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Disordered d-wave superconductors: the role of nesting and interactions in transport properties

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Doctor Philosophiæ

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Introduction

After the discovery of superconductivity at temperatures more than one hundred Kelvin degrees in doped cuprate materials [1], a lot of efforts have been devoted to find a theoretical justification of this amazing property, which effectively initiated a new research field within solid state physics. Yet, till now there is no theory able to explain the complex phenomenological aspects of these "high T_c superconductors". One of the novel features of these cuprates is the presence of gapless Landau-Bogoliubov quasiparticle excitations in the superconducting phase, due to the d-wave symmetry of the order parameter. This peculiarity makes these materials a suitable play ground for studying the role of disorder in gapless superconductors.

The present work would like to contribute to the widely studied but somehow still controversial topic of the role of disorder in these unconventional superconductors.

Motivations and main results

There were several aspects which originally attracted our theoretical interest and were the motivations for this study.

The first one is that, due to the gauge symmetry breaking, charge is not a conserved quantity in a superconducting state. Therefore although quasiparticles are gapless for a d-wave order parameter, nevertheless long-wavelength quasiparticle charge fluctuations are not diffusive in the presence of weak disorder, unlike in a normal metal. Indeed the diffusive modes just carry spin or energy, which remain conserved quantities. It is well known that, as disorder increases, quantum interference may lead to the Anderson localization in a normal metal. While it is clear that such a phenomenon may suppress spin and thermal conductivities in a d-wave superconductor, it was not at all clear to us the effects on the quasiparticle charge conductivity. For this reason we extended existing quantum field theory approaches built to deal with the truly diffusive modes in a d-wave superconductor to include the charge modes, which acquire a mass term by the onset of superconductivity. In this way we have been able to calculate how charge conductivity is modified by disorder in comparison with spin and thermal conductivities.

A second aspect which was very attractive to us had to do with some controversial results about the quasiparticle density of states at the chemical potential in the presence of disorder. Within the so called self consistent T-matrix approximation

scheme[2], it was found that, while in the pure system the density of states vanishes linearly as the Fermi energy is approached, in the presence of disorder it acquires a finite value. This was used later on as the starting point to build up a standard field-theoretical approach based on the non-linear σ-model to cope with the quantum interference corrections non included within the self-consistent T-matrix approximation[5].
This standard perturbative technique was nevertheless unsatisfactory. It was shown [3] that systems with nodes in the spectrum need a more careful analysis.
The quasiparticle spectrum in a d-wave superconductors can be described by 2-dimensional (2D) Dirac fermions, with conical spectrum. In the absence of interaction, the 2D quantum problem in the presence of disorder becomes effectively a 2D classical problem with the frequency of the single particle Green's function playing the role of an external field. On the other hand, a classical model in 2-dimensions with conical spectrum is analogous to a quantum problem of Dirac fermions in 1+1 dimension. In this language, the disorder average within the replica trick method generates an effective interaction among the one-dimensional (1D) fermions, with all the complications that are known to occur. For instance, translated in the 1D language, the self-consistent T-matrix approach which generates a finite density of states at the Fermi energy is analogous to the Hartree-Fock approximation for treating interaction in 1D, which always leads to density-wave order parameters. However, it is known that Hartree-Fock is extremely incorrect in 1D, which poses serious doubt about the validity of the T-matrix approach even as a starting point of a perturbative treatment.

In some peculiar cases, like the one analysed in Ref. [3] in which at most pairs of opposite nodes are coupled by disorder, the perturbation theory above the T-matrix saddle point solution does not contain any small parameter, like the inverse conductance in the conventional Anderson localization, hence is completely meaningless. Indeed by a more rigorous approach where both crossing and non-crossing diagrams were treated on equal footing, the authors of Ref. [3] were able to show that the density of states still vanishes at the chemical potential although with anomalous disorder-strength dependent exponent (one-node only) or universal one (pair of opposite nodes). However, in the most general case of disorder, when all four nodes are coupled, we will show that a small parameter in the perturbation theory above the saddle-point solution still exist being related to the anisotropy of the Dirac cones. This allows a conventional field-theory treatment, which nevertheless shows novel features like a renormalization of the saddle point density of states. Finally we were interested to understand if and how Wess-Zumino-Witten terms might arise in the disordered problem, and with which consequences.

A third aspect that aroused our curiosity was the role of the nesting property in these kind of systems. Given a generic eigenfunction with energy E and amplitude

$\phi(i)_E$ at site $i = (n, m)$, the operator

$$\mathcal{O}_\pi \phi(i)_E \equiv (-1)^{n+m} \phi(i)_E, \tag{1}$$

which shifts by (π, π) the momentum, generates the eigenfunction with energy $-E$ if nesting occurs. This implies an additional symmetry (chiral symmetry) at $E = 0$, when the two wavefunctions $(1 \pm (-1)^{n+m})\phi_{E \to 0^+}$, defined on different sublattices, with $n + m$ even or odd, are both eigenvectors. The nesting property occurs when the operator $\mathcal{O}_\pi$ anticommutes with the Hamiltonian, which is possible in models in which the Hamiltonian contains only terms which couple one sublattice with the other, so called *two-sublattice* models. In addition, the chiral symmetry further requires half-filling. Both conditions are quite strict and do not represent a common physical situation. Nevertheless chiral symmetry leads to quite different and somehow surprising scaling behaviors that are worth to be studied. It was seen, for instance, that this symmetry drastically change the low energy density of states. Several models presenting a chiral symmetry were found to have isolated delocalized states at the band center at low dimensions. It was argued [6] that these models corresponds to a particular class of non linear σ models and was shown that quantum corrections to the β function which controls the scaling behavior of conductivity vanish at the band center at all order in disorder strength, leading to a metallic behavior at that value of chemical potential. Moreover the β function of the density of states was found to be finite, unlike in the standard Anderson localization. These scaling laws generates a divergent behavior at low energy of density of states. The anomalous terms in the action when chiral symmetry holds were found to be connected with fluctuations of the staggered density of states [7]. The modes representing these fluctuations are massive in standard non linear σ models, while become diffusive in two sublattice cases. For this reason even retarded-retarded and advanced-advanced channels in conductance acquire diffusive poles and contribute to quantum interferences corrections. This is what we saw happening also in our two sublattices d-wave superconductive model that presents extended states at the band center which are associated with diffusive spin transport. Furthermore we found an unexpected charge conductance behavior. As we have said before, although charge modes in d-wave superconductors are not diffusive, nevertheless quantum interference corrections affect charge conductance. In particular, when chiral symmetry holds but time-reversal symmetry is broken, quasiparticle charge conductivity is suppressed, but spin and thermal conductivities stay finite, leading to a spin-metal but charge-insulator quasiparticle behavior. Moreover we saw that, even though magnetic fields or magnetic impurities introduce on-site terms in the Hamiltonian that spoil sublattice symmetry, staggered fluctuations are not totally suppressed introducing other symmetries in the model under study. We saw, for instance, that the problem of d-wave superconductors with chiral symmetry and magnetic impurities can be mapped to a U$(2n)$ non linear σ model and belongs accidentally to the same universality class of the case with d-wave superconductors far from nesting point embedded in a constant magnetic field. Moreover, the chiral symmetry can lead to all other surprising features that are known to occur in Wess-Zumino-Witten models in 1+1 dimension.

From the point of view of the cuprate d-wave superconductors, it is not unlikely that chiral symmetry may play some role, especially in underdoped systems close to the half-filled Mott insulator. Indeed it is believed that the impurity potential is close to the unitary scattering limit, in which it essentially reduces to a random nearest-neighbor hopping. Furthermore, although the band structure does not have a perfect nesting, the superexchange interaction which stabilizes a Neèl antiferromagnetic phase at half-filling may effectively reduce the energy scale at which deviations from perfect nesting get appreciable.

The last aspect that attracted our attention was the role of residual quasiparticle interaction and its effects on the conductivity and on the density of states. Following the original Finkel'stein's approach which extended effective functional method to disordered electron-electron interacting systems, we introduced effective quasiparticle scattering amplitudes in different channels, firstly considering systems without sublattice symmetry. We found that, consistently with the charge not being a conserved quantity, the singlet particle-hole channel does not contribute. On the other hand, scattering amplitudes in Cooper particle-particle channel acquires a factor 1/2 with respect to normal metal state, which correspond to the fact that only the phase of the order parameter is massless. We saw that the effective interaction, that we assumed being repulsive, has a delocalizing effect enhancing the density of states. We extended the Finkel'stein model in order to include nesting property, by introducing additional scattering amplitudes with $(\pi\pi)$ momentum transferred, and we evaluated the new corrections to the density of states and to the conductivity.

To conclude we notice an interesting fact which occurs at half-filling with a two-sublattice model. The staggered particle-hole fluctuations being diffusive lead to a log-divergent staggered susceptibility, implying a Stoner instability towards spin or charge density wave depending upon the sign of the interaction. In some sense the analogous of the Anderson's theorem for disorder in s-wave supercondutors holds for staggered fluctuations at half-filling.

Scheme of the thesis

The present thesis is divided into five chapters while some technical details are present in five appendixes and it is organized in the following way.

In Chapter 1 we will present a brief introduction to the Anderson localization that occurs in disordered systems. We will discuss how to calculate some relevant physical quantities in the presence of random impurities and we will illustrate one of the possible methods commonly used in dealing with disordered systems that resorts to the quantum field theory approach and takes advantage of the replica trick.

In Chapter 2 we will see what kind of materials are the so called "high T_c superconductors", pointing out some phenomenological properties. Then we will present a BCS model for the low temperature regime and apply to it a path integral formulation following the framework illustrated in chapter 1, paying attention to its peculiarities.

In Chapter 3 we will consider impurities into the Hamiltonian illustrated before and we will perform the average over them supposing the disorder gaussian distributed. The model, bilinear in fermionic fields, will acquire a quartic term that shall be decoupled by Hubbard-Stratonovich transformation introducing an auxiliary bosonic field. We will examine into details the various symmetry properties showed by the model with and without time reversal invariance, in the presence or absence of chiral symmetry [6], in a constant magnetic field or in the presence of magnetic impurities. Finally we will derive the non linear σ model representing transverse fluctuations, namely massless changing vacuum modes, around the saddle point of the action, finding that the coupling constant of these modes is the bare spin conductivity.

In Chapter 4 we will carry on the renormalization group within the Wilson-Polyakov procedure to derive the scaling behavior of the action obtained in the previous chapter, and calculate the quantum interference corrections to the spin conductivity and to the density of states in different universality classes. We will also evaluate corrections to charge conductivity.

In Chapter 5 we will consider the residual interaction in d-wave superconductors as previously done by Finkel'stein in the case of normal metal. We will include interactions within the path integral formulation and perform the renormalization group by Wilson-Polyakov procedure in one loop perturbation theory. This allows us to evaluate interaction corrections to the conductivity, to the density of states and to the interaction amplitudes. Moreover we will further consider interactions with $(\pi\pi)$ momentum transferred, which are relevant close to a nesting point.
At the end we will summarize the main results of this work in the final conclusions.

Chapter 1

Brief review on disordered systems

In this chapter we give a brief introduction to the scaling theory of the Anderson localization. After having defined some physical quantities of interest in transport phenomena, we present the conventional quantum field theory approach based on the replica trick commonly used to deal with disorder.

1.1 Basic concepts

In 1958 P.W. Anderson [9] pointed out that electronic wave functions in a random potential may profoundly be altered if the randomness is sufficiently strong. The traditional view had been that scattering by the random potential caused the Bloch waves to lose phase coherence on the length scale of the mean free path l but the wave function still remained extended throughout the sample.
Anderson asserted that, if the disordered is very strong in systems with dimensions greater than 2, the wave functions may become localized, that's to say the envelope of the wave function decays exponentially, while in systems with dimensions equal or less than 2 localization occurs for any amount of disorder.
The resistance in conductors is determined by elastic scattering of electrons from impurities or defects always present in the lattice. In the semiclassical Drude-Boltzman theory, the electron moves between collision as a classical free particle provided its wavelength, $\lambda = 2\pi\hbar/p$, where p is its momentum, is much less than its mean free path l. This description yields the well-known Drude expression for conductivity

$$\sigma = \frac{e^2 \tau n}{m} \tag{1.1}$$

where n is the electron concentration, m the effective mass, $\tau = l/v$ the time between collisions with v the velocity of the particle, and e the electronic charge.
The validity of this theory is based on the semiclassical hypothesis that $\lambda \ll l$, so at very low impurity concentration. When instead λ becomes of the order of

mean free path, Drude theory is unjustified and we need to include quantum effects. At sufficient high impurity concentration, in fact, when $\lambda \gg l$, the electron states become localized and do no longer contribute to conduction. The transition from delocalized to localized states occurring upon increasing disorder is called Anderson localization.

The nature of this transition resides in quantum interferences [10] and so the Anderson localization is a quantum effect. Let's consider a particle moving from a point A to a point B. Quantum mechanically, the probability P_{AB} of going from A to B is the square of the sum of the amplitudes a_i for the particle to pass along all possible paths

$$P_{AB} = |\sum_i a_i|^2 = \sum_i |a_i|^2 + 2\sum_{i\neq j} Re a_i a_j^* \tag{1.2}$$

For most of the paths the interference is not essential since their lengths differ a lot and hence also the phases of the wave functions. Therefore summing over all such paths the mean value of the interference term will vanish. However there are paths of different kind, that we can call self intersecting paths, which correspond to the passage of the loop clock and counterclockwise (Fig.1.1). These two paths have

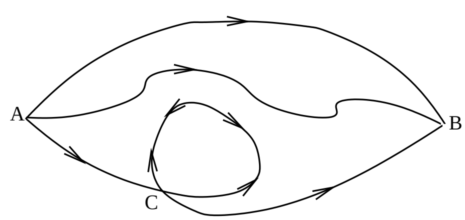

Figure 1.1: Different paths for a particle to move from point A to point B, C is the cross point of a self intersecting path.

coherent amplitudes, namely $a_1 = a_2^*$, and the interference can't be neglected since the probability to find the particle at point C is

$$|a_1|^2 + |a_2|^2 + 2Re a_1 a_2^* = 4|a_1| \tag{1.3}$$

twice the classical one, namely the one which does not include interference. Enhancing the probability to find a particle at the same point means reducing probability to find it at point B, that's to say decreasing conductivity.

Let us first analyse this problem in the quasiclassical regime $\lambda \ll l$. Because of the collisions with impurities, the electronic paths in the quasiclassical regime follow a random walk pattern and since the negative correction to conductivity is supposed to be proportional to the return probability, from diffusion equation we have

$$\frac{\delta\sigma}{\sigma} \propto -\int_{\tau}^{\tau_\phi} \frac{v\lambda^2\, dt}{(Dt)^{d/2}} \tag{1.4}$$

where D is the diffusion coefficient, τ_ϕ is the inelastic relaxation time (the time the wave function retains its coherence) [11] and d the effective dimension of the system. Integrating and defining $L = \sqrt{D\tau_\phi}$, one finds for different dimensions the following scaling relations

$$\begin{array}{lll} \delta\sigma \sim & \frac{e^2}{\hbar}\frac{1}{L} + cost & d=3 \\ \delta\sigma \sim & -\frac{e^2}{\hbar}\log(\frac{L}{l}) & d=2 \\ \delta\sigma \sim & -\frac{e^2}{\hbar}(L-l) & d=1 \end{array}$$

By defining the dimensionless quantity

$$g(L) = \sigma\, L^{d-2} \tag{1.5}$$

the above relations correspond to the following limit of the β-function in the quasi-classical regime. $g \gg 1$,

$$\beta(g) = \frac{d\log g}{d\log L} = (d-2) - \frac{a}{g} \tag{1.6}$$

In the limit of strong impurity concentration, localized states very close in energy are very far apart in space so that the hopping matrix element between them is exponentially small. Being ξ the localization length one expects that in this regime, for $L \gg \xi$,

$$g(L) \propto e^{-\frac{L}{\xi}} \tag{1.7}$$

and so the other limit of $g \ll 1$ leads to

$$\beta(g) = \log\frac{g}{g_c}. \tag{1.8}$$

If one assumes that the β-function of g only depends on g itself, so-called one parameter scaling assumption [12, 13], then the qualitative behavior of $\beta(g)$ can be analyzed in the simplest possible way by interpolating between the limiting expressions [12]

$$\lim_{g\to\infty} \beta(g) \to d-2 \tag{1.9}$$

for weakly disordered metallic phase and

$$\lim_{g\to 0} \beta(g) \to \log\frac{g}{g_c} \tag{1.10}$$

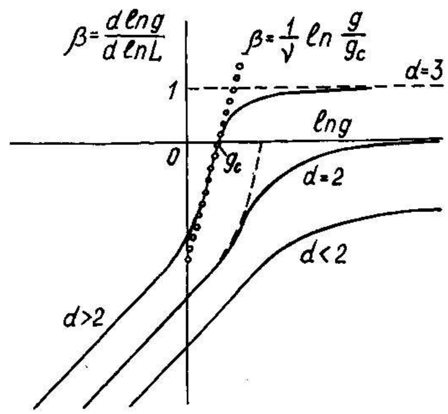

Figure 1.2: β function vs dimensionless conductance g

for the strongly disordered insulator. Following these assumptions and supposing monotonous and continuous behavior of $\beta(g)$, one can draw a tentative plot of $\beta(g)$ as shown in Fig. 1.2. We see that $\beta(g)$ has no zeros for $d < 2$. If expansion (1.6) is valid there is no zero also for $d = 2$. For $d > 2$, the β function must have a zero, $\beta(g_c) = 0$. The existence of a zero of $\beta(g)$ corresponds to existence of an unstable fixed point. The state of a system is supposedly determined by distances of the order of mean free path l. Using $g_0 = g(L = l)$ as an initial value and integrating $\beta(g) = \frac{d \log g}{d \log L}$ it is easy to find that for $g_0 > g_c$ conductivity $\sigma_L = g(L)L^{2-d}$ tends for $L \to \infty$ to a constant (metallic) value. For $g < g_c$ in the limit of $L \to \infty$ we get an insulating behavior [12, 13, 14].

1.2 Quantum Field Theory's formulation

As we have mentioned in the Introduction, in this work we will adopt a quantum field theory approach to tackle the problem of disordered system. The starting point is to express relevant physical quantities in terms of Green's functions. Then by replica trick and using Grassmann variables we will be able to build up a path integral action similar to what Efetov, Larkin and Khmel'nitsky [16] did for the standard problem.

1.2.1 Physical quantities of interest

Physical quantities like conductivity, density of states, density-density correlation function can be usually written in terms of Green's functions

$$G^{\pm}_{\varepsilon}(x,y) = \sum_k \frac{\phi^*_k(x)\phi_k(y)}{\varepsilon - E_k \pm i\eta} \tag{1.11}$$

with E_k the eigenstate of the Hamiltonian

$$H\phi_k = E_k\phi_k. \tag{1.12}$$

G^+ end G^- stand for retarded and advanced Green functions.
For instance, by the Kubo formula [15] the electrical conductivity can be written as

$$\sigma(\omega) = \frac{e^2}{4\pi m^2}\int dr dr' d\varepsilon \frac{n_{\varepsilon+\omega} - n_\varepsilon}{\omega} Tr\left(\hat{p}(G^+_{\varepsilon+\omega}(r,r') - G^-_{\varepsilon+\omega}(r,r'))\hat{p}(G^+_{\varepsilon}(r',r) - G^-_{\varepsilon}(r',r))\right) \tag{1.13}$$

where n_ϵ is the Fermi distribution function and $\hat{p}$ the momentum operator. Analogously the density of state is given by

$$\langle \rho_\varepsilon(r)\rangle = \frac{1}{2\pi i} Tr(G^+_\varepsilon(r) - G^-_\varepsilon(r)) \tag{1.14}$$

while the density density structure factor

$$\langle[\rho(r,t),\rho(r',0)]\rangle = \int d\omega d\varepsilon e^{i\omega(t-t')}\frac{n_\varepsilon - n_{\varepsilon+\omega}}{\omega} K(r,r',\varepsilon,\omega) \tag{1.15}$$

with

$$K(r,r',\varepsilon,\omega) = G^-_\varepsilon(r,r')(G^+_{\varepsilon+\omega}(r',r) - G^-_{\varepsilon+\omega}(r',r)). \tag{1.16}$$

In the presence of disorder, parametrized by some probability distribution, we need to calculate

$$\overline{G(\varepsilon)} = \overline{(\varepsilon - H \pm i\eta)^{-1}} \tag{1.17}$$

and also

$$\overline{(\varepsilon - H \pm i\eta)^{-1}(\varepsilon - H \pm i\eta)^{-1}} \tag{1.18}$$

where the overline represents the average over the disorder distribution.

1.2.2 The replica method

Calculating the average over disorder represented by a random variable s with distribution $P(s)$ of the quantum average of an operator O means by definition

$$\overline{\langle O\rangle} = \overline{\left(\frac{Tr(e^{-\beta H(s)}O)}{Tr(e^{-\beta H(s)})}\right)} = \int ds\, P(s)\,\frac{Tr(e^{-\beta H(s)}O)}{Tr(e^{-\beta H(s)})} \tag{1.19}$$

In this expression the random potential is involved both in the numerator and in the denominator and leads usually to untractable calculations. A way to avoid this difficulty is provided by the replica trick. If we introduce N independent replicas of the system, each one described by the same disordered Hamiltonian, and assume that an analytic continuation in N is meaningful so that N can be treated as a continuous variable, one easily verifies that (1.19) is equal to

$$\begin{aligned}\overline{\langle O\rangle} &= \lim_{N\to 0} \overline{\mathcal{Z}^{N-1} Tr_1(e^{-\beta H_i(s)} O_i)} \\ &\equiv \lim_{N\to 0} \frac{1}{\overline{\mathcal{Z}^N}} \int ds\, P(s)\, \mathrm{Tr}(e^{-\beta \sum_{i=1}^N H_i(s)} (O_i \otimes \mathbb{I}_{N-1}))\end{aligned} \tag{1.20}$$

where H_i and O_i are the Hamiltonian and the operator O acting on one specified replica, replica i in the example, being $\mathcal{Z} = Tr(e^{-\beta H_i(s)})$ its partition function, which is the same for all replicas. Equation (1.20) shows that the impurity averaging transform within the replica trick to a quantum averaging with an effective density matrix

$$\rho(\beta) = \int ds\, P(s) e^{-\beta \sum_{i=1}^N H_i(s)},$$

the cost being the limit $N \to 0$ which needs to be performed at the end of the calculation.

1.2.3 Path integral formulation

In real space the retarded-advanced Green functions are written in (1.11) where in the the Hamiltonian we distinguish two terms

$$H = H_0 + H_1, \tag{1.21}$$

being H_0 the regular part of the Hamiltonian and H_1 a zero average random potential, $\langle H_1\rangle = 0$. The problem of averaging Eq.1.16 over the distribution of impurities could be faced by expanding the Green's functions in terms of H_1 and take the average of each term.
Another way is to resort to a field integral formulation using the replica method previously described.
The starting point is to write the Green's functions in terms of a grassmannian path integral. One considers two families of Grassmann variables $\{c\}$ and $\{\bar{c}\}$ satisfying

$$\{c_i, \bar{c}_j\} = \{c_i, c_j\} = \{\bar{c}_i, \bar{c}_j\} = 0 \tag{1.22}$$

$$\int dc_i = \int d\bar{c}_i = 0 \qquad \int c_i dc_i = \int \bar{c}_i d\bar{c}_i = 1 \tag{1.23}$$

From the above formulas it follows that

$$\frac{1}{\varepsilon - E_k} = i \int d\bar{c}_k dc_k \, c_k \bar{c}_k e^{(-i\bar{c}_k(\varepsilon - E_k)c_k)} \Big/ \int d\bar{c}_k dc_k \, e^{(-i\bar{c}_k(\varepsilon - E_k)c_k)}. \tag{1.24}$$

Using the representation (1.11) we can express the Green function in terms of a continuum integral

$$G^{\pm}_{\varepsilon}(x,y) = i \int d\bar{c}\, dc\, c(x)\bar{c}(y) e^{\left(-i\int dz\, \bar{c}(z)(\varepsilon - H \pm i\eta)c(z)\right)} \Big/ \int d\bar{c}\, dc\, e^{\left(-i\int dz\, \bar{c}(z)(\varepsilon - H \pm i\eta)c(z)\right)} \tag{1.25}$$

with

$$\bar{c}(x) = \sum_k \bar{c}_k \phi^*_k(x) \qquad c(x) = \sum_k c_k \phi_k(x) \tag{1.26}$$

Now introducing replica indices and integrating over disorder, for one particle and two particles Green functions we have the following Lagrangian formulation

$$\overline{G^{\pm}_{\varepsilon \pm \frac{\omega}{2}}(x,y)} = \lim_{N\to 0} \frac{1}{\overline{\mathcal{Z}^N}} \int ds P(s) \int d\bar{c}\, dc\, c_1(x)\, \bar{c}_1(y)\, e^{-S(s)} \tag{1.27}$$

$$\overline{K_\omega(x,y)} = \lim_{N\to 0} \frac{1}{\overline{\mathcal{Z}^N}} \int ds P(s) \int d\bar{c}\, dc\, c_1(x)\, \bar{c}_1(y)\, c_{N+1}(y)\, \bar{c}_{N+1}(x)\, e^{-S(s)} \tag{1.28}$$

with

$$S(s) = -i \int dz \sum_{\alpha=1}^{2N} \bar{c}_\alpha(z) \left(\varepsilon - H(s) + \Lambda_\alpha \left(\frac{\omega}{2} + i\eta\right)\right) c_\alpha(z) \qquad \Lambda_\alpha = \begin{cases} +1, & \alpha \le N \\ -1, & \alpha > N \end{cases}$$

Chapter 2

The model

In this chapter we describe some phenomenological aspects of the cuprates d-wave superconductors with the aim of deriving the simplest realistic model for disorder in the superconducting phase.

2.1 High T_c superconductors: crystal structure and phenomenology

The High T_c superconductors derive from layered perovskite and are characterized by the presence of weakly coupled conducting CuO_2 planes. Fig.2.1 shows the crystal structure of YBCO compounds, one of the cuprates of the family of High T_c materials. The unit cell contains two CuO_2 planes, for which reason they are also called bilayer cuprates. The Y ions lye between these planes providing a weak tunneling among them. Between two pairs of CuO_2 planes there are two BaO planes and a plane of CuO chains, providing a much weaker coupling among the bilayers. The strong spatial anisotropy implies that all the relevant conducting and superconducting properties have to do with isolated CuO_2 planes, and that the weak interplane coupling only allows long range order to set up at finite temperatures, which would otherwise forbidden by the Hohenberg-Mermin-Wagner theorem [18] in a pure two-dimensional system.

In the stochiometric case, $Cu^{2+}\,(O^{2-})_2$, each Cu ion in the plane is in a d^9 configuration with one hole in the $d_{x^2-y^2}$ orbital, which, due to the crystal field, lyes much above the other d-orbitals. Although the valence band originating from the $d_{x^2-y^2}$ orbitals is half-filled, the system is a Mott insulator due to the strong correlations. The spins of the localized holes are coupled together by the superexchange J due to the virtual hopping through the O-ions, which is experimentally estimated to be $J \simeq 1550$ K [17], and undergo a Neèl ordering below a critical temperature T_N (see Fig. 2.2).

