SISSA - International School for Advanced Studies

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Bifurcation of free and forced vibrations for nonlinear wave and Kirchhoff equations via Nash-Moser theory

Ph.D. Thesis

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Chapter 1

Introduction

Nonlinear wave equations model the propagation of waves in a wide range of physical systems, from acoustics to electromagnetics, from seismic motions to vibrating string and elastic membranes, where oscillatory phenomena occur. Because of this intrinsic oscillatory physical structure, it is natural, from a mathematical point of view, to investigate the question of the existence of oscillations, namely periodic and quasi-periodic solutions, for the equations governing such physical systems. This is the central question of this Thesis.

In particular, our new results concern forced and autonomous[1] wave equations

$$u_{tt} - u_{xx} = f(x,t,u), \quad u_{tt} - u_{xx} = f(x,u)$$

and forced Kirchhoff equations

$$u_{tt} - \Delta u \Big(1 + \int_\Omega |\nabla u|^2 dx \Big) = g(x,t), \tag{1.1}$$

both with space-periodic and Dirichlet boundary conditions.

These equations belong to the family of nonlinear PDE having the structure of an infinite-dimensional Hamiltonian dynamical system. Many PDE of physical significance belong to this large family, for example the nonlinear Schrödinger equation, the Korteveg-de Vries equation, the Burgers equation, the Boussinesq equation, the nonlinear beam equation, the Euler equations for hydrodynamics and, including also lattice models, the Fermi-Pasta-Ulam system.

Regarding PDE as Hamiltonian systems, it is natural to pose for them a question which is typical in the study of dynamical systems, that is the description of the principal structures of phase space which are invariant under the flow, complementing the theory of the initial value problems.

[1]Autonomous, or free, is meant with respect to time.

In this approach to PDE the natural guide is the theory of finite-dimensional Hamiltonian systems, and main efforts have been devoted to develop extensions of known results from finite-dimensional to infinite-dimensional systems.

Given the vastness and the far origin of this area of analysis, we devote this introductory chapter to give a sketch of its history. The presentation of our results is in Chapter 2.

1.1 Historical preface

In the past, the study of periodic and quasi-periodic orbits for finite dimensional Hamiltonian systems was mainly motivated by celestial mechanics and the problem of the planetary system. To this purpose at the end of the 19th century Poincaré devised both perturbation and topological methods. However, periodic solutions forms an exceptional set in phase space, namely, as Poincaré explained in [94], the probability is zero for the initial conditions of a motion to be precisely those corresponding to a periodic solution.

In spite of this, Poincaré was convinced that periodic solutions play a fundamental role in the dynamics of Hamiltonian systems, and he conjectured that for every solution of the problem, for every time T and every small ε there is a periodic solution such that the distance between the two solutions remains less than ε for all $t \leq T$.

This density conjecture was the main motivation for the systematic study of periodic orbits, started by Poincaré himself and then continued by Lyapunov, Birkhoff, Moser, Weinstein, Rabinowitz and many others.

The first existence result was the hundred-years-old Lyapunov Centre Theorem [79], a bifurcation result for periodic orbits near an equilibrium ("centre") based on the Poincaré continuation method. It proves the existence of periodic solutions of a dynamical system $\dot{x} = f(x)$ near its equilibrium $x = 0$, assuming a non-degeneracy hypothesis and a *non-resonance* condition on a pair $\pm i\,\omega_1$ of purely imaginary eigenvalues of the matrix $f'(0)$ with respect to all other purely imaginary eigenvalues $\pm i\omega_2, \ldots, \pm i\omega_n$ of $f'(0)$, that is

$$\omega_1 l - \omega_j \neq 0 \quad \forall l \in \mathbb{Z}, \quad j = 2, \ldots, n. \tag{1.2}$$

Moving in the direction of the conjecture of Poincaré, in the Thirties Birkhoff and Lewis [36] proved that "sufficiently nonlinear" Hamiltonian systems have infinitely many periodic solutions with large minimal period in any neighborhood of an elliptic orbit.

Then in the Seventies Weinstein [111], Moser [85], Fadell and Rabinowitz [57] extended the Lyapunov Centre Theorem to Hamiltonian systems which violate the non-resonant condition (1.2) (*resonant* case). By variational methods in bifurcation theory they proved existence and multiplicity of periodic solutions having fixed energy (Weinstein-Moser) or fixed period (Fadell-Rabinowitz)[2].

After periodic solutions, the simplest orbits of a dynamical system are quasi-periodic motions. They are given by generalised Fourier series of the form

$$x(t) = \sum_{j \in \mathbb{Z}^d} c_j \, e^{i\, j \cdot \omega\, t}, \quad \omega = (\omega_1, \omega_2, \ldots, \omega_d) \tag{1.3}$$

where the frequencies $\omega_1, \omega_2, \ldots, \omega_d$ are rationally independent real numbers (that is, the only $j \in \mathbb{Z}^d$ such that $j \cdot \omega = 0$ is $j = 0$).

Perturbation theory of classical mechanics led to such series expansions for solutions of equations of motion already in the 19th century. However, the convergence of these series became early a notorious problem. The difficulty is due to the so-called "small divisors", which are powers of terms of the form $j \cdot \omega$, $j \in \mathbb{Z}^d \setminus \{0\}$ entering in the coefficients as denominators. Since the frequencies are rationally independent these expressions are not zero, but they become arbitrarily small (and the corresponding terms in the series arbitrarily large).

This convergence problem — which would lead to the existence of quasi-periodic solutions — has been of central interest at the end of the 19th century[3], when Poincaré indicated small divisors as the main source of chaotic dynamics, due to very complex resonance phenomena[4]. The problem remained unsolved until the work of Kolmogorov [72], Arnold [6] and Moser [83] (KAM) on "nearly integrable" Hamiltonian systems in the Fifties and Sixties.

A Hamiltonian system is "completely integrable" if there are canonical variables in which the equations of motion are explicitly integrable. In this case, all solutions are periodic or quasi-periodic. However, many important physical systems of classical mechanics are not completely integrable, but only "nearly integrable", that is, they may be written as a small perturbation of a completely integrable system.

KAM theory showed that for a set of initial conditions of positive measure, the motion of a sufficiently smooth nearly integrable Hamiltonian system remains similar to that of the unperturbed, completely integrable one;

[2]This historical review of periodic solutions in finite-dimensional systems follows [23].

[3]In 1878 Weierstraß wrote to S. Kovalevskaya that he found a series expansion for the solutions of the 3-body problem, and tried, though in vain, to prove its convergence. He was aware of a remark made by Dirichlet to Kronecker in 1858 that he had found a method to approximate solutions of the n-body problem successively. Dirichlet died soon afterwards, and no written records were found. Later Weierstraß suggested this problem to Mittag-Leffler as a prize question. This prize was awarded to Poincaré although actually he did not solve this problem. (From [86]).

[4]See [94].

in particular, for these initial conditions, the motion is quasi-periodic. In other words, a positive measure subset of the phase space consists of quasi-periodic solutions (KAM tori).

These solutions are constructed by an iterative procedure which generates canonical change of variables. In the new variables, the Hamiltonian of the system is written in a normal form for which the existence of quasi-periodic solutions is proved directly. The iterative process is a sort of Newton's method. At each step of the iteration certain initial conditions for which the inverse of the linearised operator could not be controlled are excluded. These exclusions correspond to the small divisors in the perturbation series and are the reason that one can only prove that the perturbed system behaves like the unperturbed one on some subset of all possible initial conditions.[5]

1.2 Hamiltonian PDE

At the end of the Sixties it begun the investigation of periodic orbits also for Hamiltonian PDE. Periodic solutions, interesting per se, was also viewed as a first step toward the understanding of the dynamics of infinite-dimensional systems.

The first breakthrough was the pioneering paper of Rabinowitz [99] on nonlinear wave equations. He considered one-dimensional equations for forced vibrating strings with Dirichlet boundary conditions, namely

$$u_{tt} - u_{xx} + \varepsilon f(x,t,u) = 0, \qquad u(0,t) = u(\pi,t) = 0 \tag{1.4}$$

where the nonlinearity f is 2π-periodic in time. By variational methods, he found bifurcation of 2π-periodic solutions of the problem.

Then 2π-periodic solutions of nonlinear wave equations, and more generally T-periodic solutions where $2\pi/T = \omega$ is a rational number, have been studied by Rabinowitz himself [101, 102], by Brezis, Coron, Nirenberg [43, 44] using variational methods, and by many other authors, for example [53, 109, 20, 25], proving both perturbative and global results.

These proofs work only for rational frequencies. Indeed, suppose we are looking for solutions u such that $u(0,t) = u(\pi,t) = 0$ and $u(x,t) = u(x,t+T)$, where $T = 2\pi/\omega > 0$. Decomposing in Fourier series

$$u(x,t) = \sum_{j\geq 1,\ l\in\mathbb{Z}} \hat{u}_{jl}\, e^{il\omega t} \sin jx$$

shows that the spectrum of the d'Alembertian $\partial_{tt} - \partial_{xx} =: L$ is the set

$$\sigma = \{-\omega^2 l^2 + j^2 \ :\ l \in \mathbb{Z},\ j = 1,2,\dots\}.$$

[5]This very clear description of the KAM method is drawn from: C.E. Wayne, Mathematical Review MR1668547 of [39] on AMS MathSciNet.

The nature of σ depends in a crucial way on the arithmetical nature of the number ω:

- if ω is rational, $\omega = p/q$ for some positive integers p, q, then $-\omega^2 l^2 + j^2 = 0$ for all infinitely many pairs (j, l) such that $qj = p|l|$, while for all other pairs there holds

$$(1.5) \qquad |-\omega^2 l^2 + j^2| = \frac{|-p|l| + qj|\,(p|l| + qj)}{q^2} \geq \frac{1}{q^2}\,(|l| + j).$$

- If ω is irrational and it has the "badly approximation" property that

$$(1.6)\ \exists \gamma > 0 \quad \text{such that} \quad |\omega l - j| > \frac{\gamma}{|l|} \quad \forall\, l \in \mathbb{Z} \setminus \{0\},\ j = 1, 2, \dots\,,$$

  then all eigenvalues $-\omega^2 l^2 + j^2$ are far from zero, because

$$(1.7)\ |-\omega^2 l^2 + j^2| = |-\omega|l| + j|\,(\omega|l| + j) > \frac{\gamma}{|l|}\,\omega|l| = \gamma\omega > 0 \quad \forall l \neq 0$$

  (obviously for $l = 0$ the eigenvalues equal $j^2 \geq 1$)[6].

- If ω is irrational and it does not satisfy the special property (1.6), all eigenvalues $-\omega^2 l^2 + j^2$ are not zero, but they accumulate to zero. This is the typical situation, that is, for almost every real number ω zero is an accumulation point for the spectrum[7].

When ω is a rational p/q, the functional space where the problem is set, for example the space

$$X = \Big\{ u(x,t) = \sum_{j,l} \hat{u}_{jl}\, e^{il\omega t} \sin jx \ : \ \|u\|^2 := \sum_{j,l} |\hat{u}_{jl}|^2 < \infty \Big\},$$

is split into two subspaces: the infinite-dimensional kernel V of the d'Alembertian (corresponding to pairs (j, l) such that $qj = p|l|$) and its orthogonal subspace $W = V^\perp$ (corresponding to all others (j, l)). The restricted d'Alembertian $L_{|W}$ is invertible, and its inverse is compact thanks to the growth condition (1.5) for the eigenvalues as $|(j, l)| \to \infty$. This compactness property is a key ingredient in the proof of works mentioned above.

Conversely, when ω is irrational the d'Alembert operator has kernel zero, and the inverse operator L^{-1} is defined by its spectrum

$$(1.8) \qquad \sigma^{-1} = \Big\{ \frac{1}{-\omega^2 l^2 + j^2} \ : \ l \in \mathbb{Z},\ j = 1, 2, \dots \Big\}.$$

[6]It is easy to prove that, conversely, if there is a positive constant c such that $|-\omega^2 l^2 + j^2| > c$ for all $l \in \mathbb{Z}$, $j \geq 1$, then ω satisfies (1.6).

[7]Numbers satisfying the badly approximation property (1.6) form a set of continuum cardinality and zero measure. This is proved in Appendix A by the aid of the continued fractions theory (see Remark 26).

Any growth condition like (1.5) is violated, and there is no hope to recover compactness properties.

For irrational frequencies, the more fortunate situation is when ω satisfies the badly approximation property (1.6). In this case the inverse L^{-1} is a bounded operator thanks to the separation property (1.7) of the spectrum σ from zero. This makes it possible to apply the standard implicit function theorem in functional settings, and to prove perturbative results. For example, problem (1.4) can be reformulated as the fixed point problem

$$u = \varepsilon L^{-1} f(x,t,u),$$

so that it is easy to find small amplitude solutions u_ε for all ε sufficiently small. Non-resonance conditions of the type (1.6) are used, for example, in [2, 81, 76, 52, 21, 18, 26, 27, 97, 11]. Moreover, we mention the work [16] of Bambusi, where it is given a simple proof of an infinite-dimensional extension of the Lyapunov Centre Theorem. Non-resonance conditions of badly approximation type on the eigenvalues of the linearised system are imposed, so that the proof can make use of the implicit function theorem.

On the other hand, as remarked above, almost every real ω is neither rational nor badly approximable. In this typical case, the spectrum σ of the d'Alembertian contains arbitrarily small eigenvalues, which become small denominators in the spectrum (1.8) of the inverse operator. As a consequence, L^{-1} is unbounded, and it cannot map a functional space in itself. For this reason it is impossible to apply the standard implicit function theorem.

We note an important distinction between finite- and infinite-dimensional systems: in finite dimension small divisors appear only in the construction of quasi-periodic solutions, while in infinite dimension, and in particular in PDE, they appear also in the construction of periodic solutions.

The first existence results solving these small divisors difficulties have been obtained by Kuksin [73] and Wayne [110]. They proved independently that it is possible to use the KAM theory to construct both periodic and quasi-periodic solutions of autonomous nearly-integrable Hamiltonian PDE, extending the KAM method to infinite-dimensional systems. However, both of these approaches seemed to work only for Dirichlet boundary conditions.

To avoid this restriction, Craig and Wayne [49] developed a new method, based on the Lyapunov-Schmidt decomposition and Nash-Moser theory. They proved the existence of a positive measure set of periodic solutions for autonomous wave equations of the form

$$u_{tt} - u_{xx} + f(x,u) = 0, \qquad f(x,u) = a_1(x)u + a_2(x)u^2 + \dots \tag{1.9}$$

for all f in an open, dense set of nonlinearities.

These results can be considered as infinite-dimensional extensions of the Lyapunov Centre Theorem. Indeed they impose some restrictions on the first term a_1 of the nonlinearity to obtain non-resonance conditions similar to (1.2) on the eigenvalues of the linearised problem at $u = 0$. However, in the infinite-dimensional case nonzero conditions like those in (1.2) are not sufficient: the imaginary parts ω_j of the eigenvalues form a sequence tending to infinity, so that in general nonzero terms of the form $\omega_1 l - \omega_j$ accumulates to zero. For this reason nonzero conditions (1.2) are replaced by stronger non-resonance conditions of "Diophantine" type, that is

$$|\omega_1 l - \omega_j| > \frac{\gamma}{|l|^\tau} \quad \forall\, l \in \mathbb{Z},\ j \neq 1$$

where $\gamma > 0$ and $\tau > 1$. Diophantine conditions are strong enough to control small divisors, at least for semilinear nonlinearities, and at the same time they are weak enough to be satisfied by the majority of real numbers[8].

Beyond these Diophantine conditions, infinitely many other non-resonance conditions, depending on the nonlinearity, have to be imposed[9].

The Nash-Moser theory has its origin in a paper of Nash in the Fifties [89] on isometric embeddings, whose main ideas were revisited and extended by Moser in the Sixties [82, 84].[10] The strategy is essentially based on a modified Newton's iterative method in scales of Banach spaces. Its aim is the inductive construction of a sequence of approximate solutions converging to a solution of the problem. In the version of the method developed by Craig and Wayne to deal with wave equations, at each step of the iteration some non-resonance conditions on the parameters are imposed, in order to control small divisors. For these non-resonant parameters one can invert the linearised operator, which is essentially a perturbation of the d'Alembertian. The inverse operator is not bounded, and it loses some amount of regularity. This loss of regularity, which occurs at each step of the iteration, is overcome thanks to suitable smoothing operators and to the high speed of convergence of the quadratic scheme.

Because of the exclusion of resonant parameters corresponding to small divisors, the method holds on some subset of all possible values for the parameters, which has the structure of a Cantor set. Given the presence of these "gaps" in the set of parameters, a non-degeneracy condition in the bifurcation equation is usually assumed[11]. Its analogous in the KAM method is called Arnold condition.

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[8] See Appendix A.

[9] To see a concrete example of these further restrictions, we refer to (6.16), (5.14) and (5.38).

[10] See [23, Ch. 3] for a historical review of the Nash-Moser theory and its application in a wide range of areas of mathematics. See also [107, 116, 117, 66, 64, 67, 3, 48] and Appendix B.

[11] See Remark 1.

The Nash-Moser method for PDE has been widely extended by Bourgain to construct quasi-periodic solutions [37, 39], to deal with PDE in space-dimension higher than one [38, 39, 42] and also to non-Hamiltonian PDE [40], proving the flexibility of this method with respect to the standard KAM technique. Further improvements have been obtained by Iooss, Plotnikov and Toland [91, 68] extending the Lyapunov Centre Theorem to nonlinear equations arising in the water wave problem, and by Berti and Bolle [28, 29, 30] for wave equations (1.9) when $a_1 = 0$ ("completely resonant" case), extending the Weinstein-Moser and Fadell-Rabinowitz Theorems to infinite dimension.[12] The Nash-Moser method used in [28, 29, 30] is different from that of Craig, Wayne and Bourgain, and it makes it possible to deal with more general nonlinearities, for example with low regularity and without oddness assumptions.

Recently also the Birkhoff-Lewis Theorem has been extended to infinite-dimensional systems [17, 33, 34, 35].

At the same time, the infinite-dimensional version of the KAM method has been improved by Pöschel [95], Chierchia, You [46], Eliasson, Kuksin [56] and Yuan [115].

We mention a third tool to control small divisors, the Lindstedt series method. It is essentially based on writing solutions as series expansions in the perturbative parameter, and finding suitable bounds on the series coefficients to prove their convergence. Such bounds are obtained by highlighting some delicate cancellation properties involving terms which are huge because of the presence and repetition of small divisors, see for example [61].

Currently Nash-Moser, KAM and Lindstedt series are the only three available methods to overcome the small divisors difficulty in PDE analysis. The extension of these methods to cover the largest possible range of application is a challenging area of research presenting many open directions.

One of these open questions is the study of degenerate cases, where non-degeneracy conditions in the bifurcation equation are violated. This is an extremely difficult problem. For some results in this direction see [29].

Another open question deals with the study of quasi-linear or, more generally, full-linear equations, where the nonlinearity depends on partial derivatives of the unknown having order not less than that of the linear part. For these equations, indeed, the perturbative effect of small divisors is in general so strong that it is very difficult to find a strategy to control them. We cite the work of Iooss, Plotnikov and Toland [68] for full-linear equations arising in water wave problems. Also, we believe that it could be interesting our results concerning quasi-linear Kirchhoff equations (1.1).

We point out that, despite the Kirchhoff equation is a Hamiltonian PDE, up to now it has been never studied from the point of view of periodic so-

[12] An intermediate, "partially resonant" case has been studied in [50].

lutions, but only as an intial value problem, with techniques completely different from those described here. We think that an Hamiltonian perspective could stimulate new ideas in the study of this equation.

The sources we have used in writing this Introduction are [49, 4, 86, 48, 23] and the mentioned Review by Wayne on MathSciNet. In particular, our historical presentation follows the thread of the Introduction of [23]. For a history of nearly-integrable Hamiltonian PDE, see also the mentioned Review.

Chapter 2

Results and outline of the Thesis

This Thesis is essentially divided into two main parts.

In the first part, corresponding to Chapters 3 and 4, we deal with bifurcation problems for *autonomous* completely resonant nonlinear wave equations. The main difficulties solved in this part are in the analysis of the infinite-dimensional bifurcation equation.

In the second part, corresponding to Chapters 5 and 6, we study *forced* nonlinear wave and Kirchhoff equations, where perturbative results for periodic solutions are obtained via Nash-Moser techniques.

After a short description of the problem of completely resonant wave equations, we will give a presentation of our main results.

2.1 Completely resonant autonomous equations

We consider the autonomous equation

$$u_{tt} - u_{xx} = f(x, u) \tag{2.1}$$

with Dirichlet $u(0, t) = u(\pi, t) = 0$ or space-periodic $u(x, t) = u(x + 2\pi, t)$ boundary conditions. If

$$f(x, 0) = \partial_u f(x, 0) = 0,$$

the equation is called "completely resonant". The reason is that the linearised equation at $u = 0$ is the homogeneous d'Alembert equation

$$u_{tt} - u_{xx} = 0, \tag{2.2}$$

whose solutions are all periodic in time with period 2π. This is easily verified by decomposing u in space-Fourier series, that is

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t) \sin jx \quad \text{or} \quad u(x,t) = \sum_{j\in\mathbb{Z}} u_j(t)\, e^{ijx}$$

according to Dirichlet or periodic boundary conditions. Then u solves (2.2) if and only if

$$h_j''(t) + j^2 h_j(t) = 0 \quad \forall j,$$

that is $h_j(t) = a_j \cos jt + b_j \sin jt$ for all j. All h_j are 2π-periodic, then also u.

In the language of dynamical systems, the eigenvalues of the linearised system are all purely imaginary, and in particular they are all of the form ij for some integer j. With respect to the non-resonance hypothesis (1.2) of the Lyapunov Centre Theorem, here we are in a completely opposite situation: *all* eigenvalues are resonant. If we were in finite dimension, this case would be considered by the Fadell-Rabinowitz and Weinstein-Moser Theorems.

Compared with non-resonant cases (1.9), we encounter an additional difficulty beyond small divisors, that is an infinite-dimensional bifurcation equation.

Periodic solutions of completely resonant wave equations have been found first by Lidski, Schulman [76], Bambusi and Paleari [18] for nonlinearity $f = u^3 + O(u^4)$, where the bifurcation equation is solved by an ODE analysis suitable for the cubic term. The monotonicity of the leading term u^3 is used to prove a non-degeneracy condition in the bifurcation equation.

Then the existence and multiplicity of periodic solutions have been proved by Berti and Bolle [26, 27] for general nonlinearities, solving the bifurcation equation via variational methods.

In these four papers badly approximation conditions are imposed on the frequencies, forming sets of zero measure.

Recently these results have been extended to positive measure sets of frequencies by adapting the Nash-Moser technique [28, 29, 30, 13] and the Lindstedt series method [61] to completely resonant cases. These results can be considered as an infinite-dimensional extension of Weinstein-Moser and Fadell-Rabinowitz theorems. Also, we mention [62] where related results are proved for completely resonant nonlinear Schrödinger equations.

2.2 Contents of Chapter 3

In Chapter 3 we present results obtained in [11], where we prove a bifurcation theorem for quasi-periodic solutions with two frequencies of a class of completely resonant nonlinear wave equations. In particular, we consider the problem

$$(2.3) \qquad \begin{cases} v_{tt} - v_{xx} = -v^3 + f(v) \\ v(x,t) = v(x+2\pi,t), \qquad (x,t) \in \mathbb{R}^2 \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is analytic near $v = 0$ and $f(v) = O(v^4)$ as $v \to 0$.

Since all solutions of the linearised problem

$$(2.4) \qquad v_{tt} - v_{xx} = 0, \quad v(x,t) = v(x+2\pi,t)$$

are *periodic*, it is a primary problem to understand which solution of (2.4) *quasi*-periodic solutions bifurcate from.

In the recent paper [97], Procesi proved the first existence result for quasi-periodic solutions of completely resonant wave equations. In particular, she proved the existence of small-amplitude quasi-periodic solutions of (2.3) of the form

$$(2.5) \qquad v(x,t) = u(\omega_1 t + x,\, \omega_2 t - x),$$

where u is an odd analytic function, 2π-periodic in both its arguments, the frequencies ω_1, ω_2 are close to 1 and belong to a Cantor set of zero Lebesgue measure, assuming that f is odd and $f(v) = O(v^5)$. These solutions correspond — at the first order — to the superposition of two waves, travelling in opposite directions:

$$v(x,t) = \sqrt{\varepsilon}\big[r(\omega_1 t + x) + s(\omega_2 t - x) + h.o.t.\big]$$

where $\omega_1, \omega_2 = 1 + O(\varepsilon)$.

On the contrary, in [11] we prove the existence of quasi-periodic solutions of (2.3) having a different form, that is

$$(2.6) \qquad v(x,t) = u\big(\varepsilon t,\, (1 + b\varepsilon^2)t + x\big),$$

where u is 2π-periodic in both its arguments, (b, ε) are close to $(1/2, 0)$ and the ratio $(1 + b\varepsilon^2)/\varepsilon$ is irrational. Moreover, we do not assume that f has to be odd.

We perform a Lyapunov-Schmidt reduction, leading to the usual system of a range equation and a bifurcation equation.

The small divisors problem in the range equation is avoided imposing badly approximation conditions on the parameters, see (3.12).

The delicate part of the proof is the study of the bifurcation equation, which is infinite-dimensional because the equation is completely resonant. To solve the bifurcation equation we use an ODE analysis. We cannot directly use variational methods as in [26, 27, 32] because we have to ensure the non-trivial dependence of u on each of its arguments, in order to prove that the solution v is genuinely quasi-periodic.[1]

First, we find an explicit solution of the bifurcation equation by means of Jacobi elliptic functions (following [90, 97, 61]), see Lemma 1.

Next we prove its non-degeneracy (Lemma 4). These computations are the heart of the proof. Instead of using a computer assisted proof as in [97], here we employ purely analytic arguments. See also [90] for some related computations.

The solutions we find turn out to be, at the first order, the superposition of a travelling wave (having velocity greater than 1) and a modulation of long period, depending only on time:

$$v(x,t) = \varepsilon\big[r(\varepsilon t) + s((1+b\varepsilon^2)t + x) + h.o.t.\big],$$

see Theorem 1.

Finally we show that our arguments can be also used to extend results of [97] to non-odd nonlinearities, see Theorem 2.

We mention that recently the existence of quasi-periodic solutions with n frequencies has been proved in [114]. Solutions found in [114] belong to a neighborhood of a solution $u_0(t)$ periodic in time, independent of x, so they are different from those found here.

2.3 Contents of Chapter 4

In Chapter 4 we present results obtained in [13], where we prove the bifurcation of periodic solutions for completely resonant equations of the form (2.1).

In the recent paper [28] it is proved the existence and multiplicity of small amplitude periodic solutions of (2.1) with Dirichlet boundary conditions for a Cantor set of frequencies having asymptotically full measure at $\omega = 1$. Given the presence of small divisors, a crucial step in the study of the bifurcation equation is to prove that the "zeroth order bifurcation equation" has a non-degenerate periodic solution[2]. Such key property has been verified in [28] for nonlinearities of the form $f = a_2u^2 + O(u^4)$ or $f = a_3(x)u^3 + O(u^4)$.

In Chapter 4, using the ODE techniques developed in Chapter 3, we prove this non-degeneracy property for a new class of nonlinearities, that is

$$f(x,u) = a_2u^2 + a_3(x)u^3 + O(u^4) \quad \text{and} \quad f(x,u) = a_4u^4 + O(u^5), \tag{2.7}$$

extending the results of [28], see Theorem 3.

[1] Quasi-periodic solutions with two frequencies are found via variational methods in [32] for *forced* nonlinear wave equations.

[2] See Remark 1.

We underline that our proof is purely analytic — it does not use numerical calculations. It is based on the analysis of the variational equation and properties of the Jacobi elliptic functions.

Remark 1. In presence of small divisors, the range equation can be solved only on a Cantor set of parameters. This set has a very complicated structure, being the final result of infinitely many excisions which correspond to the small divisors one encounters at each step of the iteration.

Then a difficulty appears in solving the bifurcation equation, because the unknowns have to lie in the Cantor set. In other words, the problem is to ensure an "intersection property" for the solution sets of the two equations.

By assuming a non-degeneracy condition on the nonlinearity, the bifurcation equation can be solved by the classical implicit function theorem. At the same time, the non-degeneracy assumption ensures that the smooth path of solutions we have obtained intersects "transversally" the Cantor set, at least in the vicinity of the bifurcation point. As a consequence, one obtains solutions for a positive measure set of parameters. This is the classical argument used in [49, 39]; see also [28].[3]

2.4 Contents of Chapter 5

In Chapter 5 we present results obtained in [14], where we prove the existence and regularity of periodic forced vibrations of a nonhomogeneous string with fixed endpoints, both in the case when the forcing is small and in the case when the string vibrates very rapidly. In particular, we consider the problem

$$\begin{cases} \rho(x)u_{tt} - (p(x)u_x)_x = \mu f(x, \omega t, u) \\ u(0,t) = u(\pi, t) = 0 \end{cases} \tag{2.8}$$

where $\rho(x) > 0$ is the mass per unit length, $p(x) > 0$ is the modulus of elasticity multiplied by the cross-sectional area (see [47] p.291), $\mu > 0$ is a parameter, and the nonlinear forcing term $f(x, \omega t, u)$ is $(2\pi/\omega)$-periodic in time, that is $f(x, \cdot, u)$ is 2π-periodic.

Equation (2.8) is a nonlinear model also for propagation of waves in non-isotropic media describing seismic phenomena, see for example [15].

Looking for $(2\pi/\omega)$-time periodic solutions, we note an interesting difference between the d'Alembert operator $\partial_{tt} - \partial_{xx}$ and the differential operator $L_{\rho,p} = \rho(x)\partial_{tt} - \partial_x(p(x)\partial_x)$ for nonhomogeneous models. Indeed, if the density $\rho(x)$ and the elasticity $p(x)$ are not constant, the eigenfrequencies ω_j

[3] In problems where the bifurcation equation exhibits a variational structure, it would be natural to make use of critical point theory. The main obstacle to its application is the "intersection problem" described above. For an adaptation of variational methods to this difficult situation and a first result for *degenerate* cases we refer to [29].

of the string are no longer integer numbers. They have the asymptotic expansion[4]

$$\omega_j^2 = \frac{j^2}{c^2} + b + O\Big(\frac{1}{j}\Big) \tag{2.9}$$

with c, b are constants depending on ρ and p. The spectrum of $L_{\rho,p}$ is

$$\sigma_{\rho,p} := \{-\omega^2 l^2 + \omega_j^2 \,:\, l \in \mathbb{Z},\ j = 1, 2, \dots\}.$$

If $\omega = p/q$ is rational, good separation properties of the spectrum can be recovered when $p(x) = \rho(x)$ (so that $c = 1$) and assuming the extra condition $b \neq 0$. Indeed in this case $\sigma_{\rho,p}$ contains at most finitely many zero eigenvalues, while its remaining part is far away from zero, see [19, 105].

On the other hand, if $b = 0$, the eigenvalues corresponding to $qj = p|l|$ tend to zero, even if ω is rational. As a consequence, if we are not assuming any special hypothesis on ρ, p, we cannot avoid small divisors in the spectrum of $L_{\rho,p}$.

Existence of periodic weak solutions for the nonhomogeneous string equation has been proved by Acquistapace [2] for $\rho = 1$, μ small, for a zero measure set of forcing frequencies ω for which the eigenvalues $-\omega^2 l^2 + \omega_j^2$ are far away from zero. These frequencies are essentially badly approximable numbers.

For the homogeneous string $\rho = p = 1$, existence results for rational frequencies have been mentioned in the Introduction.

Here we recall a work of Plotnikov and Yungerman [92] where classical periodic solutions are found for a positive measure set of Diophantine frequencies, for small μ, assuming that f is monotone in u. This monotonicity condition makes it possible to control the constant coefficient in the asymptotic expansion of the eigenvalues (like b in (2.9)) of some perturbed linearised operator.

Recently Fokam [58] has proved the existence of periodic solutions for large frequencies ω in a set of asymptotically full measure, in presence of a potential term, for $\mu = 1$ and for nonlinearities of the form $f = u^3 + h(x,t)$ with h a trigonometric polynomial odd in time and space.

In Chapter 5 we prove existence of classical periodic solutions of (2.8) for every $\rho(x), p(x) > 0$, for general nonlinearities $f(x, \omega t, u)$, and for (μ, ω) belonging to a large measure Cantor set B_γ, when the ratio μ/ω is small. Our Theorem 4 covers both the case $\mu \to 0$ (weak forcing) and the case $\omega \to +\infty$ (rapid forcing). Moreover, no monotonicity condition on f is assumed.

As $\mu/\omega \to 0$, the solution we find tends to a static equilibrium $v(x)$ plus smaller, zero average oscillations $w(x,t)$ of amplitude $O(\mu/\omega)$.

[4]See (5.65).

The problem is attacked by performing a Lyapunov-Schmidt decomposition, splitting the functional space in a subspace V of time-independent functions and its orthogonal W.

The limit equilibrium $v(x)$ is selected by the nonlinearity through the infinite-dimensional V-equation ("bifurcation equation"), under natural assumptions on f which ensure a non-degeneracy condition (see Remark 1). Note that this problem is not present in [58] where, thanks to the symmetry condition for f, there is no bifurcation equation.

In the W-equation ("range equation") a small divisors problem arises. We solve it with a Nash-Moser type iterative scheme. The inversion of the "linearised operators" — which is the core of the Nash-Moser method — is obtained adapting the techniques of [28] to our time-dependent case.

In the range equation there is an interaction between the forcing frequency ω and the normal modes of oscillations of the string linearised at the different positions which approximate better and better the final string configuration.

The set B_γ of "non-resonant" parameters (μ, ω) for which we solve the range equation is constructed avoiding these primary resonances. In particular the forcing frequency ω must not enter in resonance with the normal frequencies of oscillations of the string linearised at the limit solution. This could be considered as a sort of non-eigen-resonance condition.

At the end of the construction we obtain a large measure Cantor set B_γ. Outside this set the effect of resonance phenomena could in general destroy the existence of periodic solutions like those found in Theorem 4.

Remark 2. We emphasize that proving the existence of periodic solutions for wave equations, both for rational and for irrational frequencies, is more difficult when the nonlinearity is *non-monotone* in the unknown u. Indeed, as remarked in [23], in this case there is no "confinement effect" due to the potential.

Periodic solutions of forced wave equations for a large class of non-monotone nonlinearities have been found in [25] for rational frequencies via variational methods (see also [23]). The results of Chapter 5 hold for non-resonant frequencies, so that they are complementary to those of [25].

In the autonomous case, periodic solutions for non-monotone nonlinearities have been found in [26, 28].

On the other hand, a non-existence result has been obtained in [32] for even power nonlinearities and space-periodic boundary conditions. This indicates that the existence results obtained for non-monotone nonlinearities and Dirichlet boundary conditions are due to a "boundary effect".

2.5 Contents of Chapter 6

In Chapter 6 we present results obtained in [12], where we prove the existence and regularity of periodic solutions of the forced Kirchhoff equation

$$u_{tt} - \Delta u\Big(1 + \int_\Omega |\nabla u|^2 dx\Big) = \varepsilon g(x,t) \tag{2.10}$$

both for Dirichlet and for space-periodic boundary conditions, for a large measure set of parameters.

First of all, we remark some differences between this equation and those studied in the other chapters. In the nonlinear wave equations of Chapters 3, 4 and 5 the nonlinear dependence on the unknown is of composition type, that is, the nonlinear term is of the form $f(x,t,u(x,t))$ where f is a given real-valued function $f(x,t,\xi)$ of real arguments (x,t,ξ). In the Kirchhoff equation, on the contrary, the integral term makes the nonlinearity *nonlocal.*

Moreover, the Kirchhoff nonlinearity is *quasi-linear*, because the coefficient $(1 + \int_\Omega |\nabla u|^2 dx)$ of the higher order term Δu depends (quadratically) on lower order derivatives.

In addition, in Chapter 6 we set the problem in any dimension $d \geq 1$.

In (2.10) g is a given time-periodic external forcing with period $2\pi/\omega$ and amplitude ε, and the displacement $u : \Omega \times \mathbb{R} \to \mathbb{R}$ is the unknown. We consider both Dirichlet boundary conditions

$$u(x,t) = 0 \quad \forall x \in \partial\Omega,\ t \in \mathbb{R} \tag{2.11}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded, connected open set with smooth boundary, and periodic boundary conditions on $\mathbb{R}^d$

$$u(x,t) = u(x + 2\pi m, t) \quad \forall m \in \mathbb{Z}^d,\ x \in \mathbb{R}^d,\ t \in \mathbb{R} \tag{2.12}$$

where $\Omega = (0, 2\pi)^d$.

(2.10) is a Hamiltonian integro-partial differential equation, with time-depending Hamiltonian

$$H(u,v) = \int_\Omega \frac{v^2}{2}\, dx + \int_\Omega \frac{|\nabla u|^2}{2}\, dx + \Big(\int_\Omega \frac{|\nabla u|^2}{2}\, dx\Big)^2 - \int_\Omega \varepsilon g u\, dx.$$

It describes nonlinear forced vibrations of a d-dimensional body (in particular, a string for $d = 1$ and a membrane for $d = 2$).

This model has been proposed first in 1876 by Kirchhoff [71] in dimension one, without forcing terms, with Dirichlet boundary conditions, namely

$$u_{tt} - u_{xx}\Big(1 + \int_0^\pi u_x^2\, dx\Big) = 0\,, \quad u(0,t) = u(\pi,t) = 0 \tag{2.13}$$

to describe transversal free vibrations of a clamped string in which the dependence of the tension on the deformation cannot be neglected. Independently, Carrier [45] and Narasimha [88] rediscovered the same equation as a nonlinear approximation of the exact model for the stretched string.

Kirchhoff equations have been studied by many authors from the point of view of the Cauchy problem

$$u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)$$

starting from the pioneering paper of Bernstein [22]. Both local and global existence has been investigated, for initial data having Sobolev or analytic regularity, mainly by using energy estimates. See for example [54, 93, 77, 9, 10, 80] and the rich surveys [8, 108]. We remark that the global existence in Sobolev (or even C^∞) spaces is still an open problem, except for special cases (for example for $\Omega = \mathbb{R}^d$ [63, 51]).

In spite of the wide study for the Cauchy problem, to the best of our knowledge nothing is known about the existence of periodic solutions of Kirchhoff equations, except for the normal modes.

Kirchhoff himself observed that equation (2.13), thanks to its special symmetry, possesses a sequence of normal modes, that is solutions of the form $u(x,t) = u_j(t) \sin jx$, $j = 1, 2, \dots$ where $u_j(t)$ is periodic. In general, normal modes are solutions of the form $u_j(t)\varphi_j(x)$ where $\varphi_j(x)$ is an eigenfunction of the Laplacian on Ω.

In presence of a forcing term $g(x,t)$ this symmetry is broken and normal modes do not survive (except in the one-mode case $g(x,t) = g_j(t)\varphi_j(x)$). Indeed, decomposing $u(x,t) = \sum_j u_j(t)\varphi_j(x)$ shows that all components $u_j(t)$ are coupled in the integral term $\int_\Omega |\nabla u|^2 dx$, and problem (2.10) is equivalent to a system of infinitely many nonlinear coupled ODE, namely

$$u_j''(t) + \lambda_j^2 u_j(t)\Big(1 + \sum_k \lambda_k^2 u_k^2(t)\Big) = \varepsilon g_j(t), \qquad j = 1, 2, \dots$$

where $g(x,t) = \sum_j g_j(t)\varphi_j(x)$ and λ_j^2 are the eigenvalues of the Laplacian on Ω.

In Chapter 6 we prove the existence of periodic solutions of (2.10). We consider the amplitude ε and the frequency ω of the forcing term g as parameters of the problem. We prove that there exist periodic solutions of order ε and period $2\pi/\omega$ when ε is small and (ε, ω) belong to a Cantor set which has positive measure, asymptotically full for $\varepsilon \to 0$. We prove regularity estimates for the solutions, both in Sobolev and in analytic classes, and local uniqueness, see Theorem 7, Remark 18 and Theorem 8.

We point out that, to the best of our knowledge, solutions we found are the first examples not only of periodic, but also of global in time solutions of forced Kirchhoff equations with Sobolev regularity (except for special cases like $\Omega = \mathbb{R}^d$ mentioned above).

There are two main difficulties in looking for small amplitude periodic solutions of (2.10). The first one is a small divisors problem: the spectrum of the d'Alembert operator $\partial_{tt} - \Delta$ in spaces of $2\pi/\omega$-periodic functions is

$$\{-\omega^2 l^2 + \lambda_j^2 \,:\, l \in \mathbb{N},\ j = 1, 2, \ldots\},$$

and, as we have already remarked, it accumulates to zero for almost every ω.

The other difficulty is the presence of derivatives in the nonlinearity. Little is known about periodic solutions of quasi-linear wave equations. For local nonlinearities, the problem has been studied by Rabinowitz [100] in presence of a dissipative term, namely

$$u_{tt} - u_{xx} + \alpha u_t = \varepsilon f(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}),$$

for the frequency $\omega = 1$; by Craig [48] for pseudodifferential operators

$$u_{tt} - u_{xx} = a(x)u + b(x, |\partial_x|^\beta u) = 0, \quad \beta < 1\,;$$

by Bourgain [40] in cases like $u_{tt} - u_{xx} + \rho u + u_t^2 = 0$ and, for quasiperiodic solutions, [37] $u_{tt} - u_{xx} = a(x)u + \varepsilon\partial_x^{1/2}(h(x, u))$. However, the Kirchhoff nonlinearity is not covered by these results.

Our proof overcomes these two difficulties by means of a Nash-Moser scheme. We emphasize that, because of the presence of derivatives in the nonlinearity, usual KAM and Lindstedt series methods seem not to apply to the problem.

Since we deal with not only analytic, but also finite order regularity, the scheme we use here differs from those of Craig, Wayne, Bourgain, and also of [28] and Chapter 5, because it does not rely on any analyticity assumption. Such a procedure goes back directly to ideas of the original methods of [82, 84, 116] and it is developed in [23, Ch. 3]. Recently [30] this technique has made it possible to prove the existence of periodic solutions of nonlinear wave equations for nonlinearities having only C^k differentiability. We point out that some of the difficulties of [30] are not present here, thanks to the special symmetry of the Kirchhoff nonlinearity.

Moreover, the roles played here by space and time are inverted with respect to [30, 28, 14].

We remark that small divisors problems become more difficult in higher dimension. For this reason, not many works deal with such problems when the dimension is larger than one; see for example [38, 39]. In that case, indeed, λ_j have a sub-linear growth, $\lambda_j \sim j^{1/d}$ as $j \to \infty$. In general this causes further difficulties in the inversion of the linearised operators. In our case, however, the structure of the Kirchhoff nonlinear integral term makes possible the inversion in any dimension.

Finally, we note that in case of periodic boundary conditions (2.12) zero is an eigenvalue of the Laplacian. As a consequence, we have to solve a space-average equation which is not present in the Dirichlet case (2.11).

2.6 Contents of the Appendices

Appendix A is a self-contained presentation of some classical results of number theory concerning the approximation of irrationals by rationals, the continued fraction theory and measure properties of number sets.

In Appendix B we prove an abstract Nash-Moser theorem, which is proper to applications dealing with spaces of finite regularity functions.

In Appendix C we report the classical proof of the algebra property for spaces of periodic functions.

Chapter 3

Quasi-periodic solutions of the equation $v_{tt} - v_{xx} + v^3 = f(v)$

We consider completely resonant nonlinear wave equations of the type

$$\begin{cases} v_{tt} - v_{xx} = -v^3 + f(v) \\ v(t,x) = v(t, x + 2\pi) \end{cases} \tag{3.1}$$

where f is analytic in a neighborhood of $v = 0$ and $f(v) = O(v^4)$ as $v \to 0$. Motivated by the existence result in [97] for solutions of the form (2.5), we start our study looking for solutions of the form

$$v(t,x) = u(\omega_1 t + x,\, \omega_2 t + x). \tag{3.2}$$

where u is 2π-periodic in both its arguments and the frequencies ω_1, ω_2 are close to 1. Solutions $v(t,x)$ of this form are quasi-periodic in time if u depends genuinely on both its arguments and the ratio between the periods is irrational, that is $\omega_1/\omega_2 \notin \mathbb{Q}$.

We set the problem in the space $\mathcal{H}_\sigma$ defined as follows. Let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T}^2$. For doubly 2π-periodic $u : \mathbb{T}^2 \to \mathbb{R}$, we develop in Fourier series

$$u(\varphi) = \sum_{(m,n)\in\mathbb{Z}^2} \hat{u}_{mn}\, e^{im\varphi_1} e^{in\varphi_2}. \tag{3.3}$$

Let $\sigma > 0$, $s \geq 0$. We define $\mathcal{H}_\sigma$ as the space of the even 2π-periodic functions $u : \mathbb{T}^2 \to \mathbb{R}$ such that

$$\sum_{(m,n)\in\mathbb{Z}^2} |\hat{u}_{mn}|^2 \left[1 + (m^2 + n^2)^s\right] e^{2\sqrt{m^2+n^2}\,\sigma} := \|u\|_\sigma^2 < \infty.$$

Elements of $\mathcal{H}_\sigma$ are even periodic functions which admit an analytic extension to the complex strip $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < \sigma\}$.

$(\mathcal{H}_\sigma, \|\cdot\|_\sigma)$ is a Hilbert space. For $s > 1$ it is also an algebra, that is, there exists a constant $c > 0$ such that

$$\|uv\|_\sigma \le c \|u\|_\sigma \|v\|_\sigma \quad \forall\, u, v \in \mathcal{H}_\sigma,$$

see Appendix C. Moreover, the inclusion $\mathcal{H}_{\sigma,s+1} \hookrightarrow \mathcal{H}_{\sigma,s}$ is compact.

We fix $s > 1$ once and for all.

We note that all solutions of the form (3.2) can be written in the form

$$v(t,x) = u\big(\varepsilon t,\, (1 + b\varepsilon^2)t + x\big) \tag{3.4}$$

where (b, ε) are close to $(1/2, 0)$ if we set

$$\omega_1 = 1 + \varepsilon + b\varepsilon^2, \quad \omega_2 = 1 + b\varepsilon^2. \tag{3.5}$$

Forms (3.4) and (3.2) are equivalent (see Section 3.5). Thus there is no loss of generality in studying the problem for solutions of the form (3.4). The condition $\omega_1/\omega_2 \notin \mathbb{Q}$ of genuine quasi-periodicity writes as $(1 + b\varepsilon^2)/\varepsilon \notin \mathbb{Q}$.

For functions (3.4), problem (3.1) becomes

$$\begin{cases} \varepsilon \left[\varepsilon\, \partial^2_{\varphi_1} + 2(1 + b\varepsilon^2)\, \partial^2_{\varphi_1\varphi_2} + b\varepsilon(2 + b\varepsilon^2)\, \partial^2_{\varphi_2}\right](u) = -u^3 + f(u) \\ u \in \mathcal{H}_\sigma. \end{cases}$$

We define

$$M_{b,\varepsilon} := \varepsilon\, \partial^2_{\varphi_1} + 2(1 + b\varepsilon^2)\, \partial^2_{\varphi_1\varphi_2} + b\varepsilon(2 + b\varepsilon^2)\, \partial^2_{\varphi_2},$$

we rescale $u \to \varepsilon u$ and denote $f_\varepsilon(u) = \varepsilon^{-3} f(\varepsilon u)$, so that the problem writes

$$\begin{cases} M_{b,\varepsilon}[u] = -\varepsilon u^3 + \varepsilon f_\varepsilon(u) \\ u \in \mathcal{H}_\sigma. \end{cases} \tag{3.6}$$

The main result of this chapter is the existence of solutions $u_{(b,\varepsilon)}$ of (3.6) for (b, ε) in a suitable uncountable set (Theorem 1).

Remark 3. If we consider a first-order relation between the amplitude and the frequencies, that is $\omega_1 = 1 + \varepsilon$, $\omega_2 = 1 + a\varepsilon$ as in [97], we obtain a bifurcation equation different than (3.11). More precisely, there is zero instead of $-b(2 + b\varepsilon^2)\, s''$ in the left-hand term of the Q_2-equation in (3.11). With this choice, we do not find solutions which are non-trivial in both its arguments, but only solutions depending on the variable φ_1.

3.1 Lyapunov-Schmidt reduction

The operator $M_{b,\varepsilon}$ is diagonal in the Fourier basis $e_{mn} = e^{im\varphi_1}e^{in\varphi_2}$ with eigenvalues $-D_{b,\varepsilon}(m,n)$, that is, if u is written in Fourier series as in (3.3),

$$M_{b,\varepsilon}[u] = -\sum_{(m,n)\in\mathbb{Z}^2} D_{b,\varepsilon}(m,n)\,\hat{u}_{mn}\,e^{im\varphi_1}\,e^{in\varphi_2}, \tag{3.7}$$

where the eigenvalues $D_{b,\varepsilon}(m,n)$ are given by

$$\begin{aligned} D_{b,\varepsilon}(m,n) &= \varepsilon\, m^2 + 2(1+b\varepsilon^2)\,mn + b\varepsilon(2+b\varepsilon^2)\,n^2 \\ &= (2+b\varepsilon^2)\Big(\frac{\varepsilon}{2+b\varepsilon^2}\,m+n\Big)\big(m+b\varepsilon\,n\big). \end{aligned} \tag{3.8}$$

For $\varepsilon = 0$ the operator is $M_{b,0} = 2\,\partial^2_{\varphi_1\varphi_2}$. Its kernel Z is the subspace of functions of the form $u(\varphi_1,\varphi_2) = r(\varphi_1) + s(\varphi_2)$ for some $r,\, s \in \mathcal{H}_\sigma$ one-variable functions,

$$Z = \big\{u\in\mathcal{H}_\sigma : \hat{u}_{mn} = 0 \;\; \forall\,(m,n)\in\mathbb{Z}^2,\, m,n\neq 0\big\}.$$

We decompose $\mathcal{H}_\sigma$ in four subspaces setting

$$\begin{aligned} &C = \{u\in\mathcal{H}_\sigma : u(\varphi) = \hat{u}_{0,0}\} \cong \mathbb{R}, \\ &Q_1 = \{u\in\mathcal{H}_\sigma : u(\varphi) = \textstyle\sum_{m\neq 0}\hat{u}_{m,0}\,e^{im\varphi_1} = r(\varphi_1)\}, \\ &Q_2 = \{u\in\mathcal{H}_\sigma : u(\varphi) = \textstyle\sum_{n\neq 0}\hat{u}_{0,n}\,e^{in\varphi_2} = s(\varphi_2)\}, \\ &P = \{u\in\mathcal{H}_\sigma : u(\varphi) = \textstyle\sum_{m,n\neq 0}\hat{u}_{mn}\,e^{im\varphi_1}\,e^{in\varphi_2} = p(\varphi_1,\varphi_2)\}. \end{aligned} \tag{3.9}$$

Thus the kernel is the direct sum $Z = C\oplus Q_1\oplus Q_2$ and the whole space is $\mathcal{H}_\sigma = Z\oplus P$. Any element u can be decomposed as

$$\begin{aligned} u(\varphi) &= \hat{u}_{0,0} + r(\varphi_1) + s(\varphi_2) + p(\varphi_1,\varphi_2) \\ &= z(\varphi) + p(\varphi). \end{aligned} \tag{3.10}$$

We denote $\langle\,\cdot\,\rangle$ the integral average: given $g\in\mathcal{H}_\sigma$,

$$\langle g\rangle = \langle g\rangle_\varphi = \tfrac{1}{(2\pi)^2}\textstyle\int_0^{2\pi}\int_0^{2\pi} g(\varphi)\,d\varphi_1 d\varphi_2,$$

$$\langle g\rangle_{\varphi_1} = \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} g(\varphi)\,d\varphi_1, \qquad \langle g\rangle_{\varphi_2} = \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} g(\varphi)\,d\varphi_2.$$

Note that $\frac{1}{2\pi}\int_0^{2\pi} e^{ikt}\,dt = 0$ for all integers $k\neq 0$, so

$$\begin{aligned} &\langle r\rangle = \langle r\rangle_{\varphi_1} = 0 && \langle r\rangle_{\varphi_2} = r \\ &\langle s\rangle = \langle s\rangle_{\varphi_2} = 0 && \langle s\rangle_{\varphi_1} = s \\ &\langle p\rangle = \langle p\rangle_{\varphi_1} = \langle p\rangle_{\varphi_2} = 0 && \langle u\rangle = \hat{u}_{0,0} \end{aligned}$$

for all $r\in Q_1$, $s\in Q_2$, $p\in P$, $u\in\mathcal{H}_\sigma$, and by means of these averages we can construct the projections on the subspaces,

$$\Pi_C = \langle\,\cdot\,\rangle, \qquad \Pi_{Q_1} = \langle\,\cdot\,\rangle_{\varphi_2} - \langle\,\cdot\,\rangle, \qquad \Pi_{Q_2} = \langle\,\cdot\,\rangle_{\varphi_1} - \langle\,\cdot\,\rangle.$$

Let $u = z + p$ as in (3.10). We split u^3 as $u^3 = z^3 + (u^3 - z^3)$ and compute the cube $z^3 = (\hat{u}_{0,0} + r + s)^3$. Clearly the diagonal operator $M_{b,\varepsilon}$ maps every subspace of (3.9) in itself, and

$$M_{b,\varepsilon}[r] = \varepsilon r'', \quad M_{b,\varepsilon}[s] = b\varepsilon(2 + b\varepsilon^2)s'', \quad M_{b,\varepsilon}[\hat{u}_{0,0}] = 0.$$

Then project problem (3.6) on the four subspaces gives

$$\begin{aligned}
0 &= \hat{u}_{0,0}^3 + 3\hat{u}_{0,0}\left(\langle r^2\rangle + \langle s^2\rangle\right) + \langle r^3\rangle + \langle s^3\rangle + \\
&\quad + \Pi_C\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad [C\textit{-equation}] \\
-r'' &= 3\hat{u}_{0,0}^2 r + 3\hat{u}_{0,0}\left(r^2 - \langle r^2\rangle\right) + r^3 - \langle r^3\rangle + 3\langle s^2\rangle\, r + \\
&\quad + \Pi_{Q_1}\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad [Q_1\textit{-equation}] \\
-b(2 + b\varepsilon^2)\, s'' &= 3\hat{u}_{0,0}^2 s + 3\hat{u}_{0,0}\left(s^2 - \langle s^2\rangle\right) + s^3 - \langle s^3\rangle + 3\langle r^2\rangle\, s + \\
&\quad + \Pi_{Q_2}\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad [Q_2\textit{-equation}] \\
M_{b,\varepsilon}[p] &= \varepsilon\, \Pi_P\left[-u^3 + f_\varepsilon(u)\right]. \qquad [P\textit{-equation}]
\end{aligned} \tag{3.11}$$

Now we study separately the projected equations.

