Homogenization Problems for Multi-dimensional and Multi-scale Structures

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[1]Marisa and Gianfranco
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CONTENTS

Introduction 1

1 Preliminaries 11
1.1 Notation 11
1.2 Basic notions of measure theory 12
1.3 Lower semicontinuity and relaxation 13
1.4 Quasiconvexity and rank-one-convexity 15
1.5 Γ-convergence 16
1.6 Increasing set functions 18
1.7 The direct method of Γ-convergence 19
1.7.1 Compactness and measure property of the Γ-limit 19
1.7.2 Γ-limits and boundary values 22
1.7.3 Integral representation on Sobolev spaces 22
1.8 Application of the direct methods to integral functionals with standard growth conditions 27
1.8.1 Higher integrability of gradients 28
1.9 Periodic homogenization 29
1.9.1 Coercive homogenization 29
1.9.2 Non-coercive convex homogenization 31
1.10 Extensions of the direct methods 33
1.10.1 Homogenization with non-standard growth conditions 34
1.10.2 Oscillating boundaries 35
1.11 The spaces BV and BD 36

2 Homogenization of periodic multi-dimensional structures 42
2.1 Sobolev spaces with respect to a measure 42
2.2 Statement of the main result 42
2.3 Proof of the homogenization theorem 47
2.4 Limits of a class of difference schemes 53
2.5 Appendix: Sobolev inequalities in $W^{1,p}_{\mu}$ 55

3 Homogenization of periodic multi-dimensional structures: the linearly elastic/perfectly plastic case 60
3.1 The space $LD^{p}_{\lambda}(\Omega)$ 60
3.2 Choice of the measure and some examples 61
3.3 The homogenization theorem 65
3.4 Non local effects 71

4 Homogenization of oscillating boundaries 81
4.1 Scheme of the direct method 81
4.2 Homogenization of media with oscillating profile 83
4.3 An auxiliary problem. Definition of the limit domain 85
4.4 Homogenization of cylindrical domains 95
4.5 The general case 99

5 Thin films with fast-oscillating profile 103
5.1 Proof of the result 104
5.2 Convergence of minimum problems 108

6 Asymptotic analysis of periodically-perforated non-linear media 110
6.1 Statement of the main result 110
6.1.1 Periodically perforated domains 110
6.2 A joining lemma on varying domains 112
6.3 Some auxiliary energy densities 115
6.4 Proof of the liminf inequality 118
6.5 Proof of the limsup inequality 120

7 Separation of scales and almost-periodic effects in the asymptotic behaviour of perforated periodic media 124
7.1 Setting of the problem 124
7.2 A general Γ-convergence approach 126
7.2.1 The Γ-liminf inequality 126
7.2.2 The Γ-limsup inequality 130
7.3 Separation of scales 131
7.3.1 Highly-oscillating energies in perforated domains 131
7.3.2 Slowly-oscillating energies in perforated domains 135
7.4 Interaction between homogenization processes 137

References 142

INTRODUCTION

In this thesis we study some homogenization processes that may model macroscopic properties of media whose microscopic behaviour takes lower-dimensional, multi-dimensional or multi-scale structures into account. In mathematical terms, this study can be traslated into the asymptotic behaviour via Γ-convergence of integral functionals (see Chapter 1) which may model energies of these structures. In addition we face also the problem of the choice of the natural domains in which the energies before homogenization and the limit problems are set since it is not always the usual Sobolev space.

In Chapter 2 we deal with the asymptotic behaviour of integral functionals which may model energies concentrated on multi-dimensional structures. The model example we have in mind is that of composite elastic bodies composed of n-dimensional elastic grains interacting through contact forces depending on the relative displacements of their common boundaries (see Example 2.3). In a general setting, following the approach of Ambrosio, Buttazzo and Fonseca [2], we consider integrals of the form

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dDu}{d\mu_\varepsilon}\Big) d\mu_\varepsilon,$$

defined on the space $W^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^m)$ of Sobolev functions with respect to the measure μ_ε, which is the set of L^p-functions of Ω whose distributional derivative is a measure absolutely continuous with respect to μ_ε with p-summable densities. We study the limit as $\varepsilon \to 0$ of such functionals under the hypotheses that f is a Borel function 1-periodic in the first variable satisfying a standard growth condition of order p, and

$$\mu_\varepsilon(B) = \varepsilon^n \mu\Big(\frac{1}{\varepsilon}B\Big)$$

where μ is a fixed 1-periodic Radon measure. In the model example the measure μ, up to normalization, is the sum of the n-dimensional Lebesgue measure and $(n-1)$-dimensional Hausdorff measure concentrated on a 1-periodic closed set E of σ-finite $n-1$-dimensional Hausdorff measure and such that $[0,1]^n \setminus E$ has a finite number of connected component, each one with a Lipschitz boundary. We show (Theorem 2.7) that under suitable reasonably general requirements on the measure μ (we will say that μ is 'p-homogenizable', see Definition 2.4), the family (F_ε) Γ-converges as $\varepsilon \to 0$ to a functional of the form

$$F_{\rm hom}(u) = \int_\Omega f_{\rm hom}(Du)\, dx$$

on $W^{1,p}(\Omega; \mathbf{R}^m)$, where the function $f_{\rm hom}$ is described by the asymptotic formula

$$f_{\hom}(A) = \lim_{k\to+\infty} \inf\Big\{ \frac{1}{k^n}\int_{[0,k)^n} f\Big(x, \frac{dDu}{d\mu}\Big)d\mu : \\ u \in W^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^m),\ u - Ax \ \ k\text{-periodic}\Big\}.$$

This formula generalizes the usual one, corresponding to the case when μ is the Lebesgue measure (see Braides [23] and Müller [65]). Hence, in this case we deal with energies defined on multi-dimensional structures whose homogenization gives rise to integral functionals defined on full-dimensional domains. This problem had been previously studied in the case when μ is the restriction of the Lebesgue measure to a periodic set whose complement is composed by well separated bounded sets by Braides and Garroni [30] (media with stiff inclusions). Another meaningful case is when μ is the $(n-1)$-dimensional Hausdorff measure restricted to the union of the boundaries of a periodic partition of $\mathbf{R}^n$. In this case the functions in $W^{1,p}_\mu(\Omega;\mathbf{R}^m)$ are piecewise constant and the functionals F_ε can be interpreted as a finite-difference approximation of the homogenized functional (Section 2.4, see also Kozlov [58], Pankov [67] and Davini [39]).

The approach described above is somehow complementary to the "smooth approach" where the functionals F_ε are defined as

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u\Big)d\mu_\varepsilon$$

on $C^\infty(\Omega;\mathbf{R}^m)$, whose homogenization is studied by Zhikov [75] (see also Braides and Chiadò Piat [24] for the case $\mu = \chi_E$ with E periodic, Bouchitté and Fragalà [20] and Bouchitté, Buttazzo and Seppecher [19] for relaxation results in the case of general μ).

In the context of linear elasticity or perfect plasticity, in place of considering energies depending on the deformation gradient Du, it is customary to consider energy functionals depending explicitly on the *linearized strain tensor* Eu. Hence, in Chapter 3 we study the asymptotic behaviour of functionals of the type

$$F_\varepsilon(u,\Omega) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}\Big)d\mu_\varepsilon$$

defined in a particular class of *functions with bounded deformation* denoted by $LD^p_{\mu_\varepsilon}(\Omega)$ (introduced in Section 3.1). More precisely, $LD^p_{\mu_\varepsilon}(\Omega)$ is the space of functions $u \in L^p(\Omega;\mathbf{R}^n)$, whose deformation tensor Eu is a measure absolutely continuous with respect to μ_ε with p-summable density $dEu/d\mu_\varepsilon$. Using both classical and fine properties of functions with bounded deformation and the same assumptions as in Chapter 2, we prove a homogenization theorem (Theorem 3.9). Precisely, we show the existence of the Γ-limit of the functionals F_ε with respect to L^p-convergence in the Sobolev space $W^{1,p}(\Omega;\mathbf{R}^n)$, and with respect to L^1-convergence in $BD(\Omega)$ (the space of functions with bounded deformation in Ω; that is, the space of functions $u \in L^1(\Omega;\mathbf{R}^n)$ whose deformation tensor Eu is a Radon measure with finite total variation in Ω, see [3]). We show that the Γ-limit admits an integral representation

$$F_{\rm hom}(u,\Omega)=\int_\Omega f_{\rm hom}(Eu)\,dx$$

in $W^{1,p}(\Omega;\mathbf{R}^n)$; moreover, if f is convex then

$$F_{\rm hom}(u,\Omega)=\int_\Omega f_{\rm hom}(\mathcal{E}u)\,dx+\int_\Omega f^\infty_{\rm hom}\Big(\frac{dE^su}{d|E^su|}\Big)d|E^su|$$

in $BD(\Omega)$, where $\mathcal{E}u$ is the density of the absolutely continuous part and E^su is the singular part of Eu with respect to the Lebesgue measure; $f_{\rm hom}$ is described by an asymptotic formula and $f^\infty_{\rm hom}$ denotes the recession function of $f_{\rm hom}$ (see (1.1) and [15] for relaxation of functionals defined on $BD(\Omega)$).

Finally, we show that when the scaling argument leading to the functionals F_ε does not apply, non local effects can arise. More precisely, we consider functionals of the type

$$F^\gamma_\varepsilon(u,\Omega)=\varepsilon^\gamma\int_\Omega f\Big(\frac{x}{\varepsilon},\frac{dEu}{d\mu_\varepsilon}\Big)d\mu_\varepsilon\,,$$

which in the previous approach tend to the null functional when $\gamma>0$, and we construct an explicit example showing that, with a suitable choice of γ, μ_ε and of the convergence with respect to which the Γ-limit is computed, we have a limit functional of a non local nature (in the same spirit of Bellieud and Bouchitté [18]).

In Chapter 4 we prove a general Homogenization Theorem for sets with oscillating boundaries and in Chapter 5 we apply this result to the description of nonlinearly elastic thin films with a fast-oscillating profile. The behaviour of such films is governed by an elastic energy, where two parameters intervene: a first parameter ε represents the thickness of the thin film and a second one δ the scale of the oscillations. The analytic description of the elastic energy is given by a functional of the form

$$E_{\varepsilon,\delta}(u)=\int_{\Omega(\varepsilon,\delta)}W(Du)\,dx, \tag{0.1}$$

where the set $\Omega(\varepsilon,\delta)$ is of the form

$$\Omega(\varepsilon,\delta)=\Big\{x\in\mathbf{R}^3:\ |x_3|<\varepsilon\,f\Big(\frac{x_1}{\delta},\frac{x_2}{\delta}\Big),(x_1,x_2)\in\omega\Big\}, \tag{0.2}$$

with f is a bounded 1-periodic function which parameterizes the boundary of the thin film, which then has periodicity δ.

It is convenient to scale these energies by a change of variables and consider the functionals

$$E^\delta_\varepsilon(u)=\int_{\Omega(\delta)}W\Big(D_1u,D_2u,\frac{1}{\varepsilon}D_3u\Big)\,dx, \tag{0.3}$$

where now

$$\Omega(\delta) = \Big\{x \in \mathbf{R}^3 : \ |x_3| < f\Big(\frac{x_1}{\delta}, \frac{x_2}{\delta}\Big), (x_1, x_2) \in \omega\Big\}. \tag{0.4}$$

In this way we separate the effects of the two parameters ε and δ.

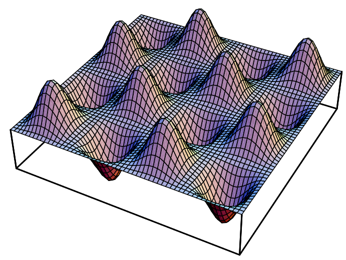

FIG. 0.1. fast oscillating boundaries

In a recent paper by Braides, Fonseca and Francfort [29] a general compactness result for functional of thin-film type has been proven which comprises energies of the form (0.3), showing that, with fixed $\delta = \delta(\varepsilon)$, upon possibly extracting a subsequence, the family $E_\varepsilon^{\delta(\varepsilon)}$ Γ-converges as $\varepsilon \to 0$ to a 2-dimensional energy, which, if $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, can be identified with a $2d$-functional of the form

$$E(u) = \int_\omega \widetilde{W}(D_1 u, D_2 u)\, dx. \tag{0.5}$$

In many cases it is possible to describe $\widetilde{W}$ explicitly in terms of W and f, and as a consequence to prove that no passage to a subsequence is necessary. When $f = C$ is constant (*i.e.*, the profile of the thin film is flat, and hence there is no real dependence on δ) the description of the energy density $\widetilde{W}$ has been given by Le Dret and Raoult [59] who proved that $\widetilde{W} = 2C\, Q_2\overline{W}$; here Q_2 denotes the operation of $2d$-quasiconvexification, and $\overline{W}$ is obtained from W by minimizing in the third component. An equivalent formula, of 'homogenization type', is given in [29] (see also Section 1.9 Theorem 1.46). If $\overline{W} \neq Q_2\overline{W}$ (*i.e.*, $\overline{W}$ is not quasiconvex). Both formulas underline the formation of microstructures generated by the passage to the limit. When f is not constant, then the function $\widetilde{W}$ depends on the behaviour of δ with respect to ε. The case when $\delta = \varepsilon$ (or more in general when δ/ε converges to a constant) has been treated in [29], where it is shown that a homogenization-type formula for $\widetilde{W}$ can be given. The same method can be used when $\delta >> \varepsilon$; in this case the recipe to obtain $\widetilde{W}$ is the following: first, keep δ fixed and apply the Le Dret and Raoult procedure, considering the thickness of the thin film as a parameter. The output of this procedure is a 2-dimensional energy of the form

$$E^{\delta}(u)=\int_{\omega}2f\Big(\frac{x_1}{\delta},\frac{x_2}{\delta}\Big)Q_2\overline{W}(D_1u,D_2u)\,dx. \tag{0.6}$$

We can then let δ tend to 0, and apply well-known homogenization procedures of Braides and Müller (see Section 1.9) obtaining a limit functional, which turns out to be the desired one. In the case $\delta << \varepsilon$ it is possible to make an *ansatz* in the same spirit, arguing that the limit E can be obtained in the following two steps:

(1) (Homogenization of sets with oscillating boundaries) First consider ε as fixed, and let $\delta \to 0$, to obtain a limit functional of the form

$$E_{\varepsilon}(u)=\int_{\omega\times(-1,1)}W_{\rm hom}\Big(x_3,D_1u,D_2u,\frac{1}{\varepsilon}D_3u\Big)\,dx$$

(we consider the normalized case $\sup f=1$).

Note that in this case an additional dependence on x_3 is introduced, which may underline a loss of coerciveness of the function $W_{\rm hom}$ for certain values of x_3. The form of $W_{\rm hom}$ will depend on W and on the sublevel sets of f;

(2) (Thin film limit) Let $\varepsilon \to 0$ and apply a suitable generalization of the method of [29] to non-coercive functionals. In this way we obtain a limit energy density

$$\overline{W}_{\rm hom}(\overline{F})=\inf_{k\in\mathbf{N}}\inf\Big\{\frac{1}{k^2}\int_{(0,k)^2\times(0,1)}W_{\rm hom}(x_3,Du+(\overline{F},0))\,dx:\\ u\in W^{1,p}_{\rm loc}((0,1)^3;\mathbf{R}^3),\ u\ k\text{-periodic in}\,(x_1,x_2)\Big\}.$$

Note that the dependence on x_3 implies that the simpler method of [59] cannot be applied to this situation.

A partial result in this case has been obtained by Kohn and Vogelius [57] who dealt with linear operators.

In Chapter 4 we give a general theory for the homogenization of non-convex energies defined on sets with oscillating boundaries by generalizing the application of the direct methods of Γ-convergence to homogenization as described in Sections 1.7 and 1.9. We clarify and prove statement (1) above, by showing that the functionals E_{ε} are defined on a 'degenerate Sobolev Space' that can be described by proving an auxiliary convex-homogenization result. The formula for $W_{\rm hom}$ can be obtained by solving a possibly degenerate localized $3d$-homogenization problem. In the case of convex W the determination of $W_{\rm hom}(t,\overline{F})$ for fixed $t\in(-1,1)$ essentially amount to solving a $2d$-homogenization problem with an energy which is coercive only on the set $E_t=\big\{(x_1,x_2)\in\mathbf{R}^2: f(x_1,x_2)>|t|\big\}$, while in the general non-convex case the problem defining $W_{\rm hom}(t,\overline{F})$ is genuinely three dimensional. We state and prove these results in a general n-dimensional setting (for some related problems in the convex setting see *e.g.* [31]).

In Chapter 5 we prove that by following steps (1) and (2) above we indeed obtain the description of $\widetilde{W}$. Even though this is an intrinsically vectorial problem, and hence the 'natural' structural condition on W is quasiconvexity, we have been able to prove this result only with the additional hypothesis that W is convex. The technical point where this assumption is needed is the separation of scales argument, which assures that, essentially, homogenization comes first, followed by the thin film $3d$–$2d$ limit. In general problems where only quasiconvexity is assumed this point is usually proved by a compactness argument which uses some equi-integrability properties of gradients of optimal sequences for the homogenization derived from the growth conditions on the energy density (see Section 1.8.1; for the use of this argument in the framework of iterated homogenization see [26] Chapter 22; for an application to heterogeneous thin films with flat profile see Shu [68]). In the case of thin films with fast-oscillating profiles, this technique cannot be used since we have a control on the gradients of optimal sequences only on varying wildly oscillating domains. In the convex case though, optimal sequences for the homogenization can be obtained simply by scaling one single periodic function, and hence their gradients automatically enjoy equi-integrability properties. Note that this difficulty is similar to those encountered when dealing with higher-order theories of thin films. In that case the necessary compactness properties can be obtained by adding a small perturbation with higher-order derivatives (as in the paper by Bhattacharya and James [17]). We do not follow this type of argument since even a singular perturbation by higher-order gradients might interact with the homogenization process, as shown by Francfort and Müller [50]. More applications of Γ-convergence arguments to thin films theory can be found in [16, 28].

In Chapter 6 and 7 we deal with the asymptotic behaviour of Dirichlet problems in perforated domains. A well-known result shows the appearance of a 'strange' extra term as the period of the perforation tends to 0. In a paper by Cioranescu and Murat [38] (see also e.g. earlier work by Marchenko and Khrushlov [62]) the following result (among others) is proved. Let Ω be a bounded open set in $\mathbf{R}^n$, $n \geq 3$ and for all $\delta > 0$ let Ω_δ be the *periodically perforated domain*

$$\Omega_\delta = \Omega \setminus \bigcup_{i \in \mathbf{Z}^n} \overline{B_i^\delta},$$

where B_i^δ denotes the open ball of centre $x_i^\delta = i\delta$ and radius $\delta^{n/(n-2)}$. Let $\phi \in H^{-1}(\Omega)$ be fixed, and let $u_\delta \in H_0^1(\Omega)$ be the solution of the problem

$$\begin{cases} -\Delta u = \phi \\ u \in H_0^1(\Omega_\delta), \end{cases}$$

extended to 0 outside Ω_δ. Then, as $\delta \to 0$, the sequence u_δ converges weakly in $H_0^1(\Omega)$ to the function u which solves the problem

$$\begin{cases} -\Delta u + Cu = \phi \\ u \in H_0^1(\Omega), \end{cases}$$

where C denotes the *capacity of the unit ball* in $\mathbf{R}^n$:

$$C = \mathrm{cap}(B_1) = \inf\Big\{\int_{\mathbf{R}^n} |D\zeta|^2 dx : \zeta \in H^1(\mathbf{R}^n),\ \zeta = 1 \text{ on } B_1(0)\Big\}.$$

This result can be easily translated in a equivalent variational form and set in the framework of Γ-convergence, since u_δ is the solution of the minimum problem

$$\min\Big\{\int_\Omega |Dv|^2\,dx - 2\langle\phi, v\rangle : v \in H^1_0(\Omega),\ v = 0 \text{ on } \Omega\setminus\Omega_\delta\Big\},$$

and the limit function u solves

$$\min\Big\{\int_\Omega (|Dv|^2 + C|v|^2)\,dx - 2\langle\phi, v\rangle : v \in H^1_0(\Omega)\Big\}.$$

In Chapter 6 we give a direct proof of the non-linear vector-valued version of this variational problem under minimal assumptions. More precisely, let Ω be a bounded open set in $\mathbf{R}^n$ and let $m \geq 1$. Let $1 < p < n$ and for all $\delta > 0$ let Ω_δ be the *periodically perforated domain* defined as above, where now B^δ_i denotes the open ball of centre $x^\delta_i = i\delta$ and radius $\delta^{n/(n-p)}$ (for notational simplicity we do not treat the case $n = p$, which can be dealt with similarly; for the necessary changes in the statements see [38]). Note that this is the only meaningful scaling for the radii of the perforation, since other choices give trivial convergence results. Let $f : \mathbf{M}^{m\times n} \to [0, +\infty)$ be a Borel function satisfying a growth condition of order p, and let (δ_j) be a sequence of strictly positive numbers converging to 0 such that there exists the limit

$$g(z) = \lim_j \delta_j^{\frac{np}{n-p}} Qf\Big(\delta_j^{-\frac{n}{n-p}} z\Big)$$

for all $z \in \mathbf{R}^m$, where Qf denotes the *quasiconvexification of* f. Note that this condition is not restrictive upon passing to a subsequence and is trivially satisfied if f is positively homogeneous of degree p. Then, if $\phi \in W^{-1,p'}(\Omega; \mathbf{R}^m)$ is fixed, the minimum values

$$m_j = \inf\Big\{\int_{\Omega_{\delta_j}} f(Du)\,dx + \langle\phi, u\rangle :\ u \in W^{1,p}_0(\Omega_{\delta_j}; \mathbf{R}^m)\Big\}$$

converge to the minimum value

$$m = \min\Big\{\int_\Omega \Big(Qf(Du) + \varphi(u)\Big)\,dx + \langle\phi, u\rangle :\ u \in W^{1,p}_0(\Omega; \mathbf{R}^m)\Big\},$$

where φ is given by the *nonlinear capacitary formula*

$$\varphi(z) = \inf\Big\{\int_{\mathbf{R}^n} g(D\zeta)dx : \zeta - z \in W^{1,p}(\mathbf{R}^n; \mathbf{R}^m),\ \zeta = 0 \text{ on } B_1(0)\Big\},$$

which agrees with those obtained in convex cases (see e.g. [13], [42], [69], [34]). Moreover, if $u_j \in W_0^{1,p}(\Omega_{\delta_j};\mathbf{R}^m)$ is such that $\int_{\Omega_{\delta_j}} f(Du_j)\,dx + \langle\phi, u_j\rangle = m_j + o(1)$ as $j \to +\infty$, then, upon extending u_j to 0 outside Ω_{δ_j}, (u_j) admits a subsequence weakly converging in $W_0^{1,p}(\Omega;\mathbf{R}^m)$ to a solution of the problem defining m.

Note that we do not assume any structure or regularity condition on f. In the case of convex and differentiable f we may recover the corresponding result for systems contained in the paper by Casado Diaz and Garroni [34], where more arbitrary geometries are also considered. Note moreover that φ may depend on the subsequence (δ_j), and as a consequence the values m_j may not converge. Furthermore, the function φ may not be positively homogeneous of degree p, as already observed by Casado Diaz and Garroni [35].

The proof of the result is based only on a new simple and direct Γ-convergence approach. The fundamental tool is a 'joining lemma for perforated domains' (Lemma 6.6), which, loosely speaking, allows us to restrict our attention to families of functions (u_δ), converging to a function u, which equal the constant $u(x_i^\delta)$ on suitable annuli surrounding B_i^δ. The contribution of these functions on such annuli easily leads to the formula defining φ. This method seems of interest also since it can be easily applied to sequences of integral functionals by considering minimum problems m_j where we replace $f(Du)$ by $f_j(x, Du)$. In Chapter 7 we examine the case $f_j(x, Du) = f(x/\varepsilon_j, Du)$. In order to highlight the effects of homogenization we only treat the case when f is positively homogeneous of degree 2 in the second variable and $n \geq 3$; the same method with minor changes applies for $n = 2$ or to p-homogeneous f and $1 < p \leq n$ (for changes in the statements see e.g. [38], [34]).

Since these problems are usually expressed in terms of G-convergence of operators (see [12]) we describe our results with that terminology. Consider problems of the general form

$$\begin{cases} -\mathrm{div}\, a_\varepsilon(x, Du) = \phi \\ u = 0 \text{ on } \partial\Omega_\delta, \end{cases} \tag{0.7}$$

(for the sake of clarity in the exposition we consider only the case of a_ε linear and symmetric). A recent compactness result by Dal Maso and Murat [43] ensures that, for a fixed choice of $\delta = \delta(\varepsilon)$, upon possibly extracting a subsequence, the solutions u_ε converge to that of a limit problem of the form

$$\begin{cases} -\mathrm{div}\,\big(a_0(x, Du)\big) + \varphi u = \phi \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{0.8}$$

where the operator $A_0 = -\mathrm{div}\, a_0(x, Du)$ is the G-limit of the sequence of operators $-\mathrm{div}\, a_\varepsilon(x, Du)$ (see e.g. [71], [66], [1], [76], [37], [67]). The G-limit operator is well defined by a compactness argument; in particular, if $a_\varepsilon(x, z) = a(x/\varepsilon, z)$ then the G-limit (homogenized) operator $A_{\rm hom} = -\mathrm{div}\, a_{\rm hom}(Du)$ is independent from the subsequence and does not depend on x. The determination of the function $\varphi \in L^\infty$ is a subtler problem and involves a complex capacitary computation.

In Chapter 7 we address the problem of the effective computation of φ in (0.8) when $a_\varepsilon(x,z)=a(x/\varepsilon,z)$ in (0.7) with a 1-periodic. We highlight various regimes, at which the oscillating Dirichlet boundary problems of the form

$$\begin{cases} -\mathrm{div}\, a\Big(\dfrac{x}{\varepsilon}, Du\Big) = \phi \\ \\ u = 0 \text{ on } \partial\Omega_\delta, \end{cases} \tag{0.9}$$

behave differently (again, for the sake of simplicity here we describe the results in the case when the function a is linear, continuous and symmetric, and $n \geq 3$ only):

(i) (*separation of scales*) if $\varepsilon << \delta^{n/(n-2)}$ or $\varepsilon >> \delta$ then the whole family of solutions $u_{\varepsilon,\delta}$ of (0.9) converges to the solution u of a problem of the form

$$\begin{cases} -\mathrm{div}\,\big(a_{\mathrm{hom}}(Du)\big) + Cu = \phi \\ u = 0 \text{ on } \partial\Omega; \end{cases} \tag{0.10}$$

i.e., $\varphi = C$. In the case $\varepsilon << \delta^{n/(n-2)}$ the constant C is given by the *homogenized capacitary problem*

$$C = \mathrm{cap}_{\mathrm{hom}}(B_1) = \inf\Big\{\int_{\mathbf{R}^n} \langle a_{\mathrm{hom}}(Du), Du\rangle\, dx : u \in H^1(\mathbf{R}^n),\ u = 1 \text{ on } B_1(0)\Big\}. \tag{0.11}$$

In a sense, we may first let $\varepsilon \to 0$ and then $\delta \to 0$. In the case $\varepsilon >> \delta$, conversely, we may let first act ε as a parameter. As a consequence the dependence on x/ε in (0.9) can be 'frozen' and we are led to consider the *parameterized capacitary problems*

$$\mathrm{cap}_y(B_1) = \inf\Big\{\int_{\mathbf{R}^n} \langle a(y, Du), Du\rangle\, dx : u \in H^1(\mathbf{R}^n),\ u = 1 \text{ on } B_1(0)\Big\}. \tag{0.12}$$

The overall effect of letting $\varepsilon \to 0$ is then obtained by averaging, and we get

$$C = \int_{(0,1)^n} \mathrm{cap}_y(B_1)\, dy. \tag{0.13}$$

(ii) (*almost-periodic effects*) in the remaining cases, the two periods ε and δ present in (0.9) interact. As a consequence in general the family of solutions $u_{\varepsilon,\delta}$ *does not converge*. The problems satisfied by converging subsequences may be of the form (0.10) with C described by a single problem (*periodic behaviour*) of the form

$$\mathrm{cap}^0(B_1) = \inf\Big\{\int_{\mathbf{R}^n} \langle b(x, Du), Du\rangle\, dx : u \in H^1(\mathbf{R}^n),\ u = 1 \text{ on } B_1(0)\Big\}, \tag{0.14}$$

with b a suitable scaled operator (in some cases independent of x), or by a formula of the type (0.13) (*almost-periodic behaviour*) with cap_y substituted by a suitable scaled and localized problem, but may even give rise to a problem of the form (0.8) with *non-constant* φ (*finely-tuned interplay between* δ *and* ε).

All the results above may be easily derived from the corresponding description of the Γ-limits (see [45], [41], [26], [22]) of the functionals $F_{\varepsilon,\delta}$ of the form

$$F_{\varepsilon,\delta}(u) = \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, Du\Big)\,dx & \text{if } u = 0 \text{ on } \bigcup_{i\in\mathbf{Z}^n} B_i^\delta \\ +\infty & \text{otherwise} \end{cases} \tag{0.15}$$

for $f(x,z) = \langle a(x,z), z\rangle$ (a linear) (see Remark 7.1(ii)). We show that the Γ-limits of converging subsequences of these functionals as $\varepsilon \to 0$ and $\delta \to 0$ are of the form

$$F(u) = \int_\Omega f_{\rm hom}(Du)\,dx + \int_\Omega \varphi|u|^2\,dx, \tag{0.16}$$

where $f_{\rm hom}$ is the *homogenized energy density of* f (see Section 1.9 Theorem 1.46) and φ is described above. Note that in the cases $\varepsilon << \delta^{n/(n-2)}$ and $\delta << \varepsilon$ then φ is constant and does not depend on the subsequence. Also in this case we propose a direct proof of all these results based on the use of the 'joining lemma on varying domains' (Lemma 7.2, see Lemma 6.6 in a general context). This technique is explained in a general framework in Section 7.2. Note that we do not make use of integral representation techniques such as those in [40].

In order to make the presentation more self-contained we introduce in Chapter 1 the necessary background on measure theory, Γ-convergence and relaxation and functions of bounded variation and deformation. Moreover we state the classical homogenization results that are used in the sequel.

Chapter 2 is the paper [8], with a modified definition of 'p-homogenizable' measure, done in collaboration with A. Braides and V. Chiadò Piat, Chapter 3 is the paper [9] in collaboration with F. B. Ebobisse, Chapters 4, 5, 6 and 7 are papers [5], [7] and [6] in collaboration with A. Braides.

1

PRELIMINARIES

We introduce the notation and the classical definitions and results that we will use in the next chapters.

1.1 Notation

Let Ω be a bounded open subset of $\mathbf{R}^n$, $\mathcal{B}(\Omega)$ denotes the family of Borel subsets of Ω and $\mathcal{B}_c(\Omega)$ the family of Borel subsets with compact closure in Ω. We denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω. In the sequel, $n, m \in \mathbf{N}$ with $n \geq 2$, $m \geq 1$. If $x \in \mathbf{R}^n$ then $x_\alpha = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ is the vector of the first $n-1$ components of x, and $D_\alpha = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\right)$. The notation $\mathbf{M}^{m\times n}$ stands for the space of $m \times n$ real matrices and $\mathbf{M}^{n\times n}_{sym}$ for the space of $n \times n$ real symmetric matrices.

Given a matrix $F \in \mathbf{M}^{m\times n}$, and following the notation introduced in [59], we write $F = (\overline{F}|F_n)$, where F_i denotes the i-th column of F, $1 \leq i \leq n$, and $\overline{F} = (F_1, \ldots, F_{n-1}) \in \mathbf{M}^{m\times n-1}$ is the matrix of the first $n-1$ columns of F. $\overline{F}$ denotes also $(\overline{F}, 0)$ when no confusion arises.

$C^k(\Omega; \mathbf{R}^m)$ is the space of k-times continuously differentiable functions $u : \Omega \to \mathbf{R}^m$, $C^k_c(\Omega; \mathbf{R}^m)$ are functions in $C^k(\Omega; \mathbf{R}^m)$ with compact support. We will use standard notation for the Sobolev and Lebesgue spaces $W^{1,p}(\Omega; \mathbf{R}^m)$ and $L^p(\Omega; \mathbf{R}^m)$, $W^{1,p}_0(\Omega; \mathbf{R}^m)$ is the closure of $C^\infty_c(\Omega; \mathbf{R}^m)$ in $W^{1,p}(\Omega; \mathbf{R}^m)$ and $W^{-1,p'}(\Omega; \mathbf{R}^m)$ is the dual space of $W^{1,p}_0(\Omega; \mathbf{R}^m)$ where p' denotes the dual exponent of $p \geq 1$; when $p = 2$ and $m = 1$ we use the notation $H^1(\Omega)$, $H^1_0(\Omega)$ and $H^{-1}(\Omega)$ respectively. The letter c will denote a strictly positive constant independent of the parameters under consideration, whose value may vary from line to line and ω a generic fixed modulus of continuity; *i.e.*, a function $\omega : [0, +\infty) \to [0, +\infty)$ continuous in 0 and with $\omega(0) = 0$. The Hausdorff k-dimensional measure and the n-dimensional Lebesgue measure in $\mathbf{R}^n$ are designated as $\mathcal{H}^k$ and $\mathcal{L}^n$ respectively. If $E \subset \mathbf{R}^n$ is a Lebesgue-measurable set then $|E|$ or $\mathcal{L}^n(E)$ is its Lebesgue measure. If E is a subset of $\mathbf{R}^n$ then χ_E is its *characteristic function*. $B_\rho(x)$ is the open ball of center x and radius ρ. The symbols $\langle \cdot, \cdot \rangle$ and $|\cdot|$ stand for the Euclidean scalar product and the Euclidean norm. For any two vectors a and b in $\mathbf{R}^n$, the symmetric tensor product $a \odot b$ is the symmetric $n \times n$ matrix defined by $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$, being the tensor product $a \otimes b$ the matrix whose entries are $a_i b_j$. We denote $[t]$ the integer part of t.

Let $f : \mathbf{R}^N \to [0, +\infty]$ be convex. We define the *recession function* f^∞ of f as

$$f^{\infty}(x) = \lim_{t\to\infty} \frac{f(tx)}{t} \qquad \text{for every } x \in \mathbf{R}^N . \tag{1.1}$$

Note that from the convexity of f it is possible to prove that the limit of $f(tx)/t$ as t tends to $+\infty$ exists so that f^{∞} is well defined. It is well-known that f^{∞} is convex and positively homogeneous of degree one.

1.2 Basic notions of measure theory

Definition 1.1 *A function $\mu : \mathcal{B}(\Omega) \to \mathbf{M}^{m\times n}$ is a* matrix-valued measure *(or a $\mathbf{M}^{m\times n}$-*valued measure*) on Ω if it is countably additive*; *i.e.*,

$$B = \bigcup_{i\in\mathbf{N}} B_i, \qquad B_i \cap B_j = \emptyset \qquad \textit{if } i \neq j \qquad\qquad \Rightarrow \qquad \mu(B) = \sum_{i\in\mathbf{N}} \mu(B_i).$$

The set of such measures will be denoted by $\mathcal{M}(\Omega; \mathbf{M}^{m\times n})$. If no confusion may arise, we denote by μ_{ij}, $i = 1, \dots, n$ and $j = 1, \dots, m$ the entries of μ.

We say that a matrix-valued measure is a vector measure *if $m = 1$ and that it is a* measure *if $n, m = 1$. We say that a* measure *is a* positive measure *if it takes its values in $[0, +\infty]$.*

The set of vector measures, measures and of positive measures on Ω will be denoted by $\mathcal{M}(\Omega; \mathbf{R}^n)$, $\mathcal{M}(\Omega)$ and $\mathcal{M}_+(\Omega)$, respectively.

A function $\mu : \mathcal{B}_c(\Omega) \to \mathbf{M}^{m\times n}$ is a matrix-valued Radon measure *on Ω if $\mu_{|\mathcal{B}(\Omega')}$ is a measure for all $\Omega' \subset\subset \Omega$. As above, we will speak of vector Radon measures, Radon measures and of positive Radon measures.*

We define the restriction $\mu \llcorner B$ of μ to $B \subset \Omega$ by

$$\mu \llcorner B(A) = \mu(B \cap A)$$

for all $A \in \mathcal{B}(\Omega)$.

Definition 1.2 *If $\mu \in \mathcal{M}(\Omega; \mathbf{M}^{m\times n})$ for all $B \in \mathcal{B}(\Omega)$ we define the* total variation *of μ on B by*

$$|\mu|(B) = \sup\Big\{ \sum_{i\in\mathbf{N}} |\mu(B_i)| \ : \ B = \bigcup_{i\in\mathbf{N}} B_i \Big\}.$$

The set function $|\mu|$ is a positive measure on Ω.

We say that $\mu \in \mathcal{M}(\Omega; \mathbf{M}^{m\times n})$ is bounded or finite if it has finite total variation in Ω; *i.e.*, $|\mu|(\Omega) < +\infty$.

Definition 1.3 *Let $\mu \in \mathcal{M}_+(\Omega)$ and $\lambda \in \mathcal{M}(\Omega; \mathbf{M}^{m\times n})$. We say that λ is* absolutely continuous *with respect to μ (and we write $\lambda << \mu$) if $\lambda(B) = 0$ for every $B \in \mathcal{B}(\Omega)$ with $\mu(B) = 0$. We say that λ is* singular *with respect to μ if there exists a set $E \in \mathcal{B}(\Omega)$ such that $\mu(E) = 0$ and $\lambda(B) = 0$ for all $B \in \mathcal{B}(\Omega)$ with $B \cap E = \emptyset$.*

Definition 1.4 *If $\mu \in \mathcal{M}(\Omega)$ we adopt the usual notation $L^p_\mu(\Omega;\mathbf{M}^{m\times n})$ to indicate the space of $\mathbf{M}^{m\times n}$-valued p-summable functions with respect to $|\mu|$ on Ω. We omit μ if it is the Lebesgue measure, and we omit $\mathbf{M}^{m\times n}$ if $n, m = 1$.*

Remark 1.5 If $f \in L^1_\mu(\Omega;\mathbf{M}^{m\times n})$ and $\mu \in \mathcal{M}(\Omega)$ then we define the measure $f\mu \in \mathcal{M}(\Omega;\mathbf{M}^{m\times n})$ by

$$f\mu(B) = \int_B f\,d\mu\,.$$

Definition 1.6. (Locally weak* convergence) *Let μ and the sequence (μ_h) be matrix-valued Radon measures; we say that (μ_h) locally weakly* converges to μ if*

$$\lim_{h\to+\infty}\int_\Omega \phi\,d\mu_h = \int_\Omega \phi\,d\mu$$

for every $\phi \in C_c(\Omega;\mathbf{M}^{m\times n})$.

Theorem 1.7. (Radon-Nikodym) *If $\lambda \in \mathcal{M}(\Omega;\mathbf{M}^{m\times n})$, and $\mu \in \mathcal{M}_+(\Omega)$, then there exists a function $f \in L^1_\mu(\Omega;\mathbf{M}^{m\times n})$ and a measure λ^s, singular with respect to μ, such that*

$$\lambda = f\mu + \lambda^s\,.$$

This will be called the Radon-Nikodym decomposition *of λ with respect to μ.*

Theorem 1.8. (Besicovitch Derivation Theorem) *Let μ, λ and f be as in Theorem 1.7. Then for μ-almost all $x \in \operatorname{spt}\mu$ there exists the limit*

$$\frac{d\lambda}{d\mu}(x) = \lim_{\rho\to 0^+}\frac{\lambda(B_\rho(x))}{\mu(B_\rho(x))},$$

and $f(x) = \frac{d\lambda}{d\mu}(x)$ for μ-almost all $x \in \operatorname{spt}\mu$.

1.3 Lower semicontinuity and relaxation

Let (X, d) be a metric space.

Definition 1.9 *A function $F : X \to [-\infty, +\infty]$ will be said to be (sequentially)* lower semicontinuous *(l.s.c. for short) at $u \in X$, if for every sequence (u_j) converging to u we have*

$$F(x) \le \liminf_j F(u_j), \tag{1.2}$$

or in other words

$$F(u) = \min\{\liminf_j F(u_j) :\ u_j \to u\}. \tag{1.3}$$

We will say that F is lower semicontinuous (on X) if it is l.s.c. at all $u \in X$.

Definition 1.10 *Let* $F : X \to [-\infty, +\infty]$ *be a function. Its* lower semicontinuous envelope $\overline{F}$ *is the greatest lower semicontinuous function not greater than* F*, i.e. for every* $u \in X$

$$\overline{F}(u) = \sup\{G(u) : \ G \ l.s.c., \ G \leq F\}. \tag{1.4}$$

$\overline{F}$ *is called also* relaxation of F *or* relaxed function.

Remark 1.11 Notice that the function $\overline{F}$ can be described as follows: for every $u \in X$

$$\overline{F}(u) = \inf\{\liminf_j F(u_j) : \ u_j \to u\}. \tag{1.5}$$

The reason for introducing this notion can be found in the following well-know theorem (see e.g. [26] Theorem 1.9).

Theorem 1.12. (Weierstrass) *Let* $F : X \to [-\infty, +\infty]$ *be such that there exists a compact set* $K \subset X$ *with* $\inf_X F(u) = \inf_K F(u)$*. Then there exists the minimum value* $\min_X \overline{F}(u)$ *and it is equals the infimum* $\inf_X F(u)$*. Moreover, the minimum points for* $\overline{F}$ *are exactly all the limits of converging sequences* u_j *such that* $\lim_j F(u_j) = \inf_X F$*.*

Let $f : \Omega \times \mathbf{R}^N \to [0, +\infty]$ be a given function, we consider functionals defined on the space $\mathcal{M}_b(\Omega; \mathbf{R}^N)$ of bounded vector measures of the form

$$F(\lambda, B) = \begin{cases} \displaystyle\int_B f(x, v)\, d\mu & \text{if } \lambda = v\mu \text{ with } v \in L^1(\Omega; \mathbf{R}^N) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.6}$$

for every $\lambda \in \mathcal{M}_b(\Omega; \mathbf{R}^N)$ amd $B \in \mathcal{B}(\Omega)$. The following theorem prove that the relaxed functional $\overline{F}$ with respect to the locally weak*-convergence in the sense of measures can be written as an integral

$$\overline{F}(\lambda, B) = \int_B \varphi\Big(x, \frac{d\lambda}{d\mu}\Big)\, d\mu + \int_B \varphi^\infty\Big(x, \frac{d\lambda^s}{d|\lambda^s|}\Big) d|\lambda^s| \tag{1.7}$$

for a suitable convex integrand φ depending on f, where φ^∞ is the recession function of φ and λ is split into the sum

$$\lambda = \frac{d\lambda}{d\mu}\mu + \lambda^s$$

by using the Radon-Nikodym Theorem 1.7 and the Besicovitch Derivation Theorem 1.8 (see [32] Theorem 3.3.1).

Theorem 1.13 *Assume the functional* $F(\cdot, \Omega)$ *defined in* (1.6) *is finite in at least one* $v_0 \in L^1(\Omega; \mathbf{R}^N)$*. Then, there exists a Borel function* $\varphi : \Omega \times \mathbf{R}^N \to [0, +\infty]$ *such that*

(i) *for μ-a.e. $x \in \Omega$ the function $\varphi(x,\cdot)$ is convex and l.s.c. on $\mathbf{R}^N$;*

(ii) *formula* (1.7) *holds for every $\lambda \in \mathcal{M}_b(\Omega;\mathbf{R}^N)$ and every $B \in \mathcal{B}(\Omega)$;*

(iii) *the recession function $\varphi^\infty(x,s)$ is l.s.c. in (x,s).*

1.4 Quasiconvexity and rank-one-convexity

Definition 1.14 *We say that $f : \mathbf{M}^{m\times n} \to \mathbf{R}$ is* quasiconvex *if f is continuous, and for all $A \in \mathbf{M}^{m\times n}$ and for every bounded open set E of $\mathbf{R}^n$*

$$|E|f(A) \le \int_E f(A + D\varphi)\,dx$$

for every $\varphi \in C_c^\infty(E;\mathbf{R}^m)$.

Definition 1.15 *A function $f : \mathbf{M}^{m\times n} \to [-\infty,+\infty]$ is* rank-1-convex *if and only if for all $A, B \in \mathbf{M}^{m\times n}$ such that* rank $(A-B) \le 1$

$$f(tA + (1-t)B) \le tf(A) + (1-t)f(B) \tag{1.8}$$

for all $t \in (0,1)$ for which the right-hand side makes sense.

Remark 1.16 (i) If f is rank-1-convex, and $0 \le f(A) \le c(1+|A|^p)$, then f satisfies a local Lipschitz condition

$$|f(A) - f(B)| \le c(1 + |A|^{p-1} + |B|^{p-1})|A-B| \tag{1.9}$$

for all $A, B \in \mathbf{M}^{m\times n}$;

(ii) if $m = 1$ or $n = 1$ then rank-1-convexity is the same as convexity;

(iii) if $1 \le p < \infty$ and f quasiconvex satisfies the growth condition from above $0 \le f(A) \le c(1+|A|^p)$ for all $A \in \mathbf{M}^{m\times n}$, then f is rank-one-convex.

Theorem 1.17 *Let $1 \le p < \infty$ and $f : \mathbf{M}^{m\times n} \to [0,+\infty)$ satisfying*

$$0 \le f(A) \le c(1+|A|^p) \qquad \text{for all } A \in \mathbf{M}^{m\times n}.$$

The functional

$$\mathcal{F}(u) = \int_\Omega f(Du)\,dx$$

is (sequentially) weakly l.s.c. on $W^{1,p}(\Omega;\mathbf{R}^m)$ if and only if f is a quasiconvex function.

(See e.g. [26] Remark 5.15, Theorem 5.16 and Proposition 4.3).

Definition 1.18 *Let $h : \mathbf{M}^{m\times n} \to [0,+\infty)$ be a Borel function. The $W^{1,p}$-quasiconvexification of h is given by the formula*

$$Qh(A) = \inf\Big\{\int_{(0,1)^n} h(A + Du)\,dx : u \in W_0^{1,p}((0,1)^n;\mathbf{R}^m)\Big\} \tag{1.10}$$

for $A \in \mathbf{M}^{m\times n}$. We say that h is $W^{1,p}$-quasiconvex if $Qh = h$.

Proposition 1.19 *If* $f : \mathbf{M}^{m\times n} \to \mathbf{R}$ *is a locally bounded Borel function, then* Qf *is quasiconvex.*

(See e.g. [26] Proposition 6.7).

Remark 1.20 If $f : \mathbf{M}^{m\times n} \to \mathbf{R}$ is a locally bounded Borel function, then

$$Qf = \sup\{g : \mathbf{M}^{m\times n} \to \mathbf{R} : \ g \text{ quasiconvex}, g \le f\}; \tag{1.11}$$

that is, Qf is the *quasiconvex envelope* of f. In fact, if we denote by h the right-hand side of (1.11), then $Qf \le h$ since Qf is quasiconvex and $Qf \le f$. On the other hand, if g is quasiconvex and $g \le f$, then $g = Qg \le Qf$, so that $h \le Qf$.

Remark 1.21 If h is a Borel function as above, and there exist constants $c_1, c_2 > 0$ such that $c_1(|A|^p - 1) \le h(A) \le c_2(|A|^p + 1)$, then the function Qh is quasiconvex (see Proposition 1.19) and the functional

$$\mathcal{H}(u) = \int_\Omega Qh(Du)\,dx$$

is the lower-semicontinuous envelope of the functional

$$H(u) = \int_\Omega h(Du)\,dx$$

on $W^{1,p}(\Omega; \mathbf{R}^m)$ with respect to the $L^p(\Omega; \mathbf{R}^m)$ convergence. In fact, by Theorem 1.46 and Remark 1.24 (v) the lower-semicontinuous envelope $\overline{H}$ of H can be written in an integral form $\overline{H}(u) = \int_\Omega \psi(Du)\,dx$, with ψ quasiconvex. Since $\psi \le h$ then by Remark 1.20 $\psi = Q\psi \le Qh$ and $\overline{H} \le \mathcal{H}$. On the other hand Qh is quasiconvex; hence, $\mathcal{H}$ is lower semicontinuous with respect to the $L^p(\Omega; \mathbf{R}^m)$ convergence (see Theorem 1.17), so that $\mathcal{H} \le \overline{H}$ (see Definition 1.10).

1.5 Γ-convergence

We introduce the definition of De Giorgi's Γ-*convergence*. Let (X, d) be a metric space. We say that a sequence of functions $F_j : X \to [-\infty, +\infty]$ Γ-converge to $F : X \to [-\infty, +\infty]$ (as $j \to +\infty$) if for all $u \in X$ we have:

(i) (*liminf inequality*) for every sequence (u_j) converging to u

$$F(u) \le \liminf_j F_j(u_j); \tag{1.12}$$

(ii) (*limsup inequality*) for all $\eta > 0$ there exists a sequence (u_j) converging to u such that

$$F(u) \ge \limsup_j F_j(u_j) - \eta\,. \tag{1.13}$$

If (i) and (ii) hold we write $F(u) = \Gamma\text{-}\lim_j F_j(u)$ and F is the Γ-limit of F_j. We also introduce the notation

$$F'(u) = \Gamma\text{-}\liminf_j F_j(u) = \inf\Big\{\liminf_j F_j(u_j) : u_j \to u\Big\},$$

$$F''(u) = \Gamma\text{-}\limsup_j F_j(u) = \inf\Big\{\limsup_j F_j(u_j) : u_j \to u\Big\},$$

so that the equality $F' = F''$ is equivalent to the existence of the $\Gamma\text{-}\lim_j F_j(u)$.

We will say that a family (F_ε) Γ-converges to F if for all sequences (ε_j) of positive numbers converging to 0 (i) and (ii) above are satisfied with F_{ε_j} in place of F_j.

Very important properties of Γ-convergence are the compactness and the convergence of the minimum values of a sequence F_j to the minimum value of F, as the following theorems show (see e.g. [26] Theorem 7.2 and Proposition 7.9):

Theorem 1.22. (Compactness) *Let (X, d) be a separable metric space, and for all $j \in \mathbf{N}$ let $F_j : X \to [-\infty, +\infty]$ be a function. Then there is an increasing sequence of integers (j_k) such that the $\Gamma\text{-}\lim_k F_{j_k}(u)$ exists for all $u \in X$.*

Theorem 1.23. (Convergence of minimum problems) *Let F_j Γ-converge to F. Let there exist a compact set $K \subset X$ such that $\inf_X F_j = \inf_K F_j$ for all $j \in \mathbf{N}$. Then*

$$\exists \min_X F = \lim_j \inf_X F_j.$$

Moreover, if (j_k) is an increasing sequence of integers and (u_k) is a converging sequence such that $\lim_k F_{j_k}(u_k) = \lim_j \inf_X F_j$ then its limit is a minimum point for F.

From the definition of Γ-convergence we immediately obtain the following properties.

Remark 1.24 (i) F' and F'' are lower semicontinuous functions on X;

(ii) If G is a continuous function and $F = \Gamma\text{-}\lim_j F_j$ then $F+G = \Gamma\text{-}\lim_j(F_j + G)$;

(iii) If each function F_j is positively homogeneous of degree p, then F' and F'' are positively homogeneous of degree p;

(iv) If each function F_j is convex then F'' is convex but in general F' is not convex;

(v) If $F_j = F$ for all $j \in \mathbf{N}$, then $\Gamma\text{-}\lim_j F_j = \overline{F}$.

(vi) $\Gamma\text{-}\liminf_j \overline{F}_j = \Gamma\text{-}\liminf_j F_j$ and $\Gamma\text{-}\limsup_j \overline{F}_j = \Gamma\text{-}\limsup_j F_j$.

