariant for every choice of the initial value $\lambda(x_2^1)$ with $(x_1^1, x_2^1) \in \tilde{\gamma}$ and such that the value of the Hamiltonian is H.
So the fact that λ_2 must be continuous at the intersection point between the x_1 axes and the trajectory $\tilde{\gamma}$ is not sufficient to determine the first switching point. But also $\lambda_1 = -\dot{\lambda}_2$ must be continuous and therefore

$$\lim_{x_2\to 0^-}\frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \lim_{x_2\to 0^+}\frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \frac{\dot{\lambda}_2}{\dot{x}_2}\Big|_{x_2=0}. \tag{4.7}$$

Assume that $(\tilde{x}_1, \tilde{x}_2) \in C$ is the first switching point (with $\tilde{x}_2 > 0$). Then we can find an expression for $\lambda_2(x_2)$, $x_2 > 0$ as in (4.5). Therefore we can apply this expression, together with (4.5) and (4.6) to the equality (4.7) obtaining the following equation

$$\begin{aligned}
&\int_{x_2^0}^{x_2(x_1=\frac{\pi}{4})}\frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} - \int_{x_2(x_1=\frac{\pi}{4})}^{x_2(x_1=\frac{\pi}{2})}\frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} - \\
&-\int_{x_2(x_1=\frac{\pi}{2})}^{x_2(x_1=\frac{3\pi}{4})}\frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} - \frac{1}{4(2-h_2^2)}\int_{x_2(x_1=\frac{3\pi}{4})}^{0}\frac{y^2\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} = \\
&= \frac{\sqrt{2-(h_2-\frac{1}{2}\tilde{x}_2^2)^2}}{\tilde{x}_2(2-h_2^2)} - \frac{1}{4(2-h_2^2)}\int_{\tilde{x}_2}^{0}\frac{y^2\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}
\end{aligned} \tag{4.8}$$

that gives $\tilde{x}_2$ (and therefore also $\tilde{x}_1$, since $h(\tilde{x}_1, \tilde{x}_2) = h_2$) in terms of $(x_1^0, x_2^0) \in \gamma^-$.
The switching curve C can be determined solving numerically previous equation, moreover it is clear that there exists a switching curve C' symmetric to C ($p \in C$ if and only if $-p \in C'$).
Notice that, for the same front of extremal trajectories, a possible switching curve that precedes C cannot be optimal. Indeed, if we assume that the x_1 coordinate for the new switching point is less than $3\pi/2$ (otherwise the corresponding extremal trajectory is obviously not optimal) and we write the equality (4.7) in this case, it turns out that the right-hand side and the left-hand side of such equation have different signs.

Consider now the extremal trajectories that reach γ^- with a coordinate $x_1^0 < \arcsin((\sqrt{2}-1)/2)$, i.e. the trajectories that are contained in region **C** before the last switching. In particular these trajectories form a front that crosses the x_1 axes with $x_1 \in [0, \pi/4)$. Now we want to see that this front cannot generate a switching curve (different from $x_1 = \frac{\pi}{2} + k\pi$) on the half-plane $x_2 > 0$. This possibility cannot be excluded a priori using the qualitative results above and in particular by the sign of f_S.
Anyway we have the following result:

Proposition 4.3 *An optimal trajectory that switches for the last time on $x^0 \in \gamma^-$ after having crossed the x_1 axes must switch, before x^0, only at the points such that $x_1 = \frac{\pi}{2} + k\pi$.*

Call Λ the strip of the trajectories that reach γ^- after crossing the axes x_1 and correspond to $u(t) = \mathrm{sgn}(\cos x_1(t))$. This strip corresponds exactly to the region **C**.
The following lemma holds:

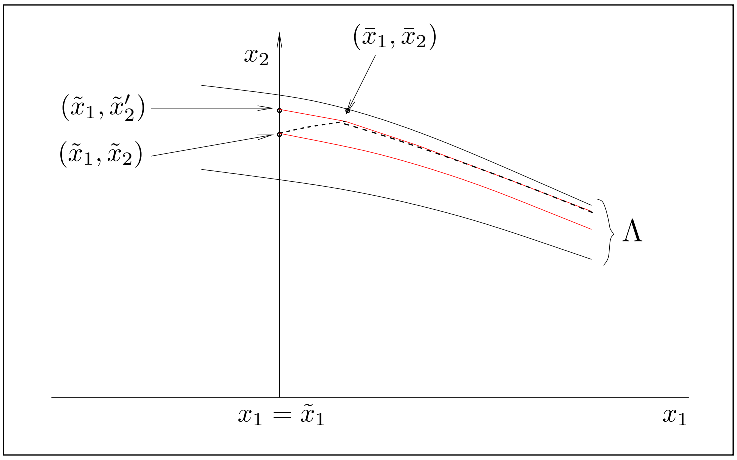

Figure 4.4: Proof of Proposition 4.3

Lemma 4.1 *Let $T(x_1, x_2)$ be the time needed to reach the origin starting from $(x_1, x_2) \in \mathbf{C}$ following the corresponding trajectory on Λ. Then $\frac{\partial T}{\partial x_2}(x_1, x_2) > 0$.*

From the lemma, Proposition 4.3 easily follows: assume by contradiction that there exists an optimal trajectory γ that switches at the point $(\bar{x}_1, \bar{x}_2)$ of Λ and then follows the trajectory of Λ until the origin and consider a point $(\tilde{x}_1, \tilde{x}_2) \in \Lambda$ of γ reached before $(\bar{x}_1, \bar{x}_2)$.
Consider the trajectory corresponding to $u(t) = \text{sgn}(\cos x_1(t))$, that passes through the point $(\bar{x}_1, \bar{x}_2)$ (this trajectories must coincide with γ at the right of $(\bar{x}_1, \bar{x}_2)$): before reaching this point, it must cross the line $x_1 = \tilde{x}_1$ in a point $(\tilde{x}_1, \tilde{x}'_2)$ with $\tilde{x}'_2 > \tilde{x}_2$ (see Figure 4.4). If T is the time needed to run the curve γ starting from $(\tilde{x}_1, \tilde{x}_2)$ then, from the equation $\dot{x}_1 = x_2$, one immediately see that $T(\tilde{x}_1, \tilde{x}'_2) < T$. Moreover by the previous lemma it must be $T(\tilde{x}_1, \tilde{x}_2) < T(\tilde{x}_1, \tilde{x}'_2)$ and therefore $T(\tilde{x}_1, \tilde{x}_2) < T$ that contradicts the optimality of γ.

To prove the lemma consider the trajectories that switch only if $x_1 = \pm\pi/2$ and assume that the last switching occurs on a point (of γ^-) with coordinate $x_1 = x_1^0$. For every point (x_1, x_2) of Λ one can associate a value x_1^0, moreover, if restricted to a line $x_1 = \bar{x}_1$, one can locally consider the inverse map $x_2(x_1^0)$, that takes values on a neighborhood of $\bar{x}_2$ (with $(\bar{x}_1, \bar{x}_2) \in \Lambda$) and which is increasing. Then it is enough to see that $\frac{\mathrm{d}T}{\mathrm{d}x_1^0}(\bar{x}_1, x_2(x_1^0)) > 0$. We know that

$$\dot{x}_1 = x_2 = \text{sgn}(x_2)\sqrt{2\big(h(x) - \sqrt{2}\cos(x_1 - u\frac{\pi}{4})\big)} \tag{4.9}$$

and, similarly

$$\dot{x}_2 = \sqrt{2}\sin\big(x_1 - u\frac{\pi}{4}\big) = \text{sgn}\Big(\sin\big(x_1 - u\frac{\pi}{4}\big)\Big)\sqrt{2 - \left(h(x) - \frac{x_2^2}{2}\right)^2}. \tag{4.10}$$

Then using the equation (4.10) it is possible to write $T(-\frac{\pi}{2}, x_2(x_1^0))$ (corresponding to the trajectories with only one switching) as a sum of two integrals:

$$T\big(-\frac{\pi}{2}, x_2(x_1^0)\big) = \int_0^{\sqrt{2-2\cos x_1^0+2\sin x_1^0}} \frac{\mathrm{d}y}{\sqrt{2-\big(1-\frac{y^2}{2}\big)^2}} + \int_{-\sqrt{2-2\cos x_1^0+2\sin x_1^0}}^{2\sqrt{1+\sin x_1^0}} \frac{\mathrm{d}y}{\sqrt{2-\big(1+2\sin x_1^0-\frac{y^2}{2}\big)^2}}.$$

Therefore it is possible to compute formally the derivative of this quantity and one can see numerically that it is larger than 9.9 for every value of x_1^0.
On the other hand it is possible to compute the time between two consecutive switchings using equation (4.9), and the corresponding derivative with respect to x_1^0 (which is negative) has the following form:

$$-2\cos x_1^0 \int_{-u\pi/2}^{\pi-u\pi/2} \frac{\mathrm{d}z}{\big(2\,(h-\cos z-u\sin z)\big)^{3/2}}.$$

If $n+1$ is the number of switchings needed to reach the origin then $h = 2n+1+2\sin x_1^0$ and the sum of the modulus of all these terms is bounded by the series $C\sum_{n=1}^{+\infty} n^{-\frac{3}{2}} < 3C$ where the constant C can be taken equal to 1.
The lemma follows immediately from these estimates.

4.1.3 Overlap curves and optimal synthesis near the downward position

Note that if x_1^0 is large enough and if P is the corresponding point on C, there exists a second extremal trajectory starting at P with control $u = 1$, then switching at $x_1 = \frac{3\pi}{2}$ and reaching the origin with an arc of γ^+.
In particular it is possible to see numerically that, if x_1^0 is larger than a value $\bar{x} \approx 0.53$, the extremal trajectory starting at $P \in C$ with $u = 1$ is time-optimal.
We deduce that the curve C is not completely optimal and there exists an overlap curve K_1 which starts at a point of C (which is a $(C,K)_1$ frame point in the notations of [22]). It is quite easy to understand that the curve K_1 passes above the point $(\frac{7\pi}{4}, 0)$ (equilibrium point for (4.2) with $u = -1$) and then continues below the trajectory γ^- and close to it.
On the other hand it is possible to see that there is a point of C corresponding to $x_1^0 \approx 0.3$, at which the tangent vector to C is parallel to the vector field corresponding to $u = 1$. At this point ($(C,K)_2$ frame point) starts a second overlap curve K_2. Associated to each point of this curve there are two optimal trajectories: the first one starts with control $u = -1$ and reaches γ^- at a point corresponding to a small value of x_1^0, while the second one starts with control $u = 1$, then switches on C and ends again on γ^-.
The curve K_2 ends at a point in which a further overlap curve K_3 is generated ((K,K) frame point). This curve contains the point $(\pi, 0)$ and the two time-optimal trajectories starting at a point of K_3 have a symmetric behaviour, in the sense that they switch for the first time respectively on C and C' and for the last time on γ^- and γ^+. The complete synthesis around the downward position $(\pi, 0)$ (that corresponds to the stable equilibrium for the uncontrolled pendulum) can be completed using the symmetry with respect to that point.
In Figure 4.5 we made a sketch of the optimal synthesis around the point $(\pi, 0)$.

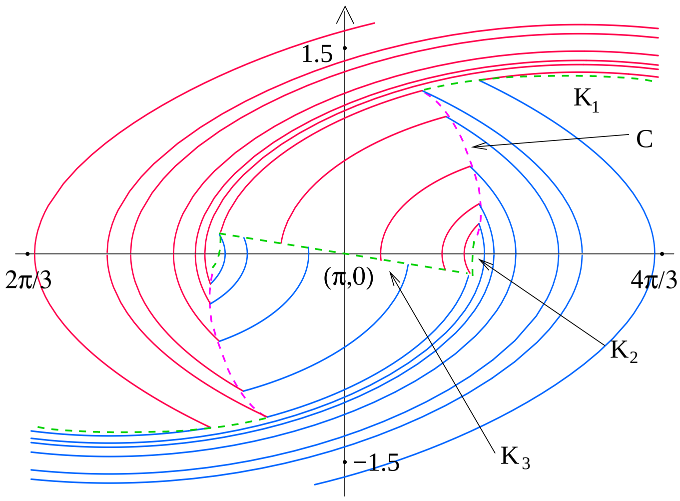

Figure 4.5: Optimal synthesis around $(\pi, 0)$

It is possible to find explicitely the equation of the curve K_1. As in the previous section, suppose that the last switching of the trajectory which reaches the origin with $u = -1$ occurs at $p_0 = (x_1^0, x_2^0) \in \gamma^-$. Then we find an equation for the corresponding point $(\bar{x}_1, \bar{x}_2) \in K_1$ with respect to x_1^0. Assume that $x_1^0 < \pi/2$ (if it is not the case one can proceed in the same way finding a similar equation).
We can apply the equations (4.9),(4.10) on each bang arc of our extremal trajectories, in which it is easy to determine the value of h in terms of x_1^0.
Moreover one can easily see that the last switching of the trajectory which reaches the origin with $u = 1$ corresponds to $x_1 = x_1^1 = 2\pi - \arcsin\left(\sin x_1^0 + \sin \bar{x}_1\right)$, and so we can finally write the equation of the overlap curve:

$$\int_0^{x_2^0} \frac{\mathrm{d}y}{\sqrt{2-(1-\frac{1}{2}y^2)^2}} + \int_{x_1^0-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_1 - 2\sqrt{2}\cos y}} + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_2 - 2\sqrt{2}\cos y}} +$$

$$+ \int_0^{2\sqrt{\sin x_1^0}} \frac{\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} + \int_0^{\bar{x}_2} \frac{\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} =$$

$$\int_{\bar{x}_1-\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_3 - 2\sqrt{2}\cos y}} + \int_{\frac{7\pi}{4}}^{x_1^1+\frac{\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_4 - 2\sqrt{2}\cos y}} + \int_0^{x_2^1} \frac{\mathrm{d}y}{\sqrt{2-(1-\frac{1}{2}y^2)^2}}$$

where the right hand side and the left hand side of the equation represent the times needed to reach the origin along the two possible extremal trajectories, and

$$x_2^0 = -\sqrt{2(1-\cos x_1^0 + \sin x_1^0)}, \quad \bar{x}_2 = \sqrt{2(h_2 - \cos \bar{x}_1 + \sin \bar{x}_1)},$$

$$x_2^1 = \sqrt{2(1-\cos x_1^1 - \sin x_1^1)}, \quad h_1 = 1 + 2\sin x_1^0, \quad h_2 = -1 + 2\sin x_1^0,$$

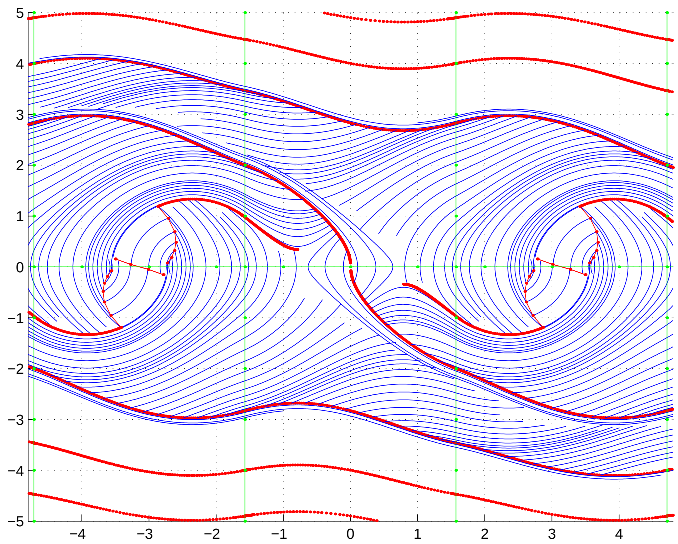

Figure 4.6: The complete time-optimal synthesis

$$h_3 = -1 + 2\sin x_1^0 + 2\sin \bar{x}_1, \quad h_4 = 1 + 2\sin x_1^0 + 2\sin \bar{x}_1.$$

It is also possible to find equation for K_2 and K_3, still using the equations (4.9) and (4.10), and using the expressions of the switching curves C and C' found above. We do not write down explicitely such equations since they are quite long and the computations are completely analogous to those of the previous case.

4.2 Conclusion

Summarizing the previous results we are now able to describe the time-optimal synthesis for the inverted pendulum on the whole cylinder $S \times \mathbb{R}$.

First, we note that the trajectories γ^+ and γ^- are optimal on $(-\infty, 0]$. Indeed, we know from Proposition 4.2 that the only extremal trajectory starting from $\gamma^+(t_0)$ with control $u = \mathrm{sgn}(\cos x_1(t_0))$ is $\gamma^+|_{[t_0,0]}$. On the other hand, from Section 4.1.2, the optimal control corresponding to a trajectory starting from $\gamma^+(t_0)$ with $u = -\mathrm{sgn}(\cos x_1)$ may switch to $u = \mathrm{sgn}(\cos x_1)$ only at a point of C or γ^+ itself. The first case is excluded from the analysis made in Section 4.1.2. In the second case the control passes from $-\mathrm{sgn}(\cos x_1)$ to $\mathrm{sgn}(\cos x_1)$ at some $\gamma^+(t_1)$, with $t_1 < t_0$, so that this trajectory can not be optimal and the only optimal trajectory starting from $\gamma^+(t_0)$ must be $\gamma^+|_{[t_0,0]}$. The same reasoning shows that γ^- is optimal on $(-\infty, 0]$.