3.2 The range equation

We write the P-equation thinking p as variable and z as a "parameter",

$$M_{b,\varepsilon}[p] = \varepsilon\, \Pi_P\left[-(z + p)^3 + f_\varepsilon(z + p)\right].$$

Our aim is to invert the operator $M_{b,\varepsilon}$. In Section 3.5 we prove that, fixed any $\gamma \in (0, 1/4)$, there exists a non-empty uncountable set $\mathcal{B}_\gamma \subseteq \mathbb{R}^2$ such that for all $(b, \varepsilon) \in \mathcal{B}_\gamma$ there holds

$$|D_{b,\varepsilon}(m, n)| > \gamma \qquad \forall\, m, n \in \mathbb{Z},\ m, n \neq 0.$$

In particular, the Cantor set $\mathcal{B}_\gamma$ is

$$\mathcal{B}_\gamma = \left\{(b, \varepsilon) \in \mathbb{R}^2 : \frac{\varepsilon}{2 + b\varepsilon^2}, b\varepsilon^2 \in \mathcal{W}_\gamma,\ \left|\frac{\varepsilon}{2 + b\varepsilon^2}\right|, |b\varepsilon^2| < \frac{1}{4},\ \frac{1 + b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}\right\}, \tag{3.12}$$

where $\mathcal{W}_\gamma$ is the set of badly approximable numbers defined by

$$\mathcal{W}_\gamma := \left\{x \in \mathbb{R} : |m + nx| > \frac{\gamma}{|n|} \quad \forall\, m, n \in \mathbb{Z},\ m \neq 0,\ n \neq 0\right\}, \tag{3.13}$$

see Appendix A. Therefore $M_{b,\varepsilon|P}$ is invertible for $(b,\varepsilon)\in\mathcal{B}_\gamma$ and by (3.7) it follows

$$(M_{b,\varepsilon|P})^{-1}[h] = -\sum_{m,n\neq 0} \frac{\hat{h}_{mn}}{D_{b,\varepsilon}(m,n)}\, e^{im\varphi_1}\, e^{in\varphi_2}$$

for all $h=\sum_{m,n\neq 0}\hat{h}_{mn}\, e^{im\varphi_1}\, e^{in\varphi_2}\in P$. Thus we obtain a bound for the inverse operators, uniformly in $(b,\varepsilon)\in\mathcal{B}_\gamma$:

$$\left\|(M_{b,\varepsilon|P})^{-1}\right\| \leq \frac{1}{\gamma}.$$

Applying the inverse operator $(M_{b,\varepsilon|P})^{-1}$, the P-equation becomes

$$p+\varepsilon(M_{b,\varepsilon|P})^{-1}\Pi_P\left[(z+p)^3 - f_\varepsilon(z+p)\right]=0. \tag{3.14}$$

We would like to apply the Implicit Function Theorem, but the inverse operator $(M_{b,\varepsilon|P})^{-1}$ is defined only for $(b,\varepsilon)\in\mathcal{B}_\gamma$ and in the set $\mathcal{B}_\gamma$ there are infinitely many holes, see Section 3.5. So we fix $(b,\varepsilon)\in\mathcal{B}_\gamma$, we introduce an auxiliary parameter μ and we consider the auxiliary equation

$$p+\mu(M_{b,\varepsilon|P})^{-1}\Pi_P\left[(z+p)^3 - f_\mu(z+p)\right]=0. \tag{3.15}$$

By the standard Contraction Mapping Theorem, there exists a positive constant c_1 depending only on f such that, if

$$(\mu,z)\in\mathbb{R}\times Z, \qquad |\mu|\,\|z\|_\sigma^2 < c_1\gamma, \tag{3.16}$$

equation (3.15) admits a solution $p_{(b,\varepsilon)}(\mu,z)\in P$ (see also Lemma 2.2 in [97]). Moreover, there exists a positive constant c_2 such that the solution $p_{(b,\varepsilon)}(\mu,z)$ respects the bound

$$\left\|p_{(b,\varepsilon)}(\mu,z)\right\|_\sigma \leq \frac{c_2}{\gamma}\,\|z\|_\sigma^3\,|\mu|. \tag{3.17}$$

Than we can apply the Implicit Function Theorem to the operator

$$\begin{aligned} &\mathbb{R}\times Z\times P \longrightarrow P \\ &\quad(\mu,z,p)\longmapsto p+\mu(M_{b,\varepsilon|P})^{-1}\Pi_P\left[(z+p)^3 - f_\mu(z+p)\right] \end{aligned}$$

at every point $(0,z,0)$. By local uniqueness we recover the regularity, and $p_{(b,\varepsilon)}$, as function of (μ,z), is at least of class $\mathcal{C}^1$.

Notice that the domain of any function $p_{(b,\varepsilon)}$ is defined by (3.16), so it does not depend on $(b,\varepsilon)\in\mathcal{B}_\gamma$.

In order to solve (3.14), we will need to evaluate $p_{(b,\varepsilon)}$ at $\mu=\varepsilon$; we will do it as last step, after the study of the bifurcation equation.

3.3 The bifurcation equation

We consider auxiliary Z-equations: we put f_μ instead of f_ε in (3.11),

$$
\begin{aligned}
0 = \ & \hat{u}_{0,0}^3 + 3\hat{u}_{0,0}\left(\langle r^2\rangle + \langle s^2\rangle\right) + \langle r^3\rangle + \langle s^3\rangle + \\
& +\Pi_C\left[(u^3 - z^3) - f_\mu(u)\right] \qquad [C - equation] \\
-r'' = \ & 3\hat{u}_{0,0}^2 r + 3\hat{u}_{0,0}\left(r^2 - \langle r^2\rangle\right) + r^3 - \langle r^3\rangle + 3\langle s^2\rangle\, r + \\
& +\Pi_{Q_1}\left[(u^3 - z^3) - f_\mu(u)\right] \qquad [Q_1 - equation] \\
-b(2+b\varepsilon^2)\, s'' = \ & 3\hat{u}_{0,0}^2 s + 3\hat{u}_{0,0}\left(s^2 - \langle s^2\rangle\right) + s^3 - \langle s^3\rangle + 3\langle r^2\rangle\, s + \\
& +\Pi_{Q_2}\left[(u^3 - z^3) - f_\mu(u)\right]. \qquad [Q_2 - equation]
\end{aligned} \tag{3.18}
$$

We substitute the solution $p_{(b,\varepsilon)}(\mu, z)$ of the auxiliary P-equation (3.15) in the auxiliary Z-equations (3.18), writing $u = z + p = z + p_{(b,\varepsilon)}(\mu, z)$, for (μ, z) in the domain (3.16) of $p_{(b,\varepsilon)}$.

We have $p_{(b,\varepsilon)}(\mu, z) = 0$ for $\mu = 0$, so the term $\left[(u^3 - z^3) - f_\mu(u)\right]$ vanishes for $\mu = 0$ and the bifurcation equations at $\mu = 0$ become

$$
\begin{aligned}
0 = \ & \hat{u}_{0,0}^3 + 3\hat{u}_{0,0}\left(\langle r^2\rangle + \langle s^2\rangle\right) + \langle r^3\rangle + \langle s^3\rangle \\
-r'' = \ & 3\hat{u}_{0,0}^2 r + 3\hat{u}_{0,0}\left(r^2 - \langle r^2\rangle\right) + r^3 - \langle r^3\rangle + 3\langle s^2\rangle\, r \\
-b(2+b\varepsilon^2)\, s'' = \ & 3\hat{u}_{0,0}^2 s + 3\hat{u}_{0,0}\left(s^2 - \langle s^2\rangle\right) + s^3 - \langle s^3\rangle + 3\langle r^2\rangle\, s.
\end{aligned} \tag{3.19}
$$

We look for a non-trivial solution $z = \hat{u}_{0,0} + r(\varphi_1) + s(\varphi_2)$ of (3.19). We rescale setting

$$
\begin{aligned}
r &= x & \hat{u}_{0,0} &= c \\
s &= \sqrt{b(2+b\varepsilon^2)}\ y & \lambda &= \lambda_{b,\varepsilon} = b(2+b\varepsilon^2),
\end{aligned} \tag{3.20}
$$

so that the equations become

$$
\begin{aligned}
& c^3 + 3c\big(\langle x^2\rangle + \lambda\langle y^2\rangle\big) + \langle x^3\rangle + \lambda^{3/2}\langle y^3\rangle = 0 \\
& x'' + 3c^2 x + 3c\big(x^2 - \langle x^2\rangle\big) + x^3 - \langle x^3\rangle + 3\lambda\langle y^2\rangle\, x = 0 \\
& y'' + 3c^2\tfrac{1}{\lambda} y + 3c\tfrac{1}{\sqrt{\lambda}}\big(y^2 - \langle y^2\rangle\big) + y^3 - \langle y^3\rangle + 3\tfrac{1}{\lambda}\langle x^2\rangle\, y = 0.
\end{aligned} \tag{3.21}
$$

In the following we show that, for $|\lambda - 1|$ sufficiently small, the system (3.21) admits a non-trivial non-degenerate solution. We consider λ as a free real parameter, recall that $Z = C \times Q_1 \times Q_2$ and define $G : \mathbb{R} \times Z \to Z$ setting $G(\lambda, c, x, y)$ as the set of three left-hand terms of (3.21).

Lemma 1. *There exist $\bar{\sigma} > 0$ and a non-trivial one-variable even analytic function β_0 belonging to $\mathcal{H}_\sigma$ for every $\sigma \in (0, \bar{\sigma})$, such that $G(1, 0, \beta_0, \beta_0) = 0$, that is $(0, \beta_0, \beta_0)$ solves (3.21) for $\lambda = 1$.*

Proof. We prove the existence of a non-trivial even analytic function β_0 which satisfies

$$\beta_0'' + \beta_0^3 + 3\langle\beta_0^2\rangle\,\beta_0 = 0, \qquad \langle\beta_0\rangle = \langle\beta_0^3\rangle = 0. \tag{3.22}$$

For any $m \in (0,1)$ we consider the Jacobi amplitude $\operatorname{am}(\,\cdot\,,m) : \mathbb{R} \to \mathbb{R}$ as the inverse of the elliptic integral of the first kind

$$I(\,\cdot\,,m) : \mathbb{R} \to \mathbb{R}, \qquad I(\varphi, m) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}}.$$

We define the Jacobi elliptic cosine setting

$$\operatorname{cn}(\xi) = \operatorname{cn}(\xi, m) = \cos(\operatorname{am}(\xi, m)),$$

see [1, Ch.16] and [113]. The elliptic cosine is a periodic function of period $4K$, where $K = K(m)$ is the complete elliptic integral of the first kind

$$K(m) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}}.$$

Jacobi cosine is even, and it is also odd-symmetric with respect to K on $[0, 2K]$, that is $\operatorname{cn}(\xi + K) = -\operatorname{cn}(\xi - K)$, just like the usual cosine. Then the averages on the period $4K$ are

$$\langle\operatorname{cn}\rangle = \langle\operatorname{cn}^3\rangle = 0.$$

Therefore it admits an analytic extension with a pole at iK', where $K' = K(1-m)$, and it satisfies $(\operatorname{cn}')^2 = -m\operatorname{cn}^4 + (2m-1)\operatorname{cn}^2 + (1-m)$, then cn is a solution of the ODE

$$\operatorname{cn}'' + 2m\operatorname{cn}^3 + (1-2m)\operatorname{cn} = 0.$$

We set $\beta_0(\xi) = V\operatorname{cn}(\Omega\xi, m)$ for some real parameters $V, \Omega > 0$, $m \in (0,1)$. β_0 has a pole at $i\frac{K'}{\Omega}$, so it belongs to $\mathcal{H}_\sigma$ for every $0 < \sigma < \frac{K'}{\Omega}$.

β_0 satisfies

$$\beta_0'' + \Big(2m\frac{\Omega^2}{V^2}\Big)\,\beta_0^3 + \Omega^2(1-2m)\,\beta_0 = 0.$$

If there holds the equality $2m\Omega^2 = V^2$, the equation becomes

$$\beta_0'' + \beta_0^3 + \Omega^2(1-2m)\,\beta_0 = 0.$$

β_0 is $\frac{4K(m)}{\Omega}$-periodic; it is 2π-periodic if $\Omega = \frac{2K(m)}{\pi}$. Hence we require

$$2m\Omega^2 = V^2, \qquad \Omega = \frac{2K(m)}{\pi}. \tag{3.23}$$

The other Jacobi elliptic functions we will use are

$$\operatorname{sn}(\xi)=\sin(\operatorname{am}(,m)),\qquad \operatorname{dn}(\xi)=\sqrt{1-m\operatorname{sn}^2(\xi)},$$

see [1, 113]. From the equality $m\operatorname{cn}^2(\xi)=\operatorname{dn}^2(\xi)-(1-m)$, with change of variable $x=\operatorname{am}(\xi)$ we obtain

$$\int_0^{K(m)} m\operatorname{cn}^2(\xi)\,d\xi=E(m)-(1-m)K(m),$$

where $E(m)$ is the complete elliptic integral of the second kind,

$$E(m)=\int_0^{\pi/2}\sqrt{1-m\sin^2\vartheta}\,d\vartheta.$$

Thus the average on $[0,2\pi]$ of β_0^2 is

$$\langle\beta_0^2\rangle=\frac{V^2}{m\,K(m)}\big[E(m)-(1-m)K(m)\big].$$

We want the equality $3\langle\beta_0^2\rangle=\Omega^2(1-2m)$ and this is true if

$$E(m)+\frac{8m-7}{6}K(m)=0. \tag{3.24}$$

The left-hand term $\psi(m):=E(m)+\frac{8m-7}{6}K(m)$ is continuous in m; its value at $m=0$ is $-(\pi/12)<0$, while at $m=1/2$, by definition of E and K,

$$\psi\Big(\frac{1}{2}\Big)=\frac{1}{2}\int_0^{\pi/2}\frac{\cos^2\vartheta}{\big(1-\frac{1}{2}\sin^2\vartheta\big)^{1/2}}\,d\vartheta>0.$$

Moreover, its derivative is strictly positive for every $m\in\big[0,\frac{1}{2}\big]$,

$$\begin{aligned}\psi'(m)&=\int_0^{\pi/2}\frac{8-\frac{5}{2}\sin^2\vartheta+3m\sin^4\vartheta-8m\sin^2\vartheta}{6(1-m\sin^2\vartheta)^{3/2}}\,d\vartheta\\&\geq\int_0^{\pi/2}\frac{3+\cos^2\vartheta}{6}\,d\vartheta>0,\end{aligned}$$

hence there exists a unique $\bar m\in\big(0,\frac{1}{2}\big)$ which solves (3.24). Thanks to the tables in [1, p. 608-609], we have $0.20<\bar m<0.21$.

By (3.23) the value $\bar m$ determines the parameters $\bar\Omega$ and $\bar V$, so the function $\beta_0(\xi)=\bar V\operatorname{cn}(\bar\Omega\xi,\bar m)$ satisfies (3.22) and $(0,\beta_0,\beta_0)$ is a solution of (3.21) for $\lambda=1$. Therefore $\beta_0\in\mathcal{H}_\sigma$ for every $\sigma\in(0,\bar\sigma)$, where $\bar\sigma=(\frac{K'}{\Omega})_{|m=\bar m}$. □

The next step will be to prove the non-degeneracy of the solution $(1,0,\beta_0,\beta_0)$, that is to show that the partial derivative $\partial_Z G(1,0,\beta_0,\beta_0)$ is an invertible operator. This is the heart of the present paper. We need some preliminary results.

Lemma 2. *Given h even 2π-periodic, there exists a unique even 2π-periodic w such that*

$$w'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)w = h.$$

This defines the Green operator $L : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$, $L[h] = w$.

Proof. We fix a 2π-periodic even function h. We look for even 2π-periodic solutions of the non-homogeneous equation

$$x'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)x = h. \tag{3.25}$$

First of all, we construct two solutions of the homogeneous equation

$$x'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)x = 0. \tag{3.26}$$

We recall that β_0 satisfies $\beta_0'' + \beta_0^3 + 3\langle\beta_0^2\rangle\,\beta_0 = 0$, then differentiating with respect to its argument ξ we obtain

$$\beta_0''' + 3\beta_0^2\beta_0' + 3\langle\beta_0^2\rangle\beta_0' = 0,$$

so β_0' satisfies (3.26). We set

$$\bar{u}(\xi) = -\frac{1}{\bar{V}\bar{\Omega}^2}\beta_0'(\xi) = -\frac{1}{\bar{\Omega}}\,\mathrm{cn}'(\bar{\Omega}\xi, \bar{m}), \tag{3.27}$$

thus $\bar{u}$ is the solution of the homogeneous equation such that $\bar{u}(0) = 0$, $\bar{u}'(0) = 1$. It is odd and 2π-periodic.

Now we construct the other solution. We indicate c_0 the constant $c_0 = \langle\beta_0\rangle$. We recall that, for any V, Ω, m the function $y(\xi) = V\mathrm{cn}(\Omega\xi, m)$ satisfies

$$y'' + \left(2m\frac{\Omega^2}{V^2}\right)y^3 + \Omega^2(1-2m)\,y = 0.$$

We consider m and V as functions of the parameter Ω, setting

$$m = m(\Omega) = \frac{1}{2} - \frac{3c_0}{2\Omega^2}, \qquad V = V(\Omega) = \sqrt{\Omega^2 - 3c_0}. \tag{3.28}$$

We indicate $y_\Omega(\xi) = V(\Omega)\,\mathrm{cn}\big(\Omega\xi, m(\Omega)\big)$, so $(y_\Omega)_\Omega$ is a one-parameter family of solutions of

$$y_\Omega'' + y_\Omega^3 + 3c_0 y_\Omega = 0.$$

We can differentiate this equation with respect to Ω, obtaining

$$(\partial_\Omega y_\Omega)'' + 3y_\Omega^2(\partial_\Omega y_\Omega) + 3c_0(\partial_\Omega y_\Omega) = 0.$$

Now we evaluate $(\partial_\Omega y_\Omega)$ at $\Omega = \bar{\Omega}$, where $\bar{\Omega}$ correspond to the value $\bar{m}$ found in Lemma 1. For $\Omega = \bar{\Omega}$ it holds $y_{\bar{\Omega}} = \beta_0$, so $(\partial_\Omega y_\Omega)_{|\Omega=\bar{\Omega}}$ satisfy (3.26). In order to normalize this solution, we compute

$$(\partial_\Omega y_\Omega)(\xi) = (\partial_\Omega V)\mathrm{cn}(\Omega\xi, m) + V\xi\mathrm{cn}'(\Omega\xi, m) + V\partial_m\mathrm{cn}(\Omega\xi, m)(\partial_\Omega m).$$

Since $\mathrm{cn}(0,m) = 1\ \forall m$, it holds $\partial_m \mathrm{cn}(0,m) = 0$; therefore $\mathrm{cn}'(0,m) = 0$ $\forall m$. From (3.28) we have $\partial_\Omega V = \frac{\Omega}{V}$, so we can normalize setting

$$\bar v(\xi) = \frac{\bar V}{\bar\Omega}\left(\partial_\Omega y_\Omega\right)_{|\Omega=\bar\Omega}(\xi).$$

$\bar v$ is the solution of the homogeneous equation (3.26) such that $\bar v(0) = 1$, $\bar v'(0) = 0$. We can write an explicit formula for $\bar v$. From the definitions it follows for any m

$$\partial_m \mathrm{am}(\xi,m) = -\mathrm{dn}(\xi,m)\frac{1}{2}\int_0^\xi \frac{\mathrm{sn}^2(t,m)}{\mathrm{dn}^2(t,m)}\,dt.$$

Therefore $\mathrm{cn}'(\xi) = -\mathrm{sn}(\xi)\,\mathrm{dn}(\xi)$; then we obtain for $(V,\Omega,m) = (\bar V,\bar\Omega,\bar m)$

$$\bar v(\xi) = \mathrm{cn}(\bar\Omega\xi) + \frac{\bar V^2}{\bar\Omega}\,\mathrm{cn}'(\bar\Omega\xi)\left[\xi + \frac{2\bar m - 1}{2}\int_0^\xi \frac{\mathrm{sn}^2(\bar\Omega t)}{\mathrm{dn}^2(\bar\Omega t)}\,dt\right]. \tag{3.29}$$

By formula (3.29) we can see that $\bar v$ is even; it is not periodic and there holds

$$\bar v(\xi + 2\pi) - \bar v(\xi) = \frac{\bar V^2 k}{\bar\Omega}\,\mathrm{cn}'(\bar\Omega\xi) = -\bar V^2 k\,\bar u(\xi), \tag{3.30}$$

where

$$k := 2\pi + \frac{2\bar m - 1}{2}\int_0^{2\pi} \frac{\mathrm{sn}^2(\bar\Omega t)}{\mathrm{dn}^2(\bar\Omega t)}\,dt. \tag{3.31}$$

From the equalities (L.1) and (L.2) of Lemma 3 we obtain

$$k = 2\pi\,\frac{-1 + 16\bar m - 16\bar m^2}{12\bar m(1-\bar m)}, \tag{3.32}$$

so $k > 0$ because $\bar m \in (0.20, 0.21)$.

We have constructed two solutions $\bar u$, $\bar v$ of the homogeneous equation; their wronskian $\bar u'\bar v - \bar u\bar v'$ is equal to 1, so we can write a particular solution $\bar w$ of the non-homogeneous equation (3.25) as

$$\bar w(\xi) = \left(\int_0^\xi h\bar v\right)\bar u(\xi) - \left(\int_0^\xi h\bar u\right)\bar v(\xi).$$

Every solution of (3.25) is of the form $w = A\bar u + B\bar v + \bar w$ for some $(A,B) \in \mathbb{R}^2$. Since h is even, $\bar w$ is also even, so w is even if and only if $A = 0$.

An even function $w = B\bar v + \bar w$ is 2π-periodic if and only if $w(\xi + 2\pi) - w(\xi) = 0$, that is, by (3.30),

$$\left(\int_\xi^{\xi+2\pi} h\bar v\right)\bar u(\xi) + \left[\left(\int_0^\xi h\bar u\right) - B\right]\bar V^2 k\,\bar u(\xi) = 0 \quad \forall\xi.$$

We remove $\bar{u}(\xi)$, derive the expression with respect to ξ and from (3.30) it results zero at any ξ. Then the expression is a constant; we compute it at $\xi = 0$ and obtain, since $h\bar{u}$ is odd and 2π-periodic, that w is 2π-periodic if and only if $B = \frac{1}{\bar{V}^2 k}\int_0^{2\pi} h\bar{v}$.

Thus, given h even 2π-periodic, there exists a unique even 2π-periodic w such that $w'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)w = h$ and this defines the operator L,

$$L[h] = \left(\int_0^{\xi} h\bar{v}\right)\bar{u}(\xi) + \left[\left(\frac{1}{\bar{V}^2 k}\int_0^{2\pi} h\bar{v}\right) - \int_0^{\xi} h\bar{u}\right]\bar{v}(\xi). \tag{3.33}$$

L is linear and continuous with respect to $\|\cdot\|_\sigma$; it is the Green operator of the equation $x'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)x = h$, so, by classical arguments, it is a bounded operator of $\mathcal{H}_{\sigma,s}$ into $\mathcal{H}_{\sigma,s+2}$; the inclusion $\mathcal{H}_{\sigma,s+2} \hookrightarrow \mathcal{H}_{\sigma,s}$ is compact, then $L : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$ is compact. □

Lemma 3. *There holds the following equalities and inequalities.*

(L.1) $\langle \mathrm{cn}^2\rangle = \frac{1-2\bar{m}}{6\bar{m}}$ *for* $m = \bar{m}$. *(Recall:* $\mathrm{cn} = \mathrm{cn}(\,\cdot\,, m)$*)*

(L.2) $\langle \frac{\mathrm{sn}^2}{\mathrm{dn}^2}\rangle = \frac{1}{1-m}\langle \mathrm{cn}^2\rangle$ *for any* m.

(L.3) $m\langle \mathrm{cn}^2 \frac{\mathrm{sn}^2}{\mathrm{dn}^2}\rangle = 1 - 2\langle \mathrm{cn}^2\rangle$ *for any* m.

(L.4) $\langle gL[h]\rangle = \langle hL[g]\rangle \quad \forall\, g, h$ *even* 2π*-periodic.*

(L.5) $1 - 3\langle \beta_0^2 L[1]\rangle = 3\langle\beta_0^2\rangle\langle L[1]\rangle$.

(L.6) $\langle \beta_0^2 L[\beta_0]\rangle = -\langle\beta_0^2\rangle\langle L[\beta_0]\rangle$.

(L.7) $3\langle \beta_0^2 L[\beta_0^2]\rangle = \langle\beta_0^2\rangle\left(1 - 3\langle L[\beta_0^2]\rangle\right)$.

(L.8) $\langle \beta_0^2 L[\beta_0]\rangle = \langle \beta_0 L[\beta_0^2]\rangle = \langle L[\beta_0]\rangle = 0$.

(L.9) $A_0 :=\ 1 - 3\langle \beta_0^2 L[1]\rangle \neq 0$.

(L.10) $B_0 :=\ 1 - 6\langle \beta_0 L[\beta_0]\rangle \neq 0$.

(L.11) $C_0 :=\ 1 + 6\langle \beta_0 L[\beta_0]\rangle \neq 0$.

(L.12) $A_0 \neq 1,\ \langle L[\beta_0^2]\rangle \neq 0$.

Proof. (L.1) By construction of β_0 we have $\bar{\Omega}^2(1 - 2\bar{m}) = 3\langle\beta_0^2\rangle = 3\bar{V}^2\langle \mathrm{cn}^2(\cdot, \bar{m})\rangle$ and $\bar{V}^2 = 2\bar{m}\bar{\Omega}^2$, see Proof of Lemma 1.

(L.2) We observe that

$$\frac{d}{d\xi}\left[\frac{\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)}\right] = \frac{(m-1)\mathrm{sn}(\xi)}{\mathrm{dn}^2(\xi)},$$

then we can integrate by parts

$$\int_0^{4K} \frac{\mathrm{sn}^2(\xi)}{\mathrm{dn}^2(\xi)}\, d\xi = \int_0^{4K} \frac{\mathrm{sn}(\xi)}{m-1} \frac{d}{d\xi}\left[\frac{\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)}\right] d\xi = \frac{1}{1-m}\int_0^{4K} \mathrm{cn}^2(\xi)\, d\xi.$$

(L.3) We compute the derivative

$$\frac{d}{d\xi}\left[\frac{\mathrm{cn}(\xi)\,\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)}\right] = 2\mathrm{cn}^2(\xi) - 1 + m\,\frac{\mathrm{sn}^2(\xi)\mathrm{cn}^2(\xi)}{\mathrm{dn}^2(\xi)}$$

and integrate on the period $[0, 4K]$.

(L.4) From the formula (3.33) of L we have

$$\begin{aligned}\langle gL[h]\rangle - \langle hL[g]\rangle = \langle \frac{d}{d\xi}\Big[\Big(\int_0^\xi h\bar{v}\Big)\Big(\int_0^\xi g\bar{u}\Big)\Big]\rangle - \langle \frac{d}{d\xi}\Big[\Big(\int_0^\xi h\bar{u}\Big)\Big(\int_0^\xi g\bar{v}\Big)\Big]\rangle \\ + \frac{1}{\bar{V}^2 k}\, 2\pi\big[\langle h\bar{v}\rangle\langle g\bar{v}\rangle - \langle g\bar{v}\rangle\langle h\bar{v}\rangle\big] = 0.\end{aligned}$$

(L.5) By definition, $L[1]$ satisfies $L[1]'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)L[1] = 1$, so we integrate on the period $[0, 2\pi]$.

(L.6),(L.7) Similarly by definition of $L[\beta_0]$, $L[\beta_0^2]$; recall that $\langle\beta_0\rangle = 0$.

(L.8) By (L.6) and (L.4), it is sufficient to show that $\langle L[\beta_0]\rangle = 0$. From the formula (3.33), integrating by parts we have

$$\langle L[\beta_0]\rangle = -\langle \beta_0\bar{v}\Big(\int_0^\xi \bar{u}\Big)\rangle - \langle\Big(\int_0^\xi \beta_0\bar{u}\Big)\bar{v}\rangle + \frac{1}{\bar{V}^2 k}\Big(\int_0^{2\pi} \beta_0\bar{v}\Big)\langle\bar{v}\rangle.$$

From the formulas (3.27), (3.29) of $\bar{u}, \bar{v}$, recalling that $\beta_0(\xi) = \bar{V}\mathrm{cn}(\bar{\Omega}\xi)$, we compute

$$\text{(3.34)}\qquad \int_0^\xi \bar{u} = -\frac{1}{\bar{\Omega}^2}\big(\mathrm{cn}(\bar{\Omega}\xi) - 1\big), \qquad \int_0^\xi \beta_0\bar{u} = -\frac{\bar{V}}{2\bar{\Omega}^2}\big(\mathrm{cn}^2(\bar{\Omega}\xi) - 1\big).$$

Observe that $\int_0^{2\pi} \mathrm{cn}(\bar{\Omega}\xi)\frac{\mathrm{sn}^2(\bar{\Omega}\xi)}{\mathrm{dn}^2(\bar{\Omega}\xi)}\, d\xi = 0$ by odd-symmetry with respect to $\frac{\pi}{2}$ on $[0, \pi]$ and periodicity. So, recalling that $\bar{V}^2 = 2\bar{m}\bar{\Omega}^2$, we compute $\langle\bar{v}\rangle = \frac{\bar{m}k}{\pi}$. We can resume the computation of $\langle L[\beta_0]\rangle$ obtaining

$$\langle L[\beta_0]\rangle = \frac{3\bar{V}}{2\bar{\Omega}^2}\langle\bar{v}(\xi)\mathrm{cn}^2(\bar{\Omega}\xi)\rangle - \frac{\bar{V}\bar{m}k}{2\pi\bar{\Omega}^2}.$$

Since $\langle\mathrm{cn}^3\rangle = \langle\mathrm{cn}^3\frac{\mathrm{sn}^2}{\mathrm{dn}^2}\rangle = 0$ by the same odd-symmetry reason, by (3.29) we have $\langle\bar{v}(\xi)\mathrm{cn}^2(\bar{\Omega}\xi)\rangle = \frac{\bar{m}k}{3\pi}$, and so $\langle L[\beta_0]\rangle = 0$.

Moreover we can remark that by (L.4) there holds also $\langle\beta_0 L[1]\rangle = 0$.

(L.9) By (L.5), it is equivalent to show that $\langle L[1]\rangle \neq 0$. From the formula (3.33), integrating by parts we have

$$\langle L[1]\rangle = \frac{2\pi}{\bar{V}^2 k}\langle \bar{v}\rangle^2 - 2\langle\Big(\int_0^\xi \bar{u}\Big)\bar{v}\rangle.$$

We know that $\langle \bar{v}\rangle = \frac{\bar{m}k}{\pi}$, so by (3.34)

$$\langle L[1]\rangle = \frac{1}{\bar{\Omega}^2}\langle \bar{v}(\xi)\big(2\mathrm{cn}(\bar{\Omega}\xi) - 1\big)\rangle.$$

From the equalities (L.1) and (L.3) we have $\langle \bar{v}(\xi)\mathrm{cn}(\bar{\Omega}\xi)\rangle = \frac{2}{3}(1-2\bar{m}) + \frac{\bar{m}k}{2\pi}$, thus

$$\langle L[1]\rangle = \frac{4(1-2\bar{m})}{3\bar{\Omega}^2} \tag{3.35}$$

and this is strictly positive because $\bar{m} < \frac{1}{2}$.

(L.10) From (3.33) integrating by parts we have

$$\langle \beta_0 L[\beta_0]\rangle = -2\langle \beta_0\bar{v}\Big(\int_0^\xi \beta_0\bar{u}\Big)\rangle + \frac{2\pi}{\bar{V}^2 k}\langle \beta_0\bar{v}\rangle^2.$$

Using (L.3), integrating by parts and recalling the definition (3.31) of k we compute

$$\langle \beta_0\bar{v}\rangle = \bar{V}\bar{m}\langle \mathrm{cn}^2\rangle + \frac{\bar{V}\bar{m}k}{2\pi} + \frac{\bar{V}(1-2\bar{m})}{2}$$

and, by (L.1) and (3.32),

$$\langle \beta_0\bar{v}\rangle = \frac{\bar{V}(7-8\bar{m})}{12(1-\bar{m})}. \tag{3.36}$$

By (3.34), $\langle \beta_0\bar{v}\Big(\int_0^\xi \beta_0\bar{u}\Big)\rangle = \frac{\bar{V}}{2\bar{\Omega}^2}\langle \beta_0\bar{v}\,\mathrm{cn}^2\rangle + \frac{\bar{V}}{2\bar{\Omega}^2}\langle \beta_0\bar{v}\rangle$. The functions β_0 and $\bar{v}$ satisfy $\beta_0'' + \beta_0^3 + 3\langle\beta_0^2\rangle\beta_0 = 0$ and $\bar{v}'' + 3\beta_0^2\bar{v} + 3\langle\beta_0^2\rangle\bar{v} = 0$, so that

$$\bar{v}''\beta_0 - \bar{v}\beta_0'' + 2\beta_0^3\bar{v} = 0. \tag{3.37}$$

Deriving (3.30) we have $\bar{v}'(2\pi) - \bar{v}'(0) = -\bar{V}^2 k$, so we can integrate (3.37) obtaining

$$\langle \beta_0^3\bar{v}\rangle = \frac{\bar{V}^3 k}{4\pi};$$

since $\langle \beta_0^3\bar{v}\rangle = \bar{V}^2\langle \beta_0\bar{v}\,\mathrm{cn}^2\rangle$, we write

$$\langle \beta_0\bar{v}\Big(\int_0^\xi \beta_0\bar{u}\Big)\rangle = -\frac{\bar{m}k}{4\pi} + \frac{\bar{V}}{2\bar{\Omega}^2}\langle \beta_0\bar{v}\rangle.$$

Thus, by (3.36) and (3.31), we can express $\langle \beta_0 L[\beta_0] \rangle$ in terms of $\bar{m}$ only,

$$\langle \beta_0 L[\beta_0] \rangle = \frac{32\bar{m}^2 - 32\bar{m} - 1}{12(16\bar{m}^2 - 16\bar{m} + 1)} = \frac{1}{6} - \frac{1}{4(16\bar{m}^2 - 16\bar{m} + 1)}. \tag{3.38}$$

The polynomial $p(m) = 16m^2 - 16m + 1$ is non-zero for $m \in \big(\frac{2-\sqrt{3}}{4}, \frac{2+\sqrt{3}}{4}\big)$ and $\bar{m} \in (0.20, 0.21)$; so $B_0 = \frac{6}{4p(\bar{m})} \neq 0$, in particular $B_0 \in (-1, -0.9)$.

(L.11) From (3.38) it follows that $C_0 \neq 0$, in particular $2.9 < C_0 = 2 - \frac{3}{2p(\bar{m})} < 3$.

(L.12) By (L.4) and (L.5), it is sufficient to show that $A_0 \neq 1$, that is $3\langle \beta_0^2 \rangle \langle L[1] \rangle \neq 1$. Recall that, by construction of $\bar{m}$, $3\langle \beta_0^2 \rangle = \bar{\Omega}^2(1 - 2\bar{m})$. So from (3.35) it follows

$$3\langle \beta_0^2 \rangle \langle L[1] \rangle = \frac{4}{3}(1 - 2\bar{m})^2,$$

and $\frac{4}{3}(1 - 2m)^2 = 1$ if and only if $16m^2 - 16m + 1 = 0$, while $\bar{m} \in (0.20, 0.21)$, like above; in particular $0.4 < 3\langle \beta_0^2 \rangle \langle L[1] \rangle < 0.5$. □

Remark 4. Approximated computations give

$$\begin{array}{lll} \bar{m} \in (0.20,\ 0.21) & \bar{\sigma} \in (2.10,\ 2.16) & \bar{\Omega} \in (1.05,\ 1.06) \\ \bar{V}^2 \in (0.44,\ 0.48) & \langle \mathrm{cn}^2 \rangle \in (2.85,\ 2.90) & \langle \beta_0^2 \rangle \in (1.27,\ 1.37). \end{array}$$

Lemma 4. *The partial derivative $\partial_Z G(1, 0, \beta_0, \beta_0)$ is an invertible operator.*

Proof. Let $\partial_Z G(1, 0, \beta_0, \beta_0)[\eta, h, k] = (0, 0, 0)$ for some $(\eta, h, k) \in Z$, that is

$$\begin{aligned} &6\eta\langle \beta_0^2 \rangle + 3\langle \beta_0^2 h \rangle + 3\langle \beta_0^2 k \rangle = 0 \\ &3\eta\big(\beta_0^2 - \langle \beta_0^2 \rangle\big) + h'' + \big(3\beta_0^2 + 3\langle \beta_0^2 \rangle\big)h - 3\langle \beta_0^2 h \rangle + 6\langle \beta_0 k \rangle \beta_0 = 0 \\ &3\eta\big(\beta_0^2 - \langle \beta_0^2 \rangle\big) + k'' + \big(3\beta_0^2 + 3\langle \beta_0^2 \rangle\big)k - 3\langle \beta_0^2 k \rangle + 6\langle \beta_0 h \rangle \beta_0 = 0. \end{aligned} \tag{3.39}$$

We evaluate the second and the third equation at the same variable and subtract; $\rho = h - k$ satisfies

$$\rho'' + \big(3\beta_0^2 + 3\langle \beta_0^2 \rangle\big)\rho - 3\langle \beta_0^2 \rho \rangle - 6\langle \beta_0 \rho \rangle \beta_0 = 0. \tag{3.40}$$

By definition of L, see Lemma 2, (3.40) can be written as

$$\rho = 3\langle \beta_0^2 \rho \rangle L[1] + 6\langle \beta_0 \rho \rangle L[\beta_0]. \tag{3.41}$$

Multiplying this equation by β_0^2 and integrating we obtain

$$\langle \beta_0^2 \rho \rangle \big(1 - 3\langle \beta_0^2 L[1] \rangle\big) = 6\langle \beta_0 \rho \rangle \langle \beta_0^2 L[\beta_0] \rangle.$$

In Lemma 3 we prove that $\big(1-3\langle\beta_0^2 L[1]\rangle\big) = A_0 \neq 0$ and $\langle\beta_0^2 L[\beta_0]\rangle = 0$, then $\langle\beta_0^2\rho\rangle = 0$.

On the other hand, multiplying (3.41) by β_0 and integrating we have

$$\langle\beta_0\rho\rangle\,\big(1-6\langle\beta_0 L[\beta_0]\rangle\big) = 3\langle\beta_0^2\rho\rangle\,\langle\beta_0 L[1]\rangle.$$

In Lemma 3 we show that $\big(1-6\langle\beta_0 L[\beta_0]\rangle\big) = B_0 \neq 0$ and $\langle\beta_0 L[1]\rangle = 0$, then $\langle\beta_0\rho\rangle = 0$. From (3.41) we have so $\rho = 0$. Thus $h = k$ and (3.39) becomes

$$\begin{aligned}&\eta\langle\beta_0^2\rangle + \langle\beta_0^2 h\rangle = 0\\ &3\eta\big(\beta_0^2 - \langle\beta_0^2\rangle\big) + h'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)h - 3\langle\beta_0^2 h\rangle + 6\langle\beta_0 h\rangle\,\beta_0 = 0.\end{aligned}$$

By substitution we have

$$h = -3\eta L[\beta_0^2] - 6\langle\beta_0 h\rangle\, L[\beta_0].$$

Multiplying, as before, by β_0^2 and by β_0 and integrating, we obtain $\langle\beta_0 h\rangle = \langle\beta_0^2 h\rangle = 0$ because $\big(1+6\langle\beta_0 L[\beta_0]\rangle\big) = C_0 \neq 0$, $\langle\beta_0 L[\beta_0^2]\rangle = 0$, and $\langle\beta_0^2\rangle - 3\langle\beta_0^2 L[\beta_0^2]\rangle = 3\langle\beta_0^2\rangle\,\langle L[\beta_0^2]\rangle \neq 0$, see Lemma 3 again. Thus $h = 0$, $\eta = 0$ and the derivative $\partial_Z G(1,0,\beta_0,\beta_0)$ is injective.

The operator $Z \to Z$, $(\eta,h,k) \mapsto ((6\langle\beta_0^2\rangle)^{-1}\eta,\, L[h],\, L[k])$ is compact because L is compact, see Lemma 2. So, by the Fredholm Alternative, the partial derivative $\partial_Z G(1,0,\beta_0,\beta_0)$ is also surjective. □

By the Implicit Function Theorem and the regularity of G, using the rescaling (3.20) we obtain, for $|b-\frac{1}{2}|$ and ε small enough, the existence of a solution close to $(0,\beta_0,\beta_0)$ for the Z-equation (3.11).

More precisely: from Lemma 1 and 4 it follows the existence of a $\mathcal{C}^1$-function g defined on a neighborhood of $\lambda = 1$ such that

$$G\big(\lambda, g(\lambda)\big) = 0,$$

that is, $g(\lambda)$ solves (3.21), and $g(1) = (0,\beta_0,\beta_0)$. Moreover, for $|\lambda-1|$ small, it holds

$$\|g(\lambda) - g(1)\|_\sigma \leq \tilde{c}|\lambda - 1| \tag{3.42}$$

for some positive constant $\tilde{c}$. In the following, we denote several positive constants with the same symbol $\tilde{c}$.

We set $\Phi_{(b,\varepsilon)} : (\hat{u}_{0,0}, r, s) \mapsto (c,x,y)$ the rescaling map (3.20) and $H_{(b,\varepsilon)} : \mathbb{R}\times Z \to Z$ the operator corresponding to the auxiliary bifurcation equation (3.18), which so can be written as

$$H_{(b,\varepsilon)}(\mu, z) = 0.$$

We define

$$z^*_{(b,\varepsilon)} = \Phi^{-1}_{(b,\varepsilon)}[g(\lambda_{(b,\varepsilon)})],$$

thus it holds $H_{(b,\varepsilon)}(0, z^*_{(b,\varepsilon)}) = 0$, that is, $z^*_{(b,\varepsilon)}$ solves the bifurcation equation (3.18) for $\mu = 0$.

We observe that $p_{(b,\varepsilon)}(0, z) = \partial_z p_{(b,\varepsilon)}(0, z) = 0$ for every z and so, in particular, for $z = z^*_{(b,\varepsilon)}$; it follows that

$$\partial_z H_{(b,\varepsilon)}(0, z^*_{(b,\varepsilon)}) = (\Phi^{-1}_{(b,\varepsilon)})^3\, \partial_z G\big(\lambda_{(b,\varepsilon)},\, g(\lambda_{(b,\varepsilon)})\big)\, \Phi_{(b,\varepsilon)}. \tag{3.43}$$

G is of class $\mathcal{C}^1$, so $\partial_Z G(\lambda, g(\lambda))$ remains invertible for λ sufficiently close to 1. Notice that $\lambda_{(b,\varepsilon)}$ is sufficiently close to 1 if $|b - \frac{1}{2}|$ and ε are small enough. Then, by (3.43), the partial derivative $\partial_z H_{(b,\varepsilon)}(0, z^*_{(b,\varepsilon)})$ is invertible. By the Implicit Function Theorem, it follows that for every μ sufficiently small there exists a solution $z_{(b,\varepsilon)}(\mu)$ of equation (3.18), that is

$$H_{(b,\varepsilon)}(\mu, z_{(b,\varepsilon)}(\mu)) = 0.$$

We define $z_0 = (0, \beta_0, \beta_0)$. The operators $\big(\partial_z H_{(b,\varepsilon)}(\mu, z)\big)^{-1}$ and $\partial_\mu H_{(b,\varepsilon)}$ (μ, z) are bounded by some constant for every (μ, z) in a neighborhood of $(0, z_0)$, uniformly in (b, ε), if $|b - 1/2|, \varepsilon$ are small enough. So the implicit functions $z_{(b,\varepsilon)}$ are defined on some common interval $(-\mu_0, \mu_0)$ for $|b - 1/2|$, ε small, and it holds

$$\|z_{(b,\varepsilon)}(\mu) - z^*_{(b,\varepsilon)}\|_\sigma \le \tilde{c}|\mu| \tag{3.44}$$

for some $\tilde{c}$ which does not depend on (b, ε).

Such a common interval $(-\mu_0, \mu_0)$ permits the evaluation $z_{(b,\varepsilon)}(\mu)$ at $\mu = \varepsilon$ for $\varepsilon < \mu_0$, obtaining a solution of the original bifurcation equation written in (3.11).

Moreover, $\|\Phi^{-1}_{(b,\varepsilon)} - \mathrm{Id}_Z\|_\sigma = |\sqrt{b(2 + b\varepsilon^2)} - 1| \le |b - \frac{1}{2}| + \varepsilon^2$, so, by (3.42) and triangular inequality,

$$\|z^*_{(b,\varepsilon)} - z_0\|_\sigma \le \tilde{c}(|b - \tfrac{1}{2}| + \varepsilon^2). \tag{3.45}$$

Thus from (3.44) and (3.45) we have

$$\|z_{(b,\varepsilon)}(\varepsilon) - z_0\|_\sigma \le \tilde{c}(|b - \tfrac{1}{2}| + \varepsilon),$$

and, by (3.17),

$$\|p\big(\varepsilon,\, z_{(b,\varepsilon)}(\varepsilon)\big)\|_\sigma \le \tilde{c}\varepsilon.$$

Remark 5. Since the solutions $z_{(b,\varepsilon)}(\varepsilon)$ are close to $z_0 = (0, \beta_0, \beta_0)$, they actually depend on the two arguments (φ_1, φ_2). This condition is necessary to obtain a genuine quasi-periodicity.

We define $u_{(b,\varepsilon)} = z_{(b,\varepsilon)}(\varepsilon) + p_{(b,\varepsilon)}\big(\varepsilon,\, z_{(b,\varepsilon)}(\varepsilon)\big)$. Renaming $\mu_0 = \varepsilon_0$, we have finally proved:

Theorem 1. *Let $\bar\sigma > 0$, β_0 as in Lemma 1, $\mathcal{W}_\gamma$ as in (3.13) with $\gamma \in (0, \frac{1}{4})$. For every $\sigma \in (0, \bar\sigma)$, there exist positive constants δ_0, ε_0, $\bar c_1$, $\bar c_2$ and the uncountable Cantor set*

$$\mathcal{B}_\gamma = \Big\{(b,\varepsilon) \in \Big(\frac{1}{2}-\delta_0,\, \frac{1}{2}+\delta_0\Big) \times (0,\varepsilon_0) : \ \frac{\varepsilon}{2+b\varepsilon^2}, b\varepsilon^2 \in \mathcal{W}_\gamma,\ \ \frac{1+b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}\Big\}$$

such that, for every $(b,\varepsilon) \in \mathcal{B}_\gamma$, there exists a solution $u_{(b,\varepsilon)} \in \mathcal{H}_\sigma$ of (3.6). According to decomposition (3.10), $u_{(b,\varepsilon)}$ can be written as

$$u_{(b,\varepsilon)}(\varphi_1,\varphi_2) = \hat u_{0,0} + r(\varphi_1) + s(\varphi_2) + p(\varphi_1,\varphi_2),$$

where its components satisfy

$$\|r-\beta_0\|_\sigma + \|s-\beta_0\|_\sigma + |\hat u_{0,0}| \le \bar c_1(|b-\tfrac{1}{2}|+\varepsilon), \qquad \|p\|_\sigma \le \bar c_2\varepsilon.$$

As a consequence, problem (3.1) admits uncountable many small amplitude, analytic, quasi-periodic solutions $v_{(b,\varepsilon)}$ with two frequencies, of the form (3.4):

$$\begin{aligned} v_{(b,\varepsilon)}(t,x) &= \varepsilon\, u_{(b,\varepsilon)}\big(\varepsilon t,\ (1+b\varepsilon^2)t + x\big) \\ &= \varepsilon\left[\hat u_{0,0} + r(\varepsilon t) + s((1+b\varepsilon^2)t+x) + \mathcal{O}(\varepsilon)\right] \\ &= \varepsilon\left[\beta_0(\varepsilon t) + \beta_0((1+b\varepsilon^2)t+x) + \mathcal{O}\big(|b-\tfrac{1}{2}|+\varepsilon\big)\right]. \end{aligned}$$

3.4 Waves travelling in opposite directions

In this section we look for solutions of (3.1) of the form (2.5),

$$v(t,x) = u(\omega_1 t + x,\, \omega_2 t - x),$$

for $u \in \mathcal{H}_\sigma$. We introduce two parameters $(a,\varepsilon) \in \mathbb{R}^2$ and set the frequencies as in [97],

$$\omega_1 = 1+\varepsilon, \qquad \omega_2 = 1 + a\varepsilon.$$

For functions of the form (2.5), problem (3.1) is written as

$$L_{a,\varepsilon}[u] = -u^3 + f(u)$$

where

$$L_{a,\varepsilon} = \varepsilon(2+\varepsilon)\,\partial^2_{\varphi_1} + 2\big(2+(a+1)\varepsilon + a\varepsilon^2\big)\,\partial^2_{\varphi_1\varphi_2} + a\varepsilon(2+a\varepsilon)\,\partial^2_{\varphi_2}.$$

We rescale $u \to \sqrt{\varepsilon}\,u$ and define $f_\varepsilon(u) = \varepsilon^{-3/2} f(\sqrt{\varepsilon}\,u)$. Thus the problem can be written as

$$L_{a,\varepsilon}[u] = -\varepsilon u^3 + \varepsilon f_\varepsilon(u). \tag{3.46}$$

For $\varepsilon = 0$, the operator is $L_{a,0} = 4\partial^2_{\varphi_1\varphi_2}$; its kernel is the direct sum $Z = C \oplus Q_1 \oplus Q_2$, see (3.9). Writing u in Fourier series we obtain an expression similar to (3.7),

$$L_{a,\varepsilon}[u] = -\sum_{(m,n)\in\mathbb{Z}^2} D_{a,\varepsilon}(m,n)\,\hat{u}_{mn}\, e^{im\varphi_1}\, e^{in\varphi_2},$$

where the eigenvalues $D_{a,\varepsilon}(m,n)$ are given by

$$\begin{aligned} D_{a,\varepsilon}(m,n) &= \varepsilon(2+\varepsilon)\, m^2 + a\varepsilon(2+a\varepsilon)\, n^2 + 2\left(2+(a+1)\,\varepsilon + a\varepsilon^2\right) mn \\ &= (2+\varepsilon)\,(2+a\varepsilon)\left(m + \frac{a\varepsilon}{2+\varepsilon}\, n\right)\left(\frac{\varepsilon}{2+a\varepsilon}\, m + n\right). \end{aligned}$$

By Lyapunov-Schmidt reduction we project the equation (3.46) on the four subspaces,

$$\begin{aligned} 0 = {} & \hat{u}^3_{0,0} + 3\hat{u}_{0,0}\left(\langle r^2\rangle + \langle s^2\rangle\right) + \langle r^3\rangle + \langle s^3\rangle + \\ & + \Pi_C\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad\qquad [C - equation] \end{aligned}$$

$$\begin{aligned} -(2+\varepsilon)\, r'' = {} & 3\hat{u}^2_{0,0} r + 3\hat{u}_{0,0}\left(r^2 - \langle r^2\rangle\right) + r^3 - \langle r^3\rangle + 3\langle s^2\rangle\, r + \\ & + \Pi_{Q_1}\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad\qquad [Q_1 - equation] \end{aligned}$$

$$\begin{aligned} -a(2+a\varepsilon)\, s'' = {} & 3\hat{u}^2_{0,0} s + 3\hat{u}_{0,0}\left(s^2 - \langle s^2\rangle\right) + s^3 - \langle s^3\rangle + 3\langle r^2\rangle\, s + \\ & + \Pi_{Q_2}\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad\qquad [Q_2 - equation] \end{aligned}$$

$$L_{a,\varepsilon}[p] = \varepsilon\, \Pi_P\left[-u^3 + f_\varepsilon(u)\right]. \qquad\qquad [P - equation]$$

We repeat the arguments of Section 3.5.1 and find a Cantor set $\mathcal{A}_\gamma$ such that $|D_{a,\varepsilon}(m,n)| > \gamma$ for every $(a,\varepsilon) \in \mathcal{A}_\gamma$. Then $L_{a,\varepsilon}$ is invertible for $(a,\varepsilon) \in \mathcal{A}_\gamma$ and the P-equation can be solved as in the section 4.