1.6 Increasing set functions

Definition 1.25 *A set function* $\alpha : \mathcal{A}(\Omega) \to [0, +\infty]$ *is called an* increasing set function *if* $\alpha(\emptyset) = 0$ *and* $\alpha(V) \leq \alpha(U)$ *if* $V \subset U$. *An increasing set function is said to be* subadditive *if*

$$\alpha(U \cup V) \leq \alpha(U) + \alpha(V) \tag{1.14}$$

for all $U, V \in \mathcal{A}(\Omega)$*;* α *is said to be* superadditive *if*

$$\alpha(U \cup V) \geq \alpha(U) + \alpha(V) \tag{1.15}$$

for all $U, V \in \mathcal{A}(\Omega)$ *with* $U \cap V = \emptyset$*;* α *is said to be* inner regular *if*

$$\alpha(U) = \sup\{\alpha(V) : \ V \in \mathcal{A}(\Omega), \ V \subset\subset U\} \tag{1.16}$$

for all $U \in \mathcal{A}(\Omega)$.

It will be useful to characterize measures as increasing set functions enjoying special properties which are often satisfied by Γ-limits. The following criterion is due to De Giorgi and Letta (see e.g. [26] Theorem 10.2).

Theorem 1.26 (Measure property criterion) *Let* $\alpha : \mathcal{A}(\Omega) \to [0, +\infty]$ *be an increasing set function. The following statements are equivalent:*
(i) α *is the restriction to* $\mathcal{A}(\Omega)$ *of a Borel measure on* Ω*;*
(ii) α *is subadditive, superadditive and inner regular;*
(iii) *the set function*

$$\beta(E) = \inf\{\alpha(U) : \ U \in \mathcal{A}(\Omega), E \subseteq U\} \tag{1.17}$$

defines a Borel measure on Ω.

The properties of increasing set functions will be used to obtain the compactness of Γ-limits as in the following theorem (see e.g. [26] Theorem 10.3).

Theorem 1.27. (Compactness) *Let* $(F_\varepsilon) : L^p(\Omega; \mathbf{R}^m) \times \mathcal{A}(\Omega) \to [0, +\infty]$ $(\varepsilon > 0)$ *be a family of functionals. Suppose that for every sequence* (ε_k) *of positive real numbers converging to* 0 *and for every* $u \in W^{1,p}(\Omega; \mathbf{R}^m)$

$$\alpha'(U) = \Gamma\text{-}\liminf_k F_{\varepsilon_k}(u, U)$$

$$\alpha''(U) = \Gamma\text{-}\limsup_k F_{\varepsilon_k}(u, U)$$

(the Γ*-limits are performed with respect to the* $L^p(U; \mathbf{R}^m)$ *convergence) define inner regular increasing set functions. Then for every sequence* (ε_j) *of positive real numbers converging to* 0 *there exists a subsequence* (ε_{j_k}) *such that the* Γ*-limit*

$$F(u, U) = \Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(u, U)$$

exists for all $U \in \mathcal{A}(\Omega)$ *and* $u \in W^{1,p}(\Omega; \mathbf{R}^m)$.

Remark 1.28 Note that fixed a sequence (ε_j) converging to 0 by Theorem 1.22 and a diagonal procedure we can extract a subsequence (not relabeled) such that $F_\varepsilon(\cdot, U)$ Γ-converges to a functional $F(\cdot, U)$ for all U in a dense family of open sets $\mathcal{R}$.

1.7 The direct method of Γ-convergence

The direct methods of Γ-convergence for the integral functionals consist in proving general abstract compactness results that assure the existence of Γ-converging sequences, and then in recovering enough information on the structure of the Γ-limits as to obtain a representation in a suitable form.

This method in the version which follows is explained in detail in the book by Braides and Defranceschi [26] (see also Dal Maso [41] and Buttazzo [32]).

1.7.1 *Compactness and measure property of the* Γ*-limit*

Definition 1.29 *Let $F_\varepsilon : L^p(\Omega; \mathbf{R}^m) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be a family of functionals. We say that (F_ε) satisfies the L^p-*fundamental estimate *as $\varepsilon \to 0$ if for all $U, U', V \in \mathcal{A}(\Omega)$ with $U' \subset\subset U$, and for all $\sigma > 0$, there exists $M_\sigma > 0$ and $\varepsilon_\sigma > 0$ such that for all $u, v \in L^p(\Omega; \mathbf{R}^m)$ and $\varepsilon < \varepsilon_\sigma$ there exists a cut-off function $\varphi \in C_c^\infty(U; [0, 1])$, $\varphi = 1$ in U', such that*

$$F_\varepsilon(\varphi u + (1 - \varphi)v, U' \cup V) \leq (1 + \sigma)(F_\varepsilon(u, U) + F_\varepsilon(v, V)) + M_\sigma \int_{(U \cap V) \setminus U'} |u - v|^p \, dx + \sigma \,. \tag{1.18}$$

The definition of fundamental estimate extends to sequences (F_j) and to functionals F with obvious changes.

We show, giving an example, how we usually proceed to prove that some functionals satisfy the fundamental estimate.

Example 1.30 Consider the functional $F : L^p(\Omega; \mathbf{R}^m) \times \mathcal{A}(\Omega) \to [0, +\infty]$ defined by

$$F(u, U) = \begin{cases} \displaystyle\int_U f(x, Du(x)) dx & \text{if } u \in W^{1,1}(\Omega; \mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.19}$$

where $f : \Omega \times \mathbf{M}^{m \times n} \to [0, +\infty)$ is a Borel function, convex in the second variable, such that there exists $C > 0$ such that

$$0 \leq f(x, A) \leq C(1 + |A|^p), \qquad f(x, 2A) \leq C(1 + f(x, A)) \tag{1.20}$$

for all $x \in \Omega$ and $A \in \mathbf{M}^{m \times n}$. In order to prove (1.18) it suffices to consider $u, v \in W^{1,1}(\Omega; \mathbf{R}^m)$, and fix $U, U', V \in \mathcal{A}(\Omega)$ with $U' \subset\subset U$. We set $\delta = \operatorname{dist}(U', \partial U)$, and we fix $N \in \mathbf{N}$, $N > 0$. With fixed $k \in \{1, \dots, N\}$ let φ be a cut-off function between $\{x \in U : N\mathrm{dist}\,(x, U') < \delta(k-1)\}$ and $\{x \in U : N\mathrm{dist}\,(x, U') < \delta k\}$ with $|D\varphi|\delta \le 2N$. We define for all $k = 1, \dots, N$

$$C_k = \{x \in U : \delta(k-1) \le N\mathrm{dist}\,(x, U') < \delta k\}.$$

We then have

$$\begin{aligned}
&F(\varphi u + (1-\varphi)v, U' \cup V) \le F(u, U) + F(v, V) \\
&\qquad + \int_{C_k \cap V} f(x, \varphi Du + (1-\varphi)Dv + (u-v)D\varphi)\, dx \\
&= F(u, U) + F(v, V) \\
&\qquad + \int_{C_k \cap V} f\big(x, 2\big(\frac{1}{2}(\varphi Du + (1-\varphi)Dv) + \frac{1}{2}(u-v)D\varphi\big)\big) dx \\
&\le F(u, U) + F(v, V) \\
&\qquad + C\int_{C_k \cap V} \Big(1 + f\big(x, \frac{1}{2}(\varphi Du + (1-\varphi)Dv) + \frac{1}{2}(u-v)D\varphi\big)\Big) dx \\
&\le F(u, U) + F(v, V) \\
&\qquad + C\int_{C_k \cap V} \big(1 + \frac{1}{2} f(x, \varphi Du + (1-\varphi)Dv) + \frac{1}{2} f(x, (u-v)D\varphi)\big)\, dx \\
&\le F(u, U) + F(v, V) \\
&\qquad + C\int_{C_k \cap V} \big(1 + \frac{1}{2}\varphi f(x, Du) + \frac{1}{2}(1-\varphi) f(x, Dv) + \frac{1}{2} f(x, (u-v)D\varphi)\big) dx \\
&\le F(u, U) + F(v, V) \\
&\qquad + C\int_{C_k \cap V} \big(1 + f(x, Du) + f(x, Dv) + C(1 + |D\varphi|^p |u-v|^p)\big)\, dx.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{k=1}^{N} \int_{C_k \cap V} (1 + C + f(x, Du) + f(x, Dv))\, dx \\
&\qquad \le \int_{(U \cap V) \setminus U'} (1 + C + f(x, Du) + f(x, Dv))\, dx \\
&\qquad \le (1 + C)|U \cap V| + F(u, U) + F(v, V)
\end{aligned}$$

we can choose k such that

$$\int_{C_k \cap V} (1 + C + f(x, Du) + f(x, Dv))\, dx \le \frac{1}{N}\Big((1+C)|U \cap V| + F(u, U) + F(v, V)\Big).$$

We then have

$$F(\varphi u + (1-\varphi)v, U' \cup V) \le (1 + \frac{C}{N})(F(u, U) + F(v, V))$$

$$
\begin{aligned}
&\quad + \frac{C}{N}(1+C)|U\cap V| + C^2\Big(\frac{2N}{\delta}\Big)^p \int_{C_k\cap V} |u-v|^p\,dx \\
&\le (1+\frac{C}{N})(F(u,U)+F(v,V)) \\
&\quad + \frac{C}{N}(1+C)|U\cap V| + C^2\Big(\frac{2N}{\delta}\Big)^p \int_{(U\cap V)\setminus U'} |u-v|^p\,dx.
\end{aligned}
$$

If we choose

$$N = N_\sigma = \Big[\max\Big\{\frac{C}{\sigma}, \frac{C}{\sigma}(1+C)|U\cap V|\Big\}\Big] + 1,$$

($[t]$ denotes the integer part of t) and

$$M_\sigma = C^2\Big(\frac{2N}{\delta}\Big)^p,$$

then (1.18) is satisfied.

From the fundamental estimate we can derive some inequalities for the Γ-limits (see e.g. [26] Proposition 11.5).

Proposition 1.31 *Let (F_ε) be a family of functionals defined on $L^p(\Omega;\mathbf{R}^m)\times\mathcal{A}(\Omega)$ with values in $[0,+\infty]$ satisfying the L^p-fundamental estimate as $\varepsilon\to 0$, and let (ε_j) be a sequence of positive real numbers converging to* 0. *If for every $u\in L^p(\Omega;\mathbf{R}^m)$ and $U\in\mathcal{A}(\Omega)$ we denote*

$$F'(u,U) = \Gamma\text{-}\liminf_j F_{\varepsilon_j}(u,U) \tag{1.21}$$

$$F''(u,U) = \Gamma\text{-}\limsup_j F_{\varepsilon_j}(u,U), \tag{1.22}$$

then we have

$$F'(u,U'\cup V) \le F'(u,U) + F''(u,V) \tag{1.23}$$

$$F''(u,U'\cup V) \le F''(u,U) + F''(u,V) \tag{1.24}$$

for all $u\in L^p(\Omega;\mathbf{R}^m)$ and $U,U',V\in\mathcal{A}(\Omega)$ with $U'\subset\subset U$.

From the previous proposition we obtain some inner regularity results, provided that a growth estimate is satisfied (see e.g. [26] Proposition 11.6).

Proposition 1.32 *Let (F_ε) be as in Proposition* 1.31, *and let F' and F'' be defined by* (1.21) *and* (1.22), *respectively. Let $q\ge 1$, and let $u\in W^{1,q}(\Omega;\mathbf{R}^m)\cap L^p(\Omega;\mathbf{R}^m)$; if $F'(u,\cdot)$ and $F''(u,\cdot)$ are increasing set functions and*

$$F''(u,U) \le c\int_U (1+|Du|^q)\,dx \tag{1.25}$$

for all $U\in\mathcal{A}(\Omega)$, then $F'(u,\cdot)$ and $F''(u,\cdot)$ are inner regular increasing set functions. Moreover, $F''(u,\cdot)$ is subadditive.

Hence, if we have a family of functionals which satisfy the L^p-fundamental estimate as $\varepsilon \to 0$ (1.18) and growth condition (1.25), by Proposition 1.32 we can apply Theorem 1.27 and we get existence of the Γ-limit $F(u, U)$ for every $U \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega; \mathbf{R}^m)$. Moreover by Proposition 1.32 we also have the subadditivity and inner regularity of the set function $F(u, \cdot)$, while the superadditivity is obvious. Hence, by the Measure property criterion (Theorem 1.26) we get that $F(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$.

1.7.2 Γ-*limits and boundary values*

If we assume that estimates of the form (1.25) are satisfied uniformly by L^p-fundamental estimate as $\varepsilon \to 0$, and we consider some boundary conditions then we get another property of the Γ-limits (see e.g. [26] Proposition 11.7).

Proposition 1.33 *Let (F_ε) be a family of functionals defined on $W^{1,p}(\Omega; \mathbf{R}^m) \times \mathcal{A}(\Omega)$ with values in $[0, +\infty]$ satisfying the L^p-fundamental estimate as $\varepsilon \to 0$ (we regard these functionals as extended to $+\infty$ on $L^p(\Omega; \mathbf{R}^m) \setminus W^{1,p}(\Omega; \mathbf{R}^m)$), and let (ε_j) be a sequence of positive real numbers converging to 0. Let*

$$F_{\varepsilon_j}(u, U) \leq c \int_U (1 + |Du|^p)\, dx \tag{1.26}$$

hold for all $U \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega; \mathbf{R}^m)$. If we take $\phi \in W^{1,p}(\Omega; \mathbf{R}^m)$, and we define $G^\phi_{\varepsilon_j} : L^p(\Omega; \mathbf{R}^m) \to [0, +\infty]$

$$G^\phi_{\varepsilon_j}(u) = \begin{cases} F_{\varepsilon_j}(u, \Omega) & \text{if } u - \phi \in W^{1,p}_0(\Omega; \mathbf{R}^m) \\ \\ +\infty & \text{otherwise,} \end{cases}$$

then we have

$$F'(u, \Omega) = \Gamma\text{-}\liminf_j F_{\varepsilon_j}(u, \Omega) = \Gamma\text{-}\liminf_j G^\phi_{\varepsilon_j}(u) \tag{1.27}$$

$$F''(u, \Omega) = \Gamma\text{-}\limsup_j F_{\varepsilon_j}(u, \Omega) = \Gamma\text{-}\limsup_j G^\phi_{\varepsilon_j}(u) \tag{1.28}$$

for all $u \in W^{1,p}(\Omega; \mathbf{R}^m)$ such that $u - \phi \in W^{1,p}_0(\Omega; \mathbf{R}^m)$.

Remark 1.34 The previous proposition is equivalent to say that if (1.26) holds, together with the L^p-fundamental estimate as $\varepsilon \to 0$, the minimizing sequences for the Γ-limits can be taken with the same boundary values as their limit.

1.7.3 *Integral representation on Sobolev spaces*

The last step of the direct methods is the integral representation of the Γ-limit; for this purpose it is important to consider the Γ-limit as a functional defined both on functions and sets and single out the properties which assure that it can be written in an integral form. Hence we introduce the "localization method" which consists in considering functionals of the form

$$F(u,U)=\int_U f(x,Du(x))\,dx,$$

with $u\in W^{1,p}(\Omega;\mathbf{R}^m)$ and $U\in\mathcal{A}(\Omega)$, and we give the Integral Representation Theorem also with its proof since it will be useful in the next chapters.

Theorem 1.35 *Let Ω be a bounded open subset of $\mathbf{R}^n$, and let $1\le p<\infty$. Let $F:W^{1,p}(\Omega;\mathbf{R}^m)\times\mathcal{A}(\Omega)\to[0,+\infty)$ be a functional satisfying the following conditions:*

(i) (locality) *F is local, i.e. $F(u,U)=F(v,U)$ if $u=v$ a.e. on $U\in\mathcal{A}(\Omega)$;*

(ii) (measure property) *for all $u\in W^{1,p}(\Omega;\mathbf{R}^m)$ the set function $F(u,\cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$;*

(iii) (growth condition) *there exists $c>0$ and $a\in L^1(\Omega)$ such that*

$$F(u,U)\le c\int_U (a(x)+|Du|^p)\,dx$$

for all $u\in W^{1,p}(\Omega;\mathbf{R}^m)$ and $U\in\mathcal{A}(\Omega)$;

(iv) (translation invariance in u) *$F(u+z,U)=F(u,U)$ for all $z\in\mathbf{R}^m$, $u\in W^{1,p}(\Omega;\mathbf{R}^m)$ and $U\in\mathcal{A}(\Omega)$;*

(v) (lower semicontinuity) *for all $U\in\mathcal{A}(\Omega)$ $F(\cdot,U)$ is sequentially lower semicontinuous with respect to the weak convergence in $W^{1,p}(\Omega;\mathbf{R}^m)$.*

Then there exists a Carathéodory function $f:\Omega\times\mathbf{M}^{m\times n}\to[0,+\infty)$ satisfying the growth condition

$$0\le f(x,A)\le c(a(x)+|A|^p) \tag{1.29}$$

for all $x\in\Omega$ and $A\in\mathbf{M}^{m\times n}$, such that

$$F(u,U)=\int_U f(x,Du(x))\,dx \tag{1.30}$$

for all $u\in W^{1,p}(\Omega;\mathbf{R}^m)$ and $U\in\mathcal{A}(\Omega)$.

Proof. *Step 1: definition of f.*

Fix $A\in\mathbf{M}^{m\times n}$; by (ii) $F(Ax,\cdot)$ can be extended to a Borel measure on Ω which, by (iii), is absolutely continuous with respect to the Lebesgue measure. Hence there exists a density function $g_A\in L^1(\Omega)$ such that

$$F(Ax,U)=\int_U g_A(x)\,dx$$

for all $U\in\mathcal{A}(\Omega)$. We set

$$f(x,A)=g_A(x)$$

for all $x \in \Omega$ and $A \in \mathbf{M}^{m\times n}$. Note that by condition (iii), with fixed $A \in \mathbf{M}^{m\times n}$

$$0 \le f(x, A) \le c(a(x) + |A|^p)$$

for a.e. $x \in \Omega$.

Step 2: integral representation on piecewise affine functions.

Let $U \in \mathcal{A}(\Omega)$ and let $u \in W^{1,p}(\Omega; \mathbf{R}^m)$ be piecewise affine on U, i.e. we can write

$$u_{|U} = \sum_{j=1}^{N} \chi_{U_j}(A_j x + z_j),$$

where the sets U_j are disjoint open sets with $|U \setminus \bigcup_{j=1}^{N} U_j| = 0$, $A_j \in \mathbf{M}^{m\times n}$, $z_j \in \mathbf{R}^m$ for $j = 1, \dots, N$. By (i), (ii), (iv) and Step 1

$$\begin{aligned} F(u, U) &= \sum_{j=1}^{N} F(u, U_j) = \sum_{j=1}^{N} F(A_j x + z_j, U_j) \\ &= \sum_{j=1}^{N} F(A_j x, U_j) = \sum_{j=1}^{N} \int_{U_j} f(x, A_j)\, dx \\ &= \sum_{j=1}^{N} \int_{U_j} f(x, Du)\, dx = \int_U f(x, Du)\, dx; \end{aligned}$$

that is, the representation (1.30).

Step 3: rank-1-convexity of f.

We want to show that, with fixed $A, B \in \mathbf{M}^{m\times n}$ such that $\operatorname{rank}(B - A) = 1$, and $t \in (0, 1)$

$$f(y, tB + (1-t)A) \le t f(y, B) + (1-t) f(y, A)$$

for all $y \in \Omega$. By definition we can take

$$f(y, A) = \limsup_{\rho\to 0+} \frac{F(Ax, B(y, \rho))}{|B(y, \rho)|}$$

for all $y \in \Omega$ and $A \in \mathbf{M}^{m\times n}$. Hence it will suffice to show that if $B(y, \rho) \subset \Omega$ then

$$F((tB + (1-t)A)x, B(y, \rho)) \le tF(Bx, B(y, \rho)) + (1-t)F(Ax, B(y, \rho)). \quad (1.31)$$

Let $a \in \mathbf{R}^m$, $b \in \mathbf{R}^n$ be vectors such that $B - A = a \otimes b$. Consider the function $v \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^n; \mathbf{R}^m)$ defined by

$$v(x) = \begin{cases} Ax + \langle b, x\rangle a - (1-t)ja & \text{if } j \in \mathbf{Z},\ j \le \langle b, x\rangle < j + t \\ \\ Ax + (1+j)ta & \text{if } j \in \mathbf{Z},\ j + t \le \langle b, x\rangle < j + 1. \end{cases}$$

We also define the sets

$$E_A = \{x \in \mathbf{R}^n : \exists j \in \mathbf{Z} :\ j + t \le \langle b, x\rangle < j+1\},$$
$$E_B = \{x \in \mathbf{R}^n : \exists j \in \mathbf{Z} :\ j \le \langle b, x\rangle < j+t\}.$$

If we set $u_j(x) = \frac{1}{j}v(jx)$, we have

$$\begin{aligned} u_j &\rightharpoonup^* (tB + (1-t)A)x && \text{weakly}^* \text{ in } W^{1,\infty}(\Omega;\mathbf{R}^m)\\ \chi_{\frac{1}{j}E_A} &\rightharpoonup^* 1-t && \text{weakly}^* \text{ in } L^\infty(\Omega)\\ \chi_{\frac{1}{j}E_B} &\rightharpoonup^* t && \text{weakly}^* \text{ in } L^\infty(\Omega); \end{aligned}$$

moreover, $Du_j = A$ on $\frac{1}{j}E_A$ and $Du_j = B$ on $\frac{1}{j}E_B$. Hence by (v) and (iv) we obtain

$$\begin{aligned} &F((tB + (1-t)A)x, B(y,\rho))\\ &\le \liminf_j F(u_j, B(y,\rho))\\ &= \liminf_j \Big(F(Ax, \frac{1}{j}E_A \cap B(y,\rho)) + F(Bx, \frac{1}{j}E_B \cap B(y,\rho))\Big)\\ &= \liminf_j \Big(\int_{B(y,\rho)} \chi_{\frac{1}{j}E_A} g_A(x)\,dx + \int_{B(y,\rho)} \chi_{\frac{1}{j}E_B} g_B(x)\,dx\Big)\\ &= t\int_{B(y,\rho)} g_B(x)\,dx + (1-t)\int_{B(y,\rho)} g_A(x)\,dx\\ &= tF(Bx, B(y,\rho)) + (1-t)F(Ax, B(y,\rho)), \end{aligned}$$

proving (1.31), and finally the rank-1-convexity of $f(y,\cdot)$, taking the $\limsup$ as $\rho \to 0+$. By Remark 1.16(i) we have that $f(y,\cdot)$ is locally Lipschitz, and hence f is a Carathéodory function.

Step 4: an inequality by continuity.

As a consequence of Step 3 and (1.29) the functional $u \mapsto \int_U f(x, Du)\,dx$ is continuous with respect to the strong convergence of $W^{1,p}(U;\mathbf{R}^m)$. If $U \subset\subset \Omega$ we can find a sequence $u_j \in W^{1,p}(\Omega;\mathbf{R}^m)$ converging strongly to u in $W^{1,p}(\Omega;\mathbf{R}^m)$, and such that their restrictions to U are piecewise affine. Then

$$\begin{aligned} F(u,U) &\le \liminf_j F(u_j, U)\\ &= \lim_j \int_U f(x, Du_j)\,dx = \int_U f(x, Du)\,dx \end{aligned}$$

by (v) and Step 3.

Step 5: equality by translation.

Let $u \in W^{1,p}(\Omega;\mathbf{R}^m)$; we consider the functional $G : W^{1,p}(\Omega;\mathbf{R}^m) \times \mathcal{A}(\Omega) \to [0,+\infty)$ defined by

$$G(v,U) = F(u+v,U)$$

that satisfies all hypotheses (i)–(v) of Theorem 1.35 (with different c and a in condition (iii)). Hence, by Steps 1–4 above, there exists a Carathéodory function $\psi : \Omega \times \mathbf{M}^{m\times n} \to [0,+\infty)$ such that

$$G(v,U) \le \int_U \psi(x,Dv)\,dx$$

for all $v \in W^{1,p}(\Omega;\mathbf{R}^m)$ and for all $U \subset\subset \Omega$ open sets, with equality for v piecewise affine on U. Let us take an open set $U \subset\subset \Omega$, and u_j piecewise affine on U converging strongly to u in $W^{1,p}(\Omega;\mathbf{R}^m)$ as in Step 4. We have

$$\begin{aligned}\int_U \psi(x,0)\,dx &= G(0,U) = F(u,U)\\ &\le \int_U f(x,Du)\,dx = \lim_j \int_U f(x,Du_j)\,dx\\ &= \lim_j F(u_j,U) = \lim_j G(u_j-u,U)\\ &\le \lim_j \int_U \psi(x,Du_j - Du)\,dx = \int_U \psi(x,0)\,dx;\end{aligned}$$

hence all inequalities are in fact equalities, and in particular

$$F(u,U) = \int_U f(x,Du)\,dx$$

for all $U \subset\subset \Omega$.

Step 6: integral representation.

By (ii) the integral representation obtained in Step 5 holds for all open subsets U of Ω. □

The following result characterizes a class of integral functionals with integrand independent of the space variable (see e.g. [26] Proposition 9.2).

Corollary 1.36 *Let $F : W^{1,p}(\Omega;\mathbf{R}^m) \times \mathcal{A}(\Omega) \to [0,+\infty)$. There exists a quasiconvex $f : \mathbf{M}^{m\times n} \to [0,+\infty)$ satisfying*

$$0 \le f(A) \le c(1+|A|^p) \qquad \forall A \in \mathbf{M}^{m\times n} \tag{1.32}$$

such that the functional F can be represented by

$$F(u,U) = \int_U f(Du)\,dx \tag{1.33}$$

if and only if conditions (i)–(v) *of Theorem* 1.35 *hold and in addition* (vi)(translation invariance in x)

$$F(Ax, B(y,\rho)) = F(Ax, B(z,\rho))$$

for all $A \in \mathbf{M}^{m\times n}$*,* $y, z \in \Omega$*, and* $\rho > 0$ *such that* $B(y,\rho) \cup B(z,\rho) \subset \Omega$*.*

1.8 Application of the direct methods to integral functionals with standard growth conditions

Let Ω be a bounded subset of $\mathbf{R}^n$, let $p \geq 1$ and let $F_\varepsilon : L^p(\Omega;\mathbf{R}^m) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be a family of functionals of the form

$$F_\varepsilon(u,U) = \begin{cases} \displaystyle\int_U f_\varepsilon(x, Du)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.34}$$

where $f_\varepsilon : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty)$ is a Borel function satisfying the *standard growth condition of order* p

$$\alpha|A|^p \leq f_\varepsilon(x,A) \leq \beta(1+|A|^p) \tag{1.35}$$

for all $x \in \Omega$ and $A \in \mathbf{M}^{m\times n}$.

Proposition 1.37 *Let* (F_ε) *be a family of functionals defined in* (1.34) *which satisfies the growth condition* (1.35)*. Then* (F_ε) *satisfies the* L^p-fundamental estimate *as* $\varepsilon \to 0$*.*

(See e.g. [26] Proposition 12.2). Hence by Section 1.7.1 we obtain the following proposition.

Proposition 1.38 *Let* (F_ε) *be a family of functionals defined in* (1.34)*. Then for every sequence* (ε_j) *of positive real numbers converging to* 0 *there exists a further subsequence* (ε_{j_k}) *such that the* Γ*-limit*

$$F(u,U) = \Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(u,U)$$

exists for all $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ *and* $U \in \mathcal{A}(\Omega)$*, and* $F(u,\cdot)$ *is the restriction of a Borel measure to* $\mathcal{A}(\Omega)$*.*

Remark 1.39 The same conclusions of Proposition 1.38 follow if we suppose in the place of (1.35) that

$$g(x,A) \leq f_\varepsilon(x,A) \leq c(1+g(x,A)),$$

with $g(x,\cdot)$ convex, $g(x,2A) \leq c(1+g(x,A))$, and $g(x,A) \leq c(1+|A|^p)$ for all $x \in \Omega$ and $A \in \mathbf{M}^{m\times n}$, taking into account Example 1.19.

By Proposition 1.38 and the Integral Representation Theorem 1.35 we get

Theorem 1.40 *Let (f_ε) be a family of Borel functions with $f_\varepsilon : \Omega \times \mathbf{M}^{m\times n} \to [0,+\infty)$ satisfying the estimate*

$$\alpha|A|^p \le f_\varepsilon(x,A) \le \beta(1+|A|^p)$$

for all $x \in \Omega$ and $A \in \mathbf{M}^{m\times n}$, and let

$$F_\varepsilon(u,U) = \int_U f_\varepsilon(x,Du)\,dx \tag{1.36}$$

if $u \in W^{1,p}(\Omega;\mathbf{R}^m)$. Then, for every sequence (ε_j) of positive real numbers converging to 0 there exists a subsequence (ε_{j_k}) and a Carathéodory function $\varphi : \Omega \times \mathbf{M}^{m\times n} \to [0,+\infty)$ satisfying the same growth estimate as f_ε such that, if we define

$$F(u,U) = \int_U \varphi(x,Du)\,dx \tag{1.37}$$

for $u \in W^{1,p}(\Omega;\mathbf{R}^m)$, we have

$$F(u,U) = \Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(u,U) \tag{1.38}$$

for all $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ and $U \in \mathcal{A}(\Omega)$.

In the case of functionals with integrands independent of the space variable Γ-convergence reduces to a pointwise convergence (see e.g. [26] Proposition 12.8).

Proposition 1.41 *Let $p > 1$, and let (f_ε) be a family of continuous functions with $f_\varepsilon : \mathbf{M}^{m\times n} \to [0,+\infty)$ satisfying the estimate*

$$\alpha|A|^p \le f_\varepsilon(A) \le \beta(1+|A|^p)$$

for all $A \in \mathbf{M}^{m\times n}$. Let, for every bounded open set U of $\mathbf{R}^n$,

$$F_\varepsilon(u,U) = \int_U f_\varepsilon(Du)\,dx \tag{1.39}$$

if $u \in W^{1,p}(U;\mathbf{R}^m)$, and let (ε_j) be a sequence of positive real numbers converging to 0. We have that $F_{\varepsilon_j}(u,U)$ Γ-converges to $F(u,U)$ for all U bounded open sets of $\mathbf{R}^n$ and $u \in W^{1,p}(U;\mathbf{R}^m)$ if and only if $Qf_{\varepsilon_j} \to f$ pointwise and

$$F(u,U) = \int_U f(Du)\,dx \tag{1.40}$$

for all U bounded open sets of $\mathbf{R}^n$ and $u \in W^{1,p}(U;\mathbf{R}^m)$.

1.8.1 *Higher integrability of gradients*

The following lemma (see [49]) allows to pass from bounded sequences to sequences with equi-integrable p-th power of the gradient. This result can be sometimes helpful since it allows us to assume that the optimal sequences for the Γ-limit can be taken with equi-integrable p-th power gradient.

Lemma 1.42 *For every bounded sequence* (u_j) *in* $W^{1,p}(\Omega;\mathbf{R}^m)$ *there exists a subsequence (not relabelled) and a sequence* (v_j) *in* $W^{1,p}(\Omega;\mathbf{R}^m)$ *such that*

$$\lim_j |\{u_j \neq v_j\} \cup \{Du_j \neq Dv_j\}| = 0$$

and $(|Dv_j|^p)$ *is equi-integrable.*

Remark 1.43 Let $f_j : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty)$ be a Borel function satisfying the standard growth condition of order p (1.35) and let (u_j) be a bounded sequence in $W^{1,p}(\Omega;\mathbf{R}^m)$ with $u_j \to u$ in $L^p(\Omega;\mathbf{R}^m)$. Then there exist a subsequence, still denoted by (u_j), and a sequence (v_j) in $W^{1,p}(\Omega;\mathbf{R}^m)$ such that $v_j \to u$ in $L^p(\Omega;\mathbf{R}^m)$, $(|Dv_j|^p)$ is equi-integrable and

$$\limsup_j \int_\Omega f_j(x, Dv_j)\,dx \le \limsup_j \int_\Omega f_j(x, Du_j)\,dx.$$

To check this, choose (v_j) as in Lemma 1.42, so that $v_j \to u$ and

$$\begin{aligned}\limsup_j \int_\Omega f_j(x, Du_j)\,dx &\ge \limsup_j \int_{\{u_j=v_j\}\cap\{Du_j=Dv_j\}} f_j(x, Du_j)\,dx \\ &= \limsup_j \int_\Omega f_j(x, Dv_j)\,dx\end{aligned}$$

the last equality following from the equi-integrability of $(|Dv_j|^p)$, and the growth conditions on f_j.

1.9 Periodic homogenization

In this section we use the direct methods of Γ-convergence to obtain a homogenization theorem.

1.9.1 *Coercive homogenization*

Let us consider

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, Du(x)\Big)\,dx,$$

where $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ and the function $f : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty)$ is a Borel function satisfying the following conditions:

(i) (*periodicity*) f is 1-periodic in the first variable;

(ii) (*standard growth condition of order* p) there exist $0 < \alpha \le \beta$ such that

$$\alpha|A|^p \le f(x, A) \le \beta(1 + |A|^p)$$

for all $x \in \mathbf{R}^n$ and $A \in \mathbf{M}^{m\times n}$.

By Theorem 1.40 we have also for this class of integral functionals a compactness and integral representation result; we still denote φ the homogenized integrand of the Γ-limit, which could depend on the sequence (ε_{j_k}).

In order to show that the whole family (F_ε) Γ-converges, we prove the following Propositions.

Proposition 1.44 *The function φ can be chosen independent of the first variable.*

Proof. By Corollary 1.36 it is sufficient to prove that if $A \in \mathbf{M}^{m\times n}$, $y, z \in \mathbf{R}^n$ and $\rho > 0$ then

$$\Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(Ax, B_\rho(y)) = \Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(Ax, B_\rho(z)).$$

By Proposition 1.33 there exists a sequence $(u_k) \subset W_0^{1,p}(B_\rho(y); \mathbf{R}^m)$ such that $u_k \to 0$ in $L^p(B_\rho(y); \mathbf{R}^m)$ and

$$\lim_k F_{\varepsilon_{j_k}}(Ax + u_k, B_\rho(y)) = \Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(Ax, B_\rho(y)).$$

We extend u_k to $\mathbf{R}^n$ by 0 outside $B_\rho(y)$. Let $r > 1$, let $\tau_k \in \mathbf{R}^n$ be given by

$$(\tau_k)_i = \varepsilon_{j_k}\Big[\frac{z_i - y_i}{\varepsilon_{j_k}}\Big]$$

and let $v_k(x) = u_k(x - \tau_k)$. Note that $\tau_k \to z - y$ and τ_k is a period for $x \mapsto f(x/\varepsilon_{j_k}, A)$ for all A, so that

$$F_{\varepsilon_{j_k}}(Ax + v_k, \tau_k + B_\rho(y)) = F_{\varepsilon_{j_k}}(Ax + u_k, B_\rho(y)).$$

Moreover, $v_k = 0$ outside $\tau_k + B_\rho(y)$. We have $v_k \to 0$ in $L^p(B_{r\rho}(z); \mathbf{R}^m)$; hence,

$$\begin{aligned}
&\Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(Ax, B_\rho(z))\\
&\le \Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(Ax, B_{r\rho}(z))\\
&\le \liminf_k F_{\varepsilon_{j_k}}(Ax + v_k, B_{r\rho}(z))\\
&\le \liminf_k F_{\varepsilon_{j_k}}(Ax + u_k, B_\rho(y)) + |B_{r\rho} \setminus B_\rho|\beta(1 + |A|^p)\\
&= \Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(Ax, B_\rho(y)) + |B_{r\rho} \setminus B_\rho|\beta(1 + |A|^p).
\end{aligned}$$

Letting $r \to 1$ we obtain the inequality

$$\Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(Ax, B_\rho(z)) \le \Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(Ax, B_\rho(y));$$

the opposite inequality is obtained by a symmetry argument. □

The following proposition is crucial in proving that the Γ-limit does not depend on the subsequence (ε_j).

Proposition 1.45 *Let $f : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0, +\infty)$ be a Borel function satisfying the periodicity condition and such that $\sup\{f(x, A) :\ x \in \mathbf{R}^n\}$ is finite for all $A \in \mathbf{M}^{m\times n}$; then the limit*

$$\lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{\int_{(0,t)^n} f(x, A + Du(x))\, dx :\ u \in W_0^{1,p}((0, t)^n; \mathbf{R}^m)\Big\} \tag{1.41}$$

exists for all $A \in \mathbf{M}^{m\times n}$.

(See e.g. [26] Proposition 14.4).

By applying the previous results we have the following Homogenization Theorem (see e.g. [26] Theorems 14.5 and 14.7) which shows that φ can be expressed by an asymptotic formula which does not depend on (ε_{j_k}).

Theorem 1.46. (Homogenization Theorem) *Let $f : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0, +\infty)$ be a Borel function satisfying the periodicity assumption and the standard growth condition of order $p \geq 1$. If Ω is a bounded open set of $\mathbf{R}^n$ and we set for all $\varepsilon > 0$*

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, Du(x)\Big)\, dx \tag{1.42}$$

for all $u \in W^{1,p}(\Omega; \mathbf{R}^m)$, then we have

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \int_\Omega f_{\rm hom}(Du(x))\, dx,$$

for all $u \in W^{1,p}(\Omega; \mathbf{R}^m)$, where $f_{\rm hom} : \mathbf{M}^{m\times n} \to [0, +\infty)$ is a quasiconvex function satisfying the asymptotic homogenization formula

$$f_{\rm hom}(A) = \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{\int_{(0,t)^n} f(x, A + Du(x))\, dx :\ u \in W_0^{1,p}((0, t)^n; \mathbf{R}^m)\Big\} \tag{1.43}$$

for all $A \in \mathbf{M}^{m\times n}$.

If in addition $f(x, \cdot)$ is convex for all $x \in \mathbf{R}^n$ then $f_{\rm hom}$ is given by the cell-problem formula

$$f_{\rm hom}(A) = \inf\Big\{\int_{(0,1)^n} f(y, A + Du(y))\, dy :\ u \in W_\#^{1,p}((0, 1)^n; \mathbf{R}^m)\Big\} \tag{1.44}$$

for all $A \in \mathbf{M}^{m\times n}$, where $W_\#^{1,p}((0, 1)^n; \mathbf{R}^m) = \{u \in W_{\rm loc}^{1,p}(\mathbf{R}^n; \mathbf{R}^m) :\ u$ 1-periodic$\}$.

For the proof of the last statement see [61].

1.9.2 *Non-coercive convex homogenization*

Theorem 1.47. (Homogenization for Non-coercive Functionals) *Let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary. Let g be a convex function satisfying the periodicity assumption and such that there exists $C > 0$ such that*

$$0 \le g(x,A) \le C(1+|A|^p), \qquad g(x,2A) \le C(1+g(x,A)) .$$

Then we have

$$\Gamma\text{-}\lim_{\varepsilon\to 0} G_\varepsilon(u) = \int_\Omega \psi(Du(x))\,dx,$$

for all $u \in W^{1,1}(\Omega;\mathbf{R}^m)$*, where* $\psi : \mathbf{M}^{m\times n} \to [0,+\infty)$ *is the convex function given by the cell-problem formula* (1.44)*. Moreover, if* ψ *satisfies*

$$\lim_{|A|\to+\infty} \frac{\psi(A)}{|A|} = +\infty, \tag{1.45}$$

then the Γ*-limit exists on the whole* $L^p(\Omega;\mathbf{R}^m)$*, and it takes the value* $+\infty$ *on* $L^p(\Omega;\mathbf{R}^m) \setminus W^{1,1}(\Omega;\mathbf{R}^m)$*.*

PROOF. By Example 1.30 the conclusions of Theorem 1.46 still hold. It remains to extend the representation of the Γ-limit G outside $W^{1,p}(\Omega;\mathbf{R}^m)$.

In the course of the proof (ρ_j) denotes a sequence of mollifiers with spt $\rho_j \subset B(0,1/j)$, and $\rho_j * v$ is the convolution between ρ_j and v. Note that since Ω has Lipschitz boundary, by the standard reflection technique near $\partial\Omega$ (see for instance Adams [1] Theorems 4.26, 4.28 and Section 4.29 for details) all functions can be extended to some $\Omega' \supset\supset \Omega$, so that we can suppose that each $\rho_j * v$ is defined on the whole Ω'. Such an extension will not influence the validity of our arguments.

For every $u \in L^p(\Omega;\mathbf{R}^m)$, let $G_\varepsilon(u,U)$ denote the localization of the functionals G_ε to the set $U \in \mathcal{A}(\Omega)$.

Step 1: $G'(u,U) \ge \int_U \psi(Du)\,dx$ *for all* $U \in \mathcal{A}(\Omega)$ *and* $u \in W^{1,1}(\Omega;\mathbf{R}^m)$*.*

Note that the Γ-limit G exists for all $u \in L^p(\Omega;\mathbf{R}^m)$ and for all R in a dense family of open sets $\mathcal{R}$ (see Remark 1.28), hence $G(\cdot,R)$ is convex for all $R \in \mathcal{R}$. $G'(u,\cdot)$ is an increasing set function for all u, and from the definition of Γ-liminf, it can be immediately checked by a translation argument that for all $U,V \in \mathcal{A}(\Omega)$, $v \in L^p(\Omega;\mathbf{R}^m)$, and $y \in \mathbf{R}^n$, if $V \subset\subset y+U$ then $G'(v^y,V) \le G'(v,U)$, where $v^y(x) = v(x-y)$.

For all $U,U' \in \mathcal{A}(\Omega)$ such that $U' \subset\subset U$ there exists $R \in \mathcal{R}$ such that $U' \subset\subset R \subset\subset U$. We can choose j large enough as to have $R \subset\subset y+U$ for all $y \in B(0,1/j)$, hence using Jensen's inequality and the properties of G' recalled above, we get

$$\begin{aligned} G(\rho_j * u, R) &\le \int_{B(0,1/j)} \rho_j(y)\,G(u^y,R)\,dy \\ &\le \int_{B(0,1/j)} \rho_j(y)\,G'(u,U)\,dy = G'(u,U) \end{aligned}$$

and

$$G(\rho_j * u, U') \le G'(u,U).$$

On the other hand, by the representation of G on $W^{1,p}(\Omega;\mathbf{R}^m)$, we have

$$G(\rho_j * u, U') = \int_{U'} \psi(D(\rho_j * u))\,dx.$$

Since the functional $v \mapsto \int_{U'} \psi(Dv)\,dx$ is lower semicontinuous with respect to the L^p-convergence, and $\rho_j * u \to u$ in $L^p(U', \mathbf{R}^m)$, we get by the previous formulae

$$\int_{U'} \psi(Du)\,dx \leq \liminf_j \int_{U'} \psi(D(\rho_j * u))\,dx \leq G'(u, U).$$

By the arbitrariness of $U' \subset\subset U$ the step is concluded.

Step 2: if ψ satisfies (1.45) then $G'(u,\Omega) = +\infty$ for all $u \in L^p(\Omega;\mathbf{R}^m) \setminus W^{1,1}(\Omega;\mathbf{R}^m)$.

We proceed exactly as in the previous step, noting that (1.45) implies that

$$\liminf_j \int_{U'} \psi(D(\rho_j * u))\,dx = +\infty.$$

Step 3: $G''(u,U) \leq \int_U \psi(Du)\,dx$ for all $U \in \mathcal{A}(\Omega)$ and $u \in W^{1,1}(\Omega;\mathbf{R}^m)$.

We have, using the lower semicontinuity of G'' and Jensen's inequality,

$$\begin{aligned}
G''(u,U) &\leq \liminf_j G''(\rho_j * u, U) = \liminf_j G(\rho_j * u, U)\\
&= \liminf_j \int_U \psi(D(\rho_j * u))\,dx\\
&\leq \liminf_j \int_U \int_{B(0,1/j)} \rho_j(y)\psi(D(u(x-y)))\,dy\,dx\\
&= \liminf_j \int_{B(0,1/j)} \rho_j(y) \int_{U+y} \psi(Du)\,dx\,dy\\
&\leq \liminf_j \int_{B(0,1/j)} \rho_j(y) \int_{U'} \psi(Du)\,dx\,dy = \int_{U'} \psi(Du)\,dx,
\end{aligned}$$

for all $U' \supset\supset U$. By the arbitrariness of U' the proof is achieved. □

1.10 Extensions of the direct methods

Convexity is crucial in establishing Theorem 1.47. In some cases though it is possible to extend that result to non-convex integrands.

From Theorem 1.47 follows that we can define

$$W^{1,\psi}(\Omega;\mathbf{R}^m) = \Big\{ u \in W^{1,1}(\Omega;\mathbf{R}^m) : \int_\Omega \psi(Du)\,dx < +\infty \Big\}$$

and

$$W^{1,p}(\Omega;\mathbf{R}^m) \subset W^{1,\psi}(\Omega;\mathbf{R}^m) \subset W^{1,1}(\Omega;\mathbf{R}^m)\,.$$

Let $F_\varepsilon : L^p(\Omega;\mathbf{R}^m) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be defined as follow

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, Du(x)\Big)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.46}$$

where $f : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty)$ is a Borel function which satisfies the periodicity assumption and the following conditions:

$$g(x,A) \le f(x,A) \le c(1 + g(x,A)), \tag{1.47}$$

with $g : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty)$ Borel function, 1-periodic in the first variable and convex in the second one, such that

$$0 \le g(x,A) \le c(1 + |A|^p) \qquad g(x,2A) \le c(1 + g(x,A)) \tag{1.48}$$

for all $x \in \Omega$ and $A \in \mathbf{M}^{m\times n}$.

As already noted in Remark 1.39, we get compactness and integral representation on $W^{1,p}(\Omega;\mathbf{R}^m)$ also under these assumptions. In general we can not extend the Γ-limit on $W^{1,\psi}(\Omega;\mathbf{R}^m)$ but we can just conclude by Theorem 1.47 that the Γ-limit is $+\infty$ on $W^{1,1}(\Omega;\mathbf{R}^m) \setminus W^{1,\psi}(\Omega;\mathbf{R}^m)$; however similar results hold in some particular cases as the following Sections 1.10.1 and 1.10.2 show.

1.10.1 *Homogenization with non-standard growth conditions*

Let Ω be a bounded subset of $\mathbf{R}^n$, let $p \le q < p^*$ and let $F_\varepsilon : L^p(\Omega;\mathbf{R}^m)\times\mathcal{A}(\Omega) \to [0,+\infty]$ be defined as follow

$$F_\varepsilon(u,U) = \begin{cases} \displaystyle\int_U f_\varepsilon(x,Du)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_\varepsilon : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty)$ is a Borel function satisfying the *non-standard growth condition*

$$\alpha|A|^p \le f_\varepsilon(x,A) \le \beta(1 + |A|^q) \tag{1.49}$$

for all $x \in \Omega$ and $A \in \mathbf{M}^{m\times n}$.

We assume that Ω has a Lipschitz boundary and by Rellich's Theorem and a generalization of the direct methods of Γ-convergence, we prove compactness and integral representation theorems on $W^{1,q}(\Omega;\mathbf{R}^m)$, which are the analogue of Proposition 1.38 and Theorem 1.40.

We restrict our attention to the case of homogenization and we study the convex case thanks to which we will then deal with the non-convex case; hence we consider

$$G_\varepsilon(u) = \int_\Omega g\Big(\frac{x}{\varepsilon}, Du\Big)\,dx$$

where $u \in W^{1,q}(\Omega;\mathbf{R}^m)$ and $g : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty)$ is a Borel function, 1-periodic in the first variable and convex in the second one, such that it satisfies the non-standard growth conditions (1.49).

Repeating the proof of Theorem 1.47 we extend the Γ-limit

$$G(u,U) = \int_U \psi(Du(x))\,dx,$$

for all $U \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega;\mathbf{R}^m)$, where ψ satisfies the homogenization formula (1.44). Hence if we define

$$W^{1,\psi}(U;\mathbf{R}^m) = \Big\{u \in W^{1,p}(U;\mathbf{R}^m) : \int_U \psi(Du)\,dx < +\infty\Big\}$$

for every bounded open subset U of $\mathbf{R}^n$, then

$$W^{1,q}(U;\mathbf{R}^m) \subset W^{1,\psi}(U;\mathbf{R}^m) \subset W^{1,p}(U;\mathbf{R}^m)$$

and the Γ-limit G is finite only in $W^{1,\psi}(\Omega;\mathbf{R}^m)$.

Now we consider

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, Du(x)\Big)\,dx$$

where $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ and $f : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty)$ is a Borel function 1-periodic in the first variable and which satisfies (1.47), with $g : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty)$ Borel function, 1-periodic in the first variable and convex in the second one, which satisfies (1.49) and $g(x,2A) \le c(1+g(x,A))$ for all $x \in \Omega$ and $A \in \mathbf{M}^{m\times n}$.

Using the results of convex homogenization, we get existence and integral representation of the Γ-limit of F_ε

$$F(u,U) = \int_U \gamma(x,Du)\,dx$$

for all $U \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega;\mathbf{R}^m)$, with F finite only in $W^{1,\psi}(\Omega;\mathbf{R}^m)$.

It is possible to prove that γ can be chosen independent of the first variable and it satisfies the homogenization formula (1.43).

For more details on this argument see Chapter 21 of [26].

1.10.2 *Oscillating boundaries*

Another case is the homogenization of non-convex functionals defined on sets with oscillating boundaries. In Chapter 4 we study this problems proving the existence of the Γ-limit on the whole $L^p(\Omega;\mathbf{R}^m)$ and we show that the integral representation holds on the whole domain of the Γ-limit that will be a 'degenerate Sobolev Space'. Also in this case the 'degenerate Sobolev Space' will be defined as domain of the Γ-limit of the convex case.

1.11 The spaces BV and BD

We summarize some definitions and basic results on functions of bounded variations and on functions with bounded deformation which will be useful in the sequel. For a general exposition of the theory of functions of bounded variation we refer to [47], [46], [51], [77] and [4]. For a general exposition of the theory of functions with bounded deformation we refer to [71], [72], [56], [63], [11], [74], [73], [3].

Given a function $u \in L^1(\Omega;\mathbf{R}^m)$, we say that $x \in \Omega$ is a Lebesgue points of u if and only if there exists $z \in \mathbf{R}^n$ such that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n}\int_{B_\rho(x)} |u(y)-z|\,dx = 0\,. \tag{1.50}$$

We denote Ω_u the set of Lebesgue points of u in Ω. If z exists then it is unique and we define $\tilde{u}(x) = z$; we call $\tilde{u}$ *precise representative* of u.

We denote by S_u the set of points which are not Lebesgue points and we call it *discontinuity set* of u; the set S_u is Lebesgue-negligible and the function $\tilde{u} : \Omega_u \to \mathbf{R}^m$ coincides with u $\mathcal{L}^n$-almost everywhere in $\Omega_u = \Omega \setminus S_u$.

Definition 1.48 *We say that $x \in \Omega$ belongs to J_u, the* jump set *of u, if and only if there exist a unit normal $\nu \in S^{n-1}$ and two vectors a and b in $\mathbf{R}^n$ $(a \neq b)$ such that*

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n}\int_{B_\rho^+(x,\nu)} |u(y)-a|\,dy = 0$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n}\int_{B_\rho^-(x,\nu)} |u(y)-b|\,dy = 0$$

where $B_\rho^\pm(x,\nu) = \{y \in B_\rho(x) \ : \ \langle y-x, \pm\nu\rangle > 0\}$.

The triplet (a,b,ν) is uniquely determined up to a change of sign of ν and a permutation of (a,b). For every $x \in J_u$ we define $u^+(x) = a$, $u^-(x) = b$ and $\nu_u(x) = \nu$.

It is easy to prove that J_u and S_u are Borel sets $(J_u \subset S_u)$ and that u^+, u^- and $\tilde{u}$ are Borel functions.