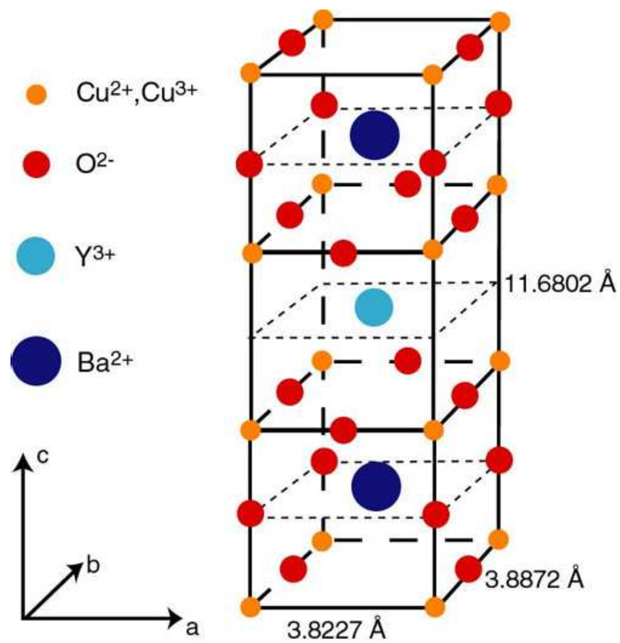

Figure 2.1: $YBa_2Cu_3O_7$

Doping introduces carriers into the Mott insulating CuO_2 planes. In the specific example of YBCO, Fig.2.1, doping is accomplished by addition of O-atoms to the stochiometric $YBa_2Cu_3O_6$ insulator. The dopant atoms go into the planes containing the CuO chains although the additional holes are mostly injected into the CuO_2 planes. Upon doping the Neèl temperature falls off quite rapidly, until, after the magnetic ordering melts into a spin glass phase, the system becomes metallic and superconducting below T_c. The superconducting transition temperature initially increases with doping (underdoped regime) but, after reaching a maximum (optimal doping) decreases (overdoping) and finally disappears at a critical doping above which the system stays metallic up to zero temperature.

At optimal doping the critical temperature for these cuprates can go from 36 K in $La_{2-x}Sr_xCuO_4$, to 92 K in YBCO up to 135 K in $HgBa_2Ca_2Cu_3O_{8+x}$ [19]. Although the precise mechanism leading to such an high T_c superconductivity is till now unclear, many experimental facts are precisely known and are sufficient to build up the minimal model for disorder in this unconventional superconductors.

The first fact is that the order parameter in the cuprates has a d-wave symmetry, which is likely due to the strong short range repulsion preventing conventional s-wave pairing.

The second fact is that, although the anomalous properties of the normal phase

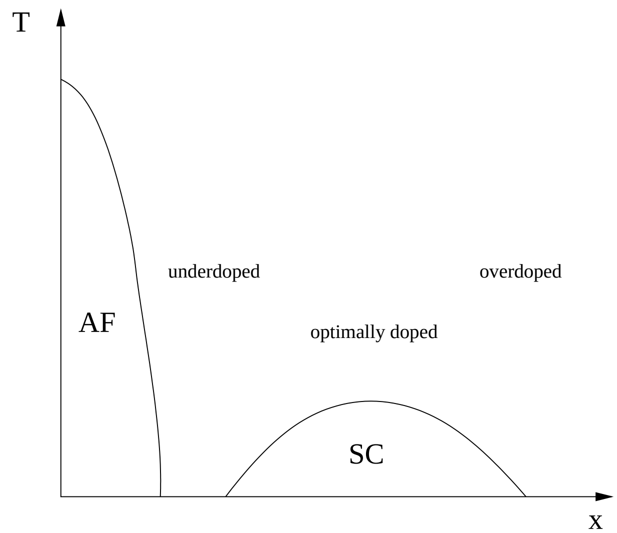

Figure 2.2: Phase diagram of high temperature superconductors. x represent doping, AF antiferromagnetically ordered phase and SC superconducting phase.

do not fit into the Landau Fermi liquid theory, Landau-Bogoliubov quasiparticles seems to re-appear below T_c, allowing a conventional description of the low-energy excitations in the superconducting phase at low temperature.

2.2 The model

The characteristic feature of a d-wave superconductor is the existence of four nodal points where the order parameter vanishes. To study the low temperature transport properties of a d-wave superconductor, we consider the following model defined in a two-dimensional square lattice of lattice constant a:

$$H = \sum_{\langle ij \rangle} (c^\dagger_{i\uparrow}, c_{i\downarrow}) \begin{pmatrix} -t_{ij} - \mu\delta_{ij} & \Delta_{ij} \\ \Delta_{ij} & t_{ij} + \mu\delta_{ij} \end{pmatrix} \begin{pmatrix} c_{j\uparrow} \\ c^\dagger_{j\downarrow} \end{pmatrix},$$

where $\langle ij \rangle$ means the sum restricted to nearest neighbor sites. The hopping parameter $t_{ij} = t_{ji}$ has a regular term, t, and a random one $\tilde{t}_{ij}$, so that the total hopping is $t_{ij} = t + \tilde{t}_{ij}$. The random variables $\tilde{t}_{ij}$ are supposed to be gaussian distributed with zero average and variance

$$\overline{\tilde{t}_{ij}\tilde{t}_{kl}} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta jk)\, u^2\, t^2.$$

The pairing term Δ_{ij} has a d-wave symmetry. This means that, for instance, $\Delta_{ij} > 0$ if $|i-j|$ parallel to x axis and $\Delta_{ij} < 0$ if $|i-j|$ parallel to y axis.
Also Δ_{ij} can be divided into a regular term and a random one. The regular term can be assumed real, without any loss of generality. The random one is generally complex, thus introducing random fluctuations in the phase of the order parameter. Since the above Hamiltonian contains only coupling terms between different sub-lattice, it posseses the nesting property discussed in the Introduction. Moreover, if the chemical potential $\mu = 0$, i.e. at half filling, the Hamiltonian possesses also the chiral symmetry. This additional symmetry is broken away from half filling, where, in spite of nesting still present, the model behaves as if no nesting were present.

We notice that, if instead of a random hopping there were random on-site impurities, that is a chemical potential μ_i random and site dependent, nesting as well as chiral symmetry would be broken. However, in the unitary scattering limit, the impurity site becomes not accessible to electrons, so that the model effectively transforms into a random hopping model in which the four hopping matrix elements into a site are randomly suppressed.

2.2.1 Without disorder

In the absence of random hopping the quasiparticle spectrum has four nodes at $(\pm k_F, \pm k_F,)$.

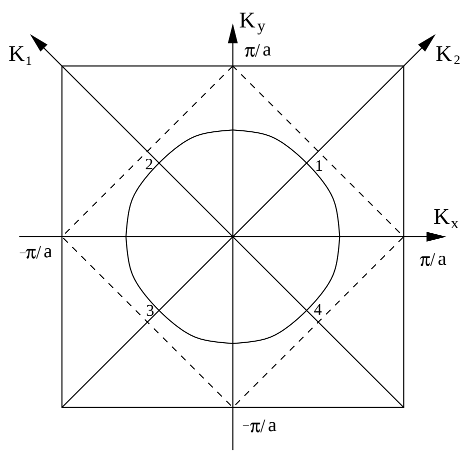

Figure 2.3: The Brillouin zone: the Fermi surface is curved while at half filling becomes a square, the diagonals are the point where $\Delta_k = 0$

In the vicinity of each gap node the Fourier transform of $(-t_{ji} - \mu)$, namely $\epsilon_k = -2t\cos(k_x a) - 2t\cos(k_y a) - \mu$, varies linearly perpendicularly to the Fermi surface while the Fourier transform of Δ_{ij}, that is $\Delta_k = 2\Delta\left(\cos(k_x a) - \cos(k_y a)\right)$, varies linearly parallel to the Fermi surface [3, 21]. Let us rotate the axes from k_x, k_y to k_1, k_2, (Fig.2.3), and define a Fermi velocity, $\mathbf{v}_1$ perpendicular to the Fermi surface, and a gap velocity $\mathbf{v}_2$ parallel to the same surface. Then, close to the nodes the quasiparticle spectrum is

$$E_k^{1,3} \simeq \sqrt{v_1^2 k_1^2 + v_2^2 k_2^2} \tag{2.1}$$

for nodes 1 and 3, see Fig.2.3,

$$E_k^{2,4} \simeq \sqrt{v_2^2 k_1^2 + v_1^2 k_2^2} \tag{2.2}$$

for nodes 2 and 4. The spectrum, in the vicinity of each gap node takes the form of a Dirac cone whose anisotropy is measured by the ratio of the two velocities. We will see that the apparently innocuous fact that the roles of v_1 and v_2 are interchanged from nodes 1 and 3 to 2 and 4, is indeed a very crucial aspect which makes the standard field theory approch to localization well justified.

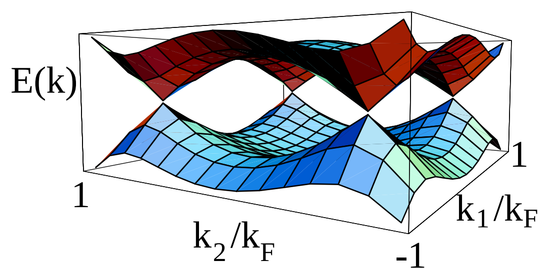

Figure 2.4: The spectrum of BCS like Hamiltonian with d wave symmetry in the order parameter near the nodes.

2.2.2 Path integral representation

Let us consider the standard BCS theory within the Nambu formalism. The Nambu spinor is defined by

$$\Psi_k = \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix}. \tag{2.3}$$

The imaginary time Green's function is therefore the matrix

$$\begin{aligned} G_k(\tau) &= -\langle T_\tau\left(\Psi_k(\tau)\Psi_k^\dagger\right)\rangle \\ &= \begin{pmatrix} -\langle T_\tau\left(c_{k\uparrow}(\tau)c_{k\uparrow}^\dagger\right)\rangle & -\langle T_\tau\left(c_{k\uparrow}(\tau)c_{-k\downarrow}\right)\rangle \\ -\langle T_\tau\left(c_{-k\downarrow}^\dagger(\tau)c_{k\uparrow}^\dagger\right)\rangle & -\langle T_\tau\left(c_{-k\downarrow}^\dagger(\tau)c_{-k\downarrow}\right)\rangle \end{pmatrix} \\ &= \begin{pmatrix} G_{11}(k,\tau) & G_{12}(k,\tau) \\ G_{21}(k,\tau) & G_{22}(k,\tau) \end{pmatrix}. \end{aligned}$$

In Matsubara frequencies, we have the following relations

$$\begin{aligned} G_{22}(k,\omega_n) &= -G_{11}(k,-\omega_n), \\ G_{11}^*(k,\omega_n) &= G_{11}(k,-\omega_n), \\ G_{21}(k,\omega_n) &= G_{12}^*(k,-\omega_n) \\ G_{12}(k,\omega_n) &= G_{12}(k,-\omega_n). \end{aligned}$$

In the absence of disorder, the Hamiltonian, in the Nambu representation, is

$$H_0 = \sum_k \Psi_k^\dagger\left(\epsilon_k\tau_3 + \Delta_k\tau_1\right)\Psi_k. \tag{2.4}$$

where $\tau_i = \tau_1, \tau_2, \tau_3$ are Pauli matrices acting on Nambu spinor (2.3), τ_0 being the unit matrix.

The inverse Green's function is given by

$$G^{-1}(k,\omega_n) = i\omega_n\tau_0 - \epsilon_k\tau_3 - \Delta_k\tau_1. \tag{2.5}$$

In other words, the path integral action is

$$-S = T\sum_n\sum_k \Psi_k^\dagger\left(i\omega_n\tau_0 - \epsilon_k\tau_3 - \Delta_k\tau_1\right)\Psi_k, \tag{2.6}$$

which is, apart from a constant,

$$\begin{aligned} -S &= T\sum_n\sum_k \left(i\omega_n - \epsilon_k\right)c_{k\uparrow}^\dagger c_{k\uparrow} + \left(-i\omega_n - \epsilon_k\right)c_{-k\downarrow}^\dagger c_{-k\downarrow} \\ &\quad -\Delta_k\left(c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + c_{-k\downarrow}c_{k\uparrow}\right). \end{aligned}$$

The structure of the action means that the pairing acts among opposite frequency fields. To be more precise, the above action, when keeping into account both spin directions, is written as

$$\begin{aligned} -S &= T\sum_n\sum_k\sum_\sigma \left(i\omega_n - \epsilon_k\right)c_{k\sigma}^\dagger(\omega_n)c_{k\sigma}(\omega_n) \\ &\quad + \left(-i\omega_n - \epsilon_k\right)c_{-k-\sigma}^\dagger(-\omega_n)c_{-k-\sigma}(-\omega_n) \\ &\quad -\Delta_k\sigma\left(c_{k\sigma}^\dagger(\omega_n)c_{-k-\sigma}^\dagger(-\omega_n) + c_{-k-\sigma}(-\omega_n)c_{k\sigma}(\omega_n)\right). \end{aligned}$$

Written in such a way, the action is explicitly SU(2) invariant, the cost being the introduction of opposite frequency fields. The BCS terms pairs (σ, k, ω_n) with $(-\sigma, -k, -\omega_n)$ fields.
Following Efetov, Larkin and Khmel'nitskii [16], introducing Grassman variables c and $\bar{c}$, we can write Green's functions in a path integral formulation, as seen in Chapter 1. We define an extended Nambu spinor

$$\Psi_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{c}_i \\ i\sigma_y c_i \end{pmatrix}, \tag{2.7}$$

as well as

$$\overline{\Psi} = [C\Psi]^t, \tag{2.8}$$

being the charge conjugacy matrix

$$C = i\sigma_y \tau_1. \tag{2.9}$$

Here and in the following, the Pauli matrices σ_b $(b = x, y, z)$ act on the spin components, s_b $(b = 1, 2, 3)$ on the frequency components, and τ_b $(b = 1, 2, 3)$ on the Nambu components $\bar{c}$ and c.
As we showed before, we need to introduce positive and negative frequency propagators, i.e. we have to add to the action a term

$$-i\frac{\omega}{2}\overline{\Psi} s_3 \Psi, \tag{2.10}$$

since paring acts among opposite frequency fields. In other words, the pairing term has to be multiplied by s_1. In this representation the action is

$$S = 2\sum_k \overline{\Psi}_k \left(\epsilon_k + i\Delta_k \tau_2 s_1 - i\frac{\omega}{2} s_3\right) \Psi_k. \tag{2.11}$$

where we have fixed Δ_k to be real. The imaginary part of the pairing parameter would have been proportional to $\tau_1 s_1$. However its introduction is essentially equivalent to have a random phase in the hopping, which becomes $e^{i\phi_{ij}\tau_3}$, with $\phi_{ij} = -\phi_{ji}$, hence breaking time reversal symmetry. If ϕ_{ij} is zero, time reversal symmetry is preserved.

Chapter 3

The effective action

In this chapter we derive the effective quantum field theory for the disordered d-wave superconducting model described in the previous section, following the work by by Efetov, Larkin, and Khmel'nitsky. As usually this field theory is a non-linear σ-model where the broken gauge symmetry enters as a reduction of the symmetry of the Q-matrix fields with respect to a normal metal.

3.1 Disorder average

The action is the sum of a regular part with fixed hopping term t plus an impurity contribution modulated by independent random hopping matrix elements $\tilde{t}_{ij}$

$$\begin{aligned} S &= -\sum_{\langle ij \rangle} \overline{\Psi}_i \left(\varepsilon\, \delta_{ij} - H_{ij} + i\frac{\omega}{2} s_3\, \delta_{ij} \right) \Psi_j \\ &= -\sum_{\langle ij \rangle} \overline{\Psi}_i \left(\varepsilon\, \delta_{ij} - H^0_{ij} + i\frac{\omega}{2} s_3\, \delta_{ij} \right) \Psi_j + S_{imp} \end{aligned} \tag{3.1}$$

with

$$S_{imp} = 2\sum_{\langle ij \rangle} \tilde{t}_{ij} \overline{\Psi}_i \Psi_j. \tag{3.2}$$

Within the replica method the generating function is

$$\mathcal{Z} = \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi\mathcal{D}\tau P(\tilde{t})\, \mathrm{e}^{-S_0 - S_{imp}}, \tag{3.3}$$

where $P(\tilde{t})$ is the gaussian probability distribution of the random bonds $\tilde{t}_{ij}$ which have zero average value and variance equal to ut. The average over disorder changes the impurity action into

$$\begin{aligned} S_{imp} &= -2\sum_{\langle ij\rangle} 2u^2t^2\left(\overline{\Psi}_i\Psi_j\right)^2 \\ &= -\sum_{\langle ij\rangle} 2u^2t^2\left(\overline{\Psi}_i\Psi_j\right)\left(\overline{\Psi}_j\Psi_i\right). \end{aligned} \tag{3.4}$$

since $\overline{\Psi}_i\Psi_j = \overline{\Psi}_j\Psi_i$. By introducing the fields

$$X_i^{\alpha\beta} = \Psi_i^\alpha\overline{\Psi}_i^\beta \tag{3.5}$$

where α and β is a multilabel for Nambu, advanced-retarded and replica components, we can write

$$S_{imp} = 2u^2t^2\sum_{\langle ij\rangle} X_i^{\alpha\beta}X_j^{\beta\alpha} = 2u^2t^2\sum_{\langle ij\rangle} Tr\left(X_iX_j\right) \tag{3.6}$$

In Fourier components it becomes

$$S_{imp} = \frac{1}{V}\sum_{q\in BZ} w_q Tr\left(X_qX_{-q}\right), \tag{3.7}$$

where BZ means the Brillouin zone and

$$w_q = 2u^2t^2\left(\cos q_xa + \cos q_ya\right) \tag{3.8}$$

a being the lattice spacing. We can decouple (3.7) by an Hubbard-Stratonovich transformation, introducing an auxiliary field. However, since $w_q = -w_{q+(\pi,\pi)}$ and $w_q > 0$ if q is restricted to the magnetic Brillouin zone (MBZ), we need to introduce two auxiliary fields defined within the MBZ, $Q_{0q} = Q^\dagger_{0-q}$ and $Q_{3q} = Q^\dagger_{3-q}$[7], through which

$$\begin{aligned} S_{imp} &= \frac{1}{V}\sum_{q\in MBZ}\frac{1}{4w_q}Tr\left[Q_{0q}Q_{0-q} + Q_{3q}Q_{3-q}\right] \\ &- \frac{i}{V}\sum_{q\in MBZ} Tr\left[Q_{0q}X^t_{-q} + iQ_{3q}X^t_{-q-(\pi,\pi)}\right]. \end{aligned} \tag{3.9}$$

The above expression shows that Q_0 corresponds to smooth fluctuations of the auxiliary field, while Q_3 to staggered fluctuations. Namely, in the long-wavelength limit, the auxiliary field in real space is

$$Q_j = Q_{0j} + i(-1)^jQ_{3j}. \tag{3.10}$$

where j is the site. In the square lattice the unit cell contains one site. However, to make sublattice symmetry more manifest, it is convenient to use a unit cell which contains two sites, one for each sublattice. Indicating with R a new unit cell vector and with A and B the labels for the two sublattices, we introduce a two component operator

$$\Psi_R = \begin{pmatrix} \Psi_{AR} \\ \Psi_{BR} \end{pmatrix}, \tag{3.11}$$

through which we can rewrite (3.10) in the following way

$$Q_R = Q_{0R}\,\gamma_0 + iQ_{3R}\,\gamma_3 \tag{3.12}$$

where γ_b ($b = 1, 2, 3$) are Pauli matrices acting on the vector (3.11). Q is not hermitian, in fact

$$Q_R^\dagger = Q_{0R}\,\gamma_0 - iQ_{3R}\,\gamma_3 = \gamma_1 Q_R \gamma_1 = \gamma_2 Q_R \gamma_2 \tag{3.13}$$

since Q_0 and Q_3 are both hermitean.

3.2 Symmetries

Since the Hamiltonian parameters couple sites of the two different sublattices, we can consider generally two different global unitary transformations, one for sublattice A and another for sublattice B

$$\Psi_A = T_A \Psi_A, \quad \Psi_B = T_B \Psi_B.$$

For those being symmetry transformations, we have to impose

$$\begin{aligned} C T_A^t C^t T_B &= 1 \\ C T_B^t C^t T_A &= 1 \end{aligned} \tag{3.14}$$

being C expressed in (2.9), valid even for non superconducting states, as well as

$$\begin{aligned} C T_A^t C^t \tau_2 s_1 T_B &= \tau_2 s_1 \\ C T_B^t C^t \tau_2 s_1 T_A &= \tau_2 s_1 \end{aligned} \tag{3.15}$$

in the presence of a real superconducting order parameter. If time reversal symmetry is broken, we must further impose that

$$\begin{aligned} C T_A^t C^t \tau_3 T_B &= \tau_3 \\ C T_B^t C^t \tau_3 T_A &= \tau_3. \end{aligned} \tag{3.16}$$

In the presence of a constant magnetic field B that introduces a Zeeman term $B_z \tau_3 \sigma_z$ in the Hamiltonian,

$$\begin{aligned} C T_A^t C^t \tau_3 T_B &= \tau_3, \quad C T_A^t C^t \tau_3 \sigma_z T_A = \tau_3 \sigma_z \\ C T_B^t C^t \tau_3 T_A &= \tau_3, \quad C T_B^t C^t \tau_3 \sigma_z T_B = \tau_3 \sigma_z \end{aligned} \tag{3.17}$$

and finally

$$\begin{aligned} CT_A^tC^t\tau_3T_A = \tau_3, \quad CT_A^tC^t\tau_3\sigma T_A = \tau_3\sigma \\ CT_B^tC^t\tau_3T_B = \tau_3, \quad CT_B^tC^t\tau_3\vec{\sigma}T_B = \tau_3\sigma \end{aligned} \tag{3.18}$$

in the presence of magnetic impurities represented by the term $\vec{S}\cdot\vec{\sigma}\tau_3$ in the Hamiltonian with $\vec{S}$ a random vector variable.
In the presence of a finite frequency, $\omega \neq 0$, we must also impose

$$CT_A^tC^ts_3T_A = s_3, \quad CT_B^tC^ts_3T_B = s_3. \tag{3.19}$$

The unitary transformations, T_A and T_B, can be written as

$$T_A = \exp\frac{W_0+W_3}{2}, \quad T_B = \exp\frac{W_0-W_3}{2}, \tag{3.20}$$

with antihermitean W's.
Moreover, if we are not at half filling or there are on-site impurities $\varepsilon \neq 0$, that means if sublattice symmetry does not hold, T_A and T_B in order to be symmetry transformations have to satisfy

$$CT_A^tC^tT_A = 1, \quad CT_B^tC^tT_B = 1. \tag{3.21}$$

Together with the conditions (3.14), the equations (3.21) imply that W_3 in (3.20) is suppressed.

Symmetries of W_0

We suppose for the moment that **sublattice symmetry does not hold**, hence we just need to consider W_0. From the above symmetry relations, W_0 has at least to satisfy

$$W_0 = -W_0^\dagger, \tag{3.22}$$

and the charge conjugacy invariance, through (3.14), that implies

$$CW_0^tC^t = -W_0. \tag{3.23}$$

In the presence of finite frequency, $\omega \neq 0$, we must impose in addition

$$[W_0, s_3] = 0 \tag{3.24}$$

which derives from (3.19). In the following we will not indicate explicitly the subscript 0. We separate singlet term from triplet one, writing

$$W = W_S + i\vec{\sigma}\cdot\vec{W}_T, \tag{3.25}$$

where the Pauli matrices σ_a, $a = x, y, z$, act on spin space. Then, (3.23) becomes

$$\tau_1 \left(W_S^t - i\vec{\sigma} \cdot \vec{W}_T^t \right) \tau_1 = -W_S - i\vec{\sigma} \cdot \vec{W}_T.$$

In addition we rewrite W in τ-components

$$W_S = W_{S0}\tau_0 + i \sum_{j=1}^{3} W_{Sj}\tau_j \tag{3.26}$$

$$\vec{W}_T = \vec{W}_{T0}\tau_0 + i \sum_{j=1}^{3} \vec{W}_{Tj}\tau_j. \tag{3.27}$$

Moreover, for each τ-component, we write, $a = 0, 1, 2, 3$

$$W_{S(T)a} = \sum_{\alpha=0}^{3} W_{S(T)a\alpha} s_\alpha. \tag{3.28}$$

Each component of W in (3.28) is a $n\times$ matrix in replica space.
• **In the absence of superconducting order parameter**, the first two conditions, (3.22) and (3.23), imply the following symmetry properties of the matrix W when $\omega = 0$ and $\omega \neq 0$,

	s_0	s_1	s_2	s_3
W_{S0}	A, R	A, R	S, I	A, R
W_{S1}	A, I	A, I	S, R	A, I
W_{S2}	A, I	A, I	S, R	A, I
W_{S3}	S, R	S, R	A, I	S, R
$\vec{W}_{T0}$	S, R	S, R	A, I	S, R
$\vec{W}_{T1}$	S, I	S, I	A, R	S, I
$\vec{W}_{T2}$	S, I	S, I	A, R	S, I
$\vec{W}_{T3}$	A, R	A, R	S, I	A, R

	s_0	s_3
W_{S0}	A, R	A, R
W_{S1}	A, I	A, I
W_{S2}	A, I	A, I
W_{S3}	S, R	S, R
$\vec{W}_{T0}$	S, R	S, R
$\vec{W}_{T1}$	S, I	S, I
$\vec{W}_{T2}$	S, I	S, I
$\vec{W}_{T3}$	A, R	A, R

Here S and A stand for symmetric and antisymmetric in replica space, while R and I for real and imaginary matrices.
The original model, with $\omega = 0$, has a Sp$(4n)$ symmetry. When the frequency is turned on, it lowers the symmetry to Sp$(2n)\times$Sp$(2n)$, briefly

$$\mathrm{Sp}(4n) \to \mathrm{Sp}(2n) \times \mathrm{Sp}(2n).$$

• **If time reversal symmetry is broken**, by the previous conditions and further imposing

$$[W, \tau_3] = 0 \tag{3.29}$$

we have, with $\omega = 0$ and $\omega \neq 0$, these components in W

	s_0	s_1	s_2	s_3
W_{S0}	A, R	A, R	S, I	A, R
W_{S3}	S, R	S, R	A, I	S, R
$\vec{W}_{T0}$	S, R	S, R	A, I	S, R
$\vec{W}_{T3}$	A, R	A, R	S, I	A, R

	s_0	s_3
W_{S0}	A, R	A, R
W_{S3}	S, R	S, R
$\vec{W}_{T0}$	S, R	S, R
$\vec{W}_{T3}$	A, R	A, R

This corresponds to this lowering of symmetry

$$\mathrm{U}(4n) \to \mathrm{U}(2n) \times \mathrm{U}(2n).$$

• **In the presence of magnetic field** we must add this condition

$$[W, \tau_3 \sigma_z] = 0. \tag{3.30}$$

and together with (3.22), (3.23) and (3.29) both with zero frequency and with finite frequency, in which case (3.24) should be imposed, we have

	s_0	s_1	s_2	s_3
W_{S0}	A, R	A, R	S, I	A, R
W_{S3}	S, R	S, R	A, I	S, R
W_{T_z0}	S, R	S, R	A, I	S, R
W_{T_z3}	A, R	A, R	S, I	A, R

	s_0	s_3
W_{S0}	A, R	A, R
W_{S3}	S, R	S, R
W_{T_z0}	S, R	S, R
W_{T_z3}	A, R	A, R

In this case finite frequency has this effect in the symmetry

$$\mathrm{U}(2n) \times \mathrm{U}(2n) \to \mathrm{U}(2n).$$

• **With magnetic impurities**, we need to add the condition

$$[W, \tau_3 \vec{\sigma}] = 0 \tag{3.31}$$

so that, for ω zero or finite,

	s_0	s_1	s_2	s_3
W_{S0}	A, R	A, R	S, I	A, R
W_{S3}	S, R	S, R	A, I	S, R

	s_0	s_3
W_{S0}	A, R	A, R
W_{S3}	S, R	S, R

This means that a finite frequency lower the symmetry according to

$$\mathrm{U}(2n) \to \mathrm{U}(n) \times \mathrm{U}(n).$$

• **In the superconducting state**, with real order parameter, the following condition has to be further imposed

$$[W, \tau_2 s_1] = 0. \tag{3.32}$$

Together with all the other conditions, we find the following properties of the W_0-matrices at $\omega = 0$ and $\omega \neq 0$

	s_0	s_1	s_2	s_3
W_{S0}	A, R	A, R		
W_{S1}			S, R	A, I
W_{S2}	A, I	A, I		
W_{S3}			A, I	S, R
$\vec{W}_{T0}$	S, R	S, R		
$\vec{W}_{T1}$			A, R	S, I
$\vec{W}_{T2}$	S, I	S, I		
$\vec{W}_{T3}$			S, I	A, R

	s_0	s_3
W_{S0}	A, R	
W_{S1}		A, I
W_{S2}	A, I	
W_{S3}		S, R
$\vec{W}_{T0}$	S, R	
$\vec{W}_{T1}$		S, I
$\vec{W}_{T2}$	S, I	
$\vec{W}_{T3}$		A, R

The original model, with $\omega = 0$, has a $\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$ symmetry. When the frequency is turned on,

$$\mathrm{Sp}(2n) \times \mathrm{Sp}(2n) \rightarrow \mathrm{Sp}(2n).$$

• **If time reversal symmetry is broken** then for $\omega = 0$ or $\omega \neq 0$

	s_0	s_1	s_2	s_3
W_{S0}	A, R	A, R		
W_{S3}			A, I	S, R
$\vec{W}_{T0}$	S, R	S, R		
$\vec{W}_{T3}$			S, I	A, R

	s_0	s_3
W_{S0}	A, R	
W_{S3}		S, R
$\vec{W}_{T0}$	S, R	
$\vec{W}_{T3}$		A, R

result if the symmetry lowering

$$\mathrm{Sp}(2n) \rightarrow \mathrm{U}(2n).$$

• **In the presence of magnetic field**, by adding the condition (3.30) to the previous ones, for ω zero or finite we have

	s_0	s_1	s_2	s_3
W_{S0}	A, R	A, R		
W_{S3}			A, I	S, R
W_{T_z0}	S, R	S, R		
W_{T_z3}			S, I	A, R

	s_0	s_3
W_{S0}	A, R	
W_{S3}		S, R
W_{T_z0}	S, R	
W_{T_z3}		A, R

In this case the finite frequency has the lowering symmetry effect

$$\mathrm{U}(2n) \rightarrow \mathrm{U}(n) \times \mathrm{U}(n).$$

• **With magnetic impurities**, adding the other condition (3.31) we have these terms in W, whether or not ω is zero

	s_0	s_1	s_2	s_3
W_{S0}	A, R	A, R		
W_{S3}			A, I	S, R

	s_0	s_3
W_{S0}	A, R	
W_{S3}		S, R

implying the symmetry lowering

$$\mathrm{O}(2n) \rightarrow \mathrm{U}(n).$$

Symmetries of W_3

In the presence of sublattice symmetry, others modes coming from W_3 have to be added to those of W_0 in the corresponding cases considered above. From the symmetry relations written above, W_3 has to satisfy

$$W_3 = -W_3^\dagger, \tag{3.33}$$

and the charge conjugacy invariance, through (3.14), that implies

$$CW_3^tC^t = W_3, \tag{3.34}$$

In the presence of a finite frequency, $\omega \neq 0$, we must impose this relation as well

$$\{W_3, s_3\} = 0 \tag{3.35}$$

deriving from (3.19). In the following we do not indicate explicitly the subscript 3. We separate singlet term from triplet one as in (3.25). Then, (3.34) becomes

$$\tau_1 \left(W_S^t - i\vec{\sigma} \cdot \vec{W}_T^t \right) \tau_1 = W_S + i\vec{\sigma} \cdot \vec{W}_T.$$

In addition we rewrite W in τ components as in (3.26) and (3.27), and each τ-component in its turn is written in energy space component as in (3.28). Each component of W in (3.28) is a $n \times n$ matrix in replica space.