Consider now an extremal trajectory starting from the region **B** with control $\mathrm{sgn}(\cos x_1)$. Then, from Proposition 4.2, the control remains $\mathrm{sgn}(\cos x_1)$ until the trajectory reaches the x_1 axis. However there are no optimal trajectories crossing the x_1 axis and starting from **B**, as it can be deduced from Sections 4.1.2, 4.1.3. On the other hand, still from 4.1.2, we know that there are no optimal trajectories switching in a point of C, so that the optimal trajectory must be the one starting with control $-\mathrm{sgn}(\cos x_1)$ and that switches to $u = \mathrm{sgn}(\cos x_1)$ when it reaches

γ^+. In the same way, one can see that every optimal trajectory starting from $\mathbf{B}'$ corresponds to the control $\text{sgn}(\cos x_1)$ until it reaches γ^-. We therefore conclude that the optimal control associated to the regions $\mathbf{B}$ and $\mathbf{B}'$ is $u = \text{sgn}(x_2 \cos x_1)$

Consider now the region $\mathbf{C}$. Let $\gamma^* : [t^*, 0] \to S \times \mathbb{R}$ be the trajectory starting from $\gamma^+(t_0)$ (for some $t_0 < 0$) with control $u = -\text{sgn}(\cos x_1)$ and switching to $u = \text{sgn}(\cos x_1)$ when it reaches again γ^+. As we proved above this trajectory is optimal on every subinterval $[t, 0] \subset [t^*, 0]$ if $\gamma^*(t) \in \mathbf{B}$ but it is not optimal on the whole interval $[t^*, 0]$. In particular, from the continuity of the minimum time function (see for instance [22]) it is not optimal on $[t, 0]$ if $t > t^*$ is close enough to t^*. We deduce that there is a strip of optimal trajectories corresponding to the control $u = \text{sgn}(\cos x_1)$ "just above" γ^+. Also, this strip of trajectories is delimited by an overlap curve K that extends the overlap curve K_1 found in Section 4.1.3 (K can be computed in an analogous way). Above K, the optimal control is $u = -\text{sgn}(\cos x_1)$, and this concludes the description of the optimal synthesis inside $\mathbf{C}$. In the region $\mathbf{C}'$ the situation is symmetric.

The most delicate regions are therefore the regions $\mathbf{A}$ and $\mathbf{D}$. However we know that every trajectory that does not cross the x_1 axis and does not cross C must switch from $u = -\text{sgn}(x_2 \cos x_1)$ to $u = \text{sgn}(x_2 \cos x_1)$ at a point of γ^+. Therefore, the synthesis inside these regions is completely determined in Sections 4.1.2 and 4.1.3 (see Figure 4.5).

The qualitative shape of the optimal synthesis is now completely clarified. After solving the equations given in the previous sections that describe the switching curves and the overlap curve, and the analogous equations that determine the curve K defined above, one can easily obtain the global shape of the synthesis, as it is depicted in Figure 4.6.

Chapter 5

Some Results on the Stability of Switched Systems

The nature of the problems treated in this chapter are rather different from the previous ones. Indeed we will deal with linear dynamical systems of the form

$$\dot{x}(t) = A_{u(t)}x(t), \quad x \in \mathbb{R}^n, \quad A_u \in \mathbb{R}^{n\times n}, \tag{5.1}$$

where n is a positive integer and $u(\cdot) : [0, \infty[\to U$ is a (measurable) function. For these systems, we will investigate the problem of asymptotic stability of the origin, uniformly with respect to $u(\cdot)$. This is a typical problem in the framework of switched systems, that, in general, are intended as dynamical systems consisting of a family of continuous-time subsystems and a rule that establish the switching between them.

5.1 Statement of the Problem and Main Results

We set $\mathbf{A} := \{A_u \ : \ u \in \ U\}$ and we call *switching function* the measurable matrix-valued map $A(\cdot) := A_{u(\cdot)}$. In this way, the switching system (5.1) reads:

$$\dot{x}(t) = A(t)x(t), \text{ with } x \in \mathbb{R}^n, \text{ and } A(\cdot) : [0, \infty[\to \mathbf{A} \text{ a measurable map.} \tag{5.2}$$

In the following, we assume that:

(H0) the set $\mathbf{A}$ is a compact subset of the set of $n \times n$ real matrices.

Moreover, the set of switching functions, denoted by $\mathcal{A}$, is the set of measurable functions $A(\cdot) : [0, \infty[\to \mathbf{A}$. With our assumptions, for every switching function $A(\cdot)$ and initial condition $x_0 \in \mathbb{R}^n$, the corresponding (Carathéodory) solution of (5.2) is defined for every $t \geq 0$. We use $\phi_t^{A(\cdot)}(x_0)$ to denote the flow of (5.2) at time $t \geq 0$ corresponding to the switching function $A(\cdot)$ and starting from x_0.

Let us recall usual notions of stability used for the system (5.2).

Definition 5.1 *Consider the switched system (5.2). We say that the origin is:*

(S) stable*, if for every* $A(\cdot) \in \mathcal{A}$ *and* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $\|\phi_t^{A(\cdot)}(x_0)\| \leq \varepsilon$ *for every* $t \geq 0$*,* $\|x_0\| \leq \delta$*.*

(US) uniformly stable*, if it is stable with* δ *not depending on* $A(\cdot)$*.*

(U) unstable *if it is not stable (i.e. if there exists* $A(\cdot) \in \mathcal{A}$ *s.t. the system* $\dot{x}(t) = A(t)x(t)$ *is unstable as a linear time-varying system.)*

(AS) asymptotically stable*, if it is stable and* attractive *(i.e. there exists* $\delta' > 0$ *so that we have* $\lim_{t\to\infty} \|\phi_t^{A(\cdot)}(x_0)\| = 0$*, for every* $A(\cdot) \in \mathcal{A}$ *and* $x_0 \in \mathbb{R}^n$ *with* $\|x_0\| \leq \delta'$*).*

(UAS) uniformly asymptotically stable *if it is uniformly stable and if, for every* $\varepsilon' > 0$ *and* $\delta' > 0$*, there exists* $T > 0$ *such that, for every switching function* $A(\cdot) \in \mathcal{A}$*,* $t \geq T$ *and* $\|x_0\| \leq \delta'$*, we have* $\|\phi_t^{A(\cdot)}(x_0)\| \leq \varepsilon'$*.*

(GUES) globally uniformly exponentially stable*, if there exist positive constants* M*,* λ *such that:* $\|\phi_t^{A(\cdot)}(x_0)\| \leq M\ e^{-\lambda t}\|x_0\|$*, for every* $x_0 \in \mathbb{R}^n$*,* $t > 0$*,* $A(\cdot) \in \mathcal{A}$*.*

Due to the fact that the dynamics is linear in the state variable, the local and global notions of stability are equivalent. More precisely, it was proved in [10] that, for system (5.2) subject to **H0**, the three notions **AS**, **UAS**, **GUES** and the notion of attractivity are all equivalent (see also the bibliographical note in [39]). In addition, if the system is unstable, then there exists a switching function $A(\cdot) \in \mathcal{A}$ and an initial condition x_0 such that $\lim_{t\to\infty} \|\phi_t^{A(\cdot)}(x_0)\| \to \infty$. In the following, we just refer to the notions of stability, instability and **GUES**.

Remark **5.1** Since for the stability issue, a system of type (5.2), subject to **H0**, is uniquely determined by a compact set $\mathbf{A}$ of $n \times n$ real matrices, we identify $\mathbf{A}$ with the corresponding system for the rest of the chapter. For instance, when we say that $\mathbf{A}$ is **GUES** , we mean that the corresponding system of type (5.2) is **GUES**.
We will often consider the problem of determining whether a system, belonging to a certain class C of systems of type (5.2) subject to **H0**, is **GUES** or not. Notice that fixing such a class of systems means to fix a set of compact subsets of $\mathbb{R}^{n\times n}$ i.e. C can be identified with a subset of $\{\mathbf{A} \subset \mathbb{R}^{n\times n} : \ \mathbf{A} \text{ compact}\}$.

For a system (5.2) subject to **H0**, it is well known that the **GUES** property is a consequence of the existence of a common Lyapunov function.

Definition 5.2 *A* common Lyapunov function *(*LF *for short)* $V : \mathbb{R}^n \longrightarrow \mathbb{R}^+$*, for a switched system* (S) *of the type (5.2), is a continuous function such that* V *is positive definite (i.e.* $V(x) > 0$, $\forall x \neq 0$, $V(0) = 0$*) and* V *is strictly decreasing along nonconstant trajectories of* (S)*.*

Vice-versa, it is known that, given a **GUES** system of the type (5.2) subject to **(H0)**, it is always possible to build a $\mathcal{C}^\infty$ common Lyapunov function (see for instance [32, 48, 49, 50] and the bibliographical note in [39]).

Anyway, the problem of finding a LF or proving the nonexistence of a LF is in general a difficult task. Sometimes, it is even easier to prove directly that a system is **GUES** or unstable. An example is provided below by bidimensional switched systems.

5.1.1 Single-Input Bidimensional Switched Systems

Consider a bidimensional system with *single input* of the type:

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \tag{5.3}$$

where $x \in \mathbb{R}^2$, A and B are two 2×2 real Hurwitz matrices and $u(\cdot)$ is a measurable function defined on $\mathbb{R}^+$ and taking values in U equal either to $[0,1]$ or $\{0,1\}$. In the sequel, *we call* Ξ the class of bidimensional systems of the above form. This class is parameterized by couples of 2×2 real Hurwitz matrices.

Remark **5.2** Whether systems of type (5.3) are **GUES** or not is independent on the specific choice $U = [0,1]$ or $U = \{0,1\}$. In fact, this is a particular instance of a more general result stating that the stability properties of systems (5.2) subject to **H0** only depend on the convex hull of the set **A**, see Proposition 5.1 and Remark 5.6 below, and Section 5.1.4.

In [61], the authors provide a necessary and sufficient condition on the pair (A, B) to share a *quadratic* LF, but it is known (see for instance [32, 51, 29, 67]) that there exist **GUES** linear bidimensional systems not admitting a quadratic LF.

In [32], Dayawansa and Martin posed the problem of finding the minimal degree of a *polynomial* LF. More precisely, the problem posed by Dayawansa and Martin is the following:

Problem P: Define $\Xi_{GUES} \subset \Xi$ as the set of **GUES** systems of the type (5.3). Find the minimal integer m such that every system of Ξ_{GUES} admits a polynomial LF of degree less or equal than m.

Remark **5.3** In the problem posed by Dayawansa and Martin, it is implicitly assumed that a **GUES** system always admits a polynomial common Lyapunov function. This fact was first proved by Molchanov and Pyatnitskii in [48, 49] under the assumption that the set **A** is of the form **A**$= \{(a_{ij})_{i,j=1,\ldots n} \ : \ a_{ij}^- \leq a_{ij} \leq a_{ij}^+\}$. In [50] the authors state the result, with no further details, under the more general hypothesis **A** just compact. In the case in which the convex hull of **A** is finitely generated, the existence of a polynomial common Lyapunov function for **GUES** systems was proved by Blanchini and Miani [12, 13], in the context of uncertain systems.

Since the proofs of Molchanov, Pyatnitskii, Blanchini and Miani need non-trivial intermediate results, in this chapter we provide a self-contained proof, based on a more direct argument, for a set **A** satisfying the weaker hypothesis that its convex hull is compact (see Theorem 5.1 and Remark 5.6)

As for the **GUES** issue, it was completely resolved in [14], where a necessary and sufficient condition for a system of type (5.3) to be **GUES** was found directly, without looking for a LF (see Section 5.3 and Theorem 5.3 for more details). This is a typical example in which it is easier to study directly the stability rather than looking for a LF.

5.1.2 Sets of Functions Sufficient to Check GUES

The concept of Lyapunov function is useful for practical purposes when one can prove that, for a certain class of systems, if a LF exists, then it is possible to find one of a certain type and possibly as simple as possible (e.g. polynomial with a bound on the degree, piecewise quadratic etc.).

More precisely, consider a class C of systems of type (5.2) in $\mathbb{R}^n$ subject to **H0**, in the sense of Remark 5.1. One would like to find a class of functions $\mathcal{S}_C$, identified by a finite number of parameters, which is sufficient to check **GUES** for systems belonging to C i.e., if a system of C admits a LF, then it admits one in $\mathcal{S}_C$. Once such a class of functions is identified, then in order to verify **GUES**, one could use numerical algorithms to check (by varying the parameters) whether a LF exists (in which case the system is **GUES**) or not (meaning that the system is not **GUES**).

For instance, a remarkable result for a given class C of systems in $\mathbb{R}^n$ could be the following:

Claim: there exists a positive integer m (depending on n) such that, whenever a system of C admits a LF, then it admits one that is polynomial of degree less than or equal to m. In other words, the class of polynomials of degree at most m is sufficient to check **GUES** for the class C.

If this result were true, one could use numerical algorithm to check, among all polynomial of degree m (varying the coefficients), if there is one that is a LF. Unfortunately, *this claim is not true*, even for the simplest non trivial case of class of systems in $\mathbb{R}^2$, namely systems of type Ξ (cf. Equation (5.3)).

The next definition formalizes the idea of class of functions sufficient to check **GUES**.

Definition 5.3 *We say that a subset $\mathcal{S}$ of $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is finitely (or q-finitely) parameterized if there exist $\Omega \subset \mathbb{R}^q$ for a positive integer q and a bijective map $\Psi : \Omega \subseteq \mathbb{R}^q \to \mathcal{S} \subset \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$. A subset $\mathcal{S}$ of $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is said to be sufficient to check* **GUES** *for a class C of systems of type (5.2) in $\mathbb{R}^n$ (***SSF** *for short), if every* **GUES** *system of C admits a LF in $\mathcal{S}$. A subset $\mathcal{S}$ of $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is said to be a* (q-parameters) finite set of functions *sufficient to check* **GUES** *for a class C of systems of type (5.2) in $\mathbb{R}^n$ (*finite-**SSF** *for short), if $\mathcal{S}$ is q-finitely parameterized and is an* **SSF** *for C.*

If a subset $\mathcal{S}$ of $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is not finitely parameterizable but is an **SSF** *for a class of systems C, we call $\mathcal{S}$ an ∞-***SSF***.*

<u>*Remark*</u> **5.4** In [13], a concept similar to those introduced in the previous definition was provided and it was called "universal class of Lyapunov functions".

Using the previous definitions, the results and the problem formulated in [32] can be rephrased in the following way:

R1 for systems (5.2) subject to **H0**, the set $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is an ∞-**SSF**;

R2 for linear bidimensional systems of the class Ξ (cf. equation (5.3)), the set of quadratic functions is not a **SSF**;

P let $\mathcal{P}^m$ be the set of polynomial functions of two variables with degree at most m. What is the minimal m such that $\mathcal{P}^m$ is a finite-**SSF** for the linear bidimensional systems of the class Ξ?

Remark **5.5** Notice that, to check numerically the existence of a LF using the concept of finite-**SSF**, one needs some regularity properties of the functions of the family, with respect to the parameters (at least continuity). Anyway, this discussion is out of the purpose of this chapter.

5.1.3 Main Results

We first provide a proof that the implicit assumption of Dayawansa and Martin (i.e. that a linear **GUES** switched system always admits a polynomial LF, cf. Remark 5.3) is correct for a set **A** just compact.

Theorem 5.1 *If the origin is a* **GUES** *equilibrium for the switched system (5.2) subject to* **H0***, then there exists a polynomial LF.*

The above result can be stated equivalently as follows.

THEOREM 1.4 BIS. *The set of polynomials from $\mathbb{R}^n$ to $\mathbb{R}$, is an ∞-***SSF** *for linear switched systems (5.2) subject to* **H0**.

The proof of Theorem 5.1 is given in Section 5.2 and the starting point is the construction of a homogeneous and convex LF W, following the corresponding argument of [32]. The main idea is then to seek for a (homogeneous) polynomial $\tilde{W}$ whose level sets approximate, in some suitable sense, those of W and, finally to show that $\tilde{W}$ is also a LF. A similar argument has also been used in [13], where the authors used an intermediate approximation with polyhedral Lyapunov functions.

Remark **5.6** In Section 5.1.4, it is proved that the **GUES** property of (5.2) depends only on the convex hull of **A** (proof of Proposition 5.1). The Claim of Section 5.1.4 can be applied to a compact set **A** being the convex hull of a set not necessarily compact. For this reason Proposition 5.1 and Theorem 5.1 hold under the weaker hypothesis:

(H0-weak) the set **A** is a measurable subset of the set of $n \times n$ real matrices, whose convex hull is compact.

The core of the chapter consists of showing that problem **P** does not have a solution, i.e. the minimum degree of a polynomial LF cannot be uniformly bounded over the set of all **GUES** systems of the form (5.3). More precisely, we have the following:

Theorem 5.2 *Let $\Xi_{GUES} \subset \Xi$ be the set of all* **GUES** *systems of the type (5.3). If (A, B) is a pair of 2×2 real matrices giving rise to a system of Ξ_{GUES}, let $m(A, B)$ be the minimum value of the degree of any polynomial LF associated to that system. Then $m(A, B)$ cannot be bounded uniformly over Ξ_{GUES}.*

The above result can be stated equivalently as follows.