Definition 1.49 *Given a Borel set $J \subset \mathbf{R}^N$, we say that J is* countably $(\mathcal{H}^{N-1}, N-1)$-rectifiable *(rectifiable for short) if*

$$J = R \cup \bigcup_{i\geq 1} K_i$$

where $\mathcal{H}^{N-1}(R) = 0$ and each K_i is a compact subset of a C^1 $(N-1)$-dimensional manifold.

Thus, for every rectifiable set J it is possible to define $\mathcal{H}^{N-1}$ a.e. a unitary normal vector field ν.

Now we have set all the tools to describe the structure of BV and BD functions.

Definition 1.50 *Let $u \in L^1(\Omega; \mathbf{R}^m)$. We say that u is a* function of bounded variation*, and we write $u \in BV(\Omega; \mathbf{R}^m)$, if all its distributional first derivatives $D_i u_j$ are Radon measure with finite total variation in Ω; i.e.,*

$$\int_\Omega u_j D_i \phi \, dx = - \int_\Omega \phi \, dD_i u_j$$

for all $\phi \in C_c^1(\Omega)$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

We denote by Du the $\mathbf{M}^{m \times n}$-valued measure whose entries are $D_i u_j$.

Definition 1.51 *For every $u \in BV(\Omega; \mathbf{R}^m)$ we consider the Radon-Nikodym decomposition $Du = D^a u + D^s u$ of Du, where $D^a u$ is absolutely continuous and $D^s u$ is singular with respect to the Lebesgue measure $\mathcal{L}^n$. We may further decompose the singular part $D^s u$ as $D^s u = D^j u + D^c u$ where $D^j u = Du \llcorner S_u$ is the jump part of Du, and $D^c u = D^s u \llcorner (\Omega \setminus S_u)$ is the Cantor part of Du. We can then write*

$$Du = D^a u + D^j u + D^c u \, .$$

Theorem 1.52 *If $u \in BV(\Omega; \mathbf{R}^m)$ then*

(1) *for $\mathcal{L}^n$-almost every $x \in \Omega$ there exists the* approximate gradient *of u, ∇u; i.e.,*

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{B_\rho(x)} \frac{|u(y) - u(x) - \langle \nabla u(x), y - x \rangle|}{|y - x|} \, dy = 0,$$

and it is the density of the absolutely continuous part of Du; i.e., $D^a u = \nabla u \mathcal{L}^n$;

(2) *S_u is rectifiable, $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ and we have*

$$D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner S_u,$$

where ν_u is defined by $Du = \nu_u |Du|$ $|Du|$-a.e. and coincides with that of Definition 1.48 on J_u $\mathcal{H}^{n-1}$-a.e. on S_u;

(3) *for any Borel set B with $\mathcal{H}^{n-1}(B) < +\infty$, we have that $|D^c u|(B) = 0$.*

For a complete proof of points (1), (2) and (3) see [4] Theorem 3.81, [21] Theorems 1.63 and 1.66, respectively.

Theorem 1.53. (Chain rule in BV) *Let $u \in BV(\Omega; \mathbf{R}^m)$ and $f \in C^1(\mathbf{R}^m; \mathbf{R}^k)$ be a Lipschitz. Then $v = f \circ u$ belongs to $BV(\Omega; \mathbf{R}^k)$ and*

$$Dv = \nabla f(\tilde{u})(\nabla u \mathcal{L}^n + D^c u) + (f(u^+) - f(u^-)) \otimes \nu_u H^{n-1} \llcorner S_u \, .$$

(See [4] Theorem 3.93 and [21] Section 1.8.2).

For any $y, \xi \in \mathbf{R}^n$, $\xi \neq 0$, and any $\Omega \subset \mathbf{R}^n$ we define

$$\begin{aligned} \pi_\xi &:= \{y \in \mathbf{R}^n \ : \ \langle y, \xi\rangle = 0\}, \\ \Omega_y^\xi &:= \{t \in \mathbf{R} \ : \ y + t\xi \in \Omega\}, \\ \Omega^\xi &:= \{y \in \pi_\xi \ : \ \Omega_y^\xi \neq \emptyset\}, \end{aligned} \tag{1.51}$$

and given $u : \Omega \mapsto \mathbf{R}^m$ the function $u_{y,\xi} : \Omega_y^\xi \mapsto \mathbf{R}$ is defined by $u_{y,\xi}(t) = u(y+t\xi)$ for every $t \in \Omega_y^\xi$; it is well know that the space $BV(\Omega; \mathbf{R}^m)$ can be characterized by means of one-dimensional sections:

Theorem 1.54 *Let $u \in BV(\Omega; \mathbf{R}^m)$ and let $\xi \in S^{n-1} = \{\zeta \in \mathbf{R}^n \ : \ |\zeta| = 1\}$. Then we have $u_{y,\xi} \in BV(\Omega_y^\xi; \mathbf{R}^m)$ for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$ and*

$$\langle D^\sigma u, \xi\rangle = \int_{\Omega^\xi} D^\sigma u_{y,\xi} \, d\mathcal{H}^{n-1} \qquad \text{for } \sigma = a, j, c\,.$$

Conversely, let $u \in L^1(\Omega; \mathbf{R}^m)$. If for every direction $\xi \in S^{n-1}$ we have $u_{y,\xi} \in BV(\Omega_y^\xi; \mathbf{R}^m)$ for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$ and

$$\int_{\Omega^\xi} |Du_{y,\xi}|(\Omega_y^\xi) \, d\mathcal{H}^{n-1}(y) < +\infty$$

then $u \in BV(\Omega; \mathbf{R}^m)$.

(See [4] Theorems 3.99 and 3.100).

Definition 1.55 *Let $u \in L^1(\Omega; \mathbf{R}^n)$, and let Eu be the symmetric part of the distributional gradient of u; i.e.,*

$$(Eu)_{ij} := (E_{ij}u), \qquad E_{ij}u := \frac{1}{2}(D_i u_j + D_j u_i).$$

The space LD(Ω) *is defined as the set of all functions $u \in L^1(\Omega; \mathbf{R}^n)$ such that $E_{ij}u \in L^1(\Omega)$ for any $i, j = 1, ..., n$.*

We say that $u \in L^1(\Omega; \mathbf{R}^n)$ is a function with bounded deformation*, and we write $u \in BD(\Omega)$, if $E_{ij}u$ is a Radon measure with finite total variation in Ω for any $i, j = 1, ..., n$.*

We denote by Eu the $\mathbf{M}^{n\times n}$-valued measure whose entries are $E_{ij}u$.

Note that the closed subspace $LD(\Omega)$ of $BD(\Omega)$ plays the same role in $BD(\Omega)$ as the one of $W^{1,1}(\Omega; \mathbf{R}^m)$ in $BV(\Omega; \mathbf{R}^m)$. One can easily see that $W^{1,1}(\Omega; \mathbf{R}^n) \subset LD(\Omega)$ with strict inclusion.

For every $\xi \in \mathbf{R}^n$, let D_ξ be the distributional derivative in the direction ξ defined by $D_\xi u = \langle Du, \xi\rangle$. For every function $u : \Omega \to \mathbf{R}^n$ let us define the function $u^\xi : \Omega \to \mathbf{R}$ by $u^\xi(x) = \langle u(x), \xi\rangle$.

Theorem 1.56 *If $u \in BD(\Omega)$ then $D_\xi u^\xi$ is a bounded Radon measure on Ω for every $\xi \in \mathbf{R}^n$ and*

$$D_\xi u^\xi = \langle Eu\xi, \xi \rangle .$$

Conversely, let $\xi_1, \dots, \xi_n$ be a basis of $\mathbf{R}^n$ and let $u \in L^1(\Omega; \mathbf{R}^n)$; then $u \in BD(\Omega)$ if $D_\xi u^\xi$ is a bounded Radon measure on Ω for every ξ of the form $\xi_i + \xi_j$, $i, j = 1, \dots, n$.

(See [73] Chapter 2 Section 2.2).

Definition 1.57 *For every $u \in BD(\Omega)$ we consider the Radon-Nikodym decomposition $Eu = E^a u + E^s u$ of Eu, where $E^a u$ is absolutely continuous and $E^s u$ is singular with respect to the Lebesgue measure $\mathcal{L}^n$. We may further decompose the singular part $E^s u$ as $E^s u = E^j u + E^c u$ where $E^j u = Eu \llcorner J_u$ is the* jump part *of Eu, and $E^c u = E^s u \llcorner (\Omega \setminus J_u)$ is the* Cantor part *of Eu. We can then write*

$$Eu = E^a u + E^j u + E^c u .$$

Theorem 1.58 *If $u \in BD(\Omega)$ then*

(1) *for $\mathcal{L}^n$-almost every $x \in \Omega$ there exists the approximate symmetric differential of u, $\mathcal{E}u$; i.e.,*

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{B_\rho(x)} \frac{|\langle u(y) - u(x) - \mathcal{E}u(y-x)\, ,\, y-x \rangle|}{|y-x|^2}\, dy = 0,$$

and it is the density of $E^a u$ with respect to $\mathcal{L}^n$; i.e., $E^a u = \mathcal{E}u \mathcal{L}^n$;

(2) *J_u is rectifiable and we have*

$$E^j u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u; \tag{1.52}$$

(3) *for any Borel set B with $\mathcal{H}^{n-1}(B) < +\infty$, we have that $|E^c u|(B) = 0$.*

For a complete proof of points (1), (2) and (3) see [3] Theorem 4.3, [3] Proposition 3.5 and [73] Chapter 2, [3] Proposition 4.4, respectively.

Note that it is not known whether $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ or not.

Now we shall see that the space $BD(\Omega)$ can be characterized using suitable one-dimensional sections. We use the notation introduced in (1.51).

Given $u : \Omega \mapsto \mathbf{R}^n$ for every $y, \xi \in \mathbf{R}^n$, $\xi \neq 0$, the function $u^\xi_y : \Omega^\xi_y \mapsto \mathbf{R}$ is defined by

$$u^\xi_y(t) = u^\xi(y + t\xi) = \langle u(y + t\xi), \xi \rangle \qquad \forall t \in \Omega^\xi_y.$$

If $u \in L^1(\Omega; \mathbf{R}^n)$, then for every $\xi \in \mathbf{R}^n$, $\xi \neq 0$, and every $y \in \Omega^\xi$ the one-dimensional section $\tilde{u}^\xi_y$ of the function $\tilde{u}$ introduced in (1.50) is defined for every $t \in (\Omega_u)^\xi_y$.

Proposition 1.59 *Let $u \in BD(\Omega)$ and let $\xi \in \mathbf{R}^n$ with $\xi \neq 0$. Then the following two conditions hold for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$:*

(i) $\tilde{u}^\xi_y$ *is defined and coincides with* u^ξ_y $\mathcal{L}^1$*-almost everywhere in* Ω^ξ_y*;*
(ii) $u^\xi_y \in BV(\Omega^\xi_y)$.
Moreover,

$$\langle Eu\xi, \xi\rangle = \int_{\Omega^\xi} Du^\xi_y \, d\mathcal{H}^{n-1}(y), \qquad |\langle Eu\xi, \xi\rangle| = \int_{\Omega^\xi} |Du^\xi_y| \, d\mathcal{H}^{n-1}(y)$$

as measures in Ω*. Conversely, let* $u \in L^1(\Omega; \mathbf{R}^n)$ *and let* $\xi_1, \dots, \xi_n$ *be a basis of* $\mathbf{R}^n$*. Assume that for every* ξ *of the form* $\xi_i + \xi_j$*,*

$$u^\xi_y \in BV(\Omega^\xi_y) \text{ for } \mathcal{H}^{n-1} - a.e.\ y \in \Omega^\xi,$$

$$\int_{\Omega^\xi} |Du^\xi_y|(\Omega^\xi_y) \, d\mathcal{H}^{n-1}(y) < +\infty.$$

Then $u \in BD(\Omega)$*.*

(See [3] Proposition 3.2).

Remark 1.60 By Proposition 1.59 we obtain that $|\langle Eu\xi, \xi\rangle|(B) = 0$ for every Borel set B with $\mathcal{H}^{n-1}(B) = 0$. Since for any basis $\xi_1, \dots, \xi_n$ there exists a constant c, depending on the basis, such that

$$|Eu| \le c \sum_{i,j=1}^{n} |\langle Eu(\xi_i + \xi_j), \xi_i + \xi_j\rangle|$$

we conclude that

$$\mathcal{H}^{n-1}(B) = 0 \quad \Rightarrow \quad |Eu|(B) = 0$$

for every Borel set B.

Let $\mathcal{R}$ be the class of the *rigid motions* in $\mathbf{R}^n$; *i.e.*, affine maps of the form $Ax + d$ with A a skew symmetric $n \times n$ matrix and $d \in \mathbf{R}^n$, or equivalently $u \in BD(\Omega)$ such that $Eu = 0$. The following "Poincaré-type" inequality for BD functions follows from Proposition 2.2 and Remark 1.1 of Chapter 2 of [73]. For a complete proof see [56] Part II, Proposition 3.11.

Theorem 1.61. (Poincaré-type inequality) *Let* Ω *be a bounded connected open set with Lipschitz boundary and let* $R : BD(\Omega) \to \mathcal{R}$ *be a continuous linear map which leaves the elements of* $\mathcal{R}$ *fixed. Then there exists a constant* $c(\Omega, R)$ *such that*

$$\int_\Omega |u - R(u)| dx \le c(\Omega, R)|Eu|(\Omega)$$

for all $u \in BD(\Omega)$*.*

Definition 1.62 *We call* intermediate topology *on* $BD(\Omega)$ *that defined by the distance*

$$\|u - v\|_{L^1(\Omega; \mathbf{R}^n)} + ||Eu|(\Omega) - |Ev|(\Omega)|. \tag{1.53}$$

Theorem 1.63 *$C^\infty(\overline{\Omega})$ is dense in the space $BD(\Omega)$ endowed with the intermediate topology.*

(See [73] Chapter 2 Theorem 3.2).

Proposition 1.64. (Korn's inequality) *For any p satisfying $1 < p < +\infty$ we define the space*

$$\{u \in L^1(\Omega; \mathbf{R}^n) \ : \ E_{ij}u \in L^p(\Omega) \ i, j = 1, ..., n\}\,. \tag{1.54}$$

If Ω has a locally Lipschitz boundary, then for all $1 < p < +\infty$ we have

$$\sum_{i,j=1}^{n} \int_\Omega |D_i u_j(x)|^p \, dx \le c \int_\Omega \Big(|u(x)|^p + |Eu(x)|^p \Big) \, dx\,. \tag{1.55}$$

for all u in the space (1.54).

The Korn's inequality implies that (1.54) is none other than $W^{1,p}(\Omega; \mathbf{R}^n)$ (see [73] Chapter 1 Section 1, [52] and [53]).

2

HOMOGENIZATION OF PERIODIC MULTI-DIMENSIONAL STRUCTURES

2.1 Sobolev spaces with respect to a measure

The following notion of Sobolev space with respect to a measure has been introduced by Ambrosio, Buttazzo and Fonseca [2].

Definition 2.1 *Let λ be a finite Borel positive measure on the open set $\Omega \subset \mathbf{R}^n$, and let $1 \leq p \leq +\infty$. The* Sobolev space with respect to λ, *$W^{1,p}_\lambda(\Omega; \mathbf{R}^m)$, is defined as*

$$W^{1,p}_\lambda(\Omega; \mathbf{R}^m) = \Big\{ u \in L^p(\Omega; \mathbf{R}^m) : u \in BV(\Omega; \mathbf{R}^m),\ Du << \lambda, \\ \frac{dDu}{d\lambda} \in L^p_\lambda(\Omega; \mathbf{M}^{m\times n}) \Big\},$$

where $L^p_\lambda(\Omega; \mathbf{R}^N)$ stands for the usual Lebesgue space of p-summable $\mathbf{R}^N$-valued functions with respect to λ.

Remark 2.2 By definition, functions in $W^{1,p}_\lambda(\Omega; \mathbf{R}^m)$ are functions of bounded variation. From the embedding of $BV(\Omega; \mathbf{R}^m)$ in $L^{n/(n-1)}(\Omega; \mathbf{R}^m)$ and the chain rule Theorem 1.53 the following two facts can be easily deduced, that are used in the sequel.

(a) $W^{1,p}_\lambda(\Omega; \mathbf{R}^m)$ is embedded in $L^{n/(n-1)}(\Omega; \mathbf{R}^m)$.

(b) If $u \in W^{1,p}_\lambda(\Omega; \mathbf{R}^m)$ and $v \in W^{1,\infty}_\lambda(\Omega)$ then $uv \in W^{1,p}_\lambda(\Omega; \mathbf{R}^m)$, and

$$\frac{dD(uv)}{d\lambda} = \tilde{v}\frac{dDu}{d\lambda} + \tilde{u} \otimes \frac{dDv}{d\lambda}. \tag{2.1}$$

Note that in (2.1) it is necessary to consider the precise representatives, since the measure λ may take into account also sets of zero Lebesgue measure.

If $u \in W^{1,p}_\lambda(\Omega; \mathbf{R}^m)$ then $Du(B) = 0$ if B is a set of zero $(n-1)$-Hausdorff measure. Hence, $W^{1,p}_\lambda(\Omega; \mathbf{R}^m) = W^{1,p}_{\lambda'}(\Omega; \mathbf{R}^m)$ if $\lambda - \lambda'$ is concentrated on a set of Hausdorff dimension lower than $n-1$; e.g., points in $\mathbf{R}^3$.

Properties of lower semicontinuity and relaxation for functionals defined on Sobolev spaces with respect to a measure have been studied in [2].

2.2 Statement of the main result

Let μ be a non-zero positive Radon measure on $\mathbf{R}^n$ which is 1-periodic; i. e.,

$$\mu(B + e_i) = \mu(B)$$

for all Borel subsets B of $\mathbf{R}^n$ and for all $i = 1, \ldots, n$. The measure μ will be fixed throughout the chapter. We will assume the normalization

$$\mu([0,1)^n) = 1\,. \tag{2.2}$$

For all $\varepsilon > 0$ we define the ε-periodic positive Radon measure μ_ε by

$$\mu_\varepsilon(B) = \varepsilon^n\, \mu\Big(\frac{1}{\varepsilon}B\Big) \tag{2.3}$$

for all Borel sets B. Note that by (2.2) the family (μ_ε) converges locally weakly* in the sense of measures to the Lebesgue measure as $\varepsilon \to 0$.

In the sequel $f : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0, +\infty)$ will be a fixed Borel function 1-periodic in the first variable and satisfying the growth condition of order $p \geq 1$: there exist $0 < \alpha \leq \beta$ such that

$$\alpha|A|^p \leq f(x, A) \leq \beta(1 + |A|^p) \tag{2.4}$$

for all $x \in \mathbf{R}^n$ and $A \in \mathbf{M}^{m\times n}$.

For every bounded open set Ω, we define the functionals at scale $\varepsilon > 0$ as

$$F_\varepsilon(u,\Omega) = \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dDu}{d\mu_\varepsilon}\Big) d\mu_\varepsilon & \text{if } u \in W^{1,p}_{\mu_\varepsilon}(\Omega;\mathbf{R}^m) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

Example 2.3 (a) (Perfectly-rigid bodies connected with springs) We take

$$E = \{y \in \mathbf{R}^n :\ \exists i \in \{1, \ldots, n\} \text{ such that } \ y_i \in \mathbf{Z}\},$$

that is, the union of all the boundaries of cubes $Q_i = i + (0,1)^n$ with $i \in \mathbf{Z}^n$. E is an $(n-1)$-dimensional set in $\mathbf{R}^n$. We take

$$\mu(B) = \frac{1}{n}\mathcal{H}^{n-1}(B \cap E)$$

for all Borel sets B. For every $\varepsilon > 0$ we have

$$\mu_\varepsilon(B) = \frac{1}{n}\varepsilon\mathcal{H}^{n-1}(B \cap \varepsilon E)\,.$$

In this case $W^{1,p}_{\mu_\varepsilon}$ consists of functions which are constant on every connected component of each $\varepsilon Q_i \cap \Omega$, since we must have $Du = 0$ on these sets. In the case that u is constant on each $\varepsilon Q_i \cap \Omega$, e.g. if Ω is convex, we have

$$\frac{dDu}{d\mu_\varepsilon} = \frac{n}{\varepsilon}\frac{dDu}{d\mathcal{H}^{n-1}} = \frac{n}{\varepsilon}(u_i - u_j) \otimes (i - j) \text{ on } \partial(\varepsilon Q_i) \cap \partial(\varepsilon Q_j) \cap \Omega\,,$$

where u_i is the value of u on εQ_i. In this case the functionals F_ε take the form

$$\varepsilon \int_{\Omega\cap\varepsilon E} g\Big(\frac{x}{\varepsilon}, \frac{1}{\varepsilon}\frac{dDu}{d\mathcal{H}^{n-1}}\Big)\, d\mathcal{H}^{n-1}.$$

Note that if Ω is bounded then $W^{1,p}_{\mu_\varepsilon}(\Omega;\mathbf{R}^m) = W^{1,\infty}_{\mu_\varepsilon}(\Omega;\mathbf{R}^m)$ for all p if the number of connected components of each $\Omega\cap\varepsilon Q_i$ is finite.

(b) (Elastic media connected with springs) Let E be as above and let

$$\mu(B) = \frac{1}{n+1}\Big(|B| + \mathcal{H}^{n-1}(E\cap B)\Big)$$
$$\mu_\varepsilon(B) = \frac{1}{n+1}\Big(|B| + \varepsilon\mathcal{H}^{n-1}((\varepsilon E)\cap B)\Big)\,.$$

In this case the functions in $W^{1,p}_{\mu_\varepsilon}(\Omega;\mathbf{R}^m)$ are functions whose restriction to each $\varepsilon Q_i\cap\Omega$ belongs to $W^{1,p}(\varepsilon Q_i\cap\Omega;\mathbf{R}^m)$, and such that the difference of the traces on both sides of $\partial(\varepsilon Q_i)\cap\partial(\varepsilon Q_j)\cap\Omega$ is p-summable for every $i,j\in\mathbf{Z}^n$. The functionals F_ε take the form

$$\frac{1}{n+1}\int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dDu}{dx}\Big)\, dx + \varepsilon\int_{\Omega\cap\varepsilon E} g\Big(\frac{x}{\varepsilon}, \frac{1}{\varepsilon}\frac{dDu}{d\mathcal{H}^{n-1}}\Big)\, d\mathcal{H}^{n-1}.$$

In order to obtain a meaningful limit of the functionals F_ε as $\varepsilon\to 0$, some requirements have to be made so that the limit functionals admit an integral representation on $W^{1,p}(\Omega;\mathbf{R}^m)$.

Definition 2.4 *A 1-periodic positive Radon measure μ on $\mathbf{R}^n$ will be called* p-homogenizable *if the following properties hold:*

(i) (existence of cut-off functions) *there exist $K>0$ and $\delta>0$ such that for all $\varepsilon>0$, for all pairs U,V of open subsets of $\mathbf{R}^n$ with $U\subset\subset V$, and* $\operatorname{dist}(U,\partial V)\ge\delta\varepsilon$, *and for all $u\in W^{1,p}_{\mu_\varepsilon}(V)$ there exists $\phi\in W^{1,\infty}_{\mu_\varepsilon}(V)$ with $0\le\phi\le 1$, $\phi=1$ on U, $\phi=0$ in a neighbourhood of ∂V, such that*

$$\int_V \Big|\frac{dD\phi}{d\mu_\varepsilon}\tilde{u}\Big|^p d\mu_\varepsilon \le \frac{K}{\big(\operatorname{dist}(U,\partial V)\big)^p}\int_{V\setminus U} |u|^p\, dx\,. \tag{2.6}$$

Such a ϕ will be called a cut-off function between U and V*;*

(ii) (existence of periodic test-functions) *for all $i=1,\dots,n$, there exists $z_i\in W^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n)$ such that $x\mapsto z_i(x)-x_i$ is 1-periodic.*

Remark 2.5 Note that the Lebesgue measure satisfies trivially all the properties of Definition 2.4. Property (i) depends on μ and p.

Example 2.6 (a) The measure μ in Example 2.3(a) is p-homogenizable for all $p\ge 1$. In fact, to prove (i) let $\delta=5\sqrt{n}$. Fixed $\varepsilon>0$, set $U_\varepsilon=\bigcup\{\varepsilon Q_i:\ \varepsilon Q_i\cap U\ne\emptyset\}$. Note that $U_\varepsilon\subset\subset V$. Choose (we use the notation $[t]$ for the integer part of t)

$$\phi(x) = 1 - \Big(\frac{1}{C}\Big[\frac{1}{\varepsilon}\inf\{|x-y|_\infty : y \in U_\varepsilon\}\Big] \wedge 1\Big),$$

where $|x-y|_\infty = \max_{1\le i\le n}|x_i - y_i|$, and

$$C = \Big[\frac{1}{\varepsilon}\inf\Big\{|x-y|_\infty : x \in U_\varepsilon,\ y \in \partial V\Big\}\Big] - 2\,.$$

Note that $|dD\phi/d\mu_\varepsilon| \le n/(C\varepsilon) \le c/\mathrm{dist}\,(U,\partial V)$ for some constant c independent of U and V. Moreover, if $u \in W^{1,p}_{\mu_\varepsilon}(V)$ then u is equal to a constant u_i on each cube εQ_i such that $D\phi \ne 0$ on $\partial(\varepsilon Q_i)$. Hence, for two such cubes

$$\varepsilon\int_{\partial\varepsilon Q_i\cap\partial\varepsilon Q_j}|\widetilde{u}|^p\,d\mathcal{H}^{n-1} \le \varepsilon\int_{\partial\varepsilon Q_i\cap\partial\varepsilon Q_j}(|u_i|^p + |u_j|^p)\,d\mathcal{H}^{n-1} = \int_{\varepsilon Q_i\cup\varepsilon Q_j}|u|^p\,dx$$

so that

$$\begin{aligned}\int_V\Big|\frac{dD\phi}{d\mu_\varepsilon}\widetilde{u}\Big|^p d\mu_\varepsilon &\le \frac{c^p\varepsilon}{\mathrm{dist}\,(U,\partial V)^p}\int_{(V\setminus U)\cap\varepsilon E\cap\mathrm{spt}D\phi}|\widetilde{u}|^p\,d\mathcal{H}^{n-1}\\ &\le 2n\frac{c^p}{\mathrm{dist}\,(U,\partial V)^p}\int_{V\setminus U}|u|^p\,dx\,.\end{aligned}$$

The proof of (i) is then complete. To verify (ii) take simply $z_i(x) = [x_i]$.

(b) The measure μ in Example 2.3(b) is p-homogenizable for all $p \ge 1$. In fact, the proof of (i) and (ii) is trivial since the Lebesgue measure is absolutely continuous with respect to μ.

The homogenization theorem for functionals in (2.5) takes the following form.

Theorem 2.7 *Let μ be a p-homogenizable measure, and for every bounded open subset Ω of $\mathbf{R}^n$ let $F_\varepsilon(\cdot,\Omega)$ be defined on $L^p(\Omega;\mathbf{R}^m)$ by (2.5). Then the Γ-limit with respect to the $L^p(\Omega;\mathbf{R}^m)$-convergence*

$$F_{\mathrm{hom}}(u,\Omega) = \Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon(u,\Omega) \tag{2.7}$$

exists for all bounded open subsets Ω with Lipschitz boundary and for all $u \in W^{1,p}(\Omega;\mathbf{R}^m)$, and it can be represented as

$$F_{\mathrm{hom}}(u,\Omega) = \int_\Omega f_{\mathrm{hom}}(Du)\,dx\,, \tag{2.8}$$

where the homogenized integrand satisfies the asymptotic formula

$$\begin{aligned}f_{\mathrm{hom}}(A) = \lim_{k\to+\infty}\inf\Big\{\frac{1}{k^n}\int_{[0,k)^n}f\Big(x,\frac{dDu}{d\mu}\Big)\,d\mu :\\ u \in W^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^m),\ u - Ax\ \ k\text{-periodic}\Big\}\,.\end{aligned} \tag{2.9}$$

If $p>1$ then $F_{\rm hom}(u,\Omega)=+\infty$ if $u\in L^p(\Omega;\mathbf{R}^m)\setminus W^{1,p}(\Omega;\mathbf{R}^m)$. Furthermore, if f is convex then the cell-problem formula holds

$$f_{\rm hom}(A)=\inf\Big\{\int_{[0,1)^n} f\Big(x,\frac{dDu}{d\mu}\Big)\,d\mu: \qquad u\in W^{1,p}_{\mu,{\rm loc}}(\mathbf{R}^n;\mathbf{R}^m),\ u-Ax \text{ 1-periodic}\Big\} \tag{2.10}$$

for all $A\in\mathbf{M}^{m\times n}$.

Remark 2.8 In formulas (2.9) and (2.10) we cannot replace the sets $[0,k)^n$ and $[0,1)^n$ by the sets $(0,k)^n$ and $(0,1)^n$, respectively, if μ charges $[0,1)^n\setminus(0,1)^n$.

Remark 2.9 If μ is not a p-homogenizable measure then $f_{\rm hom}$ may be equal to $+\infty$ for all non-zero matrices A. As an example, take

$$\mu(B)=\sum_{i\in\mathbf{Z}^n}\lambda(i+B)\,, \tag{2.11}$$

where λ is any probability measure with ${\rm spt}\,\lambda$ contained in $(0,1)^n$. Then test-functions u in (2.9) must be constant on a periodic connected component of $\mathbf{R}^n$, and hence we get that $f_{\rm hom}(A)=+\infty$ if $A\neq 0$.

Remark 2.10 Contrary to the usual homogenization results in the framework of ordinary Sobolev spaces, the hypothesis that Ω has a Lipschitz boundary (which will be used in an essential way in Step 3 of Proposition 2.13) cannot be removed from Theorem 2.7. To check this, take simply $n=2$ and

$$\Omega=\Big(\bigcup_{i=1}^\infty(q_i-2^{-i-3},q_i+2^{-i-3})\times(0,1)\Big)\cup\Big(\bigcup_{i=1}^\infty(0,1)\times(q_i-2^{-i-3},q_i+2^{-i-3})\Big),$$

where (q_i) is a numbering of $\mathbf{Q}\cap(0,1)$. Take as μ the measure of Example 2.3(a) and any f in Theorem 2.7. Note that, as $\Omega\cap\frac{1}{k}Q_i$ is connected for all sub-cubes $\frac{1}{k}Q_i$ of $(0,1)^2$, each function $u\in W^{1,p}_{\mu_{1/k}}(\Omega\cap(0,1)^2;\mathbf{R}^m)$ is constant on each such $\Omega\cap\frac{1}{k}Q_i$. Hence, the two spaces $W^{1,p}_{\mu_{1/k}}(\Omega\cap(0,1)^2;\mathbf{R}^m)$ and $W^{1,p}_{\mu_{1/k}}((0,1)^2;\mathbf{R}^m)$ are equivalent, and, as $\frac{1}{k}E\cap(0,1)^2\subset\Omega\cap(0,1)^2$,

$$F_{1/k}(u,\Omega\cap(0,1)^2)=F_{1/k}(u,(0,1)^2).$$

If the thesis of Theorem 2.7 were true, then we would easily conclude that for all $v\in W^{1,p}(\Omega\cap(0,1)^2;\mathbf{R}^m)$ with $F_{\rm hom}(u,\Omega\cap(0,1)^2)<+\infty$ there exists $u\in W^{1,p}((0,1)^2;\mathbf{R}^m)$ with $u=v$ on $\Omega\cap(0,1)^2$ and

$$F_{\rm hom}(v,\Omega\cap(0,1)^2)=F_{\rm hom}(u,(0,1)^2),$$

which is not possible for example if $f\geq 1$ since $|\Omega\cap(0,1)^2|\neq|(0,1)^2|$.

2.3 Proof of the homogenization theorem

The proof of Theorem 2.7 will be obtained at the end of the section, as a consequence of the following propositions, which adapt to this case the usual methods for the homogenization by Γ-convergence. While the usual compactness and integral representation results in Section 1.7 hold with minor modification also in this case, a more complex proof for the so-called fundamental estimate, for the growth condition from above and for the homogenization formula is necessary.

From now on, Ω will be a fixed bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary.

Proposition 2.11. (Fundamental Estimate) *For every $\sigma > 0$ there exists ε_σ and $M > 0$ such that for all U, U', V open subsets of Ω with $U' \subset U$ and* $\operatorname{dist}(U', V \setminus U) > 0$, *for all $\varepsilon < \varepsilon_\sigma \operatorname{dist}(U', V \setminus U)$ and for all $u \in W^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^m)$, $v \in W^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^m)$ there exists a cut-off function between U' and U, $\phi \in W^{1,\infty}_{\mu_\varepsilon}(U \cup V)$, such that*

$$
\begin{aligned}
F_\varepsilon(\phi u + (1-\phi)v, U' \cup V) &\le (1+\sigma)(F_\varepsilon(u,U) + F_\varepsilon(v,V)) \\
&\quad + \frac{M}{\big(\operatorname{dist}(U', V \setminus U)\big)^p} \int_{(U\cap V)\setminus U'} |u-v|^p dx + \sigma \mu_\varepsilon((U \cap V) \setminus U').
\end{aligned} \tag{2.12}
$$

Proof. Let $K > 0$ and $\delta > 0$ be the constants given by Definition 2.4(i), let $N \in \mathbf{N}$ be such that $N\delta\varepsilon \le \operatorname{dist}(U', V \setminus U)$, and let $U_k = \{x \in U : N \operatorname{dist}(x, U') < k \operatorname{dist}(U', V \setminus U)\}$, $U_0 = U'$. For each $k = 1, \dots, N$ let ϕ_k be a cut-off function between U_{k-1} and U_k, satisfying (2.6), which exists since $\operatorname{dist}(U_{k-1}, \partial U_k) \ge \delta\varepsilon$. We have, using Remark 2.2(b), (2.4) and (2.6)

$$
\begin{aligned}
&F_\varepsilon(\phi_k u + (1-\phi_k)v, U' \cup V) \\
&= \int_{U'\cup V} f\Big(\frac{x}{\varepsilon}, \tilde\phi_k \frac{dDu}{d\mu_\varepsilon} + (1-\tilde\phi_k)\frac{dDv}{d\mu_\varepsilon} + (\tilde u - \tilde v) \otimes \frac{dD\phi_k}{d\mu_\varepsilon}\Big)\, d\mu_\varepsilon \\
&\le \int_U f\Big(\frac{x}{\varepsilon}, \frac{dDu}{d\mu_\varepsilon}\Big)\, d\mu_\varepsilon + \int_V f\Big(\frac{x}{\varepsilon}, \frac{dDv}{d\mu_\varepsilon}\Big)\, d\mu_\varepsilon \\
&\qquad + 4^p \beta \int_{(U_k \setminus U_{k-1})\cap V} \Big(1 + \Big|\frac{dDu}{d\mu_\varepsilon}\Big|^p + \Big|\frac{dDv}{d\mu_\varepsilon}\Big|^p\Big) d\mu_\varepsilon \\
&\qquad + 4^p \beta \int_{(U_k \setminus U_{k-1})\cap V} \Big|(\tilde u - \tilde v) \otimes \frac{dD\phi_k}{d\mu_\varepsilon}\Big|^p d\mu_\varepsilon \\
&\le F_\varepsilon(u,U) + F_\varepsilon(v,V) \\
&\qquad + 4^p \beta \int_{(U_k \setminus U_{k-1})\cap V} \Big(1 + \Big|\frac{dDu}{d\mu_\varepsilon}\Big|^p + \Big|\frac{dDv}{d\mu_\varepsilon}\Big|^p\Big)\, d\mu_\varepsilon \\
&\qquad + 4^p \beta \frac{KN^p}{\big(\operatorname{dist}(U', V \setminus U)\big)^p} \int_{(U_k \setminus U_{k-1})\cap V} |u-v|^p\, dx
\end{aligned}
$$

where K is the constant appearing in (2.6).

Choose k such that

$$\int_{(U_k\setminus U_{k-1})\cap V}\Big(1+\Big|\frac{dDu}{d\mu_\varepsilon}\Big|^p+\Big|\frac{dDv}{d\mu_\varepsilon}\Big|^p\Big)\,d\mu_\varepsilon$$
$$+\frac{KN^p}{\big(\mathrm{dist}\,(U',V\setminus U)\big)^p}\int_{(U_k\setminus U_{k-1})\cap V}|u-v|^p\,dx$$
$$\le\frac{1}{N}\Big(\int_{(U\cap V)\setminus U'}\Big(1+\Big|\frac{dDu}{d\mu_\varepsilon}\Big|^p+\Big|\frac{dDv}{d\mu_\varepsilon}\Big|^p\Big)\,d\mu_\varepsilon$$
$$+\frac{KN^p}{\big(\mathrm{dist}\,(U',V\setminus U)\big)^p}\int_{(U\cap V)\setminus U'}|u-v|^p\,dx\Big).$$

Then, taking into account also (2.4),

$$F_\varepsilon(\phi_k u+(1-\phi_k)v,U'\cup V)$$
$$\le F_\varepsilon(u,U)+F_\varepsilon(v,V)$$
$$+\frac{4^p\beta}{N\alpha}\Big(\int_{(U\cap V)\setminus U'}f\Big(\frac{x}{\varepsilon},\frac{dDu}{d\mu_\varepsilon}\Big)\,d\mu_\varepsilon+\int_{(U\cap V)\setminus U'}f\Big(\frac{x}{\varepsilon},\frac{dDv}{d\mu_\varepsilon}\Big)\,d\mu_\varepsilon\Big)$$
$$+4^p\beta\frac{KN^{p-1}}{\big(\mathrm{dist}\,(U',V\setminus U)\big)^p}\int_{(U\cap V)\setminus U'}|u-v|^p dx+\frac{4^p\beta}{N}\mu_\varepsilon((U\cap V)\setminus U')$$
$$\le\Big(1+\frac{4^p\beta}{N\alpha}\Big)\Big(F_\varepsilon(u,U)+F_\varepsilon(v,V)\Big)$$
$$+4^p\beta\frac{KN^{p-1}}{\big(\mathrm{dist}\,(U',V\setminus U)\big)^p}\int_{(U\cap V)\setminus U'}|u-v|^p dx+\frac{4^p\beta}{N}\mu_\varepsilon((U\cap V)\setminus U')\,.$$

We can choose ε_σ satisfying

$$\frac{4^p\beta}{\sigma\min\{1,\alpha\}}+1=\frac{1}{\delta\varepsilon_\sigma},$$

so that we can find N, depending only on σ and on the constants of the problem, in such a way that (2.12) holds, with $M=4^pK\beta N^{p-1}$. □

Proposition 2.12 *For every $A\in\mathbf{M}^{m\times n}$ there exists $z_A\in W^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^m)$ such that z_A-Ax is 1-periodic and satisfies*

$$\int_{[0,1)^n}\Big|\frac{dDz_A}{d\mu}\Big|^p d\mu\le c|A|^p. \tag{2.13}$$

PROOF. Define $z_A=\sum_{i=1}^m\sum_{j=1}^n A_{ij}z_je_i$, where z_i are as in Definition 2.4(ii). Inequality (2.13) is trivial. □

We fix an infinitesimal sequence (ε_j). We define

$$F'(u,U)=\Gamma\text{-}\liminf_{j\to+\infty}F_{\varepsilon_j}(u,U)$$

$$F''(u,U) = \Gamma\text{-}\limsup_{j\to+\infty} F_{\varepsilon_j}(u,U)$$

for all $u \in L^p(\Omega;\mathbf{R}^m)$ and for all open subsets U of Ω. By Remark 1.24(i) the Γ-upper and lower limits, $F''(\cdot,U)$ and $F'(\cdot,U)$ defined above, are $L^p(\Omega;\mathbf{R}^m)$-lower semicontinuous functionals.

Proposition 2.13. (Growth Condition) *We have*

$$F''(u,U) \le c\int_U (1+|Du|^p)dx$$

for all $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ and for all open subsets U of Ω with $|\partial U| = 0$.

Proof. *Step 1: we have $F''(Ax,U) \le c|\overline{U}|(1+|A|^p)$ for all $A \in \mathbf{M}^{m\times n}$ and for all $U \in \mathcal{A}(\Omega)$.*

Let z_A be given by Proposition 2.12. We may assume that $z_j - x_j$ has mean value 0 in the periodicity cell, so that the functions $z^\varepsilon_A(x) = \varepsilon z_A(x/\varepsilon)$ converge in $L^p_{\rm loc}(\mathbf{R}^n;\mathbf{R}^m)$ to Ax, and

$$\begin{aligned} F''(Ax,U) &\le \limsup_{\varepsilon\to 0+}\int_U f\Big(\frac{x}{\varepsilon},\frac{dDz^\varepsilon_A}{d\mu_\varepsilon}\Big)\,d\mu_\varepsilon \\ &\le \beta\limsup_{\varepsilon\to 0+}\int_U\Big(1+\Big|\frac{dDz^\varepsilon_A}{d\mu_\varepsilon}\Big|^p\Big)\,d\mu_\varepsilon \le c|\overline{U}|\,(1+|A|^p). \end{aligned}$$

Step 2: we have $F''(u,U) \le c\int_U(1+|Du|^p)dx$ for all piecewise affine function $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ and for all open subsets $U \subseteq \Omega$ with $|\partial U| = 0$.

We write $u = \sum_{i=1}^N \chi_{U_i}u_i$, where $U_1,\dots,U_N$ are disjoint open subsets of U such that $|U\setminus\bigcup_i U_i| = 0$ and $|\overline{U}_i| = |U_i|$, and $u_i(x) = A_ix + c_i$ for some $A_i \in \mathbf{M}^{m\times n}$ and $c_i \in \mathbf{R}^m$. For each i we set $u^\varepsilon_i(x) = z^\varepsilon_{A_i}(x)+c_i$, as from Step 1.

We will prove Step 2 by finite induction. First, we give an estimate on $U_1\cup U_2$. For all ε sufficiently small, we can apply Proposition 2.11 choosing the sets

$$U_2^\eta = \{x \in U:\ \mathrm{dist}\,(x,U_2) < \eta\},$$

U_2 and U_1 as the sets U, U' and V in its statement, respectively, where $\eta = \eta_\varepsilon > 0$ will be determined later, and taking $\sigma = 1$, $u = u^\varepsilon_2$ and $v = u^\varepsilon_1$. We obtain then a cut-off function $\phi = \phi_\varepsilon$ between U_2 and U_2^η such that

$$\begin{aligned} F_\varepsilon(\phi_\varepsilon u^\varepsilon_2 + (1-\phi_\varepsilon)u^\varepsilon_1, U_1\cup U_2) &\le 2(F_\varepsilon(u^\varepsilon_1,U_1) + F_\varepsilon(u^\varepsilon_2,U_2^\eta)) \\ &\quad + \frac{M}{\eta^p}\int_{U_1\cap U_2^\eta}|u^\varepsilon_2 - u^\varepsilon_1|^p\,dx + \mu_\varepsilon(U_1\cap U_2^\eta)\,. \end{aligned}$$

The constant M is the one given by Proposition 2.11 with $\sigma = 1$. We can choose now $\eta = \eta_\varepsilon$, tending to 0 as $\varepsilon \to 0$, in such a way that

$$\lim_{\varepsilon\to 0}\frac{1}{\eta_\varepsilon^p}\int_{U_1\cap U_2^{\eta_\varepsilon}}|u_2^\varepsilon-u_1^\varepsilon|^p\,dx=0\,,$$

taking into account that

$$\lim_{\varepsilon\to 0}\int_{U_1\cap U_2^\eta}|u_2^\varepsilon-u_1^\varepsilon|^p\,dx=\int_{U_1\cap U_2^\eta}|u_2-u_1|^p\,dx\le c\|Du\|_\infty^p\eta^{p+1}$$

since u_i are affine and $u_2=u_1$ on $\partial U_1\cap\partial U_2$. If we define $w_1^\varepsilon=\phi_\varepsilon u_2^\varepsilon+(1-\phi_\varepsilon)u_1^\varepsilon$, we have $w_1^\varepsilon\to u$ in $L^p(U_1\cup U_2;\mathbf{R}^m)$ and

$$\limsup_{\varepsilon\to 0}F_\varepsilon(w_1^\varepsilon,U_1\cup U_2)\le c\int_{U_1\cup U_2}(1+|Du|^p)\,dx$$

as in the proof of Step 1.

We can proceed now by induction, repeating at each step the previous argument replacing U_1 by $U_1\cup\ldots\cup U_j$, U_2 by U_{j+1}, u_1^ε by the w_j^ε constructed in the preceding step, and u_2^ε by u_{j+1}^ε.

Step 3: conclusion.

To conclude the proof it suffices to recall that $F''(\cdot,U)$ is weakly lower semicontinuous and piecewise affine functions are dense in $W^{1,p}(\Omega;\mathbf{R}^m)$. □

Proposition 2.14 *There exists a subsequence of (ε_j) (not relabeled) such that for all open subsets U of Ω there exists the Γ-limit*

$$\Gamma\text{-}\lim_{j\to+\infty}F_{\varepsilon_j}(u,U)=F(u,U)\,,$$

and there exists a function $\varphi:\mathbf{M}^{m\times n}\to\mathbf{R}$ such that

$$F(u,U)=\int_U\varphi(Du)dx$$

for all $u\in W^{1,p}(\Omega;\mathbf{R}^m)$ and $U\subset\Omega$ with $|\partial U|=0$.

Proof. The proof of this proposition can be obtained using the methods of Γ-convergence, Section 1.7, outlining the necessary modifications.

Using the compactness of Γ-convergence (see Theorem 1.22) and a diagonal procedure, we extract a subsequence (not relabeled) such that the Γ-limit

$$\Gamma\text{-}\lim_{j\to+\infty}F_{\varepsilon_j}(u,U)=F(u,U)$$

exists for all $u\in L^p(\Omega;\mathbf{R}^m)$ and for all sets U in the countable family $\mathcal{R}$ of all finite unions of open rectangles of Ω with rational vertices.

Now, observe that for all open subsets $U\subseteq\Omega$ with $|\partial U|=0$ we have

$$F''(u,U)=\sup\{F''(u,V):\ V\subset\subset U,\ V\text{open}\},$$

$$F'(u,U)=\sup\{F'(u,V):\ V\subset\subset U,\ V\text{open}\}.$$

This can be shown modifying the proof of Proposition 1.32 for functionals that satisfy the conclusions of Proposition 2.11 and Proposition 2.13.

Next, we note that the Γ-limit $F(u,U)=\Gamma\text{-}\lim_{j\to+\infty}F_{\varepsilon_j}(u,U)$ exists for all $U\in\mathcal{A}(\Omega)$ with $|\partial U|=0$, and for all $u\in W^{1,p}(\Omega;\mathbf{R}^m)$ the function $F(u,\cdot)$ is the restriction to the family these open sets of a Borel measure on Ω. This result can be obtained by Theorem 1.27 and by the De Giorgi-Letta measure criterion Theorem 1.26, noting that the proof of Proposition 1.31 can be repeated using Proposition 2.11.

Eventually, the existence of $\varphi:\mathbf{M}^{m\times n}\to\mathbf{R}$ such that

$$F(u,U)=\int_U\varphi(Du)dx$$

for all $u\in W^{1,p}(\Omega;\mathbf{R}^m)$ and for all $U\in\mathcal{A}(\Omega)$ with $|\partial U|=0$ follows from the integral representation Corollary 1.36, observing that translation invariance in x can be obtained as in Proposition 1.44.

□

Proposition 2.15. (Homogenization Formula) *For all $A\in\mathbf{M}^{m\times n}$ there exists the limit in* (2.9) *and we have $\varphi(A)=f_{\rm hom}(A)$.*

Proof. In order to simplify the proof of formula (2.9), we can suppose that $\mu([0,1)^n\setminus(0,1)^n)=0$, which holds up to a translation. For all $A\in\mathbf{M}^{m\times n}$ and $k\in\mathbf{N}$ we define

$$g_k(A)=\inf\Big\{\frac{1}{k^n}\int_{(0,k)^n}f\Big(x,\frac{dDu}{d\mu}\Big)\,d\mu: u\in W^{1,p}_{\mu,\rm loc}(\mathbf{R}^n;\mathbf{R}^m),\ u{-}Ax\ k\text{-periodic}\Big\}.$$

Fixed $A\in\mathbf{M}^{m\times n}$ let $u\in W^{1,p}_{\mu,\rm loc}(\mathbf{R}^n;\mathbf{R}^m)$ with $u-Ax$ k-periodic and with mean value 0 on $(0,k)^n$. Define the sequence $u_j(x)=\varepsilon_j\,u(x/\varepsilon_j)$, and note that $u_j\to Ax$ in $L^p_{\rm loc}(\mathbf{R}^n;\mathbf{R}^m)$. We have then

$$\varphi(A)=F(Ax,(0,1)^n)\le\liminf_{j\to+\infty}F_{\varepsilon_j}(u_j,(0,1)^n)=\frac{1}{k^n}\int_{(0,k)^n}f\Big(x,\frac{dDu}{d\mu}\Big)\,d\mu.$$

Hence, $\varphi(A)\le g_k(A)$, so that

$$\varphi(A)\le\liminf_{k\to+\infty}g_k(A). \tag{2.14}$$

Conversely, let $w_j\to Ax$ be such that

$$\varphi(A)=F(Ax,(0,1)^n)=\lim_{j\to+\infty}F_{\varepsilon_j}(w_j,(0,1)^n).$$

Let $\sigma > 0$. Let $T_j = 1/\varepsilon_j$ and let $u_j(x) = T_j w_j(x/T_j)$. We use the notation $K_j = [T_j] + 1$.

If j is large enough and $N > 4$, we can use Proposition 2.11 with $\varepsilon = 1$, $U = (0, T_j)^n$, $V = (0, K_j)^n \setminus (2T_j/N, T_j - 2(T_j/N))^n$, $U' = (T_j/N, T_j - (T_j/N))^n$, $u = u_j$, and $v = z_A$. We get then

$$\begin{aligned} & F_1(\phi u + (1-\phi)v, (0, K_j)^n) \\ &= F_1(\phi u + (1-\phi)v, U' \cup V) \\ &\le (1+\sigma)(F_1(u, U) + F_1(v, V)) \\ &\quad + MN^p T_j^{-p} \int_{(U\cap V)\setminus U'} |u - v|^p dx + \sigma\mu((U \cap V) \setminus U'). \end{aligned} \tag{2.15}$$

Since $\phi u + (1-\phi)v - Ax$ is K_j-periodic, we obtain

$$\begin{aligned} & K_j^n \, g_{K_j}(A) \\ &\le (1+\sigma)(F_1(u_j, (0, T_j)^n) + F_1(z_A, V)) \\ &\quad + MN^p T_j^{-p} \int_{(0,T_j)^n \setminus (T_j/N, T_j - (T_j/N))^n} |u_j - z_A|^p dx + \sigma\mu((U \cap V) \setminus U') \\ &\le (1+\sigma)(T_j^n F_{\varepsilon_j}(w_j, (0,1)^n) + c\frac{K_j^n}{N}(1 + |A|^p) \\ &\quad + MN^p T_j^n \int_{(0,1)^n} |w_j - z_j|^p dx + \sigma c K_j^n, \end{aligned}$$

where $z_j(x) = T_j^{-1} z_A(T_j x)$. Note that $z_j \to Ax$ in $L^p((0,1)^n; \mathbf{R}^m)$; hence

$$\lim_{j\to+\infty} \int_{(0,1)^n} |w_j - z_j|^p dx = 0\,.$$

Dividing the estimate above by K_j^n, and letting first $j \to +\infty$ and then $\sigma \to 0$ and $N \to +\infty$, we get

$$\limsup_{j\to+\infty} g_{K_j}(A) \le \varphi(A). \tag{2.16}$$

By (2.14) and (2.16) we obtain then

$$\varphi(A) = \liminf_{k\to+\infty} g_k(A) = \lim_{j\to+\infty} g_{K_j}(A).$$

The first equality shows that φ is independent of the sequence (ε_j). Repeating the reasoning then with a sequence (ε_j) such that

$$\lim_{j\to+\infty} g_{K_j}(A) = \limsup_{k\to+\infty} g_k(A)$$

the proof is complete. □

PROOF OF THEOREM 2.7. The previous propositions show that the limit in (2.7) exists and (2.8) holds with $f_{\rm hom}$ given by (2.9). Formula (2.10) in the convex case follows as in Theorem 1.46.