We repeat the same procedure already shown in section 5 and solve the bifurcation equation. The only differences are:

- the parameter a tends to 1 instead of $b \to \frac{1}{2}$;
- the rescaling map is $\Psi_{(a,\varepsilon)} : (\hat{u}_{0,0}, r, s) \mapsto (c, x, y)$, where

$$r = \sqrt{2+\varepsilon}\; x \qquad \hat{u}_{0,0} = \sqrt{2+\varepsilon}\; c$$
$$s = \sqrt{a(2+a\varepsilon)}\; y \qquad \lambda = \lambda_{(a,\varepsilon)} = \tfrac{a(2+a\varepsilon)}{2+\varepsilon},$$

  instead of $\Phi_{(b,\varepsilon)}$ defined in (3.20).

We note that by means of the rescaling map $\Psi_{(a,\varepsilon)}$ we obtain just the equation (3.21). Thus we conclude:

Theorem 2. *Let $\bar\sigma > 0$, β_0 as in Lemma 1, $\mathcal{W}_\gamma$ as in* (3.13) *with* $\gamma \in (0, \frac{1}{4})$. *For every $\sigma \in (0, \bar\sigma)$, there exist positive constants δ_0, ε_0, $\bar c_1$, $\bar c_2$ and the uncountable Cantor set*

$$\mathcal{A}_\gamma = \Big\{(a,\varepsilon) \in (1-\delta_0,\, 1+\delta_0) \times (0, \varepsilon_0) : \ \frac{a\varepsilon}{2+\varepsilon},\ \frac{\varepsilon}{2+a\varepsilon} \in \mathcal{W}_\gamma,\ \ \frac{1+\varepsilon}{1+a\varepsilon} \notin \mathbb{Q}\Big\}$$

such that, for every $(a,\varepsilon) \in \mathcal{A}_\gamma$, there exists a solution $u_{(a,\varepsilon)} \in \mathcal{H}_\sigma$ of (3.46). *According to decomposition* (3.10), *$u_{(a,\varepsilon)}$ can be written as*

$$u_{(a,\varepsilon)}(\varphi_1, \varphi_2) = \hat u_{0,0} + r(\varphi_1) + s(\varphi_2) + p(\varphi_1, \varphi_2),$$

where its components satisfy

$$\|r - \beta_0\|_\sigma + \|s - \beta_0\|_\sigma + |\hat u_{0,0}| \le \bar c_1(|a-1| + \varepsilon), \qquad \|p\|_\sigma \le \bar c_2 \varepsilon.$$

As a consequence, problem (3.1) *admits uncountable many small amplitude, analytic, quasi-periodic solutions $v_{(a,\varepsilon)}$ with two frequencies, of the form* (2.5)*:*

$$\begin{aligned} v_{(a,\varepsilon)}(t,x) &= \sqrt{\varepsilon}\, u_{(a,\varepsilon)}\big((1+\varepsilon)t + x,\ (1+a\varepsilon)t - x\big) \\ &= \sqrt{\varepsilon}\,\big[\hat u_{0,0} + r\big((1+\varepsilon)t + x\big) + s\big((1+a\varepsilon)t - x\big) + \mathcal{O}(\varepsilon)\big] \\ &= \sqrt{\varepsilon}\,\big[\beta_0\big((1+\varepsilon)t + x\big)\big) + \beta_0\big((1+a\varepsilon)t - x\big) + \mathcal{O}\big(|a-1| + \varepsilon\big)\big]. \end{aligned}$$

3.5 Some calculations

We prove that (3.2) and (3.4) are equivalent.

Lemma 5. *Let $A, B \in \mathrm{Mat}_2(\mathbb{R})$ be invertible matrices such that AB^{-1} has integer coefficient. Then, given any $u \in \mathcal{H}_\sigma$, the function $v(t,x) = u\big(A(t,x)\big)$ can be written as $v(t,x) = w\big(B(t,x)\big)$ for some $w \in \mathcal{H}_\sigma$, that is $\{u \circ A : u \in \mathcal{H}_\sigma\} \subseteq \{w \circ B : w \in \mathcal{H}_\sigma\}$.*

Proof. Let $u \in \mathcal{H}_\sigma$. The function $u \circ A$ belongs to $\{w \circ B : w \in \mathcal{H}_\sigma\}$ if and only if $u \circ A \circ B^{-1} = w$ for some $w \in \mathcal{H}_\sigma$, and this is true if and only if $u \circ AB^{-1}$ is 2π periodic; since $AB^{-1} \in \mathrm{Mat}_2(\mathbb{Z})$, we can conclude. □

Lemma 6. *The set of the quasi-periodic functions of the form* (3.2) *is equal to the set of the quasi-periodic functions of the form* (3.4), *that is,*

$$\Big\{v : \ v(t,x) = u(\omega_1 t + x,\ \omega_2 t + x),\ (\omega_1, \omega_2) \in \mathbb{R}^2,$$
$$\omega_1 \neq 0,\ \omega_2 \neq 0,\ \frac{\omega_1}{\omega_2} \notin \mathbb{Q},\ u \in \mathcal{H}_\sigma\Big\}$$

$$= \Big\{v : \ v(t,x) = u\big(\varepsilon t,\ (1+b\varepsilon^2)t + x\big),\ (b,\varepsilon) \in \mathbb{R}^2,$$
$$\varepsilon \neq 0,\ (1+b\varepsilon^2) \neq 0,\ \frac{1+b\varepsilon^2}{\varepsilon} \notin \mathbb{Q},\ u \in \mathcal{H}_\sigma\Big\}.$$

Proof. Given any $\omega_1, \omega_2, b, \varepsilon,$ we define

$$A = \begin{pmatrix} \omega_1 & 1 \\ \omega_2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} \varepsilon & 0 \\ (1 + b\varepsilon^2) & 1 \end{pmatrix}.$$

Let $v(t, x)$ be any element of the set of quasi-periodic functions of the form (3.2), that is $v = u \circ A$ for some fixed $\omega_1, \omega_2 \neq 0$ such that $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$ and $u \in \mathcal{H}_\sigma$. We observe that v belongs to the set of quasi-periodic functions of the form (3.4) if $v = w \circ B$ for some (b, ε) such that $\varepsilon \neq 0$, $\frac{1+b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}$ and some $w \in \mathcal{H}_\sigma$. By Lemma 5 this holds true if we find (b, ε) such that $AB^{-1} \in \mathrm{Mat}_2(\mathbb{Z})$. We can choose

$$b = \frac{\omega_2 - 1}{(\omega_1 - \omega_2)^2}, \qquad \varepsilon = \omega_1 - \omega_2,$$

so that $AB^{-1} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We note that $\frac{1+b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}$ if and only if $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$.

Conversely, we fix (b, ε) and look for (ω_1, ω_2) such that $BA^{-1} \in \mathrm{Mat}_2(\mathbb{Z})$. This condition is satisfied if we choose the inverse transformation, $\omega_1 = 1 + \varepsilon + b\varepsilon^2$ and $\omega_2 = 1 + b\varepsilon^2$. □

3.5.1 Small divisors

Fixed $\gamma \in (0, \frac{1}{4})$, we have defined in (3.13) the set $\mathcal{W}_\gamma$ of badly approximable numbers as

$$\mathcal{W}_\gamma = \Big\{ x \in \mathbb{R} : |m + nx| > \frac{\gamma}{|n|} \quad \forall\, m, n \in \mathbb{Z},\, m \neq 0,\, n \neq 0 \Big\}.$$

$\mathcal{W}_\gamma$ is non-empty, symmetric, it has zero Lebesgue measure and it accumulates to 0. More, for every $\delta > 0$, both $\mathcal{W}_\gamma \cap (-\delta, 0)$ and $\mathcal{W}_\gamma \cap (0, \delta)$ are uncountable, see Appendix A.

Lemma 7. *Let $\gamma \in (0, \frac{1}{4})$ and $\delta \in (0, \frac{1}{2})$. Then for all $x, y \in \mathcal{W}_\gamma \cap (-\delta, \delta)$ there holds*

$$\big|(m + nx)(my + n)\big| > \gamma(1 - \delta - \delta^2) \quad \forall\, m, n \in \mathbb{Z},\, m, n \neq 0.$$

Proof. We set shortly $D = |(m + nx)(my + n)|$. There are four cases.
Case 1. $|m + nx| > 1$, $|my + n| > 1$. Then $|D| > 1$.
Case 2. $|m + nx| < 1$, $|my + n| > 1$. Multiplying the first inequality by $|y|$,

$$\begin{aligned} |y| > \; & |my + nxy| \; = \; |my + n - n(1 - xy)| \\ \geq \; & \big||n(1 - xy)| - |my + n|\big| \; \geq \; |n(1 - xy)| - |my + n|, \end{aligned}$$

so that $|my + n| > |n|(1 - xy) - |y|$ and

$$\begin{aligned} |D| > \; & \tfrac{\gamma}{|n|}\big[|n|(1 - xy) - |y|\big] = \gamma\Big[(1 - xy) - \tfrac{|y|}{|n|}\Big] \\ > \; & \gamma[(1 - \delta^2) - \delta]. \end{aligned}$$

Case 3. $|m + nx| > 1,\ |my + n| < 1.$ Analogous to case 2.
Case 4. $|m + nx| < 1,\ |my + n| < 1.$ Dividing the first inequality by $|n|$, for triangular inequality we have

$$\left|\frac{m}{n}\right| \le \left|\frac{m}{n} + x\right| + |x| < \frac{1}{|n|} + \delta,$$

and similarly $\left|\frac{n}{m}\right| < \frac{1}{|m|} + \delta$. So

$$\Big(\frac{1}{|n|} + \delta\Big)\Big(\frac{1}{|m|} + \delta\Big) > \left|\frac{n}{m} \cdot \frac{m}{n}\right| = 1.$$

If $|n|, |m| \ge 2$, then $\left(\frac{1}{|n|} + \delta\right)\left(\frac{1}{|m|} + \delta\right) < 1$, a contradiction. It follows that at least one between $|n|$ and $|m|$ is equal to 1. Suppose $|n| = 1$. Then $|m + nx| = |m \pm x| \ge |m| - \delta$ and

$$|D| > \frac{\gamma}{|m|}\big(|m| - \delta\big) = \gamma\Big(1 - \frac{\delta}{|m|}\Big) \ge \gamma(1 - \delta).$$

If $|m| = 1$ the conclusion is the same. □

Fixed $\gamma \in (0, \frac{1}{4})$ and $\delta \in (0, \frac{1}{2})$, we define the set

$$B(\gamma, \delta) = \Big\{(b, \varepsilon) \in \mathbb{R}^2 :\ \varepsilon \ne 0,\ 1 + b\varepsilon^2 \ne 0,\ 2 + b\varepsilon^2 \ne 0,$$
$$\frac{1 + b\varepsilon^2}{\varepsilon} \notin \mathbb{Q},\ \frac{\varepsilon}{2 + b\varepsilon^2},\ b\varepsilon^2 \in \mathcal{W}_\gamma \cap (-\delta, \delta)\Big\}$$

and the map

$$g : B(\gamma, \delta) \to \mathbb{R}^2, \qquad g(b, \varepsilon) = \Big(\frac{\varepsilon}{2 + b\varepsilon^2},\ b\varepsilon^2\Big).$$

$g'(b, \varepsilon)$ is invertible on $B(\gamma, \delta)$. Its imagine is the set $R(g) = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathcal{W}_\gamma \cap (-\delta, \delta),\ \frac{1}{x} - y \notin \mathbb{Q}\}$ and its inverse is

$$g^{-1}(x, y) = \left(\frac{y(1 - xy)}{2x},\ \frac{2x}{1 - xy}\right).$$

Thus $B(\gamma, \delta)$ is homeomorphic to $R(g) = \{(x, y) \in \mathcal{W}_\gamma^2 : |x|, |y| < \delta,\ \frac{1}{x} - y \notin \mathbb{Q}\}$. We observe that, fixed any $\bar{x} \in \mathcal{W}_\gamma \cap (-\delta, \delta)$, it occurs $\frac{1}{\bar{x}} - y \in \mathbb{Q}$ only for countably many numbers y. We know that $\mathcal{W}_\gamma \cap (-\delta, \delta)$ is uncountable so, removing from $[\mathcal{W}_\gamma \cap (-\delta, \delta)]^2$ the couples $\{(\bar{x}, y) : y = \frac{1}{\bar{x}} - q\ \exists q \in \mathbb{Q}\}$, it remains uncountably many other couples. Thus $R(g)$ is uncountable and so, through g, also $B(\gamma, \delta)$.

Moreover, if we consider couples $(x, y) \in [\mathcal{W}_\gamma \cap (-\delta, \delta)]^2$ such that $x \to 0$ and $(x/y) \to 1$, applying g^{-1} we find couples $(b, \varepsilon) \in B(\gamma, \delta)$ which satisfy $\varepsilon \to 0,\ b \to 1/2$. In other words, the set $B(\gamma, \delta)$ accumulates to $(1/2, 0)$.

Finally we estimate $D_{b,\varepsilon}(m,n)$ for $(b,\varepsilon)\in B(\gamma,\delta)$. We have

$$|2+b\varepsilon^2| = \frac{2}{|1-xy|} > \frac{2}{1+\delta^2},$$

so from Lemma 7 and (3.8) it follows

$$\big|D_{b,\varepsilon}(m,n)\big| = |D|\,|2+b\varepsilon^2| > \gamma(1-\delta-\delta^2)\frac{2}{1+\delta^2}.$$

The factor on the right of γ is greater than 1 if we choose, for example, $\delta = 1/4$; we define $\mathcal{B}_\gamma = B(\gamma,\delta)_{|\delta=\frac{1}{4}}$, so that there holds

$$\big|D_{b,\varepsilon}(m,n)\big| > \gamma \qquad \forall\,(b,\varepsilon)\in\mathcal{B}_\gamma.$$

We can observe that the condition $\frac{1+b\varepsilon^2}{\varepsilon}\notin\mathbb{Q}$ implies $1+b\varepsilon^2\neq 0$, that $\frac{\varepsilon}{2+b\varepsilon^2}\in\mathcal{W}_\gamma$ implies $\varepsilon\neq 0$ and $|b\varepsilon^2|<\delta$ implies $2+b\varepsilon^2\neq 0$, so that we can write

$$\mathcal{B}_\gamma = \Big\{(b,\varepsilon)\in\mathbb{R}^2:\ \frac{\varepsilon}{2+b\varepsilon^2}, b\varepsilon^2\in\mathcal{W}_\gamma,\ \Big|\frac{\varepsilon}{2+b\varepsilon^2}\Big|, |b\varepsilon^2| < \frac{1}{4},\ \frac{1+b\varepsilon^2}{\varepsilon}\notin\mathbb{Q}\Big\}.$$

We notice also that, for $|b-\frac{1}{2}|$ and ε small enough, there holds automatically $|\frac{\varepsilon}{2+b\varepsilon^2}|<\frac{1}{4}$, $|b\varepsilon^2|<\frac{1}{4}$. So, if we are interested to couples (b,ε) close to $\big(\frac{1}{2},0\big)$, say $|b-\frac{1}{2}|<\delta_0$, $|\varepsilon|<\varepsilon_0$, we can write

$$\mathcal{B}_\gamma = \Big\{(b,\varepsilon)\in\Big(\frac{1}{2}-\delta_0,\ \frac{1}{2}+\delta_0\Big)\times(0,\varepsilon_0):\ \frac{\varepsilon}{2+b\varepsilon^2}, b\varepsilon^2\in\mathcal{W}_\gamma,\ \frac{1+b\varepsilon^2}{\varepsilon}\notin\mathbb{Q}\Big\}.$$

Chapter 4

Periodic solutions of wave equations for asymptotically full measure sets of frequencies

In this Chapter we prove existence and multiplicity of small amplitude periodic solutions of the completely resonant wave equation

$$\begin{cases} \Box u + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{4.1}$$

where $\Box := \partial_{tt} - \partial_{xx}$ is the D'Alambertian operator and

$$f(x,u) = a_2 u^2 + a_3(x) u^3 + O(u^4) \qquad \text{or} \qquad f(x,u) = a_4 u^4 + O(u^5) \tag{4.2}$$

for a Cantor-like set of frequencies ω of asymptotically full measure at $\omega = 1$.

Normalizing the period to 2π, we look for solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases}$$

in the Hilbert algebra (for $s > 1/2$, $\sigma > 0$)

$$X_{\sigma,s} := \Big\{ u(t,x) = \sum_{l \geq 0} \cos(lt)\ u_l(x) \ \Big|\ u_l \in H_0^1((0,\pi),\mathbb{R}) \ \ \forall l \in \mathbb{N},$$
$$\|u\|_{\sigma,s}^2 := \sum_{l \geq 0} \exp(2\sigma l)(l^{2s}+1)\|u_l\|_{H^1}^2 < +\infty \Big\}.$$

It is natural to look for solutions which are even in time because equation (4.1) is reversible.

We look as well for solutions of (4.1) in the subalgebras

$$X_{\sigma,s,n} := \Big\{ u \in X_{\sigma,s} \mid u \text{ is } \frac{2\pi}{n}\text{-periodic} \Big\} \subset X_{\sigma,s}\,, \quad n \in \mathbb{N}$$

(they are particular 2π-periodic solutions).

The space of the solutions of the linear equation $v_{tt} - v_{xx} = 0$ that belong to $H_0^1(\mathbb{T} \times (0,\pi), \mathbb{R})$ and are even in time is

$$\begin{aligned} V &:= \Big\{ v(t,x) = \sum_{l \geq 1} \cos(lt) u_l \sin(lx) \;\Big|\; u_l \in \mathbb{R}\,,\; \sum_{l\geq 1} l^2 |u_l|^2 < +\infty \Big\} \\ &= \Big\{ v(t,x) = \eta(t+x) - \eta(t-x) \;\Big|\; \eta \in H^1(\mathbb{T},\mathbb{R}) \text{ with } \eta \text{ odd} \Big\}. \end{aligned}$$

Theorem 3. *Let*

$$f(x,u) = a_2 u^2 + a_3(x) u^3 + \sum_{k \geq 4} a_k(x) u^k \tag{4.3}$$

where $(a_2, \langle a_3 \rangle) \neq (0,0)$, $\langle a_3 \rangle := \pi^{-1} \int_0^\pi a_3(x) dx$, *or*

$$f(x,u) = a_4 u^4 + \sum_{k\geq 5} a_k(x) u^k \tag{4.4}$$

where $a_4 \neq 0$, $a_5(\pi - x) = -a_5(x)$, $a_6(\pi - x) = a_6(x)$, $a_7(\pi - x) = -a_7(x)$. *Assume moreover* $a_k(x) \in H^1((0,\pi),\mathbb{R})$ *with* $\sum_k \|a_k\|_{H^1} \rho^k < +\infty$ *for some* $\rho > 0$.

Then there exists $n_0 \in \mathbb{N}$ *such that* $\forall n \geq n_0$ *there is* $\delta_0 > 0$, $\bar\sigma > 0$ *and a* C^∞*-curve* $[0,\delta_0) \ni \delta \to u_\delta \in X_{\bar\sigma/2,s,n}$ *with the following properties:*

- *(i)* $\|u_\delta - \delta \bar v_n\|_{\bar\sigma/2,s,n} = O(\delta^2)$ *for some* $\bar v_n \in V \cap X_{\bar\sigma,s,n} \setminus \{0\}$ *with minimal period* $2\pi/n$*;*

- *(ii) there exists a Cantor set* $\mathcal{C}_n \subset [0,\delta_0)$ *of asymptotically full measure, i.e. satisfying*

$$\lim_{\varepsilon \to 0^+} \frac{\text{meas}(\mathcal{C}_n \cap (0,\varepsilon))}{\varepsilon} = 1\,, \tag{4.5}$$

such that, $\forall\, \delta \in \mathcal{C}_n$, $u_\delta(\omega(\delta)t, x)$ *is a* $2\pi/(\omega(\delta)n)$*-periodic, classical solution of (4.1) with*

$$\omega(\delta) = \begin{cases} \sqrt{1 - 2s^*\delta^2} & \text{if } f \text{ is like in (4.3)} \\ \sqrt{1 - 2\delta^6} & \text{if } f \text{ is like in (4.4)} \end{cases}$$

and[1]

$$s^* = \begin{cases} -1 & \text{if } \langle a_3 \rangle \geq \pi^2 a_2^2/12 \\ \pm 1 & \text{if } 0 < \langle a_3 \rangle < \pi^2 a_2^2/12 \\ 1 & \text{if } \langle a_3 \rangle \leq 0\,. \end{cases}$$

[1]Note how the interaction between the second and the third order terms a_2u^2, $a_3(x)u^3$ changes the bifurcation diagram, i.e. existence of periodic solutions for frequencies ω less or/and greater of $\omega = 1$.

By (4.5) also each Cantor-like set of frequencies $\mathcal{W}_n := \{\omega(\delta) \mid \delta \in \mathcal{C}_n\}$ has asymptotically full measure at $\omega = 1$.

Corollary 1. (Multiplicity). *There exists a Cantor-like set $\mathcal{W}$ of asymptotically full measure at $\omega = 1$, such that $\forall \omega \in \mathcal{C}$, equation (4.1) possesses geometrically distinct periodic solutions*

$$u_{n_0}, \ldots, u_n, \ldots u_{N_\omega}, \qquad N_\omega \in \mathbb{N}$$

with the same period $2\pi/\omega$. Their number increases arbitrarily as ω tends to 1:

$$\lim_{\omega \to 1} N_\omega = +\infty .$$

Proof. The proof is like in [28] and we report it for completeness. If δ belongs to the asymptotically full measure set (by (4.5))

$$D_n := \mathcal{C}_{n_0} \cap \ldots \cap \mathcal{C}_n , \qquad n \geq n_0$$

there exist $(n - n_0 + 1)$ geometrically distinct periodic solutions of (4.1) with the same period $2\pi/\omega(\delta)$ (each u_n has minimal period $2\pi/(n\omega(\delta))$).

There exists a decreasing sequence of positive $\varepsilon_n \to 0$ such that

$$\text{meas}(D_n^c \cap (0, \varepsilon_n)) \leq \varepsilon_n 2^{-n} .$$

Let define the set $\mathcal{C} \equiv D_n$ on each $[\varepsilon_{n+1}, \varepsilon_n)$. $\mathcal{C}$ has asymptotically full measure at $\delta = 0$ and for each $\delta \in \mathcal{C}$ there exist $N(\delta) := \max\{n \in \mathbb{N} : \delta < \varepsilon_n\}$ geometrically distinct periodic solutions of (4.1) with the same period $2\pi/\omega(\delta)$. $N(\delta) \to +\infty$ as $\delta \to 0$. $\square$

Remark 6. Corollary 1 is an analogue for equation (4.1) of the well known multiplicity results of Weinstein-Moser [111]-[85] and Fadell-Rabinowitz [57] which hold in finite dimension. The solutions form a sequence of functions with increasing norms and decreasing minimal periods. Multiplicity of solutions was also obtained in [27] (with the "optimal" number $N_\omega \approx C/\sqrt{|\omega - 1|}$) but only for a zero measure set of frequencies.

The main point for proving Theorem 3 relies in showing the existence of non-degenerate solutions of the 0th order bifurcation equation for f like in (4.2). In these cases the 0th order bifurcation equation involves higher order terms of the nonlinearity, and, for n large, can be reduced to an integro-differential equation (which physically describes an averaged effect of the nonlinearity with Dirichlet boundary conditions).

Case $f(x,u) = a_4 u^4 + O(u^5)$. Performing the rescaling

$$u \to \delta u , \qquad \delta > 0$$

we look for $2\pi/n$-periodic solutions in $X_{\sigma,s,n}$ of

$$
\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta^3 g(\delta, x, u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{4.6}
$$

where

$$
g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^4} = a_4 u^4 + \delta a_5(x) u^5 + \delta^2 a_6(x) u^6 + \dots .
$$

To find solutions of (4.6) we implement the Lyapunov-Schmidt reduction according to the orthogonal decomposition

$$
X_{\sigma,s,n} = (V_n \cap X_{\sigma,s,n}) \oplus (W \cap X_{\sigma,s,n})
$$

where

$$
V_n := \Big\{ v(t,x) = \eta(nt + nx) - \eta(nt - nx) \mid \eta \in H^1(\mathbb{T}, \mathbb{R}) \ \text{ with } \ \eta \text{ odd} \Big\}
$$

and

$$
W := \Big\{ w = \sum_{l \geq 0} \cos(lt)\ w_l(x) \in X_{0,s} \mid \int_0^\pi w_l(x) \sin(lx)\, dx = 0,\ \forall l \geq 0 \ \Big\} .
$$

Looking for solutions $u = v + w$ with $v \in V_n \cap X_{\sigma,s,n}$, $w \in W \cap X_{\sigma,s,n}$, and imposing the frequency-amplitude relation

$$
\frac{(\omega^2 - 1)}{2} = -\delta^6
$$

we are led to solve the bifurcation equation and the range equation

$$
\begin{cases} \Delta v = \delta^{-3} \Pi_{V_n} g(\delta, x, v + w) \\ L_\omega w = \delta^3 \Pi_{W_n} g(\delta, x, v + w) \end{cases} \tag{4.7}
$$

where

$$
\Delta v := v_{xx} + v_{tt}, \qquad\qquad L_\omega := -\omega^2 \partial_{tt} + \partial_{xx}
$$

and $\Pi_{V_n} : X_{\sigma,s,n} \to V_n \cap X_{\sigma,s,n}$, $\Pi_{W_n} : X_{\sigma,s,n} \to W \cap X_{\sigma,s,n}$ denote the projectors.

With the further rescaling

$$
w \to \delta^3 w
$$

and since $v^4 \in W_n$ [26, Lemma 3.4], $a_5(x)v^5$, $a_6(x)v^6$, $a_7(x)v^7 \in W_n$ because $a_5(\pi - x) = -a_5(x)$, $a_6(\pi - x) = a_6(x)$, $a_7(\pi - x) = -a_7(x)$ [28, Lemma 7.1], system (4.7) is equivalent to

$$
\begin{cases} \ \Delta v = \Pi_{V_n} \Big(4 a_4 v^3 w + \delta r(\delta, x, v, w) \Big) \\ \ L_\omega w = a_4 v^4 + \delta \Pi_{W_n} \widetilde{r}(\delta, x, v, w) \end{cases} \tag{4.8}
$$

where $r(\delta, x, v, w) = a_8(x)v^8 + 5a_5(x)v^4 w + O(\delta)$ and $\widetilde{r}(\delta, x, v, w) = a_5(x)v^5 + O(\delta)$.

For $\delta = 0$ system (4.8) reduces to $w = -a_4 \Box^{-1} v^4$ and to the 0th order bifurcation equation

$$\Delta v + 4a_4^2 \Pi_{V_n} \Big(v^3 \Box^{-1} v^4 \Big) = 0 \tag{4.9}$$

which is the Euler-Lagrange equation of the functional $\Phi_0 : V_n \to \mathbb{R}$

$$\Phi_0(v) = \frac{\|v\|_{H_1}^2}{2} - \frac{a_4^2}{2} \int_\Omega v^4 \Box^{-1} v^4 \tag{4.10}$$

where $\Omega := \mathbb{T} \times (0, \pi)$.

Proposition 1. *Let $a_4 \neq 0$. $\exists\ n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ the 0th order bifurcation equation (4.9) has a solution $\bar{v}_n \in V_n$ which is non-degenerate in V_n (i.e. $\mathrm{Ker} D^2 \Phi_0 = \{0\}$), with minimal period $2\pi/n$.*

Case $f(x, u) = a_2 u^2 + a_3(x)u^3 + O(u^4)$. Performing the rescaling $u \to \delta u$ we look for $2\pi/n$-periodic solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta g(\delta, x, u) = 0 \\ u(t, 0) = u(t, \pi) = 0 \end{cases}$$

where

$$g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^2} = a_2 u^2 + \delta a_3(x) u^3 + \delta^2 u_4(x) u^4 + \dots .$$

With the frequency-amplitude relation

$$\frac{\omega^2 - 1}{2} = -s^* \delta^2$$

where $s^* = \pm 1$, we have to solve

$$\begin{cases} -\Delta v = -s^* \delta^{-1} \Pi_{V_n} g(\delta, x, v + w) \\ L_\omega w = \delta \Pi_{W_n} g(\delta, x, v + w) . \end{cases} \tag{4.11}$$

With the further rescaling $w \to \delta w$ and since $v^2 \in W_n$, system (4.11) is equivalent to

(4.12)

$$\begin{cases} -\Delta v = s^* \Pi_{V_n} \Big(-2a_2 v w - a_2 \delta w^2 - a_3(x)(v + \delta w)^3 - \delta r(\delta, x, v + \delta w) \Big) \\ L_\omega w = a_2 v^2 + \delta \Pi_{W_n} \Big(2a_2 v w + \delta a_2 w^2 + a_3(x)(v + \delta w)^3 + \alpha_8(x) v^8 dr(\delta, x, v + \delta w) \Big) \end{cases}$$

where $r(\delta, x, u) := \delta^{-4} [f(x, \delta u) - a_2 \delta^2 u^2 - \delta^3 a_3(x) u^3] = a_4(x) u^4 + \dots$

For $\delta = 0$ system (4.12) reduces to $w = -a_2 \Box^{-1} v^2$ and the 0th order bifurcation equation

$$-s^* \Delta v = 2a_2^2 \Pi_{V_n}(v \Box^{-1} v^2) - \Pi_{V_n}(a_3(x) v^3) \tag{4.13}$$

which is the Euler-Lagrange equation of $\Phi_0 : V_n \to \mathbb{R}$

$$\Phi_0(v) := s^* \frac{\|v\|_{H^1}^2}{2} - \frac{a_2^2}{2} \int_\Omega v^2 \Box^{-1} v^2 + \frac{1}{4} \int_\Omega a_3(x) v^4 \,. \tag{4.14}$$

Proposition 2. *Let $(a_2, \langle a_3 \rangle) \neq 0$. $\exists\, n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ the 0th order bifurcation equation (4.13) has a solution $\bar{v}_n \in V_n$ which is non-degenerate in V_n, with minimal period $2\pi/n$.*

4.1 Case $f(x, u) = a_4 u^4 + O(u^5)$

We have to prove the existence of *non-degenerate* critical points of the functional

$$\Phi_n : V \to \mathbb{R}\,, \qquad \Phi_n(v) := \Phi_0(\mathcal{H}_n v)$$

where Φ_0 is defined in (4.10). Let $\mathcal{H}_n : V \to V$ be the linear isomorphism defined, for $v(t, x) = \eta(t + x) - \eta(t - x) \in V$, by

$$a_8(x) v^8 (\mathcal{H}_n v)(t, x) := \eta(n(t + x)) - \eta(n(t - x))$$

so that $V_n \equiv \mathcal{H}_n V$.

Lemma 8. (See [27]). *Φ_n has the following development: for $v(t, x) = \eta(t + x) - \eta(t - x) \in V$*

$$\Phi_n(\beta n^{1/3} v) = 4\pi \beta^2 n^{8/3} \Big[\Psi(\eta) + \alpha \frac{\mathcal{R}(\eta)}{n^2} \Big] \tag{4.15}$$

where $\beta := (3/(\pi^2 a_4^2))^{1/6}$, $\alpha := a_4^2/(8\pi)$,

$$\Psi(\eta) := \frac{1}{2} \int_{\mathbb{T}} \eta'^2(t)\, dt - \frac{2\pi}{8} \Big(\langle \eta^4 \rangle + 3 \langle \eta^2 \rangle^2 \Big)^2 , \tag{4.16}$$

$\langle\ \rangle$ denotes the average on $\mathbb{T}$, and

$$\mathcal{R}(\eta) := - \int_\Omega v^4 \Box^{-1} v^4 \, dt dx + \frac{\pi^4}{6} 4 \Big(\langle \eta^4 \rangle + 3 \langle \eta^2 \rangle^2 \Big)^2 . \tag{4.17}$$

Proof. Firstly the quadratic term writes

$$\frac{1}{2} \|\mathcal{H}_n v\|_{H^1}^2 = \frac{n^2}{2} \|v\|_{H^1}^2 = n^2 2\pi \int_{\mathbb{T}} \eta'^2(t)\, dt \,. \tag{4.18}$$

By [27, Lemma 4.8] the non-quadratic term can be developed as

$$\int_\Omega (\mathcal{H}_n v)^4 \Box^{-1} (\mathcal{H}_n v)^4 = \frac{\pi^4}{6}\langle m\rangle^2 - \frac{\mathcal{R}(\eta)}{n^2} \tag{4.19}$$

where $m : \mathbb{T}^2 \to \mathbb{R}$, $m(s_1, s_2) := (\eta(s_1) - \eta(s_2))^4$, its average is denoted by $\langle m\rangle := (2\pi)^{-2} \int_{\mathbb{T}^2} m(s_1, s_2)\, ds_1 ds_2$, and

$$\mathcal{R}(\eta) := -\int_\Omega v^4 \Box^{-1} v^4 + \frac{\pi^4}{6}\langle m\rangle^2 \tag{4.20}$$

is homogeneous of degree 8. Since η is odd, we find

$$\langle m\rangle = 2\Big(\langle\eta^4\rangle + 3\langle\eta^2\rangle^2\Big), \tag{4.21}$$

where $\langle\ \rangle$ denotes the average on $\mathbb{T}$.

Collecting (4.18), (4.19), (4.20) and (4.21), we find out

$$\Phi_n(\eta) = 2\pi n^2 \int_{\mathbb{T}} \eta'^2(t)\, dt \; - \frac{\pi^4}{3} a_4^2 \Big(\langle\eta^4\rangle + 3\langle\eta^2\rangle^2\Big)^2 + \frac{a_4^2}{2n^2}\mathcal{R}(\eta)\,.$$

By the rescaling $\eta \to \beta n^{1/3}\eta$, we get the expressions (4.16) and (4.17). $\Box$

By (4.15), in order to find for n large enough a non-degenerate critical point of Φ_n, it is sufficient to find a non-degenerate critical point of $\Psi(\eta)$ defined on

$$E := \Big\{\eta \in H^1(\mathbb{T}),\ \eta \text{ odd}\Big\},$$

namely non-degenerate solutions in E of

$$\ddot\eta + A(\eta)\big(3\langle\eta^2\rangle\eta + \eta^3\big) = 0, \qquad A(\eta) := \langle\eta^4\rangle + 3\langle\eta^2\rangle^2. \tag{4.22}$$

Proposition 3. *There exists an odd, analytic, 2π-periodic solution $g(t)$ of (4.22) which is non-degenerate in E. $g(t) = V\mathrm{sn}(\Omega t, m)$ where* sn *is the Jacobi elliptic sine and $V > 0$, $\Omega > 0$, $m \in (-1, 0)$ are suitable constants (therefore $g(t)$ has minimal period 2π).*

We will construct the solution g of (4.22) by means of the Jacobi elliptic sine in Lemma 13. The existence of a solution g follows also directly applying to $\Psi : E \to \mathbb{R}$ the Mountain-Pass Theorem [5]. Furthermore such solution is an analytic function arguing as in [28, Lemma 2.1].

4.1.1 Non-degeneracy of g

We now want to prove that g is non-degenerate. The linearised equation of (4.22) at g is

$$\begin{aligned}
&\ddot{h} + 3A(g)\Big[\langle g^2\rangle h + g^2 h\Big] + 6A(g)g\langle gh\rangle + A'(g)[h]\Big(3\langle g^2\rangle g + g^3\Big)\\
&= \ddot{h} + 3A(g)\left[\langle g^2\rangle + g^2\right] h + 6g\langle gh\rangle\Big(\langle g^4\rangle + 3\langle g^2\rangle^2\Big)\\
&\quad + 4g\big(\langle g^3 h\rangle + 3\langle g^2\rangle\langle gh\rangle\big)\big(3\langle g^2\rangle + g^2\big)\\
&= 0,
\end{aligned}$$

which we write as

$$\ddot{h} + 3A(g)\left(\langle g^2\rangle + g^2\right) h = -\langle gh\rangle I_1 - \langle g^3 h\rangle I_2, \tag{4.23}$$

where

$$\begin{cases} I_1 := 6\big(9\langle g^2\rangle^2 + \langle g^4\rangle\big)g + 12\langle g^2\rangle g^3,\\ I_2 := 12g\langle g^2\rangle + 4g^3. \end{cases} \tag{4.24}$$

For $f \in E$, let $H := L(f)$ be the unique solution belonging to E of the non-homogeneous linear system

$$\ddot{H} + 3A(g)\left(\langle g^2\rangle + g^2\right) H = f. \tag{4.25}$$

An integral representation of the Green operator L is given in Lemma 11. Thus (4.23) becomes

$$h = -\langle gh\rangle L(I_1) - \langle g^3 h\rangle L(I_2)\,. \tag{4.26}$$

Multiplying (4.26) by g and taking averages we get

$$\langle gh\rangle\left[1 + \langle gL(I_1)\rangle\right] = -\langle g^3 h\rangle\,\langle gL(I_2)\rangle, \tag{4.27}$$

while multiplying (4.26) by g^3 and taking averages

$$\langle g^3 h\rangle\big[1 + \langle g^3 L(I_2)\rangle\big] = -\langle gh\rangle\,\langle g^3 L(I_1)\rangle. \tag{4.28}$$

Since g solves (4.22) we have the following identities.

Lemma 9. *There holds*

$$2A(g)\langle g^3 L(g)\rangle = \langle g^2\rangle, \tag{4.29}$$

$$2A(g)\langle g^3 L(g^3)\rangle = \langle g^4\rangle. \tag{4.30}$$

Proof. (4.29) is obtained by the identity for $L(g)$

$$\frac{d^2}{dt^2}(L(g)) + 3A(g)\left(\langle g^2\rangle + g^2\right) L(g) = g$$

multiplying by g, taking averages, integrating by parts,

$$\langle \ddot{g} L(g)\rangle + 3A(g)\left[\langle g^2\rangle\langle L(g)g\rangle + \langle g^3 L(g)\rangle\right] = \langle g^2\rangle$$

and using that g solves (4.22).

Analogously, (4.30) is obtained by the identity for $L(g^3)$

$$\frac{d^2}{dt^2}(L(g^3)) + 3A(g)\left(\langle g^2\rangle + g^2\right) L(g^3) = g^3$$

multiplying by g, taking averages, integrating by parts, and using that g solves (4.22). □

Since L is a symmetric operator we can compute the following averages using (4.24), (4.29), (4.30):

$$\begin{cases} \langle gL(I_1)\rangle = 6\left(\langle g^4\rangle + 9\langle g^2\rangle^2\right)\langle gL(g)\rangle + 6\,A(g)^{-1}\,\langle g^2\rangle^2 \\ \langle gL(I_2)\rangle = 12\langle g^2\rangle\,\langle gL(g)\rangle + 2\,A(g)^{-1}\,\langle g^2\rangle \\ \langle g^3 L(I_1)\rangle = 9\langle g^2\rangle \\ \langle g^3 L(I_2)\rangle = 2\,. \end{cases} \tag{4.31}$$

Thanks to the identities (4.31), equations (4.27), (4.28) simplify to

$$\begin{cases} \langle gh\rangle\left[A(g) + 6\langle g^2\rangle^2\right]B(g) = -2\,\langle g^2\rangle\,B(g)\,\langle g^3 h\rangle \\ \langle g^3 h\rangle = -3\langle g^2\rangle\,\langle gh\rangle \end{cases} \tag{4.32}$$

where

$$B(g) := 1 + 6A(g)\langle gL(g)\rangle\,. \tag{4.33}$$

Solving (4.32) we get

$$B(g)\langle gh\rangle = 0\,.$$

We will prove in Lemma 12 that $B(g) \neq 0$, so $\langle gh\rangle = 0$. Hence by (4.32) also $\langle g^3 h\rangle = 0$ and therefore, by (4.26), $h = 0$. This concludes the proof of the non-degeneracy of the solution g of (4.22).

It remains to prove that $B(g) \neq 0$. The key is to express the function $L(g)$ by means of the variation of constants formula.

We first look for a fundamental set of solutions of the homogeneous equation

$$\ddot{h} + 3A(g)\left(\langle g^2\rangle + g^2\right) h = 0\,. \tag{HOM}$$

Lemma 10. *There exist two linearly independent solutions of (HOM), $\bar{u} := \dot{g}(t)/\dot{g}(0)$ and $\bar{v}$, such that*

$$\begin{cases} \bar{u} \ \text{ is even, } \ 2\pi \,\text{periodic} \\ \bar{u}(0) = 1\,, \ \dot{\bar{u}}(0) = 0 \end{cases} \qquad\qquad \begin{cases} \bar{v} \ \text{ is odd, not periodic} \\ \bar{v}(0) = 0\,, \ \dot{\bar{v}}(0) = 1 \end{cases}$$

and

$$\bar{v}(t + 2\pi) - \bar{v}(t) = \rho \bar{u}(t) \qquad \text{for some } \rho > 0\,. \tag{4.34}$$

Proof. Since (4.22) is autonomous, $\dot{g}(t)$ is a solution of the linearised equation (HOM). $\dot{g}(t)$ is even and 2π-periodic.

We can construct another solution of (HOM) in the following way. The super-quadratic Hamiltonian system (with constant coefficients)

$$\ddot{y} + 3A(g)\langle g^2 \rangle\, y + A(g)\, y^3 = 0 \tag{4.35}$$

possesses a one-parameter family of odd, $T(E)$-periodic solutions $y(E, t)$, close to g, parametrized by the energy E. Let $\bar{E}$ denote the energy level of g, i.e. $g = y(\bar{E}, t)$ and $T(\bar{E}) = 2\pi$.

Therefore $l(t) := (\partial_E y(E,t))_{|E=\bar{E}}$ is an odd solution of (HOM).

Deriving the identity $y(E, t + T(E)) = y(E, t)$ with respect to E we obtain at $E = \bar{E}$

$$l(t + 2\pi) - l(t) = -(\partial_E T(E))_{|E=\bar{E}}\, \dot{g}(t)$$

and, normalizing $\bar{v}(t) := l(t)/\dot{l}(0)$, we get (4.34) with

$$\rho := -(\partial_E T(E))_{|E=\bar{E}} \left(\frac{\dot{g}(0)}{\dot{l}(0)} \right). \tag{4.36}$$

Since $y(E, 0) = 0\ \forall E$, the energy identity gives $E = \frac{1}{2}(\dot{y}(E,0))^2$. Deriving w.r.t E at $E = \bar{E}$, yields $1 = \dot{g}(0)\dot{l}(0)$ which, inserted in (4.36), gives

$$\rho = -(\partial_E T(E))_{|E=\bar{E}}\, (\dot{g}(0))^2\,. \tag{4.37}$$

$\rho > 0$ because $(\partial_E T(E))_{|E=\bar{E}} < 0$ by the superquadraticity of the potential of (4.35). It can be checked also by a computation, see Remark 7. □

Now we write an integral formula for the Green operator L.

Lemma 11. *For every $f \in E$ there exists a unique solution $H = L(f)$ of (4.25) which can be written as*

$$L(f) = \Big(\int_0^t f(s)\bar{u}(s)\, ds + \frac{1}{\rho} \int_0^{2\pi} f\bar{v} \Big) \bar{v}(t) - \Big(\int_0^t f(s)\bar{v}(s)\, ds \Big) \bar{u}(t) \ \in E\,. \tag{4.38}$$

Proof. The non-homogeneous equation (4.25) possesses the particular solution

$$\bar{H}(t) = \Big(\int_0^t f(s)\bar{u}(s)\,ds\Big)\bar{v}(t) - \Big(\int_0^t f(s)\bar{v}(s)\,ds\Big)\bar{u}(t)$$

as can be verified noting that the Wronskian $\bar{u}(t)\dot{\bar{v}}(t) - \dot{\bar{u}}(t)\bar{v}(t) \equiv 1$, $\forall t$. Notice that $\bar{H}$ is odd.

Any solution $H(t)$ of (4.25) can be written as

$$H(t) = \bar{H}(t) + a\bar{u} + b\bar{v}\,, \qquad a, b \in \mathbb{R}\,.$$

Since $\bar{H}$ is odd, $\bar{u}$ is even and $\bar{v}$ is odd, requiring H to be odd, implies $a = 0$. Imposing now the 2π-periodicity yields

$$\begin{aligned} 0 \;=\; & \Big(\int_0^{t+2\pi} f\bar{u}\Big)\bar{v}(t+2\pi) - \Big(\int_0^{t+2\pi} f\bar{v}\Big)\bar{u}(t+2\pi) \\ & -\Big(\int_0^t f\bar{u}\Big)\bar{v}(t) + \Big(\int_0^t f\bar{v}\Big)\bar{u}(t) + b\Big(\bar{v}(t+2\pi) - \bar{v}(t)\Big) \\ =\; & \Big(b + \int_0^t f\bar{u}\Big)\Big(\bar{v}(t+2\pi) - \bar{v}(t)\Big) - \bar{u}(t)\Big(\int_t^{t+2\pi} f\bar{v}\Big) \end{aligned} \tag{4.39}$$

using that $\bar{u}$ and $f\bar{u}$ are 2π-periodic and $\langle f\bar{u}\rangle = 0$. By (4.39) and (4.34) we get

$$\rho\Big(b + \int_0^t f\bar{u}\Big) - \int_t^{t+2\pi} f\bar{v} = 0\,. \tag{4.40}$$

The left hand side in (4.40) is constant in time because, differentiating w.r.t. t,

$$\rho f(t)\bar{u}(t) - f(t)\Big(\bar{v}(t+2\pi) - \bar{v}(t)\Big) = 0$$

again by (4.34). Hence evaluating (4.40) for $t = 0$ yields $b = \rho^{-1}\int_0^{2\pi} f\bar{v}$. So there exists a unique solution $H = L(f)$ of (4.25) belonging to E and (4.38) follows. $\square$

Lemma 12. *There holds*

$$\langle gL(g)\rangle = \frac{\rho}{4\pi A(g)} + \frac{1}{2\pi\rho}\Big(\int_0^{2\pi} g\bar{v}\Big)^2 > 0$$

because $A(g)$, $\rho > 0$.

Proof. Using (4.38) we can compute

$$\begin{aligned} \langle gL(g)\rangle \;=\; & \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^t g\bar{u}\Big)\bar{v}(t)g(t)\,dt \;+\; \frac{1}{2\pi\rho}\Big(\int_0^{2\pi} g\bar{v}\Big)^2 \\ & -\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^t g\bar{v}\Big)\bar{u}(t)g(t)\,dt \\ =\; & 2\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^t g\bar{u}\Big)\bar{v}(t)g(t)\,dt \;+\; \frac{1}{2\pi\rho}\Big(\int_0^{2\pi} g\bar{v}\Big)^2 \end{aligned} \tag{4.41}$$

because, by $\int_0^{2\pi} g\bar{u} = 0$,

$$0 = \int_0^{2\pi} \frac{d}{dt}\Big[\Big(\int_0^t g\bar{v}\Big)\Big(\int_0^t g\bar{u}\Big)\Big] dt$$
$$= \int_0^{2\pi} \Big[\Big(\int_0^t g\bar{v}\Big)\bar{u}(t)g(t) + \Big(\int_0^t g\bar{u}\Big)\bar{v}(t)g(t)\Big]\, dt.$$

Now, since $\bar{u}(t) = \dot{g}(t)/\dot{g}(0)$ and $g(0) = 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \Big(\int_0^t g\bar{u}\Big)\bar{v}(t)g(t) = \frac{1}{2\pi\dot{g}(0)}\int_0^{2\pi} \Big(\int_0^t \frac{d}{d\tau}\frac{g^2(\tau)}{2}\, d\tau\Big)\bar{v}(t)g(t)$$
$$= \frac{1}{4\pi\dot{g}(0)}\int_0^{2\pi} g^3\bar{v}. \tag{4.42}$$

We claim that

$$\int_0^{2\pi} g^3\bar{v} = \frac{\rho\dot{g}(0)}{2A(g)}\,. \tag{4.43}$$

By (4.41) and (4.42), (4.43) implies the thesys.

Let us prove (4.43). Since g solves (4.22) multiplying by $\bar{v}$ and integrating

$$\int_0^{2\pi} \bar{v}(t)\ddot{g}(t) + 3A(g)\langle g^2\rangle g(t)\bar{v}(t) + A(g)g^3(t)\bar{v}(t)\, dt = 0 \tag{4.44}$$

Next, since $\bar{v}$ solves (HOM), multiplying by g and integrating

$$\int_0^{2\pi} g(t)\ddot{\bar{v}}(t) + 3A(g)\langle g^2\rangle\bar{v}(t)g(t) + 3A(g)g^3(t)\bar{v}(t)\, dt = 0\,. \tag{4.45}$$

Subtracting (4.44) and (4.45), gives

$$\int_0^{2\pi} \bar{v}(t)\ddot{g}(t) - g(t)\ddot{\bar{v}}(t) = 2A(g)\int_0^{2\pi} g^3\bar{v}\,. \tag{4.46}$$

Integrating by parts the left hand side, since $g(0) = g(2\pi) = 0,\ \bar{u}(0) = 1$ and (4.34), gives

$$\int_0^{2\pi} \bar{v}(t)\ddot{g}(t) - g(t)\ddot{\bar{v}}(t) = \dot{g}(0)[v(2\pi) - v(0)] = \rho\dot{g}(0)\,. \tag{4.47}$$

(4.46) and (4.47) give (4.43). □

4.1.2 Explicit computations

We now give the explicit construction of g by means of the Jacobi elliptic sine defined as follows. Let $\mathrm{am}(\cdot, m) : \mathbb{R} \to \mathbb{R}$ be the inverse function of the Jacobi elliptic integral of the first kind

$$\varphi \mapsto F(\varphi, m) := \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}}\,.$$

The Jacobi elliptic sine is defined by

$$\mathrm{sn}(t,m) := \sin(\mathrm{am}(t,m))\,.$$

$\mathrm{sn}(t,m)$ is $4K(m)$-periodic, where $K(m)$ is the complete elliptic integral of the first kind

$$K(m) := F\Big(\frac{\pi}{2},m\Big) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1-m\sin^2\vartheta}}$$

and admits an analytic extension with a pole in $iK(1-m)$ for $m\in(0,1)$ and in $iK\big(1/(1-m)\big)/\sqrt{1-m}$ for $m<0$. Moreover, since

$$\partial_t \mathrm{am}(t,m) = \sqrt{1-m\,\mathrm{sn}^2(t,m)}\,,$$

the elliptic sine satisfies

$$(\dot{\mathrm{sn}})^2 = (1-\mathrm{sn}^2)(1-m\,\mathrm{sn}^2)\,. \tag{4.48}$$

Lemma 13. *There exist $V>0$, $\Omega>0$, $m\in(-1,0)$ such that $g(t) := V\mathrm{sn}(\Omega t,m)$ is an odd, analytic, 2π-periodic solution of (4.22) with pole in $iK\big(1/(1-m)\big)/\big(\Omega\sqrt{1-m}\big)$.*

Proof. Deriving (4.48) we have $\ddot{\mathrm{sn}} + (1+m)\,\mathrm{sn} - 2m\,\mathrm{sn}^3 = 0$. Therefore $g_{(V,\Omega,m)}(t) := V\mathrm{sn}(\Omega t,m)$ is an odd, $(4K(m)/\Omega)$-periodic solution of

$$\ddot g + \Omega^2(1+m)g - 2m\frac{\Omega^2}{V^2}g^3 = 0\,. \tag{4.49}$$

The function $g_{(V,\Omega,m)}$ will be a solution of (4.22) if (V,Ω,m) verify

$$\begin{cases} \Omega^2(1+m) = 3A(g_{(V,\Omega,m)})\,\langle g^2_{(V,\Omega,m)}\rangle \\ -2m\Omega^2 = V^2 A(g_{(V,\Omega,m)}) \\ 2K(m) = \Omega\pi\,. \end{cases} \tag{4.50}$$

Dividing the first equation of (4.50) by the second one

$$-\frac{1+m}{6m} = \langle \mathrm{sn}^2(\cdot,m)\rangle\,. \tag{4.51}$$

The right hand side can be expressed as

$$\langle \mathrm{sn}^2(\cdot,m)\rangle = \frac{K(m)-E(m)}{mK(m)} \tag{4.52}$$

where $E(m)$ is the complete elliptic integral of the second kind

$$E(m) := \int_0^{\pi/2}\sqrt{1-m\sin^2\vartheta}\,d\vartheta = \int_0^{K(m)} 1-m\,\mathrm{sn}^2(\xi,m)\,d\xi$$

(in the last passage we make the change of variable $\vartheta = \mathrm{am}(\xi, m)$).