THEOREM 1.5 BIS. *Let $\mathcal{P}^m$ the set of polynomial functions from $\mathbb{R}^2$ to $\mathbb{R}$ of degree at most m. Then $\mathcal{P}^m$ is not a finite-***SSF** *for Ξ.*

The proof, given in Section 5.4, is based on ideas developed in [14], where necessary and sufficient conditions for **GUES** of systems (5.3) are provided. We build a sequence of **GUES** systems corresponding to a sequence of pairs of matrices (A_i, B_i), $i \geq 1$. The sequence of systems is chosen in such a way that the limit system is uniformly stable but not attractive. In particular, that limit system admits a nontrivial periodic trajectory whose support Γ is a $\mathcal{C}^1$ but not a $\mathcal{C}^2$ submanifold of the plane. To each **GUES** system of the sequence, one considers any polynomial LF $V_{\bar{A}_i,\bar{B}_i}$ whose degree is at most m. We prove that a subsequence of $(V_{\bar{A}_i,\bar{B}_i})$ converges to a non zero polynomial function V (of degree at most m) which admits Γ as a level set. Since Γ is not analytic, a contradiction is reached.

Remark **5.7** The result given by Theorem 5.2 generalizes to dimensions higher than 2 as follows. Let (A_i, B_i), $i \geq 1$, be a sequence of 2×2 matrices such that **i)** the corresponding systems of type (5.3) are **GUES**, **ii)** the limit is uniformly stable but not attractive. As explained above, for this sequence of systems it is not possible to build a sequence of polynomial LF of uniformly bounded degree. Consider now the sequence of systems in $\mathbb{R}^n$, $n \geq 2$, of the form $\dot{\bar{x}} = u\bar{A}_i\bar{x} + (1-u)\bar{B}_i\bar{x}$ corresponding to the matrices:

$$\bar{A}_i = \left(\begin{array}{c|cccc} A_i & & 0 & & \\ \hline & -1 & \dots & \dots & \vdots \\ 0 & 0 & -1 & \dots & \vdots \\ & \vdots & \vdots & \ddots & \vdots \\ & 0 & \dots & \dots & -1 \end{array}\right), \quad \bar{B}_i = \left(\begin{array}{c|cccc} B_i & & 0 & & \\ \hline & -1 & \dots & \dots & \vdots \\ 0 & 0 & -1 & \dots & \vdots \\ & \vdots & \vdots & \ddots & \vdots \\ & 0 & \dots & \dots & -1 \end{array}\right) \tag{5.4}$$

Each system of the sequence is **GUES** but the limit system is not (it is just uniformly stable). Now, if $V_{\bar{A}_i,\bar{B}_i}$, $i \geq 1$, are the corresponding polynomial LFs, then they cannot be polynomials of uniformly bounded degree since this is not true for the restriction of $V_{\bar{A}_i,\bar{B}_i}$ to the first two variables.

Remark **5.8** (EXTENSION TO PIECEWISE POLYNOMIAL FUNCTIONS (PPF)). Another class of functions commonly used to check **GUES** is that of piecewise quadratic functions or more generally piecewise polynomial functions (PPF for short). Here, by a PPF, we mean a continuous function $V \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ together with a finite number q of cones K_j, $1 \leq j \leq q$, based at zero and partitioning $\mathbb{R}^n$ so that V is a polynomial function of degree d_j on K_j, $1 \leq j \leq q$. We refer to $m(V) := \max\{q, d_1, ..., d_q\}$ as the *total degree* of V.

It is tempting to state a version of problem **P** by replacing polynomial functions of degree at most m with PPFs of total degree at most m.

Again, the PPF version of problem **P** does not have a solution for $n = 2$, i.e. the minimum total degree of a piecewise polynomial LF cannot be uniformly bounded over the set of all **GUES** system of the form (5.3). The argument is a simple extension of the proof of Theorem 5.2 and it is briefly mentioned in Remark 5.14.

The last results of the chapter concern the existence and the characterization of a finite-**SSF** for systems of the type (5.2) subject to **H0**.

Let us define the *convex semicone* generated by a set $D \subset \mathbb{R}^n$ as the set of points λx with $\lambda > 0$ and $x \in co(D)$, where $co(D)$ denotes the convex hull of the set D. With this definition, the point $x = 0$ does not belong to the convex semicone generated by a set D, if $0 \notin co(D)$.

First of all, we prove the following (see Section 5.1.4 for the argument).

Proposition 5.1 *For every compact subset* $\mathbf{A}$ *of* $\mathbb{R}^{n\times n}$ *(i.e. verifying* **H0***), let* $S_{\mathbf{A}}$ *be the system of the type (5.2) associated to* $\mathbf{A}$*. Then, for* $\mathbf{A}$ *and* $\mathbf{A}'$ *verifying* **H0** *and generating the same convex semicone,* $S_{\mathbf{A}}$ *is* **GUES** *(resp. uniformly stable) if and only if* $S_{\mathbf{A}'}$ *is* **GUES** *(resp. uniformly stable).*

Based on converse Lyapunov theorems, one can deduce some trivial existence results for finite-**SSF**s. For instance, consider a class C of systems of type (5.2) in $\mathbb{R}^n$ subject to **H0** and satisfying the following property: for every $\mathbf{A} \in C$, the convex hull of $\mathbf{A}$ is generated by at most k matrices $n \times n$, where k is a positive integer. One can build a finite-**SSF** for the class C as follows.

First of all, it follows directly from Definition 5.3 that we may simply assume that C is made of **GUES** systems. Thanks to Proposition 5.1, the class C can be parameterized by k-tuples of $n \times n$ matrices, defined up to their norm. In this way, a $k(n^2-1)$-parameters finite-**SSF** is provided for the class C. For instance, the class Ξ of two-dimensional systems of type (5.3) admits a 6-parameters finite-**SSF**.

The above construction is not explicit and therefore is not useful to check **GUES**. Similarly to Lyapunov functions, it is then clear that the real challenge for finite-**SSF**s concerns their *explicit* characterization. For classes of systems of type (5.2) in $\mathbb{R}^n$ with $n \geq 3$, that issue is completely open in general. In dimension two *we provide an explicit 5-parameters finite-***SSF** *for* Ξ, using the necessary and sufficient conditions for **GUES** given in [14]. This is the content of Section 5.5.

Clearly, Ξ can be parameterized by the pairs (A, B) of 2×2 real Hurwitz matrices, where both A and B are defined up to their norm. The construction of the explicit finite-**SSF** goes as follows. As done previously, we may assume that the pair (A, B) gives rise to a **GUES** system. By taking advantage of the complete characterization of **GUES** systems of the class Ξ given in [14], one can explicitly associate to every **GUES** pair (A, B) a LF as explained next. We start by defining, from (A, B), a pair $(\tilde{A}, \tilde{B})$ giving rise to a system of Ξ which is uniformly stable but not attractive. Such a system admits a closed trajectory whose support Γ is a simple Jordan closed curve (cf. Sections 5.3, 5.4). We then construct a homogeneous positive definite function V whose level set 1 is Γ. We finally show that V is a LF for (A, B). Since the set of $(\tilde{A}, \tilde{B})$ built from the **GUES** pairs (A, B) can be parameterized by using five parameters, we end up with a five-parameters finite-**SSF**.

5.1.4 The Stability Properties of (5.2) Only Depend on the Convex Hull of the set A

We provide here the proof of Proposition 5.1. First, let us show the following:
Claim. Consider the switched system (5.2), under **H0**, and let $\mathbf{A}'$ be a measurable subset of $\mathbf{A}$ such that the convex hull of $\mathbf{A}'$ contains $\mathbf{A}$. Then the following two conditions are equivalent:

i) the system is **GUES** (resp. uniformly stable), with $A(\cdot)$ measurable, taking values in $\mathbf{A}$,

ii) the system is **GUES** (resp. uniformly stable), with $A(\cdot)$ measurable, taking values in $\mathbf{A}'$.

Proof of the Claim. Let $\mathcal{A}$ (resp. $\mathcal{A}'$) be the set of measurable functions $A(\cdot) : [0, \infty[\to \mathbf{A}$ (resp. $A(\cdot) : [0, \infty[\to \mathbf{A}'$). Since $\mathbf{A}'$ is contained in $\mathbf{A}$ then the implication **i)** $\Rightarrow$ **ii)** is obvious.

Let us prove the other implication (that is strictly related to the classical approximability theorems in control theory). We start considering uniform stability. By contradiction, assume that we can find $\epsilon > 0$ satisfying the following. There exists a sequence of points (x_l) tending to zero and a sequence of controls $A_l(\cdot) \in \mathcal{A}$ such that the corresponding trajectory γ_l starting at x_l exits the interior of the ball of radius ϵ for some time t_l. Using classical approximability results (see for instance [4]), the trajectory γ_l can be approximated in the L^∞-norm on $[0, t_l]$ by a trajectory γ_l' corresponding to a switching function $A_l'(\cdot) \in \mathcal{A}'$ and starting at x_l. Hence γ_l' exits the interior of the ball of radius $\epsilon/2$ at time t_l. We reached a contradiction.

Now we want to prove that **GUES** holds in the case $A(\cdot) \in \mathcal{A}'$ implies **GUES** holds in the case $A(\cdot) \in \mathcal{A}$. Since $\mathbf{A}$ is compact, we know (see Definition 5.1 and below) that attractivity and **GUES** are equivalent for the corresponding switched system. Therefore, proceeding by contradiction, we can assume that there is a trajectory $\gamma(\cdot)$ of the switched system corresponding to $A(\cdot) \in \mathcal{A}$ not converging to zero. That means that there exist $\epsilon > 0$ and a sequence t_n of times tending to infinity such that $|\gamma(t_n)| > \epsilon$. As before, we can approximate $\gamma(\cdot)$ on the interval $[0, t_n]$ with a trajectory $\gamma_n(\cdot)$ corresponding to controls taking values in $\mathbf{A}'$, in such a way that $|\gamma_n(t_n)| > \epsilon/2$. But this is impossible since we have assumed **GUES** for the switched system with $A(\cdot) \in \mathcal{A}'$. □

Notice that one can provide an alternative argument for the **GUES** part of Proposition 5.1, by using LFs.

Then one immediately extend to semicones observing that the stability properties of the system (5.2), subject to the compactness hypothesis **H0**, depend only on the shape of the trajectories and not on the way in which they are parameterized.

5.2 Existence of Common Polynomial Lyapunov Functions

In this section, we prove Theorem 5.1. The starting point of the argument follows the first part of the proof of an analogous result in [32].
We define the function $V : \mathbb{R}^n \to \mathbb{R}^+$ by:

$$V(x) = \sup_{A(\cdot)\in\mathcal{A}} \int_0^{+\infty} \|\phi_t^{A(\cdot)}(x)\|^2 \mathrm{d}t \ .$$

The function V is well defined since there exist positive constants C, μ such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$:

$$\|\phi_t^{A(\cdot)}(x)\| \leq C\, e^{-\mu t}\|x\|.$$

Note that V is homogeneous of degree 2 and continuous. In addition, we next show that V is strictly convex. That fact will be crucial later in the argument. Fix $x, y \in \mathbb{R}^n$ and $x \neq y$. Let $A(\cdot)$ be a switching function. The function $x \mapsto \|\phi_t^{A(\cdot)}(x)\|^2$ is strictly convex. Moreover, for every $\lambda \in]0, 1[$, by compactness of $\mathbf{A}$, the expression:

$$\lambda \|\phi_t^{A(\cdot)}(x)\|^2 + (1-\lambda) \|\phi_t^{A(\cdot)}(y)\|^2 - \|\phi_t^{A(\cdot)}(\lambda x + (1-\lambda) y)\|^2,$$

is nonnegative for every $t \geq 0$ and is bounded from below by a positive constant on some interval $[0, \bar{t}]$, uniformly with respect to $A(\cdot)$.

Therefore, dividing the integration interval into the two intervals $[0,\bar{t}]$ and $[\bar{t},+\infty]$ and taking a maximizing sequence of switching functions for $V(\lambda\, x+(1-\lambda)\, y)$, we have:

$$V(\lambda\, x+(1-\lambda)\, y) < \lambda\, V(x)+(1-\lambda)\, V(y), \quad \forall\lambda \in]0,1[, \quad \forall x,y\in\mathbb{R}^n\,.$$

It is shown in [32] that V is a LF. Nevertheless, we need to consider at least $\mathcal{C}^1$ Lyapunov functions, therefore we define:

$$\tilde{V}(x)=\int_{SO(n)} f(R)\, V(Rx)\,\mathrm{d}R\,.$$

where $f: SO(n)\longrightarrow [0,+\infty[$ is a smooth function with support on a small neighborhood of the identity matrix and $\int_{SO(n)} f(R)\,\mathrm{d}R=1$.
In [32], it is also shown that $\tilde{V}$ is a smooth LF except at the origin. Moreover, since V is homogeneous of degree 2 and strictly convex, it follows that $\tilde{V}$ also satisfies such properties.
We consider now the function $W(x)=\sqrt{\tilde{V}(x)}$, which is a continuous, positively homogeneous LF. Therefore, $W^{-1}(1)$ is a compact set. Using the fact that the set $\{x:\, W(x)<1\}$ is strictly convex, we construct a polynomial LF $\tilde{W}$ by approximating the level sets of W. For this purpose, we need the following preliminary result which describes a continuity property of the function $\nabla W(y)\cdot Dx$ with respect to x, y, D.

Lemma 5.1 *Let us set:*

$$M := \min_{x\in W^{-1}(1)\ \ D\in\mathbf{A}} \big[-\nabla W(x)\cdot Dx\big].$$

Then, for every $\varepsilon\in(0,M)$*, there exists* $\delta\in(0,1)$ *such that, for every* $x,y\in W^{-1}(1)$ *with* $\nabla W(y)\cdot x>1-\delta$ *and every* $D\in\mathbf{A}$*, one has:*

$$\nabla W(y)\cdot Dx<-\varepsilon.$$

Proof of the Lemma. First of all, notice that M is well defined since it is the infimum of a continuous function over a compact set. Moreover, $M>0$ because W is a LF.
Since, by homogeneity, $\nabla W(y)\cdot y=W(y)=1$, we have:

$$\nabla W(y)\cdot x=1-\nabla W(y)\cdot(y-x),$$

and then the hypothesis is equivalent to $\nabla W(y)\cdot(y-x)<\delta$.
Reasoning by contradiction, assume that there exists a sequence (x_j,y_j,D_j) such that $\nabla W(y_j)\cdot D_jx_j\geq -\varepsilon$ and $\nabla W(y_j)\cdot(y_j-x_j)$ converges to 0 as j goes to infinity. By compactness, we can find a subsequence of (x_j,y_j,D_j) converging to $(\bar{x},\bar{y},\bar{D})$ and therefore, by continuity, $\nabla W(\bar{y})\cdot\bar{D}\bar{x}\geq-\varepsilon$ and $\nabla W(\bar{y})\cdot(\bar{y}-\bar{x})=0$.
Therefore $\bar{y}-\bar{x}$ belongs to the tangent space at $\bar{y}$ of the strictly convex set $W^{-1}([0,1])$. Since $\bar{x}$ also belongs to the boundary of that set, it must be $\bar{y}=\bar{x}$. It implies $\nabla W(\bar{y})\cdot\bar{D}\bar{x}=\nabla W(\bar{x})\cdot\bar{D}\bar{x}\leq -M$ and we reach a contradiction. □

Remark **5.9** Taking $-x$ instead of x, one obtains that for every $x, y \in W^{-1}(1)$ and every $D \in \mathbf{A}$, then $\nabla W(y) \cdot x < -1 + \delta \Longrightarrow \nabla W(y) \cdot Dx > \varepsilon$.