• **In absence of superconducting order parameter**, from conditions (3.33) and (3.34) for $\omega = 0$ and $\omega \neq 0$, we derive the following properties of the W_3 components

	s_0	s_1	s_2	s_3
W_{S0}	S, I	S, I	A, R	S, I
W_{S1}	S, R	S, R	A, I	S, R
W_{S2}	S, R	S, R	A, I	S, R
W_{S3}	A, I	A, I	S, R	A, I
$\vec{W}_{T0}$	A, I	A, I	S, R	A, I
$\vec{W}_{T1}$	A, R	A, R	S, I	A, R
$\vec{W}_{T2}$	A, R	A, R	S, I	A, R
$\vec{W}_{T3}$	S, I	S, I	A, R	S, I

	s_1	s_2
W_{S0}	S, I	A, R
W_{S1}	S, R	A, I
W_{S2}	S, R	A, I
W_{S3}	A, I	S, R
$\vec{W}_{T0}$	A, I	S, R
$\vec{W}_{T1}$	A, R	S, I
$\vec{W}_{T2}$	A, R	S, I
$\vec{W}_{T3}$	S, I	A, R

The group of symmetry are determined by these free components together with those of W_0 collected in the corresponding tables. The resulting groups are

$$\mathrm{U}(8n) \to \mathrm{Sp}(4n),$$

The lowering of symmetry is determined by finite frequency term.
• **In broken time reversal symmetry** case, together with conditions (3.33) and (3.34), we should impose

$$[W, \tau_3] = 0. \tag{3.36}$$

Depending on the presence of the frequency term we have, in this case,

	s_0	s_1	s_2	s_3
W_{S0}	S, I	S, I	A, R	S, I
W_{S3}	A, I	A, I	S, R	A, I
$\vec{W}_{T0}$	A, I	A, I	S, R	A, I
$\vec{W}_{T3}$	S, I	S, I	A, R	S, I

	s_1	s_2
W_{S0}	S, I	A, R
W_{S3}	A, I	S, R
$\vec{W}_{T0}$	A, I	S, R
$\vec{W}_{T3}$	S, I	A, R

Adding the symmetry properties of W_0 found previously, the resulting groups are

$$\mathrm{U}(4n) \times \mathrm{U}(4n) \to \mathrm{U}(4n).$$

In the presence of a magnetic field, we should impose

$$\{W, \tau_3\sigma_z\} = 0 \tag{3.37}$$

obtaining, with $\omega = 0$ and $\omega \neq 0$

	s_0	s_1	s_2	s_3
W_{T_x0}	A, I	A, I	S, R	A, I
W_{T_x3}	S, I	S, I	A, R	S, I
W_{T_y0}	A, I	A, I	S, R	A, I
W_{T_y3}	S, I	S, I	A, R	S, I

	s_1	s_2
W_{T_x0}	A, I	S, R
W_{T_x3}	S, I	A, R
W_{T_y0}	A, I	S, R
W_{T_y3}	S, I	A, R

namely the symmetry lowering

$$\mathrm{U}(4n) \to \mathrm{U}(2n) \times \mathrm{U}(2n).$$

• **When magnetic impurities are present**, W_3 has to satisfy the additional condition

$$\{W, \tau_3\vec{\sigma}\} = 0 \tag{3.38}$$

that leads to the results, with $\omega = 0$ and $\omega \neq 0$,

	s_0	s_1	s_2	s_3
W_{S1}	S, R	S, R	A, I	S, R
W_{S2}	S, R	S, R	A, I	S, R

	s_1	s_2
W_{S1}	S, R	A, I
W_{S2}	S, R	A, I

The corresponding groups are

$$\mathrm{Sp}(2n) \to \mathrm{U}(2n).$$

• **In the superconducting state**, with real order parameter, the additional relation has to be imposed

$$[W_3, \tau_2 s_1] = 0. \tag{3.39}$$

Together with all the other conditions, we have the following properties for W_3 matrix for $\omega = 0$ and $\omega \neq 0$

	s_0	s_1	s_2	s_3
W_{S0}	S, I	S, I		
W_{S1}			A, I	S, R
W_{S2}	S, R	S, R		
W_{S3}			S, R	A, I
$\vec{W}_{T0}$	A, I	A, I		
$\vec{W}_{T1}$			S, I	A, R
$\vec{W}_{T2}$	A, R	A, R		
$\vec{W}_{T3}$			A, R	S, I

	s_1	s_2
W_{S0}	S, I	
W_{S1}		A, I
W_{S2}	S, R	
W_{S3}		S, R
$\vec{W}_{T0}$	A, I	
$\vec{W}_{T1}$		S, I
$\vec{W}_{T2}$	A, R	
$\vec{W}_{T3}$		A, R

The group of symmetry are still determined by these components together with those of W_0 collected in the corresponding tables. The resulting groups are

$$\mathrm{U}(4n) \times \mathrm{U}(4n) \to \mathrm{U}(4n),$$

• **In the case of broken time reversal symmetry**, depending on the absence or presence of the frequency term, we have

	s_0	s_1	s_2	s_3
W_{S0}	S, I	S, I		
W_{S3}			S, R	A, I
$\vec{W}_{T0}$	A, I	A, I		
$\vec{W}_{T3}$			A, R	S, I

	s_1	s_2
W_{S0}	S, I	
W_{S3}		S, R
$\vec{W}_{T0}$	A, I	
$\vec{W}_{T3}$		A, R

By symmetry properties of W_0 found previously, the resulting groups are

$$\mathrm{U}(4n) \to \mathrm{O}(4n).$$

• **In the presence of a magnetic field**, with both $\omega = 0$ and $\omega \neq 0$, we have

	s_0	s_1	s_2	s_3
W_{T_x0}	A, I	A, I		
W_{T_x3}			A, R	S, I
W_{T_y0}	A, I	A, I		
W_{T_y3}			A, R	S, I

	s_1	s_2
W_{T_x0}	A, I	
W_{T_x3}		A, R
W_{T_y0}	A, I	
W_{T_y3}		A, R

The lowering of symmetry due to finite frequency is now

$$\mathrm{O}(4n) \rightarrow \mathrm{O}(2n) \times \mathrm{O}(2n).$$

• **If magnetic impurities are present**, we have

	s_0	s_1	s_2	s_3
W_{S1}			A, I	S, R
W_{S2}	S, R	S, R		

	s_1	s_2
W_{S1}		A, I
W_{S2}	S, R	

Together with W_0's properties, these components lead to the following groups,

$$\mathrm{U}(2n) \rightarrow \mathrm{U}(n) \times \mathrm{U}(n).$$

The lowering of the symmetry, as in all cases, is due to $\omega \neq 0$.

3.3 Saddle Point

The full action

$$\begin{aligned} S &= -\sum_{k,q} \overline{\Psi}_k \left(\mu\delta_{q0} + i\frac{\omega}{2} s_3 \delta_{q0} - H_k^{(0)} \delta_{q0} + \frac{i}{V} Q_{-q} \right) \Psi_{k+q} \\ &+ \frac{1}{V} \sum_q \frac{1}{2w_q} Tr \left[Q_q Q_q^\dagger \right], \end{aligned} \tag{3.40}$$

by integrating over the Nambu spinors, transforms into

$$S[Q] = \frac{1}{V} \sum_q \frac{1}{2w_q} Tr \left[Q_q Q_q^\dagger \right] - \frac{1}{2} Tr \ln \left[\mu + i\frac{\omega}{2} s_3 - H^{(0)} + iQ \right]. \tag{3.41}$$

where H^0 is the regular part of the Hamiltonian. In momentum space it is

$$H_k = \epsilon_k + i\Delta_k \tau_2 s_1 = E_k \mathrm{e}^{2i\theta_k \tau_2 s_1}, \tag{3.42}$$

where

$$E_k = \sqrt{\epsilon_k^2 + \Delta_k^2}, \tag{3.43}$$

and

$$\cos 2\theta_k = \frac{\epsilon_k}{E_k}, \quad \sin 2\theta_k = \frac{\Delta_k}{E_k}. \tag{3.44}$$

Let us look for a saddle point $Q_{sp}\sigma_0$ which has only both a $\tau_0 s_3$ component Σ as well as a $\tau_2 s_1$ component F, both k independent. Therefore

$$G_k^{-1} = i\omega s_3 - \epsilon_k - i\Delta_k \tau_2 s_1 + i\Sigma s_3 + iF\tau_2 s_1, \tag{3.45}$$

where we introduce explicitly a symmetry breaking term, namely ωs_3. We notice that the new pairing order parameter is $\Delta_k - F$, so that, by defining

$$\tilde{E}_k = \sqrt{\epsilon_k^2 + (\Delta_k - F)^2}, \tag{3.46}$$

as well as a modified $\tilde{\theta}_k$, we find the self-consistency equations

$$\begin{aligned} \Sigma &= i\frac{u^2\bar{t}^2}{8}\sum_k Tr\left(G_k s_3\right), \\ F &= i\frac{u^2\bar{t}^2}{8}\sum_k Tr\left(G_k \tau_2 s_1\right), \end{aligned}$$

where

$$G_k = \mathrm{e}^{-i\tilde{\theta}_k\tau_2 s_1}\frac{1}{-\tilde{E}_k + i(\omega+\Sigma)s_3}\mathrm{e}^{-i\tilde{\theta}_k\tau_2 s_1}.$$

Therefore,

$$\Sigma = \frac{u^2\bar{t}^2}{2}(\Sigma+\omega)\sum_k \frac{1}{\tilde{E}_k^2 + (\Sigma+\omega)^2}, \tag{3.47}$$

$$F = -\frac{u^2\bar{t}^2}{2}\sum_k \frac{\Delta_k - F}{\tilde{E}_k^2 + \Sigma^2} = \frac{u^2}{2}F\sum_k \frac{1}{\tilde{E}_k^2 + \Sigma^2}, \tag{3.48}$$

where the last identity holds for d-wave order parameter. Notice that, for s-wave symmetry, these equations coincide with those found by Abrikosov, Gorkov and Dzyalozinskii. The first equation implies that

$$\frac{\Sigma}{\Sigma+\omega} = \frac{u^2\bar{t}^2}{2}\sum_k \frac{1}{\tilde{E}_k^2 + (\Sigma+\omega)^2}, \tag{3.49}$$

which, inserted in the equation for F leads to

$$F\left(\frac{\omega}{\Sigma+\omega}\right) = 0. \tag{3.50}$$

Being ω non zero, although infinitesimally small, this equation has solution $F = 0$. Therefore, only $\Sigma \neq 0$ such that

$$1 = \frac{u^2 \bar{t}^2}{2} \sum_k \frac{1}{E_k^2 + \Sigma^2}. \tag{3.51}$$

The above self-consistency equation leads to

$$\Sigma = \pi u^2 \bar{t}^2 \nu = \frac{\pi}{4} w_0 \, \nu \tag{3.52}$$

with $\nu = \rho(0)$ being the density of states at the chemical potential.

3.4 Transverse modes

Now we will consider the transformations that leave the total Hamiltonian unchanged but that change the saddle point, that is to say the transformations that allows to move from a vacuum state to another one. The degrees of freedom of these transformations are the Goldstone modes which are massless in this case and whose number is equal to

$$\dim(G/H) = \dim(G) - \dim(H)$$

where G is the original symmetry group and H is the symmetry group that preserves the vacuum: the coset G/H tells us how many generators are broken. Since the saddle point has the same algebraic form of the frequency term in the action, the cosets related to Goldstone modes are obtained exactly by that transformations that are excluded in the lowering of symmetries due to finite frequency. In the following we will denote by T only those kind of transformations, represented in sublattice space language by

$$T = e^{\frac{W_0 \gamma_0 + W_3 \gamma_3}{2}} \tag{3.53}$$

where the γ's are 2×2 matrices acting on vectors (3.11). The components of W's, through (3.26), (3.27) and (3.28) are collected as follows, together with corresponding cosets.

1. **Without sublattice symmetry**

   (a) **Without superconducting order parameter**

   i. **With time reversal symmetry**

W_0	s_1	s_2
W_{S0}	A, R	S, I
W_{S1}	A, I	S, R
W_{S2}	A, I	S, R
W_{S3}	S, R	A, I
$\vec{W}_{T0}$	S, R	A, I
$\vec{W}_{T1}$	S, I	A, R
$\vec{W}_{T2}$	S, I	A, R
$\vec{W}_{T3}$	A, R	S, I

$$\mathrm{Sp}(4n)/\mathrm{Sp}(2n) \times \mathrm{Sp}(2n).$$

   ii. **Without time reversal symmetry**

W_0	s_1	s_2
W_{S0}	A, R	S, I
W_{S3}	S, R	A, I
W_{T_z0}	S, R	A, I
W_{T_z3}	A, R	S, I

$$\mathrm{U}(4n)/\mathrm{U}(2n) \times \mathrm{U}(2n).$$

   iii. **With magnetic field**

W_0	s_1	s_2
W_{S0}	A, R	S, I
W_{S3}	S, R	A, I
W_{T_z0}	S, R	A, I
W_{T_z3}	A, R	S, I

$$\mathrm{U}(2n) \times \mathrm{U}(2n)/\mathrm{U}(2n).$$

   iv. **With magnetic impurities**

W_0	s_1	s_2
W_{S0}	A, R	S, I
W_{S3}	S, R	A, I

$$\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n).$$

(b) **With superconducting order parameter**

i. **With time reversal symmetry**

W_0	s_1	s_2
W_{S0}	A, R	
W_{S1}		S, R
W_{S2}	A, I	
W_{S3}		A, I
$\vec{W}_{T0}$	S, R	
$\vec{W}_{T1}$		A, R
$\vec{W}_{T2}$	S, I	
$\vec{W}_{T3}$		S, I

$$\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)/\mathrm{Sp}(2n).$$

ii. **Without time reversal symmetry**

W_0	s_1	s_2
W_{S0}	A, R	
W_{S3}		A, I
$\vec{W}_{T0}$	S, R	
$\vec{W}_{T3}$		S, I

$$\mathrm{Sp}(2n)/\mathrm{U}(2n).$$

iii. **With magnetic field**

W_0	s_1	s_2
W_{S0}	A, R	
W_{S3}		A, I
W_{T_z0}	S, R	
W_{T_z3}		S, I

$$\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n).$$

iv. **With magnetic impurities**

W_0	s_1	s_2
W_{S0}	A, R	
W_{S3}		A, I

$$\mathrm{O}(2n)/\mathrm{U}(n).$$

2. **With sublattice symmetry**

   (a) **Without superconducting order parameter**

      i. **With time reversal symmetry**

W_0	s_1	s_2
W_{S0}	A, R	S, I
W_{S1}	A, I	S, R
W_{S2}	A, I	S, R
W_{S3}	S, R	A, I
$\vec{W}_{T0}$	S, R	A, I
$\vec{W}_{T1}$	S, I	A, R
$\vec{W}_{T2}$	S, I	A, R
$\vec{W}_{T3}$	A, R	S, I

W_3	s_0	s_3
W_{S0}	S, I	S, I
W_{S1}	S, R	S, R
W_{S2}	S, R	S, R
W_{S3}	A, I	A, I
$\vec{W}_{T0}$	A, I	A, I
$\vec{W}_{T1}$	A, R	A, R
$\vec{W}_{T2}$	A, R	A, R
$\vec{W}_{T3}$	S, I	S, I

$\mathrm{U}(8n)/\mathrm{Sp}(4n)$,

      ii. **Without time reversal symmetry**

W_0	s_1	s_2
W_{S0}	A, R	S, I
W_{S3}	S, R	A, I
W_{T_z0}	S, R	A, I
W_{T_z3}	A, R	S, I

W_3	s_0	s_3
W_{S0}	S, I	S, I
W_{S3}	A, I	A, I
$\vec{W}_{T0}$	A, I	A, I
$\vec{W}_{T3}$	S, I	S, I

$\mathrm{U}(4n) \times \mathrm{U}(4n)/\mathrm{U}(4n)$.

      iii. **With magnetic field**

W_0	s_1	s_2
W_{S0}	A, R	S, I
W_{S3}	S, R	A, I
W_{T_z0}	S, R	A, I
W_{T_z3}	A, R	S, I

W_3	s_0	s_3
W_{T_x0}	A, I	A, I
W_{T_x3}	S, I	S, I
W_{T_y0}	A, I	A, I
W_{T_y3}	S, I	S, I

$\mathrm{U}(4n)/\mathrm{U}(2n) \times \mathrm{U}(2n)$.

      iv. **With magnetic impurities**

W_0	s_1	s_2
W_{S0}	A, R	S, I
W_{S3}	S, R	A, I

W_3	s_0	s_3
W_{S1}	S, R	S, R
W_{S2}	S, R	S, R

$\mathrm{Sp}(2n)/\mathrm{U}(2n)$.

(b) **With superconducting order parameter**

i. **With time reversal symmetry**

W_0	s_1	s_2
W_{S0}	A, R	
W_{S1}		S, R
W_{S2}	A, I	
W_{S3}		A, I
$\vec{W}_{T0}$	S, R	
$\vec{W}_{T1}$		A, R
$\vec{W}_{T2}$	S, I	
$\vec{W}_{T3}$		S, I

W_3	s_0	s_3
W_{S0}	S, I	
W_{S1}		S, R
W_{S2}	S, R	
W_{S3}		A, I
$\vec{W}_{T0}$	A, I	
$\vec{W}_{T1}$		A, R
$\vec{W}_{T2}$	A, R	
$\vec{W}_{T3}$		S, I

$$\mathrm{U}(4n) \times \mathrm{U}(4n)/\mathrm{U}(4n),$$

ii. **Without time reversal symmetry**

W_0	s_1	s_2
W_{S0}	A, R	
W_{S3}		A, I
$\vec{W}_{T0}$	S, R	
$\vec{W}_{T3}$		S, I

W_3	s_0	s_3
W_{S0}	S, I	
W_{S3}		A, I
$\vec{W}_{T0}$	A, I	
$\vec{W}_{T3}$		S, I

$$\mathrm{U}(4n)/\mathrm{O}(4n).$$

iii. **With magnetic field**

W_0	s_1	s_2
W_{S0}	A, R	
W_{S3}		A, I
W_{T_z0}	S, R	
W_{T_z3}		S, I

W_3	s_0	s_3
W_{T_x0}	A, I	
W_{T_x3}		S, I
W_{T_y0}	A, I	
W_{T_y3}		S, I

$$\mathrm{O}(4n)/\mathrm{O}(2n) \times \mathrm{O}(2n).$$

iv. **With magnetic impurities**

W_0	s_1	s_2
W_{S0}	A, R	
W_{S3}		A, I

W_3	s_0	s_3
W_{S1}		S, R
W_{S2}	S, R	

$$\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n).$$

3.5 Non linear σ model

Here we derive the effective field theory describing the long wavelength transverse fluctuations of $Q(R)$ around the saddle point. In general terms we may parametrize the Q-matrix as follows

$$Q_P(R) = \tilde{T}(R)^\dagger \left[Q_{sp} + P(R)\right] T(R) \equiv Q(R) + \tilde{T}(R)^\dagger P(R) T(R) \tag{3.54}$$

where $T(R)$ involves transverse massless fluctuations and P longitudinal massive ones, $Q_{sp} = \Sigma\, s_3$ being the saddle point. Besides, we used this short notation

$$\tilde{T}^\dagger = C T^t C^t = \gamma_1 T^\dagger \gamma_1 = \gamma_2 T^\dagger \gamma_2 \tag{3.55}$$

Since only the T's are diffusive, at the moment we concentrate just on them, neglecting the P's and writing the action in terms of $Q(R) = \tilde{T}(R)^\dagger Q_{sp} T(R)$ alone, even though a term involving massless modes from integration over massive ones might appear. Afterwords we will reconsider this point. By integrating (3.40) over the Grassmann variables, we obtain the following action of Q:

$$-S[Q] = -\frac{1}{V}\sum_q \frac{1}{2w_q} Tr\left[Q_q Q_q^\dagger\right] + \frac{1}{2} Tr \ln\left[\varepsilon + i\frac{\omega}{2}s_3 - H^{(0)} + iQ\right]. \tag{3.56}$$

We can rewrite the second term of $S[Q]$ as

$$\frac{1}{2} Tr \ln\left(\varepsilon \tilde{T} T^\dagger + i\frac{\omega}{2}\tilde{T} s_3 T^\dagger - \tilde{T} H^{(0)} T^\dagger + iQ_{sp}\right) \tag{3.57}$$

Since $H^{(0)}_{RR'}$ involves either γ_1 and γ_2, while T involves γ_0 and γ_3, then

$$\begin{aligned}
\tilde{T}(R) H^0_{RR'} T(R')^\dagger &= H^0_{RR'} + \left(\tilde{T}(R')^\dagger - \tilde{T}(R)^\dagger\right) H^0_{RR'} \\
&\simeq H^0_{RR'} - \tilde{T}(R)\vec{\nabla}\tilde{T}(R)^\dagger \cdot \left(\vec{R} - \vec{R}'\right) H^0_{RR'} \\
&+ \frac{1}{2}\tilde{T}(R)\partial_{ij}\tilde{T}(R)^\dagger (R_i - R'_i)(R_j - R'_j) H^0_{RR'} \equiv H^0_{RR'} + U_{RR'}.
\end{aligned}$$

Unlike the tight-binding Hamiltonian case [7] in which the term $(\vec{R} - \vec{R}')H^0_{RR'}$ is related to the charge current vertex, in BCS Hamiltonian, since charge is not a conserved quantity unlike spin, that term is linked to the spin current vertex. This can be seen writing the continuity equation

$$i\nabla \vec{J}(R) = [H^0, \rho_{spin}(R)]$$

with $\rho_{spin}(R) = c^\dagger_{R\uparrow} c_{R\uparrow} - c^\dagger_{R\downarrow} c_{R\downarrow}$, from which we obtain the following expression for spin current on the basis (2.7), (2.8)

$$\vec{J}(R) = -i\sum_{R_1} (\vec{R} - \vec{R}_1)\bar{\Psi}_R H^0_{RR_1} \sigma_z \Psi_{R_1}$$

having chosen z as the spin quantization direction. Returning to the action, (3.57) can be written as

$$\begin{aligned} & \frac{1}{2} Tr \ln \left(\varepsilon \tilde{T} T^\dagger + i \frac{\omega}{2} \tilde{T} s_3 T^\dagger - U - H^{(0)} + i Q_{sp} \right) \\ = & -\frac{1}{2} Tr \ln G + \frac{1}{2} Tr \ln \left(1 + G\, \varepsilon \tilde{T} T^\dagger + G\, i \frac{\omega}{2} \tilde{T} s_3 T^\dagger - G\, U \right) \end{aligned} \quad (3.58)$$

where $G = (-H^{(0)} + iQ_{sp})^{-1}$ is the Green's function in the absence of transverse fluctuations. By expanding in ε and ω the following terms are found

$$\frac{\varepsilon}{2} Tr \left(G \tilde{T} T^\dagger \right) = -i \frac{\varepsilon}{w_0} Tr \left(Q_{sp} \tilde{T} T^\dagger \right) = -i \frac{\varepsilon}{w_0} Tr Q. \quad (3.59)$$

$$i \frac{\omega}{4} Tr \left(G\, \tilde{T} \hat{s} T^\dagger \right) = \frac{\omega}{2 w_0} Tr \left(s_3 Q \right). \quad (3.60)$$

The second order expansion in U contains the terms:

$$-\frac{1}{2} Tr \left(G\, U \right), \quad (3.61)$$

and

$$-\frac{1}{4} Tr \left(G\, U\, G\, U \right), \quad (3.62)$$

Taking in (3.61), the component of U containing second derivatives, we get

$$-\frac{1}{4} Tr \left\{ \tilde{T}(R) \partial_{ij} \tilde{T}(R)^{-1} \left(R_i - R'_i \right) \left(R_j - R'_j \right) H^{(0)}_{RR'} G(R', R) \right\}, \quad (3.63)$$

neglecting boundary terms coming from first derivatives in the expression of U. Now let us consider the correlation function

$$\chi_{\mu,i}(R, R'; t, t_1, t_2) = \langle T \left[c^\dagger_{R_1}(t) J^\mu_{R_1,R_2}(R) c_{R_2}(t) c^\dagger_{R_3}(t_1) J^i_{R_3,R_4}(R') c_{R_4}(t_2) \right] \rangle, \quad (3.64)$$

where $\mu = 0, 1, 2, 3$, and

$$J^0_{R_1 R_2}(R) = \delta_{RR_1} \delta_{RR_2} \sigma_z$$
$$J^i_{R_1 R_2}(R) = -i(\vec{R}_1 - \vec{R}_2) H^0_{R_1 R_2} \sigma_z \delta_{RR_2}.$$