It remains to check that $F_{\rm hom}(u,\Omega)=+\infty$ if $u\in L^p(\Omega;\mathbf{R}^m)\setminus W^{1,p}(\Omega;\mathbf{R}^m)$ when $p>1$. Clearly, it suffices to prove this for $f(A)=|A|^p$. In this case, $F_{\rm hom}$ is convex, hence it is determined by its behaviour on $W^{1,p}(\Omega;\mathbf{R}^m)$ (see [41] Chapter 23). It will be enough then to prove that $f_{\rm hom}(A)\geq c|A|^p$. Since $f_{\rm hom}$ is positively homogeneous of degree p, it is sufficient to check that $f_{\rm hom}(A)\neq 0$ if $A\neq 0$. To this aim, let $u_\varepsilon\to Ax$ be such that $F_\varepsilon(u_\varepsilon,(0,1)^n)\to f_{\rm hom}(A)$. If $f_{\rm hom}(A)=0$ then by the Poincaré inequality for BV-functions, by Hölder's inequality and a scaling argument we obtain that $u_{\frac{1}{k}}$ tends to a constant, and a contradiction. □

2.4 Limits of a class of difference schemes

In this section we show how some energies depending on finite differences can be seen as a particular case of functionals defined on Sobolev spaces with respect to the measures introduced in Example 2.3(a). For the sake of illustration we deal only with the case of integrands independent of x. We remark that in the case of quadratic functionals (i.e., $\psi_k(\xi)=c_k\xi^2$ below), our result can be framed in the theory of difference operators elaborated by Kozlov [58], where a compactness and representation theorem is given for a general class of operators.

Let $\Omega\subseteq R^n$ be an open set with Lipschitz boundary, and let

$$I_\varepsilon=\{i\in\mathbf{Z}^n : \varepsilon i+[0,\varepsilon]^n\subseteq\Omega\}\,.$$

Let $\psi_1....\psi_n$ be convex functions such that

$$|\xi|^p\leq\psi_k(\xi)\leq c(1+|\xi|^p)$$

for all $\xi\in M^{m\times n}$ and $k=1,\dots,n$. We define A_ε the set of functions

$$u:(\mathbf{Z}^n\cap\frac{1}{\varepsilon}\Omega)\to\mathbf{R}^m$$

and for all $u\in A_\varepsilon$

$$\Psi_\varepsilon(u)=\sum_{k=1}^n\sum_{i\in I_\varepsilon}\varepsilon^n\psi_k\left(\frac{u(i+e_k)-u(i)}{\varepsilon}\right).$$

If $u\in A_\varepsilon$ then we can associate to u the piecewise constant function $v_u:\Omega\longrightarrow\mathbf{R}^m$ defined by

$$v_u(x)=\begin{cases}u(i) & x\in\varepsilon i+[0,\varepsilon)^n \quad \varepsilon i\in\Omega\cap\varepsilon\mathbf{Z}^n\\ 0 & \text{otherwise}\end{cases}.$$

Definition 2.16 *Let $u_j \in A_{\varepsilon_j}$. We say that u_j converges to $u \in L^p(\Omega)$ if and only if v_{u_j} converges to u in $L^p(\Omega)$.*

Theorem 2.17 *The functionals Ψ_ε Γ-converge as $\varepsilon \to 0$ to*

$$\Psi(u) = \begin{cases} \displaystyle\sum_{k=1}^{n} \int_\Omega \psi_k\Big(\frac{\partial u}{\partial x_k}\Big)\, dx & u \in W^{1,p}(\Omega; \mathbf{R}^m) \\ +\infty & u \in L^p(\Omega; \mathbf{R}^m) \setminus W^{1,p}(\Omega; \mathbf{R}^m) \end{cases}$$

with respect to the convergence in $L^p(\Omega)$ as in Definition 2.16.

Proof. Let $f : M^{m\times n} \longrightarrow [0, +\infty)$ be defined by

$$f(\xi) = n \sum_{k=1}^{n} \psi_k\Big(\frac{\xi_k}{n}\Big)$$

where $\xi_k = \xi e_k$. If we consider μ as in Example 2.3(a), since f is convex, by formula (2.10) it follows that

$$f_{\rm hom}(\xi) = \frac{1}{n} f(n\xi) = \sum_{k=1}^{n} \psi_k(\xi_k).$$

In fact, the computation of (2.10) is trivial, since $u(x) = \sum_{k=1}^n \xi_k[x_k]$ is the unique function $u \in W^{1,p}_{\mu,{\rm loc}}(\mathbf{R}^n; \mathbf{R}^m)$, up to translations, such that $u - \xi x$ is 1-periodic. By formula (2.8)

$$F_{\rm hom}(u, \Omega) = \begin{cases} \displaystyle\int_\Omega \sum_{k=1}^{n} \psi_k\Big(\frac{\partial u}{\partial x_k}\Big)\, dx & u \in W^{1,p}(\Omega; \mathbf{R}^m) \\ +\infty & u \in L^p(\Omega; \mathbf{R}^m) \setminus W^{1,p}(\Omega; \mathbf{R}^m) \end{cases}$$

and $F_{\rm hom}(u, \Omega) = \Psi(u)$.

For all $U \subset\subset \Omega$ open set with $|\partial U| = 0$ and $\varepsilon > 0$, let

$$F_\varepsilon(u, U) = \int_U f\Big(\frac{dDu}{d\mu_\varepsilon}\Big)\, d\mu_\varepsilon\,,$$

and let $u_j \in A_{\varepsilon_j}$ converge to $u \in L^p(\Omega)$. Then

$$\begin{aligned} \liminf_{j\to+\infty} \Psi_{\varepsilon_j}(u_j) &= \liminf_{j\to+\infty} \varepsilon_j^n \sum_{k=1}^{n} \sum_{i \in I_{\varepsilon_j}} \psi_k\left(\frac{u_j(i + e_k) - u_j(i)}{\varepsilon_j}\right) \\ &\geq \liminf_{j\to+\infty} \sum_{k=1}^{n} \varepsilon_j \int_U \psi_k\Big(\frac{1}{n}\frac{dDv_{u_j}}{d\mu_{\varepsilon_j}}\Big)\, d\mathcal{H}^{n-1} \end{aligned}$$

$$
\begin{aligned}
&= \liminf_{j\to+\infty} \int_U f\Big(\frac{dDv_{u_j}}{d\mu_{\varepsilon_j}}\Big)\, d\mu_{\varepsilon_j} \\
&= \liminf_{j\to+\infty} F_{\varepsilon_j}(v_{u_j}, U) \\
&\geq F_{\rm hom}(u, U)
\end{aligned}
$$

by formula (2.7) and the definition of Γ-convergence, so that

$$
\liminf_{j\to+\infty} \Psi_{\varepsilon_j}(u_j) \geq \sup_{U\subset\subset\Omega} F_{\rm hom}(u, U) = \Psi(u)\,.
$$

By the arbitrariness of u_j

$$
\Gamma\text{-}\liminf_{\varepsilon\to 0} \Psi_\varepsilon(u) \geq \Psi(u).
$$

Conversely, suppose that $v_j \in W^{1,p}_{\mu_{\varepsilon_j}}(\Omega;\mathbf{R}^m)$ converges to u in $L^p(\Omega)$ and define

$$
u_j(i) = \limsup_{\rho\to 0^+} -\!\!\!\!\!\!\int_{B(0,\rho)\cap[0,\varepsilon_j)^n} v_j(x - \varepsilon_j i)\, dx \tag{2.17}
$$

for all $i \in \mathbf{Z}^n \cap \frac{1}{\varepsilon}\Omega$. Note that if $i \in I_\varepsilon$ or $i - e_k \in I_\varepsilon$ for some k then the average in (2.17) is constant for ρ small enough.

By definition, u_j converges to $u \in L^p(\Omega)$ and

$$
\limsup_{j\to+\infty} \Psi_{\varepsilon_j}(u_j) \leq \limsup_{j\to+\infty} F_{\varepsilon_j}(v_j, \Omega);
$$

there follows that

$$
\Gamma\text{-}\limsup_{\varepsilon\to 0} \Psi_\varepsilon(u) \leq \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u, \Omega) = \Psi(u)\,,
$$

so that

$$
\Gamma\text{-}\lim_{\varepsilon\to 0} \Psi_\varepsilon(u) = \Psi(u)\,,
$$

and the proof is concluded. □

2.5 Appendix: Sobolev inequalities in $W^{1,p}_\mu$

In this appendix we include some results about Sobolev inequalities in the spaces $W^{1,p}_\mu$. In particular, we will prove that the measures in Example 2.3 satisfy the Poincaré inequality.

Proposition 2.18 *Let μ be the measure in Example* 2.3(b). *Then for all $1 \leq q \leq n(np-2p+1)/(n-p)(n-1)$ (for any $q \geq 1$ if $p \geq n$) and for all $k \in \mathbf{N}$ there exists a constant $C(k)$ such that for all $u \in W^{1,p}_\mu((0,k)^n)$ with $\int_{(0,k)^n} u\, dx = 0$ we have*

$$
\Big(\int_{(0,k)^n} |u|^q\, dx\Big)^{1/q} \leq C(k)\Big(\int_{(0,k)^n} \Big|\frac{dDu}{d\mu}\Big|^p\, d\mu\Big)^{1/p}. \tag{2.18}
$$

Moreover, if $q = p$ then we can take $C(k) = c\,k$ with c a fixed constant.

Proof. If $n = 1$ then (2.18) follows from the Sobolev inequality for BV functions (see Remark 2.21). We will deal only with the case $p < n$ and $q > p$, which again is not a restriction. The other cases can be derived from this by applying Hölder's inequality.

We set $U = (0,k)^n$. We start by considering an inequality involving the median of a function rather than the mean. We recall that the set of the *medians of* u (in U), $\mathrm{med}(u)$, is the set of real numbers t such that

$$|U \cap \{u > t\}| \leq \frac{1}{2}|U| \qquad \text{and} \qquad |U \cap \{u < t\}| \leq \frac{1}{2}|U|.$$

Let $u \in W^{1,p}_\mu(U)$. By the Poincaré inequality for BV functions, there exists a constant $c = c(U)$ such that for $u \in BV(U)$ and $t \in \mathrm{med}(u)$

$$\|u - t\|_{L^{n/(n-1)}(U)} \leq c|Du|(U) \tag{2.19}$$

(see [77] Theorem 5.12.10). By a scaling argument it can be easily checked that c may be chosen independent of k. From now on, we denote c any constant which satisfies this property.

Let first $q \geq np/(n-1)$, and set $v = u|u|^{r-1}$ with $r > 1$. If $0 \in \mathrm{med}(u)$ then $0 \in \mathrm{med}(v)$; hence, by (2.19),

$$\|v\|_{L^{n/(n-1)}(U)} \leq c|Dv|(U).$$

We then get, by Hölder's and Minkowski's inequalities,

$$\begin{aligned}
\Big(\int_U |u|^{rn/(n-1)}\,dx\Big)^{(n-1)/n} &\leq c\int_U |u|^{r-1}|\nabla u|\,dx \\
&\quad + c\int_{U\cap E} |u^+ - u^-|(|u^+|^{r-1} + |u^-|^{r-1})\,d\mathcal{H}^{n-1} \\
&\leq c\|\nabla u\|_p \Big(\int_U |u|^{p'(r-1)}\,dx\Big)^{1/p'} \\
&\quad + c\Big(\int_{U\cap E} |u^+ - u^-|^p\,d\mathcal{H}^{n-1}\Big)^{1/p} \\
\times \Bigg(\Big(\int_{U\cap E} |u^+|^{p'(r-1)}\,d\mathcal{H}^{n-1}\Big)^{1/p'} &+ \Big(\int_{U\cap E} |u^-|^{p'(r-1)}\,d\mathcal{H}^{n-1}\Big)^{1/p'}\Bigg).
\end{aligned}$$

Let $q = rn/(n-1)$ and $\alpha = p'(r-1)$; then we can rewrite the estimate above as

$$\begin{aligned}
\Big(\int_U |u|^q\,dx\Big)^{r/q} &\leq c\|\nabla u\|_p \Big(\int_U |u|^\alpha\,dx\Big)^{(r-1)/\alpha} \\
&\quad + c\Big(\int_{U\cap E} |u^+ - u^-|^p\,d\mathcal{H}^{n-1}\Big)^{1/p}
\end{aligned}$$

$$\times\left(\left(\int_{U\cap E}|u^+|^\alpha\,d\mathcal{H}^{n-1}\right)^{(r-1)/\alpha}+\left(\int_{U\cap E}|u^-|^\alpha\,d\mathcal{H}^{n-1}\right)^{(r-1)/\alpha}\right).$$

Interpreting $u^\pm$ as traces of Sobolev functions defined on each cube of $U\setminus E$, we have

$$\left(\int_{U\cap E}|u^\pm|^\alpha\,d\mathcal{H}^{n-1}\right)^{1/\alpha}\le c\|u\|_{W^{1,p}(U\setminus E)} \tag{2.20}$$

for $p\le\alpha\le p(n-1)/(n-p)$ (see [1] Theorem 7.58). Hence,

$$\|u\|_q^r\le c\|\nabla u\|_p\|u\|_\alpha^{r-1} +c\left(\int_{U\cap E}|u^+-u^-|^p\,d\mathcal{H}^{n-1}\right)^{1/p}\left(\|u\|_p^{r-1}+\|\nabla u\|_p^{r-1}\right).$$

Note that $\alpha<q\le n(np-2p+1)/(n-1)(n-p)$. By Hölder's inequality

$$\|u\|_\alpha^{r-1}\le\|u\|_q^{r-1}|U|^{(r-1)(\frac{1}{\alpha}-\frac{1}{q})}\qquad\text{and}\qquad\|u\|_p^{r-1}\le\|u\|_q^{r-1}|U|^{(r-1)(\frac{1}{p}-\frac{1}{q})}.$$

If we denote $c_1=|U|^{(r-1)(\frac{1}{\alpha}-\frac{1}{q})}$ and $c_2=|U|^{(r-1)(\frac{1}{p}-\frac{1}{q})}$, we get

$$\begin{aligned}\|u\|_q^r&\le c_1c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{1/p}\|u\|_q^{r-1}+c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{1/p}\\&\qquad\times\left(c_2\|u\|_q^{r-1}+\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{(r-1)/p}\right)\\&\le(c_1+c_2)c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{1/p}\|u\|_q^{r-1}+c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{r/p}.\end{aligned}\tag{2.21}$$

By Young's inequality

$$\begin{aligned}&(c_1+c_2)c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{1/p}\|u\|_q^{r-1}\\&\le\frac{1}{r}\left(\left(\frac{2(r-1)}{r}\right)^{(r-1)/r}(c_1+c_2)c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{1/p}\right)^r\\&\qquad+\frac{r-1}{r}\left(\|u\|_q^{r-1}\left(\frac{r}{2(r-1)}\right)^{(r-1)/r}\right)^{r/(r-1)}\\&=\left(\frac{2(r-1)}{r}\right)^{r-1}\frac{((c_1+c_2)c)^r}{r}\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{r/p}+\frac{1}{2}\|u\|_q^r,\end{aligned}$$

so that, by (2.21),

$$\|u\|_q\le c_4c\left(\int_U\Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\right)^{1/p},$$

where $c_4 = 1+c_1+c_2$. In particular, we have that, for a general u and $t \in \operatorname{med}(u)$,

$$\|u-t\|_q \le c_4 c\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p \,d\mu\Big)^{1/p}. \tag{2.22}$$

By Minkowski's inequality and (2.22)

$$\begin{aligned}\|u\|_q &\le \|u-t\|_q + |t||U|^{1/q} \\ &\le c_4 c\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p \,d\mu\Big)^{1/p} + |t||U|^{1/q}.\end{aligned} \tag{2.23}$$

Suppose in addition that $\int_U u\,dx = 0$. We then can estimate

$$\begin{aligned}|t| = |\fint_U u\,dx - t| &\le \fint_U |u-t|\,dx \le \Big(\fint_U |u-t|^{n/(n-1)}\,dx\Big)^{(n-1)/n} \\ &\le \frac{c}{|U|^{(n-1)/n}}\int_U \Big|\frac{dDu}{d\mu}\Big|\,d\mu \le c\frac{|U|^{1/p'}}{|U|^{(n-1)/n}}\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{1/p} \\ &= c|U|^{(p-n)/np}\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{1/p},\end{aligned}$$

by (2.19) and Jensen's and Hölder's inequalities. Finally, by (2.23),

$$\Big(\int_U |u|^q\,dx\Big)^{1/q} \le c\Big(c_4 + |U|^{1/q+(p-n)/np}\Big)\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{1/p}.$$

To conclude the proof set

$$\begin{aligned}C(k) &= c\Big(c_4 + |U|^{1/q+(p-n)/np}\Big) \\ &= c\Big(1 + k^{n(r-1)(\frac{1}{\alpha}-\frac{1}{q})} + k^{n(r-1)(\frac{1}{p}-\frac{1}{q})} + k^{n/q+(p-n)/p}\Big).\end{aligned} \tag{2.24}$$

In particular if $q = np/(n-1)$ we have $\alpha = r = p$ and $C(k) = c(1+3k^{(p-1)/p})$. If $q < np/(n-1)$ an application of Hölder's inequality yields that we can take $C(k) = ck^{((p-n)/p+(n/q))}$. We obtain the last statement of the proposition when $p = q$. □

Remark 2.19 The previous proposition proves the Sobolev inequalities for the measures μ in Example 2.3, in particular the last statement proves the Poincaré inequality. In fact, the Sobolev inequalities, and hence also the Poincaré inequality, for the measures in Example 2.3(a) are a particular case of those for the measures in Example 2.3(b).

Remark 2.20 Proposition 2.18 can be proved for measures of the more general form

$$\mu(B) = \frac{1}{1+\mathcal{H}^{n-1}(E\cap[0,1)^n)}(|B| + \mathcal{H}^{n-1}(B\cap E)),$$

provided that E is a 1-periodic closed set of σ-finite $n-1$-dimensional Hausdorff measure and that $[0,1]^n \setminus E$ has a finite number of connected component, each one with a Lipschitz boundary. The proof follows the same line, remarking that the particular form of E was used only in (2.20).

Remark 2.21 The validity of a Sobolev inequality for a general μ depends on the measure μ itself and p. In particular it always holds if $n = 1$ for all p and q, or if $p < n/(n-1)$ with $q = n/(n-1)$. In fact, in this case, by the Sobolev inequality for BV-functions and Hölder's inequality

$$\begin{aligned}\Big(\int_U |u|^{n/(n-1)}\,dx\Big)^{(n-1)/n} &\le c|Du|(U) = c\int_U \Big|\frac{dDu}{d\mu}\Big|\,d\mu \\ &\le c\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p\,d\mu\Big)^{1/p}\mu(U)^{(p-1)/p}.\end{aligned}$$

Conversely, if $q > p \ge n/(n-1)$, take a 1-periodic function $u \in (BV_{\rm loc}(\mathbf{R}^n) \cap L^p((0,1)^n) \setminus \mathrm{L}^q((0,1)^n))$, and set $\mu = |Du|$. Clearly $|dDu/d\mu| = 1$, so that $u \in W^{1,p}_\mu(U)$ for all subsets U of $\mathbf{R}^n$, but we have $\int_U |u|^q\,dx = +\infty$ for each U sufficiently large.

3

HOMOGENIZATION OF PERIODIC MULTI-DIMENSIONAL STRUCTURES: THE LINEARLY ELASTIC/PERFECTLY PLASTIC CASE

3.1 The space $LD^p_\lambda(\Omega)$

In this section we define the analog of $W^{1,p}_\lambda(\Omega;\mathbf{R}^n)$ (see Definition 2.1) when the gradient is replaced by the linearized strain tensor.

Definition 3.1 *Let λ be a finite Borel positive measure on the open set $\Omega\subset\mathbf{R}^n$, and let $1\le p\le+\infty$. We define the space*

$$LD^p_\lambda(\Omega)=\Big\{u\in L^p(\Omega;\mathbf{R}^n):\ u\in BD(\Omega),\ Eu<<\lambda,\ \frac{dEu}{d\lambda}\in L^p_\lambda(\Omega;\mathbf{M}^{n\times n}_{sym})\Big\}.$$

We will use the notation $LD_\lambda(\Omega)$ instead of $LD^1_\lambda(\Omega)$.

Proposition 3.2 (i) *The spaces $LD^p_\lambda(\Omega)$ and $LD^p_\mu(\Omega)$ coincide whenever $|\lambda-\mu|(\Omega\setminus B)=0$ for some $\mathcal{H}^{n-1}$-negligible Borel subset B of Ω.*
(ii) *The measure λ in Definition* 3.1 *can always be assumed concentrated on a Borel set where its $(n-1)$-dimensional upper density is finite.*

Proof. Point (i) easily follows from the fact that BD functions do not charge $\mathcal{H}^{n-1}$-negligible sets (see Remark 1.60). Point (ii) follows from (i) since if we consider

$$E=\Big\{x\in\Omega\ :\ \limsup_{\rho\to0}\frac{\lambda(B_\rho(x))}{\rho^{n-1}}=+\infty\Big\}$$

from covering theorems (see e.g. [77]) we have $\mathcal{H}^{n-1}(E)=0$; hence if we set

$$\mu=\lambda\llcorner\Omega\setminus E$$

by (i) we have $LD^p_\lambda(\Omega)=LD^p_\mu(\Omega)$. □

In the following proposition we prove a Leibniz-type formula for the densities with respect to a measure λ. This formula will be used in the proof of the fundamental estimate, Proposition 3.11.

Proposition 3.3 *If $u\in LD^p_\lambda(\Omega)$, $v\in W^{1,\infty}_\lambda(\Omega)$ and $\tilde u\odot\frac{dDv}{d\lambda}\in L^1_\lambda(\Omega;\mathbf{M}^{n\times n})$ then $uv\in LD^p_\lambda(\Omega)$, and*

$$\frac{dE(uv)}{d\lambda}=\tilde v\frac{dEu}{d\lambda}+\tilde u\odot\frac{dDv}{d\lambda}. \tag{3.1}$$

PROOF. By definition, functions in LD^p_λ have bounded deformation. Using the characterization of the spaces $BV(\Omega)$ and $BD(\Omega)$ by means of one-dimensional sections (see Theorem 1.54 and Proposition 1.59) we have

$$u^\xi_y \in BV(\Omega^\xi_y), \qquad v_{y,\xi} \in BV(\Omega^\xi_y) \qquad \mathcal{H}^{n-1}\text{-}a.e. \quad y \in \Omega^\xi$$

where

$$u^\xi_y(t) = u^\xi(y+t\xi) = (u(y+t\xi),\xi), \qquad v_{y,\xi}(t) = v(y+t\xi) \qquad \forall t \in \Omega^\xi_y.$$

Hence by the chain rule formula for BV functions (see Theorem 1.53) we have

$$(uv)^\xi_y = u^\xi_y\, v_{y,\xi} \in BV(\Omega^\xi_y)$$

and

$$D(u^\xi_y v_{y,\xi}) = \tilde{v}_{y,\xi} D u^\xi_y + \tilde{u}^\xi_y D v_{y,\xi} \qquad \mathcal{H}^{n-1} - a.e. \quad y \in \Omega^\xi.$$

By Theorem 1.54 and Proposition 1.59, we can prove that $uv \in BD(\Omega)$ and

$$(Euv\xi,\xi) = (\tilde{v}Eu\xi,\xi) + (\tilde{u} \odot Dv\xi,\xi) \qquad \forall \xi \in \mathbf{R}^n.$$

By choosing $\xi = \xi_i + \xi_j$, where $\xi_1,\ldots,\xi_n$ is a basis of $\mathbf{R}^n$, we get

$$E(uv) = \tilde{v}Eu + \tilde{u} \odot Dv. \tag{3.2}$$

Since the measures in the left hand-side of (3.2) are absolutely continuous with respect to λ with densities in $L^p_\lambda(\Omega;\mathbf{M}^{n\times n}_{sym})$, we finally get $uv \in LD^p_\lambda(\Omega)$ and (3.1) is proved. □

Remark 3.4 Note that in (3.1) it is necessary to consider the precise representatives of u and v, since the measure λ may take into account also sets of zero Lebesgue measure.

3.2 Choice of the measure and some examples

Let μ be a non-zero positive Radon measure on $\mathbf{R}^n$ which is 1-periodic; i.e.,

$$\mu(B+e_i) = \mu(B)$$

for all Borel subsets B of $\mathbf{R}^n$ and for all $i = 1,\ldots,n$. We will assume the normalization

$$\mu([0,1)^n) = 1\,. \tag{3.3}$$

For all $\varepsilon > 0$ we define the ε-periodic positive Radon measure μ_ε by

$$\mu_\varepsilon(B) = \varepsilon^n\, \mu\Big(\frac{1}{\varepsilon}B\Big) \tag{3.4}$$

for all Borel sets B. Note that by (3.3) the family (μ_ε) converges locally weakly* in the sense of measures to the Lebesgue measure as $\varepsilon \to 0$.

In the sequel $f : \mathbf{R}^n \times \mathbf{M}^{n\times n} \to [0,+\infty)$ will be a fixed Borel function 1-periodic in the first variable and satisfying the growth condition of order $p \geq 1$: there exist $0 < \alpha \leq \beta$ such that

$$\alpha|A|^p \leq f(x,A) \leq \beta(1+|A|^p) \tag{3.5}$$

for all $x \in \mathbf{R}^n$ and $A \in \mathbf{M}^{n\times n}$.

For every bounded open set Ω, we define the functionals at scale $\varepsilon > 0$ as

$$F_\varepsilon(u,\Omega) = \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}\Big) d\mu_\varepsilon & \text{if } u \in LD^p_{\mu_\varepsilon}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

Now we consider some additional assumptions on the measure μ, in order to prove the existence and the integral representation of the Γ-limit of the functionals F_ε as $\varepsilon \to 0$. In the sequel we will point out that these conditions are necessary and sufficient.

We assume:

(i) (*existence of cut-off functions)* there exist $K > 0$ and $\delta > 0$ such that for all $\varepsilon > 0$, for all pairs U, V of open subsets of $\mathbf{R}^n$ with $U \subset\subset V$, and $\operatorname{dist}(U,\partial V) \geq \delta\varepsilon$, and for all $u \in LD^p_{\mu_\varepsilon}(V)$ there exists $\phi \in W^{1,\infty}_{\mu_\varepsilon}(V)$ with $0 \leq \phi \leq 1$, $\phi = 1$ on U, $\phi = 0$ in a neighbourhood of ∂V, such that

$$\int_V \Big|\frac{dD\phi}{d\mu_\varepsilon} \odot \tilde{u}\Big|^p d\mu_\varepsilon \leq \frac{K}{\big(\operatorname{dist}(U,\partial V)\big)^p} \int_{V\setminus U} |u|^p \, dx \,. \tag{3.7}$$

Such a ϕ will be called a *cut-off function between* U and V;

(ii) (*existence of periodic test-functions*) for all $i, j = 1,\dots,n$, there exists $z_{ij} \in LD^p_{\mu,\mathrm{loc}}(\mathbf{R}^n)$ such that $x \mapsto z_{ij}(x) - x_j e_i$ is 1-periodic.

Remark 3.5 Note that if μ is p-homogenizable in the sense of Definition 2.4; *i.e.,* if there exists $z_i \in W^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n)$ such that $x \mapsto z_i(x) - x_i$ is 1-periodic, then the functions $z_{ij} = z_j e_i$ trivially satisfy the condition (ii) above but the converse is not true.

Remark 3.6 Note that the Lebesgue measure trivially satisfies properties (i), (ii). Note that property (i) depends on μ and p.

We consider in our context the measure μ of Examples 2.3(a) and (b).

Example 3.7 (*Perfectly-rigid bodies connected by springs.*)
We consider

$$E = \{y \in \mathbf{R}^n : \ \exists i \in \{1,\dots,n\} \text{ such that } \ y_i \in \mathbf{Z}\},$$

that is, the union of all the boundaries of cubes $Q_i = i + (0,1)^n$ with $i \in \mathbf{Z}^n$. E is an $(n-1)$-dimensional set in $\mathbf{R}^n$. We set

$$\mu(B) = \frac{1}{n}\mathcal{H}^{n-1}(B \cap E)$$

for all Borel sets B. For every $\varepsilon > 0$ we have

$$\mu_\varepsilon(B) = \frac{1}{n}\varepsilon\mathcal{H}^{n-1}(B \cap \varepsilon E)\,.$$

If $u \in LD^p_{\mu_\varepsilon}(\Omega)$ then $Eu = 0$ on every connected component of each $\varepsilon Q_i \cap \Omega$, so in this case $LD^p_{\mu_\varepsilon}(\Omega)$ consists of functions which are rigid displacements on these sets; *i.e.*, $u_i = R_i x + c_i$ on each $\varepsilon Q_i \cap \Omega$ with R_i a $n \times n$ skew symmetric matrix, and $c_i \in \mathbf{R}^n$. Hence by Theorem 1.58(2), we have

$$\frac{dEu}{d\mu_\varepsilon} = \frac{n}{\varepsilon}\frac{dEu}{d\mathcal{H}^{n-1}} = \frac{n}{\varepsilon}(u_i - u_j) \odot (i-j) \text{ on } \partial(\varepsilon Q_i) \cap \partial(\varepsilon Q_j) \cap \Omega\,.$$

In this case the functionals F_ε take the form

$$\varepsilon \int_{\Omega \cap \varepsilon E} g\Big(\frac{x}{\varepsilon}, \frac{1}{\varepsilon}\frac{dEu}{d\mathcal{H}^{n-1}}\Big) d\mathcal{H}^{n-1}.$$

Note that if Ω is bounded then $LD^p_{\mu_\varepsilon}(\Omega) = \mathrm{LD}^\infty_{\mu_\varepsilon}(\Omega)$ for all p if the number of connected components of each $\Omega \cap \varepsilon Q_i$ is finite.

Comparing with Example 2.3(a), we get that $W^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^n) \subset LD^p_{\mu_\varepsilon}(\Omega)$.

The measure μ satisfies the conditions (i) and (ii) for all $p \geq 1$. In fact, to prove (i) we consider the same cut-off function in Example 2.6(a)

$$\phi(x) = 1 - \Big(\frac{1}{C}\Big[\frac{1}{\varepsilon}\inf\{|x-y|_\infty : y \in U_\varepsilon\}\Big] \wedge 1\Big),$$

where fixed $\varepsilon > 0$, $U_\varepsilon = \bigcup\{\varepsilon Q_i : \ \varepsilon Q_i \cap U \neq \emptyset\}$, $|x-y|_\infty = \max_{1\leq i\leq n}|x_i - y_i|$, and

$$C = \Big[\frac{1}{\varepsilon}\inf\Big\{|x-y|_\infty : x \in U_\varepsilon,\ y \in \partial V\Big\}\Big] - 2\,.$$

Note that $|dD\phi/d\mu_\varepsilon| \leq n/(C\varepsilon) \leq c/\mathrm{dist}\,(U, \partial V)$ for some constant c independent of U and V.

Interpreting $u^\pm$ as traces of Sobolev functions defined on each cube Q_i, we have

$$\Big(\int_{\partial Q_i} |u^\pm|^p d\mathcal{H}^{n-1}\Big)^{1/p} \leq c\|u\|_{W^{1,p}(Q_i)},$$

hence by a scaling argument and by Korn's inequality (1.55)

$$\Big(\varepsilon\int_{\partial\varepsilon Q_i} |u^\pm|^p d\mathcal{H}^{n-1}\Big)^{1/p} \leq c\Big(\int_{\varepsilon Q_i}|u|^p\,dx\Big)^{1/p} + \varepsilon\Big(\int_{\varepsilon Q_i}|Eu|^p\,dx\Big)^{1/p}$$

$$= c\left(\int_{\varepsilon Q_i} |u|^p\,dx\right)^{1/p}$$

where c depends only on the cube. If $p = 1$ we can apply the trace inequality in $LD(Q_i)$

$$\int_{\partial Q_i} |u^{\pm}| d\mathcal{H}^{n-1} \le c\int_{Q_i} |u|\,dx + |Eu|(Q_i)\,,$$

so we get

$$\varepsilon\int_{\partial\varepsilon Q_i} |u^{\pm}| d\mathcal{H}^{n-1} \le c\int_{\varepsilon Q_i} |u|\,dx.$$

Hence for all $p \ge 1$

$$\varepsilon\int_{\partial\varepsilon Q_i} |u^{\pm}|^p\,d\mathcal{H}^{n-1} \le c\int_{\varepsilon Q_i} |u|^p\,dx.$$

For two cubes

$$\varepsilon\int_{\partial\varepsilon Q_i\cap\partial\varepsilon Q_j} |\widetilde{u}|^p\,d\mathcal{H}^{n-1} \le \varepsilon\int_{\partial\varepsilon Q_i\cap\partial\varepsilon Q_j} (|u_i|^p + |u_j|^p)\,d\mathcal{H}^{n-1} \le c\int_{\varepsilon Q_i\cup\varepsilon Q_j} |u|^p\,dx$$

so that

$$\begin{aligned}\int_V \left|\frac{dD\phi}{d\mu_\varepsilon}\odot\widetilde{u}\right|^p d\mu_\varepsilon &\le \frac{c^p\varepsilon}{\mathrm{dist}\,(U,\partial V)^p}\int_{(V\setminus U)\cap\varepsilon E\cap\mathrm{spt}D\phi} |\widetilde{u}|^p\,d\mathcal{H}^{n-1}\\ &\le 2n\frac{c^p}{\mathrm{dist}\,(U,\partial V)^p}\int_{V\setminus U} |u|^p\,dx\,.\end{aligned}$$

The proof of (i) is then complete. To verify (ii) we apply Remark 3.5 to Example 2.6(a) and take simply $z_{ij}(x) = [x_j]e_i$.

Example 3.8 (*Elastic media connected by springs*).
Let E be as in the previous example and let

$$\begin{aligned}\mu(B) &= \frac{1}{n+1}\Big(|B| + \mathcal{H}^{n-1}(E\cap B)\Big)\\ \mu_\varepsilon(B) &= \frac{1}{n+1}\Big(|B| + \varepsilon\mathcal{H}^{n-1}((\varepsilon E)\cap B)\Big)\,.\end{aligned}$$

In this case the functions in $LD^p_{\mu_\varepsilon}(\Omega)$ are functions whose restriction to each $\varepsilon Q_i\cap\Omega$ belongs to $W^{1,p}(\varepsilon Q_i\cap\Omega;\mathbf{R}^n)$ when $p > 1$ by the Korn's inequality (1.55) (we suppose that $\varepsilon Q_i\cap\Omega$ has a locally Lipschitz boundary) and to $LD(\varepsilon Q_i\cap\Omega)$ when $p = 1$, while the difference of the traces on both sides of $\partial(\varepsilon Q_i)\cap\partial(\varepsilon Q_j)\cap\Omega$ is p-summable for every $i, j \in \mathbf{Z}^n$. Hence if we compare our case with Example 2.3(b), we can conclude that $W^{1,p}_{\mu_\varepsilon}(\Omega;\mathbf{R}^n) = LD^p_{\mu_\varepsilon}(\Omega)$ if $p > 1$ and $W^{1,1}_{\mu_\varepsilon}(\Omega;\mathbf{R}^n) \subset LD_{\mu_\varepsilon}(\Omega)$ if $p = 1$. The functionals F_ε take the form

$$\frac{1}{n+1}\int_\Omega f\Big(\frac{x}{\varepsilon},\frac{dEu}{dx}\Big)\,dx + \varepsilon\int_{\Omega\cap\varepsilon E} g\Big(\frac{x}{\varepsilon},\frac{1}{\varepsilon}\frac{dEu}{d\mathcal{H}^{n-1}}\Big)d\mathcal{H}^{n-1}.$$

The measure μ satisfies conditions (i) and (ii) for all $p \geq 1$ by Example 2.6(b).

3.3 The homogenization theorem

The homogenization theorem for the functionals in (3.6) takes the following form.

Theorem 3.9 *Let μ be a measure which satisfies conditions* (i) *and* (ii) *in Section* 3.2*, and for every bounded open subset Ω of $\mathbf{R}^n$ let $F_\varepsilon(\cdot,\Omega)$ be defined on $L^p(\Omega;\mathbf{R}^n)$ by* (3.6)*. Then the Γ-limit with respect to the $L^p(\Omega;\mathbf{R}^n)$-convergence*

$$F_{\rm hom}(u,\Omega) = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u,\Omega) \tag{3.8}$$

exists for all bounded open subsets Ω with Lipschitz boundary and for all $u \in L^p(\Omega;\mathbf{R}^n)$; it can be represented on $W^{1,p}(\Omega;\mathbf{R}^n)$ for $p \geq 1$ as

$$F_{\rm hom}(u,\Omega) = \int_\Omega f_{\rm hom}(Eu)\,dx\,, \tag{3.9}$$

where the homogenized integrand satisfies the asymptotic formula

$$f_{\rm hom}(A) = \lim_{k\to+\infty}\inf\Big\{\frac{1}{k^n}\int_{[0,k)^n} f\Big(x,\frac{dEu}{d\mu}\Big)d\mu : \\ u \in LD^p_{\mu,{\rm loc}}(\mathbf{R}^n),\ u - Ax \ \ k\text{-periodic}\Big\} \tag{3.10}$$

for all $A \in \mathbf{M}^{n\times n}_{sym}$.

Moreover, $F_{\rm hom}(u,\Omega) = +\infty$ if $p > 1$ and $u \in L^p(\Omega;\mathbf{R}^n)\setminus W^{1,p}(\Omega;\mathbf{R}^n)$, or if $u \in L^1(\Omega;\mathbf{R}^n)\setminus BD(\Omega)$ when $p = 1$.

Furthermore, if f is convex then the Γ-limit can be represented as

$$F_{\rm hom}(u,\Omega) = \int_\Omega f_{\rm hom}(\mathcal{E}u)\,dx + \int_\Omega f^\infty_{\rm hom}\Big(\frac{dEu^s}{d|Eu^s|}\Big)d|Eu^s|$$

for all $u \in BD(\Omega)$ when $p = 1$.

Remark 3.10 Note that we cannot replace the sets $[0,k)^n$ by the sets $(0,k)^n$ if $\mu([0,k)^n\setminus(0,k)^n) \neq 0$, see Remark 2.8.

Same examples and considerations of Remarks 2.9 and 2.10, applied to our case, show that condition (ii) for the measure μ and the assumption that Ω has a Lipschitz boundary are necessary to get a homogenization theorem. In fact, if condition (ii) fails then $f_{\rm hom}(A) = +\infty$ if $A \neq 0$; while if Ω does not have Lipschitz boundary then the equality (3.9) may not hold.

The following proposition is a usual tool to prove the existence of the Γ-limit and its integral representation (see Section 1.7).

Proposition 3.11. (Fundamental Estimate) *For every $\sigma > 0$ there exists ε_σ and $M > 0$ such that for all U, U', V open subsets of Ω with $U' \subset U$ and $\operatorname{dist}(U', V \setminus U) > 0$, for all $\varepsilon < \varepsilon_\sigma \operatorname{dist}(U', V \setminus U)$ and for all $u \in LD^p_{\mu_\varepsilon}(\Omega)$, $v \in LD^p_{\mu_\varepsilon}(\Omega)$ there exists a cut-off function between U' and U, $\phi \in W^{1,\infty}_{\mu_\varepsilon}(U \cup V)$, such that*

$$F_\varepsilon(\phi u + (1-\phi)v, U' \cup V) \le (1+\sigma)(F_\varepsilon(u,U) + F_\varepsilon(v,V)) \qquad (3.11)$$
$$+ \frac{M}{\big(\operatorname{dist}(U', V \setminus U)\big)^p} \int_{(U\cap V)\setminus U'} |u-v|^p \, dx + \sigma \mu_\varepsilon((U \cap V) \setminus U').$$

Proof. By taking (3.1) and condition (i) into account, the proof follows exactly that of Proposition 2.11.

Proposition 3.12 *For every $A \in \mathbf{M}^{n\times n}_{sym}$ there exists $z_A \in LD^p_{\mu,\mathrm{loc}}(\mathbf{R}^n)$ such that $z_A - Ax$ is 1-periodic and satisfies*

$$\int_{[0,1)^n} \Big| \frac{dEz_A}{d\mu} \Big|^p d\mu \le c|A|^p. \qquad (3.12)$$

Proof. Define $z_A = \sum_{i,j=1}^n A_{ij} z_{ij}$, where z_{ij} are as in condition (ii). Inequality (3.12) is then trivial. □

We fix (ε_j) which goes to zero. We define

$$F'(u,U) = \Gamma\text{-}\liminf_{j\to+\infty} F_{\varepsilon_j}(u,U)$$

$$F''(u,U) = \Gamma\text{-}\limsup_{j\to+\infty} F_{\varepsilon_j}(u,U)$$

for all $u \in L^p(\Omega; \mathbf{R}^n)$ and for all open subsets U of Ω.

Proposition 3.13. (Growth Condition) *We have for all open subsets U of Ω with $|\partial U| = 0$*

$$F''(u,U) \le c \int_U (1 + |Eu|^p) \, dx$$

for all $u \in W^{1,p}(\Omega; \mathbf{R}^n)$ if $p > 1$ and

$$F''(u,U) \le c(|U| + |Eu|(U))$$

for all $u \in BD(\Omega)$ if $p = 1$.

Proof. This Growth Conditions can be obtained modifying the proof of Proposition 2.13. In particular in Step 2 therein now we have to consider the affine functions $u_i(x) = A_i x + c_i$ for some $A_i \in \mathbf{M}^{n\times n}_{sym}$ and $c_i \in \mathbf{R}^n$, in Step 3 we just have to note that piecewise affine functions are dense in BD endowed with the intermediate topology (1.53) (see Theorem 1.63). □

Proposition 3.14 *There exists a subsequence of* (ε_j) *(not relabeled) such that for all open subsets* U *of* Ω *with* $|\partial U| = 0$ *there exists the* Γ*-limit*

$$\Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u,U) = F(u,U)\,,$$

for all $u \in W^{1,p}(\Omega;\mathbf{R}^n)$ *if* $p > 1$ *and for all* $u \in BD(\Omega)$ *if* $p = 1$. *There exists a function* $\varphi : \mathbf{M}^{n\times n} \to \mathbf{R}$ *such that*

$$F(u,U) = \int_U \varphi(Eu)\,dx$$

for all $u \in W^{1,p}(\Omega;\mathbf{R}^n)$ *if* $p \geq 1$*; moreover if* f *is convex*

$$F(u,U) = \int_U \varphi\Big(\mathcal{E}u\Big)\,dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big)\,d|E^s u|$$

for all $u \in BD(\Omega)$ *if* $p = 1$.

Proof. To prove the existence of the Γ-limit on $W^{1,p}(\Omega;\mathbf{R}^n)$ for $p > 1$ and $BD(\Omega)$ for $p = 1$, and the integral representation of the Γ-limit

$$F(u,U) = \int_U \varphi(Du)\,dx$$

on $W^{1,p}(\Omega;\mathbf{R}^n)$ when $p \geq 1$, we repeat the proof of Proposition 2.14 using Propositions 3.11 and 3.13. Moreover, we can prove that $\varphi(Du) = \varphi(Eu)$. In fact, let $w_j \to Ax$ be such that

$$F(Ax,\Omega) = \lim_{j\to+\infty} F_{\varepsilon_j}(w_j,\Omega)$$

and let $Rx + c$ be a rigid displacement, then

$$\begin{aligned} F(Ax + Rx + c,\Omega) &\leq \liminf_{j\to+\infty} F_{\varepsilon_j}(w_j + Rx + c,\Omega) \\ &= \lim_{j\to+\infty} F_{\varepsilon_j}(w_j,\Omega) = F(Ax,\Omega) \end{aligned}$$

so that $\varphi(A + R) \leq \varphi(A)$. The reverse inequality follows similarly, therefore for all R $(n \times n)$ skew-symmetric matrix

$$\varphi(A + R) = \varphi(A)$$

which implies $\varphi(B) = \varphi(\frac{B+B^T}{2})$ for any $B \in \mathbf{M}^{n\times n}$.

Let us prove the integral representation of the Γ-limit $F(u,\Omega)$ on $BD(\Omega)$ whenever f is convex. We consider the functional defined on $L^1(\Omega;\mathbf{R}^n)$

$$G(u,\Omega)=\begin{cases}\displaystyle\int_\Omega \varphi(Eu)\,dx & \text{if } u\in W^{1,1}(\Omega;\mathbf{R}^n)\\ +\infty & \text{otherwise;}\end{cases}$$

note that $G(u,\Omega)=F(u,\Omega)$ on $W^{1,1}(\Omega;\mathbf{R}^n)$ and that φ is convex and $\alpha|A|\le\varphi(A)\le\beta(1+|A|)$ for every $A\in\mathbf{M}^{n\times n}_{sym}$. We introduce

$$\overline{G}(u,U)=\inf\Big\{\liminf_{h\to+\infty}G(u_h,U)\ :\ u_h\to u\quad\text{in}\quad L^1(\Omega;\mathbf{R}^n)\Big\}$$

the relaxed functional of G (see (1.4) and (1.5)).

For every $u\in BD(\Omega)$, by Definition 1.55, Eu is a bounded $\mathbf{M}^{n\times n}$-valued Radon measure; hence if we define

$$\varphi Eu(U)=\int_U\varphi(\mathcal{E}u)\,dx+\int_U\varphi^\infty\Big(\frac{dE^su}{d|E^su|}\Big)d|E^su| \tag{3.13}$$

then $Eu\mapsto\varphi Eu$ takes its values in $\mathcal{M}_+(\Omega)$ and it is l.s.c. with respect to the locally weak*-convergence of measures (see [54] Theorem 3 and Section 3).

By the growth condition from below of φ and the lower semicontinuity of φEu we get, for fixed Ω, that $u\mapsto\varphi Eu(\Omega)$ is $L^1(\Omega;\mathbf{R}^n)$-l.s.c. on $BD(\Omega)$. Hence if we define

$$\Phi(u,\Omega)=\begin{cases}\varphi Eu(\Omega) & \text{if } u\in BD(\Omega)\\ +\infty & \text{otherwise,}\end{cases}$$

for all $u\in L^1(\Omega;\mathbf{R}^n)$, we have that $\Phi(u,\Omega)\le G(u,\Omega)$ on $L^1(\Omega;\mathbf{R}^n)$ which implies, by (1.4), that $\Phi(u,\Omega)\le\overline{G}(u,\Omega)$ and

$$\varphi Eu(\Omega)\le\overline{G}(u,\Omega) \tag{3.14}$$

for all $u\in BD(\Omega)$.

To prove the other inequality we use the approximation property of φEu by convolution (see [54] Theorem 4 and Theroem 4'); *i.e.*, we consider $\Omega_k=\{x\in\Omega: d(x,\partial\Omega)>\frac{1}{k}\}$, ρ_k with $\operatorname{spt}\rho_k\subset B(0,\frac{1}{k})$ and $u_k=u*\rho_k$ then

$$\varphi Eu(\Omega)=\lim_{k\to+\infty}\int_{\Omega_k}\varphi(Eu_k)\,dx\,. \tag{3.15}$$

Fix $\Omega'\subset\subset\Omega$, for k large enough $\Omega'\subset\Omega_k$ hence by (3.15) and (1.5)

$$\varphi Eu(\Omega)\ge\liminf_{k\to+\infty}\int_{\Omega'}\varphi(Eu_k)\,dx$$

$$\geq \overline{G}(u,\Omega')$$

for all $\Omega' \subset\subset \Omega$; hence by convexity and growth condition of φ we get

$$\varphi Eu(\Omega) \geq \overline{G}(u,\Omega) \tag{3.16}$$

for every $u \in BD(\Omega)$. By (3.13), (3.14) and (3.16) we obtain

$$\overline{G}(u,U) = \int_U \varphi(\mathcal{E}u)\,dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big) d|E^s u| \tag{3.17}$$

for every $u \in BD(\Omega)$.

Since $F(\cdot,U) \leq G(\cdot,U)$ in $BD(\Omega)$, by the $L^1(\Omega;\mathbf{R}^n)$-lower semicontinuity of the Γ-limit, (1.4) and (3.17) we obtain

$$F(u,U) \leq \int_U \varphi(\mathcal{E}u)\,dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big) d|E^s u| \tag{3.18}$$

for all $u \in BD(\Omega)$. The reverse inequality is obtained by a convolution argument. In fact we define as above U_k, ρ_k and u_k. For $y \in B(0,\frac{1}{k})$ and k large enough we have that $U_k \subset y + U$.

Since $F(\cdot,U)$ is convex on $BD(\Omega)$ for all $U \in \mathcal{A}(\Omega)$ and $F(u^y,U_k) \leq F(u,U)$ with $u^y(x) = u(x-y)$, by Jensen's inequality

$$F(u * \rho_k, U_k) \leq F(u,U) \tag{3.19}$$

for every $u \in BD(\Omega)$ (see proof of Theorem 1.47 Step 1).

On the other hand, by (3.15) and (3.13) we have that

$$\lim_{k\to+\infty} F(u_k,U_k) = \int_U \varphi(\mathcal{E}u)\,dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big) d|E^s u|$$

hence by (3.19) we have

$$\int_U \varphi(\mathcal{E}u)\,dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big) d|E^s u| \leq F(u,U) \tag{3.20}$$

for all $u \in BD(\Omega)$.

By (3.18) and (3.20) we can conclude that

$$F(u,U) = \int_U \varphi(\mathcal{E}u)\,dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big) d|E^s u|$$

as desired. □

Proposition 3.15. (Homogenization Formula) *For all $A \in \mathbf{M}^{n\times n}_{sym}$ there exists the limit in* (3.10) *and we have* $\varphi(A) = f_{\hom}(A)$.

PROOF. It can be obtain repeating the proof of the Proposition 2.15 but defining

$$g_k(A) = \inf\Big\{\frac{1}{k^n}\int_{(0,k)^n} f(x, \frac{dEu}{d\mu})d\mu : u \in LD^p_{\mu,\mathrm{loc}}(\mathbf{R}^n),\ u - Ax\ \ k\text{-periodic}\Big\}$$

for all $A \in \mathbf{M}^{n\times n}_{sym}$ and $k \in \mathbf{N}$. □

PROOF OF THEOREM 3.9. It remains to check the coercivity of the Γ-limit. By the growth condition on f and a comparison argument, it is enough to prove this for $f(A) = |A|^p$. We know that the Γ-limit F_{hom} exists for all $u \in L^p(\Omega;\mathbf{R}^n)$ and for all sets R in the countable family $\mathcal{R}$ of all finite unions of open rectangles of Ω with rational vertices, in this case F_{hom} is also convex. For all $U', U \in \mathcal{A}(\Omega)$ such that $U' \subset\subset U$ there exists $R \in \mathcal{R}$ such that $U' \subset\subset R \subset\subset U$. Reasoning as in the proof of Theorem 1.47 Step 1 , for $y \in B(0, \frac{1}{k})$ and k large enough we have that $R \subset y + U$ hence

$$F_{\mathrm{hom}}(u_k, R) \le F'(u, U)$$

and

$$\liminf_{k\to+\infty} F_{\mathrm{hom}}(u_k, U') \le F'(u, U) \tag{3.21}$$

with $u_k = u * \rho_k$.