To conclude the proof of the theorem, we take $\delta \in (0,1)$ corresponding to some ε as in the lemma above, and for every $y \in W^{-1}(1)$ we consider the open sets $B_y = \{x \in \mathbb{R}^n : \nabla W(y) \cdot x > 1 - \delta/2\}$. Since $y \in B_y$, we have that $\{B_y\}_{y \in W^{-1}(1)}$ is an open covering of the compact set $W^{-1}(1)$, and therefore we can find $y_1, \ldots, y_N$ points of $W^{-1}(1)$ such that the union of B_{y_k}, $k = 1, \ldots, N$, covers $W^{-1}(1)$.
Let us define:

$$\tilde{W}(x) := \sum_{k=1}^{N} (\nabla W(y_k) \cdot x)^{2p} .$$

We claim that, for an integer p large enough, $\tilde{W}$ is a polynomial LF. For $D \in \mathbf{A}$ and $x \in \mathbb{R}^n$, $x \neq 0$, we have:

$$\nabla \tilde{W}(x) \cdot Dx = 2p \sum_{k=1}^{N} \big(\nabla W(y_k) \cdot x\big)^{2p-1} \nabla W(y_k) \cdot Dx, \tag{5.5}$$

and we want to show that $\nabla \tilde{W}(x) \cdot Dx < 0$. By homogeneity, it is enough to do it for $x \in W^{-1}(1)$. Set:

$$K := \max_{x,y \in W^{-1}(1),\, D \in \mathbf{A}} \nabla W(y) \cdot Dx.$$

If, for some index k in $\{1, ..., N\}$, one has $|\nabla W(y_k) \cdot x| \leq 1 - \delta$. Then:

$$|\big(\nabla W(y_k) \cdot x\big)^{2p-1} \nabla W(y_k) \cdot Dx| \leq (1 - \delta)^{2p-1} K.$$

Otherwise, if the inequalities $1 - \delta/2 \geq |\nabla W(y_k) \cdot x| > 1 - \delta$ hold, then, by the previous lemma and remark, one has that the corresponding term in the summation must be negative.
Finally, since by the definition of the points y_k, there exist at least two distinct indices k_1 and k_2 such that $x \in B_{y_{k_1}}$ and $-x \in B_{y_{k_2}}$ we have that:

$$\big(\nabla W(y_{k_i}) \cdot x\big)^{2p-1} \nabla W(y_{k_i}) \cdot Dx < -(1 - \delta/2)^{2p-1} \varepsilon.$$

Summing up, we deduce that:

$$\begin{aligned} \nabla \tilde{W}(x) \cdot Dx &< 2p\Big(-2(1-\delta/2)^{2p-1}\varepsilon + (N-2)(1-\delta)^{2p-1} K\Big) = \\ &= -4p(1-\delta/2)^{2p-1}\varepsilon\Big(1 - \tfrac{K(N-2)}{2\varepsilon}\Big(\tfrac{1-\delta}{1-\delta/2}\Big)^{2p-1}\Big). \end{aligned}$$

For p large enough, the right-hand side of previous expression is negative, uniformly with respect to $D \in \mathbf{A}$ and $x \in W^{-1}(1)$. The theorem is proved.

Remark **5.10** One can also check that the level set $\tilde{W}^{-1}(1)$ approximates, as p tends to $+\infty$, the corresponding level set of the function $\max_{k=1,\ldots,N} |\nabla W(y_k) \cdot x|$ (which is a polytope) and, therefore, the latter is a LF as well (cf. [13]).

5.3 Necessary and Sufficient Conditions for GUES of Bidimensional Systems

Consider the following property:

($\mathcal{P}$) The bi-dimensional switched system given by:

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \text{ where } u(\cdot) : [0, \infty[\to [0, 1], \tag{5.6}$$

is **GUES** at the origin.

In this section, we recall the main ideas from [14], to get a necessary and sufficient condition on A and B under which ($\mathcal{P}$) holds, or under which we have at least uniform stability. Theorem 5.3 contains the precise statement of this result.

Remark **5.11** Recall that, by Proposition 1 (proved in Section (5.1.4)), the necessary and sufficient condition for stability of the system (5.6) are the same if we assume $u(\cdot)$ taking values in $\{0, 1\}$ or in $[0, 1]$, or if we multiply A and B by two arbitrary positive constants.

Set $M(u) := uA + (1 - u)B$, $u \in [0, 1]$. In the class of constant functions the asymptotic stability of the origin of the system (5.6) occurs if and only if the matrix $M(u)$ has eigenvalues with strictly negative real part for each $u \in [0, 1]$. So this is a necessary condition for **GUES**. On the other hand it is known that if $[A, B] = 0$ then the system (5.6) is **GUES**. So, in what follows, we always assume the conditions:

H1: Let λ_1, λ_2 (resp. λ_3, λ_4) be the eigenvalues of A (resp. B). Then $Re(\lambda_1)$, $Re(\lambda_2)$, $Re(\lambda_3)$, $Re(\lambda_4) < 0$.

H2: $[A, B] \neq 0$ (that implies that neither A nor B is proportional to the identity).

For simplicity we will also assume:

H3: A and B are diagonalizable in $\mathbb{C}$ (notice that if **H2** and **H3** hold then $\lambda_1 \neq \lambda_2$, $\lambda_3 \neq \lambda_4$).

H4: Let $\mathbf{V}_1, \mathbf{V}_2 \in \mathbb{C}P^1$ (resp. $\mathbf{V}_3, \mathbf{V}_4 \in \mathbb{C}P^1$) be the eigenvectors of A (resp. B). Then $\mathbf{V}_i \neq \mathbf{V}_j$ for $i \in \{1, 2\}$, $j \in \{3, 4\}$ (notice that, from **H2** and **H3**, the V_i's are uniquely defined, $\mathbf{V}_1 \neq \mathbf{V}_2$ and $\mathbf{V}_3 \neq \mathbf{V}_4$, and **H4** can be violated only when both A and B have real eigenvalues).

All the other cases in which **H1** and **H2** hold are the following:

- A or B are not diagonalizable. This case (in which ($\mathcal{P}$) can be true or false) can be treated with techniques entirely similar to the ones of [14].

- A or B are diagonalizable, but one eigenvector of A coincides with one eigenvector of B. In this case, using arguments similar to those of [14], it possible to conclude that ($\mathcal{P}$) is true.

We will call respectively **(CC)** the case where both matrices have non-real eigenvalues, **(RR)** the case where both matrices have real eigenvalues and finally **(RC)** the case where one matrix has real eigenvalues and the other non-real eigenvalues.

In Theorem 5.3 we will report the main result of [14], which gives necessary and sufficient conditions for the stability of the system (5.6) in terms of three (coordinates invariant) parameters given below in Definition 5.4. The first two parameters, ρ_A and ρ_B, depend on the eigenvalues of A and B respectively, and the third parameter $\mathcal{K}$ depends on $Tr(AB)$, which is a Killing-type pseudo-scalar product in the space of 2×2 matrices. As explained in [14], the parameter $\mathcal{K}$ contains the inter-relation between the two systems $\dot{x} = Ax$ and $\dot{x} = Bx$, and it has a precise geometric meaning. It is in 1–1 correspondence with the cross ratio of the four points in the projective line $\mathbb{C}P^1$ that corresponds to the four eigenvectors of A and B.

Definition 5.4 *Let A and B be two 2×2 real matrices and suppose that* **H1**, **H2**, **H3** *and* **H4** *hold. Moreover choose the labels $(_1)$ and $(_2)$ (resp. $(_3)$ and $(_4)$) so that $|\lambda_2| > |\lambda_1|$ (resp. $|\lambda_4| > |\lambda_3|$) if they are real or $Im(\lambda_2) < 0$ (resp. $Im(\lambda_4) < 0$) if they are complex. Define:*

$$\rho_A := -i\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}; \qquad \rho_B := -i\frac{\lambda_3 + \lambda_4}{\lambda_3 - \lambda_4}; \qquad \mathcal{K} := 2\frac{Tr(AB) - \frac{1}{2}Tr(A)Tr(B)}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}.$$

Moreover, define the following function of $\rho_A, \rho_B, \mathcal{K}$:

$$\mathcal{D} := \mathcal{K}^2 + 2\rho_A\rho_B\mathcal{K} - (1 + \rho_A^2 + \rho_B^2). \tag{5.7}$$

Notice that ρ_A is a positive real number if and only if A has non-real eigenvalues and $\rho_A \in i\mathbb{R}$, $\rho_A/i > 1$ if and only if A has real eigenvalues. The same holds for B. Moreover $\mathcal{D} \in \mathbb{R}$.

Under hypotheses **H1** to **H4**, using a suitable 3-parameter changes of coordinates, it is always possible to put the matrices A and B, up the their norm (cf. Remark 5.11), in the normal forms given in the following proposition, where $\rho_A, \rho_B, \mathcal{K}$ appear explicitly (see [14] for more details).

Proposition 5.2 (Normal Forms of 2×2 Matrices) *Let A, B be two 2×2 real matrices satisfying conditions* **H1**, **H2**, **H3** *and* **H4** *given in Section 5.3. In the case in which one of the two matrices has real and the other non-real eigenvalues (i.e. the* **(RC)** *case), assume that A is the one having real eigenvalues. Then there exists a 3-parameter change of coordinates and two constant $\alpha_A, \alpha_B > 0$ such that the matrices A/α_A and B/α_B (still denoted below by A and B) are in the following normal forms:*

Case in which A and B have both non-real eigenvalues (constant control case):

$$A = \begin{pmatrix} -\rho_A & -1/E \\ E & -\rho_A \end{pmatrix}, \qquad B = \begin{pmatrix} -\rho_B & -1 \\ 1 & -\rho_B \end{pmatrix}, \tag{5.8}$$

where $\rho_A, \rho_B > 0$, $|E| > 1$. In this case, $\mathcal{K} = \frac{1}{2}(E + \frac{1}{E})$. Moreover, the eigenvalues of A and B are respectively $-\rho_A \pm i$ and $-\rho_B \pm i$.

Case in which A has real and B non-real eigenvalues ((RC) case):

$$A = \begin{pmatrix} -\rho_A/i + 1 & 0 \\ 0 & -\rho_A/i - 1 \end{pmatrix}, \tag{5.9}$$

$$B = \begin{pmatrix} -\rho_B - c_\alpha/i & -\sqrt{1-c_\alpha^2} \\ \sqrt{1-c_\alpha^2} & -\rho_B + c_\alpha/i \end{pmatrix}, \tag{5.10}$$

where $\rho_B > 0$, $\rho_A/i > 1$, $\mathcal{K} \in i\mathbb{R}$. In this case, the eigenvalues of A and B are respectively $-\rho_A/i \pm 1$ and $-\rho_B \pm i$.

Case in which A and B have both real eigenvalues ((RR) case):

$$A = \begin{pmatrix} -\rho_A/i + 1 & 0 \\ 0 & -\rho_A/i - 1 \end{pmatrix}, \tag{5.11}$$

$$B = \begin{pmatrix} \mathcal{K} - \rho_B/i & 1 - \mathcal{K} \\ 1 + \mathcal{K} & -\mathcal{K} - \rho_B/i \end{pmatrix}, \tag{5.12}$$

where $\rho_A/i, \rho_B/i > 1$ and $\mathcal{K} \in \mathbb{R} \setminus \{\pm 1\}$. In this case, the eigenvalues of A and B are respectively $-\rho_A/i \pm 1$ and $-\rho_B/i \pm 1$.

The parameter $\mathcal{K}$ contains important information about the matrices A and B. They are stated in the following Proposition that can be easily proved using the normal forms given above.

Proposition 5.3 *Let A and B be as in Definition 5.4. Then:* **i)** *if A and B have both complex eigenvalues, then $\mathcal{K} \in \mathbb{R}$ and $|\mathcal{K}| > 1$;* **ii)** *if A and B have both real eigenvalues, then $\mathcal{K} \in \mathbb{R} \setminus \{\pm 1\}$;* **iii)** *$A$ and B have one complex and the other real eigenvalues if and only if $\mathcal{K} \in i\mathbb{R}$.*

Theorem 5.3, stated below for completeness, is the main result of [14], and gives necessary and sufficient conditions for $(\mathcal{P})$ holding true. Below we will point out the main ideas of the proof, since they are particularly important for our purposes

Theorem 5.3 *Let A and B be two real matrices such that* **H1**, **H2**, **H3** *and* **H4**, *given in Section 5.3, hold and define $\rho_A, \rho_B, \mathcal{K}, \mathcal{D}$ as in Definition 5.4. We have the following stability conditions:*

Case (CC) *If A and B have both complex eigenvalues then:*

- **Case (CC.1)** *if $\mathcal{D} < 0$ then $(\mathcal{P})$ is true;*
- **Case (CC.2)** *if $\mathcal{D} > 0$ then:*
  - **Case (CC.2.1)** *if $\mathcal{K} < -1$ then $(\mathcal{P})$ is false;*
  - **Case (CC.2.2)** *if $\mathcal{K} > 1$ then $(\mathcal{P})$ is true if and only if it holds the following condition:*

$$\begin{aligned} \rho_{CC} \;:=\; & \exp\left[-\rho_A \arctan\left(\frac{-\rho_A\mathcal{K} + \rho_B}{\sqrt{\mathcal{D}}}\right) - \right. \\ & \left. \rho_B \arctan\left(\frac{\rho_A - \rho_B\mathcal{K}}{\sqrt{\mathcal{D}}}\right) - \frac{\pi}{2}(\rho_A + \rho_B)\right] \times \\ & \times \sqrt{\frac{(\rho_A\rho_B + \mathcal{K}) + \sqrt{\mathcal{D}}}{(\rho_A\rho_B + \mathcal{K}) - \sqrt{\mathcal{D}}}} < 1 \end{aligned} \tag{5.13}$$

Case (CC.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ holds true or false whether $\mathcal{K} > 1$ or $\mathcal{K} < -1$.*

Case (RC) *If A and B have one of them complex and the other real eigenvalues, define $\chi := \rho_A \mathcal{K} - \rho_B$, where ρ_A and ρ_B are chosen in such a way $\rho_A \in i\mathbb{R}$, $\rho_B \in \mathbb{R}$. Then:*

Case (RC.1) *if $\mathcal{D} > 0$ then $(\mathcal{P})$ is true;*

Case (RC.2) *if $\mathcal{D} < 0$ then $\chi \neq 0$ and we have:*

Case (RC.2.1) *if $\chi > 0$ then (P) is false. Moreover in this case $\mathcal{K}/i < 0$;*

Case (RC.2.2) *if $\chi < 0$, then:*

Case (RC2.2.A) *if $\mathcal{K}/i \leq 0$ then $(\mathcal{P})$ is true;*

Case (RC2.2.B) *if $\mathcal{K}/i > 0$ then $(\mathcal{P})$ is true iff it holds the following condition:*

$$\begin{aligned} \rho_{RC} := \left(\frac{m^+}{m^-}\right)^{-\frac{1}{2}(r(\alpha)/i-1)} e^{-\rho_B \bar{t}} \times \\ \times\left(\sqrt{1-c_\alpha^2}\, m^- \sin \bar{t} - (\cos \bar{t} - \frac{c_\alpha}{i} \sin \bar{t})\right) < 1 \end{aligned} \tag{5.14}$$

where:

$$\begin{aligned} m^{\pm} &:= \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i - 1)\mathcal{K}/i} \\ \bar{t} &= \arccos \frac{-r(\alpha)/i + \rho_B c_\alpha / i}{\sqrt{(1-c_\alpha^2)(1+\rho_B^2)}} \end{aligned}$$

Case (RC.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ holds true whether $\chi < 0$ or $\chi > 0$.*

Case (RR) *If A and B have both real eigenvalues then:*

Case (RR.1) *if $\mathcal{D} < 0$ then $(\mathcal{P})$ is true. Moreover we have $|\mathcal{K}| > 1$;*

Case (RR.2) *if $\mathcal{D} > 0$ then $\mathcal{K} \neq -\rho_A \rho_B$ (notice that $-\rho_A \rho_B > 1$) and :*

Case (RR.2.1) *if $\mathcal{K} > -\rho_A \rho_B$ then (P) is false*

Case (RR.2.2) *if $\mathcal{K} < -\rho_A \rho_B$ then:*

Case (RR.2.2.A) *if $\mathcal{K} > -1$ then (P) is true;*

Case (RR.2.2.B) *if $\mathcal{K} < -1$ then (P) is true iff the following condition holds:*

$$\begin{aligned} \rho_{RR} := -f^{sym}(\rho_A, \rho_B, \mathcal{K}) f^{asym}(\rho_A, \rho_B, \mathcal{K}) \times \\ f^{asym}(\rho_B, \rho_A, \mathcal{K}) < 1, \end{aligned} \tag{5.15}$$

where:

$$\begin{aligned} f^{sym}(\rho_A, \rho_B, \mathcal{K}) &:= \frac{1 + \rho_A/i + \rho_B/i + \mathcal{K} - \sqrt{\mathcal{D}}}{1 + \rho_A/i + \rho_B/i + \mathcal{K} + \sqrt{\mathcal{D}}}; \\ f^{aym}(\rho_A, \rho_B, \mathcal{K}) &:= \left(\frac{\rho_B/i - \mathcal{K}\rho_A/i - \sqrt{\mathcal{D}}}{\rho_B/i - \mathcal{K}\rho_A/i + \sqrt{\mathcal{D}}}\right)^{\frac{1}{2}(\rho_A/i-1)}. \end{aligned}$$

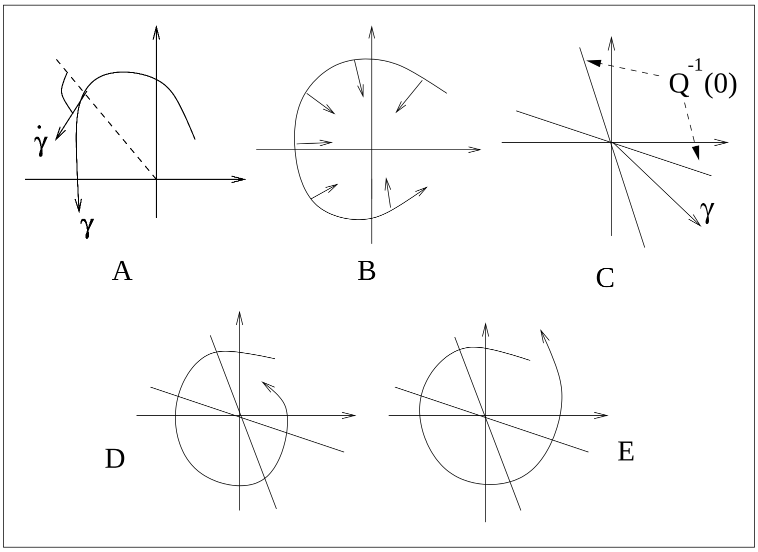

Figure 5.1: Proof of the stability conditions

Case (RR.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ holds true or false whether $\mathcal{K} < -\rho_A\rho_B$ or $\mathcal{K} > -\rho_A\rho_B$.*

Finally, if $(\mathcal{P})$ is not true, then in case **CC.2.2** *with $\rho_{CC} = 1$, case* **(RC.2.2.B)***, with $\rho_{RC} = 1$, case* **(RR.2.2.B)***, with $\rho_{RR} = 1$, case* **(CC.3)** *with $\mathcal{K} < -1$, case* **(RC.3)** *with $\chi > 0$ and case* **(RR.3)** *with $\mathcal{K} > -\rho_A\rho_B$, the origin is just stable.*
In the other cases, the system is unstable.