By means of the continuity equation as in [7] in the hydrodynamic limit we obtain

$$\sum_{RR'} \chi_{j,i}(R, R'; E) = \sum_{RR'} \left(R_i - R'_i \right) \left(R_j - R'_j \right) Tr \left(G(R, R'; E) H^0_{R',R} \right). \quad (3.65)$$

Through the Ward identity (3.65), Eq. (3.63) turns out to be

$$-\frac{\chi^{++}_{ij}}{8} Tr \left\{ \tilde{T}(R) \partial_{ij} \tilde{T}(R)^{-1} \right\}, \quad (3.66)$$

which, integrating by part, is also equal to

$$\begin{aligned} &- \frac{\chi_{ij}^{++}}{8} Tr\left\{\tilde{T}(R)\partial_i\tilde{T}(R)^{-1}\tilde{T}(R)\partial_j\tilde{T}(R)^{-1}\right\} \\ &= -\frac{1}{8}\chi_{ij}^{++}Tr\left(D_iD_j\right) \end{aligned} \tag{3.67}$$

Here we have introduced a matrix $\vec{D}(R)$ with the i-th component

$$D_i(R) = D_{0,i}(R)\gamma_0 + D_{3,i}(R)\gamma_3 \equiv \tilde{T}(R)\partial_i\tilde{T}(R)^{-1}. \tag{3.68}$$

The second term (3.62) is

$$-\frac{1}{4}Tr\left(G\,U\,G\,U\right) = \frac{1}{4}\sum_k\sum_R\; Tr\left\{\vec{D}(R)\cdot\vec{J}_k\sigma_z G(k)\,\vec{D}(R)\cdot\vec{J}_k\sigma_z G(k)\right\} \tag{3.69}$$

Since, from the properties of $\vec{J}_k$ and of $G(k)$,

$$J_{i\,k}\sigma_z G\,D_j = \frac{1}{2}(D_j + \gamma_1 s_3 D_j s_3\gamma_1)J_{i\,k}\sigma_z G^{+} + \frac{1}{2}(D_j - \gamma_1 s_3 D_j s_3\gamma_1)J_{i\,k}\sigma_z G^{-} \tag{3.70}$$

we have lastly

$$\begin{aligned} \frac{1}{4}\sum_k\sum_R\; Tr\left\{\vec{D}(R)\cdot\vec{J}_k\sigma_z G(k)\,\vec{D}(R)\cdot\vec{J}_k\sigma_z G(k)\right\} &= \frac{1}{16}\chi_{ij}^{++}Tr\left[D_iD_j + D_i s_3\gamma_1 D_j s_3\gamma_1\right] \\ &+ \frac{1}{16}\chi_{ij}^{+-}Tr\left[D_iD_j - D_i s_3\gamma_1 D_j s_3\gamma_1\right] \end{aligned}$$

which summed to (3.67) gives

$$\frac{1}{16}\underbrace{(\chi_{ij}^{+-} - \chi_{ij}^{++})}_{2\pi\sigma_{ij}}\underbrace{Tr[D_iD_j - D_i s_3\gamma_1 D_j s_3\gamma_1]}_{-\frac{1}{2\Sigma^2}Tr(\partial_i Q\partial_j Q^\dagger)} = -\frac{\pi\sigma_{ij}}{16\Sigma^2}Tr(\partial_i Q\partial_j Q^\dagger) \tag{3.71}$$

where

$$\sigma_{ij} = -\frac{1}{4\pi}\sum_k Tr\left[J_{i\,k}\left(G^{+}(k) - G^{-}(k)\right)J_{j\,k}\left(G^{+}(k) - G^{-}(k)\right)\right] \tag{3.72}$$

is the spin conductivity since $\vec{J}_k$ is the spin current vertex. Let us define the following quantity

$$\sigma = \frac{\Sigma^2}{\pi V}\sum_k Tr\left[\frac{\vec{\nabla}\epsilon_k\cdot\vec{\nabla}\epsilon_k + \vec{\nabla}\Delta_k\cdot\vec{\nabla}\Delta_k}{\left(E_k^2+\Sigma^2\right)^2}\right] \simeq \frac{1}{4\pi^2}\frac{v_1^2+v_2^2}{v_1 v_2}. \tag{3.73}$$

where v_1 and v_2 are the velocities perpendicular and parallel to the Fermi surface. Then the action can be written as

$$S[Q] = \frac{2\pi}{32\Sigma^2}\sigma \int dR\, Tr\left(\partial_\mu Q\, \alpha_{\mu\nu}\, \partial_\nu Q^\dagger\right) + \int dR\, i\frac{\varepsilon}{w_0} Tr\left(Q(R)\right) - \frac{\omega}{2w_0} Tr\left(s_3 Q(R)\right) \tag{3.74}$$

where a particular metric appears, $\mu, \nu = 1, 2$ denoting the directions k_1 and k_2, which is

$$\alpha_{\mu\nu} = \delta_{\mu\nu} \qquad \text{for 4 nodes} \tag{3.75}$$

$$\alpha_{\mu\nu} = \delta_{\mu\nu}\frac{2v_\nu}{v_1^2 + v_2^2} \qquad \text{for one node or for opposite nodes.} \tag{3.76}$$

At this point it is important to discuss the differences which occur whether one assumes that disorder couples at most two opposite nodes, or the most generic case where all nodes are coupled toghether. We anticipate that the logarithmic terms which appear upon integrating the gaussian propagator derive from the expression

$$\frac{1}{2\pi\sigma}\int \frac{d^2k}{4\pi^2}\frac{1}{k_\mu \alpha_{\mu\nu} k_\nu} \equiv g \log(\dots) \tag{3.77}$$

where the effective coupling constant which controls the perturbative expansion is gives by

$$g = \frac{1}{2\pi^2\sigma} \qquad \text{for 4 nodes} \tag{3.78}$$

$$g = \frac{1}{2\pi^2\sigma}\frac{v_1^2 + v_2^2}{2v_1 v_2} \qquad \text{for 1 or 2 nodes.} \tag{3.79}$$

We readily see that, up to terms of order u^4, the disorder strength, for one or two opposite nodes $g = 1$ so that renormalization group based on the loop expansion completely loses its meaning. This is the situation analysed in Ref. [3]. On the other hand, for the generic case, since experimentally $v_2 \simeq v_1/15$, $g < 1$ and a perturbative expansion in g is still meaningful. In the following we will consider the latter situation.

3.5.1 A term from longitudinal integration

The full expression of the Q-matrix is expressed by (3.54) where the massive modes are

$$P(R) = \left(P_{00}s_0 + P_{03}s_3\right)\gamma_0 + i\left(P_{31}s_1 + P_{32}s_2\right)\gamma_3, \tag{3.80}$$

being all P's a hermitian. Charge conjugation implies that $cP^tc^t = P$. Writing the free action of $Q_P(R)$ and expanding w_q we'll have a term that can be add to (3.71), a pure massive term and a term where massive and massless modes are mixed. Integrating over massive modes at the end [7] we find another term that we should consider in the action representing transverse fluctuations

$$-\frac{2\pi}{8\cdot 32\Sigma^4}\Pi\int dR\, Tr\left[Q^{\dagger}(R)\vec{\nabla}Q(R)\gamma_3\right]\cdot Tr\left[Q^{\dagger}(R)\vec{\nabla}Q(R)\gamma_3\right]. \tag{3.81}$$

For more details see reference [7].
Indeed one more term should be taken into account, namely a Wess-Zumino-Witten term, which is calculated in detail in Appendix A. However this term accidentally cancels out thanks to the four-fold symmetry of the Dirac nodes.

Chapter 4

The renormalization group

In this chapter we study the scaling behavior of the action that we have obtained in Chapter 3 by means of the Wilson-Polyakov renormalization group [22, 23]. Moreover we also show how it is possible to evaluate the one loop correction to the conductivity, namely to the stiffness parameter, of modes which acquire a mass term, like the charge fluctuation inside the superconducting broken symmetry phase or the spin modes when spin isotropy is broken.

4.1 Renormalization group

We have found that the final expression of the action describing the transverse massless modes in the long-wavelength limit is

$$\begin{aligned} S[Q] &= \frac{2\pi}{32\Sigma^2}\sigma \int dR\, Tr\left(\vec{\nabla}Q(R)\cdot\vec{\nabla}Q(R)^\dagger\right) \\ &+ \int dR\, i\frac{\varepsilon}{w_0}Tr\left(Q(R)\right) - \frac{\omega}{2w_0}Tr\left(s_3 Q(R)\right) \\ &- \frac{2\pi}{8\cdot 32\Sigma^4}\Pi \int dR\, Tr\left[Q^\dagger(R)\vec{\nabla}Q(R)\gamma_3\right]\cdot Tr\left[Q^\dagger(R)\vec{\nabla}Q(R)\gamma_3\right]. \end{aligned} \tag{4.1}$$

Since $Q(R) = Q_{sp}T(R)^2 = \Sigma s_3 e^W$, at the gaussian level, the first term in the action is simply

$$\frac{2\pi\sigma}{32\Sigma^2}\int dR\, Tr\left(\vec{\nabla}Q^\dagger\vec{\nabla}Q\right) \simeq -\frac{2\pi\sigma}{32}\int dR\, Tr\left(\vec{\nabla}W\vec{\nabla}W\right) \tag{4.2}$$

while the last term

$$\begin{aligned} &-\frac{2\pi\Pi}{32\cdot 8\Sigma^4}\int dR\, Tr\left[Q^\dagger(R)\vec{\nabla}Q(R)\gamma_3\right]\cdot Tr\left[Q^\dagger(R)\vec{\nabla}Q(R)\gamma_3\right] \\ &\simeq -\frac{2\pi\Pi}{64}\int dR\, Tr\left[\vec{\nabla}W_3\right]\cdot Tr\left[\vec{\nabla}W_3\right] \end{aligned} \tag{4.3}$$

Depending on whether we have a real or imaginary matrix W in replica space, which may be either symmetric or antisymmetric, we find the following gaussian propagators for the diffusive modes

$$\langle W^q_{ab}(k) W^q_{cd}(-k)\rangle = \pm D(k)\left(\delta_{ac}\delta_{bd} \pm \delta_{ad}\delta_{bc}\right) - D(k)\frac{1}{4}Tr(W^3_{aa})\frac{\Pi}{\sigma+\Pi n}\delta_{ab}\delta_{cd}\delta_{q3},$$

where $q = 0, 3$ refers to the smooth or the staggered component, the $\pm$ sign in front refers to real (R) and imaginary (I) matrices, while the $\pm$ sign inside the brackets refers to symmetric (S) or antisymmetric (A) matrices (see Sec.3.4), n is the number of replicas and

$$D(k) = \frac{1}{2\pi\sigma}\frac{1}{k^2}. \tag{4.4}$$

This propagator in two-dimensions will induce logarithmic singularities within any perturbative expansion. A standard way to handle those divergences is provided by the Renormalization Group (RG). In particular we here apply the Wilson-Polyakov RG procedure [22, 23, 24], which is particularly suitable to handle with the non-linear constraint $QQ^\dagger = Q^2_{sp}$. By this approach one assumes

$$T(R) = T_f(R)T_s(R),$$

where T_f involves fast modes with momentum $q \in [\Lambda/s, \Lambda]$, while T_s involves slow modes with $q \in [0, \Lambda/s]$, being Λ the higher momentum cut-off, and the rescaling factor $s > 1$. The following equalities hold

$$\begin{aligned} & Tr\left[\vec{\nabla}Q^\dagger\vec{\nabla}Q\right] = Tr\left[\vec{\nabla}Q^\dagger_f\cdot\vec{\nabla}Q_f\right] \\ & +2Tr\left[\vec{D}_s\gamma_1 Q_f\vec{D}_s Q^\dagger_f\gamma_1\right] - 2\Sigma^2 Tr\left[\vec{D}_s\vec{D}_s\right] \\ & +4Tr\left[\vec{D}_s Q^\dagger_f\vec{\nabla}Q_f\right], \end{aligned} \tag{4.5}$$

where $Q_f = \tilde{T}^\dagger_f Q_{sp} T_f$ and $\vec{D}_s = T_s\vec{\nabla}T^\dagger_s$, as well as

$$\begin{aligned} & \frac{1}{\Sigma^4}Tr\left[Q^\dagger\vec{\nabla}Q\gamma_3\right]\cdot Tr\left[Q^\dagger\vec{\nabla}Q\gamma_3\right] \\ & = Tr\left[\left(\vec{\nabla}W_s + \vec{\nabla}W_f\right)\gamma_3\right]\cdot Tr\left[\left(\vec{\nabla}W_s + \vec{\nabla}W_f\right)\gamma_3\right]. \end{aligned} \tag{4.6}$$

Since the fast and slow modes live in disconnected regions of momentum space, only the stiffness (4.5) generates corrections. By expanding the terms coupling slow and fast modes up to second order in W_f, one loop expansion, the stiffness generates an action term for the slow modes which, after averaging over the fast ones, is

$$\frac{2\pi\sigma}{32\Sigma^2}\int dR\, Tr\left[\vec{\nabla}Q^\dagger_s\vec{\nabla}Q_s\right] + \langle S_1\rangle_f - \frac{1}{2}\langle S^2_2\rangle_f, \tag{4.7}$$

where

$$S_1 = \frac{2\pi\sigma}{32\Sigma^2}\int dR\, 2Tr\left[\vec{D}\gamma_1 Q_{sp}\vec{D}Q_{sp}\tilde{W}_f^2\gamma_1\right] - 2Tr\left[\vec{D}\gamma_1 Q_{sp}W_f\vec{D}W_f Q_{sp}\gamma_1\right] \tag{4.8}$$

and

$$S_2 = 4\frac{2\pi\sigma}{32}\int dR\, Tr\left[\vec{D}W_f\vec{\nabla}W_f\right]. \tag{4.9}$$

After the effective action for the slow modes have been obtained, one re-scale back the slow-modes momenta according to :

$$q \in \left[0, \frac{\Lambda}{s}\right] \to \frac{q'}{s},$$

where $q' \in [0, \Lambda]$ runs over the original momentum space. In this way the model is mapped onto another model defined onto the same range of momenta with renormalized parameters $\sigma(s)$ and $\Pi(s)$. The advantage is that the low momentum cutoff λ, provided for instance by the finite size L of the system $\lambda \sim 1/L$, rescales like $\lambda \to s\lambda$. Therefore the logarithmic singularities $\ln(\Lambda/\lambda)$ in the original model transforms into less dangerous $\ln(\Lambda/s\lambda)$ which, for sufficiently large $s \sim \Lambda/\lambda$, makes perturbation theory meaningful provided the renormalized coupling constants do not blow up. In the present case the logarithmic terms are controlled by the following dimensionless coupling constants

$$g = \frac{1}{2\pi^2\sigma}, \quad c = \frac{1}{2\pi^2\Pi}, \quad \Gamma = \frac{g}{c + n\, g}. \tag{4.10}$$

When chiral symmetry holds, namely when W^3 is massless, the new coupling c has to be included. However, the combination $\sigma + n\Pi$ can be shown [6, 7] to represent the stiffness parameter of an abelian degrees of freedom connected to the $\mathrm{Tr}(W^3)$, which is finite and commutes with all other degrees of freedom. This implies that $\sigma + n\Pi$ is a constant of the RG flow, namely that

$$\beta_c = \frac{c^2}{g^2}\frac{\beta_g}{n}.$$

Since the theory is well behaved in the $n \to 0$ zero replica limit, this indirectly proves that

$$\lim_{n\to 0}\beta_g = 0,$$

namely that when chiral symmetry holds and when $\mathrm{Tr}(W^3)$ is massless the model stays metallic with a finite conductance.

The final results of the RG are collected in the Table 4.1 in which the β functions of g ($\beta_g = dg/d\ln s$) and of the density of states (DOS) ($\beta_\rho = d\rho/d\ln s$) are listed for the different universality classes. The density of states scaling behavior is obtained through the its expression in the Q matrix language

$$\rho = \frac{1}{2\pi w_0} Tr(s_3 Q) = \frac{\nu}{8\Sigma} Tr(s_3 Q), \tag{4.11}$$

which allows a very simple loop expansion.

In Table 4.1 we also list the coset spaces G/H for the different classes (*i*) time reversal invariance is preserved with chiral symmetry [27] or without [5, 28]; (*ii*) time reversal symmetry is broken by introducing random phase with chiral symmetry [27] or without [5]; (*iii*) a magnetic field is applied in the presence of chiral symmetry or without it [5, 28]; and finally (*iv*) in the presence of magnetic impurities with chiral symmetry or in its absence[20].

Table 4.1: Coset spaces and β functions for the coupling g and for the DOS ρ in the different universality classes. $\hat{T}$ is the time reversal invariance.

	Coset space	β_g	β_ρ
Yes chiral, Yes $\hat{T}$	U($4n$)×U($4n$)/U($4n$)	$8ng^2$	$(\Gamma/4 - 8n)g$
Yes chiral, No $\hat{T}$	U($4n$)/O($4n$)	$4ng^2$	$(-1 + \Gamma/4 - 4n)g$
Yes chiral, magnetic field	O($4n$)/O($2n$)×O($2n$)	$(2n-1)g^2$	$-2ng$
Yes chiral, spin flip	U($2n$)/U(n)×U(n)	ng^2	$-ng$
No chiral, Yes $\hat{T}$	Sp($2n$)×Sp($2n$)/Sp($2n$)	$2(2n+1)g^2$	$(-1-4n)g$
No chiral, No $\hat{T}$	Sp($2n$)/U($2n$)	$(2n+1)g^2$	$(-1-2n)g$
No chiral, magnetic field	U($2n$)/U(n)×U(n)	ng^2	$-ng$
No chiral, spin flip	O($2n$)/U(n)	$(n-1)g^2$	$(1-n)g$

According to Table 4.1, in the zero replica limit we obtain that, if chiral symmetry is absent and for non magnetic impurities, the conductance vanishes, and the DOS, which is finite within the simplest Born approximation, is suppressed. As shown by Ref. [8], in the localized phase the DOS vanishes as $|E|$ or E^2 depending whether time reversal symmetry holds or not. Quite surprisingly, magnetic impurities give a delocalization correction to the conductance, as well as a DOS enhancement. On the contrary, if chiral symmetry is present, the conductance stays finite, or even increases in the presence of a magnetic field. Without magnetic field and in the absence of spin flip scattering, the DOS according to the above β-function diverges approximately like, $\rho(E) \sim \exp\left[A\sqrt{-\ln E}\right]/E$, with A a model dependent constant[6, 7]. By a real space RG in the strong disorder regime[30] as well as through a supersymmetric field theory (SUSY) approach[31] it has been recently argued that the correct asymptotic expression of the DOS is instead of the form

$$\rho(E) \sim \frac{1}{E} \exp\left[A(-\ln E)^{2/3}\right].$$

The authors of Ref. [31] identify the origin of the disagreement into the existence of an infinite chain of relevant operators which are related to moments of the DOS and which are coupled together in the RG equations. Indeed we also find that the anomalous dimension of the operator Q^m is not m but m^2 times that of Q, signalling a multifractal behavior of the extended state at $E = 0$. Therefore it is quite possible that a more complete analysis which takes into account operators $\mathrm{Tr}(Q^m)$, which are generated by the inclusion of a finite energy term $E\mathrm{Tr}(Q)$, may allow to reproduce the correct behavior found by SUSY even by our field theory approach.

4.2 The action with vector potentials

The quasiparticle charge modes, as well as the spin modes when magnetic impurities or a magnetic field are present, are not described by the non linear σ-model (4.1), which only represents the truly massless diffusion modes. Nevertheless, charge and spin conductivities, σ_c and σ_s, respectively, can be still evaluated through the stiffness of the corresponding modes, although they acquire a mass term. Alternatively, σ_c and σ_s can be determined by second derivatives of the action with respect to a source field which couples to the charge or to the spin current [29].

As explained in Appendix B the source field which couples to the charge current is the vector potential

$$A_c = \lambda^s (A^0 \tau_3 s_0 + A^1 \tau_3 s_1), \tag{4.12}$$

where λ^s is a symmetric matrix in replica space, or, alternatively,

$$A_c = \lambda^a (A^0 \tau_3 \sigma_z s_0 + A^1 \tau_3 \sigma_z s_1), \tag{4.13}$$

with λ^a an antisymmetric matrix. On the other hand the spin vector potential which couples to the spin current is given by

$$A_s = \lambda^s (A^0 \tau_0 \sigma_z s_0 + A^1 \tau_0 \sigma_z s_1), \tag{4.14}$$

or alternatively

$$A_s = \lambda^a (A^0 \tau_0 s_0 + A^1 \tau_3 s_1), \tag{4.15}$$

In the hydrodynamic limit the action in the presence of a vector potential acquires a new term which, up to second order is A, is (see Appendix C)

$$S(A_c) = \frac{2\pi}{32\Sigma^2} \sigma_c Tr\left[\left(\nabla Q + i\frac{e}{c}[Q, A_c]\right)\left(\nabla Q^\dagger - i\frac{e}{c}[A_c, Q^\dagger]\right) - (\nabla Q \nabla Q^\dagger)\right] \tag{4.16}$$

for a charge vector potential, where σ_c is the bare charge conductivity

$$\sigma_c = \frac{\Sigma^2}{\pi V} \sum_k Tr\left[\frac{(\nabla_k \epsilon)^2}{(E^2+\Sigma^2)^2}\right], \tag{4.17}$$

while for the spin case

$$S(A_s) = \frac{2\pi\sigma_s}{32\Sigma^2} Tr\left[\left(\nabla Q + \frac{i}{2}[Q, A_s]\right)\left(\nabla Q^\dagger - \frac{i}{2}[A_s, Q^\dagger]\right) - (\nabla Q \nabla Q^\dagger)\right] \tag{4.18}$$

where σ_s is the bare spin conductivity, expressed by (3.73). In the presence of these terms in the action, the generating function $\mathcal{Z}(A)$ depends now on A.

$$\mathcal{Z}(A) = \int DQ e^{-S_0 - S(A)}. \tag{4.19}$$

The Kubo formula both for charge and for spin conductivities is recovered by

$$\left(\frac{\partial^2 ln\mathcal{Z}}{\partial {A^0}^2} - \frac{\partial^2 ln\mathcal{Z}}{\partial {A^1}^2}\right)\Bigg|_{A=0} \tag{4.20}$$

4.3 Perturbation theory

Let us now calculate through the one-loop corrections to spin and charge conductivity. By expanding to second order in A, the generating function is

$$\mathcal{Z}(A) = N \int DQ\, e^{-S_0 - S(A)} \simeq N \int DQ\, e^{-S_0}\left(1 - S_2(A) + \frac{1}{2} S_1(A)^2\right)$$

with

$$\begin{aligned} N^{-1} &= \int DQ\, e^{-S_0} \\ S_0 &= \frac{1}{t_0}\int dR\, Tr\left(\nabla Q(R)\, \nabla Q^\dagger(R)\right) \\ &- \frac{2\pi\Pi}{8\cdot 32\Sigma^4}\int dR Tr\left(Q^\dagger(R)\nabla Q(R)\sigma_3\right)\, Tr\left(Q^\dagger(R)\nabla Q(R)\sigma_3\right) \end{aligned}$$

$$\begin{aligned} S(A) &= \frac{1}{t}\int dR\, Tr\left[\left(\nabla Q(R) - if[A, Q(R)]\right)\left(\nabla Q^\dagger(R) + if[Q^\dagger(R), A]\right)\right] \\ &- \frac{1}{t}\int dR\, Tr\left(\nabla Q(R)\, \nabla Q^\dagger(R)\right) \end{aligned}$$

so the terms to the second order expansion are

$$S_2(A) = \frac{f^2}{t}\int dR\, Tr\left([A, Q(R)][Q^\dagger(R), A]\right)$$
$$S_1(A)^2 = -\frac{4f^2}{t^2}\int dR dR'\, Tr\left(\nabla Q(R)[Q^\dagger(R), A]\right)\, Tr\left(\nabla Q(R')[Q^\dagger(R'), A]\right)$$

where $t = t_0 = \frac{32\Sigma^2}{2\pi\sigma}$, $f = \frac{1}{2}$ in spin case and $t = \frac{32\Sigma^2}{2\pi\sigma_c}$, $f = \frac{e}{c}$ in charge case.
By expanding Q in terms of W we have

$$Q = \Sigma s_3 e^W \simeq \Sigma s_3(1 + W + \frac{1}{2}W^2) + O(W^3)$$

and taking, for spin conductivity, the gauge (4.14), the second derivatives of the generating function, $\frac{\partial^2 \mathcal{Z}(A)}{\partial {A^0}^2}$ and $\frac{\partial^2 \mathcal{Z}(A)}{\partial {A^1}^2}$, are calculated. The gauge (4.15) gives the same results. For charge conductivity we take expression (4.12) or alternatively (4.13) as vector potential and calculate the derivatives of generating function. Through (4.20) we find the one-loop quantum interference corrections for charge and for spin conductivity, which are summarized in Table (4.2).

Table 4.2: One loop corrections to to the spin and charge conductivity in $n = 0$ replica limit.

	Coset space	$\delta\sigma_s/\sigma_s$	$\delta\sigma_c/\sigma_c$
Yes chiral, Yes $\hat{T}$	U($4n$)×U($4n$)/U($4n$)	0	0
Yes chiral, No $\hat{T}$	U($4n$)/O($4n$)	0	$-2g\ln s$
Yes chiral, magnetic field	O($4n$)/O($2n$)×O($2n$)	0	0
Yes chiral, spin flip	U($2n$)/U(n)×U(n)	0	0
No chiral, Yes $\hat{T}$	Sp($2n$)×Sp($2n$)/Sp($2n$)	$-2g\ln s$	$-2g\ln s$
No chiral, No $\hat{T}$	Sp($2n$)/U($2n$)	$-g\ln s$	$-g\ln s$
No chiral, magnetic field	U($2n$)/U(n)×U(n)	0	0
No chiral, spin flip	O($2n$)/U(n)	$g\ln s/2$	$g\ln s/2$

By this procedure we find that the one loop corrections $\delta\sigma_c/\sigma_c$ and $\delta\sigma_s/\sigma_s$ coincide with $\delta\sigma/\sigma$ in the absence of sublattice symmetry. When sublattice symmetry holds, quasiparticle charge conductivity may behave differently from spin conductivity, as it happens when time reversal symmetry is broken [26]. Nevertheless it is interesting that quantum intereference corrections in the diffusive modes influence also the stiffness of modes which are on the contrary not diffusive.

Chapter 5

The residual quasiparticle interaction

Up to here we have dealt with disorder in a d-wave superconductors modeled by a BCS Hamiltonian for free Landau-Bogoliubov quasiparticles. However strong correlation is a crucial ingredient of the cuprates. Therefore it is important to understand the effects of the residual quasiparticle interactions even within the superconducting phase. In this Chapter we extend our previous analysis to include also quasiparticle interaction following the original work by Finkel'stein [32, 33, 34]. Moreover, we extend the Finkel'stein model in order to include the nesting property, which requires to add interaction amplitudes at momentum transferred (π, π).