Now, we show that system (4.50) has a unique solution. By (4.51) and (4.52)

$$(7+m)K(m) - 6E(m) = 0\,. \tag{4.53}$$

By the definitions of $E(m)$ and $K(m)$ we have

$$\psi(m) := (7+m)K(m) - 6E(m) = \int_0^{\pi/2} \frac{1+m(1+6\sin^2\vartheta)}{\big(1-m\sin^2\vartheta\big)^{1/2}}\, d\vartheta\,. \tag{4.54}$$

For $m=0$ it is $\psi(0) = \pi/2 > 0$, and, for $m=-1$, $\psi(-1) = -\int_0^{\pi/2} 6\sin^2\vartheta\,(1+\sin^2\vartheta)^{-1/2}\, d\vartheta < 0$. Since ψ is continuous, there exists a solution $\bar m \in (-1, 0)$ of (4.53). Then the third equation in (4.50) fixes $\bar\Omega$, and finally we find $\bar V$. Hence $g(t) = \bar V\, \mathrm{sn}(\bar\Omega t, \bar m)$ solves (4.22).

Analyticity and poles follow from [1, 16.2, 16.10.2, pp. 570, 573].

$\bar m$ is unique because $\psi'(m) > 0$ for $m \in (-1, 0)$, as it can be verified by (4.54). One can also compute that $\bar m \in (-0.30, -0.28)$. □

Remark 7. We compute explicitly the sign of dT/dE and ρ of (4.37) in the following way.

The functions $g_{(V,\Omega,m)}$ are solutions of the Hamiltonian system (4.35) imposing

$$\begin{cases} \Omega^2(1+m) = \alpha \\ -2m\Omega^2 = V^2\beta \end{cases} \tag{4.55}$$

where $\alpha := 3A(g)\,\langle g^2\rangle$, $\beta := A(g)$ and g is the solution constructed in Lemma 13.

We solve (4.55) w.r.t m finding the one-parameter family (y_m) of odd periodic solutions $y_m(t) := V(m)\,\mathrm{sn}(\Omega(m)t,\, m)$, close to g, with energy and period

$$E(m) = \frac{1}{2}V^2(m)\Omega^2(m) \;=\; -\frac{1}{\beta}\, m\, \Omega^4(m)\,, \qquad\qquad T(m) = \frac{4K(m)}{\Omega(m)}\,.$$

It holds

$$\frac{dT(m)}{dm} = \frac{4K'(m)\Omega(m) - 4K(m)\Omega'(m)}{\Omega^2(m)} > 0$$

because $K'(m) > 0$ and from (4.55) $\Omega'(m) = -\Omega(m)\big(2(1+m)\big)^{-1} < 0$. Then

$$\frac{dE(m)}{dm} = -\frac{1}{\beta}\,\Omega^4(m) - \frac{1}{\beta}\, m\, 4\Omega^3(m)\Omega'(m) < 0\,,$$

so

$$\frac{dT}{dE} = \frac{dT(m)}{dm}\Big(\frac{dE(m)}{dm}\Big)^{-1} < 0$$

as stated by general arguments in the proof of Lemma 10.

We can also write an explicit formula for ρ,

$$\rho = \frac{m}{m-1}\left[2\pi + (1+m)\int_0^{2\pi} \frac{\mathrm{sn}^2(\Omega t, m)}{\mathrm{dn}^2(\Omega t, m)}\, dt\right]. \tag{4.56}$$

From (4.56) it follows that $\rho > 0$ because $-1 < m < 0$.

4.2 Case $f(x,u) = a_2 u^2 + a_3(x)u^3 + O(u^4)$

We have to prove the existence of *non-degenerate* critical points of the functional $\Phi_n(v) := \Phi_0(\mathcal{H}_n v)$ where Φ_0 is defined in (4.14).

Lemma 14. (See [27]). *Φ_n has the following development. For $v(t,x) = \eta(t+x) - \eta(t-x) \in V$,*

$$\Phi_n(\beta n v) = 4\pi\beta^2 n^4\Big[\Psi(\eta) + \frac{\beta^2}{4\pi}\Big(\frac{R_2(\eta)}{n^2} + R_3(\eta)\Big)\Big], \tag{4.57}$$

where

$$\begin{aligned}
\Psi(\eta) &:= \frac{s^*}{2}\int_{\mathbb{T}} \dot\eta^2 + \frac{\beta^2}{4\pi}\Big[\alpha\Big(\int_{\mathbb{T}}\eta^2\Big)^2 + \gamma\int_{\mathbb{T}}\eta^4\Big],\\
R_2(\eta) &:= -\frac{a_2^2}{2}\Big[\int_\Omega v^2\Box^{-1}v^2 - \frac{\pi^2}{6}\Big(\int_{\mathbb{T}}\eta^2\Big)^2\Big],\\
R_3(\eta) &:= \frac{1}{4}\int_\Omega \big(a_3(x) - \langle a_3\rangle\big)(\mathcal{H}_n v)^4,
\end{aligned} \tag{4.58}$$

$\alpha := \big(9\langle a_3\rangle - \pi^2 a_2^2\big)/12$, $\gamma := \pi\langle a_3\rangle/2$, *and*

$$\beta = \begin{cases} (2|\alpha|)^{-1/2} & \text{if } \alpha \neq 0,\\ (\pi/\gamma)^{1/2} & \text{if } \alpha = 0.\end{cases}$$

Proof. By [27, Lemma 4.8] with $m(s_1,s_2) = \big(\eta(s_1) - \eta(s_2)\big)^2$, for $v(t,x) = \eta(t+x) - \eta(t-x)$ the operator Φ_n admits the development

$$\begin{aligned}
\Phi_n(v) = 2\pi s^* n^2 \int_{\mathbb{T}} \dot\eta^2(t)dt - \frac{\pi^2 a_2^2}{12}\Big(\int_{\mathbb{T}}\eta^2(t)\,dt\Big)^2 - \frac{a_2^2}{2n^2}\Big(\int_\Omega v^2\Box^{-1}v^2\\
- \frac{\pi^2}{6}\Big(\int_{\mathbb{T}}\eta^2(t)\,dt\Big)^2\Big) + \frac{1}{4}\langle a_3\rangle\int_\Omega v^4 + \frac{1}{4}\int_\Omega\big(a_3(x) - \langle a_3\rangle\big)(\mathcal{H}_n v)^4.
\end{aligned}$$

Since

$$\int_\Omega v^4 = 2\pi\int_{\mathbb{T}}\eta^4 + 3\Big(\int_{\mathbb{T}}\eta^2\Big)^2,$$

we write

$$\Phi_n(v) = 2\pi s^* n^2 \int_{\mathbb{T}} \dot\eta^2(t)dt - \frac{\pi^2 a_2^2}{12}\Big(\int_{\mathbb{T}} \eta^2\Big)^2 + \frac{1}{4}\langle a_3\rangle\Big[2\pi\int_{\mathbb{T}}\eta^4 + 3\Big(\int_{\mathbb{T}}\eta^2\Big)^2\Big] + \frac{R_2(\eta)}{n^2} + R_3(\eta),$$

where R_2, R_3 defined in (4.58) are both homogeneous of degree 4. So

$$\Phi_n(v) = 2\pi s^* n^2 \int_{\mathbb{T}} \dot\eta^2 + \alpha\Big(\int_{\mathbb{T}}\eta^2\Big)^2 + \gamma\int_{\mathbb{T}}\eta^4 + \frac{R_2(\eta)}{n^2} + R_3(\eta)$$

where α, γ are defined above. With the rescaling $\eta \to \eta\beta n$ we get decomposition (4.57). □

In order to find for n large a non-degenerate critical point of Φ_n, by (4.57) it is sufficient to find critical points of Ψ on $E = \big\{\eta \in H^1(\mathbb{T}),\ \eta \text{ odd}\big\}$ (like in [28, Lemma 6.2] also the term $R_3(\eta)$ tends to 0 with its derivatives).

If $\langle a_3\rangle \in (-\infty, 0) \cup (\pi^2 a_2^2/9, +\infty)$, then $\alpha \neq 0$ and we must choose $s^* = -\mathrm{sign}(\alpha)$, so that the functional becomes

$$\Psi(\eta) = \mathrm{sign}(\alpha)\Big(-\frac{1}{2}\int_{\mathbb{T}}\dot\eta^2 + \frac{1}{8\pi}\Big[\Big(\int_{\mathbb{T}}\eta^2\Big)^2 + \frac{\gamma}{\alpha}\int_{\mathbb{T}}\eta^4\Big]\Big).$$

Since in this case $\gamma/\alpha > 0$, the functional Ψ clearly has a mountain pass critical point, solution of

$$\ddot\eta + \langle\eta^2\rangle\eta + \lambda\eta^3 = 0, \qquad \lambda = \frac{\gamma}{2\pi\alpha} > 0. \tag{4.59}$$

The proof of the non-degeneracy of the solution of (4.59) is very simple using the analytical arguments of the previous section (since $\lambda > 0$ it is sufficient a positivity argument).

If $\langle a_3\rangle = 0$, then the equation becomes $\ddot\eta + \langle\eta^2\rangle\eta = 0$, so we find again what proved in [28] for $a_3(x) \equiv 0$.

If $\langle a_3\rangle = \pi^2 a_2^2/9$, then $\alpha = 0$. We must choose $s^* = -1$, so that we obtain

$$\Psi(\eta) = -\frac{1}{2}\int_{\mathbb{T}}\dot\eta^2 + \frac{1}{4}\int_{\mathbb{T}}\eta^4, \qquad \ddot\eta + \eta^3 = 0.$$

This equation has periodic solutions which are non-degenerate because of non-isocronicity, see Proposition 2 in [32].

Finally, if $\langle a_3\rangle \in (0, \pi^2 a_2^2/9)$, then $\alpha < 0$ and there are both solutions for $s^* = \pm 1$. The functional

$$\begin{aligned}\Psi(\eta) &= \frac{s^*}{2}\int_{\mathbb{T}}\dot\eta^2 + \frac{1}{8\pi}\Big[-\Big(\int_{\mathbb{T}}\eta^2\Big)^2 + \frac{\gamma}{|\alpha|}\int_{\mathbb{T}}\eta^4\Big] \\ &= \frac{s^*}{2}\int_{\mathbb{T}}\dot\eta^2 + \frac{1}{4}\int_{\mathbb{T}}\eta^4\Big[\lambda - Q(\eta)\Big]\end{aligned}$$

where

$$\lambda := \frac{\gamma}{2\pi|\alpha|} > 0\,, \qquad Q(\eta) := \frac{\Big(\int_{\mathbb{T}} \eta^2\Big)^2}{2\pi \int_{\mathbb{T}} \eta^4}$$

possesses Mountain pass critical points for any $\lambda > 0$ because (like in [27, Lemma 3.14])

$$\inf_{\eta \in E\setminus\{0\}} Q(\eta) = 0\,, \qquad \sup_{\eta \in E\setminus\{0\}} Q(\eta) = 1$$

(for $\lambda \geq 1$ if $s^* = -1$, and for $0 < \lambda < 1$ for both $s^* = \pm 1$).

Such critical points satisfy the Euler Lagrange equation

$$-s^* \ddot{\eta} - \langle \eta^2 \rangle \eta + \lambda \eta^3 = 0 \tag{4.60}$$

but their non-degeneracy is not obvious. For this, it is convenient to express this solutions in terms of the Jacobi elliptic sine.

Proposition 4. *(i) Let $s^* = -1$. Then for every $\lambda \in (0, +\infty)$ there exists an odd, analytic, 2π-periodic solution $g(t)$ of (4.60) which is non-degenerate in E. $g(t) = V\mathrm{sn}(\Omega t, m)$ for $V > 0$, $\Omega > 0$, $m \in (-\infty, -1)$ suitable constants.*
(ii) Let $s^* = 1$. Then for every $\lambda \in (0, 1)$ there exists an odd, analytic, 2π-periodic solution $g(t)$ of (4.60) which is non-degenerate in E. $g(t) = V\mathrm{sn}(\Omega t, m)$ for $V > 0$, $\Omega > 0$, $m \in (0, 1)$ suitable constants.

We prove Proposition 4 in several steps. First we construct the solution g like in Lemma 13.

Lemma 15. *(i) Let $s^* = -1$. Then for every $\lambda \in (0, +\infty)$ there exist $V > 0$, $\Omega > 0$, $m \in (-\infty, -1)$ such that $g(t) = V\mathrm{sn}(\Omega t, m)$ is an odd, analytic, 2π-periodic solution of (4.60) with a pole in $\frac{i}{\Omega\sqrt{1-m}} K\big(\frac{1}{1-m}\big)$.*
(ii) Let $s^* = 1$. Then for every $\lambda \in (0, 1)$ there exist $V > 0$, $\Omega > 0$, $m \in (0, 1)$ such that $g(t) = V\mathrm{sn}(\Omega t, m)$ is an odd, analytic, 2π-periodic solution of (4.60) with a pole in $iK(1 - m)/\Omega$.

Proof. We know that $g_{(V,\Omega,m)}(t) := V\mathrm{sn}(\Omega t, m)$ is an odd, $(4K(m)/\Omega)$-periodic solution of (4.49), see Lemma 13. So it is a solution of (4.60) if (V, Ω, m) verify

$$\begin{cases} \Omega^2(1 + m) = s^* V^2 \langle \mathrm{sn}^2(\cdot, m) \rangle \\ 2m\Omega^2 = s^* V^2 \lambda \\ 2K(m) = \Omega\pi\,. \end{cases} \tag{4.61}$$

Conditions (4.61) give the connection between λ and m:

$$\lambda = \frac{2m}{1 + m} \langle \mathrm{sn}^2(\cdot, m) \rangle\,. \tag{4.62}$$

Moreover system (4.61) imposes

$$\begin{cases} m \in (-\infty, -1) & \text{if } s^* = -1 \\ m \in (0,1) & \text{if } s^* = 1\,. \end{cases}$$

We know that $m \mapsto \langle \mathrm{sn}^2(\cdot, m)\rangle$ is continuous, strictly increasing on $(-\infty, 1)$, it tends to 0 for $m \to -\infty$ and to 1 for $m \to 1$, see Lemma 19. So the right-hand side of (4.62) covers $(0, +\infty)$ for $m \in (-\infty, 0)$, and it covers $(0,1)$ for $m \in (0,1)$. For this reason for every $\lambda > 0$ there exists a unique $\bar{m} < -1$ satisfying (4.62), and for every $\lambda \in (0,1)$ there exists a unique $\bar{m} \in (0,1)$ satisfying (4.62).

The value $\bar{m}$ and system (4.61) determine uniquely the values $\bar{V}$, $\bar{\Omega}$.

Analyticity and poles follow from [1, 16.2, 16.10.2, pp.570,573]. □

Now we have to prove the non-degeneracy of g. The linearised equation of (4.60) at g is

$$\ddot{h} + s^*\big(\langle g^2\rangle - 3\lambda g^2\big)h = -2s^*\langle gh\rangle g. \tag{4.63}$$

Let L be the Green operator, i.e. for $f \in E$, let $H := L(f)$ be the unique solution belonging to E of the non-homogeneous linear system

$$\ddot{H} + s^*\big(\langle g^2\rangle - 3\lambda g^2\big)\, H = f\,.$$

We can write (4.63) as

$$h = -2s^*\langle gh\rangle L(g)\,. \tag{4.64}$$

Multiplying by g and integrating we get

$$\langle gh\rangle\big[1 + 2s^*\langle gL(g)\rangle\big] = 0\,.$$

If $A_0 := 1 + 2s^*\langle gL(g)\rangle \neq 0$, then $\langle gh\rangle = 0$, so by (4.64) $h = 0$ and the non-degeneracy is proved.

It remains to show that $A_0 \neq 0$. As before, the key is to express $L(g)$ in a suitable way. We first look for a fundamental set of solutions of the homogeneous equation

$$\ddot{h} + s^*\big(\langle g^2\rangle - 3\lambda g^2\big)h = 0\,. \tag{4.65}$$

Lemma 16. *There exist two linearly independent solutions of (4.65), $\bar{u}$ even, 2π-periodic and $\bar{v}$ odd, not periodic, such that $\bar{u}(0) = 1$, $\dot{\bar{u}}(0) = 0$, $\bar{v}(0) = 0$, $\dot{\bar{v}}(0) = 1$, and*

$$\bar{v}(t + 2\pi) - \bar{v}(t) = \rho\,\bar{u}(t) \quad \forall\, t \tag{4.66}$$

for some $\rho \neq 0$. Moreover there hold the following formulae for $\bar{u}$, $\bar{v}$:

$$\bar{u}(t) = \dot{g}(t)/\dot{g}(0) = \dot{\mathrm{sn}}(\bar{\Omega}t, \bar{m}), \tag{4.67}$$

(4.68)

$$\bar{v}(t) = \frac{1}{\bar{\Omega}(1-\bar{m})}\,\mathrm{sn}(\bar{\Omega}t) + \frac{\bar{m}}{\bar{m}-1}\,\dot{\mathrm{sn}}(\bar{\Omega}t)\Big[\,t + \frac{1+\bar{m}}{\bar{\Omega}}\int_0^{\bar{\Omega}t}\frac{\mathrm{sn}^2(\xi,\bar{m})}{\mathrm{dn}^2(\xi,\bar{m})}\,d\xi\Big]\,.$$

Proof. g solves (4.60) so $\dot{g}$ solves (4.65); normalizing we get (4.67).

By (4.49), the function $y(t) = V\mathrm{sn}(\Omega t, m)$ solves

$$\ddot{y} + s^*\langle g^2\rangle y - s^*\lambda y^3 = 0 \tag{4.69}$$

if (V, Ω, m) satisfy

$$\begin{cases} \Omega^2(1+m) = s^*\langle g^2\rangle \\ 2m\Omega^2 = s^*V^2\lambda\,. \end{cases} \tag{4.70}$$

We solve (4.70) w.r.t. m finding the one-parameter family (y_m) of odd periodic solutions of (4.69), $y_m(t) = V(m)\mathrm{sn}(\Omega(m)t, m)$. So $l(t) := (\partial_m y_m)_{|m=\bar{m}}$ solves (4.65). We normalize $\bar{v}(t) := l(t)/\dot{l}(0)$ and we compute the coefficients differentiating (4.70) w.r.t. m. From the definitions of the Jacobi elliptic functions it holds

$$\partial_m \mathrm{sn}(x,m) = -\dot{\mathrm{sn}}(x,m)\frac{1}{2}\int_0^x \frac{\mathrm{sn}^2(\xi,m)}{\mathrm{dn}^2(\xi,m)}\,d\xi\,;$$

thanks to this formula we obtain (4.68).

Since $2\pi\bar{\Omega} = 4K(\bar{m})$ is the period of the Jacobi functions sn and dn, by (4.67),(4.68) we obtain (4.66) with

$$\rho = \frac{\bar{m}}{\bar{m}-1}\,2\pi\Big(\,1 + (1+\bar{m})\langle\frac{\mathrm{sn}^2}{\mathrm{dn}^2}\rangle\Big)\,.$$

If $s^* = 1$, then $\bar{m} \in (0,1)$ and directly we can see that $\rho < 0$. If $s^* = -1$, then $\bar{m} < -1$. From the equality $\langle \mathrm{sn}^2/\mathrm{dn}^2\rangle = (1-m)^{-1}\big(1 - \langle\mathrm{sn}^2\rangle\big)$ (see Lemma 3, (L.2)), it results $\rho > 0$. □

We can note that the integral representation (4.38) of the Green operator L holds again in the present case. The proof is just like in Lemma 11.

Lemma 17. *We can write $A_0 := 1 + 2s^*\langle gL(g)\rangle$ as function of λ, $\bar{m}$,*

$$A_0 = \frac{\lambda(1-\bar{m})^2 q - (1-\lambda)^2(1+\bar{m})^2 + \bar{m}q^2}{\lambda(1-\bar{m})^2 q}\,, \tag{4.71}$$

$$q = q(\lambda,\bar{m}) := 2 - \lambda\frac{(1+\bar{m})^2}{2\bar{m}} > 0.$$

Proof. First, we calculate $\langle gL(g)\rangle$ with the integral formula (4.38) of L. The equalities (4.41),(4.42) still hold, while similar calculations give

$$\int_0^{2\pi} g^3\bar{v} = -s^*\frac{\dot{g}(0)\rho}{2\lambda}$$

instead of (4.43). So

$$\langle gL(g)\rangle = -s^* \frac{\rho}{4\pi\lambda} + \frac{1}{2\pi\rho}\Big(\int_0^{2\pi} g\bar v\Big)^2 \tag{4.72}$$

and the sign of A_0 is not obvious. We calculate $\int_0^{2\pi} g\bar v$ recalling that $g(t) = \bar V \mathrm{sn}(\bar\Omega t, \bar m)$, using formula (4.68) for $\bar v$ and integrating by parts

$$\int_0^{2\pi} \mathrm{sn}(\bar\Omega t)\dot{\mathrm{sn}}(\bar\Omega t)\mu(t)\,dt = -\frac{1}{2\bar\Omega}\int_0^{2\pi} \mathrm{sn}^2(\bar\Omega t)\dot\mu(t)\,dt$$

where $\mu(t) := t + (1+\bar m)\bar\Omega^{-1}\int_0^{\bar\Omega t} \mathrm{sn}^2(\xi)/\mathrm{dn}^2(\xi)\,d\xi$. From (L.2),(L.3) in Lemma 3, we obtain the formula

$$\langle\frac{\mathrm{sn}^4}{\mathrm{dn}^2}\rangle = \frac{1+(m-2)\langle \mathrm{sn}^2\rangle}{m(1-m)}$$

and consequently

$$\int_0^{2\pi} g\bar v = \frac{\pi\bar V}{\bar\Omega(1-\bar m)^2}\left(1+\bar m - 2\bar m\langle \mathrm{sn}^2\rangle\right). \tag{4.73}$$

By the second equality of (4.61) and (4.72) we get

$$A_0 = 1 + \frac{2}{l}\Big[-\frac{\rho}{4\pi} + \frac{\pi\bar m}{\rho(1-\bar m)^4}\left(1+\bar m-2\bar m\langle \mathrm{sn}^2\rangle\right)^2\Big] \tag{4.74}$$

both for $s^* = \pm 1$. From the proof of Lemma 16 we have $\rho = -2\pi\bar m q\,(1-\bar m)^{-2}$, where q is defined in (4.71); inserting this expression of ρ in (4.74) we obtain (4.71).

Finally, for $\bar m < -1$ we have immediately $q > 0$, while for $\bar m \in (0,1)$ we get $q = 2 - (1+\bar m)\langle \mathrm{sn}^2\rangle$ by (4.62). Since $\langle \mathrm{sn}^2\rangle < 1$, it results $q > 0$. □

Lemma 18. $A_0 \neq 0$. *More precisely, sign*$(A_0) = -s^*$.

Proof. From (4.71), $A_0 > 0$ iff $\lambda(1-\bar m)^2 q - (1-\lambda)^2(1+\bar m)^2 + \bar m q^2 > 0$. This expression is equal to $-(1-\bar m)^2 p$, where

$$p = p(\lambda,\bar m) = \frac{(1+\bar m)^2}{4\bar m}\lambda^2 - 2\lambda + 1,$$

so $A_0 > 0$ iff $p < 0$. The polynomial $p(l)$ has degree 2 and its determinant is $\Delta = -(1-\bar m)^2/\bar m$. So, if $s^* = 1$, then $\bar m \in (0,1)$, $\Delta < 0$ and $p > 0$, so that $A_0 < 0$.

It remains the case $s^* = -1$. For $l > 0$, we have $p(l) < 0$ iff $l > x^*$, where x^* is the positive root of p, $x^* := 2R(1+R)^{-2}$, $R := |\bar m|^{1/2}$. By (4.62), $l > x^*$ iff

$$\langle \mathrm{sn}^2(\cdot,\bar m)\rangle > \frac{R-1}{(R+1)R}. \tag{4.75}$$

By formula (4.52) and by definition of complete elliptic integrals K and E we can write (4.75) as

$$\int_0^{\pi/2}\Big(\frac{R-1}{(R+1)R}-\sin^2\vartheta\Big)\frac{d\vartheta}{\sqrt{1+R^2\sin^2\vartheta}}<0\,. \tag{4.76}$$

We put $\sigma := R-1/(R+1)R$ and note that $\sigma<1/2$ for every $R>0$.

$\sigma-\sin^2\vartheta>0$ iff $\vartheta\in(0,\vartheta^*)$, where $\vartheta^* := \arcsin(\sqrt{\sigma})$, i.e. $\sin^2\vartheta^*=\sigma$. Moreover $1<1+R^2\sin^2\vartheta<1+R^2$ for every $\vartheta\in(0,\pi/2)$. So

$$\int_0^{\pi/2}\frac{\sigma-\sin^2\vartheta}{\sqrt{1+R^2\sin^2\vartheta}}\,d\vartheta<\int_0^{\vartheta^*}\big(\sigma-\sin^2\vartheta\big)\,d\vartheta+\int_{\vartheta^*}^{\pi/2}\frac{\sigma-\sin^2\vartheta}{\sqrt{1+R^2}}\,d\vartheta\,. \tag{4.77}$$

Thanks to the formula

$$\int_a^b\sin^2\vartheta\,d\vartheta=\frac{b-a}{2}-\frac{\sin(2b)-\sin(2a)}{4}$$

the right-hand side term of (4.77) is equal to

$$\frac{\sin(2\vartheta^*)}{4}\Big[(2\sigma-1)\Big(\frac{2\vartheta^*}{\sin(2\vartheta^*)}+\frac{1}{\sqrt{1+R^2}}\frac{\pi-2\vartheta^*}{\sin(2\vartheta^*)}\Big)+\Big(1-\frac{1}{\sqrt{1+R^2}}\Big)\Big]\,.$$

Since $2\sigma-1<0$ and $\alpha>\sin\alpha$ for every $\alpha>0$, this quantity is less than

$$\frac{\sin(2\vartheta^*)}{4}\Big[(2\sigma-1)\Big(1+\frac{1}{\sqrt{1+R^2}}\Big)+\Big(1-\frac{1}{\sqrt{1+R^2}}\Big)\Big]\,.$$

By definition of σ, the last quantity is negative for every $R>0$, so (4.76) is true. Consequently $\lambda>x^*$, $p<0$ and $A_0>0$. □

Finally, we show the properties of the function $m\mapsto\langle\mathrm{sn}^2(\cdot,m)\rangle$ used in the proof of Lemma 15.

Lemma 19. *The function* $\varphi:(-\infty,1)\to\mathbb{R}$, $m\mapsto\langle\mathrm{sn}^2(\cdot,m)\rangle$ *is continuous, differentiable, strictly increasing, and*

$$\lim_{m\to-\infty}\varphi(m)=0,\qquad\lim_{m\to1}\varphi(m)=1.$$

Proof. By (4.52) and by definition of complete elliptic integrals K and E,

$$\varphi(m)=\frac{K(m)-E(m)}{mK(m)}=\int_0^{\pi/2}\frac{\sin^2\vartheta\,d\vartheta}{\sqrt{1-m\sin^2\vartheta}}\Big(\int_0^{\pi/2}\frac{d\vartheta}{\sqrt{1-m\sin^2\vartheta}}\Big)^{-1},$$

so the continuity of φ is evident.

Using the equality $\sin^2 + \cos^2 = 1$ and the change of variable $\vartheta \to \pi/2 - \vartheta$ in the integrals which define K and E, we obtain the formulae

$$K(m) = \frac{1}{\sqrt{1-m}}\, K\Big(\frac{m}{m-1}\Big), \quad E(m) = \sqrt{1-m}\, E\Big(\frac{m}{m-1}\Big), \tag{4.78}$$

for all $m < 1$. We put $\mu := m/(m-1)$, so that

$$\varphi(m) = 1 - \frac{1}{\mu} + \frac{E(\mu)}{\mu K(\mu)}\,. \tag{4.79}$$

Since μ tends to 1 as $m \to -\infty$, $E(1) = 1$ and $\lim_{\mu\to 1} K(\mu) = +\infty$, (4.78),(4.79) give $\lim_{m\to-\infty} \varphi(m) = 0$. Since $E(m)/K(m)$ tends to 0 as $m \to 1$, (4.52) gives $\lim_{m\to 1} \varphi(m) = 1$.

Differentiating the integrals which define K and E w.r.t. m gives

$$E'(m) = \frac{E(m) - K(m)}{2m}, \quad K'(m) = \frac{1}{2m}\Big(\int_0^{\pi/2} \frac{d\vartheta}{(1 - m\sin^2\vartheta)^{3/2}} - K(m)\Big),$$

hence the derivative is

$$\varphi'(m) = \frac{1}{2m^2K^2(m)}\Big[E(m)\int_0^{\pi/2} \frac{d\vartheta}{(1 - m\sin^2\vartheta)^{3/2}} - K^2(m)\Big]\,.$$

The term in the square brackets is positive by strict Hölder inequality for $(1 - m\sin^2\vartheta)^{-3/4}$ and $(1 - m\sin^2\vartheta)^{1/4}$. □

Chapter 5

Forced vibrations of a nonhomogeneous string

We consider the nonhomogeneous string

$$\begin{cases} \rho(x)u_{tt} - (p(x)u_x)_x = \mu f(x, \omega t, u) \\ u(0,t) = u(\pi, t) = 0 \end{cases} \tag{5.1}$$

where $\rho(x) > 0$ is the mass per unit length, $p(x) > 0$ is the modulus of elasticity multiplied by the cross-sectional area, $\mu > 0$ is a parameter, and the nonlinear forcing term $f(x, \omega t, u)$ is $(2\pi/\omega)$-periodic in time, that is $f(x, \cdot, u)$ is 2π-periodic. We look for $(2\pi/\omega)$-time periodic solutions $u(x,t)$ of (5.1).

After a time rescaling we look for 2π-periodic solutions of

$$\begin{cases} \omega^2\rho(x)u_{tt} - (p(x)u_x)_x = \mu f(x, t, u) \\ u(0,t) = u(\pi, t) = 0 \end{cases} \tag{5.2}$$

where $\mu \in [0, \bar{\mu}]$ for some given $\bar{\mu} > 0$, under the 2π-periodic forcing term

$$f(x,t,u) \; = \; \sum_{l \in \mathbb{Z}} f_l(x,u) e^{ilt} = f_0(x,u) + \bar{f}(x,t,u) \tag{5.3}$$

where

$$\bar{f}(x,t,u) := \sum_{l \neq 0} f_l(x,u) e^{ilt} \, .$$

We suppose that f is analytic in (t, u), more precisely

$$f(x,t,u) = \sum_{l \in \mathbb{Z},\, k \in \mathbb{N}} f_{lk}(x)\, u^k \, e^{ilt}$$

where $f_{lk}(x) \in H^1((0,\pi); \mathbb{C})$, $f_{-l,k} = f_{lk}^*$ (the symbol z^* denotes the complex conjugate of $z \in \mathbb{C}$) and we assume the following hypothesis on the decay of the coefficients $\|f_{lk}\|_{H^1}$.

Hypothesis (F). *There exist $\sigma_0 > 0$, $r > 0$ such that*

$$\sum_{l \in \mathbb{Z}} \|f_{lk}\|^2_{H^1} (1 + l^2) e^{(2\sigma_0) 2|l|} := C_k^2(f) < \infty \quad \textit{and} \quad \sum_{k=0}^{+\infty} C_k(f)\, r^k < \infty\,.$$

For example, trigonometric polynomials in t and polynomials in u, namely

$$f(x,t,u) = \sum_{|l| \leq L, 0 \leq k \leq K} f_{lk}(x)\, u^k\, e^{ilt} \tag{5.4}$$

for some $L, K \in \mathbb{N}$, satisfy hypothesis (F) for every σ_0, r.

Remark 8. *We notice that if $f(x,t,0) = \sum_{l \in \mathbb{Z}} f_{l0}(x)\, e^{ilt} \neq 0$ equation* (5.2) *does not possess the trivial solution $u = 0$.*

We look for periodic solutions of (5.2) in the Hilbert space

$$X_{\sigma,s} := \Big\{ u \colon \mathbb{T} \to H_0^1((0,\pi);\mathbb{R}),\ u(x,t) = \sum_{l \in \mathbb{Z}} u_l(x)\, e^{ilt},\ u_l \in H_0^1((0,\pi);\mathbb{C}),$$
$$u_{-l} = u_l^*,\ \|u\|^2_{\sigma,s} := \sum_{l \in \mathbb{Z}} \|u_l\|^2_{H^1} (1 + l^{2s}) e^{2\sigma|l|} < \infty \Big\}$$

of functions 2π-periodic in time, valued in $H^1((0,\pi);\mathbb{R})$, which have a bounded analytic extension on the complex strip $|\mathrm{Im}\, t| < \sigma$, with trace function on $|\mathrm{Im}\, t| = \sigma$ belonging to $H^s(\mathbb{T}; H^1((0,\pi);\mathbb{C}))$.

For $s > 1/2$, $X_{\sigma,s}$ is a multiplicative Banach algebra:

$$\|uv\|_{\sigma,s} \leq c_s \|u\|_{\sigma,s} \|v\|_{\sigma,s} \quad \forall\, u, v \in X_{\sigma,s} \tag{5.5}$$

with

$$c_s := 2^s \Big(\sum_{n \in \mathbb{Z}} \frac{1}{1 + n^{2s}} \Big)^{1/2}, \tag{5.6}$$

see Appendix C. We shall use the notation X_σ, resp. $\| \ \|_\sigma$, for $X_{\sigma,1}$, resp. $\| \ \|_{\sigma,1}$.

5.1 The Lyapunov-Schmidt reduction

To find solutions of (5.2) we implement the Lyapunov-Schmidt reduction according to the decomposition

$$X_{\sigma,s} = V \oplus (W \cap X_{\sigma,s})$$

where

$$V := H_0^1(0,\pi)\,, \qquad W := \Big\{ w = \sum_{l \neq 0} w_l(x)\, e^{ilt} \in X_{0,s} \Big\}$$

writing every $u \in X_{\sigma,s}$ as $u = u_0(x) + \sum_{l \neq 0} u_l(x)\, e^{ilt}$.

Projecting equation (5.2) with

$$u = v + w\,, \quad v \in V,\ w \in W\,,$$

yields

$$(5.7) \qquad \begin{cases} -(pv')' = \mu \Pi_V f(v+w) & \text{bifurcation equation} \\ L_\omega w = \mu \Pi_W f(v+w) & \text{range equation} \end{cases}$$

where Π_V, Π_W denote the projectors, $f(u)(x,t) := f(x,t,u(x,t))$ and

$$L_\omega u := \omega^2 \rho(x) u_{tt} - (p(x)u_x)_x\,.$$

We shall find solutions of (5.7) when μ/ω is small. In this limit w tends to 0 and the bifurcation equation reduces to the time-independent equation

$$(5.8) \qquad -(pv')' = \mu f_0(v)$$

because, by (5.3), for $w = 0$

$$\Pi_V f(v) = \Pi_V f_0(x, v(x)) + \Pi_V \bar{f}(x,t,v(x)) = f_0(v)\,.$$

The infinite dimensional "0-th order bifurcation equation" (5.8) is a second order ordinary differential equation, which, under natural conditions on f_0, possesses non-degenerate solutions satisfying the boundary conditions $v(0) = v(\pi) = 0$.

Hypothesis (V). *The problem*

$$(5.9) \qquad \begin{cases} -(p(x)v'(x))' = \mu f_0(x, v(x)) \\ v(0) = v(\pi) = 0 \end{cases}$$

admits a solution $\bar{v} \in H_0^1(0,\pi)$ which is non-degenerate, namely the linearised equation

$$-(ph')' = \mu\, f_0'(\bar{v})h$$

possesses in $H_0^1(0,\pi)$ only the trivial solution $h = 0$.

We note that, for $\mu = 0$, the trivial solution $\bar{v} = 0$ is non-degenerate, so, by the Implicit function Theorem, hypothesis (V) is automatically satisfied for μ small. We deal also with the case μ not small, see for example Lemmas 21 and 22.

By the Implicit function Theorem, hypothesis (V) implies the existence of a smooth map

$$(\mu, w) \mapsto v(\mu, w) \in V$$

such that $v(\mu, w)$ solves the bifurcation equation in (5.7), see Lemma 23.

Remark 9. *For a discussion about the difficulties caused by a degenerate solution we refer to [29].*

Let λ_j denote the eigenvalues of the Sturm-Liouville problem

$$\begin{cases} -(p(x)y'(x))' = \lambda\rho(x)y(x) \\ y(0) = y(\pi) = 0 \end{cases} \tag{5.10}$$

and $\omega_j := \sqrt{\lambda_j}$. These are the frequencies of the free vibrations of the string (note that all the eigenvalues λ_j are positive). Physically, it is the sequence of the fundamental tone ω_1 and all its overharmonics $\omega_2, \omega_3, \ldots$ which compose the musical note of the string.

For $\gamma \in (0,1)$ we define

$$A_\gamma := \Big\{(\mu,\omega) \in (\mu_1,\mu_2) \times (\gamma,+\infty) \ : \ \frac{\mu}{\omega} < C'\gamma^5, \ \ |\omega l - \omega_j| > \frac{\gamma}{l^\tau} \quad \forall\, l = 1,\ldots,N_0, \ j \geq 1\Big\} \tag{5.11}$$

where ω_j are given by (5.10), and (μ_1,μ_2), $N_0 \in \mathbb{N}$, $C' > 0$ shall be fixed in the next theorem.

Theorem 4. ***(Existence).*** *Suppose $p(x), \rho(x) > 0$ are of class $H^3(0,\pi)$, f satisfies (F) and hypothesis (V) holds for some $\mu_0 \in [0,\bar{\mu}]$.*

Fix $\tau \in (1,2)$, $\gamma \in (0,1)$. There exist a neighborhood (μ_1,μ_2) of μ_0, $N_0 \in \mathbb{N}$, positive constants C, C' (depending on $\rho, p, f, \bar{\mu}, \bar{v}, \tau$), a map

$$\tilde{w} \in C^\infty(A_\gamma, X_{\sigma_0/2} \cap W)$$

and a Cantor set $B_\gamma \subset A_\gamma$ of positive measure such that, for all $(\mu,\omega) \in B_\gamma$

$$\tilde{u}(\mu,\omega) := v(\mu, \tilde{w}(\mu,\omega)) + \tilde{w}(\mu,\omega) \in V \oplus (W \cap X_{\sigma_0/2}) \tag{5.12}$$

is a classical solution of (5.2) and satisfies

$$\tilde{u}(\cdot,t) \in H^3(0,\pi) \cap H_0^1(0,\pi), \quad \forall t \in \mathbb{R}.$$

The Cantor set B_γ is defined in (5.14) *and satisfies the measure estimate (5.55).*

Furthermore, $\forall (\mu,\omega) \in A_\gamma$ the following estimates hold

$$\|\tilde{w}(\mu,\omega)\|_{\sigma_0/2} \leq C\frac{\mu}{\gamma\omega}, \quad \|v(\mu,\tilde{w}(\mu,\omega)) - v(\mu,0)\|_{H^1} \leq C\frac{\mu}{\gamma\omega} \tag{5.13}$$

and $\|v(\mu,0) - \bar{v}\|_{H^1} \leq C|\mu - \mu_0|$.

The neighborhood (μ_1,μ_2) of μ_0 is fixed in Lemma 23, the integer N_0 is fixed in Lemma 28 and the constant C' in Lemma 32.

Estimate (5.13) shows how close is the solution $\tilde{u}$ to the static configuration $v(\mu,0)$.

Remark 10. *We underline that the function $\tilde{w}(\mu,\omega)$, as well as $\tilde{u}(\mu,\omega)$, is defined for all the values of the parameters $(\mu,\omega)\in A_\gamma$ and not only on the Cantor set B_γ ($\tilde{w}(\mu,\omega)$ is introduced in Lemma 30). What is true is that, if $(\mu,\omega)\in B_\gamma$, then $\tilde{w}(\mu,\omega)$ solves the range equation, see Theorem 6. As a consequence, if $(\mu,\omega)\in B_\gamma$, then $\tilde{u}(\mu,\omega)$ solves equation (5.2).*

The Cantor set B_γ is explicitly defined by

(5.14)

$$B_\gamma := \Big\{(\mu,\omega)\in(\mu_1,\mu_2)\times(2\gamma,+\infty)\ :\ |\omega l-\omega_j| > \frac{2\gamma}{l^\tau}\quad \forall\, l=1,\dots,N_0,\ j\geq 1,$$
$$\frac{\mu}{\omega} < C'\gamma^5,\quad \Big|\omega l - \frac{j}{c}\Big| > \frac{2\gamma}{l^\tau},\quad |\omega l - \tilde{\omega}_j(\mu,\omega)| > \frac{2\gamma}{l^\tau}\quad \forall\, l,j\geq 1\Big\}$$

where

$$c := \frac{1}{\pi}\int_0^\pi \Big(\frac{\rho(x)}{p(x)}\Big)^{1/2}\,dx \tag{5.15}$$

and $\tilde{\lambda}_j(\mu,\omega) := \tilde{\omega}_j^2(\mu,\omega)$ denote the eigenvalues of the Sturm-Liouville problem

$$\begin{cases} -(py')' - \mu\Pi_V f'(\tilde{u}(\mu,\omega))\,y = \lambda\rho y \\ y(0)=y(\pi)=0\,. \end{cases} \tag{5.16}$$

Note that B_γ is constructed by means of the function $\tilde{u}(\mu,\omega)$ which is defined for all $(\mu,\omega)\in A_\gamma$, see Remark 10.

Remark 11. *If some $\tilde{\lambda}_j(\mu,\omega)$ is negative, then $\tilde{\omega}_j(\mu,\omega) = i\sqrt{|\tilde{\lambda}_j(\mu,\omega)|}$ is a purely imaginary complex number and the non-resonance conditions in (5.14) are trivially satisfied.*

The Cantor set B_γ is large in a measure theoretical sense, see section 5.7. In particular, for all $\mu\in(\mu_1,\mu_2)$,

$$S(\mu) := \big\{\omega : (\mu,\omega)\in\cup_{\gamma\in(0,1)}B_\gamma\big\}$$

has asymptotically full measure at $\omega\to+\infty$, i.e.

$$\lim_{\omega\to+\infty}|S(\mu)\cap(\omega,\omega+1)| = 1\,. \tag{5.17}$$

Analogously,

$$S(\omega) := \big\{\mu : (\mu,\omega)\in\cup_{\gamma\in(0,1)}B_\gamma\big\}$$

satisfies $\forall\omega'>0$, $\forall\gamma'\in(0,1)$,

$$\lim_{\mu\to 0}\Big|\Big\{\omega\in(\omega',\omega'+1) : \frac{|S(\omega)\cap(0,\mu)|}{\mu} \geq 1-\gamma'\Big\}\Big| = 1\,. \tag{5.18}$$

Finally we discuss the regularity of the solution $\tilde{u}(x,t)$ found in Theorem 4 with respect to x (by construction $\tilde{u}$ is analytic with respect to t).

Theorem 5. ***(Regularity)*** *Assume the hypotheses of Theorem 4. In addition, suppose that, for some $m \geq 3$,*

$$\rho(x) \in H^m(0,\pi)\,, \quad p(x) \in H^{m+1}(0,\pi)\,, \quad f_{lk}(x) \in H^m(0,\pi) \quad \forall\, l,k \tag{5.19}$$

and, for some $r_m > 0$,

$$\sum_{l\in\mathbb{Z},\, k\geq 0} \|f_{lk}\|_{H^m} r_m^k < +\infty\,. \tag{5.20}$$

If $\|\tilde{u}(\cdot\,,t)\|_{H^1} r_m^{-1}$ is small enough then

$$\tilde{u}(\cdot\,,t) \in H^{m+2}(0,\pi) \cap H_0^1(0,\pi)\,. \tag{5.21}$$

This conclusion holds true, for example, when $f_0(x,0) = d_u f_0(x,0) = 0$ and $\mu/\gamma\omega$ is small enough.

Note that the regularity (5.21) requires no skewsymmetry assumptions on f and requires just a smallness condition for the H^1 norm of $\tilde{u}(\cdot\,,t)$.

Remark 12. *If $f(x,t,u)$ is a trigonometric polynomial in t and a polynomial in u like in (5.4), then the series in (5.20) is a finite sum. Therefore the conclusion (5.21) is true without smallness conditions for $\tilde{u}$.*

In particular, if $\rho(x)$, $p(x)$, $f_{lk}(x)$ are C^∞ (for example $f = \cos x \cos t\,(1+u^2)$) then the solution $\tilde{u}$ is also C^∞ in the variable x (the above f does not satisfy the skewsymmetry assumption (5.23)).

The subtle problem to prove Theorem 5 is that, because of Dirichlet boundary conditions, the Sobolev regularity of a function with respect to x is *not* characterized by the rapid decaying properties of the Fourier coefficients (unless assuming skewsymmetry assumptions on the nonlinearity and restricting to solutions $u(x,t)$ odd in x, see Remark 13). Theorem 5 is proved in section 5.8 via bootstrap arguments.

5.2 Outline of the proof

In section 5.3 we prove that, under assumption (F), the composition operator induced by the nonlinearity f on $X_{\sigma,s}$ is an analytic map.

In section 5.4, we find a solution $v(\mu,w)$ of the infinite dimensional bifurcation equation in (5.7). Thanks to assumption (V) (which is verified on several examples in Lemmas 21-22), $v(\mu,w)$ is obtained in Lemma 23 by a standard Implicit function Theorem.

In sections 5.5, 5.6 and 5.7 we solve the range equation by means of an iterative Nash-Moser Implicit function Theorem. The final Theorem 6 is proved in several steps.

In section 5.5 we find inductively a sequence of approximate solutions $w_n(\mu,\omega)$ defined on smaller and smaller subsets A_n of the parameters (μ,ω) (see (5.38)). The reason of these "excisions" is to avoid resonance phenomena in order to prove the invertibility of the linearised operators obtained at each step of the iteration, see conditions (5.32)-(5.33) in Lemma 26.

In section 5.6 we extend these approximate solutions $w_n(\mu,\omega)$ to C^∞-functions $\tilde{w}_n(\mu,\omega)$ defined for all the values of the parameters (μ,ω) and converging (super-exponentially fast) to a C^∞ map $\tilde{w}$ defined for all (μ,ω), see Lemma 30. It is to prove the regularity of w_n with respect to the parameters (μ,ω) that we find convenient to define the approximate solutions w_n as exact solutions of the equations (5.40) (with $k=n$), see Remark 16.

In Lemma 31, we prove that the Cantor set B_γ, defined in (5.14) by means of $\tilde{w}$, is contained in A_n (which depends on w_{n-1}) for each n. This is a consequence of the super-exponentially fast convergence of $\tilde{w}_n$ to $\tilde{w}$, see (5.51).

In section 5.7 we prove that B_γ is a large set in a measure theoretical sense.

In all the previous steps we have to assume smallness conditions for μ/ω. The most restrictive one is $\mu/\omega < C'\gamma^5$ in the definition (5.11) of A_γ.

In section 5.8 we conclude the proof of the existence Theorem 4 and we prove the regularity Theorem 5.

In section 5.9 we study the key step for the inversion of the linearised operators. Lemma 26 is obtained by a variant of the techniques developed in [28]. In particular, the key estimate on the small divisors of Lemma 37 is reminiscent of the method of Kuksin explained in [41], pp. 90-94.

Notations. The symbols K, K_i, K_i' shall denote positive constants depending only on $\rho, p, f, \bar{\mu}, \bar{v}, \tau$.

5.3 Regularity of the composition operator

We first prove the analyticity of the composition operator

$$u(x,t) \mapsto f(x,t,u(x,t))$$

induced by f on $X_{\sigma,s}$.

By the Banach algebra property (5.5) of $X_{\sigma,s}$ the composition operator

$$u \mapsto u^k\,, \qquad \forall k \in \mathbb{N}$$

is an analytic map from $X_{\sigma,s}$ into itself. Thanks to the rapid decay of the coefficients $\|f_{lk}\|_{H^1}$ assumed in hypothesis (F), this property holds true also for the composition operator $f(u)$.

Lemma 20. *Let f satisfy assumption (F). For every $\sigma \in [0, \sigma_0]$, $s > 1/2$, the composition operator f is analytic on the ball $\{u \in X_{\sigma,s} : \|u\|_{\sigma,s} < r/c_s\}$ where c_s is defined in (5.6).*

Proof. First note that

$$\sum_{l\in\mathbb{Z}} \|u_l\|_\infty \leq \sqrt{\frac{\pi}{2}} \sum_{l\in\mathbb{Z}} \|u_l\|_{H^1} \leq \sqrt{\frac{\pi}{2}} \Big(\sum_{l\in\mathbb{Z}} \|u_l\|^2_{H^1} (1+l^{2s}) \Big)^{1/2} \Big(\sum_{l\in\mathbb{Z}} \frac{1}{1+l^{2s}} \Big)^{1/2}$$

so $\|u\|_\infty \leq c_s \|u\|_{\sigma,s}$, $\forall\, u \in X_{\sigma,s}$, $\sigma \geq 0$, $s > 1/2$, and $f(x,t,u(x,t))$ is well-defined.

By definition of the norm $\| \ \|_{\sigma,s}$, there exists $C := C(\sigma_0, s) > 0$ such that $\forall \sigma \in [0, \sigma_0]$, $\forall k \in \mathbb{N}$,

$$\Big\| \sum_{l\in\mathbb{Z}} f_{lk}(x) e^{ilt} \Big\|_{\sigma,s} \leq C \Big\| \sum_{l\in\mathbb{Z}} f_{lk}(x) e^{ilt} \Big\|_{2\sigma_0,1} .$$

Next

$$\Big\| \sum_{l\in\mathbb{Z}} f_{lk}(x) e^{ilt} \Big\|^2_{2\sigma_0,1} = \sum_{l\in\mathbb{Z}} \|f_{lk}\|^2_{H^1} (1+l^2) e^{(2\sigma_0)2|l|} =: C_k^2(f) < +\infty$$

by the assumption (F). Therefore $\sum_{l\in\mathbb{Z}} f_{lk}(x) e^{ilt} \in X_{\sigma,s}$ and

$$\Big\| \sum_{l\in\mathbb{Z}} f_{lk}(x) e^{ilt} \Big\|_{\sigma,s} \leq C\, C_k(f) . \tag{5.22}$$

Using the algebra property (5.5) of $X_{\sigma,s}$ and (5.22)

$$\begin{aligned} \|f(u)\|_{\sigma,s} &\leq \sum_{k=0}^{\infty} \Big\| \Big(\sum_{l\in\mathbb{Z}} f_{lk}(x) e^{ilt} \Big) u^k \Big\|_{\sigma,s} \\ &\leq \sum_{k=0}^{\infty} c_s \Big\| \sum_{l\in\mathbb{Z}} f_{lk}(x) e^{ilt} \Big\|_{\sigma,s} \|u^k\|_{\sigma,s} \\ &\leq C \sum_{k=0}^{\infty} C_k(f)\, (c_s \|u\|_{\sigma,s})^k < C \sum_{k=0}^{\infty} C_k(f)\, r^k < +\infty \end{aligned}$$

for $c_s \|u\|_{\sigma,s} < r$, by (F) again.

The analyticity of the composition operator f with respect to $\| \ \|_{\sigma,s}$ follows from the properties of the power series as explained in [96, Appendix A]. □

We underline that the analyticity of f as a map in $X_{\sigma,s}$ is not an assumption but it follows from (F).

Remark 13. *If $f(x,t,u)$ admits an analytic extension, which is 2π-periodic in x and skewsymmetric, namely*

$$f(-x,t,-u) = -f(x,t,u)\,, \tag{5.23}$$

then the Dirichlet problem on $[0,\pi]$ is equivalent to the 2π-periodic problem within the space of all functions odd in x. In this case also the spatial regularity is characterized by the decay properties of the Fourier coefficients. Therefore we could look for analytic solutions of (5.2)

$$u(x,t) = \sum_{l\in\mathbb{Z}} u_l(x) e^{ilt}$$

which are periodic and odd in x, more precisely with

$$u_l(x) \in Y := \Big\{ y(x) = \sum_{j\geq 1} y_j \sin(jx) \mid \sum_{j\geq 1} \exp{(2aj)} j^{2b} |y_j|^2 < +\infty \Big\}$$

for some $a \geq 0$, $b > 1/2$. Without the oddness assumption (5.23) the composition operator f does not map this subspace into itself. It is for this reason that we consider the space $X_{\sigma,s}$ of functions valued in $H^1_0(0,\pi)$: also without (5.23), f sends $X_{\sigma,s}$ into itself (Lemma 20).