Remark **5.12** Formula (5.15) is a corrected version of Formula (6), p.93, of [14] and it is proved in Subsection 5.3.1.

We next describe the main idea of the proof. All details can be found in [14].

Definition 5.5 *The* worst trajectory γ_{x_0} *i.e. the trajectory (based at x_0) having the following property. At each time t, $\dot{\gamma}_{x_0}(t)$ forms the smallest angle (in absolute value) with the (exiting) radial direction (Figure 5.1 A).*

Then the system (5.6) is **GUES** if and only if, for each $x_0 \in \mathbb{R}^2$, the worst trajectory γ_{x_0} tends to the origin. The worst trajectory is constructed as follows. We study the locus $Q^{-1}(0)$ (where $Q(x) := \det(Ax, Bx)$) where the two vector fields Ax and Bx are collinear. The quantity $\mathcal{D}$, defined in Definition 5.4, is proportional to the discriminant of the quadratic form Q. We have several cases:

- If $Q^{-1}(0)$ contains only the origin then, in the constant control and **(RC)** case, one vector field points always on the same side of the other and the worst trajectory is a trajectory of a fixed vector field (either Ax or Bx). In that case, the system is **GUES** (case **(CC.1)** and **(RC.1)** of Theorem 5.3), see Figure 5.1, case B. The situation is similar in case **(RR.1)** (the worst trajectory tends to the origin).

- If $Q^{-1}(0)$ does not contain only the origin then it is the union of two lines passing through the origin (since Q is a quadratic form). If at each point of $Q^{-1}(0)$, the two vector fields have opposite direction, then there exists a trajectory going to infinity corresponding to a constant switching function (see Figure 5.1, case C). This correspond to cases **(CC.2.1)**, **(RC.2.1)** and **(RR.2.1)** of Theorem 5.3. In that situation, there exists $u \in [0,1]$ such that the matrix $M(u) := uA + (1-u)B$, $u \in [0,1]$ admits an eigenvalue with positive real part. If at each point of $Q^{-1}(0)$, the two vector fields have the same direction, then the system is **GUES** if and only if the worst trajectory turns around the origin and after one turn the distance from the origin is decreased. (see Figure 5.1, cases D and E). The quantities $\rho_{CC}, \rho_{RC}, \rho_{RR}$ defined in Theorem 5.3 (for the three cases **(CC)**, **(RC)**, **(RR)** resp.) represent the distance from the origin of the worst trajectory (that at time zero is at distance 1), after one half turn. This correspond to cases **(CC.2.2)**, **(RC.2.2)** and **(RR.2.2)** of Theorem 5.3.

- Finally **(CC.3)**, **(RC.3)** and **(RR.3)** are the degenerate cases in which the two straight lines coincide.

5.3.1 Proof of Formula (5.14)

In this paragraph, we prove Formula (5.14), i.e. in the **(RC.2.2.B)** case, we determine an inequality defining the set of parameters $r(\alpha)$, ρ_B, $\mathcal{K}$ such that the property $(\mathcal{P})$, stated in Section 5.3, holds.

Thanks to Proposition 5.2 (see also [14], Appendix B, p.110), we can find a coordinate transformation such that (up to a rescaling of the matrices) A and B are given by equations (5.9), (5.10). In the case **(RC.2.2.B)**, we have $\mathcal{D} := \mathcal{K}^2 + 2\rho_A\rho_B\mathcal{K} - (1+\rho_A^2+\rho_B^2) < 0$, $\chi := \rho_A\mathcal{K} - \rho_B < 0$, $c_\alpha/i > 0$. Moreover, the set $Q^{-1}(0)$ is the union of two lines passing from the origin and, at each point of $Q^{-1}(0)$, the two vector fields point in the same direction. One easily checks that the slope of the two lines defining $Q^{-1}(0)$ is:

$$m^{\pm} = \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-r(\alpha)/i - 1)\sqrt{1-c_\alpha^2}}.$$

Notice that, in our case we have $m^{\pm} < 0$ and $m^+ < m^-$.

In this case, the worst trajectories are concatenations of arcs of integral curves of the vector fields Ax, Bx and rotate counterclockwise around the origin. More precisely, they are integral curves of Ax from the line $x_2 = m^+x_1$ to the line $x_2 = m^-x_1$, and integral curves of Bx otherwise.

Therefore, starting from the point $\begin{pmatrix} 1 \\ m^+ \end{pmatrix}$ (with the field Ax), we follow the worst trajectory until it touches again the line $x_2 = m^+x_1$. Property $(\mathcal{P})$ is then satisfied if and only if $\rho_{RC} < 1$, where ρ_{RC} is the absolute value of the first coordinate of the final point.

One can easily compute that the first switching time is $t_1 = \frac{1}{2}\log\frac{m^+}{m^-}$, which is positive since $\dfrac{m^+}{m^-} > 1$. Moreover, the integral curve of Bx starting from the point $\begin{pmatrix} 1 \\ m^- \end{pmatrix}$ is:

$$e^{-\rho_B t}\begin{pmatrix} -\sqrt{1-c_\alpha^2}\, m^-\, \sin t + (\cos t - \frac{c_\alpha}{i} \sin t) \\ \sqrt{1-c_\alpha^2}\, \sin t + m^-(\cos t + \frac{c_\alpha}{i} \sin t) \end{pmatrix},$$

and, setting the ratio between the second coordinate and the first one equal to m^+, one obtains that the second switching time is $t_2 = \arccos \dfrac{-r(\alpha)/i + \rho_B c_\alpha / i}{\sqrt{(1-c_\alpha^2)(1+\rho_B^2)}}$. Notice that t_2 is well defined if and only if $\mathcal{D} < 0$ (condition which is satisfied in our case). Moreover, t_2 is positive and less than π. Finally, the inequality we was seeking for is:

$$\rho_{RC} = \Big(\frac{m^+}{m^-}\Big)^{-\frac{1}{2}(r(\alpha)/i-1)} e^{-\rho_B t_2}\Big(\sqrt{1-c_\alpha^2}\, m^-\, \sin t_2 - (\cos t_2 - \frac{c_\alpha}{i} \sin t_2)\Big) < 1\,.$$

5.4 Non Existence of a Uniform Bound on the Minimal Degree of Polynomial Lyapunov Functions

In this section, we prove Theorem 5.2. The starting point of the argument is to consider a pair of matrices A and B having both non real eigenvalues (**(CC)** case) and satisfying:

$$\mathcal{D} > 0, \quad \mathcal{K} > 1, \quad \rho_{CC} = 1. \tag{5.16}$$

Such a pair exists. Indeed, Figure 5.2 translates graphically the contents of Theorem 5.3 for a fixed $\mathcal{K} > 1$, in the region of the (ρ_A, ρ_B)–plane where $\rho_A, \rho_B > 0$. The open shadowed region corresponds to values of the parameters ρ_A, ρ_B for which the system is GUES. We denote by S^+ the open subset of the shadowed region where $\mathcal{D} > 0$. The curve C represents the limit case where $\rho_{CC} = 1$. To each internal point of that curve, it is associated a system verifying (5.16), since $\mathcal{D} > 0$. A system corresponding to such a limit case is not asymptotically stable but just stable. Moreover, the worst trajectory (see Definition 5.5, and its construction in Section 5.3) is a periodic curve, whose support is of class $\mathcal{C}^1$ but not of class $\mathcal{C}^2$ (recall that the switchings occur on $Q^{-1}(0)$, i.e. when the linear vector fields corresponding to A and B are parallel).
Fix a point $(\rho_A, \rho_B) \in C$ corresponding to (A, B). Since C is a subset of the boundary of the open set S^+ in the space of parameters (see Figure 5.2), there exists a sequence of points $(\rho_{Ak}, \rho_{Bk}) \in S^+$, for $k \geq 1$, converging to (ρ_A, ρ_B). This exactly means that there exists a sequence of **GUES** pairs (A_k, B_k), $k \geq 1$, such that (A_k, B_k) tends to (A, B) as k goes to ∞.

Let $x = (x_1, x_2)$. For every $k \geq 1$, consider a polynomial LF V_k of degree at most m_k, i.e. $V_k = \sum_{1 \leq i+j \leq m_k} a_{ij}^{(k)} x_1^i x_2^j$. Arguing by contradiction, we assume that the sequence (m_k) is bounded by a positive integer m. Up to multiplication by a constant, we can choose $\sum_{1 \leq i+j \leq m_k} |a_{ij}^{(k)}| = 1$. By compactness, there exists a subsequence of (V_k) (still denoted by (V_k)) which converges (uniformly on compact subsets of $\mathbb{R}^2$) to some non-zero polynomial V with degree at most m. Note that $V(0) = 0$ since the V_k's are LFs.

Fix $x_0 \in \mathbb{R}^2$, $x_0 \neq 0$. Let $T > 0$ be the period of the worst trajectory γ_{x_0} corresponding to the pair (A, B), and starting at x_0 . Note that T is independent of x_0. The curve $\gamma_{x_0} : [0, T] \to \mathbb{R}^2$

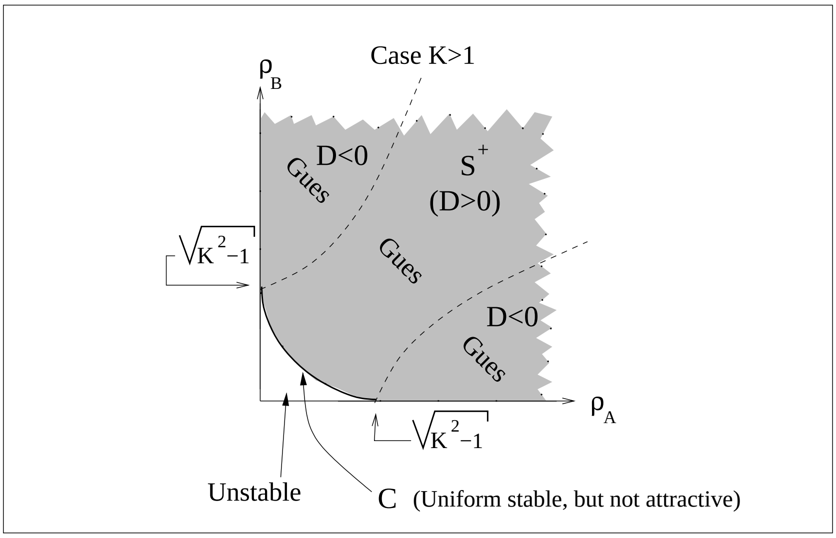

Figure 5.2: **GUES** property in the space of parameters and explicit construction of a 5-parameters **SSF** for systems of type (5.3). S^+ is the region in the (ρ_A, ρ_B)–plane in which the system is **GUES** and $\mathcal{D} > 0$.

can be seen as the concatenation of at most five arcs of integral curves of $\dot{x} = A\,x$ and $\dot{x} = B\,x$ (see Figure 5.3) and satisfies the Cauchy problem:

$$\begin{cases} \dot{x} = C(t)\,x, \\ x(0) = x_0, \end{cases}$$

where $C(t)$ is equal to A or B on subintervals of $[0, T]$.
For $k \geq 1$, consider the Cauchy problem:

$$\begin{cases} \dot{x} = C_k(t)\,x, \\ x(0) = x_0, \end{cases}$$

where $C_k(t) = A_k$ if $C(t) = A$ and $C_k(t) = B_k$ if $C(t) = B$. Then, γ_k is a trajectory of the switched system of the type (5.3) associated to (A_k, B_k). Since, the right-hand side of the previous equation is Lipschitz continuous in x and piecewise continuous in t, then the solutions γ_k converge uniformly to γ_{x_0} on $[0, T]$.
We next show that V remains constant on γ_{x_0}. For $k \geq 1$ and $t \in [0, T]$, one has:

$$\|V_k \circ \gamma_k(t) - V \circ \gamma_{x_0}(t)\| \leq \|V_k \circ \gamma_k(t) - V \circ \gamma_k(t)\| + \|V \circ \gamma_k(t) - V \circ \gamma_{x_0}(t)\|.$$

By uniform convergence of V_k to V and of γ_k to γ_{x_0}, and by continuity of V, we deduce that $V_k \circ \gamma_k(t)$ converges to $V \circ \gamma_{x_0}(t)$ for every fixed t.
Since, for every $k \geq 1$, V_k is a LF for the switched system of the type (5.3) associated to (A_k, B_k), then $V_k \circ \gamma_k$ is a decreasing function and, hence, $V \circ \gamma_{x_0}$ is non-increasing. Moreover $V \circ \gamma_{x_0}(T) = V \circ \gamma_{x_0}(0)$. Therefore, $V \circ \gamma_{x_0}$ must be constant. It implies that there exists $t_1 > 0$

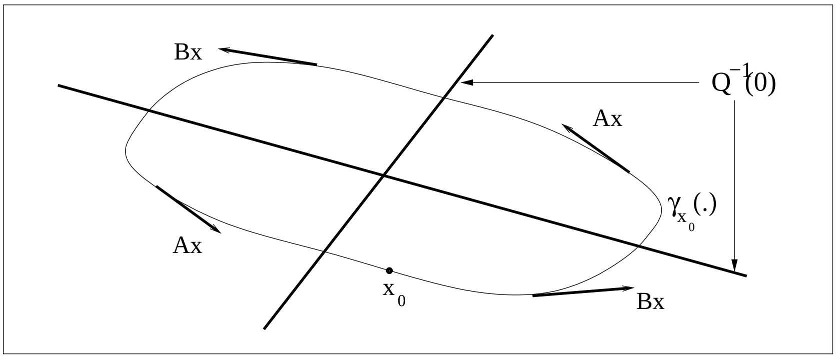

Figure 5.3: The “worst trajectory”

such that either $V(e^{At}x_0)$ or $V(e^{Bt}x_0)$ is constant on $[0, t_1]$. With no loss of generality, assume the first alternative. Since the map $t \mapsto V(e^{At}x_0)$ is real analytic, it follows that $V(e^{At}x_0)$ is constant over the whole real line. By letting t go to $+\infty$, since $e^{At}x_0 \to 0$, we deduce that $V(x_0) = V(0) = 0$. Since x_0 is an arbitrary non zero point of $\mathbb{R}^2$, we get that $V \equiv 0$, which is not possible.

Remark **5.13** The construction of the sequence (A_i, B_i) with unbounded degree for polynomial LF was performed for matrices having both non real eigenvalues (that corresponds to the **(CC)** case). The same construction can be reproduced for the **(RC)** and **(RR)** cases.

Remark **5.14** For the PPF case (see Remark 5.8), the above argument can be easily modified to get that the minimum total degree of a piecewise polynomial LF cannot be uniformly bounded over the set of all **GUES** systems of the form Ξ (cf. equation 5.3). Indeed, let V_k be the sequence of PPFs taking the value $V_k^l(x) = \sum_{1\le i+j\le m} a_{ijl}^{(k)} x_1^i x_2^j$ in the cone K_k^l, for $1 \le l \le m$. Here, to simplify the notation, we assume without loss of generality, that, for each element of the sequence, the number of cones and the degree of $V_k^l(x)$ is always m.

Each cone can be identified by a couple of angles with the x_1-direction. Therefore to each function V_k we can associate a m-uple of angles $(\alpha_1^k, \dots, \alpha_m^k)$ such that the cone K_k^l coincides with the region between the lines corresponding to α_k^l and α_k^{l+1}. In particular, up to subsequences, we can assume that the numbers α_k^l converge to α^l. Similarly to the case above, we can normalize the coefficients of the LFs V_k by $\sum_{l=1}^m \sum_{1\le i+j\le m} |a_{ijl}^{(k)}| = 1$ and consider a subsequence of the coefficients converging to a_{ijl}. Then, if we define V as the PPF such that $V(x) = V^l(x) = \sum_{1\le i+j\le m} a_{ijl} x_1^i x_2^j$ on the cone K^l defined by the angles α^l and α^{l+1}, it is easy to verify that $V_k(x)$ converges uniformly on compact subsets of $\mathbb{R}^2$ to $V(x)$. We can conclude the proof as before showing that $V^l(x_0) = V(0) = 0$ for arbitrary x_0, which leads to a contradiction.

5.5 Explicit Construction of a finite-SSF for Systems of Type (5.3)

In this section, we provide a 5-parameters finite-**SSF** for the class Ξ of bidimensional systems of type (5.3). Recall that for what concern the stability issue, Ξ can be parameterized by the 6-parameters family provided by the pairs (A, B) (of 2×2 matrices) defined up to their norm.