5.1 The action with interactions

Let us consider the following contribution to the action deriving from the residual quasiparticle interaction

$$-\sum \frac{\Gamma_1}{2} \overline{c}_n^\alpha(p_1)\, \overline{c}_m^\beta(p_2)\, c_{m-\omega}^\beta(p_2-k)\, c_{n+\omega}^\alpha(p_1+k) \tag{5.1}$$

$$-\sum \frac{\Gamma_2}{2} \overline{c}_n^\alpha(p_1)\, \overline{c}_m^\beta(p_2)\, c_{n+\omega}^\beta(p_1+k)\, c_{m-\omega}^\alpha(p_2-k) \tag{5.2}$$

with α and β spin indices and n, m and ω Matsubara frequency indices while p_1, p_2 k are the momenta involved (see Fig. 5.1).
These interactions can be rewritten distinguishing the singlet from the triplet channel through

$$-\sum \frac{\Gamma_s}{2} \overline{c}_n(p_1)\, \sigma_0\, c_{n+\omega}(p_1+k)\, \overline{c}_m(p_2)\, \sigma_0\, c_{m-\omega}(p_2-k) \tag{5.3}$$

$$\sum \frac{\Gamma_t}{2} \overline{c}_n(p_1)\, \vec{\sigma}\, c_{n+\omega}(p_1+k)\, \overline{c}_m(p_2)\, \vec{\sigma}\, c_{m-\omega}(p_2-k) \tag{5.4}$$

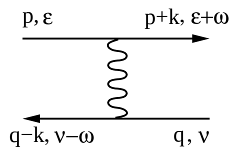

Figure 5.1: Diagram of interaction in particle-hole channel

with $\Gamma_t = \Gamma_2/2$ and $\Gamma_s = \Gamma_1 - \Gamma_t$.
By gaussian integration and using (2.7) and (2.8) we have

$$e^{\sum \frac{\Gamma}{2} \bar{c}_n(p_1)\,\sigma\, c_{n+\omega}(p_1+k)\,\bar{c}_m(p_2)\,\sigma\, c_{m-\omega}(p_2-k)} = \tag{5.5}$$
$$\int dX e^{-\frac{1}{2}\sum_\omega (X_0(\omega)X_0(-\omega)-X_3(\omega)X_3(-\omega))+2i\sum\sqrt{-\Gamma}\left(X_0(\omega)(\overline{\Psi}_n\tau_0\sigma^t\Psi_{n+\omega})+X_3(\omega)(\overline{\Psi}_n\tau_3\sigma^t\Psi_{n+\omega})\right)}$$

being $X_0(-\omega) = X_0(\omega)$, $X_3(-\omega) = -X_3(\omega)$ auxiliary Hubbard-Stratonovich fields and $\Gamma = -\Gamma_s$ for the singlet particle-hole channel with $\sigma = \sigma_0$ or $\Gamma = \Gamma_t$ with $\sigma = \vec{\sigma}$ for the triplet particle-hole channel. The full action including interaction is

$$\frac{1}{2} Tr \ln\left(\varepsilon\tilde{T}T^\dagger + i\frac{\omega}{2}\tilde{T}s_3T^\dagger - \tilde{T}H^{(0)}T^\dagger + iQ_{sp} + 2i\sqrt{-\Gamma}X_0\tilde{T}\tau_0\sigma^tT^\dagger\right). \tag{5.6}$$

By expanding in terms of non-interacting Green functions, we find new terms in the action that represents the residual interaction in the p-h channels, namely

$$-\frac{1}{2}\sum\left(X_0(\omega)^2 + X_3(\omega)^2\right) - \sum\frac{\sqrt{-\Gamma}}{2}\frac{\pi\nu}{\Sigma}\left(X_0(\omega)Tr(\tau_0\sigma Q_{n\,n+\omega}) + X_3(\omega)Tr(\tau_3\sigma Q_{n,\,n+\omega})\right) \tag{5.7}$$

Integrating over the auxiliary fields X_0 and X_3 we get for the singlet channel

$$\sum\frac{\pi^2\nu^2}{8\Sigma^2}\Gamma_s\sum_{l=0,3}\left(Tr(Q_{n,\,n+\omega}\tau_l\,\sigma_0)\,Tr(Q_{m+\omega,\,m}\tau_l\,\sigma_0)\right), \tag{5.8}$$

and for the triplet channel

$$-\sum\frac{\pi^2\nu^2}{8\Sigma^2}\Gamma_t\sum_{l=0,3}\left(Tr(Q_{n,\,n+\omega}\tau_l\,\vec{\sigma})\,Tr(Q_{m+\omega,\,m}\tau_l\,\vec{\sigma})\right). \tag{5.9}$$

In the replica space the Q matrices that are inside (5.8) and (5.9) are diagonal and have the same indices since residual interactions is present at fixed disorder. For convenience we will put upper latin indices, like Q^{ab}, to denote replicas.
In d-wave superconductors, from $[T, \tau_2 s_1] = 0$ we have

$$\tau_2 s_1 Q \tau_2 s_1 = -Q \tag{5.10}$$

together with the condition

$$C^t Q^t C = Q. \tag{5.11}$$

For the singlet and τ_0 and τ_3 components, this means

$$Q^{ab}_{S0,nm} = -Q^{ab}_{S0,-n-m}, \quad Q^{ab}_{S3,nm} = Q^{ab}_{S3,-n-m}, \tag{5.12}$$

and

$$Q^{ab}_{S0,nm} = Q^{ba}_{S0,mn}, \quad Q^{ab}_{S3,nm} = -Q^{ba}_{S3,mn} \tag{5.13}$$

having defined

$$Q = Q_S \sigma_0 + i\vec{Q}_T \cdot \vec{\sigma} \tag{5.14}$$

in spin space and

$$Q_S = Q_{S0}\tau_0 + i\sum_{j=1,2,3} Q_{Sj}\tau_j, \quad Q_T = Q_{T0}\tau_0 + i\sum_{j=1,2,3} Q_{Tj}\tau_j \tag{5.15}$$

in particle-hole space. The interaction in the p-h singlet channel is therefore

$$\begin{aligned} & \sum_{nm\omega}\sum_{a} Q^{aa}_{S0,n,n+\omega} Q^{aa}_{S0,m+\omega,m} + Q^{aa}_{S3,n,n+\omega} Q^{aa}_{S3,m+\omega,m} \\ = \quad & -\sum_{nm\omega}\sum_{a} Q^{aa}_{S0,-n-\omega,-n} Q^{aa}_{S0,m+\omega,m} + Q^{aa}_{S3,-n-\omega,-n} Q^{aa}_{S3,m+\omega,m}. \end{aligned}$$

By setting $-n \to n+\omega$ in the last term, we recover the first with opposite sign, hence the sum is zero. This means that the singlet term, with Γ_s, is suppress in d-wave superconductor by symmetry. This is physically conceivable being charge fluctuations not diffusive.

We now take into account also the interaction in the Cooper channel. The diffusive cooperon represents fluctuation in the particle-particle s-wave channel. Since the real part of the order parameter is already finite, fluctuations in the $\tau_2 s_1$ channels are indeed massive, while only fluctuations in the $\tau_1 s_1$ channel, corresponding to fluctuations of an is order parameter, stay massless. In the presence of residual interaction in the p-p channel, we must add to the action the term

$$-\sum \frac{\Gamma_c}{2}\, \overline{c}^{\alpha}_{n}(p_1)\, \overline{c}^{\beta}_{\omega-n}(k-p_1)\, c^{\beta}_{m}(p_2)\, c^{\alpha}_{\omega-m}(k-p_2) \tag{5.16}$$

By introducing one more auxiliary field, $Y^{\alpha\beta} = (Y^{\beta\alpha})^*$ with α and β spin indices, the p-p interaction can be rewritten as

$$\int dY e^{-\frac{1}{2}\sum Y^{\alpha\beta}_m(k) Y^{\beta\alpha}_m(-k) + i\sqrt{\Gamma_c}(\overline{c}^{\alpha}_n(p_1) Y^{\alpha\beta}_{\omega}(p_1+p_2)\overline{c}^{\alpha}_{\omega-n}(p_2) + c^{\beta}_n(p_1) Y^{\beta\alpha}_{\omega}(p_1+p_2) c^{\alpha}_{\omega-n}(p_2)} \tag{5.17}$$

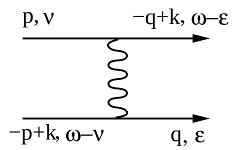

Figure 5.2: Diagram of interaction in particle-particle channel

From (2.7) and (2.8) the following equalities hold

$$\overline{\Psi}(\tau_1 + i\tau_2)\Psi = -i\, c\, \sigma_y c \tag{5.18}$$

$$\overline{\Psi}(\tau_1 - i\tau_2)\Psi = -i\, \overline{c}\, \sigma_y \overline{c} \tag{5.19}$$

so that calling $Y_R^{\alpha\beta} = \sum_\gamma Y^{\alpha\gamma}\sigma_y^{\gamma\beta}$ and $Y_L^{\alpha\beta} = \sum_\gamma \sigma_y^{\alpha\gamma} Y^{\gamma\beta}$, implying $Y_L^{\beta\alpha} = (Y_R^{\alpha\beta})^*$, (5.17) becomes

$$\int dY e^{-\frac{1}{2}\sum Y_{R\omega}^{\alpha\beta} Y_{L\omega}^{\beta\alpha} + \sqrt{\Gamma_c}(\overline{\Psi}_n^\alpha Y_{R\omega}^{\alpha\beta}\tau^+ \Psi_{\omega-n}^\beta + \overline{\Psi}_n^\beta Y_{L\omega}^{\beta\alpha}\tau^- \Psi_{\omega-n}^\alpha)} \tag{5.20}$$

where $\tau^\pm = \tau_1 \pm i\tau_2$. Integrating over fermions we find

$$\int dY e^{-\frac{1}{2}\sum Y_{R\omega}^{\alpha\beta} Y_{L\omega}^{\beta\alpha} + i\frac{\pi\nu}{2\Sigma}\sqrt{\Gamma_c}(Y_{R\omega}^{\alpha\beta} Tr(Q_{\omega-n,\,n}^{\beta\alpha}\tau^+) + Y_{L\omega}^{\beta\alpha} Tr(Q_{\omega-m,\,m}^{\alpha\beta}\tau^-))} \tag{5.21}$$

and finally after integration over Y_R, we obtain the action representing the interaction in the Cooper channel

$$-\sum \frac{\pi^2\nu^2}{4\Sigma^2}\Gamma_c Tr_{spin}\{Tr(Q_{n+\omega,\,-n}\tau^+) Tr(Q_{m+\omega,\,-m}\tau^-)\} \tag{5.22}$$

Also in this case the Q matrices are diagonal in replica space and both of them have the same replica index. By the charge conjugacy relation $C^t Q^t C = Q$, the triplet terms don't contribute since

$$Q_{T1,nm}^{ab} = -Q_{T1,mn}^{ba}, \;\; Q_{T2,nm}^{ab} = -Q_{T2,mn}^{ba} \tag{5.23}$$

so, if in (5.22) we transpose $Q_{n+\omega,\,-n}$ and put $-n \to n+\omega$ we'll have triplet terms with opposite sign. This means that at the end only the following term remains

$$-\sum \frac{\pi^2\nu^2}{8\Sigma^2}\Gamma_c \sum_{l=1,2} \left(Tr(Q_{n+\omega,\,-n}^{aa}\tau_l\,\sigma_0)\, Tr(Q_{m+\omega,\,-m}^{aa}\tau_l\,\sigma_0)\right). \tag{5.24}$$

5.2 Renormalization group

Let us calculate now the corrections to conductivity and to the density of states due to the interaction. Let us first consider the model without sublattice symmetry. The properties of massless modes in the Matsubara frequency space are the following, having imposed the conditions (3.22), (3.23), (3.32),

$$
\begin{aligned}
W^{ab}_{S0,nm} &= \ \ W^{ab*}_{S0,nm} = -W^{ba}_{S0,mn} = \ \ W^{ab}_{S0,-n-m} = -W^{ba}_{S0,-m-n},\\
W^{ab}_{S1,nm} &= -W^{ab*}_{S1,nm} = -W^{ba}_{S1,mn} = -W^{ab}_{S1,-n-m} = \ \ W^{ba}_{S1,-m-n},\\
W^{ab}_{S2,nm} &= -W^{ab*}_{S2,nm} = -W^{ba}_{S2,mn} = \ \ W^{ab}_{S2,-n-m} = -W^{ba}_{S2,-m-n},\\
W^{ab}_{S3,nm} &= \ \ W^{ab*}_{S3,nm} = \ \ W^{ba}_{S3,mn} = -W^{ab}_{S3,-n-m} = -W^{ba}_{S3,-m-n},\\
\vec{W}^{ab}_{T0,nm} &= \ \ \vec{W}^{ab*}_{T0,nm} = \ \ \vec{W}^{ba}_{T0,mn} = \ \ \vec{W}^{ab}_{T0,-n-m} = \ \ \vec{W}^{ba}_{T0,-m-n},\\
\vec{W}^{ab}_{T1,nm} &= -\vec{W}^{ab*}_{T1,nm} = \ \ \vec{W}^{ba}_{T1,mn} = -\vec{W}^{ab}_{T1,-n-m} = -\vec{W}^{ba}_{T1,-m-n},\\
\vec{W}^{ab}_{T2,nm} &= -\vec{W}^{ab*}_{T2,nm} = \ \ \vec{W}^{ba}_{T2,mn} = \ \ \vec{W}^{ab}_{T2,-n-m} = \ \ \vec{W}^{ba}_{T2,-m-n},\\
\vec{W}^{ab}_{T3,nm} &= \ \ \vec{W}^{ab*}_{T3,nm} = -\vec{W}^{ba}_{T3,mn} = -\vec{W}^{ab}_{T3,-n-m} = \ \ \vec{W}^{ba}_{T3,-m-n},
\end{aligned}
$$

where a and b are replica indices, while n and m Matsubara indices. The massless modes are obtained when n and m have opposite signs. If we take $n = -m$ we recover the symmetry properties derived in a previous chapter.
Let us introduce slow and fast modes in the spirit of Wilson Polyakov procedure, as we have seen before,

$$Q = \widetilde{U}^{\dagger}_{s} Q_f U_s = \widetilde{U}^{\dagger}_{s} \widetilde{U}^{\dagger}_{f} Q_{sp} U_f U_s \tag{5.25}$$

with $U = T = e^{\frac{W}{2}}$,

$$Q_{sp_{n\,m}} = \lambda_n \delta_{nm} \Sigma \equiv \mathrm{sign}\epsilon_n \delta_{nm} \Sigma \tag{5.26}$$

$\epsilon_n = (2n+1)\pi \mathrm{T}$ being a fermionic Matsubara frequency and

$$U_{snm} = \delta_{nm}, \quad \text{if} \quad (s_e\tau)^{-1} < |\epsilon_n| < \tau^{-1} \quad \text{or} \quad (s_e\tau)^{-1} < |\epsilon_m| < \tau^{-1} \tag{5.27}$$

where τ^{-1} is an energy cutoff and the rescaling factor $s_e > 1$. The massless fast modes satisfy by definition

$$W_{f_{nm}}(k) = 0 \quad \text{if} \quad \{Dk^2, |\epsilon_n|, |\epsilon_m|\} < (s_e\tau)^{-1} \tag{5.28}$$

with $D = \sigma/(2\nu)$ the diffusion coefficient. Now let us expand the interaction terms in the action, (5.8), (5.9) and (5.24), in terms of W_f, leaving slow U_s unexpanded. In this way, besides the terms (4.8) and (4.9), also the following contributions should be evaluated in the one loop expansion

$$S^1_{int} = -\mathcal{F}\sum \nu\Gamma\, Tr\left(\widetilde{U}^{\dagger de}_{n_1m_1}\lambda_{m_1}W^{eg}_{m_1m_2}U^{gd}_{m_2n_2}\tau_l\sigma\right) Tr\left(\widetilde{U}^{\dagger df}_{n_3m_3}\lambda_{m_3}W^{fh}_{m_3m_4}U^{hd}_{m_4n_4}\tau_l\sigma\right)$$
$$\delta(n_1 \mp n_2 \pm n_3 - n_4) \tag{5.29}$$

$$S^2_{int} = -\mathcal{F}\sum \nu\Gamma\, Tr\left(\widetilde{U}^{\dagger de}_{n_1m_1}\lambda_{m_1}W^{eg}_{m_1m_2}W^{gh}_{m_2m_3}U^{hd}_{m_3n_2}\tau_l\sigma\right) Tr\left(Q^{dd}_{n_3n_4}\tau_l\sigma\right)$$
$$\delta(n_1 \mp n_2 \pm n_3 - n_4) \tag{5.30}$$

where the upper indices are in the replica space, $\mathcal{F}$ is a coefficient equal to $\frac{\pi^2\nu}{8}$ while the other variables are respectively

for p-h singlet channel	$\Gamma = -\Gamma_s$	$l = 0, 3$	$\sigma = \sigma_0$	$\delta(n_1 - n_2 + n_3 - n_4)$
for p-h triplet channel	$\Gamma = \Gamma_t$	$l = 0, 3$	$\sigma = \vec{\sigma}$	$\delta(n_1 - n_2 + n_3 - n_4)$
for p-p Cooper channel	$\Gamma = \Gamma_c$	$l = 1, 2$	$\sigma = \sigma_0$	$\delta(n_1 + n_2 - n_3 - n_4)$

remembering that in d-wave superconductors Γ_s is suppressed by symmetry.

5.2.1 The corrections to conductivity and to density of states

To calculate one loop corrections to conductivity due to interactions, we have to consider the following averages over fast modes

$$\langle S^1_{int}\rangle + \langle S_1 S^1_{int}\rangle + \langle S_2 S^1_{int}\rangle + \frac{1}{2}\langle S_2 S_2 S^1_{int}\rangle \tag{5.31}$$

where S^1 and S^2 are defined by (4.8) and (4.9), while to calculate corrections to the density of states we would consider

$$\langle S^1_{int} S_\nu\rangle \tag{5.32}$$

where

$$S_\nu = \frac{\nu}{2}\sum Tr(\lambda_n \widetilde{U}^{\dagger ab}_{nm_1}\lambda_{m_1}W^{bc}_{m_1m_2}W^{cd}_{m_2m_3}U^{da}_{m_3n}). \tag{5.33}$$

Performing the calculation we obtain for d-wave superconductors

$$\delta\sigma/\sigma = \delta\nu/\nu = 2g\nu\left(\frac{3}{2}\Gamma_t + \frac{1}{2}\Gamma_c\right)\ln_{s_e} \tag{5.34}$$

in accordance with [35, 36], while for normal metals

$$\delta\sigma/\sigma = \delta\nu/\nu = 2g\nu\left(-\frac{1}{2}\Gamma_s + \frac{3}{2}\Gamma_t + \Gamma_c\right)\ln_{s_e}, \tag{5.35}$$

which coincides with Finkelstein's result. If time reversal symmetry is broken Γ_c disappears, while if a constant magnetic field is present

$$\Gamma_t \rightarrow \frac{1}{3}\Gamma_t$$

In the presence of magnetic impurities, only Γ_s in the normal metal remains.

We see that quasiparticle residual interaction, due to the absence of diffusive p-h singlet fluctuations, has a delocalizing effect if it is repulsive and localizing otherwise.

5.2.2 The corrections to the amplitudes

In order to calculate corrections to the interaction amplitudes at first order in g, we need to consider the sum

$$\langle S^1_{int}\rangle + \langle S^1_{int}S^2_{int}\rangle + \frac{1}{2}\langle S^1_{int}S^1_{int}\rangle + \frac{1}{2}\langle S^1_{int}S^1_{int}S^2_{int}\rangle + \frac{1}{2}\langle S^1_{int}S^2_{int}S^2_{int}\rangle. \tag{5.36}$$

Other terms can be neglected or treated within Random Phase Approximation (RPA). Even in (5.36) there are diagrams that can be obtained by RPA, so we have collected all diagrams that are not within RPA resummation in order to be sure that terms are not double counted. Using, for instance, the diagrams (Fig.5.1) and (Fig.5.3) the last four terms of (5.36) are drawn hereafter in (Fig.5.4), (Fig.5.5),

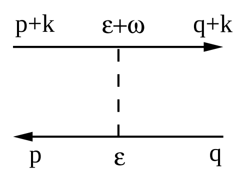

Figure 5.3: Diagram of diffusion propagator

(Fig.5.6) and (Fig.5.7). By evaluating the contributions of those diagrams, we have obtained the following corrections to the amplitudes in the superconducting case

$$\begin{aligned}\delta(\nu\Gamma_t) &= 2J_1(\nu\Gamma_t + \nu\Gamma_c) + 8\cdot 4^3 J_2\mathcal{F}\nu^2(\Gamma_t\Gamma_t + \Gamma_t\Gamma_c) + \\ &\quad 4^8\frac{1}{2\pi}J_3\mathcal{F}^2\nu^3\Gamma_t\Gamma_c\Gamma_t \qquad (5.37)\\ \delta(\nu\Gamma_c) &= 2J_1(3\nu\Gamma_t - \nu\Gamma_c) + 12\cdot 4^3 J_2\mathcal{F}\nu^2\Gamma_t\Gamma_c + O(\Gamma_c^2) \qquad (5.38)\end{aligned}$$

with

$$J_1 = \int \frac{d\vec{k}}{(2\pi)^2} D(k) = \int \frac{d\vec{k}}{(2\pi)^2} \frac{1}{4\pi\nu} \frac{1}{Dk^2} \tag{5.39}$$

$$J_2 = \int \frac{d\vec{k}}{(2\pi)^2} \frac{d\omega}{2\pi} D_\omega(k)^2 = \int \frac{d\vec{k}}{(2\pi)^2} \frac{d\omega}{2\pi} \frac{1}{(4\pi\nu)^2} \frac{1}{(Dk^2 + |\omega|)^2} \tag{5.40}$$

$$J_3 = \int \frac{d\vec{k}}{(2\pi)^2} \frac{d\omega}{2\pi} D_\omega(k)^3 |\omega| = \int \frac{d\vec{k}}{(2\pi)^2} \frac{d\omega}{2\pi} \frac{1}{(4\pi\nu)^3} \frac{|\omega|}{(Dk^2 + |\omega|)^3} \tag{5.41}$$

where the integration is done over momentum k and frequency ω in the range

$$(s_e\tau)^{-1} < Dk^2, |\omega| < \tau^{-1}.$$

Using the definition $g = \frac{1}{(2\pi)^2 \nu D}$ we find

$$J_1 = \frac{1}{4} g \ln s_e, \qquad J_2 = \frac{1}{(4\pi)^2 \nu} g \ln s_e, \qquad J_3 = \frac{1}{(4\pi)^4 \nu^2} g \ln s_e \tag{5.42}$$

and putting $\mathcal{F} = \frac{\pi^2 \nu}{8}$ in (5.37) and (5.38) we obtain in the superconducting case

$$\delta(\nu\Gamma_t) = g\Big(\nu(\Gamma_t + \Gamma_c)/2 + 4\nu^2(\Gamma_t\Gamma_t + \Gamma_t\Gamma_c) + 4\nu^3\Gamma_t\Gamma_c\Gamma_t\Big) \ln s_e \tag{5.43}$$

$$\delta(\nu\Gamma_c) = g\Big(\nu(3\Gamma_t - \Gamma_c)/2 + 6\nu^2\Gamma_t\Gamma_c\Big) \ln s_e - 2(\nu\Gamma_c)^2 \ln s_e, \tag{5.44}$$

where the last term of (5.44) comes from ladder summation. We have also considered the case of normal metal, when condition (3.32) doesn't hold, so Γ_s contributes and the propagator $D(k) \to 2D(k)$, and we have obtained

$$\delta(\nu\Gamma_t) = g\Big(\nu(\Gamma_s + \Gamma_t)/2 + \Gamma_c + 4\nu^2(\Gamma_t\Gamma_t + 2\Gamma_t\Gamma_c) + 8\nu^3\Gamma_t\Gamma_c\Gamma_t\Big) \ln s_e \tag{5.45}$$

$$\delta(\nu\Gamma_s) = g\Big(\nu(-\Gamma_s + 3\Gamma_t)/2 + \nu\Gamma_c\Big) \ln s_e \tag{5.46}$$

$$\delta(\nu\Gamma_c) = g\Big(\nu(\Gamma_s + 3\Gamma_t)/2 + 6\nu^2\Gamma_t\Gamma_c\Big) \ln s_e - 2(\nu\Gamma_c)^2 \ln s_e. \tag{5.47}$$

Putting $\Gamma \to \frac{\Gamma}{2}$ we obtain exactly the Finkel'stein equations. If time reversal symmetry is broken Γ_c disappears from the equations above. If a constant magnetic field is present we have, in superconductors,

$$\delta(\nu\Gamma_t) = -g\frac{1}{2}\nu\Gamma_t \ln s_e \tag{5.48}$$

while in normal metals

$$\delta(\nu\Gamma_t) = g\Big(\nu(\Gamma_s - \Gamma_t)/2\Big) \ln s_e \tag{5.49}$$

$$\delta(\nu\Gamma_s) = g\Big(\nu(-\Gamma_s + \Gamma_t)/2\Big) \ln s_e. \tag{5.50}$$

Finally if there are magnetic impurities only Γ_s in normal metal survives

$$\delta(\nu\Gamma_s) \quad = \quad -g\frac{1}{2}\nu\Gamma_s \ln s_e. \tag{5.51}$$

Figure 5.4: diagrams in $\langle S^1_{int} S^1_{int}\rangle$

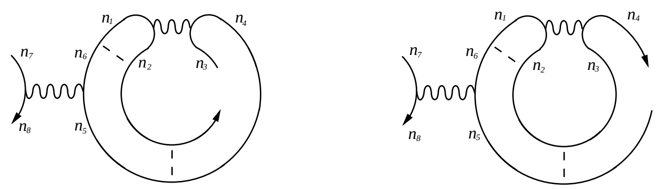

Figure 5.5: diagrams in $\langle S^1_{int} S^2_{int}\rangle$

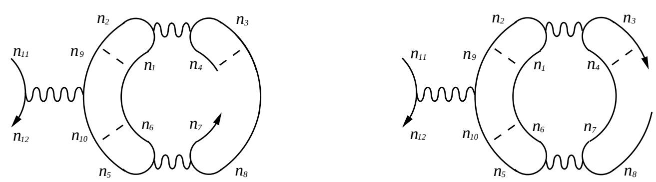

Figure 5.6: diagrams in $\langle S^1_{int} S^1_{int} S^2_{int}\rangle$

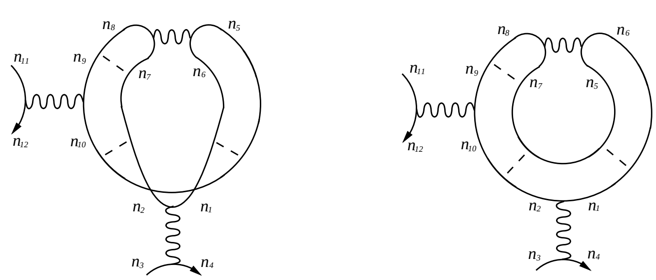

Figure 5.7: diagrams in $\langle S^2_{int} S^1_{int} S^2_{int}\rangle$

5.2.3 Interactions with (π,π) momentum transferred

Since at the nesting point staggered fluctuations become diffusive, it is interesting to include in the effective action interactions between quasiparticles with $(\pm\pi,\pm\pi)$ momentum transferred. In the sublattice representation (5.8), (5.9) and (5.24) can be rewritten like

$$\sum \frac{\mathcal{F}}{4\Sigma^2}\nu\Gamma^0\, Tr(Q^{dd}_{n_1n_2}\tau_l\sigma\gamma_0)Tr(Q^{dd}_{n_3n_4}\tau_l\sigma\gamma_0)\delta(n_1\mp n_2\pm n_3-n_4) \tag{5.52}$$

where γ_0 is the identity in the sublattice space.
Let us now consider interactions whose transferred momentum is $(\pm\pi,\pm\pi)=\vec{Q}$, which involve quasiparticles at the Fermi energy in the case of half filling,

$$-\sum \frac{\Gamma}{2}\,\overline{c}_n(p_1)\,\overline{c}_m(p_2)\,c_{m-\omega}(p_2-k-\vec{Q})\,c_{n+\omega}(p_1+k+\vec{Q}) \tag{5.53}$$

In this case we introduce other amplitudes Γ^3 related to staggered modes and consider at the end the following p-h interaction

$$\sum\sum_{p=0,3} \frac{\mathcal{F}}{4\Sigma^2}\nu\Gamma^p\, Tr(Q^{dd}_{n_1n_2}\tau_l\sigma\gamma_p)Tr(Q^{dd}_{n_3n_4}\tau_l\sigma\gamma_p)\delta(n_1\mp n_2\pm n_3-n_4) \tag{5.54}$$

The properties of W^3 in energy space derive from the conditions (3.33), (3.34), (3.39), leading to

$$\begin{aligned}
W^{ab}_{S0,nm} &= -W^{ab*}_{S0,nm} = W^{ba}_{S0,mn} = W^{ab}_{S0,-n-m} = W^{ba}_{S0,-m-n},\\
W^{ab}_{S1,nm} &= W^{ab*}_{S1,nm} = W^{ba}_{S1,mn} = -W^{ab}_{S1,-n-m} = -W^{ba}_{S1,-m-n},\\
W^{ab}_{S2,nm} &= W^{ab*}_{S2,nm} = W^{ba}_{S2,mn} = W^{ab}_{S2,-n-m} = W^{ba}_{S2,-m-n},\\
W^{ab}_{S3,nm} &= -W^{ab*}_{S3,nm} = -W^{ba}_{S3,mn} = -W^{ab}_{S3,-n-m} = W^{ba}_{S3,-m-n},\\
\vec{W}^{ab}_{T0,nm} &= -\vec{W}^{ab*}_{T0,nm} = -\vec{W}^{ba}_{T0,mn} = \vec{W}^{ab}_{T0,-n-m} = -\vec{W}^{ba}_{T0,-m-n},\\
\vec{W}^{ab}_{T1,nm} &= \vec{W}^{ab*}_{T1,nm} = -\vec{W}^{ba}_{T1,mn} = -\vec{W}^{ab}_{T1,-n-m} = \vec{W}^{ba}_{T1,-m-n},\\
\vec{W}^{ab}_{T2,nm} &= \vec{W}^{ab*}_{T2,nm} = -\vec{W}^{ba}_{T2,mn} = \vec{W}^{ab}_{T2,-n-m} = -\vec{W}^{ba}_{T2,-m-n},\\
\vec{W}^{ab}_{T3,nm} &= -\vec{W}^{ab*}_{T3,nm} = \vec{W}^{ba}_{T3,mn} = -\vec{W}^{ab}_{T3,-n-m} = -\vec{W}^{ba}_{T3,-m-n},
\end{aligned}$$

The massless modes are obtained when n and m have same signs.
By these properties we can write down the propagators (see Appendix D) and evaluate (5.31), (5.32) and (5.36) whose terms are written explicitly in Appendix E.