Throughout this paper we shall use the spaces $X_{\sigma,s}$ with $\sigma \in [\sigma_0/2, \sigma_0]$ and $s \in \mathcal{S} := \{1,\, 1 - \frac{\tau-1}{2},\, 1 + \frac{(\tau-1)\tau}{2-\tau}\}$. So we choose $\bar{c} := \max_{s\in\mathcal{S}} c_s$ as a multiplicative algebra constant for all the spaces $X_{\sigma,s}$. By Lemma 20, f is analytic in the ball

$$\Big\{ u \in X_{\sigma,s} : \|u\|_{\sigma,s} < R_0 := \frac{r}{\bar{c}} \Big\}$$

and $f, f', f'', \ldots$ are bounded, uniformly in σ, s.

5.4 The bifurcation equation

Now we give some examples in which hypothesis (V) holds.

Lemma 21. *Suppose $f_0(x,u) = u^m$ for $m \geq 3$ odd and $p(x) \equiv 1$. Then, $\forall \mu$, there exists an unbounded sequence of non-degenerate solutions v_n of (5.9).*

Proof. All the solutions of the autonomous equation $-v'' = \mu v^m$ are periodic and can be parametrized by their energy

$$E = \frac{1}{2} v'^2 + \frac{\mu}{m+1} v^{m+1}\,.$$

Let T_E denote the period of the solution v_E. We can suppose $v_E(0) = 0$, so $v'_E(0) = \sqrt{2E}$. The other boundary condition $v_E(\pi) = 0$ is satisfied iff

$$k\frac{T_E}{2} = \pi \qquad \text{for some } k \in \mathbb{N}\,. \tag{5.24}$$

By symmetry and energy conservation $v_E(T_E/4) = [(m+1)E/\mu]^{\frac{1}{m+1}}$. So

$$\begin{aligned} T_E = & 4\int_0^{\left[\frac{(m+1)E}{\mu}\right]^{\frac{1}{m+1}}} \left[2\left(E - \frac{\mu\, x^{m+1}}{m+1}\right)\right]^{-1/2} dx \\ = & \frac{4(m+1/\mu)^{\frac{1}{m+1}}}{E^{\frac{1}{2}-\frac{1}{m+1}}} \int_0^1 \frac{dy}{\sqrt{2(1-y^{m+1})}} =: \frac{C(m,\mu)}{E^{\frac{1}{2}-\frac{1}{m+1}}} \end{aligned}$$

by the change of variable $y = x\,[E(m+1)/\mu]^{-\frac{1}{m+1}}$, and (5.24) is satisfied at infinitely many energy levels. Let $\bar{E} > 0$ such that $T_{\bar{E}} = 2\pi/k$ and denote the solution $\bar{v} := v_{\bar{E}}$.

Let us prove that $\bar{v}$ is non-degenerate. Any solution h of the linearised equation at $\bar{v}$,

$$-h''(x) = \mu m\, \bar{v}^{\,m-1}(x)\, h(x)\,, \tag{5.25}$$

can be written as $h = A\bar{v}' + B\beta$, $A, B \in \mathbb{R}$, because $\bar{v}'(x)$ and $\beta(x) := (\partial_E v_E)_{|E=\bar{E}}(x)$ are solutions of (5.25); they are independent because $\bar{v}'(0) \neq 0$ while $\beta(0) = 0$. If $h(0) = 0$ then $A = 0$. We claim that $\beta(\pi) \neq 0$; as a consequence, if $h(\pi) = 0$, then $B = 0$, and so $h = 0$, i.e. $\bar{v}$ is non-degenerate. To prove that $\beta(\pi) \neq 0$, we differentiate at $\bar{E}$ the identity $v_E(kT_E/2) = 0$,

$$\beta(\pi) + \bar{v}'(\pi)(\partial_E T_E)_{|E=\bar{E}} = 0\,.$$

Since $\bar{v}'(\pi) = (-1)^k \sqrt{2E} \neq 0$ and $\partial_E T_E \neq 0$, we get $\beta(\pi) \neq 0$. □

Lemma 22. *If $f_0(x,0) = d_u f_0(x,0) = 0$, then $\bar{v} = 0$ is a non-degenerate solution of* (5.9) *for every μ.*

Proof. The linearised equation $-(ph')' = 0$, $h(0) = h(\pi) = 0$ has only the trivial solution. □

When hypothesis (V) holds at some $(\mu_0, \bar{v})$, we solve first the bifurcation equation in (5.7) using the standard Implicit function Theorem. We find, for every w small enough and μ in a neighborhood of μ_0, a unique solution $v(\mu, w)$ of the bifurcation equation.

Lemma 23. (Solution of the bifurcation equation). *There exist $0 < R < R_0$, a neighborhood $[\mu_1, \mu_2]$ of μ_0 and a C^∞ map*

$$[\mu_1, \mu_2] \times \left\{ w \in W \cap X_{\sigma,s} \,:\, \|w\|_{\sigma,s} < R \right\} \to V, \quad (\mu, w) \mapsto v(\mu, w)$$

such that $v(\mu, w)$ solves the bifurcation equation in (5.7).

Proof. The linear operator

$$h \mapsto -(ph')' - \mu_0 d_v \Pi_V f(v)[h] = -(ph')' - \mu_0\, f_0'(v)\, h$$

is invertible on $H_0^1(0,\pi)$ by hypothesis (V). Then we apply the Implicit function Theorem. □

Remark 14. *The solutions of the* 0*th order bifurcation equation (5.9) found in Lemmas 21 and 22 are non-degenerate for every* μ*, so, in that case, we can continue* $v(\mu, w)$ *for all* $[\mu_1, \mu_2] = [0, \bar{\mu}]$*.*

We denote by $\lambda_j(\mu, w) := \omega_j^2(\mu, w)$ the eigenvalues of the Sturm-Liouville problem

$$\begin{cases} -(py')' - \mu\Pi_V f'(v(\mu, w) + w)\, y = \lambda\rho y \\ y(0) = y(\pi) = 0\,. \end{cases} \tag{5.26}$$

Lemma 24. *The eigenvalues of (5.26) satisfy the continuity property*

$$|\lambda_j(\mu, w) - \lambda_j(\mu', w')| \leq K\big(|\mu - \mu'| + \|w - w'\|_{\sigma,s}\big) \tag{5.27}$$

for some constant $K > 0$ *independent of* j*.*

Proof. In Section 5.10. □

The non-degeneracy of $\bar{v} = v(\mu_0, 0)$ means that $\lambda_j(\mu_0, 0) \neq 0$, $\forall j$. By (5.27),

$$\delta_0 := \inf \Big\{ \big|\lambda_j(\mu, w)\big| \, : \, j \geq 1, \;\; \mu \in [\mu_1, \mu_2], \;\; \|w\|_{\sigma_0/2} \leq R \Big\} > 0 \tag{5.28}$$

taking, if necessary, $|\mu_2 - \mu_1|$ and R smaller in Lemma 23.

Note also that the index j_0 of the smallest positive eigenvalue is constant, independently on (μ, w).

5.5 The Nash-Moser recursive scheme

It remains to solve the range equation

$$L_\omega w = \mu\Pi_W \mathcal{F}(\mu, w) \tag{5.29}$$

where

$$\mathcal{F}(\mu, w) := f(v(\mu, w) + w)\,.$$

By Lemmas 20 and 23, $\mathcal{F}$ is C^∞ and bounded, together with its derivatives, on $[\mu_1, \mu_2] \times B_R$ where $B_R := \{w \in W \cap X_{\sigma,s} \, : \, \|w\|_{\sigma,s} < R\}$.

We define the sequence of finite-dimensional subspaces

$$W^{(n)} := \Big\{ w = \sum_{1 \leq |l| \leq N_n} w_l(x) e^{ilt} \Big\} \subset W$$

where

$$N_n := N_0 2^n$$

and $N_0 \in \mathbb{N}$ will be fixed in Lemma 28. We also set

$$W^{(n)\perp} := \Big\{ w = \sum_{|l|>N_n} w_l(x) e^{ilt} \in W \Big\}$$

and denote by P_n, resp. $P_n^{\perp}$, the projection on $W^{(n)}$, resp. $W^{(n)\perp}$. Note that $P_n \circ \Pi_W = P_n$.

Lemma 25. (Smoothing estimate). *For $w \in W^{(n)\perp}$, if $0 < \sigma'' < \sigma'$,*

$$\|w\|_{\sigma'',s} \leq \exp[-(\sigma' - \sigma'')N_n] \, \|w\|_{\sigma',s} \,. \tag{5.30}$$

Proof. It follows from the definition of the norms $\| \ \|_{\sigma,s}$ and $W^{(n)\perp}$, see e.g. [48, 28]. □

The key property for the construction of the iterative sequence is the invertibility of the linear operator

$$\begin{aligned} \mathcal{L}_n(w)h &:= -L_\omega h + \mu P_n[d_w \mathcal{F}(\mu, w)h] \\ &= -L_\omega h + \mu P_n\big[f'(v(\mu,w)+w)\big(h + d_w v(\mu,w)[h]\big)\big] \quad \forall\, h \in W^{(n)}. \end{aligned} \tag{5.31}$$

Lemma 26. (Inversion of the linear problem). *Let $\tau \in (1,2)$, $\gamma \in (0,1)$, $\sigma \in (0, \sigma_0]$. Assume $\omega > \gamma$ and the "Melnikov" non-resonance conditions:*

$$\Big|\omega l - \frac{j}{c}\Big| > \frac{\gamma}{l^\tau} \qquad \forall\, l = 1, 2, \ldots, N_n, \quad \forall\, j \geq 1 \tag{5.32}$$

where c is defined in (5.15), and

$$|\omega^2 l^2 - \lambda_j(\mu, w)| > \frac{\gamma\omega}{l^{\tau-1}} \qquad \forall\, l = 1, 2, \ldots, N_n, \ j \geq 1 \tag{5.33}$$

where $\lambda_j(\mu,w)$ are the eigenvalues of (5.26).

Let $u := v(\mu,w) + w$. There exist K_1, K_1' such that, if

$$\frac{\mu}{\gamma^3 \omega} \, \|\Pi_W f'(u)\|_{\sigma, 1 + \frac{\tau(\tau-1)}{2-\tau}} < K_1' \,, \tag{5.34}$$

then $\mathcal{L}_n(w)$ is invertible and

$$\|\mathcal{L}_n(w)^{-1} h\|_\sigma \leq \frac{K_1 N_n^{\tau-1}}{\gamma\omega} \, \|h\|_\sigma \qquad \forall\, h \in W^{(n)} \,. \tag{5.35}$$

Proof. In section 5.9. □

Remark 15. *Condition $\omega > \gamma > 0$ means that equation (5.1) is non-autonomous. Indeed, if $\omega = 0$ the nonlinearity $f(x, \omega t, u) = f(x, 0, u)$ is independent of t.*

For $\vartheta := 3\sigma_0/\pi^2$ we define the sequence

$$\sigma_{n+1} := \sigma_n - \frac{\vartheta}{(n+1)^2}, \qquad \sigma_0 > \sigma_1 > \sigma_2 > \ldots > \frac{\sigma_0}{2}. \tag{5.36}$$

Let A_0 denote the open set

$$A_0 := \Big\{ (\mu,\omega) \in (\mu_1,\mu_2) \times (\gamma,+\infty) \ : \ |\omega l - \omega_j| > \frac{\gamma}{l^\tau} \ \ \forall\, l = 1,\ldots,N_0, \ j \geq 1 \Big\}$$

where ω_j are defined by (5.10).

Lemma 27. (The approximate solution). *There exist K_2, K_2' such that, if $(\mu,\omega) \in A_0$ and $\mu N_0^{\tau-1}/\gamma\omega < K_2'$, then there exists a solution $w_0 := w_0(\mu,\omega) \in W^{(0)}$ of*

$$L_\omega w_0 = \mu P_0 \mathcal{F}(\mu, w_0)$$

satisfying $\|w_0\|_{\sigma_0} \leq \mu K_2 N_0^{\tau-1}/\gamma\omega$.

Proof. By definition of A_0, the eigenvalues of $(1/\rho)L_\omega$ satisfy

$$|\omega^2 l^2 - \lambda_j| > \frac{\gamma\omega}{l^{\tau-1}} \qquad \forall\, l = 1,2,\ldots,N_0, \ \ j \geq 1,$$

so L_ω is invertible on $W^{(0)}$ and, for some K,

$$\|L_\omega^{-1} h\|_{\sigma_0} \leq \frac{K N_0^{\tau-1}}{\gamma\omega} \|h\|_{\sigma_0} \qquad \forall\, h \in W^{(0)}. \tag{5.37}$$

Then we look for a solution $w_0 \in W^{(0)}$ of $w_0 = \mu L_\omega^{-1} P_0 \mathcal{F}(\mu, w_0)$. The right-hand side term is a contraction in $\{\|w_0\|_{\sigma_0} < R\}$ if $\mu N_0^{\tau-1}/\gamma\omega$ is sufficiently small. $\square$

Given $w_n \in W^{(n)}$, $\|w_n\|_{\sigma_n} < R$ and $A_n \subseteq A_0$, we define

$$A_{n+1} := \Big\{ (\mu,\omega) \in A_n \ : \big|\omega l - \omega_j(\mu, w_n)\big| > \frac{\gamma}{l^\tau}, \quad \Big|\omega l - \frac{j}{c}\Big| > \frac{\gamma}{l^\tau} \quad \forall\, l = 1,2,\ldots,N_{n+1}, \ j \geq 1 \Big\} \subseteq A_n \tag{5.38}$$

where $\lambda_j(\mu, w_n) = \omega_j^2(\mu, w_n)$ are defined in (5.26) with $w = w_n$.

In Lemma 27 we have constructed $h_0 := w_0$ for $(\mu,\omega) \in A_0$. Next, we proceed by induction. By means of w_0 we define the set A_1 as above, and we find $w_1 := h_0 + h_1 \in W^{(1)}$ for every $(\mu,\omega) \in A_1$ by Lemma 28 below. Then we define A_2, we find $w_2 \in W^{(2)}$ and so on. The main goal of the construction is to prove that, at the end of the recurrence, the set of parameters $(\mu,\omega) \in \cap_n A_n$ is actually a large set (see Lemmas 31 and 32).

Lemma 28. (Inductive step). *Fix* $\chi := 3/2$. *There exist* $N_0 \in \mathbb{N}$ *(depending only on* $\rho, p, f, \bar{\mu}, \bar{v}, \tau$*) and* $K_3' \leq K_2'/N_0^{\tau-1}$ *with the following property.*

Suppose that $h_i \in W^{(i)}$, $\forall i = 0, \ldots, n$, *satisfy*

$$\|h_i\|_{\sigma_i} < \frac{\mu K_3 N_0^{\tau-1}}{\gamma\omega} \exp(-\chi^i) \tag{5.39}$$

where $K_3 := eK_2$ *and* K_2 *is the constant in Lemma 27;* $\forall k = 0, \ldots, n$, *let* $w_k := h_0 + \ldots + h_k$ *satisfy* $\|w_k\|_{\sigma_k} < R$ *and*

$$L_\omega w_k = \mu P_k \mathcal{F}(\mu, w_k) \tag{5.40}$$

and suppose that $(\mu, \omega) \in A_n$, *where* A_{i+1} *is constructed by means of* w_i *as showed above.*

If $(\mu, \omega) \in A_{n+1}$ *and* $\mu/\gamma^3\omega < K_3'$, *then there exists* $h_{n+1} \in W^{(n+1)}$ *satisfying*

$$\|h_{n+1}\|_{\sigma_{n+1}} < \frac{\mu K_3 N_0^{\tau-1}}{\gamma\omega} \exp(-\chi^{n+1}) \tag{5.41}$$

such that $w_{n+1} = w_n + h_{n+1}$ *verifies* $\|w_{n+1}\|_{\sigma_{n+1}} < R$ *and*

$$L_\omega w_{n+1} = \mu P_{n+1} \mathcal{F}(\mu, w_{n+1})\,. \tag{5.42}$$

Proof. In short $\mathcal{F}(w) := \mathcal{F}(\mu, w)$ and $D\mathcal{F}(w) := d_w\mathcal{F}(\mu, w)$. Equation (5.42) for $w_{n+1} = w_n + h_{n+1}$ is $L_\omega[w_n + h_{n+1}] = \mu P_{n+1}\mathcal{F}(w_n + h_{n+1})$.

By assumption, w_n satisfies (5.40) for $k = n$, namely $L_\omega w_n = \mu P_n \mathcal{F}(w_n)$, so the equation for h_{n+1} can be written as

$$\mathcal{L}_{n+1}(w_n)h_{n+1} + \mu(P_{n+1} - P_n)\mathcal{F}(w_n) + \mu P_{n+1} Q = 0 \tag{5.43}$$

where, as defined in (5.31), $\mathcal{L}_{n+1}(w_n)h_{n+1} := -L_\omega h_{n+1} + \mu P_{n+1} D\mathcal{F}(w_n)h_{n+1}$, and Q denotes the quadratic remainder

$$Q = Q(w_n, h_{n+1}) := \mathcal{F}(w_{n+1}) - \mathcal{F}(w_n) - D\mathcal{F}(w_n)h_{n+1}\,.$$

Step 1: Inversion of $\mathcal{L}_{n+1}(w_n)$. We verify the assumptions of Lemma 26. By definition of A_{n+1}, ω satisfies (5.32). If $\lambda_j(\mu, w_n) < 0$, then $|\omega^2 l^2 - \lambda_j(\mu, w_n)| \geq \omega^2 l^2 > \gamma\omega/l^{\tau-1}$ because $\omega > \gamma$. If $\lambda_j(\mu, w_n) > 0$, we have

$$|\omega^2 l^2 - \lambda_j(\mu, w_n)| \geq |\omega l - \omega_j(\mu, w_n)|\omega l > \frac{\gamma\omega}{l^{\tau-1}} \qquad \forall\, l = 1, \ldots, N_{n+1}$$

because $(\mu, \omega) \in A_{n+1}$. In both cases the non-resonance condition (5.33) holds.

To verify (5.34) we need an estimate for w_n. Let $\eta := \tau(\tau-1)/(2-\tau)$ and $\alpha > 0$. Using the elementary inequality

$$\frac{1 + l^{2(1+\eta)}}{1 + l^2} \cdot \frac{e^{2(\sigma-\alpha)|l|}}{e^{2\sigma|l|}} \leq \frac{2l^{2\eta}}{e^{2\alpha|l|}} \leq 2\max_{y>0}(y^{2\eta}e^{-2\alpha y}) = 2\Big(\frac{\eta}{\alpha e}\Big)^{2\eta}, \quad \forall\, l \neq 0\,,$$

we deduce

$$\|h_i\|_{\sigma_{n+1},1+\eta} \le \frac{C_\eta}{(\sigma_i-\sigma_{n+1})^\eta}\,\|h_i\|_{\sigma_i}$$

where $C_\eta := \sqrt{2}(\eta/e)^\eta$. Since $\sigma_i - \sigma_{n+1} \ge \sigma_i - \sigma_{i+1}$ for every $i \le n$,

$$\|w_n\|_{\sigma_{n+1},1+\eta} \le \sum_{i=0}^{n} \|h_i\|_{\sigma_{n+1},1+\eta} \le C_\eta \sum_{i=0}^{n} \frac{\|h_i\|_{\sigma_i}}{(\sigma_i-\sigma_{i+1})^\eta} \le S_\eta\,\frac{\mu K_3 N_0^{\tau-1}}{\gamma\omega}$$

using (5.39) where $S_\eta := (C_\eta/\vartheta^\eta)\sum_{i=0}^{+\infty}(i+1)^{2\eta}\exp(-\chi^i) < +\infty$. If

$$\frac{S_\eta\,\mu K_3 N_0^{\tau-1}}{\gamma\omega} < R$$

then

$$\|f'(u_n)\|_{\sigma_{n+1},1+\eta} \le K$$

for some K, where $u_n := v(\mu, w_n) + w_n$. Hence hypothesis (5.34) is verified for $\mu/\gamma^3\omega$ sufficiently small.

Analogously we get $\|w_n\|_{\sigma_n} < R$ if $\mu N_0^{\tau-1}/\gamma\omega$ is small enough.

By Lemma 26 the operator $\mathcal{L}_{n+1}(w_n)$ is invertible on $W^{(n+1)}$ and

$$\|\mathcal{L}_{n+1}(w_n)^{-1}h\|_{\sigma_{n+1}} \le \frac{K_1 N_{n+1}^{\tau-1}}{\gamma\omega}\,\|h\|_{\sigma_{n+1}}\,, \quad \forall h \in W^{(n+1)}\,. \tag{5.44}$$

Equation (5.43) amounts to the fixed point problem

$$h_{n+1} = -\mu\mathcal{L}_{n+1}(w_n)^{-1}\left[(P_{n+1}-P_n)\mathcal{F}(w_n) + P_{n+1}Q\right] := \mathcal{G}(h_{n+1})$$

for $h_{n+1} \in W^{(n+1)}$.

Step 2: $\mathcal{G}$ is a contraction. We prove that $\mathcal{G}$ is a contraction on the ball $B_{n+1} := \{\|h\|_{\sigma_{n+1}} < r_{n+1}\}$ where $r_{n+1} := (\mu K_3 N_0^{\tau-1}/\gamma\omega)\exp(-\chi^{n+1})$, implying (5.41). By (5.30)

$$\|(P_{n+1}-P_n)\mathcal{F}(w_n)\|_{\sigma_{n+1}} \le \|\mathcal{F}(w_n)\|_{\sigma_n}\exp[-(\sigma_n-\sigma_{n+1})N_n]\,.$$

Since $\|w_n\|_{\sigma_n} < R$, we have $\|Q\|_{\sigma_{n+1}} \le K\|h_{n+1}\|^2_{\sigma_{n+1}}$. Hence, by (5.44),

$$\|\mathcal{G}(h_{n+1})\|_{\sigma_{n+1}} \le K\,\frac{\mu N_{n+1}^{\tau-1}}{\gamma\omega}\Big(\exp[-(\sigma_n-\sigma_{n+1})N_n] + \|h_{n+1}\|^2_{\sigma_{n+1}}\Big)\,.$$

Therefore $\mathcal{G}(B_{n+1}) \subseteq B_{n+1}$ if

$$\frac{\mu K N_{n+1}^{\tau-1}}{\gamma\omega}\exp[-(\sigma_n-\sigma_{n+1})N_n] < \frac{r_{n+1}}{2}\,, \qquad \frac{\mu K N_{n+1}^{\tau-1}}{\gamma\omega}\,r_{n+1}^2 < \frac{r_{n+1}}{2}\,. \tag{5.45}$$

By the definition of σ_n in (5.36) and $N_n := N_0 2^n$, the first inequality is verified for every $n \ge 0$ if $\sigma_0 N_0$ is greater than a constant depending only on χ, K, K_3. The second inequality is verified for every $n \ge 0$ if $\mu N_0^{\tau-1}/\gamma\omega$ is small enough.

The estimate for $\|\mathcal{G}h - \mathcal{G}k\|$, $h, k \in B_{n+1}$ is similar. The Lemma now follows from the Contraction Mapping Theorem. □

Remark 16. *In the previous scheme h_{n+1} is found as an exact solution of equation (5.43). We find this convenient to prove the regularity of h_{n+1} with respect to the parameters (μ,ω) in Lemma 29. However other schemes are possible. For example, we could define h_{n+1} as a solution of the linearised equation $\mathcal{L}_{n+1}(w_n)h + \mu(P_{n+1} - P_n)\mathcal{F}(w_n) = 0$.*

Corollary 2. ***(Existence)*** *Suppose $A_\infty := \cap_{n\geq 0} A_n \neq \emptyset$. If $(\mu,\omega) \in A_\infty$ and $\mu/\gamma^3\omega < K_3'$, then*

$$w_\infty(\mu,\omega) := \sum_{n\geq 0} h_n(\mu,\omega) \in W \cap X_{\sigma_0/2}$$

is a solution of the range equation (5.29) *satisfying $\|w_\infty\|_{\sigma_0/2} \leq K_\infty \mu/\gamma\omega$ for some K_∞.*

Proof. Since w_n solves (5.40) for $k = n$,

$$-L_\omega w_n + \mu \Pi_W f(u_n) = \mu P_n^\perp f(u_n) \in W^{(n)\perp}$$

where $u_n := v(\mu, w_n) + w_n$. By (5.30)

$$\lim_{n\to+\infty} \| -L_\omega w_n + \mu f(u_n)\|_{\sigma_0/2} \leq \lim_{n\to+\infty} K \exp[-(\sigma_n - \sigma_0/2)N_n] = 0\,.$$

Since $w_n \to w_\infty$ in $\| \ \|_{\sigma_0/2}$ also $f(u_n) \to f(u_\infty)$ in the same norm, while $L_\omega w_n \to L_\omega w_\infty$ in the sense of distributions. So w_∞ is a weak solution of the range equation (5.29). $\square$

Remark 17. *We shall prove, as a consequence of Lemma 31 and section 5.7, that A_∞ is actually a positive measure set. A possible way to prove it uses the Whitney extension of w_∞ of section 5.6.*

5.6 Whitney C^∞ extension

The functions h_n constructed in Lemmas 27 and 28 depend smoothly on the parameters (μ,ω).

Lemma 29. *There exist K_4 and $K_4' \leq K_3'$ such that the maps*

$$h_i : A_i \cap \{\mu/\gamma^3\omega < K_4'\} \to W^{(i)}$$

are C^∞ and

$$\|\partial_\omega h_i(\mu,\omega)\|_{\sigma_i} \leq \frac{K_4\mu}{\gamma^2\omega}\exp(-\chi_0^i), \quad \|\partial_\mu h_i(\mu,\omega)\|_{\sigma_i} \leq \frac{K_4}{\gamma\omega}\exp(-\chi_0^i)$$

where $\chi_0 := (1+\chi)/2 = 5/4$.

Proof. Since $w_0 = \mu L_\omega^{-1} P_0 \mathcal{F}(\mu, w_0)$, by the Implicit function Theorem the map w_0 is C^∞. Differentiating the identity $L_\omega(L_\omega^{-1} h) = h$ w.r.t. ω, by (5.37) we get $\|\partial_\omega L_\omega^{-1} h\|_{\sigma_0} \le (K/\gamma^2\omega)\, \|h\|_{\sigma_0}$. For $\mu/\gamma\omega$ small,

$$\|\partial_\omega w_0\|_{\sigma_0} \le \frac{K\mu}{\gamma^2\omega}\,.$$

Differentiating w.r.t. μ we get $\|\partial_\mu w_0\|_{\sigma_0} \le K'/\gamma\omega$ for some K'.

By induction, suppose that h_i depends smoothly on $(\mu, \omega) \in A_i$ for every $i = 0, \ldots, n$. For $(\mu, \omega) \in A_{n+1}$, by (5.42), h_{n+1} is a solution of

$$-L_\omega h_{n+1} + \mu P_{n+1}[\mathcal{F}(w_n + h_{n+1}) - \mathcal{F}(w_n)] + \mu(P_{n+1} - P_n)\mathcal{F}(w_n) = 0\,. \tag{5.46}$$

By the Implicit function Theorem $h_{n+1} \in C^\infty$ once we prove that

$$\mathcal{L}_{n+1}(w_{n+1})[z] := -L_\omega z + \mu P_{n+1} D\mathcal{F}(w_n + h_{n+1})[z]$$

is invertible. By (5.44), $\mathcal{L}_{n+1}(w_n)$ is invertible. Hence it is sufficient that

$$\Big\| \mathcal{L}_{n+1}^{-1}(w_n)(\mathcal{L}_{n+1}(w_{n+1}) - \mathcal{L}_{n+1}(w_n)) \Big\|_{\sigma_{n+1}} < \frac{1}{2}\,,$$

which holds true for $\mu^2/\gamma\omega$ small enough; indeed, by (5.41),

$$\|\mathcal{L}_{n+1}(w_{n+1}) - \mathcal{L}_{n+1}(w_n)\|_{\sigma_{n+1}} \le K\mu \|h_{n+1}\|_{\sigma_{n+1}} \le \frac{\mu^2 K' N_0^{\tau-1}}{\gamma\omega} \exp(-\chi^{n+1})\,.$$

Finally (5.44) implies

$$\|\mathcal{L}_{n+1}(w_{n+1})^{-1}\|_{\sigma_{n+1}} \le \frac{2K_1 N_{n+1}^{\tau-1}}{\gamma\omega}\,. \tag{5.47}$$

Differentiating (5.46) w.r.t. ω

(5.48)
$$\begin{aligned}\mathcal{L}_{n+1}(w_{n+1})[\partial_\omega h_{n+1}] = {} & 2\omega\rho(x)(h_{n+1})_{tt} - \mu(P_{n+1} - P_n) D\mathcal{F}(w_n)\partial_\omega w_n \\ & - \mu P_{n+1}\big[D\mathcal{F}(w_n + h_{n+1}) - D\mathcal{F}(w_n)\big]\partial_\omega w_n\end{aligned}$$

and, using (5.47) and (5.30),

$$\begin{aligned}\|\partial_\omega h_{n+1}\|_{\sigma_{n+1}} \le {} & \frac{K N_{n+1}^{\tau-1}}{\gamma\omega}\Big(\omega N_{n+1}^2 \|h_{n+1}\|_{\sigma_{n+1}} + \frac{\mu\|\partial_\omega w_n\|_{\sigma_n}}{\exp[(\sigma_n - \sigma_{n+1})N_n]} + \\ & + \mu\|h_{n+1}\|_{\sigma_{n+1}}\, \|\partial_\omega w_n\|_{\sigma_n}\Big)\,.\end{aligned}$$

We note that $\|\partial_\omega w_n\|_{\sigma_n} \le \sum_{i=0}^n \|\partial_\omega h_i\|_{\sigma_i}$. Using (5.45) the sequence $a_n := \|\partial_\omega h_n\|_{\sigma_n}$ satisfies

$$\begin{aligned} a_{n+1} \le {} & \frac{K N_{n+1}^{\tau-1}}{\gamma\omega}\Big(\omega N_{n+1}^2 r_{n+1} + \frac{\omega\gamma r_{n+1}}{N_{n+1}^{\tau-1}} \sum_{i=0}^n a_i + \mu r_{n+1} \sum_{i=0}^n a_i\Big) \\ \le {} & b_{n+1}\Big(1 + \sum_{i=0}^n a_i\Big) \quad \text{where} \quad b_{n+1} := \frac{K\mu}{\gamma^2\omega} N_{n+1}^{\tau+1} \exp(-\chi^{n+1}),\end{aligned}$$

recalling that $r_{n+1} = (\mu K/\gamma\omega)\exp(-\chi^{n+1})$. By induction, for $K\mu/\omega\gamma^2 < 1$, we have $a_n \le 2b_n$ and

$$\|\partial_\omega h_{n+1}\|_{\sigma_{n+1}} \le \frac{K\mu}{\gamma^2\omega}\, N_{n+1}^{\tau+1}\exp(-\chi^{n+1}) \le \frac{K'\mu}{\gamma^2\omega}\,\exp(-\chi_0^{n+1})$$

where $\chi_0 := (1+\chi)/2$. It follows that $\|\partial_\omega w_{n+1}\|_{\sigma_{n+1}} \le K\mu/\gamma^2\omega$.

Differentiating (5.46) w.r.t. μ we obtain the estimate for $\partial_\mu h_{n+1}$. □

Define, for $\nu_0 > 0$,

$$A_n^* := \Big\{(\mu,\omega) \in A_n \,:\, dist\big((\mu,\omega),\, \partial A_n\big) > \frac{\nu_0\gamma^4}{N_n^3}\Big\} \tag{5.49}$$

$$\tilde{A}_n := \Big\{(\mu,\omega) \in A_n \,:\, dist\big((\mu,\omega),\, \partial A_n\big) > \frac{2\nu_0\gamma^4}{N_n^3}\Big\} \subset A_n^*.$$

Lemma 30. (Whitney extension). *There exists a C^∞ map*

$$\tilde{w} : A_0 \cap \Big\{(\mu,\omega) : \frac{\mu}{\gamma^3\omega} < K_4'\Big\} \to W \cap X_{\sigma_0/2}$$

satisfying

$$\|\tilde{w}(\mu,\omega)\|_{\sigma_0/2} \le \frac{K_5\mu}{\gamma\omega}, \quad \|\partial_\omega\tilde{w}(\mu,\omega)\|_{\sigma_0/2} \le \frac{C(\nu_0)\mu}{\gamma^5\omega}, \tag{5.50}$$

$$\|\partial_\mu\tilde{w}(\mu,\omega)\|_{\sigma_0/2} \le \frac{C(\nu_0)}{\gamma^5\omega},$$

for some K_5 and for some $C(\nu_0) > 0$, such that, for $(\mu,\omega) \in \tilde{A}_\infty := \cap_{n\ge 0}\tilde{A}_n$, $\tilde{w}(\mu,\omega)$ solves the range equation (5.29).

Moreover there exists a sequence of C^∞ maps

$$\tilde{w}_n : A_0 \cap \Big\{(\mu,\omega) : \frac{\mu}{\gamma^3\omega} < K_4'\Big\} \to W^{(n)}$$

such that $\tilde{w}_n(\mu,\omega) = w_n(\mu,\omega)$ for $(\mu,\omega) \in \tilde{A}_n$, and

$$\|\tilde{w}(\mu,\omega) - \tilde{w}_n(\mu,\omega)\|_{\sigma_0/2} \le \frac{K_5\mu}{\gamma\omega}\,\exp(-\chi^n)\,. \tag{5.51}$$

Proof. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ be a C^∞ function supported in the open ball $B(0,1)$ of centre 0 and radius 1 and with $\int_{\mathbb{R}^2}\varphi = 1$. Let $\varphi_n : \mathbb{R}^2 \to \mathbb{R}^+$ be the mollifier

$$\varphi_n(x) := \frac{N_n^6}{\nu_0^2\gamma^8}\,\varphi\Big(\frac{N_n^3}{\nu_0\gamma^4}\,x\Big)\,.$$

Supp $(\varphi_n) \subset B(0, \nu_0\gamma^4/N_n^3)$ and $\int_{\mathbb{R}^2}\varphi_n = 1$. We define $\psi_n : \mathbb{R}^2 \to \mathbb{R}$ as

$$\psi_n(x) := \big(\varphi_n * \chi_{A_n^*}\big)(x) = \int_{\mathbb{R}^2}\varphi_n(y-x)\chi_{A_n^*}(y)\,dy$$

where $\chi_{A_n^*}$ is the characteristic function of the set A_n^*. ψ_n is C^∞,

$$|D\psi_n(x)| \le \int_{\mathbb{R}^2} |D\varphi_n(x-y)|\, \chi_{A_n^*}(y)\, dy \le \frac{N_n^3}{\nu_0\gamma^4}\, C \tag{5.52}$$

where $C := \int_{\mathbb{R}^2} |D\varphi|\, dy$,

$$0 \le \psi_n(x) \le 1, \qquad \mathrm{supp}(\psi_n) \subset A_n, \qquad \psi_n(x) = 1 \ \ \forall\, x \in \tilde{A}_n\, .$$

We define, for $(\mu,\omega) \in A_0$, the C^∞ functions

$$\tilde{h}_n(\mu,\omega) := \begin{cases} \psi_n(\mu,\omega) h_n(\mu,\omega) & \text{if} \quad (\mu,\omega) \in A_n \\ 0 & \text{if} \quad (\mu,\omega) \notin A_n \end{cases}$$

and

$$\tilde{w}_n(\mu,\omega) := \sum_{i=0}^{n} \tilde{h}_i\, , \qquad \tilde{w}(\mu,\omega) := \sum_{i\ge 0} \tilde{h}_i$$

which is a series if $(\mu,\omega) \in A_\infty := \cap_{n\ge 0} A_n$.

The estimate for $\|\tilde{w}\|_{\sigma_0/2}$ follows by $\|\tilde{h}_i\|_{\sigma_i} \le \|h_i\|_{\sigma_i}$ (because $0 \le \psi_i \le 1$) and (5.39). The estimates for the derivatives in (5.50) follow differentiating the product $\tilde{h}_i = \psi_i h_i$ and using (5.52), (5.39) and Lemma 29. Similarly it follows that $\tilde{w}$ is in C^∞, see [28] for details.

For $(\mu,\omega) \in \tilde{A}_n$, $\psi_n(\mu,\omega) = 1$, implying $\tilde{w}_n = w_n$. As a consequence, for $(\mu,\omega) \in \tilde{A}_\infty := \cap_{n\ge 0} \tilde{A}_n$, by Corollary 2, $\tilde{w} = w_\infty$ solves (5.29).

Finally, using (5.39),

$$\|\tilde{w} - \tilde{w}_n\|_{\sigma_0/2} \le \sum_{i\ge n+1} \|\tilde{h}_i\|_{\sigma_i} \le \sum_{i\ge n+1} \frac{K\mu}{\gamma\omega} \exp(-\chi^i) \le \frac{K'\mu}{\gamma\omega} \exp(-\chi^n)\, .$$

□

In the next Lemma we fix the constant ν_0 introduced in (5.49).

Lemma 31. *There exist $\nu_0 > 0$ and $K_5' \le K_4'$ such that if $\mu/\gamma^3\omega < K_5'$ then*

$$B_\gamma \subseteq \tilde{A}_n \subset A_n \quad \forall\, n \ge 0$$

where B_γ is defined in (5.14) taking $C' \le K_5'$.

Proof. By induction. Let $(\mu,\omega) \in B_\gamma$. Then $(\mu,\omega) \in \tilde{A}_0$ if A_0 contains the closed ball of centre (μ,ω) and radius $2\nu_0\gamma^4/N_0^3$. Let (ω',μ') belong to such a ball. Then, $\forall l = 1,\ldots,N_0$,

$$|\omega' l - \omega_j| \ge |\omega l - \omega_j| - |\omega - \omega'| l > \frac{2\gamma}{l^\tau} - \frac{2\nu_0\gamma^4}{N_0^3}\, l \ge \frac{\gamma}{l^\tau}$$

if $\nu_0 \le 1/2$.

Suppose now $B_\gamma \subseteq \tilde{A}_n$ and let $(\mu, \omega) \in B_\gamma$. To prove that $(\mu, \omega) \in \tilde{A}_{n+1}$, we have to show that the closed ball of centre (μ, ω) and radius $2\nu_0\gamma^4/N_{n+1}^3$ is contained in A_{n+1}. Let (μ', ω') belong to such a ball. The non-resonance condition on $|\omega' l - j/c|$ is verified, as above, for $\nu_0 \le 1/2$. For the other condition, we denote in short $\omega_j^n(\mu', \omega') := \omega_j(\mu', w_n(\mu', \omega'))$ (see (5.26) for the definition of $\omega_j(\mu, w)$). It results, $\forall l = 1, \ldots, N_{n+1}$,

$$\begin{aligned} |\omega' l - \omega_j^n(\mu', \omega')| &\ge |\omega l - \tilde{\omega}_j(\mu, \omega)| - |\omega - \omega'| l - |\omega_j^n(\mu', \omega') - \tilde{\omega}_j(\mu, \omega)| \\ &> \frac{2\gamma}{l^\tau} - \frac{2\nu_0\gamma^4 l}{N_{n+1}^3} - |\omega_j^n(\mu', \omega') - \tilde{\omega}_j(\mu, \omega)| \\ &> \frac{3\gamma}{2l^\tau} - |\omega_j^n(\mu', \omega') - \tilde{\omega}_j(\mu, \omega)| \qquad (5.53) \end{aligned}$$

if $\nu_0 \le 1/4$. Now we estimate the last term

$$|\omega_j^n(\mu', \omega') - \tilde{\omega}_j(\mu, \omega)| = \frac{|\lambda_j^n(\mu', \omega') - \tilde{\lambda}_j(\mu, \omega)|}{|\tilde{\omega}_j(\mu, \omega)| + |\omega_j^n(\mu', \omega')|} \le \frac{|\lambda_j^n(\mu', \omega') - \tilde{\lambda}_j(\mu, \omega)|}{\sqrt{\delta_0}}$$

by (5.28), both for $j < j_0$ and for $j \ge j_0$. By the comparison principle (5.27)

$$\delta_0^{-1/2} |\lambda_j^n(\mu', \omega') - \tilde{\lambda}_j(\mu, \omega)| \le K|\mu - \mu'| + K\|w_n(\mu', \omega') - \tilde{w}(\mu, \omega)\|_{\sigma_0/2} \,.$$

By Lemma 29, $\|\partial_\omega w_n\|_{\sigma_0/2}$, $\|\partial_\mu w_n\|_{\sigma_0/2} \le K/\gamma^2\omega$ for some other K, and being $\omega, \omega' > \gamma$,

$$K\|w_n(\mu', \omega') - w_n(\mu, \omega)\|_{\sigma_0/2} \le \frac{K'}{\gamma^3} \frac{\nu_0\gamma^4}{N_{n+1}^3} < \frac{\gamma}{8l^\tau} \,, \ \forall l = 1, \ldots, N_{n+1}$$

if ν_0 is small enough $(1 < \tau < 2)$. On the other hand, since $(\mu, \omega) \in \tilde{A}_n$ we have $w_n(\mu, \omega) = \tilde{w}_n(\mu, \omega)$ (Lemma 30) and, by (5.51),

$$K\|w_n(\mu, \omega) - \tilde{w}(\mu, \omega)\|_{\sigma_0/2} \le \frac{K'\mu}{\gamma\omega} \exp(-\chi^n) < \frac{\gamma}{8l^\tau} \,, \ \forall l = 1, \ldots, N_{n+1}$$

for $\mu/\gamma^2\omega$ sufficiently small. By (5.53), collecting the previous estimates,

$$|\omega' l - \omega_j^n(\mu', \omega')| > \frac{\gamma}{l^\tau} \,, \quad \forall l = 1, \ldots, N_{n+1}$$

and (μ', ω') belongs to A_{n+1}. $\square$

5.7 Measure of the Cantor set B_γ

In the following $R := (\mu', \mu'') \times (\omega', \omega'')$ denotes a rectangle contained in the region $\{(\mu, \omega) \in [\mu_1, \mu_2] \times (2\gamma, +\infty) : \mu < K_6'\gamma^5\omega\}$. Furthermore we consider $\omega'' - \omega'$ as a fixed quantity ("of order 1").

Lemma 32. *There exist K_6 and $K_6' \leq K_5'$ such that, taking $C' \leq K_6'$ in the definition* (5.14) *of B_γ, $\forall \mu \in (\mu_1, \mu_2)$ the section*

$$S_\gamma(\mu) := \{\omega : (\mu, \omega) \in B_\gamma\}$$

satisfies the measure estimate

$$|S_\gamma(\mu) \cap (\omega', \omega'')| \geq (1 - K_6\gamma)(\omega'' - \omega') . \tag{5.54}$$

As a consequence, for every $R := (\mu', \mu'') \times (\omega', \omega'')$

$$|B_\gamma \cap R| \geq |R| (1 - K_6\gamma) . \tag{5.55}$$

Proof. We consider the inequalities $|\omega l - \tilde{\omega}_j(\mu, \omega)| > 2\gamma/l^\tau$ in the definition of B_γ. The analogous inequalities are simpler because j/c and ω_j do not depend on (μ, ω).

The complementary set we have to estimate is

$$\mathcal{C} := \bigcup_{l,j \geq 1} \mathcal{R}_{lj}$$

where $\mathcal{R}_{lj} := \{\omega \in (\omega', \omega'') \, : \, |l\omega - \tilde{\omega}_j(\mu, \omega)| \leq 2\gamma/l^\tau\}$.

We claim that

$$\big|\partial_\omega \tilde{\omega}_j(\mu, \omega)\big| \leq \frac{K\mu}{\gamma^5 \omega} . \tag{5.56}$$

Indeed, by the same arguments as in the proof of Lemma 31 and the comparison principle (5.27) we have

$$|\tilde{\omega}_j(\mu, \omega) - \tilde{\omega}_j(\mu, \omega')| \leq K \|\tilde{w}(\mu, \omega) - \tilde{w}(\mu, \omega')\|_{\sigma_0/2} \leq \frac{K\mu}{\gamma^5 \omega} |\omega - \omega'|$$

using (5.50). As a consequence of (5.56)

$$\partial_\omega \big(l\omega - \tilde{\omega}_j(\mu, \omega)\big) \geq l - \frac{K\mu}{\gamma^5 \omega} \geq \frac{l}{2} \qquad \forall l \geq 1$$

for $\mu/\gamma^5\omega$ small enough; we deduce $|\mathcal{R}_{lj}| \leq 4\gamma/l^{\tau+1}$.

Furthermore the set $\mathcal{R}_{lj}$ is non-empty only if

$$\omega' l - \frac{2\gamma}{l^\tau} < \tilde{\omega}_j(\mu, \omega) < \omega'' l + \frac{2\gamma}{l^\tau} .$$

So, for every fixed l, the number of indices j such that $\mathcal{R}_{lj} \neq \emptyset$ is

$$\sharp\{j\} \leq \frac{1}{\delta} \left(l(\omega'' - \omega') + \frac{4\gamma}{l^\tau} \right) + 1 \leq K l (\omega'' - \omega')$$

where

$$\delta := \inf \Big\{ |\tilde{\omega}_{j+1}(\mu,\omega) - \tilde{\omega}_j(\mu,\omega)| \ : \ j \geq 1, \ (\mu,\omega) \in B_\gamma \Big\} .$$

For $\|\tilde{w}\|_{\sigma_0/2} \leq K'\mu/\gamma\omega < R$ we have $\delta \geq \delta_1$ where
(5.57)

$$\delta_1 := \inf \Big\{ \big|\omega_{j+1}(\mu,w) - \omega_j(\mu,w)\big| : \ j \geq 1 \, , \ \mu \in [\mu_1,\mu_2], \ \|w\|_{\sigma_0/2} \leq R \Big\} > 0$$

as proved in Section 5.10.

In conclusion, the measure of the complementary set is

$$|\mathcal{C}| \leq \sum_{l=1}^{+\infty} \frac{4\gamma}{l^{\tau+1}} \, K \, l(\omega'' - \omega') \leq K'(\omega'' - \omega')\gamma$$

and (5.54) is proved. Integrating on (μ', μ'') we obtain (5.55). $\square$

By Fubini's Theorem also the section $S_\gamma(\omega)$ is large, for ω in a large set.

Lemma 33. *Let*

$$S_\gamma(\omega) := \{\mu : \ (\mu,\omega) \in B_\gamma\} \, .$$

For every $R := (\mu', \mu'') \times (\omega', \omega'')$, $\gamma' \in (0,1)$,
(5.58)

$$\Big| \Big\{ \omega \in (\omega', \omega'') : \ \frac{|S_\gamma(\omega) \cap (\mu', \mu'')|}{\mu'' - \mu'} \geq 1 - \gamma' \Big\} \Big| \geq (\omega'' - \omega') \Big(1 - K_6 \frac{\gamma}{\gamma'} \Big) \, .$$

Proof. Consider

$$\begin{aligned} \Omega^+ &:= \big\{ \omega \in (\omega', \omega'') : \ |S_\gamma(\omega) \cap (\mu', \mu'')| \geq (\mu'' - \mu')(1 - \gamma') \big\}, \\ \Omega^- &:= \big\{ \omega \in (\omega', \omega'') : \ |S_\gamma(\omega) \cap (\mu', \mu'')| < (\mu'' - \mu')(1 - \gamma') \big\} \, . \end{aligned}$$

Using Fubini's theorem

$$\begin{aligned} |B_\gamma \cap R| \ &= \ \int_{\omega'}^{\omega''} |S_\gamma(\omega) \cap (\mu', \mu'')| \, d\omega \\ &= \ \int_{\Omega^+} |S_\gamma(\omega) \cap (\mu', \mu'')| \, d\omega + \int_{\Omega^-} |S_\gamma(\omega) \cap (\mu', \mu'')| \, d\omega \\ &\leq \ (\mu'' - \mu')|\Omega^+| + (\mu'' - \mu')(1 - \gamma')|\Omega^-| \, . \end{aligned} \tag{5.59}$$

By (5.55), $|B_\gamma \cap R| \geq (\omega'' - \omega')(\mu'' - \mu')(1 - K_6\gamma)$ and therefore, by (5.59),

$$(\omega'' - \omega')(1 - K_6\gamma) \leq |\Omega^+| + (1 - \gamma')|\Omega^-| = (\omega'' - \omega') - \gamma'|\Omega^-| \tag{5.60}$$

because $|\Omega^+| + |\Omega^-| = \omega'' - \omega'$. Then

$$|\Omega^-| \leq (\omega'' - \omega') K_6 \frac{\gamma}{\gamma'}$$

and, by the first inequality in (5.60), $|\Omega^+| \geq (\omega'' - \omega')(1 - K_6\gamma/\gamma')$, which is (5.58). □

Inequalities (5.54) and (5.58) imply the measure estimates (5.17)-(5.18).

The main conclusions of this section are summarized in the following theorem which follows by Lemmas 30, 31 and 32.

Theorem 6. (Solution of the range equation). *There exist* $\tilde{w} \in C^\infty(A_\gamma, W \cap X_{\sigma_0/2})$ *satisfying (5.50) and the large (see (5.55)) Cantor set* B_γ *defined in* (5.14) *such that, for every* $(\mu, \omega) \in B_\gamma$, *the function* $\tilde{w}(\mu, \omega)$ *solves the range equation* (5.29).

5.8 Proof of Theorems 4 and 5

Proof of Theorem 4. By Theorem 6 for all $(\mu, \omega) \in B_\gamma$ the function $\tilde{w}(\mu, \omega) \in X_{\sigma_0/2}$ solves the range equation (5.29). By Lemma 23, $v(\mu, \tilde{w}(\mu, \omega))$ solves the bifurcation equation in (5.7) and therefore

$$\tilde{u} := v(\mu, \tilde{w}(\mu, \omega)) + \tilde{w}(\mu, \omega) \in X_{\sigma_0/2}$$

is a solution of (5.2). Estimates (5.13) follow by (5.50).

Since $\tilde{u}$ solves

$$-(p(x)\tilde{u}_x)_x = \mu f(x, t, \tilde{u}) - \omega^2 \rho(x) \tilde{u}_{tt} \tag{5.61}$$

we deduce

$$-(p(x)\tilde{u}_x(t, x))_x \in H^1(0, \pi), \qquad \forall t \in \mathbb{R}.$$

This implies, since $p(x) \in H^3(0, \pi)$, that

$$\tilde{u}(t, x) \in H^3(0, \pi) \cap H^1_0(0, \pi) \subset C^2(0, \pi), \quad \forall t \in \mathbb{R}.$$

□

Proof of Theorem 5. For every fixed t, by the algebra property of H^m

$$\big\| f(x, t, u(x, t)) \big\|_{H^m} \leq \sum_{l,k} \| f_{lk}(x) u^k(x) \|_{H^m} \leq K \sum_{l,k} \| f_{lk} \|_{H^m} \, \| u^k \|_{H^m}$$

for some $K > 0$.

Using the Gagliardo-Nirenberg type inequality

$$\| u^k \|_{H^m} \leq (C_m \| u \|_{H^1})^{k-1} \| u \|_{H^m}$$

valid for every $u \in H^1_0 \cap H^m$ (see e.g. [99]-[59]), we get

$$\big\| f(x, t, u(x, t)) \big\|_{H^m} \leq K \| u \|_{H^m} \sum_{l,k} \| f_{lk} \|_{H^m} \big(C_m \| u \|_{H^1} \big)^{k-1} \tag{5.62}$$

which is convergent for $\|u\|_{H^1} < r_m/C_m$ by (5.20).

The solution $\tilde u$ satisfies (5.61) and $\tilde u(\cdot\,,t) \in H^3(0,\pi)$, $\forall t$.

By assumption $\|\tilde u\|_{H^1} < r_m/C_m$. By induction, assume $\tilde u(\cdot\,,t) \in H^k$ for $k = 3,\ldots,m$. Hence $\tilde u_{tt}(\cdot\,,t) \in H^k$ and $\rho(x)\tilde u_{tt}(\cdot\,,t) \in H^k$ because $\rho \in H^m$. Furthermore, by (5.62), $f(x,t,\tilde u) \in H^k$. We deduce, by (5.61), that $p(x)\tilde u_x \in H^{k+1}$. Finally $\tilde u \in H^{k+2}$ because $p \in H^{m+1}$.