As explained in the introduction, it is enough to construct a LF for a pair (A, B) giving rise to a **GUES** system of Ξ. We only treat the **(CC.2.2)** case since, in all the other cases, the construction is entirely similar.

In the **(CC)** case, after a three-parameters change of coordinates, the normal form for the pair (A, B) is given by (see Proposition 5.2):

$$A = \begin{pmatrix} -\rho_A & -1/E \\ E & -\rho_A \end{pmatrix}, \qquad B = \begin{pmatrix} -\rho_B & -1 \\ 1 & -\rho_B \end{pmatrix}, \qquad E > 0.$$

Moreover, in the **(CC.2.2)** case, we have $\mathcal{K} > 1$, $\mathcal{D} > 0$ and $\rho_{CC} < 1$, where $\mathcal{K} := 1/2(E+1/E)$, $\mathcal{D}$ and ρ_{CC} being respectively defined in (5.4) and (5.13). Recall that, for fixed $\mathcal{K} > 1$ (i.e fixed $E > 1$), Figure 5.2 describes, in the (ρ_A, ρ_B)-plane, the status of each point with respect to the **GUES** issue.

We now associate, to every **GUES** pair (A, B), a pair $(\tilde{A}, \tilde{B})$ corresponding to a system of the type (5.3) uniformly stable but not attractive. Consider in Figure 5.2 the line segment joining the point $(0, 0)$ to (ρ_A, ρ_B) in the S^+ region. That segment intersects the curve C in a point $(\tilde{\rho}_A, \tilde{\rho}_B)$. That results from the Jordan separation theorem and the fact that C connects the points $(\sqrt{\mathcal{K}^2 - 1}, 0)$ and $(0, \sqrt{\mathcal{K}^2 - 1})$. Therefore, there exists a $\zeta \in (0, 1)$ such that, for the system given by:

$$\tilde{A} = \begin{pmatrix} -\tilde{\rho}_A & -1/E \\ E & -\tilde{\rho}_A \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} -\tilde{\rho}_B & -1 \\ 1 & -\tilde{\rho}_B \end{pmatrix}, \qquad \tilde{\rho}_A = \rho_A \zeta, \quad \tilde{\rho}_B = \rho_B \zeta,$$

the worst trajectories γ_{x_0} are closed curves, i.e. $\rho_{CC} = 1$.
Moreover, one can easily compute:

$$\det(\tilde{A}x, Ax) = r(\alpha)(1 - \zeta)\left({x_1}^2 E + \frac{{x_2}^2}{E}\right) > 0 \ ,$$

$$\det(\tilde{B}x, Bx) = \rho_B(1 - \zeta)|x|^2 > 0 \ .$$

Therefore the vector fields Ax, Bx point inside the area delimited by a fixed worst trajectory (that is closed curve) of the modified switched system and so, passing to angular coordinates, the function:

$$V(r, \alpha) = \frac{r}{\tilde{r}(\alpha)}, \tag{5.17}$$

where $\tilde{r}(\alpha)$ is a parameterization of the fixed worst trajectory, is a LF for the system defined by (A, B).
Hence, we have provided a 5-parameter **SSF** in the **(CC.2.2)** case. The five parameters are: $\mathcal{K}$, the ratio ρ_B/ρ_A, and the three parameters involved in the change of coordinates to get the normal forms (5.9), (5.10).

Remark **5.15** Notice that, in the cases **(CC.1)** and **(CC.2.1)** (cf. Section 5.3 and Theorem 5.3), one can choose as **SSF** the set of quadratic polynomials, which actually is parameterized by two parameters.

Remark **5.16** Let us come back to the general system (5.2), subject to **H0**. Notice that the question of finding the smallest m such that there exists a m-parameters finite-**SSF**, for a certain class C of systems, has no real meaning if one does not require suitable conditions on the map Ψ in Definition 5.3. Indeed, it is always possible to build a countable **SSF** for the class of systems of type (5.2) in $\mathbb{R}^n$ subject to **H0**.

Chapter 6

Generalized Solutions for Affine Systems and Motion Planning

The aim of this chapter is to discuss the possibility of generalizing the notion of solution of an affine control system where the control is supposed to belong to a functional space such that the classical results on the local existence and uniqueness do not apply. This problem has been studied extensively by control theorists and also by probabilists for its connections with the field of stochastic processes.

The control systems under consideration here have the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} \dot{v}_i f_i(x) \tag{6.1}$$

where $x \in \mathbb{R}^n$ and the vector fields f_i are smooth for every i, and satisfy some suitable growth condition (for instance we can assume that they are sublinear or bounded). The conditions on f_i could be still sharpened, but this is not the target of this chapter. In fact we want to discuss the minimal requirements on $v(\cdot)$ in order to enable us to give a meaningful definition of solution for (6.1). Notice that the equation (6.1) is affine with respect to the derivative of the control and therefore, under the previous assumptions on f_i, it is quite natural to expect that the possible solutions have the same regularity as $v(\cdot)$.

We now define the main objects used throughout this chapter. An *admissible control* for (6.1) is an absolutely continuous function $v(\cdot) = (v_1(\cdot), ..., v_m(\cdot)) : [0, T] \to \mathbb{R}^m$. If v is an admissible control, then, setting $u = \dot{v}$, equation (6.1) becomes the control system $\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$ with $u(\cdot) \in L^1$. An *admissible trajectory* is an absolutely continuous curve $x(\cdot)$ satisfying (6.1) a.e. for some admissible control. Notice that in general, if the number of controls m is smaller than the dimension n of the state space, the class of the admissible trajectories is a proper subset of the class of absolutely continuous curves on $\mathbb{R}^n$.

Let $\mathcal{F}$ =span$\{f_0 + \sum_{i=1}^{m} u_i f_i\,,\ u \in \mathbb{R}^m\}$ and consider the vector space generated by the Lie brackets of elements of $\mathcal{F}$:

$$Lie_x(\mathcal{F}) = \mathrm{span}\{\mathrm{ad}g_1 \circ \mathrm{ad}g_2 \circ \ldots \circ \mathrm{ad}g_{k-1}(g_k)(x)\ :\ g_i \in \mathcal{F}\ \forall i = 1, \ldots, k\,,\ k > 0\} \tag{6.2}$$

If $Lie_x(\mathcal{F}) = T_x\mathbb{R}^n \simeq \mathbb{R}^n$ for every $x \in \mathbb{R}^n$, i.e. if $\mathcal{F}$ is a *bracket-generating* family of vector fields, then the attainable set for (6.1) is "full dimensional" by Krener's Theorem. In particular, in the driftless case, the attainable set coincides with $\mathbb{R}^n$. We also notice, as it is well-known, that the bracket-generating condition is a generic condition (with respect to the $\mathcal{C}^2$ topology) for families of vector fields generated at least by two vector fields. In view of this fact, the problem of generalizing the notion of solution is strictly connected with the important Motion Planning Problem (see for instance [41, 64]): given a path $x(\cdot)$ in $\mathbb{R}^n$, the aim is to find an admissible trajectory that approximates $x(\cdot)$ in some suitable sense, for instance in the uniform topology. This problem is clearly important for applications to mechanical systems and robotics.

Coming back to our original problem, it is well-known that the admissible trajectories can also be interpreted as solutions to the equation

$$dx = f_0(x)dt + \sum_{i=1}^{m} f_i(x)dv_i \tag{6.3}$$

which is integrated by means of the Stieltjes integration. It is clear that this new formulation allows one to give a meaning to the solution also in the case in which $v(\cdot)$ is not absolutely continuous, but has bounded variation. This solution can be seen as a *generalized solution* to the equation (6.1). Nevertheless, motivated by mechanical models (see for instance [26, 27]), a weaker notion of generalized solution, that applies in the case of discontinuous controls with bounded variation, has been proposed in [25]. In this framework, a generalized solution can be simply regarded as the limit of a converging sequence of admissible solutions with uniformly bounded variation (see [44]). In particular, with this new definition and in the case in which the vector fields $f_i,\ i = 1, \ldots, m$ do not commute, the uniqueness of the solution is no more guaranteed.

Now we call $\mathcal{V}_{adm}$, $\mathcal{X}_{adm}$ the spaces of the admissible controls and the admissible trajectories and we consider the *input-output map* associated to (6.1), i.e.

$$\Phi^{x_0}_{adm} : v(\cdot) \in \mathcal{V}_{adm} \mapsto x(\cdot) \in \mathcal{X}_{adm} \quad \text{where} \quad \dot{x} = f_0(x) + \sum_{i=1}^{m} \dot{v}_i f_i(x), \quad x(0) = x_0\,. \tag{6.4}$$

A natural way to define a notion of generalized solution of the equation (6.1) consists in fixing some topological spaces $\mathcal{V} \supset \mathcal{V}_{adm}$, $\mathcal{X} \supset \mathcal{X}_{adm}$ in such a way that the input-output map $\Phi^{x_0}_{adm}$ can be extended to a continuous map $\Phi^{x_0} : \mathcal{V} \to \mathcal{X}$. Then a generalized solution associated to some $v(\cdot)$ belonging to the closure of $\mathcal{V}_{adm}$ with respect to the topology of $\mathcal{V}$ can be simply defined as $\Phi^{x_0}(v)(\cdot)$. For instance it is well-known (see [62]) that in the case in which $m = 1$ the input-output map is continuous in the topology of uniform convergence and therefore in this case, associated to each continuous control $v(\cdot)$, one can define a generalized solution. This is no more true if $m > 1$ and, as one could expect, the vector fields $f_i\,,\ i > 0$ do not commute. In this case one needs to consider more refined topologies.

The notion of generalized solution is also useful in order to exploit the relation between stochastic differential equations (SDE) and ordinary differential equations of the form (6.1). In this case $v(\cdot)$ and $x(\cdot)$ represent paths of the stochastic processes that correspond respectively to the input and the output of the SDE. Notice that a path of a stochastic process is in general (with probability 1) a continuous path, so that the case of continuous, but not regular, paths, turns out to be of particular importance in the literature.

One important result in this direction has been obtained by Lyons in [42], where the spaces $\mathcal{V}$ and $\mathcal{X}$ are identified with the space of continuous functions with bounded p-variation, with $p < 2$ (for precise definitions, see the next section). More precisely in [42] the Picard iteration method has been used to prove the existence and uniqueness of the solutions of the equation (6.3). Such results are inspired by [66], where it is essentially proved that the Stieltjes integrals of the form $\int f\,dg$, where f, g have finite p and q variation and $\frac{1}{p} + \frac{1}{q} > 1$, are well-defined.

In [43] the problem of generalizing the notion of solution to the case in which $v(\cdot)$ has finite p-variation, with $p \geq 2$, has been treated. Actually this problem has been studied by means of a new space of controls, called geometric p-multiplicative functionals, that project onto controls of finite p-variation. Call π such projection, then there is a natural way to lift with π^{-1} each smooth control to a unique geometric p-multiplicative functional. Then, roughly speaking, it is shown that the input-output map can be extended continuously to this new class of controls, and in particular the generalized solution is again a geometric p-multiplicative functional. If we denote by $G_p(\mathbb{R}^m)$, $G_p(\mathbb{R}^n)$ the space of the geometric p-multiplicative functionals corresponding respectively to the spaces of the controls and of the trajectories, then

$$\Phi^{x_0} : G_p(\mathbb{R}^m) \to G_p(\mathbb{R}^n)$$

If $p < 2$ this result recover the result of [42], since in this case the lift to the class of geometric p-multiplicative functionals is simply the identity. If $p \geq 2$ then the following remarks are in order:

- Given a control $v(\cdot)$ with finite p-variation it is not always possible to find a geometric p-multiplicative functional γ such that $\pi(\gamma) = v(\cdot)$.
- Given a control $v(\cdot)$ with finite p-variation the eventual lift to the set of geometric p-multiplicative functionals could be not unique, in other words the map π is not injective.
- It is in general difficult to check if it is possible to lift a given control with finite p-variation to a geometric p-multiplicative functional.

Summing up, the main limit of this result is that it is difficult to determine the spaces of controls and paths with finite p-variation that are involved. Moreover this approach is not helpful for solving the motion planning problem. Our aim is to discuss some alternative ways of defining generalized solutions which are easier to handle. In particular we analyze some conjectures related to the Heisenberg system with the help of some examples and counterexamples.

6.1 Functional spaces and topologies

In this section we introduce some functional spaces that can be seen as generalizations of the classical spaces of Lipschitz, BV and absolutely continuous functions. The first interesting functional space is the space of the Hölder-α functions $\mathcal{C}^{0,\alpha}([0,T])$, i.e. the space of functions satisfying $\sup_{t_1,t_2\in[0,T]} \frac{|f(t_1)-f(t_2)|}{|t_1-t_2|^\alpha} < +\infty$. It is easy to verify that it is a Banach space with norm

$$\|f\|_\alpha := |f(0)| + \sup_{t_1,t_2\in[0,T]} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\alpha}\,. \tag{6.5}$$

It is interesting to investigate the properties of $\mathcal{C}^{0,\alpha}([0,T])$ as α varies. For this purpose the following simple result is useful.

Proposition 6.1 *If $f \in \mathcal{C}^{0,\alpha'}([0,T])$ with $\alpha' > \alpha$, $\lim_{n\to\infty} f_n = f$ uniformly and $\|f_n\|_{\alpha'}$ is uniformly bounded, then $\lim\limits_{n\to\infty} f_n = f$ in the topology of $\|\cdot\|_\alpha$*

Indeed, if we assume by contradiction the existence, up to a subsequence, of x_n, y_n with

$$\frac{|f_n(x_n) - f_n(y_n) - f(x_n) + f(y_n)|}{|x_n - y_n|^\alpha} > C \tag{6.6}$$

for some $C > 0$, and we suppose, still up to subsequences, that $\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty} y_n = \bar{y}$, then inequality (6.6) may hold only if $\bar{x} = \bar{y}$. This clearly contradicts our hypotheses, since the right-hand side of (6.6) is bounded by $\|f_n - f\|_{\alpha'}|x_n - y_n|^{\alpha'-\alpha}$, which tends to 0. ■

Since for every $f \in \mathcal{C}^{0,\alpha'}([0,T])$ it is easy to find a sequence of smooth functions approximating f as in the hypotheses of the previous result, we get the following corollary.

Corollary 6.1 *The closure of $\mathcal{C}^\infty([0,T])$ with respect to $\|\cdot\|_\alpha$ contains the set $\bigcup\limits_{\alpha'>\alpha} \mathcal{C}^{0,\alpha'}([0,T])$*

Notice that the closure of $\mathcal{C}^\infty([0,T])$ with respect to $\|\cdot\|_\alpha$ doesn't coincide with $\mathcal{C}^{0,\alpha}([0,T])$ itself, since for instance the function $f(t) = t^\alpha$ cannot be approximated by a smooth function in this norm.

Obviously $\mathcal{C}^{0,\alpha}([0,T])$ contains properly the space of Lipschitz functions, but it does not contain the space of absolutely continuous functions (for instance the n-th root $\sqrt[n]{t}$ is not Hölder-α for every $\alpha > 1/n$). This means that each possible definition of solution to the equation (6.1) that applies to the case in which $v(\cdot)$ is Hölder cannot be considered as a generalization of the classical notion of Carathéodory solution. Therefore we introduce new spaces that are more general.

The space BV_p of functions with finite p-variation, essentially introduced by Wiener, is the set of functions f satisfying the following condition

$$\sup_{0=t_0<t_1<\ldots<t_N=T} \sum_i |f(t_{i+1}) - f(t_i)|^p < +\infty\,. \tag{6.7}$$

In particular by the Minkowski inequality it follows trivially that the map

$$\|f\|_{BV_p} := |f(0)| + \sup_{0=t_0<t_1<\ldots<t_N=T} \Big(\sum_i |f(t_{i+1}) - f(t_i)|^p\Big)^{1/p}$$

defines a norm on BV_p and the space BV_p turns out to be a Banach space. Moreover it is clear that $\mathcal{C}^{0,1/p} \subset BV_p$.

We give a particular importance to the BV_p functions that are also continuous. In particular we mention two functional spaces that are contained in the class of continuous and BV_p functions. The first one is the set of *regular* BV_p ([42]) functions (that we denote by BV_p^{reg}), i.e. the functions f satisfying the following condition

$$\limsup_{\substack{0=t_0<t_1<\ldots<t_N=T\\ \max|t_{i+1}-t_i|\to 0}} \sum_i |f(t_{i+1}) - f(t_i)|^p = 0\,. \tag{6.8}$$

The other functional space that we introduce is a natural generalization of the space of absolutely continuous functions: we say that $f \in \mathcal{AC}_p$ if it has the following property:

$$\forall \varepsilon > 0 \quad \exists \delta \text{ s.t. } \sum_i |t_i - s_i| < \delta \Rightarrow \sum_i |f(t_i) - f(s_i)|^p < \varepsilon \,. \tag{6.9}$$

Notice that for every $p \geq 1$ we have $\mathcal{C}^{0,1/p} \subset \mathcal{AC}_p$. Moreover it is easy to see that we have the following inclusions for every $1 \leq p < p'$:

$$BV_p \cap \mathcal{C} \subset BV_{p'}^{reg} \subset \mathcal{AC}_{p'} \subset BV_{p'} \cap \mathcal{C} \tag{6.10}$$

Indeed these inclusion can be easily derived from the definitions and the uniform continuity (we are working with continuous functions defined on an interval).