5.2.4 The new corrections to conductivity and to density of states

In the presence of $(\pm\pi, \pm\pi)$ momentum transferred interactions the interaction corrections to the conductivity and to the density of states at first order in g are the following: in superconductors

$$\delta\sigma/\sigma = \delta\nu/\nu = 2g\nu \left(\frac{3}{2} (\Gamma_t^0 - \Gamma_t^3) + \frac{1}{2}\Gamma_c \right) \ln_{s_e}, \tag{5.55}$$

while in normal metals

$$\delta\sigma/\sigma = \delta\nu/\nu = 2g\nu \left(\frac{1}{2}(-\Gamma_s^0 + \Gamma_s^3) + \frac{3}{2}(\Gamma_t^0 - \Gamma_t^3) + \Gamma_c \right) \ln_{s_e} . \tag{5.56}$$

If time reversal symmetry is broken Γ_c disappears, while if a constant magnetic field is present

$$\Gamma_t^0 \to \frac{1}{3}\Gamma_t^0$$

$$\Gamma_t^3 \to \frac{2}{3}\Gamma_t^3$$

and in the normal metal case Γ_s remains invariant while Γ_s^3 disappears.
With magnetic impurities only in the normal metal the singlet p-h channel remains Γ_s^0, all the other terms disappear.

5.2.5 The new corrections to the amplitudes

In this case the p-h channel amplitudes are twice their number without sublattice symmetry and so the RG equations that we'll have are the following, for d-wave superconductors

$$\begin{aligned}
\delta(\nu\Gamma_t^0) &= g\Big(\nu(\Gamma_t^0 + \Gamma_t^3 + \Gamma_c)/2 + 4\,\nu^2(\Gamma_t^0\Gamma_t^0 + \Gamma_t^0\Gamma_c - \Gamma_t^3\Gamma_t^3 - \Gamma_t^0\Gamma_t^3) \\
&\quad +4\nu^3(\Gamma_t^0\Gamma_c\Gamma_t^0 - \Gamma_t^0\Gamma_t^3\Gamma_t^0) - \frac{g}{4c}\nu\Gamma_t^3\Big) \ln s_e && (5.57) \\
\delta(\nu\Gamma_t^3) &= g\Big(\nu(\Gamma_t^0 + \Gamma_t^3 + \Gamma_c)/2 - 2\nu^2(\Gamma_t^0\Gamma_t^3 - \Gamma_c\Gamma_t^3 + 3\,\Gamma_t^3\Gamma_t^3) \\
&\quad -\frac{g}{4c}\nu\Gamma_t^0\Big) \ln s_e + 2(\nu\Gamma_t^3)^2 \ln s_e && (5.58) \\
\delta(\nu\Gamma_c) &= g\Big(\nu(3\Gamma_t^0 + 3\Gamma_t^3 - \Gamma_c)/2 + 6\nu^2(\Gamma_t^0 - \Gamma_t^3)\Gamma_c)\Big) \ln s_e && (5.59) \\
&\quad -2(\nu\Gamma_c)^2 \ln s_e && (5.60)
\end{aligned}$$

where $c = 1/2\pi^2\Pi$, while for normal metal

$$\begin{aligned}
\delta(\nu\Gamma^0_t) &= g\Big(\nu(\Gamma^0_s+\Gamma^3_s+\Gamma^0_t+\Gamma^3_t+2\Gamma_c)/2+4\,\nu(\Gamma^0_t\Gamma^0_t+2\Gamma^0_t\Gamma_c-\Gamma^3_t\Gamma^3_t-\Gamma^0_t\Gamma^3_t) \\
&\quad +4\nu^2(2\Gamma^0_t\Gamma_c\Gamma^0_t-\Gamma^0_t\Gamma^3_t\Gamma^0_t)-\frac{g}{4c}\nu\Gamma^3_t\Big)\ln s_e && (5.61)\\
\delta(\nu\Gamma^3_t) &= g\Big(\nu(\Gamma^0_s+\Gamma^3_s+\Gamma^0_t+\Gamma^3_t+2\Gamma_c)/2-2\nu^2(\Gamma^0_t\Gamma^3_t-2\Gamma_c\Gamma^3_t+3\,\Gamma^3_t\Gamma^3_t) \\
&\quad -\frac{g}{4c}\Gamma^0_t\Big)\ln s_e+2(\nu\Gamma^3_t)^2\ln s_e && (5.62)\\
\delta(\nu\Gamma^0_s) &= g\Big(\nu(-\Gamma^0_s-\Gamma^3_s+3\Gamma^0_t+3\Gamma^3_t)/2+\nu\Gamma_c-\frac{g}{4c}\nu\Gamma^3_s\Big)\ln s_e && (5.63)\\
\delta(\nu\Gamma^3_s) &= g\Big(\nu(-\Gamma^0_s-\Gamma^3_s+3\Gamma^0_t+3\Gamma^3_t)/2+\nu\Gamma_c-\frac{g}{4c}\nu\Gamma^0_s\Big)\ln s_e && (5.64)\\
&\quad -2(\nu\Gamma^3_s)^2\ln s_e && (5.65)\\
\delta(\nu\Gamma_c) &= g\Big(\nu(\Gamma^0_s+\Gamma^3_s+3\Gamma^0_t+3\Gamma^3_t)/2+6\nu^2(\Gamma^0_t-\Gamma^3_t)\Gamma_c\Big)\ln s_e && (5.66)\\
&\quad -2(\nu\Gamma_c)^2\ln s_e && (5.67)
\end{aligned}$$

If time reversal symmetry is broken Γ_c disappears from the equations above.
If a constant magnetic field is present we have in superconductors

$$\begin{aligned}
\delta(\nu\Gamma^0_t) &= g\Big(\nu(-\Gamma^0_t/2+\Gamma^3_t)-4\nu^2\Gamma^3_t\Gamma^3_t\Big)\ln s_e && (5.68)\\
\delta(\nu\Gamma^3_t) &= g\Big(\nu\Gamma^0_t/2-2\nu^2(3\Gamma^0_t\Gamma^3_t+2\Gamma^3_t\Gamma^3_t\Big)\ln s_e+2(\nu\Gamma^3_t)^2\ln s_e && (5.69)
\end{aligned}$$

while in metals even Γ^3_s is suppressed by symmetry and we have

$$\begin{aligned}
\delta(\nu\Gamma^0_t) &= g\Big(\nu(\Gamma^0_s-\Gamma^0_t/2+\Gamma^3_t)-4\nu^2\Gamma^3_t\Gamma^3_t\Big)\ln s_e && (5.70)\\
\delta(\nu\Gamma^3_t) &= g\Big(\nu(\Gamma^0_s+\Gamma^0_t)/2-2\nu^2(3\Gamma^0_t\Gamma^3_t+2\Gamma^3_t\Gamma^3_t\Big)\ln s_e+2(\nu\Gamma^3_t)^2\ln s_e && (5.71)\\
\delta(\nu\Gamma^0_s) &= g\Big(\nu(-\Gamma^0_s+\Gamma^0_t+2\Gamma^3_t)/2\Big)\ln s_e && (5.72)
\end{aligned}$$

Finally if there are magnetic impurities only Γ^0_s in normal metal survives, as in non chiral case.

As we said, (5.34) shows that a repulsive interaction has a delocalizing effect which competes with quantum interference corrections. Moreover both particle-hole and particle-particle channels enhance the density of states. In the presence of interaction with $(\pi\,\pi)$ momentum transferred, supposing it is repulsive as well, this effect is depressed. Since with chiral symmetry the quasiparticle conductivity has not quantum interference corrections, the term in (5.34) is the only contribution.

We notice that an analogous of the Anderson's theorem for s-wave superconductors holds at half-filling for the staggered density fluctuations. Namely, since these modes are diffusive, then the staggered susceptibility remains log-divergent even in the presence of disorder. As a result, for repulsive interaction, the Stoner instability towards a spin-density-wave is not destroyed by disorder.

Conclusions

In this work we have analysed the role of disorder in d-wave superconductors, which have gapless Landau-Bogoliubov quasiparticle excitations. We have considered several universality classes, including the chiral symmetry which occurs at half-filling for a two-sublattice model. In addition, we have also studied the effects of the residual quasiparticle interaction. The main results of this work are summarized in the following.

- In the presence of non magnetic impurities the spin conductivity is suppressed by quantum intereference corrections, in agreement with Ref. [5]. The density of states vanishes in the insulating regime.

  On the contrary, magnetic impurities gives a delocalization correction to the conductivity meanwhile enhancing the density of states.

- If chiral symmetry is present, namely at half filling for a two-sublattice model, the spin stays delocalized in spite of disorder and the conductivity remains finite. The DOS diverges in the absence of magnetic field and magnetic impurities.

- The charge conductivity has in general the same behavior of the spin conductivity. However, when chiral symmetry holds and time reversal symmetry is broken, the dirty d-wave superconductor behaves like a spin metal but charge insulator, manifesting a sort of spin-charge separation.

- Charge fluctuations as well as fluctuations of the real part of the order parameter, assuming the average value to be real, are not diffusive in a superconductor. Therefore the residual quasiparticle interaction written in terms of the diffusive modes only contains the spin-triplet particle-hole channel and the Cooper channel representing s-wave fluctuations of the imaginary part of the order parameter. For repulsive interaction, particles and holes repel each other in the spin-triplet channel, hence opposing localization. In fact we find that a repulsive residual interaction gives a delocalizing correction to the conductivity and enhances the density of states.

- We have also studied (π, π) momentum transferred interactions, since they are coupled to diffusive staggered spin fluctuations at half-filling. We find that the corrections to the conductivity due to the interaction at (π, π) have opposite sign of the corrections coming from the interaction at small momentum.

Appendix A

Wess-Zumino-Witten term

Let us consider the action seen in the text

$$S = S_0 + S_{imp} = \frac{1}{V}\sum_q \frac{1}{2\omega_q} Tr\left[Q_q^\dagger Q_q\right] + \sum_R \bar{\Psi}_R \left(H_{RR'} - iQ_R\delta_{RR'}\right)\Psi_{R'}, \qquad \text{(A.1)}$$

where ω_q is the Fourier transform of the variance of the random hopping, and Q_q the Fourier transform of

$$Q_R = Q_{0R}\gamma_0 + iQ_{3R}\gamma_3,$$

with hermitean Q_{0R} and Q_{3R}.
We neglect the momentum dependence of ω_q, which just renormalizes the stiffness. By integrating over the Fermi fields, we derive the effective action for Q given by

$$S = \sum_R \frac{1}{2\omega_0} Tr\left[Q_R^\dagger Q_R\right] - \frac{1}{2} Tr\ln\left(-H + iQ\right). \qquad \text{(A.2)}$$

Let us consider the operator $H - iQ$. Since

$$\begin{aligned}
&\left(H - iQ\right)\left(H + iQ^\dagger\right) = H^2 + QQ^\dagger - iQH + iHQ^\dagger \\
&= H^2 + QQ^\dagger + V = \left[1 + V\left(H^2 + QQ^\dagger\right)^{-1}\right]\left(H^2 + QQ^\dagger\right) \\
&\left(H + iQ^\dagger\right)\left(H - iQ\right) = H^2 + Q^\dagger Q - iHQ + iQ^\dagger H \\
&= H^2 + Q^\dagger Q + U = \left(H^2 + Q^\dagger Q\right)\left[1 + \left(H^2 + Q^\dagger Q\right)^{-1} U\right],
\end{aligned}$$

then the Green's function

$$\begin{aligned}
G &= \left(H + iQ^\dagger\right)\left(H^2 + QQ^\dagger\right)^{-1}\left[1 + V\left(H^2 + QQ^\dagger\right)^{-1}\right]^{-1} \\
&= \left[1 + \left(H^2 + Q^\dagger Q\right)^{-1} U\right]^{-1}\left(H^2 + Q^\dagger Q\right)^{-1}\left(H + iQ^\dagger\right). \qquad \text{(A.3)}
\end{aligned}$$

The saddle point equation is obtained through (A.2) and reads

$$\frac{1}{\omega_0} Tr\left(Q_R^\dagger \delta Q_R\right) + \frac{i}{2} Tr\left(G_{RR}\delta Q_R\right) = 0,$$

which, for a uniform saddle point solution, gives

$$\frac{\omega_0}{2} G_{RR} = \frac{\omega_0}{2}\left(H^2 + Q^\dagger Q\right)^{-1}_{RR} = 1.$$

In general the above equation implies $Q^\dagger Q = Q_0^2$ being proportional to the unit matrix. The low energy effective field theory is obtained by projecting Q_R onto the subspace in which $Q_R^\dagger Q_R = Q_0^2$. A simple way to derive the effective field theory within the above defined subspace, is to take

$$S[Q] \simeq -\frac{1}{4} Tr \ln\left(-H + iQ\right) - \frac{1}{4} Tr \ln\left(-H - iQ^\dagger\right),$$

which selects the hermitean part of the fermionic determinant. We find

$$S[Q] = -\frac{1}{4} Tr \ln\left(H^2 + Q_0^2 + U\right) = -\frac{1}{4} Tr \ln\left(H^2 + Q_0^2\right) - \frac{1}{4} Tr \ln\left(1 + G_0 U\right),$$

where $G_0 = \left(H^2 + Q_0^2\right)^{-1}$. We notice that

$$\begin{aligned} U_{RR'} &= iQ_R^\dagger H_{RR'} - iH_{RR'} Q_{R'} = iH_{RR'}\left(Q_R - Q_{R'}\right) \\ &\simeq -\vec{J}\cdot\vec{\nabla} Q + iH_{RR'}\left(R_i - R_i'\right)\left(R_j - R_j'\right)\partial_i\partial_j Q_{R'} \\ &= -\vec{J}\cdot\vec{\nabla} Q - [R_i, J_j]\partial_i\partial_j Q_{R'}. \end{aligned}$$

By expanding in U, the second term reads

$$\begin{aligned} S[Q] &= \frac{1}{8} Tr\left(G_0 U G_0 U\right) \simeq \\ &= \frac{1}{8} Tr\left(G_0 J_\mu G_0 J_\nu \partial_\mu Q^\dagger \partial_\nu Q\right) \\ &\equiv \frac{\pi}{8Q_0^2}\int dR\, Tr\left(\hat{\sigma}_{\mu\nu}\partial_\mu Q_R^\dagger \partial_\nu Q_R\right), \end{aligned} \quad \text{(A.4)}$$

which has the standard form of a non linear σ model.

If we take into account the longitudinal fluctuations, we find an additional term of the form

$$\delta S[Q] = -\frac{2\pi}{(4Q_0)^4}\Pi \int dR\, Tr\left(Q_R^\dagger \vec{\nabla} Q_R\right)\cdot Tr\left(Q_R^\dagger \vec{\nabla} Q_R\right). \quad \text{(A.5)}$$

The non linear σ model given by the two terms (A.4) and (A.5) is invariant under $Q \leftrightarrow Q^\dagger$, although the original model was not.

In order to check whether we missed something, let us consider the Hamiltonian $H - iQ^\dagger$. One notices that under $Q \to Q^\dagger$, $V \to -U$ and $U \to -V$. Therefore the Green's function for such an Hamiltonian is

$$\begin{aligned}\tilde{G} &= (H + iQ)\left(H^2 + Q_0^2\right)^{-1}\left[1 - U\left(H^2 + Q_0^2\right)^{-1}\right]^{-1} \\ &= \left[1 - \left(H^2 + Q_0^2\right)^{-1} V\right]^{-1}\left(H^2 + Q_0^2\right)^{-1}(H + iQ).\end{aligned}$$

We are interested in the difference

$$S_\Gamma = -\frac{1}{4} Tr \ln\left[-H + iQ\right] + \frac{1}{4} Tr \ln\left[-H + iQ^\dagger\right].$$

In order to evaluate such a difference, we follow the standard trick of taking the variation of S_Γ with respect to a variation of Q or $Q^\dagger$, such that $Q^\dagger Q = Q_0^2$, and integrating starting from $Q = Q^\dagger = Q_0$ up to the actual values. Specifically

$$S_\Gamma = \frac{i}{4}\int Tr\left(G\delta Q\right) - Tr\left(\delta Q^\dagger \tilde{G}\right).$$

The operator which appears in the trace is

$$\begin{aligned}G\delta Q - \delta Q^\dagger \tilde{G} &= \left[H\delta Q + iQ^\dagger \delta Q\right]\left[G_0 - G_0 U G_0 + G_0 U G_0 U G_0\right] \\ &\quad - \left[\delta Q^\dagger H + i\delta Q^\dagger Q\right]\left[G_0 + G_0 U G_0 + G_0 U G_0 U G_0\right] \\ &= \left[H\delta Q - \delta Q^\dagger H + 2iQ^\dagger \delta Q\right]\left[G_0 + G_0 U G_0 U G_0\right] \\ &\quad - \left[H\delta Q + \delta Q^\dagger H\right] G_0 U G_0,\end{aligned}$$

where we used the fact that $Q^\dagger \delta Q + \delta Q^\dagger Q = 0$. Taking into account that H contains both γ_1 and γ_2 while Q just contains γ_0 and γ_3, the only non vanishing terms are

$$\begin{aligned}G\delta Q - \delta Q^\dagger \tilde{G} &= 2iQ^\dagger \delta Q G_0 U G_0 U G_0 \\ &\quad - \left[H\delta Q + \delta Q^\dagger H\right] G_0 U G_0.\end{aligned}$$

We also notice that

$$H\delta Q + \delta Q^\dagger H \simeq 2H\delta Q + i\vec{J}\cdot\vec{\nabla}\delta Q.$$

The only relevant term is

$$G\delta Q - \delta Q^\dagger \tilde{G} \simeq 2iQ^\dagger \delta Q G_0 U G_0 U G_0.$$

Therefore

$$S_\Gamma = -\frac{1}{2}\int Tr\left[Q^\dagger \delta Q G_0 \vec{J}\cdot\vec{\nabla} Q G_0 \vec{J}\cdot\vec{\nabla} Q G_0\right].$$

Notice that

$$\vec{J} = \vec{J}_1\gamma_1 + \vec{J}_2\gamma_2,$$

hence

$$\vec{J}\cdot\vec{\nabla}Q \simeq \vec{\nabla}Q^\dagger\cdot\vec{J},$$

and

$$\begin{aligned} J_\mu J_\nu &= J_{1\mu}J_{1\nu} + J_{2\mu}J_{2\nu} + i\gamma_3\left(J_{1\mu}J_{2\nu} - J_{2\mu}J_{1\nu}\right) \\ &= J_{1\mu}J_{1\nu} + J_{2\mu}J_{2\nu} + i\gamma_3\epsilon_{\mu\nu}\vec{J}_1\times\vec{J}_2 \\ &= \delta_{\mu\nu}\frac{1}{2}\left(\vec{J}_1\cdot\vec{J}_1 + \vec{J}_2\cdot\vec{J}_2\right) + i\tau_3\epsilon_{\mu\nu}\frac{1}{2}\left(\vec{J}_1\times\vec{J}_1 + \vec{J}_2\times\vec{J}_2\right) + i\gamma_3\epsilon_{\mu\nu}\vec{J}_1\times\vec{J}_2, \end{aligned}$$

where we have assumed that a finite magnetic field, which adds phases to the hopping matrix elements, leads to a non zero off-diagonal conductivity. The first two terms are associated to $(i = 0, 3)$

$$\begin{aligned} &Tr\left(\tau_i Q^\dagger\delta Q\partial_\mu Q^\dagger\partial_\nu Q\right) = Tr\left(\tau_i Q\delta Q^\dagger\partial_\mu Q\partial_\nu Q^\dagger\right) \\ &-Tr\left(\tau_i\delta QQ^\dagger\partial_\mu Q\partial_\nu Q^\dagger\right) = -Tr\left(Q^\dagger\tau_i\delta Q\partial_\mu Q^\dagger\partial_\nu Q\right). \end{aligned}$$

The term with $\delta_{\mu\nu}$ and τ_0 cancels. For the other we find

$$\epsilon_{\mu\nu}Tr\left(\tau_3 Q^\dagger\delta Q\partial_\mu Q^\dagger\partial_\nu Q\right) = -\epsilon_{\mu\nu}Tr\left(Q^\dagger\tau_3\delta Q\partial_\mu Q^\dagger\partial_\nu Q\right),$$

which is also zero since, if time reversal symmetry is broken, $[Q, \tau_3] = 0$. Therefore the only non vanishing term is

$$\begin{aligned} S_\Gamma &= -\frac{i}{2}\int\epsilon_{\mu\nu}Tr\left[Q^\dagger\delta Q\partial_\mu Q^\dagger\partial_\nu Q\gamma_3\vec{J}_1\times\vec{J}_2 G_0^3\right] \\ &= -i\Gamma\int\epsilon_{\mu\nu}Tr\left[\gamma_3 Q^\dagger\delta Q\partial_\mu Q^\dagger\partial_\nu Q\right], \end{aligned}$$

where

$$\begin{aligned} \Gamma &= \frac{1}{2}\int\vec{J}_1\times\vec{J}_2 G_0^3 \\ &= \frac{1}{2}\int\frac{d^2k}{4\pi^2}\epsilon_k\vec{\nabla}\epsilon_k\times\vec{\nabla}\theta_k\left(\frac{1}{\epsilon_k^2+Q_0^2}\right)^3 \\ &= -\frac{1}{8}\int\frac{d^2k}{4\pi^2}\vec{\nabla}\left(\frac{1}{\epsilon_k^2+Q_0^2}\right)^2\times\vec{\nabla}\theta_k = -\frac{n}{16\pi}\frac{1}{Q_0^4}, \end{aligned}$$

where n counts the number of vortices minus antivortices in momentum space.
In conclusion we find a Wess-Zumino-Witten term [37, 38] given by

$$S_\Gamma = \frac{n}{2}\frac{1}{24\pi Q_0^6}\int d^3R\,\epsilon_{\alpha\beta\gamma}Tr\left(\gamma_3 Q^\dagger\partial_\alpha QQ^\dagger\partial_\beta QQ^\dagger\partial_\gamma Q\right). \tag{A.6}$$

Let us write

$$Q = \frac{1}{2} Q_0 \left[\left(g + g^\dagger \right) + \gamma_3 \left(g - g^\dagger \right) \right] = \frac{1}{2} Q_0 \left[\left(1 + \gamma_3 \right) g + \left(1 - \gamma_3 \right) g^\dagger \right] , \tag{A.7}$$

with $g \propto \gamma_0$, which is compatible with $Q^\dagger Q = Q_0^2$ if $g^{-1} = g^\dagger$, namely with g unitary. We have that

$$Q^\dagger \partial Q = \frac{Q_0^2}{2} \left[g \partial g^\dagger + g^\dagger \partial g + \gamma_3 \left(g^\dagger \partial g - g \partial g^\dagger \right) \right] .$$

Upon substituting this expression into (A.6) we find

$$S_\Gamma = \frac{1}{2} \frac{n}{12\pi} \int d^3 R \, \epsilon_{\alpha\beta\gamma} Tr \left(g^\dagger \partial_\alpha g g^\dagger \partial_\beta g g^\dagger \partial_\gamma g \right) , \tag{A.8}$$

where the trace does not imply anymore the sum over the two-sublattice indices. Apart from the factor $1/2$ in front, this is the standard form of the Wess-Zumino-Witten term.

In our case the number n is equal to 0 obtaining the same result of Fukui [39]. This means that in the four Dirac nodes model the WZW term is canceled out accidentally. If we introduced in the model a term that breaks the four-fold symmetry the WZW would appear again and we'd take it into account obtaining very different scaling behaviors.

Appendix B

Gauge transformations

Let us consider an operator diagonal in the Nambu space. Namely

$$\mathcal{A}_{ij} = \begin{pmatrix} \mathcal{A}_{\uparrow,ij} & 0 \\ 0 & \mathcal{A}_{\downarrow,ij} \end{pmatrix},$$

where the matrix elements are matrices in the retarded/advanced and replica space. If we take $\mathcal{A}_{ii} = 0$, then such an operator corresponds to

$$\begin{aligned} &\sum_{ij} c^\dagger_{i\uparrow} \mathcal{A}_{\uparrow,ij} c_{j\uparrow} - c^\dagger_{i\downarrow} \mathcal{A}^t_{\downarrow,ji} c_{j\downarrow} \\ &= \sum_{ij} c^\dagger_i \left[\frac{1}{2} \left(\mathcal{A}_{\uparrow,ij} - \mathcal{A}^t_{\downarrow,ji} \right) + \frac{1}{2} \sigma_z \left(\mathcal{A}_{\uparrow,ij} + \mathcal{A}^t_{\downarrow,ji} \right) \right] c_j. \end{aligned}$$

In the path integral formalism, a generic operator diagonal in the Nambu space,

$$\sum_{ij} \bar{\Psi}_i A_{ij} \Psi_j,$$

with

$$A_{ij} = \begin{pmatrix} A_{1,ij} & 0 \\ 0 & A_{2,ij} \end{pmatrix},$$

corresponds instead to

$$\frac{1}{2} \sum_{ij} \bar{c}_i \left[A^t_{1,ji} + \sigma_y A_{2,ij} \sigma_y \right] c_j.$$

By comparison we have that

$$\left[A^t_{1,ji} + \sigma_y A_{2,ij} \sigma_y \right] = \left[\left(\mathcal{A}_{\uparrow,ij} - \mathcal{A}^t_{\downarrow,ji} \right) + \sigma_z \left(\mathcal{A}_{\uparrow,ij} + \mathcal{A}^t_{\downarrow,ji} \right) \right].$$

Suppose that the operators in question are currents. Then $\mathcal{A}_{ij} = -\mathcal{A}_{ji}$, and the above relation reads

$$\left[-A^t_{1,ij} + \sigma_y A_{2,ij} \sigma_y \right] = \left[\left(\mathcal{A}_{\uparrow,ij} + \mathcal{A}^t_{\downarrow,ij} \right) + \sigma_z \left(\mathcal{A}_{\uparrow,ij} - \mathcal{A}^t_{\downarrow,ij} \right) \right].$$

In general we can consider either a charge current, implying $\mathcal{A}_\uparrow = \mathcal{A}_\downarrow = \mathcal{A}$, or a spin current, in which case $\mathcal{A}_\uparrow = -\mathcal{A}_\downarrow = \mathcal{A}$.