It will be enough then to prove that $f_{\mathrm{hom}}(A) \ge c|A|^p$. In fact for any $u \in L^p(\Omega;\mathbf{R}^n) \setminus W^{1,p}(\Omega;\mathbf{R}^n)$ when $p > 1$ by (3.21)

$$F'(u, U) \ge c \liminf_{k\to+\infty} \int_{U'} |Du_k|^p\, dx$$

by the arbitrarity of U', we get $F_{\mathrm{hom}}(u, U) = +\infty$. Similarly, if $p = 1$ for all $u \in L^1(\Omega;\mathbf{R}^n) \setminus BD(\Omega)$ we have $|Eu|(\Omega) = +\infty$, let $\Omega' \subset\subset \Omega$ we get by (3.21) that

$$F'(u, \Omega) \ge c \liminf_{k\to+\infty} |Eu_k|(\Omega')$$

by arbitrarity of Ω' we obtain $F_{\mathrm{hom}}(u, \Omega) = +\infty$.

Since f_{hom} is positively homogeneous of degree p, to prove that $f_{\mathrm{hom}}(A) \ge c|A|^p$, it is sufficient to check that $f_{\mathrm{hom}}(A) \ne 0$ if $A \ne 0$. To this aim, let $u_\varepsilon \to Ax$ be such that $F_\varepsilon(u_\varepsilon, (0,1)^n) \to f_{\mathrm{hom}}(A)$. If $f_{\mathrm{hom}}(A) = 0$ then by a “Poincaré-type” inequality for BD functions (see Theorem 1.61), by Hölder's inequality and a scaling argument we obtain that

$$\begin{aligned} 0 = f_{\mathrm{hom}}(A) &= \lim_{\varepsilon\to 0}\int_{(0,1)^n}\Big|\frac{dEu_\varepsilon}{d\mu_\varepsilon}\Big|^p d\mu_\varepsilon \\ &\ge \lim_{\varepsilon\to 0} c\Big(\int_{(0,1)^n} |u_\varepsilon - Ru_\varepsilon|\, dx\Big)^p \end{aligned}$$

where the constant c depends only on Ω and R, and Ru_ε is a rigid displacement. Hence $Ru_\varepsilon \to Ax$ in L^1, and we get a contradiction because A is a symmetric matrix. □

3.4 Non local effects

Theorem 3.9 shows the Γ-convergence of the functionals F_ε to $F_{\rm hom}$ in $W^{1,p}(\Omega;\mathbf{R}^n)$ and that the Γ-limit is local; in fact we have represented $F_{\rm hom}$ as the integration over Ω of a local density of energy of the form $f_{\rm hom}(Eu)$.

Now, if we consider

$$F_\varepsilon^\gamma(u,\Omega) = \varepsilon^\gamma \int_\Omega f\Big(\frac{dEu}{d\mu_\varepsilon}\Big) d\mu_\varepsilon$$

then $\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon^\gamma(u,\Omega) = 0$ on $W^{1,p}(\Omega;\mathbf{R}^n)$, when $\gamma > 0$. In this case, however, no coerciveness result may hold for sequences (u_ε) with $\sup_{\varepsilon>0} F_\varepsilon^\gamma(u_\varepsilon,\Omega) < +\infty$ in any norm.

We will show with an example that a more complex notion of convergence may have to be introduced and that the Γ-limit functionals may be of a non-local nature.

Let $\Omega = \omega \times (0,1)$ be a 'cylindrical' domain where ω is a connected open subset of $\mathbf{R}^2$.

We define εD_i to be a two dimensional disk centered at $x_i = (\varepsilon i_1 + \frac{\varepsilon}{2}, \varepsilon i_2 + \frac{\varepsilon}{2})$ of radius $\varepsilon/4$

$$\varepsilon {E_i}^2 = \varepsilon D_i \times (0,1) \qquad\qquad \varepsilon E^2 = \bigcup_{i\in I_\varepsilon} \varepsilon {E_i}^2$$

where $i = (i_1,i_2) \in I_\varepsilon = \{i \in \mathbf{Z}^2 : \varepsilon {E_i}^2 \subset \Omega\}$,

$$\varepsilon E^1 = \Omega \setminus \varepsilon E^2.$$

We call $E = D_0 \times (0,1)$.

We consider the measures

$$\mu_\varepsilon(B) = \varepsilon \mathcal{H}^2(B \cap \partial\varepsilon E^2)$$

and the functionals

$$F_\varepsilon^\gamma(u,\Omega) = \varepsilon^\gamma \int_\Omega \Big|\frac{dEu}{d\mu_\varepsilon}\Big|^2 d\mu_\varepsilon\,.$$

Note that, up to normalization, μ_ε is the same measure of Example 3.7.

In this case $LD^2_{\mu_\varepsilon}(\Omega)$ consists of functions which are rigid displacements on the sets εE^1 and εE^2; *i.e.*, $u \in LD^2_{\mu_\varepsilon}(\Omega)$ if and only if there exist $a_i, b_i, c, d \in \mathbf{R}^3$ such that

$$u = c \wedge x + d \quad \text{on} \quad \varepsilon E^1$$
$$u = a_i \wedge x + b_i \quad \text{on} \quad \varepsilon {E_i}^2$$

for each $i \in I_\varepsilon$. We use the notation $x = (x_\alpha, x_3) \in \mathbf{R}^3$, $x_\alpha = (x_1,x_2)$.

Hence

$$\frac{dEu}{d\mu_\varepsilon} = \frac{1}{\varepsilon}\frac{dEu}{d\mathcal{H}^2} = \frac{1}{\varepsilon}(c \wedge x + d - a_i \wedge x - b_i) \odot \nu \text{ on } \partial(\varepsilon E_i^2)\,.$$

Definition 3.16 *Let* $u_\varepsilon \in LD^2_{\mu_\varepsilon}(\Omega)$. *We say that* u_ε *converges to* $(u_1, u_2) \in \mathrm{L}^2(\Omega; \mathbf{R}^3) \times \mathrm{L}^2(\Omega; \mathbf{R}^3)$ *if and only if*

$$\lim_{\varepsilon\to 0} \int_{\varepsilon E^1} |u_\varepsilon - u_1|^2\, dx \,= 0 \tag{3.22}$$

$$\lim_{\varepsilon\to 0} \int_{\varepsilon E^2} |u_\varepsilon - u_2|^2\, dx \,= 0\,. \tag{3.23}$$

We will study the Γ-limit F of F^γ_ε with respect to the convergence introduced in Definition 3.16 (see Theorem 3.19). The domain of F will be the set of pairs (u_1, u_2) such that u_1 is a rigid displacement and u_2 is in the space $\mathcal{U}$ of functions whose 'vertical sections are rigid displacements', introduced in the following proposition.

Let us define, for all $\eta > 0$, $T^k_\eta = Q^k_\eta \times (0,1)$ where $Q^k_\eta = k + (0,\eta)^2$ with $k = (k_1, k_2) \in J = \{k \in \mathbf{Z}^2 : T^k_\eta \cap \Omega \neq \emptyset\}$.

Proposition 3.17 *Let* $u_\varepsilon \in LD^2_{\mu_\varepsilon}(\Omega)$ *and* $u_2 \in \mathrm{L}^2(\Omega; \mathbf{R}^3)$.

$$\lim_{\varepsilon\to 0} \int_{\varepsilon E^2} |u_\varepsilon - u_2|^2\, dx \,= 0$$

if and only if $u_2 \in \mathcal{U}$ *where*

$$\mathcal{U} = \Big\{ v \in \mathrm{L}^2(\Omega; \mathbf{R}^3) : \forall \eta > 0\ \exists J \subset \mathbf{Z}^2 \ \ \textit{and}\ \ \exists A^k \wedge x + B^k \ \ \textit{on}\ T^k_\eta \quad \forall k \in J$$
$$\textit{such that} \bigcup_{k\in J} T^k_\eta \cap \Omega = \Omega \ \ \sum_{k\in J} \int_{T^k_\eta \cap \Omega} |v(x) - A^k \wedge x - B^k|^2\, dx \leq o(\eta) \Big\}\,.$$

Proof. Let $u_\varepsilon \in LD^2_{\mu_\varepsilon}(\Omega)$, by definition $u_\varepsilon = a_{\varepsilon,i} \wedge x + b_{\varepsilon,i}$ on εE^2_i. Let $h \in \mathbf{N}$ and $\eta > 0$ such that $\eta = h\varepsilon$, we extend $a_{\varepsilon,i} \wedge x + b_{\varepsilon,i}$ to T^k_η for each $i \in I_k = \{i \in \mathbf{Z}^2 : \varepsilon E^2_i \subset T^k_\eta\}$, hence we can construct a rigid displacement on T^k_η

$$A^k_\varepsilon \wedge x + B^k_\varepsilon = \frac{1}{h^2} \sum_{i\in I_k} a_{\varepsilon,i} \wedge x + b_{\varepsilon,i}\,.$$

Let us suppose that u_ε satisfies condition (3.23),

$$\int_{T^k_\eta \cap \varepsilon E^2} \Big| u_2(x) - A^k_\varepsilon \wedge x - B^k_\varepsilon \Big|^2\, dx \tag{3.24}$$

$$\leq c\Big(\sum_{j\in I_k}\int_{\varepsilon E_j^2}\Big|u_2(x)-a_{\varepsilon,j}\wedge x-b_{\varepsilon,j}\Big|^2\,dx$$
$$+\sum_{j\in I_k}\int_{\varepsilon E_j^2}\Big|a_{\varepsilon,j}\wedge x+b_{\varepsilon,j}-\frac{1}{h^2}\sum_{i\in I_k}a_{\varepsilon,i}\wedge x+b_{\varepsilon,i}\Big|^2\,dx\Big)\,.$$

Let us estimate the last term in (3.24)

$$\sum_{j\in I_k}\int_{\varepsilon E_j^2}\Big|a_{\varepsilon,j}\wedge x+b_{\varepsilon,j}-\frac{1}{h^2}\sum_{i\in I_k}a_{\varepsilon,i}\wedge x+b_{\varepsilon,i}\Big|^2\,dx$$
$$\leq c\Big(\sum_{j\in I_k}\int_{\varepsilon E_j^2}\Big|a_{\varepsilon,j}\wedge x+b_{\varepsilon,j}-u_2(x)\Big|^2\,dx$$
$$+\sum_{j\in I_k}\int_{\varepsilon E_j^2}\Big|\frac{1}{h^2}\sum_{i\in I_k}a_{\varepsilon,i}\wedge x+b_{\varepsilon,i}-u_2(x+x_i-x_j)\Big|^2\,dx$$
$$+\sum_{j\in I_k}\int_{\varepsilon E_j^2}\Big|\frac{1}{h^2}\sum_{i\in I_k}u_2(x)-u_2(x+x_i-x_j)\Big|^2\,dx\Big)\,.$$

For each $x\in\varepsilon E_j^2$ we have that $x+x_i-x_j\in\varepsilon E_i^2$, hence with a change of coordinates we get

$$\sum_{j\in I_k}\int_{\varepsilon E_j^2}\Big|a_{\varepsilon,j}\wedge x+b_{\varepsilon,j}-\frac{1}{h^2}\sum_{i\in I_k}a_{\varepsilon,i}\wedge x+b_{\varepsilon,i}\Big|^2\,dx$$
$$\leq c\Big(\sum_{j\in I_k}\int_{\varepsilon E_j^2}\Big|a_{\varepsilon,j}\wedge x+b_{\varepsilon,j}-u_2(x)\Big|^2\,dx$$
$$+\sum_{i,j\in I_k}\frac{1}{h^2}\int_{\varepsilon E_i^2}\Big|a_{\varepsilon,i}\wedge(x+x_j-x_i)+b_{\varepsilon,i}-u_2(x)\Big|^2\,dx$$
$$+\sum_{i,j\in I_k}\frac{1}{h^2}\int_{\varepsilon E_j^2}\Big|u_2(x)-u_2(x+x_i-x_j)\Big|^2\,dx\Big)$$
$$\leq c\Big(\sum_{i\in I_k}\int_{\varepsilon E_i^2}\Big|a_{\varepsilon,i}\wedge x+b_{\varepsilon,i}-u_2\Big|^2\,dx \tag{3.25}$$
$$+\sum_{i,j\in I_k}\frac{1}{h^2}\int_{\varepsilon E_i^2}\Big|a_{\varepsilon,i}\wedge(x_j-x_i)\Big|^2\,dx$$
$$+\sum_{i,j\in I_k}\frac{1}{h^2}\int_{\varepsilon E_j^2}\Big|u_2(x)-u_2(x+x_i-x_j)\Big|^2\,dx\Big)\,.$$

Now if we denote Λ the set of all translations of the type x_i-x_j with $i,j\in I_k$ we get that

$$\sum_{i,j\in I_k} \frac{1}{h^2}\int_{\varepsilon E_j^2}\Big|u_2(x)-u_2(x+x_i-x_j)\Big|^2\,dx \qquad (3.26)$$
$$\leq \sum_{\tau\in\Lambda}\frac{1}{h^2}\sum_{r\in C(k)}\int_{T_\eta^r}|u_2(x)-u_2(x+\tau)|^2\,dx$$

where $C(k)=\{(k_1,k_2),(k_1\pm 1,k_2),(k_1,k_2\pm 1),(k_1\pm 1,k_2\pm 1)\}$.

Since $|\Lambda|=c\,h^2$, by (3.26) we have

$$\sum_{k\in J}\sum_{i,j\in I_k}\frac{1}{h^2}\int_{\varepsilon E_j^2}\Big|u_2(x)-u_2(x+x_i-x_j)\Big|^2\,dx$$
$$\leq c\sum_{\tau\in\Lambda}\frac{1}{h^2}\|u_2(\cdot)-u_2(\cdot+\tau)\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^3)}$$
$$\leq c\sup_{|\tau|\leq\sqrt{2}\eta}\|u_2(\cdot)-u_2(\cdot+\tau)\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^3)}\,. \qquad (3.27)$$

Let us consider the cubes $Q^j_{\varepsilon,i}=(\varepsilon i+(0,1)^2)\times(\varepsilon j+(0,\varepsilon))$ for $i\in I_\varepsilon$, and $j\in J_\varepsilon=\{j\in\mathbf{Z}: Q^j_{\varepsilon,i}\cap\varepsilon E_i^2\neq\emptyset\}$. Since $u_2\in\mathrm{L}^2(\Omega;\mathbf{R}^3)$, we can assume that there exists a sequence $(u_{\varepsilon,2})$ which is constant on each $Q^j_{\varepsilon,i}$ such that

$$\lim_{\varepsilon\to 0}\int_\Omega|u_2-u_{\varepsilon,2}|^2\,dx=\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\sum_{j\in J_\varepsilon}\int_{Q^j_{\varepsilon,i}\cap\Omega}|u_2-u_{\varepsilon,2,i,j}|^2\,dx=0 \qquad (3.28)$$

where $u_{\varepsilon,2,i,j}$ is the value of $(u_{\varepsilon,2})$ on $Q^j_{\varepsilon,i}$.

So by (3.23)) we get

$$\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\sum_{j\in J_\varepsilon}\int_{Q^j_{\varepsilon,i}\cap\varepsilon E_i^2}|u_\varepsilon-u_{\varepsilon,2,i,j}|^2\,dx=0\,. \qquad (3.29)$$

Note that the L^2-norm on the set $\mathcal{R}$ of rigid displacements is equivalent to the norm on $\mathcal{R}$

$$\|a\wedge x+b\|_{\mathcal{R}}=(|a|^2+|b|^2)^{1/2}\,,$$

hence by (3.29)

$$\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\sum_{j\in J_\varepsilon}\varepsilon^3|a_{\varepsilon,i}|^2+\varepsilon^3|b_{\varepsilon,i}-u_{\varepsilon,2,i,j}|^2=0$$

which implies that

$$\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\varepsilon^2|a_{\varepsilon,i}|^2=0 \qquad (3.30)$$

and

$$\sum_{i\in I_\varepsilon} \varepsilon^2 |b_{\varepsilon,i}|^2 \le c \tag{3.31}$$

for each $\varepsilon > 0$ small enough.

Since $|x_j - x_i| \le \eta$, by the equivalence of the norms we have

$$\begin{aligned} \sum_{i,j\in I_k} \frac{1}{h^2} \int_{\varepsilon E_i^2} \Big| a_{\varepsilon,i} \wedge (x_j - x_i) \Big|^2 dx &\le c \sum_{i,j\in I_k} \frac{\varepsilon^2}{h^2} \eta^2 |a_{\varepsilon,i}|^2 \\ &= c\eta^2 \sum_{i\in I_k} \varepsilon^2 |a_{\varepsilon,i}|^2 \,. \end{aligned} \tag{3.32}$$

Note that $\sum_{k\in J} \sum_{i\in I_k} = \sum_{i\in I_\varepsilon}$.

Now we insert (3.32) into (3.25) and, summing up all the corresponding estimates obtained for different indices $k \in J$, by (3.27) we get

$$\begin{aligned} &\sum_{k\in J} \sum_{j\in I_k} \int_{\varepsilon E_j^2} \Big| a_{\varepsilon,j} \wedge x + b_{\varepsilon,j} - \frac{1}{h^2} \sum_{i\in I_k} a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} \Big|^2 dx \\ &\le c\Big(\sum_{i\in I_\varepsilon} \int_{\varepsilon E_i^2} \Big| a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} - u_2(x) \Big|^2 dx + \eta^2 \sum_{i\in I_\varepsilon} \varepsilon^2 |a_{\varepsilon,i}|^2 \\ &\qquad + \sup_{|\tau|\le\sqrt{2}\eta} \|u_2(\cdot) - u_2(\cdot + \tau)\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^3)} \Big) \,. \end{aligned} \tag{3.33}$$

Finally, we sum up the estimates (3.24) for $k \in J$ and insert (3.33); by (3.23) and (3.30) we get

$$\begin{aligned} &\lim_{\varepsilon\to 0} \sum_{k\in J} \int_{T_\eta^k \cap \varepsilon E^2} \Big| u_2 - A_\varepsilon^k \wedge x - B_\varepsilon^k \Big|^2 dx \\ &\le c \sup_{|\tau|\le\sqrt{2}\eta} \|u_2(\cdot) - u_2(\cdot + \tau)\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^3)} \,. \end{aligned} \tag{3.34}$$

On the other hand it is easy to see by (3.30) and (3.31) that there exists $A^k \wedge x + B^k$ such that

$$\lim_{\varepsilon\to 0} \int_{T_\eta^k} \Big| A_\varepsilon^k \wedge x + B_\varepsilon^k - A^k \wedge x - B^k \Big|^2 dx = 0$$

for each $k \in J$, hence by (3.34) we can conclude that $u_2 \in \mathcal{U}$.

Conversely, if $u_2 \in \mathcal{U}$ then $\varepsilon E^2 = \cup_{k\in J} T_\eta^k \cap \varepsilon E^2$ and we have rigid displacements $A^k \wedge x + B^k$ on each T_η^k.

We define

$$a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} = (A^k \wedge x + B^k)_{|\varepsilon E_i^2}$$

for each $i \in I_k$. Hence

$$\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|a_{\varepsilon,i}\wedge x+b_{\varepsilon,i}-u_2(x)\Big|^2\,dx=\sum_{k\in J}\int_{T_\eta^k\cap\varepsilon E^2}\Big|A^k\wedge x+B^k-u_2(x)\Big|^2\,dx$$

and by definition of $\mathcal{U}$

$$\lim_{\varepsilon\to 0}\sum_{k\in J}\int_{T_\eta^k\cap\varepsilon E^2}\Big|A^k\wedge x+B^k-u_2(x)\Big|^2\,dx \tag{3.35}$$
$$=\sum_{k\in J}|E|\int_{T_\eta^k\cap\Omega}\Big|A^k\wedge x+B^k-u_2(x)\Big|^2\,dx\le o(\eta)\,.$$

By (3.35), passing to the limit as $\eta\to 0$, we get

$$\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|a_{\varepsilon,i}\wedge x+b_{\varepsilon,i}-u_2(x)\Big|^2\,dx=0\,.$$

□

Remark 3.18 Note that, since u_ε are rigid displacements, by (3.22) it is easy to see that u_1 is a rigid displacement.

For simplicity, we will denote

$$F(u_1,u_2;\Omega)=\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon^\gamma(u_1,u_2;\Omega)$$

for $(u_1,u_2)\in\mathcal{R}\times\mathcal{U}$. We will continue to write $F_\varepsilon^\gamma(u,\Omega)$ for $u\in LD^2_{\mu_\varepsilon}(\Omega)$.

Theorem 3.19 *For $\gamma=2$ the functionals F_ε^γ Γ-converge as $\varepsilon\to 0$ to*

$$F(u_1,u_2;\Omega)=c_1\int_\Omega|(u_1)_\alpha-(u_2)_\alpha|^2\,dx+c_2\int_\Omega|(u_1)_3-(u_2)_3|^2\,dx$$

on $\mathcal{R}\times\mathcal{U}$ with respect to the convergence introduced in Definition 3.16*, where $c_1=\frac{3}{8}\pi$, $c_2=\frac{\pi}{4}$.*

Proof. By the invariance of the functionals with respect to translations of rigid displacements and by Remark 3.18 we can always assume without loss of generality that $u_\varepsilon=u_1$ on εE^1.

Let us call

$$\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i}=u_1-a_{\varepsilon,i}\wedge x-b_{\varepsilon,i}$$

hence

$$F_\varepsilon^\gamma(u_\varepsilon,\Omega)=\varepsilon^{\gamma-1}\sum_{i\in I_\varepsilon}\int_{\partial\varepsilon E_i^2}\Big|(\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i})\odot\nu\Big|^2 d\mathcal{H}^2\,.$$

Fix $x_3\in(0,1)$, we can find the following equality

$$4\varepsilon\int_{\partial\varepsilon D_i}\Big|(\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i})\odot\nu\Big|^2 d\mathcal{H}^1-16\int_{\varepsilon D_i}\Big|\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i}\Big|^2\,dx_\alpha$$

$$= \frac{\pi}{2}\varepsilon^2\Big(\Big|(\fint_{\varepsilon D_i} \alpha_{\varepsilon,i}\wedge x + \beta_{\varepsilon,i}\ dx_\alpha)_1\Big|^2 + \Big|(\fint_{\varepsilon D_i} \alpha_{\varepsilon,i}\wedge x + \beta_{\varepsilon,i}\ dx_\alpha)_2\Big|^2\Big)$$
$$+\frac{\pi}{64}\varepsilon^4((\alpha_{\varepsilon,i})_1^2 + (\alpha_{\varepsilon,i})_2^2 + 2(\alpha_{\varepsilon,i})_3^2)\,.$$

Hence, if we integrate also in x_3, we get

$$4\varepsilon\int_{\partial\varepsilon E_i^2}\Big|(\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i})\odot\nu\Big|^2 d\mathcal{H}^2 - 16\int_{\varepsilon E_i^2}\Big|\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i}\Big|^2 dx$$
$$= \int_0^1 \frac{\pi}{2}\varepsilon^2\Big(\Big|(\fint_{\varepsilon D_i} \alpha_{\varepsilon,i}\wedge x + \beta_{\varepsilon,i}\ dx_\alpha)_1\Big|^2 + \Big|(\fint_{\varepsilon D_i} \alpha_{\varepsilon,i}\wedge x + \beta_{\varepsilon,i}\ dx_\alpha)_2\Big|^2\Big)\, dx_3$$
$$+\frac{\pi}{64}\varepsilon^4((\alpha_{\varepsilon,i})_1^2 + (\alpha_{\varepsilon,i})_2^2 + 2(\alpha_{\varepsilon,i})_3^2)\,. \tag{3.36}$$

But

$$\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|(\fint_{\varepsilon D_i}\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i}\ dx_\alpha)_h\Big|^2 dx = \lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|(\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i})_h\Big|^2 dx$$

for each $h = 1, 2, 3$, and

$$\frac{\pi}{2}\varepsilon^2\Big|(\fint_{\varepsilon D_i}\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i}\, dx_\alpha)_h\Big|^2 = 8\int_{\varepsilon D_i}\Big|(\fint_{\varepsilon D_i}\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i}\ dx_\alpha)_h\Big|^2 dx_\alpha\,;$$

hence,

$$\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_0^1 \frac{\pi}{2}\varepsilon^2\Big|(\fint_{\varepsilon D_i}\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i}\, dx_\alpha)_h\Big|^2 dx_3$$
$$= 8\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|(\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i})_h\Big|^2 dx\,. \tag{3.37}$$

If we pass to the limit in (3.36), by (3.37) we obtain

$$\liminf_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\varepsilon\int_{\partial\varepsilon E_i^2}\Big|(\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i})\odot\nu\Big|^2 d\mathcal{H}^2$$
$$\geq 6\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|(\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i})_1\Big|^2 + \Big|(\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i})_2\Big|^2 dx$$
$$+4\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|(\alpha_{\varepsilon,i}\wedge x+\beta_{\varepsilon,i})_3\Big|^2 dx$$
$$+\lim_{\varepsilon\to 0}\frac{\pi}{64}\sum_{i\in I_\varepsilon}\varepsilon^4((\alpha_{\varepsilon,i})_1^2 + (\alpha_{\varepsilon,i})_2^2 + 2(\alpha_{\varepsilon,i})_3^2)\,. \tag{3.38}$$

For every sequence u_ε converging to (u_1, u_2) in the sense of Definition 3.16, by (3.30) we have that

$$\lim_{\varepsilon\to 0}\frac{\pi}{64}\sum_{i\in I_\varepsilon}\varepsilon^4((\alpha_{\varepsilon,i})_1^2+(\alpha_{\varepsilon,i})_2^2+2(\alpha_{\varepsilon,i})_3^2)=0 \tag{3.39}$$

so we insert (3.39) into (3.38) to find that

$$\liminf_{\varepsilon\to 0}\varepsilon^2\int_\Omega\Big|\frac{dEu_\varepsilon}{d\mu_\varepsilon}\Big|^2 d\mu_\varepsilon \geq 6|E|\int_\Omega |(u_1)_\alpha-(u_2)_\alpha|^2\,dx \tag{3.40}$$
$$+4|E|\int_\Omega |(u_1)_3-(u_2)_3|^2\,dx\,.$$

By the arbitrarity of u_ε, choosing $\gamma=2$

$$\Gamma\text{-}\liminf_{\varepsilon\to 0}F_\varepsilon{}^2(u_1,u_2;\Omega)\geq F(u_1,u_2;\Omega)\,. \tag{3.41}$$

Now we consider

$$u_\varepsilon=(c\wedge x+d)\,\chi_{\varepsilon E^1}+(a\wedge x+b)\,\chi_{\varepsilon E^2}$$

obviously it converges to $(c\wedge x+d, a\wedge x+b)$, and we call $\alpha\wedge x+\beta=(a-c)\wedge x+(b-d)$.

In this case

$$8\int_{\varepsilon E_i^2}\Big|(\alpha\wedge x+\beta)_h\Big|^2 dx=\int_0^1\frac{\pi}{2}\varepsilon^2\Big|\big(\fint_{\varepsilon D_i}\alpha\wedge x+\beta\,dx_\alpha\big)_h\Big|^2 dx_3+\frac{\pi}{128}\varepsilon^4\alpha_3^2$$

for $h=1,2$, hence by (3.36)

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\varepsilon\int_{\partial\varepsilon E_i^2}\Big|(\alpha\wedge x+\beta)\odot\nu\Big|^2 d\mathcal{H}^2\\
&\leq 6\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|(\alpha\wedge x+\beta)_1\Big|^2+\Big|(\alpha\wedge x+\beta)_2\Big|^2 dx\\
&\quad +4\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|(\alpha\wedge x+\beta)_3\Big|^2 dx+c\lim_{\varepsilon\to 0}\varepsilon^2|\alpha|^2\\
&=6|E|\int_\Omega\Big|(\alpha\wedge x+\beta)_\alpha\Big|^2 dx+4|E|\int_\Omega\Big|(\alpha\wedge x+\beta)_3\Big|^2 dx\,.
\end{aligned} \tag{3.42}$$

By (3.40) and (3.42) we get

$$\lim_{\varepsilon\to 0}\varepsilon^2\int_\Omega\Big|\frac{dEu_\varepsilon}{d\mu_\varepsilon}\Big|^2 d\mu_\varepsilon=6|E|\int_\Omega\Big|(\alpha\wedge x+\beta)_\alpha\Big|^2 dx \tag{3.43}$$
$$+4|E|\int_\Omega\Big|(\alpha\wedge x+\beta)_3\Big|^2 dx\,.$$

Now we fix $\eta > 0$ and consider $u_1 \in \mathcal{R}$ and v_2^η such that $v^\eta_{2\,|T^k_\eta} = A^k \wedge x + B^k$ with $k \in J$. By (3.43) we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} F^2_\varepsilon(u_1\,\chi_{\varepsilon E^1} + v_2^\eta\,\chi_{\varepsilon E^2}, \Omega)\\
&\le \sum_{k\in J} \limsup_{\varepsilon\to 0} F^2_\varepsilon(u_1\,\chi_{\varepsilon E^1} + (A^k\wedge x + B^k)\,\chi_{\varepsilon E^2}, T^k_\eta \cap \Omega)\\
&= \sum_{k\in J} 6|E| \int_{T^k_\eta\cap\Omega} \Big|(u_1(x) - A^k\wedge x - B^k)_\alpha\Big|^2\,dx\\
&\quad + \sum_{k\in J} 4|E| \int_{T^k_\eta\cap\Omega} \Big|(u_1(x) - A^k\wedge x - B^k)_3\Big|^2\,dx\\
&= 6|E| \int_\Omega \Big|(u_1(x) - v_2^\eta(x))_\alpha\Big|^2\,dx + 4|E|\int_\Omega \Big|(u_1(x) - v_2^\eta(x))_3\Big|^2\,dx \qquad (3.44)
\end{aligned}$$

If $u_2 \in \mathcal{U}$ then for all $\eta > 0$ there exists v_2^η as above such that $\|u_2 - v_2^\eta\|_{L^2(\Omega;\mathbf{R}^3)} \le o(\eta)$, since the Γ-upper limit is L^2-lower semicontinuous if we denote

$$F_2''(u_1,u_2;\Omega) = \Gamma\text{-}\limsup_{\varepsilon\to 0} {F_\varepsilon}^2(u_1,u_2;\Omega)$$

by (3.44) we get

$$\begin{aligned}
F_2''(u_1,u_2;\Omega) &\le \liminf_{\eta\to 0} F_2''(u_1,v_2^\eta;\Omega)\\
&\le \liminf_{\eta\to 0} 6|E| \int_\Omega \Big|(u_1(x) - v_2^\eta(x))_\alpha\Big|^2\,dx\\
&\qquad\quad + 4|E| \int_\Omega \Big|(u_1(x) - v_2^\eta(x))_3\Big|^2\,dx\\
&= 6|E| \int_\Omega \Big|(u_1(x) - u_2(x))_\alpha\Big|^2\,dx + 4|E| \int_\Omega \Big|(u_1(x) - u_2(x))_3\Big|^2\,dx\,.
\end{aligned}$$

It follows that given $(u_1,u_2) \in \mathcal{R}\times\mathcal{U}$

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} {F_\varepsilon}^2(u_1,u_2;\Omega) \le F(u_1,u_2;\Omega)$$

so that by (3.41)

$$\Gamma\text{-}\lim_{\varepsilon\to 0} {F_\varepsilon}^2(u_1,u_2;\Omega) = F(u_1,u_2;\Omega)$$

as desired. □

If u_ε converges to (u_1,u_2) in the sense of Definition 3.16 then u_ε converges weakly in $L^2(\Omega;\mathbf{R}^3)$ to $(1-c)u_1 + cu_2$ where $c = |E|$. If we define the energy

$$F(u,\Omega) := \inf_{\substack{u=(1-c)u_1+cu_2\\ (u_1,u_2)\in\mathcal{R}\times\mathcal{U}}} F(u_1,u_2;\Omega)$$

by Theorem 3.19

$$F(u,\Omega)=\inf_{r\in\mathcal{R}}\Big(\widetilde{c_1}\int_\Omega |r_\alpha-u_\alpha|^2\,dx+\widetilde{c_2}\int_\Omega |r_3-u_3|^2\,dx\Big)$$

where $\widetilde{c_1}=c_1/c^2$ and $\widetilde{c_2}=c_2/c^2$, which explains the non local nature of our limit.

Remark 3.20 Let us consider, up to normalization, the same measure of Example 3.8

$$\widetilde{\mu_\varepsilon}(B)=\Big(|B|+\varepsilon\mathcal{H}^2(B\cap\partial\varepsilon E^2)\Big)$$

and the functionals

$$\widetilde{F_\varepsilon^2}(u,\Omega)=\varepsilon^2\int_\Omega\Big|\frac{dEu_\varepsilon}{d\widetilde{\mu_\varepsilon}}\Big|^2 d\widetilde{\mu}_\varepsilon\,.$$

In this case by Theorem 3.19 we can deduce that the Γ- $\limsup_{\varepsilon\to0}\widetilde{F_\varepsilon^2}(u_1,u_2;\Omega)$ is finite for $(u_1,u_2)\in\mathcal{R}\times\mathcal{U}$.

In fact, since $LD^2_{\mu_\varepsilon}(\Omega;\mathbf{R}^3)\subset LD^2_{\widetilde{\mu_\varepsilon}}(\Omega;\mathbf{R}^3)$, given $(u_1,u_2)\in\mathcal{R}\times\mathcal{U}$ we have

$$\Gamma\text{-}\limsup_{\varepsilon\to0}\widetilde{F_\varepsilon^2}(u_1,u_2;\Omega)\le\Gamma\text{-}\limsup_{\varepsilon\to0}F_\varepsilon^2(u_1,u_2;\Omega)\,.$$

4

HOMOGENIZATION OF OSCILLATING BOUNDARIES

4.1 Scheme of the direct method

In the sequel we will repeatedly apply some variants of the so-called direct method of Γ-convergence to homogenization problems, which consists in combining localization and integral representation procedures to obtain compactness theorem for classes of integral functional.

The Γ-limits will be performed with respect to the $L^p(\Omega;\mathbf{R}^m)$-convergence.

Let Ω be a bounded subset of $\mathbf{R}^n$, let $p>1$ and let $F_\varepsilon : L^p(\Omega;\mathbf{R}^m)\times\mathcal{A}(\Omega)\to[0,+\infty]$ be a family of functionals of the form

$$F_\varepsilon(u,U)=\begin{cases}\displaystyle\int_U f_\varepsilon(x,Du)\,dx & \text{if } u\in X_\varepsilon(U)\\ +\infty & \text{otherwise,}\end{cases} \tag{4.1}$$

for suitable function spaces $X_\varepsilon(U)$ and $f_\varepsilon : \mathbf{R}^n\times\mathbf{M}^{m\times n}\to[0,+\infty)$ Borel functions. Suppose that there exist Borel functions $g_\varepsilon : \mathbf{R}^n\times\mathbf{R}\to[0,+\infty)$, convex and even in the second variable, with

$$g_\varepsilon(x,|F|)\le f_\varepsilon(x,F)\le C(1+g_\varepsilon(x,|F|))\le C(1+|F|^p), \tag{4.2}$$

$$g_\varepsilon(x,2t)\le C(1+g_\varepsilon(x,t)) \tag{4.3}$$

for all $F\in\mathbf{M}^{m\times n}$, $x\in\Omega$ and $t\in\mathbf{R}$. Growth conditions (4.2) and (4.3) are designed to include functions of the type $a_\varepsilon(x)|F|^p$ with the only assumption $a_\varepsilon\ge 0$, thus allowing for zones where $a_\varepsilon=0$. In the next section a_ε will be the characteristic function of a set with fast-oscillating boundary. Note that a general theory for functions satisfying

$$0\le f_\varepsilon(x,F)\le C(1+|F|^p)$$

only has not be developed yet. The aim of the direct method of Γ-convergence is to prove a compactness result for the family (F_ε), giving a representation of the limit, and, possibly, complete the description in terms of 'homogenization formulas'.

Step 1 With fixed (ε_j) extract a subsequence (not relabeled) such that $F_\varepsilon(\cdot,U)$ Γ-converges to a functional $F_0(\cdot,U)$ for all U in a dense family of open sets $\mathcal{R}$ (see Proposition 1.22);

Step 2 Thanks to (4.2) and (4.3), prove that $F_0(u,\cdot)$ is the restriction of a finite Borel measure to $\mathcal{R}$ for all $u\in W^{1,p}(\Omega;\mathbf{R}^m)$, so that by inner regularity we indeed have that $F_\varepsilon(\cdot,U)$ Γ-converges to a functional $F_0(\cdot,U)$ on $W^{1,p}(\Omega;\mathbf{R}^m)$ for all $U\in\mathcal{A}(\Omega)$. In this step is crucial the so-called *fundamental L^p-estimate*: for all $U,Y,Z\in\mathcal{A}(\Omega)$ with $Y\subset\subset U$, and for all $\sigma>0$, there exists $M>0$ such that for all $u,v\in W^{1,p}(\Omega;\mathbf{R}^m)$ one may find a cut-off function $\varphi\in C_c^\infty(U;[0,1])$, $\varphi=1$ in Y, such that

$$\begin{aligned} F_\varepsilon(\varphi u+(1-\varphi)v,Y\cup Z) &\le (1+\sigma)(F_\varepsilon(u,U)+F_\varepsilon(v,Z)) \\ &\quad +M\int_{(U\cap Z)\setminus Y}|u-v|^p\,dx+\sigma\,. \end{aligned} \tag{4.4}$$

Moreover, by again using the fundamental L^p-estimate it can be proven that if $u\in W^{1,p}(\Omega;\mathbf{R}^m)\cap X_\varepsilon(U)$ for all ε and $F_0(u,U)<+\infty$ then there exist a sequence $u_\varepsilon\in X_\varepsilon(U)$ such that

$$\lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,U)=F_0(u,U)$$

and $u_\varepsilon=u$ on a neighbourhood of ∂U (see Sections 1.6 and 1.7.1, Remark 1.39 and Section 1.7.2);

Step 3 By the locality and semicontinuity properties of Γ-limits and by Step 2 we can find a function $\varphi:\Omega\times\mathbf{M}^{m\times n}\to[0,+\infty)$ such that $0\le\varphi(x,F)\le C(1+|F|^p)$ and $F_0(u,U)=F_\varphi(u,U)$ for all $u\in W^{1,p}(\Omega;\mathbf{R}^m)$ and $U\in\mathcal{A}(\Omega)$, where

$$F_\varphi(u)=\int_\Omega\varphi(x,Du)\,dx.$$

In the proof of this step a crucial point is the passage from the identity $F_0(u)=F_\varphi(u)$ when u is piecewise affine to a general u by the continuity of F_φ with respect to a convergence in which piecewise-affine functions are dense (*e.g.* the strong $W^{1,p}$-convergence) (see Theorem 1.35);

Step 4 If $f_\varepsilon(x,F)=f(\frac{x}{\varepsilon},F)$ with f 1-periodic in the first variable then by the periodicity of f we deduce that $\varphi=\varphi(F)$ (see Proposition 1.44);

Step 5 If $g_\varepsilon(x,F)=g(\frac{x}{\varepsilon},F)$ with g 1-periodic in the first variable then we consider the auxiliary functionals

$$G_\varepsilon(u,U)=\begin{cases}\displaystyle\int_U g_\varepsilon(x,Du)\,dx & \text{if } u\in X_\varepsilon(\Omega)\\ +\infty & \text{otherwise.}\end{cases} \tag{4.5}$$

By Step 1–4 we can assume that a function ψ exists such that $G_\varepsilon(\cdot,U)$ Γ-converges to the functional $F_\psi(\cdot,U)$ on $W^{1,p}(\Omega;\mathbf{R}^m)$ for all $U\in\mathcal{A}(\Omega)$;

Step 6 Note that ψ is convex. By an argument of approximation by convolution prove that indeed the functional $G_\varepsilon(\cdot,U)$ Γ-converges to the functional $F_\psi(\cdot,U)$ on $W^{1,1}(\Omega;\mathbf{R}^m)$ for all $U\in\mathcal{A}(\Omega)$. Define the 'domain' of $F_\psi(\cdot,\Omega)$: $\mathrm{W}^{1,\psi}(\Omega;\mathbf{R}^m)=\{u\in W^{1,1}(\Omega;\mathbf{R}^m):\ F_\psi(u,\Omega)<+\infty\}$ (see Theorem 1.47);

Step 7 Repeat Step 2 and 3 substituting the space $W^{1,p}(\Omega;\mathbf{R}^m)$ by the space $W^{1,\psi}(\Omega;\mathbf{R}^m)$ thus obtaining the representation $F_0 = F_\varphi$ on $W^{1,1}(\Omega;\mathbf{R}^m)$. It is usually proved by using some additional assumptions on ψ;

Step 8 Deduce that φ and ψ do not depend on (ε_j) by proving a homogenization formula;

Step 9 Finally, the representation of F_0 on the whole $L^p(\Omega;\mathbf{R}^m)$, and not only on $W^{1,1}(\Omega;\mathbf{R}^m)$, can be obtained in some cases by a more accurate study of the properties of φ (as for example in Theorem 1.47).

We will have to modify Steps 1–9 above as to cover the case when the domain of the limit is a 'degenerate Sobolev Space'. In particular, since the function ψ obtained as in Step 5 will be degenerate, a suitable weighted Sobolev Space will have to be defined, which takes the place of $W^{1,1}(\Omega;\mathbf{R}^m)$ in Step 6 above. Moreover, we will have to deal with the fact that our functions $f_\varepsilon, g_\varepsilon$ may be periodic only in some variables, so that Step 8 will be harder to verify. We will include all the details of the reasonings which do not fall directly in this scheme, while we will feel free to refer to Chapter 1 for those procedures which have become customary.

It is worth mentioning that in some cases the arguments outlined above can be simplified by using some techniques (as blow-up arguments or the theory of Young measures) that avoid to use the complex localization procedure. As our problem is concerned those methods seem harder to apply since the energies we consider are coercive only on wildly oscillating sets.

4.2 Homogenization of media with oscillating profile

Let $f : \mathbf{R}^{n-1} \mapsto [0,1]$ be a 1-periodic lower semicontinuous function and $0 \leq \min f \leq \sup f = 1$, let $W : \mathbf{R}^{n-1} \times \mathbf{M}^{m\times n} \mapsto [0,+\infty)$ be a Borel function 1-periodic in the first variable satisfying

$$\gamma|F|^p \leq W(x_\alpha, F) \leq \beta(1+|F|^p) \tag{4.6}$$

for all $x_\alpha \in \mathbf{R}^{n-1}$ and $F \in \mathbf{M}^{m\times n}$, for some $1 < p < +\infty$, $0 < \gamma \leq \beta$. The set ω will be a fixed bounded open subset of $\mathbf{R}^{n-1}$ with Lipschitz boundary and $\Omega = \omega \times (-1,1)$.

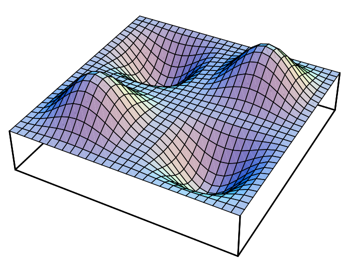

FIG. 4.1. the graph of a typical f in the unit cell

In this section we compute the Γ-limit of functionals of the form

$$J_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} W\Big(\frac{x_\alpha}{\varepsilon}, Du\Big)\,dx & \text{if } u_{|\Omega_\varepsilon} \in W^{1,p}(\Omega_\varepsilon;\mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.7}$$

where

$$\Omega_\varepsilon = \{x \in \Omega:\ |x_n| < f(x_\alpha/\varepsilon)\}. \tag{4.8}$$

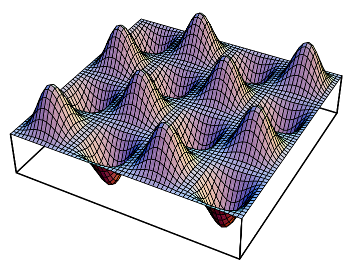

FIG. 4.2. the upper profile of Ω_ε with f as in Figure 4.1

The Γ-limit theorem will be stated and proved at the end of the chapter after some preliminary results, which are needed to define the domain of the Γ-limit and to explain the homogenization formula.

In orded to apply the method described in the previous section we introduce the localized version of the functionals J_ε: for all U open subset of Ω we define

$$J_\varepsilon(u,U) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon\cap U} W\Big(\frac{x_\alpha}{\varepsilon}, Du\Big)\,dx & \text{if } u_{|\Omega_\varepsilon\cap U} \in W^{1,p}(\Omega_\varepsilon\cap U;\mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.9}$$

so that $J_\varepsilon(u) = J_\varepsilon(u,\Omega)$.

The first proposition contains the analog of Steps 1–4 of the direct method of Γ-convergence as outlined in the previous section.

Proposition 4.1 *From every sequence (ε_j) of positive numbers converging to 0 we can extract a subsequence (not relabeled) such that the Γ-limit*

$$J_0(u,U) = \Gamma\text{-}\lim_{j\to+\infty} J_{\varepsilon_j}(u,U)$$

exists for all $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ and U open subsets of Ω. Moreover, there exists a Carathéodory function $\varphi : (-1,1)\times\mathbf{M}^{m\times n} \to [0,+\infty)$ such that

$$J_0(u,U) = \int_U \varphi(x_n, Du)\,dx$$

for all $u \in W^{1,p}(\Omega;\mathbf{R}^m)$.

Proof. The functional J_ε can be rewritten on $X_\varepsilon(U) = \{u \in L^p(\Omega;\mathbf{R}^m) : u_{|\Omega_\varepsilon \cap U} \in W^{1,p}(\Omega_\varepsilon \cap U;\mathbf{R}^m)\}$ as

$$J_\varepsilon(u,U) = \int_U \chi_{\Omega_\varepsilon}(x) W\Big(\frac{x_\alpha}{\varepsilon}, Du\Big)\,dx\,.$$

We can then apply Steps 1-3 of Section 4.1 (see Example 1.30 for the proof of the L^p-fundamental estimate). Finally, a translation argument in the x_α-plane (completely analogous, *e.g.*, to the one in the proof of Proposition 1.44) shows that

$$\int_{B_\rho(x_\alpha)\times(z-\eta,z+\eta)} \varphi(y,F)\,dy = \int_{B_\rho(x'_\alpha)\times(z-\eta,z+\eta)} \varphi(y,F)\,dy$$

for all $\rho,\eta > 0$, x_α, x'_α, z such that

$$\Big(B_\rho(x_\alpha)\times(z-\eta,z+\eta)\Big) \cup \Big(B_\rho(x'_\alpha)\times(z-\eta,z+\eta)\Big) \subset \Omega\,.$$

We then easily deduce that $\varphi(x,F) = \varphi(x_n,F)$. □

We will complete the proof of the homogenization theorem by characterizing the function φ above (showing in particular that it does not depend on the sequence (ε_j)), proving the existence of the Γ-limit J_0 on the whole $L^p(\Omega;\mathbf{R}^m)$ and showing that the integral representation in the previous proposition holds on the whole domain of J_0. In order to get to this result, we will have to define a number of auxiliary energies; we then streamline the organization of the rest of the chapter. First, in Section 4.3 we consider the case when $W(F) = \|F\|^p$. We will denote by ψ the function given by Proposition 4.1 corresponding to this particular choice of W. For fixed t the function $\psi(t,\cdot)$ is easily characterized by solving a $(n-1)$-dimensional (possibly, non coercive) homogenization problem. It is possible then to define the 'degenerate Sobolev space' $W^{1,p}_\psi(\Omega;\mathbf{R}^m)$ of functions such that $\int_\Omega \psi(x_n,Du)\,dx < +\infty$, which turns out to be the domain of the Γ-limit when $W(F) = \|F\|^p$, and hence also in the general case by (4.6). In Section 4.4, in order to describe the function φ in the general case, with fixed t we consider the case when we replace the function f with the characteristic function of $E_t = \{x_\alpha : f(x_\alpha) > |t|\}$ (i.e., we deal with cylindrical domains). The function $\varphi(t,\cdot)$ will eventually be given by the energy density of the corresponding Γ-limit. Finally, in Section 4.5 we are able to consider general W and f and obtain the oscillating-boundary homogenization Theorem 4.15 as the consequence of the previous sections.

4.3 An auxiliary problem. Definition of the limit domain

In general, the limit functional J_0 exists and is finite also outside $W^{1,p}(\Omega;\mathbf{R}^m)$. We first deal with the case of J_0 corresponding to

$$W(x,F)=\|F\|^p, \text{ where } \|F\|^p=\sum_{j=1}^{n}|F_j|^p. \tag{4.10}$$

By a careful description of the domain of the corresponding Γ-limit we will identify the domain of J_0 as a suitable 'degenerate Sobolev Space' (see Definition 4.5) which, in view of the growth condition (4.6), will also be the domain of J_0 corresponding to energy densities other than (4.10).

We recall a preliminary result.

Theorem 4.2 *Let E be a 1-periodic set in $\mathbf{R}^N$; i.e., such that χ_E is a 1-periodic function, and let*

$$J_\varepsilon^E(v,U)=\begin{cases}\displaystyle\int_{U\cap\varepsilon E}\|Dv\|^p\,dx & \text{if } v_{|U\cap\varepsilon E}\in W^{1,p}(U\cap\varepsilon E;\mathbf{R}^m)\\ +\infty & \text{otherwise.}\end{cases} \tag{4.11}$$

Then the Γ-limit

$$J_{\rm hom}^E(v,U)=\Gamma\text{-}\lim_{\varepsilon\to0}J_\varepsilon^E(v,U)$$

exists for all U bounded open subsets of $\mathbf{R}^N$ and $v\in W^{1,p}(U;\mathbf{R}^m)$. Moreover, we have

$$J_{\rm hom}^E(v,U)=\int_U\varphi_{\rm hom}^E(Dv)\,dx$$

for all $u\in W^{1,p}(U;\mathbf{R}^m)$, where $\varphi_{\rm hom}^E$ is a positively homogeneous function of degree p, satisfying the formula

$$\varphi_{\rm hom}^E(F)=\inf\Big\{\int_{E\cap(0,1)^N}\|Dv+F\|^p\,dx : v\in W^{1,p}_{\rm loc}(E;\mathbf{R}^m),\ 1\text{-periodic}\Big\}.$$

PROOF. This theorem is a particular case of Theorem 1.47, the positive homogeneity of $\varphi_{\rm hom}^E$ easily following from its definition. □

For all $t\in(-1,1)$ we define

$$\varphi_\#(t,\overline{F})=\varphi_{\rm hom}^{E_t}(\overline{F}),$$

the latter function being that given by the previous theorem, with $N=n-1$ and $E=E_t=\{x_\alpha:\ f(x_\alpha)>|t|\}$. We define also

$$\psi(t,F)=\varphi_\#(t,\overline{F})+\mathcal{L}_{n-1}(E_t\cap(0,1)^{n-1})|F_n|^p. \tag{4.12}$$

Theorem 4.3 *If $W=\|F\|^p$ and φ is given by Proposition 4.1 then we have*

$$\varphi(t,F)=\psi(t,F).$$

In particular φ does not depend on (ε_j).