In some sense this series of inclusions shows that the spaces that we have introduced are "almost" equivalent, since infinitesimal changes of the parameter p can reverse the inclusions.

We finally mention the subspace of BV_p obtained as the closure of $\mathcal{C}^\infty$ with respect to the norm $\|\cdot\|_{BVp}$. Again, it is easy to see that this space contains BV_p^{reg}.

6.2 The Heisenberg example

We consider now the simplest driftless completely non holonomic system, namely a system of the form

$$\dot{x} = \dot{v}_1 F_1(x) + \dot{v}_2 F_2(x), \tag{6.11}$$

where $x \in \mathbb{R}^3$ and F_1 and F_2 generates the Heisenberg algebra i.e.

$$F_1 = \begin{pmatrix} 1 \\ 0 \\ -x_2/2 \end{pmatrix}, F_2 = \begin{pmatrix} 0 \\ 1 \\ x_1/2 \end{pmatrix} \tag{6.12}$$

Notice that $[F_1, F_2] = (0, 0, 1)^T$ and that the system is nilpotent, in the sense that the brackets of order greater than two annihilate.

For every admissible control $v(\cdot)$ the corresponding admissible trajectory coincides with $v(\cdot)$ up to an additive constant. Moreover the third component is characterized by the equation

$$\dot{x}_3 = \frac{1}{2}(x_1\dot{x}_2 - \dot{x}_1 x_2) \tag{6.13}$$

that means that, if we set

$$\mathcal{A}[x(\cdot)](t) := \int_0^t \frac{1}{2}(x_1(s)\dot{x}_2(s) - \dot{x}_1(s)x_2(s))\, ds, \tag{6.14}$$

that is the area computed in counterclockwise sense of the region spanned by the planar curve $(x_1(\cdot), x_2(\cdot))$ (see Figure 2), then $x_3(t) = \mathcal{A}[x(\cdot)](t)$. Consider now a non admissible control $v(\cdot)$ in equation (6.11), and suppose to have a generalized solution, obtained with a continuous extension of the input-output map. Then the first two components must coincide with the corresponding controls up to constants. Therefore the problem of generalizing the notion of solution reduces to the following

Problem: *Given a continuous, but not absolutely continuous, curve $x(\cdot) = (x_1(\cdot), x_2(\cdot))$, is it possible to give a definition of the area generalizing (6.14)?*

We focus in particular on curves belonging to the spaces $\mathcal{C}^{0,\alpha}([0,T])$. We start with a simple example after which we will discuss some possible conjectures. Fix $\alpha \in]0,1[$ and consider the following sequence of admissible controls:

$$v^n_\alpha(t) = (\frac{1}{n^\alpha}\cos(n\,t), \frac{1}{n^\alpha}\sin(n\,t)). \tag{6.15}$$

It is clear that $\|v^n_\alpha(\cdot)\|_\infty \to 0$. So this is an approximating sequence of the zero function in the $\mathcal{C}^0$ topology. The area corresponding to $v^n_\alpha(\cdot)$ is easy to compute and is the following

$$\mathcal{A}[v^n_\alpha(\cdot)](t) = n^{(1-2\alpha)}t \tag{6.16}$$

Therefore we can distinguish three cases:

- If $\alpha > 1/2$ then $\mathcal{A}[v^n_\alpha(\cdot)](t)$ converges uniformly to zero as n goes to ∞,
- If $\alpha = 1/2$ then $\mathcal{A}[v^n_\alpha(\cdot)](t)$ converges uniformly to the function $v(t) = t$,
- If $\alpha < 1/2$ then $\mathcal{A}[v^n_\alpha(\cdot)](t)$ diverges.

This example with $\alpha \leq 1/2$ is enough to conclude that the $\mathcal{C}^0$ norm is too weak to make the input-output map continuous. Therefore we should look for a stronger norm. For this purpose, it is interesting to notice that the functions $v^n_\alpha(\cdot)$ are such that $\|v^n_\alpha\|_\alpha$ is uniformly bounded on n. More precisely

$$\lim_{n\to\infty} \|v^n_\alpha\|_{\alpha'} = \begin{cases} 0 & \text{if } \alpha' > \alpha \\ \frac{2}{\pi^\alpha} & \text{if } \alpha' = \alpha \\ \infty & \text{if } \alpha' < \alpha \end{cases}$$

This suggests some natural conjectures. Let $\mathcal{V}^{\alpha,K}_{adm} = \{v \in \mathcal{C}^\infty,\ \|v\|_\alpha \leq K\}$. Then:

C1 If $\alpha > 1/2$ then the input-output map can be extended continuously to the set $\{v \in \mathcal{C}^{0,\alpha},\ \|v\|_\alpha \leq K\}$, seen as the closure of $\mathcal{V}^{\alpha,K}_{adm}$ with respect to the norm $\|.\|_\infty$.

In [42] it was proved that, if we consider the topology given by $\|\cdot\|_{BV_p}$ with $p < 2$, the input-output map can be extended continuously to a map Φ^{x_0} defined on $BV_p \cap \mathcal{C}$. Combining this result with Proposition 6.1 we obtain easily that the conjecture **C1** is true:

$$\begin{aligned} v_n \overset{\|\cdot\|_\infty}{\to} v\,,\quad \|v_n\|_\alpha \leq K \quad &\Longrightarrow \quad v_n \overset{\|\cdot\|_{\alpha'}}{\to} v \quad \forall \alpha' \in (1/2, \alpha) \\ \Longrightarrow v_n \overset{\|\cdot\|_{BV_p}}{\to} v\,,\quad p := 1/\alpha' \quad &\Longrightarrow \quad \Phi^{x_0}(v_n) \overset{\|\cdot\|_{BV_p}}{\to} \Phi^{x_0}(v) \\ \Longrightarrow \quad \Phi^{x_0}(v_n) \quad &\overset{\|\cdot\|_\infty}{\to} \quad \Phi^{x_0}(v) \end{aligned}$$

As a consequence, we obtain that it is possible to generalize the definition of the area $\mathcal{A}[v(\cdot)](t)$ for $v \in \mathcal{C}^{0,\alpha}$ and $\alpha > 1/2$. For $\alpha \leq 1/2$ the example shows that the analogous conjecture is not true. However for $\alpha = 1/2$ one could still expect the possibility of defining the area on subsequences.

C2 Given $v \in \mathcal{C}^{0,\frac{1}{2}}$, for every sequence of smooth functions v_n converging uniformly to v and with $\|v_n\|_{1/2}$ uniformly bounded, there exists a subsequence such that the corresponding admissible solutions of (6.11) (or, more in general, of (6.1)) form a Cauchy sequence in the uniform topology.

If this conjecture was false than one could still try to consider a stronger norm on the space of controls, getting the following conjecture.

C3 If the space of the admissible controls is endowed with the topology given by $\| \cdot \|_{1/2}$, then it is possible to extend continuously the input-output map.

In the previous example the functions were constructed in such a way that they converge to 0, but it is clear that this is far from being a natural way of approximating the function $v = 0$. One can also try to overcome this problem by defining directly a notion of convergence of sequences of functions. For instance, given a function v and a sequence of times $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = T$, one can consider the piecewise affine approximations obtained joining successively by segments the points $v(t_h)$ and $v(t_{h+1})$, for $h = 0, \ldots, k$. Therefore another possible conjecture is the following

C4 Let $v \in \mathcal{C}^{0,\frac{1}{2}}$ and $v^{(k)}$ be a sequence of piecewise affine approximating functions defined as before and characterized by the times $t_h^{(k)}$, and assume that $\lim_{k\to\infty} \sup_h |t_{h+1}^{(k)} - t_h^{(k)}| = 0$. Then the sequence $\mathcal{A}[v^{(k)}](T)$ is a Cauchy sequence.

We discuss the above conjectures with the help of some examples.

Example A. Consider the following function

$$v : [0, 2\pi] \to \mathbb{C} \qquad v(t) = \sum_{k=0}^{+\infty} \frac{1}{2^k} e^{4^k i t}, \tag{6.17}$$

which represents a closed curve on the plane. We want to prove the following facts.

- $v(\cdot) \in \mathcal{C}^{0,\frac{1}{2}}$.
- Consider the sequence of times $t_h = \frac{2\pi h}{4^k} \quad h = 0, \ldots, 4^k - 1$ and define v_k as the piecewise affine function obtained joining the point $v(t_h)$ to $v(t_{h+1})$ for every $h = 0, \ldots, 4^k - 1$, i.e

$$v^{(k)}(t) := v(t_h) + \frac{v(t_{h+1}) - v(t_h)}{t_{h+1} - t_h}(t - t_h) \quad \text{if} \quad t \in [t_h, t_{h+1}].$$

Then $\lim_{k\to+\infty} \mathcal{A}[v^{(k)}(\cdot)](2\pi) = +\infty$.

For the first issue we first fix two real numbers x and y. Then we define an integer number Q depending on x and y:

$$Q := \Big[\log_2\Big(\frac{1}{\sqrt{|x-y|}}\Big)\Big] \iff 2^Q \le \frac{1}{\sqrt{|x-y|}} < 2^{Q+1} \iff 2^{-Q} \ge \sqrt{|x-y|} > 2^{-Q-1}.$$

We have that

$$|v(x)-v(y)| \le \Big|\sum_{k=0}^{Q} \frac{1}{2^k}\big(e^{4^k i x} - e^{4^k i y}\big)\Big| + \Big|\sum_{k=Q+1}^{+\infty} \frac{1}{2^k}\big(e^{4^k i x} - e^{4^k i y}\big)\Big|.$$

For the second term we have

$$\Big|\sum_{k=Q+1}^{+\infty} \frac{1}{2^k}\big(e^{4^k i x} - e^{4^k i y}\big)\Big| < 2\sum_{k=Q+1}^{+\infty} \frac{1}{2^k} = 2^{-Q+1} < 4\sqrt{|x-y|}$$

while we can estimate the first one using the inequality $|e^{i\alpha} - e^{i\beta}| < |\alpha - \beta|$:

$$\Big|\sum_{k=0}^{Q} \frac{1}{2^k}\big(e^{4^k i x} - e^{4^k i y}\big)\Big| \le \sum_{k=0}^{Q} \frac{1}{2^k} 4^k |x-y| = \sum_{k=0}^{Q} 2^k |x-y| < 2^{Q+1}|x-y| \le 2\sqrt{|x-y|}.$$

Finally we have found that the inequality $|v(x)-v(y)| \le 6\sqrt{|x-y|}$ holds for every x and y in the interval $[0, 2\pi]$ and therefore $v(\cdot) \in \mathcal{C}^{0,\frac{1}{2}}$.

<u>*Remark*</u> **6.1** Observe that each term in the Fourier series of $v(\cdot)$ is Hölder-$\frac{1}{2}$ with the same optimal constant.

Now we want to see that the function $v(\cdot)$ defined above, as a function from $[0, 2\pi]$ to $\mathbb{R}^2$, disproves the conjectures **C2** and **C4**.
Consider the 4^N times $t_h = \frac{2\pi h}{4^N} \quad h = 0, \dots, 4^N - 1$, then the area corresponding to the segment connecting $v(t_h)$ to $v(t_{h+1})$ is given by $\frac{1}{2}\,(\, v_2(t_h)v_1(t_{h+1}) - v_1(t_h)v_2(t_{h+1}))$, where v_1, v_2 denote the first and second components of v.
In particular we have

$$v_1(t_h) = \sum_{k=0}^{N-1} \frac{1}{2^k}\cos(4^{k-N}2\pi h) + \sum_{k=N}^{\infty} \frac{1}{2^k} = \sum_{k=0}^{N-1} \frac{1}{2^k}\cos(4^{k-N}2\pi h) + 2^{-N+1},$$

$$v_2(t_h) = \sum_{k=0}^{N-1} \frac{1}{2^k}\sin(4^{k-N}2\pi h).$$

Therefore

$$v_1(t_h)v_2(t_{h+1}) - v_2(t_h)v_1(t_{h+1}) = 2^{-N+1}\sum_{k=0}^{N-1} \frac{1}{2^k}\big(\sin(4^{k-N}2\pi(h+1)) - \sin(4^{k-N}2\pi h)\big)$$

$$+ \sum_{n,m=0}^{N-1} \frac{1}{2^{n+m}}\big(\cos(4^{n-N}2\pi h)\sin(4^{m-N}2\pi(h+1)) - \cos(4^{m-N}2\pi(h+1))\sin(4^{n-N}2\pi h)\big).$$

If we consider the sum over $h = 0, \dots 4^N - 1$ of these terms and we exchange the order of summation we obtain the area

$$\mathcal{A}[v^{(N)}(\cdot)](2\pi) = \sum_{n,m=0}^{N-1} \frac{1}{2^{n+m}} \sum_{h=0}^{4^N-1} \sin(4^{m-N}2\pi(h+1) - 4^{n-N}2\pi h)\,.$$

Now we want to see that

$$\sum_{h=0}^{4^N-1} \sin(4^{m-N}2\pi(h+1) - 4^{n-N}2\pi h) = 0 \tag{6.18}$$

if $n \neq m \quad (n, m < N)$. We rewrite $4^m(h+1) - 4^n h = h(4^m - 4^n) + 4^m = hp + q$.
Let M be the greatest common divisor of p and 4^N, and $D = \frac{4^N}{2M}$. If we prove that

$$(h+D)p + q \equiv hp + q + \frac{4^N}{2} \pmod{4^N}, \tag{6.19}$$

then $\frac{2\pi}{4^N}((h+D)p+q) \equiv \frac{2\pi}{4^N}(hp+q) + \pi$ and it is clear that the sum of $2D$ successive terms in (6.18) is 0. Therefore, since $2D$ divides 4^N, the whole series (6.18) is 0.
To prove (6.19) notice that $2Dp = 4^N \frac{p}{M} \equiv 0 \pmod{4^N}$ and then we deduce that $Dp \equiv 0 \pmod{4^N}$ or $Dp \equiv \frac{4^N}{2} \pmod{4^N}$. In the first case we would have $4^N \frac{p}{2M} \equiv 0 \pmod{4^N}$ which means that $2M$ divides p, which is impossible (from the definition of M), since $2M$ divides also 4^N, therefore the second case must hold and this immediately gives (6.19).

So we can restrict the sum to the indexes $n = m$ and therefore

$$\mathcal{A}[v^{(N)}(\cdot)](2\pi) = \sum_{n=0}^{N-1} \frac{1}{4^n} \sum_{h=0}^{4^N-1} \sin(4^{n-N}2\pi) = \sum_{n=0}^{N-1} 4^{N-n} \sin(4^{n-N}2\pi) = \sum_{m=1}^{N} 4^m \sin(4^{-m}2\pi)$$

Since $\lim_{m\to\infty} 4^m \sin(4^{-m}2\pi) = 1$ we have that $\lim_{N\to+\infty} \mathcal{A}[v^{(N)}(\cdot)](2\pi) = +\infty$.

In Figure 6.1 the graph of function $v(\cdot)$ is drawn. Notice the lack of regularity of $v(\cdot)$, which is clearly needed in order to make the area blows up.

Remark **6.2** An example similar to Example A has been discussed in [66], where the aim was to disprove the validity of an inequality connected to the Stieltjes integral $\int f\,dg$, where f and g have bounded 2-variation.

Remark **6.3** There is a easy geometric interpretation of this example. Indeed, in (6.17) each term in the summation corresponds to 4^k turns on a circle centered at the origin with radius $\frac{1}{2^k}$. The area corresponding to such path is exactly 1 and it is easy to see that the area of a finite sum of l elements in (6.17) is exactly l.

Remark **6.4** We proved that there is a sequence of piecewise affine approximations of v such that the corresponding area goes to infinity, but we don't know if this remains true for all possible converging sequences of piecewise affine approximations. However, motivated by the geometric interpretion given in Remark 6.3, we strongly believe that this is true. This is no more true for smooth approximations (also with uniformly bounded Hölder constant), since

$$v^{(N)}(t) := \sum_{k=0}^{N-1} \frac{1}{2^k} e^{4^k i\, t} + \sum_{k=N}^{2N-1} \frac{1}{2^k} e^{-4^k i\, t}$$

is such that $\mathcal{A}[v^{(N)}](2\pi) = 0 \quad \forall N > 0$.

Example B. We consider now a slightly modified example:

$$\tilde{v}(t) = \sum_{k=0}^{+\infty} \frac{1}{2^k} e^{(-4)^k i\, t}$$

In this case, exactly as before, one can prove the following.

- $\tilde{v}(\cdot)$ is Hölder-$\frac{1}{2}$.
- The sequence of times $t_h = \frac{2\pi h}{4^k} \quad h = 0, \dots 4^k - 1$ is such that the area of the corresponding piecewise affine approximation of $\tilde{v}(\cdot)$ does not converge as k goes to infinity even if it is uniformly bounded. More precisely the area of such approximation is

$$\mathcal{A}[\tilde{v}^{(k)}(\cdot)](2\pi) = (-1)^k \sum_{m=1}^{k} (-4)^m \sin(4^{-m} 2\pi)$$

  and it is easy to see that $\mathcal{A}[\tilde{v}^{(2k)}(\cdot)](2\pi)$ and $\mathcal{A}[\tilde{v}^{(2k+1)}(\cdot)](2\pi)$ converge to two different values as k goes to infinity.