If $f_0(x,0) = d_u f_0(x,0) = 0$ then, by Lemma 22, we can take $v(\mu,0) = 0$, $\forall \mu$. Therefore, by (5.13),

$$\|\tilde u(t,\cdot)\|_{H^1} \le \|\tilde u\|_{\sigma_0/2} \le \frac{2C\mu}{\gamma\omega}\,, \quad \forall t\,,$$

and, for $\mu/\gamma\omega$ small enough, we deduce the regularity in (5.21). □

5.9 Inversion of the linearised problem

Here we prove Lemma 26. Decomposing in Fourier series

$$f'(u) = \sum_{k\in\mathbb{Z}} a_k(x)e^{ikt},$$

we write, for $h = \sum_{1\le|l|\le N_n} h_l(x)e^{ilt} \in W^{(n)}$,

$$\begin{aligned}
-L_\omega h + \mu P_n[f'(u)h] &= \sum_{1\le|l|\le N_n} \big[\omega^2 l^2 \rho h_l + \partial_x\big(p\,\partial_x h_l\big)\big]e^{ilt} + \\
&\quad + \mu P_n\Big[\Big(\sum_{k\in\mathbb{Z}} a_k e^{ikt}\Big)\Big(\sum_{1\le|l|\le N_n} h_l e^{ilt}\Big)\Big] \\
&= \sum_{1\le|l|\le N_n} \big[\omega^2 l^2 \rho h_l + \partial_x\big(p\,\partial_x h_l\big) + \mu a_0 h_l\big]e^{ilt} \\
&\quad + \mu \sum_{|l|,|k+l|\in\{1,\ldots,N_n\},\,k\ne 0} a_k\,h_l\,e^{i(k+l)t}.
\end{aligned}$$

Hence $\mathcal{L}_n(w)$ defined in (5.31) can be decomposed as

$$\mathcal{L}_n(w)h = \rho\,\big(Dh + M_1 h + M_2 h\big) \tag{5.63}$$

where

$$\begin{aligned}
Dh &:= \frac{1}{\rho}\sum_{|l|=1}^{N_n} \big[\omega^2 l^2 \rho\, h_l + \big(p\, h_l'\big)' + \mu a_0\, h_l\big]e^{ilt} \\
M_1 h &:= \frac{\mu}{\rho} \sum_{|l|,|k|\in\{1,\ldots,N_n\},l\ne k} a_{k-l}\,h_l\,e^{ikt} \\
M_2 h &:= \frac{\mu}{\rho}\,P_n\big[f'(u)\,d_w v(\mu,w)[h]\big]\,.
\end{aligned} \tag{5.64}$$

Note that D is a diagonal operator in time Fourier basis. To study the eigenvalues of D, we use Sturm-Liouville type techniques.

Lemma 34. (Sturm-Liouville). *The eigenvalues $\lambda_j(\mu, w)$ of the Sturm-Liouville problem (5.26) form a strictly increasing sequence which tends to $+\infty$. Every $\lambda_j(\mu, w)$ is simple and the following asymptotic formula holds*

$$\lambda_j(\mu, w) = \frac{j^2}{c^2} + b + M(\mu, w) + r_j(\mu, w), \qquad |r_j(\mu, w)| \leq \frac{K}{j} \tag{5.65}$$

$\forall j \geq 1$, $(\mu, w) \in [\mu_1, \mu_2] \times B_R$, where

$$c := \frac{1}{\pi} \int_0^\pi \left(\frac{\rho}{p}\right)^{1/2} dx, \qquad b := \frac{1}{4\pi c} \int_0^\pi \left[\frac{(\rho p)'}{\rho \sqrt[4]{\rho p}}\right]' \frac{1}{\sqrt[4]{\rho p}}\, dx,$$
$$M(\mu, w) := -\frac{\mu}{c\pi} \int_0^\pi \frac{\Pi_V f'(v(\mu, w) + w)}{\sqrt{\rho p}}\, dx\,.$$

The eigenfunctions $\varphi_j(\mu, w)$ of (5.26) form an orthonormal basis of $L^2(0, \pi)$ with respect to the scalar product $(y, z)_{L^2_\rho} := c^{-1} \int_0^\pi yz\rho\, dx$. For K big enough

$$(y, z)_{\mu,w} := \frac{1}{c} \int_0^\pi p\, y'z' + \big[K\rho - \mu \Pi_V f'(v(\mu, w) + w)\big]\, yz\, dx$$

defines an equivalent scalar product on $H_0^1(0, \pi)$ and

$$K' \|y\|_{H^1} \leq \|y\|_{\mu,w} \leq K'' \, \|y\|_{H^1} \quad \forall y \in H_0^1\,. \tag{5.66}$$

$\varphi_j(\mu, w)$ is also an orthogonal basis of $H_0^1(0, \pi)$ with respect to the scalar product $(\ ,\ )_{\mu,w}$ and, for $y = \sum_{j \geq 1} \hat{y}_j\, \varphi_j(\mu, w)$,

$$\|y\|^2_{L^2_\rho} = \sum_{j \geq 1} \hat{y}_j^2\,, \qquad \|y\|^2_{\mu,w} = \sum_{j \geq 1} \hat{y}_j^2 \big(\lambda_j(\mu, w) + K\big)\,. \tag{5.67}$$

Proof. In Section 5.10. $\square$

We develop
$$Dh = \sum_{1 \leq |l| \leq N_n} D_l h_l e^{ilt}$$
where
$$D_l z := \frac{1}{\rho} \left[\omega^2 l^2 \rho z + \big(p\, z'\big)' + \mu a_0 z\right], \quad \forall\, z \in H_0^1(0, \pi)$$
and $a_0 = \Pi_V f(v(\mu, w) + w)$.

By Lemma 34 each D_l is diagonal w.r.t the basis $\varphi_j(\mu, w)$:

$$z = \sum_{j=1}^{+\infty} \hat{z}_j\, \varphi_j(\mu, w) \in H_0^1(0, \pi) \quad \Rightarrow \quad D_l z = \sum_{j=1}^{+\infty} \big(\omega^2 l^2 - \lambda_j(\mu, w)\big) \hat{z}_j\, \varphi_j(\mu, w)\,.$$

Lemma 35. *Suppose all the eigenvalues* $\omega^2 l^2 - \lambda_j(\mu, w)$ *are not zero. Then*

$$|D_l|^{-1/2} z := \sum_{j=1}^{+\infty} \frac{\hat{z}_j \, \varphi_j(\mu, w)}{\sqrt{|\omega^2 l^2 - \lambda_j(\mu, w)|}}$$

satisfies

$$\big\| \, |D_l|^{-1/2} z \big\|_{H^1} \leq \frac{K}{\sqrt{\alpha_l}} \, \|z\|_{H^1} \,, \quad \forall z \in H_0^1(0, \pi) \tag{5.68}$$

where $\alpha_l := \min_{j \geq 1} |\omega^2 l^2 - \lambda_j(\mu, w)| > 0$.

Proof. By (5.67) $\| \, |D_l|^{-1/2} z\|^2_{\mu,w} \leq (1/\alpha_l) \, \|z\|^2_{\mu,w}$. Hence (5.68) follows by the equivalence of the norms (5.66). □

Lemma 36. (Inversion of D). *Assume the non-resonance condition (5.33). Then* $|D|^{-1/2} : W^{(n)} \to W^{(n)}$ *defined by*

$$|D|^{-1/2} h := \sum_{1 \leq |l| \leq N_n} |D_l|^{-1/2} h_l \, e^{ilt}$$

satisfies

$$\| \, |D|^{-1/2} h \|_{\sigma,s} \leq \frac{K}{\sqrt{\gamma\omega}} \, \|h\|_{\sigma, s + \frac{\tau-1}{2}} \leq \frac{K N_n^{\frac{\tau-1}{2}}}{\sqrt{\gamma\omega}} \, \|h\|_{\sigma,s} \,, \quad \forall \, h \in W^{(n)} \,.$$

Proof. By (5.68) and $\alpha_{-l} = \alpha_l \geq \gamma\omega / |l|^{\tau-1}$

$$\begin{aligned}
\| \, |D|^{-1/2} h \|^2_{\sigma,s} &= \sum_{1 \leq |l| \leq N_n} \| \, |D_l|^{-1/2} h_l \|^2_{H^1} (1 + l^{2s}) e^{2\sigma|l|} \\
&\leq \sum_{1 \leq |l| \leq N_n} \frac{K^2 |l|^{\tau-1}}{\gamma\omega} \, \|h_l\|^2_{H^1} (1 + l^{2s}) e^{2\sigma|l|} \\
&\leq \frac{K'}{\gamma\omega} \|h\|^2_{\sigma, s + \frac{\tau-1}{2}}
\end{aligned}$$

because $|l|^{\tau-1}(1 + l^{2s}) < 2(1 + |l|^{2s+\tau-1})$, $\forall \, |l| \geq 1$. □

To prove the invertibility of $\mathcal{L}_n(w)$ we write (5.63) as

$$\mathcal{L}_n(w) = \rho |D|^{1/2} \big(U + T_1 + T_2 \big) |D|^{1/2} \tag{5.69}$$

where

$$\begin{cases} U := |D|^{-1/2} \, D \, |D|^{-1/2} \\ T_i := |D|^{-1/2} \, M_i \, |D|^{-1/2} \,, \quad i = 1, 2 \,. \end{cases} \tag{5.70}$$

With respect to the basis $\varphi_j(\mu, w)\, e^{ilt}$ the operator U is diagonal and its (l, j)-th eigenvalue is $\text{sign}(\omega^2 l^2 - \lambda_j(\mu, w)) \in \{\pm 1\}$, implying that the operator norm is

$$\|U\|_\sigma := \sup_{\|h\|_\sigma \le 1} \|Uh\|_\sigma = 1\,. \tag{5.71}$$

The smallness of T_1 requires an analysis of the small divisors. Formula (5.65) implies, by Taylor expansion, the asymptotic dispersion relation

$$\Big|\omega_j(\mu, w) - \frac{j}{c}\Big| \le \frac{K}{j} \tag{5.72}$$

and there exists K such that, for every $x \ge 0$,

$$|x^2 - \lambda_{j^*}(\mu, w)| = \min_{j \ge 1} |x^2 - \lambda_j(\mu, w)| \quad \Rightarrow \quad j^* \ge Kx\,. \tag{5.73}$$

Lemma 37. (Analysis of the small divisors). *Assume the non-resonance conditions (5.32)-(5.33) and $\omega > \gamma$. Then $\forall |k|, |l| \in \{1, \ldots, N_n\}$, $k \ne l$*

$$\alpha_l \alpha_k \ge \left(\frac{K\gamma^3 \omega}{|k-l|^{\frac{\tau(\tau-1)}{2-\tau}}} \right)^2$$

where $\alpha_l := \min_{j \ge 1} |\omega^2 l^2 - \lambda_j(\mu, w)|$.

Proof. Since $\alpha_{-l} = \alpha_l$, $\forall l$, we can suppose $l, k \ge 1$.

We distinguish two cases, if k, l are close or far one from each other. Let $\beta := (2-\tau)/\tau \in (0,1)$.

Case 1. Let $2|k-l| > (\max\{k,l\})^\beta$. By (5.33)

$$\alpha_k \alpha_l \ge \frac{(\gamma\omega)^2}{(kl)^{\tau-1}} \ge \frac{(\gamma\omega)^2}{(\max\{k,l\})^{2(\tau-1)}} \ge \frac{C(\gamma\omega)^2}{|k-l|^{\frac{2(\tau-1)}{\beta}}}\,.$$

Case 2. Let $0 < 2|k-l| \le (\max\{k,l\})^\beta$. In this case $2k \ge l \ge k/2$. Indeed, if $k > l$, then $2(k-l) \le k^\beta$, so $2l \ge 2k - k^\beta \ge k$ because $\beta \in (0,1)$. Analogously if $l > k$.

Let i, resp. j, be an integer which realizes the minimum α_k, resp. α_l, and write in short $\lambda_j(\mu) := \lambda_j(\mu, w)$, $\omega_j(\mu) := \omega_j(\mu, w)$.

If both $\lambda_i(\mu), \lambda_j(\mu) \le 0$, then $\alpha_l \ge \omega^2 l^2$, $\alpha_k \ge \omega^2 k^2$, $\alpha_l \alpha_k \ge \omega^4 > \gamma^2 \omega^2$.

If only $\lambda_j(\mu) \le 0$, then $\alpha_l \alpha_k \ge \gamma \omega^3 l^2 / k^{\tau-1} \ge 2^{1-\tau} \gamma \omega^3 \ge 2^{1-\tau} \gamma^2 \omega^2$.

The really resonant cases happen if $\lambda_i(\mu), \lambda_j(\mu) > 0$.

Suppose, for example, $\max\{k,l\} = k$. By (5.72), $|\omega_j(\mu) - (j/c)| \le K/j$, and, by (5.73), $i \ge K\omega k$, $j \ge K\omega l$. Hence, using also (5.32),

$$\begin{aligned} |(\omega k - \omega_i(\mu)) - (\omega l - \omega_j(\mu))| &= \big|\omega(k-l) - (\omega_i(\mu) - \omega_j(\mu))\big| \\ &\ge \Big|\omega(k-l) - \frac{i-j}{c}\Big| - \frac{K}{\omega l} - \frac{K}{\omega k} \\ &\ge \frac{\gamma}{(k-l)^\tau} - \frac{3K}{\omega k} \ge \frac{2^\tau \gamma}{k^{\beta\tau}} - \frac{3K}{\omega k} \end{aligned}$$

because $2(k-l) \le k^{\beta}$, $2l \ge k$. Since $\beta\tau < 1$ and $k \le 2l$,

$$\big|(\omega k - \omega_i(\mu)) - (\omega l - \omega_j(\mu))\big| \ge \frac{1}{2}\Big(\frac{\gamma}{k^{\beta\tau}} + \frac{\gamma}{l^{\beta\tau}}\Big) \qquad \forall\, k \ge \Big(\frac{K}{\omega\gamma}\Big)^{\frac{1}{1-\beta\tau}} := k^* .$$

The same conclusion if $\max\{k,l\} = l$. It follows that, for $\max\{k,l\} \ge k^*$, there holds $|\omega k - \omega_i(\mu)| \ge \gamma/2k^{\beta\tau}$ or $|\omega l - \omega_j(\mu)| \ge \gamma/2l^{\beta\tau}$. Suppose $|\omega k - \omega_i(\mu)| \ge \gamma/2k^{\beta\tau}$. Then

$$\alpha_k = |\omega^2 k^2 - \omega_i^2(\mu)| \ge |\omega k - \omega_i(\mu)|\omega k \ge \frac{\gamma\omega}{2}\, k^{1-\beta\tau} .$$

Since $l \le 2k$, for α_l we can use (5.33),

$$\alpha_k \alpha_l \ge \frac{\gamma\omega k^{1-\beta\tau}}{2}\,\frac{\gamma\omega}{l^{\tau-1}} \ge \frac{\gamma^2\omega^2}{2^\tau}\, k^{2-\tau-\beta\tau} = \frac{\gamma^2\omega^2}{2^\tau}$$

because $2-\tau-\beta\tau = 0$.

On the other hand, if $\max\{k,l\} < k^* = (K/\omega\gamma)^{1/(\tau-1)}$, we can use (5.33) for both k,l:

$$\alpha_k\alpha_l \ge \frac{(\gamma\omega)^2}{(kl)^{\tau-1}} > \frac{(\gamma\omega)^2}{(k^*)^{2(\tau-1)}} = (\gamma\omega)^2 \Big(\frac{\omega\gamma}{K}\Big)^{\frac{1}{\tau-1}\,2(\tau-1)} > \frac{\gamma^6\omega^2}{K^2}$$

(using $\omega > \gamma$). Since $\gamma < 1$, taking the minimum for all these cases we conclude. $\square$

Lemma 38. (Estimate of T_1). *Assume the non-resonance conditions (5.32)-(5.33), $\omega > \gamma$, and $\Pi_W f'(u) = \sum_{l\neq 0} a_l(x)e^{ilt} \in X_{\sigma,1+\frac{\tau(\tau-1)}{2-\tau}}$. There exists K such that*

$$\|T_1 h\|_\sigma \le \frac{K\mu}{\gamma^3\omega}\, \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau(\tau-1)}{2-\tau}}\, \|h\|_\sigma\,, \quad \forall h \in W^{(n)} .$$

Proof. $\forall h \in W^{(n)}$, $T_1 h = \sum_{1\le|k|\le N_n} (T_1 h)_k\, e^{ikt}$ where

$$\begin{aligned}(T_1 h)_k &= |D_k|^{-1/2}\big(M_1 |D|^{-1/2} h\big)_k \\ &= |D_k|^{-1/2}\Big[\sum_{1\le|l|\le N_n,\, l\neq k} \mu\,\frac{a_{k-l}}{\rho}\, |D_l|^{-1/2} h_l\Big].\end{aligned}$$

Setting $A_m := \|a_m/\rho\|_{H^1}$, using (5.68) and Lemma 37,

$$\|(T_1 h)_k\|_{H^1} \le K\,\mu \sum_{1\le|l|\le N_n,\, l\neq k} \frac{A_{k-l}}{\sqrt{\alpha_k}\sqrt{\alpha_l}}\, \|h_l\|_{H^1} \le \frac{K\mu}{\gamma^3\omega}\, S_k \tag{5.74}$$

where

$$S_k := \sum_{|l|\le N_n,\, l\neq k} A_{k-l}\, |k-l|^{\frac{\tau(\tau-1)}{2-\tau}}\, \|h_l\|_{H^1} .$$

By (5.74) we get, defining $S(t) := \sum_{|k|=1}^{N_n} S_k \, e^{ikt}$,

$$
\begin{aligned}
\|T_1 h\|_\sigma^2 &= \sum_{|k|=1}^{N_n} \|(T_1 h)_k\|_{H^1}^2 (1+k^2) e^{2\sigma|k|} \\
&\leq \Big(\frac{K\mu}{\gamma^3\omega}\Big)^2 \sum_{|k|=1}^{N_n} S_k^2 (1+k^2) e^{2\sigma|k|} = \Big(\frac{K\mu}{\gamma^3\omega}\Big)^2 \|S\|_\sigma^2 \, .
\end{aligned}
$$

Since $S = P_n(\varphi\psi)$, with $\varphi(t) := \sum_{l\in\mathbb{Z}} A_l |l|^{\frac{\tau(\tau-1)}{2-\tau}} e^{ilt}$ and $\psi(t) := \sum_{|l|=1}^{N_n} \|h_l\|_{H^1} e^{ilt}$,

$$
\|T_1 h\|_\sigma \leq \frac{K\mu}{\gamma^3\omega} \|\varphi\|_\sigma \|\psi\|_\sigma \leq \frac{K\mu}{\gamma^3\omega} \Big\|\Pi_W f'(u)\Big\|_{\sigma,1+\frac{\tau(\tau-1)}{2-\tau}} \|h\|_\sigma,
$$

because $\|\varphi\|_\sigma \leq 2\|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau(\tau-1)}{2-\tau}}$ and $\|\psi\|_\sigma = \|h\|_\sigma$. $\square$

Lemma 39. (Estimate of T_2). *Suppose that* $\Pi_W f'(u) \in X_{\sigma,1+\frac{\tau-1}{2}}$. *Then*

$$
\|T_2 h\|_\sigma \leq \frac{K\mu}{\gamma\omega} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} \|h\|_\sigma \, , \qquad \forall\, h \in W^{(n)}
$$

for some K.

Proof. By the definitions (5.70),(5.64) and Lemma 36,

$$
\begin{aligned}
\|T_2 h\|_\sigma &\leq \frac{K}{\sqrt{\gamma\omega}} \|M_2 |D|^{-1/2} h\|_{\sigma,1+\frac{\tau-1}{2}} \\
&\leq \frac{K'\mu}{\sqrt{\gamma\omega}} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} \Big\| d_w v(\mu,w)\big[\, |D|^{-1/2} h\big] \Big\|_{\sigma,1+\frac{\tau-1}{2}} \\
&= \frac{K'\mu}{\sqrt{\gamma\omega}} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} \Big\| d_w v(\mu,w)\big[\, |D|^{-1/2} h\big] \Big\|_{H^1}
\end{aligned}
$$

because $d_w v(\mu,w)[|D|^{-1/2}h] \in V$. By Lemmas 23 and 36

$$
\Big\| d_w v(\mu,w)\big[\, |D|^{-1/2} h\big] \Big\|_{H^1} \leq K \, \|\, |D|^{-1/2} h\|_{\sigma,1-\frac{\tau-1}{2}} \leq \frac{K}{\sqrt{\gamma\omega}} \|h\|_{\sigma,1}
$$

implying the thesis. $\square$

Proof of Lemma 26. By (5.71), $\|U\|_\sigma = 1$. If

$$
\|T_1 + T_2\|_\sigma < \frac{1}{2} \, , \tag{5.75}
$$

then by Neumann series $U + T_1 + T_2$ is invertible in $(W^{(n)}, \| \ \|_\sigma)$ and

$$
\|(U + T_1 + T_2)^{-1}\|_\sigma < 2 \, .
$$

By Lemmas 38 and 39, condition (5.75) is verified if

$$\|T_1\|_\sigma \le \frac{K\mu}{\gamma^3\omega} \, \|\Pi_W f'(u)\|_{\sigma, 1+\frac{\tau(\tau-1)}{2-\tau}} < \frac{1}{4} \tag{5.76}$$

and

$$\|T_2\|_\sigma \le \frac{K\mu}{\gamma\omega} \, \|\Pi_W f'(u)\|_{\sigma, 1+\frac{\tau-1}{2}} \le \frac{K\mu}{\gamma^3\omega} \, \|\Pi_W f'(u)\|_{\sigma, 1+\frac{\tau(\tau-1)}{2-\tau}} < \frac{1}{4} \tag{5.77}$$

(we recall that $\gamma \in (0,1)$ and $(\tau-1)/2 < \tau(\tau-1)/(2-\tau)$ because $\tau > 1$). Both conditions (5.76)-(5.77) are satisfied if

$$\frac{\mu}{\gamma^3\omega} \, \|\Pi_W f'(u)\|_{\sigma, 1+\frac{\tau(\tau-1)}{2-\tau}} < \frac{1}{4K} =: K_1'$$

which is condition (5.34). Hence, inverting (5.69)

$$\mathcal{L}_n(w)^{-1} h = |D|^{-1/2} \big(U + T_1 + T_2\big)^{-1} |D|^{-1/2} \Big(\frac{h}{\rho}\Big)$$

which, using Lemma 36, yields (5.35). □

5.10 Sturm-Liouville problems

Proof of Lemma 34. Let $a(x) \in L^2(0,\pi)$. Under the "Liouville change of variable"

$$x = \psi(\xi) \;\Leftrightarrow\; \xi = g(x), \qquad g(x) := \frac{1}{c} \int_0^x \Big(\frac{\rho(s)}{p(s)}\Big)^{1/2} ds \,, \tag{5.78}$$

we have that $(\lambda, y(x))$ satisfies

$$\begin{cases} -(p(x)y'(x))' + a(x)y(x) = \lambda \rho(x) y(x) \\ y(0) = y(\pi) = 0 \end{cases} \tag{5.79}$$

if and only if $(\nu, z(\xi))$ satisfies

$$\begin{cases} -z''(\xi) + [q(\xi) + \alpha(\xi)]\, z(\xi) = \nu z(\xi) \\ z(0) = z(\pi) = 0 \end{cases} \tag{5.80}$$

where

$$\nu = c^2\lambda, \qquad r(x) = \sqrt[4]{p(x)\,\rho(x)}, \qquad z(\xi) = y(\psi(\xi))\, r(\psi(\xi)),$$
$$\alpha(\xi) = c^2 \, \frac{a(\psi(\xi))}{\rho(\psi(\xi))}\,, \qquad q(\xi) = c^2 Q(\psi(\xi)), \qquad Q = \frac{p}{\rho}\,\frac{r''}{r} + \frac{1}{2}\Big(\frac{p}{\rho}\Big)'\,\frac{r'}{r}\,.$$

By [96], Theorem 4 in Chapter 2, p.35, the eigenvalues of (5.80) form an increasing sequence ν_j satisfying the asymptotic expansion

$$\nu_j = j^2 + \frac{1}{\pi}\int_0^\pi (q+\alpha)\,d\xi \; - \frac{1}{\pi}\int_0^\pi \cos(2j\xi)\big(q(\xi)+\alpha(\xi)\big)\,d\xi + r_j, \quad |r_j| \le \frac{C}{j}$$

where $C := C(\|q+\alpha\|_{L^2})$ is a positive constant. Moreover every ν_j is simple ([96], Theorem 2, p.30).

Since p, ρ are positive and belong to H^3, if $a \in H^1$ then $q, \alpha \in H^1$. Integrating by parts $|\int_0^\pi \cos(2j\xi)(q+\alpha)\,d\xi| \le \|q+\alpha\|_{H^1}/j$ and so

$$\nu_j = j^2 + \frac{1}{\pi}\int_0^\pi (q+\alpha)\,d\xi + r_j', \quad |r_j'| \le \frac{C'}{j}$$

for some $C' := C'(\|q+\alpha\|_{H^1})$. Dividing by c^2 and using the inverse Liouville change of variable we obtain the formula for the eigenvalues $\lambda_j(a)$ of (5.79)

(5.81)
$$\lambda_j(a) = \frac{j^2}{c^2} \, + \, \frac{1}{\pi c}\int_0^\pi \frac{Q\sqrt{\rho}}{\sqrt{p}}\,dx \, + \, \frac{1}{\pi c}\int_0^\pi \frac{a}{\sqrt{\rho p}}\,dx \, + \, r_j(a), \quad |r_j(a)| \le \frac{C}{j}$$

for some $C(\rho, p, \|a\|_{H^1}) > 0$. Then (5.65) follows for $a(x) = -\mu\Pi_V f'(v(\mu,w) +w)(x)$ and some algebra.

By [96], Theorem 7 p.43, the eigenfunctions of (5.80) form an orthonormal basis for L^2. Applying the Liouville change of variable (5.78) in the integrals, the eigenfunctions $\varphi_j(a)$ of (5.79) form an orthonormal basis for L^2 w.r.t. the scalar product $(\;,\;)_{L^2_\rho}$.

Finally, since $\varphi_j := \varphi_j(a)$ solves

$$-(p\varphi_j')' + (K\rho + a)\varphi_j = (\lambda_j(a) + K)\rho\varphi_j\,,$$

multiplying by φ_i and integrating by parts gives

$$(\varphi_j, \varphi_i)_{\mu,w} = \delta_{i,j}(\lambda_j(a) + K)$$

and (5.67) follows (note that $\lambda_j(a) + K > 0$, $\forall j$, for K large enough). □

Proof of Lemma 24. Let $a, b \in H^1(0,\pi)$ and consider $\alpha := c^2 a(\psi)/\rho(\psi)$, $\beta := c^2 b(\psi)/\rho(\psi)$ constructed as above via the Liouville change of variable (5.78). By [96], p.34, for every j

(5.82)
$$|\lambda_j(a) - \lambda_j(b)| = \frac{1}{c^2}|\nu_j(\alpha) - \nu_j(\beta)| \le \frac{1}{c^2}\|\alpha - \beta\|_\infty \le K\|a-b\|_{H^1}$$

and (5.27) follows by the mean value theorem because $\mu\Pi_V f(v(\mu,w)+w)$ has bounded derivatives on bounded sets. □

Proof of (5.57). By the asymptotic formula (5.72)

$$\min_{j\ge1} |\omega_{j+1}(\mu,w) - \omega_j(\mu,w)| \ge \frac{1}{c} - \frac{2K}{j} > \frac{1}{2c}$$

if $j > 4Kc$, uniformly in $\mu \in [\mu_1, \mu_2]$, $w \in B_R$. For $1 \leq j \leq 4Kc$ the minimum

$$m_j := \min_{(\mu,w)\in[\mu_1,\mu_2]\times B_R} |\omega_{j+1}(\mu, w) - \omega_j(\mu, w)|$$

is attained because $a \mapsto \lambda_j(a)$ is a compact function on H^1 by the compact embedding $H^1(0, \pi) \hookrightarrow L^\infty(0, \pi)$ and by (5.82) (see also [96], Theorem 3 p.31 and p.34). Each $m_j > 0$ because all the eigenvalues λ_j are simple. □

Chapter 6

Periodic solutions of forced Kirchhoff equations

We consider the Kirchhoff equation

$$u_{tt} - \Delta u\Big(1 + \int_\Omega |\nabla u|^2\,dx\Big) = \varepsilon g(x,t) \qquad x \in \Omega,\ t \in \mathbb{R} \tag{6.1}$$

where g is a time-periodic external forcing with period $2\pi/\omega$ and amplitude ε, and the displacement $u : \Omega \times \mathbb{R} \to \mathbb{R}$ is the unknown. We consider both Dirichlet boundary conditions

$$u(x,t) = 0 \quad \forall x \in \partial\Omega,\ t \in \mathbb{R} \tag{6.2}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded, connected open set with smooth boundary, $d \geq 1$, and periodic boundary conditions on $\mathbb{R}^d$

$$u(x,t) = u(x + 2\pi m, t) \quad \forall m \in \mathbb{Z}^d,\ x \in \mathbb{R}^d,\ t \in \mathbb{R} \tag{6.3}$$

where $\Omega = (0,2\pi)^d$. We look for solutions u with period $2\pi/\omega$.

Normalising the time $t \to \omega t$ and rescaling $u \to \varepsilon^{1/3} u$, (6.1) becomes

$$\omega^2 u_{tt} - \Delta u = \mu\Big(\Delta u \int_\Omega |\nabla u|^2\,dx + g(x,t)\Big) \tag{6.4}$$

where $\mu := \varepsilon^{2/3}$ and g, u are 2π-periodic.

6.1 Case of Dirichlet boundary conditions

Assume that $\partial\Omega$ is C^∞. Let $\lambda_j^2, \varphi_j(x)$, $j = 1, 2, \ldots$ be the eigenvalues and eigenfunctions of the boundary-value problem

$$\begin{cases} -\Delta\varphi_j = \lambda_j^2\,\varphi_j & \text{in } \Omega \\ \varphi_j = 0 & \text{on } \partial\Omega \end{cases}$$

with $\int_\Omega \varphi_j^2\,dx = 1$ and $\lambda_1 < \lambda_2 \le \dots$. Weyl's formula for the asymptotic distribution of the eigenvalues gives $\lambda_j = O(j^{1/d})$ as $j \to \infty$, thus

$$C j^{1/d} \le \lambda_j \le C' j^{1/d} \quad \forall j = 1, 2, \dots \tag{6.5}$$

for some positive C, C' depending on the dimension d and on the domain Ω (see e.g. [104, Vol. IV, XIII.15]).

By expansion in the basis $\{\varphi_j(x)\}$, we define the spaces

$$V_{\sigma,s}(\Omega) := \Big\{ v(x) = \sum_j v_j \varphi_j(x) : \sum_j |v_j|^2 \lambda_j^{2s}\, e^{2\sigma\lambda_j} < \infty \Big\}$$

for $s \ge 0$, $\sigma \ge 0$. Spaces $V_{0,s}$ with $\sigma = 0$ are used in [10]. They are the domains of the fractional powers $\Delta^{s/2}$ of the Laplace operator. See [10, 60] for a characterisation. For instance, $V_{0,2} = H^2(\Omega) \cap H_0^1(\Omega)$. We note that if $u \in V_{0,s}(\Omega)$ then $\Delta^k u \in H_0^1(\Omega)$ for all $0 \le k \le (s-1)/2$.

Spaces $V_{\sigma,0}$ with $s = 0$ are used in [9], where it is proved that $\cup_{\sigma>0} V_{\sigma,0}$ is the class of the $(-\Delta)$-analytic functions, that is, by definition, the set of functions $v(x) \in H_0^1(\Omega)$ such that

$$\Delta^k v \in H_0^1(\Omega) \quad \text{and} \quad \Big| \int_\Omega v \Delta^k v \, dx \Big|^{1/2} \le C A^k k! \qquad \forall k = 0, 1, \dots$$

for some constants C, A. In [9] it is observed that an important subset of $\cup_{\sigma>0} V_{\sigma,0}$ consists of the functions $v(x)$, analytic on some neighbourhood of $\overline{\Omega}$, such that

$$\Delta^k v = 0 \quad \text{on } \partial\Omega \qquad \forall k = 0, 1, \dots$$

This subset coincides with the whole class of $(-\Delta)$-analytic functions when $\partial\Omega$ is a real analytic manifold of dimension $(d-1)$, leaving Ω on one side [78], or when Ω is a parallelepiped [7].

Clearly $V_{\sigma,s} = \{u \in V_{\sigma,0} : \Delta^{s/2} u \in V_{\sigma,0}\}$ and $V_{\sigma,0} \subset V_{\sigma',s} \subset V_{\sigma',0}$ for all $s > 0$, $\sigma > \sigma' > 0$. Moreover, all finite sums $\sum_{j \le N} v_j \varphi_j(x)$ belong to $V_{\sigma,s}$ for all σ, s.

We set the problem in the spaces $X_{\sigma,s} = H^1(\mathbb{T}, V_{\sigma,s})$ of 2π-periodic functions $u : \mathbb{T} \to V_{\sigma,s}$, $t \mapsto u(\cdot, t)$ with H^1 regularity, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, namely

$$X_{\sigma,s} := \Big\{ u(x,t) = \sum_{j \ge 1} u_j(t) \varphi_j(x) : u_j \in H^1(\mathbb{T}, \mathbb{R}),$$
$$\|u\|_{\sigma,s}^2 := \sum_{j \ge 1} \|u_j\|_{H^1}^2 \lambda_j^{2s}\, e^{2\sigma\lambda_j} < \infty \Big\}.$$

Theorem 7. (Case of Dirichlet boundary conditions). *Suppose that $g \in X_{\sigma,s_0}$ for some $\sigma \ge 0$, $s_0 > 2d$. Let $s_1 \in (1 + d, 1 + s_0/2)$. There exist positive constants δ, C with the following properties.*

For every $\gamma \in (0,\lambda_1)$ there exists a Cantor set $\mathcal{A}_\gamma \subset (0,+\infty)\times(0,\delta\gamma)$ of parameters such that for every $(\omega,\mu)\in\mathcal{A}_\gamma$ there exists a classical solution $u(\omega,\mu)\in X_{\sigma,s_1}$ of (6.4)(6.2). Such a solution satisfies

$$\|u(\omega,\mu)\|_{\sigma,s_1} \le \frac{\mu}{\gamma}\,C, \qquad \|u(\omega,\mu)_{tt}\|_{\sigma,s_1-2} \le \frac{\mu}{\gamma\omega^2}\,C$$

and it is unique in the ball $\{\|u\|_{\sigma,s_1} < 1\}$.

The set $\mathcal{A}_\gamma$ satisfies the following Lebesgue measure property: for every $0<\bar\omega_1<\bar\omega_2<\infty$ there exists a constant $\bar C$ independent on γ such that in the rectangular region $\mathcal{R}_\gamma := (\bar\omega_1,\bar\omega_2)\times(0,\delta\gamma)$ there holds

$$\frac{|\mathcal{R}_\gamma\cap\mathcal{A}_\gamma|}{|\mathcal{R}_\gamma|} > 1-\bar C\,\gamma\,.$$

We recall that (6.4) is obtained from (6.1) by the normalisation $t\to\omega t$ and the rescaling $u\to\varepsilon^{1/3}u$. Hence, going back, the solution $u(\omega,\mu)$ of (6.4) found in Theorem 7 gives a solution of (6.1) of order ε and period $2\pi/\omega$.

Remark 18. Theorem 7 covers both Sobolev and analytic cases:

- *(Sobolev regularity).* If g belongs to the Sobolev space X_{0,s_0}, then the solution u found in the theorem belongs to the Sobolev space X_{0,s_1}.
- *(Analytic regularity).* If g belongs to the analytic space $X_{\sigma_0,0}$, then $g\in X_{\sigma_1,s_0}$ for all $\sigma_1\in(0,\sigma_0)$. Indeed,

  $$\frac{\xi^{s_0}}{\exp[(\sigma_0-\sigma_1)\xi]} \le \Big(\frac{s_0}{(\sigma_0-\sigma_1)e}\Big)^{s_0} =: C \quad \forall\xi\ge 0\,,$$

  therefore

  $$\|g\|^2_{\sigma_1,s_0} = \sum_j \|g_j\|^2_{H^1}\lambda_j^{2s_0}e^{2\sigma_1\lambda_j}\,\frac{e^{2\sigma_0\lambda_j}}{e^{2\sigma_0\lambda_j}} \le C^2\|g\|^2_{\sigma_0,0}\,.$$

  Since $g\in X_{\sigma_1,s_0}$, the solution u found in the theorem belongs to the analytic space $X_{\sigma_1,s_1}\subset X_{\sigma_1,0}$.

Remark 19. If $g(x,\cdot)\in H^r(\mathbb{T})$, $r\ge 1$, then the solution u of (6.1) found in the theorem satisfies $u(x,\cdot)\in H^{r+2}(\mathbb{T})$ by bootstrap.

Remark 20. *(Nonplanar vibrations).* We can consider the Kirchhoff equation for a string in the 3-dimensional space

$$u_{tt} - u_{xx}\Big(1+\int_0^\pi |u_x|^2\,dx\Big) = \varepsilon g(x,t), \qquad g = \binom{g_1}{g_2}, \quad u = \binom{u_1}{u_2} \tag{6.6}$$

where the forcing g and the displacement u are $\mathbb{R}^2$-vectors belonging to the plane orthogonal to the rest position of the string, see [45, 88]. In this case nonplanar vibrations of the string are permitted.

Setting $\|u_j\|^2_{H^1} := \|u_{1,j}\|^2_{H^1} + \|u_{2,j}\|^2_{H^1}$ in the definition of the spaces $X_{\sigma,s}$, Theorem 7 holds true for problem (6.6) as well.

6.2 Case of periodic boundary conditions

The eigenvalues and eigenfunctions of the Laplacian on $\mathbb{T}^d$ are $|m|^2$, $e^{im\cdot x}$ for $m \in \mathbb{Z}^d$. We consider a bijective numbering $\{m_j : j \in \mathbb{N}\}$ of $\mathbb{Z}^d$ such that $|m_j| \le |m_{j+1}|$ for all $j \in \mathbb{N} = \{0, 1, \dots\}$, and we denote

$$\tilde\lambda_j^2 := |m_j|^2, \quad \tilde\varphi_j(x) := e^{im_j\cdot x} \qquad \forall j \in \mathbb{N}.$$

We note that $\tilde\lambda_0 = 0$, $\tilde\varphi_0(x) \equiv 1$ and $\tilde\lambda_j \ge 1$ for all $j \ge 1$. Weyl's estimate (6.5) holds true for $\tilde\lambda_j$ as well, because the number of integer vectors $m \in \mathbb{Z}^d$ such that $|m| \le \lambda$ is $O(\lambda^d)$ for $\lambda \to +\infty$ (see [104, Vol. IV, XIII.15]).

We define

$$\tilde X_{\sigma,s} := \Big\{ u(x,t) = \sum_{j\ge 0} u_j(t)\tilde\varphi_j(x) \,:\, u_j \in H^1(\mathbb{T},\mathbb{R}), \\ \|u\|_{\sigma,s}^2 := \|u_0\|_{H^1}^2 + \sum_{j\ge1} \|u_j\|_{H^1}^2 \tilde\lambda_j^{2s}\, e^{2\sigma\tilde\lambda_j} < \infty \Big\}.$$

Theorem 8. (Case of periodic boundary conditions). *Suppose that $g \in \tilde X_{\sigma,s_0}$ for some $\sigma \ge 0$, $s_0 > 2d$, and*

$$\int_{(0,2\pi)^{d+1}} g(x,t)\, dxdt = 0\,. \tag{6.7}$$

Let $s_1 \in (1+d, 1+s_0/2)$. There exist positive constants δ, C with the following properties.

For every $\gamma \in (0,1)$ there exists a Cantor set $\mathcal{A}_\gamma \subset (0,+\infty)\times(0,\delta\gamma)$ of parameters such that for every $(\omega,\mu) \in \mathcal{A}_\gamma$ there exists a classical solution $u(\omega,\mu) \in \tilde X_{\sigma,s_1}$ of (6.4)(6.3) satisfying

$$\int_{(0,2\pi)^{d+1}} u(\omega,\mu)(x,t)\, dxdt = 0\,.$$

Such a solution satisfies

$$\|u(\omega,\mu)\|_{\sigma,s_1} \le \frac{\mu}{\gamma}\Big(1+\frac{1}{\omega^2}\Big) C\,, \qquad \|u(\omega,\mu)_{tt}\|_{\sigma,s_1-2} \le \frac{\mu}{\gamma\omega^2} C \tag{6.8}$$

and it is unique in the ball $\{\int_{(0,2\pi)^{d+1}} u(x,t)\, dxdt = 0,\ \|u\|_{\sigma,s_1} < 1\}$.

The set $\mathcal{A}_\gamma$ satisfies the same measure property of Theorem 7.

Remark 21. If $u(\omega,\mu)$ is a solution of (6.4)(6.3), then also $u(\omega,\mu)+c$, $c \in \mathbb{R}$, solves (6.4)(6.3).

6.3 Outline of the proof

The rest of the paper is devoted to the proof of the theorems. In Sections 6.4, 6.5, 6.6 and 6.7 we develop the details for the proof of Theorem 7, then the same calculations are used to prove Theorem 8 in Section 6.8.

In Section 6.4 we perform the Nash-Moser iteration to construct the approximating sequence (u_n), for μ small and (ω, μ) belonging to smaller and smaller "non-resonant" sets A_n. Avoiding resonances allows to invert the linearised operator at each step of the iteration.

In Section 6.5 we prove that u_n converges to a solution of the Kirchhoff equation if $(\omega, \mu) \in A_n$ for all n. Local uniqueness of the solution is also proved.

In Section 6.6 we prove that the intersection of all A_n is a nonempty set, which is very large in a Lebesgue measure sense.

In Section 6.7 we prove the invertibility of the linearised operator for $(\omega, \mu) \in A_n$ and we give an estimate on the inverse operator.

In Section 6.8 we complete the proof of Theorem 7 and we prove Theorem 8.

6.4 The iteration scheme

We fix $\sigma \geq 0$ once for all. In the following, we write in short $X_s := X_{\sigma,s}$, $\|u\|_s := \|u\|_{\sigma,s}$. We remark that all the following calculations holds true both in the Sobolev case $\sigma = 0$ and in the analytic case $\sigma > 0$. Indeed, the only index used in the present Nash-Moser method is s.

We set the iterative scheme in the Banach spaces X_s endowed with the smoothing operators P_n, defined in the following way. We consider a constant $\chi \in (1, 2)$ and denote

$$N_n := \exp(\chi^n) \tag{6.9}$$

for all $n \in \mathbb{N}$. We define the finite-dimensional space

$$W^{(n)} := \Big\{ u(x,t) = \sum_{\lambda_j \leq N_n} u_j(t) \varphi_j(x) \Big\}$$

and indicate P_n the projector onto $W^{(n)}$ (truncation operator). For all $s, \alpha \geq 0$ there holds the smoothing properties

$$\|P_n u\|_{s+\alpha} \leq N_n^\alpha \|u\|_s \quad \forall u \in X_s \tag{6.10}$$

$$\|(I - P_n)u\|_s \leq N_n^{-\alpha} \|u\|_{s+\alpha} \quad \forall u \in X_{s+\alpha} \tag{6.11}$$

where I is the identity map. We denote

$$L_\omega := \omega^2 \partial_{tt} - \Delta\,, \quad f(u) := \Delta u \int_\Omega |\nabla u|^2 \, dx,$$
$$F(u) := L_\omega u - \mu f(u) - \mu g$$

so that (6.4) can be written as

$$F(u) = 0\,. \tag{6.12}$$

Note that f is not a composition operator, because of the presence of the integral. The map f is cubic: indeed $f(u) = A[u,u,u]$ where A is the three-linear map $A[u,v,w] = \Delta u \int_\Omega \nabla v \circ \nabla w \; dx$. Moreover, since the integral term $\int_\Omega |\nabla u|^2 dx$ depends only on time, there holds

$$f(u) \in W^{(n)} \quad \forall u \in W^{(n)}.$$

The quadratic remainder of f at u is

$$\begin{aligned} Q(u,h) &:= f(u+h) - f(u) - f'(u)[h] \\ &= \Delta u \int_\Omega |\nabla h|^2 dx + \Delta h \int_\Omega \big(2\nabla u \circ \nabla h + |\nabla h|^2\big)\, dx. \end{aligned} \tag{6.13}$$

We observe that, if $a(t)$ depends only on time, then

$$\|a(t)u(x,t)\|_s \le \|a\|_{H^1} \|u\|_s$$

(we omit a factor given by the algebra constant of $H^1(\mathbb{T})$). As a consequence, by Hölder inequality it is easy to estimate $\|f(u)\|_s$, $\|f'(u)[h]\|_s$ and $\|Q(u,h)\|_s$.

We adapt the Newton's scheme with smoothing operators P_n to the special structure of problem (6.4). We will construct a sequence (u_n) defining

$$u_0 := 0, \quad u_{n+1} := u_n - F'(u_n)^{-1}[L_\omega u_n - \mu f(u_n) - \mu P_{n+1} g] \tag{6.14}$$

provided the linearised operator

$$F'(u_n) : h \mapsto L_\omega h - \mu f'(u_n)[h]$$

admits a bounded inverse $F'(u_n)^{-1}$ on $X^{(n+1)}$. In this inversion problem a small divisors difficulty arises. We will prove (Lemma 40) that $F'(u_n)$ can be inverted if the parameters (ω, μ) belong to some "nonresonant" set A_{n+1}, defined as follows. First,

$$A_0 := (0, +\infty) \times (0,1).$$

By induction, suppose we have constructed A_n and u_n. We denote

$$a_n(t) := \int_\Omega |\nabla u_n|^2 \, dx, \tag{6.15}$$

we consider the Hill's eigenvalue problem

$$\begin{cases} y'' + p^2\big(1 + \mu a_n(t)\big)\, y = 0 \\ y(t) = y(t + 2\pi) \end{cases}$$

and indicate $(p_l^{(n)})^2$ its eigenvalues, $l \in \mathbb{N}$. For

$$\tau > d, \quad \gamma \in (0, \lambda_1)$$

we define

$$(6.16) \quad A_{n+1} := \Big\{ (\omega, \mu) \in A_n \, : \, |\omega p_l^{(n)} - \lambda_j| > \frac{\gamma}{\lambda_j^\tau} \quad \forall \lambda_j \le N_{n+1}, \;\; l \in \mathbb{N} \Big\} .$$

Remark 22. Note that for all μ, n the set $A_n(\mu) := \{\omega : (\omega, \mu) \in A_n\}$ is open. Indeed, for every $0 < \bar\omega_1 < \bar\omega_2 < \infty$ the intersection $(\bar\omega_1, \bar\omega_2) \cap A_n(\mu)$ is defined by means of finitely many strict inequalities (see (6.5) and (6.50)).

We fix a positive constant R such that, if $u \in X_1$ and $\|u\|_1 < R$, then $a(t) := \int_\Omega |\nabla u|^2 \, dx$ satisfies $\|a\|_{H^1} < 1$ and $\|a\|_\infty < 1/2$.

Lemma 40. (Inversion of the linearised operator). *There exist two universal constants K_1, K_1' with the following property. Let $u \in X^{(n)}$ with $\|u\|_1 < R$. Let $(\omega, \mu) \in A_{n+1}$. If*

$$(6.17) \qquad \frac{\mu}{\gamma} \, \|u\|_{\tau+1}^2 < K_1' \, ,$$

then $F'(u)$ is invertible, $F'(u)^{-1} : X^{(n+1)} \to X^{(n+1)}$ and

$$(6.18) \qquad \|F'(u)^{-1} h\|_0 \le \frac{K_1}{\gamma} \, \|h\|_{\tau-1} \quad \forall h \in X^{(n+1)} \, .$$

Proof. In Section 6.7. □

Lemma 41. (Construction of the approximating sequence). *Let $g \in Y_{s_0}$, $s_0 > 2d$. Let $\tau \in (d, s_0/2)$. There exist a choice for χ in the definition* (6.9) *and positive constants K, b, δ_0, with $b(2-\chi) > \tau+1$, satisfying the following properties.*

(First step). *If $(\omega, \mu) \in A_1$ and $\mu/\gamma < \delta_0$, then there exists $u_1 \in X^{(0)}$ defined by* (6.14)*, and there holds*

$$(6.19) \qquad \|u_1\|_0 < K \frac{\mu}{\gamma} \, \exp(-b\chi) \, .$$

(Induction step). *Suppose we have constructed $u_1, \ldots, u_n$ by* (6.14) *for $(\omega, \mu) \in A_n$, $n \ge 1$, where each A_{k+1} is defined by means of u_k by* (6.16)*, and $u_k \in X^{(k)}$. Suppose that $\mu/\gamma < \delta_0$. Let*

$$h_{k+1} := u_{k+1} - u_k .$$

Suppose that for all $k = 1, \ldots, n$ there holds

$$(6.20) \qquad \|h_k\|_0 < K \frac{\mu}{\gamma} \, \exp(-b\chi^k) \, .$$

If $(\omega, \mu) \in A_{n+1}$ then there exists $h_{n+1} \in X^{(n+1)}$ defined by (6.14) *and there holds*

$$(6.21) \qquad \|h_{n+1}\|_0 < K \frac{\mu}{\gamma} \, \exp(-b\chi^{n+1}) \, .$$

Proof. *(First step).* Since $u_0 = 0$ and $(\omega, \mu) \in A_1$, by Lemma 40 $F'(0)$ is invertible and (6.14) defines

$$u_1 = -F'(0)^{-1}[F(0) + \mu(I - P_1)g] = \mu L_\omega^{-1} P_1 g\,. \tag{6.22}$$

By (6.18), the inequality (6.19) holds true provided

$$K_1 \|g\|_{s_0} < K \exp(-b\chi), \quad \tau - 1 \le s_0\,. \tag{6.23}$$

(Induction step). To define h_{n+1} by (6.14), we have to verify the hypotheses of Lemma 40. By (6.10) and (6.20)

$$\|h_k\|_{\tau+1} \le N_k^{\tau+1} \|h_k\|_0 \le K \frac{\mu}{\gamma} \exp[(-b + \tau + 1)\chi^k] \tag{6.24}$$

because $h_k \in X^{(k)}$. Then

$$\|u_n\|_{\tau+1} \le \sum_{k=1}^{n} \|h_k\|_{\tau+1} < K \frac{\mu}{\gamma} \sum_{k=1}^{+\infty} \exp[(-b + \tau + 1)\chi^k] \tag{6.25}$$

which is finite for $b > \tau + 1$. Thus condition (6.17) is verified provided

$$b > \tau + 1, \quad K^2 \left(\frac{\mu}{\gamma}\right)^3 C_0^2 < K_1', \quad C_0 := \sum_{k \ge 1} \exp[(-b + \tau + 1)\chi^k]\,. \tag{6.26}$$

Since $\|u_n\|_1 \le \|u_n\|_{\tau+1}$, by (6.25) we have $\|u_n\|_1 < R$ provided

$$K \frac{\mu}{\gamma} C_0 < R\,. \tag{6.27}$$

Since $(\omega, \mu) \in A_{n+1}$, we can apply Lemma 40 and we define h_{n+1} according to the scheme, namely

$$h_{n+1} := -F'(u_n)^{-1}[F(u_n) + \mu(I - P_{n+1})g]\,. \tag{6.28}$$

By (6.18) we have

$$\|h_{n+1}\|_0 \le \frac{K_1}{\gamma} \|F(u_n) + \mu(I - P_{n+1})g\|_{\tau-1}\,. \tag{6.29}$$

By construction (6.14), u_n satisfies

$$F'(u_{n-1})h_n = -L_\omega u_{n-1} + \mu f(u_{n-1}) + \mu P_n g.$$

By Taylor expansion $F(u_n) = F(u_{n-1}) + F'(u_{n-1})h_n - \mu Q(u_{n-1}, h_n)$, where Q is defined in (6.13). Thus

$$F(u_n) = -\mu[(I - P_n)g + Q(u_{n-1}, h_n)] \tag{6.30}$$

and (6.29) gives

$$\|h_{n+1}\|_0 \le \frac{K_1\mu}{\gamma}\,\|(P_{n+1}-P_n)g + Q(u_{n-1},h_n)\|_{\tau-1}\,. \tag{6.31}$$

Now $(P_{n+1}-P_n)g = (I-P_n)P_{n+1}g$, then by (6.11)

$$\|(P_{n+1}-P_n)g\|_{\tau-1} \le \frac{1}{N_n^\beta}\,\|P_{n+1}g\|_{\tau-1+\beta} \le \frac{1}{N_n^\beta}\,\|g\|_{s_0}$$

for

$$\tau-1+\beta \le s_0, \quad \beta>0\,.$$

To estimate $\|Q(u_{n-1},h_n)\|_{\tau-1}$, we note that

$$\|\Delta u_{n-1}\int_\Omega |\nabla h_n|^2\|_{\tau-1} \;\le\; \|u_{n-1}\|_{\tau+1}\|h_n\|_1^2 \;<\; K\,\frac{\mu}{\gamma}\,C_0\,\|h_n\|_1^2$$

by (6.24), and $\|h_n\|_1 \le N_n\|h_n\|_0$ by (6.10). For the second term, recalling that $2<\tau+1$,

$$\begin{aligned}\|\Delta h_n\int_\Omega \nabla(2u_{n-1}+h_n)\circ\nabla h_n\|_{\tau-1} \;&\le\; \|h_n\|_{\tau+1}\|h_n\|_0\|2u_{n-1}+h_n\|_2\\ &<\; 2K\,\frac{\mu}{\gamma}\,C_0\,\|h_n\|_{\tau+1}\|h_n\|_0\end{aligned}$$

by (6.24), and $\|h_n\|_{\tau+1} \le N_n^{\tau+1}\|h_n\|_0$ by (6.10). Then

$$\|Q(u_{n-1},h_n)\|_{\tau-1} < 3K\,\frac{\mu}{\gamma}\,C_0N_n^{\tau+1}\|h_n\|_0^2\,.$$

As a consequence, (6.21) holds true provided

$$K_1\,\frac{1}{N_n^\beta}\,\|g\|_{s_0} < \frac{1}{2}\,K\exp(-b\chi^{n+1}) \tag{6.32}$$

and

$$3K_1\,\frac{\mu}{\gamma}\,C_0\,N_n^{\tau+1}\|h_n\|_0^2 < \frac{1}{2}\,\exp(-b\chi^{n+1})\,. \tag{6.33}$$

Condition (6.32) is satisfied for

$$\beta > b\chi, \qquad K > \frac{2K_1\|g\|_{s_0}}{\exp[(\beta-b\chi)\chi]} \tag{6.34}$$

and, by (6.20), condition (6.33) is satisfied for

$$b(2-\chi) > \tau+1, \qquad \frac{\mu}{\gamma} < \left\{\frac{\exp[(b(2-\chi)-\tau-1)\chi]}{6K_1C_0K^2}\right\}^{1/3}. \tag{6.35}$$

Since $2\tau < s_0$, we can fix $\chi \in (1,2)$ so close to 1 that

$$\tau - 1 + (\tau+1)\frac{\chi}{2-\chi} < s_0 \,.$$

Now we fix b such that

$$b(2-\chi) > \tau + 1 \,, \qquad \tau - 1 + b\chi < s_0$$

and then we fix β as

$$\beta = s_0 - \tau + 1 \,.$$

So (6.34) and (6.23) are satisfied for K big enough, and we fix K in such a way. Then (6.26),(6.27) and (6.35) are satisfied for μ/γ small enough. □

6.5 The solution

Lemma 42. (Existence of a solution). *Assume the hypotheses of Lemma 41 and suppose that $(\omega, \varepsilon) \in A_n$ for all $n \in \mathbb{N}$. Then the sequence (u_n) constructed in Lemma 41 converges in $X_{\tau+1}$ to $u_\infty := \sum_{k\geq 1} h_k$. u_∞ is a solution of* (6.12) *and*

$$\|u_\infty\|_{\tau+1} \leq \frac{\mu}{\gamma} C \tag{6.36}$$

for some C. Moreover, $(u_n)_{tt}$ converges to $(u_\infty)_{tt}$ in $X_{\tau-1}$,

$$\|(u_\infty)_{tt}\|_{\tau-1} \leq \frac{\mu}{\gamma\omega^2} C \tag{6.37}$$

so u_∞ is a classical solution of (6.4).