In the former case

$$\left[-A^t_{1,ij} + \sigma_y A_{2,ij}\sigma_y\right] = \left[\left(\mathcal{A}_{ij} + \mathcal{A}^t_{ij}\right) + \sigma_z \left(\mathcal{A}_{ij} - \mathcal{A}^t_{ij}\right)\right],$$

while in the latter

$$\left[-A^t_{1,ij} + \sigma_y A_{2,ij}\sigma_y\right] = \left[\left(\mathcal{A}_{ij} - \mathcal{A}^t_{ij}\right) + \sigma_z \left(\mathcal{A}_{ij} + \mathcal{A}^t_{ij}\right)\right].$$

We therefore see that, if $\mathcal{A}$ (we assume the same property holds for A) is a symmetric matrix, the charge current operator is proportional to the identity in spin space and

$$-A_{1,ij} + A_{2,ij} = 2\mathcal{A}_{ij},$$

namely

$$A_{2,ij} = -A_{1,ij} = \mathcal{A}_{ij},$$

while the spin is proportional to σ_z and

$$-A_{1,ij} + \sigma_y A_{2,ij}\sigma_y = 2\sigma_z\mathcal{A}_{ij},$$

implying

$$A_{2,ij} = A_{1,ij} = -\sigma_z\mathcal{A}_{ij}.$$

In the opposite case of an antisymmetric $\mathcal{A}$, the charge current multiplies σ_z and

$$[A_{1,ij} + \sigma_y A_{2,ij}\sigma_y] = 2\sigma_z\mathcal{A}_{ij},$$

leading to

$$A_{1,ij} = -A_{2,ij} = \sigma_z\mathcal{A}_{ij},$$

while the spin is proportional to the identity and

$$[A_{1,ij} + \sigma_y A_{2,ij}\sigma_y] = 2\mathcal{A}_{ij},$$

leading to

$$A_{1,ij} = A_{2,ij} = \mathcal{A}_{ij}.$$

These relations imply that the charge current is always proportional to τ_3 and, if $\mathcal{A}$ is symmetric, is the identity in spin space, otherwise is proportional to σ_z. For the spin current, the opposite occurs.

Let us see now which are the vector potentials for the charge and the spin.

Charge

The vector potential on the basis of Nambu spinors in the path integral formulation that generates an U(1) gauge transformation on the vectors of the hole-particle space is $A \propto \tau_3$, so the charge current vertex is

$$J^{(ch)}_{R_1R_2} = -i(R_1 - R_2)t_{R_1R_2}\tau_3.$$

We note that the system isn't invariant neither globally for the transformation e^{τ_3}, the charge indeed is not a conserved quantity.
If we want also energy structure to be included in the definition of charge vector potential, by the considerations above we would have

replica sp.	$\lambda_{ab} = \frac{1}{\sqrt{2}}(\delta_{a1}\delta_{b2} + \delta_{a2}\delta_{b1})$	$\lambda_{ab} = \frac{i}{\sqrt{2}}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})$
A^0	$\tau_3\, s_0$	$\tau_3\, s_0\, \sigma_z$
A^1	$\tau_3\, s_1$	$\tau_3\, s_1\, \sigma_z$
A^2	$\tau_3\, s_2\, \sigma_z$	$\tau_3\, s_2$
A^3	$\tau_3\, s_3$	$\tau_3\, s_3\, \sigma_z$

Spin

The vector potential on the basis of Nambu spinors that generates an SU(2) gauge transformation on the spin vectors is $A \propto \tau_0\sigma_z$, so the spin current vertex is

$$J^{(sp)}_{R_1R_2} = -i(R_1 - R_2)(t_{R_1R_2} + i\Delta_{R_1R_2}\tau_2 s_1)\sigma_z.$$

Analogously to the charge we should have these structures for spin vector potential

replica sp.	$\lambda_{ab} = \frac{1}{\sqrt{2}}(\delta_{a1}\delta_{b2} + \delta_{a2}\delta_{b1})$	$\lambda_{ab} = \frac{i}{\sqrt{2}}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})$
A^0	$\tau_0\, s_0\, \sigma_z$	$\tau_0\, s_0$
A^1	$\tau_0\, s_1\, \sigma_z$	$\tau_0\, s_1$
A^2	$\tau_0\, s_2$	$\tau_0\, s_2\, \sigma_z$
A^3	$\tau_0\, s_3\, \sigma_z$	$\tau_0\, s_3$

We can note that the last two gauges, A^2 ed A^3, don't belong to the set of generators of the symmetry group of the system and in fact we'll see that these transformations belongs to charge modes.

We can also underline that the gauges $\lambda_{ab} = \frac{1}{\sqrt{2}}(\delta_{a1}\delta_{b2} \pm \delta_{a2}\delta_{b1})$ preserves ether sesquilinear ($g_{ij} = g^*_{ji}$) and bilinear symmetric ($g_{ij} = g_{ji}$) and antisymmetric ($g_{ij} = -g_{ji}$) metrics so it is suitable to all cases while for example a gauge like $\lambda_{ab} = \delta_{a1}\delta_{b1}$ preserves only bilinear symmetric metric (for vector space with dimension greater than 1), an so could be useful for example in superconducting chiral case with applied magnetic field or in simple metal but in the bosonic replica language, in other cases could break the symmetry in an inopportune way.

Current current correlation function

Let us suppose to calculate the current current correlation function

$$\langle T\, J(R)J(R^{'})\rangle \tag{B.1}$$

The path integral operates the time ordering, so we can forget about T. The current on the spinor Ψ is

$$\vec{J}(R) = -i\sum_{R_1}(\vec{R}-\vec{R}_1)\bar{\Psi}_R\mathcal{H}_{RR_1}\Psi_{R_1} = \sum_{R_1R_2}\bar{\Psi}_{R_1}J_{R_1R_2}(R)\Psi_{R_2} \tag{B.2}$$

with

$$J_{R_1R_2}(R) = -i(\vec{R}_1-\vec{R}_2)\mathcal{H}_{R_1R_2}\delta_{R_1R} \tag{B.3}$$

In the case of spin current $\mathcal{H} = (t+i\Delta\tau_2 s_1)\sigma_z$ while for the charge current $\mathcal{H} = t\tau_3$. The correlation function becomes

$$\langle J(R)J(R^{'})\rangle = \sum_{R_1R_2R_3R_4}\langle\bar{\Psi}_{R_1}J_{R_1R_2}(R)\Psi_{R_2}\bar{\Psi}_{R_3}J_{R_3R_4}(R^{'})\Psi_{R_4}\rangle. \tag{B.4}$$

Let us represent for simplicity the fields with Ψ^i where i is the multilabel for replica, hole particle, spin, energy position indices, so the correlation function can be written in this way

$$\langle\bar{\Psi}^iJ_{ij}\Psi^j\bar{\Psi}^lJ_{lm}\Psi^m\rangle = -J_{ij}\langle\Psi^j\bar{\Psi}^l\rangle J_{lm}\langle\Psi^m\bar{\Psi}^i\rangle + J_{ij}\langle\Psi^j\Psi^m\rangle J_{lm}\langle\bar{\Psi}^l\bar{\Psi}^i\rangle \tag{B.5}$$

or in matrix language in the following way

$$\langle\bar{\Psi}J\Psi\bar{\Psi}J\Psi\rangle = -Tr\left(J\langle\Psi\bar{\Psi}\rangle J\langle\Psi\bar{\Psi}\rangle\right) + Tr\left(J\langle\Psi\Psi^t\rangle J^t\langle\bar{\Psi}^t\bar{\Psi}\rangle\right) \tag{B.6}$$

since

$$\bar{\Psi} = (c\Psi)^t \tag{B.7}$$

with $c = i\tau_1\sigma_y$, the correlation function becomes

$$-Tr\left(J\langle\Psi\bar{\Psi}\rangle J\langle\Psi\bar{\Psi}\rangle + J\langle\Psi\bar{\Psi}\rangle\, cJ^tc^t\langle\Psi\bar{\Psi}\rangle\right) \tag{B.8}$$

in terms of single particle Green's function

$$-Tr\left(JG(J+cJ^tc^t)G\right) \tag{B.9}$$

Either for spin

$$J_{R_1R_2} = -i(R_1-R_2)(t_{R_1R_2}+i\Delta_{R_1R_2}\tau_2 s_1)\sigma_z \tag{B.10}$$

and for charge

$$J_{R_1R_2} = -i(R_1 - R_2)t_{R_1R_2}\tau_3 \tag{B.11}$$

the following relation holds

$$cJ^tc^t = J \tag{B.12}$$

therefore the correlation function can be reduced to

$$-2Tr\,(JGJG) \tag{B.13}$$

Once the potential vector is introduced, it is coupled to the current vertex and for that it needs to evaluate

$$-Tr\,\big(JAG(JA + c(JA)^tc^t)G\big) \tag{B.14}$$

In both cases, of spin and charge, if $A = A^0s_0 + A^1s_1$ the following relation holds

$$c(JA)^tc^t = JA. \tag{B.15}$$

Applying to (B.14) the second derivative with respect to A^0 menus the second derivative with respect to A^1, that's to say

$$\left(\frac{\partial^2}{\partial {A^0}^2}\Big|_{A=0} - \frac{\partial^2}{\partial {A^1}^2}\Big|_{A=0}\right) \tag{B.16}$$

we have

$$-\big(Tr(JG^+JG^+) - Tr(JG^-JG^+)\big) \propto \sigma \tag{B.17}$$

that is the conductivity of spin or of charge, depending if J is (B.10) or (B.11). Instead of using (B.16), by d-wave symmetry it is straightforward to show that (B.17) comes easily from (B.14) if we take simply $A = A^0(s_0 + i\,s_1)$.

Now let us take for example the gauge $A = A^2\tau_0s_2\sigma_z + A^3\tau_0s_3$ from the table of spin potential. In this case

$$c(J_{RR'}A)^tc^t = -i(R - R^{'})(t_{RR'} - i\Delta_{RR'}\tau_2s_1)\sigma_zA \tag{B.18}$$

therefore the correlation function becomes (we call $\delta = R - R^{'}$)

$$2Tr\,(\delta\,(t + i\Delta\tau_2s_1)\sigma_zA\,G\,\delta\,t\sigma_zA\,G) \tag{B.19}$$

to which we apply the operator

$$\left(\frac{\partial^2}{\partial {A^3}^2}\Big|_{A=0} - \frac{\partial^2}{\partial {A^2}^2}\Big|_{A=0}\right). \tag{B.20}$$

or alternatively we put in (B.19) $A = A^2(s_2\sigma_z + i\, s_3)$. Using d-wave symmetry and the relations

$$\begin{aligned} \tau_0 s_2 G \tau_0 s_2 &= \tau_3 s_1 G \tau_3 s_1 \qquad (B.21)\\ \tau_0 s_3 G \tau_0 s_3 &= \tau_3 s_0 G \tau_3 s_0 \qquad (B.22) \end{aligned}$$

that hold in every cases since the structure of the Green's function at fixed disorder is

$$G = [(\tau_0 s_0,\, \tau_3 s_0,\, \tau_2 s_1,\, \tau_1 s_1)(\gamma_1, \gamma_2)] + i\Sigma[(\tau_2 s_2,\, \tau_1 s_2)(\gamma_3)][(\tau_3 s_3,\, \tau_0 s_3)(\gamma_0)]$$

in which the terms $\tau_3 s_0,\ \tau_1 s_1,\ \tau_1 s_2,\ \tau_3 s_3$ disappears if time reversal symmetry is preserved, and developing the trace one can find that (B.20) on (B.19) gives

$$Tr\,(\delta\, t\tau_3 s_0 G\, \delta\, t\tau_3 s_0 G) - Tr\,(\delta\, t\tau_3 s_1 G\, \delta\, t\tau_3 s_1 G) \qquad (B.23)$$

that is the charge conductivity.

We have demonstrate that the gauges $A = (A^0\tau_3 s_0 + A^1\tau_3 s_1)\left\{\begin{array}{l}\lambda_s\\ \lambda_a\sigma_z\end{array}\right.$ and the gauges $A = (A^2\tau_0 s_2 + A^3\tau_0 s_3\sigma_z)\left\{\begin{array}{l}\lambda_s\\ \lambda_a\sigma_z\end{array}\right.$ are equivalent in generating charge conductivity.

The same calculation can be done to prove the equivalence of the other two remaining pairs of gauges that produce spin conductivity. At the end the possible choices for vector potential are the following:

for **charge**

replica sp.	$\lambda_{ab} = \frac{1}{\sqrt{2}}(\delta_{a1}\delta_{b2} + \delta_{a2}\delta_{b1})$	$\lambda_{ab} = \frac{i}{\sqrt{2}}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})$
A^0	$\tau_3\ s_0$	$\tau_3\ s_0\ \sigma_z$
A^1	$\tau_3\ s_1$	$\tau_3\ s_1\ \sigma_z$
A^2	$\tau_0\ s_2$	$\tau_0\ s_2\ \sigma_z$
A^3	$\tau_0\ s_3\ \sigma_z$	$\tau_0\ s_3$

for **spin**

replica sp.	$\lambda_{ab} = \frac{1}{\sqrt{2}}(\delta_{a1}\delta_{b2} + \delta_{a2}\delta_{b1})$	$\lambda_{ab} = \frac{i}{\sqrt{2}}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})$
A^0	$\tau_0\ s_0\ \sigma_z$	$\tau_0\ s_0$
A^1	$\tau_0\ s_1\ \sigma_z$	$\tau_0\ s_1$
A^2	$\tau_3\ s_2\ \sigma_z$	$\tau_3\ s_2$
A^3	$\tau_3\ s_3$	$\tau_3\ s_3\ \sigma_z$

Appendix C

Charge and spin vector potentials

Charge vector potential

Let us introduce a slow varying vector potential $\overrightarrow{A} \propto \tau_3$, so that the fields change in this way

$$c_R \longrightarrow e^{i\frac{e}{c}\int_0^R \overrightarrow{A}_{R'}\, dR'}\, c_R \simeq e^{i\frac{e}{c}\overrightarrow{A}\,\overrightarrow{R}}\, c_R \tag{C.1}$$

$$c_R^\dagger \longrightarrow c_R^\dagger\, e^{-i\frac{e}{c}\overrightarrow{A}\,\overrightarrow{R}} \tag{C.2}$$

since A is a slow varying function, nearly constant on the lattice length. In the Hamiltonian this transformation is equivalent to changing hopping term in this way

$$t_{RR'} \longrightarrow t_{RR'}\, e^{-i\frac{e}{c}\overrightarrow{A}\cdot(\overrightarrow{R}'-\overrightarrow{R})} \simeq t_{RR'}\left(1 - i\frac{e}{c}\overrightarrow{\delta}\cdot\overrightarrow{A} - \frac{e^2}{2c^2}(\overrightarrow{\delta}\cdot\overrightarrow{A})^2\right) \tag{C.3}$$

where $\overrightarrow{\delta} = \overrightarrow{R}' - \overrightarrow{R}$

The interaction term, before parametrization by Δ, is unaffected by this gauge transformation and so

$$\Delta_{RR'} \longrightarrow \Delta_{RR'} \tag{C.4}$$

The expressions (C.3) and (C.4) are a representation of the breaking U(1) gauge invariance. Now returning to (3.57) we should reconsider in $\widetilde{T}_R H^0_{RR'} T_{R'}$ the piece of transformed hopping term

$$\begin{aligned}
&\widetilde{T}_R t_{RR'}\left(1 - i\frac{e}{c}\overrightarrow{\delta}\cdot\overrightarrow{A} - \frac{e^2}{2c^2}(\overrightarrow{\delta}\cdot\overrightarrow{A})^2\right)T^\dagger_{R'} \simeq \\
&t_{RR'}\left(1 + T_R\overrightarrow{\delta}\cdot\overrightarrow{\nabla}T^\dagger_R + \frac{1}{2}T_R(\overrightarrow{\delta}\cdot\overrightarrow{\nabla})^2T^\dagger_R\right) - \widetilde{T}_R t_{RR'}\left(i\frac{e}{c}\overrightarrow{\delta}\cdot\overrightarrow{A} + \frac{e^2}{2c^2}(\overrightarrow{\delta}\cdot\overrightarrow{A})^2\right)T^\dagger_{R'} = \\
&t_{RR'} + t_{RR'}\overrightarrow{\delta}\cdot T_R\overrightarrow{\nabla}T^\dagger_R + \frac{1}{2}\sum_{ij}\delta_i\delta_j T_R\partial_{ij}T^\dagger_R - i\frac{e}{c}\widetilde{T}_R t_{RR'}\overrightarrow{\delta}\cdot\overrightarrow{A}T^\dagger_{R'} - \frac{e^2}{2c^2}\widetilde{T}_R t_{RR'}(\overrightarrow{\delta}\cdot\overrightarrow{A})^2T^\dagger_{R'}
\end{aligned}$$

Besides the standard terms in the expansion of $Tr\ln(\varepsilon - H^0 + iQ)$ given by 2° and 3° of the above expression that bring to non linear σ model as seen before, remembering that $t = t_1\gamma_1 + t_2\gamma_2$, $\Delta = \Delta_1\gamma_1 + \Delta_2\gamma_2$ and defining $G = g + i\frac{\Sigma}{E^2+\Sigma^2}s_3$ with $g(k) = -\frac{1}{E^2+\Sigma^2}[(t_1 - i\Delta_1\tau_2 s_1)\gamma_1 + (t_2 - i\Delta_2\tau_2 s_1)\gamma_2]$, therefore, for instance

$$T^{\dagger}_R\, g\,(R,R^{'})\widetilde{T}_{R'} = g(R,R^{'}) + O(\delta)$$
$$Tr\left(G(RR^{'})\widetilde{T^{'}_R}t_{R'R}\overrightarrow{\delta}\cdot\overrightarrow{A}T^{\dagger}_R\right) = Tr\left(T^{\dagger}_R g(RR^{'})\widetilde{T^{'}_R}t_{R'R}\overrightarrow{\delta}\cdot\overrightarrow{A}\right)$$
$$= Tr\left(g(RR^{'})t_{R'R}\overrightarrow{\delta}\cdot\overrightarrow{A}\right) + Tr\left(g(RR^{'})t_{R'R}\overrightarrow{\delta}\cdot\overrightarrow{A}\,\overrightarrow{\delta}\cdot\overrightarrow{\nabla}T^{\dagger}_R T_R\right),$$

we have the following other terms

1. $i\frac{e}{c}Tr\left(Gt\overrightarrow{\delta}\cdot\overrightarrow{A}\right)$
2. $i\frac{e}{c}Tr\left(G\widetilde{T}t\overrightarrow{\delta}\cdot\overrightarrow{A}\,\overrightarrow{\delta}\cdot\overrightarrow{\nabla}T^{\dagger}\right)$
3. $\frac{e^2}{2c^2}Tr\left(G\widetilde{T}t(\overrightarrow{\delta}\cdot\overrightarrow{A})^2T^{\dagger}\right)$
4. $\frac{e^2}{2c^2}Tr\left(G\widetilde{T}t\overrightarrow{\delta}\cdot\overrightarrow{A}T^{\dagger}G\widetilde{T}t\overrightarrow{\delta}\cdot\overrightarrow{A}T^{\dagger}\right)$
5. $i\frac{e}{c}Tr\left(Gt\overrightarrow{\delta}\cdot T\overrightarrow{\nabla}T^{\dagger}G\widetilde{T}t\overrightarrow{\delta}\cdot\overrightarrow{A}T^{\dagger}\right)$

The first term is always null and using:

$$Q = \widetilde{T}^{\dagger}\Sigma s_3 T, \qquad g = \frac{1}{2}(G^+ + G^-), \qquad \frac{\Sigma}{E^2+\Sigma^2} = \frac{1}{2i}(G^+ - G^-),$$

the expression (3.73), the d-wave symmetry so that odd terms in Δ are zero under momentum integration and finally the relation $(t_1\Delta_1 + \Delta_2 t_2)^2 = (\Delta_1^2 + \Delta_2^2)(t_1^2 + t_2^2)$ since $t_1\Delta_2 = t_2\Delta_1$, at the end, summing all the terms above and multiplying them for $-\frac{1}{2}$ that is the coefficient in front of $Tr\ln(\varepsilon - H^0 + iQ)$, we have the following action due to vector potential

$$\frac{2\pi}{32\Sigma^2}\sigma_c Tr\left[\left(\nabla Q + i\frac{e}{c}[Q,A]\right)\left(\nabla Q^{\dagger} - i\frac{e}{c}[A,Q^{\dagger}]\right) - (\nabla Q\nabla Q^{\dagger})\right]$$
$$+\frac{e^2}{8c^2}f_1 Tr\left(\{A,\tau_2 s_1\}^2\right) - i\frac{e}{c}f_1 Tr\left(D\,A\right) + \frac{e^2}{8c^2}f_2 Tr\left([A,\tau_2 s_1]^2\right)$$

with

$$\sigma_c = \sigma - \frac{\Sigma^2}{\pi V} \sum_k Tr \left[\frac{(\nabla_k \Delta_k)^2}{(E^2 + \Sigma^2)^2} \right] \tag{C.5}$$

$$f_1 = \frac{1}{V} \sum_k Tr \left(\frac{(\epsilon_k \nabla_k \epsilon_k) \cdot (\Delta_k \nabla_k \Delta_k)}{(E^2 + \Sigma^2)^2} \right) \tag{C.6}$$

$$f_2 = \frac{1}{V} \sum_k Tr \left(\frac{(\epsilon_k \nabla_k \epsilon_k) \cdot (\Delta_k \nabla_k \Delta_k) - (\epsilon_k \nabla_k \Delta_k)^2}{(E^2 + \Sigma^2)^2} \right) \tag{C.7}$$

The terms proportional to f_1 are zero for $A \propto s_0$ and s_1 and we have seen that this is the case since these energy structures are the only possible choices if $A \propto \tau_3$, as we have suppose to be at the beginning. Anyway at Dirac point $f_1 \propto \vec{v_F} \cdot \vec{v_\Delta} = 0$ since $\vec{v_F} \perp \vec{v_\Delta}$. So the final piece of action to be added to (4.1) in presence of gauge potential is

$$\begin{aligned} S(A) \quad = \quad & \frac{2\pi}{32\Sigma^2} \sigma_c Tr \left[\left(\nabla Q + i\frac{e}{c}[Q, A] \right) \left(\nabla Q^\dagger - i\frac{e}{c}[A, Q^\dagger] \right) - (\nabla Q \nabla Q^\dagger) \right] \\ & + \frac{e^2}{8c^2} f_2 Tr \left([A, \tau_2 s_1]^2 \right) \end{aligned} \tag{C.8}$$

Moreover $f_2 \sim \epsilon_\perp^3 \epsilon_\| / \Sigma^4$ and so goes to 0 in the vicinity of the nodes being $\epsilon_\perp$ the linear dimension perpendicular to the Fermi surface of the ellipse shaped section of Dirac cone and $\epsilon_\|$ the parallel one to the Fermi surface. The conductivity instead at the nodes have the main contribution. Anyway the last term of (C.8) doesn't affect the charge conductivity's value since it is equal to zero under the operator (B.16) or alternatively is zero if $A = A^0(s_0 + i\, s_1)$.

Bare charge conductivity

Let us suppose to have $A = A^0 s_0 + A^1 s_1$, to recover the Kubo formula we have to evaluate

$$\left. \left(\frac{\partial^2 ln\mathcal{Z}}{\partial {A^0}^2} - \frac{\partial^2 ln\mathcal{Z}}{\partial {A^1}^2} \right) \right|_{A=0} \tag{C.9}$$

with

$$\mathcal{Z}(A) = \int DQ e^{-S_0 - S(A)}$$

Since $S(A = 0) = 0$, the generating function at zero vector potential is again $\mathcal{Z}(A = 0) = \mathcal{Z}_0 = \int DQ e^{-S_0}$. In this way we can write

$$\frac{\partial^2 ln\mathcal{Z}}{\partial {A^\alpha}^2} = - \left\langle \left. \frac{\partial S(A)}{\partial A^\alpha} \right|_{A=0} \right\rangle_0^2 - \left\langle \left. \frac{\partial^2 S(A)}{\partial {A^\alpha}^2} \right|_{A=0} \right\rangle_0 + \left\langle \left(\left. \frac{\partial S(A)}{\partial A^\alpha} \right|_{A=0} \right)^2 \right\rangle_0 \tag{C.10}$$

the first term is zero, the second is the average of the operator

$$\left.\frac{\partial^2 S(A)}{\partial A^{\alpha 2}}\right|_{A=0} = \frac{e^2\pi}{8c^2\Sigma^2}\sigma_c Tr\left([Q(R),\tau_3 s_\alpha][\tau_3 s_\alpha, Q(R)^\dagger]\right) + \frac{e^2}{4c^2} f_2 Tr\left(\{s_\alpha, s_1\}^2\right) \tag{C.11}$$

while the third is the average of the square value of

$$\left.\frac{\partial S(A)}{\partial A^{\alpha}}\right|_{A=0} = i\frac{e}{c}\left(\frac{\sigma_c \pi}{4\Sigma^2} Tr(\nabla Q(R)\, Q(R)^\dagger \tau_3\, s_\alpha)\right) \tag{C.12}$$

At the saddle point

$$Q(R) = Q_{sp} = \Sigma s_3$$

(C.12) is zero and the action is

$$S(A) = \frac{\pi e^2}{4c^2}\left(\sigma_c Tr(A^{1^2}) + f_2 Tr(A^{0^2} + A^{1^2}\right)$$

so the bare conductivity given by (C.9) is simply

$$-\left.\frac{\partial^2 S(A)}{\partial A^{0^2}}\right|_{A=0} + \left.\frac{\partial^2 S(A)}{\partial A^{1^2}}\right|_{A=0} = \frac{16\pi e^2}{c^2}\sigma_c \tag{C.13}$$

where, as we have already seen, by (3.73) and (C.5)

$$\sigma_c = \frac{\Sigma^2}{\pi V}\sum_k Tr\left[\frac{(\nabla_k \epsilon_k)^2}{(E^2+\Sigma^2)^2}\right] \tag{C.14}$$

in agreement with [21].