PROOF. Let (x,F) be such that x_n is a Lebesgue point for $\varphi(\cdot,F)$. Then

$$\begin{aligned}\varphi(x_n,F) &= \lim_{\rho\to 0^+} \fint_{B_\rho(x_\alpha)\times(x_n-\rho,x_n)} \varphi(y_n,F)\,dy \\ &= \lim_{\rho\to 0^+} \frac{J_0(Fy,B_\rho(x_\alpha)\times(x_n-\rho,x_n))}{|B_\rho(x_\alpha)\times(x_n-\rho,x_n)|}\,.\end{aligned} \tag{4.13}$$

We consider the case $x_n>0$ only, the case $x_n<0$ being dealt with using a symmetric argument. Note that for $0<t<s<1$ we have $E_s\subseteq E_t$. Let $u_j\to 0$ with $u_j\in W_0^{1,p}(B_\rho(x_\alpha)\times(x_n-\rho,x_n)\cap\Omega_{\varepsilon_j})$ be such that

$$J_0(Fy,B_\rho(x_\alpha)\times(x_n-\rho,x_n)) = \lim_{j\to+\infty} J_{\varepsilon_j}(Fy+u_j,B_\rho(x_\alpha)\times(x_n-\rho,x_n))\,.$$

Then,

$$\begin{aligned}&J_{\varepsilon_j}(Fy+u_j,B_\rho(x_\alpha)\times(x_n-\rho,x_n)) \\ &= \int_{x_n-\rho}^{x_n}\int_{B_\rho(x_\alpha)} \chi_{E_{y_n}}\Big(\frac{y_\alpha}{\varepsilon_j}\Big)\|\overline{F}+D_\alpha u_j\|^p dy_\alpha\,dy_n \\ &\quad+\int_{B_\rho(x_\alpha)}\int_{x_n-\rho}^{x_n} \chi_{E_{y_n}}\Big(\frac{y_\alpha}{\varepsilon_j}\Big)|F_n+D_n u_j|^p dy_n\,dy_\alpha \\ &\ge \int_{x_n-\rho}^{x_n}\int_{B_\rho(x_\alpha)} \chi_{E_{x_n}}\Big(\frac{y_\alpha}{\varepsilon_j}\Big)\|\overline{F}+D_\alpha u_j\|^p dy_\alpha\,dy_n \\ &\quad+\rho\int_{B_\rho(x_\alpha)} \chi_{E_{x_n}}\Big(\frac{y_\alpha}{\varepsilon_j}\Big)|F_n|^p\,dy_\alpha\end{aligned}$$

by Jensen's inequality. By using the lower limit inequality for the Γ-convergence in Theorem 4.2 with $E=E_{x_n}$, and by an application of Fatou's Lemma, we get

$$\begin{aligned}J_0(Fy,B_\rho(x_\alpha)\times(x_n-\rho,x_n)) &\ge \rho\int_{B_\rho(x_\alpha)} \varphi_\#(x_n,\overline{F})dy_\alpha \\ &+\rho\mathcal{L}_{n-1}(B_\rho(x_\alpha))|F_n|^p\mathcal{L}_{n-1}(E_{x_n}\cap(0,1)^{n-1})\,.\end{aligned}$$

Letting $\rho\to 0^+$ we obtain then by (4.13)

$$\varphi(x_n,F)\ge\varphi_\#(x_n,\overline{F})+\mathcal{L}_{n-1}(E_{x_n}\cap(0,1)^{n-1})|F_n|^p.$$

Vice versa, let $v_j\to 0$ be such that $\overline{F}y_\alpha+v_j(y_\alpha)$ is a recovery sequence for $J_{\rm hom}^{E_{x_n}}(\overline{F}y_\alpha,B_\rho(x_\alpha))$ along the sequence (ε_j), and set

$$u_j(y)=Fy+(v_j(y_\alpha),0)=(\overline{F}y_\alpha+v_j(y_\alpha),F_ny_n)\,.$$

We then have

$$\int_{B_\rho(x_\alpha)\times(x_n,x_n+\rho)} \varphi(y_n,A)\,dy$$

$$
\begin{aligned}
&\le \liminf_{j\to+\infty} J_{\varepsilon_j}(u_j, B_\rho(x_\alpha)\times(x_n, x_n+\rho)) \\
&\le \liminf_{j\to+\infty} \int_{B_\rho(x_\alpha)\times(x_n,x_n+\rho)} \chi_{E_{x_n}}\Big(\frac{y_\alpha}{\varepsilon_j}\Big)\|Du_j\|^p\,dy \\
&= \lim_{j\to+\infty} \rho \int_{B_\rho(x_\alpha)} \chi_{E_{x_n}}\Big(\frac{y_\alpha}{\varepsilon_j}\Big)(\|\overline{F}+D_\alpha v_j\|^p + |F_n|^p)dy_\alpha \\
&= \rho \int_{B_\rho(x_\alpha)} \varphi_\#(x_n,\overline{F})dy_\alpha \\
&\qquad +\rho\mathcal{L}_{n-1}(B_\rho(x_\alpha))|F_n|^p\mathcal{L}_{n-1}(E_{x_n}\cap(0,1)^{n-1}\,,
\end{aligned}
$$

which gives the missing inequality by (4.13). □

Remark 4.4 With fixed t, we define the 'kernel' of $\varphi_\#(t,\cdot)$ as

$$\mathrm{Ker}\,\varphi_\# = \{\varphi_\#(t,\cdot)=0\}.$$

Then $\mathrm{Ker}\,\varphi_\#$ is a linear space and its dimension is a multiple integer of m; *i.e.*,

$$\dim \mathrm{Ker}\,\varphi_\# = km \qquad\qquad \text{for some}\quad k=0,\dots,n-1$$

and there exist $\xi_{k+1},\dots,\xi_{n-1}\in\mathbf{R}^{n-1}$ such that

$$\overline{F} = \begin{pmatrix} F^1 \\ \vdots \\ F^m \end{pmatrix} \in \mathrm{Ker}\,\varphi_\# \quad\Leftrightarrow\quad \overline{F}\xi_i = 0$$

for each $i=k+1,\dots,n-1$. (Note that k depends on t fixed and F^i denotes the i-th row of $\overline{F}$, $1\le i\le m$).

In fact, since $\overline{F}\mapsto\varphi_\#(t,\overline{F})$ is positively homogeneous of degree p, convex and even, $\mathrm{Ker}\,\varphi_\#$ is a linear space and satisfies the following properties: if $\overline{F}\in Ker\,\varphi_\#$ then

(i) for each $(s_1,\dots,s_m)\in\mathbf{R}^m$

$$\begin{pmatrix} s_1F^1 \\ \vdots \\ s_mF^m \end{pmatrix} \in \mathrm{Ker}\,\varphi_\#;$$

(ii) $P\overline{F}\in\mathrm{Ker}\,\varphi_\#$ for each permutation matrix $P\in\mathbf{M}^{m\times m}$.

Properties (i) and (ii) imply that if we fix F^1 we can construct m matrices linearly independent

$$\begin{pmatrix} F^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ F^1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ F^1 \end{pmatrix} \in \mathrm{Ker}\,\varphi_\#$$

which span a subspace $\langle F^1\rangle$ of $\operatorname{Ker}\varphi_\#$ of dimension m.

Now, if $\langle F^1\rangle \neq \operatorname{Ker}\varphi_\#$, we can single out a non-zero matrix in $\operatorname{Ker}\varphi_\#$ orthogonal to $\langle F^1\rangle$, and, by using the same argument as above taking its first row vector, find other m matrices which, together with the matrices constructed before, form a linearly independent family.

By proceeding in this way, we end up with $\eta_1,\dots,\eta_k \in \mathbf{R}^{n-1}$ such that for all $A \in \operatorname{Ker}\varphi_\#$

$$A^i = \sum_{j=1}^{k} s_{ij}\eta_j \qquad\qquad i = 1,\dots,m$$

with $s_{ij} \in \mathbf{R}$, which means that the $\dim \operatorname{Ker}\varphi_\# = km$ for some $k \in \{1,\dots,n-1\}$.

The orthogonal subspace to $\langle \eta_1,\dots,\eta_k\rangle$ is a vector subspace of $\mathbf{R}^{m(n-1)}$ $\langle \xi_{k+1},\dots,\xi_{n-1}\rangle$ and the vectors of the two basis satisfy, by definition, the conditions

$$\eta_i\xi_j = 0 \qquad i = 1,\dots,k \qquad j = k+1,\dots,n-1.$$

Hence, we can conclude that there exist vectors $\xi_{k+1},\dots,\xi_{n-1} \in \mathbf{R}^{n-1}$ such that $\overline{F} \in \operatorname{Ker}\varphi_\#$ if and only if $\overline{F}\xi_i = 0$ for each $i = k+1,\dots,n-1$.

Since $t \mapsto \varphi_\#(t,\overline{F})$ is decreasing on $(0,1)$ and it is coercive on $(0,\min f)$, there exist $0 \le \min f \le t_1 \le \ldots \le t_k \le t_{k+1} \le \ldots \le t_{n-1} \le 1$ and $\xi_{k+1},\dots,\xi_{n-1} \in \mathbf{R}^{n-1}$ such that

(i) $\varphi_\#(t,\overline{F})$ is coercive on $(0,t_1)$;

(ii) for each $k = 1,\dots,n-2$ $\varphi_\#(t,\overline{F}) = 0$ if and only if $\overline{F}\xi_i = 0$ for $i = k+1,\dots,n-1$ on (t_k,t_{k+1});

(iii) $\varphi_\#(t,\overline{F}) = 0$ on $(t_{n-1},1)$.

Definition 4.5 *We define the 'degenerate weighted Sobolev Space'* $W^{1,p}_\psi(\Omega;\mathbf{R}^m)$ *as the space of functions* $u \in L^p(\Omega;\mathbf{R}^m)$ *such that*

(i) $D_n u \in L^p_{\mathrm{loc}}(\Omega;\mathbf{R}^m)$*;*

(ii) $D_{(\xi_i,0)}u \in L^p_{\mathrm{loc}}(\omega\times(-t_i,t_i);\mathbf{R}^m)$ *for* $i = 1,\dots,n-1$*;*

(iii) *if* $\Phi : \Omega \to \mathbf{M}^{m\times(n-1)}$ *is any measurable function such that* $\Phi\xi_i = D_{(\xi_i,0)}u \in L^p_{\mathrm{loc}}(\omega\times(-t_i,t_i);\mathbf{R}^m)$ *for* $i = 1,\dots,n-1$*, then*

$$\int_\Omega \psi(x_n,\Phi|D_n u)\,dx < +\infty\,.$$

Clearly, the last integral is independent of the choice of Φ; hence, it will be denoted by

$$\int_\Omega \psi(x_n,Du)\,dx\,,$$

with a slight abuse of notation.

Remark 4.6 Note that in dimension 3 (*i.e.*, $n = 3$) the representation of the space $W^{1,p}_\psi(\Omega;\mathbf{R}^m)$ is particularly simple as, up to a rotation, we can assume that $\xi = e_2$. In this case, $W^{1,p}_\psi(\Omega;\mathbf{R}^m)$ is the space of functions $u \in L^p(\Omega;\mathbf{R}^m)$ such that

(i) $D_3 u \in L^p_{\rm loc}(\Omega;\mathbf{R}^m)$;
(ii) $D_2 u \in L^p_{\rm loc}(\omega \times (-t_2,t_2);\mathbf{R}^m)$;
(iii) $D_1 u \in L^p_{\rm loc}(\omega \times (-t_1,t_1);\mathbf{R}^m)$;
(iv) if $\Phi : \Omega \to \mathbf{M}^{m\times 2}$, $\Phi = (\Phi_1,\Phi_2)$ is any measurable function such that $\Phi_2 = D_2 u$ in $\omega \times (-t_2,t_2)$ and $\Phi_1 = D_1 u$ in $\omega \times (-t_1,t_1)$, then

$$\int_\Omega \psi(x_3, \Phi|D_3 u)\,dx < +\infty\,.$$

Example 4.7 If $n = 3$ and

$$f(x_1,x_2) = \frac{1}{2} + \frac{1}{2}\sin^2(x_1)\sin^2(x_2)\,,$$

then $\varphi_\#(t,\overline{F}) = \|\overline{F}\|^p$ if $|t| < 1/2$ and 0 otherwise, so that $t_1 = t_2 = 1/2$, and ξ is any vector. If instead

$$f(x_1,x_2) = \frac{1}{2} + \frac{1}{2}\sin^2(x_1)\,,$$

then $t_1 = 1/2$, $t_2 = 1$ and $\xi = (0,1)$.

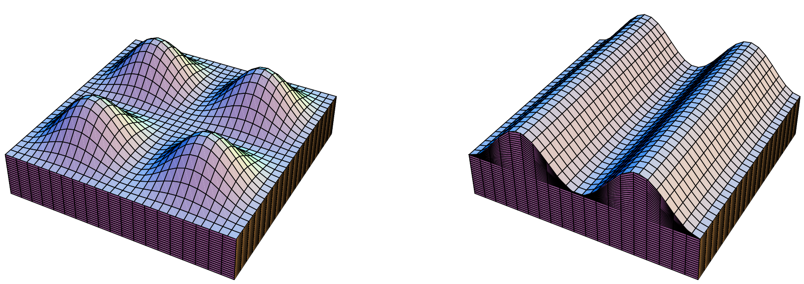

FIG. 4.3. the oscillating profiles in Example 4.7

By using a convolution argument, we can improve Proposition 4.1 to give a characterization of the Γ-limit on the whole $W^{1,p}_\psi(\Omega;\mathbf{R}^m)$ and independent of the sequence (ε_j). This result corresponds to Step 6 in Section 4.1, and its proof uses the convexity of $F \mapsto \|F\|^p$ in an essential way.

Proposition 4.8 *Let $W = \|F\|^p$, and let U be a open subset of Ω. Then*
(i) *if $u \in L^p(U;\mathbf{R}^m) \setminus W^{1,p}_\psi(U;\mathbf{R}^m)$ then there exists the Γ-limit*

$$J_0(u,U) = \Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(u,U) = +\infty;$$

(ii) *if* $u \in W^{1,p}_{\psi}(\Omega;\mathbf{R}^m)$ *then there exists the* Γ*-limit*

$$J_0(u,U) = \Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(u,U) = \int_U \psi(x_n, Du)\,dx.$$

Proof. We only outline the proof, as it closely follows that of Theorem 1.47, and details can be found therein.

Fix $u \in L^p(\Omega;\mathbf{R}^m)$ and U an open subset of Ω. In order to compute $J_0(u,U)$ it is sufficient to show that from every sequence (ε_j) we can extract a subsequence (ε_{j_k}) such that the Γ-limit along (ε_{j_k}) exists and is independent of the subsequence.

We fix a sequence (ε_j). By Theorem 4.3 the thesis of Proposition 4.1 holds with ψ in the place of φ. Upon possibly extracting a further subsequence, we may also assume that there exists the limit

$$J_0(u,U) = \Gamma\text{-}\lim_{j\to+\infty} J_{\varepsilon_j}(u,U).$$

Let (ρ_j) be a sequence of mollifiers with $\mathrm{spt}\rho_j \subset B(0,\frac{1}{j}) \subset \mathbf{R}^{n-1}$, and define

$$u_j(x) = \int_{B(0,\frac{1}{j})} \rho_j(y) u(x_\alpha - y, x_n)\,dy.$$

By the convexity of J_0 and its translation-invariance properties, we have $J_0(u_j,U') \le J_0(u,U)$ for all $U' \subset\subset U$ such that $U' \subset (y,0)+U$ for all $y \in \mathrm{spt}\rho_j$. By the convexity of ψ the functional $v \mapsto \int_{U'} \psi(x_n,Dv)\,dx$ (if $v \in L^p(U';\mathbf{R}^m) \setminus W^{1,p}_{\psi}(U';\mathbf{R}^m)$ this integral is set equal to $+\infty$) is lower semicontinuous with respect to the $L^p(U';\mathbf{R}^m)$ convergence. Hence, we have

$$\int_{U'} \psi(x_n,Du)\,dx \le \liminf_{j\to+\infty} \int_{U'} \psi(x_n,Du_j)\,dx \le J_0(u,U).$$

By the arbitrariness of U' we get

$$\int_U \psi(x_n,Du)\,dx \le J_0(u,U), \tag{4.14}$$

and in particular that $J_0(u,U) = +\infty$ if $u \in L^p(U;\mathbf{R}^m) \setminus W^{1,p}_{\psi}(U;\mathbf{R}^m)$, so that (i) is proved.

Let now $u \in W^{1,p}_{\psi}(\Omega;\mathbf{R}^m)$. We first assume that $U \subset\subset U' \subset\subset \Omega$. By using the lower semicontinuity of J_0 and Jensen's inequality, we have

$$\begin{aligned} J_0(u,U) &\le \liminf_{j\to+\infty} J_0(u_j,U) = \liminf_{j\to+\infty} \int_U \psi(x_n,Du_j)\,dx \\ &\le \liminf_{j\to+\infty} \int_U \int_{B(0,\frac{1}{j})} \rho_j(y)\psi(x_n, Du(x-(y,0)))\,dx\,dy \end{aligned}$$

$$
\begin{aligned}
&= \liminf_{j\to+\infty} \int_{B(0,\frac{1}{j})} \rho_j(y) \int_{U+(y,0)} \psi(x_n, Du)\,dx\,dy \\
&\le \liminf_{j\to+\infty} \int_{B(0,\frac{1}{j})} \rho_j(y)\,dy \int_{U'} \psi(x_n, Du)\,dx = \int_{U'} \psi(x_n, Du)\,dx.
\end{aligned}
$$

By the arbitrariness of U' we then get

$$
J_0(u, U) \le \int_U \psi(x_n, Du)\,dx, \tag{4.15}
$$

so that (ii) follows by taking (4.14) into account.

Finally, for arbitrary U, note that if $u \in W^{1,p}_\psi(\Omega; \mathbf{R}^m)$ then it can be approximated by a sequence (v_j) of functions in $W^{1,p}(\Omega; \mathbf{R}^m)$ such that $\int_\Omega \psi(x_n, Dv_j)\,dx$ are equi-bounded (we may use *e.g.* the argument in the proof of [46] Section 4.2 Theorem 3); hence, by the lower semicontinuity of $J'' = \Gamma\text{-}\limsup_j J_{\varepsilon_j}$, we have $J''(u) < +\infty$. This fact implies (as in Propositions 1.31 and 1.32) that J'' is inner-regular; i.e.,

$$
J''(u, U) = \sup\Big\{ J''(u, V) : V \subset\subset U \Big\}.
$$

Since (ii) holds with V in the place of U we easily get the thesis. □

The following proposition clarifies the structure of $W^{1,p}_\psi$, and implies that the restrictions of functions $u \in W^{1,p}_\psi(\Omega; \mathbf{R}^m)$ to relatively compact subsets of $\omega \times (t_k, t_{k+1})$ are characterized as those functions having directional derivatives $D_{k+1}, \dots, D_n$ p-summable.

Proposition 4.9 *Let $k = 1, \dots, n-2$ and $s \in (t_k, t_{k+1})$. There exist two positive constants $\alpha_k(s)$ and β_k such that*

$$
\alpha_k(s) \Big(\sum_{i=k+1}^{n-1} |\overline{F}\xi_i|^p + |F_n|^p \Big) \le \psi(t, F) \le \beta_k \Big(\sum_{i=k+1}^{n-1} |\overline{F}\xi_i|^p + |F_n|^p \Big) \tag{4.16}
$$

for all $F \in \mathbf{M}^{m\times n}$ and $t \in (t_k, s]$.

Proof. Since $\overline{F} \mapsto \varphi_\#(t, \overline{F})$ is positively homogeneous of degree p and convex, if $t \in (t_k, t_{k+1})$ we easily deduce that

$$
\varphi_\#(t, \overline{F}) \le c \sum_{i=k+1}^{n-1} \varphi_\#(t, \Xi_i) |\overline{F}\xi_i|^p
$$

where

$$\Xi_i = \begin{pmatrix} \xi_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If we denote

$$\beta'_k = \max_{i=k+1,\dots,n-1} \sup_{t\in[0,1)} c\,\varphi_\#(t,\Xi_i)$$

then

$$\varphi_\#(t,\overline{F}) \le \beta'_k \sum_{i=k+1}^{n-1} |\overline{F}\xi_i|^p. \tag{4.17}$$

On the other hand we have that

$$\begin{aligned}\frac{\varphi_\#(t,\overline{F})}{\sum_{i=k+1}^{n-1}|\overline{F}\xi_i|^p} &\ge c\,\frac{\varphi_\#(t,(\overline{F}\xi_{k+1},\dots,\overline{F}\xi_{n-1}))}{\|(\overline{F}\xi_{k+1},\dots,\overline{F}\xi_{n-1})\|^p} \\ &\ge c\inf\{\varphi_\#(t,G)\,:\,G\in S^{n-1}\cap \mathrm{Ker}\varphi_\#{}^\perp\}\end{aligned}$$

by p-homogeneity. Note that $t\mapsto c\,\inf\{\varphi_\#(t,G)\,:\,G\in S^{n-1}\cap\mathrm{Ker}\varphi_\#{}^\perp\} = c(t)$ is decreasing on $(0,1)$ and

$$\inf_{t\in(t_k,s]} c(t) = \alpha'_k(s) > 0,$$

so that we get

$$\varphi_\#(t,\overline{F}) \ge \alpha'_k(s) \sum_{i=k+1}^{n-1} |\overline{F}\xi_i|^p. \tag{4.18}$$

Let

$$\alpha_k(s) = \min\{\alpha'_k(s), \inf_{t\in(t_k,s]} \mathcal{L}_{n-1}(E_t\cap(0,1)^{n-1})\}$$

and

$$\beta_k = \max\{\beta'_k, 1\},$$

then (4.16) follows by Theorem 4.3, (4.17) and (4.18). □

Proposition 4.10 *Fix $t\in(t_k,t_{k+1})$, for $k=0,\dots,n-1$ ($t_0=0, t_n=1$). If ψ is given by* (4.12) *then*

$$\psi(t,F) = \min\Big\{\int_{(0,1)^n\cap(E_t\times(0,1))} \|Dw\|^p\,dx : \\ w\in W^{1,p}_{\mathrm{loc}}(E_t\times(0,1);\mathbf{R}^m),\ w - Fx \text{ 1-periodic}\Big\}.$$

Proof. Let w be a test function for the minimum problem above, then

$$\begin{aligned}
&\int_{(0,1)^n\cap(E_t\times(0,1))} \|Dw\|^p\,dx \\
&= \int_{(0,1)^n\cap(E_t\times(0,1))} \|D_\alpha w\|^p\,dx + \int_{(0,1)^n\cap(E_t\times(0,1))} |D_n w|^p\,dx \\
&\geq \int_0^1 \min\Big\{\int_{E_t\cap(0,1)^{n-1}} \|Dv\|^p\,dx_\alpha : \\
&\qquad\qquad v\in W^{1,p}_{\rm loc}(E_t;\mathbf{R}^m),\ v-\overline{F}x_\alpha \text{ 1-periodic}\Big\}\,dx_n \\
&\quad + \int_{E_t\cap(0,1)^{n-1}} \Big(\int_0^1 |D_n w|^p\,dx_n\Big)\,dx_\alpha \\
&\geq \varphi_{\#}(t,\overline{F}) + \mathcal{L}_{n-1}(E_t\cap(0,1)^{n-1})|F_n|^p = \psi(t,F)
\end{aligned}$$

by Jensen's inequality and the description of $\varphi_{\#}$ (see Theorem 4.2); hence,

$$\begin{aligned}
\psi(t,F) \leq \min\Big\{&\int_{(0,1)^n\cap E_t\times(0,1)} \|Dw\|^p\,dx : \\
&w\in W^{1,p}_{\rm loc}(E_t\times(0,1);\mathbf{R}^m),\ w-Fx \text{ 1-periodic}\Big\}
\end{aligned}$$

by Theorem 4.3.

Conversely, given a function v such that $v-\overline{F}x_\alpha$ is 1-periodic, we can construct a test function w, such that $w-Fx$ 1-periodic, as

$$w(x) = v(x_\alpha) - F_n x_n\,.$$

We then have

$$\begin{aligned}
&\int_{(0,1)^n\cap(E_t\times(0,1))} \|Dw\|^p\,dx \\
&= \int_{(0,1)^n\cap(E_t\times(0,1))} (\|D_\alpha v\|^p + |F_n|^p)\,dx \\
&= \int_{E_t\cap(0,1)^{n-1}} \|D_\alpha v\|^p\,dx_\alpha + \mathcal{L}_{n-1}(E_t\cap(0,1)^{n-1})|F_n|^p \\
&\geq \min\Big\{\int_{(0,1)^n\cap(E_t\times(0,1))} \|Dw\|^p\,dx : \\
&\qquad w\in W^{1,p}_{\rm loc}(E_t\times(0,1);\mathbf{R}^m),\ w-Fx \text{ 1-periodic}\Big\}
\end{aligned}$$

and hence the converse inequality

$$\psi(t,F) = \min\Big\{\int_{E_t\cap(0,1)^{n-1}} \|Dv\|^p\,dx_\alpha :$$

$$
\begin{aligned}
&\qquad v \in W^{1,p}_{\rm loc}(E_t;\mathbf{R}^m),\ v - \overline{F}x_\alpha \text{ 1-periodic}\Big\}\\
&+\mathcal{L}_{n-1}(E_t \cap (0,1)^{n-1})|F_n|^p\\
&\geq \min\Big\{\int_{(0,1)^n\cap(E_t\times(0,1))} \|Dw\|^p\,dx :\\
&\qquad w \in W^{1,p}_{\rm loc}(E_t\times(0,1);\mathbf{R}^m),\ w - Fx \text{ 1-periodic}\Big\}
\end{aligned}
$$

is obtained as desired. □

Now we can turn our attention to the case with a general W. Now that a natural domain for the limit functional is defined, we can easily state and prove a compactness result that partly improves Proposition 4.1.

Theorem 4.11 *Let J_ε be given by (4.9). Then for every sequence (ε_j) of positive numbers converging to 0 there exists a subsequence (not relabeled) such that the Γ-limit*

$$J_0(u,U) = \Gamma\text{-}\lim_{j\to+\infty} J_{\varepsilon_j}(u,U)$$

exists for all $u \in W^{1,p}_\psi(\Omega;\mathbf{R}^m)$ and U open subsets of Ω. Moreover $J_0(u,\cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$.

Proof. By (4.6) and Proposition 4.8 we deduce the condition

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} J_\varepsilon(u,U) \leq \beta\int_U (1+\psi(x_n,Du))\,dx \tag{4.19}$$

if $u \in W^{1,p}_\psi(U;\mathbf{R}^m)$ and U is an open subset of Ω. Then, we can follow the Steps 1–3 in Section 4.1 to prove the compactness of (J_ε) and that $J_0(u,\cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$. □

4.4 Homogenization of cylindrical domains

It remains now to extend the integral representation of Proposition 4.1 and characterize its integrand. We first deal with the case of 'cylindrical' domains; *i.e.,* we consider χ_E in place of f, with E a 1-periodic open subset of $\mathbf{R}^{n-1}$.

Let $t_1,\dots,t_{n-1}$ be the points in $(0,1)$ introduced to characterize the 'degenerate weighted Sobolev Space' in Definition 4.5. Since in the following we will choose $E = E_t$ (E_t defined as $\{x_\alpha :\ f(x_\alpha) > |t|\}$) we introduce the following notation: with fixed $t \in (0,1)$, $t \neq t_k$ for $k = 1,\dots,n-1$, consider the set E_t and the functional

$$J^t_\varepsilon(u,U) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon\cap U_\varepsilon} W\Big(\frac{x_\alpha}{\varepsilon}, Du\Big)\,dx & \text{if } u \in W^{1,p}(\Omega_\varepsilon\cap U_\varepsilon;\mathbf{R}^m)\\ +\infty & \text{otherwise,}\end{cases} \tag{4.20}$$

where $U_\varepsilon = U \cap (\varepsilon E_t \times (-1,1))$. Note that the integrand of J^t_ε satisfies the following growth conditions

$$\gamma g(x,A) \le \chi_{E_t \times (-1,1)} W(x_\alpha, A) \le \beta(1 + g(x,A)) \tag{4.21}$$

where $g(x,A) = \chi_{E_t \times (-1,1)}(x)\|A\|^p$ is obviously 1-periodic in x, convex in A and satisfying

$$0 \le g(x,A) \le 1 + \|A\|^p \text{ and } g(x,2A) \le c(1 + g(x,A))$$

for all $A \in \mathbf{M}^{m\times n}$.

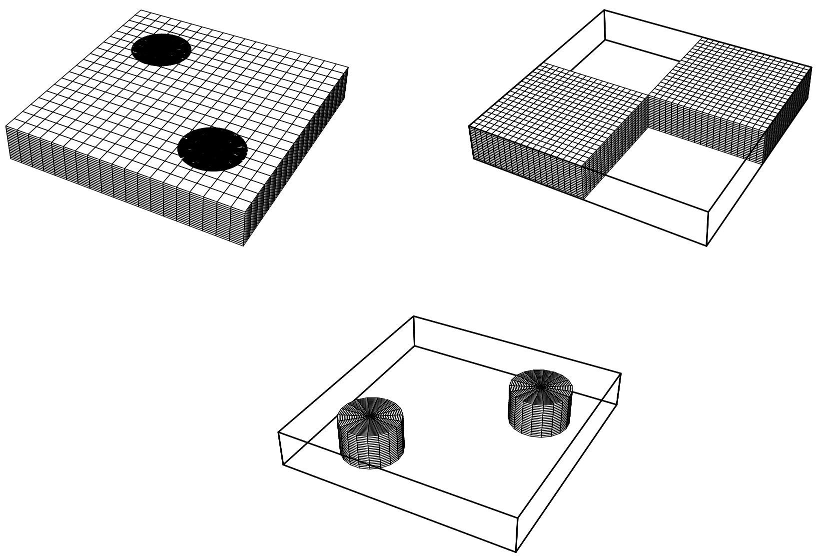

FIG. 4.4. cylindrical domains $E_t \times (-1,1)$ related to the function f in Figure 4.1 for different values of t

Remark 4.12 Note that if we fix $t \in (t_{k-1}, t_k)$ and consider χ_{E_t} in place of f then $W^{1,p}_\psi(\Omega;\mathbf{R}^m)$ turns out to be the space

$$W^{1,p}_k(\Omega;\mathbf{R}^m) = \{u \in L^p(\Omega;\mathbf{R}^m) : D_n u \in L^p(\Omega;\mathbf{R}^m), D_{(\xi_i,0)} u \in L^p(\Omega;\mathbf{R}^m) \\ i = k, \ldots, n-1\}$$

if $k = 1, \ldots, n-1$, and

$$W^{1,p}_n(\Omega;\mathbf{R}^m) = \{u \in L^p(\Omega;\mathbf{R}^m) : D_n u \in L^p(\Omega;\mathbf{R}^m)\}.$$

if $k = n$.

Theorem 4.13 *Let $t \in (t_{k-1}, t_k)$ and let $J^t_\varepsilon(\cdot, U)$ be defined by (4.20). Then the Γ-limit with respect to the $L^p(\Omega; \mathbf{R}^m)$-convergence*

$$J^t_0(u, U) = \int_U W^t_{\rm hom}(Du)\,dx$$

exists for each $u \in W^{1,p}_k(\Omega; \mathbf{R}^m)$ and U open subset of Ω, where $W^t_{\rm hom}$ is given by

$$W^t_{\rm hom}(A) = \lim_{T\to+\infty} \inf\Big\{\frac{1}{T^n}\int_{(0,T)^n} \chi_{E_t}(x_\alpha) W(x_\alpha, A + Du(x))\,dx : \\ u \in W^{1,p}_0((0,T)^n; \mathbf{R}^m)\Big\}$$

for all $A \in \mathbf{M}^{m\times n}$.

Proof. By taking Theorem 4.11 into account with χ_{E_t} in the place of f, and repeating word for word the proof of the integral representation Theorem 1.35, replacing $W^{1,p}(\Omega; \mathbf{R}^m)$ by $W^{1,p}_k(\Omega; \mathbf{R}^m)$, we obtain an integral representation on the whole $W^{1,p}_k(\Omega; \mathbf{R}^m)$. The integrand of this representation must coincide with the function $\varphi = \varphi(x_n, F)$ provided by Proposition 4.1 with χ_{E_t} in the place of f. Since the functionals are clearly invariant by translations in the direction x_n we have indeed $\varphi = \varphi(F)$. It remains to prove the asymptotic formula.

Let us fix $A \in \mathbf{M}^{m\times n}$ and for $T > 0$ set

$$h_T(A) = \inf\Big\{\frac{1}{T^n}\int_{(0,T)^n} \chi_{E_t}(x_\alpha) W(x_\alpha, A + Du(x))\,dx : \\ u \in W^{1,p}_0((0,T)^n; \mathbf{R}^m)\Big\} \tag{4.22}$$

We will prove the formula by showing first that $\varphi(A) \le \liminf_{T\to+\infty} h_T(A)$ and then that $\limsup_{T\to+\infty} h_T(A) \le \varphi(A)$.

For the first part, let $u^T \in \mathrm{W}^{1,p}_0((0,T)^n; \mathbf{R}^m)$ be such that

$$\frac{1}{T^n}\int_{(0,T)^n} \chi_{E_t}(x_\alpha) W(x_\alpha, A + Du(x))\,dx \le h_T(A) + \frac{1}{T}; \tag{4.23}$$

extend u^T trivially to $(0, [T+1])^n$, then to all of $\mathbf{R}^n$ by periodicity and set $u^T_k(x) = \varepsilon_{j_k} u^T(x/\varepsilon_{j_k})$. Note that $u^T_k \to 0$ in $L^p_{\rm loc}(\mathbf{R}^n; \mathbf{R}^m)$ as $k \to +\infty$.

Taking into account that the number of squares of side $[T+1]$ intersecting the square of side $1/\varepsilon_{j_k}$ is

$$s^n = \Big(\Big[\frac{(\varepsilon_{j_k})^{-1}}{[T+1]}\Big] + 1\Big)^n \le \Big(\frac{1}{\varepsilon_{j_k} T} + 1\Big)^n,$$

by periodicity of u^T and $W(\cdot, F)$, we have

$$\begin{aligned}
\int_{(0,1)^n} \varphi(A)\,dx &\le \liminf_k \int_{(0,1)^n} \chi_{E_t}\Big(\frac{x_\alpha}{\varepsilon_{j_k}}\Big) W\Big(\frac{x_\alpha}{\varepsilon_{j_k}}, A + Du_k^T(x)\Big)\,dx \\
&= \liminf_k \varepsilon_{j_k}^n \int_{(0,1/\varepsilon_{j_k})^n} \chi_{E_t}(x_\alpha) W(x_\alpha, A + Du^T(x))\,dx \\
&\le \liminf_k \varepsilon_{j_k}^n \int_{(0,s[T+1])^n} \chi_{E_t}(x_\alpha) W(x_\alpha, A + Du^T(x))\,dx \\
&= \liminf_k s^n \varepsilon_{j_k}^n \int_{(0,[T+1])^n} \chi_{E_t}(x_\alpha) W(x_\alpha, A + Du^T(x))\,dx \\
&\le \liminf_k \varepsilon_{j_k}^n \Big(\frac{1}{\varepsilon_{j_k} T} + 1\Big)^n \int_{(0,T)^n} \chi_{E_t}(x_\alpha) W(x_\alpha, A + Du^T(x))\,dx \\
&\qquad + c([T+1]^n - T^n)\Big(\frac{1}{T} + \varepsilon_{j_k}\Big)^n (1 + |A|^p) \\
&= \frac{1}{T^n} \int_{(0,T)^n} \chi_{E_t}(x_\alpha) W(x_\alpha, A + Du^T(x))\,dx \\
&\qquad + c\frac{1}{T^n}([T+1]^n - T^n)(1 + |A|^p).
\end{aligned}$$

Hence, by (4.23) and taking the limit as $T \to +\infty$, we get

$$\varphi(A) \le \liminf_{T\to+\infty} h_T(A). \tag{4.24}$$

For the second part, we need to use the L^p-fundamental estimate (see (1.18)). Let (u_k) be a sequence in $W^{1,p}((0,1)^n;\mathbf{R}^m)$ such that $u_k \to 0$ in $L^p((0,1)^n;\mathbf{R}^m)$ and

$$\varphi(A) = \lim_k \int_{(0,1)^n} \chi_{E_t}\Big(\frac{x_\alpha}{\varepsilon_{j_k}}\Big) W\Big(\frac{x_\alpha}{\varepsilon_{j_k}}, A + Du_k\Big)\,dx. \tag{4.25}$$

Fix $U' \subset\subset U \subset\subset Q$ with $Q = (0,1)^n$ and let $V = Q \setminus U'$. Then for every $\sigma > 0$, there exists M_σ and, with fixed k, for the functions u_k and $v = 0$ there exists $w_k = \varphi_k u_k$ (where φ_k is a cut-off function between U' and U, so that $w_k \in \mathrm{W}_0^{1,p}(Q;\mathbf{R}^m)$) such that

$$\begin{aligned}
\int_Q \chi_{E_t}\Big(\frac{x_\alpha}{\varepsilon_{j_k}}\Big) W\Big(\frac{x_\alpha}{\varepsilon_{j_k}}, A + Dw_k\Big)\,dx &\le (1+\sigma)\Big(\int_V \chi_{E_t}\Big(\frac{x_\alpha}{\varepsilon_{j_k}}\Big) W\Big(\frac{x_\alpha}{\varepsilon_{j_k}}, A\Big)\,dx \\
&\qquad + \int_U \chi_{E_t}\Big(\frac{x_\alpha}{\varepsilon_{j_k}}\Big) W\Big(\frac{x_\alpha}{\varepsilon_{j_k}}, A + Du_k\Big)\,dx\Big) \\
&\qquad + M_\sigma \int_{U\setminus U'} |u_k|^p\,dx + \sigma.
\end{aligned} \tag{4.26}$$

By a change of variable

$$\int_Q \chi_{E_t}\Big(\frac{x_\alpha}{\varepsilon_{j_k}}\Big) W\Big(\frac{x_\alpha}{\varepsilon_{j_k}}, A + Dw_k\Big)\,dx = \varepsilon_{j_k}^n \int_{(0,1/\varepsilon_{j_k})^n} \chi_{E_t}(x_\alpha) W(x_\alpha, A + Dv_k)\,dx \tag{4.27}$$

where $v_k(x) = 1/\varepsilon_{j_k} w_k(\varepsilon_{j_k} x) \in W_0^{1,p}((0,1/\varepsilon_{j_k})^n;\mathbf{R}^m)$, since $w_k \in W_0^{1,p}(Q;\mathbf{R}^m)$. Thus, by (4.22), (4.27) and (4.26) we get

$$\frac{1}{1+\sigma}\,h_{(1/\varepsilon_{j_k})}(A) \le c\,|Q\setminus U'|\,(1+|A|^p) + \int_Q \chi_{E_t}\Big(\frac{x_\alpha}{\varepsilon_{j_k}}\Big) W\big(\frac{x_\alpha}{\varepsilon_k}, A + Du_k\big)\,dx$$
$$+ \frac{M_\sigma}{1+\sigma}\int_{Q\setminus U'} |u_k|^q\,dx + \frac{\sigma}{1+\sigma}.$$

Taking limits on both sides and using the fact that $u_k \to 0$ in $\mathrm{L}^p((0,1)^n;\mathbf{R}^m)$, by (4.25) we obtain

$$\frac{1}{1+\sigma}\,\limsup_k h_{(1/\varepsilon_k)}(A) \le c\,|Q\setminus U'|\,(1+|A|^p) + \varphi(A) + \frac{\sigma}{1+\sigma}, \tag{4.28}$$

for all $\sigma > 0$ and $U' \subset\subset Q$. Let $\sigma \to 0$ and $U' \to Q$; summing up (4.28) and (4.24) we get

$$\limsup_{k\to+\infty} h_{(1/\varepsilon_{j_k})}(A) \le \varphi(A) \le \liminf_{T\to+\infty} h_T(A) \le \liminf_{k\to+\infty} h_{(1/\varepsilon_{j_k})}(A),$$

and then

$$\varphi(A) = \lim_{k\to+\infty} h_{(1/\varepsilon_{j_k})}(A) = \liminf_{T\to+\infty} h_T(A).$$

This equality proves that φ is independent of (ε_{j_k}), and that $\lim_{T\to+\infty} h_T(A)$ exists since we can choose a sequence ε_{j_k} such that

$$\limsup_{T\to+\infty} h_T(A) = \lim_{k\to+\infty} h_{(1/\varepsilon_{j_k})}(A).$$

□

4.5 The general case

We can eventually proceed to dealing with the general case.

Proposition 4.14 *Let J_ε be given by* (4.9)*. Then the Γ-limit*

$$J_0(u,U) = \Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(u,U)$$

exists for all $u \in W_\psi^{1,p}(\Omega;\mathbf{R}^m)$ and U open subsets of Ω. Moreover, for such u we have

$$J_0(u,U) = \int_U \varphi(x_n, Du)\,dx,$$

where φ is given by Proposition 4.1*.*

Proof. We have to extend the representation of J_0 given by Proposition 4.1 to $W_\psi^{1,p}(\Omega;\mathbf{R}^m)$. Note that φ is a Carathéodory function (see Theorem 1.35, Step

3). As explained in Step 3 of Section 4.1, a crucial argument used to obtain an integral representation result is the continuity in $W^{1,p}_\psi(\Omega;\mathbf{R}^m)$ of the functional

$$u \mapsto \int_U \varphi(x_n, Du)\,dx$$

along some strongly converging sequences of piecewise-affine functions. We only prove this property, as the rest of the proof follows exactly that of Theorem 1.35 (Steps 1–3, 5 and 6; the proof below replaces Step 4).

Let $U = \bigcup_{k=0}^{n-1} U_k$ where $U_k \subset\subset \omega \times (t_k, t_{k+1})$, $(t_0 = 0, t_n = 1)$; we can find functions $u_j \in W^{1,p}_\psi(\Omega;\mathbf{R}^m)$ such that their restrictions to U are piecewise affine and $u_j, D_n u_j$ converge strongly to $u, D_n u$ in $L^p(U;\mathbf{R}^m)$, respectively, while $D_{(\xi_i,0)}u_j$ converge strongly to $D_{(\xi_i,0)}u$ in $L^p(U_i;\mathbf{R}^m)$.

We will use some estimates deriving from the inequality $\varphi(t,F) \leq \beta(1 + \psi(t,F))$, which follows trivially from (4.6). By Proposition 4.9 we have that

$$\psi(x_n, Du) \leq \beta_k\Big(\sum_{i=k+1}^{n-1} |D_{(\xi_i,0)}u|^p + |D_n u|^p\Big)$$
$$\psi(x_n, Du_j) \leq \beta_k\Big(\sum_{i=k+1}^{n-1} |D_{(\xi_i,0)}u_j|^p + |D_n u_j|^p\Big)$$

on $\omega \times (t_k, t_{k+1})$. Note that by (4.19)

$$\int_U \varphi(x_n, Du_j)\,dx \leq \sum_{k=0}^{n-2} \int_{U_k \cap \omega \times (t_k,t_{k+1})} \beta\Big(1 + \sum_{i=k+1}^{N} \beta_k |D_{(\xi_i,0)}u_j|^p\Big)\,dx + \beta \int_U \beta_k |D_n u_j|^p\,dx.$$

If we use the continuity of φ in the second variable and apply Fatou's lemma to the sequences

$$\beta \int_U \beta_k |D_n u_j|^p\,dx + \sum_{k=0}^{n-2} \int_{U_k \cap \omega \times (t_k,t_{k+1})} \beta\Big(1 + \sum_{i=k+1}^{n-1} \beta_k |D_{(\xi_i,0)}u_j|^p\Big)\,dx \pm \int_U \varphi(x_n, Du_j)\,dx$$

we get that

$$\int_U \varphi(x_n, Du)\,dx = \lim_{j\to+\infty} \int_U \varphi(x, Du_j)\,dx.$$

Hence, we have proved the integral representation for sets of the type $U = \bigcup_{k=0}^{n-1} U_k$ where $U_k \subset\subset \omega \times (t_k, t_{k+1})$. A symmetric argument applies to the case where $U = \bigcup_{k=0}^{n-1} U_k$, with $U_k \subset\subset \omega \times (-t_{k+1}, -t_k)$. Since $J_0(u, \cdot)$ is a measure absolutely continuous with respect to Lebesgue measure, we conclude that the integral representation holds for all open subsets U of Ω. □

Finally, the oscillating-boundary homogenization theorem reads as follows.

Theorem 4.15 *Let J_ε be given by (4.7). Then the Γ-limit with respect to the $L^p(\Omega; \mathbf{R}^m)$-convergence*

$$J_0(u) = \Gamma\text{-}\lim_{\varepsilon \to 0} J_\varepsilon(u)$$

exists for all $u \in L^p(\Omega; \mathbf{R}^m)$, and we have

$$J_0(u) = \begin{cases} \displaystyle\int_\Omega W_{\mathrm{hom}}(|x_n|, Du)\, dx & \text{if } u \in W^{1,p}_\psi(\Omega; \mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where $W_{\mathrm{hom}}(t, A) = W^t_{\mathrm{hom}}(A)$ for a.e. $t \in (0,1)$, and W^t_{hom} is given by Theorem 4.13. Moreover, if $u \in W^{1,p}(\Omega; \mathbf{R}^m)$ there exists a family (u_ε) converging to u in $L^p(\Omega; \mathbf{R}^m)$, such that $u - u_\varepsilon$ has compact support in Ω and $J_0(u) = \lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon)$.

Proof. It is sufficient to compute the Γ-limit for $u \in W^{1,p}_\psi(\Omega; \mathbf{R}^m)$, since by comparison with Proposition 4.8(i) we immediately have $J_0(u) = +\infty$ if $u \notin W^{1,p}_\psi(\Omega; \mathbf{R}^m)$. Let φ be given by Proposition 4.1; it remains to prove that φ satisfies an asymptotic formula.

Let $x_n > 0$, let $0 < \rho < x_n$ and consider the functionals (4.20) with $t = x_n - \rho$ and $t = x_n$ so that

$$\begin{aligned} J^{x_n - \rho}_\varepsilon&(Ax, (0,1)^{n-1} \times (x_n - \rho, x_n)) \\ &\geq \int_{(0,1)^{n-1}\times(x_n-\rho,x_n)} \chi_{E_{y_n}}\Big(\frac{y_\alpha}{\varepsilon}\Big) W\Big(\frac{x_\alpha}{\varepsilon}, A\Big)\, dy \\ &\geq J^{x_n}_\varepsilon(Ax, (0,1)^{n-1} \times (x_n - \rho, x_n))\,. \end{aligned}$$

By Theorem 4.13

$$\begin{aligned} \rho\, W^{x_n-\rho}_{\mathrm{hom}}(A) &\geq \Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(Ax, (0,1)^{n-1} \times (x_n - \rho, x_n)) \\ &\geq \rho\, W^{x_n}_{\mathrm{hom}}(A). \end{aligned}$$

Taking into account that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(Ax, (0,1)^{n-1} \times (x_n - \rho, x_n)) = \int_{(0,1)^{n-1}\times(x_n-\rho,x_n)} \varphi(y_n, A)\, dy$$

we get

$$W^{x_n-\rho}_{\mathrm{hom}}(A) \leq \frac{1}{\rho}\int_{(x_n-\rho,x_n)} \varphi(y_n, A)\, dy_n \leq W^{x_n}_{\mathrm{hom}}(A).$$

Since $t \mapsto W^t_{\rm hom}(A)$ and $t \mapsto \varphi(t, A)$ are decreasing functions on $(0, 1)$, there exists a subset M of $(0, 1)$, $|M| = 0$, such that they are continuous on $(0, 1) \setminus M$; hence, by passing to the limit as $\rho \to 0$ we get

$$\varphi(x_n, A) = W^{x_n}_{\rm hom}(A)$$

for every $x_n \in (0, 1) \setminus M$. For $x_n < 0$ it suffices to apply a symmetric argument.

The last statement follows by a well-known argument of stability of Γ-convergence by compatible boundary data due to De Giorgi (see Proposition 1.33) □

5

THIN FILMS WITH FAST-OSCILLATING PROFILE

In this chapter we prove that the Γ-limit with respect to the L^p-convergence of functionals

$$E_{\varepsilon,\delta}(u) = \int_{\Omega(\varepsilon,\delta)} W(Du)\,dx,$$

where the set $\Omega(\varepsilon,\delta)$ is of the form

$$\Omega(\varepsilon,\delta) = \Big\{ x \in \mathbf{R}^n : \ |x_n| < \varepsilon\, f\Big(\frac{x_\alpha}{\delta}\Big), x_\alpha \in \omega \Big\}, \tag{5.1}$$

when $\varepsilon \to 0$ and $\delta << \varepsilon$, is given by first applying the theory constructed in Chapter 4 with ε as a parameter and letting $\delta \to 0$, and subsequently letting $\varepsilon \to 0$. The final result can be summarized as follows, in a n-dimensional setting.

Theorem 5.1 *Let $f : \mathbf{R}^{n-1} \to [0,1]$ be a 1-periodic lower semicontinuous function with $0 < \min f \le \sup f = 1$, let $W : \mathbf{M}^{m\times n} \to [0,+\infty)$ be a convex function satisfying*

$$\gamma|F|^p \le W(F) \le \beta(1 + |F|^p)$$

for all $F \in \mathbf{M}^{m\times n}$ and for some $1 < p < +\infty$, $0 < \gamma \le \beta$. Let $\delta : (0,+\infty) \to (0,+\infty)$ be such that

$$\lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\varepsilon} = 0.$$

Let ω be a bounded open subset of $\mathbf{R}^{n-1}$ and let $\Omega_\varepsilon \subset \omega \times (-1,1)$ be defined by

$$\Omega_\varepsilon = \Big\{ x \in \mathbf{R}^n : \ |x_n| < f\Big(\frac{x_\alpha}{\delta(\varepsilon)}\Big), x_\alpha \in \omega \Big\}. \tag{5.2}$$

Define $E_\varepsilon : L^p(\omega \times (-1,1); \mathbf{R}^m) \to [0,+\infty]$ by

$$E_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} W\Big(D_\alpha u, \frac{1}{\varepsilon} D_n u\Big)\,dx & \text{if } u\big|_{\Omega_\varepsilon} \in W^{1,p}(\Omega_\varepsilon; \mathbf{R}^m) \\ +\infty & \text{otherwise.} \end{cases} \tag{5.3}$$

Then the Γ-limit with respect to the L^p-convergence as $\varepsilon \to 0$ of E_ε is given by

$$E(u) = \begin{cases} \displaystyle\int_{\omega\times(-1,1)} \overline{W}_{\rm hom}(D_\alpha u)\,dx & \text{if } u \in W^{1,p}(\omega\times(-1,1); \mathbf{R}^m) \text{ and } D_n u = 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{5.4}$$

where $\overline{W}_{\rm hom} : \mathbf{M}^{m\times(n-1)} \to [0,+\infty)$ *is given by*

$$\overline{W}_{\rm hom}(\overline{F}) = \int_0^1 \inf_{F_n} W_{\rm hom}(t, \overline{F}|F_n)\, dt, \tag{5.5}$$

and $W_{\rm hom}$ *by*

$$W_{\rm hom}(t,F) = \inf\Big\{\int_{(0,1)^n} \chi_{E_t}(x_\alpha)\, W(F + Du(x))\, dx : \\ u \in W^{1,p}_{\rm loc}(\mathbf{R}^n;\mathbf{R}^m) \text{ 1-periodic}\Big\} \tag{5.6}$$

for all $t \in (0,1)$ *and* $F \in \mathbf{M}^{m\times n}$, *where* $E_t = \{f > t\}$.

5.1 Proof of the result

In order to simplify the proof without loosing sight of the main intricacies of the argument, we deal only with the case where $\varepsilon = 1/j$ and $\delta = \varepsilon^2$. The general case can be dealt with similarly, by introducing some error terms. We define, with a slight abuse of notation,

$$\Omega_k = \{x \in \Omega : \ |x_n| < f(kx_\alpha)\}$$

and for $k = j^2$, $j \in \mathbf{N}$

$$E_j(u,U) = \int_{\Omega_{j^2}\cap U} W(D_\alpha u|jD_n u)\, dx$$

for all $u_{|\Omega_{j^2}\cap U} \in W^{1,p}(\Omega_{j^2}\cap U;\mathbf{R}^m)$.