Proof. By (6.24), the series $\sum_k \|h_k\|_{\tau+1}$ converges, u_n converges to u_∞ in $X_{\tau+1}$ and (6.36) holds true.

By (6.22) there holds $\omega^2(h_1)_{tt} = \Delta h_1 + \mu P_0 g$. By (6.10) and (6.20)

$$\|\Delta h_1\|_{\tau-1} + \mu\|P_0 g\|_{\tau-1} \leq \frac{K\mu}{\gamma} \exp[(-b+\tau+1)\chi] + \mu\|g\|_{s_0},$$

so that

$$\|(h_1)_{tt}\|_{\tau-1} \leq C\,\frac{\mu}{\gamma\omega^2} \tag{6.38}$$

for some C (recall that $\gamma < \lambda_1$). For $n \geq 1$, by (6.28) and (6.30)

$$F'(u_n)h_{n+1} = \mu[(P_{n+1} - P_n)g + Q(u_{n-1}, h_n)], \tag{6.39}$$

thus

$$\omega^2(h_{n+1})_{tt} = \Delta h_{n+1} + \mu\big(f'(u_n)[h_{n+1}] + (P_n - P_{n-1})g + Q(u_{n-1}, h_n)\big).$$

By (6.24),(6.31),(6.32) and (6.33) we get

$$\|(h_{n+1})_{tt}\|_{\tau-1} \le C\,\frac{\mu}{\gamma\omega^2}\,\exp[(-b+\tau+1)\chi^{n+1}]. \tag{6.40}$$

It follows that $(u_n)_{tt}$ converges in $X_{\tau-1}$, $(u_\infty)_{tt} \in X_{\tau-1}$, so that u_∞ has regularity $H^3 \subset C^2$ in time and (6.37) holds true. As a consequence $F(u_n)$ converges to $F(u_\infty)$ in $X_{\tau-1}$.

On the other hand, by (6.30),(6.32) and (6.33)

$$\|F(u_n)\|_{\tau-1} < \frac{K\mu}{K_1}\,\exp(-b\chi^{n+1}),$$

and $F(u_n) \to 0$ in $X_{\tau-1}$. Then $F(u_\infty) = 0$. □

Remark 23. We will prove in Lemma 45 that the set $\{(\omega,\mu) \in A_n\ \forall n \in \mathbb{N}\}$ is nonempty and has positive, large measure. As a consequence, the sequence (u_n) of Lemma 42 is defined for all (ω,μ) in that large set.

Lemma 43. (Uniqueness of the solution). *Assume the hypotheses of Lemma 42. There exists $\delta_1 \in (0,\delta_0]$ such that, for $\mu/\gamma < \delta_1$, u_∞ is the unique solution of* (6.12) *in the ball* $\{v \in X_{\tau+1} : \|v\|_{\tau+1} < 1\}$.

Proof. Suppose v is another solution of (6.12), with $\|v\|_{\tau+1} < 1$. Let $v_n := P_n v$. Projecting the equation $F(v) = 0$ on $X^{(n)}$ gives

$$L_\omega v_n = \mu(f(v_n) + R_n(v) + P_n g), \qquad R_n(v) := \Delta v_n \int_\Omega |\nabla(v - v_n)|^2\,dx.$$

Since u_n solves (6.14), that is

$$F'(u_n)h_{n+1} = -L_\omega u_n + \mu(f(u_n) + P_{n+1}g),$$

the difference $w_n := v_n - u_n$ satisfies

$$L_\omega w_n - \mu(f(v_n) - f(u_n)) - \mu R_n(v) + \mu(P_{n+1} - P_n)g = F'(u_n)h_{n+1}\,.$$

Since $f(v_n) - f(u_n) = f'(u_n)[w_n] + Q(u_n, w_n)$, applying $F'(u_n)^{-1}$

$$w_n = h_{n+1} + \mu F'(u_n)^{-1}[Q(u_n,w_n) + R_n(v) - (P_{n+1} - P_n)g]\,. \tag{6.41}$$

Now, by (6.18)

$$\|\mu F'(u_n)^{-1}Q(u_n,w_n)\|_0 \le K_1\,\frac{\mu}{\gamma}\,\|Q(u_n,w_n)\|_{\tau-1}\,.$$

By assumption and (6.36)

$$\|w_n\|_2 \le \|w_n\|_{\tau+1} \le \|v\|_{\tau+1} + \|u_\infty\|_{\tau+1} < C$$

and by (6.24) $\|u_n\|_2 \le \|u_n\|_{\tau+1} < C$, so that $\|Q(u_n, w_n)\|_{\tau-1} \le C\|w_n\|_0$ and

$$\|\mu F'(u_n)^{-1} Q(u_n, w_n)\|_0 \le K_1 \frac{\mu}{\gamma} C \|w_n\|_0 \le \frac{1}{2} \|w_n\|_0$$

provided μ/γ is small enough. Thus (6.41) gives

$$\frac{1}{2} \|w_n\|_0 \le \|h_{n+1}\|_0 + K_1 \frac{\mu}{\gamma} \|R_n(v) - (P_{n+1} - P_n)g\|_{\tau-1} .$$

By (6.21) and (6.11) the right-hand side tends to 0 as $n \to \infty$, so that $\|v_n - u_n\|_0 \to 0$. Since v_n converges to v and u_n to u_∞ in X_0, it follows that $v = u_\infty$. □

6.6 The Cantor set of parameters

Lemma 44. (Regular dependence on the parameter ω). *Assume the hypotheses of Lemma 41. There exist $\delta_2 \in (0, \delta_1]$ such that all the maps*

$$h_n : A_n \cap \{(\omega, \mu) : \mu/\gamma < \delta_2\} \to X^{(n)} , \quad (\omega, \mu) \mapsto h_n(\omega, \mu)$$

are differentiable w.r.t. ω and

$$\|\partial_\omega u_n\|_0 \le C \frac{\mu}{\gamma^2 \omega} \tag{6.42}$$

for some C.

Proof. $u_1 = h_1 \in X^{(1)}$ is defined for $(\omega, \mu) \in A_1$ and it solves $\omega^2 (h_1)_{tt} = \Delta h_1 + \mu P_0 g$. Recalling Remark 22 and Lemma 40, by the classical implicit function theorem it follows that h_1 is differentiable w.r.t. ω. Differentiating w.r.t. ω gives

$$2\omega (h_1)_{tt} + L_\omega[\partial_\omega h_1] = 0.$$

We apply L_ω^{-1} and by (6.18) and (6.38)

$$\|\partial_\omega h_1\|_0 \le C \frac{\mu}{\gamma^2 \omega}$$

for some C.

Assume that for $n \ge 1$

$$\|\partial_\omega h_k\|_0 \le \bar{C} \frac{\mu}{\gamma^2 \omega} \exp[(-b + \tau + 1)\chi^k] \quad \forall k = 1, \dots, n. \tag{6.43}$$

h_{n+1} solves (6.39), then it is differentiable w.r.t. ω. Differentiating (6.39) gives

$$F'(u_n)[\partial_\omega h_{n+1}] = -2\omega (h_{n+1})_{tt} + \mu f''(u_n)[\partial_\omega u_n, h_{n+1}] + \mu \partial_\omega (Q(u_{n-1}, h_n)).$$

We apply $F'(u_n)^{-1}$ and observe that

$$\|F'(u_n)^{-1}[\omega(h_{n+1})_{tt}]\|_0 \le \frac{K_1\omega}{\gamma}\,\|(h_{n+1})_{tt}\|_{\tau-1} \le \frac{C\mu}{\gamma^2\omega}\,\exp[(-b+\tau+1)\chi^{n+1}]$$

by (6.40). To estimate

$$K_1\frac{\mu}{\gamma}\,\|f''(u_n)[\partial_\omega u_n, h_{n+1}] + \partial_\omega(Q(u_{n-1},h_n))\|_{\tau-1} \tag{6.44}$$

we write all the integral terms and apply (6.10) and (6.43) to each of them. We write here the calculations for two terms, the other ones are analogous. First,

$$\|\Delta(\partial_\omega u_n)\int_\Omega \nabla u_n \circ \nabla h_{n+1}\|_{\tau-1} \le \|\partial_\omega u_n\|_{\tau+1}\|u_n\|_2\|h_{n+1}\|_0 \tag{6.45}$$

and $\|\partial_\omega u_n\|_{\tau+1} \le \|\partial_\omega u_n\|_0\, N_n^{\tau+1}$,

$$\begin{aligned}
&\|\partial_\omega u_n\|_0 \le \sum_{k=1}^{n}\|\partial_\omega h_k\|_0 \le C\,\frac{\mu}{\gamma^2\omega}\,\sum_{k\ge1}\exp[(-b+\tau+1)\chi^k] = \frac{C'\mu}{\gamma^2\omega}\,,\\
&\|u_n\|_2 \le \|u_n\|_{\tau+1} < C\mu/\gamma \quad \text{by (6.24)},\\
&N_n^{\tau+1}\|h_{n+1}\|_0 \le \frac{K\mu}{\gamma}\,\exp[(-b+\tau+1)\chi^{n+1}] \quad \text{by (6.21)},
\end{aligned}$$

so that (6.45) $\le C(\mu/\gamma)^2(\mu/\gamma^2\omega)\exp[(-b+\tau+1)\chi^{n+1}]$. As second example,

$$\|\Delta h_{n+1}\int_\Omega \nabla u_n \circ \nabla(\partial_\omega u_n)\|_{\tau-1} \le \|h_{n+1}\|_0\, N_{n+1}^{\tau+1}\|u_n\|_2\|\partial_\omega u_n\|_0\,.$$

At the end we have

$$(6.44) \le C\left(\frac{\mu}{\gamma}\right)^3\frac{\mu}{\gamma^2\omega}\,\exp[(-b+\tau+1)\chi^{n+1}]$$

for some C, so that (6.43) holds true for $k=n+1$ provided μ/γ is small enough, independently on n.

Finally, $\|\partial_\omega u_n\|_0 \le \sum_{k=1}^{n}\|\partial_\omega h_k\|_0$ and (6.43) implies (6.42). □

Lemma 45. (The Cantor set). *There exist $\delta_3 \le \delta_2$ such that the Cantor set $\mathcal{A}_\gamma := A_\infty \cap \{(\omega,\mu) : \mu < \delta_3\gamma\}$, $\gamma \in (0,\lambda_1)$, has the following measure property.*

For every interval $I = (\bar\omega_1, \bar\omega_2)$ with $0 < \bar\omega_1 < \bar\omega_2 < \infty$ there is a constant $\bar C$ depending on I such that, denoted by $\mathcal{R}_\gamma$ the rectangular region $\mathcal{R}_\gamma = I \times (0,\delta_3\gamma)$,

$$\frac{|\mathcal{R}_\gamma \cap \mathcal{A}_\gamma|}{|\mathcal{R}_\gamma|} > 1 - \bar C\gamma\,.$$

Proof. We fix μ, we recall that $A_n(\mu) := \{\omega : (\omega,\mu) \in A_n\}$ and define

$$E_n := A_n(\mu) \setminus A_{n+1}(\mu), \quad n \in \mathbb{N}.$$

We have to prove that $\cup_{n\in\mathbb{N}} E_n$ has small measure. As a consequence, its complementary set $A_\infty(\mu) := \cap_{n\in\mathbb{N}} A_n(\mu)$ will be a large set. Let

$$\Omega^n_{j,l} := \Big\{ \omega : |\omega p_l^{(n)}(\omega,\mu) - \lambda_j| \le \frac{\gamma}{\lambda_j^\tau} \Big\} .$$

We note that $\Omega^n_{j,0} = \emptyset$ for all j, n because $\gamma < \lambda_1$ and $p_l^{(n)} = 0$ for $l = 0$. Suppose that $\omega \in \Omega^n_{j,l}$. Then $C\lambda_j < \omega l < C'\lambda_j$ for some C, C' by (6.50). Moreover

$$|\partial_\omega p_l^{(n)}| \le 2l\mu \|\partial_\omega a_n\|_\infty$$

by (6.51). Recalling the definition (6.15) of a_n, by (6.42) and (6.24)

$$\|\partial_\omega a_n\|_{H^1} = \|2\int_\Omega \nabla u_n \circ \nabla(\partial_\omega u_n)\, dx\|_{H^1} \le 2\|u_n\|_2 \|\|\partial_\omega u_n\|_0 \le C\,\frac{\mu^2}{\gamma^3\omega}$$

for some C, so that

$$|\partial_\omega p_l^{(n)}| \le C\, l\, \frac{\mu^3}{\gamma^3\omega} .$$

By (6.50) it follows that

$$\partial_\omega(\omega\, p_l^{(n)}(\omega,\mu)) \ge p_l^{(n)} - \omega C\, l\, \frac{\mu^3}{\gamma^3\omega} > \frac{l}{4} \tag{6.46}$$

provided μ/γ is small enough, say $\mu/\gamma < \delta_3$. Fix $0 < \bar\omega_1 < \bar\omega_2 < \infty$. If $\Omega^n_{j,l} \cap (\bar\omega_1, \bar\omega_2)$ is nonempty, then

$$|\Omega^n_{j,l}| < \frac{8\gamma}{l\lambda_j^\tau} < C\bar\omega_2\, \frac{\gamma}{\lambda_j^{\tau+1}}\,, \quad l \in \Big(\frac{C'}{\bar\omega_2}\,\lambda_j\,,\, \frac{C''}{\bar\omega_1}\,\lambda_j\Big) =: \Lambda(j) \tag{6.47}$$

for some C, C', C''. Since $E_0 = \cup_{\lambda_j \le N_1,\, l\ge 1} \Omega^0_{j,l}$, we have the estimate

$$|E_0 \cap (\bar\omega_1,\bar\omega_2)| \le \sum_{\lambda_j \le N_1} \sum_{l\in\Lambda(j)} |\Omega^0_{j,l}| < \gamma\tilde C \sum_{\lambda_j\le N_1} \frac{1}{\lambda_j^\tau} \tag{6.48}$$

for some $\tilde C$ depending on $(\bar\omega_1,\bar\omega_2)$.

To estimate $|E_n \cap (\bar\omega_1,\bar\omega_2)|$, $n \ge 1$, we notice that

$$E_n = \bigcup_{\lambda_j \le N_{n+1},\, l\ge 1} \Omega^n_{j,l} \cap A_n(\mu) .$$

For the sets $\Omega^n_{j,l}$ with $N_n < \lambda_j \le N_{n+1}$ we use (6.47) and we get

$$\Big| \bigcup_{\substack{N_n<\lambda_j\le N_{n+1}\\ l\in\Lambda(j)}} \Omega^n_{j,l} \cap (\bar\omega_1,\bar\omega_2) \Big| < \gamma\tilde C \sum_{N_n<\lambda_j\le N_{n+1}} \frac{1}{\lambda_j^\tau}$$

where $\tilde{C}$ is the constant of (6.48). To estimate the remaining sets, suppose that $\omega \in \Omega^n_{j,l}$ for some $\lambda_j \le N_n$, $l \ge 1$. Then by (6.51)

$$\begin{aligned} |\lambda_j - \omega p_l^{(n-1)}| &\le |\lambda_j - \omega p_l^{(n)}| + \omega\, |p_l^{(n)} - p_l^{(n-1)}| \\ &\le \frac{\gamma}{\lambda_j^\tau} + 2\omega l \mu \|a_n - a_{n-1}\|_\infty . \end{aligned}$$

Since $\omega l \le C\lambda_j$ and $\|a_n - a_{n-1}\|_{H^1} \le \|h_n\|_0 \|2u_{n-1} + h_n\|_2$, by (6.20),(6.24) we have

$$|\lambda_j - \omega p_l^{(n-1)}| \le \frac{\gamma}{\lambda_j^\tau} + C\lambda_j \frac{\mu^3}{\gamma^2} \exp(-b\chi^n).$$

Thus

$$\Omega^n_{j,l} \cap A_n(\mu) \subseteq \Big\{ \omega \,:\, \frac{\gamma}{\lambda_j^\tau} < |\lambda_j - \omega p_l^{(n-1)}| \le \frac{\gamma}{\lambda_j^\tau} + C\lambda_j \frac{\mu^3}{\gamma^2} \exp(-b\chi^n) \Big\}$$

and by (6.46)

$$|\Omega^n_{j,l} \cap A_n(\mu)| \le C\bar{\omega}_2 \frac{\mu^3}{\gamma^2} \exp(-b\chi^n) .$$

It follows that

$$\Big| \bigcup_{\substack{\lambda_j \le N_n \\ l \in \Lambda(j)}} \Omega^n_{j,l} \cap A_n(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2) \Big| \le \gamma C \frac{\mu^3}{\gamma^3} \sum_{\lambda_j \le N_n} \lambda_j \exp(-b\chi^n)$$

for some C depending on $(\bar{\omega}_1, \bar{\omega}_2)$. Now, by (6.5) $\lambda_j \le N_n$ implies $j \le (N_n/C)^d$, then

$$\sum_{\lambda_j \le N_n} \lambda_j \le \sum_{j \le (N_n/C)^d} C' j^{1/d} \le C'' \int_0^{(N_n/C)^d} \xi^{1/d} d\xi \le C''' N_n^{d+1}$$

for some C'''. As a consequence

$$\sum_{\lambda_j \le N_n} \lambda_j \exp(-b\chi^n) \le C \exp[(-b + d + 1)\chi^n].$$

Then

$$|E_n \cap (\bar{\omega}_1, \bar{\omega}_2)| \le \gamma C \Big(\sum_{N_n < \lambda_j \le N_{n+1}} \frac{1}{\lambda_j^\tau} + \exp[(-b + d + 1)\chi^n] \Big)$$

for some C, and

$$\Big| \bigcup_{n \in \mathbb{N}} E_n \cap (\bar{\omega}_1, \bar{\omega}_2) \Big| \le \gamma C \Big(\sum_{j \ge 1} \frac{1}{\lambda_j^\tau} + \sum_{n \in \mathbb{N}} \exp[(-b + d + 1)\chi^n] \Big) .$$

The first series converges because by (6.5)

$$\sum_{j\geq 1} \frac{1}{\lambda_j^\tau} \leq C \sum_{j\geq 1} \frac{1}{j^{\tau/d}} < \infty$$

being $\tau > d$. The second series converges because $b > d+1$. Thus

$$\Big| \bigcup_{n\in\mathbb{N}} E_n \cap (\bar{\omega}_1, \bar{\omega}_2) \Big| \leq C\gamma$$

and the relative measure of $A_\infty(\mu)$ in $(\bar{\omega}_1, \bar{\omega}_2)$ satisfies

$$\frac{|A_\infty(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2)|}{\bar{\omega}_2 - \bar{\omega}_1} \geq 1 - C\gamma$$

for some C.

Finally, we integrate in μ in the interval where $\mu/\gamma < \delta_3$,

$$|A_\infty \cap R_\gamma| = \int_0^{\delta_3\gamma} |A_\infty(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2)| \, d\mu \, .$$

□

6.7 Inversion of the linearised operator

In this section we prove Lemma 40. Let $u \in X^{(n)}, h \in X^{(n+1)}$. The linearised operator is

$$F'(u)h = L_\omega h - \mu f'(u)[h] = Dh + Sh$$

where we split $F'(u)$ in a diagonal part

$$Dh := \omega^2 h_{tt} - \Delta h \Big(1 + \mu \int_\Omega |\nabla u|^2 \, dx\Big)$$

and a "projection" part

$$Sh := -\mu \Delta u \int_\Omega 2\nabla u \circ \nabla h \, dx \, .$$

We recall here some results on Hill's problems. The proof is in Section 6.9.

Lemma 46. (Hill's problems). *Let $\alpha(t)$ be 2π-periodic and $\|\alpha\|_\infty < 1/2$. The eigenvalues p^2 of the periodic problem*

$$\begin{cases} y'' + p^2(1+\alpha(t))\, y = 0 \\ y(t) = y(t+2\pi) \end{cases} \tag{6.49}$$

form a sequence $\{p_l^2\}_{l\in\mathbb{N}}$ *such that*

$$\frac{1}{3}\,l \le p_l \le 2l \quad \forall l \in \mathbb{N}. \tag{6.50}$$

For α, β 2π*-periodic,* $\|\alpha\|_\infty, \|\beta\|_\infty < 1/2$*,*

$$\big|p_l(\alpha) - p_l(\beta)\big| \le 2l\,\|\alpha - \beta\|_\infty \quad \forall l \in \mathbb{N}. \tag{6.51}$$

The eigenfunctions $\psi_l(t)$ *of* (6.49) *form an orthonormal basis of* $L^2(\mathbb{T})$ *w.r.t. the scalar product*

$$(u,v)_{L^2_\alpha} = \int_0^{2\pi} uv(1+\alpha)\,dt$$

and also an orthogonal basis of $H^1(\mathbb{T})$ *w.r.t. the scalar product*

$$(u,v)_{H^1_\alpha} = \int_0^{2\pi} u'v'\,dt + (u,v)_{L^2_\alpha}.$$

The corresponding norms are equivalent to the standard Sobolev norms,

$$\frac{1}{2}\,\|y\|_{L^2} \le \|y\|_{L^2_\alpha} \le 2\,\|y\|_{L^2}, \quad \frac{1}{2}\,\|y\|_{H^1} \le \|y\|_{H^1_\alpha} \le 2\,\|y\|_{H^1}. \tag{6.52}$$

Lemma 47. (Inversion of D). *Let* $u \in X^{(n)}$*,* $a(t) := \int_\Omega |\nabla u|^2\,dx \in H^1(\mathbb{T})$*,* $\|a\|_{H^1} < 1$*,* $\|a\|_\infty < 1/2$*. Let* p_l^2 *be the eigenvalues of the Hill's problem*

$$\begin{cases} y'' + p^2(1+\mu a(t))\,y = 0 \\ y(t) = y(t+2\pi)\,. \end{cases} \tag{6.53}$$

If (ω,μ) *satisfy the non-resonant condition*

$$|\omega p_l - \lambda_j| > \frac{\gamma}{\lambda_j^\tau} \quad \forall \lambda_j \le N_{n+1}, \quad l \in \mathbb{N}\,,$$

then D *is invertible,* $D^{-1} : X^{(n+1)} \to X^{(n+1)}$ *and*

$$\|D^{-1}h\|_0 \le \frac{C}{\gamma}\,\|h\|_{\tau-1} \quad \forall h \in X^{(n+1)} \tag{6.54}$$

for some constant C*.*

Proof. If $h = \sum h_j(t)\,\varphi_j(x)$, then $Dh = \sum D_j h_j(t)\,\varphi_j(x)$, where

$$D_j z(t) = \omega^2 z''(t) + \lambda_j^2 z(t)\rho(t), \quad \rho(t) := 1 + \mu a(t).$$

Using the eigenfunctions $\psi_l(t)$ of (6.53) as a basis of $H^1(\mathbb{T})$,

$$D_j z(t) = \sum_{l\in\mathbb{N}} (\lambda_j^2 - \omega^2 p_l^2)\hat{z}_l\,\psi_l(t)\,\rho(t), \qquad z = \sum_{l\in\mathbb{N}} \hat{z}_l\,\psi_l(t),$$

and $K_j := (1/\rho)D_j$ is the diagonal operator $\{\lambda_j^2 - \omega^2 p_l^2\}_{l\in\mathbb{N}}$. Since $|\lambda_j^2 - \omega^2 p_l^2| > \gamma/\lambda_j^{\tau-1}$ for all $\lambda_j \le N_{n+1}$, we have that K_j is invertible and

$$\|K_j^{-1}z\|^2_{H^1_{\mu a}} = \sum_{l\in\mathbb{N}} \Big(\frac{\hat{z}_l}{\lambda_j^2 - \omega^2 p_l^2}\Big)^2 \|\psi_l\|^2_{H^1_{\mu a}} \le \frac{\lambda_j^{2(\tau-1)}}{\gamma^2} \|z\|^2_{H^1_{\mu a}}.$$

By (6.52) $\|K_j^{-1}z\|_{H^1} \le 4(\lambda_j^{\tau-1}/\gamma)\|z\|_{H^1}$. Since $D_j^{-1}z = K_j^{-1}(z/\rho)$ and $\|1/\rho\|_{H^1}$ is smaller than a universal constant,

$$\|D_j^{-1}z\|_{H^1} \le \frac{C\lambda_j^{\tau-1}}{\gamma} \|z\|_{H^1}.$$

Since $D^{-1}h = \sum_j D_j^{-1}h_j(t)\varphi_j(x)$ we obtain (6.54). □

Lemma 48. (Control of S). *For all $s \ge 0$, if $u \in X_{s+2}$ then $S : X_0 \to X_s$ is bounded and*

$$\|Sh\|_s \le \mu\|u\|_{s+2}\|u\|_2\|h\|_0 \qquad \forall h \in X_0.$$

Proof. Since $\int_\Omega \nabla u \circ \nabla h\, dx$ does not depend on x,

$$\Big\|\Delta u \int_\Omega \nabla u \circ \nabla h\, dx\Big\|_s \le \|\Delta u\|_s \Big\|\int_\Omega \nabla u \circ \nabla h\, dx\Big\|_{H^1(\mathbb{T})}.$$

$\int_\Omega \nabla u \circ \nabla h\, dx = \sum_j \lambda_j^2 u_j(t)h_j(t)$, so $\|\int_\Omega \nabla u \circ \nabla h\, dx\|_{H^1(\mathbb{T})} \le \|u\|_2\|h\|_0$ by Hölder inequality. □

Proof of Lemma 40. $F'(u) = D + S = (I + SD^{-1})D$ where I is the identity map. Since D^{-1} satisfies (6.54), we have to prove the invertibility of $I + SD^{-1}$ in norm $\|\ \|_{\tau-1}$. By Neumann series it is sufficient to show that

$$\|SD^{-1}h\|_{\tau-1} \le \frac{1}{2}\|h\|_{\tau-1} \quad \forall h \in X^{(n+1)}. \tag{6.55}$$

By Lemmas 47 and 48

$$\|SD^{-1}h\|_{\tau-1} \le \mu\|u\|_{\tau-1+2}\|u\|_2\|D^{-1}h\|_0 \le \frac{C\mu}{\gamma}\|u\|^2_{\tau+1}\|h\|_{\tau-1}$$

because $\|u\|_2 \le \|u\|_{\tau+1}$. Thus the condition

$$\frac{\mu}{\gamma}\|u\|^2_{\tau+1} \le \frac{1}{2C} =: K_1'$$

implies (6.55) and by Neumann series $\|(I + SD^{-1})^{-1}h\|_{\tau-1} \le 2\|h\|_{\tau-1}$. □

6.8 Proof of the theorems

Proof of Theorem 7. Let $g \in X_{\sigma,s_0}$ and $2d < 2(s_1 - 1) < s_0$ as assumed in the theorem. We apply Lemma 41 with

$$\tau := s_1 - 1\,.$$

The construction of the sequence (u_n) is possible provided the parameters (ω, μ) belong to A_n for all $n \in \mathbb{N}$. Lemma 45 assures that, for μ/γ sufficiently small, the set $\mathcal{A}_\gamma$ of parameters satisfying this property is a nonempty set, which is very large in a Lebesgue measure sense. Lemmas 42 and 43 complete the proof. □

Proof of Theorem 8. Let $g \in X_{\sigma,s_0}$ with $2d < 2(s_1 - 1) < s_0$ as assumed in the theorem. We consider a Lyapunov-Schmidt reduction splitting the space $\tilde{X}_{\sigma,s}$ in two subspaces $\tilde{X}_{\sigma,s} = Y \oplus (W \cap \tilde{X}_{\sigma,s})$,

$$Y := \{y(t) \in H^1(\mathbb{T}, \mathbb{R})\}\,, \quad W := \Big\{w \in \tilde{X}_{0,0} \,:\, w(x,t) = \sum_{j\geq 1} w_j(t)\tilde{\varphi}_j(x)\Big\}\,.$$

We denote Π_Y, Π_W the projectors on Y, W, and observe that Π_Y is the map

$$u \mapsto \int_{(0,2\pi)^d} u(x,t)\,dx.$$

We define

$$g_0(t) := \Pi_Y g, \quad \bar{g}(x,t) := \Pi_W g.$$

We decompose $u(x,t) = y(t) + w(x,t)$, $y \in Y$, $w \in W$, and note that

$$f(u) = f(y + w) = f(w) \in W.$$

Then projecting equation (6.4) on Y gives

$$\omega^2 y''(t) = \mu g_0(t) \quad (Y \textit{ equation}) \tag{6.56}$$

while projecting it on W

$$L_\omega w = \mu(f(w) + \bar{g}) \quad (W \textit{ equation}). \tag{6.57}$$

Equation (6.56) is an ODE. With direct calculations (or by Fourier series) we see that (6.56) admits 2π-periodic solutions if and only if

$$\int_0^{2\pi} g_0(t)\,dt = 0 \tag{6.58}$$

and (6.58) is just assumption (6.7). We note that, if $y(t)$ solves (6.56), then also $y(t) + c$ solves (6.56), for all $c \in \mathbb{R}$. Moreover, the unique solution $y(t)$ of (6.56) such that $\int_0^{2\pi} y(t)\,dt = 0$ satisfies

$$\|y\|_{H^1} \leq \|y''\|_{H^1} \leq \frac{\mu}{\omega^2}\,\|g_0\|_{H^1}.$$

To solve (6.57), we consider all the calculations in Sections 6.4,6.5,6.6,6.7 replacing $X_{\sigma,s}$ with $\tilde{X}_{\sigma,s} \cap W$ and $\lambda_j, \varphi_j(x)$ with $\tilde{\lambda}_j, \tilde{\varphi}_j(x)$, $j \geq 1$. It follows the existence of a unique solution $w \in \tilde{X}_{\sigma,s_1} \cap W$ of (6.57) satisfying

$$\|w\|_{\sigma,s_1} \leq \frac{\mu}{\gamma} C, \quad \|w_{tt}\|_{\sigma,s_1-2} \leq \frac{\mu}{\gamma\omega^2} C.$$

Then $u = y + w$ solves (6.4)(6.3). Since

$$\|u\|^2_{\sigma,s_1} = \|y\|^2_{H^1} + \|w\|^2_{\sigma,s_1}, \quad \|u_{tt}\|^2_{\sigma,s_1-2} = \|y''\|^2_{H^1} + \|w_{tt}\|^2_{\sigma,s_1-2},$$

we obtain estimates (6.8). □

6.9 Hill's problems

Proof of Lemma 46. The proof follows from classical results in [55, 69]. First, if $y'' + p^2(1+\alpha)y = 0$, then

$$\int_0^{2\pi} y'^2 \, dt = p^2 \int_0^{2\pi} (1+\alpha) y^2 \, dt \,,$$

so that $p^2 \geq 0$ because $1 + \alpha$ is positive. $p_0^2 = 0$ is an eigenvalue, the corresponding eigenfunctions are the constants, and all the other eigenvalues are positive.

By [55, Theorem 2.2.2, p.23], for every $k \in \mathbb{N}$ both p_{2k+1} and p_{2k+2} satisfy

$$\frac{2}{3}(k+1)^2 \leq p^2 \leq 2(k+1)^2$$

and (6.50) follows.

The equivalence (6.52) and the orthogonality of (ψ_l) w.r.t. $(\, , \,)_{H^1_\alpha}$ can be verified by direct calculations.

To prove (6.51), we define

$$q(\vartheta)(t) = q(t) := 1 + \alpha(t) + \vartheta(\beta(t) - \alpha(t)), \quad \vartheta \in [0,1]$$

and the "Liouville's change of variable" $t \to \xi$

$$t = f(\xi) \quad \Leftrightarrow \quad \xi = g(t) := \frac{1}{c} \int_0^t \sqrt{q(s)} \, ds, \qquad c := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{q(t)} \, dt.$$

We note that $p^2, y(t)$ satisfy

$$\begin{cases} y''(t) + p^2 q(t) y(t) = 0 \\ y(t) = y(t + 2\pi) \end{cases} \tag{6.59}$$

if and only if $p^2, z(\xi)$ satisfy

$$\begin{cases} z''(\xi) + c^2[p^2 - Q(f(\xi))] z(\xi) = 0 \\ z(\xi) = z(\xi + 2\pi) \end{cases} \tag{6.60}$$

where

$$z(\xi) := y(f(\xi))\sqrt[4]{q(f(\xi))}, \quad Q(t) := -\frac{5}{16}\frac{q'(t)^2}{q(t)^3} + \frac{1}{4}\frac{q''(t)}{q(t)^2}\,.$$

The operators $T(\vartheta) : z \mapsto -z'' + c^2 Q(f)z$ are selfadjoint in $L^2(0, 2\pi)$. We apply [69, Theorem 3.9, VII.3.5, p. 392] to the holomorphic family $\{T(\vartheta) : \vartheta \in [0, 1]\}$ (see [69, Definition VII.2.1, p. 375, Example 2.12, VII.2.3, p. 380 and Example 6.13, III.6.8, p. 187] to verify the hypotheses of the Theorem in the present case) to prove that the eigenvalues and eigenfunctions of (6.60) are analytic in ϑ. As a consequence, the eigenvalues and eigenfunctions of (6.59) are analytic in ϑ as well. This allows us to differentiate the equation

$$\psi_l(\vartheta)'' + p_l(\vartheta)^2(1 + \alpha + \vartheta(\beta - \alpha))\psi_l(\vartheta) = 0$$

w.r.t. ϑ. Recalling that $\int_0^{2\pi} q\psi_l^2\, dt = 1$, multiplying by $\psi_l(\vartheta)$ and integrating

$$\partial_\vartheta p_l(\vartheta) = -\frac{1}{2}\, p_l(\vartheta) \int_0^{2\pi} (\beta - \alpha)\psi_l(\vartheta)^2\, dt\,.$$

Since $p_l(\vartheta) \le 2l$ and $q(\vartheta) \ge 1/2$,

$$|\partial_\vartheta p_l(\vartheta)| \le l \int_0^{2\pi} \frac{|\beta - \alpha|}{q(\vartheta)}\, q(\vartheta)\psi_l(\vartheta)^2\, dt \le 2l\|\beta - \alpha\|_\infty$$

and

$$|p_l(\beta) - p_l(\alpha)| \le \int_0^1 |\partial_\vartheta p_l(\vartheta)|\, d\vartheta \le 2l\|\beta - \alpha\|_\infty.$$

□

Chapter 7

Perspectives

We mention some natural problems related to the results we have proved in this Thesis.

Regarding results of Chapter 3, an interesting direction would be to study the bifurcation of quasi-periodic solutions with two frequencies via variational methods. This would make it possible to deal with more general non-linearities, and hopefully to remove the hypothesis of monotonicity on the leading term of the nonlinearity (in Chapter 3 such a term is u^3). A first study could assume non-resonance conditions of badly approximation type, and focus on the variational analysis of the bifurcation equation.

Another direction would be the extension to positive measure sets of parameters, that is to prove a Nash-Moser version of Chapter 3.

Up to now, no existence result has been proved for quasi-periodic solutions of completely resonant wave equations with Dirichlet boundary conditions. Then it is a completely open question — possibly very difficult — the study of the equation of Chapter 3 with Dirichlet boundary conditions, that is

$$v_{tt} - v_{xx} + v^3 = O(v^4), \quad v(0,t) = v(\pi,t) = 0.$$

A general problem in the study of nonlinear wave equations is the presence of derivatives in the nonlinearity. We have mentioned the few known results in Section 2.5. We think that this wide, open question is important both from a purely mathematical point of view in the study of PDE and with respect to applications, because in careful models of physical systems it arises often a nonlinear dependence on partial derivatives. In this difficult direction, in fact, almost every question is open.

However, the results of Chapter 6 on the quasi-linear Kirchhoff equation indicate that for equations where the nonlinearity has a special structure some existence results can be proved with currently available techniques. Then it seems not completely unreasonable to hope to extend results like those obtained in Chapters 4 and 5 to (which?) nonlinearities depending on derivatives.

Another general problem for nonlinear wave equations is the study in dimension higher than one. Because of the sub-quadratic asymptotic behaviour of the eigenvalues of the Laplacian, small divisors difficulties are heavier than in dimension one. Some results for periodic and quasi-periodic solutions with space-periodic boundary conditions have been obtained by Bourgain and Yuan, see for example [38, 42, 115], but the study is far from being concluded.

In this direction, it could be interesting to extend results of Chapter 5 to forced nonlinear wave equations in higher dimension.

Regarding the Kirchhoff equation, a natural question is to find periodic solutions for more general forcing term depending also on the displacement u, that is

$$u_{tt} - \Delta u\Big(1 + \int_\Omega |\nabla u|^2 \, dx\Big) = \varepsilon g(x,t,u).$$

Another interesting question is to find solutions of (6.1) which are perturbations of a nonzero normal mode v of the free Kirchhoff equation. The problem is not a mere corollary of results we have proved. Instead of the d'Alembertian, the unperturbed operator is

$$u \;\mapsto\; u_{tt} - \Delta u\Big(1 + \int_\Omega |\nabla v|^2 \, dx\Big) - 2\Delta v \int_\Omega \nabla u \cdot \nabla v.$$

In wave equations with a *dissipative* term like for example

$$u_{tt} - u_{xx} + a u_t = f(x,t,u), \quad a \neq 0$$

the term au_t eliminates all small divisors. Indeed, the effect of the dissipation on the eigenvalues of the problem is to add a purely imaginary part which is always far from zero, so that, even if real parts can be arbitrarily small, the spectrum is far from zero.

For the Kirchhoff equation, instead, a small divisor problem appears even in presence of a dissipative term, because of the interaction with the derivatives in the nonlinearity. Here resonances involve the "spectral gaps" of the associated Hill's problem. As a consequence, it seems interesting to study this problem by the Nash-Moser method.

Appendix A

Diophantine approximation and measure theory of numbers

This Appendix is a self-contained presentation of some classical results of number theory, based on Khinchin [70], Schmidt [106] and Hardy-Wright [65]. See also [18, 23].

Notation. Given a real number x, we denote $[[x]]$ its integer part, which is the greatest integer $\leq x$, and $\{x\} = x - [[x]]$ its fractional part.

Definition. Numbers ω such that

$$\text{(A.1)} \qquad \exists\, \gamma > 0 \quad \text{such that} \quad |\omega q - p| > \frac{\gamma}{q} \quad \forall p, q \in \mathbb{Z},\ q \geq 1$$

are called *badly approximable.* We denote $\mathcal{B}$ the set of such numbers and define

$$\mathcal{B}_\gamma := \left\{ \omega \in \mathbb{R} : |\omega q - p| > \frac{\gamma}{q} \quad \forall p, q \in \mathbb{Z},\ q \geq 1 \right\}, \quad \mathcal{B} := \bigcup_{\gamma > 0} \mathcal{B}_\gamma \,.$$

The reason for this name is that badly approximable numbers are "the most far" from rational numbers (see Corollary 4). In general, it is possible to describe how well rational numbers approximate a given irrational number ω by studying the validity of the "Diophantine inequality"

$$\text{(A.2)} \qquad |\omega q - p| > \frac{\gamma}{q^\tau} \quad \forall p, q \in \mathbb{Z},\ q \geq 1$$

with $\gamma, \tau > 0$.

Definition. Irrational numbers satisfying (A.2) are called (γ, τ)-*Diophantine.* We denote $\mathcal{D}_{\gamma,\tau}$ the set of such numbers, namely

$$\mathcal{D}_{\gamma,\tau} := \left\{ \omega \in \mathbb{R} : |\omega q - p| > \frac{\gamma}{q^\tau} \quad \forall p, q \in \mathbb{Z},\ q \geq 1 \right\},$$

and define

$$\mathcal{D}_\tau := \bigcup_{\gamma>0} \mathcal{D}_{\gamma,\tau}\,.$$

For $\tau = 1$ we have $\mathcal{D}_{\gamma,1} = \mathcal{B}_\gamma$ and $\mathcal{D}_1 = \mathcal{B}$. Note that $\mathcal{D}_{\gamma,\tau} \cap \mathbb{Q} = \emptyset$ for all positive γ, τ. We recall the following classical results.

Theorem 9. (Dirichlet 1842). *Let $\omega \in \mathbb{R}$ and $Q > 1$. Then there exist integers p, q such that*

$$|\omega q - p| \le \frac{1}{Q}\,, \quad 1 \le q < Q.$$

Proof. Suppose that Q is an integer. We consider the following $Q + 1$ numbers

$$0,\ 1,\ \{\omega\},\ \{2\omega\},\ \ldots,\ \{(Q-1)\omega\}\,.$$

They all lie in the unit interval $[0,1]$. We divide $[0,1]$ into Q subintervals

$$\Big[0, \frac{1}{Q}\Big),\ \ \Big[\frac{1}{Q}, \frac{2}{Q}\Big),\ \ \ldots\ \Big[\frac{k}{Q}, \frac{k+1}{Q}\Big),\ \ \ldots\ \Big[\frac{Q-1}{Q}, 1\Big]\,.$$

At least one among such subintervals contains two (or more) of the $Q + 1$ numbers above. Hence there are integers r_1, r_2, s_1, s_2 with $r_1, r_2 \in [0, Q)$ and $r_1 \ne r_2$ such that

$$|(r_1\omega - s_1) - (r_2\omega - s_2)| \le \frac{1}{Q}\,.$$

If, say, $r_1 > r_2$, we put $q = r_1 - r_2$ and $p = s_1 - s_2$. Then $1 \le q \le Q - 1$ and $|\omega q - p| \le 1/Q$, proving the theorem when Q is an integer.

Next, suppose Q is not an integer. We apply what has already been proved to $Q' = [[Q]] + 1$. Then $1 \le q \le Q' - 1 = [[Q]]$, whence $1 \le q < Q$. □

Remark 24. There is no loss of generality assuming that p, q in Dirichlet's Theorem are relatively prime. If not, we have $p = ma$ and $q = mb$ for some integer factor $m > 1$, then $1 \le b < Q$ and $|\omega b - a| \le 1/Q$.

Corollary 3. *Suppose that ω is irrational. Then there exist infinitely many distinct rationals p/q such that*

$$|\omega q - p| < \frac{1}{q}\,. \tag{A.3}$$

Proof. Fix $Q_1 > 1$. By Theorem 9 and Remark 24 we find p_1, q_1 relatively prime such that $\delta_1 := |\omega q_1 - p_1| \le 1/Q_1 < 1/q_1$. Since ω is irrational, $\delta_1 > 0$. Fix $Q_2 > 1/\delta_1$. We find p_2, q_2 relatively prime such that $\delta_2 := |\omega q_2 - p_2| \le 1/Q_2 < 1/q_2$. Now, suppose that $p_2/q_2 = p_1/q_1$. Then $p_1 = p_2$

and $q_1 = q_2$ because p_i, q_i are relatively prime, therefore $\delta_1 = \delta_2$. However, $\delta_2 \leq 1/Q_2 < \delta_1$, a contradiction. It follows that p_1/q_1 and p_2/q_2 are distinct.

Since ω is irrational, $\delta_2 > 0$ and we can go on inductively. □

Note that Corollary 3 is not valid for ω rational. Indeed, suppose that $\omega = k/h \in \mathbb{Q}$. If $p/q \neq k/h$, then

$$|\omega q - p| = \left|\frac{k}{h}\, q - p\right| = \frac{|kq - ph|}{h} \geq \frac{1}{h}$$

and therefore (A.3) can be satisfied by only finitely many rationals p/q.

Corollary 4. *$\mathcal{D}_{\gamma,\tau} = \emptyset$ for all $\tau \in (0, 1)$, for all $\gamma > 0$. In other words, $\tau = 1$ is the smallest possible exponent in* (A.2)*, and badly approximable numbers are those achieving the largest possible distance from $\mathbb{Q}$.*

Proof. Suppose $\omega \in \mathcal{D}_{\gamma,\tau}$. Then ω is irrational. By (A.3)

$$\frac{\gamma}{q^\tau} < \frac{1}{q}$$

for infinitely many integers q, and this is impossible for $\tau < 1$. □

Also, by Corollary 3 $\mathcal{B}_\gamma = \emptyset$ for all $\gamma \geq 1$. However, this bound can be improved:

Theorem 10. (Hurwitz 1891). *(i) For every irrational number ω there exist infinitely many distinct rationals p/q such that*

$$|\omega q - p| < \frac{1}{\sqrt{5}\, q}\,.$$

(ii) This is not true if $\frac{1}{\sqrt{5}}$ is replaced by a smaller constant $\gamma < \frac{1}{\sqrt{5}}$.

For a proof see for example [106, p.6]. As a consequence,

$$\mathcal{B}_\gamma = \emptyset \quad \forall \gamma \geq \frac{1}{\sqrt{5}}\,.$$

A special family of badly approximable numbers are irrational algebraic numbers of degree two, that is irrational roots of quadratic polynomials with integer coefficients.

Theorem 11. (Liouville). *Let a, b, c be integers, $a \neq 0$. If $\omega \notin \mathbb{Q}$ satisfies*

$$a\omega^2 + b\omega + c = 0,$$

then $\omega \in \mathcal{B}$.

Proof. We define $P(x) := ax^2+bx+c$. Suppose that a rational p/q satisfies $|\omega - p/q| \leq 1$. Then

$$\Big|P\Big(\frac{p}{q}\Big)\Big| = \Big|P\Big(\frac{p}{q}\Big) - P(\omega)\Big| \leq \int_{\omega}^{p/q} |P'(\xi)|\, d\xi \leq M\Big|\omega - \frac{p}{q}\Big|$$

where

$$M := \max_{\xi\in[\omega-1,\,\omega+1]} |P'(\xi)| > 0.$$

Moreover

$$\Big|P\Big(\frac{p}{q}\Big)\Big| = \frac{|ap^2+bpq+cq^2|}{q^2} \geq \frac{1}{q^2}$$

because $ap^2+bpq+cq^2$ is an integer and, if it would be zero, then p/q would be a root of the polynomial P. But ω is irrational, then both the two roots of P are irrational, so $ap^2+bpq+cq^2 \neq 0$. It follows that

$$\Big|\omega - \frac{p}{q}\Big| \geq \frac{1}{Mq^2}\,.$$

On the other hand, for rationals p/q such that $|\omega - p/q| > 1$, we have immediately $|\omega - p/q| > 1/q^2$ for all $q \geq 1$. Then for all $p \in \mathbb{Z}$, $q \geq 1$

$$\Big|\omega - \frac{p}{q}\Big| > \frac{\gamma}{q^2}\,, \qquad \gamma := \frac{1}{2}\,\min\Big\{\frac{1}{M}\,,1\Big\},$$

that is ω is badly approximable. □

To investigate how large are sets $\mathcal{B}_\gamma$, namely to estimate their cardinality and measure, we need the continued fractions theory.

A.1 Continued fractions

Let $(a_0, a_1, a_2, \ldots)$ be a real sequence with $a_0 \in \mathbb{R}$ and $a_k \in [1,+\infty)$ for all $k \geq 1$. We define

$$[a_0] = a_0, \quad [a_0,a_1] = a_0 + \frac{1}{a_1}\,, \quad [a_0,a_1,a_2] = a_0 + \frac{1}{a_1+\frac{1}{a_2}}\,, \quad \ldots$$

and we call

$$[a_0,\ldots,a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\ldots + \frac{1}{a_n}}}$$

the (finite) continued fraction of order n.

We define a "canonical representation" of all $[a_0,\ldots,a_n]$ as fraction p/q by induction. For the continued fraction of order zero $[a_0] = a_0$ we put

$$p = a_0, \quad q = 1\,,$$

so that $[a_0] = p/q$. Suppose now that the canonical representation is defined for continued fractions of order $\leq n-1$. Let p'/q' the canonical representation of $[a_1, \ldots, a_n]$. Then we define p, q setting

$$p = a_0 p' + q', \quad q = p' \,. \tag{A.4}$$

$[a_0, \ldots, a_n] = p/q$ because $[a_1, \ldots, a_n] = p'/q'$ and

$$[a_0, \ldots, a_n] = a_0 + \frac{1}{[a_1, \ldots, a_n]} \,.$$

We denote p_n/q_n the canonical representation of $[a_0, \ldots, a_n]$ and we call it its *convergent* of order n.

Lemma 49. (Rule for the formation of the convergents). *For all* $n \geq 2$

$$p_n = a_n p_{n-1} + p_{n-2} \tag{A.5}$$

$$q_n = a_n q_{n-1} + q_{n-2} \,. \tag{A.6}$$

Proof. For $n = 2$ the formulas are easily verified directly. Suppose that they are true for all $k \leq n-1$. We consider the continued fraction $[a_1, \ldots, a_n]$ and we denote p'_k/q'_k its kth convergent, $0 \leq k \leq n-1$. By definition (A.4) of canonical representations we have

$$p_n = a_0 p'_{n-1} + q'_{n-1} \,, \quad q_n = p'_{n-1} \,.$$

By hypothesis

$$p'_{n-1} = a_n p'_{n-2} + p'_{n-3} \,, \quad q'_{n-1} = a_n q'_{n-2} + q'_{n-3}$$

(here we have a_n rather than a_{n-1} because the fraction $[a_1, \ldots, a_n]$ begins with a_1 and not with a_0). It follows that

$$\begin{aligned} p_n &= a_0(a_n p'_{n-2} + p'_{n-3}) + (a_n q'_{n-2} + q'_{n-3}) \\ &= a_n(a_0 p'_{n-2} + q'_{n-2}) + a_0 p'_{n-3} + q'_{n-3} \\ &= a_n p_{n-1} + p_{n-2} \,, \\ q_n &= a_n p'_{n-2} + p'_{n-3} = a_n q_{n-1} + q_{n-2} \end{aligned}$$

by (A.4). □

As a consequence, since any continued fraction coincides with its convergent of the same order,

$$[a_0, \ldots, a_n] = \frac{p_n}{q_n} = \frac{p_{n-1} a_n + p_{n-2}}{q_{n-1} a_n + q_{n-2}} \,.$$

Also, we note that $p_{n-1}, p_{n-2}, q_{n-1}, q_{n-2}$ depend only on the elements a_0, a_1, …, a_{n-1}, then

$$[a_0, \ldots, a_{n-1}, \lambda] = \frac{p_{n-1}\lambda + p_{n-2}}{q_{n-1}\lambda + q_{n-2}} \quad \forall \lambda \geq 1. \tag{A.7}$$

Moreover, since $q_0 = 1$, $q_1 = a_1$ and $a_n \geq 1$ for all $n \geq 1$, by (A.6) it follows

$$1 = q_0 \leq q_1 < q_2 < q_3 < \dots \tag{A.8}$$

Lemma 50. *For all $n \geq 1$*

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n . \tag{A.9}$$

Proof. Since $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0 a_1 + 1$ and $q_1 = a_1$, the formula is verified directly for $n = 1$.