Spin vector potential

Let us now introduce a vector potential of this kind $\overrightarrow{A} \propto \tau_0 \sigma_z$, so that

$$c_{R\uparrow} \longrightarrow e^{i\frac{1}{2}\overrightarrow{A}\overrightarrow{R}}\, c_{R\uparrow} \tag{C.15}$$

$$c_{R\downarrow} \longrightarrow e^{-i\frac{1}{2}\overrightarrow{A}\overrightarrow{R}}\, c_{R\downarrow} \tag{C.16}$$

This time the Hamiltonian's parameters becomes

$$t_{RR'} \rightarrow t_{RR'}\, e^{-\frac{i}{2}\overrightarrow{A}\cdot(\overrightarrow{R}'-\overrightarrow{R})} \simeq t_{RR'}\left(1 - \frac{i}{2}\overrightarrow{\delta}\cdot\overrightarrow{A} - \frac{1}{4}(\overrightarrow{\delta}\cdot\overrightarrow{A})^2\right)$$

$$\Delta_{RR'} \rightarrow \Delta_{RR'}\frac{1}{2}\left(1 + e^{-i\overrightarrow{A}\cdot(\overrightarrow{R}'-\overrightarrow{R})}\right) \simeq \Delta_{RR'}\left(1 - \frac{i}{2}\overrightarrow{\delta}\cdot\overrightarrow{A} - \frac{1}{4}(\overrightarrow{\delta}\cdot\overrightarrow{A})^2\right)$$

Although the parameters transforms themselves differently the two expansions are equal to second order and the piece of action due to spin vector potential is

$$S(A) = \frac{2\pi\sigma_s}{32\Sigma^2} Tr \left[\left(\nabla Q + \frac{i}{2}[Q, A] \right) \left(\nabla Q^\dagger - \frac{i}{2}[A, Q^\dagger] \right) - (\nabla Q \nabla Q^\dagger) \right] \tag{C.17}$$

where the bare spin conductivity is $\sigma_s = \sigma$ whose explicit expression is (3.73).

Appendix D

Useful formulæ for interaction's perturbative corrections

Let us write the gaussian propagator in this way

$$
\begin{aligned}
&\left\langle W^{ab}_{p\frac{S}{T}i,nm} W^{cd}_{p\frac{S}{T}i,rq} \right\rangle = \frac{1}{2}\Big(1-(-)^p\lambda_n\lambda_m\Big) \\
&\Big[(-)^p(\pm)D^p_{nm}\Big(\delta^{ac}_{nr}\delta^{bd}_{mq}(-)^p[\pm]\delta^{ad}_{nq}\delta^{bc}_{mr}(-)^i\delta^{ac}_{n-r}\delta^{bd}_{m-q}(-)^p(-)^i[\pm]\delta^{ad}_{n-q}\delta^{bc}_{m-r}\Big) \\
&-\frac{1}{4}Tr(W^{aa}_{3\frac{S}{T}i,nn})\Pi_{nr}(k)\delta_{p3}\delta^{ab}_{nm}\delta^{cd}_{rq}\Big]
\end{aligned}
\tag{D.1}
$$

where $p = 0, 3$, $(\pm)$ are related to real or imaginary matrix elements of W_0 listed in Section 5.2, $[\pm]$ for symmetric or antisymmetric matrix, $(-)^i$ the sign that W acquires changing the signs of Matsubara frequencies and this occurs only for modes proportional to τ_1 and τ_3, and finally

$$D^0_{nm}(k) = \frac{1}{4\pi\nu}\frac{1}{Dk^2 + |\epsilon_n - \epsilon_m|} \qquad \text{with } \lambda_n = -\lambda_m \tag{D.2}$$

$$D^3_{nm}(k) = \frac{1}{4\pi\nu}\frac{1}{Dk^2 + |\epsilon_n + \epsilon_m|} \qquad \text{with } \lambda_n = \lambda_m \tag{D.3}$$

$$\Pi_{n,r}(k) = D^3_{nn}(k)\frac{\Pi k^2}{Dk^2 + 2|\epsilon_r|} \qquad \text{in the 0 replica limit.} \tag{D.4}$$

By this formulation and for this symmetry property of U

$$\tau_1\sigma_y U^t \tau_1\sigma_y = \widetilde{U}^\dagger \tag{D.5}$$

we have the following relation

$$Tr\left(U^{ab}_{m_1m_2}\tau_l\sigma\widetilde{U}^{\dagger cd}_{m_3m_4}\tau_i\sigma_j\right) = (-)^{\bar{l}}[\pm]\,Tr\left(U^{dc}_{m_4m_3}\tau_l\sigma\widetilde{U}^{\dagger ba}_{m_2m_1}\tau_i\sigma_j\right) \tag{D.6}$$

with

$$(-)^{\bar{l}} = \begin{cases} -(-)^l & \text{in p-h singlet channel, } (l = 0, 3,\ \sigma = \sigma_0) \\ (-)^l & \text{in p-h triplet channel, } (l = 0, 3,\ \sigma = \vec{\sigma}) \\ - & \text{in p-p Cooper channel, } (l = 1, 2,\ \sigma = \sigma_0) \end{cases}$$

where all σs are Pauli matrices in spin space.
Moreover if we have a term like $A = \nabla \widetilde{U} \widetilde{U}^\dagger$, so that it is possible to write

$$\int dr\, Tr(AA) = -\sum_k k^2 Tr(\widetilde{U}(k) \widetilde{U}^\dagger(-k)), \tag{D.7}$$

by the following symmetry condition

$$\tau_1 \sigma_y A^t \tau_1 \sigma_y = -\gamma_1 A \gamma_1 \tag{D.8}$$

where γ_1 is the first Pauli matrix on sublattice space, we can write the this relation

$$Tr\left(\tau_i \sigma_j \gamma_q \tau_{i'} \sigma_{j'} \gamma_p A^{ab}_{nm}\right) = -(-)^p (-)^q [\pm]_{ij} [\pm]_{i'j'} Tr\left(A^{ba}_{mn} \tau_{i'} \sigma_{j'} \gamma_p \tau_i \sigma_j \gamma_q\right) \tag{D.9}$$

useful in the evaluation of $\langle S_2 S^1_{int} \rangle$ and $\frac{1}{2} \langle S_2 S_2 S^1_{int} \rangle$, where γ_q and γ_p are γ_0 or γ_3 in sublattice space while σ_j and $\sigma_{j'}$ are identities or Pauli matrices in spin space.
Defining the quaternions $\bar{\tau}_i = \tau_0, i\tau_1, i\tau_2, i\tau_3$ and $\bar{\sigma}_j = \sigma_0, i\sigma_x, i\sigma_y, i\sigma_z$ we have also this sum rule

$$\sum_{i,j} (\pm)_{ij} [\pm]_{ij} Tr\left(M \bar{\tau}_i \bar{\sigma}_j\right) Tr\left(N \bar{\tau}_i \bar{\sigma}_j\right) = -4 Tr(MN). \tag{D.10}$$

where M and N are generic 4×4 matrices.

Appendix E

Perturbative terms for the renormalization of amplitudes in superconductive case

Here we write down the values averaged over fast modes of action's terms in the one loop expansion, useful to calculate corrections to amplitudes of interaction in d-wave superconductor case.
In the following expressions the symbol Tr means trace over all degrees of freedom except for sublattice space over which we shall trace by tr.

$$
\begin{aligned}
\left\langle S^1_{int}\right\rangle \quad = \quad & -\sum 4\mathcal{F}\nu\Gamma^q_{(n_1,n_2,n_3,\tau_l,\sigma)} D^{k_1}_{m_1m_2}\Big(1-(-)^{k_1}\lambda_{m_1}\lambda_{m_2}\Big)\,\delta(n_1\mp n_2\pm n_3-n_4)\\
&\Big\{\mathrm{Tr}\Big(\lambda_{m_2}\mathrm{tr}(U^{gd}_{m_2n_2}\tau_l\sigma\widetilde{U}^{\dagger de}_{n_1m_1}\gamma_{k_1}\gamma_q)\lambda_{m_1}\mathrm{tr}(U^{ed}_{m_1n_4}\tau_l\sigma\widetilde{U}^{\dagger dg}_{n_3m_2}\gamma_{k_1}\gamma_q)\Big)\\
&-(-)^l\mathrm{Tr}\Big(\lambda_{m_2}\mathrm{tr}(U^{gd}_{m_2n_2}\tau_l\sigma\widetilde{U}^{\dagger de}_{n_1m_1}\gamma_{k_1}\gamma_q)\lambda_{m_1}\mathrm{tr}(U^{ed}_{m_1-n_4}\tau_l\sigma\widetilde{U}^{\dagger dg}_{-n_3m_2}\gamma_{k_1}\gamma_p)\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle S^1_{int}S^2_{int}\right\rangle \quad = \quad & \sum 4^3\mathcal{F}^2\nu^2\Gamma^q_{(n_1,n_2,n_3,\tau_l,\sigma)}\Gamma^p_{(n_5,n_6,n_7,\tau_{l'},\sigma')} D^{k_1}_{m_1m_2}D^{k_2}_{m_2m_3}\Big(1-(-)^{k_1}\lambda_{m_1}\lambda_{m_2}\Big)\Big(1-(-)^{k_2}\lambda_{m_2}\lambda_{m_3}\Big)\\
&\mathrm{Tr}\Big(\mathrm{tr}(Q^{dd}_{n_7n_8}\tau_{l'}\sigma'\gamma_p)\Big)\Big\{\mathrm{Tr}\Big(\lambda_{m_3}\mathrm{tr}(U^{gd}_{m_3n_6}\tau_{l'}\sigma'\widetilde{U}^{\dagger de}_{n_5m_1}\gamma_{k_1}\gamma_{k_2}\gamma_p)\lambda_{m_1}\\
&\mathrm{tr}(U^{ea}_{m_1n_1}\tau_l\sigma\widetilde{U}^{\dagger af}_{n_2m_2}\gamma_{k_1}\gamma_q)\lambda_{m_2}\mathrm{tr}(U^{fa}_{m_2n_3}\tau_l\sigma\widetilde{U}^{\dagger ag}_{n_4m_3}\gamma_{k_2}\gamma_q)\Big)\\
&-(-)^l\mathrm{Tr}\Big(\lambda_{m_3}\mathrm{tr}(U^{gd}_{m_3n_6}\tau_{l'}\sigma'\widetilde{U}^{\dagger de}_{n_5m_1}\gamma_{k_1}\gamma_{k_2}\gamma_p)\lambda_{m_1}\mathrm{tr}(U^{ea}_{m_1n_1}\tau_l\sigma\widetilde{U}^{\dagger af}_{n_2m_2}\gamma_{k_1}\gamma_q)\\
&\lambda_{m_2}\mathrm{tr}(U^{fa}_{m_2-n_3}\tau_l\sigma\widetilde{U}^{\dagger ag}_{-n_4m_3}\gamma_{k_2}\gamma_q)\Big)\Big\}\,\delta(n_1\mp n_2\pm n_3-n_4)\delta(n_5\mp n_6\pm n_7-n_8)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\left\langle S^1_{int}S^1_{int}\right\rangle \;\; = \;\; & \sum 4^2\mathcal{F}^2\nu^2\Gamma^q_{(n_1,n_2,n_3,\tau_l,\sigma)}\Gamma^p_{(n_5,n_6,n_7,\tau_{l'},\sigma')}D^{k_1}_{m_1m_2}D^{k_2}_{m_3m_4}\Big(1-(-)^{k_1}\lambda_{m_1}\lambda_{m_2}\Big)\Big(1-(-)^{k_2}\lambda_{m_3}\lambda_{m_4}\Big)\\
& \Big\{\mathrm{Tr}\Big(\lambda_{m_2}\mathrm{tr}(U^{gd}_{m_2n_2}\tau_l\sigma\widetilde{U}^{\dagger de}_{n_1m_1}\gamma_{k_1}\gamma_q)\lambda_{m_1}\mathrm{tr}(U^{ea}_{m_1n_6}\tau_{l'}\sigma'\widetilde{U}^{\dagger ag}_{n_5m_2}\gamma_{k_1}\gamma_p)\Big)\\
& -(-)^{l'}\mathrm{Tr}\Big(\lambda_{m_2}\mathrm{tr}(U^{gd}_{m_2n_2}\tau_l\sigma\widetilde{U}^{\dagger de}_{n_1m_1}\gamma_{k_1}\gamma_q)\lambda_{m_1}\mathrm{tr}(U^{ea}_{m_1-n_6}\tau_{l'}\sigma'\widetilde{U}^{\dagger ag}_{-n_5m_2}\gamma_{k_1}\gamma_p)\Big)\Big\}\\
& \Big\{\mathrm{Tr}\Big(\lambda_{m_4}\mathrm{tr}(U^{gd}_{m_4n_4}\tau_l\sigma\widetilde{U}^{\dagger de}_{n_3m_3}\gamma_{k_2}\gamma_q)\lambda_{m_3}\mathrm{tr}(U^{ea}_{m_3n_8}\tau_{l'}\sigma'\widetilde{U}^{\dagger ag}_{n_7m_4}\gamma_{k_2}\gamma_p)\Big)\\
& -(-)^{l'}\mathrm{Tr}\Big(\lambda_{m_4}\mathrm{tr}(U^{gd}_{m_4n_4}\tau_l\sigma\widetilde{U}^{\dagger de}_{n_3m_3}\gamma_{k_2}\gamma_q)\lambda_{m_3}\mathrm{tr}(U^{ea}_{m_3-n_8}\tau_{l'}\sigma'\widetilde{U}^{\dagger ag}_{-n_7m_4}\gamma_{k_2}\gamma_p)\Big)\Big\}\\
& \delta(n_1\mp n_2\pm n_3-n_4)\delta(n_5\mp n_6\pm n_7-n_8)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\left\langle S^1_{int}S^1_{int}S^2_{int}\right\rangle \;\; = \;\; & \sum \frac{4^5}{2}\mathcal{F}^3\nu^3\Gamma^{q_1}_{(n_1,n_2,n_3,\tau_l,\sigma)}\Gamma^{q_2}_{(n_5,n_6,n_7,\widetilde{\tau}_l,\widetilde{\sigma})}\Gamma^p_{(n_9,n_{10},n_{11},\tau_{l'},\sigma')}D^{k_1}_{m_1m_2}D^{k_2}_{m_2m_5}D^{k_3}_{m_3m_4}\\
& \Big(1-(-)^{k_1}\lambda_{m_1}\lambda_{m_2}\Big)\Big(1-(-)^{k_2}\lambda_{m_2}\lambda_{m_5}\Big)\Big(1-(-)^{k_3}\lambda_{m_3}\lambda_{m_4}\Big)\\
& \mathrm{Tr}\Big(\mathrm{tr}(Q^{dd}_{n_{11}n_{12}}\tau_{l'}\sigma'\gamma_p)\Big)\mathrm{Tr}\Big(\lambda_{m_3}\mathrm{tr}(U^{ea}_{m_3n_8}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger af}_{n_7m_4}\gamma_{k_3}\gamma_{q_1})\lambda_{m_4}\mathrm{tr}(U^{fg}_{m_4n_4}\tau_l\sigma\widetilde{U}^{\dagger ge}_{n_3m_3}\gamma_{k_2}\gamma_q)\Big)\\
& \Big\{\mathrm{Tr}\Big(\lambda_{m_1}\mathrm{tr}(U^{hg}_{m_1n_2}\tau_l\sigma\widetilde{U}^{\dagger gi}_{n_1m_2}\gamma_{k_1}\gamma_{q_1})\lambda_{m_2}\mathrm{tr}(U^{ia}_{m_2n_6}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger al}_{n_5m_5}\gamma_{k_2}\gamma_{q_2})\\
& \lambda_{m_5}\mathrm{tr}(U^{ld}_{m_5n_{10}}\tau_{l'}\sigma'\widetilde{U}^{\dagger dh}_{n_9m_1}\gamma_{k_1}\gamma_{k_2}\gamma_p)\Big)-(-)^{\widetilde{l}}\mathrm{Tr}\Big(\lambda_{m_1}\mathrm{tr}(U^{hg}_{m_1n_2}\tau_l\sigma\widetilde{U}^{\dagger gi}_{n_1m_2}\gamma_{k_1}\gamma_{q_1})\\
& \lambda_{m_2}\mathrm{tr}(U^{ia}_{m_2-n_6}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger al}_{-n_5m_5}\gamma_{k_2}\gamma_{q_2})\lambda_{m_5}\mathrm{tr}(U^{ld}_{m_5n_{10}}\tau_{l'}\sigma'\widetilde{U}^{\dagger dh}_{n_9m_1}\gamma_{k_1}\gamma_{k_2}\gamma_p)\Big)\\
& -(-)^{l}\mathrm{Tr}\Big(\lambda_{m_1}\mathrm{tr}(U^{hg}_{m_1-n_2}\tau_l\sigma\widetilde{U}^{\dagger gi}_{-n_1m_2}\gamma_{k_1}\gamma_{q_1})\lambda_{m_2}\mathrm{tr}(U^{ia}_{m_2n_6}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger al}_{n_5m_5}\gamma_{k_2}\gamma_{q_2})\\
& \lambda_{m_5}\mathrm{tr}(U^{ld}_{m_5n_{10}}\tau_{l'}\sigma'\widetilde{U}^{\dagger dh}_{n_9m_1}\gamma_{k_1}\gamma_{k_2}\gamma_p)\Big)+(-)^{l}(-)^{\widetilde{l}}\mathrm{Tr}\Big(\lambda_{m_1}\mathrm{tr}(U^{hg}_{m_1-n_2}\tau_l\sigma\widetilde{U}^{\dagger gi}_{-n_1m_2}\gamma_{k_1}\gamma_{q_1})\\
& \lambda_{m_2}\mathrm{tr}(U^{ia}_{m_2-n_6}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger al}_{-n_5m_5}\gamma_{k_2}\gamma_{q_2})\lambda_{m_5}\mathrm{tr}(U^{ld}_{m_5n_{10}}\tau_{l'}\sigma'\widetilde{U}^{\dagger dh}_{n_9m_1}\gamma_{k_1}\gamma_{k_2}\gamma_p)\Big)\Big\}\\
& \delta(n_1\mp n_2\pm n_3-n_4)\delta(n_5\mp n_6\pm n_7-n_8)\delta(n_9\mp n_{10}\pm n_{11}-n_{12})
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\left\langle S_{int}^2 S_{int}^1 S_{int}^2\right\rangle \;=\;& \sum \frac{4^5}{2}\mathcal{F}^3\nu^3\Gamma^{p_1}_{(n_1,n_2,n_3,\tau_l,\sigma)}\Gamma^{q}_{(n_5,n_6,n_7,\widetilde{\tau}_l,\widetilde{\sigma})}\Gamma^{p_2}_{(n_9,n_{10},n_{11},\tau_{l'},\sigma')}D^{k_1}_{m_1m_5}D^{k_2}_{m_5m_6}D^{k_3}_{m_2m_3}\\
&\Big(1-(-)^{k_1}\lambda_{m_1}\lambda_{m_5}\Big)\Big(1-(-)^{k_2}\lambda_{m_5}\lambda_{m_6}\Big)\Big(1-(-)^{k_3}\lambda_{m_2}\lambda_{m_3}\Big)\\
&\mathrm{Tr}\Big(\mathrm{tr}(Q^{aa}_{n_3n_4}\tau_l\sigma\gamma_{p_1})\Big)\mathrm{Tr}\Big(\mathrm{tr}(Q^{dd}_{n_{11}n_{12}}\tau_{l'}\sigma'\gamma_{p_2})\Big)\\
&\Big\{(-)^{\bar{l}}\mathrm{Tr}\Big(\lambda_{m_1}\mathrm{tr}(U^{ha}_{m_1n_1}\tau_l\sigma\widetilde{U}^{\dagger ai}_{n_2m_3}\gamma_{k_1}\gamma_{k_3}\gamma_{p_1})\lambda_{m_3}\mathrm{tr}(U^{ig}_{m_3n_8}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger gl}_{n_7m_2}\gamma_{k_3}\gamma_q)\delta^{lf}_{m_2m_5}\\
&\lambda_{m_5}\mathrm{tr}(U^{fg}_{m_5n_6}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger gj}_{n_5m_6}\gamma_{k_2}\gamma_q)\lambda_{m_6}\mathrm{tr}(U^{jd}_{m_6n_{10}}\tau_{l'}\sigma'\widetilde{U}^{\dagger dh}_{n_9m_1}\gamma_{k_1}\gamma_{k_2}\gamma_{p_2})\Big)\\
&(-)^{\widetilde{l}}(-)^{\bar{l}}\mathrm{Tr}\Big(\lambda_{m_1}\mathrm{tr}(U^{ha}_{m_1n_1}\tau_l\sigma\widetilde{U}^{\dagger ai}_{n_2m_3}\gamma_{k_1}\gamma_{k_3}\gamma_{p_1})\lambda_{m_3}\mathrm{tr}(U^{ig}_{m_3-n_8}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger gl}_{-n_7m_2}\gamma_{k_3}\gamma_q)\delta^{lf}_{m_2m_5}\\
&\lambda_{m_5}\mathrm{tr}(U^{fg}_{m_5n_6}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger gj}_{n_5m_6}\gamma_{k_2}\gamma_q)\lambda_{m_6}\mathrm{tr}(U^{jd}_{m_6n_{10}}\tau_{l'}\sigma'\widetilde{U}^{\dagger dh}_{n_9m_1}\gamma_{k_1}\gamma_{k_2}\gamma_{p_2})\Big)\\
&-\mathrm{Tr}\Big(\lambda_{m_1}\delta^{hl}_{m_1n_1}\mathrm{tr}(U^{lg}_{m_2n_8}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger gi}_{n_7m_3}\gamma_{k_3}\gamma_q)\lambda_{m_3}\mathrm{tr}(U^{ia}_{m_3n_2}\tau_l\sigma\widetilde{U}^{\dagger af}_{n_1m_5}\gamma_{k_1}\gamma_{k_3}\gamma_{p_1})\\
&\lambda_{m_5}\mathrm{tr}(U^{fg}_{m_5n_6}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger gj}_{n_5m_6}\gamma_{k_2}\gamma_q)\lambda_{m_6}\mathrm{tr}(U^{jd}_{m_6n_{10}}\tau_{l'}\sigma'\widetilde{U}^{\dagger dh}_{n_9m_1}\gamma_{k_1}\gamma_{k_2}\gamma_{p_2})\Big)\\
&-(-)^{\widetilde{l}}\mathrm{Tr}\Big(\lambda_{m_1}\delta^{hl}_{m_1n_1}\mathrm{tr}(U^{lg}_{m_2-n_8}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger gi}_{-n_7m_3}\gamma_{k_3}\gamma_q)\lambda_{m_3}\mathrm{tr}(U^{ia}_{m_3n_2}\tau_l\sigma\widetilde{U}^{\dagger af}_{n_1m_5}\gamma_{k_1}\gamma_{k_3}\gamma_{p_1})\\
&\lambda_{m_5}\mathrm{tr}(U^{fg}_{m_5n_6}\widetilde{\tau}_l\widetilde{\sigma}\widetilde{U}^{\dagger gj}_{n_5m_6}\gamma_{k_2}\gamma_q)\lambda_{m_6}\mathrm{tr}(U^{jd}_{m_6n_{10}}\tau_{l'}\sigma'\widetilde{U}^{\dagger dh}_{n_9m_1}\gamma_{k_1}\gamma_{k_2}\gamma_{p_2})\Big)\Big\}\\
&\delta(n_1\mp n_2\pm n_3-n_4)\delta(n_5\mp n_6\pm n_7-n_8)\delta(n_9\mp n_{10}\pm n_{11}-n_{12})
\end{aligned}
$$

Bibliography

[1] J.G. Bednorz, K.A. M*ü*ller, Z. Phys. B**64**, 189 (1986).

[2] P.A. Lee, Phys. Rev. Lett. **71**, 1887 (1993).

[3] A.A. Nersesyan, A.M. Tsvelik, and F. Wenger, Nucl. Phys. B **438**, 561 (1995).

[4] C. Pepin and P.A. Lee, Phys. Rev. Lett. **81**, 2779 (1998); Phys. Rev. B **63**, 054502 (2001).

[5] T. Senthil. M.P.A. Fisher, L. Balents and C. Nayak, Phys. Rev. Lett. **81**, 4704 (1998).

[6] R. Gade and F. Wegner, Nucl. Phys. B **360**, 213 (1991); R. Gade, Nucl. Phys. B **398**, 499 (1993).

[7] M. Fabrizio and C. Castellani, Nucl. Phys. B **583**, 542 (2000).

[8] T. Senthil and M.P.A. Fisher, Phys. Rev. B **60**, 6893 (1999).

[9] P.W. Anderson Phy. Rev. **109**, 1492 (1958).

[10] B. L. Altshuler, A. G. Aronov. In "Electron-Electron Interactions in Disordered Systems". Ed. by A. L. Efros and M. Pollak. Elsevier Science Publishers B.V., p. 1, 1985.

[11] B. L. Altshuler, A. G. Aronov, D. E. Khmelnitskii, A. I. Larkin. In "Quantum Theory of Solids". Ed. by I. M. Lifshits. Mir Publishers, Moscow 1982, p. 130.

[12] E. Abrahams, P.W. Anderson, D.C. Licciardello, T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

[13] F.J. Wegner Z. Phys. B **35**, 207 (1979).

[14] P.A. Lee, T.V. Ramakrishnan Rev. Mod. Phys. **57**, 287 (1985).

[15] A. A. Abrikosov, L. P. Gorkov, I. E. Dzyaloshinskii. "Methods of Quantum Field Theory in Statistical Physics", Pergamon Press, Oxford 1965.

[16] K.B. Efetov, A.I.Larkin, and D.E. Khmel'nitsky, Zh. Eksp. Teor. Fiz. **79**, 1120 (1979) [Sov. Phys. JETP **52**, 568 (1980)].

[17] E. Manousakis, Rev. Mod. Phys. **63**, 1 (1991).

[18] P. C. Hohenberg, Phys. Rev. **158**, 383 (1967); N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

[19] C.P.Poole Jr., "Handbooke of superconductivity", Accademic Press (2000).

[20] A. Altland and M.R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997); A. Altland, B.D. Simons, and M.R. Zirnbauer, cond-mat/0006326.

[21] A.C. Durst and P.A. Lee, Phys. Rev. B **62**, 1270 (2000).

[22] K.G. Wilson, J. Kogut., Phys. Rep. C **12**, 75 (1974).

[23] A.M. Polyakov, Phys. Lett. B **59**, 79 (1975).

[24] D.J. Amit, "Field Theory, the Renormalization Group and Critical Phenomena", (1984).

[25] S. Hikami, Phys. Lett. **98**B, 208 (1981).

[26] M. Fabrizio, L. Dell'Anna, C. Castellani, Phys. Rev. Lett. **88**, 076603 (2002).

[27] T. Fukui, cond-mat/9905388.

[28] T. Senthil, J.B. Marston, and M.P.A. Fisher, Phys. Rev. B **60**, 4245 (1999); I.A. Gruzberg, A.W.W. Ludwig, and N. Read, Phys. Rev. Lett. **82**, 4254 (1999).

[29] C. Castellani and G. Kotliar, Phys. Rev. B **36**, 7407 (1987).

[30] O. Motrunich, K. Damle, and D.A. Huse, Phys. Rev. B **65**, 064206 (2001).

[31] C. Mudry, S. Ryu, and A. Furusaki, cond-mat/0207723.

[32] A.M. Finkel'stein, Zh. Eksp. Teor. Fiz. **84**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)]; Z. Phys. B **56**, 189 (1984).

[33] A. M. Finkelstein., Zs. Phys. **B56**, 189 (1984).

[34] C. Castellani, C. Di Castro, P.A. Lee, M.Ma, S. Sorella, T. Tabet, Phys. Rev. B **30**, 1596 (1984).

[35] D.V. Khveshchenko, A.G. Yashenkin, and I.V. Gornyi, Phys. Rev. Lett. **86**, 4668 (2001).

[36] M. Jeng, A.W.W. Ludwig, T. Senthil, C. Chamon, unpublished.

[37] A. Polyakov, P.B. Wiegmann, Phys. Lett. **131B**, 121 (1983).

[38] E. Witten, Com. Math. Phys. **92**, 455 (1984).

[39] T. Fukui, cond-mat/0002002.