By the compactness result Theorem 2.5 in [29] we can suppose that there exists $W_0 : \mathbf{M}^{m\times(n-1)} \to [0,+\infty)$ such that $E_j(u,U)$ Γ-converge for all sets of the form $U = U' \times (-1,1)$ or $U = U' \times (0,1)$ to the functional given by

$$E_0(u,U) = \begin{cases} \displaystyle\int_U W_0(D_\alpha u)\, dx & \text{if } u \in W^{1,p}(U;\mathbf{R}^m) \text{ and } D_n u = 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{5.7}$$

Proposition 5.2 *For all* $\overline{F} \in \mathbf{M}^{m\times(n-1)}$ *define*

$$\widetilde{W}_{\rm hom}(\overline{F}) = \inf\Big\{\int_{(0,1)^n} W_{\rm hom}(x_n, Du + \overline{F})\, dx : \\ u \in W^{1,p}_{\rm loc}(\mathbf{R}^n;\mathbf{R}^m),\ u \text{ 1-periodic in } x_\alpha\Big\}. \tag{5.8}$$

Then

$$\overline{W}_{\rm hom}(\overline{F}) = \int_0^1 \widetilde{W}_{\rm hom}(t,\overline{F})\, dt, \tag{5.9}$$

where

$$\widetilde{W}_{\text{hom}}(t,\overline{F}) = \inf_{F_n} W_{\text{hom}}(t,\overline{F}|F_n) \tag{5.10}$$

and $\overline{F} \mapsto \widetilde{W}_{\text{hom}}(t,\overline{F})$ *is convex.*

PROOF. It can be easily proved that $\overline{F} \mapsto \widetilde{W}_{\text{hom}}(t,\overline{F})$ is convex.

With fixed $\eta > 0$, by the Measurable Selection Criterion (see e.g [36]), we can find $G_n(t)$ a measurable function such that

$$W_{\text{hom}}(t,\overline{F}|G_n) \leq \inf_{F_n} W_{\text{hom}}(t,\overline{F}|F_n) + \eta.$$

We can consider

$$u(x_\alpha, x_n) = \int_0^{x_n} G_n(s)\,ds$$

as test function in (5.8). We then get

$$\overline{W}_{\text{hom}}(\overline{F}) \leq \int_0^1 W_{\text{hom}}(x_n,\overline{F}|G_n(x_n))\,dx_n$$

and so

$$\overline{W}_{\text{hom}}(\overline{F}) \leq \int_0^1 \inf_{F_n} W_{\text{hom}}(t,\overline{F}|F_n)\,dt + \eta = \int_0^1 \widetilde{W}_{\text{hom}}(t,\overline{F})\,dt + \eta.$$

Conversely,

$$\begin{aligned}
\overline{W}_{\text{hom}}(\overline{F}) &\geq \inf\Big\{\int_{(0,1)^n} \widetilde{W}_{\text{hom}}(x_n, D_\alpha u + \overline{F})\,dx : \\
&\qquad u \in W^{1,p}_{\text{loc}}((0,1)^n;\mathbf{R}^m),\ u \text{ 1-periodic in } x_\alpha\Big\} \\
&\geq \int_0^1 \Big(\inf\Big\{\int_{(0,1)^{n-1}} \widetilde{W}_{\text{hom}}(t, D_\alpha u + \overline{F})\,dx_\alpha : \\
&\qquad u_{|(0,1)^{n-1}} \in W^{1,p}_{\text{loc}}((0,1)^{n-1};\mathbf{R}^m),\ u \text{ 1-periodic in } x_\alpha\Big\}\Big)\,dt \\
&\geq \int_0^1 \widetilde{W}_{\text{hom}}(t,\overline{F})\,dt
\end{aligned}$$

by Jensen's inequality. □

Theorem 5.3 *For all* $\overline{F} \in \mathbf{M}^{m\times(n-1)}$ *we have* $W_0(\overline{F}) = \overline{W}_{\text{hom}}(\overline{F})$.

PROOF. With fixed $\eta > 0$ let v be a test function for (5.8) such that

$$\int_{(0,1)^n} W_{\text{hom}}(x_n, Dv + \overline{F})\,dx \leq \overline{W}_{\text{hom}}(\overline{F}) + \eta\,.$$

By Theorem 4.15 there exists a sequence v_j converging to v such that $v_j = v$ on $\partial(0,1)^n$ (and, hence, in particular v_j is 1-periodic in x_α) and

$$\int_{(0,1)^n} W_{\rm hom}(x_n, Dv + \overline{F})\,dx = \lim_{j\to+\infty}\int_{\Omega_j\cap(0,1)^n} W(Dv_j + \overline{F})\,dx\,. \tag{5.11}$$

If we define $u_j(x_\alpha, x_n) = \frac{1}{j}v_j(jx_\alpha, x_n)$ then $u_j \to 0$ in $L^p((0,1)^n;\mathbf{R}^m)$ and

$$\begin{aligned}\int_{\Omega_j\cap(0,1)^n} W(Dv_j + \overline{F})\,dx &= \frac{1}{j^{n-1}}\int_{\Omega_j\cap((0,j)^{n-1}\times(0,1))} W(Dv_j + \overline{F})\,dx\\ &= \int_{\Omega_{j^2}\cap(0,1)^n} W(Dv_j(jy_\alpha, y_n) + \overline{F})\,dy\\ &= \int_{\Omega_{j^2}\cap(0,1)^n} W(D_\alpha u_j + \overline{F}|jD_n u_j)\,dy\\ &= E_j(u_j + \overline{F}x_\alpha, (0,1)^n);\end{aligned} \tag{5.12}$$

hence, we can conclude that

$$\begin{aligned}W_0(\overline{F}) &\le \liminf_{j\to+\infty} E_j(u_j + \overline{F}x_\alpha, (0,1)^n)\\ &= \liminf_{j\to+\infty}\int_{\Omega_j\cap(0,1)^n} W(Dv_j + \overline{F})\,dx\\ &\le \overline{W}_{\rm hom}(\overline{F}) + \eta\end{aligned}$$

by (5.7), (5.12), (5.11) and (5.8).

Now we prove the converse inequality. Let $u_j \to 0$ be such that

$$W_0(\overline{F}) = \lim_{j\to+\infty} E_j(u_j + \overline{F}x_\alpha, (0,1)^n).$$

By [29] Lemma 2.6 we can choose u_j 1-periodic in x_α; let v_j be defined by $v_j(x) = ju_j(x_\alpha/j, x_n)$. With fixed $j, N \in \mathbf{N}$, $(0,1)^n = \bigcup_{m=1}^N (0,1)^{n-1}\times((m-1)/N, m/N)$; we can define a function $v_{j,m}$ by setting

$$v_{j,m}(x_\alpha, x_n) = \begin{cases} v_j(x_\alpha, x_n + \frac{2k}{N}) & \text{if } \frac{m-1}{N} - \frac{2k}{N} < x_n < \frac{m}{N} - \frac{2k}{N}\\ \\ v_j(x_\alpha, \frac{2m}{N} - x_n - \frac{2k+2}{N}) & \text{if } \frac{m-1}{N} - \frac{2k+1}{N} < x_n < \frac{m}{N} - \frac{2k+1}{N}\end{cases}$$

for $k \in \mathbf{Z}$, which is 1-periodic in x_α and $2/N$-periodic in x_n. Hence, we can construct

$$w_{j,k}{}_{|(0,1)^{n-1}\times((m-1)/N, m/N)} = v_{j,m,k}(x)$$

where $v_{j,m,k}(x) = \frac{j}{k}v_{j,m}(\frac{k}{j}x)$, such that $w_{j,k}$ is $\frac{j}{k}$-periodic in x_α and

$$w_{j,k}{}_{|(0,1)^{n-1}\times((m-1)/N,m/N)} \to \Big(0, \Big(\int_{(0,1)^n} D_n v_{j,m} dx\Big) x_n\Big) = w^m$$

as $k \to +\infty$, in $L^p((0,1)^n; \mathbf{R}^m)$. In this case the functions $w_{j,k}$ defined as above belong to $W^{1,p}(\Omega_k \cap (0,1)^n; \mathbf{R}^m)$.

Finally, we define w such that

$$w_{|(0,1)^{n-1}\times((m-1)/N,m/N)} = w^m$$

which is 1-periodic in x_α. Let

$$A_j^{m/N} = \Omega_j \cap \{x_n = m/N\}$$

and

$$A_k^{m/N} = \Omega_k \cap \{x_n = m/N\},$$

we define

$$E_j^N = \bigcup_{m=1}^{N} A_j^{m/N} \times ((m-1)/N, m/N)$$

and

$$E_k^N = \bigcup_{m=1}^{N} A_k^{m/N} \times ((m-1)/N, m/N).$$

We restrict our analysis to the case where k/j odd, the other case being dealt with by introducing a small error term. Hence, if we use the notation

$$I_l(u, (0,1)^n) = \int_{E_l^N \cap (0,1)^n} W(Du)\, dx$$

($l = j$ or k) we have that

$$I_j(v_j + \overline{F} x_\alpha, (0,1)^n) = I_k(w_{j,k} + \overline{F} x_\alpha, (0,1)^n). \tag{5.13}$$

Reasoning as in Theorems 4.11 and 4.13 we get that

$$\begin{aligned} I_{\rm hom}(w + \overline{F} x_\alpha, (0,1)^n) &= \Gamma\text{-}\lim_{k\to+\infty} I_k(w + \overline{F} x_\alpha, (0,1)^n) \\ &= \sum_{m=1}^{N} \int_{(0,1)^{n-1}\times((m-1)/N,m/N)} W_{\rm hom}(m/N, Dw + \overline{F})\, dx \\ &= \sum_{m=1}^{N} \int_{(0,1)^{n-1}\times((m-1)/N,m/N)} W_{\rm hom}\Big(\frac{[x_n N]+1}{N}, Dw + \overline{F}\Big)\, dx \\ &= \int_{(0,1)^n} W_{\rm hom}\Big(\frac{[x_n N]+1}{N}, Dw + \overline{F}\Big)\, dx \end{aligned}$$

$$\geq \int_0^1 \widetilde{W}_{\rm hom}\Big(\frac{[x_n N]+1}{N}, \overline{F}\Big)\,dx_n$$

by (5.10). Taking the limit as $N \to +\infty$, we obtain

$$I_{\rm hom}(w + \overline{F}x_\alpha, (0,1)^n) \geq \overline{W}_{\rm hom}(\overline{F}) \tag{5.14}$$

by Proposition 5.2. Hence,

$$\begin{aligned} E_j(u_j + \overline{F}x_\alpha, (0,1)^n) &= \int_{\Omega_j\cap(0,1)^n} W(Dv_j + \overline{F})\,dx \\ &\geq \liminf_{k\to+\infty} I_k(w_{j,k} + \overline{F}x_\alpha, (0,1)^n) \\ &\geq \overline{W}_{\rm hom}(\overline{F}) \end{aligned}$$

by (5.12)-(5.14). By the choice of (u_j) we get the desired inequality. □

The proof of Theorem 5.1 will be complete once we observe that in the convex case formula (5.6) simplifies that in Theorem 4.13 (see Theorem 1.46).

5.2 Convergence of minimum problems

As an application of the Γ-convergence result of the previous section, we describe the asymptotic behaviour of problems of the form

$$\begin{aligned} m_{\varepsilon,\delta} = \min\Big\{ \int_{\Omega(\varepsilon,\delta)} W(Du)\,dx : u \in L^p(\omega\times(-\varepsilon,\varepsilon);\mathbf{R}^m), \\ u_{|\Omega(\varepsilon,\delta)} \in W^{1,p}(\Omega(\varepsilon,\delta);\mathbf{R}^m),\ u = \phi \text{ on } (\partial\omega)\times(-\varepsilon,\varepsilon)\Big\}, \end{aligned} \tag{5.15}$$

where $\phi = \phi(x_\alpha) \in W^{1,p}(\omega;\mathbf{R}^m)$, $\Omega(\varepsilon,\delta)$ is given by (5.1) and f and W satisfy the hypotheses of Theorem 5.1. By using Poincaré's inequality it can immediately be checked that problem (5.15) admits at least one solution for each choice of $\varepsilon, \delta > 0$. The asymptotic behaviour of these solutions when $\varepsilon \to 0$ and $\delta << \varepsilon$ is given by the following result.

Proposition 5.4 *Let ε and $\delta = \delta(\varepsilon)$ satisfy the hypotheses of Theorem* 5.1*, and for each ε let u_ε be a solution of* (5.15)*. Then, upon extracting a subsequence, there exist a sequence (v_ε) in $L^p(\omega\times(-1,1);\mathbf{R}^m)$ and a function $w \in W^{1,p}(\omega;\mathbf{R}^m)$ such that*

(i) *$v_\varepsilon = u_\varepsilon$ on $\Omega(\varepsilon,\delta(\varepsilon))$,*

(ii) *if $w_\varepsilon(x_\alpha,x_n) = v_\varepsilon(x_\alpha,\varepsilon x_n)$, then w_ε converges (with the identification $w(x) = w(x_\alpha)$) to w in $L^p((\omega\times(-1,1);\mathbf{R}^m)$,*

(iii) *w is a solution of the minimum problem*

$$\widetilde{m}_0 = \min\Big\{\int_\omega 2\overline{W}_{\rm hom}(D_\alpha u)\,dx_\alpha : u \in L^p(\omega;\mathbf{R}^m),\ u = \phi \textit{ on } \partial\omega\Big\}, \tag{5.16}$$

where $\overline{W}_{\rm hom}$ is defined by (5.5) *and* (5.6)*,*

(iv) $m_{\varepsilon,\delta(\varepsilon)}/\varepsilon$ *converges to* $\widetilde{m}_0$.

Proof. Note that, in the notation of Theorem 5.1, $\widetilde{u}_\varepsilon$ defined by $\widetilde{u}_\varepsilon(x_\alpha, x_n) = u_\varepsilon(x_\alpha, \varepsilon x_n)$ is a solution of

$$\widetilde{m}_\varepsilon = \frac{1}{\varepsilon} m_{\varepsilon,\delta(\varepsilon)} = \min\Big\{ \int_{\Omega_\varepsilon} W\Big(D_\alpha u, \frac{1}{\varepsilon} D_n u\Big)\, dx : u \in L^p(\omega \times (-1,1); \mathbf{R}^m),$$
$$u_{|\Omega_\varepsilon} \in W^{1,p}(\Omega_\varepsilon; \mathbf{R}^m),\ u = \phi \text{ on } (\partial\omega) \times (-1,1) \Big\}. \tag{5.17}$$

By [29] Remark 2.3, upon extracting a subsequence, there exist $w_\varepsilon \in L^p((\omega \times (-1,1); \mathbf{R}^m)$ converging to some w in $L^p((\omega \times (-1,1); \mathbf{R}^m)$, $D_n w = 0$ and $w_\varepsilon = \widetilde{u}_\varepsilon$ on Ω_ε. By the well-known property of the convergence of minima and minimizers of Γ-converging functionals (see Theorem 1.23), (iii) and (iv) follow from Theorem 5.1, since the Γ-limit is not influenced by the boundary value ϕ (see [29] Lemma 2.6). □

6

ASYMPTOTIC ANALYSIS OF PERIODICALLY-PERFORATED NONLINEAR MEDIA

6.1 Statement of the main result

In all that follows $p > 1$, $m \geq 1$, $n > p$ are fixed ($m, n \in \mathbf{N}$); the Γ-limit of a sequence (Φ_j) of functionals defined on $W^{1,p}(\Omega; \mathbf{R}^m)$ will be performed with respect to $L^p(\Omega; \mathbf{R}^m)$-convergence.

6.1.1 *Periodically perforated domains*

For all $\delta > 0$ we consider the lattice $\delta\mathbf{Z}^n$ whose points will be denoted by $x_i^\delta = \delta i$ ($i \in \mathbf{Z}^n$). Moreover, for all $i \in \mathbf{Z}^n$

$$B_i^\delta = B_{\delta^{n/(n-p)}}(x_i^\delta)$$

denotes the ball of centre x_i^δ and radius $\delta^{n/(n-p)}$. The main result of the chapter is the following.

Theorem 6.1 *Let Ω be a bounded open subset of $\mathbf{R}^n$ with $|\partial\Omega| = 0$. Let $f : \mathbf{M}^{m\times n} \to [0, +\infty)$ be a Borel function such that $f(0) = 0$ and satisfying a* growth condition of order p: *there exist two constants $c_1, c_2 > 0$ such that*

$$c_1(|A|^p - 1) \leq f(A) \leq c_2(|A|^p + 1) \qquad \textit{for all } A \in \mathbf{M}^{m\times n}. \tag{6.1}$$

Let (δ_j) be a sequence of strictly positive numbers converging to 0. Then, upon possibly extracting a subsequence, for all $A \in \mathbf{M}^{m\times n}$ there exist the limit

$$g(A) = \lim_j \delta_j^{\frac{np}{n-p}} Qf\Big(\delta_j^{-\frac{n}{n-p}} A\Big), \tag{6.2}$$

where Qf denotes the quasiconvexification of f, so that the value

$$\varphi(z) = \inf\Big\{\int_{\mathbf{R}^n} g(D\zeta)dx : \zeta - z \in W^{1,p}(\mathbf{R}^n; \mathbf{R}^m),\ \zeta = 0 \textit{ on } B_1(0)\Big\} \tag{6.3}$$

is well defined for all $z \in \mathbf{R}^m$. Moreover, the functionals $F_j : W^{1,p}(\Omega; \mathbf{R}^m) \to [0, +\infty]$ defined by

$$F_j(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\, dx & \textit{if } u = 0 \textit{ a.e. on } \bigcup_{i\in\mathbf{Z}^n} B_i^{\delta_j} \cap \Omega \\ +\infty & \textit{otherwise} \end{cases} \tag{6.4}$$

Γ-converge with respect to the $L^p(\Omega;\mathbf{R}^m)$-convergence to the functional $F : W^{1,p}(\Omega;\mathbf{R}^m) \to [0,+\infty)$ defined by

$$F(u) = \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx. \tag{6.5}$$

Corollary 6.2 *If f is positively homogeneous of degree p then the limit is independent of the subsequence and*

$$\varphi(z) = \inf\Big\{\int_{\mathbf{R}^n} f(D\zeta)\,dx : \zeta - z \in W^{1,p}(\mathbf{R}^n;\mathbf{R}^m),\ \zeta = 0 \text{ on } B_1(0)\Big\} \tag{6.6}$$

for all $z \in \mathbf{R}^m$.

Proof. It suffices to remark that in this case formula (6.2) gives $g = Qf$ and that we may replace Qf by f in (6.3) by using Remark 1.21 and Theorem 1.12. □

Corollary 6.3 (Convergence of minimum problems) *Let (δ_j) satisfy the thesis of Theorem 6.1. Then for all $\phi \in W^{-1,p}(\Omega;\mathbf{R}^m)$ the minimum values*

$$m_j = \inf\Big\{F_j(u) + \langle\phi,u\rangle :\ u \in W_0^{1,p}(\Omega;\mathbf{R}^m)\Big\}$$

converge to

$$m = \min\Big\{F(u) + \langle\phi,u\rangle :\ u \in W_0^{1,p}(\Omega;\mathbf{R}^m)\Big\}.$$

Moreover, if u_j is such that $F_j(u_j) + \langle\phi,u_j\rangle = m_j + o(1)$ as $j \to +\infty$, then it admits a subsequence weakly converging in $W_0^{1,p}(\Omega;\mathbf{R}^m)$ to a solution of the problem defining m.

Proof. By a cut-off argument near $\partial\Omega$ (see Section 1.7.2) if $u \in W_0^{1,p}(\Omega;\mathbf{R}^m)$ then the sequences in (1.13) of the definition of Γ-convergence can be taken in $W_0^{1,p}(\Omega;\mathbf{R}^m)$ as well, while by the growth condition (6.1) we have $u_j \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega;\mathbf{R}^m)$. This fact, together with the continuity of $G(u) = \langle\phi,u\rangle$ with respect to the weak convergence in $W_0^{1,p}(\Omega;\mathbf{R}^m)$, implies that the functionals

$$\Phi_j(u) = \begin{cases} F_j(u) + G(u) & \text{if } u \in W_0^{1,p}(\Omega;\mathbf{R}^m) \\ +\infty \end{cases}$$

Γ-converge to

$$\Phi_0(u) = \begin{cases} F(u) + G(u) & \text{if } u \in W_0^{1,p}(\Omega;\mathbf{R}^m) \\ +\infty \end{cases}$$

on $W^{1,p}(\Omega;\mathbf{R}^m)$. We can then apply Theorem 1.23 with $K = \{u \in W_0^{1,p}(\Omega;\mathbf{R}^m) : \|Du\|_{L^p(\Omega;\mathbf{R}^m)} \le c\}$ for a suitable $c > 0$. □

Remark 6.4 (*Non-spherical holes*) The results are easily extended to non-spherical geometries, by fixing any bounded set $E \subset \mathbf{R}^n$ and considering $x_i^\delta + \delta^{n/(n-p)}E$ in place of B_i^δ. The same conclusion follows, upon replacing $B_1(0)$ by E in the definition of φ.

Remark 6.5 In general, the function g depends on the subsequence (δ_j), and so does φ. In this case, the Γ-limit as $\delta \to 0$ of the functionals

$$F_\delta(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx & \text{if } u = 0 \text{ a.e. on } \bigcup_{i\in\mathbf{Z}^n} B_i^\delta \cap \Omega \\ +\infty & \text{otherwise} \end{cases} \tag{6.7}$$

does not exist.

The proof of Theorem 6.1 will be obtained in the next sections.

6.2 A joining lemma on varying domains

In this section we prove a technical result which allows to modify sequences of functions near the sets B_i^δ. Its proof is close in spirit to the method introduced by De Giorgi to match boundary conditions for minimizing sequences (see [44]). For future reference we state this lemma in a general form.

Let (δ_j) be a sequence of positive numbers converging to 0, and let $f_j : \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0, +\infty)$ be Borel functions satisfying the growth conditions (6.1) uniformly in j. In the following sections we will simply take $f_j(x, z) = f(z)$.

Note that in this section and the following ones sometimes we simply write δ in place of δ_j not to overburden notation.

Lemma 6.6 *Let (u_j) converge weakly to u in $W^{1,p}(\Omega;\mathbf{R}^m)$, and let*

$$Z_j = \{i \in \mathbf{Z}^n : \ \mathrm{dist}\,(x_i^\delta, \mathbf{R}^n \setminus \Omega) > \delta_j\}. \tag{6.8}$$

Let $k \in \mathbf{N}$ be fixed. Let (ρ_j) be a sequence of positive numbers with $\rho_j < \delta_j/2$. For all $i \in Z_j$ there exists $k_i \in \{0, \dots, k-1\}$ such that, having set

$$C_i^j = \Big\{x \in \Omega : \ 2^{-k_i-1}\rho_j < |x - x_i^\delta| < 2^{-k_i}\rho_j\Big\}, \tag{6.9}$$

$$u_j^i = |C_i^j|^{-1} \int_{C_i^j} u_j\,dx \quad \textit{(the mean value of } u_j \textit{ on } C_i^j\textit{)}, \tag{6.10}$$

and

$$\rho_j^i = \frac{3}{4} 2^{-k_i}\rho_j \quad \textit{(the middle radius of } C_i^j\textit{)}, \tag{6.11}$$

there exists a sequence (w_j), with $w_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbf{R}^m)$ such that

$$w_j = u_j \ \textit{on } \Omega \setminus \bigcup_{i\in Z_j} C_i^j \tag{6.12}$$

$$w_j(x) = u_j^i \text{ if } |x - x_i^\delta| = \rho_j^i \tag{6.13}$$

and

$$\int_\Omega \Big| f_j(x, Dw_j) - f_j(x, Du_j) \Big| \, dx \le c\frac{1}{k}. \tag{6.14}$$

Moreover, if $\rho_j = o(\delta_j)$ and the sequence $(|Du_j|^p)$ is equi-integrable, then we can choose $k_i = 0$ for all $i \in Z_j$.

Proof. For all $j \in \mathbf{N}$, $i \in Z_j$ and $h \in \{0, ..., k-1\}$ let

$$C_{i,h}^j = \Big\{ x \in \Omega : \ 2^{-h-1}\rho_j < |x - x_i^\delta| < 2^{-h}\rho_j \Big\},$$

and let

$$u_j^{i,h} = |C_{i,h}^j|^{-1} \int_{C_{i,h}^j} u_j \, dx,$$

and

$$\rho_j^{i,h} = \frac{3}{4} 2^{-h} \rho_j.$$

Consider a function $\phi = \phi_{i,h}^j \in C_0^\infty(C_{i,h}^j)$ such that $\phi = 1$ on $\partial B_{\rho_j^{i,h}}(x_i^\delta)$ and $|D\phi| \le c/2^{-h}\rho_j = c/\rho_j^{i,h}$. Let $w_j^{i,h}$ be defined on $C_{i,h}^j$ by

$$w_j^{i,h} = u_j^{i,h}\phi + (1-\phi)u_j \text{ on } C_{i,h}^j,$$

with $\phi = \phi_{i,h}^j$ as above. We then have, by the growth conditions on f_j,

$$\begin{aligned}\int_{C_{i,h}^j} f_j(x, Dw_j^{i,h}) \, dx &= \int_{C_{i,h}^j} f_j(x, D\phi(u_j^{i,h} - u_j) + (1-\phi)Du_j)) \, dx \\ &\le c \int_{C_{i,h}^j} (1 + |D\phi|^p |u_j - u_j^{i,h}|^p + |Du_j|^p) \, dx.\end{aligned}$$

By the Poincaré inequality and its scaling properties we have

$$\int_{C_{i,h}^j} |u_j - u_j^{i,h}|^p \, dx \le c(\rho_j^{i,h})^p \int_{C_{i,h}^j} |Du_j|^p \, dx, \tag{6.15}$$

so that, recalling that $|D\phi| \le c/\rho_j^{i,h}$,

$$\int_{C_{i,h}^j} f_j(x, Dw_j^{i,h}) \, dx \le c \int_{C_{i,h}^j} (1 + |Du_j|^p) \, dx.$$

Since by summing up in h we trivially have

$$\sum_{h=0}^{k-1}\int_{C^j_{i,h}}(1+|Du_j|^p)\,dx\le|B_{\rho_j}(x^\delta_i)|+\int_{B_{\rho_j}(x^\delta_i)}|Du_j|^p\,dx,$$

there exists $k_i\in\{0,\dots,k-1\}$ such that

$$\int_{C^j_{i,k_i}}(1+|Du_j|^p)\,dx\le\frac{1}{k}\Big(|B_{\rho_j}(x^\delta_i)|+\int_{B_{\rho_j}(x^\delta_i)}|Du_j|^p\,dx\Big),\tag{6.16}$$

There follows that

$$\int_{C^j_{i,k_i}}f_j(x,Dw^{i,k_i}_j)\,dx\le\frac{c}{k}\Big(|B_{\rho_j}(x^\delta_i)|+\int_{B_{\rho_j}(x^\delta_i)}|Du_j|^p\,dx\Big).\tag{6.17}$$

By (6.16) and (6.17) we get

$$\begin{aligned}\int_{C^j_{i,k_i}}|f_j(x,Du_j)-f_j(x,Dw_j)|\,dx&\le\int_{C^j_{i,k_i}}(f_j(x,Du_j)+f_j(x,Dw_j))\,dx\\&\le\frac{c}{k}\Big(|B_{\rho_j}(x^\delta_i)|+\int_{B_{\rho_j}(x^\delta_i)}|Du_j|^p\,dx\Big).\end{aligned}$$

Note that if $(|Du_j|^p)$ is equi-integrable and $\rho_j=o(\delta_j)$ then we do not need to use this argument, and may simply choose $k_i=0$ for all $i\in Z_j$.

With this choice of k_i for all $i\in Z_j$, conditions (6.12)–(6.14) are satisfied by choosing $h=k_i$ in the definitions above, i.e. with $C^j_i=C^j_{i,k_i}$, $u^i_j=u^{i,k_i}_j$ $\rho^i_j=\rho^{i,k_i}_j$ and w_j defined by (6.12) and

$$w_j=u^i_j\phi+(1-\phi)u_j\ \text{on}\ C^j_i,$$

with $\phi=\phi^j_{i,k_i}$.

Finally we prove the convergence of w_j to u in $L^p(\Omega;\mathbf{R}^m)$. By (6.15)

$$\begin{aligned}\int_\Omega|w_j-u|^p\,dx&=\int_{\Omega\setminus\bigcup_{i\in Z_j}C^j_i}|u_j-u|^p\,dx\\&\quad+\int_{\bigcup_{i\in Z_j}C^j_i}|u^i_j\phi^j_{i,k_i}+(1-\phi^j_{i,k_i})u_j-u|^p\,dx\\&\le\int_{\Omega\setminus\bigcup_{i\in Z_j}C^j_i}|u_j-u|^p\,dx\\&\quad+c\sum_{i\in Z_j}\int_{C^j_i}|u_j-u^i_j|^p\,dx+c\int_{\bigcup_{i\in Z_j}C^j_i}|u_j-u|^p\,dx\\&\le c\int_\Omega|u_j-u|^p\,dx+c\rho^p_j\sum_{i\in Z_j}\int_{C^j_i}|Du_j|^p\,dx\end{aligned}$$

$$\leq c \int_\Omega |u_j - u|^p \, dx + c\rho_j^p \sup_j \int_\Omega |Du_j|^p \, dx.$$

Hence passing to the limit as j tends to $+\infty$ we get the desired convergence. In particular, since (w_j) is bounded in $W^{1,p}(\Omega; \mathbf{R}^m)$, we get that (w_j) weakly converges to u in $W^{1,p}(\Omega; \mathbf{R}^m)$. □

6.3 Some auxiliary energy densities

It will be convenient to approximate the function φ defined in (6.3) by suitable energy densities defined by minimum problems on bounded sets so as to use the properties of convergence of minima by Γ-convergence (Theorem 1.23). In this section we define such energies and list some of their properties.

We begin by proving in the following remark the existence of g in (6.2).

Remark 6.7 We can consider the functions $g_j : \mathbf{M}^{m\times n} \to [0, +\infty)$ defined by

$$g_j(A) = \delta_j^{\frac{np}{n-p}} Qf\Big(\delta_j^{-\frac{n}{n-p}} A\Big). \tag{6.18}$$

Since g_j are quasiconvex and satisfy uniformly a growth condition of order p they are equi-locally Lipschitz continuous on $\mathbf{M}^{m\times n}$: there exists C depending only on c_1, c_2, p such that

$$|g_j(A) - g_j(B)| \leq C(1 + |A|^{p-1} + |B|^{p-1})|A - B| \tag{6.19}$$

for all $A, B \in \mathbf{M}^{m\times n}$ (see Remark 1.16(i) and (iii)). Hence, there exists a subsequence (not relabeled) converging pointwise to some limit function g. We may therefore assume that (6.2) holds. Note that this convergence implies that for all subsets U of $\mathbf{R}^n$ the functionals $G_j(\cdot, U)$ defined on $W^{1,p}(U; \mathbf{R}^m)$ by

$$G_j(u, U) = \int_U g_j(Du) \, dx \tag{6.20}$$

Γ-converge to the functional $G(\cdot, U)$ defined on $W^{1,p}(U; \mathbf{R}^m)$ by

$$G(u, U) = \int_U g(Du) \, dx \tag{6.21}$$

(see Proposition 1.41).

Using the notation of the remark above, we set

$$\varphi_{N,j}(z) = \inf\Big\{\int_{B_N(0)} g_j(D\zeta) \, dy : \ \zeta - z \in W_0^{1,p}(B_N(0); \mathbf{R}^m), \ \zeta = 0 \text{ on } B_1(0)\Big\}. \tag{6.22}$$

Note that by the Γ-convergence in Remark 6.7 and Theorem 1.23, arguing as in the proof of Corollary 6.3, we easily deduce that $\varphi_{N,j}$ converge pointwise as $j \to +\infty$ to the function φ_N, defined by

$$\varphi_N(z) = \inf\Big\{\int_{B_N(0)} g(D\zeta)\,dy:\ \zeta - z \in W_0^{1,p}(B_N(0);\mathbf{R}^m),\ \zeta = 0 \text{ on } B_1(0)\Big\}. \tag{6.23}$$

We briefly examine some properties of the functions $\varphi_{N,j}$ and φ_N which are easily deduced from the growth conditions satisfied by g_j and g.

Remark 6.8 (i) For all $N \in \mathbf{N}$ and $\eta > 0$ there exists $c_{N,\eta}$ such that

$$\begin{aligned}|\varphi_{N,j}(z) - \varphi_{N,j}(w)| \le c_{N,\eta}\,\delta_j^{n(p-1)/(n-p)}\,|z-w|(1+|w|^{p-1}+|z|^{p-1})\\ +c|z-w|(|w|^{p-1}+|z|^{p-1})\end{aligned} \tag{6.24}$$

for all $|z|, |w| > \eta$ and j. This can be easily checked if we consider a linear similitude ϕ such that $\phi(z) = w$ and $\zeta \in z + W_0^{1,p}(B_N(0);\mathbf{R}^m)$ such that $\zeta = 0$ on $B_1(0)$ and

$$\varphi_{N,j}(z) = \int_{B_N(0)} g_j(D\zeta)\,dy\,.$$

The existence of ζ follows from the quasiconvexity of g_j. If we define $\tilde\zeta = \phi(\zeta)$ then $\tilde\zeta \in w + W_0^{1,p}(B_N(0);\mathbf{R}^m)$ and $\tilde\zeta = 0$ on $B_1(0)$. By using $\tilde\zeta$ as a test function we can estimate $\varphi_{N,j}(w)$ taking into account the following inequality

$$|g_j(A) - g_j(B)| \le C(\delta_j^{n(p-1)/(n-p)} + |A|^{p-1} + |B|^{p-1})|A-B|\,,$$

which refines (6.19). By a symmetric argument we deduce the estimate on $|\varphi_{N,j}(z) - \varphi_{N,j}(w)|$.

(ii) From (i) we deduce that $\varphi_{N,j} \to \varphi_N$ uniformly on compact sets of $\mathbf{R}^m \setminus \{0\}$ by Ascoli Arzela's Theorem.

(iii) By comparison with the well-known case $g_j(A) = |A|^p$, in which case we have $\varphi_{N,j}(z) = c|z|^p$, we deduce that

$$\varphi_{N,j}(z) \le c_N \delta_j^{np/(n-p)} + c|z|^p. \tag{6.25}$$

(iv) Note that $c_1|A|^p \le g(A) \le c_2|A|^p$, so that, again by comparison with the case $g(A) = |A|^p$, we have $c_1 c|z|^p \le \varphi_N(z) \le c_2 c|z|^p$. Taking this into account and arguing as in (i) for fixed $\eta > 0$ we also have

$$|\varphi_N(z) - \varphi_N(w)| \le c\,(\eta^p + |z-w|(|w|^{p-1} + |z|^{p-1})) \tag{6.26}$$

for all $w, z \in \mathbf{R}^m$.

(v) Arguing as in (ii) and taking (iv) into account, we deduce that $\varphi_N \to \varphi$ uniformly on compact sets of $\mathbf{R}^m$.

Proposition 6.9 *Let (u_j) be a bounded sequence in $L^\infty(\Omega;\mathbf{R}^m)$ converging to u weakly in $W^{1,p}(\Omega;\mathbf{R}^m)$, let (C_i^j) $(i \in Z_j)$ be a collection of annuli of the form*

(6.9) *for an arbitrary choice of* k_i, *let* u_j^i *be defined by* (6.10), *and let* ψ_j *be defined by*

$$Q_i^\delta = x_i^\delta + \Big(-\frac{\delta_j}{2}, \frac{\delta_j}{2}\Big)^n, \qquad \psi_j = \sum_{i\in Z_j} \varphi_{N,j}(u_j^i)\chi_{Q_i^\delta}. \tag{6.27}$$

Then we have

$$\lim_j \int_\Omega |\psi_j - \varphi_N(u)|\,dx = 0. \tag{6.28}$$

PROOF. Let $\eta > 0$ be fixed. If $\eta < |z| \le \sup_j \|u_j\|_\infty$ then we have, by Remark 6.8(ii),

$$|\varphi_{N,j}(z) - \varphi_N(z)| \le o(1)$$

as $j \to +\infty$, uniformly in z, while, if $|z| < \eta$ then, by Remark 6.8(iii),

$$|\varphi_{N,j}(z) - \varphi_N(z)| \le c_N \delta_j^{np/(n-p)} + 2c\eta^p.$$

Set

$$\hat\psi_j = \sum_{i\in Z_j} \varphi_N(u_j^i)\chi_{Q_i^\delta}. \tag{6.29}$$

By the arbitrariness of η and the convergence of $\varphi_N(u_j)$ to $\varphi_N(u)$ in $L^1(\Omega)$, we deduce that the limit in (6.28) equals the limits

$$\begin{aligned}\lim_j \int_\Omega |\hat\psi_j - \varphi_N(u)|\,dx &= \lim_j \int_\Omega |\hat\psi_j - \varphi_N(u_j)|\,dx \\ &= \lim_j \sum_{i\in Z_j} \int_{Q_i^\delta} |\varphi_N(u_j^i) - \varphi_N(u_j)|\,dx \\ &\le c\Big(\eta^p + \lim_j \Big(\sup_j \|u_j\|^p_{L^\infty(\Omega;\mathbf{R}^m)}\Big) \sum_{i\in Z_j} \int_{Q_i^\delta} |u_j^i - u_j| dx\Big)\end{aligned} \tag{6.30}$$

by (6.26). By Hölder's and Poincaré's inequalities, we have

$$\begin{aligned}\int_{Q_i^\delta} |u_j^i - u_j|\,dx &\le \delta_j^{n(p-1)/p}\Big(\int_{Q_i^\delta} |u_j^i - u_j|^p\,dx\Big)^{1/p} \\ &\le \delta_j^{n(p-1)/p} c\delta_j \Big(\int_{Q_i^\delta} |Du_j|^p\,dx\Big)^{1/p},\end{aligned}$$

so that

$$\sum_{i\in Z_j} \int_{Q_i^\delta} |u_j^i - u_j|\,dx \le c\delta_j \Big(\int_\Omega |Du_j|^p\,dx\Big)^{1/p},$$

which proves the convergence to 0 of the limits in (6.30) by the arbitrariness of η. □

6.4 Proof of the liminf inequality

Let $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ and let $u_j \to u$ in $L^p(\Omega;\mathbf{R}^m)$ be such that $\sup_j F_j(u_j) < +\infty$. Note that by (6.1) $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbf{R}^m)$.

We can use a sequence (w_j) constructed as in Lemma 6.6 to estimate the liminf inequality for (F_j). We fix $k, N \in \mathbf{N}$ with $N > 2^k$, and define w_j as in Lemma 6.6 with

$$\rho_j = N\delta_j^{n/(n-p)}. \tag{6.31}$$

Note that with this choice of ρ_j we always have $w_j = u_j = 0$ on B_i^δ. Let $E_j = E_j^{k,N}$ be given by

$$E_j = \bigcup_{i\in Z_j} B_i^j, \qquad \text{where} \qquad B_i^j = B_{\rho_j^i}(x_i^\delta)$$

for all $i \in Z_j$ (Z_j given by (6.8) and ρ_j^i by (6.11)). We first deal with the contribution of the part of Du_j outside the set E_j.

Proposition 6.10 *We have*

$$\liminf_j \int_{\Omega\setminus E_j} f(Du_j)\,dx \ge \int_\Omega Qf(Du)\,dx - \frac{c}{k} \tag{6.32}$$

Proof. Let

$$v_j(x) = \begin{cases} u_j^i & \text{if } x \in B_i^j \\ w_j(x) & \text{if } x \in \Omega\setminus E_j. \end{cases}$$

Note that by Lemma 6.6 (v_j) is bounded in $W^{1,p}(\Omega;\mathbf{R}^m)$ and that $\lim_j |\{x \in \Omega:\ u_j(x) \ne v_j(x)\}| = 0$. We deduce that $v_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbf{R}^m)$ so that

$$\begin{aligned}\liminf_j \int_{\Omega\setminus E_j} f(Du_j)\,dx + \frac{c}{k} &\ge \liminf_j \int_{\Omega\setminus E_j} f(Dw_j)\,dx \\ &= \liminf_j \int_\Omega f(Dv_j)\,dx \ge \int_\Omega Qf(Du)\,dx,\end{aligned}$$

the last inequality following from Remark 1.21. □

We now turn to the estimate of the contribution on E_j. With fixed $j \in \mathbf{N}$ and $i \in Z_j$, let

$$\zeta(y) = w_j\Big(x_i^\delta + \delta_j^{n/(n-p)} y\Big)$$

be defined on $B_{\frac{3}{4}2^{-k_i}N}(0)$, and extended to u_j^i outside this ball. Note that

$$\zeta - u_j^i \in W_0^{1,p}(B_N(0);\mathbf{R}^m) \qquad \text{and} \qquad \zeta = 0 \text{ on } B_1(0). \tag{6.33}$$

By a change of variables we obtain

$$\int_{B_i^j} f(Dw_j)\,dx = \delta_j^n \int_{B_N(0)} \delta_j^{np/(n-p)} f(\delta_j^{-n/(n-p)} D\zeta)\,dx \geq \delta_j^n \varphi_{N,j}(u_j^i) \quad (6.34)$$

by (6.22); hence, to give the estimate on E_j we have to compute the limit

$$\liminf_j \sum_{i \in Z_j} \delta_j^n \varphi_{N,j}(u_j^i) = \liminf_j \int_\Omega \psi_j\,dx, \quad (6.35)$$

where ψ_j is defined as in (6.27).

Proposition 6.11 *We have*

$$\Gamma\text{-}\liminf_j F_j(u) \geq \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx$$

for all $u \in W^{1,p}(\Omega;\mathbf{R}^m)$.

Proof. Let $u_j \to u$ in $L^p(\Omega;\mathbf{R}^m)$. We can assume, upon possibly passing to a subsequence, that there exists the limit

$$\lim_j F_j(u_j) < +\infty,$$

so that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbf{R}^m)$. By [27] Lemma 3.5, upon passing to a further subsequence, for all $M \in \mathbf{N}$ and $\eta > 0$ there exists $R_M > M$ and a Lipschitz function Φ_M of Lipschitz constant 1 such that $\Phi_M(z) = z$ if $|z| < R_M$ and $\Phi_M(z) = 0$ if $|z| > 2R_M$, and

$$\lim_j F_j(u_j) \geq \liminf_j F_j(\Phi_M(u_j)) - \eta. \quad (6.36)$$

From Lemma 6.6, (6.35), and Proposition 6.9, applied to $(\Phi_M(u_j))$ in place of (u_j), we get that

$$\begin{aligned}\liminf_j \int_{E_j} f(D\Phi_M(u_j))\,dx + \frac{c}{k} &\geq \liminf_j \sum_{i\in Z_j} \delta_j^n \varphi_{N,j}((\Phi_M(u))_j^i) \\ &= \int_\Omega \varphi_N(\Phi_M(u))\,dx \\ &\geq \int_\Omega \varphi(\Phi_M(u))\,dx. \quad (6.37)\end{aligned}$$

Summing up (6.37) and (6.32) and by the arbitrariness of k, we then obtain

$$\liminf_j F_j(\Phi_M(u_j)) \geq \int_\Omega Qf(D\Phi_M(u))\,dx + \int_\Omega \varphi(\Phi_M(u))\,dx. \quad (6.38)$$

By (6.36) we then have

$$\lim_j F_j(u_j) + \eta \geq \int_\Omega Qf(D\Phi_M(u))\,dx + \int_\Omega \varphi(\Phi_M(u))\,dx.$$

We can let $M \to +\infty$ and note that $\Phi_M(u) \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbf{R}^m)$ to get

$$\lim_j F_j(u_j) + \eta \geq \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx.$$

The thesis is obtained by letting $\eta \to 0$. □

6.5 Proof of the limsup inequality

The limsup inequality is obtained by suitably modifying a recovery sequence for the lower semicontinuous envelope of $\int_\Omega f(Du)\,dx$.

Proposition 6.12 *If* $|\partial\Omega| = 0$ *then we have*

$$\Gamma\text{-}\limsup_j F_j(u) \leq \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx$$

for all $u \in W^{1,p}(\Omega;\mathbf{R}^m)$.

Proof. Let $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ and let (v_j) be a sequence converging to u weakly in $W^{1,p}(\Omega;\mathbf{R}^m)$ such that

$$\lim_j \int_\Omega f(Dv_j)\,dx = \int_\Omega Qf(Du)\,dx \tag{6.39}$$

We preliminarily note that we may assume that $(|Dv_j|^p)$ is equi-integrable on Ω (see Section 1.8.1). With fixed $N \in \mathbf{N}$, by Lemma 6.6 applied with $u_j = v_j$,

$$\rho_j = \frac{4}{3} N \delta_j^{n/(n-p)},$$

and taking the equi-integrability of $|Dv_j|^p$ into account we may also suppose that v_j equals a constant v_i^j on $\partial B_{\rho'_j}(x_i^\delta)$ for all $i \in Z_j$, where

$$\rho'_j = N\delta_j^{n/(n-p)}.$$

Step 1. We first assume that in addition (v_j) is a bounded sequence in $L^\infty(\Omega;\mathbf{R}^m)$.

Let $\eta > 0$ be fixed. We now modify the sequence (v_j) to obtain functions $u_j \in W^{1,p}(\Omega;\mathbf{R}^m)$ such that

$$u_j = v_j \text{ on } \Omega \setminus \bigcup_{i\in\mathbf{Z}^n} B_{\rho'_j}(x_i^\delta), \qquad u_j = 0 \text{ on } \Omega \cap \bigcup_{i\in\mathbf{Z}^n} B_i^\delta$$

and

$$\limsup_j \int_{\Omega\cap\bigcup_{i\in\mathbf{Z}^n} B_{\rho'_j}(x_i^\delta)} f(Du_j)\,dx \le \int_\Omega \varphi(u)\,dx + \eta|\Omega|. \tag{6.40}$$

The sequence (u_j) will then be a recovery sequence for the limsup inequality. In fact, clearly $u_j \to u$ in $L^p(\Omega;\mathbf{R}^m)$ since $\lim_j |\{u_j \neq v_j\}| = 0$ and (u_j) is bounded in $W^{1,p}(\Omega;\mathbf{R}^m)$, and

$$\begin{aligned}\limsup_j \int_\Omega f(Du_j)\,dx &\le \limsup_j \int_{\Omega\setminus\bigcup_{i\in\mathbf{Z}^n} B_{\rho'_j}(x_i^\delta)} f(Dv_j)\,dx \\ &\quad + \limsup_j \int_{\Omega\cap\bigcup_{i\in\mathbf{Z}^n} B_{\rho'_j}(x_i^\delta)} f(Du_j)\,dx \\ &\le \lim_j \int_\Omega f(Dv_j)\,dx + \int_\Omega \varphi(u)\,dx + \eta|\Omega| \\ &= \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx + \eta|\Omega|. \end{aligned}\tag{6.41}$$

We now define u_j on each $B_{\rho'_j}(x_i^\delta)\cap\Omega$. We treat separately the cases $i\in Z_j$ and $i\in\mathbf{Z}^n\setminus Z_j$. We first treat the case $i\in Z_j$. Let

$$M = \sup_j \|v_j\|_{L^\infty(\Omega;\mathbf{R}^m)}.$$

By Remark 6.8(v) we can choose N such that

$$\varphi(z) \ge \varphi_N(z) - \frac{\eta}{3} \tag{6.42}$$

for all $|z|\le M$. Recall moreover that $\varphi_{N,j}$ converges uniformly on compact sets of $\mathbf{R}^m$ to φ_N as $j\to+\infty$; we may therefore assume that

$$|\varphi_{N,j}(z) - \varphi_N(z)| \le \frac{\eta}{3} \tag{6.43}$$

for all $|z|\le M$ and $j\in\mathbf{N}$.

Let $\zeta_j^i \in v_j^i + W_0^{1,p}(B_N(0);\mathbf{R}^m)$ be such that $\zeta_j^i = 0$ on $B_1(0)$ and

$$\int_{B_N(0)} \delta_j^{np/(n-p)} f(\delta_j^{-n/(n-p)} D\zeta_j^i)\,dx \le \varphi_{N,j}(v_j^i) + \frac{\eta}{3} \le \varphi(v_j^i) + \eta, \tag{6.44}$$

the last inequality being a consequence of (6.42) and (6.43), taking into account that $|v_j^i|\le M$.

We define u_j on $B_{\rho'_j}(x_i^\delta)$ by

$$u_j(x) = \zeta_j^i\Big((x - x_i^\delta)\delta_j^{-n/(n-p)}\Big).$$

By a change of variables we then have

$$\int_{B_{\rho'_j}(x_i^\delta)} f(Du_j)\,dx = \delta_j^n \int_{B_N(0)} \delta_j^{np/(n-p)} f(\delta_j^{-n/(n-p)} D\zeta_j^i)\,dx \le \delta_j^n \varphi(v_j^i) + \delta_j^n \eta. \tag{6.45}$$

If $i \notin Z_j$ it is not possible to use the construction above since $B_{\rho'_j}(x_i^\delta)$ might intersect $\partial\Omega$. We then consider a scalar $\zeta \in W^{1,p}(B_N(0))$ such that $\zeta - 1 \in W_0^{1,p}(B_N(0))$, $0 \le \zeta \le 1$ and $\zeta = 0$ on $B_1(0)$, and simply define

$$u_j(x) = v_j(x)\,\zeta\Big((x - x_i^\delta)\delta_j^{-n/(n-p)}\Big)$$

on $B_{\rho'_j}(x_i^\delta) \cap \Omega$. We then have

$$\begin{aligned}
&\int_{B_{\rho'_j}(x_i^\delta)\cap\Omega} f(Du_j)\,dx \\
&\le c_2 \int_{B_{\rho'_j}(x_i^\delta)\cap\Omega} (1 + |Du_j|^p)\,dx \\
&\le c \int_{B_{\rho'_j}(x_i^\delta)\cap\Omega} \Big(1 + |Dv_j|^p + \delta_j^{-np/(n-p)} \Big|D\zeta\Big((x - x_i^\delta)\delta_j^{-n/(n-p)}\Big)\Big|^p |v_j|^p\Big)\,dx \\
&\le c\delta_j^n \Big(1 + M \int_{B_N(0)} |D\zeta|^p\,dx\Big) + c \int_{B_{\rho'_j}(x_i^\delta)\cap\Omega} |Dv_j|^p\,dx.
\end{aligned} \tag{6.46}$$

Let

$$E'_j = \bigcup_{i\in\mathbf{Z}^n\setminus Z_j} B_{\rho'_j}(x_i^\delta) \cap \Omega \quad \text{and} \quad \Omega'_j = \bigcup_{i\in\mathbf{Z}^n\setminus Z_j} Q_i^\delta.$$

Then (6.46) above implies that

$$\int_{E'_j} f(Du_j)\,dx \le c|\Omega'_j| + c\int_{E'_j} |Dv_j|^p\,dx = o(1), \tag{6.47}$$

by the equi-integrability of $(|Dv_j|^p)$ and the fact that $\lim_j |\Omega'_j| = |\partial\Omega| = 0$.

Taking (6.45) and (6.47) into account, we have

$$\limsup_j \int_{\Omega\cap\bigcup_{i\in\mathbf{Z}^n} B_{\rho'_j}(x_i^\delta)} f(Du_j)\,dx \le \limsup_j \sum_{i\in Z_j} \delta_j^n \varphi(v_j^i)\,dx + \eta|\Omega|,$$

so that (6.40) is proved by Proposition 6.9.

Step 2. We now remove the boundedness assumption. First assume that $u \in L^\infty(\Omega;\mathbf{R}^m)$. Then let $M = 4\|u\|_{L^\infty(\Omega;\mathbf{R}^m)}$ and let $\Phi : \mathbf{R}^m \to \mathbf{R}^m$ be a Lipschitz function of Lipschitz constant 1 such that $\Phi(z) = z$ if $|z| \le M/2$ and $\Phi(z) = 0$

if $|z| \geq M$. Let (v_j) be a sequence converging to u weakly in $W^{1,p}(\Omega; \mathbf{R}^m)$ such that (6.39) holds and $(|Dv_j|^p)$ is equi-integrable on Ω, and define $v_j^M = \Phi(v_j)$. We have $v_j^M \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbf{R}^m)$ and $\lim_j |\{v_j \neq v_j^M\}| = 0$. Hence, by the equi-integrability of $(|Dv_j|^p)$, we obtain that

$$\lim_j \int_\Omega f(Dv_j^M)\,dx = \lim_j \int_\Omega f(Dv_j)\,dx = \int_\Omega Qf(Du)\,dx.$$

We can then repeat all the reasonings above with (v_j^M) in the place of (v_j).