Next, we multiply (A.5) by q_{n-1} and (A.6) by p_{n-1}. Subtracting we obtain

$$q_n p_{n-1} - p_n q_{n-1} = -(q_{n-1} p_{n-2} - p_{n-1} q_{n-2}).$$

□

As a consequence,

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n-1}} . \tag{A.10}$$

(A.8) and (A.10) imply that even-order convergents form an increasing and odd-order convergents a decreasing sequence. Also, every odd-order convergent is greater than any even-order convergent. These two sequences converge because they are monotone, and their limits coincide by (A.10) (note that $q_n q_{n-1} \to \infty$ by (A.8)). So the entire sequence (p_n/q_n) converges. Its limit ω is "the value" of the infinite continued fraction

$$[a_0, a_1, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} .$$

Also, ω is greater than any of its even-order convergents and is less than any of its odd-order convergents,

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \dots < \frac{p_{2k}}{q_{2k}} < \dots < \omega < \dots < \frac{p_{2k+1}}{q_{2k+1}} < \dots < \frac{p_3}{q_3} < \frac{p_1}{q_1} . \tag{A.11}$$

Lemma 51. *The value $\omega = [a_0, a_1, \dots]$ satisfies*

$$\frac{1}{q_n(q_n + q_{n+1})} < \left|\omega - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \qquad \forall n \in \mathbb{N}. \tag{A.12}$$

Proof. By (A.11) and (A.10)

$$\left|\omega - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n q_{n+1}} .$$

To prove the other inequality, we note that

$$\left|\omega - \frac{p_n}{q_n}\right| > \left|\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right| .$$

Now, by (A.11) and (A.10)

$$\left|\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| - \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_{n+2}}{q_{n+2}}\right| = \frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1}q_{n+2}}.$$

We use (A.6) to substitute q_{n+2} and we obtain

$$\frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1}q_{n+2}} = \frac{a_{n+2}}{q_n(a_{n+2}q_{n+1} + q_n)}.$$

Finally $q_n(a_{n+2}q_{n+1} + q_n) \leq q_n a_{n+2}(q_{n+1} + q_n)$ because $a_{n+2} \geq 1$. □

We note that

$$a_{n+1}q_n \leq q_{n+1} \leq (a_{n+1} + 1)q_n \quad \forall n \in \mathbb{N}. \tag{A.13}$$

For $n = 0$ this can be verified directly. For $n \geq 1$ it follows from (A.6) and (A.8). Then (A.12) and (A.13) imply the following estimate.

Corollary 5. *The value* $\omega = [a_0, a_1, \ldots]$ *satisfies*

$$\frac{1}{q_n^2(a_{n+1} + 2)} < \left|\omega - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2\, a_{n+1}} \quad \forall n \in \mathbb{N}. \tag{A.14}$$

From now on, we will consider continued fractions where all elements a_n are integer numbers,

$$a_0 \in \mathbb{Z}, \quad a_n \in \mathbb{Z} \cap [1, +\infty) \quad \forall n \geq 1.$$

As a consequence, all p_n and q_n are integers and by (A.9) all convergents p_n/q_n are irreducible fractions.

Lemma 52. (Representation of real numbers as continued fractions). *To every real number ω there corresponds a unique continued fraction with value equal to ω. This fraction is finite, i.e. $\omega = [a_0, \ldots, a_n]$, if ω is rational, and it is infinite, i.e. $\omega = [a_0, a_1, \ldots]$, if ω is irrational.*

If ω is rational, the last element a_n is greater than 1, unless the trivial case when ω is an integer ≤ 1.

Proof. Fix any real number ω. We define $a_0 = [[\omega]]$ its integer part. If ω is integer then simply $\omega = [a_0]$. If ω is not an integer, we define r_1 by the relation

$$\omega = a_0 + \frac{1}{r_1} \tag{A.15}$$

and we note that $r_1 > 1$. We define $a_1 = [[r_1]]$. If r_1 is integer then $\omega = [a_0, a_1]$. If r_1 is not an integer, we go on. In general, we set $a_n = [[r_n]]$ and, if r_n is not an integer, we define r_{n+1} by

$$r_n = a_n + \frac{1}{r_{n+1}}. \tag{A.16}$$

Clearly $r_{n+1} > 1$.

Equation (A.15) shows that $\omega = [a_0, r_1]$. Suppose that, in general,

$$\omega = [a_0, \ldots, a_{n-1}, r_n]. \tag{A.17}$$

Then from (A.16) we have

$$\omega = [a_0, \ldots, a_{n-1}, a_n, r_{n+1}]$$

and (A.17) is valid for all n (in case $r_1, r_2, \ldots r_{n-1}$ are not integers).

If ω is rational, all the r_n will be rational. Then our process will stop after a finite number of steps. Indeed, suppose $r_n = a/b \in \mathbb{Q} \setminus \mathbb{Z}$ (if r_n is integer we have finished). Then

$$r_n - a_n = \frac{a - ba_n}{b} =: \frac{c}{b}$$

and $0 < c < b$ because $r_n - a_n = \{r_n\} \in (0, 1)$. Equation (A.16) then gives

$$r_{n+1} = \frac{b}{c}$$

and so r_{n+1} has a smaller denominator than does r_n. It follows that if we consider $r_1, r_2, \ldots$ we must eventually come to an integer $r_n = a_n$, and so $\omega = [a_0, \ldots, a_n]$.

Note that $a_n > 1$, except for the trivial cases when ω is an integer ≤ 1. Indeed, if $n \geq 1$ and $a_n = 1$ were such a last element, we would have $r_{n-1} = a_{n-1} + 1$ and $a_{n-1} \neq [[r_{n-1}]]$.

If ω is irrational, then all r_n are irrational. Then our process is infinite and it defines a sequence $(a_0, a_1, \ldots)$ of integers with $a_n \geq 1$ for all $n \geq 1$. We consider the infinite continued fraction $[a_0, a_1, \ldots]$ and its convergents p_n/q_n. For all $n \geq 2$, by (A.17) and (A.7)

$$\omega = \frac{p_{n-1} r_n + p_{n-2}}{q_{n-1} r_n + q_{n-2}}.$$

By (A.5) and (A.6)

$$\frac{p_n}{q_n} = \frac{p_{n-1} a_n + p_{n-2}}{q_{n-1} a_n + q_{n-2}}$$

so that

$$\omega - \frac{p_n}{q_n} = \frac{(p_{n-1} q_{n-2} - q_{n-1} p_{n-2})\,(r_n - a_n)}{(q_{n-1} r_n + q_{n-2})\,(q_{n-1} a_n + q_{n-2})}$$

and consequently

$$\left|\omega - \frac{p_n}{q_n}\right| < \frac{1}{(q_{n-1}r_n + q_{n-2})\,(q_{n-1}a_n + q_{n-2})} < \frac{1}{q_n^2}\,.$$

Thus $p_n/q_n \to \omega$ as $n \to \infty$, namely $\omega = [a_0, a_1, \ldots]$.

It remains to show the uniqueness of of the expansion. Suppose that

$$\omega = [a_0, a_1, \ldots] = [a'_0, a'_1, \ldots]$$

where the two continued fractions may be either finite or infinite. Since both a_0 and a'_0 are defined as the integer part $[[\omega]]$ of ω, obviously $a_0 = a'_0$. Furthermore, if $a_k = a'_k$ for all $k = 1, \ldots, n$, then (in analogous notation)

$$p_k = p'_k, \quad q_k = q'_k \quad \forall k = 1, \ldots, n.$$

By formula (A.7)

$$\omega = \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} = \frac{p'_n r'_{n+1} + p'_{n-1}}{q'_n r'_{n+1} + q'_{n-1}} = \frac{p_n r'_{n+1} + p_{n-1}}{q_n r'_{n+1} + q_{n-1}}$$

so that $r_{n+1} = r'_{n+1}$. Since $a_{n+1} = [[r_{n+1}]]$ and $a'_{n+1} = [[r'_{n+1}]]$, we have $a_{n+1} = a'_{n+1}$. As a consequence the two fractions coincide completely. □

Definition. A rational a/b is called a *best approximation* of the real number ω if

$$|\omega b - a| < |\omega d - c|$$

for all rationals $c/d \neq a/b$ with $0 < d \leq b$.

We can assume without loss of generality that best approximants are irreducible fractions. Indeed, if $a/b = p/q$, $a = mp$, $b = mq$, then obviously

$$|\omega q - p| = \frac{|\omega b - a|}{|m|} \leq |\omega b - a|.$$

Lemma 53. *Every best approximation is a convergent.*

Proof. Let a/b be a best approximation of the number $\omega = [a_0, a_1, \ldots]$, and let p_k/q_k be the convergents of ω. If $a/b < a_0$, since $a_0 = [[\omega]] \leq \omega$ and $b \geq 1$ we would obtain

$$|\omega - a_0| < \left|\omega - \frac{a}{b}\right| = \frac{|\omega b - a|}{b} \leq |\omega b - a|$$

so that $|\omega \cdot 1 - a_0| < |\omega b - a|$ and a/b would not be a best approximation. Thus, $a/b \geq a_0$.

Suppose ω is an integer, $\omega = a_0$. If a/b would be greater than a_0, we would have $0 = |\omega \cdot 1 - a_0| < |\omega b - a|$, so that, as above, a/b would not be a best approximation. It follows that $a/b = a_0 = p_0/q_0$ and we have finished.

Thus, let ω be not an integer. Then, if the fraction a/b does not coincide with one of the convergents, recalling (A.11) there are three possible situations:

(*i*) a/b is between two convergents p_{k-1}/q_{k-1} and p_{k+1}/q_{k+1} for some $k \geq 1$;

(*ii*) $\omega = [a_0, \ldots, a_n]$, $n \geq 1$ and a/b is between the last and the last but one convergents p_n/q_n and p_{n-1}/q_{n-1};

(*iii*) $a/b > p_1/q_1$.

In the case (*i*),

$$\left|\frac{a}{b} - \frac{p_{k-1}}{q_{k-1}}\right| = \frac{|aq_{k-1} - p_{k-1}b|}{bq_{k-1}} \geq \frac{1}{bq_{k-1}}$$

and

$$\left|\frac{a}{b} - \frac{p_{k-1}}{q_{k-1}}\right| < \left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| = \frac{1}{q_k q_{k-1}}$$

so that $b > q_k$. On the other hand,

$$\left|\omega - \frac{a}{b}\right| \geq \left|\frac{p_{k+1}}{q_{k+1}} - \frac{a}{b}\right| \geq \frac{1}{bq_{k+1}}$$

and hence $|\omega b - a| \geq 1/q_{k+1}$. Since $|\omega q_k - p_k| < 1/q_{k+1}$ by (A.12), we have

$$|\omega q_k - p_k| < |\omega b - a|, \quad 0 < b < q_k,$$

that is a/b is not a best approximation.

In the case (*ii*), ω coincides with its last convergent p_n/q_n, so that

$$\left|\omega - \frac{a}{b}\right| = \left|\frac{p_n}{q_n} - \frac{a}{b}\right| \geq \frac{1}{q_n b}$$

and

$$\left|\omega - \frac{a}{b}\right| < \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_n q_{n-1}}\,.$$

It follows that $q_{n-1} < b$ and

$$|\omega q_{n-1} - p_{n-1}| = q_{n-1}\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_n} \leq |\omega b - a|,$$

that is a/b is not a best approximation.

In the case (*iii*), since $\omega \leq p_1/q_1 < a/b$,

$$\left|\omega - \frac{a}{b}\right| \geq \left|\frac{p_1}{q_1} - \frac{a}{b}\right| \geq \frac{1}{q_1 b}$$

so that $|\omega b - a| \geq 1/q_1$. On the other hand, ω may be either the fraction $[a_0, a_1]$ or a number $[a_0, r_1]$ with $r_1 = [a_1, \ldots] > a_1$. In any case,

$$|\omega - a_0| \leq \frac{1}{a_1}\,.$$

Since $q_1 = a_1$, we have

$$|\omega \cdot 1 - a_0| \leq \frac{1}{q_1} \leq |\omega b - a|, \quad 1 \leq b$$

which again contradicts the definition of a best approximation. □

Lemma 54. (Legendre). *Let a/b be an irreducible rational fraction such that*

$$\left|\omega - \frac{a}{b}\right| < \frac{1}{2b^2}.$$

Then a/b is a convergent of ω.

Proof. By Lemma (53) it is sufficient to show that the fraction a/b is a best approximation of ω. Suppose that

$$|\omega d - c| \leq |\omega b - a| < \frac{1}{2b}, \qquad d > 0, \quad \frac{c}{d} \neq \frac{a}{b}.$$

Then

$$\left|\omega - \frac{c}{d}\right| < \frac{1}{2bd}$$

and, consequently,

$$\left|\frac{c}{d} - \frac{a}{b}\right| \leq \left|\frac{c}{d} - \omega\right| + \left|\omega - \frac{a}{b}\right| < \frac{1}{2bd} + \frac{1}{2b^2} = \frac{b+d}{2b^2 d}.$$

On the other hand, since $c/d \neq a/b$, we have

$$\left|\frac{c}{d} - \frac{a}{b}\right| = \frac{|cb - ad|}{bd} \geq \frac{1}{bd}.$$

Therefore

$$\frac{1}{bd} < \frac{b+d}{2b^2 d},$$

so that $d > b$. Thus a/b is a best approximation of ω. □

Theorem 12. (Characterization of badly approximable numbers). *An irrational number ω is badly approximable if and only if the elements a_n, $n \geq 1$ of its expansion $[a_0, a_1, \ldots]$ in continued fractions are bounded. More precisely, for all positive γ*

$$\omega \in \mathcal{B}_\gamma \quad \Rightarrow \quad |a_n| < \frac{1}{\gamma} \quad \forall n \geq 1$$

and for all $M \geq 1$

$$|a_n| \leq M \quad \forall n \geq 1 \quad \Rightarrow \quad \omega \in \mathcal{B}_\gamma\,, \quad \gamma = \frac{1}{M+2}.$$

Proof. Let $\omega \in \mathcal{B}_\gamma$ for some γ. Then

$$\left|\omega - \frac{p_n}{q_n}\right| > \frac{\gamma}{q_n^2} \quad \forall n \in \mathbb{N}.$$

By (A.14) it follows immediately

$$a_{n+1} < \frac{1}{\gamma} \quad \forall n \in \mathbb{N}.$$

Conversely, suppose $a_n \leq M$ for all $n \geq 1$. For all rationals a/b such that

$$\left|\omega - \frac{a}{b}\right| \geq \frac{1}{2b^2}$$

we have immediately $|\omega - a/b| > \gamma/b^2$ because $1/2 > \gamma = 1/(M+2)$. On the other hand, if

$$\left|\omega - \frac{a}{b}\right| < \frac{1}{2b^2},$$

by Lemma 54 $a/b = p_n/q_n$ for some n. Then by (A.14)

$$\left|\omega - \frac{a}{b}\right| = \left|\omega - \frac{p_n}{q_n}\right| > \frac{1}{(a_{n+1}+2)\, q_n^2} \geq \frac{1}{(M+2)\, q_n^2} = \frac{\gamma}{b^2}.$$

□

Corollary 6. *For every $0 < \gamma \leq 1/4$, the set $\mathcal{B}_\gamma$ has continuum cardinality.*

Proof. Taking $M = 2$ in Theorem 12, the set $\mathcal{B}_\gamma$ with $\gamma \leq 1/4 = 1/(M+2)$ contains all numbers $\omega = [a_0, a_1, \ldots]$ with $a_n \in \{1, 2\}$ for all $n \geq 1$. □

Theorem 13. ($\mathcal{B}$ has zero measure). *For almost all ω the elements a_n are unbounded, that is, the Lebesgue measure of $\mathcal{B}$ is zero.*

Proof. Let $k_0 \in \mathbb{Z}$. We consider n fixed positive integers $k_1, \ldots, k_n$ and define $J_n = J_n(k_0, k_1, \ldots, k_n)$ as the set of numbers ω such that

$$a_0 = k_0, \quad a_1 = k_1, \quad \ldots, \quad a_n = k_n. \tag{A.18}$$

Except for the finite fraction $[k_0, k_1, \ldots, k_n] = p_n/q_n$, all other numbers $\omega \in J_n$ can be written as

$$\omega = [k_0, k_1, \ldots, k_n, r_{n+1}] \tag{A.19}$$

where $r_{n+1} = [a_{n+1}, \ldots]$ takes all possible values in $(1, +\infty)$. Conversely, every number ω of the form (A.19) with $r_{n+1} \in (1, +\infty)$ satisfies (A.18) and so it belongs to J_n. Then by (A.7) elements of J_n are

$$\omega = \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} \tag{A.20}$$

where r_{n+1} runs through $(1, +\infty)$ and p_n, p_{n-1}, q_n, q_{n-1} remain fixed because they depend only on $k_0, k_1, \ldots, k_n$. It follows that J_n is the interval of endpoints

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} \qquad \text{and} \qquad \frac{p_n}{q_n},$$

and by (A.9) its measure is

$$|J_n| = \left|\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n}{q_n}\right| = \frac{|p_n q_{n-1} - p_{n-1} q_n|}{q_n\,(q_n + q_{n-1})} = \frac{1}{q_n\,(q_n + q_{n-1})}\,.$$

Now for any positive integer s we define $J_{n+1}^{(s)} \subset J_n$ as the set of ω such that

$$a_0 = k_0, \quad a_1 = k_1, \quad \ldots, \quad a_n = k_n, \quad a_{n+1} = s.$$

Every $\omega \in J_{n+1}^{(s)}$ satisfies (A.19) with $[[r_{n+1}]] = s$ and hence

$$s \le r_{n+1} < s + 1.$$

Conversely, among all the points (A.20) of the interval J_n, those for which $s \le r_{n+1} < s$ belong to $J_{n+1}^{(s)}$. It follows that $J_{n+1}^{(s)}$ is the interval of endpoints

$$\frac{p_n s + p_{n-1}}{q_n s + q_{n-1}} \qquad \text{and} \qquad \frac{p_n(s+1) + p_{n-1}}{q_n(s+1) + q_{n-1}}$$

and by (A.9) its measure is

$$|J_{n+1}^{(s)}| = \left|\frac{p_n s + p_{n-1}}{q_n s + q_{n-1}} - \frac{p_n(s+1) + p_{n-1}}{q_n(s+1) + q_{n-1}}\right| = \frac{1}{(q_n s + q_{n-1})(q_n(s+1) + q_{n-1})}\,.$$

Since $0 < q_{n-1}/q_n \le 1$ and $s \ge 1$, there holds

$$1 + \frac{q_{n-1}}{q_n} \ge 1 + \frac{q_{n-1}}{s q_n}\,, \quad 1 + \frac{1}{s} + \frac{q_{n-1}}{s q_n} \le 3$$

so that

$$\text{(A.21)} \qquad \frac{|J_{n+1}^{(s)}|}{|J_n|} = \frac{q_n^2\big(1 + \frac{q_{n-1}}{q_n}\big)}{s^2 q_n^2\big(1 + \frac{q_{n-1}}{s q_n}\big)\big(1 + \frac{1}{s} + \frac{q_{n-1}}{s q_n}\big)} \ge \frac{1}{3s^2}\,.$$

We note that sets $J_{n+1}^{(s)}$ form a partition of J_n, that is

$$\text{(A.22)} \qquad J_{n+1}^{(s)} \cap J_{n+1}^{(s')} = \emptyset \quad \forall s \ne s', \qquad \bigcup_{s \in \mathbb{Z} \cap [1, +\infty)} J_{n+1}^{(s)} = J_n$$

(with a little abuse of notation, we are ignoring the question if either endpoints are excluded or included in the intervals because in any case that does not change the measure of those intervals).

Now we fix a positive integer M. We denote $E_M = E_M(k_0)$ the set of numbers in the interval (k_0, k_0+1) all of whose elements $a_1, a_2, \ldots$ are less than M. We consider a set J_n as defined in (A.18) where all $k_1, \ldots, k_n$ are less than M. By (A.22) and (A.21) we have

$$\Big|\bigcup_{s\geq M} J_{n+1}^{(s)}\Big| = \sum_{s\geq M} |J_{n+1}^{(s)}| \geq \frac{|J_n|}{3}\sum_{s\geq M}\frac{1}{s^2} > \frac{|J_n|}{3}\int_M^{+\infty}\frac{d\xi}{\xi^2} = \frac{|J_n|}{3M}.$$

By (A.22)

$$|J_n| = \Big|\bigcup_{s<M} J_{n+1}^{(s)}\Big| + \Big|\bigcup_{s\geq M} J_{n+1}^{(s)}\Big|$$

so that

$$\Big|\bigcup_{s<M} J_{n+1}^{(s)}\Big| < \mu|J_n|, \qquad \mu := 1 - \frac{1}{3M} < 1. \tag{A.23}$$

We denote

$$E_M^{(n)} = \{\omega : a_0 = k_0,\ a_j < M\ \forall j = 1, \ldots, n\}.$$

Clearly all $J_n(k_0, k_1, \ldots, k_n)$ having $k_1, \ldots, k_n$ less than M form a partition of $E_M^{(n)}$, so that

$$\big|E_M^{(n)}\big| = \sum_{k_1,\ldots,k_n<M} |J_n(k_0, k_1, \ldots, k_n)|.$$

Analogously all $J_{n+1}(k_0, k_1, \ldots, k_n, s)$ having $k_1, \ldots, k_n, s$ less than M form a partition of $E_M^{(n+1)}$. Then by (A.23)

$$\begin{aligned} |E_M^{(n+1)}| &= \Big|\bigcup_{k_1,\ldots,k_n<M}\Big(\bigcup_{s<M} J_{n+1}^{(s)}(k_0, \ldots, k_n)\Big)\Big| \\ &< \sum_{k_1,\ldots,k_n<M} \mu|J_n(k_0, k_1, \ldots, k_n)| = \mu|E_M^{(n)}|. \end{aligned}$$

Successive application of this inequality gives

$$\big|E_M^{(n+1)}\big| < \mu^n\big|E_M^{(1)}\big| \tag{A.24}$$

so that $|E_M^{(n)}| \to 0$ as $n \to \infty$, being $\mu < 1$. Since $E_M \subset E_M^{(n)}$ for all n,

$$\big|E_M\big| = 0.$$

Now the set of all numbers in (k_0, k_0+1) with bounded elements is the countable union

$$E(k_0) = \bigcup_{M\geq 1} E_M(k_0)$$

so that $|E(k_0)| = 0$. Finally, we take the union of all $E(k_0)$ for $k_0 \in \mathbb{Z}$ and we obtain that $|\mathcal{B}| = 0$. □

Now we define

$$\mathcal{W}_\gamma := \Big\{\omega \in \mathbb{R} : |\omega q - p| > \frac{\gamma}{q} \quad \forall p, q \in \mathbb{Z},\ p \neq 0,\ q \geq 1\Big\}. \tag{A.25}$$

Sets $\mathcal{W}_\gamma$ have been introduced in [18]; we have used them in Chapter 3.

Lemma 55. *For all* $0 < \gamma \leq 1/4$,
(i) $\mathcal{W}_\gamma$ *has zero measure,*
(ii) $\mathcal{W}_\gamma \cap (0, \delta)$ *has continuum cardinality for all* $\delta > 0$.

Proof. (i) We note that if $\omega \in \mathcal{W}_\gamma$ and $|\omega| > \gamma$ then $|\omega q| > \gamma/q$ for all $q \geq 1$, so that $\omega \in \mathcal{B}_\gamma$. In other words,

$$\mathcal{W}_\gamma \cap \{|\omega| > \gamma\} = \mathcal{B}_\gamma \cap \{|\omega| > \gamma\} = \mathcal{B}_\gamma.$$

It remain to study the intersection $\mathcal{W}_\gamma \cap [-\gamma, \gamma]$. We note that $0 \in \mathcal{W}_\gamma$. We define

$$I_n = \Big\{\omega : \frac{\gamma}{n+1} < |\omega| \leq \frac{\gamma}{n}\Big\}, \quad n \geq 1$$

so that (I_n) is a partition of $0 < |\omega| \leq \gamma$. Now, if $\omega \in \mathcal{W}_\gamma \cap I_n$, then for all $q \geq 1$

$$|\omega| q > \frac{\gamma}{n+1}\, q \geq \frac{\gamma}{(n+1)q} = \frac{\gamma_n}{q}, \qquad \gamma_n := \frac{\gamma}{n+1}.$$

Since $\gamma_n < \gamma$, there holds

$$|\omega q - p| > \frac{\gamma_n}{q} \quad \forall p \in \mathbb{Z},\ q \geq 1,$$

and we have proved that

$$\mathcal{W}_\gamma \cap I_n \subseteq \mathcal{B}_{\gamma_n}.$$

Since $\mathcal{B}_{\gamma_n} \subset \mathcal{B}$, by Theorem 13 $|\mathcal{B}_{\gamma_n}| = 0$ for all n. Thus $\mathcal{W}_\gamma$ is union of countably many sets having zero measure, so that $|\mathcal{W}_\gamma| = 0$.

(ii) First, we show that $\mathcal{W}_\gamma$ contains all irrationals of the form

$$\omega = [0, k, a_2, a_3, \ldots], \qquad k \geq 1, \quad a_n \in \{1, 2\} \quad \forall n \geq 2. \tag{A.26}$$

Let ω be a number of this form. For all rationals p/q such that

$$\Big|\omega - \frac{p}{q}\Big| \geq \frac{1}{2q^2}$$

we have immediately $|\omega q - p| > \gamma/q$ because $\gamma \leq 1/4$. On the other hand, if $p \neq 0$ and $q \geq 1$ satisfy

$$\Big|\omega - \frac{p}{q}\Big| < \frac{1}{2q^2}\,,$$

then by Lemma 54 $p/q = p_n/q_n$ for some $n \in \mathbb{N}$. Since $p_0/q_0 = a_0 = 0$ while $p/q \neq 0$, the index n has to be at least 1. Then by (A.14) and (A.26)

$$\left|\omega - \frac{p}{q}\right| = \left|\omega - \frac{p_n}{q_n}\right| > \frac{1}{q_n^2(a_{n+1}+2)} \geq \frac{1}{4q_n^2} \geq \frac{\gamma}{q^2}.$$

We have proved that all irrationals (A.26) belong to $\mathcal{W}_\gamma$.

Now, given any $\delta > 0$, we fix an integer $k \geq 1/\delta$. All irrationals (A.26) lie between their zero-th and first convergents, that is $\omega \in (0, 1/k) \subseteq (0, \delta)$. Then

$$\mathcal{W}_\gamma \cap (0,\delta) \supseteq \left\{\omega = [0, k, a_2, a_3, \ldots] \, : \, a_n \in \{1,2\} \quad \forall n \geq 2\right\}$$

so that $\mathcal{W}_\gamma \cap (0,\delta)$ contains a subset of continuum cardinality. □

Remark 25. $\mathcal{W}_\gamma$ is symmetric, that is $\omega \in \mathcal{W}_\gamma$ iff $-\omega \in \mathcal{W}_\gamma$. As a consequence, also $(-\delta, 0) \cap \mathcal{W}_\gamma$ has continuum cardinality for all $\delta > 0$.

Remark 26. Since $\mathcal{W}_\gamma \subset \mathcal{W}_{\gamma'}$ for all $\gamma > \gamma'$, there holds

$$\mathcal{W} := \bigcup_{\gamma>0} \mathcal{W}_\gamma = \bigcup_{n\in\mathbb{N}} \mathcal{W}_{\gamma_n}, \quad \gamma_n := \frac{1}{n+1}.$$

$\mathcal{W}$ is the union of countable many sets of zero measure, then $|\mathcal{W}| = 0$.

A.2 Diophantine numbers

We have seen that $\mathcal{D}_\tau$ is empty for all $\tau < 1$, while for $\tau = 1$ $\mathcal{D}_1$ is the set $\mathcal{B}$ of badly approximable numbers, which is an uncountable set of zero measure. Now we prove that, for $\tau > 1$, $\mathcal{D}_\tau$ is a set of full measure, that is, almost every real number belongs to $\mathcal{D}_\tau$. The proof does not need continued fractions.

Lemma 56. *Let $\tau > 1$. Then:*

(i) For every interval (a,b) with $a < b$ there exists a positive constant C, depending only on the length $b-a$ and on τ, with the following property: for any $\gamma \in (0,1)$

$$|\mathcal{D}_{\gamma,\tau} \cap (a,b)| > b - a - C\,\gamma.$$

This means that as smaller is γ, as larger is the portion of the interval (a,b) occupied by $\mathcal{D}_{\gamma,\tau}$.

(ii) Almost every $\omega \in \mathbb{R}$ belongs to $\mathcal{D}_\tau$.

Proof. (i) Let $\tau > 1$, $\gamma \in (0,1)$ and $a < b$. For all integers p, q with $q > 0$ we define

$$\mathcal{R}_{p,q} := \left\{\omega \in \mathbb{R} : \left|\omega - \frac{p}{q}\right| \leq \frac{\gamma}{q^{1+\tau}}\right\}$$

so that the complementary $\mathcal{D}^c_{\gamma,\tau}$ of $\mathcal{D}_{\gamma,\tau}$ is the union of all $\mathcal{R}_{p,q}$, and

$$\mathcal{D}^c_{\gamma,\tau} \cap (a,b) = \bigcup_{p,\,q} \mathcal{R}_{p,q} \cap (a,b).$$

Now, $\mathcal{R}_{p,q} \cap (a,b)$ is nonempty only if

$$aq - \frac{\gamma}{q^\tau} < p < bq + \frac{\gamma}{q^\tau}\,.$$

It follows that, for any fixed positive integer q, the number of integers p such that $\mathcal{R}_{p,q} \cap (a,b)$ is nonempty satisfies the bound

$$\sharp\big\{p \in \mathbb{Z} :\ \mathcal{R}_{p,q} \cap (a,b) \neq \emptyset\big\} \ <\ (b-a)q + \frac{2\gamma}{q^\tau} + 1.$$

Every $\mathcal{R}_{p,q}$ has measure

$$\big|\mathcal{R}_{p,q}\big| \leq \frac{2\gamma}{q^{1+\tau}}\,.$$

Then

$$\big|\mathcal{D}^c_{\gamma,\tau} \cap (a,b)\big| \leq \sum_{q=1}^{\infty} \frac{2\gamma}{q^{1+\tau}} \Big((b-a)q + \frac{2\gamma}{q^\tau} + 1\Big) < C\gamma$$

where

$$C = 2(b-a+3) \sum_{q=1}^{\infty} \frac{1}{q^\tau} < \infty.$$

The series converges because $\tau > 1$.

(ii) We fix $m \in \mathbb{Z}$ and we apply (i) to the interval $(m, m+1)$. Since the complementary $\mathcal{D}^c_\tau$ of $\mathcal{D}_\tau$ is contained in all $\mathcal{D}^c_{\gamma,\tau}$, there holds

$$|\mathcal{D}^c_\tau \cap (m, m+1)| \leq |\mathcal{D}^c_{\gamma,\tau} \cap (m, m+1)| \leq C\gamma$$

for all $\gamma > 0$, so that $|\mathcal{D}^c_\tau \cap (m, m+1)| = 0$. Taking the union for all integers m gives

$$|\mathcal{D}^c_\tau| = 0,$$

that is almost every real number belongs to $\mathcal{D}_\tau$. □

Appendix B

A Nash-Moser theorem for finite regularity spaces

In this Appendix we prove an abstract Nash-Moser theorem which is modeled for applications to spaces of finite regularity functions. It is drawn from [23, Ch. 3], which we refer to for a very rich bibliography on the topics and its many applications. See also the two papers of Moser [82, 84] where he developed the core of the method, Zehnder [116] where a Nash-Moser theorem for low regularity cases is proved, and [107, 117, 66, 64, 67, 3, 48] for further developments.

We consider a Banach scale (X_s), $s \geq 0$, that is a one-parameter family of Banach spaces such that

$$0 < s < s' \quad \Rightarrow \quad X_{s'} \subset X_s \subset X_0 \quad \text{and} \quad \|u\|_s \leq \|u\|_{s'} \quad \forall u \in X_{s'},$$

and define $X_\infty := \bigcap_s X_s$. We assume that the scale admits a family of "smoothing operators"

$$S(t) : X_0 \to X_\infty, \quad t \geq 0$$

such that

$$\|S(t)u\|_{s+p} \leq C_{s,p}\, t^p \|u\|_s \quad \forall u \in X_s \tag{B.1}$$

$$\|(I - S(t))u\|_s \leq C_{s,p}\, t^{-p}\, \|u\|_{s+p} \quad \forall u \in X_{s+p} \tag{B.2}$$

for some positive constants $C_{s,p}$, where I is the identity map.[1]

[1] We have in mind, as a concrete example, spaces

$$X_s := \Big\{ u = \sum_{j \geq 1} u_j\, \varphi_j(x) \ : \ u_j \in H, \quad \|u\|^2 := \sum_{j \geq 1} |u_j|^2 \lambda_j^{2s} < \infty \Big\}$$

where H is a Banach algebra (in Chapter 6 is $H = H_0^1(\Omega)$). These spaces form a Banach

Let α, δ, K be positive constants, and consider a mapping F with the following properties.

- (H1) (Tame estimate). F maps $X_{s+\alpha}$ into X_s for all $s \geq 0$ and there holds
$$\|F(u)\|_s \leq K(1 + \|u\|_{s+\alpha}) \quad \forall u \in X_{s+\alpha}.$$
- (H2) (Taylor estimate). $F : X_{s+\alpha} \to X_s$ is differentiable for all $s \geq 0$ and
$$\|F'(u)[h]\|_s \leq K\|h\|_{s+\alpha}.$$
Moreover, denoting
$$Q(u,h) := F(u+h) - F(u) - F'(u)[h]$$
there holds
$$\|Q(u,h)\|_s \leq K\|h\|^2_{s+\alpha}.$$
- (H3) (Right inverse losing δ). For every $u \in X_\infty$ there exists a bounded linear operator $L(u)$ mapping $X_{s+\delta}$ into X_s for all $s \geq 0$, that is
$$\|L(u)[h]\|_s \leq K\|h\|_{s+\delta} \quad \forall h \in X_{s+\delta},$$
such that
$$F'(u) \circ L(u)[h] = h \quad \forall h \in X_{s+\delta}.$$

Theorem 14. *Let F satisfy (H1), (H2), (H3), and fix any $s_0 > \alpha + \delta$. If $\|F(0)\|_{s_0+\delta}$ is sufficiently small (depending on α, δ, K, s_0) then there exists a solution $u \in X_{s_0}$ of the equation $F(u) = 0$.*

Proof. We consider two constants $\lambda > 0$ and $\chi > 1$. We define

$$N_n := \exp(\lambda \chi^n), \quad S_n := S(N_n) \tag{B.3}$$

and the modified Newton's scheme

$$u_0 := 0, \quad u_{n+1} := u_n - S_n L(u_n) F(u_n). \tag{B.4}$$

We define $h_0 := 0$ and denote $h_{n+1} := u_{n+1} - u_n$. By construction, the difference h_n belongs to X_∞ for all $n \geq 1$, and also $h_0 = 0 \in X_\infty$. Since $u_n = h_0 + \ldots + h_n$, also $u_n \in X_\infty$ for all $n \geq 0$.

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scale, and truncation operators

$$S(t) : u = \sum_{j \geq 1} u_j \, \varphi_j(x) \ \mapsto \ S(t)u = \sum_{\lambda_j \leq t} u_j \, \varphi_j(x)$$

satisfy (B.1),(B.2) with constants $C_{s,p} \equiv C$ independent on s, p.

Another example of Banach scales is the class of Hölder functions. Smoothings can be constructed via convolution with mollifiers [107] or diadic decomposition [3].

By (B.4) and (B.1) we have

$$\|h_{n+1}\|_{s_0} = \|S_n L(u_n) F(u_n)\|_{s_0} \le C_{s_0-\alpha-\delta,\,\alpha+\delta}\, N_n^{\alpha+\delta} \|L(u_n)F(u_n)\|_{s_0-\alpha-\delta}$$

and by (H3)

$$\|L(u_n)F(u_n)\|_{s_0-\alpha-\delta} \le K \|F(u_n)\|_{s_0-\alpha}.$$

By Taylor expansion, for all $n \ge 1$

$$\|F(u_n)\|_{s_0-\alpha} \le \|F(u_{n-1}) + F'(u_{n-1})[h_n]\|_{s_0-\alpha} + \|Q(u_{n-1}, h_n)\|_{s_0-\alpha}.$$

Now, by (B.4) the linear part is

$$F(u_{n-1}) + F'(u_{n-1})[h_n] = F'(u_{n-1})\,(I - S_{n-1})\,L(u_{n-1})\,F(u_{n-1}),$$

so that by (H3) and (B.2)

$$\begin{aligned} \|F(u_{n-1}) + F'(u_{n-1})[h_n]\|_{s_0-\alpha} &\le K \|(I - S_{n-1})\,L(u_{n-1})\,F(u_{n-1})\|_{s_0} \\ &\le K C_{s_0,\beta}\, N_{n-1}^{-\beta}\, B_{n-1} \end{aligned}$$

where

$$B_{n-1} := \|L(u_{n-1})\,F(u_{n-1})\|_{s_0+\beta}.$$

The quadratic part can be estimated by (H2)

$$\|Q(u_{n-1}, h_n)\|_{s_0-\alpha} \le K \|h_n\|_{s_0}^2.$$

Then we have

$$\|h_{n+1}\|_{s_0} \le C_1\, N_n^{\alpha+\delta} \big(N_{n-1}^{-\beta}\, B_{n-1} + \|h_n\|_{s_0}^2 \big). \tag{B.5}$$

where

$$C_1 := K^2\, C_{s_0-\alpha-\delta,\,\alpha+\delta}\ \max\{1,\, C_{s_0,\beta}\}$$

and $\beta > 0$. In estimate (B.5) we have the quadratic part, which is typical of the classical Newton's scheme, plus an additional part B_{n-1} which is due to the presence of the smoothings in the scheme. Our goal is to ensure that B_n goes to zero so rapidly that it does not affect the super-exponential convergence of the Newton's scheme. The key ingredient to prove it is to give an *a-priori* estimate for the divergence of the B_n independent on β.

By (H3) and (H1)

$$B_n = \|L(u_n)\,F(u_n)\|_{s_0+\beta} \le K \|F(u_n)\|_{s_0+\beta+\delta} \le K^2 \big(1 + \|u_n\|_{s_0+\beta+\delta+\alpha} \big)$$

for all $n \ge 0$. For $n \ge 1$, since $u_n = \sum_{k=1}^n h_k$,

$$\|u_n\|_{s_0+\beta+\delta+\alpha} \le \sum_{k=1}^{n} \|h_k\|_{s_0+\beta+\delta+\alpha} \le \sum_{k=1}^{n} \|S_{k-1}\,L(u_{k-1})\,F(u_{k-1})\|_{s_0+\beta+\delta+\alpha}$$

by (B.4). Then by (B.1)

$$\begin{aligned}\|S_{k-1}\, L(u_{k-1})\, F(u_{k-1})\|_{s_0+\beta+\delta+\alpha} &\le C_{s_0+\beta,\,\delta+\alpha}\, N_{k-1}^{\delta+\alpha}\, \|L(u_{k-1})\, F(u_{k-1})\|_{s_0+\beta} \\ &= C_{s_0+\beta,\,\delta+\alpha}\, N_{k-1}^{\delta+\alpha}\, B_{k-1}.\end{aligned}$$

As a consequence,

$$B_n \le C_2\Big(1 + \sum_{k=0}^{n-1} N_k^{\delta+\alpha}\, B_k\Big) \quad \forall n \ge 1 \tag{B.6}$$

where

$$C_2 := K^2 \max\{1,\, C_{s_0+\beta,\,\delta+\alpha}\}.$$

Now, given a positive constant ν, we study under which conditions we are able to prove the following properties:

$$\begin{aligned} &(n;i) &&B_n \le N_n^{\nu} = \exp(\lambda\nu\chi^n) \\ &(n;ii) &&\|h_{n+1}\|_{s_0} \le N_n^{-\nu} = \exp(-\lambda\nu\chi^n). \end{aligned}$$

Since by (H3)

$$B_0 := \|L(0)F(0)\|_{s_0+\beta} \le K\|F(0)\|_{s_0+\beta+\delta},$$

for $n = 0$ the first condition holds true if

$$K\|F(0)\|_{s_0+\beta+\delta} \le \exp(\lambda\nu). \tag{B.7}$$

To verify condition $(0;ii)$, we note that

$$\|h_1\|_{s_0} = \|S_0 L(0)F(0)\|_{s_0} \le C_{s_0,0}\|L(0)F(0)\|_{s_0} \le C_{s_0,0}K\|F(0)\|_{s_0+\delta}$$

by (B.4), (B.1) and (H3). Then $(0;ii)$ is verified if

$$C_{s_0,0}K\|F(0)\|_{s_0+\delta} \le \exp(-\lambda\nu). \tag{B.8}$$

Now, let $n \ge 0$ and suppose that the conditions $(k;i), (k;ii)$ hold true for all $k \le n$. By (B.6), (B.3) and $(k;i)$, $k = 0, \dots, n$ we have

$$B_{n+1} \le C_2\Big(1 + \sum_{k=0}^{n} N_k^{\delta+\alpha}\, B_k\Big) \le C_2\Big(1 + \sum_{k=0}^{n} \exp[(\delta+\alpha+\nu)\lambda\chi^k]\Big).$$

By induction, it is easy to prove the general fact that given any $a > 0$, $\chi > 1$, if we take a constant M such that

$$M > 1, \quad \frac{M}{M-1} \le \exp[a(\chi - 1)],$$

then there holds

$$\sum_{k=0}^{n} \exp(a\chi^k) \leq M \exp(a\chi^n) \quad \forall n \geq 0.$$

We fix M taking $a = \delta + \alpha$ (so that M depends on α, δ, χ). We assume that $\lambda \geq 1$, so that $(\delta+\alpha+\nu)\lambda \geq (\delta+\alpha)$. Then there exists a constant C_3 which does not depend on ν, λ such that

$$\sum_{k=0}^{n} \exp[(\delta + \alpha + \nu)\lambda\chi^k] \leq C_3 \exp[(\delta + \alpha + \nu)\lambda\chi^n] \quad \forall n \geq 0.$$

It follows that

$$B_{n+1} \leq C_4 \exp[(\delta + \alpha + \nu)\lambda\chi^n] \quad \forall n \geq 0$$

where C_4 depends on $\delta, \alpha, s_0, \beta, K$ and it does not depend on ν, λ. Thus, $(n+1; i)$ is verified provided

$$\text{(B.9)} \qquad C_4 \exp\{[(\delta + \alpha - \nu(\chi - 1)]\lambda\chi^n\} \leq 1.$$

We prove $(n+1; ii)$. Using (B.5), (B.3), $(n; i)$ and $(n; ii)$ we obtain

$$\begin{aligned} \|h_{n+2}\|_{s_0} \ \leq \ & C_1 \exp\left\{\lambda\chi^n[\chi(\alpha + \delta) - \beta + \nu]\right\} \\ & + C_1 \exp[\lambda\chi^n[\chi(\alpha + \delta) - 2\nu)]. \end{aligned}$$

Then condition $(n+1; ii)$ is verified provided

$$\text{(B.10)} \qquad C_1 \exp\left\{\lambda\chi^n[\chi(\alpha + \delta) - \beta + \nu]\right\} \leq \frac{1}{2} \exp(-\lambda\nu\chi^{n+1})$$

and

$$\text{(B.11)} \qquad C_1 \exp[\lambda\chi^n[\chi(\alpha + \delta) - 2\nu)] \leq \frac{1}{2} \exp(-\lambda\nu\chi^{n+1}).$$

Now, conditions (B.9), (B.10) and (B.11) are satisfied if

$$\nu(\chi - 1) > \delta + \alpha, \quad \beta > \nu(1 + \chi) + \chi(\alpha + \delta), \quad \nu(2 - \chi) > \chi(\alpha + \delta)$$

and λ is large enough. All conditions are satisfied with the choice

$$\beta := 15(\alpha + \delta), \quad \nu := 4(\alpha + \delta), \quad \chi := \frac{3}{2},$$

λ larger than a constant depending on K, s_0, α, δ, and at last $\|F(0)\|_{s_0+\delta}$ small by (B.8).

We have proved, by induction, that $(n; i)$, $(n; ii)$ hold for all $n \geq 0$. Condition $(n; ii)$ implies that (u_n) is a Cauchy sequence in X_{s_0}, so that it converges in X_{s_0} to some $u \in X_{s_0}$.

In addition, by previous inequalities it follows that

$$\|F(u_n)\|_{s_0-\alpha} \leq KC_{s_0,\beta}N_{n-1}^{-\beta}B_{n-1} + K\|h_n\|_{s_0}^2.$$

Then $(n; i)$ and $(n; ii)$ imply that $\|F(u_n)\|_{s_0-\alpha} \to 0$. By the continuity of F we have $F(u) = 0$. □

Remark 27. The scheme we have used in Chapter 6 to deal with Sobolev regularity spaces is a simpler version of Theorem 14. Indeed, the Kirchhoff nonlinearity

$$u \;\mapsto\; \Delta u \int_\Omega |\nabla u|^2 \, dx$$

has a special symmetry, a sort of "diagonal" structure. As a consequence, using the truncations P_n as smoothings operators, the general Newton's scheme (B.4) can be written in the special form (6.14).

Appendix C

Algebra property for spaces of periodic functions

Let H be a Banach algebra, that is $(H, |\cdot|_H)$ is a Banach space and there exists a positive constant C_H such that

$$h, k \in H \;\Rightarrow\; hk \in H, \quad |hk|_H \leq C_H |h|_H |k|_H .$$

Let d be a positive integer and $s > d/2$. Consider a sequence $(w_j)_{j \in \mathbb{Z}^d}$ of positive numbers ("weights") such that

$$w_{j+k} \leq w_j w_k \quad \forall j, k \in \mathbb{Z}^d . \tag{C.1}$$

We define $X_{w,s}$ as the space of periodic functions

$$u : \mathbb{T}^d \to H, \quad u(\varphi) = \sum_{j \in \mathbb{Z}^d} u_j \, e^{ij\varphi}, \quad u_j \in H \quad \forall j$$

such that

$$\sum_{j \in \mathbb{Z}^d} |u_j|_H^2 (1 + |j|^{2s}) \, w_j =: \|u\|_{w,s}^2 < \infty$$

where $j\varphi = j_1\varphi_1 + \ldots + j_d\varphi_d$ and $|j|^2 = j_1^2 + \ldots + j_d^2$. In this way we are dealing with simultaneously several important spaces:

$$\begin{cases} w_j = 1 & \text{Sobolev spaces} \\ w_j = e^{2|j|^a} & \text{Gevray spaces} \\ w_j = e^{2\sigma|j|} & \text{Abel spaces (analytic functions)} \end{cases}$$

where $0 < a < 1$ and $\sigma > 0$.

Lemma 57. (Algebra property). *Let $s > d/2$ and let (w_j) satisfy* (C.1). *Then*

$$u, v \in X_{w,s} \quad \Rightarrow \quad uv \in X_{w,s} \, , \quad \|uv\|_{w,s} \leq C \|u\|_{w,s} \|v\|_{w,s}$$

where

$$C = 2^s C_H \Big(\sum_{k\in\mathbb{Z}^d} \frac{1}{1+|k|^{2s}} \Big)^{1/2}.$$

We note that the algebra constant C does not depend on the weights w_j.

Proof. Let $u, v \in X_{w,s}$. The product uv is

$$uv = \sum_j \Big(\sum_k u_{j-k} v_k \Big) e^{ij\varphi},$$

so that its norm in $X_{w,s}$, if it converges, is

$$\|uv\|^2_{w,s} = \sum_j \Big| \sum_k u_{j-k} v_k \Big|^2_H \big(1+|j|^{2s}\big)\, w_j.$$

We define $a_{jk} > 0$ as

$$a^2_{jk} = \frac{\big(1+|j-k|^{2s}\big)\big(1+|k|^{2s}\big)}{\big(1+|j|^{2s}\big)}.$$

Given any $(x_k)_k \subset H$, by Hölder inequality

$$\Big| \sum_k x_k \Big|^2_H \le \Big(\sum_k \frac{1}{a_{jk}}\, a_{jk} |x_k|_H \Big)^2 \le c_j^2 \sum_k a^2_{jk} |x_k|^2_H, \tag{C.2}$$

where

$$c_j^2 := \sum_k \frac{1}{a^2_{jk}}.$$

We recall that, fixed $p \ge 1,$ there holds

$$(a+b)^p \le 2^{p-1}\left(a^p + b^p\right) \quad \forall\, a, b \ge 0.$$

Then

$$\begin{aligned} 1+|j|^{2s} &\le 1 + \big(|j-k| + |k|\big)^{2s} \\ &\le 1 + 2^{2s-1}\left(|j-k|^{2s} + |k|^{2s}\right) \\ &< 2^{2s-1}\left(1+|j-k|^{2s} + 1 + |k|^{2s}\right) \end{aligned}$$

and

$$\frac{1}{a^2_{jk}} < 2^{2s-1} \Big(\frac{1}{1+|j-k|^{2s}} + \frac{1}{1+|k|^{2s}} \Big).$$

The series

$$\sum_{k\in\mathbb{Z}^d} \frac{1}{1+|k|^p} \qquad \Big(\sim \int_{\mathbb{R}^d} \frac{1}{1+|x|^p}\, dx \Big)$$

converges for $p > d$. By assumption $s > d/2$, then

$$\begin{aligned} c_j^2 &< 2^{2s-1}\Big(\sum_k \frac{1}{1+|j-k|^{2s}} + \sum_k \frac{1}{1+|k|^{2s}}\Big) \\ &= 2^{2s} \sum_{k\in\mathbb{Z}^d} \frac{1}{1+|k|^{2s}} =: c^2 < \infty. \end{aligned}$$

Since $c_j < c$ for all j, taking $x_k = u_{j-k}v_k$ in (C.2) gives

$$\begin{aligned} \Big|\sum_k u_{j-k}v_k\Big|_H^2 \big(1+|j|^{2s}\big) &\leq c^2 \sum_k a_{jk}^2 |u_{j-k}v_k|_H^2 \big(1+|j|^{2s}\big) \\ &= c^2 \sum_k |u_{j-k}v_k|_H^2 \big(1+|j-k|^{2s}\big)\big(1+|k|^{2s}\big) \end{aligned}$$

by definition of a_{jk}. Then

$$\begin{aligned} \|uv\|_{w,s}^2 &= \sum_j \Big|\sum_k u_{j-k}v_k\Big|_H^2 \big(1+|j|^{2s}\big)\, w_j \\ &\leq \sum_j c^2 \sum_k C_H^2 |u_{j-k}|_H^2 |v_k|_H^2 \big(1+|j-k|^{2s}\big)\big(1+|k|^{2s}\big)\, w_{j-k}w_k \\ &= c^2 C_H^2 \sum_k \Big(\sum_j |u_{j-k}|_H^2 \big(1+|j-k|^{2s}\big)\, w_{j-k}\Big) |v_k|_H^2 \big(1+|k|^{2s}\big)\, w_k \\ &= c^2 C_H^2 \|u\|_{w,s}^2 \|v\|_{w,s}^2. \end{aligned}$$

□

Lemma 57 applies, for example, to the spaces $\mathcal{H}_\sigma$ defined in Chapter 3 taking $d = 2$ and $H = \mathbb{R}$. Elements of $\mathcal{H}_\sigma$ are real analytic functions $u(\varphi_1, \varphi_2)$ which are 2π-periodic in both their arguments (φ_1, φ_2).

Lemma 57 applies also to the spaces $X_{\sigma,s}$ defined in Chapter 5 taking $d = 1$ and $H = H_0^1(0,\pi)$. Elements of $X_{\sigma,s}$ are functions $u(x,t)$ satisfying Dirichlet boundary conditions in space and periodic in time, that is

$$u(0,t) = u(\pi,t) = 0, \quad u(x,t) = u(x,t+2\pi) \quad \forall (x,t) \in (0,\pi)\times\mathbb{R},$$

defined by series

$$u(x,t) = \sum_{j\in\mathbb{Z}} u_j(x)\, e^{ijt}, \qquad u_j \in H_0^1(0,\pi) \quad \forall j,$$

having regularity H^1 in x and analytic in t given by

$$\sum_{j\in\mathbb{Z}} \|u_j\|_{H_0^1(0,\pi)}^2 (1+|j|^{2s})\, e^{2\sigma|j|} < \infty.$$

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