The graph of function $\tilde{v}$ is depicted in Figure 6.2.

Example C. Consider now the following function.

$$\hat{v} : [0, 2\pi] \to \mathbb{C} \qquad \hat{v}(t) = \sum_{k=0}^{+\infty} \frac{1}{\sqrt{k} 2^k} e^{4^k i\, t}.$$

This function satisfies the same properties as v in the first example, and such properties can be proved exactly in the same way. The most interesting feature of this example is that the approximating sequence $\hat{v}^{(k)} = \sum_{k=0}^{k} \frac{1}{\sqrt{k}2^k} e^{4^k i\, t}$ converges to $\hat{v}$ in the norm $\| \cdot \|_{1/2}$, while we still have $\lim_{k \to +\infty} \mathcal{A}[\hat{v}^{(k)}](2\pi) = +\infty$. Similarly to Remark 6.4 it is easy to construct an approximating sequence converging to $\hat{v}$ in the norm $\| \cdot \|_{1/2}$ and such that the corresponding area is uniformly bounded.

Therefore also conjecture **C3** turns out to be false.

We have seen that it is not easy to find a definition of generalized solution that applies to the case of Hölder-$\frac{1}{2}$ controls. However notice that the example we have constructed is very particular, since the components v_1 and v_2 seem to "cooperate" in order to increase the area. Therefore it seems quite natural to look for some particular "non-resonance" condition on the space of controls, that could also represent the case of two paths of independent stochastic processes.

Then one could try to define a generalized solution of (6.11) extending continuously the input-output map relatively to the class of controls satisfying this non-resonance condition. Our future research on this field will be based on this new approach.

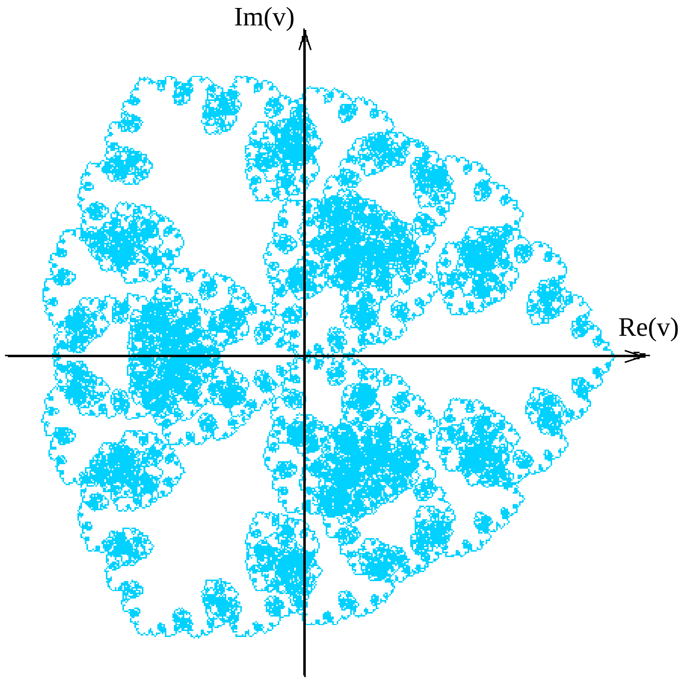

Figure 6.1: The graph of v

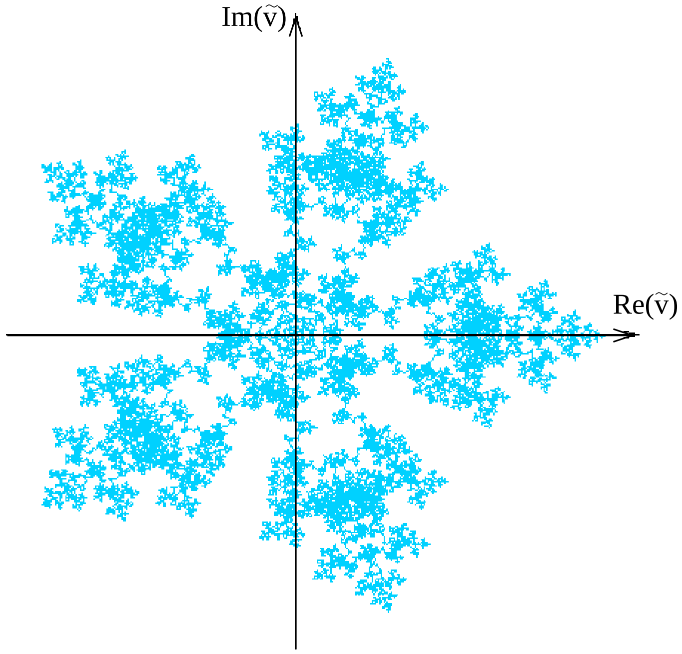

Figure 6.2: The graph of $\tilde{v}$

Bibliography

[1] A.A. Agrachev and T. Chambrion, *An estimation of the controllability time for single-input systems on compact groups*, to appear on ESAIM COCV (2006).

[2] A.A. Agrachev and R.V. Gamkrelidze, *Symplectic Geometry for Optimal Control*, in "Non-Linear Controllability and Optimal Control", Monogr. Textbooks Pure Appl. Math., 133, Dekker, New York, pp. 263-277 (1990).

[3] A. A. Agrachev and D. Liberzon, *Lie-algebraic stability criteria for switched systems*, SIAM J. Control Optim., vol. 40, no. 1, 253–269 (2001).

[4] A.A. Agrachev, Yu.L. Sachkov, *Control Theory from the Geometric Viewpoint*, Encyclopedia of Mathematical Sciences, Vol.87, Springer (2004).

[5] A.A. Agrachev, M. Sigalotti, *On the Local Structure of Optimal Trajectories in* $\mathbb{R}^3$, SIAM J. Control Optimization, Vol. 42, No. 2, pp. 513–531 (2003).

[6] F. Albertini, D. D'Alessandro, *Notions of controllability for bilinear multilevel quantum systems*, IEEE Trans. Automat. Control 48, no. 8, 1399–1403 (2003).

[7] L. Allen, J. H. Eberly, *Optical Resonance and Two-Level Atoms*, Wiley, New York 1975.

[8] C. Altafini. *Controllability of quantum mechanical systems by root space decomposition of su(N)*, Journal of Mathematical Physics, 43(5): 2051-2062 (2002).

[9] D. Angeli, *Almost global stabilization of the inverted pendulum via continuous state feedback* Automatica, to appear.

[10] D. Angeli, *A note on stability of arbitrarily switched homogeneous systems*, Systems Control Lett., to appear.

[11] K.J. Åström,K. Furuta, *Swinging up a pendulum by energy control*, Automatica, vol. 36, pages 287-295, May 1999.

[12] F. Blanchini, *Nonquadratic Lyapunov function for robust control*, Automatica J. IFAC, vol. 31, no. 3, pp. 451–461 (1995).

[13] F. Blanchini, S. Miani, *A new class of universal Lyapunov functions for the control of uncertain linear systems*, IEEE Trans. Automat. Control, vol. 44, no. 4, pp. 641–647 (1999).

[14] U. Boscain: *Stability of planar switched systems: the linear single input case*, SIAM J. Control Optim. vol. 41, no. 1, pp. 89–112 (2002).

[15] U. Boscain, T. Chambrion, G. Charlot, *Nonisotropic 3-level Quantum Systems: Complete Solutions for Minimum Time and Minimal Energy*, Discrete and Continuous Dynamical Systems-B, Vol. 5, n. 4, pp. 957-990 (2005).

[16] U. Boscain, T. Chambrion and J.-P. Gauthier, *On the K+P problem for a three-level quantum system: Optimality implies resonance*, Journal of Dynamical and Control Systems, 8, 547-572 (2002).

[17] U. Boscain, G. Charlot, *Resonance of Minimizers for n-level Quantum Systems with an Arbitrary Cost*, ESAIM COCV, 593-614 (2004).

[18] U. Boscain, G. Charlot, J.-P. Gauthier, Stéphane Guérin and Hans–Rudolf Jauslin, *Optimal Control in laser-induced population transfer for two- and three-level quantum systems*, Journal of Mathematical Physics, v.43, 2107-2132 (2002).

[19] U. Boscain, Y. Chitour, *Time Optimal Synthesis for Left–Invariant Control Systems on SO(3)*, SIAM J. Control and Optim., Vol. 44, No. 1, pp. 111-139 (2005).

[20] U. Boscain, P. Mason, *Time Minimal Trajectories for a Spin $1/2$ Particle in a Magnetic Field*, J. Math. Phys. 47, 062101 (2006).

[21] U. Boscain and B. Piccoli, *Extremal syntheses for generic planar systems*, Journal of Dynamical and Control Systems, Vol.7 No.2, pp. 209-258 (2001).

[22] U. Boscain, B, Piccoli, *Optimal Synthesis for Control Systems on 2-D Manifolds,* Springer, SMAI, Vol.43 (2004).

[23] A. Bressan, *The Generic Local Time-Optimal Stabilizing controls in Dimension 3*, SIAM J. Control Optim.,Vol. 24, pp. 177–190 (1986).

[24] A. Bressan and B. Piccoli, *A Generic Classification of Time Optimal Planar Stabilizing Feedbacks*, SIAM J. Control and Optimization, Vol.36 No.1, pp. 12-32 (1998).

[25] Alberto Bressan, F. Rampazzo: *On Differential Systems with Vector-Valued Impulsive Controls*, Bollettino U.M.I (7) Vol. 2-B, 641-656 (1988).

[26] Aldo Bressan: *Hyperimpulsive motions and controllizable coordinates for Lagrangian Systems*, Atti Accad. Naz. dei Lincei, Memorie classe di Scienze Mat. Fis. Nat. Serie 9 Vol. 1, 149-196 (1989).

[27] Aldo Bressan: *On some recent results in control theory, for their application to Lagrangian systems*, Atti Accad. Naz. dei Lincei, Memorie classe di Scienze Mat. Fis. Nat. Serie 8 Vol. 19, 249-287 (1989).

[28] M. Broucke, P. Mason, B. Piccoli, *Time optimal swing-up of the planar pendulum*, preprint SISSA 50/2006/M.

[29] G. Chesi, A. Garulli, A. Tesi, A. Vicino, *Homogeneous Lyapunov functions for systems with structured uncertainties*, Automatica J. IFAC, Vol. 39, no. 6, pp. 1027–1035 (2003).

[30] M. A. Dahleh, A. P. Peirce and H. Rabitz, *Optimal control of quantum-mechanical systems: Existence, numerical approximation, and applications*, Phys. Rev. A Vol. 37, 4950–4964 (1988).

[31] D. D'Alessandro and M. Dahleh, *Optimal control of two-level quantum systems*, IEEE Transactions on Automatic Control, v.46, No. 6, 866-876 (2001).

[32] W. P. Dayawansa, C. F. Martin, *A converse Lyapunov theorem for a class of dynamical systems which undergo switching*, IEEE Trans. Automat. Control, vol. 44, no. 4, pp. 751–760 (1999).

[33] J.P. Gauthier and G. Bornard. *Controlabilite des sytemes bilineaires*, SIAM J. Control Optim., 20, 377–384 (1982).

[34] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.

[35] V. Jurdjevic and I.K. Kupka, *Control Systems on Semisimple Lie Groups and Their Homogeneous Spaces*, Ann. Inst. Fourier, Vol.31, pp. 151–179 (1981).

[36] N. Khaneja, R. Brockett and S. J. Glaser, *Time optimal control in spin systems*, Phys. Rev. A v.63 (2001).

[37] A.J. Krener, H. Schättler, *The structure of small-time reachable sets in low dimensions*, SIAM J. Control Optim. 27, no. 1, pp. 120–147 (1989).

[38] B. Kuipers, S. Ramamoorthy, *Qualitative modeling and heterogeneous control of global system behavior* Hybrid systems computation and control, pp. 294-307, Stanford, California, March 2002.

[39] D. Liberzon, *Switching in Systems and Control*, Volume in series Systems and Control: Foundations and Applications. Birkhauser, Boston, MA (2003).

[40] D. Liberzon and A. S. Morse, *Basic problems in stability and design of switched systems*, IEEE Control Systems Magazine, vol. 19, no. 5, pp. 59–70 (1999).

[41] W. Liu, *Averaging theorems for highly oscillatory differential equations and iterated Lie brackets*, SIAM J. Control Optim. Vol. 35, no. 6, 1989–2020 (1997).

[42] T. Lyons , *Differential equations driven by rough signals (I): an extension of an inequality of L. C. Young*, Math. Res. Lett., Vol. 1, no. 4, pp. 451-464 (1994).

[43] T. Lyons , *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana, Vol. 14, no. 2, pp. 215–310 (1998).

[44] P. Mason *Sistemi di Controllo Impulsivi su Varietà*, Tesi di Laurea, Università degli studi di Padova, 2002.

[45] P. Mason, *Generalized solutions for affine systems and motion planning*, preprint SISSA 51/2006/M

[46] P. Mason, R. Salmoni, U. Boscain, Y. Chitour, *Limit time optimal synthesis for a control-affine system on S^2*, preprint SISSA 48/2006/M.

[47] P. Mason, U. Boscain, Y. Chitour *Common polynomial Lyapunov functions for linear switched systems*, SIAM J. Control Optim. Vol. 45, no. 1, 226–245 (2006).

[48] A. P. Molchanov, E. S. Pyatnitskii, *Lyapunov functions that specify necessary and sufficient conditions of absolute stability of nonlinear nonstationary control systems.* Part I, Automat. Remote Control, vol. 47 no.3, pp. 344–354 (1986).

[49] A. P. Molchanov, E. S. Pyatnitskii, *Lyapunov functions that specify necessary and sufficient conditions of absolute stability of nonlinear nonstationary control systems.* Part II, Automat. Remote Control, vol. 47 no. 4, pp. 443–451 (1986).

[50] A. P. Molchanov, E. S. Pyatnitskii, *Criteria of asymptotic stability of differential and difference inclusions encountered in control theory.* Systems Control Lett. vol. 13, no. 1, pp. 59–64 (1989).

[51] A. Olas, *On robustness of systems with structured uncertainties* Proc. 4th Workshop on Control Mechanics University of Southern California (1991).

[52] B. Piccoli. *Time-optimal control problems for the swing and the ski*, Int. J. Control, vol. 62, no. 6, pp. 1409-1429 (1995).

[53] B. Piccoli, *Regular Time-Optimal Synthesis for Smooth Planar Systems*, Rend. Sem. Mat. Univ. Padova, Vol.95, pp. 59-79 (1996).

[54] B. Piccoli, *Classifications of Generic Singularities for the Planar Time-Optimal Synthesis*, SIAM J. Control and Optimization, Vol.34 No.6, 1914-1946 (1996).

[55] B. Piccoli and H.J. Sussmann, *Regular Synthesis and Sufficiency Conditions for Optimality*, SIAM J. Control Optim., Vol. 39 No. 2 pp. 359-410 (2000).

[56] L.S. Pontryagin, V. Boltianski, R. Gamkrelidze and E. Mitchtchenko, *The Mathematical Theory of Optimal Processes*, John Wiley and Sons, Inc 1961.

[57] V. Ramakrishna, H. Rabitz, M.V. Salapaka, M. Dahleh and A. Peirce. *Controllability of Molecular Systems.* Phys. Rev. A, 62, 960-966 (1995).

[58] Y. Sachkov, *Controllability of Invariant Systems on Lie Groups and Homogeneous Spaces,* J. Math. Sci., Vol.100, n.4, 2355-2427 (2000).

[59] H. Schattler, *On the Local Structure of Time Optimal Bang-Bang Trajectories in $\mathbb{R}^3$*, SIAM J. Control Optim. Vol. 26, No. 1, pp. 186–204 (1988).

[60] A. Shiriaev, O. Egeland, H. Ludvigsen. *Global stabilization of unstable equilibrium point of pendulum* Proceedings of IEEE conference on decision and control, pp. 4584-4585, Tampa, Florida, December (1998).

[61] R. Shorten and K. Narendra, *Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for two stable second order linear time–invariant systems*, Proceeding 1999, American Control Conf. pp. 1410–1414 (1999).

[62] H.J. Sussmann, *On the gap between deterministic and stochastic ordinary differential equations*, Ann. Probability, Vol. 6, no. 1, pp. 19–41 (1978)

[63] H.J. Sussmann, *Regular synthesis for time optimal control of single–input real–analytic systems in the plane*, SIAM J. Control Optim., 25, pp.1145-1162 (1987).

[64] H. J. Sussmann and W. Liu, *Limits of highly oscillatory controls and the approximation of general paths by admissible trajectories*, Proceedings of the 30th Conference on Decision and Control, Brighton, UK (1991).

[65] Y. Xu. M. Iwase, K. Furuta. *Time optimal swing-up control of single pendulum* Transactions of the ASME, vol. 123, pages 518-527, September 2001.

[66] L.C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math., Vol. 67, pp. 251–282 (1936).

[67] A. L. Zelentsovsky, *Nonquadratic Lyapunov functions for robust stability analysis of linear uncertain systems*, IEEE Trans. Automat. Control, vol. 39, no. 1, pp. 135–138 (1994).