Finally, for arbitrary $u \in W^{1,p}(\Omega; \mathbf{R}^m)$, simply note that it can be approximated by a sequence of functions $u_k \in W^{1,p}(\Omega; \mathbf{R}^m) \cap L^\infty(\Omega; \mathbf{R}^m)$ with respect to the strong convergence of $W^{1,p}(\Omega; \mathbf{R}^m)$. By the lower semicontinuity of $F''(u) = \Gamma\text{-}\limsup_j F_j(u)$ with respect to the $L^p(\Omega; \mathbf{R}^m)$ convergence (see Remark 1.24(i)) we then have $F''(u) \leq \liminf_k F''(u_k) = \lim_k F(u_k) = F(u)$ as desired. $\square$

7

SEPARATION OF SCALES AND ALMOST-PERIODIC EFFECTS IN THE ASYMPTOTIC BEHAVIOUR OF PERFORATED PERIODIC MEDIA

7.1 Setting of the problem

In all that follows Ω is a bounded open subset of $\mathbf{R}^n$, $n \geq 3$; the Γ-limit of a sequence of functionals F_j defined on $H^1_0(\Omega)$ will be performed with respect to the $L^2(\Omega)$-convergence.

The functionals we consider are defined as follows. Let $f : \mathbf{R}^n \times \mathbf{R}^n \to [0, +\infty)$ be a Borel function satisfying

(H1) (*periodicity*) $f(\cdot, z)$ is 1-periodic for all $z \in \mathbf{R}^n$;

(H2) (*positive homogeneity*) $f(x, \cdot)$ is positively homogeneous of degree 2 for all $x \in \mathbf{R}^n$;

(H3) (*growth conditions*) there exist two constants $c_1, c_2 > 0$ such that $c_1|z|^2 \leq f(x, z) \leq c_2|z|^2$ for all x, z.

It is well known (see Theorem 1.46) that the Γ-limit G_0 of the functionals (G_ε) defined by

$$G_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, Du\Big)\, dx \tag{7.1}$$

on $H^1_0(\Omega)$ exists and can be represented as

$$G_0(u) = \int_\Omega f_{\rm hom}(Du)\, dx, \tag{7.2}$$

where

$$f_{\rm hom}(z) = \inf\Big\{\int_{(0,1)^n} f(y, Du + z)\, dy : u \in H^1_{\rm loc}(\mathbf{R}^n) \text{ 1-periodic}\Big\} \tag{7.3}$$

for $z \in \mathbf{R}^n$ defines a convex function positively homogeneous of degree 2 (see Remark 1.16(ii) and (iii) and Remark 1.24(iii)).

For all $\delta > 0$ we will consider the lattice $\delta\mathbf{Z}^n$ whose points will be denoted $x^\delta_i = \delta i$ $(i \in \mathbf{Z}^n)$. Moreover, for all $i \in \mathbf{Z}^n$

$$B^\delta_i = B_{\delta^{n/(n-2)}}(x^\delta_i).$$

For all $\varepsilon, \delta > 0$ we consider $F_{\varepsilon,\delta} : H^1_0(\Omega) \to [0, +\infty]$ defined by

$$F_{\varepsilon,\delta}(u) = \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, Du\Big)\, dx & \text{if } u = 0 \text{ on } \bigcup_{i\in\mathbf{Z}^n} B_i^\delta \\ +\infty & \text{otherwise.} \end{cases} \tag{7.4}$$

With fixed $\delta = \delta(\varepsilon)$ we will study the Γ-limits of sequences (F_j) with

$$F_j = F_{\varepsilon_j,\delta(\varepsilon_j)}. \tag{7.5}$$

We will separately consider the following cases:

(1) (Section 7.3.1) $\varepsilon << \delta^{n/(n-2)}$. In this case the Γ-limit does not depend on (ε_j) and can be written in the form

$$F_0(u) = \int_\Omega f_{\text{hom}}(Du)\, dx + C\int_\Omega |u|^2\, dx \tag{7.6}$$

on the whole $H_0^1(\Omega)$. The characterization of C is described in Theorem 7.6;

(2) (Section 7.3.2) $\varepsilon >> \delta$. The same conclusion of (1) above holds with a different characterization of C (see Theorem 7.7);

(3) (Section 7.4) In the remaining cases in general the Γ-limit does not exist, but we may have converging sequences (F_j) both to functionals of the form (7.6) with different C or to functionals of the form

$$F_0(u) = \int_\Omega f_{\text{hom}}(Du)\, dx + \int_\Omega \varphi|u|^2\, dx \tag{7.7}$$

for some strictly positive $\varphi \in L^\infty(\Omega)$.

Remark 7.1 (i) Since the functionals we consider are weakly equi-coercive on $H_0^1(\Omega)$ (more precisely, if $\sup_j(F_j(u_j)) < +\infty$ and (u_j) is bounded in $L^2(\Omega)$ then it is weakly pre-compact in $H_0^1(\Omega)$) in the Γ-liminf inequality above we may consider only sequences (u_j) weakly converging in $H_0^1(\Omega)$;

(ii) if H is a continuous functional on $L^2(\Omega)$ then F_j+H Γ-converge to F_0+H. By the well-known property of convergence of minima of Γ-limits (see Theorem 1.23) we deduce for instance in case (1) above that for all fixed $h \in L^2(\Omega)$ the values

$$m_\varepsilon = \inf\Big\{\int_{\Omega_{\delta(\varepsilon)}} f\Big(\frac{x}{\varepsilon}, Du\Big)\, dx - \int_{\Omega_{\delta(\varepsilon)}} hu\, dx : u = 0 \text{ on } \partial\Omega_{\delta(\varepsilon)}\Big\},$$

where Ω_δ denotes the δ-periodically perforated set

$$\Omega_\delta = \Omega \setminus (B_{\delta^{n/(n-2)}}(0) + \mathbf{Z}^n) = \Omega \setminus \bigcup_{i\in\mathbf{Z}^n} B_i^\delta, \tag{7.8}$$

converge to

$$m = \min\Big\{\int_\Omega \Big(f_{\text{hom}}(Du) + C|u|^2 - hu\Big)\, dx : u = 0 \text{ on } \partial\Omega\Big\}$$

as $\varepsilon \to 0$.

Furthermore, if f is convex in the second variable, for each ε a solution u_ε of m_ε exists, the family (u_ε) (extended to 0 on $\Omega \setminus \Omega_{\delta(\varepsilon)}$) is weakly precompact in $H^1_0(\Omega)$ and every its limit is a solution for m. If $f(x,z) = \langle a(x,z), z\rangle$ (a linear) we may then restate this Γ-convergence result in terms of convergence of solutions of elliptic PDE as in the Introduction.

7.2 A general Γ-convergence approach

In this section we describe a general procedure to compute the Γ-limit of functionals defined on perforated domains. In the following sections we specialize this approach to the cases (1)–(3) highlighted in the previous section.

Let $f_j : \mathbf{R}^n \times \mathbf{R}^n \to [0, +\infty)$ be Borel functions satisfying the positive homogeneity condition (H2) and the growth conditions (H3) uniformly in j. We suppose that the sequence of functionals (G_j) defined on $H^1_0(\Omega)$ by

$$G_j(u) = \int_\Omega f_j(x, Du)\, dx \tag{7.9}$$

Γ-converges to a functional G_0 of the form

$$G_0(u) = \int_\Omega f_0(x, Du)\, dx. \tag{7.10}$$

In our case $f_j(x,z) = f(x/\varepsilon_j, z)$ and $f_0 = f_{\rm hom}$.

Let (δ_j) be a sequence of positive numbers converging to 0 and let (F_j) be defined on $H^1_0(\Omega)$ by

$$F_j(u) = \begin{cases} G_j(u) & \text{if } u = 0 \text{ on } \bigcup_{i\in\mathbf{Z}^n} B_i^\delta \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{7.11}$$

Note that sometimes we use the notation $\delta = \delta_j$ not to overburden notation.

7.2.1 *The Γ-liminf inequality*

Let (u_j) converge weakly to u in $H^1_0(\Omega)$. We can suppose that $\sup_j F_j(u_j) < +\infty$. We wish to separate the contribution due to Du_j 'near the balls B_i^δ' and 'far from them'. The latter will be estimated simply by $G_0(u)$, while the former will be described by a limit capacitary formula.

The way to discriminate between 'near' and 'far' contribution is formalized by the following lemma, whose proof, together with a slightly more general statement can be found in Chapter 6.

Lemma 7.2 *Let u_j be a sequence weakly converging to u in $H^1_0(\Omega)$ as above, and let $N, k \in \mathbf{N}$. Let (δ_j) be a sequence of positive numbers converging to* 0 *and let*

$$Z_j = \{i \in \mathbf{Z}^n : \ \operatorname{dist}(x_i^\delta, \mathbf{R}^n \setminus \Omega) > \delta_j\}.$$

For all $i \in Z_j$ there exists $k_i \in \{0, \dots, k-1\}$ such that, having set

$$C_i^j = \Big\{x \in \Omega : \; 2^{-k_i-1} N \delta_j^{n/(n-2)} < |x - x_i^\delta| < 2^{-k_i} N \delta_j^{n/(n-2)}\Big\}, \tag{7.12}$$

$$u_j^i = |C_i^j|^{-1} \int_{C_i^j} u_j \, dx \quad (\textit{the } \text{mean value } \textit{of } u_j \textit{ on } C_i^j), \tag{7.13}$$

and

$$\rho_j^i = \frac{3}{4} 2^{-k_i} N \delta_j^{n/(n-2)} \quad (\textit{the } \text{middle radius } \textit{of } C_i^j), \tag{7.14}$$

there exists a sequence (w_j), with $w_j \rightharpoonup u$ in $H_0^1(\Omega)$ such that

$$w_j = u_j \; \textit{on } \Omega \setminus \bigcup_{i \in Z_j} C_i^j \tag{7.15}$$

$$w_j(x) = u_j^i \; \textit{if } |x - x_i^\delta| = \rho_j^i \tag{7.16}$$

and

$$\int_\Omega \Big| f_j(x, Dw_j) - f_j(x, Du_j) \Big| \, dx \le c \frac{1}{k}. \tag{7.17}$$

Moreover if $u_j = v_j$ with $|Dv_j|^2$ equi-integrable, setting

$$C_i^j = \Big\{x \in \Omega : \; \frac{1}{2} N \delta_j^{n/(n-2)} < |x - x_i^\delta| < \frac{3}{2} N \delta_j^{n/(n-2)}\Big\}, \tag{7.18}$$

$$v_j^i = |C_i^j|^{-1} \int_{C_i^j} v_j \, dx \quad (\textit{the } \text{mean value } \textit{of } v_j \textit{ on } C_i^j), \tag{7.19}$$

and

$$\rho_j = N \delta_j^{n/(n-2)} \quad (\textit{the } \text{middle radius } \textit{of } C_i^j), \tag{7.20}$$

we get the same conclusions above.

By this lemma we can use the sequence (w_j) to estimate the Γ-liminf inequality for (F_j). We first deal with the contribution of the part of Du_j 'external' to the annuli C_i^j; i.e., outside the set

$$E_j = \bigcup_{i \in Z_j} B_i^j, \qquad \text{where} \qquad B_i^j = B_{\rho_j^i}(x_i^\delta) \tag{7.21}$$

for all $i \in Z_j$.

Let k, N be fixed, let u_j^i be constructed as in (7.13). We define

$$\psi_j = \sum_{i \in Z_j} |u_j^i|^2 \chi_{Q_i^\delta}, \tag{7.22}$$

where

$$Q_i^\delta = x_i^\delta + \Big(-\frac{\delta_j}{2}, \frac{\delta_j}{2}\Big)^n.$$

The following lemma describes the asymptotic behaviour of ψ_j.

Lemma 7.3 *The sequence ψ_j converges to $|u|^2$ strongly in $L^1(\Omega)$.*

PROOF. By the Poincaré inequality

$$\int_{Q_i^\delta} |u_j - u_j^i|^2 \, dx \le c(k_i)\, \delta_j^2 \int_{Q_i^\delta} |Du_j|^2 \, dx,$$

where $c(l)$ depends only on $l \in \{0, \ldots, k-1\}$; since $k \in \mathbf{N}$ is fixed we get

$$\sum_{i \in Z_j} \int_{Q_i^\delta} |u_j - u_j^i|^2 \, dx \le c\, \delta_j^2 \int_\Omega |Du_j|^2 \, dx, \tag{7.23}$$

where $c := \max_{k_i = 0, \ldots, k-1} c(k_i)$. Since $\bigcup_{i \in Z_j} Q_i^\delta$ invades Ω and $u_j \to u$ in $L^2(\Omega)$ as $j \to +\infty$, by (7.23) we have that

$$\begin{aligned} \limsup_{j \to +\infty} \int_\Omega \psi_j \, dx &\le \limsup_{j \to +\infty} 2\Big(\sum_{i \in Z_j} \int_{Q_i^\delta} |u_j^i - u_j|^2 + \int_\Omega |u_j|^2 \, dx \Big) \\ &= 2 \int_\Omega |u|^2 \, dx\,, \end{aligned} \tag{7.24}$$

and, by (7.24), (7.23) and Hölder's inequality

$$\begin{aligned} \limsup_{j \to +\infty} \int_\Omega |\psi_j - |u|^2| \, dx &\le c \limsup_{j \to +\infty} \Big(\sum_{i \in Z_j} \int_{Q_i^\delta} |u_j^i - u_j|^2 \, dx \Big)^{1/2} \\ &\qquad \times \limsup_{j \to +\infty} \Big(\int_\Omega (\psi_j + |u_j|^2) \, dx \Big)^{1/2} \\ &\le c \Big(\int_\Omega |u|^2 \, dx \Big)^{1/2} \lim_{j \to +\infty} \delta_j \Big(\int_\Omega |Du_j|^2 \, dx \Big)^{1/2} = 0 \end{aligned}$$

as desired. □

Proposition 7.4 *Let (u_j) be as above. Let $k, N \in \mathbf{N}$ and let (w_j) be given by Lemma 7.2. Then we have*

$$\liminf_{j \to +\infty} \int_\Omega f_j(x, Du_j) \, dx \ge \int_\Omega f_0(x, Du) \, dx + \liminf_{j \to +\infty} \int_{E_j} f_j(x, Dw_j) \, dx - \frac{c}{k}. \tag{7.25}$$

PROOF. We define

$$v_j^{k,N} = \begin{cases} u_j^i & \text{on } B_i^j,\ i \in Z_j \\ w_j & \text{otherwise}\,. \end{cases}$$

The sequence $(v_j^{k,N})_j$ is bounded in $H_0^1(\Omega)$; hence, it is pre-compact in $L^2(\Omega)$. Since $\mathcal{L}^n(\{v_j^{k,N} - w_j\}) \to 0$ and $w_j \to u$ in $L^2(\Omega)$ as $j \to +\infty$, $v_j^{k,N}$ converges strongly to u in $L^2(\Omega)$.

By Lemma 7.2 and condition (H2)

$$\begin{aligned} F_j(u_j) + c\frac{1}{k} \geq F_j(w_j) &= \int_{\Omega\setminus E_j} f_j(x, Dw_j)\,dx + \int_{E_j} f_j(x, Dw_j)\,dx \\ &= \int_{\Omega} f_j(x, Dv_j^{k,N})\,dx + \int_{E_j} f_j(x, Dw_j)\,dx \\ &= G_j(v_j^{k,N}) + \int_{E_j} f_j(x, Dw_j)\,dx\,. \end{aligned} \tag{7.26}$$

By the Γ-liminf inequality of the functionals G_j (7.9)

$$\liminf_{j\to+\infty} G_j(v_j^{k,N}) \geq \int_{\Omega} f_0(x, Du)\,dx \tag{7.27}$$

and (7.25) follows immediately. □

We now turn to the estimate of the contribution due to Du_j on E_j. From now on, we suppose that $N > 2^k$ so that the construction of w_j in Lemma 7.2 keeps $w_j = u_j$ on B_i^δ. With fixed $j \in \mathbf{N}$ and $i \in Z_j$ such that $u_j^i \neq 0$ let $\zeta : B_N(0) \to \mathbf{R}$ be defined by

$$\zeta(y) = \begin{cases} \dfrac{1}{u_j^i}\Big(u_j^i - w_j\Big(x_i^\delta - \delta_j^{n/(n-2)}y\Big)\Big) & y \in B_{\frac{3}{4}2^{-k_i}N}(0) \\ 0 & \text{otherwise}\,. \end{cases}$$

If $u_j^i = 0$ we simply set $\zeta = 0$. Note that

$$\zeta \in H_0^1(B_N(0)) \qquad \text{and} \qquad \zeta = 1 \text{ on } B_1(0). \tag{7.28}$$

By a change of variables we obtain

$$\int_{B_i^j} f_j(x, Dw_j)\,dx = \delta_j^n |u_j^i|^2 \int_{B_N(0)} f_j(x_i^\delta - \delta_j^{n/(n-2)}x, D\zeta)\,dx; \tag{7.29}$$

hence, if we set

$$\varphi_{N,j}(x) = \inf\Big\{\int_{B_N(0)} f_j(x - \delta_j^{n/(n-2)}y, D\zeta)\,dy :\ \zeta \in H_0^1(B_N(0)),\ \zeta = 1 \text{ on } B_1(0)\Big\} \tag{7.30}$$

the computation of the liminf on the right hand side of (7.25) is translated into computing the limit

$$\liminf_{j\to+\infty} \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \varphi_{N,j}(x_i^\delta). \tag{7.31}$$

By considering the functions ψ_j and φ_j^N defined by (7.22) and by

$$\varphi_j^N = \sum_{i \in Z_j} \varphi_{N,j}(x_i^\delta)\chi_{Q_i^\delta}, \tag{7.32}$$

respectively, the limit (7.31) is translated into

$$\liminf_{j\to+\infty} \int_\Omega \varphi_j^N \psi_j \, dx. \tag{7.33}$$

By Lemma 7.3 it is sufficient to compute the weak* limit φ^N in $L^\infty(\Omega)$ of the functions φ_j^N as $j \to +\infty$. For our problem this will be done differently in the cases (1)–(3) described in Section 7.1. We then have

$$\liminf_{j\to+\infty} \int_{E_j} f_j(x, Dw_j)\, dx \geq \int_\Omega \varphi^N |u|^2 \, dx, \tag{7.34}$$

and a Γ-liminf inequality is achieved by taking the supremum in N.

7.2.2 *The Γ-limsup inequality*

The Γ-limsup inequality is obtained by suitably modifying a recovery sequence for the Γ-limit of G_j. Let $u \in H_0^1(\Omega)$ and let (v_j) be a sequence converging to u weakly in $H_0^1(\Omega)$ such that $\lim_j G_j(v_j) = G_0(u)$. Let

$$\Omega(\delta_j) = \{x \in \Omega : \ \mathrm{dist}\,(x, \partial\Omega) > \delta_j\};$$

we may assume that $\mathrm{spt}\, v_j \subset \Omega(\delta_j)$ (see Proposition 1.33) and that $|Dv_j|^2$ is equi-integrable (see Section 1.8.1).

By Lemma 7.2, taking the equi-integrability of $|Dv_j|^2$ into account, we may also suppose that v_j equals a constant v_j^i on $\partial B_{\rho_j}(x_i^\delta)$, where

$$\rho_j = N\delta_j^{n/(n-2)}.$$

The construction of a recovery sequence will be then obtained easily if, fixed η, we construct functions ζ_j^i in $H_0^1(B_N(0))$ with $\zeta_j^i = 1$ on $B_1(0)$ such that, setting

$$u_j(x) = \begin{cases} v_j(x) & \text{on } \Omega \setminus \bigcup_{i\in Z_j} B_{\rho_j}(x_i^\delta) \\ v_j^i\Big(1 - \zeta_j^i\Big(\frac{x - x_i^\delta}{\delta_j^{n/(n-2)}}\Big)\Big) & \text{on } B_{\rho_j}(x_i^\delta), \end{cases} \tag{7.35}$$

we have

$$\limsup_j \int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f_j(x, Du_j)\, dx \leq \int_\Omega \varphi |u|^2 \, dx + \eta,$$

where $\varphi = \sup_N \varphi^N$ is suggested by the liminf inequality. Indeed, with this choice of (u_j), we obtain

$$\limsup_{j\to+\infty} \int_\Omega f_j(x, Du_j)\, dx \leq \int_\Omega f_0(x, Du)\, dx + \limsup_{j\to+\infty} \int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f_j(x, Du_j)\, dx$$

$$\leq \int_\Omega f_0(x, Du)\, dx + \int_\Omega \varphi |u|^2\, dx + \eta\,, \tag{7.36}$$

and the Γ-limsup inequality is verified.

7.3 Separation of scales

In this section we study the extreme cases $\varepsilon << \delta^{n/(n-2)}$ and $\varepsilon >> \delta$. In both cases the Γ-limit of the whole family $(F_{\varepsilon,\delta})$ exists and it is described by an extra term of the form $C \int_\Omega |u|^2\, dx$, whose computation highlights a separation of scales effect.

7.3.1 *Highly-oscillating energies in perforated domains*

We treat the case $\varepsilon << \delta^{n/(n-2)}$ first. In this case the limit is computed as if by first letting $\varepsilon \to 0$, thus obtaining a homogenized functional, and then applying the theory of perforated domains for a fixed functional.

Remark 7.5 We define

$$\mathrm{cap}_{\mathrm{hom}}(B_1) = \inf\Big\{ \int_{\mathbf{R}^n} f_{\mathrm{hom}}(D\zeta)\, dz \ :\ \zeta \in H^1(\mathbf{R}^n), \quad \zeta = 1 \text{ on } B_1(0) \Big\}.$$

It can be easily checked that

$$\begin{aligned}\mathrm{cap}_{\mathrm{hom}}(B_1) &= \lim_{N\to+\infty} \min\Big\{ \int_{B_{N+\frac{1}{N}}(0)} f_{\mathrm{hom}}(D\zeta)\, dz \ :\ \zeta \in H^1(B_{N+\frac{1}{N}}(0)) \\ &\qquad\qquad \zeta = 1 \text{ on } \partial B_{N+\frac{1}{N}}(0) \quad \zeta = 0 \text{ on } B_{1-\frac{1}{N}}(0) \Big\} \\ &= \lim_{N\to+\infty} \min\Big\{ \int_{B_{N-\frac{1}{N}}(0)} f_{\mathrm{hom}}(D\zeta)\, dz \ :\ \zeta \in H^1(B_{N-\frac{1}{N}}(0)) \\ &\qquad\qquad \zeta = 1 \text{ on } \partial B_{N-\frac{1}{N}}(0) \quad \zeta = 0 \text{ on } B_{1+\frac{1}{N}}(0) \Big\}.\end{aligned}$$

Theorem 7.6 *Let f satisfy* (H1)–(H3) *and let $F_{\varepsilon,\delta}$ be given by* (7.4). *Let $\delta : (0,+\infty) \to (0,+\infty)$ be such that*

$$\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0 \qquad \lim_{\varepsilon\to 0} \frac{\delta^{n/n-2}(\varepsilon)}{\varepsilon} = +\infty;$$

then there exists the Γ-limit with respect to the $L^2(\Omega)$-convergence

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta(\varepsilon)}(u) = \int_\Omega f_{\mathrm{hom}}(Du)\, dx + \mathrm{cap}_{\mathrm{hom}}(B_1) \int_\Omega |u|^2\, dx$$

for all $u \in H^1_0(\Omega)$.

Proof. We fix a sequence (ε_j) of positive numbers converging to 0 and let $\delta_j = \delta(\varepsilon_j)$. Let $F_j = F_{\varepsilon_j,\delta_j}$. Note that we sometime simply write δ in place of δ_j.

We first deal with the Γ-liminf inequality. Let u_j be weakly converging to u in $H^1_0(\Omega)$, such that $\sup_j F_j(u_j) < \infty$. Let $k \in \mathbf{N}$ and $N > 2^k$, and let w_j be as in Lemma 7.2; by Proposition 7.4 to compute the Γ-liminf inequality we have to study the contribution on the set E_j given by (7.21).

For all $i \in \mathbf{Z}^n$ let $y_i^\varepsilon = \varepsilon_j[\frac{x_i^\delta}{\varepsilon_j}]$, so that $x_i^\delta \in y_i^\varepsilon + [0,\varepsilon_j)^n$. Taking into account that $\varepsilon_j << \delta_j^{n/(n-2)}$, we deduce the following inclusions

$$B_{(1-\frac{1}{N})\delta_j^{\frac{n}{n-2}}}(y_i^\varepsilon) \subset B_i^\delta \tag{7.37}$$

and

$$B_i^j \subset B_{\rho_j}(x_i^\delta) \subset B_{(N+\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \tag{7.38}$$

for j large enough. There follows that w_j can be extended outside B_i^j as

$$w_{j,i} = \begin{cases} w_j & \text{on } B_i^j \\ u_j^i & \text{on } B_{(N+\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \setminus B_i^j\,. \end{cases} \tag{7.39}$$

Let $u_j^i \neq 0$. By (7.39) and conditions (H1) and (H2), by a change of variables, we get

$$\begin{aligned}\int_{B_i^j} f\Big(\frac{x}{\varepsilon_j}, Dw_j\Big)\,dx &= \int_{B_{(N+\frac{1}{N})\delta_j^{n/n-2}(y_i^\varepsilon)}} f\Big(\frac{x}{\varepsilon_j}, Dw_{j,i}\Big)\,dx \\ &= \delta_j^n |u_j^i|^2 \int_{B_{(N+\frac{1}{N})}(0)} f\Big(z\frac{\delta_j^{n/n-2}}{\varepsilon_j}, D\zeta_j^i\Big)\,dz, \end{aligned} \tag{7.40}$$

where

$$\zeta_j^i(z) = w_{j,i}(z\delta_j^{n/n-2} + y_i^\varepsilon)/u_j^i\,.$$

Note that by (7.37) and (7.38) $\zeta_j^i(z) = 1$ on $\partial B_{(N+\frac{1}{N})}(0)$ and $\zeta_j^i = 0$ on $B_1(0)$.

If we denote $\eta_j = \varepsilon_j/\delta_j^{n/n-2}$, by (7.40) we have

$$\begin{aligned}&\int_{\bigcup_i B_i^j} f\Big(\frac{x}{\varepsilon_j}, Dw_j\Big)\,dx \\ &\geq \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \min\Big\{\int_{B_{(N+\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta(z)\Big)\,dz \ :\ \zeta \in H^1(B_{N+\frac{1}{N}}(0)) \\ &\qquad \zeta = 1 \text{ on } \partial B_{(N+\frac{1}{N})}(0)\ \zeta = 0 \text{ on } B_{(1-\frac{1}{N})}(0)\Big\}; \end{aligned} \tag{7.41}$$

hence, by (7.41), Lemma 7.3 and the Γ-convergence of the functionals (7.1) to that in (7.2), we have

$$\liminf_{j\to+\infty} \int_{\bigcup_i B_i^j} f\Big(\frac{x}{\varepsilon_j}, Dw_j\Big)\,dx$$

$$\geq \min\Big\{\int_{B_{(N+\frac{1}{N})}(0)} f_{\rm hom}(D\zeta(z))\,dz \ :\ \zeta\in H^1(B_{N+\frac{1}{N}}(0))$$

$$\zeta=1 \text{ on } \partial B_{(N+\frac{1}{N})}(0)\ \zeta=0 \text{ on } B_{(1-\frac{1}{N})}(0)\Big\}\int_\Omega |u|^2\,dx\,.$$

Passing to the limit in the inequality given by Proposition 7.4 first as N and then as k tend to $+\infty$, by Remark 7.5 we have that

$$\liminf_{j\to+\infty} F_j(u_j) \geq \int_\Omega f_{\rm hom}(Du)\,dx + {\rm cap}_{\rm hom}(B_1)\int_\Omega |u|^2\,dx$$

as desired. By the arbitrariness of u_j the Γ-liminf inequality is proved.

Now we pass to compute the Γ-limsup inequality. Given $u\in H^1_0(\Omega)$ we want to construct a recovery sequence (u_j) for the Γ-limit of F_j. Following the approach of Section 7.2.2, it remains to define u_j on $B_{\rho_j}(x_i^\delta)$.

We denote

$$m^N_\eta = \min\Big\{\int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta}, D\zeta(z)\Big)\,dz \ :\ \zeta\in H^1(B_{N-\frac{1}{N}}(0))$$

$$\zeta=1 \text{ on } \partial B_{(N-\frac{1}{N})}(0)\ \zeta=0 \text{ on } B_{(1+\frac{1}{N})}(0)\Big\}$$

and

$$m^N = \min\Big\{\int_{B_{(N-\frac{1}{N})}(0)} f_{\rm hom}(D\zeta(z))\,dz \ :\ \zeta\in H^1(B_{N-\frac{1}{N}}(0))$$

$$\zeta=1 \text{ on } \partial B_{(N-\frac{1}{N})}(0)\ \zeta=0 \text{ on } B_{(1+\frac{1}{N})}(0)\Big\},$$

and fix $M\in\mathbf{N}$; by Remark 7.5 and for N large enough

$$m^N \leq {\rm cap}_{\rm hom}(B_1) + \frac{1}{M}\,.$$

By the Γ-convergence of the functionals (7.1) to that in (7.2), we have that m^N_η converges to m^N as η tends to 0 (see Theorem 1.23). Considering $\eta_j=\varepsilon_j/\delta_j^{n/n-2}$, from the convergence of minima we deduce that there exists a sequence $\zeta_j\in H^1(B_{N-\frac{1}{N}}(0))$ with $\zeta_j=1$ on $\partial B_{(N-\frac{1}{N})}(0)$ and $\zeta_j=0$ on $B_{(1+\frac{1}{N})}(0)$ such that

$$\lim_{j\to+\infty}\int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta_j(z)\Big)\,dz \leq {\rm cap}_{\rm hom}(B_1)+\frac{1}{M}\,. \tag{7.42}$$

By a change of variables we get

$$\int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta_j(z)\Big)\,dz = \frac{1}{\delta_j^n}\int_{B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon)} f\Big(\frac{x}{\varepsilon_j}, D\tilde\zeta^i_j(x)\Big)\,dx, \tag{7.43}$$

where

$$\tilde{\zeta}_j^i(x) = \zeta_j\Big(\frac{x - y_i^\varepsilon}{\delta_j^{n/n-2}}\Big).$$

Reasoning as for the Γ-liminf inequality we may suppose that

$$B_i^\delta \subset B_{(1+\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \qquad \text{and} \qquad B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \subset B_{\rho_j}(x_i^\delta). \tag{7.44}$$

Since $\tilde{\zeta}_j^i(x) = 1$ on $\partial B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon)$ and $\tilde{\zeta}_j^i(x) = 0$ on $B_{(1+\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon)$, by (7.44) we can define

$$\zeta_j^i(x) = \begin{cases} \tilde{\zeta}_j^i & \text{on } B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \\ 1 & \text{on } B_{\rho_j}(x_i^\delta) \setminus B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \end{cases}$$

so that $\zeta_j^i = 1$ on $\partial B_{\rho_j}(x_i^\delta)$ and $\zeta_j^i = 0$ on B_i^δ. By (7.43) and condition (H2), we get

$$\int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta_j(z)\Big)\, dz = \frac{1}{\delta_j^n} \int_{B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, D\zeta_j^i(x)\Big)\, dx. \tag{7.45}$$

Now we can construct the recovery sequence u_j by setting

$$u_j = \begin{cases} v_j & \text{on } \Omega \setminus \bigcup_i B_{\rho_j}(x_i^\delta) \\ v_j^i\, \zeta_j^i(x) & \text{on } B_{\rho_j}(x_i^\delta), \end{cases} \tag{7.46}$$

and prove that it converges weakly to u in $H^1(\Omega)$. In fact (u_j) is bounded in $H^1(\Omega)$ and $v_j - u_j$ tends to 0 in measure. Since $v_j \to u$ in $L^2(\Omega)$, then also $u_j \to u$ in $L^2(\Omega)$ and hence weakly in $H^1(\Omega)$.

By (7.36), (7.46), (7.45), Lemma 7.3 and (7.42) we have

$$\begin{aligned} \limsup_{j\to+\infty} F_j(u_j) \le & \int_\Omega f_{\text{hom}}(Du)\, dx \\ & + \limsup_{j\to+\infty} \sum_{i\in Z_j} \delta_j^n |v_j^i|^2 \int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta_j(z)\Big)\, dz \\ \le & \int_\Omega f_{\text{hom}}(Du)\, dx + \Big(\text{cap}_{\text{hom}}(B_1) + \frac{1}{M}\Big) \int_\Omega |u|^2\, dx. \end{aligned}$$

By the arbitrariness of M we conclude the Γ-limsup inequality; hence, the Γ-convergence of the functionals $F_{\varepsilon,\delta(\varepsilon)}$ as $\varepsilon \to 0$. □

7.3.2 *Slowly-oscillating energies in perforated domains*

Now we treat the case $\varepsilon >> \delta$. In this case the limit is computed as if first applying the limit process to functionals in which x/ε acts as a parameter, and then averaging the outcome.

We consider for the sake of simplicity the case of continuous f:

(H4) (*continuity*) $f(\cdot, z)$ is continuous for all $z \in \mathbf{R}^n$.

This condition can be easily dropped, at the expense of a much heavier notation.

Theorem 7.7 *Let f satisfy* (H1)–(H4) *and let $F_{\varepsilon,\delta}$ be given by* (7.4)*. Let $\delta : (0,+\infty) \to (0,+\infty)$ be such that*

$$\lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\varepsilon} = 0\,.$$

There exists the Γ-limit with respect to the $L^2(\Omega)$-convergence

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta(\varepsilon)}(u) = \int_\Omega f_{\rm hom}(Du)\,dx + \int_{(0,1)^n} a(x)\,dx \int_\Omega |u|^2\,dx$$

for all $u \in H^1_0(\Omega)$, where

$$a(x) = \inf\Big\{\int_{\mathbf{R}^n} f(x, D\zeta)\,dy :\ \zeta \in H^1(\mathbf{R}^n),\ \zeta = 1 \text{ on } B_1(0)\Big\}\,. \tag{7.47}$$

Before proving Theorem 7.7 we make some general observations from which the Γ-limsup inequality will easily follow, and that will be used also in the next sections.

Remark 7.8 In this Section and in the next one, we will consider several cases for which the Γ-limsup inequality will be obtained by considering the recovery sequence (7.35) introduced in Section 7.2.2, but the functions ζ^i_j will be constructed in a different way with respect to the previous section. In this case the function $\varphi_{N,j}$ defined as in (7.30) takes the form

$$\varphi_{N,j}(x) = \inf\Big\{\int_{B_N(0)} f\Big(x - \frac{\delta_j^{n/(n-2)}}{\varepsilon_j} y, D\zeta\Big)\,dy :\ \zeta \in H^1_0(B_N(0)),\ \zeta = 1 \text{ on } B_1(0)\Big\}\,. \tag{7.48}$$

With fixed $j \in \mathbf{N}$ and $i \in Z_j$ we take ζ^i_j in $H^1_0(B_N(0))$ with $\zeta^i_j = 1$ on $B_1(0)$ such that

$$\int_{B_N(0)} f\Big(\frac{x^\delta_i}{\varepsilon_j} - \frac{\delta_j^{n/(n-2)}}{\varepsilon_j} y, D\zeta^i_j\Big)\,dy \le \varphi_{N,j}\Big(\frac{x^\delta_i}{\varepsilon_j}\Big) + \frac{1}{j}\,. \tag{7.49}$$

By a change of variables we obtain

$$\frac{1}{\delta_j^n}\int_{B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, D\zeta_j^i\Big(\frac{x-x_i^\delta}{\delta_j^{n/(n-2)}}\Big)\Big)\,dx \le \varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)+\frac{1}{j}$$

and

$$\int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, D\Big(v_j^i\Big(1-\zeta_j^i\Big(\frac{x-x_i^\delta}{\delta_j^{n/(n-2)}}\Big)\Big)\Big)\Big)\,dx \le \sum_{i\in Z_j}\delta_j^n|v_j^i|^2\varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)+\frac{1}{j}\,.$$

Hence, if we define

$$\varphi_j^N = \sum_{i\in Z_j}\varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)\chi_{Q_i^\delta}, \tag{7.50}$$

where $\varphi_{N,j}$ is given by (7.48), and

$$\psi_j = \sum_{i\in Z_j} v_j^i\chi_{Q_i^\delta} \tag{7.51}$$

with v_j^i given by (7.19), we have

$$\begin{aligned}
&\limsup_{j\to+\infty}\int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, Du_j\Big)\,dx\\
&=\limsup_{j\to+\infty}\int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, D\Big(v_j^i\Big(1-\zeta_j^i\Big(\frac{x-x_i^\delta}{\delta_j^{n/(n-2)}}\Big)\Big)\Big)\Big)\,dx\\
&\le \limsup_{j\to+\infty}\sum_{i\in Z_j}\delta_j^n|v_j^i|^2\varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) = \limsup_{j\to+\infty}\int_\Omega \psi_j\varphi_j^N\,dx\,.
\end{aligned}\tag{7.52}$$

Proof of Theorem 7.7. We fix a sequence (ε_j) of positive numbers converging to 0 and let $\delta_j=\delta(\varepsilon_j)$. We have already shown in Section 7.2.2 that to get the Γ-liminf inequality we have to study (in the notation of that section) the weak* convergence in $L^\infty(\Omega)$ of the functions φ_j^N to φ^N, as $j\to+\infty$. In our case φ_j^N is given by (7.50).

If we define

$$a_N(x)=\inf\Big\{\int_{B_N(0)} f(x,D\zeta)\,dy:\ \zeta\in H_0^1(B_N(0)),\ \zeta=1 \text{ on } B_1(0)\Big\} \tag{7.53}$$

by hypothesis (H4), we have that

$$\|\varphi_{N,j}-a_N\|_\infty \le \omega\Big(N\frac{\delta_j^{n/(n-2)}}{\varepsilon_j}\Big) \tag{7.54}$$

and

$$\left|a_N\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) - a_N\Big(\frac{y}{\varepsilon_j}\Big)\right| \le \omega\Big(\frac{\delta_j}{\varepsilon_j}\Big) \tag{7.55}$$

for all $y \in Q_i^\delta$. Hence, if we define

$$a_j^N = \sum_{i\in Z_j} a_N\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)\chi_{Q_i^\delta} \qquad \text{and} \qquad \bar a_N = \int_{(0,1)^n} a_N(y)\,dy$$

since a_N is 1-periodic, we have $a_N(\frac{\cdot}{\varepsilon_j}) \rightharpoonup^* \bar a_N$ in L^∞ and by (7.55) also $a_j^N \rightharpoonup^* \bar a_N$ as $j \to +\infty$. By (7.54) $\varphi_j^N \rightharpoonup^* \varphi^N = \bar a_N$ and hence

$$\lim_{N\to+\infty} \varphi^N = \int_{(0,1)^n} a(x)\,dx\,. \tag{7.56}$$

By Proposition 7.4, (7.34), Lemma 7.3 and (7.56) we get the Γ-liminf inequality.

The Γ-limsup inequality is obtained by considering the recovery sequence (7.35) with ζ_j^i constructed by (7.49), and recalling (7.52) and Lemma 7.3. □

7.4 Interaction between homogenization processes

In this section we treat the remaining cases when ε is between the scales $\delta^{n/n-2}$ and δ. We will suppose that (δ_j) and (ε_j) are such that

$$\lim_{j\to\infty} \frac{\delta_j^{n/(n-2)}}{\varepsilon_j} = q \in [0,+\infty) \qquad \lim_{j\to\infty} \frac{\varepsilon_j}{\delta_j} < +\infty \tag{7.57}$$

hold. We define the *localized capacitary formula*

$$a^q(x) = \inf\Big\{\int_{\mathbf{R}^n} f(x-qy, D\zeta)\,dy:\ \zeta \in H^1(\mathbf{R}^n),\ \zeta = 1 \text{ on } B_1(0)\Big\}\,. \tag{7.58}$$

Note that when $q = 0$, a^0 coincides with the function a defined in (7.47).

Theorem 7.9 (Periodic interaction of scales) *Let f satisfy* (H1)–(H4) *and let $F_j = F_{\varepsilon_j,\delta_j}$ with $F_{\varepsilon,\delta}$ as in* (7.4). *Let $\varepsilon_j \to 0$ and let $\delta_j \to 0$ be such that* (7.57) *holds. Suppose that $\delta_j = \frac{k_j}{M}\varepsilon_j$ with $k_j \in \mathbf{N}$ prime with $M \in \mathbf{N}$. Then there exists the Γ-limit with respect to the $L^2(\Omega)$-convergence*

$$\Gamma\text{-}\lim_{j\to+\infty} F_j(u) = \int_\Omega f_{\rm hom}(Du)\,dx + C\int_\Omega |u|^2\,dx,$$

on $H_0^1(\Omega)$, where

$$C = \frac{1}{M^n} \sum_{h\in\{0,\dots,M-1\}^n} a^q\Big(\frac{h}{M}\Big)\,. \tag{7.59}$$

PROOF. Let $\varphi_{N,j}$ be the 1-periodic function defined as in (7.48), and let

$$a^q_N(x) = \inf\Big\{\int_{B_N(0)} f(x-qy, D\zeta)\,dy:\ \zeta\in H^1_0(B_N(0)),\ \zeta=1 \text{ on } B_1(0)\Big\}. \tag{7.60}$$

As $\delta_j = \frac{k_j}{M}\varepsilon_j$ then

$$\frac{x^\delta_i}{\varepsilon_j} = i\frac{k_j}{M} = k+\frac{h}{M} \qquad k\in\mathbf{Z}^n,\quad h\in\{0,...,M-1\}^n. \tag{7.61}$$

By (7.61) and the periodicity of $\varphi_{N,j}$

$$\sum_{i\in Z_j}\delta^n_j|u^i_j|^2\varphi_{N,j}\Big(\frac{x^\delta_i}{\varepsilon_j}\Big) = \sum_{h\in\{0,\dots,M-1\}^n}\Big(\sum_{i\in I_h}\delta^n_j|u^i_j|^2\Big)\varphi_{N,j}\Big(\frac{h}{M}\Big) = \int_\Omega \psi_j\varphi^N_j\,dx, \tag{7.62}$$

where

$$I_h = \frac{h}{M}+\mathbf{Z}^n\cap Z_j$$

and ψ_j, φ^N_j are defined in (7.22) (7.50), respectively. Note that

$$\varphi^N_j(x) = \sum_{h\in\{0,\dots,M-1\}^n}\sum_{i\in I_h}\varphi_{N,j}\Big(\frac{h}{M}\Big)\chi_{Q^\delta_i}(x)$$

and $\|\varphi_{N,j}-a^q_N\|_\infty\to 0$ as $j\to+\infty$; hence,

$$\varphi^N_j \rightharpoonup^* \varphi^N = \sum_{h\in\{0,\dots,M-1\}^n}\frac{1}{M^n}a^q_N\Big(\frac{h}{M}\Big)$$

and

$$\lim_{N\to+\infty}\varphi^N = \sum_{h\in\{0,\dots,M-1\}^n}\frac{1}{M^n}a^q\Big(\frac{h}{M}\Big).$$

Recalling Lemma 7.3 we obtain the Γ-liminf inequality.

In order to obtain the Γ-limsup inequality, by (7.52) it is sufficient to use the scheme of Section 7.2.2 with ζ^i_j as in (7.49). □

Remark 7.10 In the particular case when $\delta_j/\varepsilon_j\in\mathbf{N}$ (i.e., $M=1$) the constant C is given by the single problem defining $a^q(0)$.

Theorem 7.11 (Almost-periodic interaction of scales) *Let f satisfy* (H1)–(H4) *and let $F_j = F_{\varepsilon_j,\delta_j}$ with $F_{\varepsilon,\delta}$ as in* (7.4)*. Let $\varepsilon_j\to 0$ and let $\delta_j\to 0$ be such that*

(7.57) holds. Suppose that $\delta_j = (k_j + r)\varepsilon_j$ with $k_j \in \mathbf{N}$ and $r \notin \mathbf{Q}$. Then there exists the Γ-limit with respect to the $L^2(\Omega)$-convergence

$$\Gamma\text{-}\lim_{j\to+\infty} F_j(u_j) = \int_\Omega f_{\rm hom}(Du)\,dx + C\int_\Omega |u|^2\,dx$$

on $H^1_0(\Omega)$, where

$$C = \int_{(0,1)^n} a^q(x)\,dx\,. \tag{7.63}$$

PROOF. The sequence φ_j^N defined in (7.50) is bounded in $L^\infty(\Omega)$; hence, up to subsequences, there exists $\varphi^N \in L^\infty(\Omega)$ such that $\varphi_j^N \rightharpoonup^* \varphi^N$ in $L^\infty(\Omega)$ as $j \to +\infty$. In order to identify the limit φ^N, it suffices to test it against characteristic functions of n-cubes. Hence, if we prove that

$$\int_A \varphi_j^N\,dx \to \mathcal{L}^n(A)C \tag{7.64}$$

for every n-cube A, we have $\varphi^N = C$.

We define

$$\tilde\varphi_j^N = \sum_{i\in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n}\varphi_{N,j}(\frac{x_i^\delta}{\varepsilon_j})\chi_{Q_{\frac{\varepsilon}{M}}(x_i^\delta)+\mathbf{Z}^n},$$

where

$$Q_{\frac{\varepsilon}{M}}(x_i^\delta) = x_i^\delta + \Big(-\frac{\varepsilon_j}{M}, \frac{\varepsilon_j}{M}\Big)^n.$$

Note that also $\tilde\varphi_j^N \rightharpoonup^* \varphi^N$ in $L^\infty(\Omega)$. By the continuity of $\varphi_{N,j}$

$$\Big|\varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) - \varphi_{N,j}\Big(\frac{x}{\varepsilon_j}\Big)\Big| \le \omega\Big(\frac{1}{M}\Big)$$

if $x \in Q_{\frac{\varepsilon}{M}}(x_i^\delta) + \mathbf{Z}^n$; hence, we study the weak* convergence of

$$x \mapsto \sum_{i\in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n}\varphi_{N,j}\Big(\frac{x}{\varepsilon_j}\Big)\chi_{Q_{\frac{\varepsilon}{M}}(x_i^\delta)+\mathbf{Z}^n}(x)\,.$$

Let A be an n-cube with edges parallel to the coordinate axes and of side length l, we compute

$$\begin{aligned}&\int_A \sum_{i\in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n}\varphi_{N,j}\Big(\frac{x}{\varepsilon_j}\Big)\chi_{Q_{\frac{1}{M}}(\frac{x_i^\delta}{\varepsilon_j})+\mathbf{Z}^n}\Big(\frac{x}{\varepsilon_j}\Big)\,dx\\ &= \varepsilon_j^n \int_{\frac{1}{\varepsilon_j}A} \sum_{i\in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n}\varphi_{N,j}(z)\chi_{Q_{\frac{1}{M}}(\frac{x}{\varepsilon_j})+\mathbf{Z}^n}(z)\,dz\end{aligned}$$

$$= [(l/\varepsilon_j) - 1]^n \varepsilon_j^n \int_{(0,1)^n} \varphi_{N,j}(z) \sum_{i \in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n} \chi_{Q_{\frac{1}{M}}(\frac{x_i^\delta}{\varepsilon_j}) + \mathbf{Z}^n}(z)\, dz$$

$$+\varepsilon_j^n \int_{R_j} \varphi_{N,j}(z) \sum_{i \in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n} \chi_{Q_{\frac{1}{M}}(\frac{x_i^\delta}{\varepsilon_j}) + \mathbf{Z}^n}(z)\, dz, \tag{7.65}$$

where we have decomposed $(1/\varepsilon_j)A$ as the union of $[(l/\varepsilon_j) - 1]^n$ unit cubes and of a set R_j, with $\mathcal{L}^n(R_j) \leq 2n(l/\varepsilon_j)^{n-1}$.

By an application of Birkhoff's Theorem (see e.g. [60]) as in [25] Appendix A and (7.65) we deduce

$$\lim_{j \to +\infty} \int_A \sum_{i \in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n} \varphi_{N,j}\Big(\frac{x}{\varepsilon_j}\Big) \chi_{Q_{\frac{\varepsilon_j}{M}}(x_i^\delta) + \mathbf{Z}^n}(x)\, dx$$

$$= \mathcal{L}^n(A) \int_{(0,1)^n} a_N^q(z)\, dz \int_{(0,1)^n} \chi_{Q_{\frac{1}{M}}}(z) M^n\, dz = \mathcal{L}^n(A) \int_{(0,1)^n} a_N^q(z)\, dz,$$

where a_N^q is defined by (7.60). By (7.64) we have

$$\varphi^N = \int_{(0,1)^n} a_N^q(z)\, dz;$$

hence, by Lemma 7.3

$$\lim_j \int_\Omega \varphi_j^N \psi_j\, dx = \varphi^N \int_\Omega |u|^2\, dx, \tag{7.66}$$

where ψ_j is defined as in (7.22), and

$$\lim_{N \to +\infty} \varphi^N = \int_{(0,1)^n} a^q(x)\, dx\,. \tag{7.67}$$

By (7.34)

$$\liminf_{j \to +\infty} \int_{E_j} f\Big(\frac{x}{\varepsilon_j}, Dw_j\Big)\, dx \geq \int_{(0,1)^n} a^q(x)\, dx \int_\Omega |u|^2\, dx \tag{7.68}$$

and we obtain the Γ-liminf inequality.

Recalling (7.52), we choose ζ_j^i as in (7.49), and by (7.66), (7.67) we get the Γ-limsup inequality. □

Corollary 7.12 (Non-existence) *If $\delta : (0,+\infty) \to (0,+\infty)$ is a continuous function such that*

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta(\varepsilon)} = 0, \qquad \textit{and} \qquad \lim_{\varepsilon \to 0} \frac{\delta^{n/(n-2)}(\varepsilon)}{\varepsilon} = q \in [0,+\infty),$$

then the Γ-limit of the functionals $F_{\varepsilon,\delta(\varepsilon)}$ as $\varepsilon \to 0$ does not exist.

Remark 7.13 The case $\delta_j = \varepsilon_j$ (more generally, $\delta_j = s\varepsilon_j$ with fixed $s > 0$) is covered by Theorem 7.9 and Theorem 7.11. Note that the condition $\lim_{j\to+\infty} \delta_j/\varepsilon_j = 1$ does not allow to conclude the existence of the Γ-limit of F_j as shown by Example 7.14 below.

Example 7.14 (Finely-tuned interplay between scales) We finally give an example when the extra term in the limit is not described by a constant: if $\delta_j = \varepsilon_j + \varepsilon_j^2$ then

$$\Gamma\text{-}\lim_{j\to+\infty} F_j(u) = \int_\Omega f_{\mathrm{hom}}(Du)\,dx + \int_\Omega a(x)|u(x)|^2\,dx\,.$$

In fact, by the periodicity of $\varphi_{N,j}$ defined as in (7.48)

$$\sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) = \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \varphi_{N,j}(i\varepsilon_j)$$

If we consider the function a_N defined by (7.53), by condition (H4)

$$\Big| a_N\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) - a_N(x_i^\delta)\Big| \le \omega(\varepsilon_j^2); \tag{7.69}$$

hence, by (7.54) and (7.69), we have that

$$\begin{aligned}\lim_{j\to+\infty} \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) &= \lim_{j\to+\infty} \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 a_N(x_i^\delta)\\ &= \int_\Omega a_N(x)|u(x)|^2\,dx \end{aligned} \tag{7.70}$$

and

$$\lim_{N\to+\infty} \int_\Omega a_N(x)|u(x)|^2\,dx = \int_\Omega a(x)|u(x)|^2\,dx\,. \tag{7.71}$$

Reasoning as in the proof of Theorems 7.9 and 7.11 we get the Γ-liminf and the Γ-limsup inequalities.

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